STRENGTH OF MATERIALS

[For Engineering Degree, Diploma and A.M.I.E. Students]

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By the Same Author

Design of Reinforced Concrete Structures
Design of Steel Structures
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Simple Stresses and Strains

§1. Introduction

Materials which we come across may be classified into elastic, plastic and rigid materials. An elastic material undergoes a deformation when subjected to an external loading such that, the deformation disappears on the removal of the loading. A plastic material undergoes a continuous deformation during the period of loading and the deformation is permanent and the material does not regain its original dimensions on the removal of the loading. A rigid material does not undergo any deformation when subjected to an external loading.

In practice no material is absolutely elastic nor plastic nor rigid. We attribute these properties when the deformations are within certain limits. Generally we handle a member in its elastic range. Structural members are all generally designed so as to remain in the elastic condition under the action of the working loads.

§2. Resistance to Deformation

A material when subjected to an external load system undergoes a deformation. Against this deformation the material will offer a resistance which tends to prevent the deformation. This resistance is offered by the material as long as the member is forced to remain in the deformed condition. This resistance is offered by the material by virtue of its *strength*. In the elastic stage, the resistance offered by the material is proportional to the deformation brought about on the material by the external loading. The material will have the ability to offer the necessary resistance when the deformation is within a certain limit. A loaded member remains in equilibrium when the resistance offered by the member against the deformation and the applied load are in equilibrium. When the member is incapable of offering the necessary resistance against the external forces, the deformation will continue leading to the failure of the member.

§3. Stress

The force of resistance offered by a body against the deformation is called the *stress*. The external force acting on the body is called the *load*. The load is *applied* on the body while the stress is *induced* in the material of the body.

Types of stresses. Fig. 1 (*a*) shows a rod of uniform sectional area A and subjected to axial loads P at the ends A and B.

Consider a section XX normal to the longitudinal axis of the member.

Let the member be taken to consist of two parts C and D into which it is divided by the section XX.

Let us consider the equilibrium of the part C.

See Fig. 1 (*b*). This part is subjected to the external load P at the end A. In order to keep this part in equilibrium, the part D offers a resistance or reaction R at the section XX. Similarly the part D [Fig 1(*c*)] is subjected to the external load P at the end B, and in order to keep this part in equilibrium, the part C offers a resistance R at the section XX. In other words, we may say, that section XX is offering the resistance R against a possible separation at the section XX. This resistance R against the deformation is the *stress.*

Obviously for the case mentioned above, the resistance R is equal and opposite to the load. If the resistance offered by the section

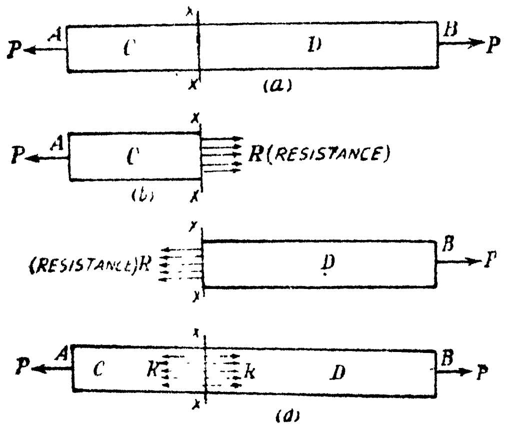


Fig. 1

against the deformation be assumed to be uniform across the section, the intensity of the resistance per unit area of the section is called the *intensity of stress* or *unit stress.* In common usage the word *stress* is used to mean the intensity of stress.

$$\therefore \quad \text{Intensity of stress} = p = \frac{R}{A} = \frac{P}{A}.$$

Let due to the application of the load the length of the member change from l to $l+dl$.

The ratio of the change in length to the original length of the member is called *strain.*

$$\therefore \quad \text{strain} = e = \frac{dl}{l}$$

A material is capable of offering the following types of stresses.

(*i*) *Tensile stress.* When the resistance offered by a section of a member is against an increase in length, the section is said to offer a *tensile stress.* For example, the stress offered by the section XX of the rod in Fig. 1 is a tensile stress. The intensity of the tensile stress is given by

$$p=\frac{R}{A}=\frac{P}{A}$$

The corresponding strain is called a *tensile strain.*

$$\therefore \quad \text{Tensile strain}=\frac{\text{Increase in length}}{\text{Original length}}$$

$$\therefore \quad e=\frac{dl}{l}.$$

(*ii*) *Compressive stress.* If the bar AB of Fig. 1 be subjected to pushing axial loads as shown in Fig. 2, a resistance is set up by any section such as XX against a decrease in length.

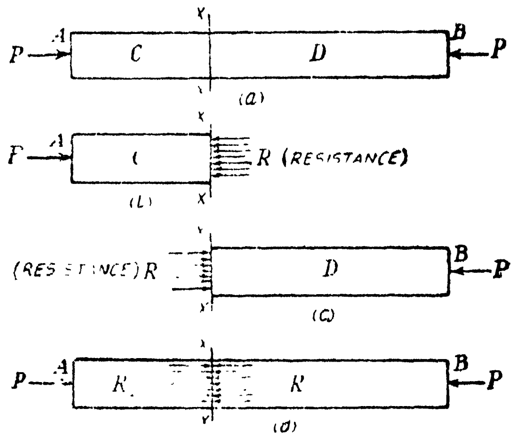


Fig. 2

This resistance is called a compressive resistance. The intensity of the compressive resistance or stress is given by

$$p=\frac{R}{A}=\frac{P}{A}$$

Let due to the external loading the length of the member decrease by dl. The ratio of the decrease in length to the original length is called a *compressive strain.*

$$\therefore \text{Compressive strain}=\frac{\text{Decrease in length}}{\text{Original length}}$$

$$\therefore \quad e=\frac{dl}{l}.$$

(*iii*) *Shear stress.* Fig. 3 shows a rectangular block of height l and length L and width unity.

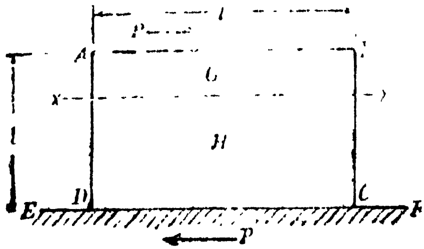


Fig. 3

Let the bottom face of the block be fixed to a surface EF. Let a force P be applied tangentially along the top face of the block. Such a force acting tangentially along a surface is called a shear force.

For the equilibrium of the block, the surface EF will offer a tangential reaction P equal and opposite to the applied tangential force P. Let the block be taken to consist of two parts G and H to which it is divided by a section XX. Consider the equilibrium of the part G.

See Fig. 4. In order the part G may not move from left to right, the part H will offer a resistance R along the section XX such that $R=P$.

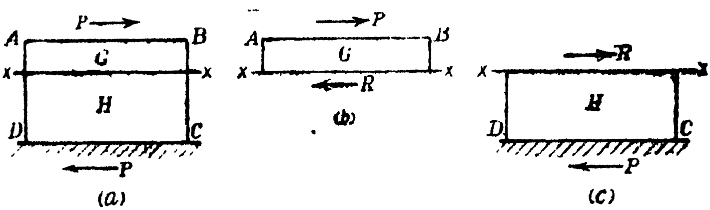


Fig. 4

Similarly, considering the equilibrium of the part H, we find that the part G will offer a resistance R along the section XX such that $R=P$.

The resistance R along the section XX is called a *shear resistance.*

Fig. 5 shows a failure at the section XX caused by the tangential loads acting on the top and bottom faces of the block. This type of failure is called a shear failure. In a shear failure, the two parts into which the block is separated, slide over each other. Hence if such a shear failure should not occur, the section XX must be able to offer tangential resistances along the section opposing the force P at the top face and the force P at the bottom face. For the equili-

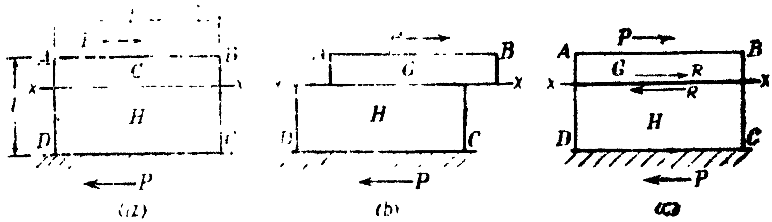


Fig. 5

brium of the system the shear resistance R should be equal to the tangential load P.

$$\therefore \qquad R=P.$$

The intensity of the shear resistance along the section XX is called the *shear stress*.

$$\therefore \qquad \text{Shear stress}=q=\frac{R}{A}=\frac{\text{Shear resistance}}{\text{Shear area}}=\frac{R}{L\times 1}=\frac{P}{L\times 1}$$

Shear deformation. Fig. 6 (a) shows a rectangular block subjected to shear forces P on its top and bottom faces.

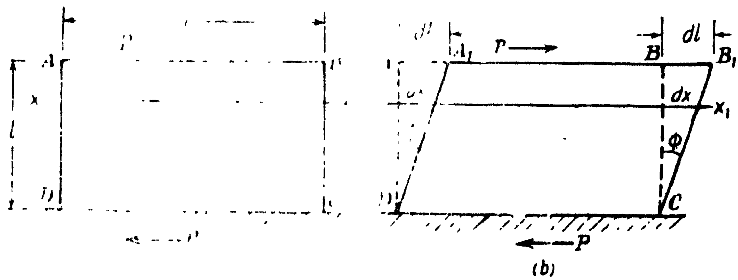


Fig. 6

When the block does not fail in shear, a shear deformation occurs as shown in Fig. 6 (b). If the bottom face of the block be fixed, it can be realized that the block has deformed to the position A_1B_1CD. Or we can say, that the face $ABCD$ has been distorted to the position A_1B_1CD through the angle $BCB_1=\phi$.

Let us now imagine that the block consists of a number of horizontal layers. These horizontal layers have undergone horizontal displacements by different amounts with respect to the bottom face. We can assume that the horizontal displacement of any horizontal layer is proportional to its distance from the lower face of the block.

Let the horizontal displacement of the upper face of the block be dl. Let the height of the block be l.

The ratio $\dfrac{dl}{l}=\dfrac{\text{Transverse displacement}}{\text{Distance from the lower face}}$ is called the *shear strain.* We could have considered any other horizontal layer say the layer XX which is at a distance x from the lower face. Let dx be the horizontal displacement of the layer XX.

Then shear strain $= \frac{dx}{x}$

Since ϕ is very small, $\phi = \tan\phi = \frac{dl}{l} =$ shear strain.

Hence, the angular deformation ϕ in radian measure represents the shear strain.

§4. Elastic Limit

A material is said to be elastic when it undergoes a deformation on the application of a loading such that the deformation disappears on the removal of the loading. When a member is subjected to an axial loading, its section will offer a resistance or stress. When the loading is removed, obviously the stress will vanish and the deformation will also vanish. But this is true when the deformation caused by the loading is within a certain limit. For every material the property of assuming or regaining its previous shape and size is exhibited on the removal of the loading, when the intensity of stress is within a certain limit called the *elastic limit*. If the loading is so large that the intensity of stress exceeds the elastic limit, the member loses to some extent its property of elasticity. If after exceeding the elastic limit the loading is removed, the member will not regain its original shape and a residual strain or permanent set remains.

Hooke's law. It is observed that when a material is loaded such that the intensity of stress is within a certain limit, the ratio of the intensity of stress to the corresponding strain is a constant which is characteristic of that material.

i.e., $\frac{\text{Intensity of stress}}{\text{strain}} = \text{constant}.$

In the case of axial loading, the ratio of the intensity of tensile or compressive stress to the corresponding strain is constant. This ratio is called Young's Modulus or Modulus of Elasticity, and is denoted by E

$$\therefore \qquad \frac{p}{e} = E$$

In the case of shear loading also, the ratio of the shear stress to the corresponding shear strain is found to be a constant when the shear deformation is within a certain limit. This ratio is called Shear Modulus or Modulus of Rigidity and is denoted by C, N or G.

§5. Units

In this book SI and MKS units are adopted to express quantities of various magnitudes. The following nomenclature is adopted :

KILO 10^3	*MILLI* 10^{-3}
MEGA 10^6	*MICRO* 10^{-6}
GIGA 10^9	*NANA* 10^{-9}
TERRA 10^{12}	*PICA* 10^{-12}

In the *SI units* force is generally expressed in *newtons*. The *kilo-newton* (*kN*) means 1000 *newtons*.

In the *MKS units* force is expressed in *kg* (the earlier practice was to express force in *kg wt*).

Stress intensity is expressed in various forms like,

$$newton/mm^2,\ newton/m^2,\ kg/cm^2,\ kg/m^2$$

$1\ N/metre^2 = 10^{-6}\ N/mm^2$

$1\ N/mm^2 = 10^6\ N/metre^2 = 1\ mega\ Newton/metre^2$

Problem 1. *A rod of steel is 25 mm in diameter and 200 cm long. The rod is subjected to an axial pull of 4500 kg. Find (i) the intensity of stress, (ii) the strain, and (iii) elongation. Take* $E = 2{\cdot}1 \times 10^6$ *kg/cm²*.

Solution. Diameter of rod $= d = 25\ mm = 2{\cdot}5\ cm$

Length of rod $= l = 200\ cm$

Pull $= P = 4500\ kg$

Area of the section $= A = \frac{\pi d^2}{4} = \frac{\pi \times 2{\cdot}5^2}{4} = 4{\cdot}909\ cm^2$

(*i*) *Intensity of stress* $= f = \frac{P}{A} = \frac{4500}{4{\cdot}909} = 916{\cdot}7\ kg/cm^2$

(*ii*) *Strain* $= e = \frac{f}{E} = \frac{916{\cdot}7}{2{\cdot}1 \times 10^6} = 0{\cdot}0004365$

(*iii*) *Elongation* $= dl =$ Strain × original length

$= 0{\cdot}0004365 \times 200\ cm$

$= 0{\cdot}0873\ cm.$

Problem 2. (SI) *A steel rod 25 mm in diameter and 2 metre long is subjected to an axial pull of 45 kN. Find (i) the intensity of stress, (ii) the strain, and (iii) elongation. Take* $E = 200\ GN/metre^2$.

Solution. Diameter of rod $= 25\ mm$

Length of rod $= 2\ metre$

Pull $P = 45\ kN = 45000\ N$

Area of rod $= \frac{\pi}{4}(25)^2 = 490{\cdot}9\ mm^2$

$= 490{\cdot}9 \times 10^{-6}\ metre^2$

(*i*) *Intensity of stress* $= f = \frac{P}{A}$

$$=\frac{45000}{490\cdot9\times10^{-6}}\ N/metre^2$$

$$=91\cdot67\times10^6\ N/metre^2=91\cdot67\ N/mm^2$$

$$=91\cdot67\ MN/metre^2$$

(ii) *Strain* $=e=\frac{f}{E}=\frac{91\cdot67\times10^6}{200\times10^9}=0\cdot0004583$

(iii) *Elongation* $=dl=$ Strain × original length

$=0\cdot0004583\times2$ *metre*

$=0\cdot0009166$ *metre*

$=0\cdot9166$ *mm.*

Problem 3. *A wooden tie is 7·5 cm wide, 15 cm deep and 1·50 metre long. It is subjected to an axial pull of 4500 kg. The stretch of the member is found to be 0·0638 cm. Find the Young's Modulus for the tie material.*

Solution. Area of tie $=A=7\cdot5\times15=112\cdot5\ cm^2$

Pull $=P=4500\ kg$

∴ Stress $=f=\frac{P}{A}=\frac{4500}{112\cdot5}=40\ kg/cm^2$

Strain $=e=\frac{\text{Change in length}}{\text{Original length}}=\frac{0\cdot0638}{150}=0\cdot0004253$

∴ *Young's Modulus* $=E=\frac{f}{e}$

$$=\frac{40}{0\cdot0004253}=94051\ kg/cm^2.$$

Problem 4. *A load of 400 kg has to be raised at the end of a steel wire. If the unit stress in the wire must not exceed 800 kg/cm² what is the minimum diameter required? What will be the extension of 3·50 metre length of wire? Take $E=2\times10^6\ kg/cm^2$.*

Solution. Load on the wire $=W=400\ kg$

Area required $=A=\frac{W}{f}=\frac{400}{800}=0\cdot5\ cm^2$

Let the diameter of the wire be d *cm*

∴ $\frac{\pi d^2}{4}=0\cdot5$

∴ $d=\sqrt{\frac{2}{\pi}}=0\cdot7979\ cm$

$=7\cdot979\ mm$

Extension $=dl=$ Strain × original length

$=\frac{f}{E}l$

$$= \frac{800}{2\times10^6} \times 350\ cm$$

$$= 0{\cdot}14\ cm$$

$$= 1{\cdot}4\ mm$$

Problem 5. (SI) *A wooden tie is 60 mm wide, 120 mm deep and 1·50 metres long. It is subjected to an axial pull of 30 kN. The stretch of the member is found to be 0·625 mm. Find the Young's Modulus for the tie material.*

Solution.

Area of the tie $= A = 60\times120 = 7200\ mm^2 = 7200\times10^{-6}\ metre^2$

Pull $= P = 30\ kN = 30{,}000\ N$

$$\therefore \quad \text{Stress} = f = \frac{P}{A} = \frac{30{,}000}{7200\times10^{-6}} = 4{\cdot}167\times10^6\ N/metre^2$$

$$\text{Strain} = e = \frac{\text{Change in length}}{\text{Original length}} = \frac{0{\cdot}625}{1{\cdot}5\times1000}$$

$$= 4{\cdot}167\times10^{-4}.$$

$$\therefore \quad \textit{Young's Modulus} = E = \frac{f}{e}$$

$$= \frac{4{\cdot}167\times10^6}{4{\cdot}167\times10^{-4}}\ N/metre^2$$

$$= 10^{10}\ N/metre^2$$

$$= 10\times10^9\ N/metre^2$$

$$= 10\ GN/metre^2.$$

Problem 6 (SI). *A 20 mm diameter brass rod was subjected to a tensile load of 40 kN. The extension of the rod was found to be 254 divisions in the 200 mm extensometre. If each division is equal to 0·001 mm, find the electric modulus of brass.*

Solution.

$$\text{Area of the rod} = A = \frac{\pi}{4}(20)^2\ mm^2$$

$$= 314{\cdot}16\ mm^2$$

$$= 314{\cdot}16\times10^{-6}\ metre^2$$

Pull $= P = 40\ kN = 40{,}000\ N$

$$\therefore \quad \text{Stress} = f = \frac{P}{A} = \frac{40{,}000}{314{\cdot}16\times10^{-6}}\ N/metre^2$$

$$= 1{\cdot}2732\times10^8\ N/metre^2$$

Length of specimen $= l = 200\ mm$

Extension $= dl = 254\times0{\cdot}001 = 0{\cdot}254\ mm$

$$\therefore \quad \text{Strain} = e = \frac{dl}{l} = \frac{0{\cdot}254}{200} = 0{\cdot}00127$$

$\therefore$ *Young's Modulus* $= E = \dfrac{f}{e} = \dfrac{1{\cdot}2732\times 10^8}{0{\cdot}00127}$ *N/metre*^{2}

$= 1002{\cdot}5\times 10^8$ *N/metre*^{2}

$= 100{\cdot}25\times 10^9$ *N/metre*^{2}

$= 100{\cdot}25$ *GN/metre*^{2}.

Problem 7. *A hollow steel column has to carry an axial load of 200,000 kg. If the external diameter of the column is 25 cm, nd the internal diameter. Take the ultimate stress for the steel column to be 4800 kg/cm² and allow a load factor of 4.*

Solution.

External diameter $= D = 25$ *cm*

Internal diameter $= d$

Load on the column $= W = 200{,}000$ *kg*

Ultimate stress $= 4800$ *kg/cm*^{2}

Factor of safety $= 4$

$\therefore$ Safe stress $= f = \dfrac{\text{Ultimate stress}}{\text{Factor of safety}}$

$= \dfrac{4800}{4} = 1200$ *kg/cm*^{2}

$\therefore$ Sectional area required $= A$

$= \dfrac{W}{f} = \dfrac{200{,}000}{1200} = 166{\cdot}67$ *cm*^{2}

$\therefore \quad \dfrac{\pi}{4}(25^2 - d^2) = 166{\cdot}67$

$625 - d^2 = 212{\cdot}21$

$d^2 = 412{\cdot}79$

$\therefore \quad d = 20{\cdot}30$ *cm.*

Problem 8. (SI) *The following data refer to a tensile test conducted on a mild steel bar.*

(*i*) *Diameter of the steel bar* $= 30$ *mm*

(*ii*) *Gauge length* $= 200$ *mm*

(*iii*) *Extension at a load of 100 kN* is $= 0{\cdot}139$ *mm*

(*iv*) *Load at elastic limit* $= 230$ *kN*

(*v*) *Maximum load* $= 360$ *kN*

(*vi*) *Total extension* $= 56$ *mm*

(*vii*) *Diameter of the rod at failure* $= 22{\cdot}25$ *mm*

Calculate (*a*) *The Young's modulus* (*b*) *The stress at elastic limit* (*c*) *The percentage elongation and* (*d*) *The percentage decrease in area.*

Solution.

Area of the rod $= A = \frac{\pi}{4}(30)^2\ mm^2 = 706{\cdot}86\ mm^2$

$= 706{\cdot}86 \times 10^{-6}\ metre^2$

(*a*) *Young's Modulus*

$$\text{Stress} = f = \frac{P}{A} = \frac{100 \times 1000}{706{\cdot}86 \times 10^{-6}}\ N/metre^2$$

$$= 141{\cdot}47 \times 10^6\ N/metre^2$$

$$\text{Strain} = e = \frac{dl}{l} = \frac{0{\cdot}139}{200} = 0{\cdot}000695$$

∴ Young's Modulus

$$E = \frac{f}{E} = \frac{141{\cdot}47 \times 10^6}{0{\cdot}000695}\ N/metre^2$$

$$= 203{\cdot}55 \times 10^9\ N/metre^2$$

$$= 203{\cdot}55\ GN/metre^2$$

(*c*) *Percentage elongation*

$$= \frac{\text{Increase in length}}{\text{Original length}} \times 100\%$$

$$= \frac{56}{200} \times 100\%$$

$$= 28\%.$$

(*d*) *Percentage decrease in area*

$$= \frac{\frac{\pi}{4}(d^2 - d'^2)}{\frac{\pi d^2}{4}} \times 100\%$$

$$= \frac{d^2 - d'^2}{d^2} \times 100\%$$

$$= \frac{30^2 - 22{\cdot}25^2}{30^2} \times 100\%$$

$$= 44{\cdot}99\%.$$

§6. Bars of Varying Sections

Fig. 7 shows a bar which consists of three lengths l_1, l_2 and l_3 with sectional areas A_1, A_2 and A_3 and subjected to an axial load P.

Even though the total force on each section is the same, the intensities of stress will be different for the three sections.

For instance,

Intensity of stress for the portion $AB = f_1 = \frac{P}{A_1}$

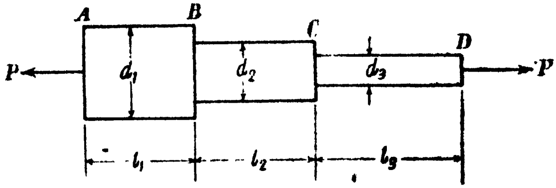


Fig. 7.

Intensity of stress for the portion $BC=f_2=\frac{P}{A_2}$

and, Intensity of stress for the portion $CD=f_3=\frac{P}{A_3}$

Let E be the Young's Modulus.

$\therefore$ Strain of the part $AB=e_1=\frac{f_1}{E}$

Strain of the part $BC=e_2=\frac{f_2}{E}$

and Strain of the part $CD=e_3=\frac{f_3}{E}$

$\therefore$ Change in length of the part $AB=dl_1=e_1l_1$

Change in length of the part $BC=dl_2=e_2l_2$

and Change in length of the part $CD=dl_3=e_3l_3$

$\therefore$ Total change in length of the bar$=dl=dl_1+dl_2+dl_3$.

Problem 9. *Fig. 8 shows a bar consisting of three lengths. Find the stresses in the three parts and the total extension of the bar for an axial pull of 4 tonnes. Take* $E=2\times10^6$ *kg/cm²*.

Solution. Load $P=4$ tonnes$=4000$ *kg*.

Intensity of stress on the part AB

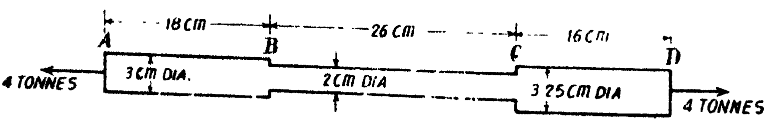


Fig. 8

$$=f_1=\frac{P}{A_1}$$

$$=\frac{4000}{\frac{\pi}{4}(3)^2}\ kg/cm^2.$$

$$=565\cdot8\ kg/cm^2.$$

Intensity of stress on the part $BC = f_2 = \dfrac{P}{A_2}$

$$\therefore \quad f_2 = \frac{4000}{\frac{\pi}{4}(2)^2} = 1274 \text{ kg./cm}^2.$$

Intensity of stress on the part $CD = f_3 = \dfrac{P}{A_3}$

$$\therefore \quad f_3 = \frac{4000}{\frac{\pi}{4}(3{\cdot}25)^2} = 480 \text{ kg./cm}^2.$$

Total extension $\quad dl = dl_1 + dl_2 + dl_3$

$$= \frac{f_1}{E} l_1 + \frac{f_2}{E} l_2 + \frac{f_3}{E} . l_3$$

$$= \frac{1}{E}(f_1 l_1 + f_2 l_2 + f_3 l_3)$$

$$= \frac{1}{2 \times 10^6}[565{\cdot}8 \times 18 + 1274 \times 26 + 480 \times 16] \text{ cm.}$$

$$= 0{\cdot}0255 \text{ cm.}$$

Problem 10. *A brass bar having a cross-sectional area of 10 sq. cm. is subjected to axial forces shown in fig. 9. Find the total change in length of the bar. Take* $E_b = 1{\cdot}05 \times 10^6$ *kg./cm*^{2}.

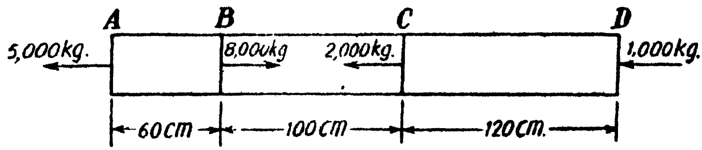


Fig. 9.

Solution. *Part AB.* The section of the bar in this part is subjected to a tension of 5000 *kg.*

$\therefore$ Extension of $AB = \dfrac{P_1}{AE} l_1$

$$= \frac{5000 \times 60}{10 \times 1{\cdot}05 \times 10^6} \text{ cm.}$$

$$= \frac{1}{35} \text{ cm. (Extension)}$$

Part BC. The section of the bar in this part is subjected to a compression of 8000 − 5000 = 3000 *kg.*

$\therefore$ Contraction of $BC = \dfrac{P_2 l_2}{AE}$

$$=\frac{3000\times100}{10\times1\cdot05\times10^6}\ cm.$$

$$=\frac{1}{35}cm.\ (contraction)$$

Part CD. The section of the bar in this part is subjected to a compression of 1000 *kg*.

∴ Contraction of *CD*

$$=\frac{P_3l_3}{AE}$$

$$=\frac{1000\times120}{10\times1\cdot05\times10^6}\ cm.$$

$$=\frac{6}{525}\ cm.=0\cdot0114\ cm.\ (contraction)$$

∴ Change in length of the member

$$=\frac{1}{35}-\frac{1}{35}-0\cdot0114=-0\cdot0114\ cm.$$

$$=0\cdot0114\ cm.\ (Decrease\ in\ length).$$

Problem 11 (SI). *A member ABCD is subjected to poin loads P_1, P_2, P_3, and P_4 as shown in Fig. 10. Calculate the force P_3 necessary for equilibrium if $P_1=120$ kN, $P_2=220$ kN and $P_4=160$ kN. Determine also the net change in length of the member. Take $E=200$ GN/m².*

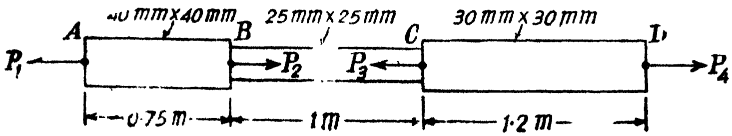


Fig. 10.

Solution. Resolving the forces on the rod along its axis, we have

$$P_1+P_3=P_2+P_4$$

$$120+P_3=220+160$$

$$\therefore\quad P_3=260\ kN$$

Part AB

Force on the cross-section $=P_1=120$ kN $=120{,}000$ *N* (*tensile*)

$$\therefore\quad Extension\ of\ AB=+\frac{120{,}000\times0\cdot75}{[1600\times10^{-6}][200\times10^9]}\ metre$$

$$=+0\cdot00028\ metre=+0\cdot28\ mm$$

Part BC

Force on the cross-section $=P_1-P_2=120-220=-100$ kN

$=-100{,}000\ N$ (*compressive*)

$$\therefore \quad \textit{Contraction of } BC=-\frac{100000\times 1}{[625\times 10^{-6}][200\times 10^{9}]}\ \textit{metre}$$

$=-0{\cdot}0008$ *metre* $=-0{\cdot}80$ *mm*

Part CD

Force on the cross-section $=P_4=160$ kN

$=160000\ N$ (*tensile*)

$$\therefore \quad \textit{Extension}=+\frac{160000\times 1.2}{[900\times 10^{-6}][200\times 10^{9}]}\ \textit{metre}$$

$=+0{\cdot}00107$ *metre* $=+1{\cdot}07$ *mm*

∴ *Net change in length of the member*

$=+0{\cdot}28-0{\cdot}80+1{\cdot}07$

$=+0{\cdot}55$ *mm* (*extension*).

Problem 12. *A member ABCD is subjected to point loads, P_1, P_2, P_3 and P_4 as shown in figure 11. Calculate the force P_2 necessary for equilibrium, if $P_1=4500$ kg., $P_3=45000$ kg., $P_4=13000$ kg. Determine the total elongation of the member, assuming the modulus of elasticity to be $2{\cdot}1\times 10^6$ kg./cm^2.*

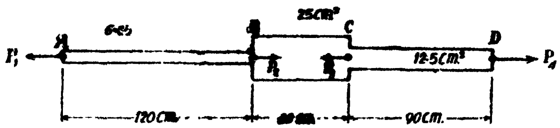


Fig. 11

Solution. Resolving the forces on the rod along its axis, we have,

$$P_1+P_3=P_2+P_4$$

But $P_1=4500$ *kg.*, $P_3=45000$ *kg.* and $P_4=13000$ *kg.*

$\therefore \quad 4500+45000=P_2+13000$

$\therefore \quad P_2=36500$ *kg.*

Part AB

Force on the cross-section $=P_1$

$=4500$ *kg.* (*tensile*)

$$\therefore \quad \text{Extension of } AB=\frac{P_1 l_1}{A_1 E}$$

$$=\frac{4500\times 120}{6{\cdot}25\ E}$$

$$=\frac{86400}{E}\ (\textit{extension})$$

Part BC

Force on the cross-section $=P_1-P_2$

$$=4500-36500 \; kg.$$

$$= -32000 \; kg. \; (compressive)$$

$\therefore$ Contraction of $BC = -\dfrac{32000\times 60}{25\,E}\, cm.$

$$= -\frac{76800}{E}\, cm. \; (contraction)$$

Part CD

Force on the cross-section $=P_4=13000$ *kg. (tensile)*

$\therefore$ Extension of $CD=\dfrac{P_4 l_4}{A_4 E}$

$$=\frac{13000\times 90}{12{\cdot}5\,E}\, cm.$$

$$=\frac{93600}{E}\, cm. \; (extension)$$

$\therefore$ Total change in length of the member

$$=\frac{86400}{E}+\frac{76800}{E}+\frac{93600}{E}\, cm.$$

$$=\frac{103200}{E}\, cm.$$

$$=\frac{103200}{2{\cdot}1\times 10^6}\, cm.$$

$$=0{\cdot}049143 \; cm. \; (extension)$$

Problem 13. *The bar shown in fig. 12 is subjected to a tensile load of 15200 kg. Find the diameter of the middle portion if the stress there is to be limited to 1400 kg./cm².*

Find also the length of the middle portion if the total elongation of the bar is to be 0·016 cm. Take $E=2\times 10^6$ *kg./cm².*

Solution. Area required for the middle portion

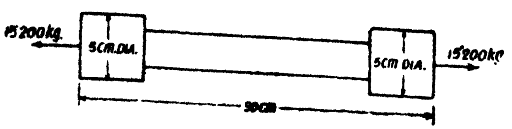


Fig. 12.

$$=A=\frac{\text{Load}}{\text{Stress}}=\frac{p}{f}$$

$$=\frac{15200}{1400}\, cm^2.$$

$$=\frac{76}{7}\, cm^2.$$

Let the diameter of the middle portion be d *cms.*

$$\therefore \quad \frac{\pi d^2}{4} = \frac{76}{7}$$

$$\therefore \quad d^2 = \frac{76}{7} \times \frac{4}{\pi}$$

$$\therefore \quad d = 3.718 \text{ cms}$$

Let the length of the middle portion be x *cms.*

Stress in the end portions

$$= f' = \frac{15200}{\left(\frac{\pi \times 5^2}{4}\right)} \text{ kg./cm}^2.$$

$$= 774.1 \text{ kg/cm}^2.$$

Total extension of the rod $= 0.016$ *cm.*

$\therefore$ Extension of the end portions + Extension of the middle portion $= 0.016$ *cm*

$$\therefore \frac{f'}{E}(30 - x) + \frac{f}{E}x = 0.016$$

$$\therefore \quad f'(30 - x) + fx = 0.016 \times E$$

$$= 0.016 \times 2 \times 10^6$$

$$= 32000$$

$$\therefore \quad 774.1(30 - x) + 1400x = 32000$$

$$\therefore \quad 23223 - 774.1x + 1400x = 32000$$

$$\therefore \quad 625.9x = 8777$$

$$\therefore \quad x = \frac{8777}{625.9} \text{ cms.}$$

$$\therefore \quad x = 14.03 \text{ cms.}$$

Problem 14. *A gradually applied load* $W = \frac{1}{2}$ *tonne is suspended by ropes as shown in Fig. 13 (a) and (b). In both cases, the ropes have a cross-sectional area of 5 cm² and the value of E is 9800 kg/cm².*

In (a) the rope ABC is continuous and W is suspended from a small frictionless pulley. In (b) AB and CB are separate ropes joined to a block from which W is suspended in such a way that both ropes stretch by the same amount.

Find, for both (a) and (b), the stresses in the ropes and find the downward movement of the pulley and the block due to the gradual application of the load. (*London University*)

Solution. (*a*) In this arrangement the tension in the rope is uniform since the pulley is smooth. Let the tension in the rope be P *kg*. Hence for the equilibrium of the system,

$$2P=W$$

$$\therefore\ P=\frac{W}{2}=\frac{1}{4}\ tonne$$

$$=250\ kg.$$

$\therefore$ Intensity of stress on the rope section

$$=p=\frac{P}{A}$$

$$=\frac{250}{8}\ kg/cm^2.$$

$$=31{\cdot}25\ kg/cm^2.$$

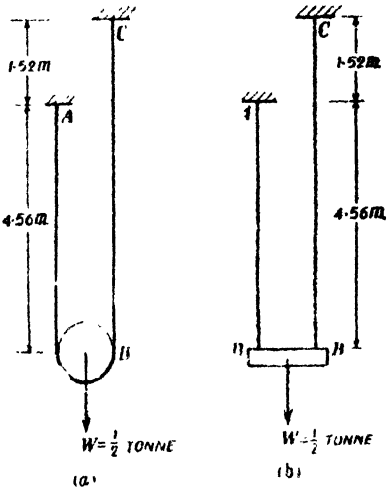


Fig. 13.

Length of the rope $=4{\cdot}56+4{\cdot}56+1{\cdot}52=10{\cdot}64\ m.$

$\therefore$ The increase in length of the rope

$$=\frac{p}{E}\cdot l$$

$$=\frac{31{\cdot}25}{9800}\times10{\cdot}64\times100\ cm.$$

$$=3{\cdot}394\ cm$$

Let the downward movement of the pulley be δ *cm*.

$$\therefore\quad 2\delta=3{\cdot}394\ cm$$

$$\therefore\quad \delta=1{\cdot}697\ cm.$$

(*b*) In the second arrangement the loads shared by the two ropes AB and CB should be such that they extend by the same amount. Let the stress intensity on the sections of the ropes AB and CB be p_1 and p_2 respectively.

Equating the extensions of the two ropes we have,

$$\frac{p_1}{E}l_1=\frac{p_2}{E}l_2$$

$$\therefore\quad \frac{p_1}{p_2}=\frac{l_2}{l_1}=\frac{6{\cdot}08}{4{\cdot}56}=\frac{4}{3}$$

$$\therefore\quad p_1=\frac{4}{3}p_2$$

But tension in AB+tension in $CB=W$

$$\therefore\quad p_1A+p_2A=500\ kg.$$

$\therefore \quad p_1+p_2=\frac{500}{8}$

$\therefore \quad \frac{4}{3}\cdot p_2+p_2=\frac{500}{8}$

$\therefore \quad \frac{7}{3}\,p_2=\frac{500}{8}$

$\therefore \quad p_2=26{\cdot}8\ kg/cm^2.$

and $\quad p_1=\frac{4}{3}\times 26{\cdot}8=35{\cdot}7\ kg/cm^2.$

$\therefore$ Downward movement of the pulley

$\doteq$ Extension of the rope AB or CB

$$=\frac{p_1}{E}\cdot l_1=\frac{35{\cdot}7}{9800}\times 4{\cdot}56\times 100\ cm.$$

$$=1{\cdot}662\ cm.$$

Problem 15. *A steel tie rod 4 cm. in diameter and 2 m. long is subjected to a pull of 8t. To what length the bar should be bored centrally so that the total extension will increase by 20% under the same pull, the bore being 2 cm. diameter.*

Take $E=2000\ t/cm^2$.

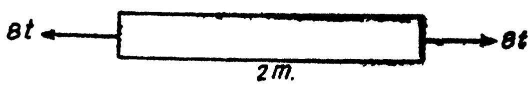


Fig. 14.

Solution.

$$A=\frac{\pi}{4}(4)^2=4\pi\ cm^2.$$

$$f=\frac{8}{4\pi}=\frac{2}{\pi}\ t/cm^2$$

$$\delta=\frac{f}{E}\cdot L=\frac{2}{\pi\times 2000}\times 200=\frac{1}{5\pi}$$

Extension after the bore is made $=1{\cdot}2\times\frac{1}{5\pi}$

$$=\frac{6}{25\pi}\ cm.$$

Let the bar be bored to a length of l metres. Area at the reduced section

$$A'=4\pi-\frac{\pi}{4}(2)^2$$

$$=4\pi-\pi=3\pi\ cm^2.$$

Fig. 15.

$\therefore$ Extension of the rod

$$=\frac{2}{\pi\times2000}(2-l)\,100+\frac{8l}{3\pi\times2000}\times100\quad\frac{6}{25\pi}$$

$$\frac{(2-l)}{10\pi}+\frac{}{15\pi}\,l=\frac{6}{25\pi}$$

$$\frac{2-l}{10}+\frac{2l}{15}=\frac{6}{25}$$

$$30-15l+20l=36$$

$$5l=6$$

$$l=1\cdot2\ m.$$

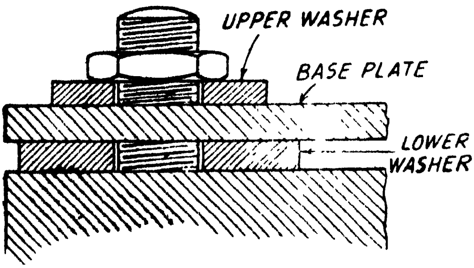


Fig. 16

Problem 16 *A rectangular base plate is fixed at each of its four corners by a 20 mm. diameter bolt and nut as shown in Fig. 16.*

The plate rests on washers of 22 mm. internal diameter and 50 mm. external diameter. Upper washers which are placed between nut and plate are of 22 mm. internal diameter and 44 mm. external diameter.

If the base plate carries a load of 12 t (including self-weight which is equally distributed at the four corners) calculate the stress on the l er washers before the nuts are tightened.

What would be the stress in the upper and lower washers when the nuts are tightened so as to produce a tension of 0·5t on each bolt ?

(AMIE May 1971)

Solution. Area of the lower washer $=\frac{\pi}{4}(5^2-2\,2^2)$ *cm.*^{2}

$=15\cdot83$ *cm.*^{2}

Load transmitted to one lower washer $=\frac{12}{4}\quad 3t=3000$ *kg.*

$\therefore$ Stress intensity in the lower washer $=\frac{3000}{15\,83}$ *kg./cm.*^{2}

$=189\cdot4$ *kg./cm.*^{2}

When the nuts are tightened the compressive load in the upper washer = tension in the bolt $=0\cdot5t=500$ *kg.*

Area of the upper washer $=\frac{\pi}{4}(4\cdot4^2-2\cdot2^2)$ *cm*^{2}.

$=11\cdot40$ *cm*^{2}.

$\therefore$ Stress intensity in the lower washer $=\frac{500}{11\cdot40}$ *kg./cm*^{2}.

$=43\cdot85$ *kg./cm* ^{2}

Now the compressive load on the lower washer

$= 3000+500=3500\ kg.$

$\therefore$ Stress intensity in the lower washer

$$\frac{3500}{15{\cdot}83}=221\ kg./cm^2.$$

Problem 17. *Fig. 17 shows a rigid bar ABC hinged at A and suspended at two points B and C by two bars BD and CE made of aluminium and steel respectively. The bar carries a load of 2000 kg. midway between B and C. The cross-sectional area of the aluminium bar BD is 3 sq. mm. and that of the steel bar CE is 2 sq. mm. Determine the load taken by the two bars BD and CE.*

(A.M.I.E. Nov., 1965)

Modulus of Elasticity for aluminium $E_{al}=7000\ kg/mm.^2$

Modulus of Elasticity for steel $E_s=20{,}000\ kg./mm.^2$

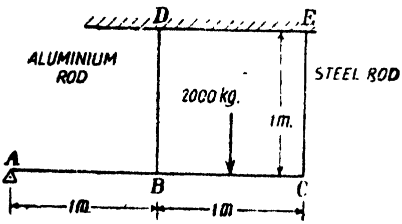


Fig. 17.

Solution. A rigid rod will not bend. It will remain straight. But the aluminium and steel rods will extend due to their elasticity. Fig. 17 shows the position of the rigid bar after the aluminium and steel rods have undergone their extensions.

Let the tensions in the aluminium and steel rods be R *kg.* and S *kg.* respectively.

Since the rigid bar remains straight, the extensions of CE and BD are proportional to their distances from A.

Let the extension of BD be $BB_1=\delta$

$$\therefore\ \text{Extension of } CE=CC_1=\frac{2}{1}\times\delta=2\delta$$

$$\text{But extension of } BD=\delta=\frac{Rl}{A_aE_a}$$

$$\text{and extension of } CE=2\delta=\frac{S}{A_sE_s}\cdot l$$

$$\therefore\ \delta=\frac{Rl}{A_aE_a}=\frac{Sl}{2A_sE_s}$$

$$\therefore\ \frac{R}{S}=\frac{1}{2}\cdot\frac{A_a}{A_s}\cdot\frac{E_a}{E_s}$$

$$\therefore\ \frac{R}{S}=\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{7000}{20000}$$

$$\therefore\ \frac{R}{S}=\frac{21}{80}$$

$$\therefore\ R=\frac{21}{80}S \quad \ldots(i)$$

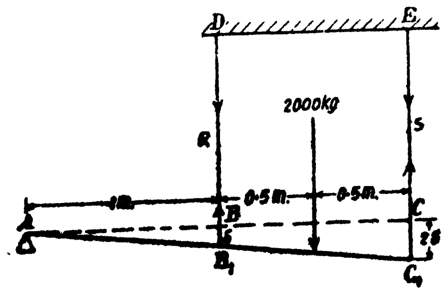


Fig. 18.

Now consider the equilibrium of the rigid bar.

Taking moments of the forces on the rigid bar, about the end A, we have,

$$R \times 1 + S \times 2 = 2000 \times 1 \cdot 5$$

$$\therefore \quad R + 2S = 3000 \qquad \ldots(ii)$$

But $$R = \frac{21}{80} S \qquad \ldots(i)$$

$$\therefore \quad \frac{21}{80} S + 2S = 3000$$

$$\therefore \quad \frac{181}{80} S = 3{,}000$$

$$\therefore \quad S = \frac{3000 \times 80}{181} = 1326 \text{ kg. (tensile)}$$

$$\therefore \quad S = \frac{21}{80} \times 1326 = 348 \text{ kg. (tensile)}$$

§7. Extension of a tapering rod

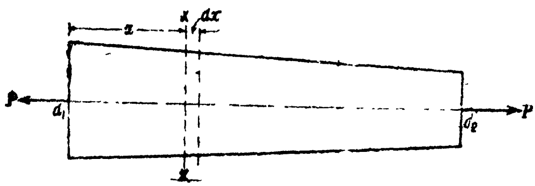


Fig. 19

Fig. 19 shows a bar uniformly tapering from a diameter d_1 at one end to a diameter d_2 at the other end.

Let the member be subjected to an axial tensile load P.

Consider an elemental length dx of the bar at a distance x from the larger end. Let the diameter of the bar be d' at distance x from the larger end.

$$\therefore \quad d' = d_1 - \left(\frac{d_1 - d_2}{l} \right) x$$

Let $$\left(\frac{d_1 - d_2}{l} \right) = k$$

$$\therefore \quad d' = d_1 - kx.$$

$\therefore$ Cross-sectional area at distance x from the larger end

$$= A' = \frac{\pi d'^2}{4} = \frac{\pi}{4}(d_1 - kx)^2$$

Intensity of stress on the section

$$= p' = \frac{P}{A'} = \frac{4P}{\pi(d_1 - kx)^2}$$

$\therefore$ Strain $= e' = \dfrac{p'}{E} = \dfrac{4P}{\pi E(d_1 - kx)^2}$

$\therefore$ Extension of the elemental length dx

$$= e'dx$$

$$= \frac{4P}{\pi E(d_1 - kx)^2} dx$$

$\therefore$ Total extension of the bar

$$= \delta = \frac{4P}{\pi E} \int_0^l \frac{dx}{(d_1 - kx)^2}$$

$$= \frac{4P}{\pi Ek} \left\{ \frac{1}{d_1 - kx} \right\}_0^l$$

$$= \frac{4P}{\pi Ek} \left\{ \frac{1}{d_1 - kl} - \frac{1}{d_1} \right\}$$

But $\quad k = \dfrac{d_1 - d_2}{l}$

$$\delta = \frac{4Pl}{\pi E(d_1 - d_2)} \left\{ \frac{1}{d_1 - d_1 + d_2} - \frac{1}{d_1} \right\}$$

$$= \frac{4Pl}{\pi E(d_1 - d_2)} \left(\frac{1}{d_2} - \frac{1}{d_1} \right)$$

$$= \frac{4Pl}{\pi E(d_1 - d_2)} \cdot \frac{d_1 - d_2}{d_1 d_2}$$

$\therefore \quad \delta = \dfrac{4Pl}{\pi E d_1 d_2}$

For the particular case when the rod is of uniform diameter,

$$d_1 = d_2 = d$$

and for this case

$$\delta = \frac{4Pl}{\pi E d^2}$$

Problem 18. *A rod tapers uniformly from 30 mm. to 15 mm. diameter in a length of 30 centimeters. If the rod be subjected to an axial load of 600 kg, find the extension of the rod. Take $E = 2 \times 10^6$ kg./cm^2.*

Solution. The extension of the rod is given by,

$$\delta = \frac{4Pl}{\pi E d_1 d_2}$$

In our case, $P=600\ kg.$

$l=30\ cm.$

$d_1=30\ mm.=3\ cm.$

$d_2=15\ mm=1{\cdot}5\ cm.$

and $E=2\times10^6\ kg./cm^2.$

$$\delta=\frac{4\times600\times30}{\pi\times2\times10^6\times3\times1{\cdot}5}\ cm$$

$$=0{\cdot}002547\ cm$$

Problem 19. *If a tension test bar is found to taper uniformly from $(D-a)$ cm. diameter to $(D+a)$ cm. diameter prove that the error involved in using the mean diameter to calculate the Young's Modulus is $\left(\frac{10\,a}{D}\right)^2$ percent.* *(A.M.I.E. May, 1965)*

Solution. Diameter at the larger end $=d_1=(D+a)$

Diameter at the smaller end $=d_2=(D-a)$

Let the length of the bar be l

Let the Young's Modulus be E

Let the extension of the member be δ

$$\therefore\quad \delta=\frac{4Pl}{\pi E d_1 d_2}$$

$$\therefore\quad E=\frac{4Pl}{\pi d_1 d_2 \delta}$$

But $d_1=D+a$ and $d_2=D-a$

$$\therefore\quad E=\frac{4Pl}{\pi(D^2-a^2)\delta}.$$

If the mean diameter be adopted let E' be the computed Young's Modulus. Obviously E' is erroneous.

$$\therefore\quad \delta=\frac{4Pl}{\pi D^2 E'}$$

$$\therefore\quad E'=\frac{4Pl}{\pi D^2\delta}$$

$\therefore$ Percentage error in the computation of the Young's Modulus, when the mean diameter is adopted

$$=\left(\frac{E-E'}{E}\right)100\ percent$$

$$=\frac{\frac{4Pl}{\pi(D^2-a^2)\delta}-\frac{4Pl}{\pi D^2\delta}}{\frac{4Pl}{\pi(D^2-a^2)\delta}}\times100\ percen$$

$$\frac{\frac{1}{D^2-a^2}-\frac{1}{D^2}}{\frac{1}{D^2-a^2}}\times 100 \textit{ percent}$$

$$=\frac{a^2}{D^2}\times 100 \textit{ percent}$$

$$=\left(\frac{10\,a}{D}\right)^2 \textit{percent.}$$

Problem 20. *A bar of steel is of length l and is of uniform thickness t. The width of the bar varies uniformly from a at one end to b at the other end. Find the extension of the rod when it carries an axial pull P.*

Solution. Fig. 20 shows the tapering rod.

Consider any section $X-X$ distant x from the bigger end.

Width of the section $= a - \frac{a-b}{l}x$

$= a - kx$ where $k = \frac{a-b}{l}$

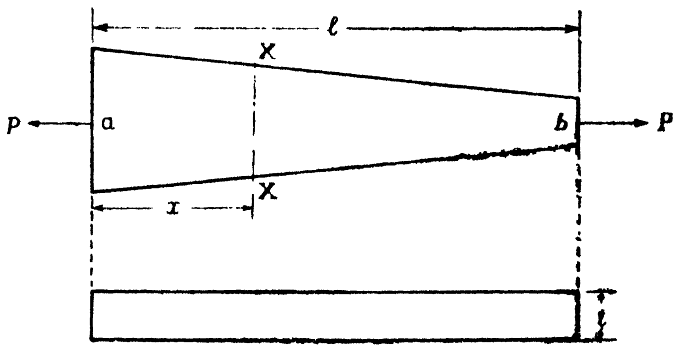


Fig. 20

Thickness at the section $= t$

$\therefore$ Area of the section $= t\,(a-kx)$

$\therefore$ Stress on the section $= \frac{P}{t\,(a-kx)}$

$\therefore$ Extension of an elemental length dx

$$= \frac{P\,dx}{t\,(a-kx)\,E}$$

$\therefore$ Total extension of the rod $=\delta=\dfrac{P}{tE}\displaystyle\int_0^l \dfrac{dx}{a-kx}$

$$=-\frac{P}{tE}\cdot\frac{1}{K}\cdot\log_e\Big[(a-kx)\Big]_0^l$$

$$=-\frac{P}{tEk}\Big[\log_e(a-Kl)-\log_e a\Big]$$

$$=\frac{P}{tEk}\log_e\left\{\frac{a}{a-Kl}\right\}$$

But $\qquad k=\dfrac{a-b}{l}$

$$\delta=\frac{Pl}{Et\,(a-b)}\log_e\frac{a}{b}.$$

Problem 21. *A straight bar of steel rectangular in section is 300 cms. long and is of uniform thickness 1·5 cm. The width of the rod varies uniformly from 10 cms. at one end to 4 cms. at the other. If the rod is subjected to an axial tensile load of 3000 kg., find the extension of the rod. Take $E_s=2\times10^6$ kg./cm^2.*

Solution.

Extension of the rod $\delta=\dfrac{Pl}{Et\,(a-b)}\log_e\dfrac{a}{b}$

In our case $P=3000$ kg.

$l=300$ cms.

$t=1\cdot5$ cms.

$a=10$ cms.

$b=4$ cms

and $\qquad E=2\times10^6$ kg./cm^2.

$\therefore \qquad \delta=\dfrac{3000\times300}{2\times10^6\times1\cdot5\,(10-4)}\log_e.\dfrac{10}{4}$ cm.

$$=\frac{3000\times300\times0\cdot9163}{2\times10^6\times1\cdot5\times6}\text{ cm.}=0\cdot045815\text{ cm.}$$

§8. Bars of Composite Sections

Suppose the cross-section of a member consists of different materials, the load applied on the member will be shared by the various components of the section. For instance suppose a column consists of an outer tube of area A_1 and Young's Modulus E_1 and an inner tube of area A_2 and Young's Modulus E_2. Let the length of the column be l. Suppose a load P be applied on the column. Let the unit stresses on the outer and inner tube sections be p_1 and p_2.

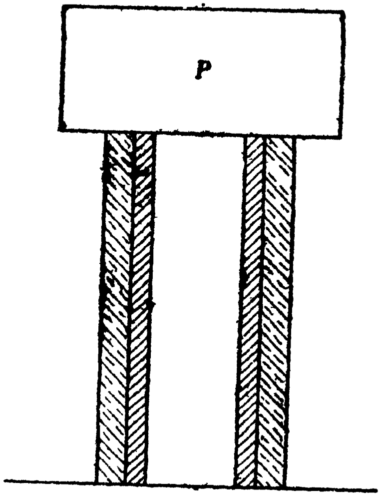


Fig. 21

Load on outer tube + load on the inner tube

= Total load on the column

$$\therefore \quad p_1 A_1 + p_2 A_2 = P \quad \ldots(i)$$

Let dl be the decrease in length of the column.

$\therefore$ Strain of each tube

$$= e = \frac{dl}{l}$$

But
$$e = \frac{p_1}{E_1} = \frac{p_2}{E_2} \quad \ldots(ii)$$

From equations (i) and (ii) the stresses p_1 and p_2 and may be computed. From Eqn. (ii) $p_1 = \frac{E_1}{E_2} p_2$*.

Problem 22. *A compound tube consists of a steel tube 15 cms. internal diameter and 1 cm. thickness and an outer brass tube 17 cm. internal diameter and 1 cm. thickness. The two tubes are of the same length. The compound tube carries an axial load of 100 tonnes. Find the stresses and the load carried by each tube and the amount it shortens. Length of each tube is 15 cms. Take $E_s = 2 \times 10^6$ kg./cm.² and $E_b = 1 \times 10^6$ kg/cm².*

Solution Area of steel tube $= A_s = \frac{\pi}{4}(17^2 - 15^2)$ cm.²

$= 50{\cdot}27$ cm ²

* The ratio $\frac{E_1}{E_2}$ is called the modular ratio between the materials of the outer and inner tubes.

Area of brass tube $= A_b = \frac{\pi}{4}(19^2 - 17^2)\ cm.^2$

$= 56{\cdot}55\ cm.^2$

Let the stresses in steel and brass be p_s and $p_b\ kg./cm.^2$ respectively.

Strain in steel = Strain in brass

$$\frac{p_s}{E_s} = \frac{p_b}{E_b}$$

$$\therefore \quad p_s = \frac{E_s}{E_b} p_b \quad \therefore \quad p_s = 2p_b$$

Load on steel + load on copper = Total load

i.e., $p_s A_s + p_b A_b = P$

$\therefore \quad 2p_b \times 50{\cdot}27 + p_b \times 56{\cdot}55 = 100{,}000\ kg.$

$\therefore \quad 157{\cdot}09\ p_b = 100{,}000\ kg.$

$$\therefore \quad p_b = \frac{100{,}000}{157{\cdot}09}\ kg./cm.^2$$

$\therefore \quad p_b = 636{\cdot}5\ kg./cm.^2$

$\therefore \quad p_s = 2 \times 636{\cdot}5 = 1273\ kg/cm.^2$

$\therefore$ Load on the brass tube $= P_b = p_b A_b = 636{\cdot}5 \times 56{\cdot}55$

$= 35990\ kg.$

Load on the steel tube $= P_s = p_s A_s = 1273 \times 50{\cdot}27$

$= 64010\ kg.$

Decrease in length of the compound tube

= Decrease in length of either of the tubes

= Decrease in length of brass tube

$$= \frac{p_b}{E_b} \cdot l$$

$$= \frac{636{\cdot}5}{1 \times 10^6} \times 15\ cm. = 0{\cdot}009548\ cm.$$

Problem 23. *A reinforced concrete column is 30 cm. × 30 cm. in section. The column is provided with 8 bars of 20 mm. diameter. The column carries a load of 18 tonnes Find the stresses in concrete and the steel bars. Take* $E_s = 2{\cdot}1 \times 10^6\ kg./cm.^2$ *and* $E_c = 0{\cdot}14 \times 10^6\ kg./cm.^2$

Solution. Area of steel bars

$A_s = 8 \times 3{\cdot}142 = 25{\cdot}136\ cm.^2$

say $25{\cdot}14\ cm.^2$

Actual area of concrete

$= A_c = 30^2 - 25{\cdot}14 = 874{\cdot}86\ cm^2.$

Let the stresses in concrete and steel be p_c and p_s respectively.

Strain in concrete = Strain in steel

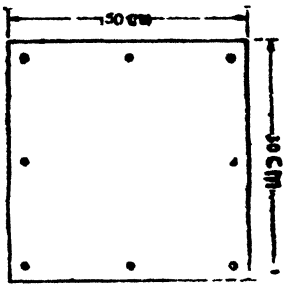


Fig. 22

$$\frac{p_c}{E_c}=\frac{p_s}{E_s}$$

$$\therefore \quad p_s=\frac{E_s}{E_c}\cdot p_c$$

$$=\frac{2{\cdot}1}{0{\cdot}14}\,p_c$$

$$\therefore \quad p_s=15\,p_c \qquad \ldots(i)$$

Load on steel + load on concrete

= Total load on the column.

i.e., $p_sA_s+p_cA_c=P$

$$\therefore \quad 15p_c\times25{\cdot}14+p_c\times874{\cdot}86=P=18000 \text{ kg.}$$

$$\therefore \quad 1251{\cdot}96\,p_c=18000$$

$$\therefore \quad p_c=\frac{18000}{1251{\cdot}96}\text{ kg./cm.}^2$$

$$=14{\cdot}37 \text{ kg./cm.}^2$$

$$\therefore \quad p_s=15\times14{\cdot}37 \text{ kg./cm.}^2$$

$$=215{\cdot}55 \text{ kg /cm.}^2$$

Problem 24. *A load of 30,000 kg. is applied on a short concrete column 25 cm. × 25 cm. The column is reinforced by steel bars of total area 56 cm.² If the modulus of elasticity for steel is 15 times that of concrete find the stresses in concrete and steel.*

If the stress in concrete should not exceed 40 kg./cm.², find the area of steel required so that the column may support a load of 60,000 kg.

Solution. *When the column carries a load of 30,000 kg.*

Area of steel $=A_s=56$ cm.²

Area of concrete $=A_c=25^2-56=569$ cm.²

Let the stresses in steel and concrete be p_s and p_c respectively.

Strain in steel = Strain in concrete

$$\therefore \quad \frac{p_s}{E_s}=\frac{p_c}{E_c}$$

$$\therefore \quad p_s=\frac{E_s}{E_c}\cdot p_c$$

$$\therefore \quad p_s=15\,p_c \qquad \ldots(i)$$

Load on steel + load on concrete

= Load on column.

$$p_sA_s+p_cA_c=P$$

$$\therefore \quad 15\,p_c\times56+p_c\times569$$

$= 30,000$ *kg.*

$\therefore \quad 1409\, p_c = 30,000$

$\therefore \quad p_c = \frac{30,000}{1409}$ *kg./cm.*^{2}

$\therefore \quad p_c = 21{\cdot}29$ *kg./cm.*^{2}

$\therefore \quad p_s = 15 \times 21{\cdot}29 = 319{\cdot}35$ *kg./cm.*^{2}

When the column carries a load of 60,000 kg.

Let the area of steel bars be A_s *cm.*^{2}

$\therefore$ Area of concrete $= A_c = (625 - A_s)$ *cm.*^{2}

Strain in steel = Strain in concrete

$\therefore \quad \frac{p_s}{E_s} = \frac{p_c}{E_c}$

$\therefore \quad p_s = \frac{E_s}{E_c}\, p_c$

$\therefore \quad p_s = 15\, p_c$

But $\quad p_c = 40$ *kg./cm*^{2}

$\therefore \quad p_s = 15 \times 40 = 600$ *kg./cm.*^{2}

Load on steel + load on concrete

= Load on column

$\therefore \quad p_s A_s + p_c A_c = P$

$\therefore \quad 600\, A_s + 40\,(625 - A_s)$

$= 60,000$ *kg.*

$\therefore \quad 560\, A_s = 35000$

$A_s = \frac{35000}{560} = 62{\cdot}5$ *cm*^{2}.

Problem 25. *A compound tube is made by shrinking a thin steel tube on a thin brass tube. A_s and A_b are the sectional areas of the steel and brass tubes, and E_s and E_b are the corresponding values of Young's Modulus. Show that for any tensile load the extension of the compound tube is equal to that of a single tube of the same length and total cross-sectional area, but having a Young's Modulus of* $\frac{E_s A_s + E_b A_b}{A_s + A_b}$.

Solution. Let the load on the compound tube be P

Area of steel tube $= A_s$

Area of brass tube $= A_b$

Let the stresses in steel and brass be p_s and p_b respectively.

$\therefore \quad \frac{p_s}{E_s} = \frac{p_b}{E_b}$

$\therefore \quad p_s = \frac{E_s}{E_b} \cdot p_b$

Load on steel+load on brass

$$=\text{Total load on the compound tube}$$

i.e., $$p_sA_s+p_bA_b=P$$

$$\therefore \quad \frac{E_s}{E_b}p_bA_s+p_bA_b=P$$

$$\therefore \quad p_b\left[\frac{E_s}{E_b}A_s+A_b\right]=P$$

$$\therefore \quad p_b\left[\frac{E_sA_s+E_bE_b}{E_b}\right]=P$$

$$\therefore p_b=\left[\frac{E_b}{E_sA_s+E_bA_b}\right]P$$

$\therefore$ Extension of the compound tube

$$=dl$$

$$=\text{Extension of steel or brass tube}$$

$$\therefore \quad dl=\frac{p_b}{E_b}\,l$$

$$dl=\left(\frac{P}{E_sA_s+E_bA_b}\right)l$$

Let E be Young's Modulus of a tube of area (A_s+A_b) carrying the same load and undergoing the same extension.

$$\therefore \quad dl=\left(\frac{Pl}{(A_s+A_b)E}\right)$$

$$\therefore \quad \frac{Pl}{(A_s+A_b)E}=\frac{Pl}{(E_sA_b+E_bA_b)}$$

$$\therefore \quad E=\frac{E_sA_s+E_bA_b}{A_s+A_b}.$$

Problem 26. *A tube of aluminium 4 cm. external diameter and 2 cm. internal diameter is snugly fitted on to a solid steel rod of 2 cm. diameter. The composite bar is loaded in compression by an axial load P. Find the stress in aluminium when the load is such that the stress in steel is 700 kg./cm.² What is the value of P ?* $E_s=2\times10^6$ *kg./cm.*^{2}, $E_a=7\times10^5$ *kg./cm.*^{2} *(A.M.I.E. May 1969)*

Solution. $$A_a=\frac{\pi}{4}\left(4^2-2^2\right)=3\pi\ cm^2$$

$$A_s=\frac{\pi}{4}(2)^2=\pi\ cm.^2$$

Modular ratio $$=m=\frac{E_s}{E_a}=\frac{20}{7}$$

$$f_s=\frac{20}{7}\times f_a$$

$\therefore \quad f_b = \frac{7}{20} \times 700 \ kg/cm.^2 = 245 \ kg./cm.^2$

Total load $\quad P = f_b A_b + f_s A_s$

$= 245 \times 3\pi + 700 \times \pi = 1435 \ \pi \ kg.$

$= 4508 \ kg.$

Problem 27. *A compound bar consists of a central steel strip 2·5 cm wide and 0·64 cm. thick placed between two strips of brass each 2·5 cm wide and t cm thick. The strips are firmly fixed together to form a compound bar of rectangular section 2·5 cm. wide and (2t+0·64) cm thick. Determine (a) the thickness of the brass strips which will make the apparent modulus of elasticity of compound bar 1570 tonnes/cm² and (b) the maximum axial pull the bar can then carry if the stress is not to exceed 1·57 tonne/cm², in either the brass or the steel. Take the values of E for steel and brass as 2070 tonnes/cm.² and 1140 tonnes/cm².* (*London University*)

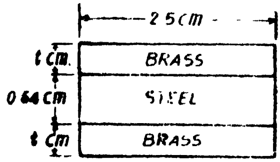


Fig. 23

Solution. Let the load on the compound bar be *P tonnes.*

Let the stresses in brass and steel be p_b and p_s *tonnes/cm* ² respectively.

Strain in steel = Strain in brass

$$\therefore \quad \frac{p_s}{E_s} = \frac{p_b}{E_b}$$

$$\therefore \quad p_s = \frac{E_s}{E_b} \cdot p_b = \frac{2070}{1140}$$

$$\therefore \quad p_s = 1·816 \ p_b$$

But load on steel + load on brass = Total load

$$p_s A_s + p_b A_b = P$$

$$1·816 \ p_b (2·5 \times ·64) + p_b \times 2t \times 2·5 = P \ tonnes.$$

$$\therefore \quad P = p_b (2·905 + 5t) \ \text{tonnes}.$$

Area of the composite section

$$= A = 2·5 \times 0·64 + 2·5 \times 2t \ cm.^2$$

$$= (1·60 + 5t) \ cm.^2$$

Apparent Young's Modulus

$$E = 1570 \ tonnes/cm.^2$$

$$\therefore \quad \text{Strain} = e = \frac{P}{AE}$$

$$= \frac{p_b(2·905 + 5t)}{(1·60 + 5t)1570}$$

This must be equal to the strain of brass or steel

$$= \frac{p_b}{E_b} = \frac{p_b}{1140}$$

$$\therefore \quad \frac{p_b(2{\cdot}905+5t)}{(1{\cdot}60+5t)1570}=\frac{p_b}{1140}$$

$$\therefore \quad \frac{2{\cdot}905+5t}{1{\cdot}60+5t}=\frac{1570}{1140}=1{\cdot}377$$

$$\therefore \quad 2{\cdot}905+5t=2{\cdot}203+6{\cdot}885t$$

$$\therefore \quad 1{\cdot}885t=0{\cdot}702$$

$$\therefore \quad t=\frac{0{\cdot}702}{1{\cdot}885}=0{\cdot}372 \text{ cm}$$

Since $p_s=1{\cdot}816\ p_b$ and since the stress in either brass or steel should not exceed $1{\cdot}570$ *tonne/cm*^{2}

Let $\quad p_s=1{\cdot}570$ *tonne/cm.*^{2}

$$\therefore \quad p_b=\frac{1{\cdot}570}{1{\cdot}815}=0{\cdot}865 \text{ tonne/cm.}^2$$

$\therefore$ Load on the bar $=P=p_sA_s+p_bA_b$

$$=1{\cdot}57\times2{\cdot}5\times0{\cdot}64+0{\cdot}865\times2{\cdot}5\times2\times0{\cdot}372 \text{ tonnes}$$

$$=2{\cdot}512+1{\cdot}608=4{\cdot}12 \text{ tonnes.}$$

Problem 28. *Two vertical wires are suspended at a distance of 50 cm. apart as shown in Fig. 24. Their upper ends are firmly secured and their lower ends support a rigid horizontal bar which carries a load W. The left hand wire has a diameter of 1·6 mm. and is made of copper and the right hand wire has a diameter of 0·9 mm. and is made of steel. Both wires initially are 4·5 m. long*

(*a*) *Determine the position of the line of action of W if due to W, both wires extend by the same amount*

(*b*) *Determine the slope of the rigid bar if a load of 20 kg. is hung at the centre of the bar. Neglect the weight of the bar.*

Take $E_s=2{\cdot}1\times10^6$ *kg./cm.*^{2} and $E_c=1{\cdot}3\times10^6$ *kg./cm.*^{2}

(*A.M.I.E. Nov., 1968*)

Solution. $A_s=0{\cdot}00636$ *cm.*^{2}

$A_c=0{\cdot}02010$ *cm*^{2}. $\qquad f_s=mf_c$

$$m=\frac{2{\cdot}1}{1{\cdot}3}=1{\cdot}615 \qquad f_c=\frac{W}{A_c+mA_s}$$

$$f_s=\frac{mW}{A_c+mA_s}$$

$$T_s=f_sA_s=\left[\frac{mA_s}{A_c+mA_s}\right]W=\left[\frac{1{\cdot}615\times0{\cdot}00636}{0{\cdot}0201+1{\cdot}615\times0{\cdot}00636}\right],$$

$\therefore \quad T_s = 0.338\ W$

Taking moments about copper

$0.338\ W \times 50 = W x$

$x = 16.900\ cm.$

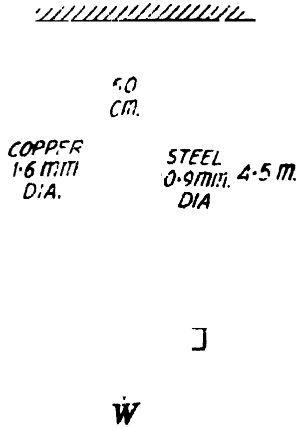


Fig. 24

Case (b)

When 20 *kg*. load is at mid span

Load in each wire = 10 *kg*.

$$\delta_c = \frac{10 \times 450}{0.0201 \times 1.3 \times 10^6} = 0.1722\ cm.$$

$$\delta_s = \frac{10 \times 450}{0.00636 \times 2 \times 10^6} = 0.3538\ cm.$$

Let θ be the inclination of the rigid bar with the horizontal.

$$\tan \theta = \frac{0.3538 - 0.1722}{50}$$

$$\theta = \frac{0.1816}{50} = 0.003632$$

$$\theta = 0°\ 12'.$$

Problem 29. *Two vertical rods one of steel and the other of copper are each rigidly fixed at the top and are 50 cm apart. Diameter and length of each rod are 2 cm and 4 metres respectively. A cross bar fixed to the rods at the lower ends carries a load of 500 kg such that the cross bar remains horizontal even after loading. Find the stress in each rod and the position of the load on the bar.*

Take $E_s = 2 \times 10^6\ kg./cm.^2$ *and* $E_c = 1 \times 10^6\ kg/cm^2$.

(*A.M.I.E. Winter 1978*)

Solution.

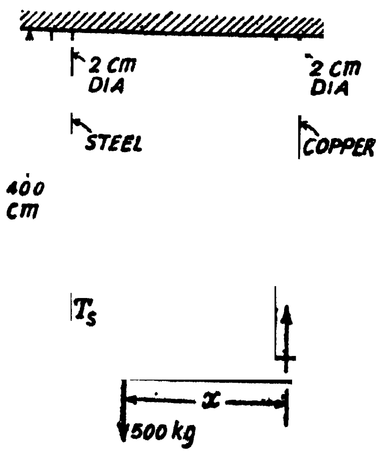


Fig. 25

Area of steel rod

$$= A_s = \frac{\pi}{4}(2)^2 = \pi\ cm^2$$

Area of copper rod

$$= A_c = \pi\ cm^2$$

Modular ratio

$$= m = \frac{E_s}{E_c} = \frac{2 \times 10^6}{1 \times 10^6} = 2$$

Let f_s and f_c be the stresses in steel and copper respectively. Since the cross bar remains horizontal, the extensions of the steel and copper rods are equal. Since these rods have the same original length the strains of these rods are equal.

Strain in steel = Strain in copper

$$e_s = e_c$$

$$\therefore \quad \frac{f_s}{E_s} = \frac{f_c}{E_c}$$

$$\therefore \quad f_s = \frac{E_s}{E_c} \cdot f_c = m\, f_c, \text{ but } m = 2$$

$$\therefore \quad f_s = 2\, f_c$$

Let T_s and T_c be the tensions in the steel and copper rods.

$$\therefore \quad T_s = f_s A_s = 2 f_c A_s = 2 f_c\, \pi$$

$$T_c = f_c A_c = f_c\, \pi$$

$$\therefore \quad T_s = 2 T_c$$

But $\quad T_s + T_c = 500\ kg$

$$\therefore \quad 2T_c + T_c = 500\ kg.$$

$$\therefore \quad T_c = \frac{500}{3}\ kg. \text{ and } T_s = \frac{1000}{3}\ kg.$$

Let the 500 kg load be at a distance x from the copper rod.

Considering the equilibrium of the cross bar, and taking moments about the right end,

$$500\, x = T_s \times 50$$

$$500\, x = \frac{1000}{3} \times 50$$

$$\therefore \quad x = 33{\cdot}33\ cm.$$

Problem 30. *Two vertical rods are each fastened at the upper end at a distance of 63 cms. apart. Each rod is 300 cms. long and 12 mm. in diameter. A horizontal rigid cross bar connects the lower ends of the rods and on it is placed a load of 450 kg. so that the cross bar remains horizontal. Find the position of the load on the cross bar and the stresses in each rod. One rod is of steel for which* $E = 1{\cdot}96 \times 10^6\ kg./cm.^2$ *and the other of bronze for which* $E = 0{\cdot}63 \times 10^6\ kg./cm.^2$

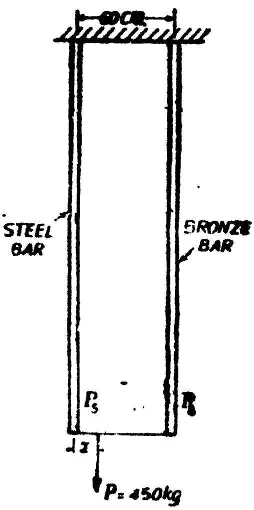


Fig. 26

Solution. Area of each bar

$$= A = \frac{\pi}{4}(1{\cdot}2)^2\ cm.^2$$

$$= 1{\cdot}131\ cm.^2$$

Let the stresses in steel and bronze be p_s and p_b $kg./cm.^2$ respectively.

Since the rigid bar remains horizontal, the extensions of the steel and bronze bars are equal.

$\therefore$ Strain in steel = Strain in brass.

$$\frac{p_s}{E_s}=\frac{p_b}{E_b}$$

$\therefore$ $$p_s=\frac{E_s}{E_b}\ p_b$$

$$p_s=\frac{1{\cdot}96}{0{\cdot}63}p_b$$

$\therefore$ $$p_s=3{\cdot}111\ p_b.$$

Load on steel + load on bronze = Total load

$\therefore$ $$p_sA_s+p_bA_b=P$$

But $$A_s=A_b=1{\cdot}311\ cm.^2$$

$\therefore$ $$3{\cdot}111\ p_b\times 1{\cdot}311+p_b\times 1{\cdot}311=450$$

$\therefore$ $$4{\cdot}650p_b=450$$

$$p_b=\frac{450}{4{\cdot}65}\ kg./cm^2$$

$$=96{\cdot}80\ kg./cm^2.$$

$\therefore$ $$p_s=3{\cdot}111\times 96{\cdot}8=301{\cdot}1\ kg./cm.^2$$

Consider the equilibrium of the rigid rod, see Fig. 26. Let the load be applied at x *cms.* from the steel bar.

Taking moments about the left end we have

$$P_b\times 60=450\times x$$

$\therefore$ $$96{\cdot}8\times 1{\cdot}131\times 60=450\ x$$

$\therefore$ $$x=\frac{96{\cdot}8\times 1{\cdot}131\times 60}{450}\ cms.$$

$\therefore$ $$x=14{\cdot}6\ cms.$$

Hence the load must be applied on the rigid bar at a distance of 14·6 cms. from the steel bar.

Problem 31. *A solid steel bar 50 cm. long and 7 cm. diameter is placed inside an aluminium tube having 7·5 cms. inside diameter and 10 cms. outside diameter. The aluminium cylinder is 0·015 cm. longer than the steel cylinder. An axial load of 60,000 kg. is applied to the bar and the cylinder through rigid cover plates as shown in Fig. 27. Find the stresses developed in the steel bar and the aluminium tube. Assume* $E_s=2{\cdot}2\times 10^6$ *kg./cm.*^{2} *and* $E_a=0{\cdot}7\times 10^6$ *kg. cm*^{2}.

(*A.M.I.E., May 1964*)

Solution.

Area of the steel bar $=A_s=\frac{\pi}{4}\times 7^2\ cm.^2$

$$=38{\cdot}49\ cm.^2$$

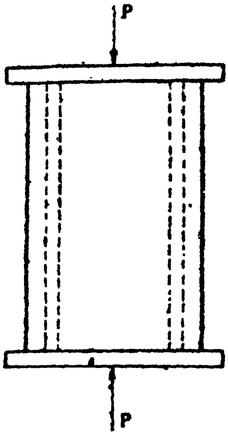


Fig. 27

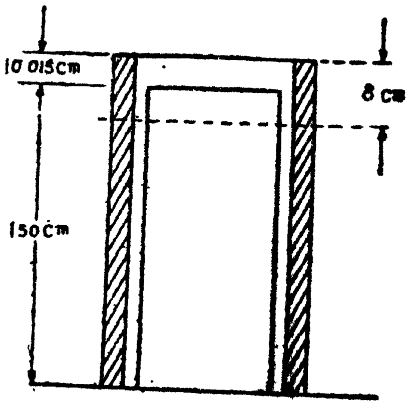


Fig. 28

Area of the aluminium tube

$$=A_a=\frac{\pi}{4}(10^2-7\cdot5^2)\ cm.^2$$

$$=34\cdot36\ cm.^2$$

Fig. 28 shows the original dimensions of the steel bar and the aluminium tube.

Let the aluminium tube be compressed by δ *cm.*

$\therefore$ The steel bar is compressed by $(\delta-0\cdot015)$ *cm.*

$$\therefore \text{ Strain in steel}=e_s=\frac{\text{change in length}}{\text{original length}}$$

$$=\frac{\delta-0\cdot015}{50}$$

$$\text{Strain in aluminium}=e_a=\frac{\text{change in length}}{\text{original length}}$$

$$=\frac{\delta}{50\cdot015}$$

$$\therefore \text{ Stress in steel}=p_s=E_se_s=\left(\frac{\delta-0\cdot015}{50}\right)2\cdot2\times10^6\ kg/cm^2$$

$$\therefore \quad p_s=(\delta-0\cdot015)\ 44000\ kg./cm.^2$$

$$\text{Stress in aluminium}=p_a=E_ae_a=\left(\frac{\delta}{50\cdot015}\right)0\cdot7\times10^6\ kg./cm.^2$$

$$p_a=14000\ \delta\ kg./cm^2.$$

Load on steel+load on aluminium

=total load on the composite :

$$p_sA_s+p_aA_a=P$$

$$(\delta-0\cdot015)\ 44000\times38\cdot49+14000\delta\times34\cdot36=60000\ kg.$$

$\therefore \quad 1693\delta+481{\cdot}04\delta=60+25{\cdot}4$

$\therefore \quad 2174{\cdot}04\delta=85{\cdot}4$

$\therefore \quad \delta=\frac{85{\cdot}4}{2174{\cdot}04}\ cm.$

$\therefore \quad \delta=0{\cdot}03928\ cm.$

$\therefore \quad p_s=(\delta-0{\cdot}015)\ 44000\ kg./cm.^2$

$=(0{\cdot}03928-0{\cdot}015)\ 44000\ kg./cm.^2$

$=1068{\cdot}32\ kg./cm.^2$

and $\quad p_a=14000\ \delta$

$=14000\times0{\cdot}03928\ kg./cm^2.$

$=549{\cdot}92\ kg./cm.^2$

§9. Equivalent Area of a Compound Section

Suppose a compound column consists of a concrete column reinforced with steel bars.

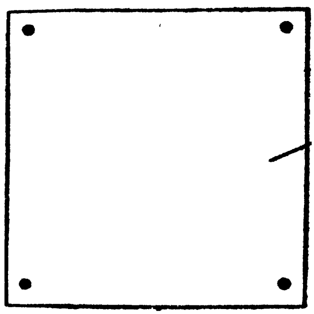


Fig. 29

Let the gross area of the column be A.

Let the area of steel be A_s.

$\therefore$ Actual area of concrete $=A_c=(A-A_s)$.

Let the stresses in concrete and steel be p_c and p_s respectively.

Strain in Concrete = Strain in Steel

$$\therefore \quad \frac{p_c}{E_c}=\frac{p_s}{E_s}$$

$$\therefore \quad p_s=\frac{E_s}{E_c}p_c$$

$$\therefore \quad p_s=mp_c \qquad ...(i)$$

where $\quad m=\frac{E_s}{E_c}$

This ratio m is called the *modular ratio* between steel and concrete.

Load on steel + load on concrete = load on column

i.e. $\quad p_sA_s+p_cA_c=P$

$\therefore \quad mp_cA_s+p_cA_c=P$

$\therefore \quad p_c(A_c+mA_s)=P$

$$p_c=\frac{P}{A_c+mA_s}$$

But $$A_c = (A - A_s)$$

$\therefore$ $$p_c = \frac{P}{A - A_s + mA_s}$$

$\therefore$ $$p_c = \frac{P}{A + (m-1)A_s}$$

Suppose in place of the composite section, a plain concrete column of area $A + (m-1)A_s$ had been provided the stress in concrete

$$= p_c = \frac{P}{A + (m-1)A_s}$$

Hence for determining the stress in concrete, we may consider that the given reinforced concrete column is equivalent to a plain concrete column whose sectional area

$$= A_e = A + (m-1)A_s$$

This area $A_e = A + (m-1)\ A_s$ is called the *equivalent concrete area.*

$\therefore$ Stress in concrete $= \dfrac{\text{Load}}{\text{Equivalent concrete area}}$

Stress in steel $=$ modular ratio $\times$ stress in concrete

The above principle, for instance, can be applied to problem 23, page 28.

In the problem, Size of the column $= 30\ cm. \times 30\ cm.$

Load on the column $= 18000\ kg.$

Area of steel $A_s = 8 \times 3{\cdot}142 = 25{\cdot}14\ cm^2.$

Modular ratio $= m = \dfrac{E_s}{E_c} = 15$

$\therefore$ Equivalent concrete area

$$= A_e = A + (m-1)\ A_s$$
$$= 30^2 + (15-1) \times 25{\cdot}14\ cm.^2$$
$$= 1251{\cdot}96\ cm.^2$$

$\therefore$ Stress in concrete

$$= p_c = \frac{\text{Load on the column}}{\text{Equivalent concrete area}}$$
$$= \frac{18000}{1251{\cdot}96}\ kg./cm.^2$$
$$= 14{\cdot}37\ kg./cm.^2$$

$\therefore$ Stress in steel $= p_s = mp_c = 15 \times 14{\cdot}37\ kg./cm.^2$

$$= 215{\cdot}55\ kg./cm.^2$$

Problem 32. *A steel strip of cross-section 40 mm. × 10 mm. is bolted to two copper strips one on either side, each of cross-section 40 mm. × 7·5 mm. to transfer the load. There are two bolts on the line of the pull. Show that neglecting friction and the deformation of the* *bolts a pull applied to the joint will be shared by the bolts in the ratio of 3 to 4. Assume E for steel is twice that of copper.*

Solution.

Figure 30 shows the details of connection of the members mentioned in the problem.

Let the load applied on the connection be P *kg.*, *i.e.*, the load of P *kg.* is applied at the end of the steel plate. Let the load transferred to the bolt A be P_1 *kg.* Hence between the two bolts the load in the steel plate will be $(P-P_1)$ *kg.* This load will be transferred to the bolt B.

The load P_1 transferred to the bolt A will be transferred to the two copper plates between the two bolts. Now consider the plates between the two bolts.

Load on the steel plate $=(P-P_1)$ *kg.*

Load on the copper plates $=P_1$ *kg.*

Strain in steel = Strain in copper.

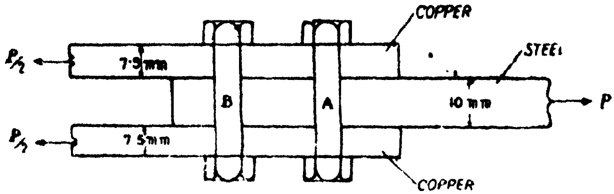


Fig. 30

$$=\frac{P-P_1}{A_sE_s}=\frac{P_1}{A_cE_c}$$

$$\therefore \quad \frac{P_1}{P-P_1}=\frac{A_cE_c}{A_sE_s}=\frac{A_c}{A_s}\cdot\frac{E_c}{E_s}$$

$$\therefore \quad \frac{P_1}{P-P_1}=\frac{2\times40\times7{\cdot}5}{40\times10}\cdot\frac{1}{2}$$

$$\therefore \quad \frac{P_1}{P-P_1}=\frac{3}{4}$$

But load on the bolt $A=P_1$

and the load on the bolt $B=P-P_1$

$\therefore$ Ratio of the loads shared by the bolts A and $B=3:4$.

Problem 33. *Two copper rods and a steel rod, together support a rigid uniform beam weighing P kg as shown in Fig. 31. The stresses in copper and steel are not to exceed 600 kg./cm² and 1200 kg./cm². respectively. Find the magnitude of the load P that can be safely supported. Young's Modulus for steel is twice that of copper.*

Solution. Area of copper component

$$=A_c=2[4\times4]=32\ cm^2.$$

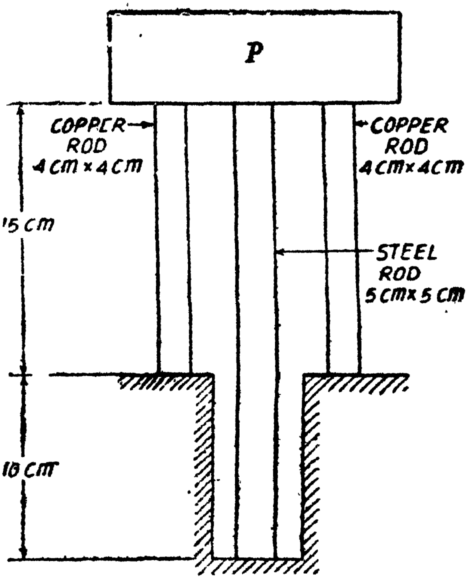


Fig. 31.

Area of steel component

$=A_s=5\times5=25\ cm^2$.

Length of copper component $=l_c=15\ cm$.

Length of steel component $=l_s=25\ cm$.

Decrease in length of copper = Decrease in length of steel $=\delta$

$\therefore\quad \delta=e_cl_c=e_sl_s$

$$\therefore\quad \frac{e_s}{e_c}=\frac{l_c}{l_s}=\frac{15}{25}=0{\cdot}6$$

Stress in steel $=p_s=e_sE_s$

Stress in copper $=p_c=e_cE_c$

$$\frac{p_s}{p_c}=\frac{e_s}{e_c}\cdot\frac{E_s}{E_c}=0{\cdot}6\times2=1{\cdot}2$$

$\therefore\quad p_s=1{\cdot}2\ p_c$

when p_c reaches 600 $kg./cm^2$. p_s will reach $1{\cdot}2\times600=720\ kg./cm^2$. which is less than its permissible value.

$$\therefore\quad P=p_cA_c+p_sA_s$$
$$=(600\times32)+(720\times25)\ kg.$$
$$=37200\ kg.$$

Problem 34. *Two copper rods and one steel rod together support a load of 25000 kg. as shown in Fig. 32. Find the stresses in the rods.*

Take $E_s = 2\times 10^6$ *kg./cm.*^{2} *and* $E_c = 1\times 10^6$ *kg./cm.*^{2}

Solution. Each rod will be compressed by the same amount.

Let the decrease in length of each rod be δ *cm.*

Let the strain in copper and steel be e_c and e_s respectively.

$$\therefore \quad e_c l_c = e_s l_s = \delta$$

$$\therefore \quad e_s = \frac{l_c}{l_s} \cdot e_c$$

$$= \frac{14}{21} e_c$$

$$\therefore \quad e_s = \frac{2}{3} e_c$$

Let the stresses in steel and copper be p_s and p_c respectively.

$$\therefore \quad p_s = e_s E_s \text{ and } p_c = e_c E_c$$

$$\therefore \quad \frac{p_s}{p_c} = \frac{e_s}{e_c} \cdot \frac{E_s}{E_c}$$

$$\therefore \quad \frac{p_s}{p_c} = \frac{2}{3} \times 2 = \frac{4}{3}$$

$$\therefore \quad p_s = \frac{4}{3} p_c \qquad \ldots(i)$$

Load on steel + Load on copper = Total load applied

i.e., $$p_s A_s + p_c A_c = P$$

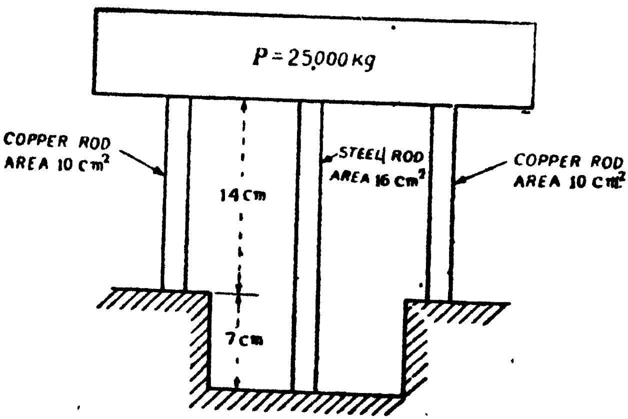


Fig. 32.

$$\therefore \quad \frac{4}{3} p_c \times 16 + p_c \times (2\times 10) = 25000 \text{ kg.}$$

$$\therefore \quad \frac{64}{3} p_c + 20\, p_c = 25000 \text{ kg.}$$

$$\therefore \quad \frac{124}{3} p_c = 25000 \ kg.$$

$$\therefore \quad p_c = \frac{25000 \times 3}{124} \ kg./cm.^2$$

$$\therefore \quad p_c = 604{\cdot}84 \ kg./cm.^2$$

$$\therefore \quad p_s = \frac{4}{3} \times 604{\cdot}84 = 806{\cdot}45 \ kg./cm.^2$$

Problem 35. *Two copper rods and one steel rod together support a load as shown in Fig. 33. If the stresses in copper and steel are not to exceed 600 kg./cm.² and 1200 kg./cm.² find the safe*

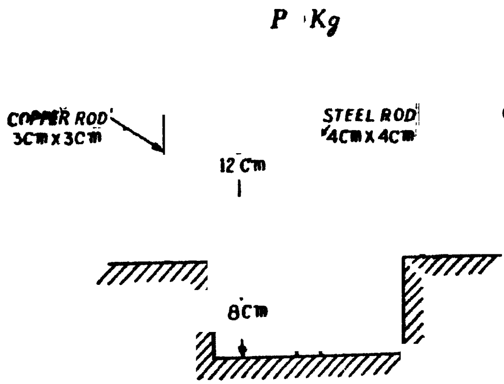


Fig. 33.

load that can be supported. Young's Modulus for steel is twice that of copper.

Solution. Each rod will be compressed to the same extent.

Let the decrease in length of each rod be δ *cm.*

Let the strain in steel and copper be e_s and e_c respectively.

$$\therefore \quad \delta = e_s . l_s = e_c l_c$$

$$\therefore \quad \frac{e_s}{e_c} = \frac{l_c}{l_s} = \frac{12}{20} = 0{\cdot}6$$

Let the stresses in steel and copper be p_s and p_c respectively.

$$\therefore \quad p_s = e_s E_s \text{ and } p_c = e_c E_c$$

$$\therefore \quad \frac{p_s}{p_c} = \frac{e_s}{e_c} \cdot \frac{E_s}{E_c} = 0{\cdot}6 \times 2 = 1{\cdot}2$$

$$\therefore \quad p_s = 1{\cdot}2 \ p_c.$$

Suppose steel is permitted to reach its safe stress of 1200 kg./cm.² the corresponding stress in copper will be $\frac{1200}{1{\cdot}2} = 1000$

kg./cm.^{2} which exceeds the safe stress of 600 *kg./cm.*^{2} for copper. Therefore let copper be allowed to reach its safe stress of 600 *kg/cm.*^{2} Corresponding stress in steel will be $600 \times 1{\cdot}2 = 720$ *kg./cm.*^{2}

$\therefore$ Total load $= P =$ load on steel + load on copper

$$= p_s A_s + p_c A_c$$

$$= 720 \times 16 + 600 \times 2 \times 9 \text{ kg.}$$

$$= 22320 \text{ kg.}$$

Problem 36. *Three vertical rods equal in length and each 12 mm. in diameter are equispaced in a vertical plane and together support a load of 1000 kg. the rods being so adjusted as to share the load equally. If now an additional load of 1000 kg. be added determine the stress in each rod. The middle rod is of copper and the outer rods are of steel. Take* $E_s = 2 \times 10^6$ *kg./cm.*^{2} *and* $E_c = 1 \times 10^6$ *kg./cm*^{2}

Solution. (*i*) *Stresses due to the initial load of 1,000 kg.*

Area of each bar $= \frac{\pi}{4} \times 1{\cdot}2^2 \text{ cm.}^2 = 1{\cdot}13 \text{ cm.}^2$

Initially the load on each bar

$$= \frac{1000}{3} \text{ kg.}$$

The initial stress in each bar

$$= p_0 = \frac{1000}{3 \times 1{\cdot}13} \text{ kg./cm}^2.$$

$\therefore$ $$p_0 = 294{\cdot}9 \text{ kg./cm.}^2$$

(*ii*) *Stresses due to additional load of 1000 kg.*

Let the stresses in copper and steel be p_c and p_s due to the additional load of 1000 *kg.*

Strain in copper = Strain in steel.

$\therefore$ $$\frac{p_c}{E_c} = \frac{p_s}{E_s}.$$

$\therefore$ $$p_s = \frac{E_s}{E_c} \cdot p_c = 2p_c$$

Load on steel + load on copper = Total load.

$$p_s A_s + p_c A_c = P$$

$\therefore$ $$2p_c(2 \times 1{\cdot}13) + p_c \times 1{\cdot}13 = 1000 \text{ kg.}$$

$\therefore$ $$5{\cdot}65\, p_c = 1000 \text{ kg.}$$

$\therefore$ $$p_c = \frac{1000}{5{\cdot}65} \text{ kg./cm.}^2$$

$\therefore$ $$p_c = 177 \text{ kg./cm.}^2$$

and $$p_s = 2 \times 177 = 354 \text{ kg./cm}^2.$$

$\therefore$ Final stress in copper $=p_o+p_c=294{\cdot}9+177=471{\cdot}9$ *kg./cm²*.

Final stress in steel $=p_o+p_s=294{\cdot}9+354=648{\cdot}9$ *kg./cm²*.

Problem 37. *A steel rod 18 mm. in diameter passes centrally through a steel tube 25 mm. in internal diameter and 30 mm. in external diameter. The tube is 75 cm. long and is closed by rigid washers of negligible thickness which are fastened by nuts threaded on the rod. The nuts are tightened until the compressive load on the tube is 2 tonnes. Calculate the stresses in the tube and the rod.*

Find the increase in these stresses when one nut is tightened by one quarter of a turn relative to the other. There are 4 threads per cm. Take E=2000 tonnes/cm².

Solution. When the nuts are tightened the tube will be compressed and the rod will be elongated. Since no external forces have been applied, the compressive load on the tube must be equal to the tensile load on the rod.

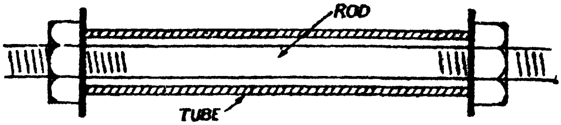


Fig. 34

Area of the tube $=A_t=\frac{\pi}{4}(3^2-2{\cdot}5^2)=\frac{11\pi}{16}\ cm^2$

Area of the rod $=A_r=\frac{\pi}{4}\times(1{\cdot}8)^2=\frac{81}{100}\pi\ cm^2$.

Let the stresses in the rod and tube be p_r and p_t *tonnes/cm.²* respectively.

Tensile load on the rod = compressive load on the tube

$$\therefore \quad p_rA_r=p_t.A_t$$

$$\therefore \quad p_r=\frac{A_t}{A_r}.p_t$$

$$\therefore \quad p_r=\frac{11\pi}{16}\cdot\frac{100}{81\pi}p_t$$

$$\therefore \quad p_r=\frac{275}{324}p_t$$

(*i*) *When the compressive load on the tube is 2 tonnes.*

$$\therefore \text{Stress in the tube}=p_t=\frac{2}{\frac{11\pi}{16}}\ tonnes/cm^2.$$

$$=\frac{32}{11\pi}\ tonnes/cm^2$$

$$=0{\cdot}926\ tonnes/cm^2\ (compressive)$$

$\therefore$ Stress in the rod $= p_r = \dfrac{275}{324} \times 0{\cdot}926$ *tonnes/cm.*^{2}

$= 0{\cdot}786$ *tonnes/cm*^{2}. (*tensile*)

(*ii*) *When one nut is tightened by one quarter of a turn.*

Let f_t and f_r be the stresses due to tightening of the nut by one quarter of a turn.

Obviously f_t is a compressive stress and f_r is tensile, and

$$f_r = \frac{275}{324} f_t$$

$\therefore$ Reduction in the length of the tube

$$= \frac{f_t}{E} \cdot l$$

$$= \frac{f_t \times 75}{2000} \text{ cm.}$$

Extension of the rod

$$= \frac{f_r}{E} l$$

$$= \frac{f_r \times 75}{2000} \text{ cm.}$$

$$= \frac{275}{324} \cdot \frac{f_t \times 75}{2000} \text{ cm.}$$

$$= \frac{68{\cdot}66 f_t}{2000} \text{ cm.}$$

But contraction of the tube + extension of the rod = axial advance of the nut.

$$\therefore \frac{75 f_t}{2000} + \frac{63{\cdot}66 f_t}{2000} = \tfrac{1}{4}\left(\frac{1}{4}\right) = 0{\cdot}0625 \text{ cm.}$$

$$\therefore \quad 75 f_t + 63{\cdot}66 f_t = 125$$

$$\therefore \quad 138{\cdot}66 f_t = 125$$

$$\therefore \quad f_t = \frac{125}{138{\cdot}66} \text{ t/cm}^2.$$

$$\therefore \quad f_t = 0{\cdot}90 \text{ t/cm}^2. \text{ (compressive)}$$

$$\therefore \quad f_r = \frac{275}{324} \times 0{\cdot}9 = 0{\cdot}764 \text{ t/cm}^2. \text{ (tensile)}$$

Problem 38. *A solid uniform metal bar of diameter D and length l is hanging vertically from its upper end. Obtain the total elongation of the bar due to its own weight if* γ *is the specific weight and E the Young's Modulus of the material of the bar.*

(*A.M.I.E. May, 1975*)

Solution. Consider any section XX of the rod distant x from the lower end.

Weight of the rod below the section XX

$$=\gamma A x,$$

where A=sectional area of the rod.

$$\text{Stress at the section } XX = f = \frac{\gamma A x}{A} = \gamma x$$

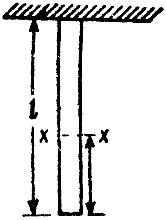


Fig. 35

Consider an elemental length dx of the rod from XX.

$$\text{Extension of the elemental length of rod} = \frac{f}{E}\,dx$$

$$= \frac{\gamma x}{E}\,dx$$

$$\therefore \quad \text{Total extension of the rod} = \delta = \int_0^l \frac{\gamma x}{E}\,dx$$

$$\therefore \qquad \delta = \frac{\gamma l^2}{2E}$$

§10. Bar of Uniform Strength

Fig. 36 shows a bar subjected to an external tensile load P. If the bar had been of uniform cross-section, the tensile stress intensity at any section would be constant only if the self weight of the member be ignored. If the weight of the member is also considered, the intensity of stress increases on sections at higher levels.

It is possible to maintain a uniform stress on all the sections by increasing the area from the lower end to the upper end.

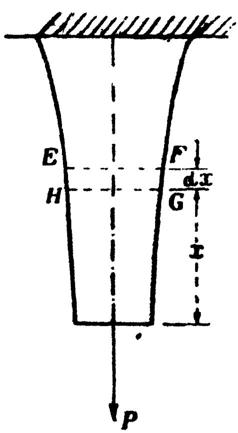


Fig. 36

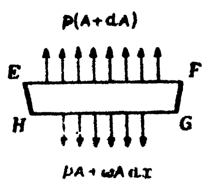


Fig. 37

Suppose the area of the upper and lower ends be A_1 and A_2 respectively.

Let the area of the section be A at a distance x from the lower end. Let the area be $A+dA$ at a distance $x+dx$ from the lower end.

Let the weight per unit volume of the member be w

Consider the equilibrium of the strip $EFGH$

Total force acting upwards = Total force acting downwards

Let p be the uniform stress intensity.

$$\therefore \quad p(A+dA)=pA+wA\,dx$$

$$\therefore \quad p.\,dA=wA\,.\,dx$$

$$\therefore \quad \frac{dA}{A}=\frac{w}{p}dx$$

Integrating, we get

$$\log_e A=\frac{w}{p}x+C_1$$

where $\quad C_1$ = constant of integration

At $\quad x=0, \quad A=A_2$

$$\therefore \quad \log_e A_2=C_1$$

$$\therefore \quad \log_e A=\frac{w}{p}x+\log_e A_2$$

$$\therefore \quad \log_e\frac{A}{A_2}=\frac{w}{p}x$$

$$\therefore \quad \frac{A}{A_2}=e^{\frac{wx}{p}}$$

$$\therefore \quad A=A_2e^{\frac{wx}{p}} \qquad ...(i)$$

Obviously $\quad A_1=A_2\,e^{\frac{wl}{p}} \qquad ...(ii)$

Eq. (i) may also be written as

$$2{\cdot}3\log_{10}\frac{A}{A_2}=\frac{w}{p}x \qquad ...(iii)$$

Problem 39. *A vertical tie of uniform strength is 18 metres long. If the area of the bar at the lower end is 5 cm² find the area at the upper end when the tie is to carry a load of 70000 kg. The material of the tie weighs 8 gms./cm.³*

Solution. *Area of the tie at the bottom* $= A_2 = 5\ cm^2$.

Intensity of stress $= p = \dfrac{70{,}000}{5} = 14000\ kg./cm^2$.

Wt. per unit volume $= w = \dfrac{8}{1000}\ kg./cm^3$.

The area at any distance x from the bottom end is given by

$$2{\cdot}3 \cdot \log_{10} \frac{A}{A_2} = \frac{w}{p}\, x$$

Let the area at the upper end be A_1

$$\therefore \qquad 2{\cdot}3 \log_{10} \frac{A_1}{A_2} = \frac{w}{p}\, l$$

$$= \frac{8 \times 1800}{1000 \times 14000}$$

$$= \frac{9}{8750}$$

$$\therefore \qquad \log_{10} \frac{A_1}{A_2} = \frac{9}{8750 \times 2{\cdot}3} = 0{\cdot}0004$$

$$\therefore \qquad \frac{A_1}{A_2} = 1{\cdot}001$$

$$\therefore \qquad A_1 = 1{\cdot}001 \times 5 = 5{\cdot}005\ cm^2.$$

Problem 40. *Fig. 38 shows a rigid square platform of negligible weight and of side l supported by four identical elastic pillars each of height h. If a load P be applied at a point distant a and b from the adjacent sides AB and AD find the pressure on each pillar and the depression of the centre of the platform.*

Solution. Let the pressures on the legs AA_1, BB_1, CC_1, DD_1 be P_a, P_b, P and P_d respectively.

For the equilibrium of the platform,

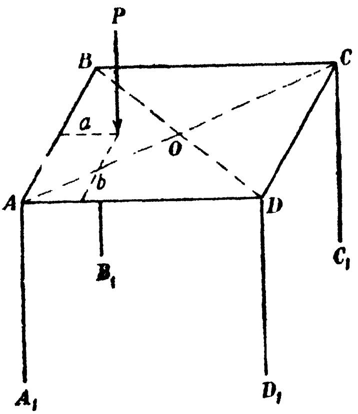


Fig. 38

$$P_a + P_b + P_c + P_d = P \qquad \text{..}(i)$$

Taking moments about AB, we have,

$$P_c l + P_d l = Pa$$

$$\therefore \qquad P_c + P_d = \frac{Pa}{l} \qquad \text{...}(ii)$$

Taking moments about AD, we have

$$P_b l + P_c l = Pb$$

$$\therefore \quad P_b + P_c = \frac{Pb}{l} \qquad \ldots(iii)$$

Let δ_a, δ_b, δ_c and δ_d be the depressions of the pillars AA_1, BB_1, CC_1 and DD_1. These depressions are proportional to the respective pressures on the pillars.

$\therefore \quad \delta_a \propto P_a \quad \therefore \quad \delta_a = K.P_a.$ where K is constant

$$\delta_b = K P_b$$

$$\delta_c = K P_c$$

$$\delta_d = K P_d$$

Depression of O = Average of depressions of A and C

$$= \frac{\delta_a + \delta_c}{2} = \frac{K(P_a + P_c)}{2}$$

= Average of depressions of B and D

$$= \frac{\delta_b + \delta_d}{2} = \frac{K(P_b + P_d)}{2}$$

$$\therefore \quad P_a + P_c = P_b + P_d \qquad \ldots(iv)$$

Rewriting the above equations,

$$P_a + P_b + P_c + P_d = P \qquad \ldots(i)$$

$$P_c + P_d = \frac{Pa}{l} \qquad \ldots(ii)$$

$$P_b + P_c = \frac{Pb}{l} \qquad \ldots(iii)$$

$$P_a + P_c = P_b + P_d \qquad \ldots(iv)$$

From equations (i) and (iv), we get

$$P_a + P_c = P_b + P_d = \frac{P}{2}$$

Subtracting eq. (ii) from eq. (iii), we get

$$P_b - P_d = \frac{P}{l}(b - a) \qquad \ldots(v)$$

But $$P_b + P_d = \frac{P}{2} \qquad \ldots(vi)$$

Adding equations (v) and (vi), we get

$$2P_b = \frac{P}{2l}\left\{l + 2(b - a)\right\}$$

$$\therefore \quad P_b = \frac{P}{4l}\left\{l + 2(b - a)\right\}$$

Substituting the value of P_b in equation (*vi*)

$$P_d=\frac{P}{2}-\frac{P}{4l}\left\{l+2\ (b-a)\right\}$$

$$\therefore \qquad P_d=\frac{P}{4l}\left\{l-2\ (b-a)\right\}$$

Substituting the value of P_d in equation (*ii*)

$$P_c=\frac{P_a}{l}-\frac{P}{4l}\left\{1-2\ (b-a)\right\}$$

$$\therefore \qquad P_c=\frac{P}{4l}\left\{2\ (a+b)-l\right\}$$

Substituting the value of P_c in equation (*iv*)

$$P_a=P_b+P_d-P_c=\frac{P}{2}-\frac{P}{4l}\left\{2(a+b)-l\right\}$$

$$\therefore \qquad P_a=\frac{P}{4l}\left\{3\ l-2(a+b)\right\}$$

Thus the pressures on the pillars are.

$$P_a=\frac{P}{4l}\left\{3l-2\ (a+b)\right\}$$

$$P_b=\frac{P}{4l}\left\{l+2\ (b-a)\right\}$$

$$P_c=\frac{P}{4l}\left\{2\ (a+b)-l\right\}.$$

and $$P_d=\frac{P}{4l}\left[l-2\ (b-a)\right]$$

Let the Young's modulus of the material of pillars be E

Depression of the centre O of the platform

$=\delta=\frac{1}{2}$ (Depression of the pillar AA_1+ depression of the pillar CC_1)

$$=\frac{1}{2}\left[\frac{P_a}{AE}h+\frac{P_c}{AE}h\right]$$

where A is the sectional area of each pillar.

$$\because \qquad \delta=\frac{h}{2EA}\left(P_a+P_c\right)$$

But $$P_a+P_c=\frac{P}{2}$$

$$\therefore \qquad \delta=\frac{Ph}{4EA}$$

Problem 41. *Three ropes AD, BD and CD support a load P as shown in Fig. 39. If the three ropes are of the same sectional area,* *the middle rope be vertical and the other ropes be at θ with the vertical, find the load carried by each rope.*

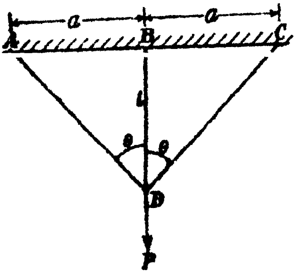


Fig. 39

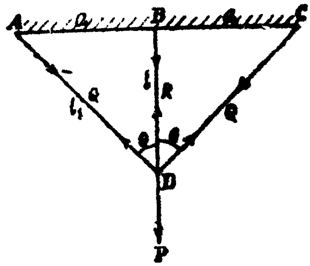


Fig. 40

Solution. Let the length of AD be l_1

Let the tension in the middle rope be R

Let the tension in each of the ropes AD and CD be Q

Resolving the forces at D vertically we have,

$$R+2\,Q\cos\theta=P$$

$$\therefore \quad Q=\tfrac{1}{2}(P-R)\sec\theta \qquad ..(i)$$

Increase in length of AD

$$=dl_1=\frac{Q}{AE}\,l_2$$

Increase in length of BD

$$=dl=\frac{R}{AE}\,l$$

But $\quad l_1^2=l^2+a^2$

$\therefore$ Differentiating, we get,

$$2l_1dl_1=2l.\,dl$$

$$\therefore \quad \frac{dl_1}{dl}=\frac{l}{l_1}=\cos\theta$$

$$\therefore \quad \frac{\frac{Q}{AE}\cdot l_1}{\frac{R}{AE}\cdot l}=\cos\theta$$

$$\therefore \quad \frac{Q}{R}=\frac{l}{l_1}\cos\theta=\cos^2\theta$$

$$\therefore \quad Q=R\cos^2\theta \qquad .(ii)$$

But $\quad Q=\tfrac{1}{2}(P-R)\sec\theta \qquad ..(i)$

$$\therefore \quad R\cos^2\theta=\frac{P-R}{2}\sec\theta$$

$$\therefore \quad 2R\cos^3\theta=(P-R)$$

$$\therefore \quad R(1+2\cos^3\theta)=P$$

$$\therefore \quad R=\frac{P}{1+2\cos^3\theta}$$

and $$Q=\frac{P\cos^2\theta}{1+2\cos^2\theta}$$

Problem 42*. *A rigid horizontal beam of length 2l is carried by three wires each of length l but of sectional areas A_1, A_2, A_3 as shown in Fig. 41. If a load W be placed at a distance Kl from the left end, find the tensions in the three wires neglecting the weight of the beam.*

Solution. Let the tension in the wire AA' be P,

and tension in the wire CC' be Q

and tension in the wire BB' be R.

For the equilibrium, we have

$$P+Q+R=W \qquad \ldots(i)$$

Taking moments about the left end of the beam, we have

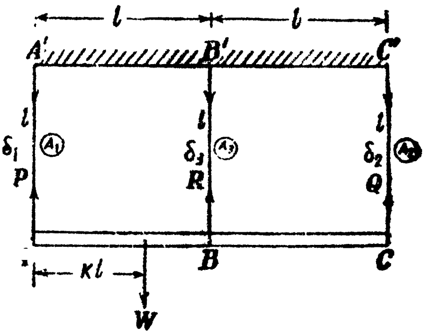


Fig 41.

$$Rl+Q2l=WKl$$

$$\therefore\quad R+2Q=WK$$

$$\therefore\qquad Q=\tfrac{1}{2}(WK-R) \qquad \ldots(ii)$$

$$\therefore\qquad P=W-Q-R$$

$$=W-\tfrac{1}{2}(WK-R)-R$$

$$=W-\tfrac{1}{2}WK-\frac{R}{2}$$

$$=\frac{W}{2}(2-K)-\frac{R}{2}$$

$$P=\frac{W(2-K)-R}{2} \qquad \ldots(iii)$$

Since the beam will always remain straight

Extension of BB' = mean of the extensions of AA' and CC'.

Let the extensions of AA', CC' and BB' be δ_1, δ_2 and δ_3.

$$\therefore\qquad \delta_3=\frac{\delta_1+\delta_2}{2} \qquad \ldots(iv)$$

But $$\delta_1=\frac{P}{A_1E}\,l$$

$$\delta_2=\frac{Q}{A_2E}\,l$$

and $$\delta_3=\frac{R}{A_3}\,l$$

*Junior students may leave this numerical.

Since $\delta_3=\dfrac{\delta_1+\delta_2}{2}$, we have

$$\frac{Rl}{A_3E}=\frac{1}{2}\left[\frac{P}{A_1E}l+\frac{Q}{A_2E}l\right]$$

$$\therefore\quad \frac{R}{A_3}=\tfrac{1}{2}\left[\frac{P}{A_1}+\frac{Q}{A_2}\right]$$

Substituting for P and Q, we have,

$$\frac{R}{A_3}=\tfrac{1}{2}\left[\frac{W(2-K)-R}{2A_1}+\frac{WK-R}{2A_2}\right]$$

$$\therefore\quad \frac{R}{A_3}=\tfrac{1}{4}\left[\frac{W(2-K)-R}{A_1}+\frac{WK-R}{A_2}\right]$$

$$\therefore\quad R\left[\frac{1}{4A_1}+\frac{1}{4A_2}+\frac{1}{A_3}\right]=\frac{W}{4}\left(\frac{2-K}{A_1}+\frac{K}{A_2}\right)$$

$$\therefore\quad R=\left[\frac{\dfrac{2-K}{A_1}+\dfrac{K}{A_2}}{\dfrac{1}{4A_1}+\dfrac{1}{4A_2}+\dfrac{1}{A_3}}\right]\frac{W}{4}$$

R being known the tensions P and Q can be determined from equations (*ii*) and (*iii*).

Problem 43. *A uniform rope of length l units hangs vertically. Find the extension of the first a units of length of the rope from the top due to the weight of the rope itself. Find also the total extension of the rope.*

Solution. Consider an elemental length dx of the rope at a distance x from the bottom of the rope.

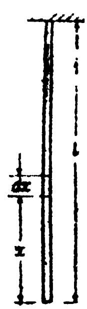


Fig. 42

Let the weight per unit volume of the rope be ρ.

Let the cross-sectional area of the rope be A.

$\therefore$ Force on the cross-section of the elemental part

$$=\rho Ax$$

$\therefore$ Stress on the section of the elemental part

$$=p=\frac{\rho Ax}{A}=\rho x$$

$\therefore$ Extension of the elemental part

$$=\frac{p}{E}dx=\frac{\rho x}{E}dx$$

$$\therefore\quad \text{Total extension}=\delta=\frac{\rho}{E}\int_0^l x\,dx$$

$$\therefore \qquad \delta = \frac{\rho l^2}{2E}$$

Extension of the top a units of length

$= \text{Total extension} - \text{extension of the bottom } (l-a) \text{ units of length.}$

$$= \frac{\rho l^2}{2E} - \frac{\rho (l-a)^2}{2E}$$

$$= \frac{\rho}{2E}\left[l^2 - (l-a)^2 \right]$$

$$= \frac{\rho}{2E}\left(2\,la - a^2 \right)$$

$$= \frac{\rho}{2E}\, a(2l-a)$$

§ 11. Temperature Stresses

When the temperature of a material changes there will be corresponding change in the dimension. When a member is free to expand or contract due to rise or fall of temperature, no stresses will be induced in the member. But, if the natural change in length due to rise or fall of temperature be prevented, stresses will be offered.

Suppose a rod AB of length l be fixed at the ends A and B. Let the temperature rise by T. If the member was free to expand, the free expansion of the member would be $BB' = \alpha Tl$ where α is the coefficient of linear expansion. If the member is allowed to freely expand no stresses will be induced. But if the member is prevented from expanding, compressive stresses will be induced.

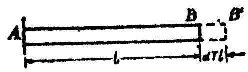


Fig. 43

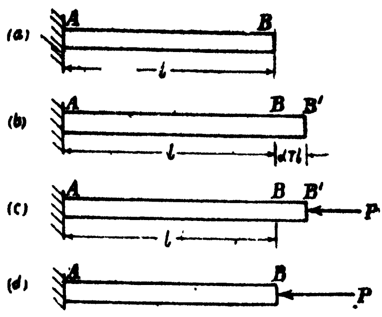


Fig. 44

This can be realized as follows. Let the rod AB be of length l. Let its ends A and B be fixed.

Suppose there is a rise of temperature. The rod tends to expand by αTl.

Suppose the fixture at the end B is removed so that the rod freely expands by αTl so that $BB' = \alpha Tl$.

Let now an external load be applied at B so that the rod is decreased in its length from $(l+\alpha Tl)$ to l.

Let a be the sectional area of the rod.

$$\therefore \quad \text{Compressive stress} = p = \frac{P}{a}$$

Strain $=\frac{\alpha Tl}{l+\alpha Tl} \approx \frac{\alpha Tl}{l} = \alpha T$

$$\frac{\text{Stress}}{\text{Strain}} = E \text{ (Young's modulus)}$$

$$\therefore \quad \frac{p}{\alpha T} = E$$

$$\therefore \quad P = \alpha TE$$

Hence when the rod is prevented from expanding temperature stress $= \alpha TE$.

The thrust on the rod section

$$= P = pa$$

$$= \alpha TEa$$

In general, the temperature strain

$$= \frac{\text{Expansion or contraction prevented}}{\text{original length}}$$

Suppose a rod of length l, when subjected to a rise of temperature is permitted to expand only by δ, the temperature strain

$$e = \frac{\text{Expansion prevented}}{\text{Original length}}$$

$$= \frac{\alpha Tl - \delta}{l}$$

$\therefore$ Temperature stress

$$= P = Ee$$

$$= \frac{E(\alpha Tl - \delta)}{l}$$

Problem 44. *A rod is 2 metres long at 10°C. Find the expansion of the rod when the temperature is raised to 80°C. If this expansion is prevented, find the stress in the material. Take $E = 1 \times 10^6$ kg./cm.^{2} and $\alpha = 0{\cdot}000012$ per °C.* *(AMIE, May 1974)*

Solution. Rise in temperature $= T = 80 - 10 = 70$°C.

Free expansion $= \alpha Tl$

$$= 0{\cdot}000012 \times 70 \times 2 \times 100 \text{ cm.}$$

$$= 0{\cdot}168 \text{ cm.}$$

Temperature stress $= \alpha TE$

$$= 0{\cdot}000012 \times 70 \times 1 \times 10^6 \text{ kg./cm.}^2$$

$$= 840 \text{ kg./cm.}^2$$

Problem 45. (S.I.) *A rod of steel is 20 metres long at a temperature of 20°C. Find the free expansion of the length when the temperature is raised to 65°C. Find the temperature stress produced (i) when* *the expansion of the rod is prevented (ii) when the rod is permitted to expand by 5·8 mm. Take* $\alpha = 12 \times 10^{-6}$ *per °C and* $E = 200\ GN/m^2$

Solution.

Free expansion of the rod $= \alpha Tl$

$$= 12 \times 10^{-6}\,(65-20)\,20 \text{ metre}$$

$$= 0{\cdot}0108 \text{ metre}$$

$$= 10{\cdot}8 \text{ mm.}$$

(i) *When the expansion is fully prevented*

Temperature stress $= \alpha TE$

$$= 12 \times 10^{-6} \times (65-20) \times 200 \times 10^{9}\ N/metre^2$$

$$= 108 \times 10^{6}\ N/metre^2$$

$$= 108\ MN/metre^2$$

(ii) *When the rod is permitted to expand by 5·8 mm.*

In this case, expansion prevented $= 10{\cdot}8 - 5{\cdot}8 = 5$ *mm.*

$$\therefore \quad \text{Strain} = \frac{\text{Expansion prevented}}{\text{Original length}}$$

$$= \frac{5}{20 \times 1000} = \frac{1}{4000}$$

$\therefore$ *Temperature stress* $=$ Strain $\times E$

$$= \frac{1}{4000} \times 200 \times 10^{9}$$

$$= 50 \times 10^{6}\ N/metre^2$$

$$= 50\ MN/metre^2$$

Problem 46. (S.I.). *A 15 mm. diameter steel rod passes centrally through a copper tube 50 mm. external diameter and 40 mm. internal diameter. The tube is closed at each end by rigid plates of negligible thickness. The nuts are tightened lightly home on the projecting parts of the rod. If the temperature of the assembly is raised by 60°C. Calculate the stresses developed in copper and steel.*

Take $E_s = 210\ GN/metre^2$; $E_c = 105\ GN/metre^2$;

$\alpha_s = 12 \times 10^{-6}$ *per °C* ; $\alpha_c = 17{\cdot}5 \times 10^{-6}$ *per °C.*

Solution.

Area of the steel rod $= A_s = \frac{\pi}{4}(15)^2\ mm.^2$

$$= 56{\cdot}25\,\pi\ mm.^2$$

Area of the copper tube $= A_c = \frac{\pi}{4}(50^2 - 40^2)\ mm.^2$

$$= 225\,\pi\ mm.^2$$

Free expansion of steel $= \alpha_s Tl$

Free expansion of copper $= \alpha_c Tl$

Let the actual expansion of each component be δ

$$\alpha_c Tl > \delta > \alpha_s Tl$$

$\therefore$ steel is in tension and copper is in compression.

Let f_s and f_c be the stresses in steel and copper.

For the equilibrium of the system

Tension in steel = Compression in Copper.

$$f_s A_s = f_c A_c$$

$$f_s = \left[\frac{A_c}{A_s}\right] f_c$$

$$\therefore \quad f_s = \left[\frac{225\pi}{56{\cdot}25\pi}\right] f_c$$

$$\therefore \quad f_s = 4f_c$$

Actual expansions of steel = Actual expansion of copper

$$\alpha_s Tl + \frac{f_s}{E_s}\, l = \alpha_c Tl - \frac{f_c}{E_c}\, l$$

$$\therefore \quad \alpha_s T + \frac{f_s}{E_s} = \alpha_c T - \frac{f_c}{E_c}$$

But $f_s = 4f_c$ and substituting for α_s, α_c, E_s and E_c, we get

$$12 \times 10^{-6} \times 60 + \frac{4f_c}{210 \times 10^9} = 17{\cdot}5 \times 10^{-6} \times 60 - \frac{f_c}{105 \times 10^9}$$

$$720 \times 10^3 + \frac{4f_c}{210} = 1050 \times 10^3 - \frac{f_c}{105}$$

$$\therefore \quad \frac{4f_c}{210} + \frac{f_c}{105} = 330 \times 10^3$$

$$\frac{3f_c}{105} = 330 \times 10^3$$

$$f_c = 330 \times 10^3 \times 35 \ N/metre^2$$

$$= 11{\cdot}55 \times 10^6 \ N/metre^2$$

$$= 11{\cdot}55 \ MN/metre^2$$

$$\therefore \quad f_s = 4f_c = 4 \times 11{\cdot}55 = 46{\cdot}2 \ MN/metre^2.$$

Problem 47. *A gun metal rod 22 mm. diameter screwed at the ends passes through a steel tube 25 mm. and 30 mm. internal and external diameters. The temperature of the whole assembly is raised to 126°C and the nuts on the rod are then screwed lightly home on the ends of the tube. Find the intensity of stress in the rod when the common temperature has fallen to 16°C.*

Coefficient of expansion for steel $= 12 \times 10^{-6}$ *per °C*

Coefficient of expansion for gun metal $= 20 \times 10^{-6}$ *per °C*

Modulus of Elasticity for steel $= 2.1 \times 10^6$ *kg./cm².*
Modulus of Elasticity for gun metal $= 0.94 \times 10^6$ *kg./cm².*
(A.M.I.E., May 1966)

Solution. Area of steel tube

$$= A_s = \frac{\pi}{4}\left[3^2 - 2.5^2\right] cm^2.$$

$$= \frac{11}{16}\pi \; cm.^2$$

Area of the gun metal rod $= A_g = \frac{\pi}{4} \times 2.2^2 \; cm^2.$

$$= 1.21\,\pi \; cm^2.$$

Let the length of the rod and tube between the nuts be *l cms.*

If the two members had been free to contract,

free contraction of the gun metal rod $= \alpha_g Tl$

free contraction of the steel tube $= \alpha_s Tl$

Since α_g is greater than α_s the free contration of the gun metal rod is greater than the free contraction of the steel tube. But, since

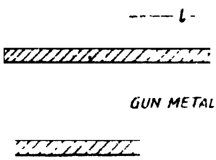


Fig. 45

the ends of the rod have been provided with nuts the two members are not free to contract fully, each of the members will contract by the same amount. Let δ *cm.* be the final contraction of each rod. The free contraction of the gun metal rod is greater than δ, while the free contraction of the steel tube is less than δ. Hence the steel tube will be subjected to compressive stress while the gun metal rod will be subjected to tensile stress. Let p_s and p_g be the stresses in steel and gun metal.

For the equilibrium of the whole system.

Total compressive force in steel

= Total tension in gun metal.

$$\therefore \quad p_s A_s = p_g A_g$$

$$p_s = \frac{A_g}{A_s} p_g = \frac{1.21\pi}{\left(\frac{11}{16}\pi\right)} p_g$$

$$p_s = 1.76\, p_g$$

Final contraction of steel

=Final contraction of gun metal

$$\therefore \quad \alpha_s Tl + \frac{p_s}{E_s} l = \alpha_g Tl - \frac{p_g}{E_g} l$$

$$\therefore \quad \alpha_s T + \frac{p_s}{E_s} = \alpha_g T - \frac{p_g}{E_g}$$

$$T = 126 - 16 = 110^\circ C$$

$$\therefore \; 12 \times 10^{-6} \times 110 + \frac{1.76 p_g}{2.1 \times 10^6} = 20 \times 10^{-6} \times 110 - \frac{p_g}{0.94 \times 10^6}$$

$$\therefore \; 12 \times 110 + \frac{1.76}{2.1} p_g = 20 \times 110 - \frac{p_g}{0.94}$$

$$\therefore \; p_g \left(\frac{1.76}{2.1} + \frac{1}{0.94} \right) = 8 \times 110$$

$$p_g = 462.7 \text{ kg./cm}^2 \; (tensile)$$

$$\therefore \quad p_s = 1.76 \times 462.7 \text{ kg./cm}^2.$$

$$\therefore \quad = 814.3 \text{ kg./cm}^2. \; (compressive)$$

Problem 48. *A steel bar is placed between two copper bars each having the same area and length as the steel bar at 15°C. At this stage they are rigidly connected together at both the ends. When the temperature is raised to 315°C, the length of the bars increases by 0.15 cm. Determine the original length and the final stresses in the bars.*

Take $E_s = 2.1 \times 10^6$ *kg./cm.*^{2} ; $E_c = 1 \times 10^6$ *kg./cm.*^{2} ;

$\alpha_s = 0.000012$ *per °C* ; $\alpha_c = 0.0000175$ *per °C*

(*AMIE, Summer 1978*)

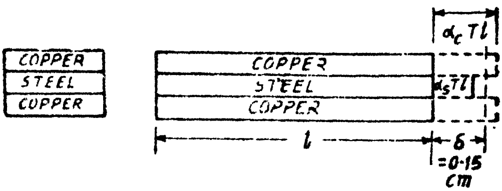


Fig. 46

Solution.

Let the sectional area of the steel component be A cm.^{2}

$\therefore$ Sectional area of the copper component $= 2A$ cm.^{2}

Free expansion of steel component $= \alpha_s Tl$

Free expansion of copper component $= \alpha_c Tl$

Let δ be the actual expansion of each bar.

$$\alpha_s Tl < \delta < \alpha_c Tl$$

$\therefore$ Steel is in tension and copper is in compression.

Let f_s and f_c be the stresses in steel and copper respectively.

For the equilibrium of the system

Tension in steel = compression in copper.

$$\therefore \quad f_s A = f_c (2A)$$

$$\therefore \quad f_s = 2f_c$$

Actual expansion of steel = Actual expansion of copper.

$$\therefore \quad \alpha_s Tl + \frac{f_s}{E_s} l = \alpha_c Tl - \frac{f_c}{E_c} l$$

$$\therefore \quad \alpha_s T + \frac{f_s}{E_s} = \alpha_c T - \frac{f_c}{E_c}$$

$$T = 315 - 15 = 300°C$$

$$\therefore \quad 0{\cdot}000012 \times 300 + \frac{2f_c}{2{\cdot}1 \times 10^6} = 0{\cdot}0000175 \times 300 - \frac{f_c}{1 \times 10^6}$$

$$\therefore \quad 12 \times 300 + \frac{2f_c}{2{\cdot}1} = 17{\cdot}5 \times 300 - f_c$$

$$1{\cdot}9524\, f_c = 1650$$

$$\therefore \quad f_c = 845{\cdot}11 \textit{ kg./cm.}^2 \textit{ (compressive)}$$

$$\therefore \quad f_s = 2 \times 845{\cdot}11 = 1690{\cdot}22 \textit{ kg./cm.}^2 \textit{ (tensile)}$$

Now consider say the steel bar.

Actual expansion of the steel bar

$$= \alpha_s Tl + \frac{f_s}{E_s} l = 0{\cdot}15$$

$$0{\cdot}000012 \times 300\, l + \frac{1690{\cdot}22}{2{\cdot}1 \times 10^6} l = 0{\cdot}15$$

$$\therefore \quad 12 \times 300\, l + \frac{1690{\cdot}22}{2{\cdot}1} l = 150000$$

$$\therefore \quad 3600\, l + 804{\cdot}87\, l = 150000$$

$$\therefore \quad 4404{\cdot}87\, l = 150000$$

$$\therefore \quad l = 34{\cdot}05 \textit{ cm.}$$

Problem 49. *A 12 mm. diameter steel rod passes centrally through a copper tube 48 mm. external and 36 mm. internal diameter and 2·50 metres long. The tube is closed at each end by 24 mm. thick steel plates which are secured by nuts. The nuts are tightened until the copper tube is reduced in length by 0·508 mm. The whole assembly is then raised in temperature by 60°C. Calculate the stress in copper and steel before and after the rise of temperature, assuming that the thickness of the plates remains unchanged.*

Take $E_s = 2{\cdot}1 \times 10^6$ *kg./cm.²,* $E_c = 1{\cdot}05 \times 10^6$ *kg./cm².,*

$\alpha_s = 12 \times 10^{-6}/°C$ *and* $\alpha_c = 17{\cdot}5 \times 10^{-6}/°C$

Solution. Area of the steel rod $= A_s = \frac{\pi}{4}(1\cdot2)^2\ cm^2. = 0\cdot36\pi\ cm^2.$

Area of the copper tube

$$= A_c = \frac{\pi}{4}(4\cdot8^2 - 3\cdot6^2)\ cm.^2 = 2\cdot52\pi\ cm^2.$$

Case (i). Stresses due to tightening the nuts. When the nuts are tightened the steel rod will be subjected to tensile stress and the copper tube will be subjected to compressive stress. Let p_c and p_s be the stresses in copper and steel.

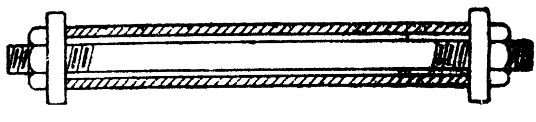


Fig. 47

Total compression in copper = Total tension in steel.

$$p_c A_c = p_s A_s$$

$$\therefore \quad p_s = \frac{A_c}{A_s}\, p_c = \frac{2\cdot52\pi}{0\cdot36\pi}\, p_c$$

$$\therefore \quad p_s = 7 p_c$$

Strain in copper $= e_c = \dfrac{\text{change in length}}{\text{original length}}$

$$e_c = \frac{0\cdot0508}{250}$$

Stress in copper $= p_c = e_c E_c$

$$= \frac{0\cdot0508}{250} \times 1\cdot05 \times 10^6\ kg./cm.^2$$

$$= 213\cdot4\ kg./cm.^2\ (compressive)$$

$\therefore$ Stress in steel $= p_c = 7p_s = 7 \times 213\cdot4$

$$= 1493\cdot8\ kg./cm.^2$$

Case (ii). Stresses due to rise of temperature.

If the two members had been free to expand,

Free expansion of steel $= \alpha_s T l_s$

Free expansion of copper $= \alpha_c T l_c$

Since α_c is greater than α_s the free expansion of copper is greater than the free expansion of steel. But since the ends of the rod are provided with washers and nuts the members are not free to expand fully. Final expansion of each of the members will be the same. Let this final expansion be δ. The free expansion of copper is greater than δ while the free expansion of steel is less than δ. Hence the steel rod will be subjected to a tensile stress while the copper tube will be subjected to a compressive stress. Let f_s and f_c be the stresses in steel and copper. For the equilibrium of the whole system, Total tension in steel = Total compression in copper

$\therefore \quad f_s A_s = f_c A_c$

$\therefore \quad f_s = \dfrac{A_c}{A_s} f_c \qquad \therefore f_s = 7 f_c$

Final expansion of steel

= Final expansion of copper

$\therefore \quad \alpha_s T l_s + \dfrac{f_s}{E_s} \cdot l_s = \alpha_c T l_c - \dfrac{f_c}{E_c} l_c$

But $T = 60°C$,

$l_s = 250 + 4·8 = 254·8$ *cms.*

and $l_c = 250$ *cms.*

$\therefore \quad 12 \times 10^{-6} \times 60 \times 254·8 + \dfrac{7 f_c \times 254·8}{2·1 \times 10^6}$

$= 17·5 \times 10^{-6} \times 60 \times 250 - \dfrac{f_c \times 250}{1·05 \times 10^6}$

$\therefore \quad f_c = 72·64$ *kg./cm.*^{2} *(compressive)*

$\therefore \quad f_s = 7 \times 72·64$ *kg./cm.*^{2}

$\therefore \quad f_s = 508·48$ *kg /cm.*^{2} *(tensile)*

$\therefore$ Final stresses due to tightening the nuts and rise of temperature :

Stress in copper $= p_c + f_c = 213·4 + 72·64 = 286·04$ *kg./cm.*^{2} *(compressive)*

Stress in steel $= p_s + f_s = 1493·8 + 508·41 = 2002·28$ *kg./cm.*^{2} *(tensile)*

Problem. 50. *A steel rod 20 mm. diameter and 6 metre long is connected to two grips one at each end at a temperature of 120°C. Find the pull exerted when the temperature falls to 40°C (i) if the ends do not yield (ii) if the ends yield by 0 11 cm. Take $E = 2 \times 10^6$ kg./cm.^{2} and $\alpha = 12 \times 10^{-6}/°C$.*

Solution. Length of the rod $= l = 600$ *cms*

Fall of temperature $= T = 120 - 40 = 80°C$

Case (i) When the ends do not yield.

$\therefore$ Temperature stress $= \alpha T E$

$= 12 \times 10^{-6} \times 80 \times 2 \times 10^6$ *kg./cm.*^{2}

$= 1920$ *kg./cm.*^{2} *(tensile)*

$\therefore$ Pull in the rod $=$ Stress $\times$ area

$= 1920 \times \dfrac{\pi}{4} \times 2^2$ *kg*

$= 6033$ *kg.*

Case (ii). When the ends yield by 0·11 cm.

$$\text{Temperature strain} = \frac{\text{Contraction prevented}}{\text{Original length}}$$

$$= \frac{\alpha T l - \delta}{l}$$

$$= \frac{12 \times 10^{-6} \times 80 \times 600 - 0{\cdot}11}{600}$$

$$= \frac{0{\cdot}466}{600}$$

$\therefore$ Temperature stress = Strain × Young's Modulus

$$= \frac{0{\cdot}466}{600} \times 2 \times 10^6 \ kg./cm.^2$$

$$= 1553 \ kg./cm.^2$$

$\therefore$ Pull in the rod = Stress × area

$$= 1553 \times \frac{\pi}{4} \times 2^2 \ kg. = 4878 \ kg.$$

Problem 51. *A steel tube 4·5 cm. external diameter and 3 mm. thick encloses centrally a solid copper bar of 3 cm. diameter. The bar and the tube are rigidly connected together at the ends at a temperature of 30°C. Find the stress in each metal when heated to 180°C. Also find the increase in length if the original length of the assembly is 30 cm. Coefficients of expansion for steel and copper are $1{\cdot}08 \times 10^{-5}$ and $1{\cdot}7 \times 10^{-5}$ respectively per degree centigrade. $E = 2{\cdot}1 \times 10^6$ kg./cm.² for steel and $1{\cdot}1 \times 10^6$ kg./cm.² for copper.*

Solution.

Area of steel tube $A_s = \frac{\pi}{4}(4{\cdot}5^2 - 3{\cdot}9^2) = 3{\cdot}959 \ cm^2$.

Area of copper bar $= A_c = \frac{\pi}{4}(3)^2 = 7{\cdot}069 \ cm^2$.

Since the coefficient of expansion for copper is greater than that of steel, the free expansion of the copper bar is greater than the free expansion of the steel tube. Since the two components are rigidly connected together at the ends,

actual expansion of steel = actual expansion of copper.

Let δ be the actual expansion of each component.

Obviously, δ is greater than the free expansion of the steel tube and less than the free expansion of copper. Hence steel is in tension and the copper bar is in compression. For the equilibrium of the system.

Total tension in steel = Total compression in copper

Let the stresses in steel and copper be f_s and f_c respectively.

$\therefore \quad f_s A_s = f_c A_c$

$\therefore \quad f_s = \frac{A_c}{A_s} f_c = \frac{7{\cdot}069}{3{\cdot}959} f_c$

$\therefore \quad f_s = 1{\cdot}785 f_c$

Actual expansion of steel = Actual expansion of copper

$$\alpha_s Tl + \frac{f_s}{E_s} l = \alpha_c Tl - \frac{f_c}{E_c} l$$

$$\therefore \quad \alpha_s T + \frac{f_s}{E_s} = \alpha_c T - \frac{f_c}{E_c}$$

$$T = 180 - 30 = 150°C$$

$$\therefore \quad 1{\cdot}08 \times 10^{-5} \times 150 + \frac{f_s}{2{\cdot}1 \times 10^6} = 1{\cdot}70 \times 10^{-5} \times 150 - \frac{f_c}{1{\cdot}1 \times 10^6}$$

$$\therefore \quad 1620 + \frac{1{\cdot}785 f_c}{2{\cdot}1} = 2550 - \frac{f_c}{1{\cdot}1}$$

$$1620 + 0{\cdot}85 f_c = 2550 - 0{\cdot}9091 f_c$$

$$\therefore \quad 1{\cdot}7591 f_c = 930$$

$$\therefore \quad f_c = \frac{930}{1{\cdot}7591} = 528{\cdot}8 \text{ kg./cm.}^2 \quad \textit{(compressive)}$$

$$\therefore \quad f_s = 1{\cdot}785 \times 528{\cdot}8 = 943{\cdot}9 \text{ kg./cm.}^2 \quad \textit{(tensile)}$$

Increase in length of either component

$$= \alpha_s Tl + \frac{f_s}{E_s} l$$

$$= \left(\alpha_s T + \frac{f_s}{E_s} \right) l$$

$$= \left[1{\cdot}08 \times 10^{-5} \times 150 + \frac{943{\cdot}9}{2 \times 10^6} \right] 30 \text{ cm.}$$

$$= 0{\cdot}062 \text{ cm.}$$

Problem 52. *A weight of 20 tonnes is supported by three short pillars, each 5 cm.² in section. The central pillar is of steel and the outer ones are of copper. The pillars are so adjusted that at a temperature of 15°C each carries equal load. The temperature is then raised to 115°C. Estimate the stress in each pillar at 15°C and 115°C. Take* $E_s = 2 \times 10^6$ *kg./cm.²;* $E_c = 0{\cdot}8 \times 10^6$ *kg./cm.²;* $\alpha_s = 12 \times 10^{-6}/°C$ *and* $\alpha_c = 18{\cdot}5 \times 10^{-6}/°C$. *(A.M.I.E. May, 1965)*

Solution. Area of each pillar $= A = 5$ *cm.²*

Initial stresses

At 15°C each pillar carries $\frac{20}{3}$ *tonnes.*

Stress in each pillar $= \frac{20}{3 \times 5} = \frac{4}{3}$ *tonnes/cm².*

$$= 1333{\cdot}33 \text{ kg./cm.}^2$$

Stresses due to rise of temperature alone. Let the stresses due to rise of temperature alone be p_c *kg/cm.*^{2} (*compressive*) in copper and p_s kg./*cm.*^{2} (*tensile*) in steel.

Let the change in length of each member be δ.

$$\therefore \quad \alpha_s Tl + \frac{p_s}{E_s} l = \alpha_c Tl - \frac{p_c}{E_c} l$$

$$\therefore \quad \alpha_s T + \frac{p_s}{E_s} = \alpha_c T - \frac{p_c}{E_c}$$

$$\therefore \quad \frac{p_s}{E_s} + \frac{p_c}{E_c} = T(\alpha_c - \alpha_s) = (115-15)(18{\cdot}5-12)10^{-6}$$

$$\therefore \quad \frac{p_s}{2\times10^6} + \frac{p_c}{0{\cdot}8\times10^6} = 100\times6{\cdot}5\times10^{-6}$$

$$\therefore \quad \frac{p_s}{2} + \frac{p_c}{0{\cdot}8} = 650$$

$$\therefore \quad 2p_s + 5p_c = 2600 \qquad \ldots(i)$$

Also $\quad p_s A_s = p_c A_c$

$$\therefore \quad p_s \times 5 = p_c \times 10$$

$$\therefore \quad p_s = 2p_s \qquad \ldots(ii)$$

Substituting in Eqn. (*i*), we get,

$$2\times 2p_c + 5p_c = 2600$$

$$\therefore \quad p_c = \frac{2600}{9} = 288{\cdot}89 \textit{ kg./cm.}^2 \quad (\textit{compressive})$$

and $\quad p_c = 2\times288{\cdot}89 = 577{\cdot}78$ *kg./cm.*^{2} (*tensile*)

$\therefore$ Final stress in copper

$$= 1333{\cdot}33 + 288{\cdot}89 = \textit{1622·22 kg./cm.}^2$$

Final stress in steel $= 1333{\cdot}33 - 577{\cdot}78 = $ *755·55 kg./cm.*^{2}

Problem 53. *A flat bar of aluminium alloy 24 mm. wide and 6 mm. thick is placed between two steel bars each 24 mm. wide and 9 mm. thick to form a composite bar 24 mm. × 24 mm. as shown in Fig. 48. The three bars are fastened together at their ends when the temperature is 10°C. Find the stress in each of the materials when the temperature of the whole assembly is raised to 50°C.*

If at the new temperature a tensile load of 2000 kg. is applied to the composite bar what are the final stresses in steel and alloy? Take $E_s = 2\times10^6$ *kg./cm.*^{2}, $E_a = \frac{2}{3}\times10^6$ *kg./cm.*^{2}, $\alpha_s = 12\times10^{-6}$ *per °C and* $\alpha_a = 24\times10^{-6}$ *per °C.*

Solution. Area of aluminium

$A_a = 2{\cdot}4 \times 0{\cdot}6 = 1{\cdot}44 \ cm.^2$

Area of steel

$A_s = 2 \times 2{\cdot}4 \times 0{\cdot}9 = 4{\cdot}32 \ cm^2$

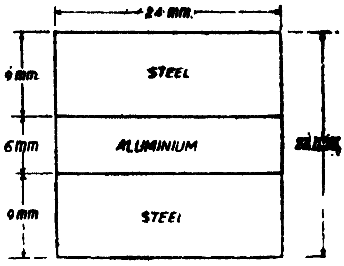


Fig. 48

(*i*) *Stresses due to rise of temperature.* If the two members had been free to expand,

free expansion of steel $= \alpha_s Tl$

free expansion of aluminium $= \alpha_a Tl$

But since the members are fastened to each other at the ends final expansion of each member would be the same. Let this expansion be δ. The free expansion of aluminium is greater than δ while the free expansion of steel is less than δ. Hence steel is subjected to tensile stress while aluminium is subjected to compressive stress. Let p_s and p_a be the stresses in steel and aluminium.

For the equilibrium of the whole system

Total tension in steel = Total compression in aluminium

i.e., $p_s A_s = p_a A_a$

$\therefore \quad p_s \times 4{\cdot}32 = p_a \times 1{\cdot}44$

$\therefore \quad p_s = \frac{p_a}{3}$

Final increase in length of steel = Final increase in length of aluminium.

$$\therefore \quad \alpha_s Tl + \frac{p_s}{E_s} l = \alpha_a Tl - \frac{p_a}{E_a} l$$

$$\therefore \quad \alpha_s T + \frac{p_s}{E_s} = \alpha_a T - \frac{p_a}{E_a}$$

But $T = 50 - 10 = 40°C$

$$\therefore \quad 12 \times 10^{-6} \times 40 + \frac{p_s}{2 \times 10^6} = 24 \times 10^{-6} \times 40 - \frac{p_a}{\frac{2}{3} \times 10^6}$$

$$\therefore \quad 480 + \frac{p_s}{2} = 960 - \frac{3}{2} p_a$$

$$\therefore \quad \frac{p_s}{2} + \frac{3}{2} p_a = 480$$

$$\therefore \quad p_s + 3p_a = 1920$$

But $p_s = \frac{p_a}{3}$

$$\therefore \quad \frac{p_a}{3} + 3p_a = 1920$$

$$\therefore \quad p_a = 576 \ kg./cm.^2 \ (compressive)$$

$\therefore \quad p_s = \frac{576}{3}$ *kg./cm.*^{2}

$\therefore \quad p_s = 192$ *kg./cm.*^{2} *(tensile)*

(ii) Stresses due to external load of 2000 kg.

Let the stresses due to the external loading be f_s and f_a in steel and aluminium.

Strain in steel = strain in aluminium.

$$\therefore \quad \frac{f_s}{E_s} = \frac{f_a}{E_a}$$

$$\therefore \quad f_s = \frac{E_s}{E_a} f_a = 3 f_a$$

But load on steel + Load on aluminium = Total load

$\therefore \quad f_a A_a + f_s A_s = P$

$\therefore \quad 1{\cdot}44 f_a + 4{\cdot}32 f_s = 2000$

$\therefore \quad 1{\cdot}44 f_a + 4{\cdot}32 \times 3 f_a = 2000$

$\therefore \quad 14{\cdot}40 f_a = 2000$

$\therefore \quad f_a = \frac{2000}{14{\cdot}4}$ *kg./cm.*^{2}

$= 69{\cdot}44$ *kg./cm.*^{2} *(compressive)*

$\therefore \quad f_s = 3 \times 69{\cdot}44$ *kg./cm.*^{2}

$= 208.32$ *kg./cm.*^{2} *(compressive)*

Final stresses due to rise of temperature and loading

Stress in aluminium $= 576 + 69{\cdot}44$ *kg./cm*^{2}

$= 645{\cdot}44$ *kg./cm.*^{2} *(compressive)*

Stress in steel $= 208{\cdot}32 - 192$ *kg./cm.*^{2}

$= 16{\cdot}32$ *kg./cm.*^{2} *(Compressive)*

Problem. 54. *Two steel rods one of 8 cm. diameter and the other of 6 cm. diameter are joined end to end by means of a turn buckle. The other end of each rod is rigidly fixed and there is initially a small tension in the rods. If the effective length of each rod is 4 metres, find the increase in this tension when the turn buckle is turned by one-quarter of a turn. On the rod of bigger diameter there are 1·5 threads per centimetre while there are 2 threads per centimetre on the other rod. Neglect the extension of the turn buckle. Find also what rise in temperature would nullify the increase in tension.*

Take $E = 2 \times 10^6$ *kg./cm.*^{2} *and* $\alpha = 12 \times 10^{-6}/°C$.

Solution. Cross-sectional area of the smaller bar

$$= A_1 = \frac{\pi}{4} \times 6^2 \text{ cm.}^2$$

$$= 28{\cdot}28 \text{ cm.}^2$$

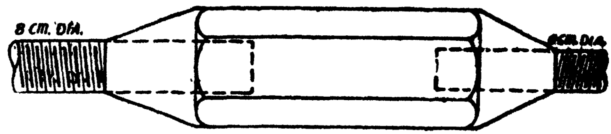


Fig. 49.

Cross-sectional area of the bigger bar

$$=A_2=\frac{\pi}{4}\times 8^2\ cm.^2$$

$$=50{\cdot}27\ cm.^2$$

When the turn buckle is turned by one quarter of a turn,

Extension of the smaller bar $=\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}\ cm.$

Extension of the bigger bar $=\frac{1}{4}\quad\frac{1}{1{\cdot}5}=\frac{1}{6}\ cm.$

$\therefore$ Total extension of the two rods $=\frac{1}{8}+\frac{1}{6}\ cm.$

$$=\frac{7}{24}\ cm.$$

Let the tension in each bar be $T\ kg$.

$\therefore$ Total extension in the two bars

$$=\frac{P}{A_1E}l_1+\frac{P}{A_2E}l_2$$

$$=\frac{400P}{2\times 10^6}\left(\frac{1}{28{\cdot}28}+\frac{1}{50{\cdot}27}\right)cms.$$

$$=\frac{400}{2\times 10^6}\times\frac{78{\cdot}55P}{28{\cdot}28\times 50{\cdot}27}\quad\frac{7}{24}$$

$\therefore$ $P=26380\ kg.$

In order this tension must be nullified by rise of temperature total expansion of the two rods must be equal to $\frac{7}{24}\ cm.$

Let the rise of temperature be $T°C$

$\therefore$ $\alpha Tl=\frac{7}{24}$

$\therefore\ 12\times 10^{-6}\times T\times 800=\frac{7}{24}$

$\therefore$ $T=\frac{7\times 10^6}{24\times 12\times 800}$

$$=30{\cdot}38°C.$$

Problem 55. *A steel rod 32 mm. in diameter is fixed concentrically in a brass tube which has outside and inside diameters of 48 mm. and 34 mm. respectively. Both the rod and the tube are 40 cms. long and their ends are level. The compound rod is held between two stops which are exactly 40 cms. apart and the temperature of the bar is then raised by 60°C.*

(*a*) *Find the stresses in the rod and tube if the distance between the stops* (*i*) *remains constant* (*ii*) *is increased by 0·025 cm.*

(*b*) *Find the increase in the distance between the stops if the force exerted between them is 8000 kg.*

Take $E_s = 2 \times 10^6$ kg./cm.^{2} $\quad E_b = 0{\cdot}9 \times 10^6$ kg./cm.^{2}

$\alpha_s = 12 \times 10^{-6}$ per°C and $\alpha_b = 21 \times 10^{-6}$ per°C.

Solution. (*a*) (*i*) *When the distance between the stops remains constant.*

Stress in steel $= \alpha_s T E_s$

$= 12 \times 10^{-6} \times 60 \times 2 \times 10^6$ kg./cm.^{2}

$= 1440$ kg./cm.^{2} (*compressive*)

Stress in brass $= \alpha_b T E_b$

$= 21 \times 10^{-6} \times 60 \times 0{\cdot}9 \times 10^6$ kg./cm.^{2}

$= 1134$ kg./cm.^{2} (*compressive*)

(*ii*) *When the distance between the stops is increased by 0·025 cm.*

Strain in steel $= e_s = \dfrac{\text{expansion prevented}}{\text{original length}}$

$$= \frac{\alpha_s T l - \delta}{l}$$

$$= \frac{12 \times 10^{-6} \times 60 \times 40 - 0{\cdot}025}{40}$$

$= 0{\cdot}000095$

∴ Stress in steel $= E_s e_s = 2 \times 10^6 \times 0{\cdot}000095$ kg./cm.^{2}

$= 190$ kg./cm.^{2} (*compressive*)

Strain in brass $= e_b = \dfrac{\text{expansion prevented}}{\text{original length}}$

$$= \frac{\alpha_b T l - \delta}{l}$$

$$= \frac{21 \times 10^{-6} \times 60 \times 40 - 0{\cdot}025}{40}$$

$= 0{\cdot}000635$

∴ Stress in brass $= E_b e_b$

$= 0{\cdot}9 \times 10^6 \times 0{\cdot}000635$ kg./cm.^{2}

$= 571{\cdot}5$ kg./cm.^{2} (*compressive*) *(b) When the force exerted between the stops is 8000 kg.*

Let the expansion of the composite member be δ *cm*.

Strain in steel $= e_s = \dfrac{\alpha_s Tl - \delta}{l}$

$$= \frac{12 \times 10^{-6} \times 60 \times 40 - \delta}{40}$$

$$= \left(720 \times 10^{-6} - \frac{\delta}{40}\right)$$

∴ Stress in steel $= p_s = E_s e_s$

$$= 2 \times 10^{6} \left(720 \times 10^{-6} - \frac{\delta}{40}\right) \text{ kg./cm.}^2$$

$$= \left(1440 - \frac{10^{6}\,\delta}{20}\right) \text{ kg./cm.}^2$$

Similarly strain in brass

$$= e_b = \frac{\alpha_b\, Tl - \delta}{l}$$

$$= \frac{21 \times 10^{-6} \times 60 \times 40 - \delta}{40}$$

$$= \left(1260 \times 10^{-6} - \frac{\delta}{40}\right)$$

∴ Stress in brass $= p_b = E_b\, e_b$

$$= 0{\cdot}9 \times 10^{6} \left(1260 \times 10^{-6} - \frac{\delta}{40}\right) \text{ kg./cm.}^2$$

$$= \left(1134 - \frac{0{\cdot}9}{40} \times 10^{6}\,\delta\right) \text{ kg./cm.}^2$$

Area of steel $= A_s = \dfrac{\pi}{4} \times 3{\cdot}2^2 \text{ cm.}^2$

$$= 8{\cdot}041 \text{ cm}^2.$$

Area of brass $= A_b = \dfrac{\pi}{4}\left(4{\cdot}8^2 - 3{\cdot}4^2\right) \text{ cm.}^2 = 9{\cdot}016 \text{ cm}^2.$

Load on steel + load on brass

= Total load between the stops

$$p_s A_s + p_b A_b = P$$

$$\left(1440 - \frac{10^{6}\delta}{20}\right) 8{\cdot}041 + \left(1134 - \frac{0{\cdot}9}{40} \times 10^{6}\,\delta\right) 9{\cdot}016 = 8000 \text{ kg.}$$

∴ $11580 + 10220 - 605000\,\delta = 8000$

∴ $$\delta = \frac{13800}{605000} = 0{\cdot}02281 \text{ cm.}$$

§12. Hoop Stress.

Fig. 50 shows a thin steel tyre of internal diameter d. Such a tyre can be shrunk on to a wheel of slightly bigger diameter D. The steel tyre is heated so that its diameter exceeds D. In this stage the steel tyre is slipped on to the wheel. If now the tyre be cooled it is prevented from assuming its original diameter d. Hence it will grip the wheel.

Hence a tensile stress is induced circumferentially along the tyre. Such a stress is called a hoop stress.

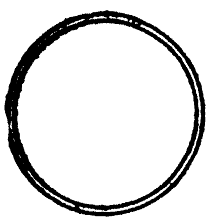


Fig. 50

Temperature strain

$$=e=\frac{\text{contraction prevented}}{\text{original length}}$$

$$=\frac{\pi D-\pi d}{\pi d}$$

$$=\frac{D-d}{d}$$

Hoop stress due to fall of temperature

$$=p=eE$$

$$=\left(\frac{D-d}{d}\right)E.$$

Problem 56. *A rigid wheel is 3 metres in diameter. It is desired to shrink on to the wheel a thin steel tyre. Find the internal diameter of the tyre if after fitting the hoop stress in the tyre is 900 kg./cm.² Find also the least temperature to which the tyre must be heated above that of the wheel.*

Take $E=2\times10^6$ kg./cm.² and $\alpha=12\times10^{-6}$ per °C.

Solution. Hoop stress $p=\left(\frac{D-d}{d}\right)E=900$ kg./cm.²

$$=\left(\frac{300-d}{d}\right)2\times10^6=900$$

$$\therefore \quad \frac{300-d}{d}=\frac{900}{2\times10^6}$$

$$\therefore \quad \frac{300}{d}=1+\frac{900}{2\times10^6}$$

$$\therefore \quad \frac{d}{300}=\left(1+\frac{900}{2\times10^6}\right)^{-1}$$

$$\therefore \quad \frac{d}{300}=1-\frac{900}{2\times10^6} \text{ approximately}$$

$$=1-0\cdot00045=\cdot99955$$

$$\therefore \quad d=\cdot99955\times300 \text{ cm.}=299\cdot865 \text{ cms.}$$

Let the rise of temperature to which the tyre must be subjected be T°C.

$$\therefore \quad \pi D = \pi d(1+\alpha T)$$

$$1+\alpha T = \frac{D}{d}$$

$$\therefore \quad \alpha T = \frac{D}{d} - 1 = \frac{D-d}{d} = \frac{p}{E}$$

$$\alpha T = \frac{p}{E}$$

$$\therefore \quad T = \frac{p}{\alpha E}$$

$$= \frac{900}{12 \times 10^{-6} \times 2 \times 10^{6}}$$

$$= \frac{900}{24} = 35{\cdot}83°C.$$

Problem 57 (SI). *A rigid wheel 1·25 metre in diameter is to be provided with a thin steel tyre. If the stress in the steel tyre is not to exceed 140 MN/metre², find the minimum diameter of the tyre. Find also the minimum temperature to which the tyre is to be raised so that it can be fitted over the wheel.*

Take $E = 200$ *GN/metre² and* $\alpha = 12 \times 10^{-6}$ *per °C.*

Solution. D = Diameter of the rigid wheel

d = Least diameter of the steel tyre.

Strain in the tyre $= \dfrac{D-d}{d}$

Stress in the tyre $= \left(\dfrac{D-d}{d}\right)E = f$

$$\frac{D}{d} - 1 = \frac{f}{E} = \frac{140 \times 10^{6}}{200 \times 10^{9}} = {\cdot}0007$$

$$\therefore \quad \frac{D}{d} = 1{\cdot}0007$$

$$\therefore \quad \frac{d}{D} = 0{\cdot}9993$$

$$\therefore \quad d = 0{\cdot}9993 \times 1{\cdot}25 = 1{\cdot}24912 \text{ metre.}$$

Let the steel be subjected to a temperature rise of T °C

$$\therefore \quad \pi D = \pi d\,[1+\alpha T]$$

$$\therefore \quad 1+\alpha T = \frac{D}{d} = 1{\cdot}0007$$

$$\therefore \quad \alpha T = 0{\cdot}0007$$

$$\therefore \quad T = \frac{0{\cdot}0007}{12 \times 10^{-6}} = 58{\cdot}33 \text{ °C.}$$

§13. Lateral Strain and Poisson's Ratio

Suppose a cylindrical rod be subjected to an axial tensile load P, the length of the rod will obviously increase. But at the same time the diameter of the rod will decrease. In other words, a longitudinal stress will not only produce a strain in its own direction, but will also produce a lateral strain.

Similarly suppose a rectangular bar of width b, depth d and length l be subjected to an axial tensile load. The deformation of the member will take place such that the length of the member will increase while the lateral dimensions namely the breadth and the depth will decrease. Let δl be the increase in length and let δb and δd be the decrease in width and depth.

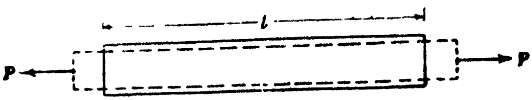


Fig. 51. Shows a rod of length l and diameter d.

The ratio $\frac{\delta l}{l}$ is called the longitudinal strain while the strain $\frac{\delta b}{b}$ or $\frac{\delta d}{d}$ is called the lateral strain.

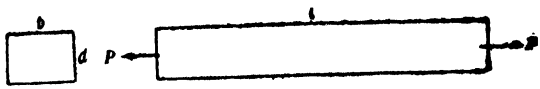


Fig. 52.

When the deformation of the member is within the elastic limit it is found that the ratio of the lateral strain to the longitudinal strain is a constant for a given material. This ratio is called Poisson's ratio and is usually denoted by $\frac{1}{m}$. For most of the metals we come across m lies between 3 and 4.

$$\text{Hence} \quad \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{1}{m} = \text{Poisson's ratio.}$$

§14. Volumetric Strain

When a member is subjected to forces deforming it, it undergoes changes in its dimensions and hence its volume will be subjected to changes. The ratio of the change in volume to the original volume is called the volumetric strain. This is usually denoted by e_v

$$\therefore \quad e_v = \frac{\text{Change in volume}}{\text{Original volume}}.$$

§15. Volumetric Strain of a Rectangular Bar

Let a rectangular bar l units long, b units wide and d units deep undergo small changes by δl, δb and δd respectively in length, width and depth.

Original volume $= V = lbd$

Final volume $=(l+\delta l)(b+\delta b)(d+\delta d)$

$=lbd+bd\,\delta l+lb\,\delta d+dl\,\delta d$

(ignoring products of small quantities)

$\therefore$ Change in volume

$=\delta V=bd\,\delta l+lb\,\delta d+dl\,\delta b$

$\therefore$ Volumetric strain

$$=e_v=\frac{\text{Change in volume}}{\text{Original volume}}$$

$$=\frac{bd\,\delta l+lb\,\delta d+dl\,\delta b}{lbd}$$

$$=\frac{\delta l}{l}+\frac{\delta d}{d}+\frac{\delta b}{b}$$

$$=e_l+e_d+e_b$$

$\therefore$ Volumetric strain = Strain of the length + Strain of the depth + Strain of the width.

§16. Volumetric Strain of a Cylindrical Rod

Let a rod be l units long. Let its diameter be d.

Let the length and diameter change by δl and δd respectively.

Original volume $=V=\dfrac{\pi d^2 l}{4}$

Final volume $=\dfrac{\pi}{4}(d+\delta d)^2(l+\delta l)$

$=\dfrac{\pi}{4}(d^2l+d^2\delta l+2ld\,\delta d)$

ignoring products and higher powers of small quantities.

$\therefore$ Change in volume $=\delta V=\dfrac{\pi}{4}(d^2\delta l+2ld\,\delta d)$

$\therefore$ Volumetric strain

$$=e_v=\frac{\text{change in volume}}{\text{original volume}}=\frac{\delta V}{V}$$

$$=\frac{d^2\delta l+2ld\,\delta d}{d^2 l}$$

$$\therefore \quad e_v=\frac{\delta l}{l}+2\frac{\delta d}{d}$$

$\therefore$ Volumetric strain = Strain of the length + Twice the strain of the diameter.

§17. Volumetric Strain of a Sphere

Let the diameter of a solid sphere be d. Let its diameter increase to $d+\delta d$.

Original volume of the sphere

$$=V=\frac{\pi d^3}{6}$$

Final volume of the sphere

$$=\frac{\pi}{6}(d+\delta d)^3 \qquad =\frac{\pi}{6}(d^3+3d^2\delta d)$$

ignoring higher powers of δd.

∴ Change in volume $=\delta V=\frac{\pi}{6}\cdot 3d^2\delta d$

∴ Volumetric strain

$$=e_v=\frac{\text{Change in volume}}{\text{Original volume}}=\frac{\delta V}{V}$$

$$=\frac{3d^2\, d\delta}{d^3}=3\frac{\delta d}{d}=3e_d$$

∴ Volumetric strain = three times the strain of the diameter.

Problem 58. *A steel bar 50 mm. wide, 12 mm. thick and 30 cms. long is subjected to an axial pull of 8400 kg. Find the change in the length, width, thickness and the volume of the bar.*

Take $E=2\times 10^6$ kg./cm.^{2} and Poisson's ratio = 0·32.

Solution. Longitudinal strain $=p=\frac{P}{A}$

$$=\frac{8400}{5\times 1\cdot 2}\ kg./cm.^2$$

$$=1400\ kg./cm.^2$$

∴ Longitudinal strain

$$=e=\frac{p}{E}$$

$$=\frac{1400}{2\times 10^6}=0\cdot 0007$$

Lateral strain = Poisson's ratio × Longitudinal strain

$$=0\cdot 32\times 0\cdot 0007=0\cdot 000224$$

∴ Increase in length of the bar

$$=\delta l=el$$

$$=0\cdot 0007\times 30\ cm.=0\cdot 0210\ cm.$$

Decrease in width $=\delta b=$ Lateral strain × Original width

$=0{\cdot}000224 \times 50$ *cm.*

$=0{\cdot}001120$ *cm.* (−)

Decrease in thickness $=\delta d=$ Lateral strain × Original thickness

$=0{\cdot}000224 \times 1{\cdot}2$ *cm.*

$=0{\cdot}0002688$ *cm.* (−)

Volumetric strain $=e_v=e_l+e_b+e_d$

$=0{\cdot}0007-2\times 0{\cdot}000224=0{\cdot}000252$ (+)

∴ Increase in volume $=e_v.V$

$=0{\cdot}000252 \times 30 \times 5 \times 1{\cdot}2$ *cm.*^{3}

$=0{\cdot}04536$ *cm.*^{3}

Problem 59. *A bar of uniform rectangular section A is subjected to an axial tensile load P. Show that the volumetric strain is given by* $e_v=\frac{P}{AE}\left(1-\frac{2}{m}\right)$ *where E is the Young's Modulus and* $\frac{1}{m}$ *is the Poisson's ratio.*

Solution. Strain of the length $=e_l=\frac{P}{AE}$

∴ Lateral strain $=$ Strain of width $=$ Strain of depth

$$=-\frac{1}{m}e_l=-\frac{P}{mAE}$$

∴ Volumetric strain $=$ Strain of length $+$ Strain of width $+$ Strain of depth

$$\therefore \quad e_v=\frac{P}{AE}-\frac{P}{mAE}-\frac{P}{mAE}$$

$$\therefore \quad e_v=\frac{P}{AE}\left(1-\frac{2}{m}\right)$$

Problem 60. *A steel rod 400 cms. long and 20 mm. diameter is subjected to an axial tensile load of 4500 kg. Find the change in length, diameter and the volume of the rod. Take* $E_s=2\times 10^6$ *kg./cm.*^{2} *and Poisson's ratio* $=\frac{1}{4}$.

Solution. Area of the rod $=A=\frac{\pi}{4}\times 2^2$ *cm.*^{2} $=3{\cdot}142$ *cm.*^{2}

∴ Tensile stress $p=\frac{P}{A}=\frac{4500}{3{\cdot}142}$ *kg./cm.*^{2}

1432 *kg./cm.*^{2}

Strain of length $=\frac{p}{E}=\frac{1432}{2\times 10^6}$

$0{\cdot}000716$ (+)

$\therefore$ Increase in length $= 0{\cdot}000716 \times 400$ *cm.*

$= 0{\cdot}2864$ *cm.* (+)

Lateral strain = Strain of diameter

$= \frac{1}{4} \times 0{\cdot}000716$ (−)

$= 0{\cdot}000179$ (−)

$\therefore$ Decrease in diameter

$= 0{\cdot}000179 \times 2$ *cm.*

$= 0{\cdot}000358$ *cm.* (−)

Volumetric strain = *Strain of length + Twice the strain of the diameter.*

$= 0{\cdot}000716 - 2 \times 0{\cdot}000179$

$= 0{\cdot}000358$ (+)

$\therefore$ Increase in volume

$= 0{\cdot}000358 \times \frac{\pi}{4} \times 2^2 \times 400$ $cm.^3$

$= +0{\cdot}45$ $cm.^3$

§18. Rectangular Block Subject to Normal Stresses on all its Faces.

Fig. 53 shows a rectangular block of dimensions x, y and z so that $AB = x$, $BF = y$ and $BC = z$.

Let reference axes OX, OY and OZ be imagined paralleled to AB, BF and BC.

Let the stresses on the various faces be p_x, p_y and p_z acting parallel to the reference axes OX, OY and OZ.

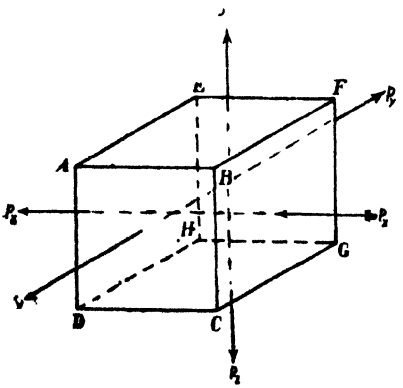


Fig. 53.

Now the strain of each dimension can be determined as the algebraic sum of the strains produced by the stresses p_x, p_y and p_z.

$\therefore$ Strain of AB

e_x = Strain along the X axis

= Algebraic sum of the strains along the X axis due to the stresses p_x, p_y and p_z.

$$\therefore \quad e_x = \frac{p_x}{E} - \frac{p_y}{mE} - \frac{p_z}{mE} \quad \ldots(i)$$

Similarly strain of $BF = e_y$ = Strain along the Y axis

$$\therefore \quad e_y = \frac{p_y}{E} - \frac{p_z}{mE} - \frac{p_x}{mE} \qquad ...(ii)$$

Similarly,

Strain of $BC = e_z$ = Strain along the Z axis

$$\therefore \quad e_z = \frac{p_z}{E} - \frac{p_x}{mE} - \frac{p_y}{mE} \qquad ...(iii)$$

$\therefore$ Change in the length of $AB = \delta_x = e_x\, x$

Change in the length of $BF = \delta_y = e_y.\, y$

and Change in the length of $BC = \delta_z = e_z\, z$

Volumetric strain $= e_v = e_x + e_y + e_z$

$$\therefore \quad e_v = \frac{p_x}{E} + \frac{p_y}{E} + \frac{p_z}{E} - \frac{2p_x}{mE} - \frac{2p_y}{mE} - \frac{2p_z}{mE}$$

$$= \frac{1}{E}\left(p_x + p_y + p_z\right) - \frac{2}{mE}\left(p_x + p_y + p_z\right)$$

$$\therefore \quad e_v = \left(p_x + p_y + p_z\right)\left(1 - \frac{2}{m}\right)\frac{1}{E}$$

Since for the metals we come across m lies between 3 and 4 the quantity $\left(1 - \frac{2}{m}\right)$ is always positive and is never equal to zero.

Hence $\qquad e_v = 0$ if $p_x + p_y + p_z = 0$

This is no doubt possible if the stresses p_x, p_y and p_z may not all be like stresses Regarding tensile stresses as positive and compressive stresses as negative the volumetric strain can be determined. According to the above convention if e_v is positive there will be an increase in volume of the block and if e_v is negative there will be a decrease in volume of the block.

Change in volume of the block $= e_v \times$ original volume of the block.

§19. Stresses on Oblique Sections of a Bar Carrying Axial Load.

Fig 54 (a) shows a rectangular bar of cross-sectional area A subjected to axial tensile load P Suppose we consider a normal cross-section 1—1 (section normal to the axis of the member). The intensity of stress on this normal cross-section 1—1 $= p = \frac{P}{A}$. The direction of this stress on this normal cross-section is entirely normal and no tangential stresses (shear stresses) are induced on the section 1—1.

Suppose we now consider a section 2—2 at an angle θ with the normal cross-section.

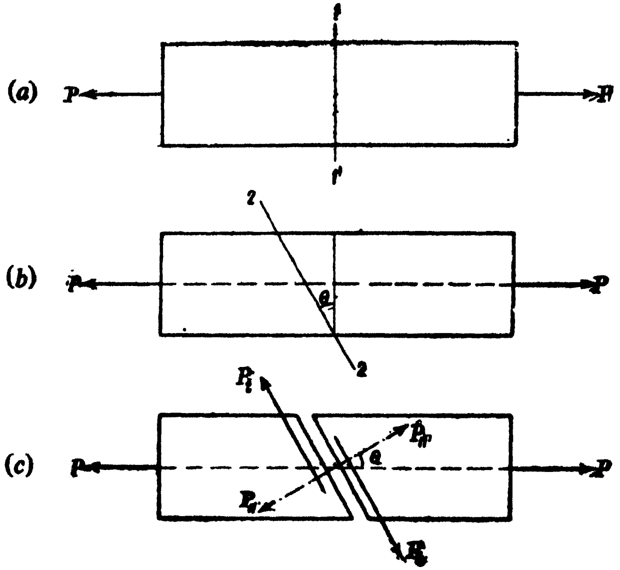


Fig. 54.

The area corresponding to the section 2—2 = A sec θ

∴ Intensity of stress on this section

$$=p'=\frac{P}{A \sec \theta}$$

i.e., $p'=p \cos \theta$

This stress being parallel to the axis of the member is not normal to the section 2—2.

Now the pull P applied can be resolved into a normal component $P_n=P\cos\theta$ and a tangential component $P_t=P \sin\theta$. Hence the normal and tangential stress intensities are,

$$p_n=\frac{P_n}{A \sec \theta}=\frac{P \cos \theta}{A \sec \theta}=p \cos^2 \theta$$

and
$$p_t=\frac{P_t}{A \sec \theta}=\frac{P \sin \theta}{A \sec \theta}=p \sin \theta \cos \theta$$

The above expressions for the normal and tangential stress intensities may also be obtained as follows :

Consider unit area of the section 2—2.

Resultant force on this area

$$=p' \times 1=p \cos \theta$$

acting along the axis of the member.

∴ Normal stress intensity

$$p_n=p' \cos \theta=p \cos^2 \theta$$

and tangential stress $p_t=p' \sin \theta=p \sin \theta \cos \theta$

Suppose the magnitude of the tensile force P be increased till failure occurs on the section 2—2.

The failure may be due to excessive normal stress p_n or due to excessive tangential stress p_t.

Since $p_n = p\cos^2\theta$, p_n is maximum when $\theta = 0$

Hence the greatest normal stress occurs on the normal cross-section. Maximum normal stress $= p$

Since $p_t = p\sin\theta\cos\theta = \frac{p}{2}\sin 2\theta$,

the greatest shear stress occurs when

$$2\theta = 90° \text{ or } 270°$$

or $\theta = 45°$ or $135°$

Hence on planes at 45° or 135° with the normal cross-section the maximum shear stress occurs.

Maximum shear stress $= q_{max} = \frac{p}{2}$

It is easily seen that there are two planes perpendicular to each other carrying the greatest shear stress and these planes are at 45° with the plane carrying the maximum normal stress.

§20. Element in a State of Simple Shear

Fig. 55 shows an elemental rectangular block $ABCD$ whose thickness normal to the plane of the drawing is unity. Let shear stresses of intensity q be set up on the faces AD and BC. Hence the forces acting on these faces, each of which equals $q.AD$, will form a couple. If the block should be in equilibrium, shear stresses of intensity say q' must be set up on the faces BA and DC. The forces acting on these faces, each of which equals $q'\ AB$ will form a restoring couple.

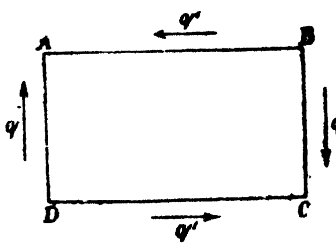


Fig. 55.

Hence for the equilibrium of the elemental block, equating the couples, we have

$$qAD \times AB = q'\ AB \times AD$$

$$\therefore \quad q = q'$$

Hence a set of shear stresses is always accompanied by a transverse set of shear stresses of the same intensity. This principle is called the principle of complementary shear stresses. The element offering these stresses is said to be in a state of simple shear.

§21. Element in a State of Simple Shear—Stresses on Oblique Sections

Consider an elemental rectangular block $ABCD$ (fig. 56) whose thickness perpendicular to the plane of the drawing is unity. Let this element be in a state of simple shear offering the shear stresses of intensity q across the faces BA, DC and the faces DA, BC.

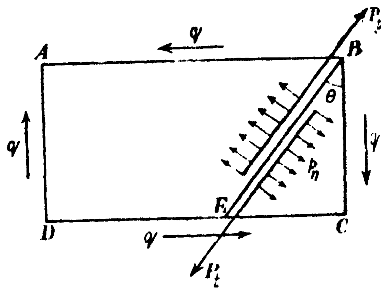


Fig. 56.

Consider a plane BE at angle θ with the face BC. Consider the equilibrium of the wedge BEC. This is subjected to the following forces.

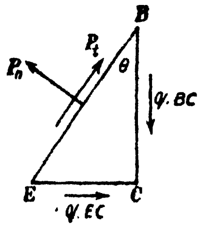


Fig. 57.

(*i*) a force qBC acting along the face BC ↓

(*ii*) a force qEC acting along the face EC →

(*iii*) a force P_n normal to the plane EB

(*iv*) a force P_t tangential to the plane EB

Resolving these forces normal to the plane BE and along the plane BE we have,

$$P_n = qBC \sin\theta + qEC \cos\theta$$

and $$P_t = qBC \cos\theta - qEC \sin\theta$$

Area of the section corresponding to the sectional plane $BE = A = AE \times 1$

The normal and tangential stresses on the plane BE are given by

$$p_n = \frac{P_n}{A} = \frac{qBC \sin\theta + qEC \cos\theta}{BE}$$

$$= 2q \sin\theta \cos\theta$$

$$= q \sin 2\theta$$

and $$p_t = \frac{P_t}{A} = \frac{qBC \cos\theta - qEC \sin\theta}{BE}$$

$$= q \cos^2\theta - q \sin^2\theta$$

$$= q \cos 2\theta$$

Hence the normal and tangential stresses on the plane BE are

$$p_n = q \sin 2\theta$$

and $$p_t = q \cos 2\theta$$

For the planes carrying the maximum normal stress, p_n should be a maximum

p_n will be a maximum when $\sin 2\theta = \pm 1$

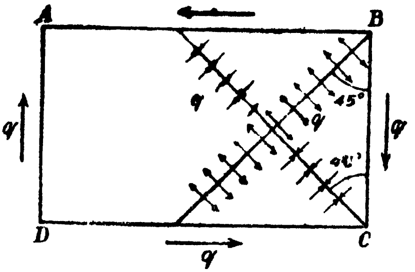


Fig. 58.

i.e., $2\theta = 90°$ or $-90°$

or $\theta = 45°$ or $-45°$

when $\theta = 45°$, $p_n = +q$

(positive sign indicates that the normal stress is tensile)

when $\theta = -45°$, $p_n = -q$

(Negative sign indicates that the normal stress is compressive)

Corresponding to $\theta = \pm 45°$, we find $p_t = 0$

Hence the planes carrying the maximum normal stresses do not carry any shear stress.

Obviously for p_t to be a maximum

$$\cos 2\theta = \pm 1$$

$$\therefore \quad 2\theta = 0° \text{ or } 180°$$

$$\text{or} \quad \theta = 0° \text{ or } 90°$$

On these planes corresponding to the maximum shear stress the normal stre..es are zero.

Hence, we come to a very important conclusion.

When an element is in a state of simple shear, maximum direct stresses are induced on mutually perpendicular planes which are at 45° to the planes of pure shear. One of the maximum direct stresses is tensile while the other maximum direct stress is compressive. These direct maximum tensile and compressive stress intensities are of the same magnitude as the intensity of shear stress on the planes of pure shear.

The above observation may also be made by considering a square block *ABCD* (Fig. 59) whose thickness perpendicular to the plane of the drawing is unity.

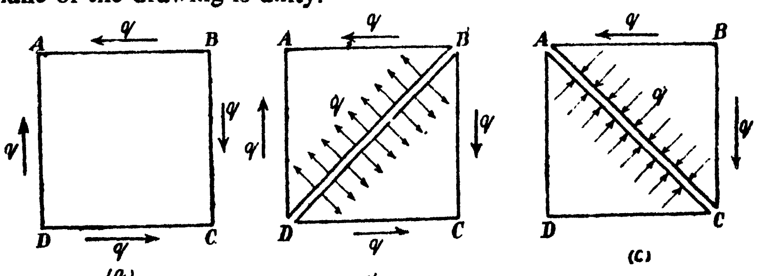


Fig. 59.

Let us imagine for a moment that the block is of two parts *ABD* and *CBD*. Consider the forces on the part *ABD*. This is subjected to a force of $qAB \leftarrow$ on the face *AB* and a force $qDA \uparrow$ on the face *DA*.

$\therefore$ Resultant force on the part

$$ABD=\sqrt{q^2\,AB^2+q^2\,DA^2}\quad=\sqrt{2}\,aq$$

where a is the side of the square.

It can be easily seen that this resultant force $\sqrt{2}\,aq$ is acting normal to BD and is acting in such a manner as to cause a separation of the part ABD from the part CBD.

Hence when there is equilibrium the force $\sqrt{2}\,aq$ is resisted by a tensile resistance by the section BD.

Hence the tensile stress on the plane BD

$$p_n=\frac{\sqrt{2}\,aq}{BD\times 1}$$

But $\quad BD=\sqrt{2}\,a$

$$\therefore \quad p_n=\frac{\sqrt{2}\,aq}{\sqrt{2}\,a}$$

$$\therefore \quad p_n=q \text{ (tensile)}$$

Similarly, if the block $ABCD$ had been taken to consist of the parts ABC and ADC it can be seen that the forces acting on these two parts tend to press the two parts towards each other and it can be easily seen that the direct compressive stress is equal to q.

The pure direct tensile and compressive stresses acting on the diagonal planes BD and AC are called diagonal tensile and diagonal compressive stresses.

Suppose the elemental block $ABCD$ is of a material very poor in offering tensile stresses, then, as the magnitude of the shear stress q goes on increasing the block will fail due to excessive diagonal tensile stress. On the contrary if the material is very poor in offering compressive stresses, then a failure may occur by crushing due to excessive diagonal compressive stress.

§22. Relation between the modulus of elasticity and the modulus of rigidity.

Consider a square block $ABCD$ of side a and of thickness unity perpendicular to the plane of the drawing (Fig. 60).

Let the block be subjected to shear stresses of intensity q as shown in the figure.

Due to these stresses the block will be subjected to a deformation such that the diagonal AC is elongated and the diagonal BD is shortened. Consider the diagonal AC.

The increase in length of the diagonal can be computed by considering the effect of the diagonal tensile and diagonal compressive stresses. We know that these diagonal tensile and compressive stresses are also of intensity q.

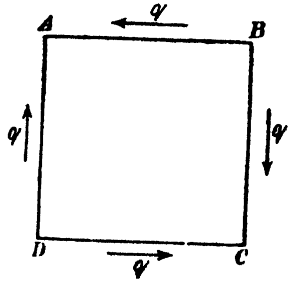


Fig. 60.

Strain in the length of the diagonal AC=Strain in the length of AC due to diagonal tensile stresses on the plane BD (Fig. 59 b) + Strain in length of AC due to diagonal compressive stresses on the plane AC (Fig. 59 c).

$$\therefore \quad \text{Strain of } AC = \frac{q}{E} + \frac{q}{mE} = \frac{q}{E}\left(1+\frac{1}{m}\right)$$

Hence the strain of the diagonal $AC = \frac{q}{E}\left(1+\frac{1}{m}\right)$

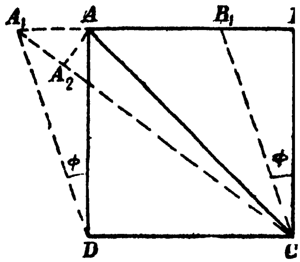


Fig. 61.

The strain of the diagonal AC may also be determined from the geometry of the distorted shape of the block (Fig. 61).

Let the block $ABCD$ deform to the position A_1B_1CD through the angle ϕ relative to the face DC.

Increase in length of the diagonal AC

$$= A_1C - AC$$

Let AA_2 be perpendicular to A_1C

Since the angle ACA_2 is very small, $AC = A_2C$

$\therefore$ Increase in length of the diagonal AC

$$= CA_1 - CA_2$$
$$= A_1A_2$$
$$= AA_1 \cos AA_1A_2$$

But the angle AA_1A_2 is nearly equal to $BAC = 45°$

$\therefore$ Increase in length of the diagonal AC

$$= AA_1 \cos 45°$$
$$= \frac{AA_1}{\sqrt{2}}$$

But shear strain $= \phi = \frac{AA_1}{AD}$

$$= \frac{AA_1}{a}$$

$$\therefore \quad AA_1 = a\,\phi$$

$\therefore$ Increase in length of the diagonal AC

$$= \frac{a\,\phi}{\sqrt{2}}$$

But the length of the diagonal $AC = a\sqrt{2}$

$\therefore$ Strain of the diagonal $AC = \frac{\text{Increase in length}}{\text{Original length}}$

$$=\frac{a\phi}{\sqrt{2}}\cdot\frac{1}{a\sqrt{2}}$$

$$=\frac{\phi}{2}$$

But we have also found that the strain of the diagonal

$$AC=\frac{q}{E}\left(1+\frac{1}{m}\right)$$

$$\therefore\qquad \frac{\phi}{2}=\frac{q}{E}\left(1+\frac{1}{m}\right)$$

But $\qquad \dfrac{\text{shear stress}}{\text{shear strain}}=\text{modulus of rigidity}=C$

$$\therefore\qquad \frac{q}{\phi}=C$$

$$\therefore\qquad \phi=\frac{q}{C}$$

$\therefore$ Strain of the diagonal AC

$$=\frac{\phi}{2}=\frac{q}{2C}\qquad\left(1+\frac{1}{m}\right)$$

$$\therefore\qquad E=2C\left(1+\frac{1}{m}\right)$$

The above is the relationship between the Young's Modulus and the modulus of rigidity.

§23. Bulk Modulus

Suppose a body is subjected to like and equal direct stresses along three mutually perpendicular directions. We find that the ratio of this direct stress to the corresponding volumetric strain is found to be a constant for a given material. When the deformation is within a certain limit, this ratio is called the bulk modulus and is usually denoted by K.

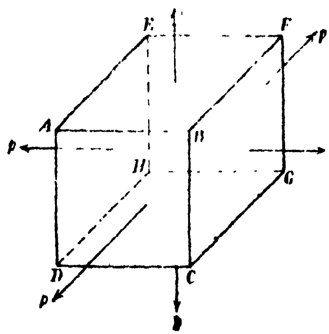


Fig. 62.

Fig. 62 shows a cube $A\ B\ C\ D\ E\ F\ G\ H$ of side a.

Let the faces of the cube be subjected to a direct stress of intensity p. Let E be the Young's Modulus, and $\frac{1}{m}$ the Poisson's ratio.

Let us now consider the strain of one of the edges, say, AB.

Strain of AB due to stresses on the faces $AEHD$ and $BFGC$

$$=\frac{p}{E}$$

Strain of AB due to stresses on the faces $AEFB$ and $DHGC$

$$=-\frac{p}{mE}$$

Strain of AB due to stresses on the faces $ABCD$ and $EFGH$

$$=-\frac{p}{mE}$$

$\therefore$ Total strain of $AB=\frac{p}{E}\left(1-\frac{2}{m}\right)$

Original volume of the cube

$$=V=a^3$$

Let the side of the cube change by δa so that the volume changes by δV.

$$\therefore \qquad \delta V=3a^2\,\delta a$$

$\therefore$ Volumetric strain $=\frac{\delta V}{V}=3.\frac{\delta a}{a}$

$\therefore$ Volumetric strain $=3\times$ Strain of AB

$$=\frac{3p}{E}\left(1-\frac{2}{m}\right)$$

But bulk modulus $K=\frac{\text{Stress}}{\text{Volumetric strain}}$

$$\therefore \qquad K=\frac{p}{\frac{3p}{E}\left(1-\frac{2}{m}\right)}$$

$$E=3K\left(1-\frac{2}{m}\right)$$

Hence we have

$$E=2C\left(1+\frac{1}{m}\right) \qquad \ldots(i)$$

and

$$=3K\left(1-\frac{2}{m}\right) \qquad \ldots(ii)$$

$$1+\frac{1}{m}=\frac{E}{2C} \qquad \ldots(iii)$$

and

$$1-\frac{2}{m}=\frac{E}{3K} \qquad \ldots(iv)$$

Multiplying eq. (*iii*) by 2 and adding eq. (*iv*), we have,

$$3=\frac{E}{C}+\frac{E}{3K}$$

$$=\frac{E(3K+C)}{3KC}$$

$$\therefore \quad E=\frac{9\,KC}{3K+C}$$

Problem 61. *A rectangular block 25 cm.×10 cm.×8 cm. is subjected to axial load as follows :*

48 tonnes tensile in the direction of its length

90 tonnes tensile on the 25 cm.×8 cm. faces

100 tonnes compressive on the 25 cm.×10 cm. faces

Assuming Poisson's ratio as 0·25 find in terms of the modulus of elasticity E of the material the strains in the direction of each force.

If $E=2\times10^6$ kg./cm.² find the values of the modulus of rigidity and bulk modulus for the material of the block. Also calculate the change in the volume of the block due to the application of the loading specified above.

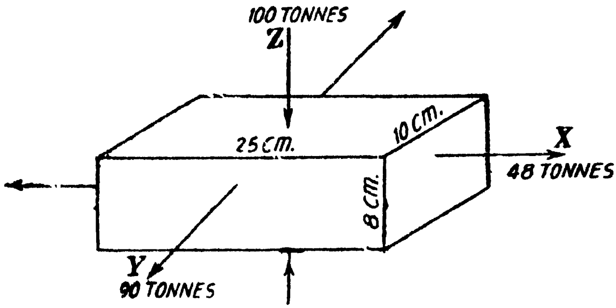


Fig. 63

Solution. Fig. 63 shows the block with its 25 *cm.*, 10 *cm.* and 8 *cm.* dimensions parallel to *X*, *Y* and *Z* axes.

The stresses in the directions of these axes are,

$$f_x=\frac{48\times1000}{10\times8}=600\ kg./cm^2\ (tensile)$$

$$f_y=\frac{90\times1000}{25\times8}=450\ kg./cm.^2\ (tensile)$$

$$f_z=\frac{100\times1000}{25\times10}=400\ kg./cm.^2\ (compressive)$$

The strains along the three principal directions are,

$$e_x=\frac{600}{E}-\frac{450}{mE}+\frac{400}{mE}$$

$$=\frac{1}{E}\left(600-\frac{450}{m}+\frac{400}{m}\right)$$

$$=\frac{1}{E}\left(600-\frac{50}{m}\right)$$

$$=\frac{1}{E}(600-50\times0{\cdot}25)$$

$$=+\frac{587{\cdot}5}{E}$$

$$e_y=\frac{450}{E}-\frac{600}{mE}+\frac{400}{mE}$$

$$=\frac{1}{E}\left(450-\frac{200}{m}\right)$$

$$=\frac{1}{E}\left(450-200\times0{\cdot}25\right)$$

$$=+\frac{400}{E}$$

$$e_z=-\frac{400}{E}-\frac{600}{mE}-\frac{450}{mE}$$

$$=-\frac{1}{E}\left(400+\frac{1050}{m}\right)$$

$$=-\frac{1}{E}\left(400+1050\times0{\cdot}25\right)$$

$$=-\frac{662{\cdot}5}{E}$$

Volumetric strain

$$=e_v=e_x+e_y+e_z$$

$$=+\frac{587{\cdot}5}{E}+\frac{400}{E}-\frac{662{\cdot}5}{E}$$

$$=+\frac{325}{E}$$

$$=+\frac{325}{2\times10^6}$$

$\therefore$ Increase in volume $=e_v\times V$

$$=\frac{325}{2\times10^6}\times\left(25\times10\times8\right)\ cm.^3$$

$$=0{\cdot}325\ cm.^2$$

We know the following relations between the Young's Modulus E and the modulus of rigidity C and the bulk modulus K.

$$E=2C\left(1+\frac{1}{m}\right)$$

and

$$E=3K\left(1-\frac{2}{m}\right)$$

$\therefore \quad 2\times10^6=2C\,(1+0{\cdot}25)=3K\,(1-2\times0.25)$

$\therefore \quad C=0{\cdot}8\times10^6\ kg./cm.^2$

and $\quad K=1{\cdot}33\times10^6\ kg\ /cm.^2$

Problem 62. *The modulus of rigidity of a material is 0·8 × 10⁶ kg./cm² When a 6 mm. × 6 mm. rod of this material was subjected to an axial pull of 360 kg. it was found that the lateral dimension of the rod changed to 5·9991 mm. × 5·9991 mm. Find the Poisson's ratio and the modulus of elasticity.*

Solution. Area of the section of the rod

$$=0{\cdot}6\times0{\cdot}6\ cm.^2$$

$$=0{\cdot}36\ cm.^2$$

$\therefore$ Stress $\quad =p=\dfrac{P}{A}=\dfrac{360}{0{\cdot}36}\,kg./cm.^2$

$$=1000\ kg./cm.^2$$

Lateral strain $\quad =\dfrac{\text{Change in lateral dimension}}{\text{Original lateral dimension}}$

$$=\frac{6-5{\cdot}9991}{6}$$

$$=\frac{0{\cdot}0009}{6}=0{\cdot}00015$$

But lateral strain

$$=\frac{p}{mE}={\cdot}00015$$

$$\therefore \quad mE=\frac{1000}{{\cdot}00015}=\frac{20000000}{3} \quad \ldots(i)$$

$$=\frac{2\times10^7}{3}$$

Also, $\quad E=2C\left(1+\dfrac{1}{m}\right)$

$\therefore \quad mE=2C(m+1)$

$\therefore \quad \dfrac{2\times10^7}{3}=2\times0{\cdot}8\times10^6\ (m+1)$

$\therefore \quad m+1=4{\cdot}167$

$\therefore \quad m=3{\cdot}167$

$\therefore$ Poisson's ratio $=\dfrac{1}{3{\cdot}167}=0{\cdot}3158$

$$E=\frac{mE}{m}=\frac{2\times10^7}{3\times3{\cdot}167}=\frac{20}{9{\cdot}501}\times10^6$$

$$=2{\cdot}1\times10^6\ kg./cm.^2$$

Problem 63. *For a given material the Young's Modulus is 1100 tonnes/cm.² and the modulus of rigidity is 430 tonnes/cm.² Find the bulk modulus and the lateral contraction of a round bar of 40 mm. diameter and 2·5 m. long when stretched by 2·5 mm.* (*A.M.I.E*)

Solution.

$$E=1100\ t/cm.^2$$

$$C=430\ t/cm.^2$$

$$E=\frac{9\,C\,K}{3K+C}$$

$$1100=\frac{9\times430\ K}{3K+430}$$

$$\therefore\quad K=830\ t/cm.^2\ \text{(Bulk-modulus)}$$

$$E=2C\left(1+\frac{1}{m}\right)$$

$$\therefore\quad 1+\frac{1}{m}=\frac{E}{2C}=\frac{1100}{2\times430}=1{\cdot}28$$

$$\therefore\quad \frac{1}{m}=0{\cdot}28=\text{Poisson's ratio}$$

Lateral strain $=$ Poisson's ratio $\times$ Linear strain

$$=0{\cdot}28\times\frac{0{\cdot}25}{250}$$

$$=0{\cdot}00028$$

$\therefore$ Lateral contraction

$$=0{\cdot}00028\times4\ cm.$$

$$=0{\cdot}00112\ cm.$$

Problem 64. *A bar of metal 10 cms. × 5 cms. in cross-section is 25 cms long. It carries a tensile load of 40 tonnes in the direction of its length, a compressive load of 400 tonnes on its 10 cms. × 25 cms. faces and a tensile load of 200 tonnes on its 5 cms. × 25 cms. faces. If E=2000 tonnes per cm.² and Poisson's ratio is 0·25, find the change in volume of the bar.*

What change must be made in the 400 tonnes load in order that there shall be no change in volume of the bar.

Solution. Let reference axes X, Y and Z be taken as shown in Fig. 64. The stresses along these axes are,

$$f_x=+\frac{40}{10\times5}$$

$$=+0{\cdot}80\ tonnes/cm.^2\ (tensile)$$

$$f_y = +\frac{200}{25\times 5}$$
$$= +1{\cdot}60 \ tonnes/cm.^2 \ (tensile)$$

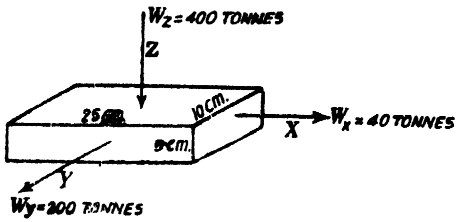


Fig. 64

$$f_z = -\frac{400}{25\times 10} = -1{\cdot}60 \ tonnes/cm.^2 \ (compressive)$$

The Volumetric strain is given by

$$e_v = \left(f_x + f_y + f_z\right)\left(1 - \frac{2}{m}\right)\frac{1}{E}$$

$$= \left(0{\cdot}8 + 1{\cdot}60 - 1{\cdot}00\right)\left(1 - 0{\cdot}5\right)\frac{1}{2000}$$

$$= \frac{1}{5000}$$

∴ Change in volume $= e_v \times$ original volume,

∴ Change in volume

$$= \frac{1}{5000} \times 25 \times 10 \times 5 \ cm.^3$$
$$= 0{\cdot}25 \ cm.^3$$

In order that there may not be any change in volume $f_x + f_y + f_z$ must be equal to zero

∴ $0{\cdot}80 + 1{\cdot}60 + f_z = 0$

∴ $f_z = -2{\cdot}4$ *tonnes/cm.*^{2} *(compressive)*

Hence the compressive load on the 25 *cm.* × 10 *cm.* faces

$= 2{\cdot}4 \times 25 \times 10$ *tonnes* = 600 *tonnes*

But the existing compressive load on this face = 400 *tonnes.*

∴ Additional compressive load that must be applied on this face $= 600 - 400 = 200$ *tonnes compressive.*

Problem 65. *A bar of steel is of square section 60 mm. × 60 mm. and 18 cms. long. It is subjected to an axial load of 30 tonnes. The lateral strain is prevented by the application of uniform external pressure. If* $\frac{1}{m} = 0{\cdot}3$ *and* $E = 2 \times 10^6$ *kg./cm.*^{2}*., find the alteration in the length of the bar.*

If, however, only one-half the lateral strain is prevented what would be the alteration in the length of the bar ?

Solution. Fig. 65 shows the bar with its axis placed vertical.

Axial stress on the normal section $=p_1=\frac{30}{6\times6}=\frac{5}{6}$ *tonnes/cm.*^{2}

Let the compressive stresses applied on the side face be p_2 and p_3. Due to symmetry $p_2=p_3$

Strain of either side

$$=\frac{p_2}{E}-\frac{p_2}{mE}-\frac{p_1}{mE}=0$$

$$=\frac{1}{E}\left[p_2-0{\cdot}3p_2-0{\cdot}3\times\frac{5}{6}\right]=0$$

$$\therefore\quad 0{\cdot}7p_2=\frac{1}{4}$$

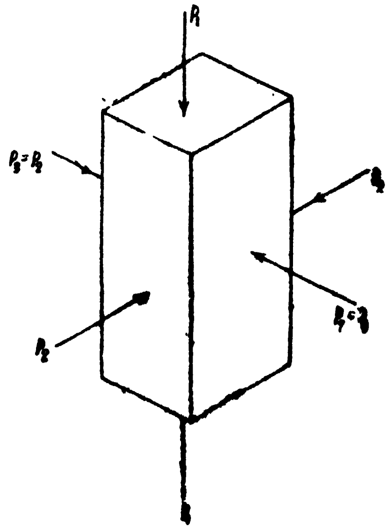


Fig. 65

$$\therefore\qquad p_2=\frac{1}{2{\cdot}8}=\frac{5}{14}\ \textit{tonne/cm.}^2\ (\textit{compressive})$$

$$\text{Strain of the length}=\frac{p_1}{E}-2\frac{p_2}{mE}$$

$$=\frac{1}{E}\left[\frac{5}{6}-2\times0{\cdot}3\times\frac{5}{14}\right]$$

$$=\frac{1}{2000}\left[\frac{5}{6}-\frac{3}{14}\right]$$

$$=\frac{13}{21\times2000}\ (\textit{compressive strain})$$

$$\therefore\ \text{Decrease in length}=\frac{13}{21\times2000}\times18\ cm.$$

$$=0{\cdot}000557\ cm.$$

When only half the lateral strain is prevented.

Suppose the lateral stresses had not been applied the lateral strain would be $\frac{p_1}{mE}$. Since half this strain is prevented by the application of the lateral stresses, we have

$$\text{Lateral strain}\qquad=\frac{p_2}{E}-\frac{p_2}{mE}-\frac{p_1}{mE}=-\frac{1}{2}\cdot\frac{p_1}{mE}$$

$$\therefore \quad p_2 - \frac{p_2}{m} = \frac{1}{2} \cdot \frac{p_1}{m}$$

$$\therefore \quad p_2\left(1 - \frac{1}{m}\right) = \frac{1}{2} \cdot \frac{p_1}{m}$$

$$\therefore \quad p_2\left(1 - 0{\cdot}3\right) = \frac{1}{2} \times 0{\cdot}3 \times \frac{5}{6}$$

$$0{\cdot}7\, p_2 = \frac{1}{8}$$

$$\therefore \quad p_2 = \frac{5}{28} \textit{ tonnes/cm.}^2 \textit{ (compressive)}$$

$\therefore$ Strain of the length

$$= \frac{p_1}{E} - 2\frac{p_2}{mE}$$

$$= \frac{1}{E}\left(\frac{5}{6} - 2 \times 0{\cdot}3 \times \frac{5}{28}\right)$$

$$= \frac{1}{2000} \times \frac{61}{81}$$

(compressive strain)

$$\therefore \text{ Decrease in length} = \frac{1}{2000} \times \frac{61}{81} \times 18 \textit{ cm.}$$

$$= 0{\cdot}0067 \textit{ cm.}$$

Problem 66. *A bar of steel is 4 cm. × 4 cm. in section and is 12 cm. long. It is subjected to tensile load of 20 tonnes along the longitudinal axis and tensile load of 50 tonnes and 40 tonnes on the lateral faces.*

(a) Find the change in the dimensions of the bar and the change in volume.

(b) Find also what axial longitudinal tensile load acting alone can produce the same longitudinal strain as in (a).

Take $E = 2 \times 10^3$ *tonnes/cm.*^{2}

and $\frac{1}{m} = 0{\cdot}3$

Solution. Fig. 66 shows the bar subjected to the given load system.

Let reference axes X, Y and Z be chosen as shown in the figure.

Then stresses along these axes are

$$p_x = +\frac{50}{4 \times 12} = +\frac{25}{24} \quad \textit{tonne/cm.}^2 \textit{ (tensile)}$$

$$p_y = +\frac{40}{4 \times 12} = +\frac{5}{6} \quad \textit{tonne/cm}^2\textit{. (tensile)}$$

$$p_z = +\frac{20}{4\times4} = +\frac{5}{4} \qquad tonne/cm^2; \ (tensile)$$

Strain along the X-axis

$$= e_x = \frac{p_x}{E} - \frac{p_y}{mE} - \frac{p_z}{mE}$$

$$= \frac{1}{2\times10^3}\left(\frac{25}{24} - 0{\cdot}3\times\frac{5}{6} - 0{\cdot}3\times\frac{5}{4}\right)$$

$$= +\frac{5}{24\times10^3}$$

$\therefore$ Increase in the dimension parallel to X-axis

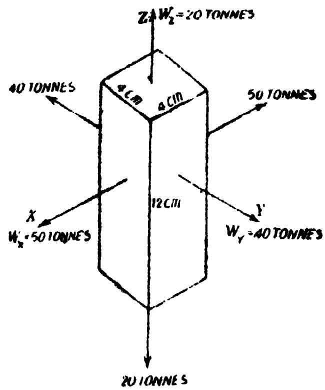


Fig. 66

$$= \frac{5}{24\times10^3}\times4 \ cm.$$

$= 0{\cdot}000833$ *cm.*

Strain along the Y-axis

$$= e_y = \frac{p_y}{E} - \frac{p_x}{mE} - \frac{p_z}{mE}$$

$$= \frac{1}{E}\left(p_y - \frac{p_x+p_z}{m}\right)$$

$$= \frac{1}{2\times10^3}\left\{\frac{5}{6} - 0{\cdot}3\left(\frac{25}{24} + \frac{5}{4}\right)\right\}$$

$$= 0{\cdot}0000729$$

$\therefore$ Increase in dimension parallel to the Y-axis.

$= 0{\cdot}0000729\times4$ *cm.*

$= 0{\cdot}0002916$ *cm.*

Strain along the Z-axis

$$= e_z = \frac{p_z}{E} - \frac{p_x}{mE} - \frac{p_y}{mE}$$

$$= \frac{1}{E}\left(p_z - \frac{p_x+p_y}{m}\right)$$

$$= \frac{1}{2\times10^3}\left\{\frac{5}{4} - 0{\cdot}3\left(\frac{25}{24} + \frac{5}{6}\right)\right\}$$

$$= 0{\cdot}00034375$$

$\therefore$ Increase in dimension parallel to the Z-axis, *i.e.*, increase in length $= 0{\cdot}00034375\times12$ *cm.*

$= 0{\cdot}004125$ *cm.*

Let p the axial stress acting alone longitudinally *i.e.*, along the Z-axis to produce the same longitudinal strain.

$$\therefore \qquad \frac{p}{E} = 0{\cdot}00034375$$

$\therefore \quad p = 0{\cdot}00034375 \times 2 \times 10^3$ *tonne/cm²*

$= 0{\cdot}6865$ *tonne/cm.²*

$\therefore$ **Longitudinal load** $= 0{\cdot}6865 \times 4 \times 4$ *tonnes.*

$= 10\ 984$ *tonnes.*

Problem 67. *A 60 mm. diameter bar carries an axial tensile load of 18 tonnes.*

Find the normal and tangential stress intensities across planes at 30°, 45° and 60° with the normal section of the bar.

Solution. Area of the normal section of the bar

$$= \frac{\pi}{4} \times 6^2 \text{ cm.}^2$$

$$= 28{\cdot}27 \text{ cm}^2.$$

$\therefore$ Stress on the normal cross-section

$$p = \frac{P}{A} = \frac{18}{28{\cdot}27} \text{ tonne/cm.}^2$$

$$= 0{\cdot}6368 \text{ tonne/cm.}^2$$

The normal stress on any plane at an angle θ with the normal section is given by

$$p_n = p \cos^2 \theta$$

When $\theta = 30°$, $p_n = 0{\cdot}6368 \cos^2 30° = 0{\cdot}4776$ *tonne/cm.² tensile*

When $\theta = 45°$, $p_n = 0{\cdot}6368 \cos^2 45° = 0{\cdot}3184$ *tonne/cm.² tensile*

When $\theta = 60°$, $p_n = 0{\cdot}6368 \cos^2 60° = 0{\cdot}1592$ *tonne/cm.² tensile*

The tangential stress on any plane at an angle θ with the normal section is given by

$$p_t = \frac{p}{2} \sin 2\theta$$

When $\theta = 30°$, $\quad p_t = \dfrac{0{\cdot}6368}{2} \sin 60° = 0{\cdot}2757$ *tonne/cm.²*

When $\theta = 45°$, $\quad p_t = \dfrac{p}{2} = 0{\cdot}3184$ *tonne/cm.²*

When $\theta = 60°$, $\quad p_t = \dfrac{0{\cdot}6368}{2} \sin 120° = 0{\cdot}2757$ *tonne/cm.²*

Problem 68. *A bar of 30 mm. diameter is subjected to a pull of 6 t. The measured extension on gauge length of 20 cms. is 0·09 mm. and the change in diameter is 0·0039 mm. Calculate the Poisson's ratio and the values of the three moduli.*

Solution. Diameter of the bar $= 30$ *mm.* $= 3$ *cms.*

Area of the bar $\quad = A = \dfrac{\pi}{4} \times 3^2$ *cm.²*

$= 7{\cdot}0686\ cm.^2$

Tensile load $= 6$ *tonnes*

∴ Intensity of tensile stress

$$= p = \frac{P}{A}$$

$$= \frac{6}{7{\cdot}0686}\ tonne/cm.^2$$

$$= 0{\cdot}8487\ tonne/cm.^2$$

Change in length $= \delta = 0{\cdot}09\ mm = 0{\cdot}009\ cm.$

Longitudinal strain $= e = \frac{\delta}{l} = \frac{0{\cdot}009}{20} = 0{\cdot}00045$

∴ Young's Modulus $= E = \frac{p}{e} = \frac{0{\cdot}8487}{0{\cdot}00045}$

$$= 1886\ tonnes/cm.^2$$

Lateral strain $= \frac{\text{Change in lateral dimension}}{\text{Original lateral dimension}}$

$$= \frac{0{\cdot}0039}{30} = 0{\cdot}00013$$

∴ Poisson's ratio $= \frac{1}{m} = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$

$$= \frac{0{\cdot}00013}{0{\cdot}00045} = \frac{13}{45}$$

Let C be the Modulus of rigidity,

we know,

$$E = 2C\left(1 + \frac{1}{m}\right)$$

$$1886 = 2C\left(1 + \frac{13}{45}\right) = 2\,C \times \frac{58}{45}$$

$$\therefore \quad C = \frac{1886 \times 45}{2 \times 58}\ tonnes/cm.^2$$

$$= 731{\cdot}8\ tonne/cm.^2$$

Let K be the bulk modulus.

we know, $E = 3K\left(1 - \frac{2}{m}\right)$

$$1886 = 3K\left(1 - \frac{26}{45}\right)$$

$$= 3K \times \frac{19}{45}$$

$$K = \frac{1886 \times 45}{3 \times 19}\ tonne/cm.^2$$

$$= 496{\cdot}4\ tonne/cm.^2$$

Problem 69. *A steel bar 40 mm. × 40 mm. 300 cm. long in section is subjected to an axial pull of 12·8 tonnes. Taking $E=2\times10^6$ kg./cm² ond poisson's ratio as 0·3, calculate the alterations in the length and sides of the bar during the extension.* (*AMIE, Winter 1974*)

Solution.

Tensile stress $\cdot f=\frac{P}{A}=\frac{12800}{4\times4}=800\ kg./cm^2$

Linear or longitudinal strain$=e_x=\frac{f}{E}=\frac{800}{2\times10^6}=4\times10^{-4}$

Lateral strain$=e_y=e_z=-\frac{f}{mE}$

$=-0{\cdot}3\times4\times10^{-4}$

$=-1{\cdot}2\times10^{-4}$

∴ *Increase in length* $=e_x\times$ original length

$=(4\times10^{-4})\ 300=0{\cdot}12\ cm.$

Decrease in side (or lateral dimension)

$=e_y\times$ original lateral dimension

$=(1{\cdot}2\times10^{-4})\ 4=0{\cdot}00048\ cm.$

Problem 70. *A cylindrical bar is 2 cm. in diameter and 100 cm long. During a tensile test it is found that the longitudinal strain is 4 times the lateral strain Calculate the modulus of rigidity and the bulk modulus, if its elastic modulus is 1×10^6 kg./cm.². Find the change in volume, when the bar is subjected to a hydrostatic pressure of 1000 kg./cm.².*

Solution. $d=2\ cm.$; $l=100\ cm.$

Longitudinal strain =4 times lateral strain

$E=1\times10^6\ kg./cm.^2$ $p=1000\ kg./cm.^2$

Rigidity modulus $C=?$ Bulk modulus $K=?$

$\delta V=?$

Poisson's ratio $=\frac{1}{m}=\frac{\text{lateral strain}}{\text{longitudinal strain}}=\frac{1}{4}=0{\cdot}25$

$$E=2C\left(1+\frac{1}{m}\right)$$

$$C=\frac{E}{2\left(1+\frac{1}{m}\right)}=\frac{1\times10^6}{2(1+0{\cdot}25)}$$

$$=4\times10^5\ kg/cm.^2$$

$$E=3K\left(1-\frac{2}{m}\right)$$

$$\nu = \frac{E}{3\left(1-\frac{2}{m}\right)} = \frac{1\times10^6}{3\,(1-2\times0{\cdot}25)}$$

$$=\frac{2}{3}\times10^6\ kg/cm.^2$$

Volumetric strain

$$=e_v=\frac{p}{K}=\frac{1000}{\frac{2}{3}\times10^6}=1{\cdot}5\times10^{-3}$$

∴ *Decrease in volume*

$$\delta V=e_v\,V$$

$$=1{\cdot}5\times10^{-3}\times\frac{\pi}{4}\,(2)^2\,100\ cm.^3$$

$$=0{\cdot}471\ cm.^3.$$

Problem 71. *Find the elongation in cm. of a straight bar of steel 12 metres long due to its own weight if hung. The value of the modulus of elasticity of the material is unknown. However, it is known that the modulus of rigidity of the material is $0{\cdot}88\times10^6$ kg./cm.^{2} and Poisson's ratio is $0{\cdot}25$. Take specific weight of steel equal to $0{\cdot}0083$ kg./cm.^{3}*

Solution.

$$E=2C\left(1+\frac{1}{m}\right)$$

$$=2\times0{\cdot}88\times10^6\,(1+0{\cdot}25)\ kg./cm.^2$$

$$=2{\cdot}2\times10^6\ kg./cm.^2$$

Extension of the bar due to its own weight

$$=\delta=\frac{\gamma l^2}{2E}$$

$$=\frac{0{\cdot}0083\,(1200)^2}{2\times2{\cdot}2\times10^6}\ cm.$$

$$=0{\cdot}0027163\ cm.$$

Examples on Chapter 1

(1) A rod of steel 6 *cm.* wide and 1·5 *cm.* thick is 800 *cm* long. It extends by 0·558 *cm.* when an axial pull of 12 00 *kg.* is applied. Find the modulus of elasticity of steel. ($2{\cdot}007\times10^6$ *kg /cm.*^{2})

(2) A rectangular base-plate is fixed at each of its corners by a 20 *mm.* diameter bolt and nut as shown in Fig. 67 At each corner the plate rests on washers of internal diameter 22 *mm.* and external diameter 50 *mm.* The washers provided between the nuts and the plate are 22 *mm.* internal diameter and 40 *mm.* external diameter.

Find the stress in the washers when the base plate carries a load of 20,000 *kg.* assuming that the load is equally distributed to the four corners.

Find also what would be the stress in the top and bottom washers when the nuts are tightened so as to produce a tension of 2000 *kg.* in each bolt.

(*Before tightening nuts*) :

Stress in top washer $=0$

Stress in bottom washer $=315{\cdot}9$ *kg./cm.*^{2}

after tightening nuts :

Stress in top washers $=228{\cdot}1$ *kg./cm.*^{2}

Stress in bottom washers $=442{\cdot}2$ *kg/cm.*^{2}

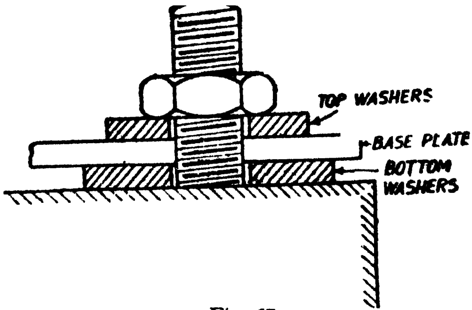


Fig. 67

(3) A steel column is 12 *cms.* in diameter and 3 *metres* long. Find the intensity of stress and the strain when it carries an axial compressive load of 95 *tonnes.* Take $E_s=2\times10^6$ *kg./cm.*^{2}

(*839·9 kg./cm.*^{2} ; *0·00042*)

(4) Find the extension of the conical rod shown in Fig. 68 due to its own weight, the weight per unit volume of the material being *w*

$$\left(\frac{wl^2}{6E}\right)$$

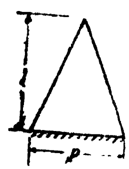


Fig. 68

(5) Find the maximum diameter of a steel wire with which a load of 350 *kg* can be raised so that the stress in the wire may not exceed 1300 *kg /cm*^{2} For the diameter chosen, find the extension of the wire if it is 4 *metres* long. Take $E=2\times10^6$ *kg./cm.*^{2}

(*5·86 mm.* ; *2·596 mm.*)

(6) A tie bar has enlarged ends of square section 6 *cm.* × 6 *cm.* as shown in Fig. 69. If the middle portion of the bar is also of square section find the size and length of the middle portion if the stress there is 1400 *kg /cm*^{2}, and the total extension of the bar is 0·014 *cm.* Take $E=2\times10^6$ *kg./cm*^{2}

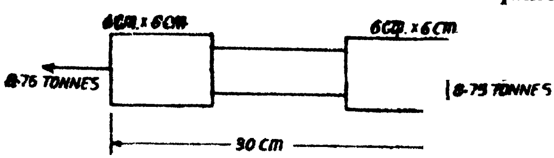


Fig. 69

[*2·5 cm.* × *2·5 cm* , *17·9 cm.*]

(7) A rod circular in section tapers from 2 *cms.* diameter at one end to 1 *cm.* diameter at the other end and is 20 *cms.* long. On applying an axial pull of 600 *kg.* it was found to extend by 0·0068 *cm.* Find the Young's modulus of the material of the rod

(*1·123* × *10*^{6} *kg./cm*^{2}).

2

Strain Energy—Impact Loading

§24. Strain Energy

When a load is applied on a member, the member is deformed. The member offers a resistance against this deformation. We know that the internal resistance offered by the member is the total stress in the member. It is very important to note that only when the member has the capacity to offer a resistance to the deformation, a stress will be induced in the member, when the member is subjected to a deformation.

Fig. 70 (a) shows a bar of length l and of uniform cross-section A. Suppose the top end A of the rod is fixed and the bottom end B is pulled down, so that the rod is extended by $\delta = BB_1$.

In this position if R is the resistance offered by the member against the extension,

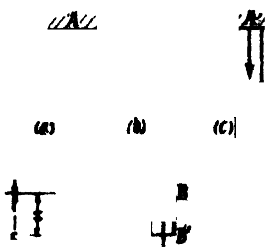


Fig. 70

We have,

Stress intensity $= p = \dfrac{R}{A}$

Strain $= e = \dfrac{p}{E}$

and extension $\delta = el = \dfrac{p}{E}\, l$

Since the bar is in equilibrium after the extension BB_1 has taken place

We have $R = P$

Stress intensity $= p = \dfrac{R}{A} = \dfrac{P}{A}$.

Now let us consider as to what may be the resistance offered by the member *during the process of extension*. The resistance at any instant depends upon the extension at that instant. Suppose at any instant during the process of extension, the extension of the member is $BB' = x$

Strain $= e_x = \dfrac{x}{l}$

$\therefore$ Stress intensity $= e_x E = \frac{x}{l} E$

$\therefore$ Resistance $= R' = \frac{x}{l} AE$

But when the total extension δ has taken place, resistance

$$= R = \frac{\delta}{l} AE = P$$

Hence the resistance R' offered by the member goes on increasing as the extension x goes on increasing. This resistance therefore increases gradually from zero when the bar is not stretched, to $R=P$ when the bar has stretched fully by δ.

Hence in order to extend the member by δ the average resistance put forth by the member

$$= \frac{R}{2} = \frac{P}{2}$$

Hence work done against the resistance to extend the member
= Average resistance × distance for which the resistance is overcome.

$$= \frac{P}{2} \times \delta$$

$$= \frac{1}{2} P \delta$$

This work done on the member will be stored by the member as energy and is called strain energy.

Strain energy is the energy absorbed or stored by a member when work is done on it to deform it.

Suppose a member is of such a material that it does not offer any resistance at all against the deformation. Such a member cannot regain its original shape after the load producing the deformation is removed. In other words when the member allows itself to be deformed without offering any resistance at all, there will be no stress in the member. Hence it is very important to recognize this fact that the stress induced by a member is entirely due to its capacity to offer a resistance against the deformation. A member which allows itself to be deformed without any resistance is said to be *plastic*. When a member is in a plastic stage, no stress will be induced by its section due to any deformation of the member.

A member which allows itself to be deformed, but will offer a resistance to the deformation is said to be *elastic*. Hence when an elastic member is kept in a deformed position the member will be offering a resistance. When the external load producing the deformation is removed, the member will regain its original dimension. A member which does not allow itself to be deformed at all is said to be *rigid*.

Structural members, we come across, are neither plastic nor rigid. In general, structural members are elastic allowing themselves to be deformed and offering a resistance against the deformation when the deformation is within the elasic limit. If the member is deformed beyond the elastic limit, the member will certainly allow itself to be deformed without offering any more resistance.

Let us now consider the elastic member in Fig. 70.

Let P be the load causing the deformation. Let p be the stress in the member when the full extension δ has taken place.

$\therefore$ Stress $\quad p=\dfrac{R}{A}$

$\therefore \quad R=pA=P$

Work done against the resistance, on the member

= Strain energy stored by the member

= Average resistance × displacement

$$=\frac{R}{2}\cdot\delta$$

But $\quad \delta=el \quad$ where e is the strain

and $\quad R=pA$

$\therefore$ Strain energy stored by the member

$$=\frac{pA}{2}\cdot el$$

$$=\frac{1}{2}\cdot p\cdot e\,(Al)$$

$$=\frac{1}{2}\cdot \text{Stress}\times\text{Strain}\times\text{Volume of the member}$$

But $\quad e=\dfrac{p}{E}$

$\therefore$ Strain energy stored by the member

$$=\frac{1}{2}\,p\;\frac{p}{E}\cdot Al$$

$$=\frac{p^2}{2E}\,(Al)$$

$\therefore$ Strain energy stored per unit volume

$$=\frac{p^2}{2E}$$

§25. Stresses due to various types of Axial Loads

Gradually applied load

Let a load of magnitude P be applied axially on a member of length l and uniform cross-sectional area A. Let δl be the extension of the rod. Let p be the stress intensity in the rod.

Strain Energy stored by the member

$$=\frac{p^2}{2E}\times\text{volume of the rod}$$

$$=\frac{p^2}{2E}\cdot Al$$

Work done by the external load = Average load × extension.

$$=\frac{1}{2}P\cdot\delta l$$

Equating the strain energy stored by the member to the work done by the loading.

We get, $$\frac{p^2}{2E}Al=\frac{1}{2}P\cdot\delta l$$

But $$\delta l=\frac{p}{E}l$$

$\therefore$ $$\frac{p^2}{2E}Al=\frac{1}{2}P\cdot\frac{p}{E}l$$

$\therefore$ $$p=\frac{P}{A}.$$

Suddenly applied load. Let the load P be suddenly applied. Let the extension of the member be δl In this case the magnitude of the load is constant at P throughout the process of extension. Let p be the maximum stress induced.

Equating the strain energy stored by the member to the work done.

We get, $$\frac{p^2}{2E}Al=P\cdot\delta l$$

But $$\delta l=\frac{p}{E}l$$

$\therefore$ $$\frac{p^2}{2E}Al=P\cdot\frac{p}{E}l$$

$\therefore$ $$p=2\frac{P}{A}$$

Hence the maximum stress intensity due to a suddenly applied load is twice the stress intensity produced by the load of the same magnitude applied gradually.

Impact load.

In this case the load P is dropped from a height h before it commences to stretch the bar. Fig. 71 shows a vertical bar whose upper end is fixed at the top and a collar is provided at the lower end. The load P drops by a height h on the collar and will thus extend the member by δl.

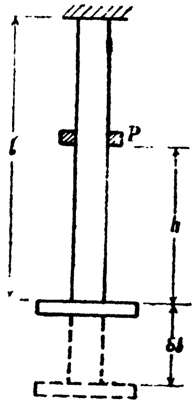


Fig 71.

Let p be the maximum intensity of stress produced in the bar.

$\therefore$ Extension of the bar

$$=\delta l=\frac{p}{E}l$$

Equating the loss of potential energy to the strain energy stored by the rod,
we have,

$$P\left(h+\delta l\right)=\frac{p^2}{2E}Al$$

$$\therefore \quad P\left(h+\frac{pl}{E}\right)=\frac{p^2}{2E}Al$$

$$\therefore \quad \frac{p^2}{2E}Al-\frac{Ppl}{E}=Ph$$

$$\therefore \quad p^2-2\frac{Pp}{A}=\frac{2PhE}{Al}$$

Adding $\frac{P^2}{A^2}$ to both sides of this equation,

we get,

$$p^2-2\frac{Pp}{A}+\frac{P^2}{A^2}=\frac{P^2}{A^2}+\frac{2PhE}{Al}$$

$$\therefore \quad \left(p-\frac{P}{A}\right)^2=\frac{P^2}{A^2}+\frac{2PhE}{Al}$$

$$\therefore \quad p-\frac{P}{A}=\sqrt{\frac{P^2}{A^2}+\frac{2PhE}{Al}}$$

$$\therefore \quad p=\frac{P}{A}+\sqrt{\frac{P^2}{A^2}+\frac{2PhE}{Al}} \qquad \ldots(i)$$

When δl is very small in comparison with h

Loss of potential energy $=Ph$

Equating the loss of energy to the strain energy,

we get, $$Ph=\frac{p^2}{2E}Al$$

$$\therefore \qquad p=\sqrt{\frac{2PhE}{Al}} \qquad \text{...(ii)}$$

When $h=0$, from eq. (*i*),

we get,

$p=\frac{2P}{A}$ which is the case of the suddenly applied load.

Having determined the maximum instantaneous stress, the instantaneous extension is given by

$$\delta l=\frac{p}{E}\,l$$

Problem 72. *A steel rod 5 cm. in diameter is 3 metres long. Find the maximum instantaneous stress induced when a pull of 10 tonnes is suddenly applied to it. Find also the instantaneous elongation. Take $E=2\times 10^6$ kg/cm.²*

Solution. Area of the rod $=\frac{\pi}{4}\times 5^2$ cm.²

$$=\frac{25}{4}\pi \text{ cm.}^2$$

Since the load is suddenly applied the maximum stress is double the stress due to gradual application.

$$\therefore \quad \text{Max. stress}=p=2\,\frac{P}{A}$$

$$=\frac{2\times 10}{\left(\frac{25\pi}{4}\right)} \text{ tonne/cm.}^2$$

$$=1{\cdot}019 \text{ tonne/cm.}^2$$

Max. instantaneous elongation

$$=\frac{p}{E}\,l$$

$$=\frac{1{\cdot}019}{2\times 10^3}\times 300 \text{ cm.}$$

$$=0{\cdot}1528 \text{ cm.}$$

Problem 73. *An unknown weight falls by 3 cms. on to a collar rigidly attached to the lower end of a vertical bar 4 metres long and 10 cm.² in section. If the maximum instantaneous extension is found to be 0·366 cm., find the corresponding stress and the value of the unknown weight. Take $E=2\times 10^6$ kg/cm.²*

Solution. Maximum stress $=p=E\times$ Max. strain

$$=2\times 10^6\times \frac{0{\cdot}366}{400} \text{ kg./cm.}^2$$

$$=1830 \text{ kg/cm.}^2$$

Let P be the load.

Equating the loss of petential energy to the strain energy stored by the rod, we have,

$$P\left(h+\delta l\right)=\frac{p^2}{2E}Al$$

$$\therefore\ P\left(3+0{\cdot}366\right)=\frac{(1830)^2}{2\times2\times10^6}\times10\times400$$

$$P=\frac{(1830)^2\times10\times400}{2\times2\times10^6\times3{\cdot}366}\ kg.$$

$$=995{\cdot}1\ kg.$$

Problem 74. *A steel specimen 1·5 cm^2 in crosss-ection stretches by 0·005 cm. over a 5 cm. gauge length under an axial load of 3000 kg. Calculate the strain energy stored in the specimen at this stage. If the load at the elastic limit for the specimen is 5000 kg., calculate the elongation at elastic limit and the proof resilience*

(AMIE, Winter, 1977)

Solution.

$$A=1{\cdot}5\ cm.^2$$

$$\delta l=0{\cdot}005\ cm.$$

$$l=5\ cm.$$

$$P=3000\ kg.$$

Load at elastic limit

$$=5000\ kg.$$

Strain energy stored $=$ work done

$=\frac{1}{2}P\delta$

$=\frac{1}{2}(3000)(0{\cdot}005)=7{\cdot}5\ kg.\ cm.$

Elongation due to 3000 $kg.$ $=0{\cdot}005\ cm.$

$\therefore$ Elongation due to 5000 $kg.=\frac{5000}{3000}\ (0{\cdot}005)=\frac{1}{120}\ cm.$

$=$*elongation at elastic limit.*

Proof resilience (Maximum strain energy in the elastic condition)

$=$Work done at this stage

$=\frac{1}{2}$(load at elastic limit)

(Extension)

$=\frac{1}{2}\times5000\times\frac{1}{120}\ kg.\ cm.$

$=20{\cdot}833\ kg.\ cm.$

Problem 75. *A bar 100 cm. in length is subjected to a pull such that the maximum stress is equal to 1500 kg/cm^2. Its area of* *cross-section is 2 cm.² over a length of 95 cm. and for the middle 5 cm. length the sectional area is 1 cm². If $E=2\times10^6$ kg./cm²., calculate the strain energy stored in the bar.* (*AMIE, Summer 1975*)

Solution.

Let the stress in the larger part $=f_1$ *kg./cm.*²

Stress in the smaller part $=f_2=1500$ *kg./cm.*²

$A_1=2$ *cm.*² ; $A_2=1$ *cm.*²

$f_1A_1=f_2A_2$

$\therefore f_1=750$ *kg./cm.*²

Total strain energy stored

$$=\frac{f_1^2}{2E}\times\text{volume of the larger part}$$

$$+\frac{f_2^2}{2E}\times\text{volume to the smaller part}$$

$$=\frac{1}{2\times2\times10^6}[(750^2\times2\times95)+(1500^2\times1\times5)]\ kg.\ cm.$$

$=29\cdot53$ *kg. cm.*

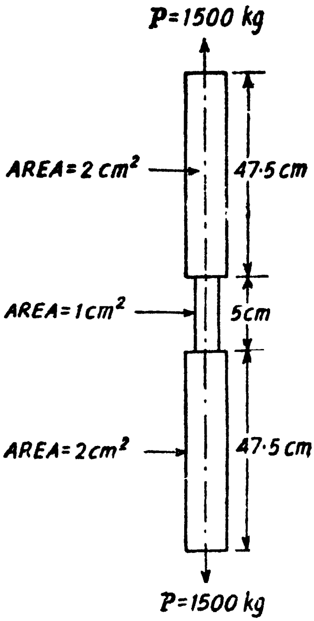


Fig. 72

Problem 76. *An object of weight 10 kg. falls by gravity a vertical distance of 5 metres when it is suddenly stopped by a collar at the end of a vertical rod of length 10 metres, and diameter 2 cm. The top of the bar is rigidly fixed to a support. Calculate the maximum stress and strain induced in the bar due to the impact. Take $E=2\times10^6$ kg./cm.²* (*AMIE, Winter 1979*)

Solution.

Falling weight $=W=10$ *kg.*

$h=5$ *metres* $=500$ *cm.*

$l=10$ *metres* $=1000$ *cm.*

$d=2$ *cm.*

$E=2\times10^6$ *kg./cm.*²

Maximum stress

$$f=\frac{W}{A}+\sqrt{\frac{W^2}{A^2}+\frac{2EWh}{Al}}$$

$$=\frac{W}{A}+\sqrt{\left(\frac{W}{A}\right)^2+2E\left(\frac{W}{A}\right)\left(\frac{h}{l}\right)}$$

But $$\frac{W}{A}=\frac{10}{\frac{\pi}{4}(2)^2}=3{\cdot}183\ kg./cm.^2$$

$$\left(\frac{W}{A}\right)^2=3{\cdot}183^2=10{\cdot}131$$

and $$\frac{h}{l}=\frac{500}{1000}=0{\cdot}5$$

$\therefore$ $$f=3{\cdot}183+\sqrt{10{\cdot}131+2\times 2\times 10^6\times 3{\cdot}183\times 0{\cdot}5}$$
$$=2526{\cdot}278\ kg./cm.^2$$

Maximum strain $$=e=\frac{f}{E}$$

$$=\frac{2526{\cdot}278}{2\times 10^6}=0{\cdot}0012631$$

Problem 77. (*SI*) *A load of 100 Newtons falls by gravity a vertical distance of 300 cms. when it is suddenly stopped by a collar at the end of a vertical rod of length 6 metres and diameter 2 cms. The top of the bar is rigidly fixed to a ceiling. Calculate the maximum stress and the strain induced in the bar. Take* $E=1{\cdot}96\times 10^7 N/cm.^2$

Solution.

Falling weight $=W=100$ *Newtons*

$$h=300\ cm.$$
$$l=6\ m=600\ cm.$$
$$d=2\ cm.$$
$$E=1{\cdot}96\times 10^7\ N/cm.^2$$

Maximum stress $$=f=\frac{W}{A}+\sqrt{\frac{W^2}{A^2}+\frac{2EWh}{AL}}$$

$$=\frac{W}{A}+\sqrt{\left(\frac{W}{A}\right)^2+2E\left(\frac{W}{A}\right)\left(\frac{h}{l}\right)}$$

But $$\frac{W}{A}=\frac{100}{\frac{\pi}{4}(2)^2}=31{\cdot}83\ N/cm.^2$$

$\therefore$ $$\left(\frac{W}{A}\right)^2=\left(31{\cdot}83\right)^2=1013{\cdot}1$$

and $$\frac{h}{l}=\frac{300}{600}=0{\cdot}5$$

$\therefore$ $$f=31{\cdot}83+\sqrt{1013{\cdot}1+2\times 1{\cdot}96\times 10^7\times 31{\cdot}83\times 0{\cdot}5}$$
$$=24977\ N/cm^2.$$

$$Maximum\ strain = e = \frac{f}{E}$$

$$= \frac{24977}{1{\cdot}96 \times 10^7} = 0{\cdot}001274$$

Problem 78. *A 10 mm. diameter mild steel bar of length 1·50 metres is stressed by a weight of 12 kg. dropping freely through 2 cms. before commencing to stretch the bar. Find the maximum instantaneous stress and the elongation produced in the bar. Take $E = 2 \times 10^6$ kg./cm.²*

Solution. Area of the bar $= \frac{\pi}{4} \times 1^2 = 0{\cdot}7854$ cm.²

Let the maximum instantaneous stress be p kg./cm.²

∴ Maximum elongation

$$= \delta l = \frac{p}{E} l$$

Equating the loss of potential energy to the strain energy stored by the member, we have

$$P(h + \delta l) = \frac{p^2}{2E} Al$$

$$\therefore\ 12\left(2 + \frac{p}{E} l\right) = \frac{p^2}{2E} Al$$

$$\therefore\ 24 + \frac{12pl}{E} = \frac{p^2}{2E} Al$$

$$\therefore\ p^2 - \frac{24p}{A} = \frac{24 \times 2E}{Al}$$

$$\therefore\ p^2 - \frac{24}{0{\cdot}7854} = \frac{24 \times 2 \times 2 \times 10^6}{0{\cdot}7854 \times 150}$$

$$\therefore\ p^2 - 30{\cdot}56\, p = 814700$$

$$\therefore\ (p - 15{\cdot}28)^2 = (15{\cdot}28)^2 + 814700$$

$$= 814933$$

$$\therefore\ p - 15{\cdot}28 = 902{\cdot}8$$

$$p = 918{\cdot}08 \text{ kg./cm.}^2$$

Alternatively, $$p = \frac{P}{A} + \sqrt{\frac{P^2}{A^2} + 2\frac{P}{A}\frac{hE}{l}}$$

But $$\frac{P}{A} = \frac{12}{0{\cdot}7854} = 15{\cdot}28 \text{ kg./cm.}^2$$

$$\therefore\ p = 15{\cdot}28 + \sqrt{15{\cdot}28^2 + \frac{2 \times 15{\cdot}28 \times 2 \times 2 \times 10^6}{150}}$$

$$= 918{\cdot}08 \text{ kg./cm.}^2$$

∴ Maximum elongation

$$=\delta l=\frac{p}{E}l$$

$$=\frac{918{\cdot}08}{2\times10^6}\times150\ cm.$$

$$=0{\cdot}06885\ cm.$$

Problem 79. *A steel wire 2·5 mm. diameter is firmly held in a clamp from which it hangs vertically. An anvil, the weight of which may be neglected, is secured to the wire 1·8 m. below the clamp. The wire is to be tested allowing a weight bored to slide over the wire to drop freely from 1 metre above the anvil. Calculate the weight required to stress the wire to 100 kg./mm.² assuming the wire to be elastic up to this stress. Take E=2·1×10⁶ kg./cm.²*

Solution.

Maximum stress $=p=100\ kg./mm.^2=100\times100\ kg./cm.^2$

$$=10{,}000\ kg./cm.^2$$

Area of the wire $=A=\frac{\pi}{4}(0{\cdot}25)^2=0{\cdot}04909\ cm.^2$

Instantaneous extension

$$=\delta l=\frac{10{,}000}{2{\cdot}1\times10^6}\times180\ cm.=0{\cdot}8572\ cm.$$

Let P be falling load.

Equating the loss of potential energy to the strain energy stored by the wire, we have,

$$P\,(h+\delta l)=\frac{p^2}{2E}Al$$

$$P\,(100+0{\cdot}8572)=\frac{(10{,}000)^2}{2\times2{\cdot}1\times10^6}\times0{\cdot}04909\times180$$

$$P=\frac{(10{,}000)^2\times0{\cdot}04909\times180}{100{\cdot}8572\times2\times2{\cdot}1\times10^6}\ kg.$$

$$=2{\cdot}086\ kg.$$

Problem 80. *A weight W falls a distance h before beginning to stretch a bar of length l and cross sectional area A. Derive expressions for the maximum stress induced in the bar when*

(a) the maximum extension is negligible compared with h

(b) the maximum extension is of the same order as h

(c) A bar 3 metres long and 5 cm. in diameter hangs vertically and has a collar securely attached to the lower end. Find the maximum stress induced :

(i) when a weight of 250 kg. falls by 15 cm. on the collar

(ii) when a weight of 2500 kg. falls by 1·5 cm. on the collar.

Take $E=2{\cdot}1\times10^6$ kg./cm.²

Solution. Let δl be the extension and let p be the maximum stress

$$\delta l = \frac{p}{E} l$$

(*a*) *when the maximum extension is small compared with h*

Loss of potential energy of the load

$$= W(h+\delta l) = Wh \text{ (since } \delta l \text{ may be ignored)}$$

Strain energy stored $= \dfrac{p^2}{2E} Al$

Equating the loss of potential energy to strain energy stored

$$Wh = \frac{p^2}{2E} Al$$

$$\therefore \qquad p = \sqrt{\frac{2WhE}{Al}}$$

(*b*) *when the maximum extension is of the same order as h*

Loss of potential energy of the load

$$= W(h+\delta l) = W\left(h + \frac{p}{E} l\right)$$

Strain energy $= \dfrac{p^2}{2E} Al$

Equating the loss of potential energy to the strain energy

$$W\left(h + \frac{p}{E} l\right) = \frac{p^2}{2E} Al$$

$$p^2 - 2\frac{Wp}{A} = \frac{2WhE}{Al}$$

Solving as a quadratic in p we have,

$$p = \frac{W}{A} + \sqrt{\frac{W^2}{A^2} + \frac{2WhE}{Al}}$$

For the numerical, $W = 250$ *kg*.

$$A = \frac{\pi}{4} \times 5^2 = 19{\cdot}63 \ cm^2.$$

(*i*) *when a weight of 250 kg. falls by 15 cm. on the collar.*

For this case $W = 250$ *kg*. and $h = 15$ *cm*.

The maximum extension being small compared with h

$$p = \sqrt{\frac{2WhE}{Al}}$$

$$= \sqrt{\frac{2 \times 250 \times 15 \times 2{\cdot}1 \times 10^6}{19{\cdot}63 \times 300}} \ kg./cm.^2$$

$$= 1635 \ kg./cm.^2$$

(ii) when a weight of 2500 kg. falls by 1·5 cm. on the collar

For this case $W=2500$ *kg.* and $h=1·5$ *cm.*

The maximum extension being of the same order as that of h

$$p=\frac{W}{A}+\sqrt{\frac{W^2}{A^2}+\frac{2WhE}{Al}}$$

$$=\frac{2500}{19·63}+\sqrt{\left(\frac{2500}{19·63}\right)^2+\frac{2\times2500\times1·5\times2·1\times10^6}{19·63\times300}}\ kg./cm.^2$$

$$=127+1640\ kg./cm.^2=1767\ kg./cm.^2$$

Problem 81. *A vertical round steel rod, 1·82 metres long is securely held at its upper end and a weight sliding freely on the rod falls on to a stop at the lower end of the rod. When the weight falls from a height of 3 cm. the maximum stress reached in the rod is estimated to be 1570 kg./cm.² Determine the stress if the load had been applied gradually and also the maximum stress if the load had fallen from a height of 4·5 cms. Take $E=2·1\times10^6$ kg./cm.²*

Solution. Maximum stress$=p=1570$ *kg./cm.²*

∴ Maximum extension

$$=\delta l=\frac{p}{E}l$$

$$=\frac{1570}{2·1\times10^6}\times182\ cm.$$

$$=0·1361\ cm.$$

Strain energy stored by the rod

$$=\frac{p^2}{2E}Al\ kg.\ cm.$$

Let the falling weight be W *kg.*

Loss of potential energy of the weight

$$=W(h+\delta l)\ kg.\ cm.$$

$$=W(3+0·1361)\ kg.\ cm.$$

$$=3·1361\ W\ kg.\ cm.$$

Equating this loss of potential energy to the strain energy stored, we have,

$$3·1361\,W=\frac{p^2}{2E}Al$$

$$=\frac{p}{2}\cdot\left(\frac{p}{E}l\right)\cdot A$$

$$=\frac{1570}{2}\times0·1361\,A$$

$$\therefore \quad \frac{W}{A} = \frac{1570}{2} \times \frac{0{\cdot}1361}{3{\cdot}1351} \ kg./cm.^2$$

$$= 34{\cdot}07 \ kg./cm.^2$$

If the load W had been gradually applied the intensity of stress would be $\frac{W}{A} = 34{\cdot}07 \ kg./cm.^2$

If the load had fallen from a height of 4·5 *cms.* the maximum stress is given by

$$p = \frac{W}{A} + \sqrt{\frac{W^2}{A^2} + \frac{2WhE}{Al}}$$

But $\quad \frac{W}{A} = 30{\cdot}07 \ kg./cm.^2$

$$h = 4{\cdot}5 \ cm.$$

$$\therefore \quad p = 34{\cdot}07 + \sqrt{(34{\cdot}07)^2 + 2 \times 34{\cdot}07 \times \frac{4{\cdot}5}{182} \times 2{\cdot}1 \times 10^6} \ kg./cm.^2$$

$$= 1915 \ kg./cm.^2$$

Problem 82. *A vertical tie, fixed rigidly at the top end consists of a steel rod 2·5 metres long and 20 mm. diameter encased throughout in a brass tube 20 mm. internal diameter and 30 mm. external diameter. The rod and the casing are fixed together at both ends. The compound rod is suddenly loaded in tension by a weight of 1000 kg. falling freely through 3 mm. before being arrested by the tie. Calculate the maximum stresses in steel and brass. Take $E_s = 2 \times 10^6$ kg./cm.² and $E_b = 1 \times 10^6$ kg./cm.²* (*A.M.I.E.*)

Solution. A_s = Area of steel rod

$$= \frac{\pi}{4} \times 2^2 \ cm.^2$$

$$= 3{\cdot}142 \ cm.^2$$

A_b = Area of brass tube

$$= \frac{\pi}{4}\left(3^2 - 2^2\right) cm.^2$$

$$= 3{\cdot}927 \ cm.^2$$

Since the elongations of the steel rod and brass tube are equal,

we have,

Strain in steel = strain in brass

Let p_s and p_b be the stress intensities in steel and brass.

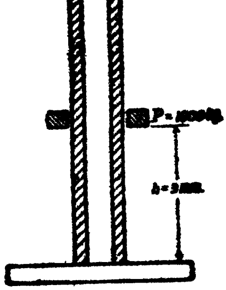


Fig. 73

$$\therefore \quad \frac{p_s}{E_s} = \frac{p_b}{E_b}$$

$$\therefore \quad p_s = \frac{E_s}{E_b} p_b = \frac{2\times10^6}{1\times10^6} p_b$$

$$\therefore \quad p_s = 2p_b$$

Extension of each bar $= \delta l = \frac{p_s}{E_s} l = \frac{p_b}{E_b} l$

Loss of potential energy of the load $= P\,(h+\delta l)$

Strain energy stored by steel rod $= \frac{p_s^2}{2E} A_s l$

Strain energy stored by brass tube $= \frac{p_b^2}{2E_b} A_b l$

Equating the loss of potential energy of the load to the strain energy stored by the steel rod and brass tube.

$$\therefore \quad P\left(h+\delta l\right) = \frac{p_s^2}{2E_s} A_s l + \frac{p_b^2}{2E_b} A_b l$$

$$P\left(h+\delta l\right) = \frac{(2p_b)^2}{2E_s} . A_s l + \frac{p_b^2}{2E_b} A_b l$$

$$\therefore \; 1000\left(0{\cdot}3 + \frac{p_b}{10^6} 250\right) = \frac{2p_b^2}{2\times10^6} \times 3{\cdot}142\times250$$

$$+ \frac{p_b^2}{2\times10^6} \times 3{\cdot}927\times250$$

$$\therefore \quad 300 + 0{\cdot}250\, p_b = 0{\cdot}0007856\, p_b^2 + 0{\cdot}0004909\, p_b^2$$

$$\therefore \quad 0{\cdot}0012765\, p_b^2 - 0{\cdot}250\, p_b = 300$$

$$\therefore \quad p_b^2 - 195{\cdot}8\, p_b = 235100$$

$$\therefore \; (p_b - 97{\cdot}9)^2 = 235100 + (97{\cdot}9)^2 = 244685$$

$$\therefore \quad p_b - 97{\cdot}9 = 494{\cdot}6$$

$$\therefore \quad p_b = 592{\cdot}5 \textit{ kg./cm.}^2$$

$$\therefore \quad p_s = 2\,p_b = 1185 \textit{ kg./cm.}^2$$

Problem 83. *A member formed by connecting a steel bar to an aluminium bar is shown in Fig. 74. Assuming that the bars are* *prevented from buckling sidewise, calculate the magnitude of the force P that will cause the total length of the member to decrease by 0·25 mm. The values of the elastic modulus for steel and aluminium are 2100 t/cm³ and 700 t/cm² respectively. What is the total work done by the force P ?* (*A.M.I.E.*)

Solution. Area of the steel bar

$$=A_s=5\times5=25\ cm^2$$

Area of the aluminium bar

$$=A_a=10\times10=100\ cm.^2$$

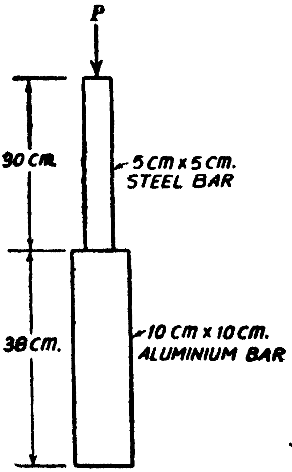


Fig. 74.

Total change in length

$$=\frac{Pl_s}{A_sE_s}+\frac{Pl_a}{A_aE_a}=\delta$$

$$\therefore\quad P\left[\frac{l_s}{A_sE_s}+\frac{l_a}{A_aE_a}\right]=\delta$$

$$\therefore\quad P\left[\frac{30}{25\times2100}+\frac{38}{100\times700}\right]$$

$$=0\ 025\ cm.$$

$$\frac{78P}{700\times100}=0·025$$

$$\therefore\quad P=\frac{0·025\times700\times100}{78}\ tonnes$$

$$\therefore\quad P=22·44\ tonnes$$

Total work done $=\frac{1}{2}\times$ load $\times$ deformation

$$=\tfrac{1}{2}\times22·44\times0·025\ tonne\ cm.$$

$$=0·2805\ tonne\ cm.$$

Problem 84. *A vertical steel rod, 130 cm. long, is rigidly secured at its upper end and a weight of 8 kg. is allowed to slide freely on the rod through a distance of 10 cm. on to the stop at the lower end. The upper 70 cm. length of the rod has a diameter of 20 mm. while the lower 60 cm. length is 16 mm. in diameter.*

Calculate the maximum stress induced in the bar ignoring the extension of the bar in determining the potential energy given up by the weight. $E=2\times10^6\ kg./cm.^2$. (*London University*)

Solution. Area of the upper part

$$=A_1=\frac{\pi}{4}\times2^2\ cm.^2=3·14\ cm.^2$$

Area of the lower part

$$=A_2=\frac{\pi}{4}\times1·6^2\ cm.^2=2·01\ cm.^2$$

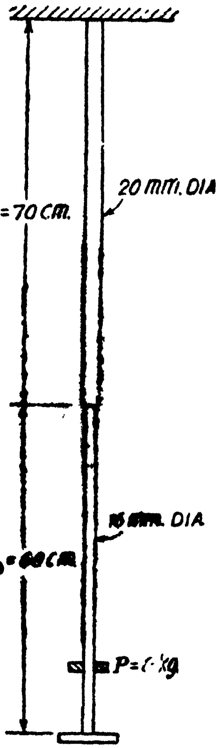


Fig. 75.

Let the stresses in the upper and lower portions be p_1 and p_2.

We have, $p_1 A_1 = p_2 A_2$

$$\therefore \quad p_1 = \frac{A_2}{A_1} p_2$$

$$= \left(\frac{16}{20}\right)^2 p_2$$

$$\therefore \quad p_1 = 0{\cdot}64\, p_2 \qquad (i)$$

Strain energy stored by the rod = Strain energy stored by the upper and lower parts

$$= \frac{p_1^2}{2E} A_1 l_1 + \frac{p_2^2}{2E} A_2 l_2$$

$$= \frac{1}{2E}\left[\left(0\ 6\, p_2\right)^2 \times 3{\cdot}14 \times 70 + p_2^2 \times 2{\cdot}01 \times 60\right] \; kg./cm.$$

$$= \frac{105{\cdot}3}{E} p_2^2$$

Ignoring the extension of the rod the potential energy given up by the falling weight $= 8 \times 10$ *kg./cm.*

Equating the total strain energy stored to the loss of potential energy, we have,

$$\frac{105{\cdot}3}{E} p_2^2 = 8 \times 10$$

$$\therefore \quad p_2^2 = \frac{8 \times 10 \times 2 \times 10^6}{105{\cdot}3}$$

$$\therefore \quad p_2 = 1232\ kg.cm.^2$$

Examples in Chapter 2

(1) A bar is 3 *metres* long and 6 *cms.* diameter. It is subjected to a tensile load of 20,000 *kg.* Find the stress and the elongation when the load is applied gradually. What would be the maximum stress and the maximum elongation if the load had been suddenly applied ? Take $E = 2 \times 10^6$ *kg /cm²*. (*707·3 kg./cm²., 0·1061 cm., 1414·6 kg./cm²., 0·2122 cm.*)

(2) A vertical steel rod, 5/8" diameter is rigidly secured at its upper end and a weight of 15 *lbs.* is allowed to slide freely on the rod through a distance of 3 *in.* on to a stop at the lower end of the

rod which is 50 *in.* below the fixed end. Find the maximum stress induced in the rod.

What would be the maximum stress reached if the upper length of the rod had a diameter of 3/4" with the lower 20" length remaining at 5/8" diameter ?

In both cases take $E=30\times10^6$ *lbs./in*^{2}. and ignore the extension of the rod in the determination of the potential energy given by the weight. (*London University, 13330 lbs./in*^{2}.; *14700 lbs./in*^{2}.)

(3) An unknown weight falls by 2·20 *cm.* on to a collar rigidly attached to the lower end of a vertical bar 3·25 *metres* long and 6 *cm.*^{2} in section. If the maximum instantaneous extension is known to be 0·25 *cm.*, find the corresponding stress and the magnitude of the falling weight. $E=2\times10^6$ *kg./cm.*^{2}

(*1538 kg./cm.*^{2}; *470·6 kg.*)

(4) A bar of a certain material $1\frac{1}{2}$" diameter and 4 ft. long has a collar securely fitted to one end. It is suspended vertically with the collar at the lower end and a tensile load of 5000 *lbs.* is gradually applied to the collar, producing an extension in the bar of 0·011 in. Find the height from which this load could be dropped on to the collar if the maximum tensile stress in the bar is to be 6 tons per square inch. (*London University*) (*0·07175 in.*)

(5) A vertical steel rod of 2·5 *cm.* diameter checks the fall on its end of a weight of 230 *kg.* which drops through a distance of 0·38 *cm.* before it strikes the rod. Find the shortest length of the rod which will bear the impact if the stress is not to exceed 1260 *kg./cm*^{2}. Take $E=2\times10^6$ *kg./cm*^{2}. Verify that the length found is the least possible length. (*48·48 cm.*)

3

Centre of Gravity and Moment of Inertia

§ 26. Centre of Gravity

Definition. *The centre of gravity of a body is that point through which the resultant of the system of parallel forces formed by the weights of all the particles of the body passes, for all positions of the body.* A given body has a definite centre of gravity.

Centre of gravity or centroid of a lamina.

Fig. 76 shows a lamina of definite area. The lamina may

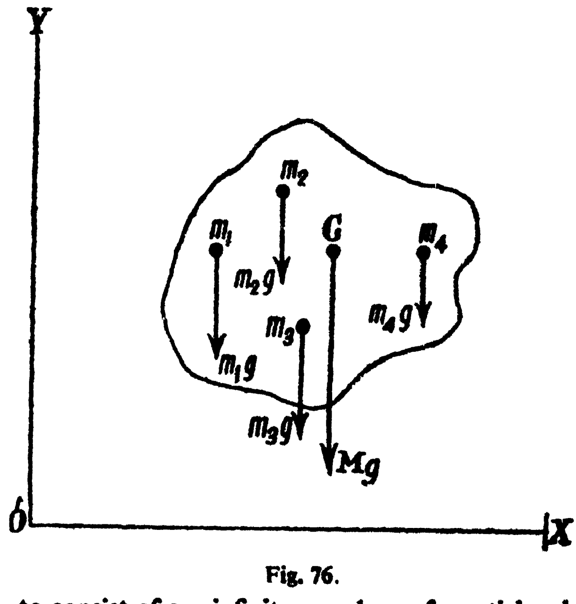


Fig. 76.

be taken to consist of an infinite number of particles lying in the plane of the lamina. Suppose the masses of the various particles be m_1, m_2, m_3, etc.

The weights of these particles form a system of parallel forces like m_1g, m_2g, m_3g and m_4g...*etc.* Let the coordinates of the various particles be (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), *etc.* referred to a set of reference axes OX and OY. Let the mass of the whole lamina be M so that the weight of the whole lamina is Mg. Let G be the centre of gravity or centroid of the lamina.

Let the coordinates of G be $(\bar{x}, \bar{y})$.

Hence Mg is the resultant of the forces, m_1g, m_2g, m_3g, m_4g *etc.* Since the sum of the moments of a system of coplanar forces equals the moment of the resultant we have, taking moments about O,

$$m_1g\,x_1+m_2g\,x_2+m_3g\,x_3+m_4g\,x_4+\ldots\ldots$$
$$=Mg\bar{x}$$

$$\therefore \quad \bar{x}=\frac{m_1x_1+m_2x_2+m_3x_3+m_4x_4+\ldots}{M}$$

By a similar reasoning imagining the lamina and the reference axes as turned by 90°, it can be shown that

$$\therefore \quad \bar{y}=\frac{m_1y_1+m_2y_2+m_3y_3+m_4y_4+\ldots}{M}$$

Uniform lamina. This means a lamina where particles within equal areas of the lamina are of equal weight. If a uniform lamina has a symmetrical shape the centroid of the lamina will be the geometric centre of the lamina.

Moment of an area about a point. This means the product of the area and its centroidal distance from the point.

Centroid of a uniform lamina. Fig. 77 shows a uniform lamina of surface density ρ per unit area. Let the total area of the lamina be A. Let G be the centroid of the lamina. Hence the weight of the lamina ρAg acts through G.

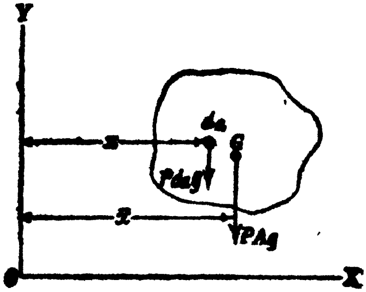


Fig. 77.

Consider an elemental area da of the lamina at a distance x from the axis OY. The weight of the elemental part is $\rho\, da\, g$. The moment of this force about the axis $OY=\rho da\, gx$.

$\therefore$ Total moment of the weight of lamina

$$=\rho Ag\bar{x}=\Sigma\rho dag\,x$$
$$=\rho g\,\Sigma\,da\,x$$

$$\therefore \quad \bar{x}=\frac{\Sigma\,da.x}{A}$$

Hence if a lamina be split up into smaller areas a_1, a_2, a_3,...*etc.*

$$\bar{x}=\frac{\text{Moment of the individual areas about } OY}{\text{Total area}}$$

or
$$\bar{x}=\frac{\Sigma\,ax}{\Sigma a}$$

Similarly
$$\bar{y}=\frac{\Sigma\,ay}{\Sigma a}$$

where x_1, x_2, x_3...are the centroidal distance of the areas a_1, a_2, a_3...from the axis OY and y_1, y_2, y_3...are the centroidal distances of the areas a_1, a_2, a_3...from the axis OX.

Problem 85. *Find the centroid of the lamina shown in Fig. 78.*

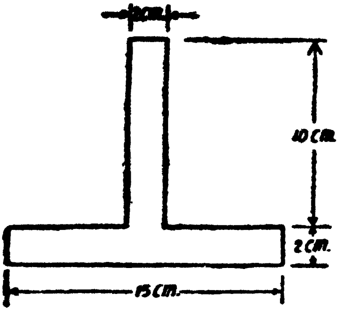


Fig. 78.

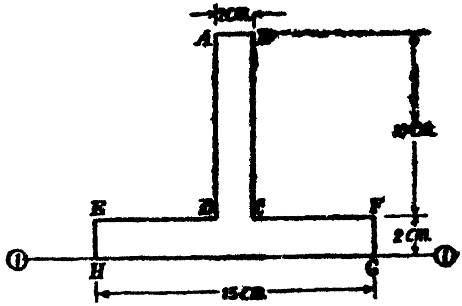


Fig. 79.

Solution. The lamina will be split up into two rectangular areas $ABCD$ and $EFGH$ as shown in Fig. 79 of areas.

$2 \times 10 = 20\ cm^2$.

and $15 \times 2 = 30\ cm^2$. respectively.

Centroidal distance of $ABCD$ from the axis $1-1 = 7$ *cms.*

Centroidal distance of $EFGH$ from the axis $1-1 = 1$ *cm.*

Let $\bar{y}$ be the height of the centroid of the lamina from the axis $1-1$.

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{20 \times 7 + 30 \times 1}{20 + 30}\ cms.$$

$$= 3{\cdot}4\ cms. \text{ above the axis } 1-1.$$

The above computation may be conveniently worked out in a tabular form as shown below :

Component	*Area a cm^2.*	*Centroidal distance from 1—1 y cm.*	*ay cm^3*
ABCD	20	7	140
EFGH	30	1	30
Total	50		170

$$\therefore \quad \bar{y} = \frac{\Sigma ay}{\Sigma a}$$

$$= \frac{170}{50} = 3{\cdot}4\ cms.$$

Problem 86. *Find the centroid of the lamina shown in Fig. 80.*

Solution. The given lamina will be split up into a number of components. The areas of the various components and their controidal distances from axis 1—1 and the moments of the individual components about the axis 1—1 are shown in the following table.

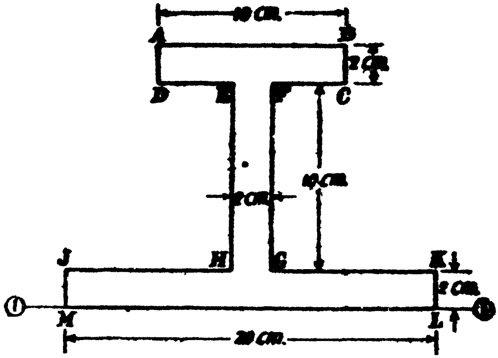


Fig. 80

Component	*Area a* $cm.^2$	*Centroidal distance from 1—1 y cm.*	*ay* $cm.^3$
ABCD 10×2	20	13	260
EFGH 10×2	20	7	140
JKLM 20×2	40	1	40
Total	80		440

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{440}{80} \text{ cms.}$$

$= 5{\cdot}50$ *cms.* above the axis 1—1.

Problem 87. *Find the centroid of the lamina shown in Fig. 81.*

Solution. The given lamina may be split up into two rectangles *ABCD* and *EFGC* as shown in Fig. 82.

The position of the centroid of the lamina with respect to the axis 1—1 and 2—2 will now be worked out. The relevant computations are shown in the following table.

Component	*Area a* cm^2	*Centroidal Distance y from 1—1 cm.*	*Centroidal Distance x from 2—2 cm.*	*a y* $cm.^3$	*a x* $cm.^3$
ABCD 2×10	20	5	1	100	20
EFGC 6×2	12	1	5	12	60
Total	32			112	80

$$\bar{y}=\frac{\Sigma ay}{\Sigma a}=\frac{112}{32}=3{\cdot}50 \text{ cms.}$$

$$\bar{x}=\frac{\Sigma ax}{\Sigma a}=\frac{80}{32}=2{\cdot}50 \text{ cms.}$$

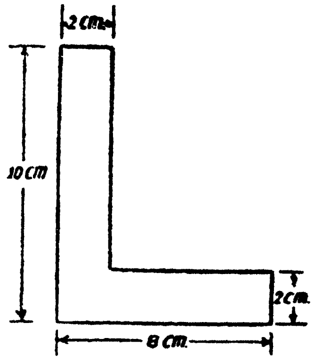


Fig. 81.

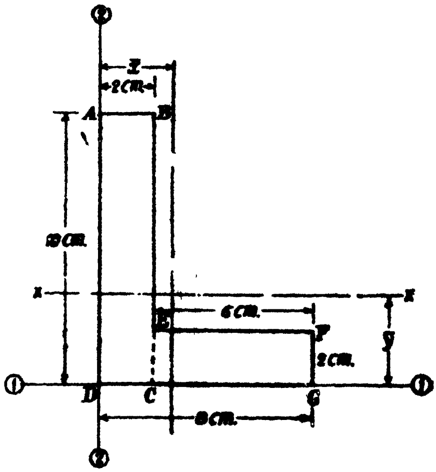


Fig. 82.

Problem 88. *In a rectangular lamina 10 cms. × 12 cms., a rectangular opening PQRS 3 cms × 4 cms. is made as shown in Fig. 83. Find the centroid of the lamina after the opening is made.*

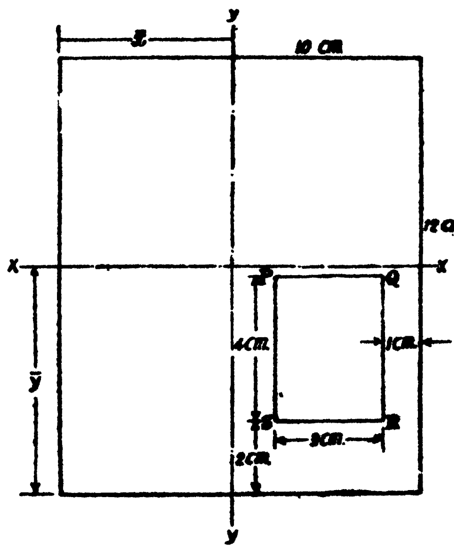


Fig. 83.

Solution. The computation is made in the following table.

$$\bar{y} = \frac{\Sigma\, ay}{\Sigma\, a} = \frac{672}{108} = 6{\cdot}22 \text{ cms.}$$

$$\bar{x} = \frac{\Sigma\, ax}{\Sigma\, a} = \frac{510}{108} = 4{\cdot}72 \text{ cms.}$$

Component	*Area cm.^{2}*	*Centroidal Distance y from bottom edge (cm.)*	*Centroidal Distance x from left edge (cm.)*	*a y cm.^{3}*	*a x cm.^{3}*
Area *ABCD* 10×12	120	6	5	720	600
Deduct for opening *PQRS* 4×3	12	4	7·5	48	90
Net quantity	108			672	510

Problem 89. *Determine the centroid of the Section shown in Fig. 84.*

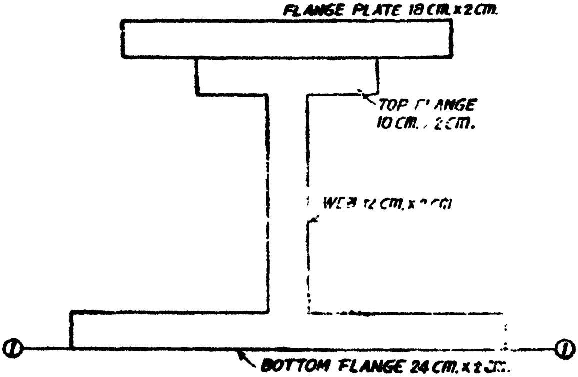


Fig. 84.

Solution. The computation is made in the following table.

Component	*Area a cm.^{2}*	*Centroidal distance from 1—1 y cms.*	*ay (cm.)^3*
Top Flange plate	36	17	612
Top flange	20	15	300
Web	24	8	192
Bottom flange	48	1	48
Total	128		1152

$$\therefore \quad \bar{y}=\frac{\Sigma\, ay}{\Sigma\, a}=\frac{1152}{128}=9 \text{ cms.}$$

§27. Centroid of a triangular lamina.

Let ABC be a triangular lamina.

Let this lamina be divided into a number of narrow strips like B_1C_1 parallel to BC

The centre of gravity of the strip B_1C_1 is at the middle point M of B_1C_1.

If AD be the median we find that M lies on AD. This holds good for the centre of gravity of any other strip parallel to BC Hence the centres of gravity of all such parallel strips lie on the median AD. Hence the centre of gravity of the whole lamina lies on AD. Similarly it can be shown that the centre of gravity should also be on median BE. Hence the centre of gravity lies at the point of intersection of the medians of the triangle.

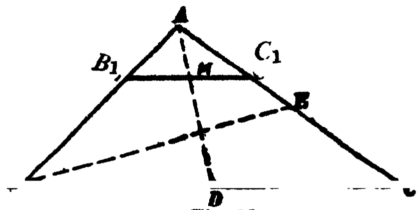


Fig. 85.

Some important cases :

(*i*) *Right-angled triangle of base b and altitude h.*

Let G be the centroid.

Let the distance of the centroid from AC be $\bar{x}$.

By similar triangles AGE and AHC, we have

$$\frac{GE}{HC}=\frac{AG}{AH}=\frac{2}{3}$$

$$\therefore \quad GE=\bar{x}=\frac{2}{3}HC$$

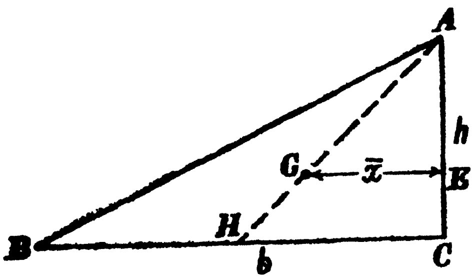


Fig. 86

$$=\frac{2}{3}\cdot\frac{b}{2}=\frac{b}{3}$$

$\therefore$ The centroid is at $\frac{b}{3}$ from the end C and $\frac{2}{3}\,b$ from the end B.

(*ii*) *Any triangle*

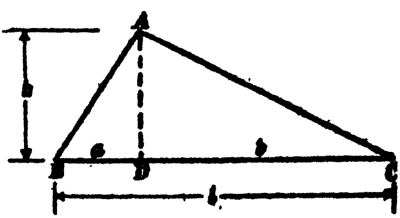
Let ABC be any triangle of base l and altitude h.

Let AD be perpendicular to BC.

Let $BD=a$ and $DC=h$

$\therefore \quad a+b=l$

Let the centroid of the triangle be at a horizontal distance of $\bar{x}$ from B.



Fiρ. 87.

Sum of the moments of the areas ABD and ADC about B
=Moment of the area ABC about B.

$$\therefore \quad \frac{1}{2}ah\,\frac{2}{3}a+\frac{1}{2}bh\left(a+\frac{b}{3}\right)$$

$$=\frac{1}{2}h\bar{x}$$

$$\therefore \quad \frac{2}{3}a^2+ab+\frac{b^2}{3}=l\bar{x}$$

$$\therefore \quad \frac{2a^2+3ab+b^2}{3}=l\bar{x}$$

$$\frac{(2a+b)(a+b)}{3}=l\bar{x}$$

But $\quad (a+b)=l$

$$\therefore \quad \bar{x}=\frac{2a+b}{3}$$

$$\therefore \quad \bar{x}=\frac{a+b+a}{3}$$

$$\therefore \quad \bar{x}=\frac{l+a}{3}$$

Hence the centroidal distance from the left end

$$=\frac{l+a}{3}$$

Similarly, the centroidal distance from the right end

$$=\frac{l+b}{3}$$

§28. Centroid of a Trapezium

Let $ABCD$ be a trapezium.

Let the parallel sides AB and DC be a and b respectively.

Let h be the distance between the parallel sides.

Let AA_1 be perpendicular to DC.

and BB_1 be perpendicular to DC.

Let the centroidal distance from DC be $\bar{y}$.

Total Area $\quad =\frac{a+b}{2}h$

The moment of the trapezium about DC

=Sum of the moments of triangle ADA_1 triangle BB_1C and rectangle ABB_1A_1, about DC.

$$\frac{a+b}{2}h\bar{y}=\frac{1}{2}h(DA_1+CB_1)\frac{h}{3}+\frac{ah^2}{2}$$

$$\therefore \quad \frac{a+b}{2}h\bar{y}=\frac{h^2}{6}(b-a)+\frac{ah^2}{2}$$

$$=\frac{h^2}{6}(b-a+3a)$$

$$\therefore \quad \frac{(a+b)}{2}h\bar{y}=\frac{h^2}{6}(2a+b)$$

$$\therefore \quad \bar{y}=\frac{b+2a}{b+a}\cdot\frac{h}{3}$$

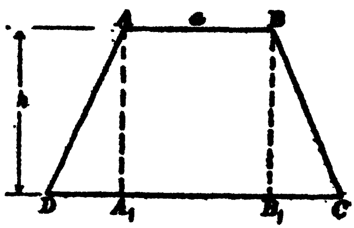


Fig. 88.

Hence centroidal distance from DC

$$=\frac{b+2a}{b+a}\cdot\frac{h}{3}$$

Similarly, centroidal distance from AB

$$=\frac{a+2b}{a+b}\cdot\frac{h}{3}$$

Problem 90. *Find the centre of gravity of a uniform plate in the form of a symmetrical trapezium whose parallel sides are 3 metres and 1·5 metres in length and 2 metres apart.*

If it has a rectangular extension of the same weight per square metre attached to the 1·5 metres edge and 1·5 metres long so as just to fit that edge, find what should be the height of the rectangular piece if the centre of gravity of the whole is on the 1·5 metre edge of the trapezium. (*I. Mech. E.*)

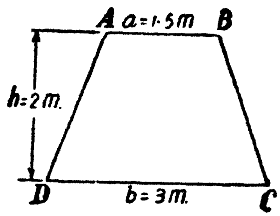


Fig. 89.

Solution. *Case 1. Trapezoidal plate.*

Distance of the centroid from the side AB

$$=\frac{a+2b}{a+b}\cdot\frac{h}{3}$$

$$=\frac{1·5+6}{4·5}\times\frac{2}{3}\text{ metre}$$

$$=\frac{10}{9}=1·11\text{ metre}$$

Case 2

Let the size of the rectangular piece be 1·5 $m\times y$ *metres*

Since AB is the centroidal axis.

Moment of the area $FEBA$ about AB
= Moment of the area $ABCD$ about AB

$$\therefore \quad 1·5\,y.\frac{y}{2}=\left(\frac{4·5}{2}\right)\times2\times\frac{10}{9}$$

$$y=2·58\ m.$$

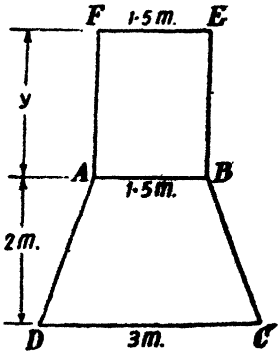


Fig. 90.

Problem. 91 *Find the centroid of a semicircular lamina of radius r.*

Solution. Consider the elemental radial area OPP' the angle POP' being $d\theta$. This can be taken as a triangle of area $\frac{1}{2}r^2\,d\theta$. The distance of the centroid of this elemental area is $\frac{2}{3}r$ from O.

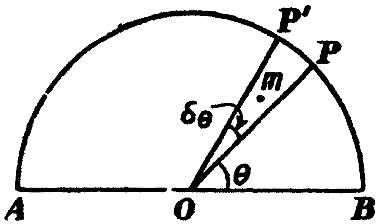


Fig. 91.

Hence the height of the centroid of the elemental area above AB

$$=\frac{2}{3}r\sin\theta$$

$\therefore$ Moment of the elemental area about AB

$$=\frac{1}{2}r^2 d\theta\times\frac{2}{3}r\sin\theta$$

$$=\frac{1}{3}r^3\sin\theta\,d\theta$$

$\therefore$ Moment of the whole area about AB

$$=\frac{1}{3}r^3\int_0^{\pi}\sin\theta\,d\theta$$

$$=\frac{1}{3}r^3\,2\int_0^{\pi/2}\sin\theta\,d\theta$$

$$=\frac{2}{3}r^3$$

Let $\bar{y}$ be the height of the centroid above AB

Area of the semicircle

$$=\frac{\pi r^2}{2}$$

$$\therefore\quad \frac{\pi r^2}{2}\bar{y}=\frac{2}{3}r^3$$

$$\therefore\quad \bar{y}=\frac{4r}{3\pi}$$

Problem. 92. *Determine the position of the centre of gravity of the plane figure shown in Fig. 92.*

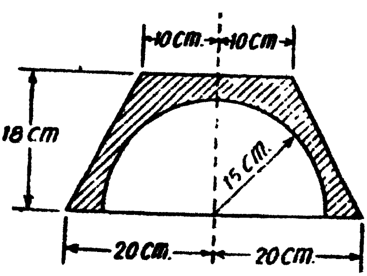


Fig. 92.

Solution. Area of the trapezium.

$$=A_1=\frac{18}{2}(20+40)\ cm^2.$$

$$=540\ cm^2.$$

Height of the centroid of the trapezium above the base

$$y_1 = \frac{b+2a}{b+a} \cdot \frac{h}{3}$$

$$= \frac{40+2\times 20}{40+20} \times \frac{18}{3} \text{ cm.}$$

$$= 8 \text{ cm.}$$

Area of the semicircle $= A_2 = \frac{\pi r^2}{2} = \frac{\pi \times 15^2}{2} \text{ cm.}^2$

$$= 353{\cdot}5 \text{ cm.}^2$$

Height of the centroid of the semicircle above the base

$$= y_2 = \frac{4r}{3\pi} = \frac{4\times 15}{3\pi} \text{ cm.} = 6{\cdot}365 \text{ cm.}$$

∴ Distance of centroid of the net section

$$= y = \frac{A_1 y_1 - A_2 y_2}{A_1 - A_2}$$

$$= \frac{540\times 8 - 353{\cdot}5\times 6{\cdot}365}{540 - 353{\cdot}5} \text{ cm.}$$

$$= 11{\cdot}10 \text{ cm.}$$

Problem 93. *Find the height of the centroid above the axis 1—1 for the lamina shown in Fig. 93.*

Solution. $FI = FG \cos GFH$

$$= 8 \times \frac{8}{10} = 6{\cdot}40 \text{ cm.}$$

$$IH = 10 - 6{\cdot}4 = 3{\cdot}6 \text{ cm.}$$

The lamina can be split up into a number of components. The areas of the various components, their centroidal distances from the axis 1—1, and the moments of the various components about the axis 1—1 are tabulated below.

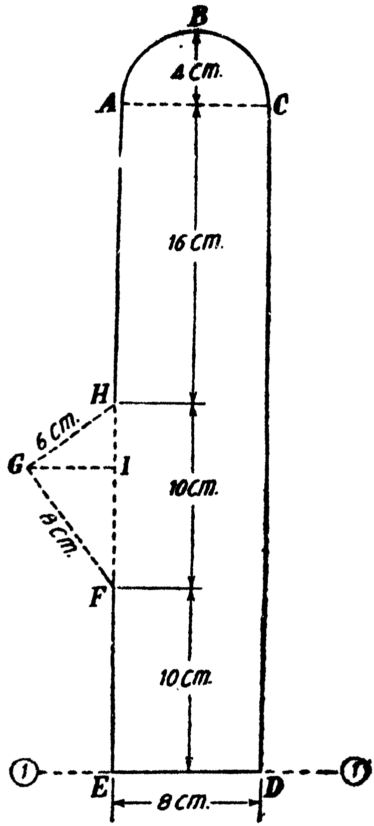


Fig. 93.

Component	*Area a* *cm.*^{2}	*Centroidal distance y from 1 1* (*cm.*)	*a y* (*cm.*)^3
Semicircle *ABC*	$\frac{\pi\times4^2}{2}=25{\cdot}13$	$36+\frac{4\times4}{3\pi}=37\ 70$	947 2
Rectangle *ACDE*	$36\times8=288$	18	5184
Triangle *FGH*	$\frac{8}{2}\times6=24$	$10+\frac{10+6\ 4}{3}=15{\cdot}46$	371
Total	337·12		6502·2

$\therefore$ Distance of the centroid from the axis 1—1

$$=\bar{y}=\frac{\Sigma ay}{\Sigma a}=\frac{6502{\cdot}2}{337{\cdot}12}=19{\cdot}29\ cm.$$

Problem 94. *Determine the centroid of the lamina shown in Fig. 94.*

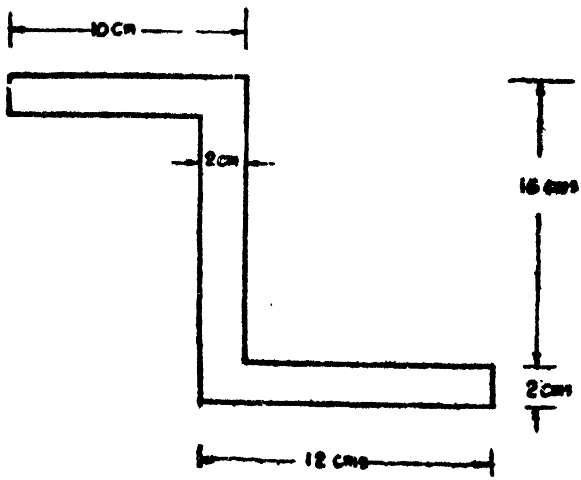


Fig. 94

Solution. The position of the centroid with respect to axes 1—1 and 2—2 will be determined. The lamina is conveniently split into separate components *ABCD*, *LCKE* and *EFGH*, as shown in Fig. 95.

The areas of these components, their centroidal distances from the axes 1—1 and 2—2 and the moments of the areas of the individual components about the axes 1—1 and 2—2 are tabulated below :

mponents	Area a $cm.^2$	Centroidal distance y from 1 1 cm.	Centroidal distance x from 2 2 cm.	ay $cm.^3$	ax $cm.^3$
ABCD	20	17	15	340	300
LCKE	28	9	11	252	308
EFGH	24	1	6	24	144
Total	72			616	752

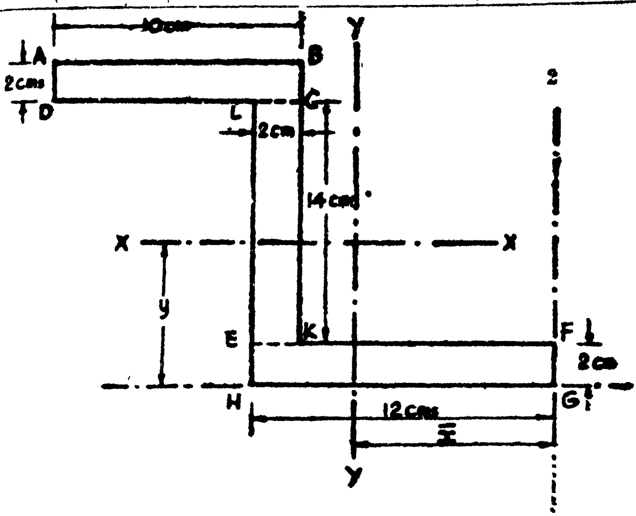


Fig. 95

$$\bar{y}=\frac{\Sigma ay}{\Sigma a}$$

$$=\frac{616}{72}=8{\cdot}56 \text{ cms. from the axis } 1-1$$

and,

$$\bar{x}=\frac{\Sigma ax}{\Sigma a}$$

$$=\frac{752}{72}=10{\cdot}44 \text{ cms. from the axis } 2-2$$

ical Solution

To find $\bar{x}$. The given section is split up into three rectangles own. The areas of the three rectangles are respectively 20 $cm.^2$, ^{2} and 24 cm^2. Now forces of magnitude 20 units, 28 units, units are assumed acting vertically through the centroids of pective rectangles. Let the forces be represented by *AB*, *BC* *CD*. Adopting a convenient scale mark off *ab*=20 units, units and *cd*=24 units.

A pole O is chosen and ao, bo, co, and do are joined. Starting from any point on the load line AB the funicular polygon is drawn, and the point O_1 is obtained. Through O_1 a vertical line is drawn. This line represents the vertical centroidal axis. This axis is at a distance of 10·4 *cm.* from the right end.

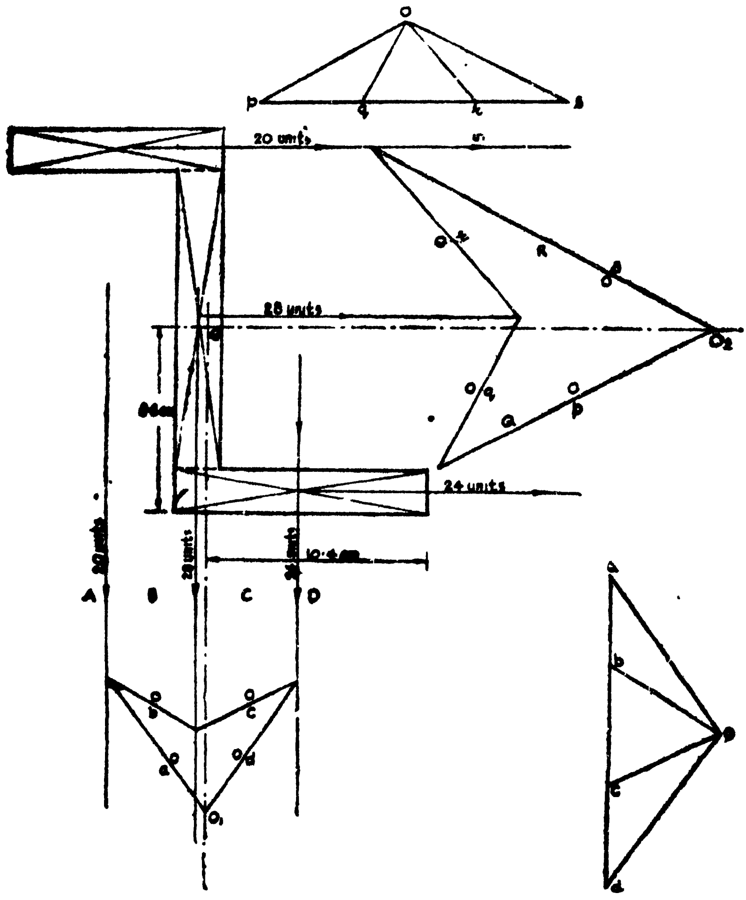


Fig. 96

Similarly taking three horizontal forces of 20 units, 28 units and 24 units through the centroids of the respective rectangles, a polar diagram and the corresponding funicular polygon are drawn and the point O_2 is obtained. Now a horizontal line is drawn through O_2. This represents the horizontal centroidal axis. This axis is found to be 8·6 *cms* above the bottom edge.

Problem 95. *Fig. 97 shows the cross-section of a masonry dam. Determine the distance of the centroid from the vertical face.*

Solution. The figure will be split up into a rectangle $ABCD$ and a triangle BCE as shown in Fig. 98. Area of the rectangle $ABCD = ah$.

Centroidal distance of the rectangle $ABCD$ from the vertical side

$$\frac{a}{2}$$

Area of the triangle BCE

$$\frac{(b-a)}{}$$

Centroidal distance of the triangle BCE, from the vertical side

$$=a+\frac{b-a}{3}$$

$$2a+b$$

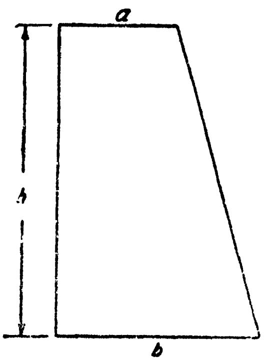


Fig. 97

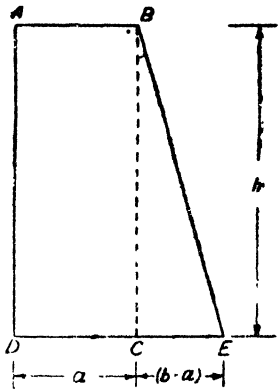


Fig. 98

Moment of the cross-sectional area about the vertical side

$$=ah\ \frac{a}{2}+\frac{(b-a)}{2}h\ \frac{(2a+b)}{3}$$

$$=\frac{h}{6}\ (3a^2+b^2+ab-2a^2)$$

$$=\frac{a^2+ab+b^2}{6}h$$

Area of the whole section

$$=\frac{h}{2}(a+b)$$

Let the centroid of the section be at $\bar{x}$ from the vertical side

$$\therefore \quad \frac{h}{2}\ (a+b)\ \bar{x}=\frac{a^2+ab+b^2}{6}\ h$$

$$\therefore \quad \bar{x}=\frac{a^2+ab+b^2}{3(a+b)} \text{ or } \frac{\quad}{3(b^2-a^2)}$$

Problem 96. *Determine the centroid of the dam section shown in Fig. 99.*

Solution. The section may be conveniently divided into various components as shown in Fig. 100.

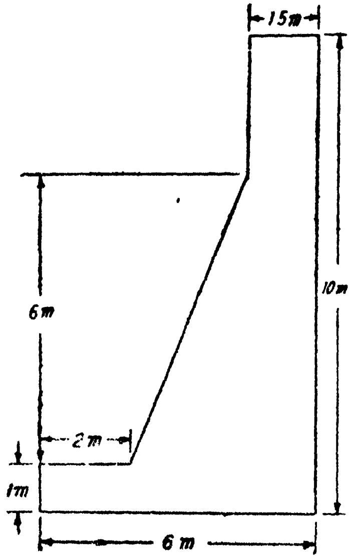


Fig. 99

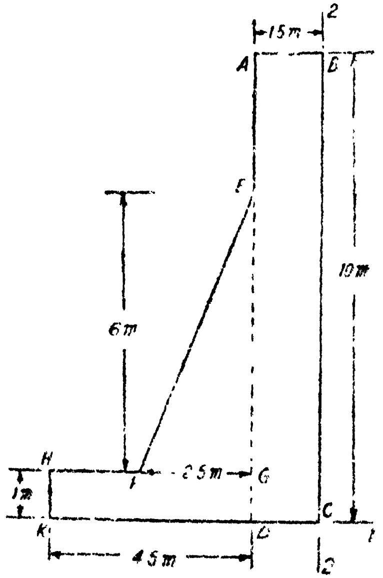


Fig. 100

The areas of the various components, their centroidal distances from the axes 1—1 and 2—2 and the moments of the areas of the various components about the axes 1—1 and 2—2 are tabulated below :

Component	*Area a* (m^2)	*Centroidal distance y from 1—1* (m)	*Centroidal distance x from 2 - 2* (m)	*Ay* (m^3)	*Ax* (m^3)
ABCD 1.5×10	15.00	5.00	0.75	75.00	11.25
EFG $\frac{2.5}{2}\times6$	7.50	$1+\frac{6}{3}=3.00$	$1.50+\frac{2.5}{3}=\frac{7}{3}$	22.50	17.50
HGDK 4.5×1	4.50	$\frac{1}{2}$	$1.5+\frac{4.5}{2}=\frac{15}{4}$	2.25	16.88
Total	27.00			99.75	45.63

∴ Height of centroid above the axis 1—1

$$=\bar{y}=\frac{\Sigma ay}{\Sigma a}$$

$$=\frac{99.75}{27}=3.694\ m$$

Distance to the centroid from the axis 2—2

$$=\bar{x}=\frac{\Sigma ax}{\Sigma a}$$

$$=\frac{45.63}{27}=1.69\ m$$

Problem 97. *In a circular sheet of metal of radius R a hole of radius r is made as shown in Fig. 101. Determine the centroid of the remaining sheet.*

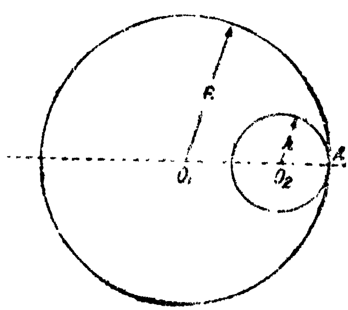


Fig. 101

Solution Area of the sheet before making the hole $=\pi R^2$

Its centroidal distance from $A=R$.

Area of the hole $=\pi r^2$

Its centroidal distance from $A=r$.

Moment of the area of the sheet about A before driving the hole

$=\pi R^2\times R=\pi R^3$

Moment of the area of the hole about A

$=\pi r^2\times r=\pi r^3$.

Net area of the sheet after driving the hole

$$=\pi R^2-\pi r^2=\pi(R^2-r^2)$$

Net moment of the area of the sheet about A after driving the hole

$$=\pi R^3-\pi r^3=\pi(R^3-r^3)$$

Let the centroidal distance of the sheet after making the hole be $\bar{x}$ from A

$$\therefore \quad \pi(R^2-r^2)\bar{x}=\pi(R^3-r^3)$$

$$\therefore \quad \bar{x}=\frac{R^3-r^3}{R^2-r^2}$$

§29. Moment of Inertia of a Lamina

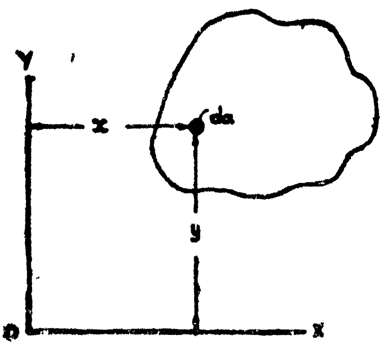


Fig. 102

Fig. 102 shows a lamina of area A. The lamina may be split up into an infinite number of elemental components each of area da. Suppose these infinite components be at distances y_1, y_2, y_3 *etc.* from the axis OX, the quantity

$da\, y_1^2+da\, y_2^2+da\, y_3^2+\ldots etc$ *i.e.*, $\Sigma da y^2$ is called the *moment of inertia* or the *second moment of area of the lamina* about the axis OX.

Similarly, if x_1, x_2, x_3, *etc.* are the distances of the various elemental components of area da each from the axis OY, then

$da\, x_1^2+da\, x_2^2+da\, x_3^2+\ldots$ *i e.*, $\Sigma da\, x^2$ is called the moment of Inertia of the lamina about the axis OY.

In general if r_1, r_2, $r_3\ldots etc.$ are the distances of the elemental components each of area da of the lamina from a given axis, then $da\, r_1^2+da\, r_2^2+da\, r_3^2+\ldots$ *i.e.* $\Sigma da\, r^2$ is called the moment of Inertia of the lamina about the given axis.

Radius of gyration of a given lamina about a given axis.

Fig. 103 (*i*) shows a lamina of area A. Let this area be split up into infinite components of area da each. Let these components be at distances r_1, r_2, r_3 *etc.* from a given axis AB. By definition, moment of inertia of the given lamina about the given axis AB

$$=I_{ab}=\Sigma da\, r_1^2$$

Let the infinite components of the lamina be arranged at the same distance k from the axis AB, the distance k being such that the moment of inertia about the axis AB remains unchanged. In this second position, each elemental area is at the same distance k from AB.

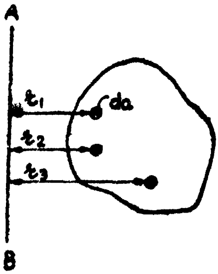
Moment of inertia of the lamina about the axis AB

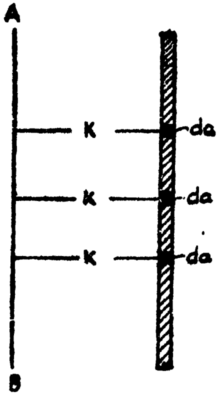
$$=I_{ab}=da\, k^2+da\, k^2+da\, k^2+\ldots$$

$$=k^2\Sigma da=Ak^2$$

If in the two positions the moment of inertia should be the same $I_{ab} = Ak^2$

k is called the radius of gyration of the lamina about the axis AB.





(i) Fig 103 (ii)

The radius of gyration of a given lamina about a given axis is that distance from the given axis at which all elemental parts of the lamina would have to be placed so as not to alter the moment of inertia about the given axis.

§30. The perpendicular axes theorem

If I_{ox} and I_{oy} be the Moments of Inertia of a lamina about mutually perpendicular axis OX and OY in the plane of the lamina and I_{oz} be the Moment of Inertia of the lamina about an axis normal to the lamina and passing through the point of intersection of the axes OX and OY,

$$I_{oz} = I_{ox} + I_{oy}$$

Fig. 104 shows a lamina of area A. Let OX and OY be two mutually perpendicular axes lying in the plane of the lamina.

Let OZ be an axis normal to the plane of the lamina and passing through O. Consider an elemental component of area da of the lamina. Let the distance of this elemental component from the axis OZ, *i.e.* from O be r.

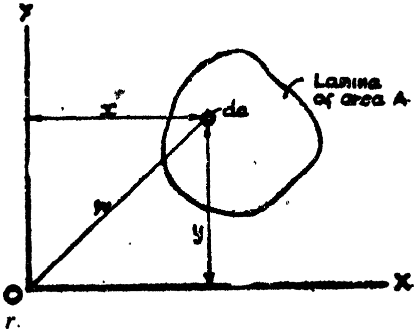


Fig. 104

∴ Moment of inertia of the elemental component about OZ

$$= da\, r^2$$

If the coordinates of the elemental component be (x, y) referred to the axes OX and OY, we have

$$r^2 = x^2 + y^2$$

$$I_{xx}=\frac{bd^3}{12}$$

$$\therefore \qquad I_{1-1}=\frac{bd^3}{12}+bd.\left(\frac{d}{2}\right)^2$$

$$=\frac{bd^3}{12}+\frac{bd^3}{4}$$

$$=\frac{bd^3}{3}$$

Similarly the moment of inertia about the axis 2—2 (Fig. 106) is given by

$$I_{2-2}=\frac{db^3}{3}$$

If G be the centroid of the lamina the axis through-the centroid and normal to the plane of the lamina is called the *polar axis*. Let I_p be the moment of inertia about the polar axis. I_p is called the polar moment of inertia.

By the perpendicular axes theorem.

$$I_p=I_{xx}+I_{yy}=\frac{bd^3}{12}+\frac{db^3}{12}$$

An important particular case

Suppose the lamina $ABCD$ be a square of side b

$$I_{xx}=I_{yy}=\frac{b^4}{12}$$

Polar Moment of Inertia

$$=I_p=\frac{b^4}{6}$$

$$I_{ac}=I_{bd}=\frac{b^4}{12}$$

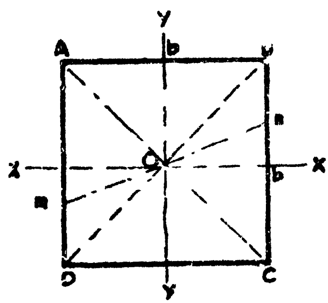


Fig. 107

In general if mn be any centroidal axis in the plane of the lamina

$$I_{mn}=\frac{b^4}{12}$$

(*ii*) *Rectangular lamina with a centrally situated rectangular hole.*

Let in a rectangular lamina $B\times D$ a rectangular hole $b\times d$ be made centrally.

Moment of inertia of the lamina about any axis = Moment of inertia of the bigger rectangle — Moment of inertia of the smaller rectangle.

For example $\qquad I_{xx}=\frac{BD^3}{12}-\frac{bd^3}{12}$

(iii) I-section and channel section

Fig. 109 shows an *I*-section. Let the overall dimension of the *I*-section be *B* units wide and *D* units deep.

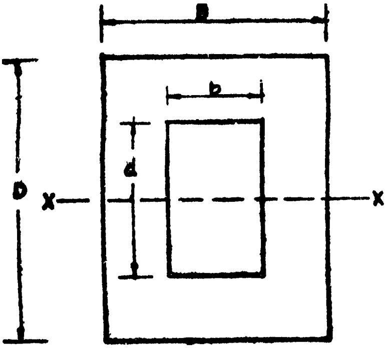


Fig. 108

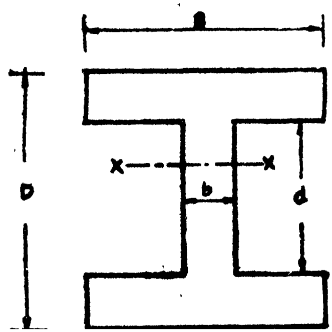


Fig. 109

Let the dimensions of the web be *b* units wide and *d* units deep.

Moment of inertia about the axis *XX*

= Moment of inertia of the rectangle $B \times D$ − moment of inertia of the hollow rectangular part.

$$\therefore I_{xx} = \frac{BD^3}{12} - \frac{(B-b)d^3}{12}$$

Fig. 110 shows a channel section of overall width *B* and depth *D*. Let the web be *b* units wide and *d* units deep.

Moment of inertia of this lamina about the axis *XX* =

Moment of inertia of the rectangle $B \times D$ − Moment of inertia of the rectangle $(B-b) \times d$

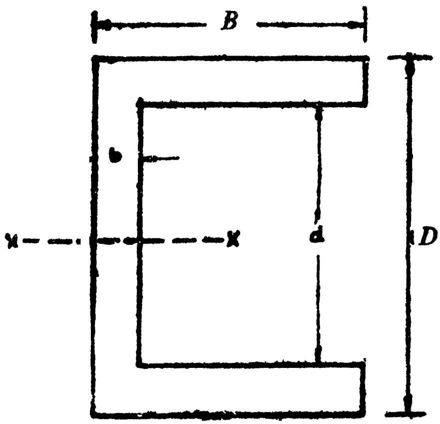


Fig. 110

$$\therefore \quad I_{xx} = \frac{BD^3}{12} - \frac{(B-b)d^3}{12}$$

(iv) Moment of Inertia of a thin ring

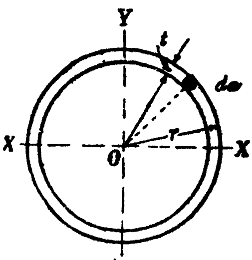


Fig. 111

Fig. 111 shows a ring of mean radius *r* and of thickness *t*.

Consider an element component *da* of the lamina. The moment of inertia of this elemental component about the polar axis of the lamina $= dar^2$.

Since all the elemental components are at the same distance *r* from the polar axis, moment of inertia of the ring about the polar axis

$= I_p = \Sigma da r^2 = r^2 \Sigma da$

$= r^2 \times$ area of the whole ring

$= r^2 \times 2\pi r t$

Moment of inertia about the axis XX

= Moment of inertia about the axis YY

I_p

(v) *Moment of Inertia of a circular lamina*

Fig. 112 shows a circular lamina of radius R.

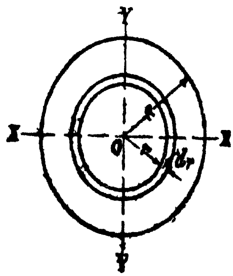


Fig. 112

The lamina may be considered as consisting of infinite number of elemental concentric rings.

Consider one such elemental ring at radius r and having a thickness dr.

The moment of inertia of the elemental ring about the polar axis

= area of the ring $\times$ (radius)^2

$= 2\pi r\, dr \,.\, r^2$

$= 2\pi r^3\, dr$

$\therefore$ Polar moment of Inertia of the whole lamina

$$= I_p = \int_0^R 2\pi r^3\, dr$$

$$\therefore \quad I_p = \frac{2\pi R^4}{4} = \frac{\pi R^4}{2}$$

If D be the diameter of the lamina

$$D = 2R$$

$$\therefore \quad I_p = \frac{\pi}{2}\left(\frac{D}{2}\right)^4$$

$$= \frac{\pi D^4}{32}$$

But, $\quad I_{xx} = I_{yy}$

and $\quad I_{xx} + I_{yy} = I_p = \dfrac{\pi D^4}{32}$

$$\therefore \quad I_{xx} = I_{yy} = \frac{\pi D^4}{64}$$

(vi) *Moment of inertia of circular lamina with a cen-ılly situated circular hole.*

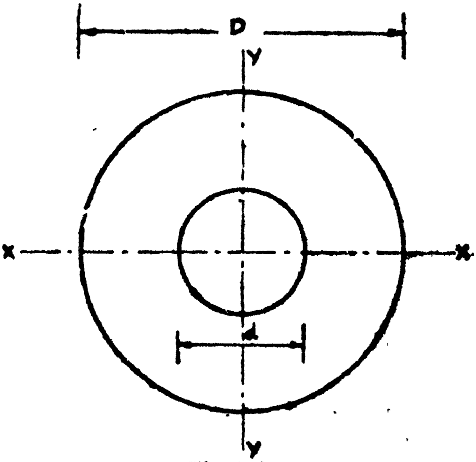


Fig 113

Let D be the external ameter and d the internal ameter of the lamina.

Polar moment of Inertia

I_p = Polar moment of Inertia of the bigger circle − Polar moment of Inertia of the smaller circle

$$I_p = \frac{\pi D^4}{32} - \frac{\pi d^4}{32}$$

$$= I_{yy} = \frac{I_p}{2} = \frac{\pi D^4}{64} - \frac{\pi d^4}{64}$$

(vii) *Moment of Inertia of a semicircular lamina*

Fig. 114 shows a semicircle of radius R. Let AB be the base the semicircle.

The moment of inertia of a circular lamina about a diameter

$$AB = \frac{\pi R^4}{4}$$

$$= \frac{\pi D^4}{64}$$

∴ Moment of inertia of the semicircle about AB

$$= I_{ab} = \frac{\pi R^4}{8}$$

$$= \frac{\pi D^4}{128}$$

ṙe the c .dal axis parallel to the base AB. Let $\bar{y}$ be betwee .e axis XX and AB.

we

$$\bar{y} = \frac{4R}{3\pi}$$

$$\frac{4}{3\pi} \cdot \frac{D}{2}$$

$$\frac{2D}{3\pi}$$

We have by the parallel axes theorem.

$$I_{ab} = I_{xx} + A\bar{y}^2$$

The area of the lamina $= A$

$$= \frac{\pi R^2}{2}$$

$$= \frac{\pi D^2}{8}$$

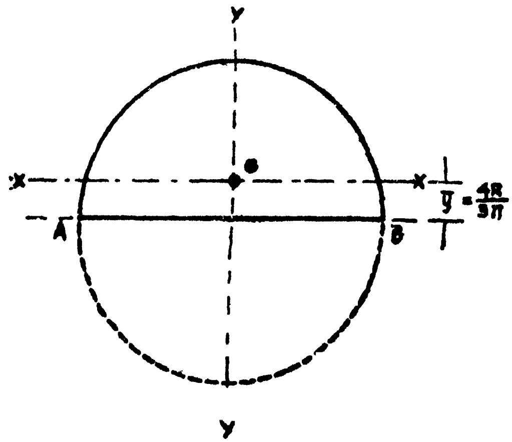


Fig. 114

$$\therefore \frac{\pi R^4}{8} = I_{xx} + \frac{\pi R^2}{2}\left(\frac{4R}{3\pi}\right)^2$$

$$\therefore I_{xx} = \frac{\pi R^4}{8} - \frac{8\pi R^4}{9\pi^2}$$

$$= \frac{\pi R^4}{8} - \frac{8R^4}{9\pi}$$

$$= 0{\cdot}11 R^4$$

$$I_{yy} = \frac{\pi R^4}{8}$$

$$\frac{\pi D^4}{128}$$

(*viii*) *Moment of Inertia of a quadrant*

Let AOC be a quadrant of a circular plate of radius R.

Moment of inertia of the area AOC about the axis AB=one-fourth of the moment of inertia of the circular area about axis AB.

$$\therefore I_{ab} = \frac{1}{4} \cdot \frac{\pi R^4}{4}$$

$$= \frac{\pi R^4}{16}$$

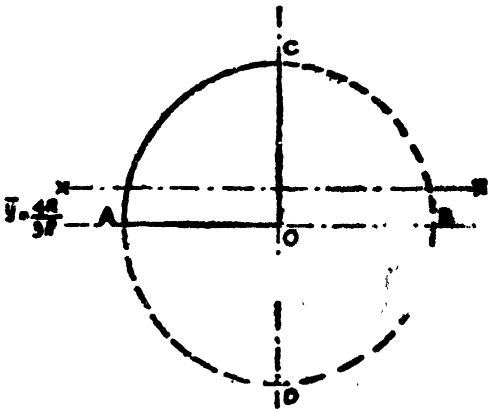


Fig. 115

Consider the semi-circle ABC. Distance of its centroid from

$$AB = \frac{4R}{3\pi}$$

The distance of the centroid of the quadrant AOC from AB is also equal to

$$\frac{4R}{3\pi}$$

i.e., the axis XX is centroidal axis of the quadrant AOC as well as the semicircle ACB

$\therefore$ Moment of inertia of the quadrant about the axis XX equals one-half the moment of inertia of the semicircle about XX.

$$I_{xx} = \frac{1}{2}\left[\ 0{\cdot}11\ R^4\ \right] = 0{\cdot}055\ R^4$$

(*v*) *Moment of Inertia of a triangular lamina*

Let *ABC* be a triangle of base *b* and altitude *h*.

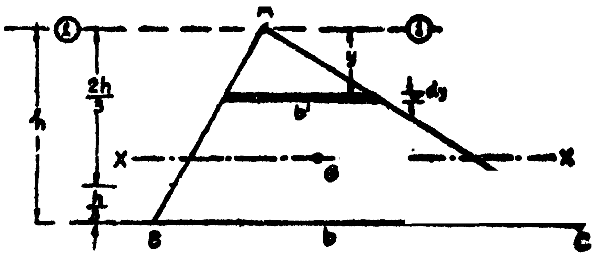


Fig. 116.

(*a*) *Moment of Inertia of a triangle about an axis 1—1 through the vertex and parallel to the base.*

The triangle may be taken to consist of a number of infinitely small elemental components parallel to the base.

Consider one such elemental component at a distance y from the vertex and of thickness dy. Width of the elemental component

$$=b'=\frac{by}{h}$$

∴ Area of the elemental component

$$=b'\ dy=\frac{b}{h}y\ dy.$$

Moment of inertia of the elemental component about the axis 1—1

$$=\frac{by}{h}dy\ y^2=\frac{b}{h}y^3\ dy$$

∴ Moment of inertia of the lamina about the axis 1—1

$$=I_{1-1}=\frac{b}{h}\int^{h} y^3\ dy=\frac{bh^3}{4}$$

(*b*) *Moment of inertia of a triangle about the centroidal axis rallel to the base.*

Let XX be the centroidal axis. This axis is at a distance of $\frac{2}{3}h$ the vertex.

Applying the parallel axis theorem, we have,

$$I_{1-1}=I_{xx}+A\left(\frac{2}{3}h\right)^2$$

$$\frac{bh^3}{4}=I_{xx}+\frac{bh}{2}\cdot\frac{4}{9}h^2$$

$$\therefore \quad I_{xx}=\frac{bh^3}{4}-\frac{2}{9}bh^3$$

$$=\frac{bh^3}{36}$$

(*c*) ***Moment of inertia of a triangle about the base.***
Applying the parallel axes theorem again, we have

$$I_{bc}=I_{xx}+A\left(\frac{h}{3}\right)^2$$

$$\frac{bh^3}{36}+\frac{bh}{2}\quad\frac{h^2}{9}$$

$$\frac{bh^3}{36}+\frac{bh^3}{18}$$

$$\frac{bh^3}{12}$$

Problem 98. *Find the moment of inertia of the area shaded in Fig. 117 about the axis AB.*

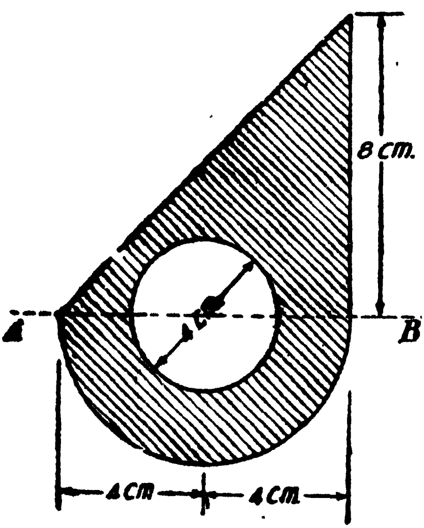


Fig. 117

Solution. Moment of inertia of the triangle about the axis AB

$$=\frac{\text{Base}\times(\text{altitude})^3}{12}$$

$$=\frac{8\times 8^3}{12}\,cm^4$$

$$=341{\cdot}33\ cm^4$$

Moment of inertia of the semicircle of 8 cm. diameter about the axis AB.

$$=\frac{\pi\times(\text{diameter})^4}{128}$$

$$=\frac{\pi\times 8^4}{128}\,cm^4$$

$$=100{\cdot}53\ cm^4.$$

$\therefore$ Moment of the gross area about the axis AB

$$=341{\cdot}33+100{\cdot}53$$
$$=441{\cdot}86\ cm^4.$$

$\therefore$ From this we have to subtract the moment of inertia of the hollow part which is a circle of 4 *cm*. diameter.

Moment of inertia of the circle of 4 *cm*. diameter about the axis AB

$$=\frac{\pi\times 4^4}{64}=12\ 56\ cm^4$$

∴ The moment of inertia of the *net* area about the axis AB
$=441{\cdot}86-12{\cdot}56=429{\cdot}30\ cm^4$.

§32. Moment of Inertia of composite sections.

Fig. 118 shows an I-section.

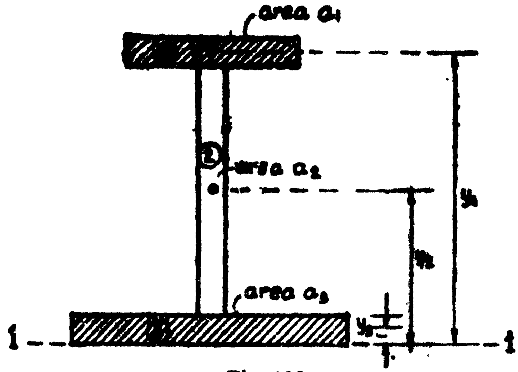


Fig. 118

Let it be required to find the moment of inertia of the section about an axis 1—1 passing through the bottom edge.

The section may be split up into components like 1, 2 and 3 of areas a_1, a_2 and a_3. Let the centroidal distances of these areas, from the axis 1—1 be y_1, y_2 and y_3. Let I_{s1}, I_{s2}, I_{s3} be the Moments of Inertia of the individual components about their respective individual centroidal axes parallel to the axis 1—1.

Now the moment of Inertia of the section about the axis 1—1

$=I_{1-1}$

= Moment of Inertia of component 1
+ Moment of Inertia of component 2
+ Moment of Inertia of component 3

$=I_{s1}+a_1y_1^2$
$+I_{s2}+a_2y_2^2$
$+I_{s3}+a_3y_3^2$

∴ $I_{1-1}=\Sigma a_1y_1^2+\Sigma I_{s1}$

i.e., $I_{1-1}=\Sigma ay^2+\Sigma I_{self}$

where I_{self} = Moment of Inertia of any component about its centroidal axis parallel to the axis 1—1.

The computations may be conveniently made in a tabular form shown below :

Components	*Area* a	*Centroidal distance from the axis 1—1*	ay	ay^2	I_{self}
1	a_1	y_1	a_1y_1	a_1y^2	I_1
2	a_2	y_2	a_2y_2	$a_2y_2^2$	I_{s2}
3	a_3	y_3	a_3y_3	$a_3y_3^2$	I_{s3}
Total	Σa		Σay	Σay^2	ΣI_{self}

Hence for any complicated composite section the above quantities may be entered as shown.

Moment of Inertia about the axis 1—1

$$=I_{1-1}=\Sigma a\, y^2+\Sigma I_{self}$$

i.e., I_{1-1} = Sum of the last two columns of the table

Distance of the centroidal axis XX from 1—1

$$=\bar{y}=\frac{\Sigma a\, y}{\Sigma a}$$

The moment of inertia about the centroidal axis XX is given by the relation

$$I_{1-1}=I_{xx}+(\Sigma a)\bar{y}^2$$

$$\therefore \quad I_{xx}=I_{1-1}-(\Sigma a)\,\bar{y}^2$$

Problem 99. *Find the moment of inertia of the section shown in Fig. 119 about the centroidal axis XX perpendicular to the web.*

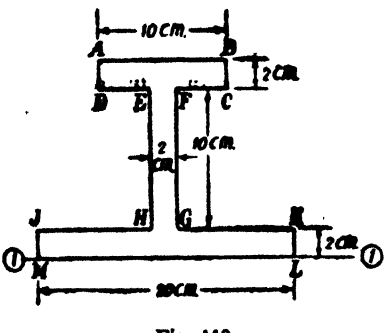


Fig. 119

Solution. The given section will be split up into components *ABCD*, *EFGH* and *JKLM*. The areas of the individual components, their centroidal distances from the axis 1—1 through the bottom edge and the moments of inertia of the individual components about their respective centroidal axes parallel to the axis 1—1 are given in the following table.

Component	*Area* *a* cm^2	*Centroidal distance y from 1—1* *cm*	*ay* cm^3	ay^2 cm^4	I_{self} cm^4
ABCD	20	13	260	3380	$\frac{10\times 2^3}{12}$ = 6·67
E'GH	20	7	140	980	$\frac{2\times 10^3}{12}$ = 16 ·67
JKLM	40	1	40	40	$\frac{20\times 2^3}{12}$ = 13·33
Total	80		440	4400	186·67

Distance of the centroidal axis XX from the axis $1-1$

$$=\bar{y}=\frac{\Sigma ay}{\Sigma a}=\frac{440}{80}=5\text{·}50 \; cms.$$

Moment of inertia about the axis $1-1$

$$=I_{1-1}=\Sigma I_{self}+\Sigma ay^2$$
$$=186\text{·}67+4400 \; cm.^4$$
$$=4586\text{·}67 \; cm.^4$$

But $\quad I_{1-1}=I_{xx}+(\Sigma a)\,\bar{y}^2$

$\therefore \quad 4586\text{·}67=I_{xx}+10(5\text{·}5)^2$

$\therefore \quad I_{xx}=2166\text{·}67 \; cm.^4$

Problem 100. *Find the moment of inertia about the centroidal axes XX and YY of the section shown in Fig. 120.*

Solution. *To find the moment of inertia about the axis XX.*

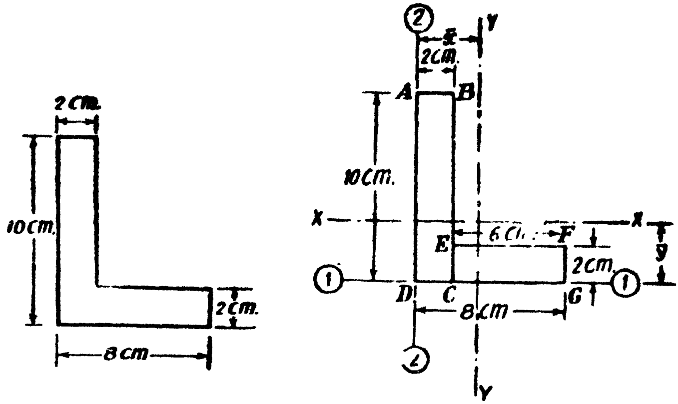


Fig. 120

The following table shows the relevant computations.

Component	*Area* a cm^2	*Centroidal distance cm from 1—1*	ay cm^3	ay^2 cm^4	I_{self} cm^4
ABCD	20	5	100	500	$\frac{2\times 10^3}{12}=166\text{·}7$
EFGC	12	1	12	12	$\frac{6\times 2^3}{12}=4\text{·}0$
Total	32		112	512	170·7

$$\therefore \quad \bar{y}=\frac{\Sigma ay}{\Sigma a}=\frac{112}{32}=3\text{·}50 \; cms.$$

Moment of inertia about the axis $1-1=I_{1-1}=\Sigma I_{self}+\Sigma ay^2$

But $\quad I_{1-1}=170\text{·}7+512=682\text{·}7 \; cm.^4$

$$\therefore \quad I_{1-1} = I_{xx} + (\Sigma a)\bar{y}^2$$

$$682{\cdot}7 = I_{xx} + 32(3{\cdot}5)^2 \quad \therefore \; I_{xx} = 297{\cdot}7 \text{ cm.}^4$$

To find the moment of inertia about the axis YY.

The following table shows the relevant computations.

Component	*area* a	*Centroidal distance* x *from* 2—2	$a\,x$	ax^2	I_{self}
ABCD	20	1	20	20	$\frac{10\times 2^3}{12} = 6{\cdot}67$
EFGH	12	5	60	300	$\frac{2\times 6^3}{12} = 36{\cdot}00$
Total	32		80	320	42·67

$$\therefore \quad \bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{80}{32} = 2{\cdot}50 \text{ cms.}$$

Moment of inertia about the axis 2—2

$$= I_{2-2} = \Sigma I_{self} + \Sigma ax^2$$

$$\therefore \quad I_{2-2} = 42{\cdot}67 + 320 = 362{\cdot}67 \text{ cm.}^4$$

But $\quad I_{2-2} = I_{yy} + (\Sigma a)(\bar{x})^2$

$$\therefore \quad 362{\cdot}67 = I_{yy} + 32(2{\cdot}5)^2$$

$$\therefore \quad I_{yy} = 162{\cdot}67 \text{ cm.}^4$$

Problem 101. *Calculate the moment of Inertia about the horizontal and vertical gravity axes (I_{xx} and I_{yy}) of the section shown in Fig. 121* *(A.M.I.E.)*

Solution. The given section will be split up into two components. The areas of the individual components, their centroidal distances from the axis 1—1 through the top edge and the Moments of Inertia of the individual components about their centroidal axes parallel to the axis 1—1 are tabulated below.

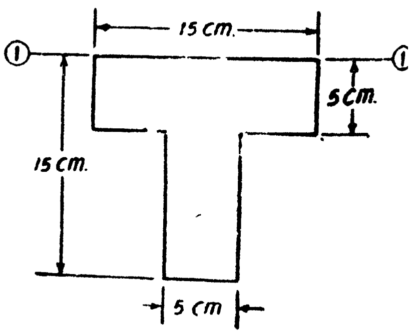


Fig. 121

Properties of the Section

Component	Area a $(cm)^2$	Centroidal distance y (cm)	ay $(cm)^3$	ay^2 $(cm)^4$	I_{self} (cm^4)
Top flange	75	2·5	187·50	468·75	$\frac{15\times5^3}{12}=156\cdot25$
Web	50	10	500·00	5000·00	$\frac{5\times10^3}{12}=416\cdot67$
Total	125		687·5	5468·75	572·92

Distance of the centroidal axis XX from 1 – 1

$$=\bar{y}=\frac{\Sigma ay}{\Sigma a}=\frac{687\cdot5}{125}$$

$$=5\cdot5\ cm.$$

$$I_{1-1}=\Sigma I_{self}+\Sigma ay^2=572\cdot92+5468\cdot75$$
$$=6041\cdot67\ cm.^4$$

But $\quad I_{1-1}=I_{xx}+(\Sigma a)\bar{y}^2$

$\therefore \quad 6041\cdot67=I_{xx}+(125)(5\cdot5)^2$

$\therefore \quad I_{xx}=2260\cdot42\ cm.^4$

$$I_{yy}=\frac{5\times15^3}{12}+\frac{10\times5^3}{12}=1510\cdot4\ cm.^4$$

Problem 102. *Find the moment of inertia about the centroidal axis XX for the lamina shown in Fig. 122.*

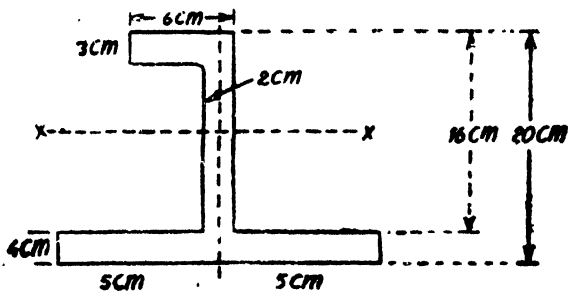


Fig. 122

Solution. The given section will be split up into three rectangular components, *i.e.*, the top flange, the web and the bottom flange.

The properties of the components are tabulated below :

Components	Area a (cm^2)	Centroidal distance y from 1–1 (cm)	ay (cm^3)	ay^2 (cm^4)	I_{self} (cm^4)
Top flange	18	18·5	333	6160·50	$\frac{6\times 3^3}{12}=13·5$
Web	26	10·5	273	2866·50	$\frac{2\times 13^3}{12}=366·167$
Bottom flange	40	2	80	160	$\frac{10\times 4^3}{12}=53·333$
Total	84		686	9187	433

Distance of centroidal axis XX from the bottom edge $1-1$

$$=\bar{y}=\frac{\Sigma ay}{\Sigma a}=\frac{686}{84}=8·17\ cm.$$

Moment of inertia about the axis $1-1$

$$=I_{1-1}=\Sigma I_{self}+\Sigma ay^2$$

$$=433+9187=9620\ cm^4$$

But $\quad I_{1-1}=I_{xx}+A\bar{y}^2$

$\therefore \quad 9620=I_{xx}+84\times 8·17^2$

$\therefore \quad I_{xx}=4015·38\ cm.^4$

Problem 103. *A T-beam is made up of two plates and two angles as shown. Determine the moment of Inertia of the T-section about*

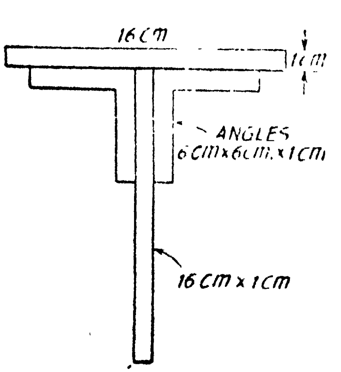


Fig. 123.

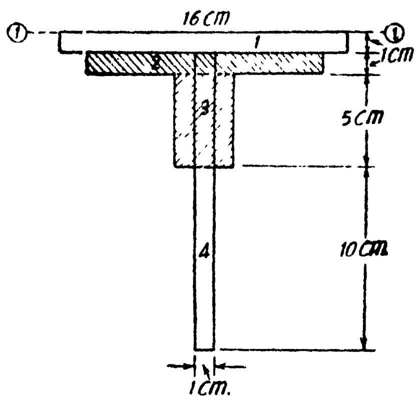


Fig. 124.

an axis passing through the centroid of the section and parallel to the top plate. (*A.M.I.E.*)

Solution. The given section will be split up into different components. The areas of the individual components 1, 2, 3, 4 and their centroidal distances from an axis 1—1 through the upper edge, their moments about the axis 1—1 and their moments of inertia about their respective centroidal axes parallel to the axis 1—1 are tabulated below :

Component	Area a $(cm)^2$	Centroidal distance y from the axis 1—1 (cm)	$a\,y$ $(cm)^3$	$a\,y^2$ $(cm)^4$	I_{self} $(cm)^4$
1	$16\times1=16{\cdot}00$	0·5	8·00	4·00	$\frac{16\times1^3}{12}=1{\cdot}333$
2	$13\times1=13{\cdot}00$	1·5	19·50	29·25	$\frac{13\times1^3}{12}=1{\cdot}083$
3	$3\times5=15{\cdot}00$	4·5	67·50	303·75	$\frac{3\times5^3}{12}=31{\cdot}250$
4	$1\times10=10{\cdot}00$	12·0	120·00	1440·00	$\frac{1\times10^3}{12}=83\ 333$
Total	54		215	1777	116.999 say 117

Distance of centroid of the section from the axis 1 — 1

$$=\bar{y}=\frac{\Sigma\,ay}{\Sigma a}=\frac{215}{54}\ cm.=3{\cdot}98\ cm.$$

Moment of inertia about the axis 1—1

$$=I_{1-1}=\Sigma\ I_{self}+\Sigma ay^2$$
$$=117+1777=1894\ cm^4.$$

But $\quad I_{1-1}=I_{xx}+(\Sigma a)\bar{y}^2$

$\therefore\quad 1894=I_{xx}+54(3{\cdot}98)^2$

$\therefore\quad I_{xx}=1037{\cdot}98\ cm^4.$

say *1038 cm*^{4}.

The method of tabulation adopted in the above problems can be used with great advantage even when the section is complicated consisting of several components. The following problem illustrates an interesting case. The student may note that any other method to analyse such a section will be tedious.

Problem 104. *Find the moment of Inertia of the built up section about the centroidal axis X-X shown in Fig. 125.* (*A.M.I.E.*)

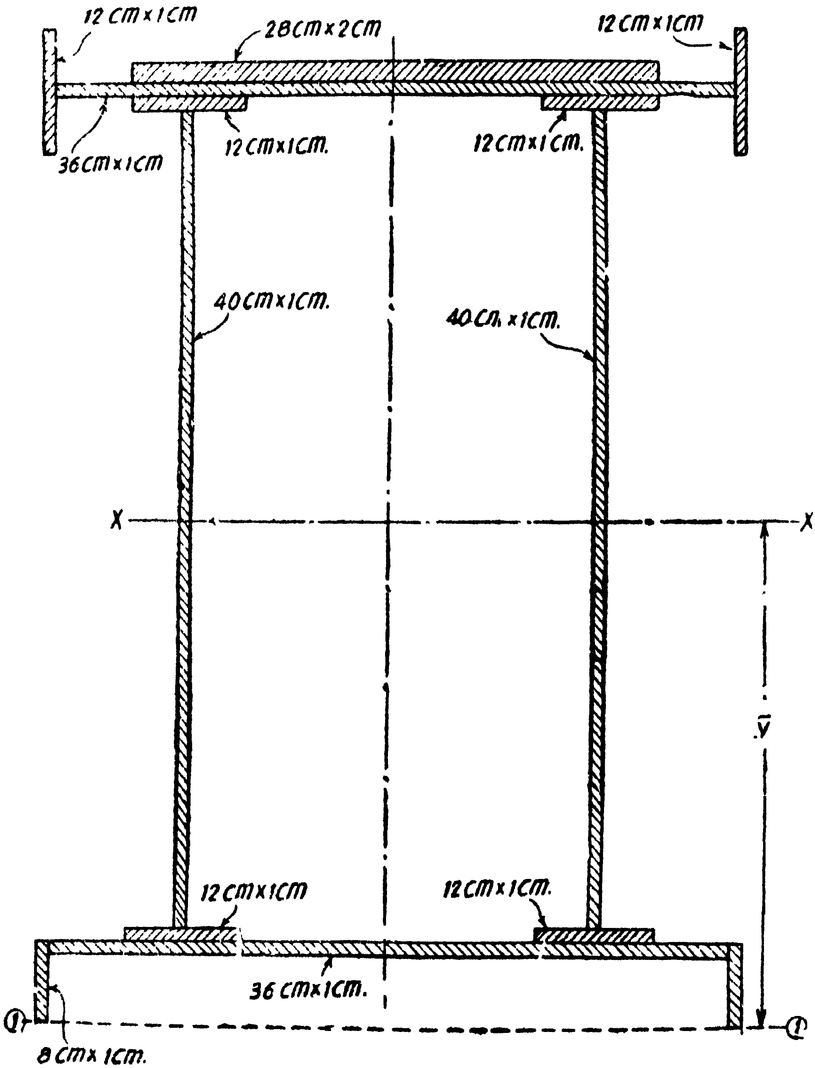


Fig. 125.

Solution. The section consists of several components. Consider an axis 1 – 1 through the bottom edge. The areas of the various components, their centroidal distances from 1 – 1, their moments about the axis 1 – 1, and the moments of inertia of the individual components about their own centroidal axes parallel to the axis 1 – 1 are tabulated below :

	Component	Area a (cm)^2	Centroidal distance y from 1—1 (cm)	ay (cm)^3	ay^2 (cm)^4	I_{self} (cm)^4
Top Flange	(*i*) One plate 28 cm × 2 cm	56	52	2912	151424	$\frac{28(2)^3}{12}=18{\cdot}67$
	(*ii*) Two plates 12 cm × 1 cm	24	49·5	1188	58806	$\frac{2\times12(1)^3}{12}=2{\cdot}00$
	(*iii*) Two plates 12 cm × 1 cm	24	50·5	1212	61206	$\frac{2\times1(12)^3}{12}=288{\cdot}00$
	(*iv*) One plate 36 cm × 1 cm	36	50·5	1818	91809	$\frac{36(1)^3}{12}=3{\cdot}00$
Web	Two plates 40 cm × 1 cm	80	29	2320	67280	$\frac{2\times1\ (40)^3}{12}=10666{\cdot}67$
Bottom Flange	(*i*) Two plates 12 cm × 1 cm	24	8·5	204	1734	$\frac{2\times12\ (1)^3}{12}=2{\cdot}00$
	(*ii*) Two plates 8 cm × 1 cm	16	4	64	256	$\frac{2\times1\ (8)^3}{12}=85{\cdot}33$
	(*iii*) One plate 36 cm × 1 cm	36	7·5	270	2025	$\frac{36\times1^3}{12}=3{\cdot}00$
	Total	296		9988	434540	11068·67

Distance of the centroidal axis XX from the axis 1—1

$$=\bar{y}=\frac{\Sigma\, a\, y}{\Sigma\, a}=\frac{9988}{296}=33{\cdot}74\ cm.$$

Moment of inertia about the axis 1 — 1

$$=I_{1-1}=\Sigma I_{self}+\Sigma ay^2$$

$$=11068{\cdot}67+434540=445608{\cdot}67\ cm.$$

But $\qquad I_{1-1}=I_{xx}+(\Sigma a)\bar{y}^2$

$\therefore \qquad 445608{\cdot}67=I_{xx}+296(33{\cdot}74)^2$

$\therefore \qquad I_{xx}=336995{\cdot}12\ cm^1$

Examples in chapter 3

(1) Determine the centroidal distances $\bar{x}$ and $\bar{y}$ for the section shown in Fig. 126. ($\bar{y}=2{\cdot}49$ *cm.*, $\bar{x}=1{\cdot}15$ *cm.*)

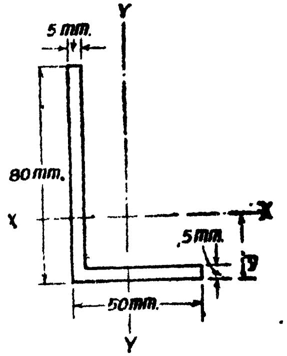


Fig. 126

(2) Find the centroidal axis of the lamina shown in Fig. 127 parallel to the base. Find also the moment of inertia about this centroidal axis. (4·08 *cms.* ; 385.33 *cm.*^{4})

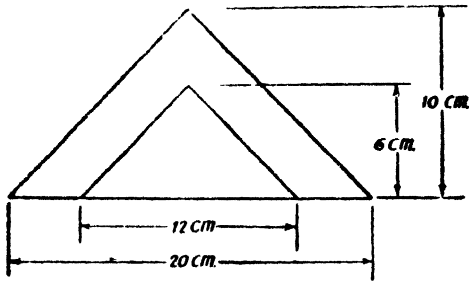


Fig. 127

(3) For the lamina shown in Fig. 128 find the moment of inertia about the centroidal axis $X—X$ parallel to the base.

(18·91 *cm.*, 183193·34 *cm*^{4}.)

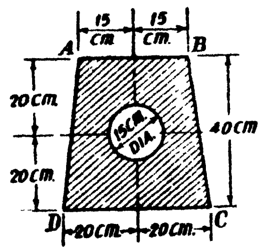


Fig. 128

(4) Find the moment of inertia of the lamina shown in Fig. 129 about the centroidal axes XX and YY.

($\bar{y}$=6·345 *cm.* from top edge

I_{xx}=3340·51 *cm*^{4}.

I_{yy}=1199·67 *cm.*^{4})

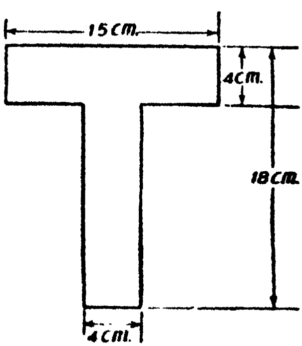


Fig. 129

(5) Find the moment of inertia of the box girder section shown in Fig. 130 about its centroidal axis XX. (*London University*)

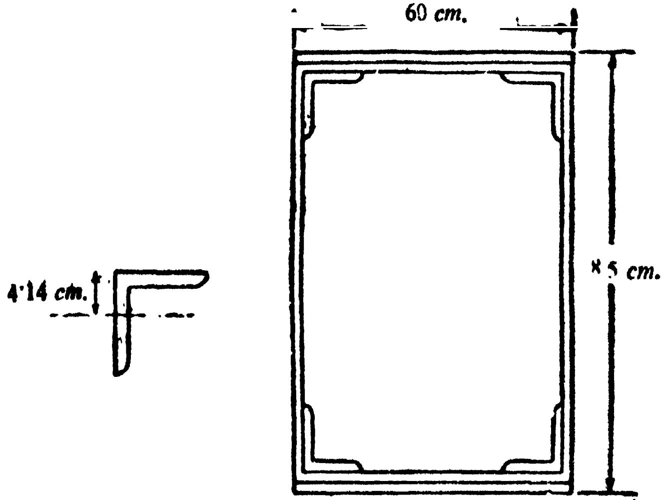


Fig. 130

2 flange plates at top each 60 *cm.* × 1·2 *cm.*

2 flange plates at bottom each 60 *cm* × 1·2 *cm.*

2 web plates each 80 *cm.* × 1 *cm.*

4 angles each 14 *cm.* × 15 *cm.* × 1·2 *cm.*

For one angle $A = 34{\cdot}59$ $cm.^2$

$I_{xx} = 735{\cdot}4$ $cm.^4$

4

Shear Forces and Bending Moments

Definitions. A beam is a structural member subjected to a system of external forces at right angles to its axis.

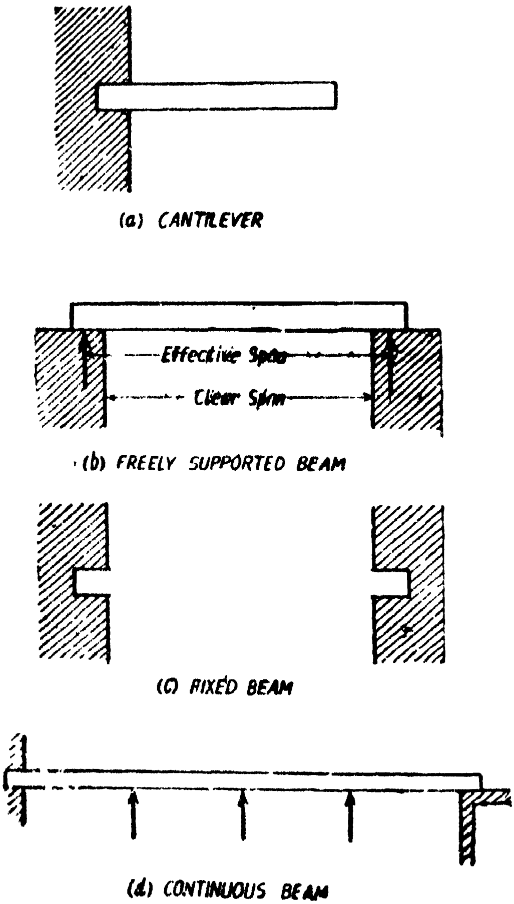


Fig. 131

If such a member is *fixed* or *built-in* at one end while its other end is free, the member is called a *cantilever*. (Fig. 131*a*).

If the ends of a beam are made to freely rest on supports the beam is called a *freely or simply supported beam.* Fig. 131*b* shows such a beam. In this case the beam is resting freely on brick masonry walls. The clear horizontal distance between the walls is called the *clear span* of the beam. The horizontal distance between the centres of the end bearings is called the *effective span* of the beam. If the intensity of the bearing reaction is not uniform the effective span is the horizontal distance between the lines of action of the end reactions.

If a beam is *fixed* at both its ends, it is called a *built-in*, or *encastred* or *fixed beam* (Fig. 131*c*).

A beam which is provided with more than two supports is called a *continuous* beam (Fig. 131*d*).

§33. Conception of Shear Force and Bending Moment.

Fig. 132 shows a cantilever *AB* whose end *A* is fixed. Let the cantilever carry a vertical load of 4000 *kg* at *C*.

For the equilibrium of the cantilever the fixed support at *A* will provide a *vertical reaction* vertically upwards, of magnitude $V_a = 4000$ *kg*.

Taking moments about *A*, we have a *clockwise* moment of $4000 \times 2 = 8000$ *kg. m.*

Hence for the equilibrium of the cantilever the fixed support at *A* must also provide a reacting moment or fixing moment of 8000 *kg. m.* of an *anti-clockwise* order.

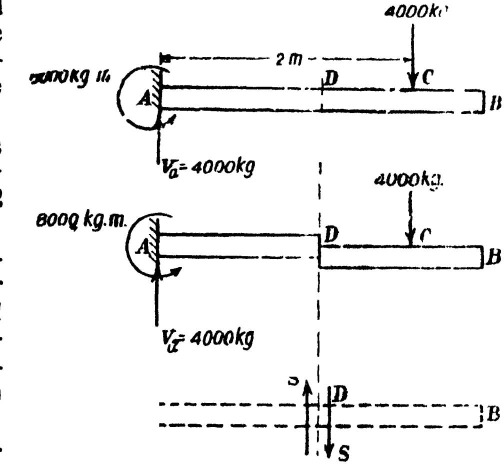


Fig. 132

Now consider a section *D*. At this section there is a possibility of failure by shear as shown in Fig. 132. If such a failure will occur at section *D*, the cantilever is liable to be sheared off into two parts. It is clear that the force acting normal to the centre line of the member on each part equals $S = 4000$ *kg*. The force acting on the *right part* of the section *D* is *downward*, while the force acting on the *left part* is *upwards*. The resultant force acting on any one of the parts normal to the axis of the member is called the *shear force* at the section *D*. For the case illustrated above the resultant force normal to the axis of the member on the right part of the section is downwards while the resultant force normal to the axis of the member on the left part of the section is upwards Such a shear force will be regarded as a *positive shear force.*

Let us now study another effect of the load applied on the cantilever. The cantilever is liable to *bend* due to the load on it.

For instance, the cantilever has a tendency to rotate in a clockwise direction about *A* (Fig. 133). Hence, the fixed support at *A* has to offer a resistance against this rotation.

Taking moments about *A* we find that the applied load of 4000 *kg.* has a clockwise moment of $4000 \times 2 = 8000$ *kg. m.* about *A*. Hence, for the equilibrium of the cantilever, the fixed support at *A* will provide a reacting or resisting *anticlockwise* moment of 8000 *kg. m.* If the support *A* is not able to provide such a resisting moment, the cantilever will not be in equilibrium and will, therefore, rotate about *A* in the clockwise order.

The magnitude of the reacting moment at *A* depends on (*i*) the magnitude of the load and (*ii*) the position of the load. We say that the support *A* provides the necessary fixing or reacting moment at *A*, and that at the section *A* of the beam, there is a bending moment of $4000 \times 2 = 8000$ *kg. m.*

Now consider, for instance, the section *D*. Suppose the part *DB* was free to rotate about *D*, obviously the load on the part *DB* would cause the part *DB* to rotate in a *cockwise order* about *D*. Considering the part *DB*, taking moments about *D*, we find that there is a *clockwise* moment of $4000 \times 0{\cdot}8 = 3200$ *kg. m.* about *D*. Hence for the equilibrium of the part *DB* it is necessary that the part *DA* of the cantilever should provide a reacting or restoring *anti-clockwise* moment of 3200 *kg. m.* about *D*.

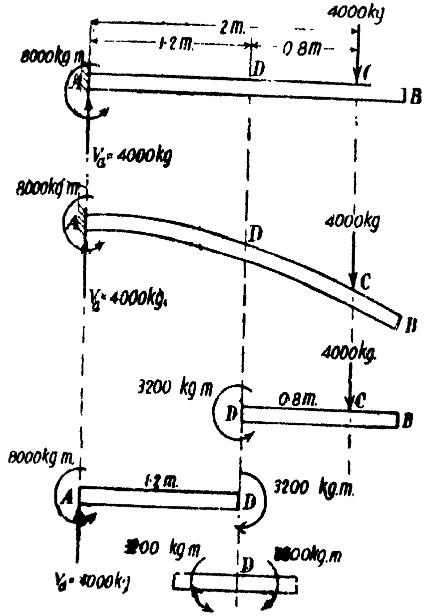


Fig. 133

Let us now discuss the equilibrium of the part *AD* (Fig. 133). Taking moments about *D*, we have following moments about *D*.

(*i*) $V_a \times AD = 4000 \times 1{\cdot}2 = 4800$ *kg. m.* (*clockwise*)

(*ii*) couple $= 8000$ *kg. m.* (*anticlockwise*)

$\therefore$ Net moment about $D = 8000 - 4800 = 3200$ *kg. m.* (*anticlockwise*).

Hence, for the equilibrium of the part *AD*, the part *DB* should provide a clockwise moment of 3200 *kg. m.*

Hence, we find that at the section *D*,

The part *DB* provides a clockwise moment of 3200 *kg. m.* and the part *DA* provides an anti-clockwise moment of 3200 *kg. m.* We say that at the section *D* there is a *bending moment* of 3200 *kg. m.*

The bending moment at the section *D* is the algebraic sum of the moments of forces and reactions acting on one side of the section.

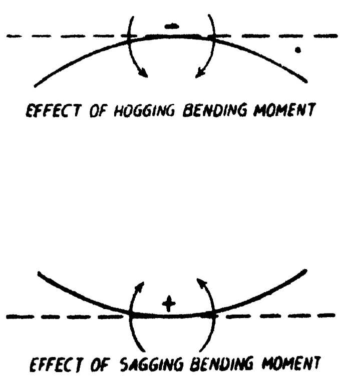


Fig. 134

In the case illustrated above the tendency of the bending moment at *D* is to bend it so as to produce convexity above the centre line. Such a bending moment is called a *hogging bending moment*. It is quite possible that the bending moment at a section may bend the member at the section so as to produce concavity above the centre line. Such a bending moment is called a *sagging bending moment.*

In our discussion we will consider sagging moments as positive and hogging moments as negative.

Now let us consider the cantilever *AB* carrying three point loads as shown (Fig. 135).

Total load on the cantilever being 3000 *kg.* the vertical reaction at *A* = 3000 *kg. upwards.*

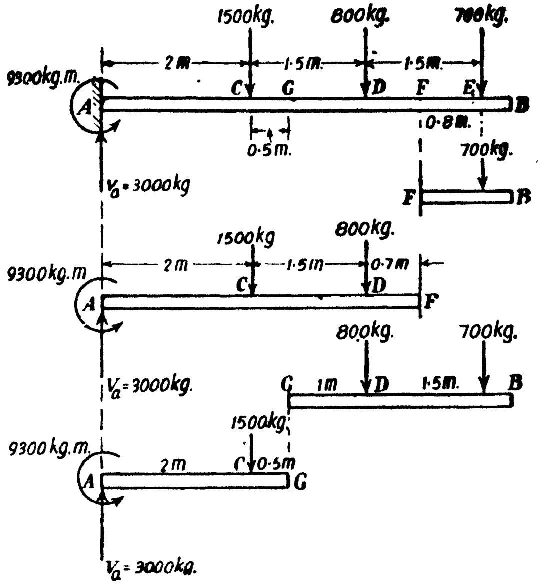


Fig. 135

Taking moments about *A* we have the following moments :

(*i*) 1500 × 2 = 3000 *kg. m.* (*clockwise*)

(*ii*) 800 × 3·5 = 2800 *kg. m.* (*clockwise*)

(*iii*) 700 × 5 = 3500 *kg. m.* (*clockwise*)

Total moment about *A*

= 9300 *kg. m.* (*clockwise*)

Hence the fixed support at *A* must provide a reacting or fixing moment of 9300 *kg. m.* (*anticlockwise*).

Now consider the section *F*.

To find the shear force at this section, consider the force acting on any one side of the section.

∴ S.F. at F = Total force normal to the member on the right hand side of $F = +700$ *kg.*

Alternatively S.F. at F = Total force normal to the member on the left hand side of F

$= +3000 - 1500 - 800 = +700$ *kg.*

Bending moment at F = Algebraic sum of moments and reactions acting on one side of the section F, about F

∴ B.M. at F = Algebraic sum of moments of forces on the right hand side of F about F

$= -700 \times 0.8 = -560$ *kg. m.* (*hogging b.m.*)

Alternatively Bending moment at F = Algebraic sum of moments and reactions on the left side of F about F

$= 3000 \times 4.2 - 1500 \times 2.2 - 800 \times 0.7 - 9300$ *kg. m.*

$= -560$ *kg. m.* (*hogging*)

Similarly

S.F. at G = Total force normal to the member on the right hand side of G

$= +700 + 800 = +1500$ *kg.*

Alternatively, S.F. at G = Total force normal to the member on the left side of G

$= +3000 - 1500 = +1500$ *kg.* (*hogging*)

B.M. at G = Algebraic sum of the moments of forces acting on the right hand side of G about G

$= -700 \times 2.5 - 800 \times 1 = -2550$ *kg. m.* (*hogging*)

Alternatively, B.M. at G = Algebraic sum of the moments of forces and reactions acting on the left hand side of G about G

$= +3000 \times 2.5 - 1500 \times 0.5 - 9300$ *kg. m.*

$= -2550$ *kg. m.* (*hogging*)

Some important hints to be noted :

SHEAR FORCE

In general if we have to calculate the shear force at a section the following procedure may be adopted :

(*i*) Consider the left or the right part of the section.

(*ii*) Add the forces normal to the member on one of the parts.

If the right part of the section is chosen, force on the right part acting downwards is positive while a force on the right part acting upwards is negative. For instance, if the S.F. at a section X of a beam is required and if the right part XB be considered the forces P and Q are positive while the force R is negative.

$\therefore$ S.F. at $X=P+Q-R$

If the left part of the section be chosen, a force on the left part acting upwards is positive and a force on the left part downwards is negative. For instance if the S.F. at X of a beam is required and if XA is the left part, the force Q is positive while the forces W_1 and W_2 are negative.

$\therefore$ S.F. at $X=Q-W_1-W_2$

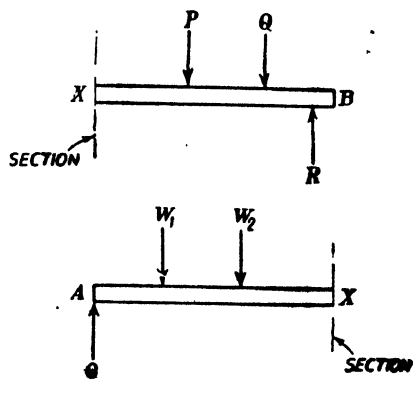


Fig 136.

BENDING MOMENT

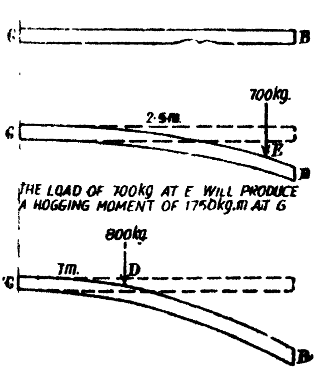
To find the bending moment at a section of a beam the following procedure may be adopted :

(*i*) Consider the left or right part of the section.

(*ii*) Remove all restraints on the part selected.

(*iii*) Now introduce each force or reacting element one at a time and find its effect at the section (*i.e.*, find whether the moment produces a hogging or sagging effect at the section). Treat sagging moments as positive and hogging moments as negative. *Note that the moment due to every downward force is negative and the moment due to every upward force is positive.* For instance, let the bending moment at the section G of the cantilever AB (Fig. 135) be required.

If the right part of the section be selected



The load of 800 kg at D will produce a hogging moment of 800 kg m at G.
Fig. 137.

Remove the restraints on the part GB

Introduce the load of 700 *kg.* at E

The independent effect of the load is to produce a hogging moment of $-700\times2{\cdot}5=-1750$ *kg.m.*

Now consider the independent effect of the 800 *kg.* load, at D.

Obviously this will also produce a hogging moment of -800×1

$=-800$ *kg.m.*

$\therefore$ Resultant bending moment at $G=-1750-800==2550$ *kg m.*(*hogging*)

§ 34. Shear Force and Bending Moment Diagrams

A shear force diagram for a structural member is a diagram which shows the values of shear forces at various sections of the member.

A bending moment diagram for a member is a diagram which shows the values of bending moment at various sections of the member. We shall now construct these diagrams for members carrying different load systems.

A. CANTILEVERS

(i) Cantilever of length l carrying a concentrated load W at the free end.

Fig. 138 shows a cantilever *AB* fixed at *A* and free at *B* and carrying the load *W* at the free end *B*.

Consider a section *X* at a distance of *x* from the free end *B*.

S.F. at $X = S_x = +W$

B.M. at $X = M_x = -Wx$

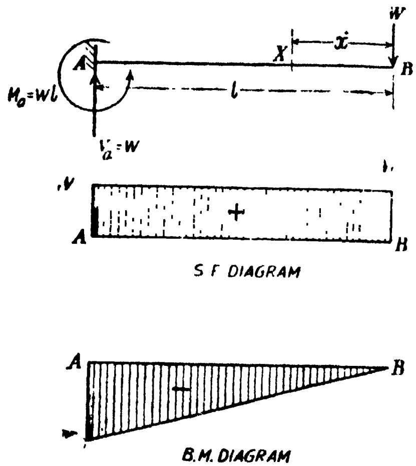


Fig. 138.

Hence we find that the S.F. is constant at all sections of the member between *A* and *B*.

But the B.M. at any section is proportional to the distance of the section from the free end.

At $x=0$, *i.e.*, at *B*, B.M. $=0$
At $x=l$, *i.e.*, at *A*,
B.M.$= Wl$

Fig. 138 shows the S.F. and B.M. diagrams.

(ii) Cantilever carrying several concentrated loads.

Suppose a cantilever *AE* is 2 *metres* long and is subjected to the forces shown in Fig. 139.

At any section between D and E, distant x from E,

S.F. $= S_x = +500$ *kg.*
B.M. $= M_x = -500x$
At $x=0$, $M_x=0$
At $x=0{\cdot}5$ *m.*
$M_x = -500 \times 0{\cdot}5$ *kg. m.*
$= -250$ *kg m.*

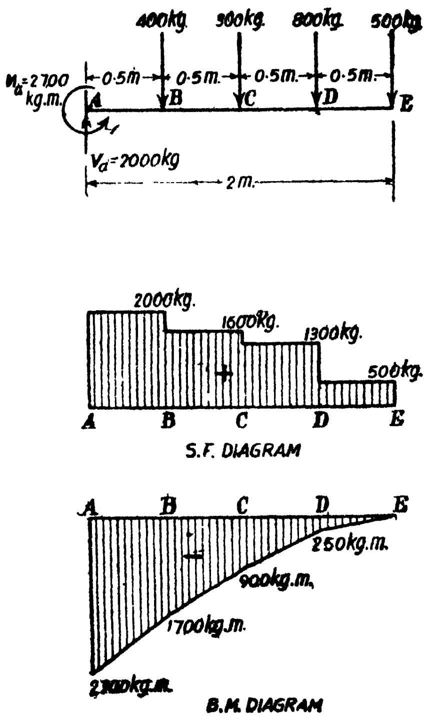


Fig. 139.

At any section between C and D, distant x from E,

S.F. $= S_x = +500 + 800 = +1300$ *kg.*

B.M. $= M_x = -500x - 800\,(x - 0{\cdot}5) = -1300x + 400$

At $x = 0{\cdot}5$, $M_x = -1300 \times 0{\cdot}5 + 400$ *kg. m.* $= -250$ *kg. m.*

At $x = 1$ *m*, $M_x = -1300 + 400$ *kg. m.* $= -900$ *kg. m.*

At any section between B and C distant x from E,

S.F. $= S_x = +500 + 800 + 300$ *kg.* $= 1600$ *kg.*

B.M. $= M_x = -500x - 800(x - 0{\cdot}5) - 300(x - 1)$ *kg. m.*

$= -1600x + 700$ *kg. m.*

At $x = 1$ *m*, $M_x = -1600 + 700 = -900$ *kg. m.*

At $x = 1{\cdot}5$ *m*, $M_x = -1600 \times 1{\cdot}5 + 700 = -1700$ *kg. m.*

At any section between A and B distant x from E,

S.F. $= S_x = +500 + 800 + 300 + 400 = 2000$ *kg.*

B.M. $= M_x = -500x - 800(x - 0{\cdot}5) - 300(x - 1) - 400\,(x - 1{\cdot}5)$

$= -2000\,x + 1300$ *kg. m.*

At $x = 1{\cdot}5$ *m*, $M_x = -2000 \times 1{\cdot}5 + 1300$ *kg. m.* $= -1700$ *kg. m.*

At $x = 2$ *m.* $M_x = -2000 \times 2 + 1300 = -2700$ *kg. m.*

In the above example the S.F. and B.M. were computed considering the forces on the right hand side of any section. In fact the computation could be made by considering the left hand side of a section also.

Suppose we consider a section between B and C distant x from E.

$\therefore$ Distance between A and the section $= (2 - x)$ *metres.*

S.F. at the section, considering the left side of the section

$$= S_x = V_a - 400.$$

$\therefore$ $S_x = 2000 - 400 = +1600$ *kg.* (as obtained before)

B M. at the section, considering the force and reactions on the left side of the section

$$= M_x = +V_a(2 - x) - M_a - 400\,(2 - x - 0{\cdot}5)$$

$$= +2000(2 - x) - 2700 - 400(1{\cdot}5 - x)$$

$\therefore$ $M_x = -1600x + 700$ *kg. m.* (as obtained before).

(*iii*) *Cantilever of length l carrying a uniformly distributed load of w per unit run over the whole length.*

Fig. 140 shows a cantilever AB fixed at A and free at B carrying a uniformly distributed load of w per unit run over the whole span.

Consider any section X distant x from the end B.

S.F. at $X = S_x = +wx$

B.M. at $X = M_x = -wx \cdot \dfrac{x}{2}$

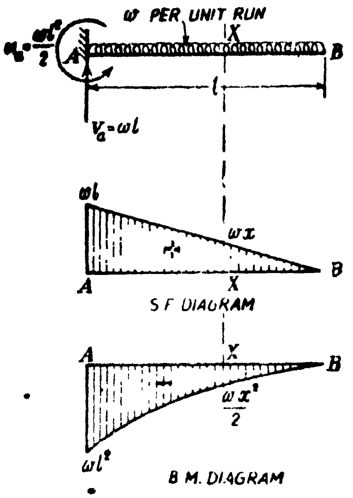


Fig. 140.

$$M_x = - \frac{w}{2} x^2$$

Hence we find that the variation of the shear force is according to a linear law, while the variation of the bending moment is according to a parabolic law.

At	$x=0,$	$S_x=0\ ;$
At	$x=0,$	$M_x=0\ ;$
At	$x=l,$	$S_x=+wl$
At	$x=l,$	$M_x=-\frac{wl^2}{2}$

(*iv*) *Cantilever of length l carrying a uniformly distributed load of w per unit run over the whole length and a concentrated load W at the free end.*

Fig. 141 shows a cantilever *AB* fixed at *A* and free at *B* and carrying the load system mentioned above Consider any section *X* distant *x* from the end *B*. The S.F. and the B.M. at the section *X*, are respectively given by

$$S_x = wx + W$$

and $$M_x = -\left(\frac{wx^2}{2} + Wx\right)$$

At, $x=0$, *i e*. at *B*,

$$S_x = +W$$

$$M_x = 0$$

At $x=l$, *i.e.* at *A*,

$$S_x = +(wl+W)$$

$$M_x = -\left(\frac{wl^2}{2} + Wl\right)$$

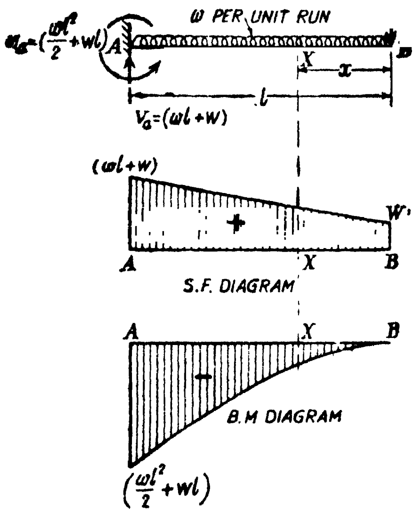


Fig. 141.

S F. varies following a linear law while B.M. varies following a parabolic law.

(*v*) *Cantilever of length l carrying a uniformly distributed load of w per unit run for a distance a from the free end.*

Fig. 142 shows a cantilever *AB* fixed at *A* and free at *B* and carrying the load system mentioned above.

Consider any section between *D* and *B* distant *x* from the free end *B*.

S.F. and B.M. at this section are given by

$$S_x = +wx$$

and $$M_x = -\frac{wx^2}{2}$$

The above relations hold good for all values of x between $x=0$ and $x=a$ (*i.e.*, between B and D).

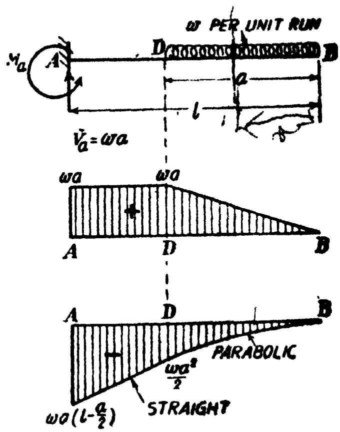


Fig. 142.

Hence for this range the S.F. varies following a linear law while the B.M. varies following a parabolic law.

At $x=0$ $S_x=0$

and $M_x=0$

At $x=a$, $S_x=+wa$

and $M_x=-\frac{wa^2}{2}$

Now consider any section between D and A, distant x from the end B.

The S.F. and B.M. at this section are given by,

$$S_x = +wa$$

and $$M_x = -wa\left(x-\frac{a}{2}\right)$$

Hence between A and D, $S.F.$ is constant at $+wa$ but the B.M. varies according to a linear law

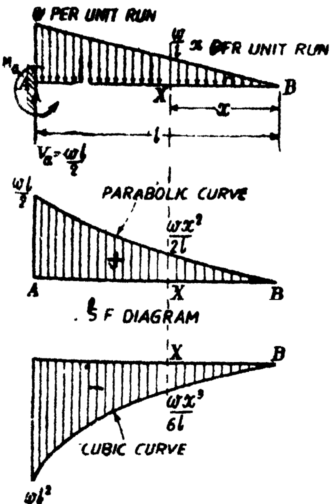


Fig. 143.

At $x=a$,

$$M_x = -wa\left(a-\frac{a}{2}\right) = -\frac{wa^2}{2}$$

At $x=l$,

$$M_x = -wa\left(l-\frac{a}{2}\right)$$

(*vi*) *Cantilever of length l carrying a load whose intensity varies uniformly from zero at the free end to w per unit at the fixed end.*

Fig. 143 shows a cantilever AB of length l fixed at A and free at B carrying the load system mentioned above.

Let the intensity of loading at X, at a distance x from the free end B be w_x per unit run.

$\therefore$ $w_x = \frac{x}{l}w$ since the intensity of load increases uniformly from zero at the free end to w at the fixed end.

$\therefore$ Load acting for an elemental distance dx from $X = w_x.dx$.

Hence the total load acting for any distance between $x=a$ and $x=b$

$$= \sum_{x=a}^{x=b} w_x.dx$$

= area of the load diagram between $x=a$ and $x=b$.

Hence we come to a very important point, namely that the total distributed load acting on any segment equals the area of the load diagram on that segment.

S.F. and B.M. at X are given by,

S_x = area of the load diagram between X and B.

$$= +\frac{1}{2} \cdot x.w_x = +\frac{1}{2} \cdot x.\frac{w}{l}x$$

$$\therefore \qquad S_x = +\frac{wx^2}{2l}$$

and, M_x = Moment of the load acting on XB about X

= area of the load diagram between X and $B \times$ Distance of the centroid of this diagram from X

$$= -\frac{wx^2}{2l} \cdot \frac{x}{3}$$

$$\therefore \qquad M_x = -\frac{wx^3}{6l}$$

At $\quad x=0, \quad S_x=0$

and $\quad M_x=0$

At $\quad x=l, \quad S_x = +\frac{wl}{2}$

and $\quad M_x = -\frac{wl^2}{6}$

The S.F. changes following parabolic law while the B.M. changes following a cubic law.

(*vii*) *Cantilever carrying a load whose intensity varies uniformly from zero at the fixed end to w per unit run at the free end*

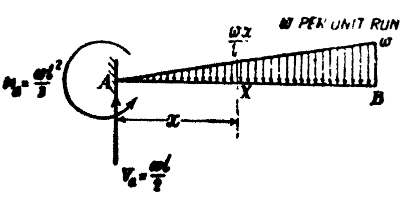


Fig. 144

Fig. 144 shows a cantilever AB of length l and fixed at A and free at B and carrying the load system mentioned above.

It is convenient to find the S.F. and B.M. at any section by considering the left part of the section.

Let M_a be the reacting moment or fixing moment at A.

$\therefore$ M_a = moment of the total load about A.

$$=\frac{wl}{2}\cdot\frac{2l}{3}=\frac{wl^2}{3}$$

V_a = Vertical reaction at A
= Total load on the cantilever

$\therefore$ $V_a=\frac{wl}{2}$

Consider any section X distant x *from the fixed end* A

S.F. at X = Algebraic sum of forces on AX

$\therefore$ $S_x=\frac{wl}{2}-\frac{x}{2}\cdot\frac{wx}{l}$

$\therefore$ $S_x=\frac{wl}{2}-\frac{wx^2}{2l}$

B.M. at X = Algebraic sum of moments of forces and reactions on AX above X

$\therefore$ $M_x=\frac{wl}{2}x-\frac{wx^2}{2l}\cdot\frac{x}{2}-M_a$

$$=\frac{wl}{2}x-\frac{wx^3}{6l}-\frac{wl^2}{3}$$

At $x=0$, *i.e.*, at A

$S_x=+\frac{wl}{2}$

and $M_r=-\frac{wl^2}{3}$

At $x=l$, *i e.*, at B,

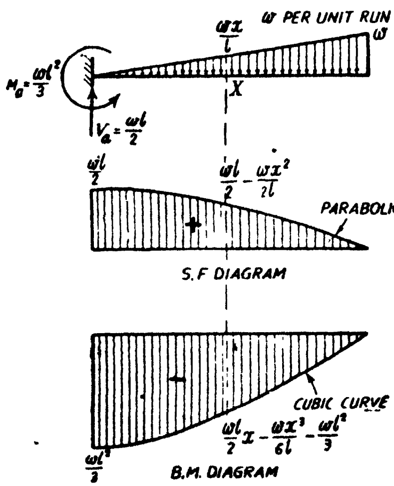


Fig. 145

$$S_x = wl - \frac{wl^2}{2l} = 0$$

and $$M_x = \frac{wl}{2}l - \frac{wl^3}{6l} - \frac{wl^2}{3} = 0$$

Problem 105. *Draw Shear force and Bending moment diagrams for the cantilever shown in Fig. 146.*

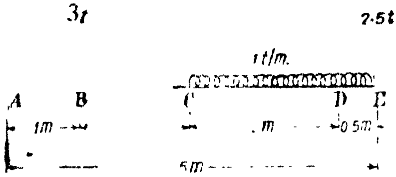


Fig. 146

Solution. Let V_a be the vertical reaction at A. Since there is only one support

V_a = Total load on the span

$\therefore V_a = 3 + 1\times 2 + 2{\cdot}5$

$= 7{\cdot}5$ *tonnes (upwards)*

There will also be a reacting moment or fixing moment at A in an *anticlockwise order* which will be equal and opposite to the moment of the forces on the cantilever about A.

$\therefore$ Reacting moment $= 3\times 1 + 1\times 2\times 3{\cdot}5 + 2{\cdot}5\times 5 = 22{\cdot}5$ *tm.*

S.F. Calculations.

S.F. between A and B

$= +7{\cdot}5\ t$

S.F. between B and C

$= +7{\cdot}5 - 3 = +4{\cdot}5\ t$

S.F. at $D = 2{\cdot}5\ t$

From C to D the S.F. will change uniformly from $+4{\cdot}5t$ to $+2{\cdot}5t$,

S.F between D and E

$= +2{\cdot}5\ t$

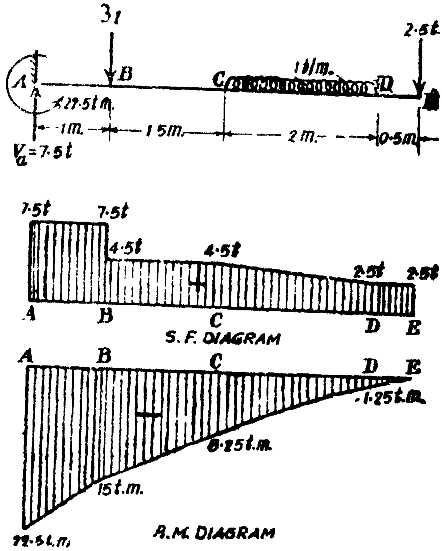


Fig. 147

B.M. Calculations

B.M. at $E = 0$

B.M. at $D = -2{\cdot}5\times 0{\cdot}5$

$= -1{\cdot}25$ *tm*

B.M. at $C = -2{\cdot}5\times 2{\cdot}5 - 1\times 2\times 1 = -8{\cdot}25$ *tm*

or alternatively,

$-22{\cdot}5 + 7{\cdot}5\times 3{\cdot}5 - 3\times 1{\cdot}5$

$= -8{\cdot}25$ *tm*

B.M. at $B = -2{\cdot}5\times 4 - 1\times 2\times 2{\cdot}5 = -15$ *tm.*

or alternatively,

$-22{\cdot}5 + 7{\cdot}5\times 1 = -15$ *tm*

B.M. at $A = -22{\cdot}5$ *tm*

From C to D the B.M. will vary according to a ***parabolic law.***

§ 35. Beams freely supported at the two ends

(i) Simply supported beam of span l carrying a concentrated load at mid span.

Fig. 148 shows a beam *AB* simply supported at the ends *A* and *B*. Let the span of the beam be *l* and let the beam carry a concentrated load *W* at mid span.

Since the load is symmetrically placed on the span, reaction at each support

$$=\frac{W}{2}$$

$$\therefore\ V_a=V_b=\frac{W}{2}$$

For any section between *A* and *C*

$$\text{S.F.}=S_x=+\frac{W}{2}$$

For any section between *C* and *B*

$$\text{S.F.}=S_x=-\frac{W}{2}$$

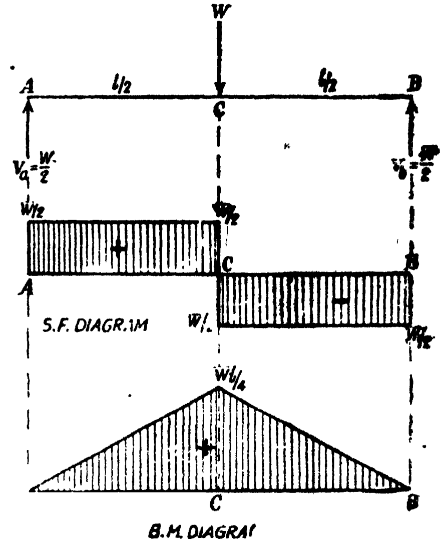


Fig. 148

At the section *C* the S.F. changes from $+\frac{W}{2}$ to $-\frac{W}{2}$.

At any section between *A* and *C* distant *x* from the end *A*, the bending moment is given by,

$$M_x=+\frac{W}{2}x\ (\textit{sagging moment})$$

At $\quad x=0, \quad M_x=0$

and at $\quad x=\frac{l}{2} \quad M_x=\frac{Wl}{4}$

Hence the B.M. increases uniformly from zero at *A* to $\frac{Wl}{4}$ at *C*.

Similarly the B M. decreases uniformly from $\frac{Wl}{4}$ at *C* to zero at *B*.

Maximum bending moment occurs at mid span, *i.e.*, at *C* where the shear force changes its sign.

(ii) Simply supported beam carrying a concentrated load placed eccentrically on the span.

Fig. 149 shows a simply supported beam *AB* of span *l* carrying a concentrated load *W* at *D* eccentrically on the span.

Let $\quad AD=a$

and $\quad DB=b$

Let V_a and V_b be the vertical reactions at A and B.

For the equilibrium of the beam,

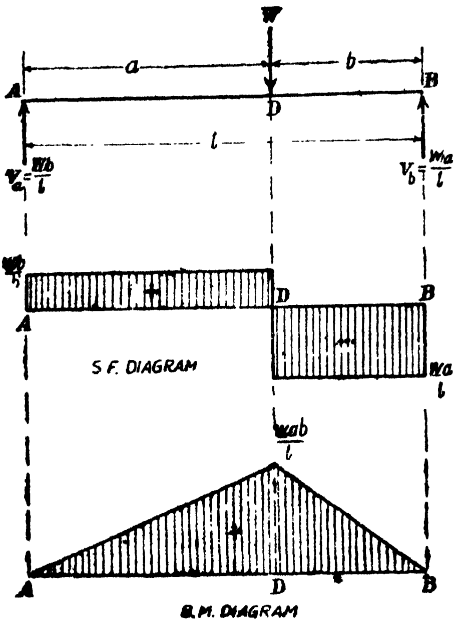


Fig. 149

Taking moments of the forces on the beam about A, we have

$$V_b l = Wa$$

$$\therefore \quad V_b = \frac{Wa}{l}$$

$$\therefore \quad V_a = W - \frac{Wa}{l} = \frac{W(l-a)}{l}$$

$$\therefore \quad V_a = \frac{Wb}{l}$$

since $a+b=l$

For any section between A and D the shear force

$$= S_x = V_a = +\frac{Wb}{l}$$

For any section between D and B, the shear force

$$= S_x = -V_b = -\frac{Wa}{l}$$

At any section between A and D distant x from A, the bending moment is given by

$$M_x = +\frac{Wb}{l}x \ (sagging)$$

At	$x=0,$	$M_x=0$
At	$x=a,$	$M_x=\frac{Wab}{l}$

Hence the B.M. increases uniformly from zero at the left end A to $\frac{Wab}{l}$ at D. Similarly the B.M. will decrease uniformly from $\frac{Wab}{l}$ at D to zero at the right end B.

It may be observed from the S.F. and B.M. diagrams that the maximum B.M. occurs at D where the S.F. changes its sign.

(*iii*) *Simply supported beam carrying a number of concentrated loads.*

Fig. 150 shows a simply supported beam AB of span 8 *metres* carrying concentrated loads of 4 *tonnes*, 10 *tonnes*, and 7 *tonnes* at distances of 1·5 *metres*, 4 *metres* and 6 *metres* from the left support. Let us construct the S.F. and B.M. diagrams.

Let V_a and V_b be the vertical reactions at the supports A and B respectively.

For the equilibrium of the beam, taking moments of the forces on the beam about the left support, we have,

$$V_b \times 8 = 4 \times 1.5 + 10 \times 4 + 7 \times 6$$

$$= 88 \textit{ tonne metres}$$

$$\therefore \; V_b = \frac{88}{8} = 11 \textit{ tonnes}$$

$\therefore \; V_a =$ Total load on the beam $- V_b$
$= 21 - 11 = 10$ *tonnes*

S.F. between A and C
$= +10$ *tonnes*

S.F. between C and D
$= +10 - 4 = +6$ *tonnes*

S.F. between D and E
$= +10 - 4 - 10 = -4$ *tonnes*

or alternatively
$= -11 + 7 = -4$ *tonnes*

S.F. between E and B
$= +10 - 4 - 10 - 7 = -11$ *tonnes*

or alternatively
$= -V_b = -11$ *tonnes.*

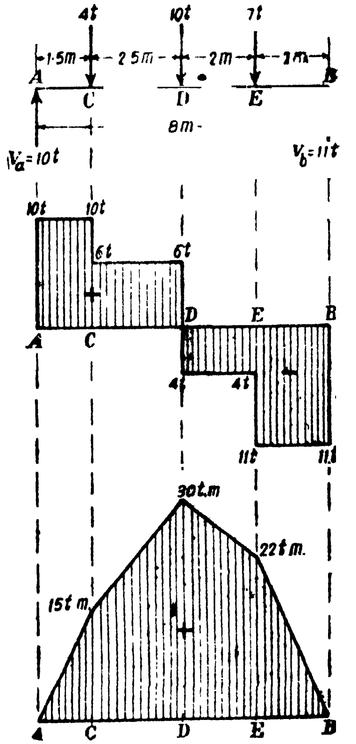


Fig. 150

B.M. at $A = 0$

B.M. at $C = +10 \times 1.5 = +15$ *tonne metres (sagging)*

B.M. at $D = +10 \times 4 - 4 \times 2.5 = +30$ *tonne metres (sagging)*

B.M. at $E = +11 \times 2 = +22$ *tonne metres (sagging)*

It may be observed from the S.F. and B.M diagrams that the maximum B.M. occurs at D where the S.F. changes its sign.

(iv) *Simply supported beam carrying a uniformly distributed load of w per unit run over the whole span.*

Fig. 151 shows a simply supported beam AB of span

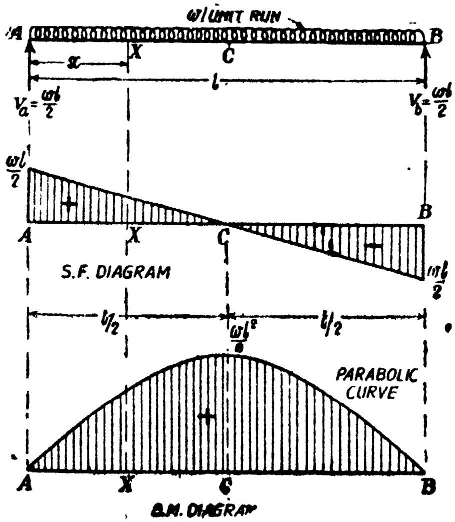


Fig. 151

l carrying a uniformly distributed load w per unit run over the whole span. Let V_a and V_b the vertical reactions at the supports A and B respectively.

Since the loading is symmetrical on the span, each vertical reaction equals half the total load on the span

$$\therefore \qquad V_a = V_b = \frac{wl}{2}$$

Consider any section X distant x from the left end A.

S.F. and B.M. at the section X are given by,

$$S_x = +V_a - wx = +\frac{wl}{2} - wx$$

and

$$M_x = +V_a x - \frac{wx^2}{2}$$

$$= \frac{wl}{2}x - \frac{wx^2}{2}$$

$$\therefore \qquad M_x = +\frac{w}{2}x(l-x)$$

At $\qquad x=0,\ S_x = +\frac{wl}{2}$ and $M_x = 0$

At $\qquad x=l,\ S_x = \frac{wl}{2} - wl = -\frac{wl}{2}$ and $M_x = 0$

At $\qquad x = \frac{l}{2},\ S_x = \frac{wl}{2} - \frac{wl}{2} = 0$ and

$$M_x = +\frac{w}{2}\cdot\frac{l}{2}\left(l - \frac{l}{2}\right) = +\frac{wl^2}{8}$$

The S.F. diagram is a straight line. The S.F. *uniformly* changes from $+\frac{wl}{2}$ at A to $-\frac{wl}{2}$ at B and obviously the S.F. at mid span is zero.

The B.M. diagram is a parabola. The B.M. increases according to a *parabolic law* from zero at A to $+\frac{wl^2}{8}$ at the mid span C and from this value the B.M. decreases to zero at B following the *parabolic law*.

(*v*) *Simply supported beam carrying a uniformly distributed load over part of its span.*

(*a*) *When the beam carries a uniformly distributed load for a certain distance from one end.*

Fig. 152 shows a simply supported beam AB of span 9 *metres* carrying a uniformly distributed load of 1800 *kg. per metre* for a distance of 4 *metres* from the left support A.

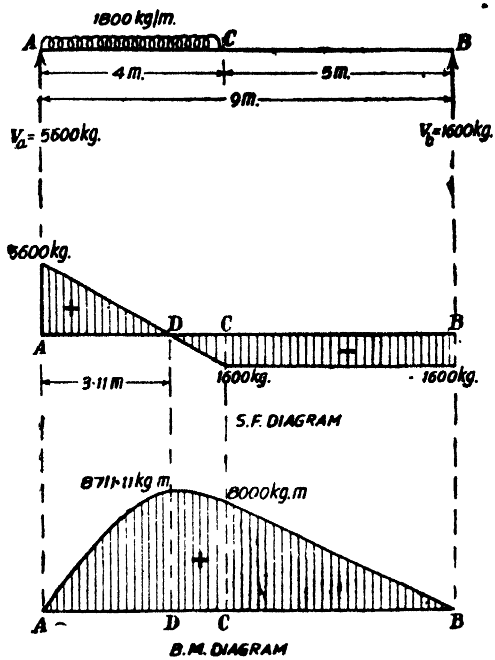


Fig. 152.

Let V_a and V_b be the vertical reactions at A and B. For the equilibrium of the beam, taking moments about the left support A, we have,

$$V_b \times 9 = 1800 \times 4 \times \frac{4}{2}$$

$$\therefore \quad V_b = 1600 \ kg.$$

$$\therefore \quad V_a = 1800 \times 4 - 1600 \ kg.$$

$$= 5600 \ kg.$$

At any section between C and B, S.F. $= -1600 \ kg.$

Consider any section between A and C distant x from A.

S. F. at the section is given by,

$$S_x = +5600 - 1800 \ x$$

At $x = 0$,

$$S_x = +5600 \ kg.$$

At $x = 4m$, $\quad S_x = +5600 - 1800 \times 4 = -1600 \ kg.$

Let the S.F. be zero at x *metres* from A. Equating the S.F. to zero, we get,

$$5600 - 1800 \ x = 0$$

$$\therefore \quad x = \frac{28}{9} \ m = 3{\cdot}11 \ m. \text{ from } A.$$

At any section in AC distant x from A the B.M. is given by,

$$M_x = +5600 \ x - 1800 \ \frac{x^2}{2} = 5600 \ x - 900 \ x^2$$

At $\quad x = 0, \quad M_x = 0$

At $\quad x = 4m. \quad M_x = +5600 \times 4 - 900 \times 4^2 = +8000 \ kg. \ m.$

At $\quad x = \frac{28}{9} \ m, \ M_x = 5600 \times \frac{28}{9} - 900 \times \left(\frac{28}{9}\right)^2$

$$\frac{78400}{9} \ k.g.m = +8711{\cdot}11 \ kg. \ m.$$

B.M. decerases from $+8000 \ kg. \ m.$ at C to zero at B according to a linear law.

Max. B.M. occurs at D where S.F. $= 0$: *i.e.*, where the S.F. changes sign.

(*b*) *Simply supported beam carrying a uniformly distributed load on an intermediate part of the span.*

Fig. 153 shows a simply supported beam AB of span 9 *m* carrying a uniformly distributed load 1800 *kg. per metre* run on the part CD of the span so that $AC = 2\ m$, $CD = 4\ m$ and $DB = 3m$. Let us construct the S.F. and B.M. diagrams for this beam.

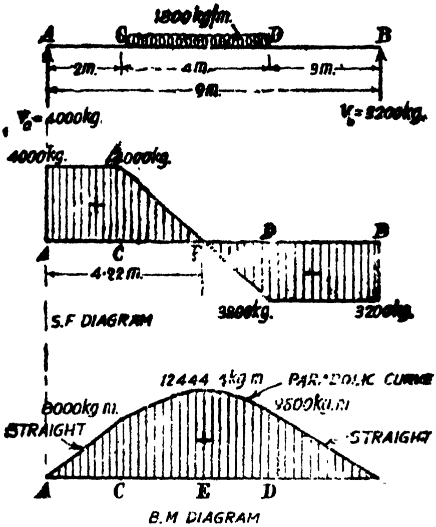


Fig. 153

Let V_a and V_b be the vertical reactions at A and B respectively. For the equilibrium of the beam, taking moments about the end A, we have,

$$V_b \times 9 = 1800 \times 4 \times (2+2)$$

$$\therefore\ V_b = 3200\ kg.$$

$$\therefore\ V_a = 1800 \times 4 - 3200 = 4000\ kg.$$

At any section between A and C

S.F. = +4000 *kg*.

At any section between D and B, S.F. = −3200 *kg*. Consider any section between C and D distant *x metres* from A.

S.F. at this section is given by

$$S_x = +4000 - 1800\,(x-2)$$

At $x = 2\ m$, $S_x = +4000\ kg.$

At $x = 6\ m$, $S_x = +4000 - 1800 \times 4 = -3200\ kg.$

Let the S.F. be zero *x metres* from A. Equating the S.F. to zero, we get

$$4000 - 1800\,(x-2) = 0$$

$$x - 2 = \frac{4000}{1800} = \frac{20}{9}$$

$$\therefore\ x = \frac{38}{9}\ m = 4{\cdot}22\ m.$$

B.M. at $A = 0$

B.M. at $B = 0$

B.M. at $C = +4000 \times 2 = +8000\ kg.\ m.$

B.M. at $D = +3200 \times 3 = +9600\ kg.\ m.$

At any section in CD distant x from A, the B.M. is given by

$$M_x = 4000\,x - 1800\,\frac{(x-2)^2}{2}$$

$$=4000\,x-900\,(x-2)^2$$

At $\frac{38}{9}$ m.

B.M. $$=+4000\times\frac{38}{9}-900\left(\frac{38}{9}-2\right)^2\ kg.\ m.$$

$$=+\frac{112000}{9}\ kg.\ m.$$

$$=12444{\cdot}4\ kg.\ m.$$

B.M. will uniformly vary from 0 at A to 8000 *kg. m* at C. Between C and D the B.M. varies following a parabolic law reaching the maximum value at E. From D to B the B.M. will uniformly decrease from 9600 *kg. m.* to zero.

Problem 106. *Draw the shear force and bending moment diagrams for the beam shown in Fig. 154. Also find the position and magnitude of the maximum bending moment.*

Solution. Let the salient points of the beam be named as shown in the figure.

Let V_a and V_b be the vertical reactions at the left and right supports respectively.

Taking moments about the left support, we have

$$V_b\times 5=400\times 1{\cdot}5\,(1{\cdot}5+0{\cdot}75)$$

$\therefore$ $V_b=270\ kg.$

$\therefore$ $V_a=400\times 1{\cdot}5-270$
$=330\ kg.$

S.F. Between A and C
$=+330\ kg.$

S.F. Between D and B
$=-270\ kg.$

Let the S.F. be zero at E.

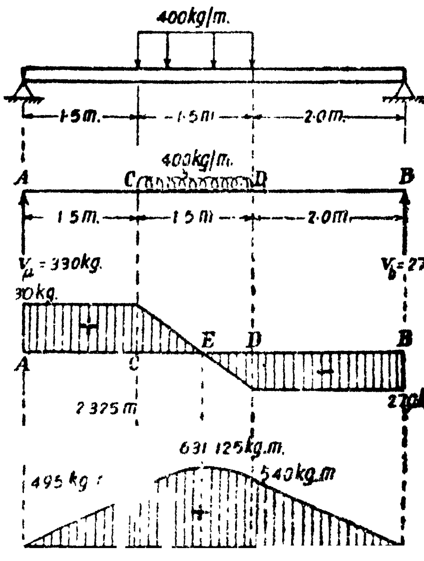


Fig. 154.

Let $AE=x$

Equating the S.F. at E to zero we have

$$330-400\,(x-1{\cdot}5)=0$$

$\therefore$ $x=2{\cdot}325\ m.$

B.M. calculations

B.M. at $A=M_a=0$

B.M. at $C=M_c=330\times 1{\cdot}5=495\ kg.\ m.$

B.M. at $D=M_d=270\times 2=540\ kg.\ m.$

B.M. at 2·325 m from A

$$= M_c = 330 \times 2.325 - 400 \frac{(2.325-1.5)^2}{2} \; kg. m.$$

$$= 631.125 \; kg \; m.$$

Problem 107. *A beam AB 10 metres long has supports at its ends A and B. It carries a point load of 5 t. at 3 metres from A and a point load of 5 t. at 7 metres from A and a uniformly distributed load of 1 tonne per metre between the point loads. Draw SF and BM diagram for the beam.*

Solution.

***Reactions*:**

Since the loading is symmetrical reaction at each support equals half the total load

$$\therefore \qquad V_a = V_b = \frac{5+5+1\times 4}{2} = 7t.$$

S.F. Analysis

S.F. at any section in $AD = +7\ t$.

S.F. just on RHS of $D = +7-5 = +2\ t$.

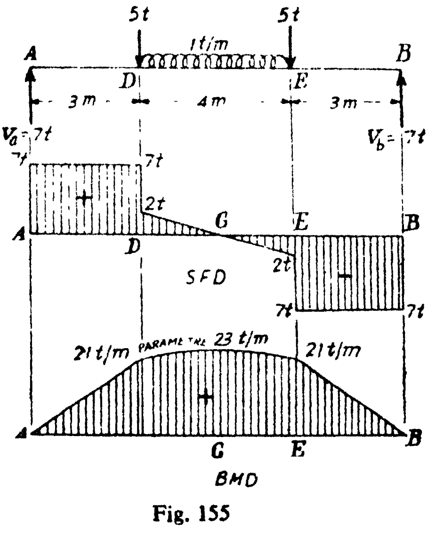


Fig. 155

S.F. just on LHS of $E = -7+5 = -2\ t$

S.F. at any section in $EB = -7\ t$.

S.F. at centre $G = 0$

B.M. Analysis

B.M. at $A = BM$ at $B = 0$.

B.M. at $D = BM$ at $E = +7\times 3 = +21\ tm$.

B.M. at $G=7\times5-5\times2-1\times\frac{2^2}{2}=35-10-2=+23$ *tm.* (*maximum bending moment*).

The B.M. diagram is linear for the parts AD and BE and is parabolic for the part DE

(*vi*) *Simply supported beam carrying a load whose intensity varies uniformly from zero at each end to w per unit run at the mid span.*

Fig. 156 shows a simply supported beam AB of span l carrying the loading mentioned above.

Total load on the beam
$=$area of the load diagram
$=\frac{1}{2}l\cdot w=\frac{wl}{2}$

$\therefore$ Each vertical reaction
$=$half the total load

$\therefore\ V_a=V_b=\frac{wl}{4}$

Consider any section X in AC distant x from the end A.

Rate of loading at X
$=\frac{x}{\left(\frac{l}{2}\right)}w=\frac{2w}{l}x$

$\therefore$ External load on the length AX
$=$area of the load diagram between A and X

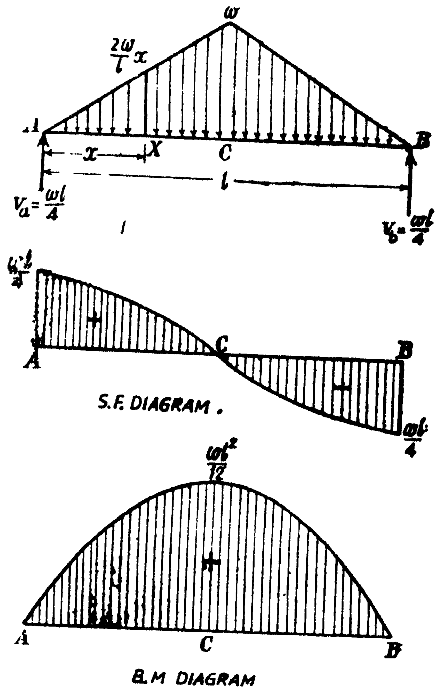


Fig. 156

$$=\frac{x}{2}\cdot\frac{2wx}{l}=\frac{w}{l}x^2$$

acting at $\frac{x}{3}$ from X.

$\therefore$ S.F. at X is given by,

$$S_x=+\frac{wl}{4}-\frac{w}{l}x^2$$

$$x=0,$$

$$S_x=+\frac{wl}{4};$$

At $\qquad x=\frac{1}{2}l$

$$S_x = \frac{wl}{4} - \frac{w}{l} \cdot \frac{l^2}{4} = 0$$

At $\quad x = l, \quad S_x = -\frac{wl}{4}$

B.M. at X is given by,

$$M_x = \frac{wl}{4}x - \frac{wx^2}{l} \times \frac{x}{3}$$

$\therefore \quad M_x = \frac{wl}{4}x - \frac{w}{3l}x^3$

At $\quad x = 0.$

$M_x = 0;$

At $\quad x = \frac{l}{2},$

$$M_x = \frac{wl}{4} \cdot \frac{l}{2} - \frac{w}{3l} \cdot \frac{l^3}{8}$$

$$= \frac{wl^2}{12}$$

The loading being symmetrical, the S.F. and B.M. diagrams can be easily drawn.

Max. B.M. occurs at mid span and is equal to

$$= +\frac{wl^2}{12}$$

Total load on the span

$$= W = \frac{wl}{2}$$

$\therefore$ Max. B.M. $= \frac{wl}{2} \cdot \frac{l}{6}$

$$= \frac{Wl}{6}$$

$$= \frac{\text{Total load} \times \text{span}}{6}$$

(vii) *Simply supported beam carrying a load whose intensity varies uniformly from zero at one end to w per unit run at the other end.*

Fig. 157 shows a simply supported beam AB of span l carrying a load whose intensity varies uniformly from zero at the left end A to w per unit run at the right end B.

Let V_a and V_b be the vertical reactions at A and B.

For the equilibrium of the beam, taking moments about A, we have,

$$V_b \cdot l = \frac{w}{2} l \times \frac{2}{3} l$$

$$V_b = wl$$

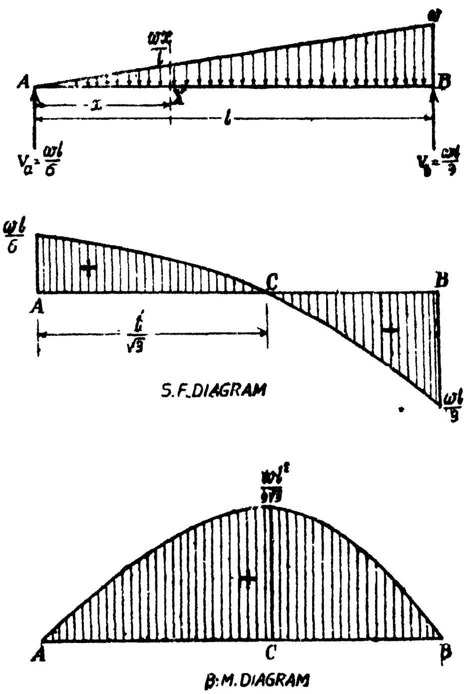


Fig. 157

$$\therefore \qquad V_a = \frac{wl}{2} - \frac{wl}{3} = \frac{wl}{6}$$

Consider any section X distant x from the end A. S.F. and B.M. at this section are given by

$$S_x = \frac{wl}{6} - \frac{x}{2} \cdot \frac{wx}{l} = \frac{wl}{6} - \frac{wx^2}{2l}$$

$$M_x = \frac{wl}{6}x - \frac{wx^2}{2l} \cdot \frac{x}{3} = \frac{wl}{6}x - \frac{wx^3}{6l}$$

Hence the S.F. diagram is a *parabolic curve* and the B.M. diagram is a *cubic* curve

At $\qquad x=0,$

$$\text{S.F.} = \frac{wl}{6};$$

At $\qquad x=l,$

$$\text{S.F.} = \frac{wl}{6} - \frac{wl^2}{2l} = -\frac{wl}{3}$$

At $\qquad x=0,$

$$M_x = 0$$

At $x=l,$

$$M_x=\frac{wl}{6}\cdot l-\frac{wl^3}{6l}=0$$

Let the S.F. be zero at a distance x from A.

$\therefore$ Equating the S.F. to zero, we have,

$$\frac{wl}{6}-\frac{wx^2}{2l}=0$$

$\therefore$ $$x=\frac{l}{\sqrt{3}}$$

$\therefore$ Max. B.M. occurs at $x=\frac{l}{\sqrt{3}}$ from A

$\therefore$ $$M_{max}=\frac{wl}{6}\cdot\frac{l}{\sqrt{3}}-\frac{w}{6l}\left(\frac{l}{\sqrt{3}}\right)^3$$

$$=+\frac{wl^2}{9\sqrt{3}}$$

Problem 108. *The intensity of loading on a simply supported beam of 5 metres span increases uniformly from 800 kg./m. at one end to 1600 kg./m. at the other end. Find the position and magnitude of the maximum bending moment. Also draw shear force and Bending Moment diagrams.*

Solution. The trapezoidal loading on the beam consists of a uniformly distributed loading and a triangular loading as shown in Fig. 158.

Reactions:

Taking moments about the end A, we have,

$$V_b\times 5-\left[800\times 5\times\frac{5}{2}\right]+\left[\frac{1}{2}\times 800\times 5\times \; 10\right.$$

$\therefore$ $$V_b=\frac{10,000}{3}\ kg.$$

$\therefore$ $$V_a=\text{Total load}-V_b$$

$$=\left[800\times 5+\frac{1}{2}\times 800\times 5\right]-\frac{10,000}{3}$$

$$=6,000-\frac{10,000}{3}$$

$$=\frac{8,000}{3}\ kg.$$

Consider any section XX at a distance x from A.

Load intensity at the section XX

$$=800+\frac{x}{5}\times 800=800+160\,x.$$

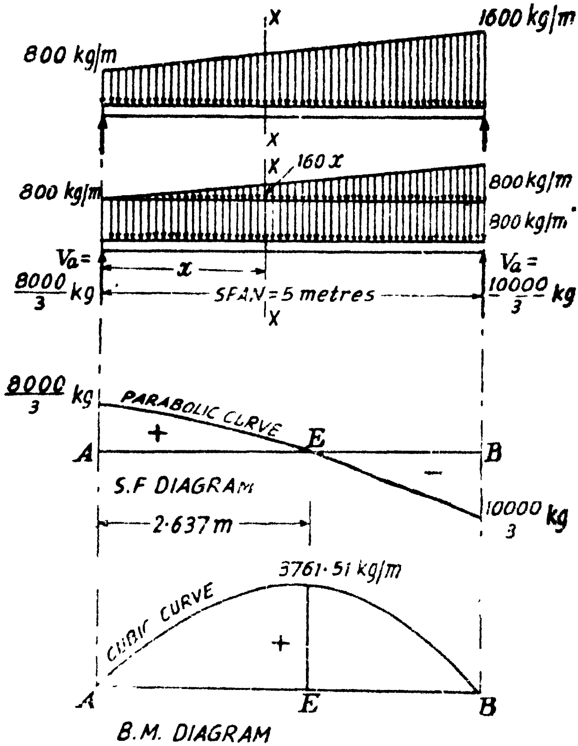


Fig. 158

S.F. Analysis

S.F. at the section XX-

$$S=\frac{8000}{3}-800x-\frac{1}{2}\cdot x\,(160\,x)$$

$$\therefore \qquad S=\frac{8000}{3}-800\,x-80\,x^2$$

The S.F. diagram follows a parabolic law

At $\qquad x=0$ *i.e.*, at A,

$$S=+\frac{8000}{3}\ kg,$$

At $\qquad x=5\ m.$, *i.e.*, at B,

$$S=\frac{8000}{3}-800\times 5-80\times 5^2=-\frac{10000}{3}\ kg.$$

Section of zero shear

Equating the general expression for shear force to zero,

$$\frac{8000}{3}-800x-80x^2=0$$

$\therefore \quad 3x^2+30x-100=0.$

Solving, we get $\quad x=2{\cdot}637$ *m.*

B.M. Analysis

B.M. at the section XX

$$=M=\frac{8000}{3}x-800\frac{x^2}{2}-\frac{1}{2}x\,(160\,x)\frac{x}{3}$$

$$\therefore \quad M=\frac{8000}{3}x-400\,x^2-\frac{80}{3}x^3$$

The B.M. diagram follows a cubic law.

Maximum bending moment occurs at the section of zero shear, *i.e.*, at a distance of 2·637 *metres* from A.

$\therefore$ *Maximum bending moment*

$$=\frac{8000}{3}\times 2{\cdot}637-400\times 2{\cdot}637^2-\frac{80}{3}\times 2{\cdot}637^3 \text{ kgm.}$$

$$=3761{\cdot}51 \textit{ kgm.}$$

Problem 109. *A beam of length* $(l+2a)$ *has supports* l *apart with an overhang* a *on each side. The beam carries a concentrated load* W *at each end. Construct shear force and bending moment diagrams.*

Solution. Let $DABC$ be the beam of length $(l+2a)$. Let the supports be at A and B so that

$$DA=BC=a$$

$$\therefore \quad AB=l$$

Each vertical reaction $=W$

$$\therefore \quad V_a=V_b=W$$

S.F. at any section between D and $A=-W$

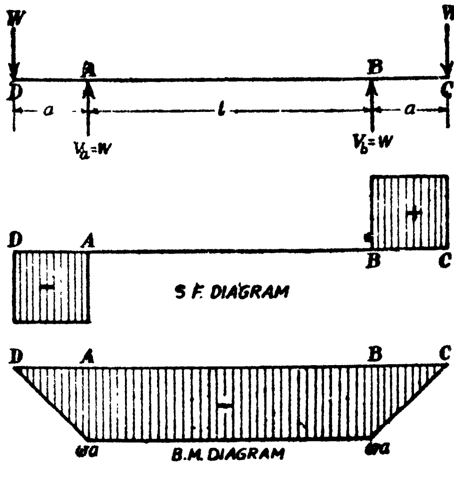


Fig. 159

S.F. at any section between B and $C=+W$
S.F. at any section between A and $B=0$
B.M. at $D=0$ B.M. at $A=-Wa$

At any section in AB distant x from D the B.M. is given by

$$M_x=-Wx+W(x-a)=-Wa$$

B.M. at $B=-Wa$ B.M. at $C=0$

The B.M. throughout the length is of the hogging type.

(*viii*) *Beam with overhang at one end and carrying a uniformly distributed load over the whole length.*

Fig. 160 shows a simply supported beam ABC with supports at A and B, 6 *metres* apart with an overhang BC 2 *metres* long. Let us construct the S.F. and B.M. diagrams for this beam.

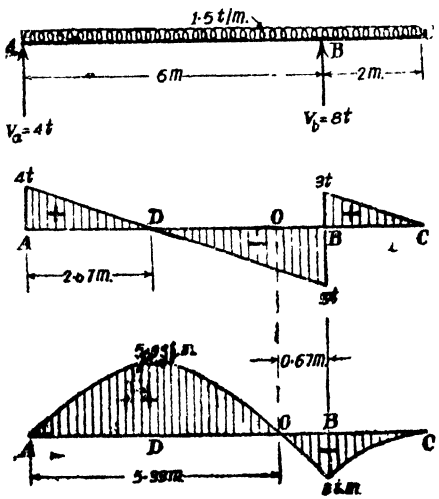


Fig. 160

Let V_a and V_b be the vartical reactions at A and B.

For the equilibrium of the beam, taking moments about A, we have,

$$V_b\times 6=1{\cdot}5\times 8\times 4$$

$\therefore$ $V_b=8$ *tonnes.*

$\therefore$ $V_a=1{\cdot}5\times 8-8=4$ *tonnes*

S.F. at the left end $=+4\ t.$

S.F. just on the left hand side of B

$$=+4-1{\cdot}5\times 6=-5\ t$$

S.F. just on the right hand side of B

$$=+1{\cdot}5\times 2=3\ t$$

S.F. at $C=0$

Fig. 160 shows the S.F. diagram.

Let the S.F. be zero at x *metres* from A. Equating the S.F. to zero, we get,

$$S_x=4-1{\cdot}5\ x=0$$

$$x=\frac{8}{3}\ metres=2{\cdot}67\ m.$$

B.M. at A $=0$

At any section in AB distant x from A, the B.M. is given by

$$M_x=4x-1{\cdot}5\frac{x^2}{2}$$

Hence the B.M. diagram is parabolic

$$\text{B.M. at } x=\frac{8}{3}m \text{ is } M_{max}=4\times\frac{8}{3}-\frac{1{\cdot}5}{2}\left(\frac{8}{3}\right)^2$$

$$=+\frac{16}{3}\ tm$$

$$=+5{\cdot}33\ tm$$

B.M. at $x=6\ m$ *i.e.*, at B

$$=4\times6-\frac{1{\cdot}5}{2}\times6^2$$

$$=-3\ tm$$

or alternatively B.M. at B,

taking moments of the forces on the right hand side of B,

$$M_b=-1{\cdot}5\times2\times\left(\frac{2}{3}\right)=-3\ tm$$

Point at which the B M. is zero.

Since at $x=\frac{8}{3}\ m$ the B M. is $+5{\cdot}33\ tm$ and at $x=\ m$ the B.M is $-3\ tm$ there must be a section where the B.M. is zero. This section can be determined by equating the generel expression for B.M. to zero.

i.e., by the equation,

$$4x-1{\cdot}5\frac{x^2}{2}=0$$

$$\therefore\quad x(4-0{\cdot}75\,x)=0$$

$$\therefore\quad x=0$$

and $$x=\frac{16}{3}\ m=5{\cdot}33\ m$$

Let the B.M. be zero at O (Fig. 160)

$$\therefore\quad AO=\frac{16}{3}\ m.$$

The point O where the B M. is zero is called the point of *contra-flexure* or *point of inflexion*

For all sections from A to O the B.M. is of the *sagging* type while for all sections between O and C the B M. is of the *hogging* type.

Some interesting observations

For the beam discussed above the B.M. at A is zero and the B M. at O is also zero and the distance $AO=\frac{16}{3}\ m.$

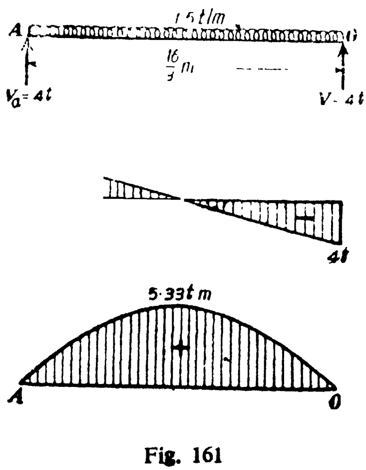


Fig. 161

Suppose a simply supported beam AO has a span of $\frac{16}{3}\ m.$ and is subjected to a uniformly distributed load of 1·5 *t/m* over the span, each vertical reaction would be $\frac{1}{2}\left(1{\cdot}5\times\frac{16}{3}\right)=4$ *tonnes.*

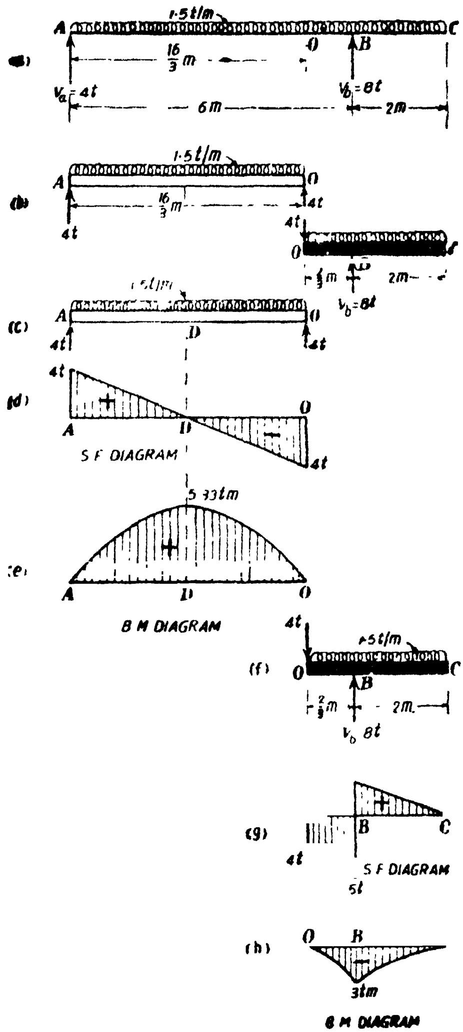


Fig. 162

Max. sagging B.M. will occur at the cenre and its magnitude

$$=+\frac{1{\cdot}5}{8}\times\left(\frac{16}{3}\right)^2 tm.$$

$$=+\frac{16}{3}\ tm.=+5\cdot33\ tm.$$

It may be noted that the S.F. and B.M. diagrams for this beam *AO* are exactly the same as the S.F. and B.M. diagrams for the portion *AO* of the beam *ABC*.

Hence for analysis purposes we may consider *AO* as a separate simply supported beam supported at *A* and at the end *O* of a double cantilever *OBC*. The S.F and B M. diagrams for the portion *OBC* of the given beam can be drawn easily by drawing the corresponding diagrams for the double cantilever *OBC*.

Problem 110. *A simply supported beam ABC with supports at A and B, 6 metres apart and with an overhang BC 2 metres long carries a uniformly distributed load of 1·5 tonne per metre over the whole length as shown in Fig 163 Draw S.F. and B.M. diagrams.*

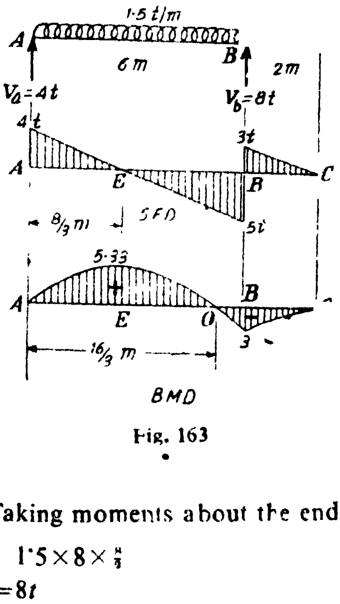


Fig. 163

Solution.

Reactions. Taking moments about the end *A*,

$V_b\times6=1\cdot5\times8\times\frac{8}{2}$

$\therefore\ V_b=8t$

$\therefore\ V_a=\text{Total load}-V_b$

$=(1\cdot5\times8)-8$

$=12-8=4t$

S.F. analysis. At any section in *AB* distant *x* from *A*, shear orce

$=S=4-1\cdot5\,x$

At $x=0$, $S=+4t$

At $x=6\ m$, $S=4-1\cdot5\times6=-5t$

Section of zero shear. Equating the general expression for shear force to zero,

$$4-1\cdot5x=0$$

$$\therefore \quad x=\frac{4}{1\cdot5}=\frac{8}{3}\ m=2\cdot67\ metre$$

At any section in CB, distant x from C

Shear force $=S=+1\cdot5\ x$

At $x=0$, $\quad S=0$

At $x=2$, $\quad S=1\cdot5\times2=+3t$

B.M. Analysis. At any section in AB distant x from A, Bending Moment

$$=M=4x-1\cdot5\ \frac{x^2}{2}$$

$\therefore \quad M=4x-0\cdot75x^2 \quad$ This is a parabolic law.

At $x=0$, $\quad M=0$,

At $x=6\ m$. $\quad M=4\times6-0\cdot75\times6^2=-3\ tm$.

Maximum bending moment, which occurs at the section of zero shear

$$=M_e=4\times\frac{8}{3}-0\cdot75\left(\frac{8}{3}\right)^2$$

$$=+\frac{16}{3}=+5\cdot33\,tm.$$

At any section in CB distant x from C

Bending moment

$$M=-1\cdot5\ .\ \frac{x^2}{2}\quad -0\cdot75\ x^2$$

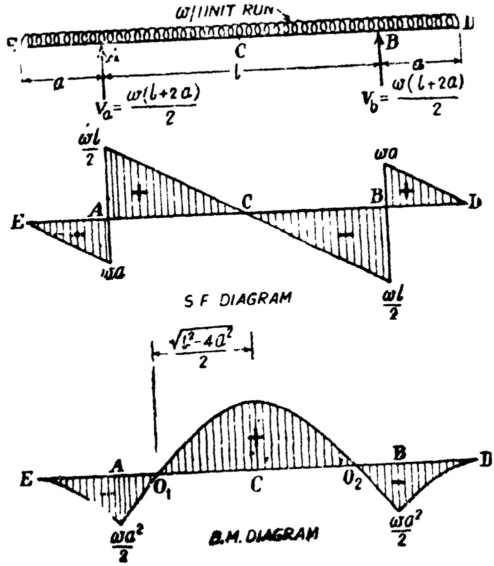
At $x=0$, $\quad M=0$

and at $x=2\ m$. $\quad M=-0\cdot75\times2^2=-3\ tm$.

(ix) Simply supported beam with equal overhangs and carrying a uniformly distributed load of w per unit run over the whole length.

Fig. 164 shows a beam $EABD$ of length $(l+2a)$ with supports at A and B so that $AB=l$ and $AE=BD=a$.

Let the beam carry a uniformly distributed load of w per unit run over the whole length.



Since the loading is symmetrical on the beam, each vertical reaction equals half the total load on the beam.

Let V_a and V_b be the reactions at A and B.

$$\therefore \quad V_a = V_b = \frac{w(l+2a)}{2}$$

S.F. at any section in EA distant x from E is given by $S_x = -wx$.

At $x=0$, *i.e.*, at E, $S_x = 0$

At $x=a$, *i.e.*, just on the left hand side of A, $S_x = -wa$

Fig. 164.

At any section in AB distant x from E the S.F. is given by

$$S_x = V_a - wx$$

$$= \frac{w}{2}(l+2a) - wx$$

At $x=a$, *i.e.*, just on the right hand side of A,

$$S_x = \frac{w}{2}(l+2a) - wa = +\frac{wl}{2}$$

At $x=(a+l)$, *i.e.*, just on the left hand side of B

$$S_x = \frac{w}{2}(l+2a) - w(a+l)$$

$$= -\frac{wl}{2}$$

Hence from A to B the S.F. changes uniformly from $-\frac{wl}{2}$ to $+\frac{wl}{2}$

Hence the S.F. at C the middle point of the beam is zero.

For the portion BD, the S.F. will change uniformly from $+wa$ at B to zero at D.

At any section in EA, distant x from E, the B.M. is given by,

$$M_x = -\frac{wx^2}{2} \text{ (hogging)}$$

At $x=0$, *i.e.*, at E,

B.M.$=0$

At $x=a$, *i.e.*, at A,

$$\text{B.M.} = M_a = -\frac{wa^2}{2} \text{ (hogging)}$$

Hence from E to A the B.M. increases from zero at E to $\frac{wa^2}{2}$ (*hogging*) at A following a parabolic law. Similarly the B.M. for the portion DB will vary from zero at D to $\frac{wa^2}{2}$ (*hogging*) at B following a parabolic law.

At any section in AB, distant x from E the B.M. is given by,

$$M_x = V_b(x-a) - \frac{wx^2}{2}$$

$$\therefore \quad M_x = \frac{w}{2}(l+2a)(x-a) - \frac{wx^2}{2}$$

$$M_x = \frac{w}{2}\left\{(l+2a)(x-a) - x^2\right\}$$

At $x=a$ and at $x=a+l$ *i.e.*, at A and B

$$M_x = -\frac{wa^2}{2}$$

At $x=a+\frac{l}{2}$ *i.e.*, at the middle point of the beam (where the S.F. is zero) the B.M. is given by

$$M_c = \frac{w}{2}\left\{(l+2a)\left(a+\frac{l}{2}-a\right) - \left(a+\frac{l}{2}\right)^2\right\}$$

$$\therefore \quad M_c = \frac{w}{8}(l^2 - 4a^2)$$

Some important observations :

We find from the above discussion that the B.M. for the overhanging parts *EA* and *BD* are of the *hogging* type.

The B.M. at the middle point $C = M_c = \frac{w}{8}(l^2 - 4a^2)$

Case (a). When $l^2>4a^2$ i.e., $l>2a$

For this case (l^2-4a^2) is positive. Hence M_c is positive and the B.M. diagram will be as shown in Fig. 164.

For this case there will be two points of contraflexure O_1 and O_2 between A and B.

The positions of these points can be determined by equating the general expression for B.M. for any section in AB to zero.

We know at any section in AB distant x from E the B.M is given by,

$$M_x=\frac{w}{2}\left\{(l+2a)(x-a)-x^2\right\}$$

For the points of contraflexure,

we have,

$$\therefore \quad \frac{w}{2}\left\{(l+2a)(x-a)-x^2\right\}=0$$

$$\therefore \quad (l+2a)(x-a)-x^2=0$$

$$x^2-(2a+l)x+a(2a+l)=0$$

$$x=\frac{(2a+l)\pm\sqrt{(2a+l)^2-4a(2a+l)}}{2}$$

$$\therefore \quad x=\left(a+\frac{l}{2}\right)\pm\frac{1}{2}\sqrt{(2a+l)(2a+l-4a)}$$

$$x=\left(a+\frac{l}{2}\right)\pm\frac{1}{2}\sqrt{(2a+l)(l-2a)}$$

$$\therefore \quad x=\left(a+\frac{l}{2}\right)\pm\frac{1}{2}\sqrt{l^2-4a^2}$$

But $\quad EC=a+\dfrac{l}{2}$

Hence the points of contraflexure O_1 and O_2 are at the distance $\frac{1}{2}\sqrt{l^2-4a^2}$ from the middle point C of the beam.

Hence distance between the two points of contraflexure $O_1O_2=\sqrt{l^2-4a^2}$

Hence the part O_1O_2 alone can be regarded as a separate simply supported beam, the maximum B M. at the centre being

$$w\times\frac{(O_1O_2)^2}{8}=\frac{w}{8}(l^2-4a^2) \text{ as obtained before}$$

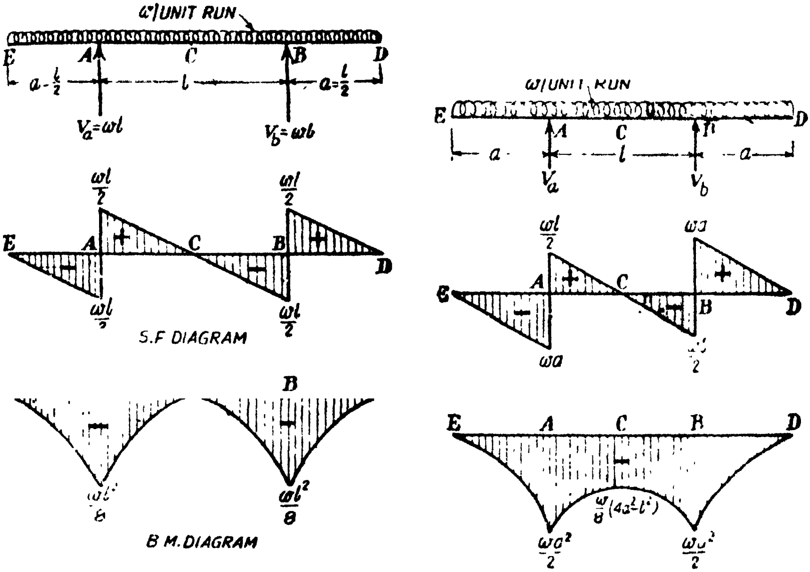
Case (b). When $l^2=4a^2$ i.e., $l=2a$

$$\text{B.M. at } C=M_c=\frac{w}{8}(l^2-4a^2)$$

$$=0$$

The B.M. diagram between A and B will just touch the span at the middle point. The beam is subjected to only hogging bending moments. The points of contraflexure O_1 and O_2 will coincide with C.

Fig. 165 (a) shows the S.F. and B.M. diagram for this case.



(a) Fig. 165. (b)

Case (c). *When* $l^2<4a^2$ *i.e.,* $l<2a$

B.M. at $C=M_c=\frac{w}{8}(l^2-4a^2)$

M_c is negative since $l^2<4a^2$

$$M_c=-\frac{w}{8}(4a^2-l^2).$$

Hence for this case, the B.M. will be zero only at the ends A and D and at all other sections the B.M. will be of the hogging type.

Problem 111. *Calculate the reactions at the supports A and B of the beam shown in Fig. 166. Draw bending moment and shearing force diagrams. Determine also the points of contraflexure within the span AB and show their positions on the bending moment diagram.*

Solution. Let reactions at A and B be V_a and V_b respectively. Taking moments about A, we have, $V_b\times 7+1000\times 2$

$$=2400\times 4+1500\times 10$$

$$V_b=\frac{22600}{}\ kg.$$

$$=3228{\cdot}6\ kg.$$

$\therefore \quad V_a = \text{Total load} - V_b$
$= 4900 - 3228{\cdot}6 \; kg.$
$= +1671{\cdot}4 \; kg.$

S.F. Calculations

S.F. between D and $A = -1000 \; kg.$

S.F. between A and $C = -1000 + 1671{\cdot}4 \; kg.$
$= +671{\cdot}4 \; kg.$

S.F. between C and $B = +1500 - 3228{\cdot}6 \; kg.$
$= -1728{\cdot}6 \; kg.$

S.F. between B and $E = +1500 \; kg.$

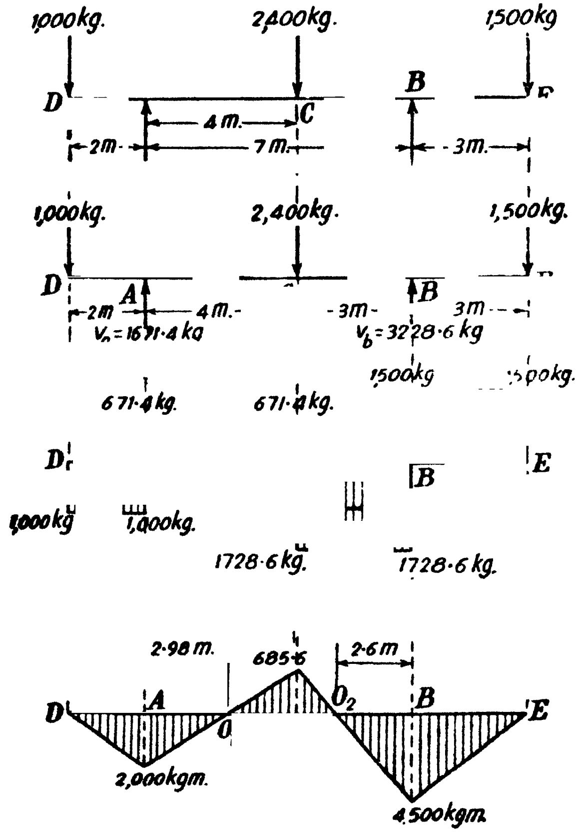


Fig. 166

B.M. Calculations

B.M. at $D = M_d = 0$

B.M. at $A = M_a = -1000 \times 2 = -2000 \; kg. \; m.$

B M. at $C = M_c = +1671{\cdot}4 \times 4 - 1000 \times 6 \; kg. \; m.$
$= +685{\cdot}6 \; kg. \; m.$

B.M. at B$=M_b=-1500\times\cdot3=-4500$ *kg. m.*
B.M. at E$=M_e=0$

Points of contraflexure. There will be two points of contraflexure O_1 and O_2. One of them lies between A and C and the other lies between C and B.

Point of contraflexure O_1 between A and C. Let this point be x *metres* from A. Equating the bending moment to zero

$$1671\cdot4x-1000(x+2)=0$$
$$\therefore \qquad x=2\cdot98 \text{ metres from } A.$$

Point of contraflexture O_2 between C and B. Let this point be x *metres* from B. Equating the bending moment to zero,

$$3228\cdot6x-1500(x+3)=0$$
$$\therefore \qquad x=2\cdot6 \text{ metres from } B.$$

Problem 112. *A beam of length L is simply supported on two intermediate supports, movable along the length, with equal overhangs on either side. The supports are so adjusted that the maximum B.M. is the minimum possible. Determine the position of the supports and draw the B.M and S F diagrams for this position. The beam carries a uniformly distributed load of w per unit length over the entire length.*

(*A.M.I.E., November 1965*)

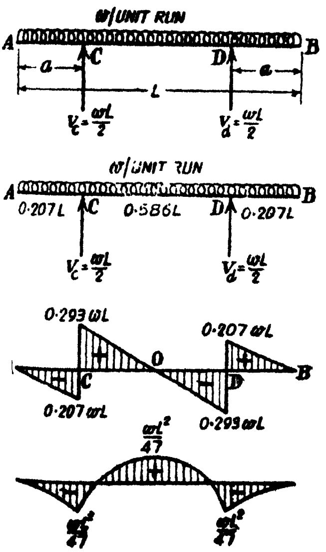


Fig. 167

Solution When the overhanging lengths are small there will be a sagging B M. at midspan and a hogging B M. over the supports. If the maximum B.M. should be the minimum possible the length of the overhanging portions must be so adjusted that the sagging B.M. at midspan equals in magnitude the hogging B.M. over the supports.

Let the overhanging length of each side be a.

Fig. 167 shows the beam carrying the distributed load over the whole length.

Each vertical reaction
=half the total load

$$=\frac{wL}{2}$$

$$\therefore \quad V_c=V_d=\frac{wL}{2}$$

Hogging B.M. over the supports

$$=\frac{wa^2}{2}$$

Sagging B.M. at the centre of the span

$$=\frac{wL}{2}\cdot\left(\frac{L}{2}-a\right)-\frac{wL^2}{8}$$

$$=\frac{wL}{2}\left(\frac{L}{4}-a\right)$$

Equating the sagging B.M. at midspan to the hogging B.M. over the supports, we have,

$$\frac{wL}{2}\left(\frac{L}{4}-a\right)=\frac{wa^2}{2}$$

$$\frac{L^2}{4}-La=a^2$$

$$\therefore \quad a^2+La=\frac{L^2}{4}$$

Completing the square, we get,

$$\left(a+\frac{L}{2}\right)^2=\frac{L^2}{4}+\frac{L^2}{4}=\frac{L^2}{2}$$

$$\therefore \quad a+\frac{L}{2}=\frac{L}{\sqrt{2}}$$

$$a=\frac{L}{\sqrt{2}}-\frac{L}{2}$$

$$=\frac{L}{2}(\sqrt{2}-1)$$

$$\therefore \quad a=\left(\frac{\sqrt{2}-1}{2}\right)L$$

$$=\left(\frac{1.414-1}{2}\right)L$$

$$a=0.207L$$

Sagging B.M. at midspan

$$=\text{Hogging B.M. over the supports}$$

$$=\frac{wa^2}{2}$$

$$=\frac{w}{2}(0.207L)^2=0.02\ \ wL^2$$

$$=\frac{wL^2}{47}\ \textit{nearly}$$

S.F Calculations. S.F. at A and $B=0$

S.F. just on the left side of $C=-w\times0.207L=-0.207\ wL$

S.F. just on the right side of $C=-0.207wL+0.5wL=+0.293wL$

S.F. at midspan $=0$

S.F. just on the left side of $D=-0.293\ wL$

S.F. just on the right side of $D=+0.207\ wL$

Problem 113. *Draw the B.M and S.F diagrams for the overhanging beam carrying loads as shown in Fig 168. Make the values of the principal ordinates and locate the point of contraflexure, if any* (A M I E May ...)

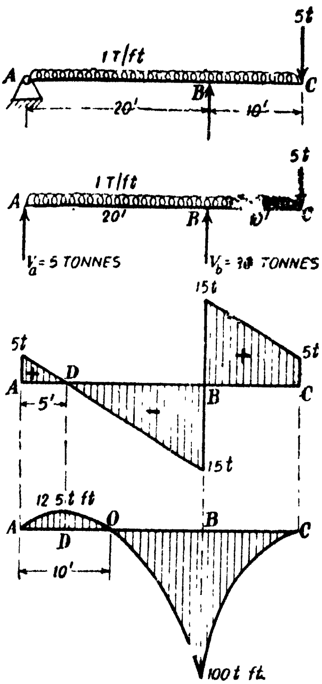


Fig. 168

Solution. Let the reactions at A and B be V_a and V_b respectively.

Taking moments about, A we have,

$$V_b \times 20 = 1 \times 30 \times \frac{30}{2} + 5 \times 30$$

$\therefore \quad V_b = 30\ t$

$\therefore \quad V_a = 1 \times 30 + 5 - 30 = 5\ t$

S F. Calculations

S.F. just on the right hand side of $A = +5$ *tons.*

S.F. just on the left hand side of $B = +5 - 20 = -15$ *tons.*

S.F. just on the right hand side of $B = +5 + 1 \times 10 = +15$ *tons.*

S.F. just on the left hand side of $C = +5$ *tons.*

Let the S.F. be zero at x ft. from A

Equating the S.F. to zero, we get

$$5 - 1 \times x = 0$$

$\therefore \qquad x = 5$ ft.

B.M. Calculations B.M. at $A = 0$

B.M. at $B = -5 \times 10 - \frac{1 \times 10^2}{2} = -100$ *ton. ft. (max. negative b.m.)*

B.M. at $C = 0$ B.M. at $x = 5$ *ft.*

$$M_d = 5 \times 5 - \frac{1 \times 5^2}{2} = +12\ 50 \textit{ ton. ft. (max. positive b.m.)}$$

Point of contraflexure. Let the B.M. at a distance x from A (between A and B) be zero. Equating the bending moment to zero, we have

$$5x - 1\frac{x^2}{2} = 0.$$

$x^2 - 10x = 0 \qquad \therefore\ x(x-10) = 0$ or $x = 10.$

Problem 114. *A beam AB, 20 metres long supported on two intermediate props 12 metres apart carries a uniformly distributed load 0·6 tonne per metre together with concentrated loads of 3 tonnes at the left end A and 5 tonnes at the right end B. The props are so located that the reaction is the same at each support Determine the position of the props and draw B.M. and S.F. diagrams. Mark the values of the maximum B.M. and S.F.*

Solution. Let the left support be at C and the right support be at D.

Let $AC = a$ metres

$\therefore \quad BD = 20 - 12 - a$
$= (8 - a)$ *metres*

Let V_1 and V_2 be the reactions at the left and right supports respectively.

Total load on the beam
$= 3 + 5 + 0{\cdot}6 \times 20$ *tonnes*
$= 20$ *tonnes*

Since the reactions at the supports are given to be equal, we have

$V_1 = V_2 = 10$ *tonnes.*

Taking moments about A, we have

$$5 \times 20 + 0{\cdot}6 \times 20 \times 10 = 10a + 10(12 + a)$$

$$\therefore \quad 100 + 120 = 10a + 120 + 10a$$

$$\therefore \quad 20a = 100$$

$$\therefore \quad a = 5 \text{ metres}$$

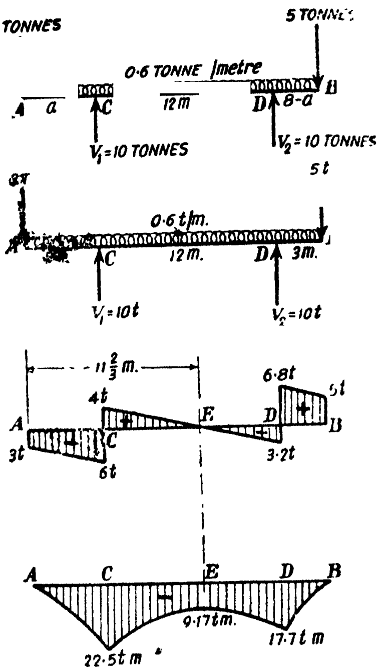


Fig. 169.

$\therefore$ The left support is at 5 *metres* from A and the right support is at $8 - 5 = 3$ *metres* from B.

Shear force calculations

S.F. just on the right hand side of $A = -3$ *tonnes*
S.F. just on the left hand side of $C = -3 - 0{\cdot}6 \times 5 = -6$ *tonnes*
S.F. just on the right hand side of $C = -6 + 10 = +4$ *tonnes*
S.F. just on the right hand side of $D = +5 + 0{\cdot}6 \times 3 = +6{\cdot}8$ *tonnes*
S.F. just on the left hand side of $D = +6{\cdot}8 - 10{\cdot}0 = -3{\cdot}2$ *tonnes.*

Let the S.F. be zero at a distance of x *metres* from A (between the two supports)

Equating the S.F. to zero, we have,

$$10 - 3 - 0{\cdot}6x = 0$$

$$\therefore \quad 0{\cdot}6x = 7$$

$$\therefore \quad x = \frac{7}{0{\cdot}6} \text{ metres}$$

$$= 11\frac{2}{3} \text{ metres.}$$

Bending moment calculations.

B.M. at $A = 0$

$$\text{B.M. at } C = M_c = -3\times5-0{\cdot}6\times\frac{5^2}{2} \text{ tonne metres}$$
$$= -22{\cdot}5 \text{ tonne metre}$$

$$\text{B.M. at } D = M_d = -5\times3-0{\cdot}6\times\frac{3^2}{2} \text{ tonne metres}$$
$$= -17{\cdot}7 \text{ tonne metres}$$

B.M. at a distance of $11\frac{2}{3}$ *metres* from A

$$= M_c = 10\times6\frac{2}{3} - 3\times11\frac{2}{3} - \frac{0{\cdot}6\left(11\frac{2}{3}\right)^2}{2} \text{ tonne metres}$$

$$= -9\frac{1}{6} \text{ tonne metres}$$

$$= -9{\cdot}17 \text{ tonne metres.}$$

Problem 115. *Draw the shear force and bending moment diagrams for the beam shown in Fig. 170. Indicate on the diagrams the values of shear force and bending moment (with proper units) at significant points. Also show the location and magnitude of the maximum bending moment.*

(*A.M.I.E., May 1964*)

Solution. Let the reactions at the left and right supports be V_b and V_d respectively.

Taking moments about the left support, we have,

$V_d\times12=2\times12\times3+8\times9$

$\therefore\quad V_d=12$ *tonnes*

$\therefore\quad V_b=2\times12+8-12$

$=20$ *tonnes.*

Shear force calculations

Shear force at $A=0$

S.F. just on the left hand side of B

$=-2\times3=-6$ *tonnes*

S.F. just on the right hand side of B

$=-6+20=+14$ *tonnes*

S.F. just on the right hand side of C

$=-12$ *tonnes.*

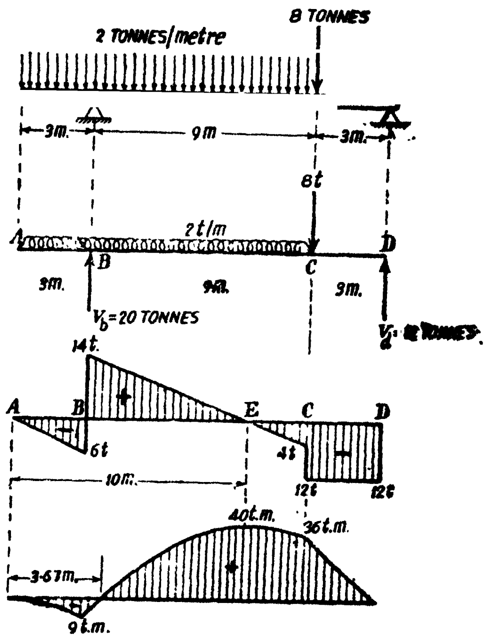


Fig. 170

S.F. just on the left hand side of C

$=-12+8=-4$ *tonnes.*

Let the S.F. be zero at x *metres* form A (between B and C)

Equating the S.F. to zero, we have

$$20-2x=0$$

$$x=10 \textit{ metres.}$$

Bending moment calculations.

B.M. at $A=M_a=0$

B.M. at B $M_b=-2\times\frac{3^2}{2}=-9$ *tonnes metres*

B.M. at C $M_c=+12\times 3=+36$ *tonnes metres*

B.M. at $D=M_d=0$

B.M. at 10 metres from A

$$=M_e=20\times 7-2\times\frac{10^2}{2}=+40 \textit{ tonnes metres}$$

Point of contraflexure.

Let the B.M. be zero at x *metres* from A (between the two supports).

Equating the bending moment to zero, we get,

$$20(x-3)-2\frac{x^2}{2}=0$$

$$\therefore \quad x^2-20x+60=0$$

The practical value of x should be between 3 and 12.

Solving the equation we get $x=3{\cdot}67$ *metres.*

Problem 116. *Determine the support reactions for the beam shown in Fig. 171 and construct the bending moment diagram and shear force diagram marking the values of the various ordinates.* (*Bombay, 1966*)

Solution. Let V_a and V_b be the vertical reactions A and B.

Taking moments about the end A, we have

$$V_b\,L+W.\frac{3}{4}L=2W.\frac{L}{4}+\frac{W}{2}.\frac{3L}{2}$$

$$\therefore \quad V_b=\frac{W}{2}.$$

$$\therefore \quad V_a=\frac{3W}{2}-\frac{W}{2}=W$$

S.F. ***computations***

S.F. between A and $B=+W$

S.F. between C and $D=+W-2W=-W$

S.F. between D and $B=+W-2W+W=0$

S.F. between B and $E=+\frac{W}{2}$

B.M. Computations. B.M. at $A=0$

B.M. at $C = W \times \frac{L}{4} = + \frac{WL}{4}$ (*sagging*)

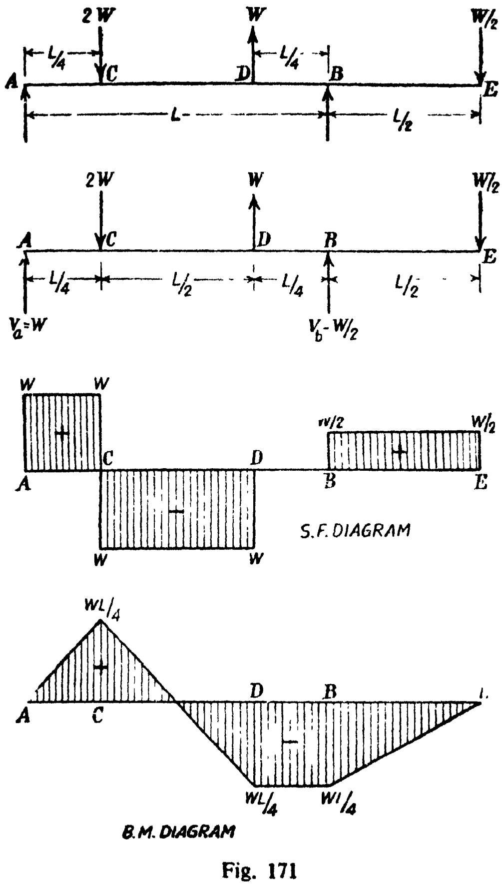


Fig. 171

B.M. at $D = W \times \frac{3L}{4} - 2W \times \frac{L}{2} = -\frac{WL}{4}$ (*hogging*)

B.M. at $B = -\frac{W}{2} \times \frac{L}{2} = -\frac{WL}{4}$ (*hogging*)

B.M. at $E=0$.

Problem 117. *Calculate the reactions for the beam shown in Fig. 172. Construct the bending moment and shear force diagrams. Determine the location of the maximum bending moment and mark it clearly on each of the diagrams.*

Solution Let V_a and V_c be the reactions at the supports A and C respectively.

Taking moments about A, we have

$$V_c \times 6 = 8 \times \frac{3}{2} + 1{\cdot}5 \times 8$$

$$\therefore \quad V_c = 4 \text{ tonnes}$$

$$V_a = 8 + 1{\cdot}5 - 4 = 5{\cdot}5 \ t$$

S.F. Calculations

S.F. at $A = +5{\cdot}5\ t$

S.F. at $B = 5{\cdot}5 - 8\ t = -2{\cdot}5\ t$

S.F. between B and C

$= -2{\cdot}5\ t$

S.F. between C and D

$= +1{\cdot}5\ t$

Let the S.F. be zero at x *metres* from A.

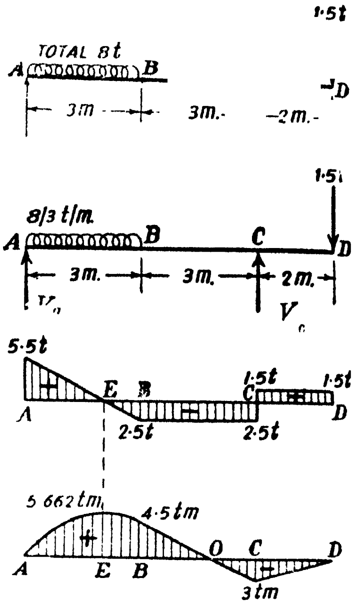


Fig 172

Equating the S.F. to zero we have

$$5{\cdot}5 - \frac{8}{3}x = 0$$

$$\therefore \quad x = \frac{3}{8} \times 5{\cdot}5$$

$$= 2{\cdot}0625\ m \text{ from } A$$

B M Calculations

B.M. at $A = M_a = 0$

B.M. at $B = M_b = 5{\cdot}5 \times 3 - 8 \times \frac{3}{2}$ *tonne meters*

$= +4{\cdot}5$ *tonne metre*

B.M. at E *i.e.*, $2{\cdot}0625$ *metres* from A

$$= 5{\cdot}5 \times 2{\cdot}0625 - \frac{8}{3} \times \left(\frac{2{\cdot}0625}{2}\right)^2 \text{ tonne meter}$$

$= +5{\cdot}662$ *tonne metre*

B.M. at $C = M_c = -1{\cdot}5 \times 2 = -3{\cdot}00$ *tonnes metre*

B.M. at $D = M_d = 0$

Point of contraflexure

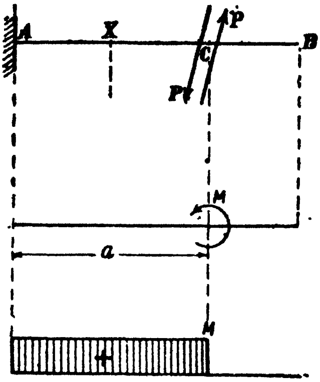
There will be a point of contraflexture between B and C. Let this point of contraflexure be at x *metres* from C.

Equating the B.M. to zero, we get,

$4x-1.5(2+x)=0$

$x=1.2\ m$

§36. B.M. at a section due to a couple



B.M. Diagram
Fig. 173

Fig. 173 shows a cantilever AB of length l. Let an anti-clockwise couple $M=Pp$ be applied at a section C. distant a from A. The couple here consists of the equal and parallel forces P with a lever arm p between them.

It is obvious that the moment of the individual forces constituting the couple about any point in the plane of the couple is Pp.

Hence at any section X, in AC

B.M. = Moment of the individual forces P of the couple.

= anticlockwise moment Pp

= Sagging moment $Pp=M$

Hence at every section between A and C there will be a sagging moment M

Due to the couple alone there will be no shear force.

Fig. 173 shows the B.M. diagram for the cantilever.

Problem. 118. *Draw the B.M. diagram for the cantilever shown in Fig. 174.*

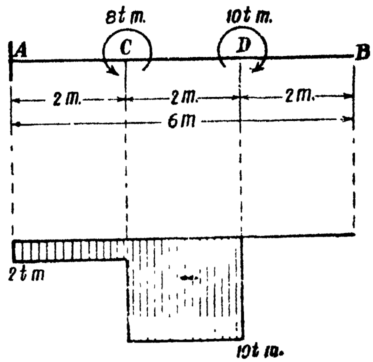


Fig. 174

Solution.

B.M. between D and $B=0$.

B.M. between C and D

$=-10\ tm$ (*hogging*)

B.M between A and C

$=-10+8=-2\ tm$

(*hogging*)

Problem 118. *Find the reaction at the fixed end of the cantilever loaded as shown in Fig. 175. Draw also the shear force and the bending moment diagrams*

Solution. Total vertical load on the cantilever $=3+2=5$ *tonnes*

$\therefore$ Vertical reaction at $A=5$ *tonnes* (*upwards*)

Taking moments about A

We have the following moments :

(*i*) Couple at $B=2.00$ *tm.* (*anticlockwise*)

(*ii*) Moment due to $3t=3\times4=12$ *tm* (*clockwise*)

(*iii*) Couple at $D=3.00$ *tm* (*clockwise*)

(iv) Moment due to 2t
$= 2 \times 8 = 16$ *tm* (*clockwise*)

∴ Net moment = 29 *tm*
(*clockwise*)

Hence at *A* the fixed support will provide a balancing or reacting moment of 29 *tm*.
(*anticlockwise*)

Hence the reaction at *A* will consist of an upward reacting force of 5 *tonnes* and an *anticlockwise* reacting moment of 29 *tonne metres*.

S.F. between
A and $C = +5$ *tonnes*.

S.F. between
C and $E = +2$ *tonnes*.

B.M. Calculations

B.M. at $E = 0$

B.M. just on the right hand side of $D = -2 \times 2 = -4$ *tm*.

B.M. just on the left hand side of $D = -4 - 3 = -7$ *tm*.

B.M. at $C = -2 \times 4 - 3 = -11$ *tm*.

B.M. just on the right hand side of *B*

$= -2 \times 6 - 3 - 3 \times 2$
$= -21$ *tm*.

B.M. just on the left hand side of *B*
$= -21 + 2 = -19$ *tm*

B M at
$A = -2 \times 8 - 3 - 3 \times 4 + 2$
$= -29$ *tm*.

= reacting moment at *A*.

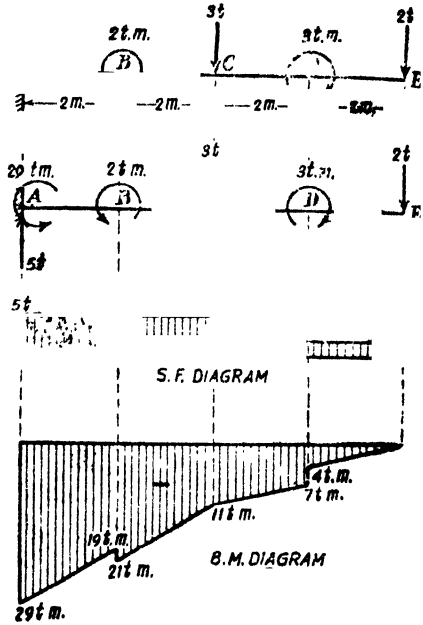


Fig. 175

Problem 120. *Find the reaction at the fixed end and draw the shear force and bending moment diagrams for the cantilever shown in Fig. 176.*

Solution. Total vertical load on the cantilever
$= 1 + 2 - 2 = 1$ *tonne*.
(*downwards*)

∴ Vertical reaction at *A*
$= V_a = 1$ *tonne* (*upwards*)

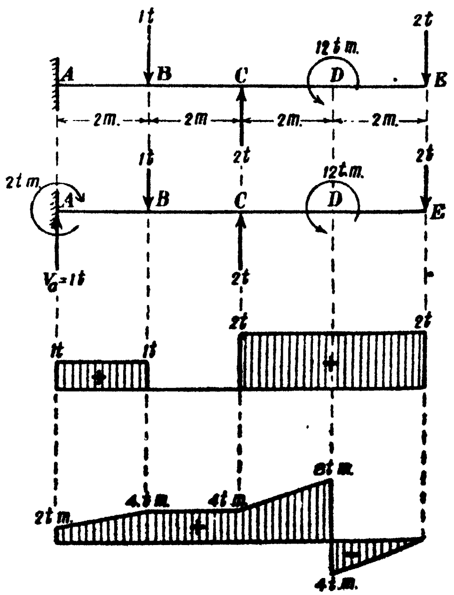


Fig. 176

Taking moments about A, we have the following moments :

(i) $2\times8=16$ *tm.* (*clockwise*)
(ii) 12 *tm.* (*anticlockwise*)
(iii) $2\times4=8$ *tm.* (*anticlockwise*)
(iv) $1\times2=2$ *tm.* (*clockwise*)

$\therefore$ Net moment $=2$ *tm.* (*anticlockwise*)

Hence the fixed support at A will provide the necessary reacting moment of 2 *tm* (*clockwise*)

S.F. between E and $C=+2$ *tonnes*
S.F. between C and $B=0$
S.F. between B and $A=+1$ *tonne.*

B.M. at $E=0$

B.M. just on the right hand side of $D=-2\times2=-4$ *tm.*

B.M. just on the left hand side of $D=-4+12=+8$ *tm.*

B.M. at $C=-2\times4+12=+4$ *tm.*

B.M. at $B=-2\times6+12+2\times2=+4$ *tm.*

B.M. at $A=-2\times8+12+2\times4-1\times2=+2$ *tm.*

Problem 121. *Calculate the reactions at A and B for the beam shown in Fig. 177 and draw the bending moment and shear force diagrams.* (*A.M.I.E., November 1967*)

Solution. Let the reactions at A and B be V_a and V_b respectively. Taking moments about A

we have,

$$V_bL+WL=\frac{WL}{3}+W\frac{4L}{3}$$

$$\therefore \quad V_b=\frac{2}{3}W\uparrow$$

$$\therefore \quad V_a=2W-\frac{2}{3}W$$

$$=\frac{4}{3}W\uparrow$$

S.F. Calculations

S.F. between A and C

$$=+\frac{4}{3}W$$

S.F. between C and B

$$=+\frac{4}{3}W-W=+\frac{W}{3}$$

S.F. between B and $E=+W$

B.M. Calculations

B.M. at $A=M_a=0$

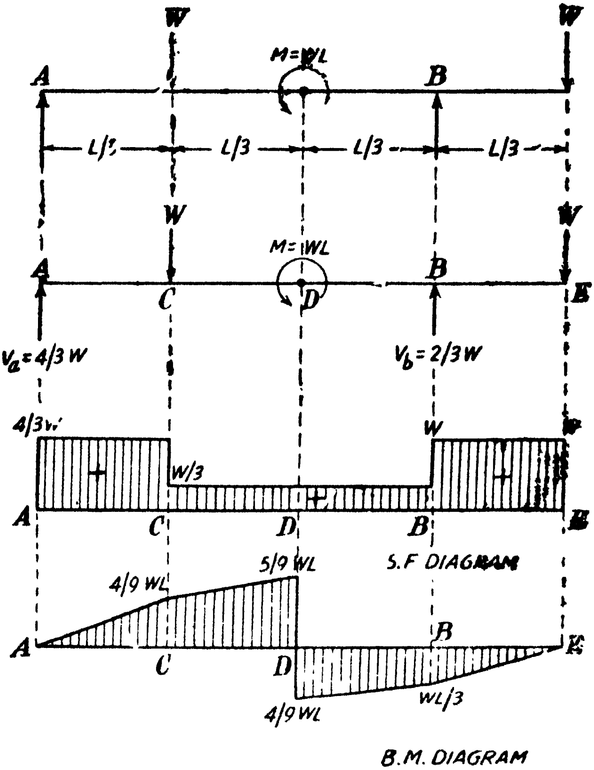


Fig. 177

B.M. at $C = M_c = +\frac{4}{3}W.\frac{L}{3} = +\frac{4}{9}WL$

B.M. just on the left side of D

$$= M_{dc} = +\frac{4}{3}W\left(\frac{2L}{3}\right) - \frac{WL}{3} = +\frac{5}{9}WL$$

B.M. just on the right side of D

$$= M_{db} = +\frac{5}{9}WL - WL = -\frac{4}{9}WL$$

B.M. at $B = -\frac{WL}{3}$

§37. Beam with a couple at an intermediate point

Fig. 178 shows a beam AB of span l hinged at the ends A and B and subjected to a clockwise couple M *tonne metres* at C distant a and b from A and B respectively.

Taking moments about A, we have

$$V_b l = M$$

$$\therefore \quad V_b = \frac{M}{l} \text{ tonnes (upwards)}$$

Since there is no external vertical load,

$$V_a = \frac{M}{l} \text{ tonne (downwards)}$$

$\therefore$ S.F. between A and $C = -\frac{M}{l}$

S.F. between C and $B = -\frac{M}{l}$

B.M. at $A = 0$

B.M. just on the left hand side of C

$$= -\frac{M}{l} \cdot a$$

B.M. just on the right hand side of C

$$= +\frac{M}{l} b$$

B.M. at $B = 0$

Fig. 178 shows the S.F. and B.M. diagrams for the beam.

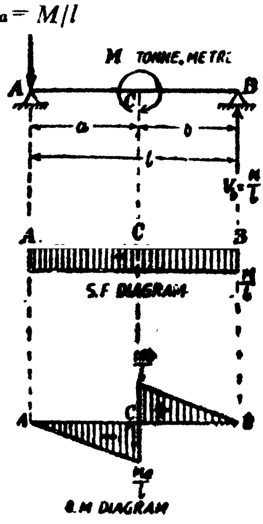


Fig. 178

Problem 22. *Draw shear force and bending moment diagrams for the beam shown in Fig. 179.*

Solution. Let the vertical reactions at A and B be V_a and V_b.

Taking moments about A we have the following moments.

(*i*) $1 \times 5 \times \frac{5}{2}$

$= 12.5$ *tm.* (*clockwise*)

(*ii*) couple $= 15.00$ *tm.* (*anticlockwise*)

net moment $= 2.5$ *tm.* (*anticlockwise*)

Hence $V_b \times 10 = 2.5$ *tm.* (*clockwise*)

$\therefore$ $V_b = 0.25$ *tonnes* (*downwards*)

$\therefore$ $V_a = 1 \times 5 + 0.25$

$= 5.25$ *tonne* (*upwards*)

S.F. between C and B

$= +0.25$ *t.*

From A to C, the S.F. will change from $+5.25$ to $+0.25$ *t.*

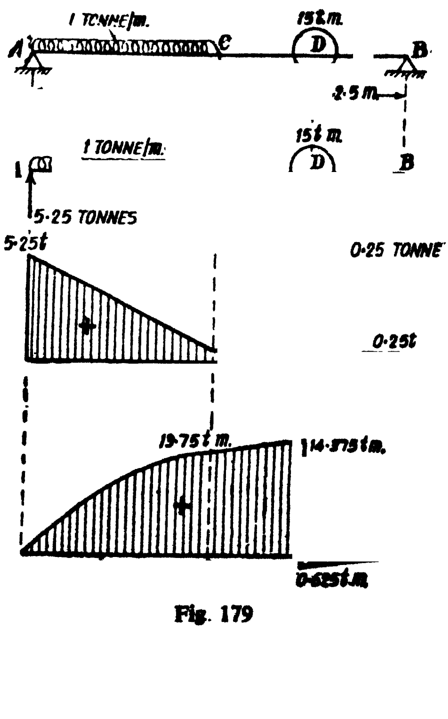


Fig. 179

B.M. at $A=0$

B.M. at $C=5{\cdot}25\times5-1\times\frac{5^2}{2}$ *tm.*

$=13{\cdot}75$ *tm.*

B.M. at $B=0$

B.M. just on the right hand side of D

$=-0{\cdot}25\times2{\cdot}5$ *tm*

$=-0{\cdot}625$ *tm.*

B.M. just on the left hand side of D

$=-0{\cdot}625+15$ *tm.*

$=+14{\cdot}375$ *tm.*

Problem 123. *Construct the bending moment diagram and shear force diagram for the beam shown in Fig. 180 and mark the values of the important ordinates.* *(A.M.I.E., November 1966)*

Solution. Taking moments about the end A, we have

$V_b\times6+12=6\times4+6\times7$

$\therefore \quad 6\,V_b=54$

$\therefore \quad V_b=9$ *tonnes*

$\therefore \quad V_a=12-9=3$ *tonnes*

Shear force calculations.

Shear force just on the right hand side of $A=+3$ *tonnes.*

Shear force just on the left hand side of $D=+3$ *tonnes.*

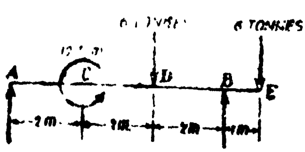


Fig. 180

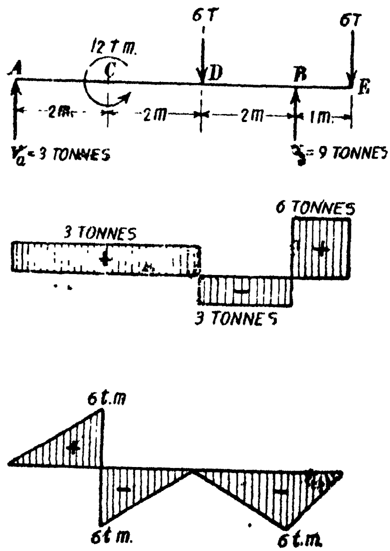


Fig. 181

Shear force just on the right hand side of $D=+3-6$
$=-3$ *tonnes.*

Shear force just on the left hand side of $B=-3$ *tonnes.*

Shear force just on the right hand side of $B=+6$ *tonnes.*

Fig. 181 shows the S.F. diagram

Bending Moment Calculations :

Bending moment at $A=0$

Bending moment just on the left hand side of C
$=+3\times2=+6$ *tonne metres*

Bending moment just on the right hand side of C
$=+3\times2-12=-6$ *tonne metres*

B.M. at $D=+3\times4-12=0$

or alternatively,

B.M. at $D=+9\times2-6\times3=0$

B.M at $B=-6\times1=-6$ *tonne metres*

B.M. at $E=0.$

Fig. 181 shows the B.M. diagram.

Problem 124. *Explain the inter-relation between 'bending moment' and 'shear force' in a beam.* (*A.M.I E., Nov 1966*)

Solution. Fig. 182 shows a beam subjected to an external loading. Consider the equilibrium of the portion of the beam between sections $1-1$ and $2-2$, dx apart, at a distance x from the left support.

Let the shear force at the sections $1-1$ and $2-2$ be S and $S+\delta S$ respectively.

Let the bending moments at the sections $1-1$ and $2-2$ be M and $M+\delta M$ respectively.

The forces and moments keeping the portion of the beam between the sections $1-1$ and $2-2$ in equilibrium consist of the following :

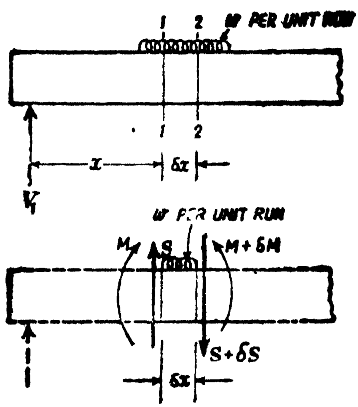


Fig. 182

(*i*) upward force S at section $1-1$

(*ii*) downward force $S+\delta S$ at section $2-2$

(*iii*) downward load $w\delta x$

(*iv*)) moments M and $(M+\delta M)$.

Resolving the forces on this part vertically, we have,

$$S+\delta S+w\delta x=S$$

$$\therefore \quad \delta S+w\delta x=0 \qquad \ldots(i)$$

Taking moments of the forces and couples about the section 2–2, we have,

$$M+\delta M=M+S\,.\,\delta x-\frac{w(\delta x)^2}{2}$$

(If instead of a uniformly distributed load of w per unit run there had been a load varying according to some law the last expression in the above relation would be of the form $-w\delta x\,.\,\frac{\delta x}{n}$ where the quantity n depends on the law of variation of the load).

Ignoring higher powers of small quantities, and simplifying the above relation, we get,

$$\delta M=S\,.\,\delta x$$

$$\therefore \qquad \frac{\delta M}{\delta x}=S$$

i.e., the rate of change of bending moment is equal to the shear force.

Similarly, from equation (i)

$$\delta S+w\delta x=0$$

$$\therefore \qquad \frac{\delta S}{\delta x}=-w$$

i.e., the rate of change of shear force equals the rate of loading.

For instance for the beam shown in Fig. 183.

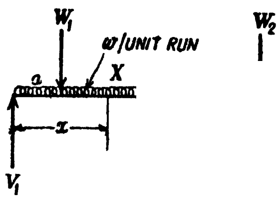


Fig. 183

Shear force at the section X distant x from the left support is given by

$$S=V_1-wx-W_1$$

B.M. at the section X is given by

$$M=V_1x-\frac{wx^2}{2}-W_1(x-a)$$

we have

$$\frac{dM}{dx}=V_1-w\,.\,\frac{2x}{2}-W_1=V_1-wx-W_1=S$$

and $$\frac{dS}{dx}=-w=\text{rate of loading at the section } X.$$

§38. Members with Oblique Loading

Fig. 184 (a) shows a beam AB of span 8 *metres* carrying three point loads applied in an oblique manner. Let the end A be hinged while the end B is placed on rollers.

The various forces can be resolved into their vertical and horizontal components [Fig. 184 (b)].

Total external horizontal load on the beam

$$=(4+5{\cdot}2-1{\cdot}73)\ \textit{tonnes}$$

$$=7{\cdot}47\ t\leftarrow$$

The roller support at B cannot provide any horizontal reaction.

Hence the hinged support will provide a horizontal reaction of $H_a = 7{\cdot}47\ t \rightarrow$.

Let V_a and V_b be the vertical reactions at A and B.

Taking moments about A, we get,

$$V_b \times 8 = 3 \times 2 + 3 \times 4 + 4 \times 6$$

$$\therefore \quad V_b = \frac{42}{8} = 5{\cdot}25\ t$$

$$\therefore \quad V_a = \text{Total vertical load} - V_b = 10 - 5{\cdot}25 = 4{\cdot}75\ t$$

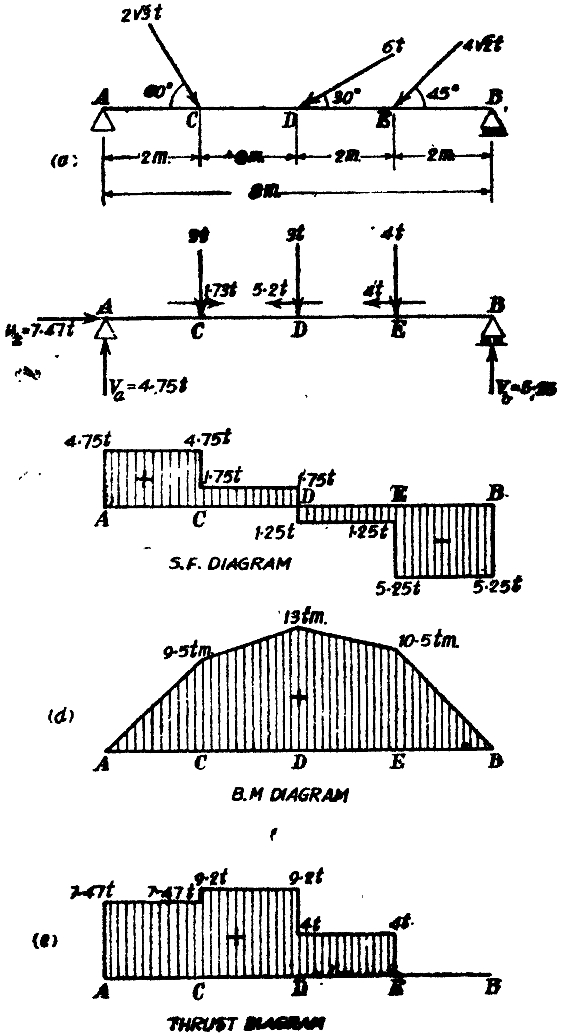


Fig. 184

The beam being horizontal the shear forces and bending moments are only due to the vertical components of the external loading.

∴ S.F. between A and $C = +4\cdot75\ t$.
S.F. between C and $D = +4\cdot75 - 3 = +1\cdot75\ t$.
S.F. between D and $E = -5\cdot25 + 4 = -1\cdot25\ t$.
S.F. between E and $B = -5\cdot25\ t$.
B.M. at $A = 0$
B.M. at $C = M_c = +4\cdot75 \times 2 = +9\cdot50\ tm$.
B.M. at $D = M_d = +4\cdot75 \times 4 - 3 \times 2 = +13\ tm$.
B.M. at $E = M_e = +5\cdot25 \times 2 = +10\cdot50\ tm$.
B.M. at $B = 0$.

The Thrust Diagram

The horizontal components of the loads on the beam will introduce axial loads or thrust in the member.

Axial load or thrust between A and $C = 7\cdot47\ t$ (compressive)
Thrust between C and $D = 7\cdot47 + 1\cdot73 = 9\cdot20\ t$. (compressive)
Thrust between D and $E = 4\ t$. (compressive)
Thrust between E and $B = 0$.

A diagram which shows the variation of the axial load for all sections of the span is called the thrust diagram. The thrust diagram for the above beam is shown in Fig 184 (*e*).

Problem 125. *A simply supported beam carries inclined loads 100 kg, 200 kg. and 300 kg inclined at 30°, 45°, and 60° to the vertical as shown in Fig. 185. These loads act 1 metre, 2 metres and 3 metres from the left support respectively. If the span is 4 metres, draw shear force, Bending Moment and Thrust diagrams. (A M I E, Summer 1979)*

Solution. The inclined forces are replaced by their vertical and horizontal components.

Now taking moments about the hinged end A,

$$V_b \times 4 = (86\cdot6 \times 1) + (141\cdot4 \times 2) + (150 \times 3)$$

$$\therefore \quad V_b = 204\cdot85\ kg \uparrow$$

$$\therefore \quad V_a = (86\cdot6 + 141\cdot4 + 150) - (204\cdot85)$$

$$= 173\cdot15\ kg \uparrow$$

Resolving horizontally,

$$H_a = 50 + 141\cdot4 + 259\cdot8$$

$$\therefore \quad H_a = 451\cdot20\ kg \rightarrow$$

S.F. Calculations

S.F. at any section in AC
$= +173\cdot15\ kg$.

S.F. at any section in CD
$= +173\cdot15 - 86\cdot60 = +86\cdot55\ kg$.

S.F. at any section in DE
$= +150 - 204\cdot85 = -54\cdot85\ kg$.

S.F. at any section in EB
$= -204\cdot85\ kg$

B.M. Calculations

B.M. at $A = 0$
B.M. at $C = +173\cdot15 \times 1 = +173\cdot15\ kg.\ m$.

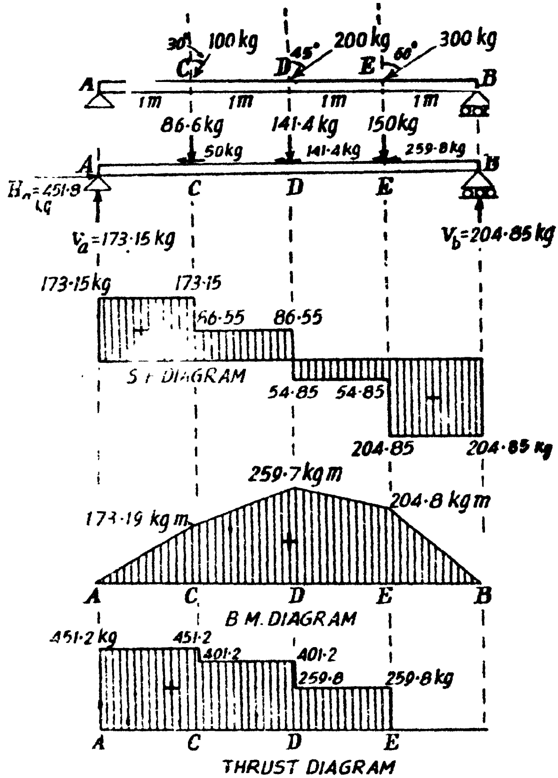


Fig. 185

B.M. at $D = +(173{\cdot}15\times2)-(86{\cdot}6\times1) = +259{\cdot}7$ *kg.-m.*

B.M. at $E = +204{\cdot}85\times1 = +204{\cdot}85$ *kg-m.*

Thrust Calculations

Thrust at any section in $AC = +451{\cdot}2$ kg.

Thrust at any section in $CD = +451{\cdot}20-50 = +401{\cdot}2$ *kg.*

Thrust at any section in $DE = +259{\cdot}8$ *kg.*

Problem 126 *The beam ABC shown in Fig. 186 is hinged to the wall at A. A vertical bracket BD is firmly fixed to the beam at B and a tie DE is hinged to the bracket at D and to the wall at E. Draw S.F. and B M diagrams for the beam ABC when it carries a uniformly distributed load of 4 tonnes per metre run over the whole length and a point load of 2 tonnes at C.*

Solution. Let V_e and H_e be the vertical and horizontal reactions at E. Let V_a and H_a be the vertical and horizontal reactions at A.

For the equilibrium of the whole structure, taking moments about E, we get

$$H_a\times2=\frac{4\times3^2}{2}+2\times3$$

$$\therefore \quad H_a=12 \text{ tonnes.}$$

Resolving the forces on the structure horizontally, we get,

$$H_e = H_a = 12 \text{ tonnes.}$$

Let the tension in the tie ED be T

Resolving the forces at E horizontally and vertically, we get,

$$T \cos \theta = H_e$$

$$T \sin \theta = V_e$$

$$\therefore \quad \tan \theta = \frac{V_e}{H_e} = \frac{1.5}{2} = \frac{3}{4}$$

$$\therefore \quad V_e = \frac{3}{4} H_e = \frac{3}{4} \times 12 = 9 \text{ tonnes.}$$

Resolving the forces on the whole structure vertically, we get,

$$V_a = 4 \times 3 + 2 - 9 = 5t$$

Now consider the equilibrium of the beam ABC. This part is in equilibrium under the action of the following forces :

(*i*) External loading on the beam.

(*ii*) Vertical reaction at $A = V_a = 5\ t$

(*iii*) Horizontal reaction at $A = H_a = 12\ t$

(*iv*) Vertical component of the tension T at $D = V_d$

$$= T \sin \theta = 9\ t$$

(*v*) Horizontal component of the tension T at $D = H_d$

$$= T \cos \theta = 12\ t$$

The effect, of the forces $H_d = 12\ t$ and $V_d = 9t$, is the same as that of an upward force of $9t$ at B and an anticlockwise couple of $12 \times 0.5 = 6\ tm$ at B.

S.F. Calculations

At any section in AB distant x from A the S.F. is given by,

$$S = 5 - 4x$$

At $x = 0$,

$$S = +5 \text{ tonnes.}$$

At $x = 2\ m$, *i.e.*, just on the left side of B

$$S = 5 - 4 \times 2 = -3 \text{ tonnes.}$$

S.F. is zero at a distance from A given by the condition

$$5 - 4x = 0$$

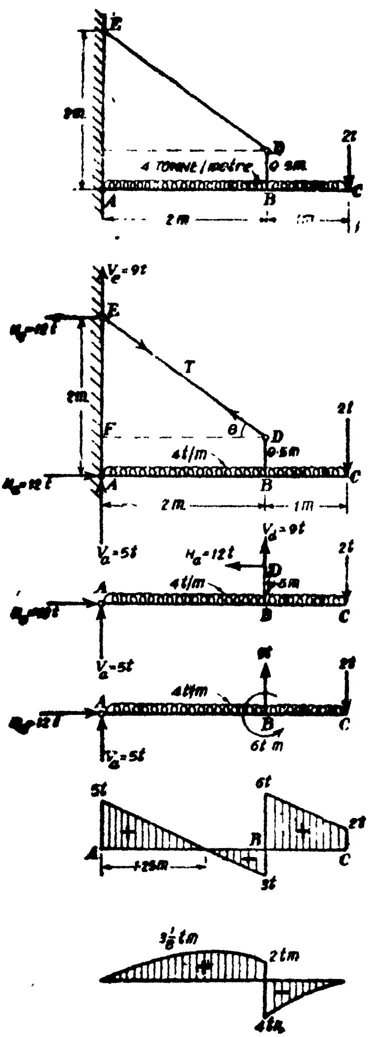


Fig. 186.

i.e, at $x=\frac{5}{4}$ m

S.F. just on the right side of $B=-3+9=+6$ *tonnes.*

From B to C the S.F. will change uniformly from $+6$ *tonnes to* $+2$ *tonnes.*

B.M. calculations

At any section in AB distant x from A the B.M. is given by,

$$M=5x-\frac{4x^2}{2}=5x-2x^2$$

At A *i.e.*, at $x=0$ $M=0$

at $x=2m$ *i.e.*, just on the left side of B.

$$M=5\times2-2\times2^2=+10-8=+2\ tm.$$

B.M. at $x=\frac{5}{4}$ m where the S.F. is zero

$$=\frac{5\times5}{4}-2\times\frac{25}{16}=3\frac{1}{8}\ tm.$$

B.M. just on the right side of $B=+2-6=-4$ *tm* or alternatively$=2+1\times4\times\frac{1^2}{2}=4$ *tm.*

B.M. at $C=0$.

Problem 127. *The structure shown in Fig. 187 is supported on horizontal rollers at D and F and a vertical roller at E. Four loads are applied as shown.*

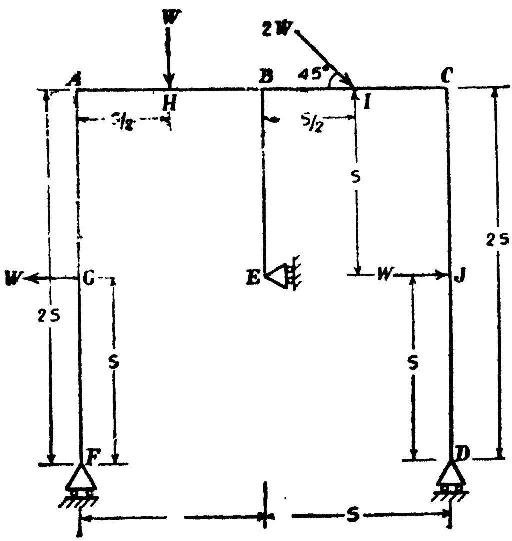


Fig. 187.

Calculate—

(*i*) *the reactions at D, E and F.*
(*ii*) *the bending moments at A, C and J*
(*iii*) *the direct force in portions CD, BI and IC*
(*iv*) *the shear force in portion AH.* (*Bombay*)

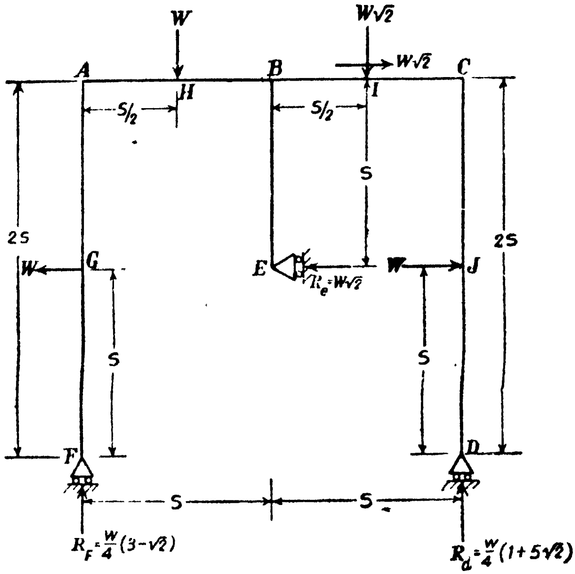


Fig. 188.

Solution. (*i*) *Reactions.* The inclined force $2W$ at I may be split into a vertically downward force $W\sqrt{2}$ and a horizontal force $W\sqrt{2}$. Let the reactions at D, E and F be R_d, R_e and R_f.

Resolving the forces on the structure horizontally, we get

$$R_e = W\sqrt{2}$$

Taking moments of the forces on the structure about F, we have,

$$W \times \frac{S}{2} + W\sqrt{2} \times \frac{3S}{2} + W\sqrt{2} \times 2S + W \times S$$

$$= W \times S + W\sqrt{2} \times S + R_d \times 2S$$

$$\therefore \quad R_d = \frac{W}{4}(1+5\sqrt{2})$$

$$\therefore \quad R_f = W + W\sqrt{2} - \frac{W}{4}(1+5\sqrt{2})$$

$$\therefore \quad R_f = \frac{W}{4}(3-\sqrt{2})$$

(*ii*) ***Bending moments at A, C and J***

B.M at $A = M_a = +W \times S = +WS$ (*producing concavity on the outside*)

B.M. at $C = M_c = +W \times S = +WS$ *(producing concavity on the outside)*

B.M. at $J = 0$

(iii) *Direct force in portions CD, BI and IC*

Direct force in $CD = \frac{W}{4}(1+5\sqrt{2})$ (*compressive*)

Direct force in $BI = W + W\sqrt{2} = W(1+\sqrt{2})$ (*tensile*)
Direct force in $IC = W$ (*tensile*)

(iv) *Shear force in portion AH*

$$= \frac{W}{4}(3-\sqrt{2})$$

Problem 128. *A vertical pile AB is hinged at the base B and subjected to the variable load (due to earth pressure) as shown. The pile is anchored by a tie pin connected at C. Draw the shear force and bending moment diagrams for the pile AB.*

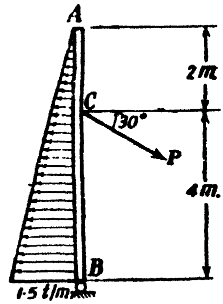


Fig. 189.

Solution. Taking moments of the forces on AB about B, we have,

$$P\cos 30° \times 4 = \frac{6\times 1\cdot5}{2} \times \frac{6}{3}$$

$$\therefore \quad P = \frac{9}{4} \times \frac{2}{\sqrt{3}}$$

$$= \frac{9}{2\sqrt{3}}$$

Resolving the forces on the member horizontally we get, horizontal reaction at B

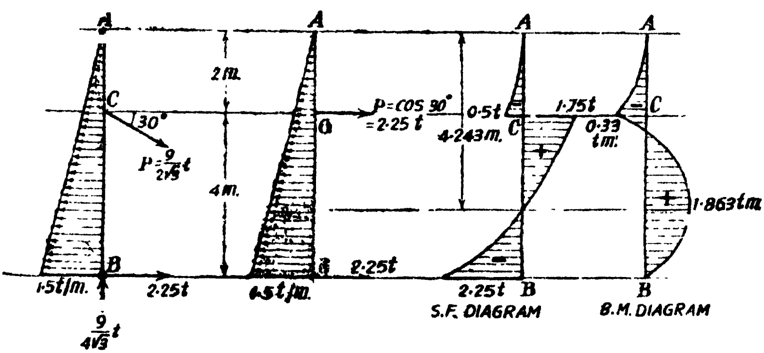


Fig. 190.

$$\frac{6\times 1\cdot5}{2} - \frac{9}{2\sqrt{3}}\cos 30°$$

$$\frac{9}{2} - \frac{9}{4} = \frac{9}{4} = 2\cdot25 \textit{ tonnes.}$$

Vertical reaction at $C=\frac{y}{2\sqrt{3}}\cdot\sin 30°$

$$=\frac{\quad}{4\sqrt{3}}\,t$$

Now consider the horizontal forces on the member. The B.M. or S.F. at any section of the member is only due to the horizontal forces. These forces are shown in Fig. 190.

S.F. Calculations.

At any section in AC distant x from A, the S.F. is given by

$$S=-\frac{x}{2}\cdot\frac{x}{4}=-\frac{x^2}{8}$$

At A, *i.e.*, at $x=0$, $S=0$

S.F. just above $C=-\frac{2^2}{8}=-0{\cdot}5\ t.$

At any section in CB distant x from A, the S.F. is given by

$$S=-\frac{x^2}{8}+2{\cdot}25$$

S.F. just below C

$$=-\frac{2^2}{8}+2{\cdot}25=+1{\cdot}75\ t.$$

S.F. at B, *i.e.*, 6 m from A

$$=-\frac{6^2}{8}+2{\cdot}25$$

$$=-2{\cdot}25\ t$$

Point of zero shear

Equating the S.F. to zero,

$$-\frac{x^2}{8}+2{\cdot}25=0$$

$\therefore$ $x^2=18$

$\therefore$ $x=4{\cdot}243$ *metres* from A.

B.M. Calculations

At any section in AC distant x from A the B.M. is given by

$$M=-\frac{x^2}{8}\cdot\frac{x}{3}$$

$$=-\frac{x^3}{24}$$

At $x=0$, $M=0$,

and at $x=2\ m$

$M=-\frac{1}{3}\ tm.$

At any section in CB distant x from A the B.M. is given by

$$M = -\frac{x^3}{24} + 2.25\,(x-2)$$

At $x = 2\ m.$

$$M = -\tfrac{1}{3}\ tm.$$

At $x = 6\ m.$

$$M = -\frac{6^3}{24} + 2.25\,(6-2) = 0$$

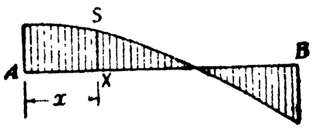
At $x = 4.243\ m$

$$M = -\frac{4.243^3}{24} + 2.25 \times 2.243\ tm.$$

$$= +1.863\ tm.$$

§39. **To obtain the B.M. diagram given the S.F. diagram**

Fig 191 shows the S.F. diagram for a beam AB.



S.F DIAGRAM

Fig. 191.

Let at a section distant x from A the $B.M$ and S.F. be M and S respectively.

we have

$$S = \frac{dM}{dx}$$

$\therefore$ $$dM = S.\,dx$$

Integrating between A and X

we have

$$\int_{x=0}^{x=x} dM = \int_0^x S dx$$

But $\int_0^x S dx$ is the area of the shear force diagram between A and X

Completing the integration, we get

$(M - M_a) =$ area of the S.F. diagram between A and X. If at A the B.M. is zero, we have

$$M_a = 0$$

$\therefore$ $M_x =$ area of the S.F. diagram between A and X

Hence the B.M. at any section is numerically equal to the area of the S.F. diagram between A and X.

Problem 129. *The diagram shown is the shear force diagram for a beam which rests on two supports one being at the left hand end. Deduce directly from the S.F. diagram*

(a) B.M. at 2 m intervals along the beam
(b) Loading on the beam
(c) Position of the second support.

Solution.

(*a*) *B.M. at 2 m intervals*

Since there is a support at the left end A the B.M. at $A=0$

At any section X in AB distant X from A the ordinate of the S.F. diagram

$$XE=10-\frac{4\cdot5}{6}x$$

$$=(10-0\cdot75\,x)$$

$\therefore$ Area of the S.F. diagram between A and X

$$=\frac{(10+10-0\cdot75\,x)\,x}{2}$$

$$10\,x-0\cdot375\,x^2$$

But the area of the S.F. diagram between A and X

$$M_x-M_a$$

Since $M_a=0$, we have

$$M_x=10\,x-0\cdot375\,x^2$$

At $x=2\ m$,

$$M_2=10\times2-0\cdot375\times4=18\cdot5\ tm.$$

At $x=4\ m$,

$$M_4=10\times4-0\cdot375\times16=34\cdot0\ tm.$$

At $x=6\ m$,

$$M_6=10\times6-0\cdot375\times6^2=46\cdot5\ tm.$$

Now consider a section X' in BC distant x' from B

The ordinate of the S.F. diagram at $X'=$

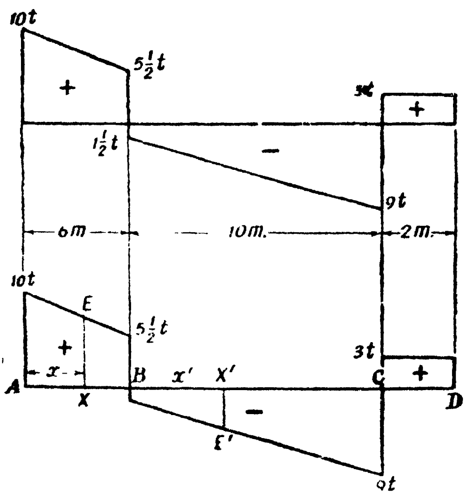


Fig 192

$$X'E'=1\cdot5+\frac{7\cdot5}{10}x'$$

$$X'E'=1\cdot5+0\cdot75\,x'$$

$\therefore$ Area of the S.F. diagram between B and X'

$$= -\left(\frac{1.5+1.5+0.75\,x'}{2}\right)x'$$

$$= -(1.5x' + 0.375x'^2)$$

But the area of the S.F. diagram between B and X'

= the difference between B.M. at X'

and B.M. at $B = M'_x - M_b$

$\therefore$ $M'_x - M_b = -(1.5x + 0.375x'^2)$

$\therefore$ $M'_x = -(1.5x' + 0.375x'^2) + M_b$

But $M_b = 46.5$ *tm*.

$\therefore$ $M'_x = 46.5 - 1.5x' - 0.375x'^2$

At 2 *metres* from B, putting $x' = 2m$,

$$M'_x = 46.5 - 1.5 \times 2 - 0.375 + 2^2 = 42 \ tm.$$

At 4 *metres* from B, puting $x' = 4\ m$

$$M'_x = 46.5 - 1.5 \times 4 - 0.375 \times 4^2 = 34.5 \ tm.$$

At 6 *metres* from B, putting $x' = 6\ m$,

$$M'_x = 46.5 - 1.5 \times 6 - 0.375 \times 6^2 = 24 \ tm.$$

At 8 *metres* from B, putting $x' = 8\ m$,

$$M'_x = 46.5 - 1.5 \times 8 - 0.375 \times 8^2 = 10.5 \ tm.$$

At 10 *metres* from B, putting $x' = 10\ m$,

$$M'_x = 46.5 - 1.5 \times 10 - 0.375 \times 10^2 = -6 \ tm.$$

Now consider the general expression for the B.M.

$$M'_x = 46.5 - 1.5x' - 0.375x'^2$$

For M'_x to be zero

$46.5 - 1.5x' - 0.375x'^2 = 0$

Solving we get $x' = 9.31\ m.$

(*b*) *Loading on the beam.*

(*i*) Since the shear force uniformly changes from A to B there should be a uniformly distributed load in this range.

The intensity of the load = Slope of the shear force diagram between A and B

$$= \frac{10 - 5.5}{5} = 0.75 \ t/m$$

(*ii*) At B there is an abrupt change in S.F. Hence at B there must be a concentrated load.

Magnitude of the concentrated load

= Abrupt change in S F.

$$= \left(5\frac{1}{2} + 1\frac{1}{2}\right) = 7 \ \textit{tonnes}.$$

(*iii*) Since the S.F. uniformly changes from B to C,

there must be a uniformly distributed load between B and C.
Intensity of the distributed load

=Slope of the S.F. diagram between B and C

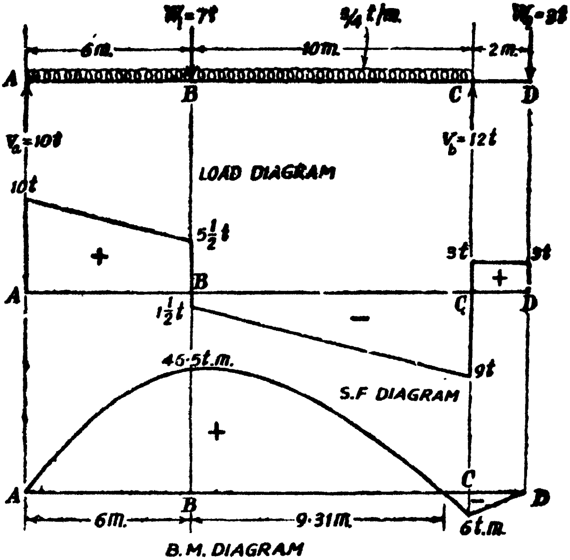


Fig. 193

$$\frac{9-1\cdot5}{10}\quad 0\cdot75\ t/m.$$

(*iv*) Since the S.F. is constant between C and D at 3 *tonnes*, there must be a point load of 3*t* at D.

(*c*) *Position of second support*

Since the S.F. abruptly changes at C the second support is at C. Reaction at C=abrupt change in S.F. at $C=9+3=12$ *tonnes*.

Problem 130. *Draw S.F. and B.M. diagrams for the members ABC and DEF shown in Fig. 194.*

Solution Consider the equilibrium of the member DEF. This member is supported at F as a support and at D on the vertical member DC.

Let V_d be the vertical reaction exerted by the member DC at D. Now DF can be considered as a simply supported beam.

Taking moments about F, we get,

$$V_d \times 4 = 8 \times 1$$

$$\therefore \quad V_d = 2 \text{ tonnes.}$$

S.F. between D and $E=2t$

S.F. between E and $F=-6t$

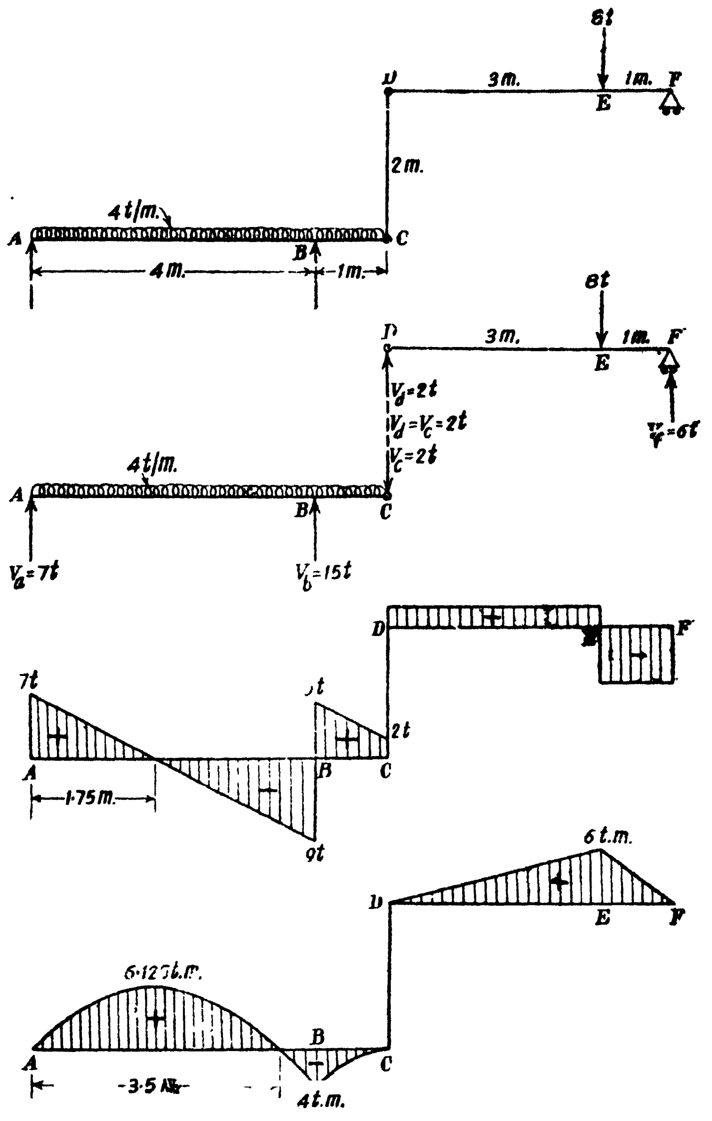


Fig. 194

B.M. at $D=0$

B.M. at $F=0$

B.M. at $E=2\times3=+6$ *tm. (sagging)*

Now consider the member *ABC*. The loading on the member consists of a uniformly distributed load of 4*t* per *metre* over the whole length together with a downward point load $V_c=2t$ at *C* exerted by the member, CD. Considering the equilibrium of this member, taking moments about A, we get

$$V_b \times 4 = 4 \times 5 \times \tfrac{5}{2} + 2 \times 5$$

$\therefore$ $V_b = 15$ *tonnes*

$\therefore$ $V_a = 4 \times 5 + 2 - 15 = 7$ *tonnes.*

S.F. at $A = +7\ t$

S.F. just on the left side of B

$$= 7 - 4 \times 4 = -9\ t$$

S.F. just on the right side of B

$$= -9 + 15 = +6t$$

S.F. at $C = 2t$

Let the S.F. be zero at a section in AB distant x from A.

Equating the S.F. to zero, we get

$$7 - 4x = 0$$

$\therefore$ $x = 1.75$ *metres*

B.M. at $A = 0$

B.M. at $B = -\dfrac{4 \times 1^2}{2} - 2 \times 1 = -4$ *tm.* (*hogging*)

B.M. at 1.75 *m.* from A

$$= M_{max} = 7 \times 1.75 - 4 \times 1.75 \times \frac{1.75}{2}$$

$$= 12.25 - 6.125 = 6.125\ tm.$$

Point of contraflexure. Let at a section in AB distant x from A the B.M. be zero.

$$7x - \frac{4x^2}{2} = 0$$

$\therefore$ $x(7 - 2x) = 0$

$\therefore$ $x = 3.5$ *meters.*

Examples in Chapter 4

1. A beam 6 *metres* long is simply supported at the ends and carries a uniformly distributed load of 3 *tonnes* per *metre* run for a distance of 4 *metres* from the left end. Find the maximum shear force and bending moment and draw the S.F. and B.M. diagrams.

[Max. +ve S.F. = 8*t*,
Max −ve S.F. = 4*t*

Max. sagging B.M. = 10.667 *tonne metres* at $2\frac{2}{3}$ *metres* from left end.]

2. A beam 5 *metres* long, supported at the ends carries point loads of 14 *tonnes*, 6 *tonnes* and 8 *tonnes* at distances 0.5 *metre*, 2.5 *metres* and 3.5 *metres* respectively from the left end. Find the maximum S.F. and B.M. Draw the S.F. and B.M. diagrams.

(**Max. +ve S.F. = 8** *tonnes*,

Max.−ve S.F.=4 *tonnes*

Max. Sagging B.M.=17 *tonne metres.*)

3. A beam *ABC* 5 *metres* long has one support at the end *A* and the other support at *B* 3 *metres* from *A*. It carries a point load of $4t$ at the middle point of *AB* and a point load of $3t$ at *C*. Draw S.F. and B.M. diagrams.

(Max.+ve S.F.=$3t$,
Max.−ve S.F.=4t
Max. hogging B.M.=6 *tm*. at 3*m* from *A*.)

4. Draw S.F. and B.M. diagrams for the beam shown in Fig. 195.

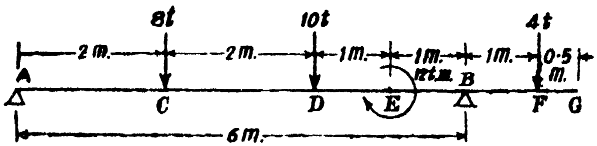


Fig. 195

S.F. between *A* and $C=+6t$

S.F. between *C* and $D=-2t$

S.F. between *D* and $B=-12t$

S.F. between *B* and $E=+4t$

$M_a=0$ B.M. at *E* abruptly changes from $+8$ *tm* to -4 *tm*

$M_c=+12$ *tm*

$M_d=+8$ *tm* $M_b=-4$ tm

$M_f=0$

5. Draw S.F. and B.M. diagrams for the beam shown in Fig. 196

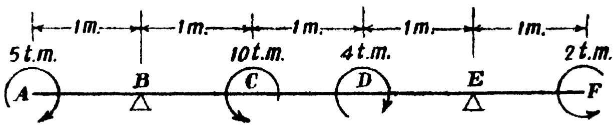


Fig. 196

S.F. between *A* and $B=0$

S.F. between *B* and $E=+1t$

S.F. between *E* and $F=0$

[$M_a=5$ *tm*, $M_b=+5$ *tm*, $M_{cb}=+6$ *tm*,
$M_{cd}=-4$ *tm*, $M_{dc}=-3$ *tm*, $M_{de}=+1$ *tm*, $M_e=+2$ *tm*,
$M_{fe}=+2$ *tm*.]

6. Draw S.F. and B.M. diagrams for the beam shown in Fig. 197

S.F. at $A=0$ S.F. between *C* and $E=-6t$

$S_{ba}=-2t$ S.F. between *E* and $F=+3t$

$S_{bc}=+8t$ $M_a=0$, $M_b=-1$ *tm*, $M_b=+11$ *tm*,
$M_{dc}=+5$ *tm*.

$S_{cb}=+4t$ $M_{de}=+12$ *tm*, $M_e=-6$ *tm*, $M_f=0$

$S_{cd}=-6t$

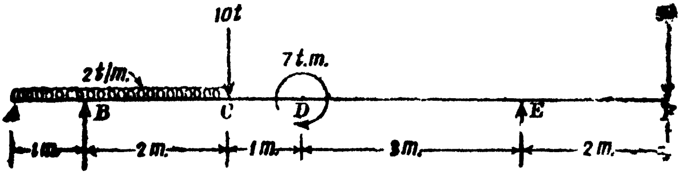


Fig. 197

Point of contraflexure at 1·127 *m* from *A* and 3 *m* from *F.*

7. Draw S.F. and B.M. diagrams for the members *ABCD* and *EFGH* shown in Fig. 198.

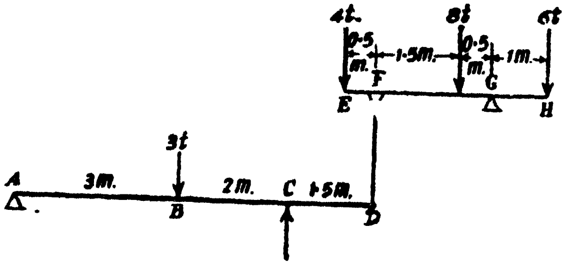


Fig. 198

(Member *ABCD* : S.F. between *A* and *B*=0 $M_a=0$

S.F. between *B* and *C*=−3*t* $M_b=0$

S.F. between *C* and *D*=+4*t* $M_c=-6$ *tm.*

$M_d=0$

Member *EFGH* : S.F. between *E* and *F*=4*t* $M_e=0$

S.F. between *F* and *I*=0 $M_f=-2$ *tm.*

S.F. between *I* and *G*=−8*t* $M_i=-2$ *tm.*

S.F. between *G* and *H*=+6*t* $M_g=-6$ *tm.*

$M_h=0$)

8. A beam simply supported and with equal overhangs, carries

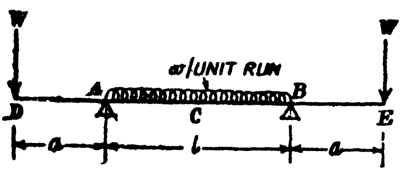


Fig. 199

loads as shown in Fig. 199. If $W=wl$ find the ratio $\frac{a}{l}$ for which the bending moment at mid span will be zero. (0·125)

9. Find the normal force, shear force and bending moment at the section D of the beam shown in Fig. 200.

$$\left(P, -\frac{P}{7}, -\frac{P}{14}\right)$$

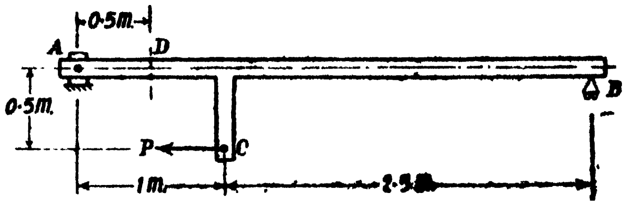
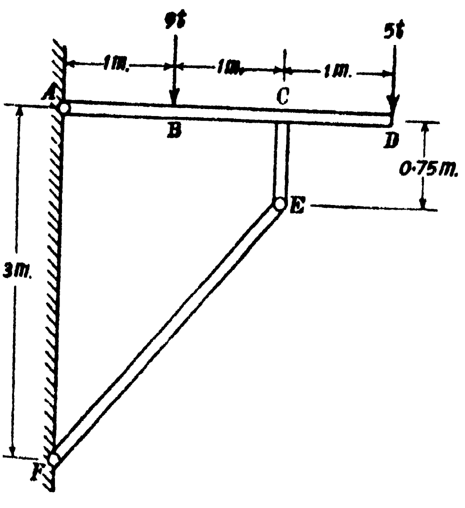


Fig. 200

10. A beam of length l has one support at one end and the other support at a distance a from the other end. The beam carries a uniformaly distributed load of w per unit run over the whole length. Find the value of a, (i) if the maximum sagging bending moment equals the maximum hogging bending moment, (ii) if the middle point of the beam is a point of inflexion·

$$\left[(i)\ 0{\cdot}293l,\ (ii)\ \frac{l}{3} \right]$$

11. The beam $ABCD$ shown in Fig. 201 is 3 *metres* long and is hinged to a wall at A and supported horizontally by the strut EF hinged to the wall at F and at E to the part CE which is rigidly attached to the beam at C. For the loading shown find the reactions at the hinges A and F and the thrust in EF and draw S.F. and B.M. diagram for the beam $ABCD$.



(V_a = 5t upwards
$H_a = 8t$
$V_f = 9t$
$H_f = 8t$
thrust in $EF = 12{\cdot}04t$
$M_a = 0$
$M_b = +5$ tm.
$M_{cb} = +1$ tm.
$M_{cd} = -6$ tm.
$M_d = 0$)

Fig. 201

5

Stresses in Beams

A member subjected to bending moment and shear force undergoes certain deformations. The material of the member will offer resistance or stresses against these deformations. It is possible to estimate these stresses with certain assumptions. In this chapter we will discuss about these stresses. A bending moment *bends* a member. Stresses introduced by bending moment are called *Bending Stresses.* Similarly a shear force will introduce stresses called *Shear stresses.*

§40. Pure Bending

Fig. 202 shows a beam $EABF$ with supports at A and B, l units apart. Let the overhangs $EA = BF = a$. Let a point load W be applied at each end of the beam. It is easily seen that between A and B the B.M. is constant and there is no shear force at all between A and B *i.e.*, between A and B the beam is absolutely free from shear but is subjected to a bending moment Wa. This condition of the beam between A and B is called *pure bending* or *simple bending.*

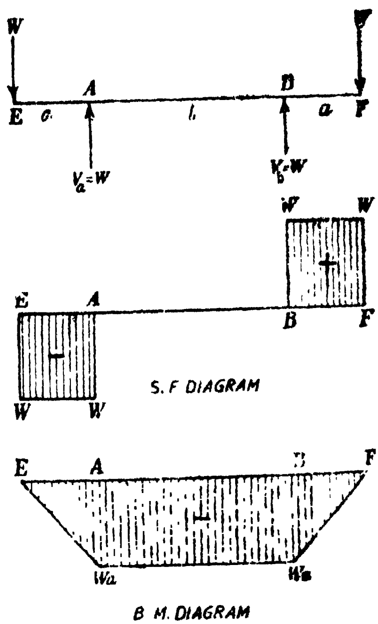


Fig. 202

§41. Theory of simple bending

Fig. 203 (*a*) shows a part of a beam subjected to *pure bending.* The part of length δx being subjected to pure bending has deformed to the shape shown in Fig. 203 (*c*).

A fibre such as AC has deformed to the shape A_1C_1. This fibre has been shortened in its length. The fibre BD on the contrary has been elongated and has taken the shape B_1D_1. Similarly, the fibre GH has been elongated and has taken the shape G_1H_1. Hence, if the beam for the length δx be taken to consist of a large number

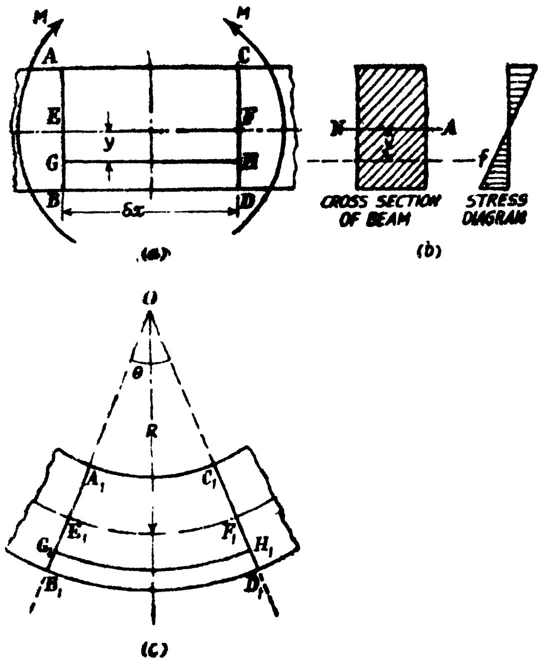


Fig. 203

of fibres, we find that all of them have changed their shape ; some of them have been shortened while some of them are elongated. At a level between the top and bottom of the beam there will be a layer of fibres which are neither shortened nor extended. Fibres in this layer are not stressed at all. This layer is called the *neutral layer* or the *neutral surface*. The line of intersection of the neutral surface on a cross-section is called the *neutral axis*.

If now all the fibres between the two transverse sections AB and CD be considered, the extremities of these fibres will remain on the planes A_1B_1 and C_1D_1 after the deformation. Let A_1B_1 and C_1D_1 meet at O. Let the angle between the planes A_1B_1 and C_1D_1 be θ. Let the radius of the neutral surface be R. Consider the fibre GH distant y from the neutral layer.

Original length of this fibre $GH=\delta x$

After deformation this fibre will deform and take the position G_1H_1, the new length of the fibre being $(R+y)\,\theta$.

The fibre EF in the neutral layer takes the position E_1F_1 without undergoing any change in length

$$\therefore \quad EF=E_1F_1=\delta x$$

$$\therefore \quad \delta x=R\theta$$

$\therefore$ Change in length of the fibre GH

$$=G_1H_1-GH$$

$$=(R+y)\,\theta-\delta x$$
$$=(R+y)\,\theta-R\theta$$
$$=y\theta$$

∴ Strain of the fibre GH

$$=e=\frac{\text{change in length}}{\text{original length}}=\frac{y\theta}{R\theta}$$

∴ $$e=\frac{y}{R}$$

Suppose the stress intensity in the fibre be f, we have strain of the fibre $=\frac{f}{E}$

where E is the Young's Modulus.

∴ $$e=\frac{f}{E}=\frac{y}{R}$$

or $$\frac{f}{y}=\frac{E}{R}$$

∴ $$f=\frac{E}{R}\cdot y$$

Hence the stress intensity in any fibre is proportional to the distance of the fibre from the neutral axis.

For the case explained above all fibres below the neutral layer are subjected to tensile stresses while those above the neutral layer are subjected to compressive stresses. The stress distribution diagram is shown in Fig. 203 (*b*).

§42. Neutral axis

As mentioned earlier, the neutral axis of any transverse section

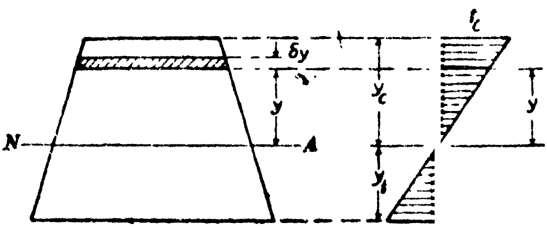


Fig. 204

of a beam is the line of intersection of the neutral layer with the transverse section.

Fig. 204 shows the cross-section of a beam. Let R be the radius of curvature of the neutral layer at this section. Hence the stress at any point distant y from the neutral axis is given by

$$f=\frac{E}{R}\cdot y$$

where E is the Young's Modulus. If the section be subjecte to pure sagging moment, this stress will be compressive at any point above the neutral axis and tensile below the neutral axis.

Now consider an elemental area δa distant y from the neutral axis.

Stress on the elemental area

$$=f=\frac{E}{R}y$$

∴ Thrust on the elemental area

$$=f\delta a=\frac{E}{R}\cdot y\ \delta a$$

∴ Total thrust on the beam section

$$=\frac{E}{R}\sum_{y_t}^{y_c} y\ \delta a$$

Since no axial load has been applied, the total thrust on the beam section equals zero.

$$\therefore \qquad \frac{E}{R}\sum_{y_t}^{y_c} y\ \delta a=0$$

$$\therefore \qquad \sum_{y_t}^{y_c} y\ \delta a=0$$

This is possible only when the neutral axis is a centroidal axis.

§43. Moment of resistance

Fig. 205 shows a beam subjected to an external loading. Le V_a and V_b be the end reactions.

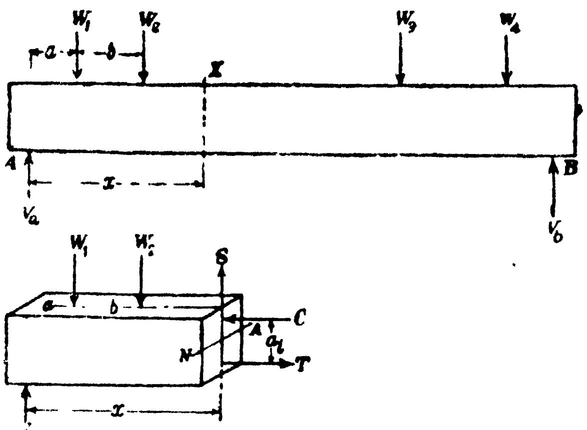


Fig. 205

Consider any section X distant x from the support A.

Fig. 205 shows the forces keeping the part AX in equilibrium. This part is in equilibrium under the action of the following forces :

(*i*) Vertical reaction V_a
(*ii*) Downward loads W_1, W_2
(*iii*) Shear resistance S offered by the section X
(*iv*) Compressive resistance C offered by the section X
(*v*) Tensile resistance T offered by the section X

Since C and T are the only longitudinal forces on the beam we have

$$C = T$$

Let the distance between the lines of action of C and T be a_l.

The two equal and opposite resistances C and T will form a couple $T\,a_l$ or $C\,a_l$. This couple is called the moment of resistance.

For the equilibrium of the part AX,

taking moments of the forces on this part about the N.A. of the section X, we have

$$C.\ a_l = V_a x - W_1(x-a) - W_2(x-a-b)$$

But the expression on the right hand side of the above relation is the bending moment at X.

Hence the moment of resistance offered by the section is equal to the bending moment.

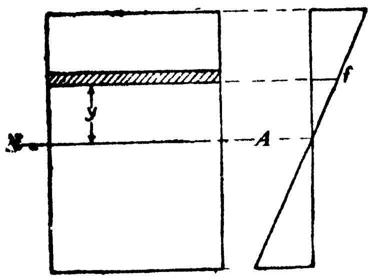


Fig. 206

Now consider an elemental area δa at a distance y from the neutral axis.

The stress intensity on the elemental area $= f = \frac{E}{R}y$.

Thrust on the elemental area $= f\,\delta a$

$$= \frac{E}{R}\,y\,\delta a$$

Moment of resistance offered by the elemental area
= moment of the thrust about the N.A.

$$= \frac{E}{R}\,y^2\,\delta a$$

$\therefore$ Total moment of resistance offered by the beam section

$$= M = \frac{E}{R}\sum_{y_t}^{y_c} y^2\,\delta a$$

But $\sum_{y_t}^{y_c} y^2\,\delta a$ is the *moment of inertia* of the beam section about

the neutral axis. Let this moment of inertia be I

$$\therefore \qquad M=\frac{E}{R}I$$

$$\therefore \qquad \frac{M}{I}=\frac{E}{R}$$

But we know $\quad \frac{f}{y}=\frac{E}{R}$

Hence $\quad \frac{M}{I}=\frac{f}{y}=\frac{E}{R}$

Assumptions in the theory of pure bending

The following assumptions have been made in the theory of pure bending :

(*i*) The value of the Young's Modulus is the same for the beam material in tension as well as compression.

(*ii*) A transverse section of the beam, which is a plane before bending will remain a plane after bending.

(*iii*) The material of the beam is *homogeneous* and *isotropic* (*Isotropic* means having the same elastic properties *in all the directions*).

(*iv*) The Elastic limit is not exceeded.

(*v*) The resultant pull or thrust on a transverse section of the beam is zero.

(*vi*) The transverse section of the beam is symmetrical about an axis passing through the centroid of the section and parallel to the plane of bending.

§44. Practical application of bending equation

The bending equation $\frac{M}{I}=\frac{f}{y}=\frac{E}{R}$ is based on the theory of *pure bending*. In practice, in a beam subjected to a loading the bending moment varies from section to section. Further the bending moment at a section is accompanied by a shearing force. But in pure bending a member is absolutely free from shear force and is subjected to constant bending moment Hence it may appear that the bending equation which was obtained for the case of a member subjected to constant bending moment not accompanied by shear force is not applicable to the practical member which is subjected to different bending moments at different sections and accompanied by shear forces. But it is generally observed that the shear force is zero where the bending moment is maximum and hence the conditions of pure or simple bending may be taken to be satisfied at such a section.

The stresses produced due to the maximum bending moment are the most important stresses in a beam and hence if the stresses be determined by pure bending equation the results are fairly correct since at these sections the shear forces are generally zero.

Problem 131. *A steel plate is bent into a circular arc of radius 10 metres. If the plate section be 12 cms wide and 2 cms. thick find the maximum stress induced and the bending moment which can produce the stress. Take $E=2\times 10^6$ kg /cm ^{2}*

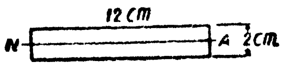


Fig. 07

Solution. Moment of inertia of the section about the neutral axis

$$=I=\frac{12\times 2^3}{10}\ cm.^4=8\ cm.^4$$

$$\frac{M}{I}=\frac{f}{y}=\frac{E}{R}$$

$$\therefore\quad f=\frac{E}{R}\cdot y$$

$$\therefore\quad f_{max}=\frac{E}{R}y_{max}$$

$$=\frac{2\times 10^6}{10\times 100}\times\left(\frac{2}{2}\right)\ kg./cm.^2$$

$$=2000\ kg./cm.^2$$

$$M=\frac{E}{R}\cdot I$$

$$=\frac{2\times 10^6}{10\times 100}\times 8\ kg.\ cm.$$

$$=16000\ kg.\ cm.$$

§45. Section Modulus

Let M be the moment of resistance of a section of the beam, and I the moment of inertia of the section about the neutral axis.

The stress at any point on the section distant y from the neutral axis is given by

$$f=\frac{M}{I}\cdot y$$

The maximum stress occurs at the greatest distance from the neutral axis.

Let y_{max} be the distance of the most distant point of the section from the neutral axis. Let f_{max} be this stress at this distance, we have

$$f_{max}=\frac{M}{I}\ y_{max}$$

or

$$M=f_{max}\cdot\frac{I}{y_{max}}$$

$$=f_{max}\cdot Z$$

where

$$Z=\frac{I}{y_{max}}$$

$$=\frac{\text{Moment of inertia about the neutral axis}}{\text{Distance of the most distant point from the neutral axis}}$$

This ratio is called the *section modulus*.

Hence if the maximum stress offered by a section is known we can easily compute the moment of resistance that can be offered by the section. Hence for a beam of a given material the greatest moment of resistance the beam section can offer is given by

$$M = f_{safe} \cdot Z.$$

where f_{safe}=permissible bending stress (which occurs at the point most distant from the neutral axis).

§46. Section modulus for various shapes of beam sections

(*i*) *Rectangular section.* Fig. 208 shows a rectangular section of width b and depth d. Let the horizontal centroidal axis be the neutral axis

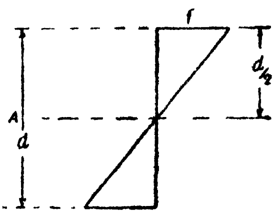


Fig. 208

Section modulus$=Z$

$$= \frac{\text{Moment of inertia about the neutral axis}}{\text{Distance of the most distant point of the section from the neutral axis}}$$

$$= \frac{I}{y_{max}}$$

But $\quad I = \frac{bd^3}{12}$ and $y_{max} = \frac{d}{2}$.

$$\therefore \quad Z = \frac{bd^3}{12} \cdot \frac{2}{d} = \frac{bd^2}{6}$$

Let f be the maximum stress offered by the beam section.

$\therefore$ Moment of resistance$=M=fZ$

$$= M = f\frac{bd^2}{6}$$

or $\quad M = \frac{1}{6} fbd^2$

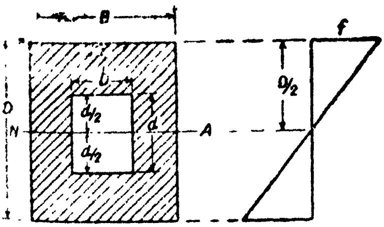


Fig. 209

(*ii*) *Hollow rectangular section.* Fig. 209 shows a hollow rectangular section. Let the overall width and depth be B and D. Let the width and depth of the centrally situated rectangular hole be b and d.

Moment of inertia about the neutral axis

$$I = \frac{BD^3}{12} - \frac{bd^3}{12} = \frac{1}{12}\left[BD^3 - bd^3\right]$$

$$y_{max}=\frac{D}{2}$$

$$\therefore \text{ Sectional modulus}=Z=\frac{I}{y_{max}}=\frac{1}{12}\left(BD^3-bd^3\right)\frac{2}{D}$$

$$\therefore \qquad Z=\frac{BD^3-bd^3}{6D}$$

If f be the maximum bending stress the moment of resistance $=M=fZ$

$$M=\frac{1}{6}f\left(\frac{BD^3-bd^3}{D}\right)$$

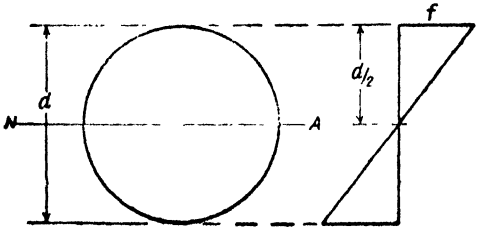


Fig. 210

(*iii*) *Circular section.* Let the diameter of the section be d.

Moment of inertia of the section about the neutral axis $=I$

$$=\frac{\pi d^4}{64}$$

$$y_{max}=\frac{d}{2}$$

$$\therefore \quad \text{Section modulus}=Z=\frac{I}{y_{max}}=\frac{\pi d^3}{32}$$

If f be the maximum stress offered by the section, the moment of resistance $=M=fZ$

$$=f\frac{\pi d^4}{32}$$

(*iv*) *Hollow circular section.* Fig. 211 shows a hollow circular section of external diameter D and internal diameter d.

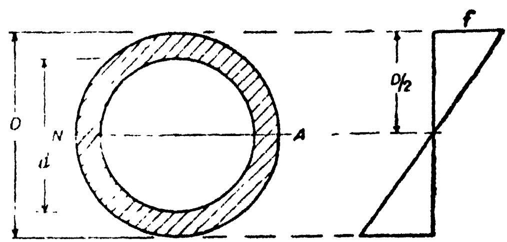


Fig. 211

Moment of inertia about the neutral axis

$$=I=\frac{\pi}{64}\left[D^4-d^4\right].$$

$$y_{max}=\frac{D}{2}$$

$$\therefore \quad \text{Section modulus}=Z=\frac{I}{y}=\frac{\pi(D^4-d^4)}{32D}$$

If f be the maximum stress offered by the section, the moment of resistance $=M=fZ$

$$M = f\frac{\pi(D^4 - d^4)}{32\,D}$$

Problem 132. *A cast iron test beam 2 cm. × 2 cm. in section and 1 metre long and supported at the ends fails when a central load of 64 kg. is applied. What uniformly distributed load will break a cantilever of the same material 5 cm. wide 10 cm. deep and 2 metres long ?*

Solution. Let us first consider the test beam.

Fig. 212

Maximum bending moment $= M = \frac{WL}{4}$

$$64 \times 1 \times 100 \textit{ kg. cm.}$$

$$= 1600 \textit{ kg. cm.}$$

Moment of resistance $1 \;\; f\,bd^2$

$$= \frac{1}{6} f\,2 \times 2^2 \textit{ kg. cm.}$$

$$= \frac{4}{3} f \textit{ kg. cm.}$$

Equating the moment of resistance to the max. bending moment

$$\frac{4}{3} f = 1600 \textit{ kg. cm.}$$

$$\therefore \quad f = \frac{3}{4} \times 1600 \textit{ kg./cm.}^2$$

$$= 1200 \textit{ kg./cm.}^2$$

Now let us consider the cantilever.

Let the distributed load on the cantilever be *w kg.* per *metre* run so as to break it.

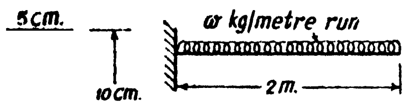


Fig. 213

Maximum bending moment

$$= M = \frac{wL^2}{2}$$

$$= \frac{w \times 2^2}{2} \times 100 \textit{ kg. cm.}$$

$$= 200\,w \textit{ kg. cm.}$$

M.R. of the section $= \frac{1}{6} f\,bd^2$

$$\frac{1}{6} \times 1200 \times 5 \times 10^2 \textit{ kg. cm.}$$

$$=100{,}000 \ kg. \ cm.$$

Equating the maximum bending moment to moment of resistance we have, 200 $w = 100{,}000$

$$\therefore \quad w = 500 \ kg. \ per \ metre \ run.$$

Problem 133. *The moment of inertia of a beam section 50 cm. deep is 69490 cm.⁴ Find the longest span over which a beam of this section, when simply supported, could carry a uniformly distributed load af 5000 kg per metre run. The maximum flange stress in the material is not to exceed 1100 kg./cm.²*

Solution. Section modulus of section

$$=Z=\frac{I}{y_{max}}=\frac{69490}{25} \ cm.^3$$

$$\therefore \quad Z=2780 \ cm.^3$$

Let the maximum span be *l metre*

$\therefore$ Max. bending moment

$$=M=\frac{wl^2}{8}$$

$$=\frac{5000\times l^2}{8}\times 100 \ kg. \ cm.$$

Moment of resistance of the section corresponding to the max. bending stress of 1100 *kg.* per *cm.²*

$$=fZ=1100\times 2780 \ kg. \ cm.$$

Equating the max. bending moment to the moment of resistance, we get

$$5000 \ \frac{l^2}{8} \ 100=1100\times 2780$$

$$\therefore \quad l^2=\frac{1100\times 2780\times 8}{5000\times 100}=48{\cdot}928$$

$$\therefore \quad l=6{\cdot}99 \ metres$$

say 7 *metres*

Problem 134. *A rolled steel joist of I section has the following dimensions :*

flange : 250 mm. wide and 24 mm. thick.

web : 12 mm. thick

overall depth : 600 mm.

If this beam carries a uniformly distributed load of 5 tonnes per metre run on a span of 8 metres, calculate the maximum stress produced due to bending.

Solution. Moment of inertia about the neutral axis,

$$=\frac{25\times 60^3}{12}-\frac{23{\cdot}8\times 55{\cdot}2^3}{12} \ cm.^4$$

$$=116410 \ cm.^4$$

Maximum B.M.

$$=M=\frac{wl^2}{8}$$

$$=\frac{5\times 8^2}{8}\times 100 \text{ tonne cm.}$$

$$=4000 \text{ tonne cm.}$$

$$\frac{M}{I}=\frac{f}{y}$$

$$\therefore \quad f=\frac{M}{I}y$$

$$\therefore f_{max}=\frac{M}{I}y_{max}$$

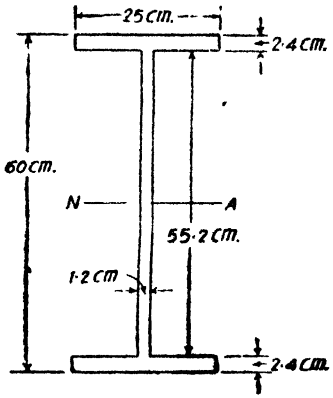


Fig. 214

$$=\frac{4000}{116410}\times 30 \text{ tonne/cm.}^2$$

$$=1{\cdot}03 \text{ tonne/cm.}^2$$

Problem 135. *A timber beam is freely supported on supports 6 metres apart. It carries a uniformly distributed load of 1200 kg. per metre run and a concentrated load of 900 kg. at 2 5 metres from the left support. If the stress in timber is not to exceed 80 kg./cm.² design a suitable section making the depth twice the width.*

Solution.

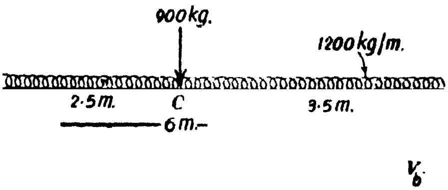


Fig. 215

Fig. 215 shows the beam carrying the loading mentioned in the problem. Let V_a and V_b be the reactions at the left and right supports.

Taking moments about the left support *A*, we have

$$V_b\times 6=1200\times 6\times 3+900\times 2{\cdot}5$$

$$\therefore \quad V_b=3975 \text{ kg.}$$

$$\therefore \quad V_a=1200\times 6+900-3975=4125 \text{ kg.}$$

Let the S.F. be zero at *x metres* from *B*.

Equating the S.F. at this section to zero, we have

$$1200\,x-3975=0$$

$$\therefore \quad x=\frac{3975}{1200}=3{\cdot}3125 \text{ metres.}$$

$\therefore$ Max. B.M. will occur at 3·3125 *metres* from B

$$\therefore \text{ Max. B.M.} = M = 3975 \times 3{\cdot}3125 - \frac{1200 \times (3{\cdot}3125)^2}{2} \ kg.m.$$

$$= 6583{\cdot}6 \ kgm.$$
$$= 658360 \ kg.cm.$$

But the M.R. of the section

$$= \frac{1}{6} fbd^2$$

Since $f = 80 \ kg./cm.^2$ and

$b = \frac{d}{2}$, we have,

$$\frac{1}{6} \times 80 \times \frac{d}{2} . d^2 = 658360$$

$$\therefore \quad d^3 = \frac{658360 \times 12}{80}$$

$$\therefore \quad d = 46{\cdot}2 \ cm. \text{ say } 46 \ cm.$$

$$b = \frac{d}{2} = 23 \ cm.$$

$\therefore$ The beam section is *23 cm × 46 cm.*

Problem 136. *A timber beam is 16 cm. wide and 30 cm. deep and is simply supported on a span of 5 metres. It carries a uniformly distributed load of 300 kg. per metre run over the whole span and three equal concentrated loads W kg. each placed at mid span and quarter span points. If the stress in timber is not to exceed 80 kg. per cm.² find maximum value of W.*

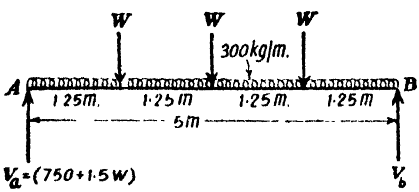


Fig. 216

Solution. Fig. 216 shows the beam carrying the loading mentioned in the problem.

Each vertical reaction $= \frac{1}{2}$ total load on the span

$$= \frac{300 \times 5 + 3W}{2} = (750 + 1{\cdot}5W) \ kg.$$

Max. B.M. will occur at mid span.

$$\text{Max. B.M.} = M = (750 + 1{\cdot}5\ W) \times \frac{5}{2} - 300 \times \frac{2{\cdot}5^2}{2} - W \times 1{\cdot}25$$

$$= 1875 + 3{\cdot}75\ W - 937{\cdot}5 - 1{\cdot}25W$$
$$= (937{\cdot}5 + 2{\cdot}5W) \ kg.m.$$
$$= (937{\cdot}5 + 2{\cdot}5W)100 \ kg.cm.$$

M.R. of the section $= \frac{1}{6} fbd^2$

$$=\frac{1}{6}\times 80\times 16\times 30^2 \; kg.cm.$$

Equating the maximum B.M. to the moment of resistance we have,

$$(937{\cdot}5+2{\cdot}5W)100=\frac{1}{6}\times 80\times 16\times 30^2$$

$$\therefore \quad 937{\cdot}5+2{\cdot}5W=1920$$

$$\therefore \quad W=393 \; kg.$$

Problem 137. *A water main of 120 cms. internal diameter and 12 mm. thick is running full. If the bending stress is not to exceed 560 kg. per cm.², find the greatest span on which the pipe may be freely supported. Steel and water weigh 7680 kg./m.³ and 1000 kg./m³ respectively.*

Solution. Consider 1 *metre* run of the main.

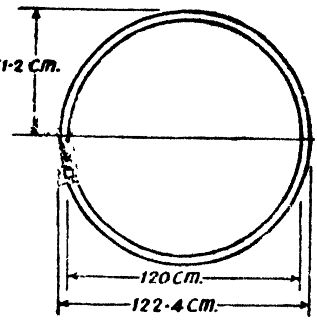


Fig. 217

Area of the pipe section

$$A_p=\frac{\pi}{4}(D^2-d^2)$$

$$D=1{\cdot}224 \; m.$$

$$d=1{\cdot}200 \; m.$$

$$A_p=\frac{\pi}{4}[1{\cdot}224^2-1{\cdot}2^2] \; metre^2$$

$$=0{\cdot}0456 \; metre^2$$

Area of the water section

$$=A_w=\frac{\pi}{4}d^2$$

$$=\frac{\pi}{4}\times 1{\cdot}2^2 \; metre^2$$

$$=1{\cdot}131 \; metre^2$$

Weight of the pipe for one *metre* run

$$=0{\cdot}0456\times 1\times 7680 \; kg.$$

$$=350{\cdot}2 \; kg.$$

Weight of water for one *metre* run of the pipe

$$=1{\cdot}131\times 1\times 1000 \; kg.$$

$$=1131 \; kg.$$

$\therefore$ Total load on the pipe for one *metre* run

$$=350{\cdot}2+1131 \; kg.$$

$$=1481{\cdot}2 \; kg. \; per \; metre.$$

Let the maximum span be *l metres.*

$\therefore$ Maximum bending moment

$$=M=\frac{1481{\cdot}2\times l^2}{8} \; kg \; m.$$

$$=\frac{1481 \cdot 2 \times l^2 \times 100}{8} \ kg.cm.$$

$$=18515 \ l^2 \ kg.cm.$$

Moment of inertia of the pipe section about the neutral axis

$$=I=\frac{\pi}{64}[D^4-d^4]$$

$$=\frac{\pi}{64}[122 \cdot 4^4-120^4] \ cm^4.$$

$$=836600 \ cm^4.$$

But $$\frac{M}{I}=\frac{f}{y}$$

$$\therefore \quad \frac{18515 \ l^2}{836600}=\frac{560}{61 \cdot 2}$$

$$\therefore \quad l^2=\frac{560 \times 836600}{61 \cdot 2 \times 18515}$$

$$\therefore \quad l=20 \cdot 33 \ metres$$

Problem 138. *Compare the section moduli of two beams of the same weight if the first beam is a solid circular beam of diameter d and the second is a circular tube of outer diameter D_1 and inner diameter D_2.*

Solution. Cross-sectional area of each beam

$$=A=\frac{\pi d^2}{4}=\frac{\pi}{4}(D_1^2-D_2^2) \ldots\ldots(i)$$

Section modulus of the solid section

$$=Z_s=\frac{\pi d^3}{32}=\frac{\pi d^2}{4} \cdot \frac{d}{8}$$

$$\therefore \quad Z_s=A\frac{d}{8} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (ii)$$

Section modulus of the tubular section

$$=Z_t=\frac{\frac{\pi}{64}(D_1^4-D_2^4)}{\frac{D_1}{2}}$$

$$=\frac{\pi}{32} \ \frac{D_1^4-D_2^4}{D_1}$$

$$=\frac{\pi}{4}(D_1^2-D_2^2)\frac{(D_1^2+D_2^2)}{8D_1}$$

$$=\frac{A}{8}\left(\frac{D_1^2+D_2^2}{D_1}\right)$$

$$\therefore \quad Z_t=\frac{AD_1}{8}\left(1+\frac{D_2^2}{D_1^2}\right)$$

But from equation (*i*) $D_2^2=D_1^2-\frac{4A}{\pi}$

$$\therefore \quad Z_t = \frac{AD_1}{8}\left[1 + \frac{D_1^2 - \frac{4A}{\pi}}{D_1^2}\right]$$

$$\therefore \quad Z_t = \frac{AD_1}{8}\left[2 - \frac{4A}{\pi D_1^2}\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots (iii)$$

$\therefore$ Ratio of the section moduli of the two beam sections

$$= \frac{Z_t}{Z_s} = \frac{AD_1}{8}\left[2 - \frac{4A}{\pi D_1^2}\right] \times \frac{8}{Ad}$$

$$\therefore \quad \frac{Z_t}{Z_s} = \frac{D_1}{d}\left[2 - \frac{4A}{\pi D_1^2}\right]$$

n important observation :

When the tube is very thick, D_1 approaches d so that Z_t will proach Z_s. When the tube is very thin, D_1 is very large compared ith d and hence the ratio $\frac{Z_t}{Z_s}$ will approach the value $2\frac{D_1}{d}$.

Problem. 139. *Find the width x of the flange of a cast iron beam ving the section shown in Fig. 218 so that the maximum compressive ress is three times the maximum tensile stress the member being in bending subjected to sagging moment. The depth h of the beam 10 cm., the thickness t of the web and flange is 2·5 cm.*

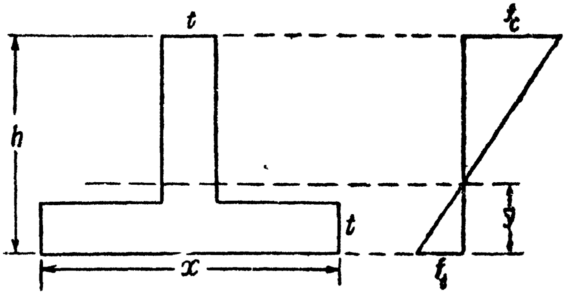


Fig. 218

Solution. Let the distance of the centroidal neutral axis from bottom edge be y. Let f_c and f_t be the maximum compressive nd maximum tensile stress

$$\therefore \quad \frac{f_c}{f_t} = \frac{h-y}{y} = 3 \qquad\qquad h - y = 3y$$

$$\therefore \quad y = \frac{h}{4}$$

But from the geometry of the section,

$$y = \frac{th\,\frac{h}{2} + (x-t)t\,.\,\frac{t}{2}}{th + (x-t)t} \qquad \frac{h}{4}$$

$$\therefore \qquad x=t+\frac{h^2}{h-2t}$$

For the numerical, $t=2.5$ *cm*, and $h=10$ *cm*.

$$\therefore \qquad x=2.5+\frac{10^2}{10-2\times2.5}\ cm.$$

$$\therefore \qquad x=22.5\ cm.$$

Problem 140. *A cast iron bracket subject to bending has a cross-section of I-form with unequal flanges. The total depth of the section is 28 cm. and the metal is 4 cm. thick throughout. The top flange is 20 cm. wide and the bottom flange is 12 cm. wide. Find the position of the neutral axis and the moment of inertia of the section about the neutral axis and determine the maximum bending moment that should be imposed on this section if the tensile stress in the top flange is not to exceed 200 kg./cm.² What is then the value of the compressive stress in the bottom flange ?*

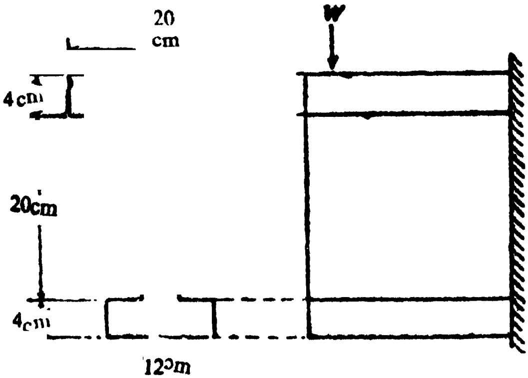


Fig. 219

Solution. Fig. 219 shows the section of the bracket. The section may be conveniently split into three rectangular components.

Component	*Area a* (cm^2)	*Centroidal distance from top edge y* (cm)	*ay* (cm^3)	ay^2 (cm^4)	I_{self} (cm^4)
Top flange	80	2	160	320	$\frac{20\times4^3}{12}=\frac{320}{3}$
Web	80	14	1120	15680	$\frac{4\times20^3}{12}=\frac{8000}{3}$
Bottom flange	48	26	1248	32448	$\frac{12\times4^3}{12}=64$
Total	208		2'28	48448	2837'33

[T]he areas of the individual components, their centroidal distances [fr]om the top edge and their moments about the top edge and their [m]oments of inertia about their centroidal axes are tabulated above.

∴ Distance of the neutral axis from the upper edge

$$=\bar{y}=\frac{\Sigma ay}{\Sigma a}=\frac{2528}{208}=12{\cdot}15\ cm.$$

Moment of inertia about the upper edge

$$=I_{uu}=\Sigma I_{self}+\Sigma ay^2$$

$$=2837{\cdot}33+48448=51285{\cdot}33\ cm.^4$$

But $$I_{uu}=I_{xx}+(\Sigma a)\bar{y}^2$$

∴ $$51285{\cdot}33=I_{xx}+208\times 12{\cdot}15^2$$

∴ $$I_{xx}=51285{\cdot}33-30715{\cdot}20=20570{\cdot}13\ cm.^4$$

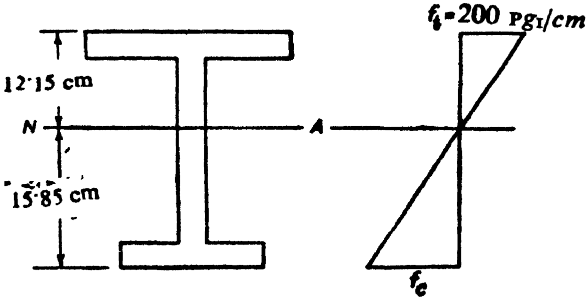


Fig. 220

Let the maximum bending moment be M kg. cm.

$$\frac{M}{I}=\frac{f_t}{y}$$

$$M=\frac{f_t}{y_t}\times I$$

$$=\frac{200}{12{\cdot}15}\times 20570{\cdot}13\ kg.\ cm.$$

$$=338600\ kg.\ cm.$$

Let the max. compressive stress be f_c kg./cm.²

∴ $$f_c=\frac{15{\cdot}85}{12{\cdot}15}\times 200\ kg./cm.^2$$

∴ $$f_c=260{\cdot}9\ kg./cm.^2$$

Problem 141. *A cast iron beam section is of I section with a [to]p flange 8 cms. × 2 cms. thick, bottom flange 16 cms. × 4 cms. thick [a]nd the web 20 cms. deep and 2 cms. thick. The beam is freely [su]pported on a span of 5 metres. If the tensile stress is not to exceed [2]00 kg./cm²., find the safe uniformly distributed load which the beam [ca]n carry. Find also the maximum compressive stress.*

Solution. Fig. 221 shows the cross-section of the beam. Let us first determine the neutral axis and the moment of inertia about the neutral axis.

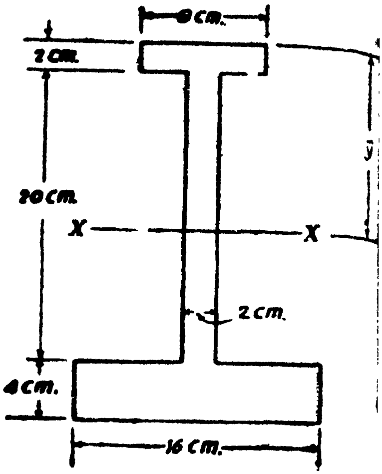


Fig. 221

The section may be split up into three components—top flange : 8 *cm.* × 2 *cm.* web : 2 *cm.* × 20 *cm.*, bottom flange : 16 *cm.* × 4 *cm.* The areas of the individual components, their centroidal distances from the top edge, and their moments of inertia about their own centroidal axes parallel to the neutral axis are shown in the following table.

Component	*Area a* (cm^2)	*Centroidal distance y from top edge* (cm)	*ay* (cm^3)	ay^2 (cm^4)	I_{self} (cm^4)
Top flange	16	1	16	16	$\frac{8\times2^3}{12}=5{\cdot}33$
Web	40	12	480	5760	$\frac{2\times20^3}{12}=1333{\cdot}33$
Bottom flange	64	24	1536	36864	$\frac{16\times4^3}{12}=85{\cdot}33$
Total	120		2032	42640	1423·99 say 1424

∴ Distance of the neutral axis *XX* from the top edge

$$=\bar{y}=\frac{\Sigma ay}{\Sigma a}$$

$$=\frac{2032}{120}=16{\cdot}93\ cm.$$

Moment of inertia about the upper edge of the section

$$=I_{uu}=\Sigma I_{self}+\Sigma ay^2$$

$$=1424+42640$$

$$=44064\ cm.^4$$

But $\qquad I_{uu}=I_{xx}+(\Sigma a)\bar{y}^2$

$$44064 = I_{xx} + 120(16{\cdot}93)^2$$

$$\therefore \quad I_{xx} = 9669 \ cm^4.$$

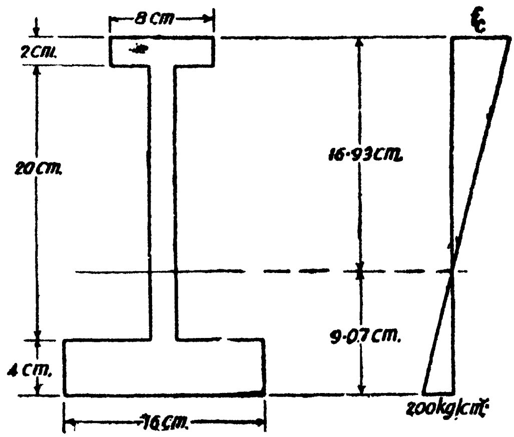


Fig. 222

Fig. 222 shows the stress diagram for the beam section. The maximum tensile stress $= 200 \ kg./cm.^2$ Let the maximum compressive stress be f_c $kg./cm^2$.

$$\therefore \quad f_c = \frac{16{\cdot}93}{9{\cdot}07} \times 200 \ kg./cm.^2$$

$$\therefore \quad f_c = 373{\cdot}4 \ kg./cm.^2$$

Let the uniformly distributed load on the beam be w $kg./metre$.

$\therefore$ Max. bending moment

$$= \frac{wl^2}{8}$$

$$= \frac{w \times 5^2}{8} \times 100 \ kg. \ cm.$$

$$= \frac{2500}{8} w \ kg. \ cm.$$

$$\frac{M}{I} = \frac{f}{y}$$

$$\therefore \quad \frac{2500 \ w}{8 \times 9669} = \frac{200}{9{\cdot}07}$$

$$\therefore \quad w = \frac{200 \times 8 \times 9669}{9{\cdot}07 \times 2500} \ kg/m.$$

$$\therefore \quad w = 682{\cdot}3 \ kg./m.$$

Problem 142. *Find the width and depth of the strongest beam that can be cut out of a cylindrical log of wood whose diameter is 50 cms.*

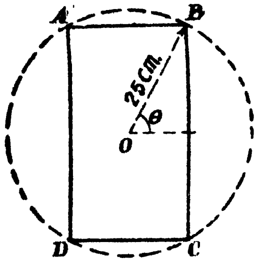


Fig. 223

Solution. Let $ABCD$ be the rectangular section cut out of the cylindrical log of wood. Let OB be at θ with the horizontal diameter

$\therefore$ $BC=d=2\times 25\sin\theta=50\sin\theta$ and $AB=b=2\times 25\cos\theta=50\cos\theta$

Section modulus of the rectangular section

$$=Z=\frac{bd^2}{6}=\frac{50\cos\theta\,.\,(50\sin\theta)^2}{6}$$

$$\therefore\quad Z=\frac{50^3}{6}\,.\,\sin^2\theta\cos\theta$$

For the beam to be strongest the section modulus must be a maximum.

Hence for Z to be a maximum

$$\frac{dZ}{d\theta}=0$$

$$\therefore\quad \frac{dZ}{d\theta}=\frac{50^3}{6}\left(-\sin^3\theta+\cos\theta\times 2\sin\theta\cos\theta\right)=0$$

$\therefore$ $\sin\theta(2\cos^2\theta-\sin^2\theta)=0$

$$\therefore\quad 2\cos^2\theta=\sin^2\theta$$

$$\therefore\quad \tan^2\theta=2$$

$$\tan\theta=\sqrt{2}$$

$$\therefore\quad \sin\theta=\frac{\sqrt{2}}{\sqrt{3}}\text{ and }\cos\theta=\frac{1}{\sqrt{3}}$$

$\therefore$ Width of the beam $=b=50\cos\theta$

$$=50\times\frac{1}{\sqrt{3}}=28{\cdot}87\text{ cms}$$

and depth of the beam $=d=50\sin\theta$

$$=50\times\frac{\sqrt{2}}{\sqrt{3}}=40{\cdot}83\text{ cms.}$$

Problem 143 *Three beams have the same length, same allowable bending stress and are subjected to the same maximum bending moment. The cross-sections of the beams are a circle, a square and a rectangle with depth twice the width Find the ratio of the weights of the circular and rectangular beams with respect to the square beam.*

(A.M.I.E., November 1963)

Solution. Fig. 224 shows the three sections.

Let the circular section be of diameter d
Let the square section be of side x
Let the rectangular section be of width b and depth $2b$.

For the conditions mentioned in the problem the three sections must have the same section modulus.

Section modulus of the circular section

$$=\frac{\frac{\pi d^4}{64}}{\frac{d}{2}}=\frac{\pi d^3}{32}$$

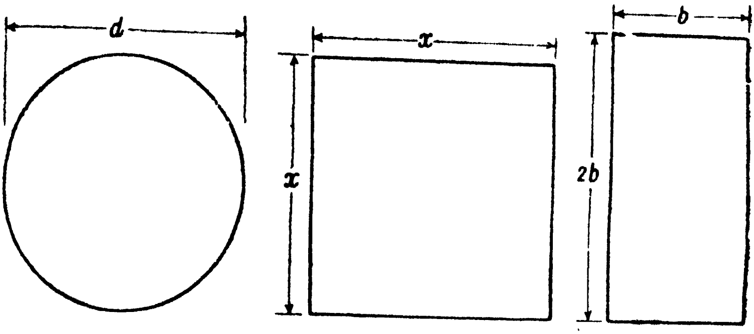


Fig. 224

Section modulus of the square section

$$=\frac{x^3}{6}$$

Section modulus of the rectangular section

$$=\frac{b\,(2b)^2}{6}=\frac{2}{3}\,b^3$$

$$\therefore \qquad \frac{\pi d^3}{32}=\frac{x^3}{6}=\frac{2}{3}\,b^3$$

$$\therefore \qquad d=1{\cdot}193\ x \text{ and } b=0{\cdot}6299\ x$$

$$\therefore \quad \frac{\text{wt. of circular beam}}{\text{wt. of square beam}}=\frac{\text{area of circular section}}{\text{area of square section}}=\frac{\pi d^2}{4x^2}$$

$$=\frac{\pi}{4}\,(1{\cdot}193)^2=1{\cdot}118$$

$$\frac{\text{wt. of rectangular beam}}{\text{wt. of square beam}}=\frac{2b^2}{x^2}=2\times(0{\cdot}6299)^2=0{\cdot}7936$$

Problem 144. *A horizontal beam of the section shown in Fig. 225 is 3 metres long and is simply supported at the ends. Find the maximum uniformly distributed load it can carry if the compressive and tensile stresses must not exceed 560 kg./cm.² and 300 kg./cm.² respectively. Draw a diagram showing the variation of stress over the mid span section of the beam.*

Solution. Depth of neutral axis from the top edge

$$=\frac{15\times 12\times 6-10\times 9{\cdot}5\times 4{\cdot}75}{15\times 12-10\times 9{\cdot}5}\ cm.$$

$$=7{\cdot}397\ cm.$$

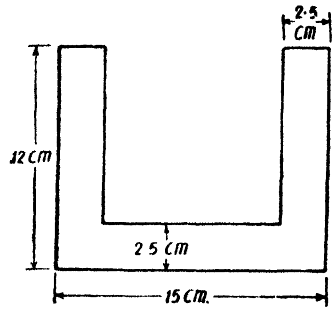


Fig. 225

∴ Moment of inertia of the section about the axis through the top edge parallel to the neutral axis

$$=I_o=\frac{15\times12^3}{3}-\frac{10\times9{\cdot}5^3}{3}\ cm.^4$$

$$=5782{\cdot}08\ cm.^4$$

Let the moment of inertia about the neutral axis be I

∴ By the parallel axes theorem, we have

$$I_o=I+A\bar{y}^2$$

∴ $I=5782{\cdot}08-85\times7{\cdot}397^2\ cm.^4$

∴ $I=5782{\cdot}08-4650{\cdot}86\ cm^4$

$I=1131{\cdot}22\ cm.^4$

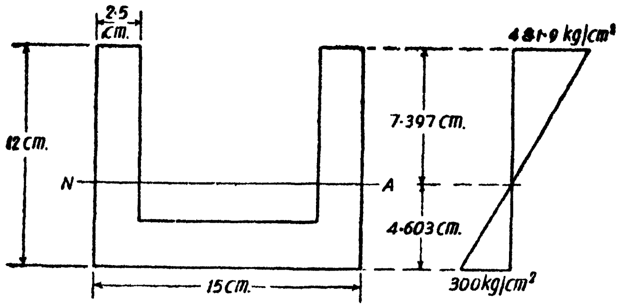


Fig. 226

Suppose the maximum stress in compression is 560 $kg./cm.^2$ the corresponding maximum stress in tension.

$$=\frac{4{\cdot}603}{7{\cdot}397}\times560\ kg./cm.^2$$

$$=348{\cdot}5\ kg./cm.^2$$

But the permissible tensile stress is only 300 $kg./cm.^2$ Hence let the maximum stress in tension be allowed to reach 300 $kg./cm.^2$ Corresponding maximum compressive stress

$$=\frac{7{\cdot}397}{4{\cdot}603}\times300\ kg./cm.^2$$

$$=481{\cdot}9\ kg./cm.^2$$

Fig. 226 shows the variation of the stress on the beam section.

Let the uniformly distributed load on the span be w *kg.* per *metre* run.

$$\therefore\ \text{Max. B.M.}=M=\frac{wl^2}{8}$$

$$=\frac{w\times 3^2}{8}\times 100\ kg.\ cm.$$

$$=\frac{900}{8}w\ kg.\ cm.$$

$$\frac{M}{I}=\frac{f}{y}$$

$$\therefore\quad M=\frac{f}{y}\,.\,I$$

$$\therefore\quad \frac{900}{8}w=\frac{300}{4{\cdot}603}\times 1131{\cdot}22$$

$$\therefore\quad w=\frac{8}{900}\times\frac{300\times 1131{\cdot}22}{4{\cdot}603}\ kg./metre$$

$$=655{\cdot}2\ kg./metre$$

Problem 145. *A cast iron beam has a section as shown in Fig. 227. Find the position of the neutral axis XX and the moment of inertia about the neutral axis.*

When subjected to a bending moment the tensile stress at the bottom edge is 250 kg./cm.², find

(*a*) *the value of the bending moment*

(*b*) *the stress at the top edge.*

Solution The given section may be split up into

(*i*) a rectangular part 16 *cm.* × 2 *cm.*

(*ii*) Two rectangular parts each 2 *cm.* × 15 *cm.*

(*iii*) Two rectangular parts each 9 *cm.* × 4 *cm.*

The areas of the various parts, their centroidal distances from the upper edge U-U and the moments of inertia of the individual parts about their centroidal axes parallel to the top edge U-U are shown in the table on page 252.

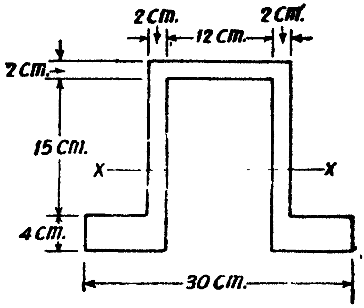


Fig. 227

Centroidal distance from the top edge $=\bar{y}$

$$=\frac{\Sigma ay}{\Sigma a}=\frac{1970}{164}=12{\cdot}01\ cm.$$

Moment of inertia of the section about the upper edge

$$I_{uu}=\Sigma I_{self}+\Sigma ay^2$$

$$=1231{\cdot}67+31439=32670{\cdot}67\ cm.^4$$

But $\quad I_{uu}=I_{xx}+(\Sigma a)\bar{y}^2$

$$\therefore\quad 32670{\cdot}67=I_{xx}+164\times(12{\cdot}01)^2$$

$$\therefore\quad I_{xx}=9015{\cdot}29\ cm.^4$$

say $9015\ cm.^4$

Component	Area a (cm^2)	Centroidal distance from top edge y (cm)	ay (cm^3)	ay^2 cm^4	I_{self} cm^4
(i) 16×2	32	1	32	32	$\frac{16\times2^3}{12}=10·67$
(ii) 2×(15×2)	60	9·5	570	5415	$2\times\frac{2\times15^3}{12}=1125$
(iii) 2×9×4	72	19	1368	25992	$\frac{2\times9\times4^3}{12}=96$
Total	164		1970	31439	1231·67

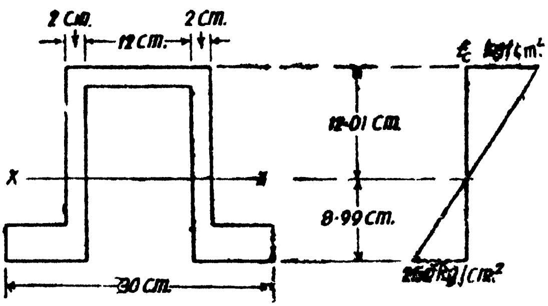


Fig 228

Let the bending moment at the section be M *kg. cm.*

$$\frac{M}{I}=\frac{f}{y}$$

$$M=\frac{f}{y}\cdot I$$

$$=\frac{250}{8·99}\times 9015 \ kg.\ cm.$$

$$=250600 \ kg \ cm.$$

Let the stress at the top edge be f_c $kg/cm.^2$

$\therefore$ $$f_c=\frac{12·01}{8·99}\times 250 \ kg/cm^2$$

$\therefore$ $$f_c=333·9 \ kg/cm^2$$

Problem 146. *Fig. 229 shows the section of an inverted steel channel used as a beam. The beam is simply supported over a span of 3 metres and carries two equal concentrated loads at points distant 30 cms from each support.*

Find the value of each of these loads if the maximum tensile *stress is not to exceed 940 kg. per cm.² Find also the corresponding maximum compressive stress.*

Solution. Let each point load be W *kg*

∴ Each vertical reaction $= W$ *kg*

∴ Max. B M.

$= M = W \times 30$ *kg. cm.*

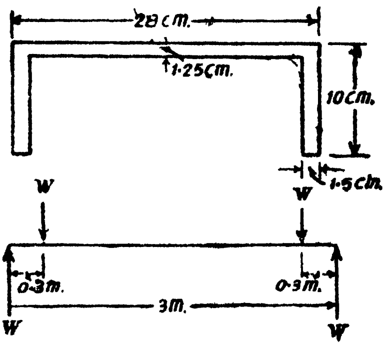


Fig. 229

Distance of the neutral axis from the top edge.

$$= y = \frac{28 \times 10 \times 5 - 25 \times 8{\cdot}75 \times 5{\cdot}625}{28 \times 10 - 25 \times 8{\cdot}75} \text{ cm.}$$

$$= \frac{169{\cdot}53}{61{\cdot}25} \text{ cm.}$$

$= 2{\cdot}767$ *cm.*

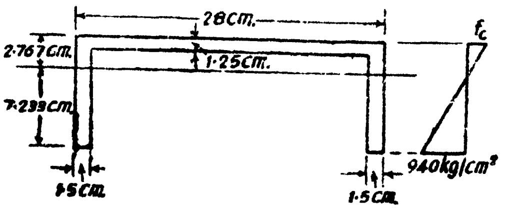


Fig. 230

∴ Distance of the neutral axis from the bottom edge.

$= 10 - 2{\cdot}767$ *cm.*

$= 7{\cdot}233$ *cm.*

When the maximum tensile stress is 940 *kg./cm.*² corresponding maximum compressive stress

$$= f_c = \frac{2{\cdot}767}{7{\cdot}233} \times 940 \text{ kg./cm.}^2$$

$$\therefore \quad f_c = 359{\cdot}6 \text{ kg./cm}^2$$

Moment of inertia about the neutral axis

$$= I = \frac{28 \times 2{\cdot}767^3}{3} - 25 \times \frac{(2{\cdot}767 - 1{\cdot}25)^3}{3} + \frac{2 \times 1{\cdot}5 \times 7{\cdot}233^3}{3} \text{ cm.}^4$$

$$= 197{\cdot}70 - 29{\cdot}09 + 378{\cdot}4 \text{ cm.}^4$$

$$= 547{\cdot}01 \text{ cm.}^4$$

$$\frac{M}{I} = \frac{f_t}{y_t}$$

$$\therefore \quad \frac{W \times 30}{547{\cdot}01} = \frac{940}{7{\cdot}233}$$

$$\therefore \quad W = \frac{940 \times 547{\cdot}01}{7{\cdot}233 \times 30} \text{ kg.}$$

$W = 2370$ *kg.*

Problem 147. *A simply supported beam 3 metres long has a section symmetrical about the y-y axis as shown in Fig. 231. The beam carries two loads of equal magnitude as shown in Fig. 232. If the maximum compressive and tensile stresses are not to exceed 1000 kg/cm.² and 1200 kg./cm.², determine the maximum value of*

W. Also state clearly the values of the maximum compressive and tensile stresses and where they occur.

Solution.

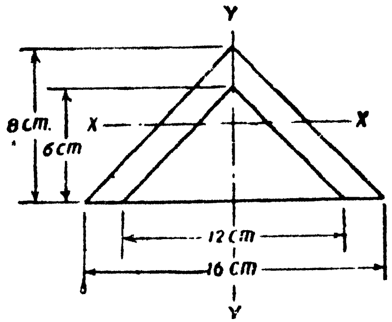


Fig. 231

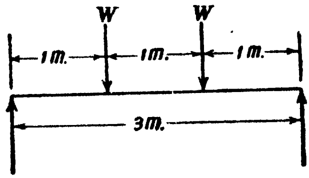


Fig. 232

Area of section $= A = 16 \times \frac{8}{2} - 12 \times \frac{6}{2} = 28\ cm.^2$

Height of neutral axis above the base

$$= \bar{y} = \frac{64 \times \frac{8}{3} - 36 \times \frac{6}{3}}{28}\ cm.$$

$= 3{\cdot}52\ cm.$

Moment of inertia about the base

$$= I_b = \frac{BH^3}{12} - \frac{bh^3}{12}$$

$$= \frac{1}{12}\left[16 \times 8^3 - 12 \times 6^3\right] cm.^4$$

$= 466{\cdot}67\ cm.^4$

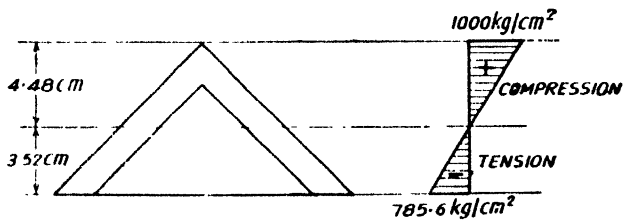


Fig. 233

But $I_b = I_{cx} + A\bar{y}^2$

$\therefore$ $466{\cdot}67 = I_{xx} + 28 \times 3{\cdot}52^2$

$\therefore$ $I_{xx} = 119{\cdot}75\ cm.^4$

Suppose the maximum tensile stress is allowed to reach 1200 *kg./cm.*² the corresponding maximum stress in compression

$$= \frac{4{\cdot}48}{3{\cdot}52} \times 1200 = 1528\ kg./cm.^2$$

But the permissible compressive stress is only 1000 $kg./cm.^2$ Hence the maximum tensile stress should not be allowed to reach 1200 $kg./cm.^2$

Let the maximum compressive stress be allowed to reach 1000 $kg./cm.^2$

∴ Corresponding maximum tensile stress

$$=\frac{3{\cdot}52}{4{\cdot}48}\times 1000 \ kg./cm.^2$$

$$=785{\cdot}6 \ kg./cm.^2$$

Now consider the beam carrying the two point loads,

∴ Maximum B.M. $=M=W\times 1 \ kg.\ m.$

$$=W\times 100 \ kg.\ cm.$$

But
$$\frac{M}{I}=\frac{f}{y}$$

∴
$$\frac{100W}{119{\cdot}75}=\frac{10000}{4{\cdot}48}$$

$$W=267{\cdot}4 \ kg.$$

Problem 148 *A cast iron beam has a section shown in Fig. 234. The beam is simply supported on a span of 1·25 metres and is used to carry a downward point load at midspan. Find the magnitude of the load if the maximum tensile stress on the beam section is 300 kg./cm². Determine also the maximum compressive stress.*

Solution. Distance of the centroidal axis XX from the bottom edge 1—1

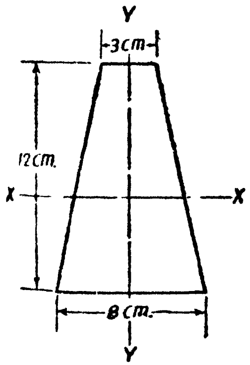


Fig. 234

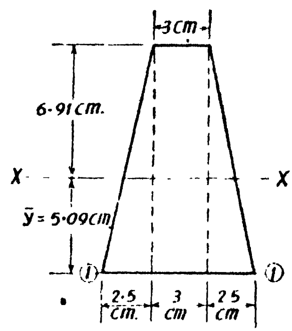


Fig. 235

$$=\bar{y}=\frac{8+2\times 3}{8+3}\times\frac{12}{3} \ cm.$$

$$=5{\cdot}09 \ cm.$$

M.I. about the bottom edge 1—1

$=I_{1-1}=$ M.I. of the two triangles + M.I. of the central rectangle

$$=2\times\frac{2.5\times12^3}{12}+\frac{3\times12^3}{3}\ cm^4$$

$$=2448\ cm^4.$$

But $I_{1-1}=I_{xx}+A\bar{y}^2$

$\therefore$ $2448=I_{xx}+\left(\frac{11}{2}\times12\right)\times5.09^2.$

$\therefore$ $I_{xx}=738.1\ cm^4.$

Let the central point load on the span be W kg.

$\therefore$ Max. B.M. $=M=\frac{WL}{4}$

$$=\frac{W\times100}{4}=25\ W\ kg.\ cm.$$

When the maximum stress in tension is 300 $kg./cm.^2$

$$\frac{M}{I}=\frac{f}{y}$$

$\therefore$ $\frac{25\ W}{738.1}=\frac{300}{5.09}$

$\therefore$ $W=\frac{300\times738.1}{25\times5.09}$

$\therefore$ $W=1741$ kg.

When the maximum stress in tension is 300 $kg./cm^2$.
the corresponding maximum compressive stress is given by

$$f_c=\frac{6.91}{5.09}\times300\ kg./cm^2.$$

$$f_c=407.2\ kg\ /cm^2.$$

Problem 149. *The cross-section of a cast iron machine-element used as a beam is shown in Fig. 236. The beam resists bending moments about the horizontal neutral axis The permissible stresses in tension and compression are 220 kg./cm.² and 880 kg./cm.² respectively.*

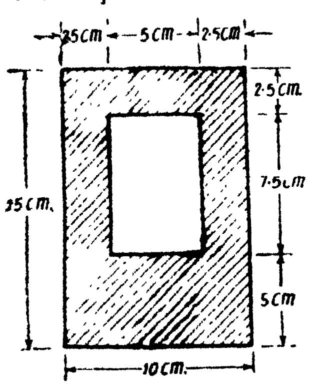


Fig 236.

Calculate the moment of resistance of the section about the horizontal neutral axis for both positive and negative bending moments. *(A.M.I.E., Nov., 1966)*

Solution. The computation of the position of the horizontal neutral axis and the moment of inertia about the horizontal neutral axis may be made as shown in the following table.

Component	Area a (cm²)	Centroidal distance from the bottom edge y (cm)	ay (cm³)	ay² (cm⁴)	I_{self} M.I. about respective centroidal axis cm⁴
Rectangle 10 cm × 15 cm	150	7·5	1125	8437·50	$\frac{10\times 15^3}{12}=2812\cdot50$
Deduct for rectangular hole 5 cm × 7·5 cm	37·5	8·75	328·125	2871·094	$\frac{5\times 7\cdot5^3}{12}=175\cdot781$
Net values	112·5		796·875	5566·406	2636·719

∴ Distance of the centroid of the net section from the bottom edge

$$=\bar{y}=\frac{\Sigma ay}{\Sigma a}$$

$$=\frac{796\cdot875}{112\cdot5}=7\cdot082\ cm.$$

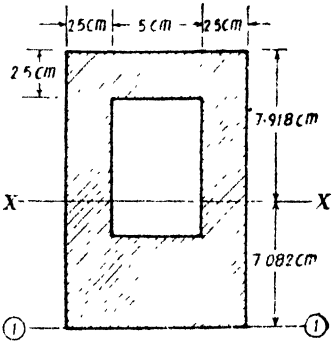


Fig. 237

M.I. about the botton edge

$$=I_{1-1}=\Sigma I_{self}+\Sigma ay^2$$

$$=2636\cdot719+5566\cdot406\ cm.^4$$

$$=8203\cdot125\ cm^4$$

But $I_{1-1}=I_{xx}+(\Sigma a)\bar{y}^2$

∴ $8203\cdot125=I_{xx}+112\cdot5(7\cdot082)^2$

∴ $I_{xx}=2563\cdot95\ cm^4.$

Analysis for positive and negative bending moments.

Moment of resistance for maximum positive (sagging) bending moment.

Let the maximum tensile stress be permitted to reach 220 $kg./cm^2$.

The corresponding maximum compressive stress

$$=\frac{7\cdot918}{7\cdot082}\times 220\ kg/cm.^2$$

$$=246\ kg./cm^2.$$

Obviously the compressive stress should not be permitted to exceed 246 $kg./cm.^2$ since for greater compressive stress the corresponding tensile stress will exceed the safe stress of 220 kg/cm^2. Hence corresponding to the safe critical condition the extreme stress in tension and compression will be 220 $kg./cm^2$. and 246 $kg./cm^2$. respectively.

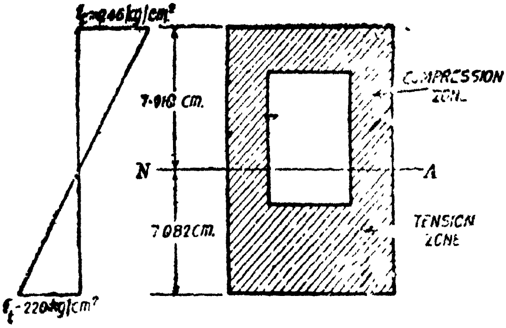


Fig. 238

Let M be the max. B.M. (*sagging moment*)

$$\frac{M}{I}=\frac{f}{y}$$

$$\therefore \quad M=\frac{f}{y}\cdot I$$

$$=\frac{220}{7{\cdot}082}\times 2563{\cdot}95\ kg.\ cm.$$

$$=79660\ kg.\ cm.\ (\textit{sagging moment})$$

Moment of resistance for maximum negative (hogging) bending moment

Let the maximum tensile stress be 220 $kg./cm^2$.

Corresponding maximum compressive stress

$$=\frac{7{\cdot}082}{7{\cdot}918}\times 220\ kg./cm.^2$$

$$=196{\cdot}7\ kg/cm.^2$$

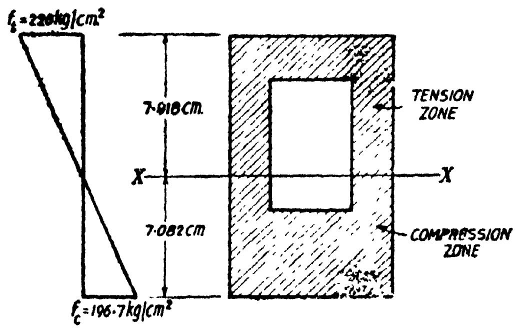


Fig. 239

Let M **be the maximum** negative bending moment (*hogging moment*).

$$\frac{M}{I}=\frac{f}{y}$$

$$M = \frac{f}{y} \cdot I$$

$$= \frac{220}{7.918} \times 2563.95 \text{ kg. cm}$$

$$= 71240 \text{ kg. cm. (hogging moment)}$$

Problem 150. *Two wooden planks 5 cm. × 15 cm. each are connected together to form a cross-section of a beam as shown in Fig. 240. If a bending moment of 340 kgm. is applied about the horizontal neutral axis find the stresses at the extreme fibres of the cross-section. Also calculate the total tensile force on the cross-section.*

(A.M.I.E., May 1967)

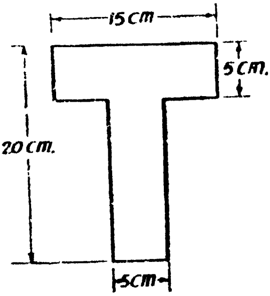


Fig. 240.

Solution Let us first locate the horizontal neutral axis and the moment of inertia about the neutral axis. The relevant calculations are shown below :

Component	*Area a (cm)²*	*Centroidal distance from top edge (cm) y*	*ay (cm)³*	*ay² (cm)⁴*	*I_{self} (cm)⁴*
Top flange	75	2·50	187·50	468·75	$\frac{15\times5^3}{12} = 156.25$
Web	75	12·5	937·5	11718·75	$\frac{5\times15^3}{12} = 1406.25$
Total	150		1125·00	12187·50	1562·50

Distance of the centroidal axis *XX* from the top edge

$$= \bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{1125}{150} = 7.5 \text{ cm.}$$

Moment of inertia about the top edge

$$= I_o = \Sigma I_{self} + \Sigma ay^2$$

$$= 1562.50 + 12187.50 \text{ cm.}^4$$

$$= 13750 \text{ cm}^4$$

But $\qquad I_o = I_{xx} + (\Sigma a)\bar{y}^2$

$\therefore \qquad 13750 = I_{xx} + 150 \times 7.5^2$

$\therefore \qquad I_{xx} = 5312.5 \text{ cm}^4.$

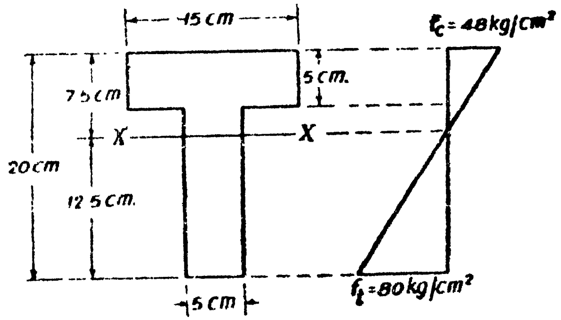


Fig. 241

Let the maximum tensile and compressive stresses at the extreme fibres be f_t and f_c respectively.

We have

$$\frac{M}{I}=\frac{f}{y}$$

$$\therefore \quad \frac{340\times100}{5312 \cdot 5}=\frac{f_c}{7 \cdot 5}=\frac{f_t}{12 \cdot 5}$$

$$\therefore \quad f_c=48 \text{ kg./cm.}^2$$

$$f_t=80 \text{ kg./cm}^2.$$

Total tensile force = average tensile stress × area of the tension zone

$$=\frac{80}{2}\times 12 \cdot 5\times 5 \text{ kg.}$$

$$=2500 \text{ kg.}$$

Problem 151. *The cross-section of a conveyor beam is as shown in Fig. 242. The beam is subjected to a bending moment in the plane YY. Determine the maximum permissible bending moment (a) for bottom flange in tension (b) for bottom flange in compression. Safe bending stress in tension and compression are 300 kg. per cm.² and 1500 kg./cm.² respectively*

Solution. Let us first determine the neutral axis and the moment of inertia of the section about the neutral axis

The given section may be split up into three rectangular components as shown in Fig. 242. The areas of the individual components, their centroidal distances from the bottom edge, and their moments of inertia about their centroidal axes parallel to the bottom edge are tabulated below :

∴ Distance of the neutral axis from the bottom edge

$$=\bar{y}=\frac{\Sigma ay}{\Sigma a}=\frac{2968 \cdot 75}{287 \cdot 5}=10 \cdot 33 \text{ cm.}$$

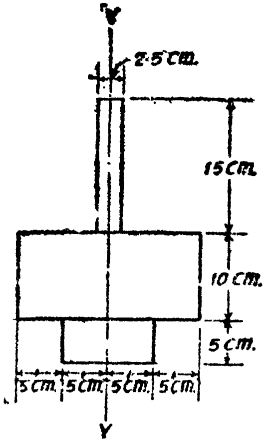


Fig. 242

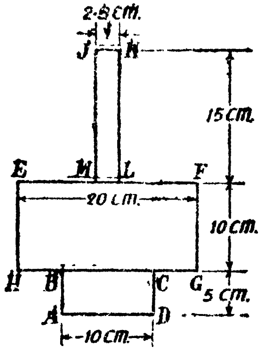


Fig. 243

Component	Area a cm^2	Centroidal distance from the bottom edge y cm.	ay cm^3	ay^2 cm^4	I_{self} cm^4
ABCD	50	2·5	125	312·50	$\frac{10\times5^3}{12}=\frac{1250}{12}=104{\cdot}17$
EFGH	200	10	2000	20,000	$\frac{20\times10^3}{12}=\frac{5000}{3}=1666{\cdot}67$
JKLM	37·5	22·5	843·75	18984·37	$\frac{2{\cdot}5\times15^3}{12}=703{\cdot}12$
Total	287·5		2968·75	39296·87	2473·96

∴ Moment of inertia of the section about the bottom edge

$= I_{ad} = \Sigma I_{self} + \Sigma ay^2$

$= 2473{\cdot}96 + 39296{\cdot}87$

$= 41770{\cdot}83\ cm^4.$

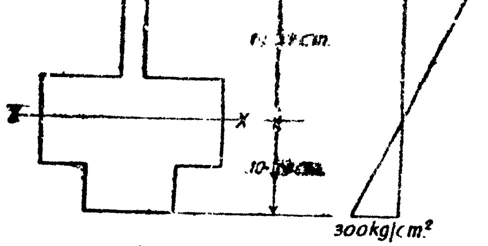
Let the moment of inertia about the neutral axis be I_{xx}

∴ $I_{ad} = I_{xx} + (\Sigma a)\bar{y}^2$

∴ $41770{\cdot}83 = I_{xx} + 287{\cdot}5 \times 10{\cdot}33^2$

∴ $I_{xx} = 11092{\cdot}07\ cm.^4$

say $11092\ cm^4$.



BOTTOM FLANGE IN TENSION

Fig. 244

(*a*) *When the bottom flange is in tension*

Let the max. stress in tension be 300 $kg./cm.^2$

Corresponding maximum compressive stress

$$=\frac{19{\cdot}67}{10{\cdot}33}\times 300 \; kg./cm.^2$$

$$=571 \; kg./cm^2.$$

Let M be the max. bending moment corresponding to this condition

$$\frac{M}{I}=\frac{f_t}{y_t}$$

$$\therefore \quad M=I\frac{f_t}{y_t}$$

$$=\frac{11092\times 300}{10{\cdot}33} \; kg. \; cm.$$

$$=322100 \; kg. \; cm.$$

(*b*) *When the bottom flange is in compression.* The max compressive stress should not be allowed to reach 1500 $kg./cm^2$. Because corresponding to this condition the maximum tensile stress will exceed the safe tensile stress. Hence let the maximum tensile stress be allowed to reach 3'0 $kg./cm^2$

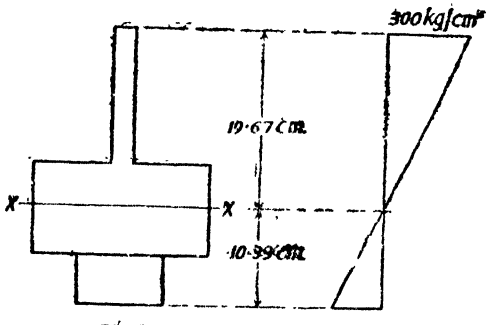


Fig. 245

Corresponding maximum compressive stress :

$$=\frac{10{\cdot}33}{19{\cdot}67}\times 300 \; kg./cm.^2$$

$$=157{\cdot}5 \; kg./cm.^2$$

Let M be the maximum bending moment corresponding to this condition.

$$\frac{M}{I}=\frac{f_t}{y_t}$$

$$M=I\frac{f_t}{y_t}$$

$$=\frac{11092\times 300}{19{\cdot}67} \; kg. \; cm.$$

$$=169100 \; kg. \; cm.$$

Problem 152. *Two planks of thickness t_1 and t_2 rest just one on another forming a beam as shown in Fig. 246 and support a uniformly distributed load of w kg. per metre run. Find the ratio of the maximum stress in the two beams.*

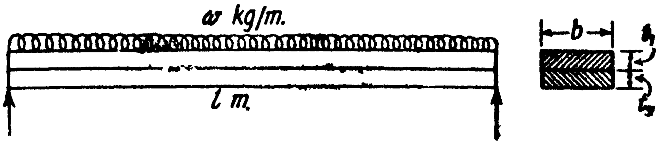


Fig. 246

Solution. Since one plank just rests on another, each plank will behave as a separate beam section, having its own neutral axis. Since the thicknesses of the planks are small we may further assume that the radii of curvature of the neutral layer is the same for each plank at any section.

Let maximum bending moment for the first plank $= M_1$

Maximum bending moment for the second plank $= M_2$

Radius of curvature of each plank at centre $= R$

Moment of inertia of the section for the first plank $= I_1$

Moment of inertia of the section for the second plank $= I_2$

$$\therefore \quad \frac{1}{R} = \frac{M_1}{EI_1} = \frac{M_2}{EI_2}$$

$$\frac{M_1}{M_2} = \frac{I_1}{I_2} = \frac{bt_1^3/12}{bt_2^3/12} = \frac{t_1^3}{t_2^3} \qquad \ldots(i)$$

Let f_1 and f_2 be the maximum stress in the two planks at the midspan section.

$$\therefore \quad M_1 = \frac{1}{6} f_1 \, bt_1^2$$

and $$M_2 = \frac{1}{6} f_2 \, bt_2^2$$

$$\therefore \quad \frac{M_1}{M_2} = \frac{f_1}{f_2} \cdot \frac{t_1^2}{t_2^2} \qquad \ldots(ii)$$

From equations (*i*) and (*ii*), we have,

$$\frac{t_1^3}{t_2^3} = \frac{f_1}{f_2} \cdot \frac{t_1^2}{t_2^2}$$

$$\therefore \quad \frac{f_1}{f_2} = \frac{t_1}{t_2}$$

Hence the ratio of the maximum stresses in the two planks equals the ratio of the thicknesses of the two planks.

Problem 153. *An I-section beam, shown in Fig. 247, is simply supported over a span of 10 metres. If the maximum permissible bending stress is 800 kg./cm.², what concentrated load can be carried at a distance of 3·50 metres from one support ?*

Solution Moment of inertia of the section about the neutral axis

$$=I=\frac{10\times(22{\cdot}5)^3}{12}-\frac{9{\cdot}25\times(20{\cdot}2)^3}{12}\ cm.^4$$

$$=3138{\cdot}66\ cm^4$$

Section modulus $=Z=\frac{3138{\cdot}66}{11{\cdot}25}=278{\cdot}99\ cm.^3$

Let the maximum concentrated load that can be placed on the beam at a distance of 3·50 *metres* from one support be W *kg.*

Maximum bending moment

$$=M=\frac{Wab}{L}$$

$$=\frac{W\times3{\cdot}50\times6{\cdot}50}{10}\times100\ kg./cm.$$

$$=227{\cdot}5\ W\ kg.\ cm.$$

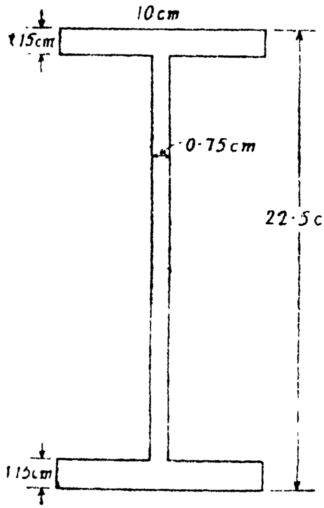


Fig. 247

But, $M=fZ$

$227{\cdot}5\ W=800\times278{\cdot}99$

$\therefore$ $W=981{\cdot}06\ kg.$

Problem 154. *A rolled steel joist of 4·5 metres span is freely supported at both ends. The flanges of the joist are 20 cms. × 1·54 cms. and the distance between the outside faces is 45 cms. The flanges are strengthened by two 30 cms. × 1·5 cm. plates, one riveted to each flange. The moment of inertia of the joist alone is 35060 cm.⁴ Find (i) the* *moment of inertia of the enlarged section (ii) the greatest central point load the beam will carry if the bending stress is not to exceed 1260 kg/cm.² (iii) the minimum length of the 30 cm. × 1·5 cm. plates.*

Solution. (*i*) Fig. 248 shows the section with the plates connected to the flanges.

Moment of inertia of the section

$=I=$ Moment of inertia of the I section

+ Moment of inertia of the plates

$=35060+$

$$2\left\{\frac{30\times 1{\cdot}5^3}{12}+30\times 1{\cdot}5\times 23{\cdot}25^2\right\} cm^4$$

$=35060+48657\ cm.^4$

$\therefore\quad I=83717\ cm.^4$

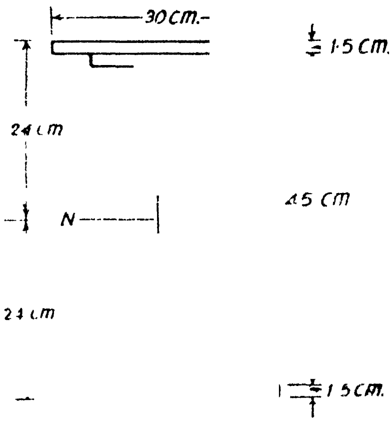


Fig. 248

$\therefore$ Section modulus of the section

$$=Z=\frac{I}{y}$$

$$=\frac{83717}{24}\ cm.^3$$

$$=3488\ cm.^3$$

(*ii*) Let the maximum point load at mid span be W kg.

$\therefore\quad$ Max. B.M. $=\dfrac{WL}{4}$

$$=\frac{W\times 4{\cdot}5\times 100}{4}\ kg.\ cm.$$

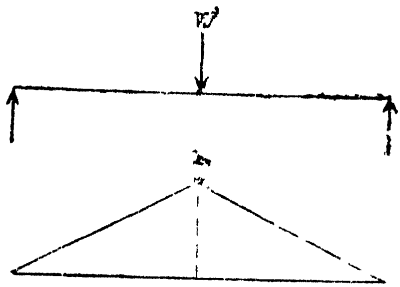


Fig. 249

But M.R. of the section $=fZ=1260\times 3488$ kg. cm.

$\therefore$ Equating the max B.M. to the moment of resistance, we have,

$$\frac{W\times 4{\cdot}5\times 100}{4}=1260\times 3488$$

$\therefore\quad W=39066\ kg.$

(*iii*) Moment of resistance of the I section alone

$$=fZ$$

$$=1260\times\frac{35060}{22{\cdot}5}\ kg.\ cm.$$

$$=1963000\ kg.\ cm.$$

Suppose the cover plates are absent for a distance of x *metres* from each support. Then at these points the bending moment must not exceed 1963000 *kg. cm.*

$\therefore$ B.M. at x *metres* from each support

$$=\frac{W}{2}.x$$

$$=\frac{39066}{2}x\times100=1963000$$

$$\therefore \quad x=\frac{1963000\times2}{39066\times100} \text{ metres}$$

$$\therefore \quad x=1.005 \text{ metres}$$

say 1 *metre.*

Hence leaving 1 *metre* from each support, for the middle $4.5-2=2.5$ *metres* the cover plates should be provided.

Problem 155. *A beam is of square section of side b. If the permissible bending stress is f, find the moment of resistance when the beam section is placed such that (i) Two sides are horizontal, (ii) one diagonal is vertical.*

Find also the ratio of the flexural strengths of the two positions.

Solution (*i*) Fig 250 shows the beam section when two sides are horizontal.

Moment of inertia of the section about the neutral axis

$$I=\frac{b(b)^3}{12}=\frac{b^4}{12}$$

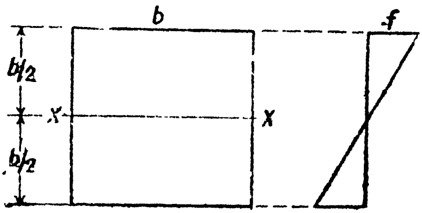


Fig. 250

$\therefore$ Section modulus $Z_1=\frac{I}{y_{max}}$

$$=\frac{\frac{b^4}{12}}{\frac{b}{2}}=\frac{b^3}{6}$$

$\therefore$ Corresponding to the maximum flexural stress f, the moment of resistance

$$=M_1=fZ_1=\frac{fb^3}{6}$$

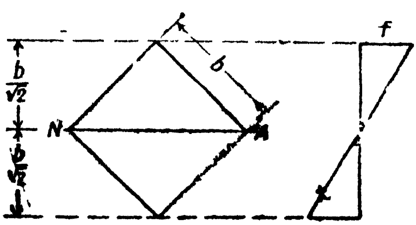


Fig. 251

(*ii*) Fig. 251 shows the beam section when one diagonal is vertical.

$\therefore$ Moment of inertia of the section about the neutral axis

$$=I=\frac{b^4}{12}$$

$\therefore$ Section modulus $=Z_2=\frac{I}{y_{max}}$

$$=\frac{\left(\frac{b^4}{12}\right)}{\left(\frac{b}{\sqrt{2}}\right)}=\frac{\sqrt{2}}{12}b^3$$

$\therefore$ Corresponding to the maximum flexural stress f, the moment of the resistance $=M_2=fZ_2=\frac{\sqrt{2}}{12}fb^3$

(*iii*) Ratio of the flexural strengths in the two positions

$$\frac{M_1}{M_2}=\frac{fZ_1}{fZ_2}=\frac{Z_1}{Z_2}=\frac{b^3}{6}\times\frac{12}{\sqrt{2}b^3}=\sqrt{2}=1\text{·}414.$$

§ 47. Various shapes of beam sections

We know that at any beam section the bending stress at any point is proportional to its distance from the neutral axis. Hence the maximum tensile and compressive stresses in a beam section are proportional to the distances of the most distant tensile and compressive fibres from the neutral axis. Suppose the tensile and compressive strength of the material are the same, it is justified to provide such a shape for the cross-section so that the centroidal axis is at the middle of the depth of the beam For this case the factors of safety for fibres subjected to maximum tensile and compressive stresses will be equal. This is the basis in the selection of sections which are symmetrical about the neutral axis, for materials like structural steel having the same yield stress in tension as well as compression. In case the section has to be unsymmetrical with respect to the neutral axis (for instance a rail section) the distribution of material should be such that the centroid lies at the middle of the height.

Some materials have a small tensile strength but a relatively high compressive strength. Cast iron and concrete *etc.*, are such materials In a case like this, the cross-section must be so chosen that it will not be symmetrical with respect to the neutral axis and also that the distances y_t and y_c of the most remote tension and compression fibres, from the neutral axis are in the same ratio of the tensile and compressive srengths of the material For instance if f_t and f_c are the permissible stresses in tension and compression respectively then the section must be shaped to satisfy the condition,

$$\frac{f_t}{f_c}=\frac{y_t}{y_c}$$

A very interesting case

For a given cross-section the maximum stress to which the section is subjected due to a given bending moment depends upon the section modulus of the section. There are some cases where an increase in the sectional area does not result in a decrease in stress. It may so happen that in some cases a slight increase in the area may result in a decrease in section modulus which results in an increase of stress to resist the same bending moment.

The following examples illustrate this case

Problem 156 *A horizontal beam subjected to pure bending is of square section with a diagonal vertical The beam carries bending moment in the vertical plane through the vertical diagonal. Show that by cutting off the top and bottom corners shown shaded in Fig. 252 the section modulus can be increased.*

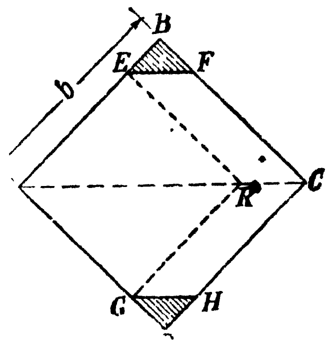


Fig 252

Solution For the square section, moment of inertia about the neutral axis $= I = \frac{b^4}{12}$

$\therefore$ Section modulus $= Z$

$$= \frac{b^4}{12} \div \frac{b}{\sqrt{2}} = \frac{\sqrt{2}}{12} b^3$$

Let the portions BEF and DGH be cut off.
Let $BE = \alpha b$ where α is a fraction.
Now the resulting cross-section consists of

(*i*) a square $AERG$ of side $b\,(1-\alpha)$
(*ii*) two parallelograms $EFCR$ and $GHCR$.

Moment of inertia of the new section

$$= I' = \frac{b^4(1-\alpha)^4}{12} + 2\frac{\alpha b\sqrt{2}}{3}\left[\frac{b(1-\alpha)}{\sqrt{2}}\right]^3$$

$$= (1+3\alpha)(1-\alpha)^3 \frac{b^4}{12}$$

$\therefore$ Section modulus of the new section

$$= Z' = \frac{I'}{\left(\frac{b(1-\alpha)}{\sqrt{2}}\right)}$$

$$Z' = \frac{\sqrt{2}}{12}(1+3\alpha)(1-\alpha)^2 b^3$$

$$= \frac{\sqrt{2}}{12} b^3 \cdot \left[1-\alpha-5\alpha^2+3\alpha^3\right]$$

For Z' to be a maximum,

$$\frac{dZ'}{d\alpha} = 0$$

$$\frac{dZ'}{d\alpha}=\frac{\sqrt{2}}{12}\cdot b^3\left[-1-10\alpha+9\alpha^2\right]=0$$

$$9\alpha^2-10\alpha-1=0$$

$$(9\alpha-1)(\alpha-1)=0$$

$$\therefore \qquad \alpha=\frac{1}{9}$$

Hence the maximum value of the section modulus is obtained when $\alpha=\frac{1}{9}$.

$$\therefore \qquad Z_{max}=\frac{\sqrt{2}}{12}\left(1+\frac{3}{9}\right)\left(1-\frac{1}{9}\right)^2 b^3$$

$$=\frac{\sqrt{2}}{12}\times\frac{256}{243}.b^3$$

But the section modulus of the square section is equal to

$$Z=\frac{\sqrt{2}}{12}b^3$$

$\therefore$ Increase in section modulus

$$=Z_{max}-Z$$

$$=\frac{\sqrt{2}}{12}b^3\left(\frac{256}{243}-1\right)$$

$$=\frac{\sqrt{2}}{12}\times\frac{13}{243}b^3$$

$\therefore$ Percentage increase in section modulus

$$=\frac{Z_{max}-Z}{Z}\times 100\%$$

$$=\frac{13}{243}\times 100\%$$

$$=5{\cdot}35\%$$

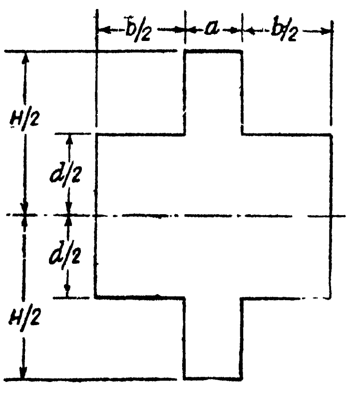


Fig. 253

Hence if the new section is adopted in place of the square section the maximum bending stress would be reduced by 5·35%.

Problem 157. *Find the condition at which any further decrease of the depth H of the beam section shown in Fig. 253 will result in an increase in section modulus.*

Solution. Moment of inertia of the section about in neutral axis

$$=I=\frac{bd^3}{12}+\frac{aH^3}{12}$$

∴ Section modulus of the section

$$=Z=\frac{I}{\frac{H}{2}}$$

$$\therefore \quad Z=\frac{bd^3}{6H}+\frac{aH^2}{6}$$

∴ Rate of change of Z with respect to H is given by

$$\frac{dZ}{dH}=-\frac{bd^3}{6H^2}+\frac{aH}{3}$$

The condition for Z to increase when H decreases is given by

$$\frac{bd^3}{6H^2}>\frac{aH}{3}$$

$$\therefore \quad \frac{b}{2a}>\frac{H^3}{d^3}$$

The above examples show how in some cases the section modulus increases with decrease in depth.

Similarly it can be shown, for the circular section (Fig. 254) it is possible to increase the section modulus by 0·7% by cutting off the segmental portions shaded. The depth of segment δ should be 0·011 times the diameter of the section.

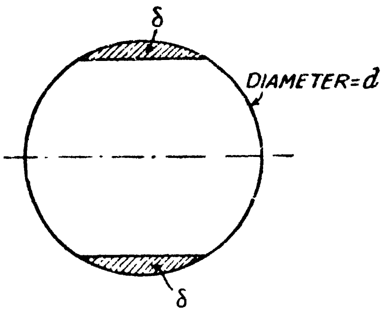


Fig. 254

Similarly the section modulus of the triangular section can be increased by cutting off the corner shaded.

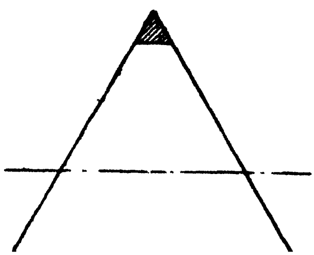


Fig. 255

Problem 158. *An over-hung steel crank pin journal is so designed that the pressure on the journal is limited to 40 kg. per cm.² of projected area. The total load on the journal is 27220 kg. and the maximum bending stress is limited to 700 kg. per cm.² Find the diameter and the length of the journal*

Solution. Let the diameter and length of the journal be d *cm.* and l *cm.* respectively.

Projected area of the journal resisting the load $=dl$ *cm.*²

Since the stress on the projected area is 40 *kg./cm.*² we have

$$\frac{27220}{d}=40$$

$$\therefore \quad dl=680{\cdot}5 \qquad \ldots(i)$$

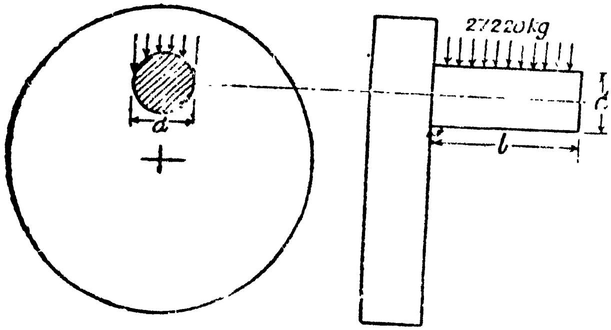


Fig. 256

Maximum bending moment for the journal

$$=M=27220\times\frac{l}{2}\ kg.\ cm.$$

$$=13610\ l\ kg.\ cm.$$

Section modulus of the journal section

$$=\frac{\pi d^3}{32}\ cm.^3$$

Since the bending stress is to be limited to 700 $kg./cm.^2$, we have,

$$M=fZ$$

$$13610\ l=700\times\frac{\pi d^3}{32}\qquad l=0{\cdot}005052\ d^3\qquad ...(ii)$$

Substituting in eq. (*i*), we get

$$0{\cdot}005052\ d^4=680{\cdot}5$$

$$\therefore\qquad d^4=\frac{680{\cdot}5}{0\ 005052}\qquad d=19{\cdot}15\ cm.$$

$$\therefore\qquad l=\frac{680{\cdot}5}{19{\cdot}15}=35{\cdot}54\ cms.$$

say the pin may be 19 *cm*. diameter and 36 *cm*. long.

Problem 159. *A simple bridge is formed of telegraph poles laid side by side with all the butt ends on the one abutment. If Z is the modulus of the section at the butt end, and Z_x is the modulus of the section x inches from the butt end. $Z_x=Z-0{\cdot}3\ x$. Span=30 ft. Diameter of poles at butt ends is 12 in. Find the position of the most highly stressed section when the bridge is uniformly loaded throughout its length.* (*London University*)

Solution. Fig. 257 shows one of the poles.

Let the load on the pole be *w lbs*. per foot run.

Each vertical reaction

$$=\frac{w\times 30}{2}=15\ w\ lbs.$$

B M. at a section distant x feet from the butt end

$$=M=15\,wx-\frac{wx^2}{2}$$

$$=\frac{w}{2}\left(30x-x^2\right)\ lb.\,ft.$$

$$=\frac{w}{2}\left(30x-x^2\right)\times 12\ lb.\ in.$$

$$=6w\,(30x-x^2)\ lb.\ in.$$

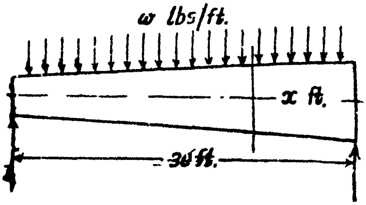


Fig. 257

Section modulus of the butt end

$$=Z=\frac{\pi\times 12^3}{32}=54\pi\ in.^3$$

∴ Section modulus at section X distant x feet from the butt end

$$=Z_x=Z-0{\cdot}3\,(x\times 12)\ in.^3$$

∴ $$Z_x=54\pi-3{\cdot}6x\ in^3$$
$$=3{\cdot}6\,(15\pi-x)\ in^3$$

∴ Extreme stress at the section X

$$=f=\frac{M}{Z_x}=\frac{6w\,(30x\times x^2)}{3{\cdot}6(15-x)}$$

∴ for f to be a maximum

$$\frac{df}{dx}=0$$

∴ $(15\pi-x)(30-2x)-(30x-x^2)(-1)=0$

∴ $450\pi-30\pi x+x^2=0$

∴ $x^2-94{\cdot}25x+1411{\cdot}72=0$

Solving this quadratic, we get

$$x=18{\cdot}69\ ft.$$

Problem 160. *A uniformly tapering vertical post of height H having a diameter D at the base and a diameter d at the top is fixed at its base. A horizontal force P is applied at the top of the post. Determine the maximum bending stress for the post and state where it occurs.*

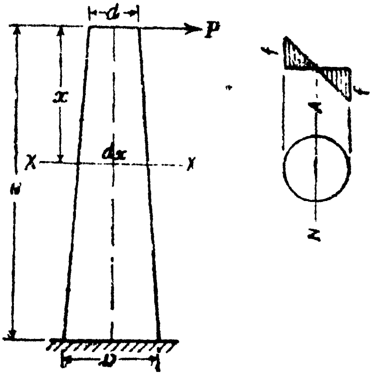


Fig. 258

Solution. Consider a section XX distant x below the top of the post. Let the diameter at the section XX be d_x.

Section modulus of the section.

$$Z=\frac{\pi d_x^3}{32}$$

B.M. at the section $=M=Px$

Let the extreme stress at the section XX be

But $$M=fZ$$

$\therefore$ $$Px=f\frac{\pi d_x^3}{32}$$

$\therefore$ $$f=\frac{32P}{\pi}\frac{x}{d_x^3}$$

But $d_x=d+\frac{D-d}{H}x=d+Kx$ where $K=\frac{D-d}{H}$

$\therefore$ $$f=\frac{32P}{\pi}\cdot\frac{x}{(d+Kx)^3}$$

For f to be a maximum

$$\frac{df}{dx}=0$$

$\therefore$ $$\frac{df}{dx}=\frac{32P}{\pi}\frac{(d+Kx)^3-x(3)(d+Kx)^2X}{(d+Kx)^6}=0$$

$\therefore$ $(d+Kx)^2(d+Kx-3Kx)=0$

$$d-2Kx=0$$

$\therefore$ $$x=\frac{d}{2K}$$

But $$K=\frac{D-d}{H}$$

$\therefore$ $$x=\frac{d}{2(D-d)}\cdot H$$

Putting $x=\frac{d}{2K}$ in the expression for f, we get

$$f_{max}=\frac{32P}{\pi}\cdot\frac{\frac{d}{2K}}{\left(d+\frac{d}{2}\right)^3}$$

$$=\frac{32P}{\pi}\times\frac{8}{27}\cdot\frac{d}{2Kd^3}$$

$$=\frac{32\times8}{\pi\times27\times2}\times\frac{Pd\,H}{(D-d)d^3}$$

$\therefore$ $$f_{max}=\frac{128}{27\pi}\frac{PH}{(D-d)d^2}$$

Problem 161. *A vertical flag staff standing 10 metres above the ground is of square section throughout, the dimensions being 8 cm.×8 cm. at the top tapering uniformly to 16 cm.×16 cm. at the ground. A* *horizontal pull of 30 kg. is applied at the top, the direction of the loading being along a diagonal of the section. Find the maximum stress due to bending.*

Solution. Consider a horizontal section XX at a depth of x *metres* from the top. Let the sectional dimension at this level be d *cm.* × d *cm.*

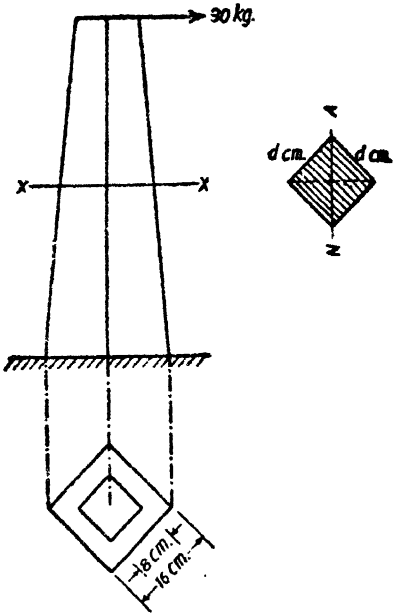


Fig. 259

$$\therefore \quad d=8+\frac{x}{10}(16-8) \text{ cm.}$$

$$=8+0{\cdot}8\,x \text{ cm.}$$

$$=0{\cdot}8\,(x+10) \text{ cm.}$$

Moment of inertia of the section about the neutral axis

$$=I=\frac{d^4}{12} \text{ cm.}^4$$

Maximum distance from the neutral axis

$$=\frac{d}{\sqrt{2}} \text{ cm.}$$

∴ Section modulus of the section

$$=Z=\frac{I}{y}=\frac{d^3}{6\sqrt{2}} \text{ cm}^3$$

$$\therefore \qquad Z=\frac{\{0{\cdot}8(x+10)\}^3}{6\sqrt{2}} \text{ cm.}$$

$$=\frac{0{\cdot}512}{6\sqrt{2}}(x+10)^3$$

B.M. at the section $=M=30\times x$ *kg. m.*

$$=30\times x\times 100 \text{ kg. cm.}$$

$$=3000\,x \text{ kg. cm.}$$

∴ Extreme fibre stress at the section

$$=f=\frac{M}{Z}$$

$$=\frac{3000\,x\,6\sqrt{2}}{0{\cdot}512(x+10)^3} \text{ kg./cm.}^2$$

$$\therefore \qquad f=\frac{18000\sqrt{2}}{0{\cdot}512}\,\frac{x}{(x+10)^3} \text{ kg./cm.}^2$$

For f to be a maximum,

$$\frac{df}{dx}=0$$

$$\frac{df}{dx}=\frac{18000\sqrt{2}}{0{\cdot}512}\,\frac{(x+10)^3-x3\,(x+10)^2}{(x+10)^6}=0$$

$\therefore \quad (x+10)^2 (x+10-3x)=0$

$$10-2x=0$$

$\therefore \quad x=5$ *metres.*

Substituting $x=5$ *metres* in the general expression for the extreme stress, we have,

$$f_{max}=\frac{18000\sqrt{2}}{0\cdot512}\cdot\frac{5}{15^3}\ kg./cm.^2$$

$$=73\cdot65\ kg./cm.^2$$

§48. Force on a partial area of a beam section

Fig. 260 shows a beam section. Let due to the bending moment at the section the maximum bending stress be f_{max}. Let y_{max} be the distance of the most distant fibre from the neutral axis.

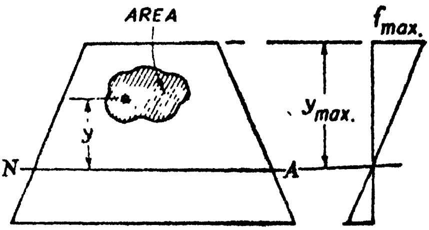


Fig. 260

Let it be required to find the normal force on a partial area A (shown shaded). Consider an elemental area da of the shaded area at a distance y from the neutral axis. Let the stress on the elemental area be f

$$\therefore \quad f=\frac{y}{y_{max}}\times f_{max}$$

$\therefore$ Force on elemental area

$$=f\,da=\frac{f_{max}}{y_{max}}\,.\,yda$$

$\therefore$ Total force on the shaded area

$$=P=\frac{f_{max}}{y_{max}}\,\Sigma yda$$

$$\therefore \quad P=\frac{f_{max}}{y_{max}}\,A\,\bar{y}$$

where $\bar{y}$ is the distance of the centroid of the shaded area from the neutral axis.

§49. Moment of the force on a partial area of a beam section about the neutral axis.

Consider again the beam section shown in Fig. 260.

The stress on the elemental area da at a distance y from the neutral axis

$$=f=\frac{f_{max}}{y_{max}}\,.\,y$$

Force on the elemental area

$$=fda=\frac{f_{max}}{y_{max}}\,.\,da\,.\,y$$

Moment of this force about the neutral axis

$$= \frac{f_{max}}{y_{max}} \cdot da \cdot y^2$$

$\therefore$ Moment of the force on the shaded area about the neutral axis

$$= \frac{f_{max}}{y_{max}} \Sigma\, day^2$$

$$= \frac{f_{max}}{y_{max}} I_o$$

where I_o = moment of inertia of the shaded area about the neutral axis.

Problem 162. *Fig. 261 shows a rectangular beam section 10 cm. wide and 20 cm. deep. If the maximum flexural stress is 80 kg/cm.² find (i) the total force on the area shaded, (ii) and the moment of this force about the neutral axis.*

Solution. (*i*) Force on the shaded area

$$= \frac{f_{max}}{y_{max}} A\bar{y}$$

$$= \frac{80}{10} (5 \times 5)(5+2{\cdot}5) \text{ kg.}$$

$$= 1500 \text{ kg. (compressive)}$$

Moment of this force about the neutral axis

$$= M = \frac{f_{max}}{y_{max}} I_o$$

$$I_o = 5 \times \frac{5^3}{12} + 25 \times 7{\cdot}5^2 = \frac{17500}{12}$$

$$\therefore\ M = \frac{80}{10} \times \frac{17500}{12}$$

$$= 11667 \text{ kg. cm.}$$

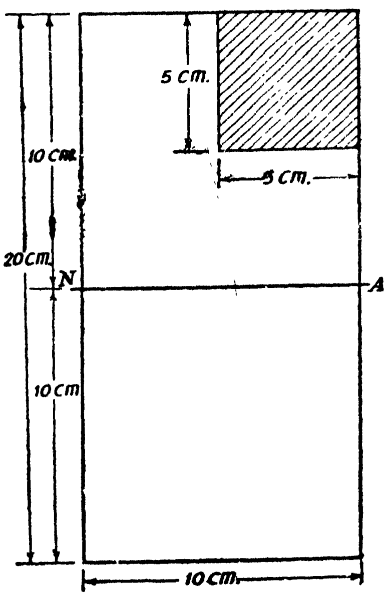


Fig. 261

Problem 163. *The beam section shown in Fig. 262 is subjected to a maximum bending stress of 20 kg./cm.². Find (i) the force on the area shaded and (ii) the moment of this force about the neutral axis.*

Solution.

(*i*) *Force on the shaded area*

$$= \frac{f_{max}}{y_{max}} A\bar{y}$$

$$=\frac{90}{12}\left(\frac{15}{2}\times 12\right)\left(\frac{2}{3}\times 12\right) kg.$$

$=5400\ kg.$

(ii) Moment of the force on the shaded area about the neutral axis

$$=\frac{f_{max}}{y_{max}}\times I_o$$

$$=\frac{90}{12}\times\frac{15\times 12^3}{4}\ kg.\ cm.$$

$=48600\ kg.\ cm.$

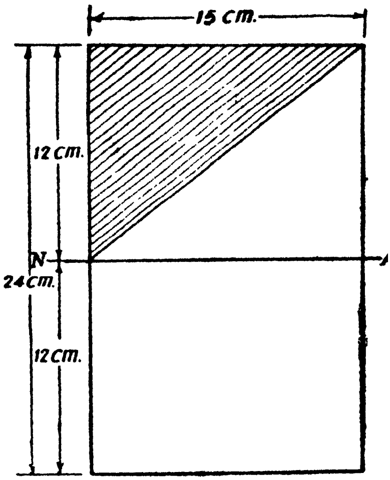


Fig. 262

Problem 164. *A beam section 10 cm. × 20 cm. (Fig. 263) is subjected to a sagging moment producing a maximum bending stress of 80 kg./cm.². Find (i) the total force on the area shaded and (ii) the moment of this force about the neutral axis.*

Solution. (*i*) Force on the shaded area

$$=\frac{f_{max}}{y_{max}}A\bar{y}$$

$$=\frac{80}{10}\times(15\times 5)\times 2{\cdot}5\ kg.$$

$=1500\ kg.$ *(compressive)*

This force is compressive since the centroid of the shaded area is in the compressive zone on the section.

(*ii*) Moment of the force on the shaded area about the neutral axis

$$=\frac{f_{max}}{y_{max}}\times I_o$$

$$=\frac{80}{10}\left[\frac{5\times 15^3}{12}+75\times 2{\cdot}5^2\right]\ kg.\ cm$$

$=15000\ kg.\ cm.$

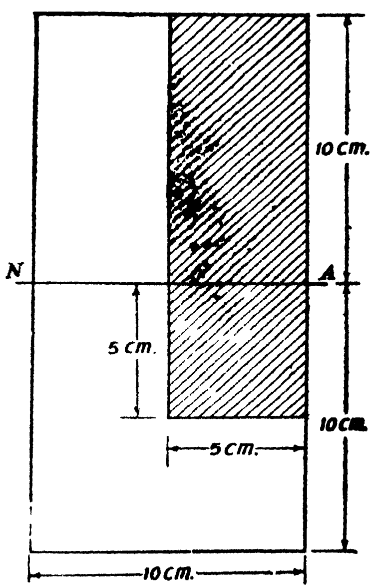


Fig. 263

Problem 165. *A box beam is made from 5 mm. × 15 mm. pieces screwed together as shown in Fig. 264. If the maximum flexural stress is 85 kg./cm.² compute (i) the force acting on the shaded portion and (ii) the moment of this force about the neutral axis.*

(A.M.I.E., May 1966)

Solution. Area shaded

$$=\frac{15\times 5}{100}=0{\cdot}75\ cm.^2$$

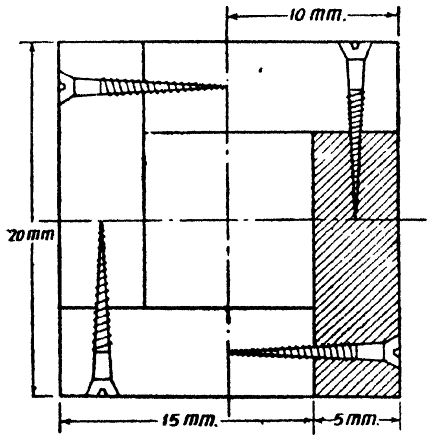


Fig. 264

(*i*) Force on the shaded area,

$$=\frac{f_{max}}{y_{max}}\,A\bar{y}$$

$$\frac{85}{1}\times 0{\cdot}75\times\frac{2{\cdot}5}{10}\ kg.$$

$$=15{\cdot}94\ kg.\ (tensile)$$

The force on the shaded area is tensile since the centroid of the shaded area is in the tension zone.

(*ii*) Moment of the force on the shaded area about the neutral axis

$$M=\frac{f_m}{y_{max}}\times I_0$$

$$I_0=\frac{0{\cdot}5\times 1{\cdot}5^3}{12}+0{\cdot}5\times 1{\cdot}5\times(0{\cdot}25)^2\ cm.^4$$

$$=\frac{-}{16}\ cm.^4$$

$$M=\frac{85}{1}\times\frac{3}{16}\ kg.\ cm.$$

$$M=15{\cdot}94\ kg.\ cm.$$

§ 49. Flitched Beams

A flitched beam means a beam of composite section consisting of a wooden beam strengthened by mild steel plates Fig. 265 shows one possible arrangement for a flitched beam. The steel and the wooden components are so connected that there will not be any slipping between them *i.e.*, the components act together behaving as one beam.

Let at a distance y from the neutral axis the stresses in wood and steel be p_w and p_s.

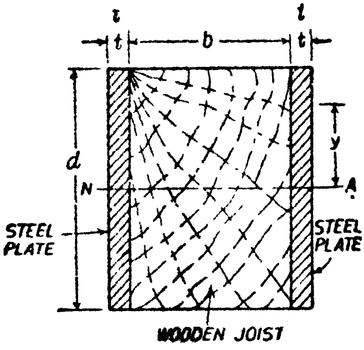


Fig. 265

In order a slip may not occur between the steel and wood at this level the strains in steel and wood at this level must be equal.

$$\therefore \quad \text{Strain}=e=\frac{p_w}{E_w}=\frac{p_s}{E_s}$$

where E_w and E_s are the Modulus of Elasticity for wood and steel

$$\therefore \qquad p_s=\frac{E_s}{E_w}\cdot p_w$$

$$\therefore \qquad p_s=m\,p_w$$

$$\text{where} \qquad m=\frac{E_s}{E_w}$$

The ratio m is called the modular ratio between steel and wood.

Moment of resistance of the section

Let M_r be the moment of resistance of the section. Let M_w and M_s be the moments of resistance of wood and steel. Let the wooden joist be b units wide and d units deep. Let each steel plate be t units thick and d units deep. Let f_w and f_s be the extreme stresses in wood and steel.

$$\therefore \qquad M=M_w+M_s$$

$$\therefore \qquad M_r=\frac{1}{6}f_w.bd^2+2\times\frac{1}{6}f_s td^2$$

$$\text{But} \qquad f_s=m\,f_w$$

$$\therefore \qquad M_r=\frac{1}{6}f_w\,bd^2+2\times\frac{1}{6}mf_w td^2$$

$$=\frac{1}{6}f_w\,(b+m\,2t)\,d^2$$

H.nce the moment of resistance of the section is the same as that of a wooden member of breadth $b+m(2t)$ and depth d. This rectangular section $b+m(2t)$ units wide and d units deep is called the equivalent wooden section.

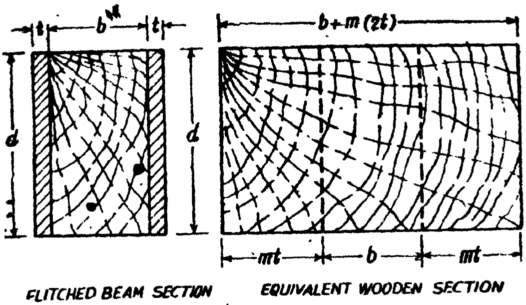


Fig. 266

The moment of resistance of the flitched beam section may therefore be determined by considering the equivalent wooden section. In some cases however the moments of resistances of the individual components may also be computed and the total moment of resistance may be determined. The following examples show how the moment of resistance of a flitched beam can be determined.

Problem 166. *A flitched beam consists of a wooden joist 15 cm. wide and 30 cm deep strengthened by a steel plate 12 mm. thick and 30 cm. deep one on either side of the joist. If the maximum stress in the wooden joist is 70 kg. per cm.² find the corresponding maximum stress attained in steel. Find also the moment of resistance of the section. Take* $E_s = 20\ E_w$.

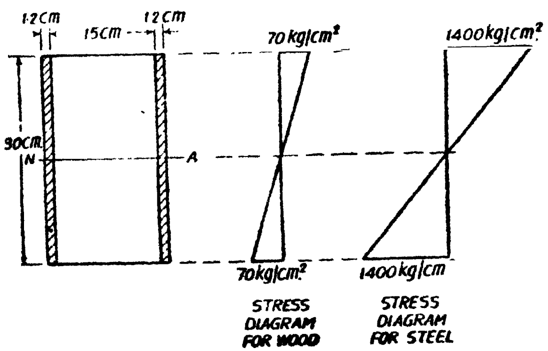


Fig. 267

Solution. Maximum stress in wood $= 70\ kg./cm.^2$

Corresponding maximum stress in steel

$$= 70 \times 20 = 1400\ kg./cm.^2$$

Moment of resistance of wooden joist

$$=\frac{1}{6} f_w bd^2$$

$$=\frac{1}{6} \times 70 \times 15 \times 30^2 \ kg.\ cm.$$

$$=157500 \ kg.\ cm.$$

Moment of resistane of steel plates

$$=2\times \frac{1}{6} f_s \ . \ td^2$$

$$=2\times \frac{1}{6} \times 1400 \times 1{\cdot}2 \times 30^2 \ kg.\ cm.$$

$$=504000 \ kg.\ cm.$$

∴ **Total moment of resistance**

$$=157500+504000 \ kg.\ cm.$$

$$=661500 \ kg.\ cm.$$

Alternatively the moment of resistance of the beam can also be determined by considering the equivalent wooden section.

Width of the equivalent wooden section

$$=b+m\ (2t)$$

$$=15+20\ (2\times 1{\cdot}2)\ cm.$$

$$=15+48\ cm.=63\ cm.$$

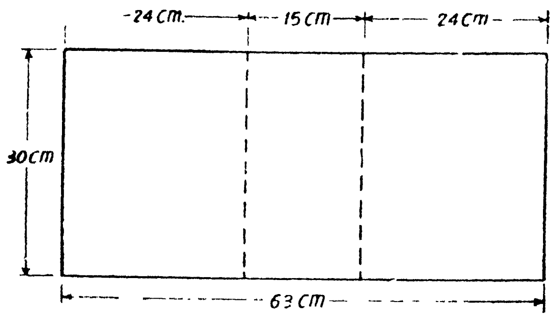


Fig. 268

∴ M.R. of the section

$$=\text{M.R. of the equivalent wooden section}$$

$$=\frac{1}{6} f_w Bd^2$$

$$=\frac{1}{6} \times 70 \times 63 \times 30^2 \ kg.\ cm.$$

$$=661500 \ kg.\ cm.$$

Problem 167. *A flitched beam consists of two wooden joists 10 cms. wide and 20 cms. deep with a steel plate 14 cm. deep and*

10 mm. thick placed symmetrically between them. If the maximum stress in the wooden joist be 70 kg./cm.², find the corresponding maximum stress reached in steel. Find also the moment of resistance of the section. Take $E_s = 20\ E_w$.

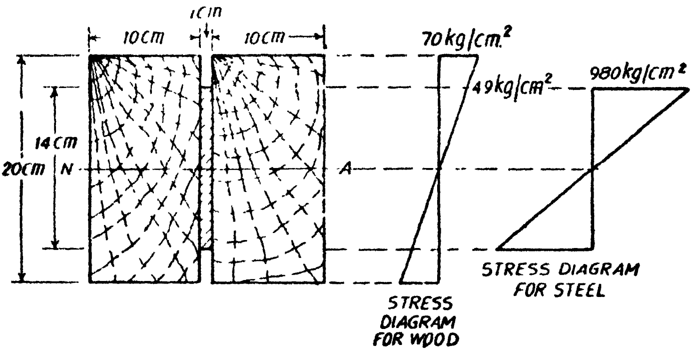


Fig. 269.

Solution. Maximum stress in wood occurs at 10 *cm.* from the N.A.

∴ Stress in wood at 10 *cm.* from the N.A.

$$= 70 \ kg./cm.^2$$

∴ Stress in wood at 7 *cm.* from the N.A.

$$= \frac{7}{10} \times 70 = 49 \ kg./cm.^2$$

∴ Maximum stress in steel

$$= 49 \times 20 = 980 \ kg./cm.^2$$

Moment of resistance of the wooden joists

$$= 2 \times \frac{1}{6} f_w \ BD^2$$

$$= 2 \times \frac{1}{6} \times 70 \times 10 \times 20^2 \ kg. \ cm.$$

$$= \frac{280000}{3} \ kg. \ cm.$$

Moment of resistance of the steel plate

$$= \frac{1}{6} f_s t d^2$$

$$= \frac{1}{6} \times 980 \times 1 \times 14^2 \ kg. \ cm.$$

$$= \frac{96040}{3} \ kg. \ cm.$$

∴ Total moment of resistance

$$= \frac{280000}{3} + \frac{96040}{3} kg. \ cm.$$

$$= 125346{\cdot}7 \ kg. \ cm.$$

Problem 168. *A composite beam consists of two timber beams of rectangular section each having a breadth B and a depth D together with a steel plate having a breadth (i.e., thickness) b and depth d. The plate is placed between the timber beams and all the three are joined together so that the whole section is symmetrical about the horizontal axis. If the maximum allowable stress in the steel and timber are 1260 kg./cm.² and 70 kg./cm.² respectively and these are reached simultaneously, find (a) the ratio of D and d and (b) the ratio of B and b in order that the moment of resistance of timber alone shall be equal to that of the steel alone, E for steel = 2·10 × 10⁶ kg./cm.² and E for timber = 0·105 × 10⁶ kg./cm.²*

If the timber beams are 25 cm. deep and 7·5 cm. wide find the size of the steel plate. (London University)

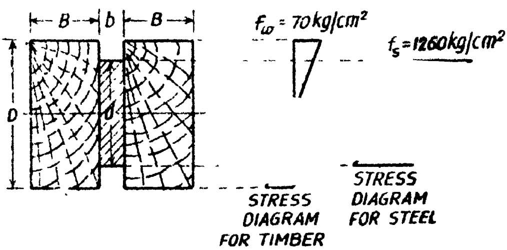


Fig. 270

Solution. Fig. 270 shows the composite beam.

Let the maximum stresses in timber and steel be f_w and f_s respectively.

$\therefore$ $f_w = 70$ kg./cm.²

and $f_s = 1260$ kg./cm.²

But $$f_s = \frac{E_s}{E_w} \cdot \frac{d}{D} f_w$$

$\therefore$ $$1260 = \frac{2{\cdot}1}{0{\cdot}105} \cdot \frac{d}{D} \times 70$$

$$= 20 \times 70 \times \frac{d}{D}$$

$\therefore$ $$\frac{d}{D} = \frac{1260}{1400} = \frac{9}{10}$$

$\therefore$ $$\frac{D}{d} = \frac{10}{9}$$

Moment of resistance of wood

$$= M_w = 2 \times \frac{1}{6} f_w BD^2$$

$$= 2 \times \frac{1}{6} \times 70\ BD^2$$

$$=\frac{70}{3}\ BD^2$$

Moment of resistance of steel

$$=M_s=\frac{1}{6}f_s\ bd^2$$

$$=\frac{1}{6}\times 1260\ bd^2=210\ bd^2$$

If the moments of resistance of the timber and steei are equal, we have,

$$\frac{70}{3}BD^2=210\ bd^2$$

$$\therefore \qquad \frac{B}{b}=\frac{210\times 3}{70}\cdot\frac{d^2}{D^2}$$

$$=\frac{210\times 3}{70}\times\frac{9^2}{10^2}$$

$$=7{\cdot}29$$

If $\qquad D=25\ cm.$

and $\qquad B=7{\cdot}5\ cm.$

$$d=\frac{9}{10}\times 25=22{\cdot}5\ cm.$$

and $\qquad b=\frac{7{\cdot}5}{7{\cdot}29}=1{\cdot}03\ cm.$

Problem 169. *A flitched beam consists of a wooden joist 15 cm. wide and 25 cm deep strengthend by steel plate 1 cm. thick and 20 cms deep one on either side of the ioist. If the stresses in wood and steel are not to exceed 70 kg. per cm² and 1200 kg. per cm², find the moment of resistance of the section of the beam. Take $E_s=20\ E_w$.*

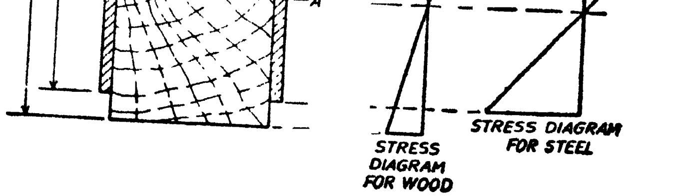


Fig. 271

Solution. Let the maximum stress in timber be 70 $kg./cm.^2$

$\therefore$ Stress in timber at 12·5 *cms.* from the N A.

$$=70 \ kg/cm.^2$$

$\therefore$ Stress in timber at 10 *cms.* from the N.A.

$$=\frac{10}{12·5}\times 70 \ kg./cm.^2$$

$$=56 \ kg./cm.^2$$

$\therefore$ Stress in steel at 10 *cms.* from the N.A.

$$=20\times 56 \ kg./cm.^2$$
$$=1120 \ kg./cm.^2$$

This is less than the permissible stress of **1200** $kg./cm.^2$ for steel.

$\therefore$ Moment of resistance of wooden joist

$$=\frac{1}{6} f_w \ BD^2$$

$$=\frac{1}{6}\times 70\times 15\times 25^2 \ kg. \ cm.$$

$$=109375 \ kg. \ cm.$$

Moment of resistance of steel plate

$$=2\times \frac{1}{6} f_s \ td^2$$

$$=2\times \frac{1}{6}\times 1120\times 1\times 20^2 \ kg. \ cm.$$

$$=149333·3 \ kg. \ cm.$$

$\therefore$ Total moment of resistance

$$=258708·3 \ kg. \ cm.$$

Problem 170 *A flitched beam consists of a wooden joist 12 cm. wide and 20 cm. deep strengthened by a steel plate 1 cm thick and 18 cm. deep, one on either side of the joist. If the stresses in wood and*

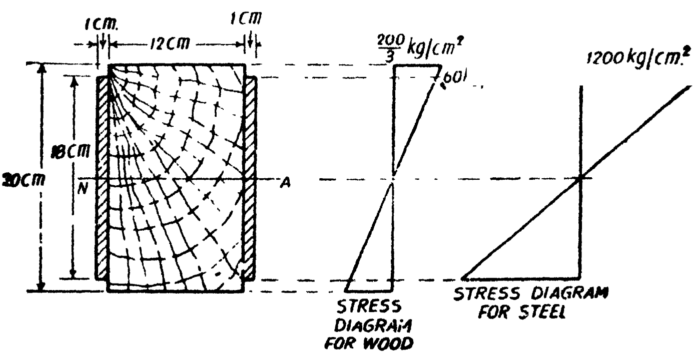


Fig. 272

steel are not to exceed 70 kg./cm.² and 1200 kg./cm.² find the moment of resistance of the section af the beam. Take $E_s = 20\ E_w$.

Solution. Let the maximum stress in wood be allowed to reach 70 *kg.* per *cm.*²

∴ Stress in wood at 10 *cm.* from the N.A.

$$=70\ kg./cm.^2$$

∴ Stress in wood at 9 *cm.* from the N.A.

$$\frac{9}{10}\times 70\ kg./cm.^2$$

$$=63\ kg./cm.^2$$

∴ Stress in steel at 9 *cm* from the N.A.

$$=20\times 63\ kg./cm.^2$$

$$=1260\ kg./cm.^2$$

But the permissible stress in steel is only 1200 *kg./cm.*²

Hence the maximum stress in wood should not be allowed to reach 70 *kg./cm.*²

Let the maximum stress in steel be allowed to reach 1200 *kg. per cm.*²

∴ Stress in steel at 9 *cm.* for the N.A.

$$=1200\ kg./cm.^2$$

∴ Stress in wood at 9 *cm.* from the N.A.

$$=\frac{1200}{20}=60\ kg./cm.^2$$

∴ Stress in wood at 10 *cm.* from the N.A.

$$=\frac{10}{9}\times 60=\frac{200}{3}\ kg./cm.^2$$

This stress is less than the safe stress of 70 *kg./cm.*²

Moment of resistance of the wooden joist

$$=\frac{1}{6}f_w\ BD^2$$

$$=\frac{1}{6}\times\frac{200}{3}\times 12\times 20^2\ kg.\ cm.$$

$$=53333{\cdot}3\ kg.\ cm.$$

Moment of resistance of the steel plate

$$=2\times\frac{1}{6}f_s\ td^2$$

$$=2\times\frac{1}{6}\times 1200\times 1\times 18^2\ kg.\ cm.$$

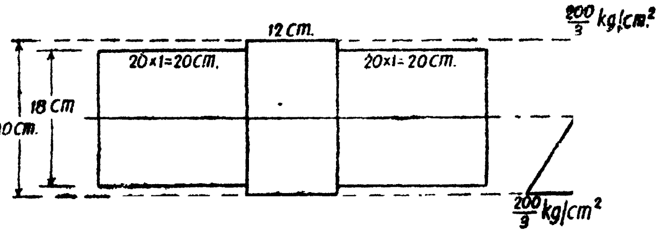
$$=129600\ kg.\ cm.$$

∴ Total moment of resistance

$$=53333{\cdot}3+129600\ kg.\ cm.$$

$$=182933{\cdot}3\ kg.\ cm.$$

Alternative solution



EQUIVALENT WOODEN SECTION

Fig. 273

Fig. 273 shows the equivalent wooden section. Each steel plate is replaced by a wooden joist having the same depth as that of the steel plate but having a width equal to the modular ratio times the thickness of the plate.

Moment of inertia of the equivalent section about the N.A.

$$=2\times\frac{20\times18^3}{12}+\frac{12\times20^3}{12}\ cm.^4$$

$$=27440\ cm.^4$$

$$\frac{M}{I}=\frac{f}{y}$$

$$M=\frac{f}{y}.\ I$$

$$=\frac{\frac{200}{3}}{10}\times27440\ kg.\ cm.$$

$$=\frac{200}{3\times10}\times27440\ kg.\ cm.$$

$$=182933{\cdot}3\ kg.\ cm.$$

Problem 171. *A flitched beam is made up of two timber joists 10 cm. wide and 22 cms. deep with a 2 cm. thick steel plate 16 cm. deep placed symmetrically between them and firmly attached to both. The plate is recessed into grooves cut in the inner faces of the joists so that the overall dimensions of the built-up section may be taken as 20 cms. by 22 cms.*

Calculate the moment of resistance of the combined section when the maximum bending stress in timber is 80 kg./cm.² What is then the maximum stress in steel ? Take $E_s=20\ E_w$ where E_s and E_w are the Young's Modulus for steel and timber.

Solution. Let the maximum stress in timber be 80 *kg./cm.*²

Hence stress in timber at 11 *cm.* from the neutral axis

$$=80\ kg./cm^2$$

∴ Stress in timber at 8 *cm.* from the neutral axis

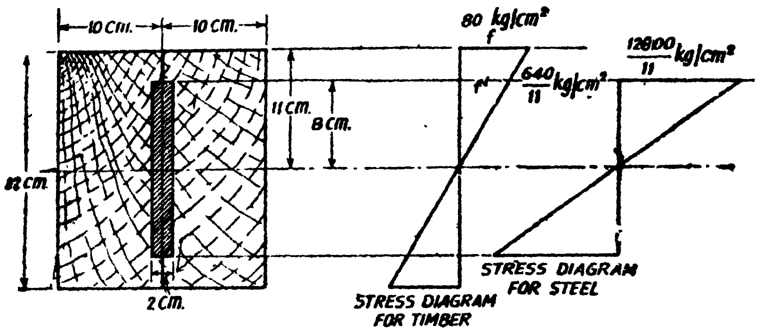


Fig. 274

$$=\frac{8}{11}\times 80=\frac{640}{11}\ kg./cm.^2$$

∴ Maximum stress in steel

$$=20\times\frac{640}{11}=\frac{12800}{11}\ kg./cm.^2$$

Moment of resistance of timber

= M R. of 20 *cm.* by 22 *cm.* rectangular timber section.

− M.R. of 2 *cm.* by 16 *cm.* rectangular timber section.

$$=\frac{1}{6}fBD^2-\frac{1}{6}f'td^2$$

$$=\frac{1}{6}\times 80\times 20\times 22^2$$

$$-\frac{1}{6}\times\frac{640}{11}\times 2\times 16^2\ kg.\ cm.$$

= *124102 kg. cm.*

Moment of resistance of steel

$$=\frac{1}{6}f_s\,td^2$$

$$=\frac{1}{6}\times\frac{12800}{11}\times 2\times 16^2$$

= *99297 kg. cm.*

∴ Total moment of resistance

= 124102 + 99297 *kg. cm.*

= *223399 kg. cm.*

Problem 172. *A timber beam 15 cm. wide and 20 cm. deep is to be reinforced by bolting on two steel flitches each 15 cm. by 1·25 cm. in section. Find the moment of resistance when (a) the flitches are attached symmetrically at top and bottom : and (b) the flitches are attached symmetrically at the sides. Allowable stress in timber is 60 kg./cm.² What is the maximum stress in steel in each case ? Take $E_s = 20\ E_w$.*

Solution.

Case (a). When the flitches are attached symmetrically at top and bottom.

Fig. 275 shows the stress diagram for wood and steel for this case.

Modular ratio $= m = \frac{E_s}{E_w} = 20$

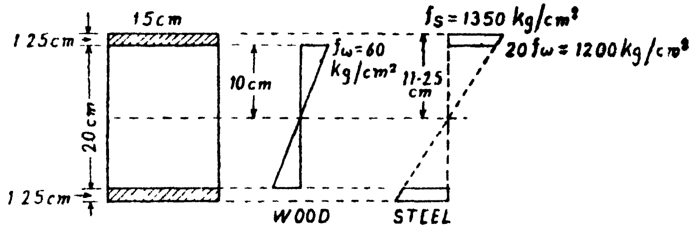


Fig. 275

Let f_w be the extreme fibre stress for wood

f_s be the extreme fibre stress for steel

$$f_s = \frac{11 \cdot 25}{10}(20\ f_w) = 22 \cdot 5\ f_w$$

∴ When $f_w = 60$ *kg./cm.²*, $f_s = 22 \cdot 5 \times 60 =$ *1350 kg./cm.²*

M.R. of the section = M.R. of the wooden component + M.R. of the steel component

$$= \frac{1}{6} \times 60 \times 15 \times 20^2 + \frac{1}{6} \times 1350 \times 15 \times 22 \cdot 5^2$$

$$- \frac{1}{6} \times 1200 \times 15 \times 20^2$$

$= 60{,}000 + 1708593 \cdot 60 - 1200000$

$=$ *568593·60 kg. cm.*

Case (b). When the flitches are attached symmetrically at the sides.

For this case when the maximum stress in wood is

$$f_w = 60 \text{ kg./cm.}^2$$

The maximum stress in steel

$$= f_s = \left[\frac{7 \cdot 5}{10} \times 60\right] 20 = 45 \times 20 = 900 \text{ kg./cm.}^2$$

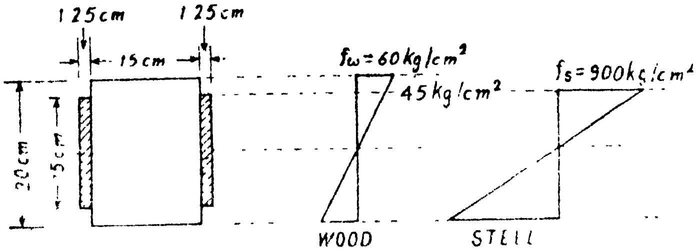


Fig. 276

M.R. of the section = M.R. of the wooden component + M.R. of the steel component

$$= \frac{1}{6} \times 60 \times 15 \times 20^2 + 2\left[\frac{1}{6} \times 900 \times 1{\cdot}25 \times 15^2\right] \textit{kg. cm}$$

$$= 60{,}000 + 84375 \textit{ kg. cm.}$$

$$= 144375 \textit{ kg. cm.}$$

Problem 173. *Two rectangular plates, one of steel and the other of brass each 3·75 cm. by 1 cm. are placed together to form a beam 3·75 cm. wide by 2 cm. deep, on two supports 75 cm. apart, the brass component being on top of the steel component. Determine the maximum central load if the plates are (i) separate and can bend independently (ii) firmly secured throughout their length. Permissible stresses for brass and steel are 700 kg./cm.² and 1000 kg./cm². Take $E_b = 0{\cdot}875 \times 10^6$ kg./cm.² and $E_s = 2{\cdot}10 \times 10^6$ kg./cm.².*

Solution.

Case (i). When the two plates are separate and can bend independently.

For this case it may be realized that each plate will have its own neutral axis.

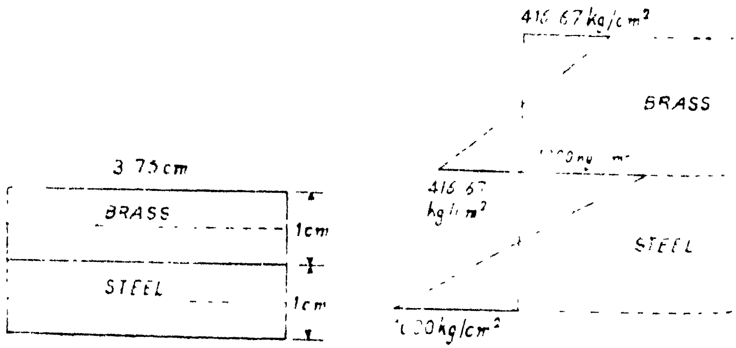


Fig 277

It will be assumed that the radius of curvature R is the same for both the components

$$\frac{f_s}{y_s} = \frac{E_s}{R} \quad \text{and} \quad \frac{f_b}{y_b} = \frac{E_b}{R}$$

$$\therefore \quad \frac{f_s}{f_b} = \frac{E_s y_s}{E_b y_b} = \frac{E_s}{E_b}$$

Since $y_s = y_b$

$$\therefore \quad \frac{f_s}{f_b} = \frac{2{\cdot}10 \times 10^6}{0{\cdot}875 \times 10^6} = 2{\cdot}4$$

Allowable stresses in brass and steel are given to be 700 *kg./cm.*² and 1000 *kg./cm*²

Obviously in this case we will allow $f_s = 1000$ *kg./cm.*²

$$f_b = \frac{1000}{2{\cdot}4} = 416{\cdot}67 \ kg./cm.^2$$

Fig. 277 shows the stress distribution.

Moment of resistance of brass component

$$= \frac{1}{6} f_b b d^2$$

$$= \frac{1}{6} \times 416{\cdot}67 \times 3{\cdot}75 \times 1^2 = 260{\cdot}42 \ kg.\,cm.$$

Moment of resistance of steel component

$$= \frac{1}{6} f_s b d_s^2$$

$$= \frac{1}{6} \times 1000 \times 3{\cdot}75 \times 1^2 = 625 \ kg.\,cm.$$

$\therefore$ Total moment of resistance

$$= 260{\cdot}42 + 625 = 885{\cdot}42 \ kg.cm.$$

Let the maximum concentrated load applied at the centre be *W kg*

Equating the maximum bending moment to the moment of resistance

$$\frac{Wl}{4} = 885{\cdot}42 \ kg.\ cm.$$

$$\therefore \quad \frac{W \times 75}{4} = 885{\cdot}42$$

$$\therefore \quad W = 47{\cdot}22 \ kg.$$

Case (ii). When the two plates are firmly secured throughout.

In this case the two components act as a single unit and will have a single neutral axis. Fig. 278 shows the equivalent brass section corresponding to the given composite section.

The width of the steel component is magnified by the modular ratio 2·4.

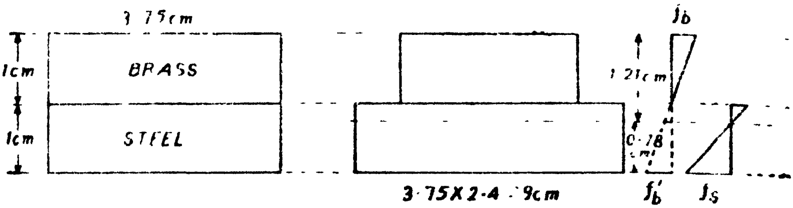


Fig. 278

Distance of the centroidal axis from the bottom edge

$$=\bar{y}=\frac{(9\times1\times0{\cdot}5)+(3{\cdot}75\times1\times1{\cdot}5)}{(9\times1)+(3{\cdot}75\times1)}=0{\cdot}79\ cm.$$

Moment of inertia of the equivalent brass section about the neutral axis

$$=I=\frac{9\times0{\cdot}79^3}{3}+\frac{9\times0{\cdot}21^3}{3}+\frac{3{\cdot}75\times1^3}{12}+3{\cdot}75\times1\times0{\cdot}71^2$$

$$=1{\cdot}479+0{\cdot}028+0{\cdot}313+1{\cdot}890$$

$$=3{\cdot}71\ cm^4.$$

Fig 278 shows the stress diagram.

$$\frac{f_b'}{f_b}=\frac{0{\cdot}79}{1{\cdot}21}$$

But $$f_b'=\frac{f_s}{m}=\frac{f_s}{2{\cdot}4}$$

$$\therefore\quad \frac{f_s}{2{\cdot}4f_b}=\frac{0{\cdot}79}{1{\cdot}21}$$

$$\therefore\quad \frac{f_s}{f_b}=1{\cdot}56$$

Taking $f_s=1000\ kg./cm.^2$

$$f_b=\frac{1000}{1\ 567}=638{\cdot}16\ kg./cm^2.$$

Let $M=M.R.$ of the section

$$\frac{M}{I}=\frac{f}{y}$$

$$M=\frac{f}{y}.I=\frac{638{\cdot}16}{1{\cdot}21}\times3{\cdot}71=1956{\cdot}67\ kg.cm.$$

If W kg. be the central point load,

$$\frac{Wl}{4}=\frac{W\times75}{4}=1956{\cdot}67$$

$$\therefore\quad W=104{\cdot}35\ kg.$$

Problem 174. *A composite beam is made by bolting an ISJC 150 steel channel to a 15 cm.×7·5 cm. wooden beam as shown in Fig. 280. The composite beam is freely supported over a span of 3* *metres. Find (a) the neutral axis of the composite section ; (b) the maximum uniformly distributed load that the beam may safely carry. Assume allowable stress in timber and steel as 70 kg./cm.^{2} and 1500 kg./cm.^{2}. Also assume E for steel $= 2\times 10^6$ kg./cm^2 and E for timber $= 0.1 \times 10^6$ kg./cm.^{2}, and for the steel channel $I_{xx} = 471.1$ cm^4 and $I_{yy} = 37.9$ cm.^{4} area $= 12.65$ cm.^{2}. Distance of the centroid from the back of channel $= 1.66$ cm.* (*A.M.I.E.*)

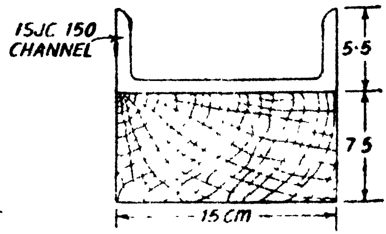


Fig. 279

Solution. It is convenient to convert the given composite section into the *equivalent steel section.* This is done by replacing the timber section by steel section having a depth equal to 7·5 *cm.* but having a width equal to

$$\frac{\text{width of timber component}}{\text{modular ratio}}$$

In our case modular ratio

$$= \frac{2\times 10^6}{0.1\times 10^6} = 20$$

Width of steel component equivalent to the timber section

$$= \frac{15}{20} = 0.75 \ cm.$$

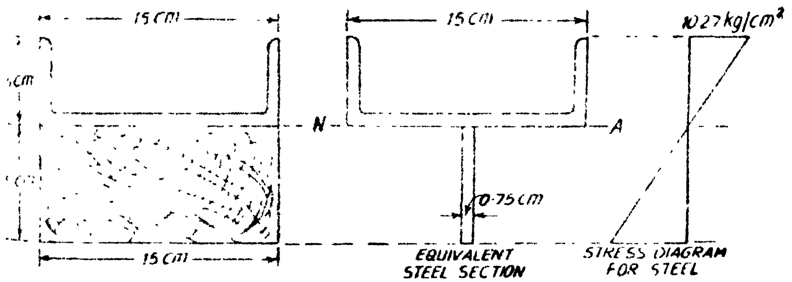


Fig. 280.

Now consider the equivalent steel section.

$\therefore$ Distance of the centroidal axis from the bottom edge

$$= \frac{0.75\times\frac{7.5^2}{2} + 12.65(7.5+1.66)}{0.75\times 7.5+12.65} = 7.50 \ cm.$$

Moment of inertia of the equivalent steel section about the neutral axis

$$= I = \frac{0.75\times 7.5^3}{3} + 37.9 + 12.65(1.66)^2 \ cm.^4$$

$$= \mathit{178.24 \ cm.^4}$$

Suppose the stress in steel be allowed to reach 1500 *kg./cm.*^{2}

$\therefore$ Stress in steel at 5·5 *cm*, from the neutral axis

$$= 1500 \ kg./cm^2$$

$\therefore$ Stress in timber at 7·5 *cm* from the neutral axis

$$= \frac{7·5}{5·5} \times \frac{1500}{20} kg./cm.^2 = 102·2 \ kg./cm.^2$$

This is greater than the permissible stress of 70 $kg./cm.^2$ for timber

Hence let us allow timber to reach a stress of 70 $kg./cm.^2$

$\therefore$ Maximum stress in steel

$$= \left\{\frac{5·5}{7·5} \times 70\right\} \times 20 \ kg./cm^2$$

$$= 1027 \ kg./cm^2.$$

Let the safe distributed load on the beam be *w kg. per metre* run.

$$\therefore \text{ Max. B.M.} = M = \frac{wl^2}{8}$$

$$M = w \times \frac{3^2}{8} \times 100 \ kg. \ cm.$$

$$= \frac{900}{8} w \ kg. \ cm.$$

But $\frac{M}{I} = \frac{f}{y}$

$$\therefore \frac{900}{8} \times \frac{w}{178·25} = \frac{1027}{5·5}$$

$$\therefore w = 296 \ kg./m.$$

§51. Beams of uniform strength

Beams we come across are usually of the same sectional area throughout the span. But the bending moment for the beam due to the load system on it is not the same at all the sections. If the beam section be designed so that at the section where the greatest bending moment occurs the maximum fibre stress reaches the permissible stress, obviously at other sections the extreme fibre stresses are less than the permissible stress. Hence it is uneconomical to provide a beam of uniform section. As the bending moment decreases towards the support, the section of the beam may also be reduced so that at every section the extreme fibre stress reaches the permissible stress. A beam so designed is called a *beam of uniform strength*. The section of the beam may be varied (*i*) by maintaining constant depth and varying the width; (*ii*) by maintaining constant width and varying the depth or (*iii*) by varying the width and depth.

Problem 175. *A beam of span l carries a point load W at mid span. Find the shape of the beam of uniform strength (a) if the depth be maintained constant, (b) if the breadth be maintained constant.*

Solution. (*a*) *Beam of uniform strength—constant depth.*

Let the permissible stress be f.

The B.M. at any section X distant x from the support $(x<l/2)$, given by

$$M=\frac{W}{2}x$$

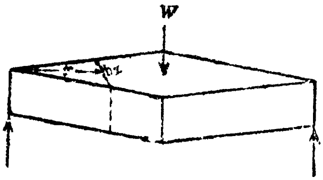


Fig. 281

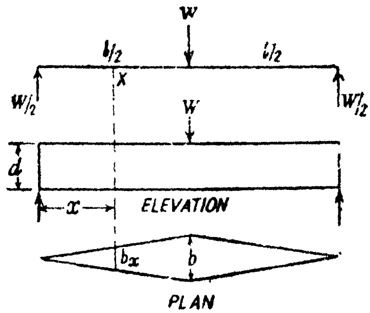


Fig. 282

Let the depth and breadth at the mid-span section be d and b respectively.

Let the breadth of the beam at X be b_x while the depth is constant at d

Equating the moment of resistance to the bending moment at X, we have

$$\frac{1}{6}f\,b_xd^2=\frac{W}{2}x$$

$$\therefore \qquad b_x=\left(\frac{3W}{fd^2}\right)x$$

Hence the width should proportionately change with x.

At mid-span *i.e.*, at $x=\frac{l}{2}$, the width of the beam

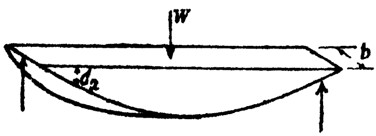


Fig. 283

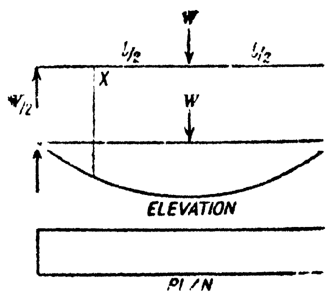


Fig. 284

$$=b=\frac{3W}{fd^2}\cdot\frac{l}{2}=\frac{3Wl}{2fd^2}$$

(*b*) ***Beam of uniform strength—constant width.***

Let the depth and width at the midspan section be d and b respectively.

Let the depth of the beam at any section X distant x from the support be d_x.

Equating the moment of resistance to the bending moment at X we have,

$$\frac{1}{6}f\,b\,d_x^2=\frac{W}{2}x$$

$$\therefore \quad d_x^2=\frac{3W}{fb}x$$

$$\therefore \quad d_x=\sqrt{\frac{3W}{fb}x}$$

At midspan, depth $=d=\sqrt{\frac{3W}{fb}\cdot\frac{l}{2}}$

$$=\sqrt{\frac{3Wl}{2fb}}$$

Problem 176. *A beam of rectangular section carries a point load W which can be placed anywhere on the span of length l. How should the depth of the beam vary in order to have a form of equal strength if the width be of the section remains constant along the beam Ignore the self weight of the beam.*

Fig. 285

Solution. The greatest bending moment occurs for the beam at mid-span when the load is at mid-span.

Let the depth of the beam at mid-section be d_m.

Let the maximum stress at this section be f.

Maximum bending moment $=M=\frac{Wl}{4}$

Moment of resistance $=\frac{1}{6}fb\,d_m^2$

$$\therefore \quad \frac{1}{6}fb\,d_m^2=\frac{Wl}{4}$$

$$\therefore \quad f=\frac{3}{2}\,\frac{Wl}{bd_m^2}.$$

Now consider any section X distant x from the end A. Maximum bending moment at this section will occur when the load is at the section itself.

For this condition,

Max. B.M. at $X = \dfrac{W x(l-x)}{l}$

Let the depth of the beam at this section be d.

The extreme stress at this section should also be the same as at the mid-span section.

$$f = \frac{M}{Z} = \frac{W x(l-x)}{l} \div \frac{bd^2}{6} = \frac{3}{2} \cdot \frac{Wl}{bd_m^2}$$

$$f = \frac{Wx(l-x)}{l} \cdot \frac{6}{bd^2} = \frac{3}{2} \cdot \frac{Wl}{bd_m^2}$$

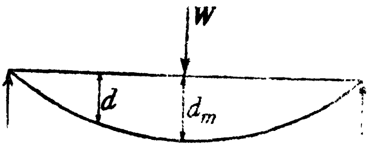


Fig. 28b

$$\therefore \quad d^2 = \frac{4d_m^2 x(l-x)}{l^2}$$

$$\text{or} \quad d = \frac{2d_m}{l} \sqrt{x(l-x)}$$

This is the law for the variation of the depth of the beam along the length of the beam.

§ 52. Shear stress distribution on a beam section

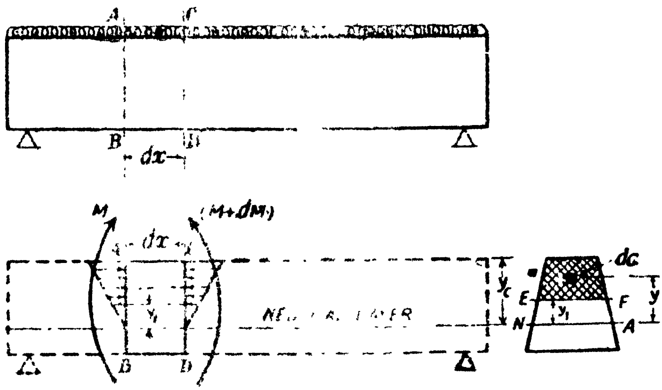


Fig. 28c

Any transverse section of a beam is subjected to a bending moment and shear force. We have earlier studied how the bending moment is resisted by the section. We will now discuss the shear resistance.

Let at any section AB the bending moment and shear force be M and S respectively. Let at another section CD distant dx from the section AB the B.M. and S.F. be $(M+dM)$ and $(S+dS)$ respectively.

Let it be required to find the shear stress intensity on the section AB at a distance y_1 from the neutral axis. On the cross-section of the beam, let EF be a line distant y_1 from the neutral axis. Now consider the part of the beam above the level EF and between the sections AB and CD. This part of the beam may be taken to consist of an infinite number of elemental cylinders each of area da and length dx. Consider one such elemental cylinder at a distance y from the neutral layer. The intensity of stress on the end of the elemental cylinder on the section $AB = f = \frac{M}{I}y$ where I = Moment of inertia of the beam section about the neutral axis. Similarly, the intensity of stress on the end of the elemental cylinder on the section CD

$$= f + df = \frac{M+dM}{I} \cdot y$$

Hence the forces on the ends of the elemental cylinder are respectively $f . da$ i.e., $\frac{M}{I}y\,da$ and $(f+df)\,da$ i.e., $\left(\frac{M+dM}{I}\right)y\,da$.

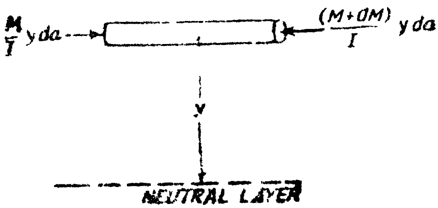


Fig. 288

Hence unbalanced force on the elemental cylinder $= \frac{dM}{I} \cdot y\,da$. Considering all the elemental cylinders between the sections AB and CD and above the level EF

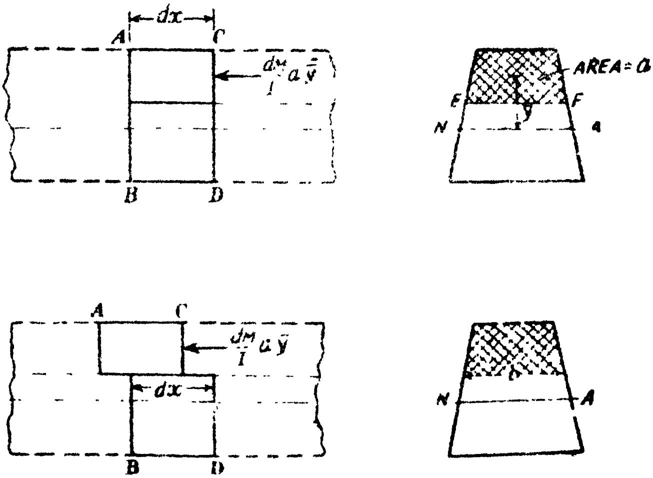


Fig. 289

Total unbalanced force above the level EF and between the two sections AB and CD

$$= \Sigma \frac{dM}{I} \cdot y\,da$$

$$= \frac{dM}{I} \sum_{y=y_1}^{y=y_c} y \, da$$

$$= \frac{dM}{I} a\bar{y}$$

where a=area of the section above the level EF and $\bar{y}$=Distance of the centroid of the area *above* the level EF *about* the neutral axis.

Hence in order the part of the beam above the level EF and between the sections AB and CD may not fail by shear due to the unbalanced force of $\frac{dM}{I} a\bar{y}$ the horizontal section of the beam at level EF must offer a shear resistance. If the width of the beam at the level EF is b, the intensity of horizontal shear at the level EF

$$\therefore q = \frac{\text{Unbalanced force}}{\text{Shear area}}$$

$$= \frac{dM}{I} \cdot \frac{a\bar{y}}{b\,dx}$$

$$= \frac{dM}{dx} \cdot \frac{a\bar{y}}{Ib}$$

but $\frac{dM}{dx} = S$

$$\therefore \quad q = \frac{Sa\bar{y}}{Ib}$$

This shear stress is the horizontal shear stress at the distance y_1 from the neutral axis. By the principle of *complementary shear*, this horizontal shear stress is accompanied by a vertical shear stress q of the same intensity.

§ 53 Shear stress distribution for beam sections of various shapes

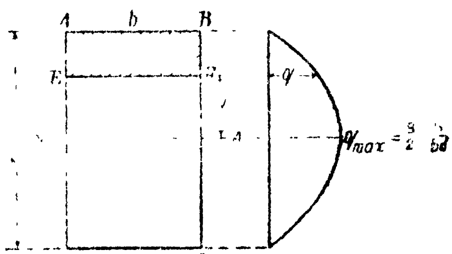


Fig. 290

(a) *Rectangular section.* Fig. 290 shows a rectangular section of width b and depth d. Let the section be subjected to a shear force S.

Consider a level EF at a distance y from the neutral axis.

The intensity of shear stress at this level is given by

$$q = \frac{Sa\bar{y}}{Ib}$$

where $a\bar{y}$ is the moment of the area above EF shown shaded about the neutral axis.

$$\therefore \qquad a\bar{y} = b\left(\frac{d}{2}-y\right)\cdot\frac{1}{2}\left(\frac{d}{2}+y\right)$$

$$=\frac{b}{2}\left(\frac{d^2}{4}-y^2\right)$$

$$\therefore \qquad q=\frac{S}{Ib}\cdot\frac{b}{2}\left(\frac{d^2}{4}-y^2\right)$$

But $\qquad I=\frac{bd^3}{12}$

$$\therefore \qquad q=\frac{12}{bd^3}\cdot\frac{S}{b}\cdot\frac{b}{2}\left(\frac{d^2}{4}-y^2\right)$$

$$\therefore \qquad q=\frac{6S}{bd^3}\left(\frac{d^2}{4}-y^2\right)$$

At the top edge *i.e.*, at $y=\frac{d}{2}$, $q=0$

At the neutral axis, *i.e.*, at $y=0$,

$$q=\frac{6S}{bd^3}\cdot\frac{d^2}{4}=\frac{3}{2}\cdot\frac{S}{bd}$$

Average shear stress $q_{avg}=\frac{S}{bd}$

$$\therefore \qquad q_{max}=\frac{3}{2}q_{avg}.$$

Hence the *maximum shear stress intensity for a rectangular section is 1·5 times the average shear stress*

Problem 177 *A rectangular beam 10 cm. wide is subjected to a maximum shear force of 5000 kg, the corresponding maximum shearing stress being 30 kg./cm.² Find the depth of the beam.*

Solution Let the depth of the beam be d cm.

Maximum shear stress $=\frac{3}{2}\times$ average shear stress

$$\therefore \qquad 30=\frac{3}{2}\times\frac{5000}{10\times d}\ kg/cm.^2$$

$$\therefore \qquad d=\frac{3}{2}\times\frac{5000}{10\times 30}\ cm.$$

$$=25\ cm.$$

Problem 178. *A timber beam is simply supported at the ends and carries a concentrated load at mid span. The maximum longitudinal stress is f and the maximum shearing stress is q. Find the ratio of the span to the depth of the beam ignoring the self-weight of the beam.*

If $f=120$ kg./cm.² and $q=10$ kg./cm.², find the ratio of the span to the depth.

Solution. Let the width and depth of the section be b and d.

Let l be the span.

Let the concentrated load placed on the beam at the mid span be W

Maximum shear force $= S = \frac{W}{2}$

Maximum shear stress $= q = \frac{3}{2} \times$ Average shear stress

$$= \frac{3}{2} \cdot \frac{W}{2bd}$$

$$\therefore \quad q = \frac{3W}{4bd} \qquad \ldots(i)$$

Maximum bending moment $= M = \frac{Wl}{4}$

Section modulus $= Z = \frac{bd^2}{6}$

$\therefore$ Maximum bending stress $= f = \frac{M}{Z} = \frac{Wl}{4} \cdot \frac{6}{bd^2}$

$$\therefore \quad f = \frac{3Wl}{2bd^2} \qquad \ldots(ii)$$

Dividing *eq* (*i*) by *eq*. (*ii*), we have

$$\frac{q}{f} = \frac{3W}{4bd} \cdot \frac{2bd^2}{3Wl} = \frac{d}{2l}$$

$$\therefore \quad \frac{l}{d} = \frac{f}{2q}$$

When $f = 120$ *kg./cm.*^{2} and $q = 10$ *kg./cm.*^{2}

$$\frac{l}{d} = \frac{120}{2 \times 10} = 6$$

Problem 179. *A timber beam 10 cm. wide and 15 cm. deep supports a uniformly distributed load over a span of 2 metres. If the safe stresses are 280 kg./cm.^{2} longitudinally and 20 kg./cm.^{2} in transverse shear, calculate the maximum load which can be supported by the beam.*

Solution. Let the maximum total uniformly distributed load on the beam be W *kg*.

(*i*) *Bending stress consideration*

Maximum bending moment $= M = \frac{Wl}{8}$

$$= W \times \frac{200}{8} \text{ kg. cm.}$$

$$= 25\, W \text{ kg. cm.}$$

But, M.R. = B.M.

$$\frac{1}{6}fbd^2=\frac{1}{6}\times 280\times 10\times 15^2=25\ W$$

$$W=\frac{280\times 10\times 15^2}{6\times 25}=4200\ kg.$$

(*ii*) *Shear stress consideration*

Maximum shear force $S=\frac{W}{2}\ kg.$

Maximum shear stress $=\frac{3}{2}\times$ Average shear stress

$$=\frac{3}{2}\cdot\frac{\frac{W}{2}}{10\times 15}\ kg./cm.^2$$

$$=\frac{W}{200}\ kg./cm.^2$$

Limiting the shear stress to 20 $kg./cm.^2$

$$\frac{W}{200}=20$$

$\therefore$ $W=4000\ kg.$

$\therefore$ *Safe total load on the beam*$=4000\ kg$

Safe intensity of the load$=\frac{4000}{2}=$*1000 kg./m.*

Problem 180. *A simply supported timber beam is 10 cm. wide and 20 cm deep carries a point load W at the middle point of the span. The permissible stress in flexure and shear are 100 kg./cm.² and 15 kg./cm.² respectively. Ignoring the self-weight of the beam, calculate the span length below which the shear stress will govern the safe load and above which the bending stress will govern the safe load.*

Solution Let W kg be the safe point load at mid span.

Let the span be l *metres*

Maximum shear force $S=\frac{W}{2}\ kg.$

Maximum shear stress $=\frac{3}{2}$ average shear stress $=15\ kg./cm^2$

$$\frac{3}{2}\cdot\frac{S}{10\times 20}=15$$

$\therefore$ $S=2000\ kg.$

$\therefore$ $W=2S=2\times 2000=4000\ kg.$

Maximum bending moment

$$=M=\frac{4000\times l}{4}\times 100\ kg.\ cm$$

$$=100{,}000\ l\ kg.\ cm.$$

Equating the maximum bending moment to the moment of resistance of $\frac{1}{6} fbd^2$, we have,

$$100{,}000\, l = \frac{1}{6} \times 100 \times 10 \times 20^2$$

$$\therefore \quad l = \frac{2}{3} \text{ metre}$$

$\therefore$ If the span exceeds $\frac{2}{3}$ *metre* the bending stress will govern the safe load.

If the span is less than $\frac{2}{3}$ *metre* the shearing stress will govern the safe load.

Problem 181. *A simply supported wooden beam of span 130 cm. having a cross section 15 cm. wide by 25 cm. deep carries a concentrated load W at the centre. Allowable working stresses are*

$$\sigma_b = 70 \text{ kg/cm}^2 \text{ (bending)}$$
$$q = 10 \text{ kg/cm}^2 \text{ (shear)}$$

What is the safe load W? (*A.M.I.E. November, 1972*)

Solution

Maximum B.M. $\quad M = \frac{Wl}{4} = \frac{W \times 130}{4}$ *kg. cm.*

Maximum S.F. $\quad S = \frac{W}{2}$ *kg.*

Bending stress consideration

Equating the max. B.M. to the moment of resistance

$$W \times \frac{130}{4} = \frac{1}{6} \times 70 \times 15 \times 25^2$$

$$\therefore \quad W = 3365 \text{ kg.}$$

Shear stress consideration

Max. Shearing stress $= \frac{3}{2} \times$ average shear stress

$$\therefore \quad 10 = \frac{3}{2} \cdot \frac{W}{2 \times 15 \times 25}$$

$$\therefore \quad W = 5000 \text{ kg.}$$

$\therefore$ Safe load $\quad W = 3365$ *kg.*

Problem 182. *Determine the concentrated load which when placed at the free end of a cantilever of length 1 m would produce a shear stress of 15 kg/cm.² at the level of the neutral axis of the section carrying the maximum shear. Assume that the beam has uniform rectangular cross-section 20 cm. × 40 cm.*

Hence compute the maximum compressive and tensile stresses due to bending over the cross-section at the fixed end of the cantilever.

Solution. Let W be the concentrated load applied at the free end of the cantilever.

Max. S.F. $=S=W$

Max. shear stress $=\frac{3}{2}\times$ average shear stress

$$=\frac{3}{2}\cdot\frac{W}{20\times 40}=15\ kg/cm^2$$

$\therefore$ $W=8000\ kg$

Max. B.M. $=M=8000\times 1\times 100\ kg.\ cm.$

$=800000\ kg\ cm.$

Equating the moment of resistance to the maximum bending moment

$$\frac{1}{6}fbd^2=M$$

$$\frac{1}{6}\times f\times 20\times 40^2=800000$$

$\therefore$ $f=150\ kg./cm^2$

(b) *Solid circular section*

Fig. 291 shows a solid circular section of radius r.

Consider any level EF at a distance y from the neutral axis.

Width of the section at the level EF

$=2\sqrt{r^2-y^2}$

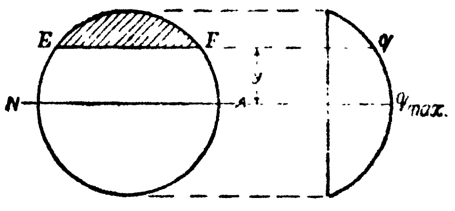


Fig. 291

Moment of the area above EF about the neutral axis.

$$a\bar{y}=\int_y^r 2y\sqrt{r^2-y^2}\,dy$$

Let $2\sqrt{r^2-y^2}=u$

$\therefore$ $4(r^2-y^2)=u^2$

$\therefore$ $-8y\,dy=2u\,du$

$\therefore$ $y\,dy=-\frac{1}{4}u\,du$

$\therefore$ $$a\bar{y}=\int_u^0 -\frac{1}{4}u^2\,du$$

$$=\frac{1}{4}\int_0^u u^2\,du=\frac{u^3}{12}$$

$\therefore$ Shear stress at the level EF

$$=q=\frac{Sa\bar{y}}{Ib}$$

where b = width of the section at the level EF

$$=EF=2\sqrt{r^2-y^2}=u$$

$\therefore$ $$q=\frac{S}{I}\cdot\frac{u^3}{12u}$$

$$=\frac{S}{12I}u^2$$

$$q=\frac{S}{12I}\cdot 4(r^2-y^2)$$

$$=\frac{S}{3I}(r^2-y^2)$$

But $$I=\frac{\pi r^4}{4}$$

$\therefore$ $$q=\frac{S\times 4}{3\pi r^4}(r^2-y^2)$$

$\therefore$ $$q=\frac{4}{3}\cdot\frac{S}{\pi r^4}(r^2-y^2)$$

Hence the shear stress distribution is according to a parabolic law.

At $y=r$, *i.e.*, at the extreme distance from the neutral axis,

$$q=0$$

At $y=0$, *i.e.*, at the neutral axis, the shear stress

$$=q_{max}=\frac{4}{3}\cdot\frac{S}{\pi r^2}$$

But the average shear stress

$$=q_{avg}=\frac{S}{\pi r^2}$$

$\therefore$ $$q_{max}=\frac{4}{3}q_{avg}$$

(*c*) *Shear stress distribution in an I section.*

Fig. 292 shows an I section. Let B and D be the width of the flange and overall depth. Let b and d be the thickness of the web and its depth.

Shear stress distribution in the flange

Width of the section at a distance y from the neutral axis

$$=B$$

Area shaded

$$=B\left(\frac{D}{2}-y\right)$$

Centroidal distance of this area from the neutral axis

$$=\frac{\left(\frac{D}{2}+y\right)}{2}$$

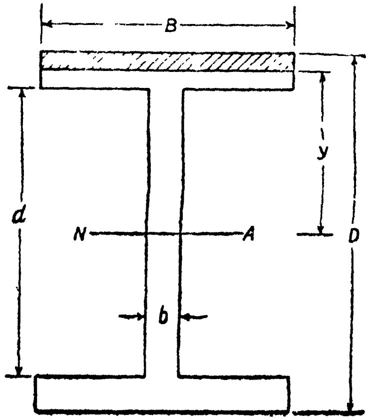


Fig. 292

∴ Shear stress

$$=q=\frac{Sa\bar{y}}{IB}$$

$$=\frac{S}{IB}\cdot B\left(\frac{D}{2}-y\right)\cdot\frac{\left(\frac{D}{2}+y\right)}{2}$$

$$q=\frac{S}{2I}\left(\frac{D^2}{4}-y^2\right)$$

Hence in the flange, the shear stress follows a parabolic law.

At $$y=\frac{D}{2}$$

$$q=0$$

At $$y=\frac{d}{2},$$

$$q=\frac{S}{2I}\left(\frac{D^2}{4}-\frac{d^2}{4}\right)$$

$$=\frac{S}{8I}(D^2-d^2)$$

Shear stress distribution in the web

Width of the section at a distance y from the neutral axis$=b$

$$q=\frac{Sa\bar{y}}{Ib}$$

But $a\bar{y}$ = moment of the flange area about the neutral axes

+ moment of the shaded area of the web about the neutral axis

$$=B\left(\frac{D-d}{2}\right)\times\frac{1}{2}\left(\frac{D}{2}+\frac{d}{2}\right)$$

$$+b\left(\frac{d}{2}-y\right)\times\frac{1}{2}\left(\frac{d}{2}+y\right)$$

$$=\frac{B}{8}(D^2-d^2)+\frac{b}{2}\left(\frac{d^2}{4}-y^2\right)$$

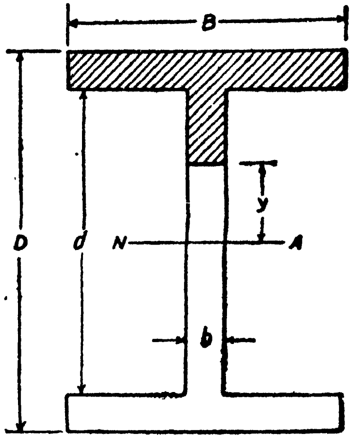


Fig. 293

∴ Shear stress

$$=q=\frac{S}{Ib}\left\{\frac{B(D^2-d^2)}{8}+\frac{b}{2}\left(\frac{d^2}{4}-y^2\right)\right\}$$

∴ $$q=\frac{B}{b}\cdot\frac{S}{8I}(D^2-d^2)+\frac{S}{2I}\left(\frac{d^2}{4}-y^2\right)$$

Hence in the web also the shear stress follows a parabolic law.

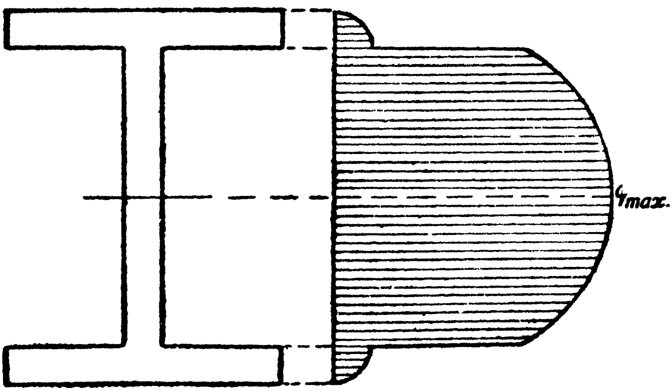


Fig. 294

At $y=\frac{d}{2}$

$$q=\frac{B}{b}\cdot\frac{S}{8I}(D^2-d^2)$$

At $y=0$, *i.e.*, at the neutral axis,

$$q_{max}=\frac{B}{b}\,\frac{S}{8I}(D^2-d^2)+\frac{Sd^2}{8I}$$

Fig. 294 shows the shear stress distribution diagram. It may be noted that at the junction of the flange and the web the shear stress abruptly changes from

$$\frac{S}{8I}(D^2-d^2) \text{ to } \frac{B}{b}\cdot\frac{S}{8I}(D^2-d^2).$$

Problem 183. *A beam of I section 50 cm. deep and 19 cm. wide has flanges 2·5 cm. thick and web 1·5 cm. thick. It carries a shearing force of 40 tonnes at a section. Calculate the maximum intensity of shear stress in the section assuming the moment of inertia to be 64500 cm.⁴. Also calculate the total shear force carried by the web and sketch the shear stress distribution across the section.*

(*A.M.I.E.*)

Solution. Maximum intensity of shear stress will occur at the neutral axis. This shear stress is given by

$$q_{max}=\frac{Sa\bar{y}}{Ib}$$

where S = maximum *S.F.*

$a\bar{y}$ = moment of the area above the neutral axis, about the neutral axis.

I = moment of inertia of the whole section about the neutral axis.

b = breadth of the web.

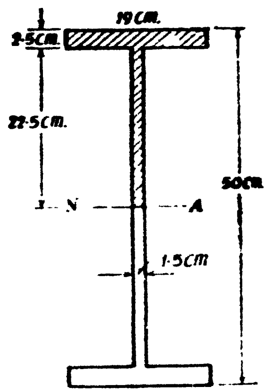


Fig. 295

In our case,

$$a\bar{y}=19\times2{\cdot}5\times23{\cdot}75+1{\cdot}5\times22{\cdot}5\times11{\cdot}25 \text{ cm.}^3$$
$$=1128{\cdot}125+379{\cdot}688 \text{ cm}^3$$
$$=1507{\cdot}813 \text{ cm.}^3$$

$$q_{max}=\frac{40\times1507{\cdot}813}{64500\times1{\cdot}5}\ \text{tonne/cm.}^2$$
$$=0{\cdot}6233 \text{ tonne/cm.}^2$$

Shear stress in the flange at a distance y from the neutral axis

$$=q=\frac{Sa\bar{y}}{IB}$$

$$a\bar{y}=B(25-y)\frac{(25+y)}{2}$$

$$=\frac{B}{2}(625-y^2)$$

$$\therefore\quad q=\frac{S}{IB}\cdot\frac{B}{2}(625-y^2)$$

$$q=\frac{S}{2I}(625-y^2)$$

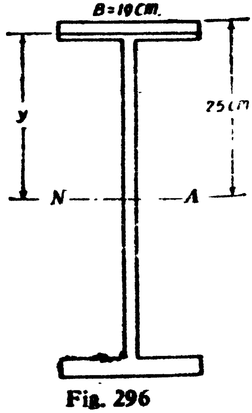


Fig. 296

∴ Shear resistance offered by an elemental strip of the flange 19 *cm.* wide and *dy cm.* deep.

$$= q\,da = q \times (19\,dy)$$
$$= \frac{S}{2I}(625 - y^2)\,19\,dy$$
$$= \frac{19}{2} \cdot \frac{S}{I}(625 - y^2)\,dy$$

$\therefore$ **Shear resistance of one flange**

$$= \frac{19S}{2I}\int_{22.5}^{25}(625 - y^2)\,dy$$
$$= \frac{19S}{2I}\left\{625\,(25 - 22.5) - \frac{1}{3}(25^3 - 22.5^3)\right\}$$
$$= \frac{19S}{2I} \times 151$$
$$= \frac{19 \times 40 \times 151}{2 \times 64500}\ \textit{tonne.}$$
$$= 0.89\ \textit{tonne.}$$

$\therefore$ **Total shear resistance of the two flanges**

$$= 0.89 \times 2 = 1.78\ \textit{tonnes.}$$

$\therefore$ **Total shear resistance of the web**

$$= 40 - 1.78\ \textit{tonnes}$$
$$= 38.22\ \textit{tonnes.}$$

Shear stress distribution across the section

Shear stress in the flange 22·5 *cm*. from the N.A.

$$\frac{S}{IB} \times \text{Moment of the flange area about the N.A.}$$
$$= \frac{40}{64500 \times 19} \times 19 \times 2.5 \times 23.75\ \textit{tonne/cm.}^2$$
$$= 0.03681\ t/cm.^2\ (\text{say } 0.037\ t/cm.^2).$$

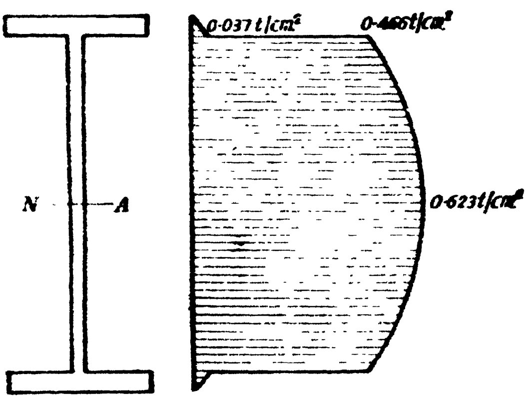


Fig. 297

Shear stress in the web at 22·5 *cm.* from the N.A.

$$=\frac{19}{1\cdot5}\times0\cdot03681\ t/cm.^2$$

$$=0\cdot4662\ t/cm.^2$$

$$\text{say } 0\cdot466\ t/cm.^2$$

Problem 184. *An I section has flanges of width b and the overall depth is 2b. The flanges and the web are of uniform thickness t. Find the ratio of the maximum shear stress intensity to the mean shear stress intensity.*

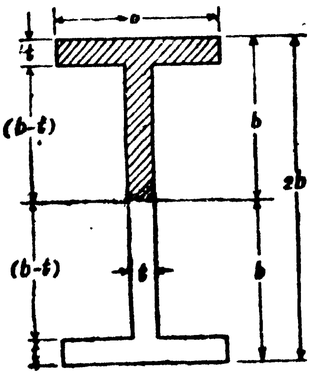


Fig. 298

Solution. Let the S.F. on the section be S. Area of the beam section,

$$A=2bt+t\,(2b-2t)$$

$$A=2bt+2bt-2t^2$$

$$A=4bt+2t^2=4bt$$

since $2t^2$ is a small quantity.

Moment of inertia of the section about the neutral axis

$$=I=\frac{b(2b)^3}{12}-\frac{(b-t)(2b-2t)^3}{12}$$

$$=\frac{2}{3}b^4-\frac{2}{3}(b-t)^4$$

$$=\frac{2}{3}\left[b^4-(b-t)^4\right]$$

$$=\frac{2}{3}\left[b^4-(b^4-4b^3t)\right]$$ omitting terms involving t^2, t^3 and t^4

$$=\frac{2}{3}\times4b^3t$$

$$=\frac{8}{3}b^3t$$

Shear stress is maximum at the neutral axis.

$$q_{max}=\frac{Sa\bar{y}}{It}$$

$a\bar{y}$ = moment of the area shaded about the neutral axis (Fig. 298)

$$=bt\left(b-\frac{t}{2}\right)+\frac{(b-t)^2\,t}{2}$$

$$=b^2t-\frac{bt}{2}+\frac{b^2t}{2}-bt^2+\frac{t^3}{2}$$

$$=\frac{3}{2}b^2t$$ neglecting terms involving t^2 and t^3

$$q_{max}=\frac{S\times\frac{3}{2}b^2t}{\frac{8}{3}b^3t\times t}$$

$$=\frac{9}{16}\frac{S}{bt}$$

The average shear stress

$$=q_{average}=\frac{S}{A}$$

$$=\frac{S}{4bt}$$

$$\frac{q_{max}}{q_{average}}=\frac{9}{16}\cdot\frac{S}{bt}\cdot\frac{4bt}{S}$$

$$=\frac{9}{4}=2{\cdot}25$$

Problem 185. *A beam of span L metres simply supported by the ends, carries a central load W. The beam section has an overall depth of 29 cm., with horizontal flanges each 15 cm.× 2 cm. and a vertical web 25 cm.× 1 cm. If the maximum shear stress is to be 450 kg./cm.² when the maximum bending stress is 1500 kg./cm.², calculate the value of the centrally applied point load W and the span L.*

Solution. Fig. 299 shows the section of the beam.

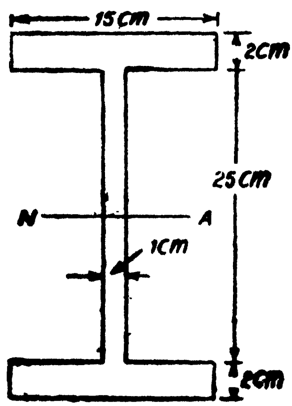


Fig. 299

Moment of inertia of the section about the neutral axis

$$=I=\frac{15\times29^3}{12}-\frac{14\times25^3}{12}\ cm.^4$$

$$=12260\ cm.^4$$

Span $=L$ *metres*

Point load at mid span $=W$ *kg*.

∴ Maximum shear force

$$=S=\frac{W}{2}\ kg.$$

Maximum shear stress occurs at the neutral axis

$$q_{max}=\frac{S\,a\bar{y}}{Ib}=450\ kg./cm.^2$$

$a\bar{y}$ = moment of the area above the neutral axis about the neutral axis

$$=(15\times2\times13.5)+(1\times12.5\times6.25)\ cm.^3$$
$$=405+78.12=483.12\ cm.^3$$

$$\therefore\quad \frac{Sa\bar{y}}{Ib}=\frac{W}{2}\times\frac{483.12}{12260\times1}=450$$

$$W=\frac{450\times2\times12260\times1}{483.12}\ kg.$$

$$W=22840\ kg.$$

Maximum bending moment

$$=M=\frac{WL}{4}$$

$$=\frac{22840\,L}{4}\ kg.\ m.$$

$$=\frac{22840\,L}{4}\times100\ kg.\ cm.$$

$$=571000\ L\ kg.\ cm.$$

$\therefore$ **Maximum bending stress**

$$=f=\frac{M}{I}y=1500\ kg./cm.^2$$

$$\therefore\quad \frac{571000\,L}{12260}\times\frac{29}{2}=1500$$

$$\therefore\quad L=\frac{1500\times12260\times2}{571000\times29}\ m.$$

$$\therefore\quad L=2.221\ m.$$

Problem 186. *The T-shaped cross-section of a beam shown in Fig. 300 is subjected to a vertical shear force of 10 t. Calculate the shear stress at the neutral axis and at the junction of the web and the flange. Moment of inertia about the horizontal neutral axis is 11340 cm.⁴* *(A.M.I.E., May 1967)*

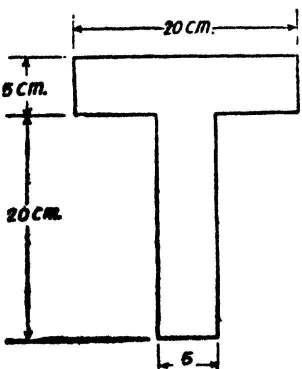


Fig. 300

Solution. Distance of the neutral axis from the top edge

$$=\frac{20\times5\times\frac{5}{2}+20\times5\times15}{200}\ cm.$$

$$=8.75\ cm.$$

Shear stress at the neutral axis

$$=q_{na}=\frac{Sa\bar{y}}{Ib}$$

$$=\frac{10\times\left(100\times6.25+3.75\times5\times\frac{3.75}{2}\right)}{11340\times5}$$

$$=0.1164\ t/cm^2.$$

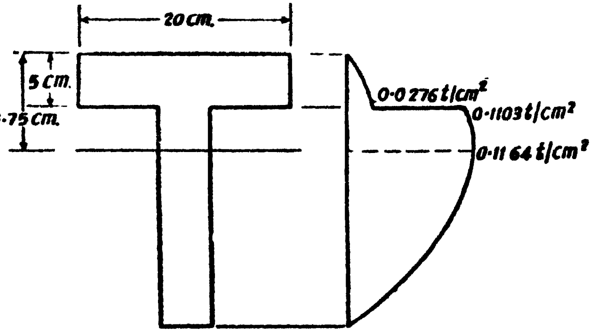


Fig. 301

Shear stress in the web just at the junction of the web and flange

$$=\frac{10\times 100\times 6{\cdot}25}{11340\times 5}=0{\cdot}1103\ t/cm.^2$$

Shear stress in the flange just at the junction of the flange and web

$$=\frac{5}{20}\times 0{\cdot}1103$$

$$=0{\cdot}0276\ t/cm.^2$$

Problem 187. *A simply supported beam carries a uniformly distributed load of intensity 30 kg./metre over the entire span of 1 metre. The cross-section of the beam is a T-section having the dimensions as shown in Fig. 302. Calculate the maximum shear stress for the section of the beam.* (*A.M.I.E., November 1972*)

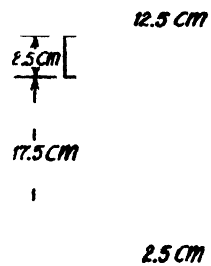


Fig. 302

Solution We will first determine the position of the neutral axis and the moment of inertia of the section about the neutral axis. The relevant calculations are shown in the Table below.

∴ Distance of centroidal axis form the top edge

$$=\bar{y}=\frac{\Sigma ay}{\Sigma a}=\frac{531{\cdot}25}{75}=7{\cdot}08\ cm.$$

Moment of inertia about the top edge 1--1

$$=I_{1-1}=\Sigma I_{self}+\Sigma ay^2$$

$$=1132{\cdot}28+5585{\cdot}97=6718{\cdot}25\ cm.^4$$

Component	Area a $(cm)^2$	Centroidal distance y from $1-1$ (cm)	$a\,y$ $(cm)^3$	$a\,y^2$ $(cm)^4$	I_{self} $(cm)^4$
Flange 12.5×2.5	31·25	1·25	39·06	48·83	$\frac{12.5\times2.5^3}{12}=16.28$
Web 17.5×2.5	43·75	11·25	492·19	5537·14	$\frac{2.5\times17.5^3}{12}=1116$
Total	75		531·25	5585·97	1132·28

But $I_{1-1}=I_{xx}+A\bar{y}^2$

$\therefore$ $6718.25=I_{xx}+75\times7.0^2$

$\therefore$ $I_{xx}=2959\ cm.^4$

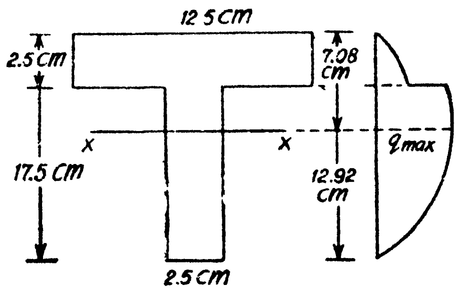


Fig. 303

Max. shear stress will occur at the neutral axis

$$q_{max}=\frac{Sa\bar{y}}{Ib}$$

Max S.F. $=S$

$$=\frac{30\times100}{2}=1500\ kg.$$

$$\therefore \quad q_{max}=\frac{1500\times2.5\times12.92\times6.46}{2959\times2.5}\ kg./cm.^2$$

$$=42.32\ kg./cm.^2$$

Problem 188. *A beam is triangular in section having a base b and an altitude h. It is placed with its base horizontal. If at a certain section of the beam the shear force is S, find the maximum shear stress and the shear stress at the neutral axis.*

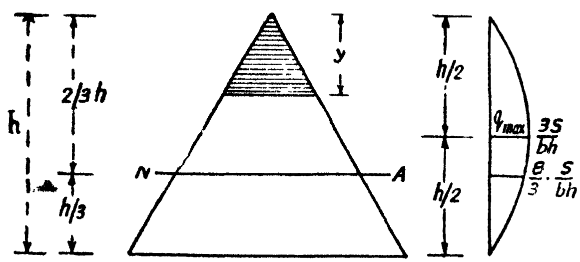


Fig. 304

Solution.

Moment of Inertia about the neutral axis

$$=I=\frac{bh^3}{36}$$

Let the shear stress intensity be q at a depth y from the top. Width of the beam at a depth y from the top

$$=b'=\frac{b}{h}y$$

$$\therefore \quad q=\frac{Sa\bar{y}}{Ib'}$$

$$=\frac{S\left(\frac{y}{2}\frac{by}{h}\right)\left(\frac{2}{3}h-\frac{2}{3}y\right)}{\left(\frac{bh^3}{36}\right)\left(\frac{by}{h}\right)}$$

$$\therefore \quad q=\frac{12\,S}{bh^3}\,y\left(h-y\right)$$

For q to be a maximum,

$$\frac{dq}{dy}=\frac{12\,S}{bh^3}\left(h-2y\right)=0$$

$$\therefore \quad y=\frac{h}{2}$$

$$\therefore \quad q_{max}=\frac{12\,S}{bh^3}\cdot\frac{h}{2}\cdot\frac{h}{2}=\frac{3\,S}{bh}$$

To find the shear stress at the neutral axis, put

$$y=\frac{2}{3}h$$

$$\therefore \quad q_{na}=\frac{12\,S}{bh^3}\,\frac{2}{3}h\cdot\frac{h}{3}=\frac{8\,S}{3\,bh}$$

Problem 189. *A beam of square section is placed horizontally with one diagonal placed horizontally. If the shear force at a section of the beam is S, draw the shear stress distribution diagram for the section.*

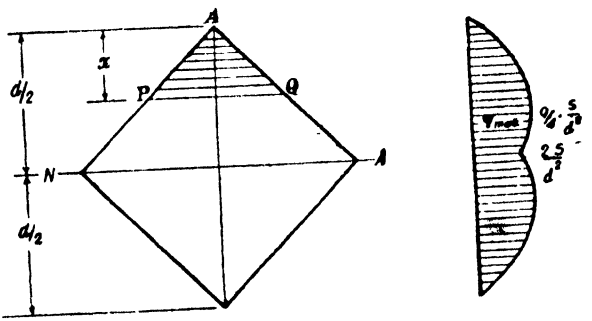


Fig. 305

Solution. Fig. 305 shows the square beam section with one diagonal placed horizontally. Let the length of the diagonal be d.

Moment of inertia of the beam section about the neutral axis

$$=I=2\,\frac{d\left(\frac{d}{2}\right)^3}{12}=\frac{d^4}{48}$$

Consider a point in the section at a depth x from A. Shear stress at this point is given by

$$q=\frac{Sa\bar{y}}{Ib}$$

where a=area above the level PQ

$\bar{y}$=centroidal distance of the shaded area from the neutral axis

b=width of the beam at a depth x from A

$\therefore \quad a=x^2 \quad \text{and } \bar{y}=\left(\frac{d}{2}-\frac{2}{3}x\right)=\frac{1}{6}\left(3d-4x\right)$

$b=2x$

$$\therefore \quad q=Sx^2\frac{\frac{1}{6}\left(3d-4x\right)}{\left(\frac{d^4}{48}\right)\times 2x}$$

$$\therefore \quad q=\frac{4S}{d^4}x\left(3d-4x\right)$$

At $\quad x=0$, *i.e.*, at A, $q=0$

At $\quad x=\frac{d}{2}$, *i.e.*, at the neutral axis,

$$q_{na}=\frac{4S}{d^4}\cdot\frac{d}{2}\left(3d-\frac{4d}{2}\right)=\frac{2S}{d^2}$$

Average shear stress $=q_{avg}=\dfrac{S}{\text{area of beam section}}$

$$=\frac{S}{\left(\frac{d^2}{2}\right)}=\frac{2S}{d^2}$$

$$\therefore \quad q_{avg}=q_{na}$$

For the shear stress to be a maximum

$$\frac{dq}{dx}=\frac{4S}{d^4}\left(3d-8x\right)=0$$

i.e., $x=\frac{3}{8}d$

Hence at a distance of $\frac{3}{8}d$ from A maximum stress occurs.

Putting $x=\frac{3}{8}d$ in the general expression for the shear stress. the maximum shear stress is given by

$$q_{max}=\frac{4S}{d^4}\cdot\frac{3}{8}d\left(3d-4\times\frac{3}{8}d\right)$$

$$=\frac{9}{4}\cdot\frac{S}{d^2}$$

$$=\frac{9}{8}\cdot\left(\frac{2S}{d^2}\right)$$

$$q_{max}=\frac{9}{8}q_{avg}.$$

Problem 190. *A laminated wooden beam 10 cm. wide and 15 cm. deep is made of three 10 cms. × 5 cms. planks glued together to resist longitudinal shear. The beam is simply supported over a span of 2 metres. If the allowable shearing stress in the glued joints is 4 kg. per cm.², find the safe concentrated load that the beam may carry at its centre.* (A.M.I.E.)

Solution. Let the safe concentrated load at mid span be W kg.

Moment of inertia of the section about the neutral axis

$$=I=\frac{10\times15^3}{12}\ cm.^4$$

$$=2812\cdot5\ cm.^4$$

Maximum $S.F.=S=\frac{W}{2}$ kg.

Shear stress at the glued joint

$$=q=\frac{Sa\bar{y}}{Ib}=4 \text{ where } a\bar{y} \text{ is}$$

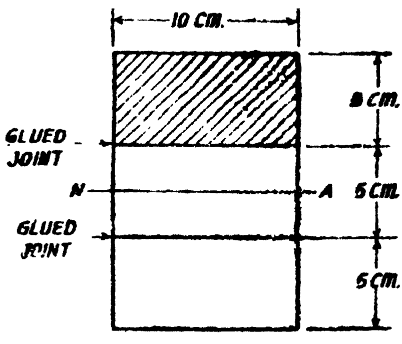


Fig. 306

the moment of the area shaded about the neutral axis.

$$\frac{W}{2}\times\frac{10\times5\times5}{2812\cdot5\times10}=4$$

$$\therefore \quad W=\frac{4\times2\times2812\cdot5\times10}{10\times5\times5}\ kg.=900\ kg.$$

Problem 191. *For the section shown in Fig. 307 determine the shearing stresses at A, B, C and D for a shearing force of 20 tonnes and find the ratio of the maximum to mean shear stress.*

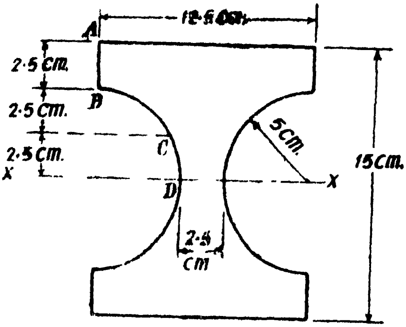


Fig. 307

Solution. Moment of inertia about the neutral axis xx

$$=I=\frac{12\cdot5\times15^3}{12}-\frac{\pi\times10^4}{64}\,cm.^4$$

$$=3025\ cm.^4$$

Shear stress at A

$a\bar{y}=0 \quad \therefore\ q=0$

Shear stress at B

$$a\bar{y}=12\cdot5\times2\cdot5\times6\cdot25\ cm.^3$$

$$=195\cdot31\ cm.^3$$

$$q=\frac{Sa\bar{y}}{Ib}=\frac{20\times195\cdot31}{3025\times12\cdot5}$$

$$=0\cdot1033\ tonne/cm.^2$$

Shear stress at C

$$a\bar{y}=(12\cdot5\times5)5-\int_{y=2\cdot5}^{y=5}2x.\ dy.\ y$$

$$=312\cdot5-\int_{2\cdot5}^{5}\sqrt{25-y^2}\,2y\,dy$$

$$=312\cdot5+\frac{2}{3}\Big[(25-y^2)^{3/2}\Big]_{2\cdot5}^{5}$$

$$=312\cdot5+\frac{2}{3}\Big[\,0-(18\cdot75)^{3/2}\Big]$$

$$=312\cdot50-54\cdot12=258\cdot38\ cm.^3$$

Width of the beam at C

$$=12\cdot5-2\sqrt{25-2\cdot5^2}\ cm.$$

$$=12\cdot5-8\cdot66\ cm\ =3\cdot84\ cm.$$

$$\therefore\qquad q=\frac{Sa\bar{y}}{Ib}$$

$$=\frac{20\times258\cdot38}{3025\times3\cdot84}\ tonne/cm.^2$$

$$=0\cdot445\ tonne/cm.^2$$

Shear stress at D

$$a\bar{y}=12\cdot5\times7\cdot5\times3\cdot75-\int^{5}\sqrt{25-y^2}\,2y\,dy$$

$$=351{\cdot}56-\frac{2}{3}(25)^{3/2}$$

$$=351{\cdot}56-83{\cdot}33=268{\cdot}23\ cm.^3$$

$$q=\frac{Sa\bar{y}}{Ib}$$

$$=\frac{20\times268{\cdot}23}{3025\times2{\cdot}5}\ tonne/cm.^2$$

$$=0{\cdot}7092\ tonne/cm.^2$$

Problem 192. *A beam section is a regular hexagon of side a and is placed so that one diagonal is horizontal as shown in Fig. 308. If the beam section be subjected to a shear force S obtain an expression for the shearing stress q at any distance x from the diagonal, and hence find the ratio of :*

(*i*) *Shearing stress at the neutral axis to the average shearing stress;*

(*ii*) *Shearing stress at a distance* $\frac{a}{2}$ *from the neutral axis to the average shearing stress.* (*London University*)

Solution. Moment of inertia about the neutral axis

$=I=$M.I. of the rectangle

$=$M.I. of four triangles about apex

Fig. 308

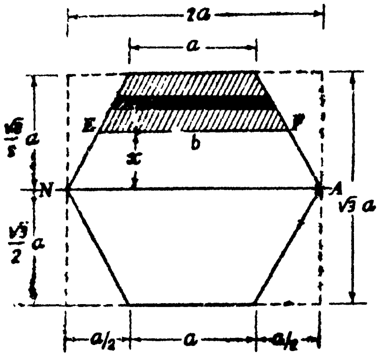


Fig. 309

$$=\frac{2a(\sqrt{3}a)^3}{12}-4\left[\frac{a}{2}\left(\frac{\sqrt{3}}{2}a\right)^3\cdot\frac{1}{4}\right]$$

$$=\frac{5\sqrt{3}}{16}a^4$$

At any level EF distant x from the neutral axis width of the section

$$=b=2a-\frac{2}{\sqrt{3}}x$$

$$\frac{2}{\sqrt{3}}(a\sqrt{3}-x)$$

Shear stress at the level EF

$$=q=\frac{SA\bar{y}}{Ib}$$

$A\bar{y}$ = moment of the shaded area about N.A.

Moment of an elemental area about the neutral axis $=bx\,dx$

$$=\frac{2}{\sqrt{3}}x(a\sqrt{3}-x)\,dx$$

∴ Moment of the shaded area about N.A.

$$=A\bar{y}$$

$$=\frac{2}{\sqrt{3}}\int_{x}^{\frac{\sqrt{3}}{2}a}(a\sqrt{3}x-x^2)\,dx$$

$$=\frac{2}{\sqrt{3}}\left[\frac{a^3\sqrt{3}}{4}-\frac{a\sqrt{3}}{2}x^2+\frac{x^3}{3}\right]$$

$$\therefore \quad q=\frac{SA\bar{y}}{Ib}$$

$$=\frac{S\dfrac{2}{\sqrt{3}}\left[\dfrac{a^3\sqrt{3}}{4}-\dfrac{a\sqrt{3}}{2}x^2+\dfrac{x^3}{3}\right]}{\dfrac{5\sqrt{3}}{16}a^4\dfrac{2}{\sqrt{3}}(a\sqrt{3}-x)}$$

$$\therefore \quad q=\frac{16S}{5\sqrt{3}a^4}\left(\frac{\dfrac{a^3\sqrt{3}}{4}-\dfrac{a\sqrt{3}}{2}x^2+\dfrac{x^3}{3}}{a\sqrt{3}-x}\right)$$

Shear stress at N.A.

Putting $x=0$ in the expression for the shear stress we get

$$q_{na}=\frac{16S}{5\sqrt{3}a^4}\left[\frac{a^3\sqrt{3}}{4a\sqrt{3}}\right]$$

$$=\frac{4}{5\sqrt{3}}\frac{S}{a^2}$$

$$=0{\cdot}4618\frac{S}{a^2}$$

Shear stress at $x=\dfrac{a}{2}$

Putting $x=\dfrac{a}{2}$ in the expression for the shear stress

$$q_{0.5a}=\frac{16S}{5\sqrt{3}a^4}\left(\frac{\dfrac{a^3\sqrt{3}}{4}-\dfrac{a\sqrt{3}}{2}\cdot\dfrac{a^2}{4}+\dfrac{1}{3}\cdot\dfrac{a^3}{8}}{a\sqrt{3}-\dfrac{a}{2}}\right)$$

$$=0.386\frac{S}{a^2}$$

$$q_{average}=\frac{S}{\text{area}}$$

$$=\frac{S}{\frac{1}{2}\,3\sqrt{3}a^2}$$

$$=0.386\frac{S}{a^2}$$

$$\therefore\qquad \frac{q_{na}}{q_{avg}}=\frac{0.4618}{0.386}=1.2$$

and $$\frac{q_{0.5a}}{q_{avg}}=\frac{0.386}{0.386}=1.$$

Problem 193. *Three planks each 5 cm. × 20 cm. are arranged to form an I section as shown in Fig. 310. The section is subjected to a shear force of 1400 kg. Suggest an alternative rectangular section of the same material so that the same maximum shearing stress is produced due to the same shear force. The width of the rectangular section shall be two-thirds the depth.*

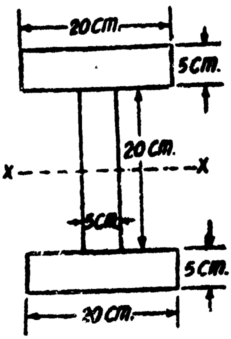


Fig. 310

Solution. Moment of inertia of the I section

$$I=\frac{1}{12}\,[20\times30^3-15\times20^3]\ cm.^4$$

$$=35{,}000\ cm.^4$$

The maximum shear stress will occur at the neutral axis.

$$a\bar{y}=20\times5\times12.5+5\times10\times5\ cm.^4$$

$$=1500\ cm.^3$$

$\therefore$ Maximum shear stress

$$=q_{max}=\frac{Sa\bar{y}}{Ib}$$

$$=\frac{1400\times1500}{35000\times5}\ kg./cm.^2$$

$$=12\ kg./cm.^2$$

For the rectangular section, let the depth be d *cm.*

$\therefore$ breadth $=\frac{2}{3}\,d\ cm.$

Sectional area $=\frac{2}{3}\,d^2$

Maximum shear stress $=\frac{3}{2}\times$ average shear stress

$$=\frac{3}{2}\times\frac{1400}{\frac{2}{3}\,d^2}\cdot 12\ kg./cm.^2$$

$$d^2=\frac{3}{2}\times\frac{3}{2}\times\frac{1400}{12}$$

$$d=16{\cdot}2\ cm.$$

$$\therefore \qquad b=\frac{2}{3}\,d=\frac{2}{3}\times 16{\cdot}2=10{\cdot}8\ cm.$$

Problem 194. *A cantilever 3 metres long is fabricated from five wooden planks 5 cm. × 15 cm. fastened together by vertical bolts 19 mm. diameter as indicated in Fig. 311 which shows the cross-section of the cantilever. The bolts are provided at a spacing of 12·5 cm. The cantilever carries a uniformly distributed load of 330 kg. per metre including its own weight. Find the shear stresses in a bolt located 1·5 metres from the support. Make the calculations of all the four planes of contact of the planks.* (*A.M.I.E.*)

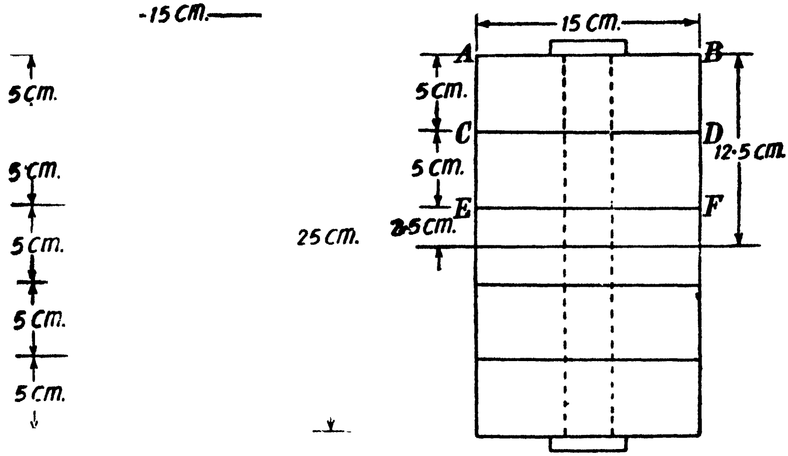


Fig. 311

Solution. Moment of inertia of the section

$$=\frac{15\times 25^3}{12}\ cm^4.$$

$$=19531\ cm^4.$$

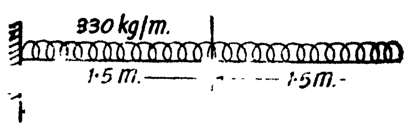


Fig. 312

S.F. at 1·5 *metres* from the support

$$=330\times 1{\cdot}5\ kg.=495\ kg.$$

Shear stresses at the planes of contact are as follows

$$q_{cd}=\frac{Sa\bar{y}}{Ib}=\frac{495\times 15\times 5\times 10}{12531\times 15}\ kg./cm.^2$$

$$=1{\cdot}268\ kg./cm.^2$$

$$q_{ef}=\frac{Sa\bar{y}}{Ib}=\frac{495\times15\times10\times7\cdot5}{19531\times15}\ kg./cm.^2$$

$$=1\cdot901\ kg./cm.^2$$

Shear stress in the bolt at the plane of contact CD

Horizontal shear per pitch length

$$=q_{cd}\times p\times b$$

$$=1\cdot268\times12\cdot5\times15\ kg.$$

Area of bolt $=\frac{\pi}{4}\times1\cdot9^2\ cm.^2$

$$=2\cdot835\ cm.^2$$

$\therefore$ Shear stress in the bolt $=\frac{1\cdot268\times12\cdot5\times15}{2\cdot835}$

$$=83\cdot87\ kg./cm.^2$$

Shear stress in the bolt at the plane of contact EF

Horizontal shear per pitch length

$$=q_{ef}\times p\times b=1\cdot901\times12\cdot5\times15\ kg.$$

$\therefore$ Shear stress in the bolt $=\frac{1\cdot901\times12\cdot5\times15}{2\cdot835}\ kg./cm.^2=125\cdot7\ kg.\ cm.^2$

Problem 195. *Three planks, each 20 cm. × 6 cm. are bolted together by 12 mm. diameter bolts to form an I section beam as shown in Fig. 313. The beam carries a central point load of 2500 kg. If the shear stress in the bolts is not to exceed 800 kg./cm.² find the pitch of the bolts.*

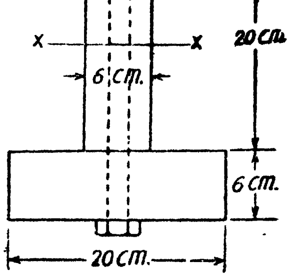


Fig. 313

Solution. Moment of inertia of the section about the neutral axis

$$=I=\frac{20\times32^3}{12}-\frac{14\times20^3}{12}\ cm.^4$$

$$=45840\ cm.^4$$

Maximum shear force $=\frac{2500}{2}$

$$=1250\ kg.$$

At the junction of the flange and web, the shear stress

$$\frac{Sa\bar{y}}{Ib}$$

$$=\frac{1250\times20\times6\times13}{45840\times6}\ kg./cm.^2$$

$$=7\cdot087\ kg./cm.^2$$

Let the pitch of the bolts be p cm.

Horizontal shear per pitch length

$$=b\times p\times q'$$

$$=6\times p\times 7{\cdot}087\ kg.$$
$$=42{\cdot}522\ p\ kg.$$

Safe shear per bolt

$$=f_s\times\frac{\pi d^2}{4}$$

$$=800\times\frac{\pi}{4}\times 1{\cdot}2^2\ kg.$$

$$\therefore\quad 42{\cdot}522\ p=800\times\frac{\pi}{4}\ 1{\cdot}2^2$$

$$p=21{\cdot}28\ cm.$$
say 20 *cm.*

Problem 196. *The box beam shown in Fig. 314 is made up of four 15 cm. × 2·5 cm. wooden planks connected by screws. Each screw can safely transmit a shear force of 125 kg. Estimate the minimum necessary spacing of screws along the length of the beam if the maximum shear force transmitted by the cross-section is 500 kg. Sketch the corresponding shear stress distribution across the section*

(*A.M.I.E.*)

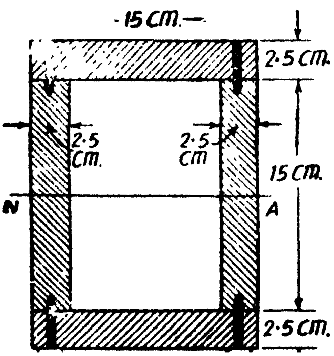


Fig. 314

Solution. Moment of inertia of the beam section

$$I=\frac{15\times 20^3}{12}-\frac{10\times 15^3}{12}\ cm.^4$$
$$=7187{\cdot}5\ cm.^4$$

Shear stress distribution

Shear stress at 10 *cm.* from *N.A.* =0

Shear stress in the flange at 7·5 *cm.* from *N.A.*

$$=\frac{Sa\bar{y}}{IB}$$
$$=\frac{500\times 15\times 2{\cdot}5\times 8{\cdot}75}{7187{\cdot}5\times 15}\ kg./cm.^2$$
$$=1{\cdot}52\ kg./cm^2$$

Shear stress in the web at 7·5 *cm* from *N A.*

$$=\frac{15}{5}\times 1{\cdot}52\ kg./cm.^2$$
$$=4{\cdot}56\ kg./cm^2$$

Shear stress at the neutral axis

$$=\frac{500(15\times 2{\cdot}5\times 8{\cdot}75+2\times 2{\cdot}5\times 7{\cdot}5\times 3{\cdot}75)}{7187{\cdot}5\times(2\times 2{\cdot}5)}\ kg./cm.^2$$

$$=6{\cdot}52\ kg./cm.^2$$

Minimum pitch of screws connecting the flange and the web planks

Horizontal shear stress in the web at the junction of flange and web

$$=4{\cdot}56\ kg./cm.^2$$

Let the pitch of screws be *p cm.*

Consider one pitch length.

Horizontal shear force at this level for one pitch length

$$=4.56\times(2\times2.5)\ p\ kg.$$

$$=22.80\ p\ kg.$$

Equating the horizontal shear per pitch length to the shear strength of the two bolts, we have,

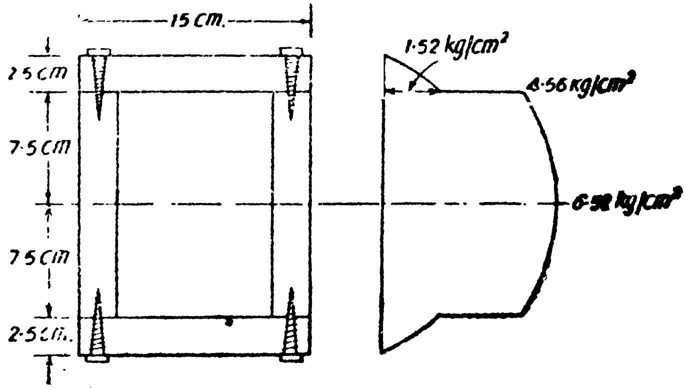


Fig. 315. Shear stress distribution across the section.

$$22.80\ p=2\times125$$

$$p=\frac{2\times125}{22.8}\ cm.=11\ cm.\ centers$$

Problem 197. *Show that the difference between the maximum and the mean shear stress down the rectangular web of an I-joist is* $\frac{Sd^2}{24I}$.

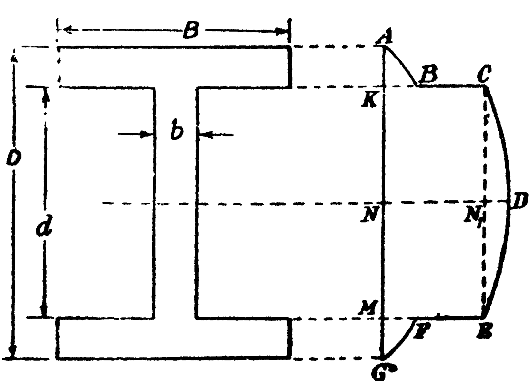


Fig. 316
Shear stress distribution

where S is the shear force at a section, d the depth of the web and I, the moment of inertia of the section about the neutral axis.

(University of Mysore)

Solution. Let B be the breadth of each flange

Let D be the overall depth.

Shear stress in the web at the junction of the flange and web

$$=KC=\frac{S}{Ib}\left[B\frac{(D-d)}{2}\left(\frac{D}{2}+\frac{d}{2}\right)\frac{1}{2}\right]$$

$$=\frac{SB}{8Ib}\cdot D^2-d^2]$$

Maximum shear stress in the web

$$=ND=\frac{S}{Ib}\left[B\frac{(D-d)}{2}\cdot\left(\frac{D}{2}+\frac{d}{2}\right)\frac{1}{2}\right.$$

$$\left.+b\frac{d}{2}\cdot\frac{d}{4}\right]$$

$$=\frac{SB}{8Ib}[D^2-d^2]+\frac{Sd^2}{8I}$$

Shear stress in the web changes from KC to ND following a parabolic law.

$\therefore$ Average shear stress in the web

$$=q_{mean}=KC+\frac{2}{3}N_1D$$

But $\quad N_1D=ND-KC$

$$=\frac{Sd^2}{8I}$$

$\therefore$ $$q_{mean}=\frac{SB}{8Ib}(D^2-d^2)+\frac{2}{3}\cdot\frac{Sd^2}{8I}$$

$$=\frac{SB}{8Ib}(D^2-d^2)+\frac{Sd^2}{12I}$$

But $$q_{max}=ND=\frac{SB}{8Ib}(D^2-d^2)+\frac{Sd^2}{8I}$$

$\therefore$ $$q_{max}-q_{mean}=\frac{Sd^2}{8I}-\frac{Sd^2}{12I}$$

$$=\frac{Sd^2}{24I}.$$

Problem 198. *A cantilever of I section 30 cm. × 15 cm. with a uniform thickness of flange and web equal to 3 cm. carries a uniformly distributed load. Find the length of the cantilever if the maximum bending stress is four times the maximum shearing stress.*

Solution. Let the length of the cantilever be *l metres.*

Let the loading on the cantilever be *w kg./metre* run.

Let the moment of inertia of the section about the neutral axis be *I cm.*^{4}

Maximum shear force

$=S= wl\ kg.$

Maximum shearing stress

$$q_{max}=\frac{Sa\bar{y}}{Ib}$$

$$=\frac{wl}{3I}[15\times3\times13\cdot5+3\times12\times6]\ kg./cm.^2$$

$$=\frac{823\cdot5}{3I}\ wl\ kg./cm.^2$$

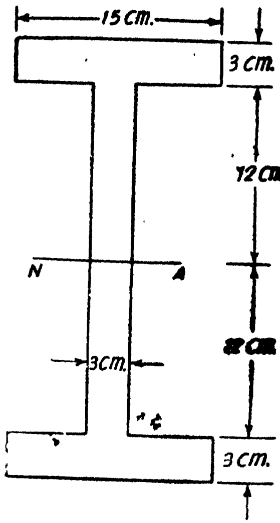


Fig. 317

Maximum bending moment$=M$

$$=\frac{wl^2}{2}\times100\ kg.\ cm.$$

$\therefore$ Maximum bending stress$=f_{max}=\dfrac{M}{I}\cdot y$

$$=\frac{wl^2}{2}\times\frac{100}{I}\times15\ kg./cm.^2$$

$$=750\ \frac{wl^2}{I}\ kg./cm.^2$$

But $\qquad f_{max}=4\ q_{max}$

$\therefore \qquad 750\ \dfrac{wl^2}{I}=\dfrac{4\times823\cdot5}{3I}\ wl$

$\therefore \qquad l=1\cdot468$ *metre*

Problem 199. *An I section beam has 20 cm. wide flanges and an overall depth of 50 cm. Each flange is 2·5 cm. thick while the web is 2 cm. thick. At a certain section the bending moment is M kg. cm. and the shear force is S kg. Find what percentage of M and S are resisted by the flanges and the web.*

Solution. Moment of inertia of the beam section

$$=I=\frac{20\times50^3}{12}-\frac{18\times45^3}{12}\ cm.^4$$

$$=71646\ cm.^4$$

Resistance to bending moment

Maximum bending stress in the web

$$=f=\frac{M}{I}\cdot\left(\frac{d}{2}\right)$$

$$=\frac{M\times 22\cdot5}{71646}\ kg./cm.^2$$

$\therefore$ Moment of resistance of the web $=\frac{1}{6}fbd^2$

$$=\frac{1}{6}\times\frac{M\times22\cdot5}{71646}\times2\times45^2\ kg.\ cm.$$

$$=0\cdot212\ M\ kg.\ cm.$$

$\therefore$ Moment of resistance of the two flanges

$$=M-0\cdot212\ M=0\cdot788\ M$$

i.e. the web will resist only 21·2 % *M* while the two flanges together will resist 78·8% of *M*

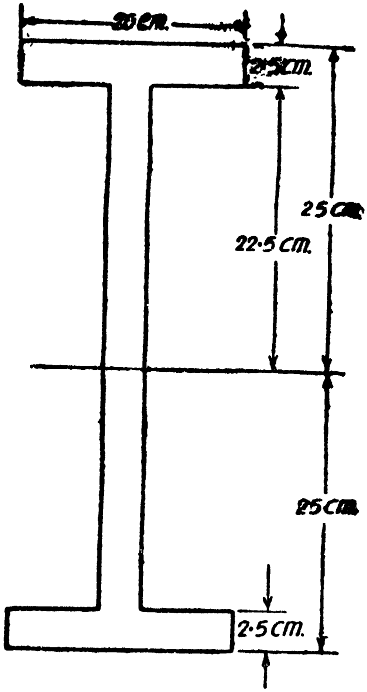


Fig. 318

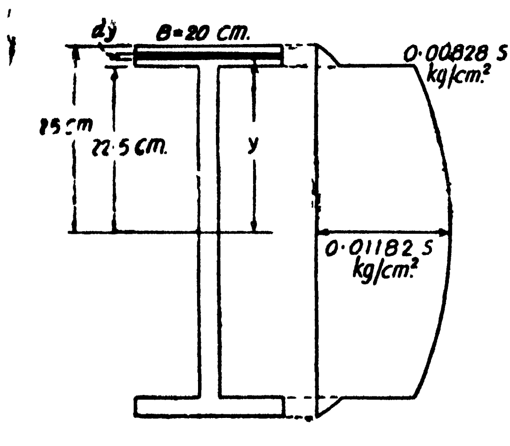


Fig. 319

Resistance to shear force S

Shear stress in the flange at any distance y from the neutral axis

$$=q=\frac{Sa\bar{y}}{IB}$$

$$=\frac{S}{IB}B(25-y)\frac{(25+y)}{2}$$

$$=\frac{S}{2I}(625-y^2)$$

Consider an elemental flange area of width $B=20$ *cm.* and height dy at a distance y from the neutral axis

Shear resistance of this elemental flange area

$$=\frac{S}{2I}(625-y^2)\,20\,dy$$

$$=\frac{10\,S}{I}(625-y^2)dy$$

∴ Shear resistance of one flange

$$=\frac{10\,S}{I}\int_{22\cdot5}^{25}(625-y^2)\,dy$$

$$=\frac{10\,S}{I}\left[625\,(25-22\cdot5)-\tfrac{1}{3}\,(25^3-22\cdot5^3)\right]$$

$$=\frac{10\,S}{I}\,(151\cdot04)\,kg.$$

$$=\frac{10\times151\cdot04}{71646}\,S\,kg.$$

$$=0\cdot021\;S\;kg.$$

Since there are two flanges, the total shear resistance of the two flanges

$$=0\cdot021\;S\times2\;kg.$$

$$=0\cdot042\;S\;kg.$$

∴ Shear resistance of the web

$$=S-0\cdot042\;S\;kg.$$

$$=0\cdot958\;S\;kg.$$

i.e., the two flanges together will only resist 4·2% of S while the web alone will resist 95·8% of S.

Alternative method of computing the shear resistances of flange and web.

Shear stress in the web at the junction of flange and web

$$=q=\frac{Sa\bar{y}}{Ib}$$

$$=\frac{S\times20\times2\cdot5\times23\cdot75}{71646\times2}\,kg./cm^2$$

$$=0\cdot008285\;S\;kg./cm.^2$$

Maximum shear stress (which occurs at the neutral axis)

$$=q_{max}=\frac{S}{71646\times 2}\Big[20\times 2\cdot 5\times 23\cdot 75+2\times 22\cdot 5\times 11\cdot 25\Big] \; kg./cm.^2$$

$$=0\cdot 01182\ S\ kg./cm.^2$$

The shear stress in the web follows a parabolic law varying from 0·008285 S $kg./cm.^2$ at the junction of the web and flange to 0·01182 S $kg./cm.^2$ at the neutral axis.

∴ Average stress in the web

$$=0\cdot 008285\ S+\tfrac{2}{3}\ [0\cdot 01182\ S-0\cdot 008285\ S]\ kg./cm.^2$$

$$=0\cdot 01064\ S\ kg./cm.^2$$

∴ Shear resistance of the web

=Area of the web × average shear stress in the web

$$=(45\times 2)\times 0\cdot 01064\ S\ kg.$$

$$=0\cdot 958\ S \text{ or } 95\cdot 8\% \text{ of } S$$

Shear resistance of the two flanges together

$$=4\cdot 2\% \text{ of } S.$$

§54. Shearing stresses in a channel section—Shear centre

Let a channel be used as a cantilever subjected to external vertical loading. Fig. 320 (*i*) shows a short length dx of the channel. Let the bending moment increase from M to $M+\delta M$ over this short length. Let us now consider a small rectangular elemental part $ABCDEFG$ from the top flange. Let $AB=CD=z$. Let the thickness of the flange be t. Let d be the depth of the channel from the centre of top flange to the centre of bottom flange. Let I be the moment of inertia of the section about the neutral axis.

Since the thickness of the flange is small the stress intensity due to bending may be taken to be uniform.

∴ Tensile stress intensity of the face $ABCD=\left(\frac{M+\delta M}{I}\right)\frac{d}{2}$

and tensile stress intensity on the opposite face $EFG=\left(\frac{M}{I}\right)\frac{d}{2}$

∴ Tensile force on the face $ABCD=P_1=\left(\frac{M+\delta M}{I}\right)\frac{d}{2}\,tz$

and Tensile force on the opposite face $EFG=P_2=\left(\frac{M}{I}\right)\frac{d}{2}\,tz$

∴ Net force on the rectangular element due to bending

$$\delta P=P_1-P_2=\frac{\delta M}{I}\frac{dtz}{2}$$

For the condition of equilibrium this force must be balanced by an equal and opposite force which is provided as shear resistance along $BCGF$. Let the intensity of this shear stress be q_1. Since the thickness of the flange is small, this shear stress intensity q_1 may be assumed to be uniform over the thickness.

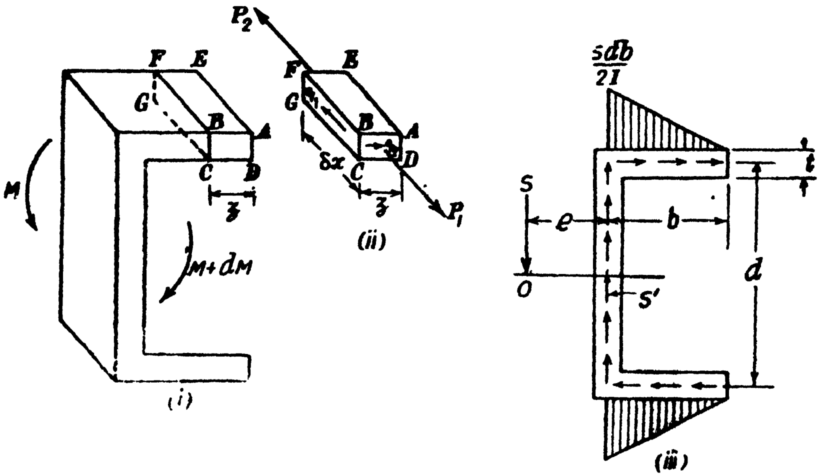


Fig. 320

$$\therefore \quad q_1 \times \text{area } BCGF = P_1 - P_2 = \delta P$$

$$\therefore \quad q_1\, t\, \delta x = \frac{\delta M}{I}\frac{dtz}{2}$$

$$\therefore \quad q_1 = \frac{\delta M}{\delta x} \cdot \frac{dz}{2I}$$

But in the limit as δx tends to zero,

$$\frac{\delta M}{\delta x} = \frac{dM}{dx} = S = \text{shear force}$$

$$\therefore \quad q_1 = \left(\frac{Sd}{2I}\right)$$

By the principle of complementary shear, the above shear stress q_1 will also be accompanied by an equal complementary shear stress q_2 acting along the flange

$$\therefore \quad q_2 = \left(\frac{Sd}{2I}\right) z$$

$\therefore$ q_2 varies directly with the distance z

At $\quad z=0 \quad\quad q_2=0$

At $\quad z=b \quad\quad q_2 = \dfrac{Sdb}{2I}$

In an exactly similar manner at any distance z a shear stress of the same intensity will also be induced in the bottom flange. But, since the bending stress in the bottom flange being compressive, the directionof the shear stress will be opposite to that in the top flange.

Average shear stress in each flange

$=$ half the maximum shear stress in the flange

$$=\frac{1}{2}\frac{Sdb}{2I}=\frac{Sdb}{4I}$$

$\therefore$ Shear force on the top flange

$=$ Flange area $\times$ average shear stress

$$=(bt)\frac{Sdb}{4I}=\frac{Sdb^2t}{4I}$$

The shear force on the top flange and the shear force on the bottom flange will therefore form a twisting couple equal to

$$\frac{Sdb^2t}{4I}\times d=\frac{Sd^2b^2t}{4I}$$

The beam will therefore have a tendency to twist in a clockwise direction.

Now consider the shear resistance offered by the web. This shear intensity at any point can be determined from the usual relation $q=\frac{Sa\bar{y}}{Ib}$.

Let S' be the vertical shear resistance of the web which should be equal and opposite to the applied vertical shear force S. These two forces will also form a couple equal to Se where e is the horizontal distance from the centre of the web at which the shear force S is applied. This couple is easily seen to be anticlockwise.

If the beam should be free from twisting, the above mentioned couples must balance.

$$\therefore \qquad Se=\frac{Sd^2b^2t}{4I}$$

$$\therefore \qquad e=\frac{d^2b^2t}{4I}$$

For this value of e, the point O [Fig. 320 (*iii*)] through which the shear force S should act in order the member may be free from torsion, is called the *shear centre*.

Examples in Chapter 5

1. A steel plate is bent into a circular arc of radius 12 *metres*. If the plate section be 10 *cms*. wide and 2 *cms*. thick find the maximum stress induced and the bending moment which can produce this stress.

Take $E=2\times10^6$ *kg./cm.*^{2}

(*1666 67 kg./cm.*^{2} ; *11, 111 kg cm.*)

2. A timber beam is 12 *cm*. wide and 20 *cm*. deep and is used on a span of 4 *metres*. If the stress due to bending is not to exceed 70 *kg*. per *cm*.^{2}, find the safe uniformly distributed load on the beam.

(*280 kg./metre* run)

3. A cast iron beam has a section as shown in Fig. 321. The beam is simply supported on a span of 4 *ft.* and is used to carry a downward point load at mid span. Find the magnitude of the load if the maximum tensile stress on the beam is 2 *tons* per *sq. in.* Determine also the greatest compressive stress in the beam. (*0·852 ton* ; *2·667 tons/in.*^{2})

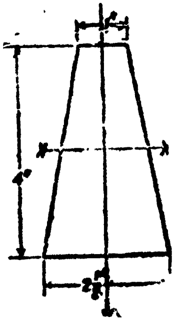


Fig. 321

4. A uniformly tapering vertical post of height 10 *metres* has a diameter 30 *cms.* at the base and a diameter of 15 *cms.* at the top. A horizontal pull of 25 *kg.* is applied at the top of the post. Find the maximum bending stress for the post and state where it occurs.

(*11·18 kg/cm.*^{2} *5 metres* from the top)

5. A plank of timber 240 *cms.* long, 5 *cms.* thick has a width varying uniformly from 46 *cms.* at one end to 25 *cms.* at the other end. It is supported at its ends with its length and width horizontal. If the weight of the timber is 880 *kg.* per cubic *metre*, find the maximum bending moment in the plank due to its own weight and find the maximum longitudinal stress produced by this bending moment.

(*1130 kg. cm.* at *114·1 cm.* from the bigger end, *7·532 kg./cm.*^{2})

6. A cantilever specimen for a fatigue-testing machine, is of circular cross-section throughout its length, but in a length of 8 *cms.* the diameter decreases from 1 *cm.* at the fixed end to 0·5 *cm.* at the free end. Calculate the maximum stress due to bending when a static load of 30 *kg. wt.* is applied at the free end in a direction perpendicular to the length of the specimen.

(*2897 kg./cm.*^{2} at *4 cm.* from the fixed end)

7. A cantilever of mild steel 6 *cm.* wide and 2 *cm.* deep is 100 *cms.* long. If at the free end of the cantilever there is a clockwise couple of 800 *kg. cm.*, find the radius to which the cantilever will be bent. Find also the vertical displacement of the free end. Ignore the self weight of the member. Take $E=2\times10^6$ *kg./cm.*^{2}

(*100 m., 0·5 cm.*)

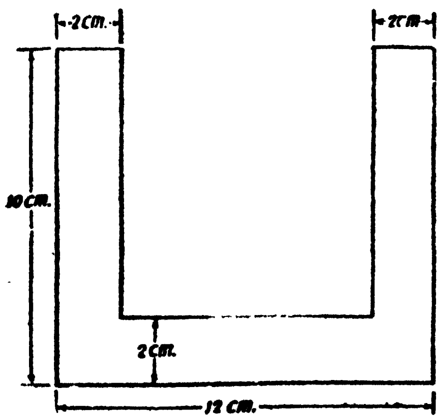


Fig. 322

8. A horizontal beam of the section shown in Fig. 322 is 4 *metres* long and is simply supported at the ends. Find the maxi-

mum uniformly distributed load it can carry if the compressive and tensile stresses must not exceed 600 $kg. cm^2$ and 300 $kg/cm.^2$ respectively. Draw a diagram showing variation of stress over the mid span section of the beam.

(*203·4 kg./m.* ; f_c=*477·2 kg./cm.*^{2} and f_t=*300 kg./cm.*^{2})

9. A cast iron beam of *I* section is simply supported on a span of 6 *metres* The section consists of a top flange 8 *cm.*×2 *cm.* thick, web 20 *cm.* deep and 2 *cm.* thick and bottom flange 16 *cm.*×4 *cm* thick. Find the safe uniformly distributed load on the beam if the tensile stress shall not exceed 300 $kg./cm^2$. Find also the corresponding maximum compressive stress.

710·8 kg./m. ; f_t *300 kg./cm*^{2} f_c=*560·1 kg./cm.*^{2})

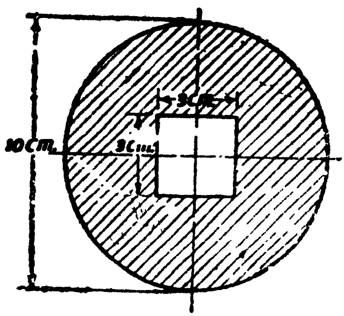


Fig. 323

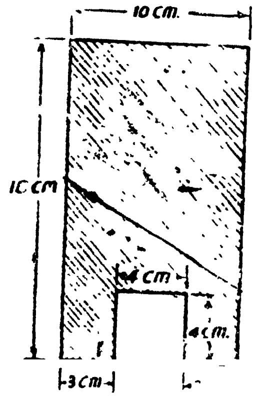


Fig. 324

10. Find the maximum bending moment which the section shown in Fig. 323 can resist if the bending stress is not to exceed 500 $kg./cm^2$.

(*48415 kg. cm.*)

11. A groove 4 *cm.*×4 *cm.* is cut symmetrically at the bottom of a rectangular beam section as shown in Fig. 324. If the tensile stress shall not exceed 250 $kg/cm.^2$ find the safe uniformly distributed load which the beam can carry, on a simply supported span of 4 *metres.* (*515 kg./m.*)

12 Fig. 325 shows the section of a beam. If the stress due to bending is not to exceed 300 $kg./cm.^2$ find the maximum bending moment which the section can resist.

(*32178 kg./cm.*)

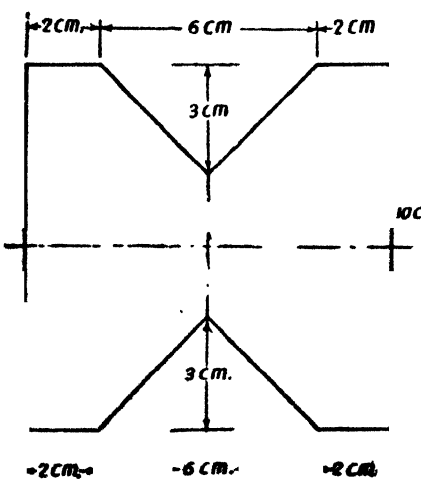


Fig. 325

13. The cross-section of a cast-iron beam consists of :

top flange 4 *in.* × 1 *in.*
web 12 *in.* × 1 *in.*
bottom flange 8 *in.* × 3 *in.*

The beam is 22 *ft.* long and simply supported at points 6 *ft.* and 18 *ft.* from the left hand end. Determine the maximum uniformly distributed load the beam can carry over the whole length if the stress due to bending shall not exceed 1 *ton/in.*^{2} in tension and 13 *t/in.*^{2} in compression. (*0·46 ton* per foot)

14. A composite beam is made by bolting a 15 *cm.* × 5·5 *cm.* steel channel to a 15 *cm.* × 7·50 *cm.* wooden beam as shown in Fig. 326. The composite beam is freely supported over a span of 3 *metres*. Find (*a*) the neutral axis of the composite section (*b*) the maximum uniformly distributed load that the beam may safely carry. Assume allowable stress in timber and steel as 70 *kg./cm.*^{2} and 1400 *kg./cm*^{2}. Also assume $E_s=2\times10^6$ *kg/cm.*^{2} and $E_w=0\cdot1\times10^6$ *kg./cm*^{2}. For the steel channel $I_{xx}=471\cdot10$ *cm.*^{4}, $I_{yy}=37\cdot90$ *cm.*^{4} area $=12\cdot65$ *cm.*^{4} Distance of the centroid from the back of the channel $=1\cdot66$ *cm.* (*7·50 cms.* from the bottom edge ; *295·7 kg./m.*)

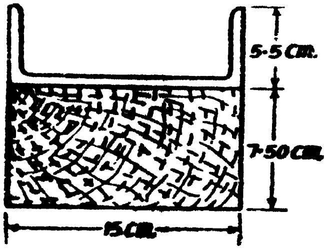


Fig. 326

15. A flitched beam consists of a wooden joist 20 *cm.* wide and 30 *cm.* deep strengthened by a steel plate 12 *mm.* thick and 20 *cm.* deep, one on either side of the joist. If the maximum stress in the wooden joist is 60 *kg./cm.*^{2}, find the corresponding maximum stress in steel. Find also the moment of resistance of the section. Take $E_s=20\ E_w$. (*800 kg./cm.*^{2} ; *308,000 kg. cm.*)

16. A flitched beam consists of a wooden joist 15 *cm.* wide and 30 *cm.* deep strengthened by a steel plate 2 *cm.* thick and 20 *cm.* deep one on either side of the joist arranged symmetrically. If the maximum stress in timber is 60 *kg./cm.*^{2}, find the maximum stress in steel. Find also the moment of resistance of the section. Take $E_s=20\ E_w$. (*800 kg/cm.*^{2}, *348, 333 kg. cm.*)

17. A flitched beam consists of a timber joist 15 *cm.* wide and 30 *cm.* deep reinforced by two vertical plates 25 *cm.* deep and 2 *cm.* thick one on each side and arranged symmetrically. If the stresses in timber and steel are not to exceed 70 *kg./cm.*^{2} and 1500 *kg/cm.*^{2} find the moment of resistance of the section. Take $E_s=18\ E_w$.

(*595,000 kg. cm.*)

18. A flitched beam consists of two wooden joists 15 *cm.* wide and 30 *cm.* deep with a steel plate 25 *cm.* deep and 10 *mm.* thick placed symmetrically between them. If the stresses in steel and timber shall not exceed 1400 *kg. cm*^{2}. and 60 *kg./cm.*^{2} respectively, find the moment of resistance of the section. Take the modular ratio between steel and timber as 20. (*374167 kg. cm.*)

19. A beam of rectangular section is 10 *cm.* wide and 20 *cm.* deep. If the section is subjected to a maximum shear force of 1000 *kg.*, find the maximum shear stress. (*7·5 kg./cm.*^{2})

20. A beam of circular section is 15 *cm* diameter. If the beam be subjected to a maximum shear force of 700 *kg*, find the maximum shear stress. (*5·28 kg./cm* ^{2})

21. A beam of triangular section has a base of width 15 *cm.* and an altitude of 9 *cm.* If the section is subjected to a shear force of 300 *kg.*, find the maximum shear stress. (*6·67 kg./cm.*^{2})

22. A beam of square section, side 10 *cm.* is placed with one diagonal vartically. If the shear force at a section is 1000 *kg*, find the maximum shear stress. (*22·5 kg./cm*^{2}.)

23. For the beam section shown in Fig. 327 find the total force on the shaded area and the moment of this force about the neutral axis if the extreme fibre stress is 90 *kg./cm.*^{2}

(*6750 kg., 45000 kg. cm.*)

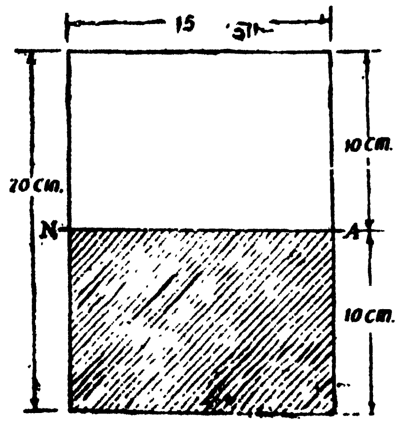


Fig. 327

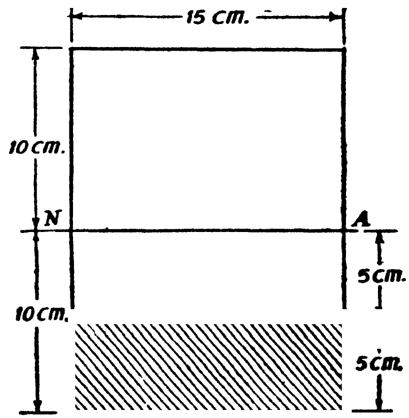


Fig. 328

24. Find the total force on the shaded area of the beam section shown in Fig. 328 if the maximum fibre stress is 80 *kg./cm.*^{2}

(*4500 kg., 35000 kg. cm.*)

25. Deduce an expression for the intensity of shearing stress at any point in the cross-section of a loaded beam and criticise the assumptions made.

A beam of symmetrical *I* section has the following dimensions: overall depth *D*, inside depth measured between the flanges *d*, width of flanges *B* and thickness of web *b*. If the flanges are assumed to be rectangular and q_1 and q_2 are the shearing stresses at the middle and top of the web respectively, find an expression for the ratio $\frac{q_1 - q_2}{q_2}$ and show that it is equal to 0·34 approximately when $\frac{B}{b}=14$ and $\frac{D}{d}=1{\cdot}1$

$$\left[q_1=\frac{B}{b}\ \frac{S}{8I}(D^2-d^2)+\frac{Sd^2}{8I} \qquad q_2=\frac{B}{b}\ \frac{S}{8I}(D^2-d^2)\right]$$

26. Explain why a single channel section with its web vertical subjected to vertical loading as a beam, will be in torsion unless the load is applied through a point outside the section known as the 'shear centre'. Find the approximate position for a channel section 40 *mm.* × 40 *mm.* outside dimensions by 3 *mm.* thick. (*1·652 cm.*)

27. A beam of square section of side a is used with a diagonal in the vertical position. If the vertical shearing force at the cross-section is S, show that the shear stress at the neutral axis is equal to the mean shear stress. Also find where the shear stress is maximum. Find also the ratio of the maximum shear stress to the mean shear stress.

$$\left[q_{na}=q_{avg}=\frac{S}{a^2}\ ; \right.$$

$$q_{max}=\frac{9}{8}\ \frac{S}{a^2} \text{ at } \frac{a}{4\sqrt{2}} \text{ from N.A.}$$

$$\left. \frac{q_{max}}{q_{avg}}=\frac{9}{8} \right]$$

28. A beam has a symmetrical triangular section of breadth B and depth D and is subjected at a certain section to a vertical shearing force S acting in the direction of the axis of symmetry. Deduce in terms of B, D and S the shearing stress q at any depth d from the vertex of the triangular section. Plot a graph showing how q varies over the depth of the section and find the ratio of the average shearing stress over the section to the maximum shearing stress. (*London University*)

$$\left[q=\frac{12\,S}{BD^3}\,d(D-d)\ ;\ \text{Ratio}=\frac{2}{3} \right]$$

29. Show from first principles that if a beam of rectangular section is subjected to a transverse shearing force the maximum shear stress at a cross-section is 1·5 times the mean shear stress.

A timber beam is simply supported at the ends and carries a concentrated load at midspan. The maximum longitudinal stress due to bending is 126 *kg./cm.*² and the maximum shearing stress is 10·5 *kg./cm.*² Find the ratio of the span to the depth of the beam.

$$\left[\frac{L}{d}=6 \right]$$

30. A beam 7 *metres* long supported at two points equidistant from the ends is loaded with a uniformly distributed load of w *kg.* per *metre* run. Calculate the length of the overhangs on each side, if the maximum bending moment for the beam has the least value. If the beam is an *I*-section 10 *cm.* × 25 *cm.* overall, with 2 *cm.* thick flange and web and the maximum bending stress is limited to 1200 *kg./cm.*² find the value of w. (*1·45 m ; w=6·250 kg./m.*)

6

Direct and Bending Stresses

Very often we come across members like a column or a tie rod mainly subjected to a longitudinal thrust or a pull. Sometimes such members are also subjected to bending stresses. In this chapter we shall study some important cases of members subjected to axial and bending stresses.

§55. Rectangular Section

Fig. 329 shows a short column of rectangular section, of area A carrying a vertical point load W axially. Obviously the intensity of stress on the section is uniform.

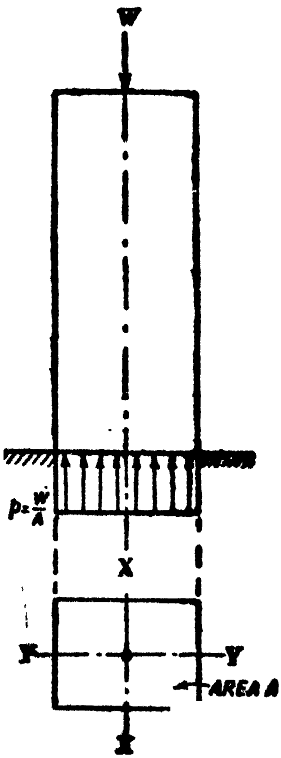


Fig. 329

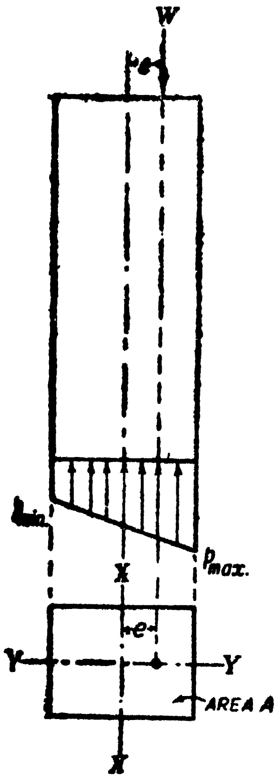


Fig. 330

Uniform stress intensity $= p = \frac{W}{A}$

Fig. 330 shows a short column of rectangular section, of area A carrying a vertical point load W eccentrically. Let the load be eccentric with respect to the axis XX. Let the eccentricity of the load be e. The section of the column is subjected to direct and bending stresses.

The load W produces a direct stress $p_o = \frac{P}{A}$

Due to the eccentricity of the load, the section of the member is subjected to a moment $M = P.e.$

At any point distant y from the neutral axis XX, the stress intensity due to the moment M is given by

$$p = \pm \frac{M}{I}$$

where I = Moment of inertia of the section of the member about the neutral axis XX.

This stress due to bending may be compressive or tensile depending on the situation of the point with respect to the neutral axis.

Hence the resultant stress at any point distant y from the neutral axis is given by

$$p_o \pm p \text{ or } \left(\frac{W}{A} \pm \frac{M}{I} . y \right)$$

$$= \frac{W}{A} \pm \frac{(We)}{I} y$$

Let p_b be the maximum stress intensity due to bending. This obviously occurs at the extremities of the section.

$$\therefore \quad p_b = \pm \frac{We}{I} . y_{max}$$

$$. We$$

where $\frac{I}{y_{max}}$ = section modulus.

Hence the resultant stresses at the extremities of the section are given by

$$p_{max} = \frac{W}{A} + \frac{We}{Z}$$

$$= p_o + p_b$$

$$p_{min} = \frac{W}{A} - \frac{We}{Z}$$

$$= p_o - p_b$$

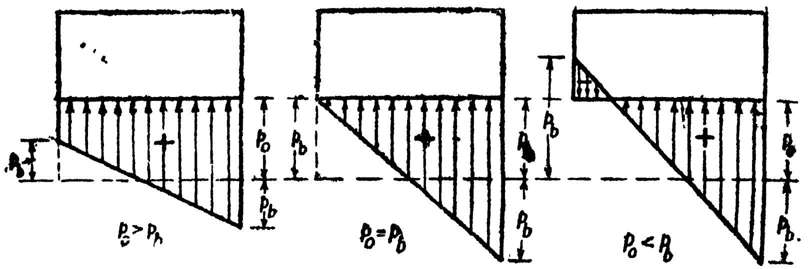


Fig. 331

If $p_o > p_b$, the stresses on the section are wholly compressive

If $p_o = p_b$, we have,

$$p_{max} = p_o + p_b = 2p_o$$

$$p_{min} = p_o - p_b = 0$$

For this case, the stresses on the section are wholly compressive and the stress intensity varies uniformly from zero at one extremity to a maximum value at the other extremity.

If $p_o < p_b$, we have,

$$p_{max} = p_o + p_b$$

$$p_{min} = p_o - p_b = -(p_b - p_o)$$

For this case p_{max} is compressive and p_{min} is tensile.

Condition for the stresses to remain wholly compressive

If the stresses are to be wholly compressive, *i.e.*, if tensile stress should not occur.

$$p_b \leqslant p_o$$

$$\therefore \quad \frac{We}{Z} \leqslant \frac{W}{A}$$

$$e \leqslant \frac{Z}{A}$$

Consider the rectangular column section in Fig. 332.

For the section shown

$$Z = \frac{ab^2}{6}$$

and $$A = ab$$

Hence, the condition that tensile stress should not occur is

$$e \leqslant \frac{ab^2}{6ab} = \frac{b}{6}$$

$\therefore$ e must be less than or equal to $\frac{b}{6}$. Hence the greatest eccentricity of the load is $\frac{b}{6}$ from the axis XX. Hence if the load is applied at any distance less than $\frac{b}{6}$ from the axis, on any side of the axis XX the stresses are wholly compressive. Hence the range within which the load can be applied so as not to produce any tensile stress, is *within the middle third of the base.*

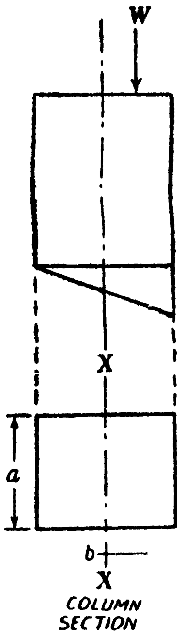


Fig. 332

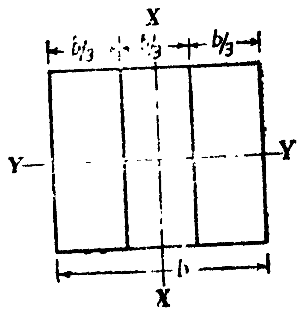


Fig. 333

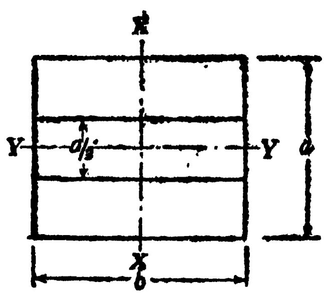


Fig. 334

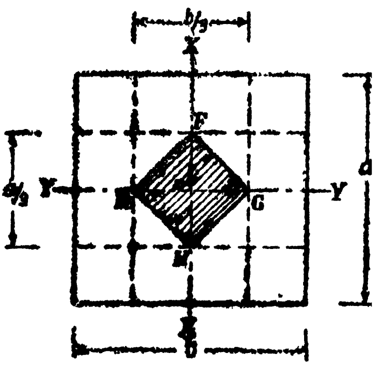


Fig. 335

Similarly, if the load had been eccentric with respect to the axis YY, the condition that tensile stress will not occur is when the eccentricity of the load with respect to the axis YY does not exceed $\frac{a}{6}$.

Hence the range within which the load may be applied is within the middle $\frac{a}{3}$.

If it is possible that the load is likely to be eccentric about both the axes XX and YY the condition that tensile stress will not occur is when the load is applied anywhere within the rhombus $EFGH$ whose diagonals are $FH=\frac{a}{3}$ and $EG=\frac{b}{3}$. This figure $EFGH$ within which the load may be placed so as not to produce tensile stress is called the *core* or *kernel* of the section.

§56. Solid Circular Section

For this case, in order tensile stresses are not developed

$$e \leq \frac{Z}{A}$$

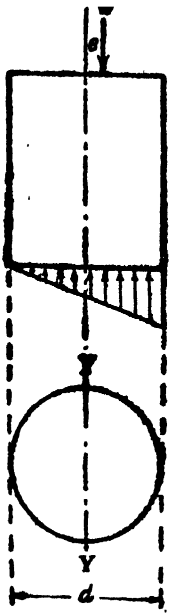


Fig. 336

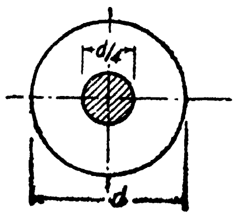


Fig. 337

Let the diameter of the section be d.

Section modulus

$$=Z=\frac{\pi d^3}{32}$$

$$A=\frac{\pi d^2}{4}$$

$$e \leq \frac{\pi d^3}{32} \cdot \frac{4}{\pi d^2}$$

$$\therefore \qquad e \leqq \frac{d}{8}$$

Hence in general, if the load be applied anywhere within a concentric circle of diameter $\frac{d}{4}$, the stresses will wholly remain compressive.

Following the same principle for any other shape of the column section also, the range within which the load can be applied so as not to produce tensile stresses may be determined.

Problem 200. *A short masonry pillar is 60 cm. × 60 cm. in section. The pillar carries a point load of 1,00,000 kg. acting on the centroidal axis of the section shown in Fig. 338 and at an eccentricity of 8 cms. from the longitudinal axis. Find the maximum and minimum stresses on the section.*

Solution. Load on the section $= W$

$= 100{,}000$ *kg.*

Sectional area $= A = 60 \times 60$ $cm.^2$

Moment due to eccentricity

$= M = 100{,}000 \times 8$ *kg. cm.*

Section modulus $= Z = \frac{1}{6} bd^2$

$= \frac{1}{6} \times 60 \times 60^2$ $cm.^3$

$= 36{,}000$ $cm.^3$

$\therefore$ Stress due to direct load

$$\text{,,} = \frac{W}{A} = + \frac{100{,}000}{60 \times 60} \ kg./cm.^2$$

$= 27{\cdot}8$ $kg./cm.^2$

Stress due to moment

$$= P_b = \pm \frac{M}{Z}$$

$$= \pm \frac{100{,}000 \times 8}{36000} \ kg./cm.^2$$

$= \pm 22{\cdot}2$ $kg./cm.^2$

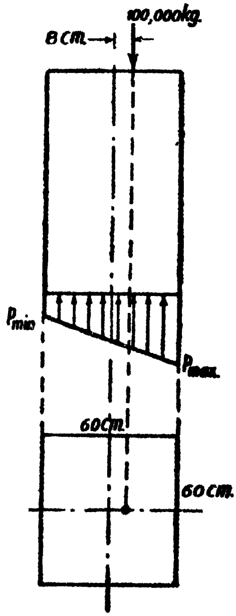


Fig. 338

$\therefore$ **Maximum stress** $= 27{\cdot}8 + 22{\cdot}2 = 50$ $kg./cm.^2$ *(compressive)*

Minimum stress $= 27{\cdot}8 - 22{\cdot}2 = 5{\cdot}6$ $kg./cm.^2$ *(compressive)*.

Problem 201. *The line of thrust in a compression testing specimen 1·432 cm. diameter is parallel to the axis of the specimen but is displaced from it. Calculate the distance of the line of thrust from the axis when the maximum stress is 15% greater than the mean stress on a normal section.*

Solution. Let the load on the section be *W kg.*

Area of the section $\frac{\pi d^2}{4}$ *cm.*^{2}

Section modulus $= \frac{\pi d^3}{32}$ *cm.*^{3}

Moment $= M = We$ *kg. cm.*

Stress due to direct load

$$=p_o = \frac{W}{A} = \frac{4W}{\pi d^2} \text{ kg./cm.}^2$$

Stress due to moment

$$=p_b = \pm\frac{M}{Z} = \frac{32\,We}{\pi d^3}$$

Maximum stress $= p_{max} = p_o + p_b$

$$= \frac{4W}{\pi d^2} + \frac{32We}{\pi d^3}$$

Mean stress $= p_o = \frac{4W}{\pi d^2}$

Since the maximum stress is 15% greater than mean stress, we have,

$$\frac{4W}{\pi d^2} + \frac{32We}{\pi d^3} = \frac{115}{100}\frac{4W}{\pi d^2}$$

$\therefore$ $$1 + \frac{8e}{d} = 1{\cdot}15$$

$\therefore$ $$\frac{8e}{d} = 0{\cdot}15$$

$\therefore$ $$e = 0{\cdot}15 \times \frac{d}{8} = \frac{0{\cdot}15 \times 0{\cdot}432}{8} \text{ cm.}$$

$$= 0{\cdot}02685 \text{ cm.}$$

Problem 202. *A short column of I section 25 cm. × 20 cm. has a cross-sectional area 52·05 cm.*^{2} *and maximum radius of gyration of 10·69 cm. A vertical load W kg. acts through the centroid of the section together with a parallel load of* $\frac{W}{4}$ *kg. acting through a point on the centre line of the web distant 8 cms. from the centroid Calculate the greatest allowable value of W if the maximum stress is not to exceed 800 kg./cm.*^{2} *What is then the minimum stress ?*

Area of the section $A = 52{\cdot}05$ *cm.*^{2}

Total load on the section

$$= W + \frac{W}{4} = 1{\cdot}25\, W \text{ kg.}$$

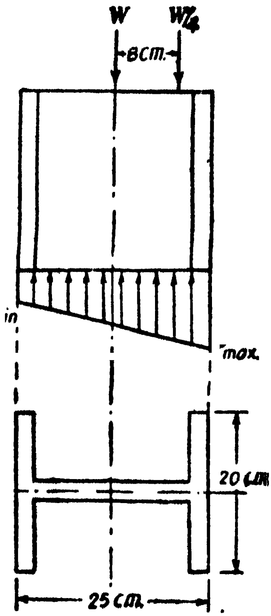


Fig. 339

Moment on the section

$$=M=\frac{W}{4}\times 8=2W \text{ kg. cm.}$$

Moment of inertia about the neutral axis

$$=I=AK^2$$
$$=52{\cdot}05(10{\cdot}69)^2 \text{ cm.}^4$$

Stress due to direct load

$$=p_o=+\frac{1{\cdot}25W}{52{\cdot}05} \text{ kg./cm.}^2$$

Stress due to moment

$$p_b=\pm\frac{M}{I}y$$

$$=\pm\frac{2W\times 12{\cdot}5}{52{\cdot}05(10{\cdot}69)^2} \text{ kg./cm.}^2$$

Equating the maximum stress to 800 *kg./cm.*² we have

$$\frac{1{\cdot}25W}{52{\cdot}05}+\frac{2W\times 12{\cdot}5}{52{\cdot}05\times 10{\cdot}69^2}=800$$

$$\therefore \quad \frac{W}{41{\cdot}64}+\frac{W}{238}=800$$

$$W=800\times\frac{238\times 41{\cdot}64}{(238+41{\cdot}64)} \text{ kg.}$$
$$=28360 \text{ kg.}$$

Minimum stress $=P_o-p_b$

$$=\frac{W}{41{\cdot}64}-\frac{W}{238} \text{ kg./cm.}^2$$

$$=\frac{(238-41{\cdot}64)}{238\times 41{\cdot}64} W \text{ kg./cm.}^2$$

$$=\frac{196{\cdot}36\times 28360}{238\times 41{\cdot}64} \text{ kg./cm.}^2$$

$$=562 \text{ kg./cm.}^2 \text{ (compressive)}$$

Problem 203. *A tie rod of constant circular cross-section is required to withstand a maximum tension of 50 tonnes, but the end fixing is such that the line of action is off the axis of the member by 0·75 cm. Find the minimum diameter of the tie rod if the maximum allowable stress is 1·25 t/cm.²*

Solution. Let the diameter of the rod be *d cm.* Stress due to direct load

$$P_o=\frac{50}{\left(\frac{\pi d^2}{4}\right)} \text{ tonnes/cm.}^2$$

$$=\frac{200}{\pi d^2} \text{ tonnes/cm.}^2$$

Moment due to eccentricity of the load

$$= M = 50 \times 0{\cdot}75 \text{ tonne cm.}$$

Stress due to moment

$$p_b = \frac{M}{Z}$$

$$= \frac{50 \times 0.75}{\left(\frac{\pi d^3}{32}\right)} \ t/cm.^2$$

$$= \frac{50 \times 32 \times 0{\cdot}75}{\pi d^3} \ t/cm.^2$$

$\therefore$ Maximum stress $= p_o + p_b = 1{\cdot}25 \ t/cm.^2$

$$\therefore \quad \frac{200}{\pi d^2} + \frac{50 \times 32 \times 0{\cdot}75}{\pi d^3} = 1{\cdot}25$$

$$\therefore \quad \frac{1}{\pi d^3}[200d + 1200] = 1{\cdot}25$$

$$\therefore \quad 200d + 1200 = 1{\cdot}25 \times \pi d^3$$

i.e., $d^3 = 50{\cdot}93d + 305{\cdot}6$

Solving by trial and error

$$d = 9{\cdot}18 \text{ cm.}$$

Problem 204. *A short column of external diameter D and internal diameter d carries an eccentric load W. Find the greatest eccentricity which the load can have without producing tension on the cross-section of the column.*

Solution. Stress due to direct load

$$= p_o = \frac{W}{\frac{\pi}{4}\left(D^2 - d^2\right)}$$

$$\therefore \quad p_o = \frac{4W}{\pi(D^2 - d^2)}$$

Section modulus

$$Z = \frac{\pi(D^4 - d^4)}{32D}$$

Moment $= M = We$

$\therefore$ Stress due to moment

$$= p_b = \frac{M}{Z}$$

$$= \frac{32WeD}{\pi(D^4 - d^4)}$$

If tension is to be just avoided

$$p_o = p_b$$

$$\therefore \quad \frac{4W}{\pi(D^2 - d^2)} = \frac{32WeD}{\pi(D^4 - d^4)}$$

$$\therefore \quad e = \frac{D^2 + d^2}{8D}$$

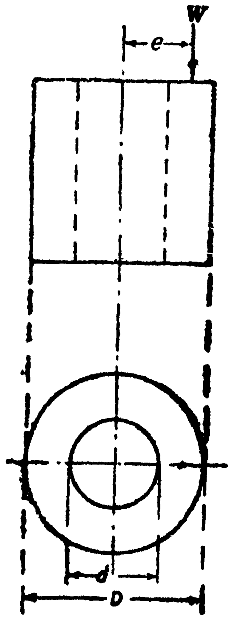


Fig. 340

Problem 205. *A short hollow pier 1·25 metre square outside and 0·70 metre square inside, supports a vertical point load of 12 tonnes located on a diagonal and 0·7 metre from the vertical axis of the pier. Neglecting the self weight of the pier, calculate the normal stresses at the four outside corners on a horizontal section of the pier.*

Solution. Stress due to direct load

$$=p_o=\frac{W}{A}$$

$$=\frac{12}{(1{\cdot}25^2-0{\cdot}75^2)}\ t/m^2$$

$$=12\ t/m^2.$$

Moment due to eccentricity $=M=We=12\times0{\cdot}7=8{\cdot}4\ tm.$

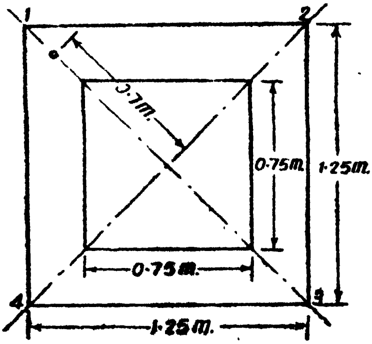


Fig. 341

Moment of inertia about the diagonal

$$I=\frac{1}{12}\Big[1{\cdot}25^4-0{\cdot}75^4\Big]\ metre^4$$

$$=\frac{2{\cdot}125}{12}\ metre^4$$

$\therefore$ Section modulus $=Z=\dfrac{I}{y_{max}}$

$$=\frac{2{\cdot}125}{12\times\left(\dfrac{1{\cdot}25}{\sqrt{2}}\right)}\ metre^3$$

$$=\frac{2{\cdot}125\sqrt{2}}{15}\ metre^3$$

$\therefore$ Stress due to moment at the corners 1 or 3

$$=p_b=\pm\frac{M}{Z}$$

$$=\pm\frac{8{\cdot}4\times15}{2{\cdot}125\sqrt{2}}\ t/m^2$$

$$=\pm41{\cdot}94\ t/m^2$$

At the corners 2 or 4 there will be no bending stress.

Hence the stresses at the various corners are as follows.
Stress at corner 1 $=+12+41{\cdot}94=+53{\cdot}94\ t/m^2$ (*compressive*).
Stress at corner 2 or 4 $=+12\ t/m^2$ (*compressive*)
Stress at corner 3 $=+12-41{\cdot}94=-29{\cdot}94\ t/m^2$ (*tensile*).

Problem 206. *Fig. 342 shows the section of a short 200 mm. × 140 mm. I section column carrying an axial load W_1 kg. at O and a load W_2 kg. at P in a direction parallel to that of W_1. The stress at the edge BC is 150 kg./cm.², compressive and that at DE is 900 kg./cm.² compressive. Determine the magnitudes of W_1 and W_2. Take for the I section, $A=36{\cdot}71$ cm.² and $I_{xx}=2624$ cm.⁴*

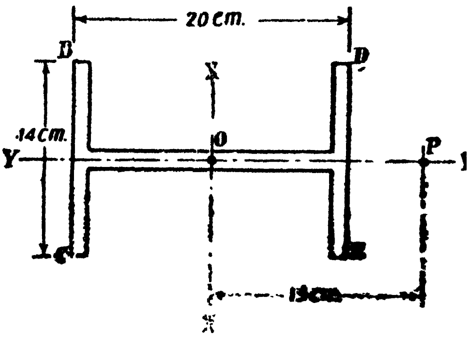


Fig. 342

Solution. Stress due to direct load

$$=p_0=\frac{W_1+W_2}{36\cdot71}\ kg./cm.^2$$

Stress due to the eccentricity of the load W_2 at P

$$=p_b=\pm\frac{W_2\times15}{2624}\times\frac{20}{2}\ kg./cm.^2$$

$$=\pm\frac{150}{2624}W_2\ kg./cm.^2$$

$\therefore$ Resultant stress at the edge DE

$$=\frac{W_1+W_2}{36\cdot71}+\frac{150\ W_2}{2624}=900\ kg./cm.^2 \quad ...(i)$$

Resultant stress at the edge BC

$$=\frac{W_1+W_2}{36\cdot71}-\frac{150W_2}{2624}=150\ kg./cm.^2 \quad ...(ii)$$

Subtracting Eq. (*ii*) from Eq. (*i*) we have

$$\frac{300W_2}{2624}=750$$

$$W_2=\frac{750\times2624}{300}\ kg.=6561\ kg.$$

Substituting in Eq. (*ii*)

$$\frac{W_1+6561}{36\cdot71}-\frac{150\times6561}{2624}=150$$

$$W_1=12720\ kg.$$

Problem 207. *A tension member consists of a T-section symmetrical about the vertical centre line having the following dimensions :*

Top flange : 10 cm wide and 2 cm. thick.

Web : 8 cm. deep and 2 cm. thick.

The member transmits a longitudinal pull P which acts on the section at a point on the centre line and 4 cm. from the bottom edge of the web.

Find : (a) the magnitude of P if the greatest tensile stress on the section is 1400 kg./cm.² and

(b) the minimum stress on the section when P is being transmitted.

Solution. Let us first determine the position of the centroidal axis and the moment of inertia about this axis. The relevent calculations are shown in the following table.

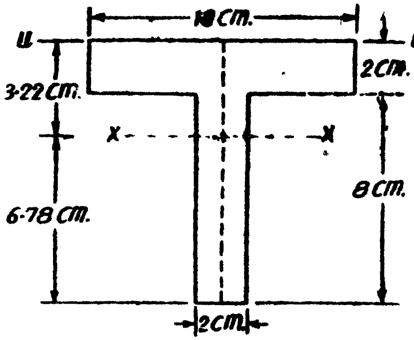


Fig. 343

Component	*Area* (*a*) *cm.*²	*Centroidal distance from top edge u-u* (*y*) *cm.*	*ay* *cm.*³	*ay*² *cm.*⁴	I_{self} *cm.*⁴
Top flange	20	1	20	20	$\frac{10\times2^3}{12}=6{\cdot}67$
Web	16	6	96	576	$\frac{2\times8^3}{12}=85{\cdot}33$
Total	36		116	596	92

Distance of centroidal axis *xx* from the top edge *uu*

$$=\bar{y}=\frac{\Sigma ay}{\Sigma a}=\frac{116}{36}\ cm.$$

$$=3{\cdot}22\ cm.$$

Moment of inertia about the top edge $=I_{uu}=\Sigma I_{self}+\Sigma ay^2$

$$=92+596=688\ cm.^4$$

Let the moment of inertia about the centroidal axis *XX* be *I*

$$\therefore \quad I_{uu}=I+(\Sigma a)\bar{y}^2$$

$$\therefore \quad 688=I+36(3{\cdot}22)^2$$

$$\therefore \quad I=314{\cdot}7\ cm.^4$$

Let the pull on the section be *P kg*.

Eccentricity $=6{\cdot}78-4{\cdot}00=2{\cdot}78\ cm.$

$\therefore$ Moment on the section$=Pe=2{\cdot}78\ P\ kg.\ cm.$

Equating the maximum tensile stress to the given permissible limit, we have,

$$\frac{P}{A}+\frac{M}{I}y_t=1400\ kg./cm.^2$$

$$\frac{P}{36}+\frac{2{\cdot}78P\times6{\cdot}78}{314{\cdot}7}=1400$$

$$P[0{\cdot}0278+0{\cdot}05988]=1400$$
$$\therefore \quad 0{\cdot}08768P=1400$$
$$\therefore \quad P=.5970 \ kg.$$

$$\text{Minimum stress}=\frac{P}{A}-\frac{M}{I}y_c$$

$$=\frac{15970}{36}-\frac{(2{\cdot}8\times15970)}{314{\cdot}7}\times3{\cdot}22 \ kg./cm.^2$$
$$=443{\cdot}6-454{\cdot}2=-10{\cdot}6 \ kg./cm.^2 \ (compressive)$$

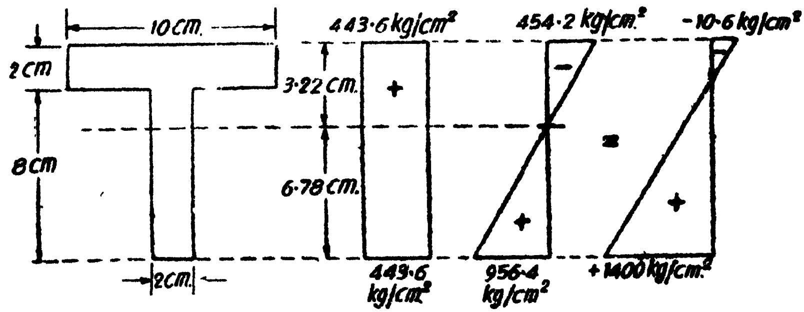


Fig. 344

Problem 208. *A concrete block has the cross-section shown in Fig. 345. The block weighs 9 tonnes and carries a vertical downward load of 2 tonnes on the axis XX but eccentric about the YY axis. Calculate the distance of the point P from the axis YY if the pressure under the block along the edge AD is just twice the pressure under the edge BC and determine these pressures.*

Solution. Stress due to direct load

$$p_o=\frac{9+2}{2\times2} \ t/m^2$$
$$=2{\cdot}75 \ t/m^2.$$

Let the distance between P and the YY axis be *e metres*.

$\therefore$ eccentricity $=e$ *metres*

Moment due to the eccentricity of the $2t$ load

$=M=2e$ *tonne metre*

Section modulus $=Z$

$$=\frac{2\times2^2}{6} \ m^3$$

$$=\frac{4}{3}m^3$$

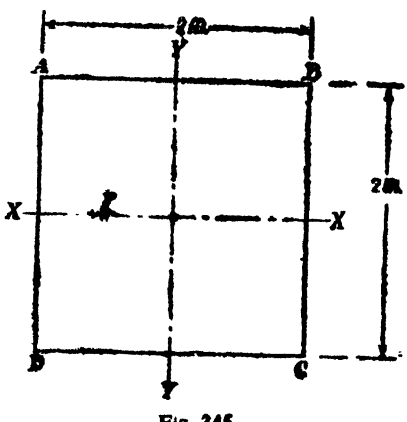


Fig. 345

Extreme stresses due to moment

$$=p_b=\pm\frac{M}{Z}$$

$$=\pm\frac{2e\times 3}{4}\ t/m^2$$

$$=\pm 1{\cdot}5e\ t/m^2$$

$\therefore \quad p_{max}=p_o+p_b=(2{\cdot}75+1{\cdot}5e)\ t/m^2$

$p_{min}=p_o-p_b=(2{\cdot}75-1{\cdot}5e)\ t/m^2$

But $\quad p_{max}=2p_{min}$

$\therefore \quad 2{\cdot}75+1{\cdot}5e=2\,(2{\cdot}75-1{\cdot}5e)$

$\therefore \quad 2{\cdot}75+1{\cdot}5e=5{\cdot}5-3e$

$\therefore \quad 4{\cdot}5e=2{\cdot}75$

$\therefore \quad e=\dfrac{2{\cdot}75}{4{\cdot}5}$ *metre*

$\therefore \quad e=0{\cdot}61$ *metre*

$\therefore\ p_{max}=2{\cdot}75+1{\cdot}5\times 0{\cdot}61=3{\cdot}665\ t/m^2$

and $p_{min}=2{\cdot}75-1{\cdot}5\times 0{\cdot}61=1{\cdot}835\ t/m^2$.

Problem 209. *A masonry pier 3 m × 4 m supports a vertical load of 60t as shown in Fig. 346*

(*a*) *Find the stresses at the corners of the pier.*

(*b*) *What additional load should be applied at the centre of the pier so that there is no tension anywhere in the pier section ?*

(*c*) *What are the stresses at the four corners with the additional load at the centre ?*

Solution.

(*a*) Area of the section

$=A=4\times 3$

$=12$ *metre*^{2}

Section modulus about the *xx* axis

$=Z_{xx}=\dfrac{4\times 3^2}{6}$

$=6$ *metre*^{3}

Section modulus about the *yy* axis

$Z_{yy}=\dfrac{3\times 4^2}{6}$

$=8$ *metre*^{3}

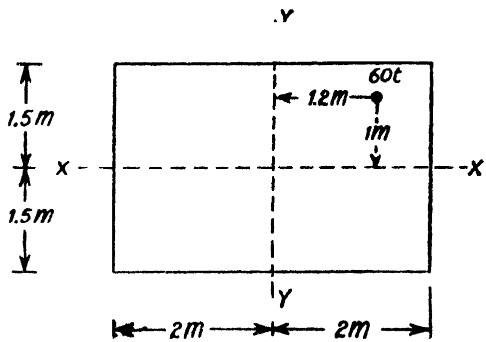


Fig. 346

Uniform Direct Stress due to load$=\dfrac{W}{A}=\dfrac{60}{12}=+5\ t/m^2$

(*compressive*)

Max. bending stress due to eccentricity of the load about the *xx* axis

$$=\pm\frac{M_{xx}}{Z_{xx}}=\pm\frac{60\times 1}{6}=\pm 10\ t/m^2$$

compressive at *A* and *B* and *tensile* at *C* and *D*.

Maximum bending stress due to eccentricity of the load about the *yy* axis

$$= \frac{M_{yy}}{Z_{yy}}$$

$$= \pm \frac{60 \times 1.2}{8} = \pm 9 \ t/m^2 \text{ compressive at } B$$

and *C* and tensile at *A* and *D*.

Hence the resultant stresses at the four corners are as follows :

Stress at $A = +5+10-9 = +6 \ t/m^2$ (*compressive*)
Stress at $B = +5+10+9 = +24 \ t/m^2$ (*compressive*)
Stress at $C = +5-10+9 = +4 \ t/m^2$ (*compressive*)
Stress at $D = +5-10-9 = -14 \ t/m^2$ (*tensile*)

(*b*) In order to avoid tension the additional axial load should produce a compressive stress of 14 t/m^2. Hence additional load required

$$= +14 \times 12 = 168 \ t$$

(*c*) After the above additional load is applied the final Stresses at the four corners will be as follows :

Stress at $A = 6+14 = +20 \ t/m^2$ (*compressive*)
Stress at $B = +24+14 = +38 \ t/m^2$ (*compressive*)
Stress at $C = +4+14 = +18 \ t/m^2$ (*compressive*)
Stress at $D = 0$.

Problem 210. *An R.C.C. footing rectangular in plan 2 m×3 m carries four vertical concentrated loads of 10t, 20t, 30t and 40t which are located as shown in Fig. 347. Neglecting the self-weight of the footing.*

(*i*) *Calculate the intensity of loading on the foundation at each of the corners A, B, C and D.*

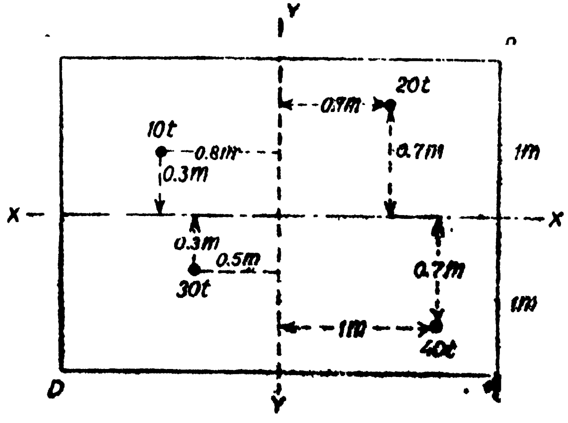


Fig. 347

(ii) Determine the location of another 50t load with reference to the axis XX and YY which will make the intensity of loading uniform at all corners.

Solution.

Area of the footing $=3\times2=6$ *metre*^{2}

Section modulus about the axis XX

$$=Z_{xx}=\frac{3\times2^2}{6}=2 \text{ metre}^3$$

Section modulus about the axis YY

$$=Z_{yy}=\frac{2\times3^2}{6}=3 \text{ metre}^3$$

(*i*) Direct stress due to direct loading

$$=\frac{\text{Total load}}{\text{area}}=+\frac{10+20+30+40}{6}$$

$$=+16{\cdot}67 \; t/m^2 \quad \textit{(compressive)}$$

B.M. due to loads eccentric about the axis XX

$$=M_{xx}=\{(30\times0{\cdot}3)+(40\times0{\cdot}7)\}-\{(10\times0{\cdot}3)+(20\times0{\cdot}7)\}$$

$$=20 \; tm$$

∴ Max stress due to the above B.M.

$$=\pm\frac{M_{xx}}{Z_{xx}}=\pm\frac{20}{2}=\pm10 \; t/m^2$$

compressive at C and D and tensile at A and B.

B.M. due to loads eccentric about the axis YY

$$=M_{yy}=\{(20\times0{\cdot}7)+(40\times1)\}-\{(10\times0{\cdot}8)+(30\times0{\cdot}5)\}$$

$$=41 \; tm$$

∴ Max. stress due to the above B.M.

$$=\pm\frac{M_{yy}}{Z_{yy}}=\pm\frac{41}{3}=\pm13{\cdot}67 \; t/m^2$$

compressive at *B and C and tensile at A and D*

Hence the resultant stress at the corners are as follows :

Stress at $A=+16{\cdot}67-10-13{\cdot}67=-7 \; t/m^2$ *(tensile)*

Stress at $B=+16{\cdot}67-10+13{\cdot}67=+20{\cdot}34 \; t/m^2$ *(compressive)*

Stress at $C=+16{\cdot}67+10+13{\cdot}67=+40{\cdot}34 \; t/m^2$ *(compressive)*

Stress at $D=+16{\cdot}67+10-13{\cdot}67=+13 \; t/m^2$. *(compressive)*

(*ii*) If it is desired to make the stress uniform at all the four corners the condition to be satisfied is $M_{xx}=0$ and $M_{yy}=0$.

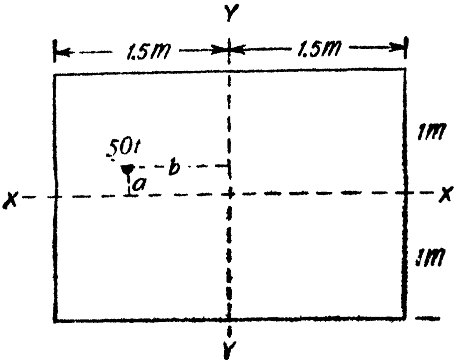


Fig. 348

Let the additional 50t load be placed as shown in Fig. 348 at a distance a from the axis XX and b from the axis YY

Since $M_{xx}=0$,

$50\times a=20 \quad \therefore\ a=0{\cdot}4\ m$

Since $M_{yy}=0$,

$50\times b=41 \quad \therefore\ b=0{\cdot}82\ m$

Problem 211. *The section shown in Fig. 349 is subjected to a compressive load of 60 tonnes acting at the load point L. Find the maximum compressive and tensile stress intensities across the section.*

Solution. Let us first determine the position of the centroidal axis XX and the moment of inertia about the centroidal axis.

This calculation is shown in the following table.

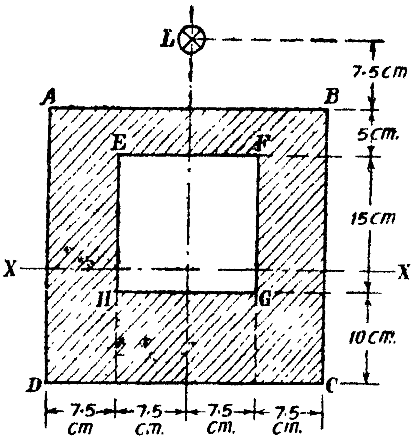


Fig. 349

Component	Area a cm^2	Centroidal distance y from CD cm	ay cm^3	ay^2 cm^4	I_{self} cm^4
ABCD	900	15	13500	202500	$\frac{30^4}{12}=67500$
Deduct for EFGH	225	17·5	3937·5	68906·25	$\frac{15^4}{12}=4218{\cdot}75$
Total	675		9562·5	133593·75	63281·25

∴ Centroidal distance from the edge CD

$$=\bar{y}=\frac{\Sigma ay}{\Sigma a}$$

$$=\frac{9562{\cdot}5}{675}\ cm.$$

$$=14{\cdot}17\ cm.$$

Moment of inertia about the axis CD

$$I_{cd}=\Sigma I_{self}+\Sigma ay^2$$

$$=63281{\cdot}25+133593{\cdot}75\ cm.^4$$

$$=196875\ cm^4$$

But, $I_{cd}=I_c+(\Sigma a)\bar{y}^2$

∴ $196875=I_{xx}+675\times14{\cdot}17^2$

∴ $I_{xx}=196875-135500\ cm.^4$

$$=61375\ cm^4$$

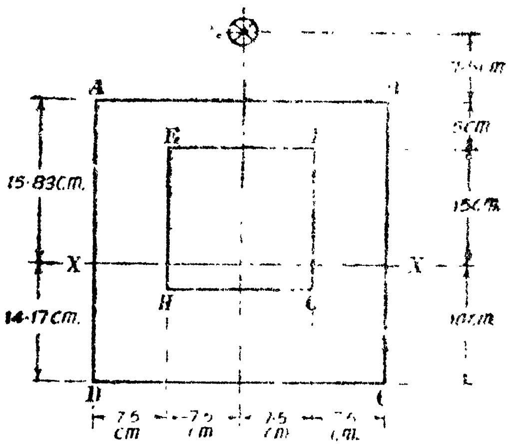


Fig. 350

Eccentricity of the load from the XX axis

$$=e=7{\cdot}5+15{\cdot}83\ cm.$$

$$=23{\cdot}33\ cm$$

∴ Moment on the section

$$=M=60\times23{\cdot}33\ t\ cm.$$

∴ Stress at the edge AB due to moment

$$=+\frac{60\times23{\cdot}33}{61375}\times15{\cdot}83\ t/cm.^2$$

$$=+0{\cdot}3612\ t/cm.^2\ (compressive)$$

Stress at the edge CD due to moment

$$=-\frac{60\times23{\cdot}33}{61375}\times14{\cdot}17\ t/cm.^2$$

$$=-0{\cdot}3232\ t/cm.^2\ (tensile)$$

Stress due to direct load

$$=\frac{60}{675}\ t/cm.^2$$

$$=+0.0889\ t/cm^2\ (compressive)$$

Resultant stress at the edge AB

$$=+0.3612+0.0889\ t/cm.^2$$

$$=+0.4501\ t/cm^2\ (compressive)$$

and Resulant stress at the edge CD

$$=-0.3232+0.0889\ t/cm^2$$

$$=-0.2343\ t/cm^2\ (tensile)$$

Problem 212. *A column section 30 cm. external diameter and 15 cm. internal diameter supports an axial load of 260 tonnes and an eccentric load of P tonnes at an eccentricity of 40 cms. If the compressive and tensile stresses are not to exceed 1400 kg./cm.² and 600 kg./cm.² respectively, find the magnitude of the load P.*

Solution. Area of the column section

$$=\frac{\pi}{4}\left(30^2-15^2\right)$$

$$=530.2\ cm.^2$$

$$I=\frac{\pi}{64}\left(30^4-15^4\right)$$

$$=37270\ cm^4.$$

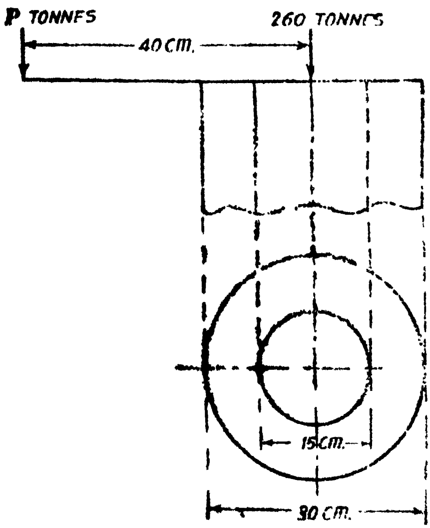


Fig. 351

Equating the maximum compressive stress to 1400 $kg./cm.^2$

$$\frac{260+P}{530.2}+\frac{P\times40}{37270}\times\frac{30}{2}=1.400$$

$$0.4904+\frac{P}{530.2}+\frac{600}{37270}P=1.400$$

$$\therefore\ \frac{P}{530.2}+\frac{60}{3727}P=0.91$$

$$\therefore\ P=50.59\ tonnes.$$

Equating the maximum tensile stress to 600 $kg./cm.^2$

$$\frac{P\times40}{37270}\times\frac{30}{2}-\frac{260+P}{530.2}=0.6\ t/cm.^2$$

$$\frac{60}{3727}P-0.4904-\frac{P}{530.2}=0.6$$

$$\therefore\ \frac{60}{3727}P-\frac{P}{530.2}=1.0904$$

$$\therefore \quad \frac{28085}{3727 \times 530\cdot2} P = 1\cdot0904$$

$$P = \frac{1\cdot0904 \times 3727 \times 530\cdot2}{28085} \text{ tonnes}$$

$$= 76\cdot68 \text{ tonnes}$$

Taking the smaller value of P,

$$P = 50\cdot39 \text{ tonnes.}$$

Problem 213. *A beam 2·5 metres long is simply supported at the two ends, and carries a point load of 4 tonnes at the centre, and a longitudinal axial tensile load of 10 tonnes. If the section of the beam is rectangular, 30 cms. wide and 50 cms. deep, find the maximum and minimum stresses at the mid span section of the beam. Neglect the self-weight of the beam.*

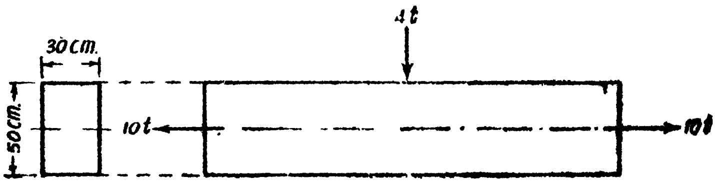


Fig. 352

Solution. Area of the beam section

$$= A = 30 \times 50 = 1500 \text{ cm.}^2$$

Stress due to direct load

$$= p_o = \frac{10 \times 1000}{1500} \text{ kg./cm.}^2$$

$$= 6\cdot67 \text{ kg./cm.}^2 \text{ (tensile)}$$

Maximum bending moment

$$= M = \frac{Wl}{4}$$

$$= \frac{4 \times 1000 \times 2\cdot5 \times 100}{4} \text{ kg. cm}$$

$$= 250000 \text{ kg. cm.}$$

Section modulus

$$= Z = \frac{30 \times 50^2}{6} \text{ cm.}^3$$

$$= 12500 \text{ cm.}^3$$

$\therefore$ Maximum stress due to bending moment

$$= p_b = \pm \frac{M}{Z}$$

$$= \pm \frac{250000}{12500} \text{ kg./cm.}^2$$

$$= \pm 20 \text{ kg./cm.}^2$$

$\therefore$ Stress in the bottommost fibre

$$= -6{\cdot}67 - 20 = -26{\cdot}67 \ kg./cm.^2 \ (tensile)$$

and stress in the uppermost fibre

$$= -6{\cdot}67 + 20$$
$$= +13{\cdot}33 \ kg./cm.^2 \ (compressive)$$

Problem 214. *A beam of rectangular section, 45 cm. wide and 75 cm. deep has a span of 6 metres. The beam is subjected to a uniformly distributed load of 2000 kg. per metre run (including the self-weight of the beam) over the whole span. The beam is also subjected to a longitudinal axial compressive load of 150 tonnes. Find the extreme fibre stresses at the mid span section.*

Solution. Stress due to direct load

$$= p_o = \frac{P}{A} = \frac{150 \times 1000}{45 \times 75} \ kg./cm.^2$$
$$= 44 \ kg./cm.^2 \ (compressive)$$

Maximum bending moment

$$= M = \frac{wl^2}{8}$$
$$= \frac{2000 \times 6^2}{8} \times 100 \ kg. \ cm.$$
$$= 900{,}000 \ kg. \ cm.$$

Section modulus $= Z = \frac{bd^2}{6}$

$$= \frac{45 \times 75^2}{6} \ cm.^3$$

$\therefore$ Max. stress due to bending moment

$$= p_b = \pm \frac{M}{Z}$$
$$= \pm \frac{900{,}000 \times 6}{45 \times 75^2} \ kg./cm.^2$$
$$= \pm 21 \ kg./cm.^2$$

$\therefore$ The extreme stresses are,

$44 + 21 = 65 \ kg./cm.^2$ *(compressive)* in the top fibre

and, $44 - 21 = 23 \ kg./cm.^2$ *(compressive)* in the bottom fibre.

Problem 215. *A beam of rectangular section 45 cm. wide and 75 cm. deep has a span of 6 metres. The beam carries a uniformly distributed load of 2000 kg. per metre run including the self-weight of the beam. The beam is also subjected to a longitudinal compressive force of 160 tonnes located at the lower third point as shown in Fig. 353. Find the extreme fibre stresses at the mid span section.*

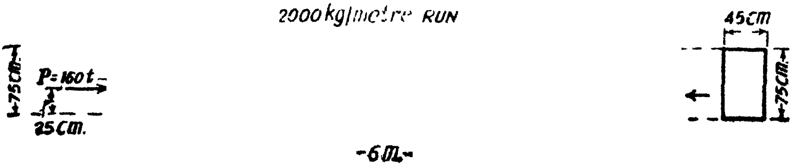


Fig. 353

Solution. Stress due to direct longitudinal load

$$=\frac{P}{A}=+\frac{160\times1000}{4}\ kg./cm.^2$$

$$=+47\ kg./cm.^2\ (compressive)$$

Section modulus $=Z=\frac{bd^2}{6}$

$$45\times75^2\ cm.^3$$

B.M. due to external loads normal to the span

$$=\frac{wl^2}{}=\frac{2000\times6^2}{8}\times100\ kg.\ cm.$$

$$=900{,}000\ kg.\ cm.\ (sagging\ moment)$$

Max. stress due to sagging moment

$$=\pm\frac{900{,}000\times6}{45\times75^2}\ kg./cm.^2$$

$$=\pm21\ kg./cm.^2$$

B.M. due to the eccentricity of the longitudinal load

$$=P.e$$

$$=160\times1000\,(37{\cdot}5-25)\ kg.\ cm.$$

$$=160\times1000\times12{\cdot}5\ kg.\ cm.\ (hogging\ moment)$$

Max stress due to hogging moment

$$=\mp\frac{160\times1000\times12{\cdot}5\times6}{45\times75^2}\ kg./cm.^2$$

$$=\mp47\ kg./cm.^2$$

Hence the resultant extreme stresses are

$$+47+21-47=+21\ kg./cm.^2\ (compressive)\ \textbf{at the top,}$$

and $$+47-21+47=+73\ kg./cm.^2\ (compressive)\ \textbf{at the bottom.}$$

Problem 216. *The bent member ABCD shown in Fig 354 is 10 cm. in diameter. If the member carries a point load of 2000 kg. at the free end A, find the maximum tensile stress on the section of the part BC of the member.*

Solution. Area of the member

$$A=\frac{\pi}{4}\times 10^2\ cm.^2$$

$$=78{\cdot}54\ cm.^2$$

Consider the part BC. This is subjected to a tensile load of 2000 kg. and a bending moment of 2000×75 $kg./cm$.

∴ Stress due to the tensile load

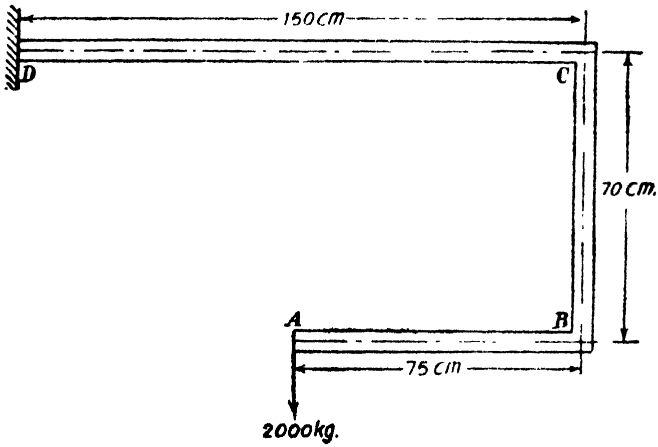


Fig. 354

$$=\frac{2000}{78{\cdot}54}\ kg./cm.^2$$

$$=25{\cdot}46\ kg./cm.^2\ (tensile)$$

Section modulus $=Z=\dfrac{\pi d^3}{32}=\dfrac{\pi\times 10^3}{32}\ cm.^3$

∴ Max. stress due to bending moment

$$=\pm\frac{M}{Z}$$

$$=\pm\frac{2000\times 75\times 32}{\pi\times 10^3}\ kg./cm.^2$$

$$=\pm 1528\ kg/cm.^2$$

∴ Maximum tensile stress

$$=-25{\cdot}46-1528\ kg./cm.^2$$

$$=-1553{\cdot}46\ kg/cm.^2\ (tensile)$$

Problem 217. *Fig. 355 shows a cross-section of the vertica standard of a radial drilling machine, in which the drill thrust imposel on the stand rd a vertical pull whose line of action passes through thes point D. Find by what percentage the maximum tensile stress induced differs from the value which would be obtained if the cored hole had been concentric with the outside diameter of the standard.*

Solution. Sectional area of the standard

$$=A=\frac{\pi}{4}\left[10^2-6^2\right] cm.^2$$

$$=64\ \pi\ cm.^2$$

Let a be the distance of the centroid of the section from B

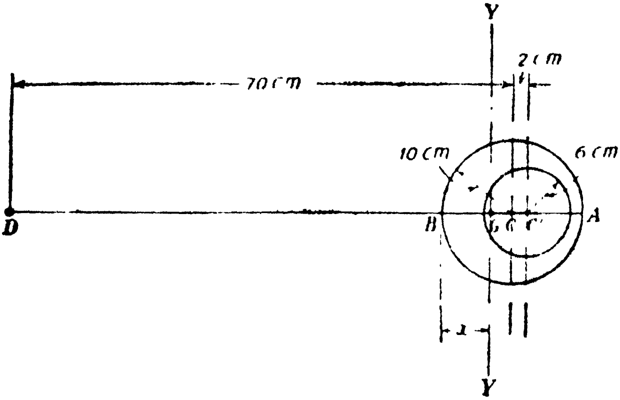


Fig. 355

$$\therefore \qquad A\bar{x}=64\pi x$$

$$=\pi\times 10^2\times 10-\pi\times 6^2\times 12$$

$$\therefore \qquad x=\frac{10^3-36\times 12}{64}\ cm.$$

$$=\frac{71}{8}\ cm.$$

$$=8{\cdot}88\ cm.$$

Let the tensile load be P *kg.*

∴ Stress due to direct load

$$=p_0=\frac{P}{64\pi}\ kg./cm.^2$$

$$=\frac{0{\cdot}015625\ P}{\pi}\ kg./cm.^2$$

Moment of inertia about the centroidal axis YY

$$=\left[\frac{\pi 10^4}{4}+\pi\times 10^2(10-8{\cdot}88)^2\right]$$

$$-\left[\frac{\pi\times 6^4}{4}+\pi\times 6^2(12-8{\cdot}88)^2\right] cm.^4$$

$$=1951\pi\ cm.^4$$

$$\text{B.M.}=(60+\bar{x})P\ kg.\ cm.$$

$$=68{\cdot}88\ P\ kg.\ cm.$$

∴ Tensile stress due to bending moment

$$=p_b=\frac{68.88\,P}{1951\pi}\times 8.88\ kg./cm.^2$$

$$=\frac{0.3134\ P}{\pi}\ kg./cm.^2$$

∴ Maximum tensile stress

$$=p_o+p_b$$

$$=0.015625\ \frac{P}{\pi}+0.3134\frac{P}{\pi}\ kg./cm.^2$$

$$=0.3290\ \frac{P}{\pi}\ kg./cm.^2$$

If the core had been central,

$$A=64\pi\ cm.^2$$

∴ Stress due to direct load

$$=f_o=\frac{P}{64\pi}\ kg./cm.^2$$

$$=0.015625\frac{P}{\pi}\ kg./cm.^2$$

$$I=\frac{\pi}{4}\left[10^4-6^4\right]\ cm.^4$$

$$=2176\pi\ cm.^4 \qquad \text{B.M.}=70\ P\ kg.\ cm.$$

Maximum bending stress

$$f_b=\frac{70\ P}{2176\pi}\times 10\ kg./cm.^2$$

$$=0.3217\ \frac{P}{\pi}\ kg./cm.^2$$

Maximum tensile stress

$$=f_o+f_b$$

$$=0.015625\ \frac{P}{\pi}+0.3217\frac{P}{\pi}\ kg./cm.^2$$

$$=0.3373\ \frac{P}{\pi}\ kg./cm.^2$$

∴ Percentage change in the maximum tensile stress

$$=\frac{0.3373-0.3290}{0.3373}\times 100\%$$

$$=2.461\%.$$

Problem 218. *The cross-section of a short masonry pier is 60 cm. × 120 cm. The force action across the section consists of a normal compressive load of 30 tonnes at A and a bending moment of 15 tonne metre which causes tension above XX. For this load condition find the maximum and minimum stress across the section.*

Solution. Stress due to direct load

$$p_o = \frac{W}{A}$$

$$= \frac{30\times 1000}{60\times 120}\ kg./cm.^2$$

$$= 4{\cdot}25\ kg./cm.^2$$

$$M = (30\times 60 - 15\times 100)\ 1000\ kg.cm$$

$$= 300\times 1000\ kg.\ cm.$$

$\therefore$ Stress due to net moment

$$= \pm \frac{M}{Z}$$

$$= \frac{300\times 1000\times 6}{60\times 120\times 120}$$

$$= \pm 2{\cdot}08\ kg./cm.^2$$

$$\therefore\ p_{max} = 4{\cdot}25 + 2{\cdot}08$$

$$= 6{\cdot}33\ kg./cm.^2$$

and $p_{min} = 4{\cdot}25 - 2{\cdot}08$

$$= 2{\cdot}17\ kg./cm^2$$

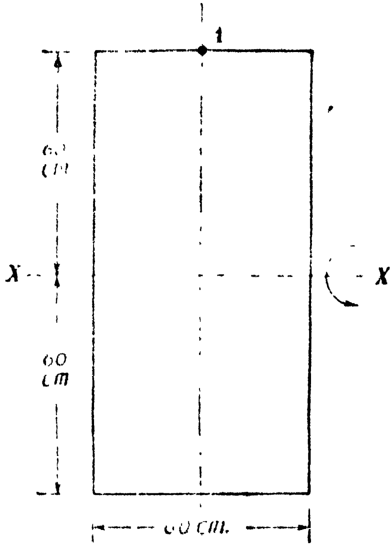


Fig. 356

Problem 219. *Find the maximum tensile and compressive stresses on the section AB of the clamp shown in Fig. 357 when a pressure of 250 kg. is exerted by the screw. The section is rectangular 3 cm. × 1 cm.*

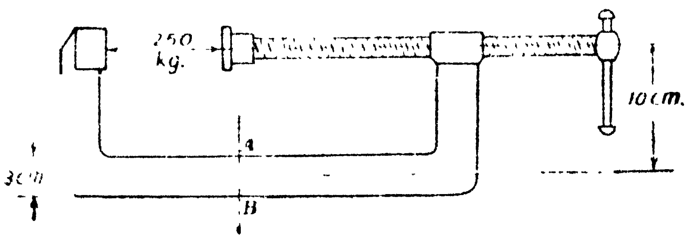


Fig. 357

Solution. The section AB is subjected to a tension of 250 *kg.* as well as bending moment of $250\times 10 = 2500$ *kg. cm.*

Sectional area $= A = 3\times 1 = 3$ *cm.*²

Stress due to direct load

$$= p = \frac{250}{3}\ kg./cm.^2$$

$$= 83{\cdot}33\ kg./cm.^2\ (tensile)$$

Section modulus $Z = \frac{bd^2}{6}$

1×3^2 ... cm.³

∴ Maximum stress due to bending moment

$$=p_b=\frac{M}{Z}$$

$$=\frac{2500}{1\cdot5}\ kg./cm.^2$$

$$=1666\cdot67\ kg./cm.^2$$

∴ Maximum tensile stress on the section

$$=1666\cdot67+83\cdot33$$

$$=1750\ kg./cm.^2$$

Maximum compressive stress on the section

$$=1666\cdot67-83\cdot33$$

$$=1583\cdot34\ kg./cm.^2$$

§57. Walls and Pillars subjected to wind pressure

Fig. 358 shows a masonry pillar of plan dimensions $b\times a$ and of height h. Let the weight per unit volume of masonry be ρ.

∴ Weight of pillar

$$=W=\rho bah$$

∴ Intensity of stress on the base due to the weight of masonry

$$=p_o=\frac{W}{A}=\frac{W}{ba}$$

Let the vertical face $a\times h$ be subjected to a uniform wind pressure of intensity p_w per unit area of the vertical surface.

∴ Total wind pressure

$$=P=p_w ah$$

∴ Moment on the base due to wind pressure

$$=M=P\frac{h}{2}$$

∴ Stress caused on the base section due to the moment

$$=p_b=\pm\frac{M}{Z}$$

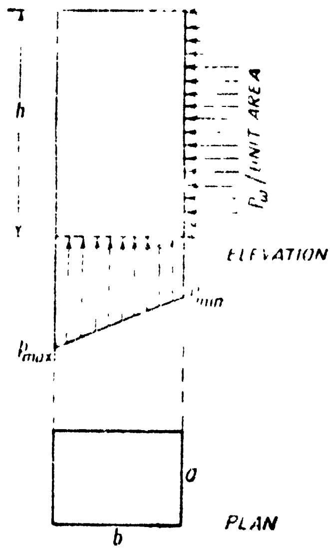


Fig. 358

where Z is the section modulus of the base section.

∴ The extreme stresses on the base section are,

$$p_{max}=p_o+p_b$$

and $$p_{min}=p_o-p_b$$

Problem 220. *A masonry pillar 8 metres high is 1·5 metres × 2·5 metres in section. A horizontal wind pressure of 140 kg. per metres² acts on the 2·5 m × 8 m face. Find the maximum and minimum stress intensities induced on the base section. The weight of masonry is 2250 kg. per cubic metre.*

Solution. Weight of the pillar $= W = 1{\cdot}5 \times 2{\cdot}5 \times 8 \times 2250$ *kg.*
Area of the section $= A = 1{\cdot}5 \times 2{\cdot}5$ *metre*^{2}

∴ Stress due to direct load

$$p_o = \frac{W}{A} = \frac{1{\cdot}5 \times 2{\cdot}5 \times 8 \times 2250}{1{\cdot}5 \times 2{\cdot}5} \ kg/metre^2$$

$$= 18000 \ kg./metre^2$$

Total wind pressure

$$= P = 140 \times 2{\cdot}5 \times 8 \ kg. = 2800 \ kg.$$

$$\therefore \quad M = P\frac{h}{2}$$

$$= 2800 \times \frac{8}{2} \ kg. \ m.$$

$$= 11200 \ kg. \ m.$$

∴ Section modulus $= Z = \dfrac{2{\cdot}5 \times 1{\cdot}5^2}{6}$ *metre*^{3}

∴ Stress due to moment

$$= p_b = \pm\frac{M}{Z}$$

$$= \pm\frac{11200 \times 6}{2{\cdot}5 \times 1{\cdot}5^2} \ kg./metre^2$$

$$= \pm 11950 \ kg./metre^2$$

∴ The extreme stresses at the base are,

$$p_{max} = 18000 + 11950 = 29950 \ kg./metre^2$$

$$p_{min} = 18000 - 11950 = 6050 \ kg./metre^2.$$

Problem 221. *A masonry chimney having the shape of a frustum of a cone is 25 metres high. The external diameter at the top and the internal diameter at the bottom is 2 metres. The chimney is 0·5 metre thick at the base. If the weight of the chimney is 180 tonnes find the uniform horizontal wind pressure that may act per unit projected area of the chimney in order tension at the base may be just avoided.*

Solution Area of the base section

$$= A = \frac{\pi}{4}[3^2 - 2^2] \ m^2$$

$$= 3{\cdot}927 \ metre^2$$

Moment of inertia of the base section about a diameter

$$I = \frac{\pi}{64}[3^4 - 2^4] \ metre^4$$

$$= 3{\cdot}191 \ metre^4$$

∴ Section modulus of the base section

$$= Z = \frac{3{\cdot}191}{\left(\frac{3}{2}\right)} = 2{\cdot}1273 \ metre^3$$

Direct stress due to weight of the chimney

$$p_c = \frac{W}{A} = \frac{180 \times 1000}{3.927} \ kg./metre^2$$

$= 45830 \ kg./metre^2$

Let the uniform intensity of wind pressure be p kg./metre² of projected area of the chimney.

Projected area of the chimney = area of trapezium $ABCD$

$$= \frac{25}{2} \times (2+3) \ metre^2$$

$= 62.5 \ metre^2$

Total wind pressure $= P = 62.5\,p$ kg.

This resultant pressure acts at the level of the centroid of the trapezium $ABCD$.

Height of centroid of the trapezium $ABCD$ above the base

$$= \bar{y} = \frac{3+2 \times 2}{3+2} \times \frac{25}{3} \ metres$$

$= 11.67$ metres

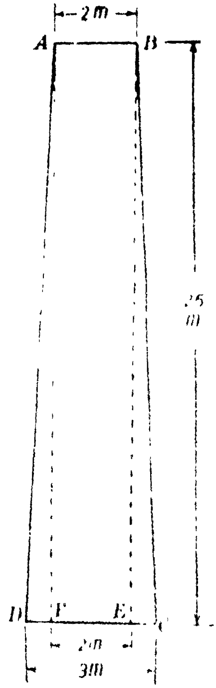


Fig. 359

∴ Moment due to wind pressure

$= M = P\bar{y}$

$= 62.5\,p \times 11.67$ kg. m.

∴ Stress due to moment

$$= p_b = \pm \frac{M}{Z}$$

$$= \pm \frac{62.5 \times 11.67p}{2.1273} \ kg./m^2$$

In order tension at the base is to be just avoided,

$$p_b = p_c$$

$$\therefore \quad \frac{62.5 \times 11.67p}{2.1273} = 45830$$

$$\therefore \quad p = \frac{45830 \times 2.1273}{62.5 \times 11.67} \ kg./metre^2$$

$= 133.7$ kg./m².

Problem 219. *A 20 metres high masonry chimney is 2 metre square at the base and tapers to 1 metre square at the top. The tapered central flue is circular in cross-section and 1 metre diameter at the base.*

If the total weight of the brickwork above the base is 130 tonnes find for what uniform intensity of wind pressure on one face of the chimney the stress distribution across the base just cease to be wholly compressive.

Solution. Area of the base section

$A = 2^2 - \frac{\pi}{4} \times 1^2$ *metre*^{2}

$= 3.215$ *metre*^{2}

$\therefore$ Stress to the direct load

$= \frac{130 \times 1000}{3.215}$ *kg./metre*^{2}

$= 40420$ *kg./m.*^{2}

Projected area exposed to wind

= area of the trapezium *ABCD*

$\frac{1}{2} \times 20 \times (1 + 2)$ *metre*^{2}

$= 30$ *metre*^{2}

Let the intensity of wind pressure be *p. kg./merte*^{2} of projected area

$\therefore$ Total wind pressure

$P = 30\,p$ *kg*

This pressure acts at the level of the centroid of the trapezium *ABCD*, *i.e.*, at a height of

$\frac{2 + 2 \times 1}{2 + 1} \times \frac{20}{3}$ *metres* above the base

i.e., 8.89 *metres* above the base

$\therefore$ Moment due to wind pressure

$M = 30\,p \times 8.89$ *kg. metre*

$= 266.70\,p$ *kg. metre*

Moment of inertia of the base section

$I = \frac{2^4}{12} - \frac{\pi \times 1^4}{64}$ *metre*^{4}

$= 1.333 - 0.049$ *metre*^{4}

$= 1.284$ *metre*^{4}

Stress due to moment

$p_b = \frac{M}{I} \times y$

$= \frac{266.7\,p}{1.284} \times \left(\frac{2}{2}\right)$ *kg*

$= \frac{266.7}{1.284}\,p$ *kg./m*^{2}

If tension is to be just avoided,

$p_b = p_o$

$\therefore \frac{266.7}{1.284}\,p = 40420$ *kg./metre*^{2}

$p = \frac{40420 \times 1.284}{266.7}$ *kg./metre*^{2}

$= $ ***194.6 kg./metre***^{2}

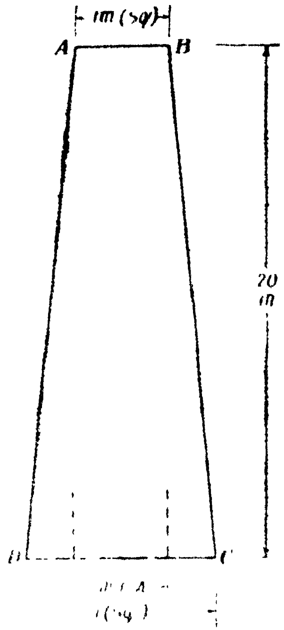


Fig. 360

Corresponding to this condition, maximum compressive stress on the base

$$=p_o+p_b=2p_o$$
$$=2\times 40420\ kg./m^2$$
$$=80840\ kg./m^2.$$

Problem 223. *Fig. 361 shows the cross-section of a masonry chimney with three flues. If wind pressure of 140 kg./metre² acts normal to the longer side, calculate the stresses on the windward and leeward side given that the height of the chimney is 1 metre and the weight of masonry is 1920 kg./metre³.*

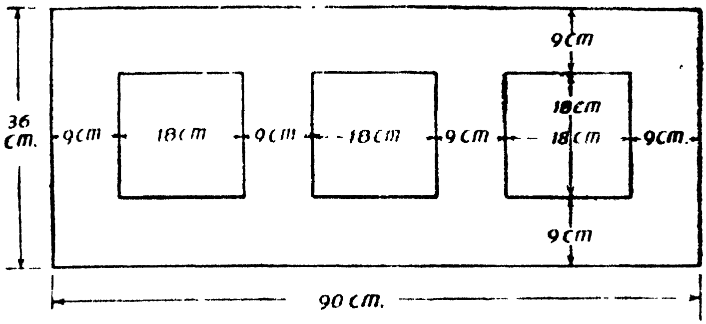


Fig. 361

Solution. Area of chimney section

$$=0{\cdot}36\times 0{\cdot}9-3\times 0{\cdot}18^2=0.2268\ metre^2$$

∴ Weight of the chimney

$$=W=0{\cdot}2268\times 1\times 1920\ kg.$$

∴ Stress due to direct load

$$=p_o=\frac{W}{A}$$

$$=\frac{0{\cdot}2268\times 1920}{0{\cdot}2268}\ kg./metre^2$$

$$=1920\ kg/metre^2$$

Moment of inertia of the chimney section about the longitudinal centroidal axis

$$=\frac{0{\cdot}9\times 0{\cdot}36^3}{12}-\frac{0{\cdot}54\times 0{\cdot}18^3}{12}\ metre^4$$

$$=0{\cdot}003237\ metre^4$$

Wind load $=P=0{\cdot}9\times 1\times 140=126\ kg.$

∴ Moment due to wind load

$$=M=126\times \tfrac{1}{2}=63\ kg.\ m$$

∴ **Stress due to moment**

$$=p_b=\pm\frac{M}{I}.y=\pm\frac{63\times0{\cdot}18}{0{\cdot}003237}$$

$$=\pm3503\ kg./m^2$$

∴ Resultant stress on leeward side

$$=p_o+p_b$$

$$=1920+3503\ kg./metre^2$$

$$=5423\ kg./metre^2$$

Resultant stress on the windward side

$$=p_o-p_b$$

$$=1920-3503\ kg./metre^2$$

$$=1583\ kg./metre^2\ (tensile)$$

Examples in Chapter 6

1. The vertical post of a crane consists of an *I* section 550 *mm.* × 190 *mm.* When a load of 6 *tonnes* was lifted by the crane the distance of the load line from the centroid of the section is 400 *cms.* Find the extreme stresses for the section. Take for the 550 *mm.* × 190 *mm.* *I* section area of the section $=109{\cdot}97\ cm.^2$ $I_{xx}=53161{\cdot}4\ cm.^4$.

(*1297 kg./cm.² compressive ; 1187 kg./cm.² tensile*)

2. The section of a short standard is shown in Fig. 362. The section is subjected to a compressive force of 60 *tonnes* action at the load point *L*. Determine the maximum compressive and tensile stress intensities across the section. Find also the stress intensity at *A*.

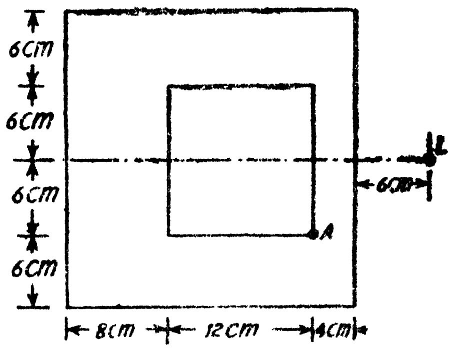


Fig. 362

3. A masonry pier 6 *metres* high is a hollow rectangle in section. The external dimensions are 5 *m* × 2*m* while the internal dimensions are 4 *m* × 1 *m*. If the pier is subjected to a horizontal thrust of 2600 *kg.* at its top in the vertical plane bisecting the length, find the extreme stresses on the base section. Take the weight of masonry as 2200 *kg./metres³*.

(*18400 kg./m² ; 8000 kg./m² compressive*).

4. The line of pull in a tension specimen 0·564 *in.* diameter is parallel to the axis of the specimen but is displaced from it. Calculate the distance of the line of pull from the axis when the maximum stress is 15% greater than the mean stress on a section normal to the axis. (*London University*) (*a=0·01058 in.*)

5. A short cast iron column is of hollow section of uniform thickness, the external diameter being 25 *cm.* and the internal diameter 15 *cm.* A vertical compressive load acts at an eccentricity of 5 *cms.* from the axis of the column. If the maximum permissible stress is 900 $kg./cm.^2$ in compressions calculate the greatest allowable load. *(263·3 tonnes)*

6. The cross-section of a short masonry pier is 2 $m \times 4$ m. The force action across the section consists of a normal compressive load of 32 *t* at *A* and a bending moment of 48 *tm* which causes tension about *XX*. For this load condition determine the maximum and minimum stresses across the section. *(7 t/m^2, 1 t/m^2 compressive)*

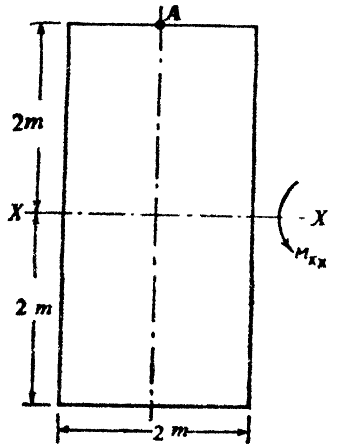


Fig. 363

7. A tie rod of constant circular cross-section is required to withstand a maximum tension of 50·3 *tons*, but the end fixing is such that the line of action must be offset 0·3 *in.* from the axis. Determine to the nearest $\frac{1}{8}$ of an inch, the maximum diameter of the tie rod if the maximum allowable stress is 8 $tons/in.^2$ *(London University) (3·75 in.)*

8. A short column of external diameter 15 *cms.* and internal diameter 10 *cms.* carries an eccentric load. Find the greatest eccentricity which the load can have without producing tension on the cross-section of the column. *(2·71 cms.)*

9. A concrete block has the cross-section shown in Fig 364. The block weighs 9 *t* and carries a vertical downward load of 2 *tons* at point *P* on the axis *XX* but eccentric about the axis *YY*. Calculate the distance of the point *P* if it is known that the pressure under the block along the edge *AD* is just twice the pressure under the edge *BC*. Determine also these pressures.

(22 *in.*;
Max. stress=0·408 $t/ft.^2$
Min. stress=0·204 $t/ft.^2$)

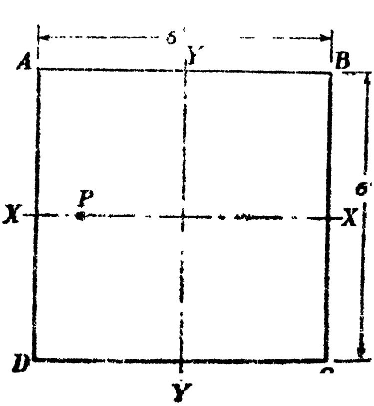


Fig. 364

7

Masonry Dams

§58. Analysis of a Masonry Dam

Fig. 365 shows the cross-section of a masonry dam of trapezoidal section. Let the top and bottom width of the section be a and b. Let H be the height of the dam. Let the face of the dam exposed to water be vertical. Let the height of water impounded be h.

Consider unit length of the dam. The forces acting on this part of the dam are the following :

(*i*) *Weight W of the dam*

$$W=\rho\left(\frac{a+b}{2}\right)H,$$

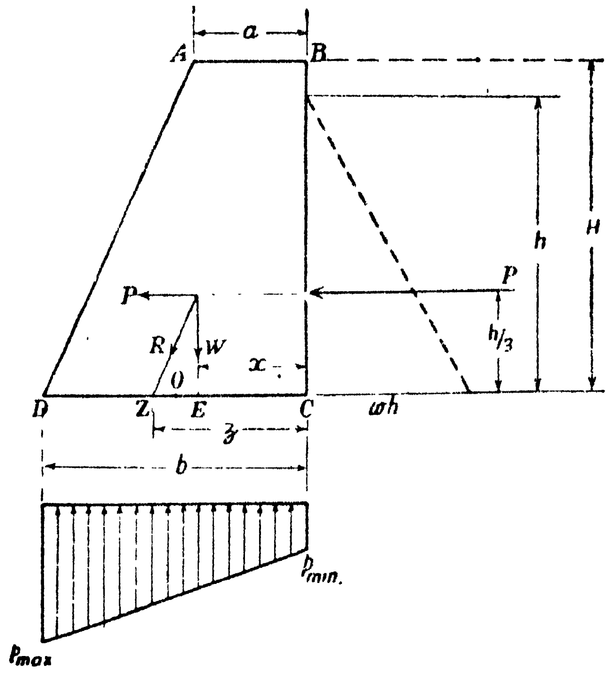


Fig. 365

where ρ is the weight of masonry per unit volume.

The weight of the dam acts at a distance $\bar{x}$ from the vertical face BC so that,

$$\bar{x}=\frac{a^2+ab+b^2}{3(a+b)}$$

or

$$=\frac{b^3-a^3}{3(b^2-a^2)}$$

(*ii*) *Horizontal water pressure P*

At any depth x from the free surface of water the intensity of water pressure is wx. Hence the pressure intensity uniformly increases from zero at the free surface to wh at the bottom.

∴ Total water pressure on unit length of the wall

$$=P=\text{average pressure}\times\text{height of free surface}$$

$$=\frac{wh}{2}\cdot h$$

or

$$P=\frac{wh^2}{2}$$

This total water pressure acts at a height $\frac{h}{3}$ above the base.

(*iii*) *Reaction at the base*

For the equilibrium of the dam, the resultant of W and P must be counteracted by the reaction at the base.

Let the resultant R of P and W meet the base at Z. This resultant force R acting at Z may be resolved into its vertical and horizontal components. Obviously the vertical and horizontal components of R acting at Z are equal to W and P respectively.

The vertical component W of the force R acting at Z is resisted by the normal reaction at the base.

The horizontal component P of the force R acting at Z is resisted by the friction between the bottom of the dam and the soil on which it is resting.

The vertical component W acting at Z is an eccentric load. Let the distance ZC, *i.e.*, the distance of the point of application of the resultant force R on the base from C, be z. Let O be the middle point of the base.

∴ Eccentricity of the vertical component W is equal to OZ.

∴ Eccentricity $=e=OZ=CZ-CO=\left(z-\frac{b}{2}\right)$.

The position of the point Z where the resultant meets the base can be determined by taking moments of the components P and W about C and equating this sum to the moment of the resultant force R about C.

i.e., Moment of W about C + moment of P about C

= Moment R about C ...(*i*)

The force R acting at Z consists of the vertical component W acting at Z and a horizontal component P acting at Z.

Hence moment of R about $C=Wz$ since the horizontal component P acting at Z has no moment about C.

∴ From Eq. (i)

$$W\bar{x}+P\frac{h}{3}=Wz$$

$$\therefore \quad z=\bar{x}+\frac{h}{3}\cdot\frac{P}{W}$$

Stresses across the section

Stress on the section due to the direct load

$$=p_o=\frac{W}{A}=\frac{W}{b\times 1}=\frac{W}{b}$$

Moment on the base section

$$=M=We$$

Section modulus of the base section

$$=Z=\frac{1\times b^2}{6}=\frac{b^2}{6}$$

∴ Extreme bending stress

$$=p_b=\pm\frac{M}{Z}$$

$$=\pm\frac{We}{\left(\frac{b^2}{6}\right)}=\pm\frac{6We}{b^2}$$

∴ The extreme resultant stresses are

$$p_{max}=p_o+p_b=\frac{W}{b}+\frac{6We}{b^2}$$

$$=\frac{W}{b}\left(1+\frac{6e}{b}\right)$$

at the edge away from the water.

$$p_{min}=p_o-p_b=\frac{W}{b}-\frac{6We}{b^2}$$

$$=\frac{W}{b}\left(1-\frac{6e}{b}\right) \text{ at the water edge.}$$

§59. Stability of a Dam

A dam is liable to fail (i) by sliding on the soil on which it rests or (ii) by overturning. (iii) due to tensile stresses developed (iv) due to excessive compressive stresses.

In order that the failure by sliding may not occur the maximum available frictional resistance should be greater than the horizontal water pressure P. If the weight of the structure per unit run be W the maximum available frictional resistance is μW where μ is the coefficient of friction between the masonry dam and the soil on which it rests. Hence, for safety against sliding, The condition $\mu W > P$ should be satisfied. The ratio $\frac{\mu W}{P}$ is called the factor of safety against sliding. It is usual to design the structure such that the factor of safety against sliding is at least 1·5.

In order the structure may not overturn it is necessary that resultant R of the weight W of the structure and the horizontal water pressure P, must strike the base within its width *i.e.*, the point Z must lie within the base CD.

For the dam shown in Fig. 365 taking moments about D,

Overturning moment

$$=\frac{Ph}{3}.$$

Available restoring moment $= W(b-\bar{x})$

The ratio of the available restoring moment to the overturning moment is called the factor of safety against overturning. A failure may also occur due to tensile stresses induced in masonry. To safeguard the structure from this sort of failure, the dam section must be designed such that the resultant force on the base meets the base within the middle third.

i.e., z shall not be greater than $\frac{2}{3}b$.

A failure may also occur when the maximum compressive stress exceeds the permissible compressive stress for masonry.

§60. Minimum bottom width required for a dam section

In order that tension may not be developed in the base section it is necessary that the bottom width of the dam section shall not be less than a certain limit. Let us now determine the minimum width required for sections of various shapes.

(*i*) *Dam of Triangular Section*

Fig. 366 shows a triangular dam section Let the width at the bottom be b. Let the height of the dam be H.

In order that tension may be just avoided, the resultant of water pressure P and the weight of the structure W should strike the base at Z such that the distance

$$CZ = z = \frac{2}{3}b$$

Consider unit run of d m.

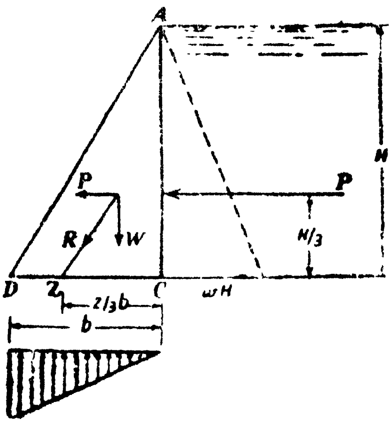


Fig. 365

Weight of the structure

$$=W=\frac{b}{2}H\rho$$

Water pressure $=P=\frac{wH^2}{2}$

But $z=\bar{x}+\frac{H}{3}\cdot\frac{P}{W}$

$\therefore$ $$\upsilon=\frac{b}{3}+\frac{H}{3}\cdot\frac{wH^2}{2}\cdot\frac{2}{bH\rho}$$

$\therefore$ $$b^2=\frac{w}{\rho}H^2$$

$\therefore$ $$b^2=\frac{H^2}{S}$$

where $S=\frac{\rho}{w}$ = specific gravity of masonry

$$b=\frac{H}{\sqrt{S}}$$

(*ii*) *Dam of rectangular section*

Fig. 367 shows a dam of rectangular section b units wide H units deep.

Consider unit run of the dam.

Corresponding to the full reservoir condition,

$P=\frac{wH^2}{2}$

$W=bH\rho$

If tension is just avoided

$z=\frac{2}{3}b$

But $z=\bar{x}+\frac{H}{3}\cdot\frac{P}{W}$

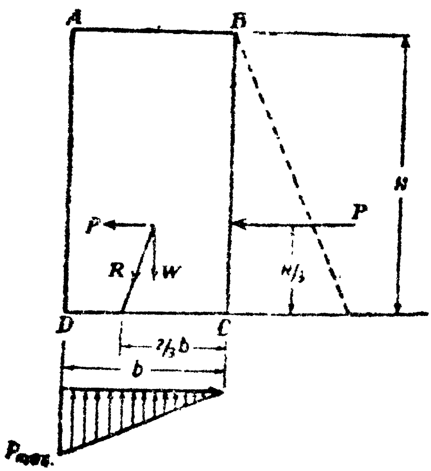


Fig 367

$\therefore$ $$\frac{2}{3}b=\frac{b}{2}+\frac{H}{3}\cdot\frac{wH^2}{2}\cdot\frac{1}{bH\rho}$$

$\therefore$ $$\frac{b}{6}=\frac{wH^2}{6b\rho}$$

$\therefore$ $$b^2=\frac{w}{b}H^2=\frac{H^2}{S}$$

where $S=\frac{\rho}{w}$ = specific gravity of masonry

$$b=\frac{H}{\sqrt{S}}$$

Hence the minimum bottom width required to avoid tension is $\frac{H}{\sqrt{S}}$ whether the dam section is triangular or rectangular.

(iii) Trapezoidal section

Let a dam section be trapezoidal having a top width a and a bottom width b and height H.

Let the water face be vertical

We have, $$\bar{x}=\frac{a^2+ab+b^2}{3(a+b)}$$

Consider unit length of the dam

Weight of the structure

$$W=\rho\left(\frac{a+b}{2}\right)H$$

When the reservoir is full, the maximum water pressure

$$=P=\frac{wH^2}{2}$$

In order tension may be just avoided,

$$z=\bar{x}+\frac{H}{3}\cdot\frac{P}{W}$$

$$=\frac{2}{3}b$$

$$\therefore \quad \frac{a^2+ab+b^2}{3(a+b)}+\frac{H}{3}\cdot\frac{wH^2}{2\rho\frac{(a+b)}{2}H}=\frac{2}{3}b$$

$$\therefore \quad a^2+ab+b^2+\frac{w}{\rho}H^2=2b(a+b)$$

$$\therefore \quad ab+b^2=a^2+\frac{w}{\rho}H^2$$

For the above relation the minimum bottom width required may be computed.

For the case of the triangular section,

$$a=0$$

$$\therefore \quad b^2=\frac{w}{\rho}H^2=\frac{H^2}{S}$$

$$\therefore \quad b=\frac{H}{\sqrt{S}}$$

For the case of the rectangular section,

$$a=b$$

$$\therefore \quad b^2+b^2=b^2+\frac{w}{\rho}H^2$$

$$\therefore \quad b^2=\frac{H^2}{S}$$

$$\therefore \quad b=\frac{H}{\sqrt{S}}$$

Minimum width to avoid sliding

In order the structure may not slide, the condition to be satisfied is,

$$\mu W>P$$

$$\therefore \quad \mu\frac{(a+b)}{2}H\rho>\frac{wH^2}{2}$$

$$a+b>\frac{w}{\rho}\;\frac{H}{\mu}$$

$$\therefore \quad a+b>\frac{H}{\mu S}$$

where $S=\frac{\rho}{w}$ = Specific gravity of masonry,

For the critical condition

$$a+b=\frac{H}{\mu S}$$

or

$$b=\frac{H}{\mu S}-a$$

Minimum width from consideration of maximum normal stress

The maximum compressive stress is given by

$$\frac{W}{b}\left(1+\frac{6e}{b}\right)=p_{max}$$

From this relation, the width required in order the maximum normal stress may not exceed a given limit may be computed.

Hence for the stability the minimum width b required (*i*) to avoid tension (*ii*) to avoid sliding and (*iii*) to avoid excessive normal stress may thus be computed.

For the structure to be safe the greatest of the three values of b may be chosen in order failure may not occur.

Problem 224. *A masonry dam 8 metres high, 1·5 metres wide at the top and 5 metres wide at the base retains water to a depth of 7·5 metres, the water face of the dam being vertical Find the maximum and minimum stress intensities at the base. The weight of water is 1000 kg./cu. metre while the weight of masonry is 2240 kg./cu. metre.*

Solution. Consider one *metre* run of the dam.

Weight of the structure

$$=W=\frac{a+b}{2}H\rho$$

$$=\frac{(1{\cdot}5+5)}{2}\times8\times2240\ kg.$$

$$\therefore\quad W=58240\ kg.$$

Maximum water pressure

$$=P=\frac{wh^2}{2}\ kg.$$

$$=\frac{1000\times7{\cdot}5^2}{2}\ kg.$$

$$P=28125\ kg.$$

$\therefore$ Distance of the point of application of the resultant force on the base, from the extreme water edge

$$=z=\bar{x}+\frac{h}{3}\cdot\frac{P}{W}$$

$$\bar{x}=\frac{a^2+ab+b^2}{3(a+b)}$$

$$=\frac{1{\cdot}5^2+1{\cdot}5\times5+5^2}{3(1{\cdot}5+5)}\ metres$$

$$\therefore\quad \bar{x}=1{\cdot}78\ metres$$

$$\therefore\quad z=1{\cdot}78+\frac{7{\cdot}5}{3}\times\frac{28125}{58240}\ metres$$

$$=1{\cdot}78+1{\cdot}21\ metres\qquad=2{\cdot}99\ metres$$

$$\therefore\quad \text{eccentricity}=e=z-\frac{b}{2}$$

$$=2{\cdot}99-\frac{5}{2}=0{\cdot}49\ metre$$

$\therefore$ The maximum and minimum pressures are given by

$$p_{max}=\frac{W}{b}\left(1+\frac{6e}{b}\right)$$

$$=\frac{58240}{5}\left(1+\frac{6\times0{\cdot}49}{5}\right)kg./metre^2$$

$$=18490\ kg./metre^2$$

$$p_{min}=\frac{W}{b}\left(1-\frac{6e}{b}\right)$$

$$=\frac{58240}{5}\left(1-\frac{6\times0{\cdot}49}{5}\right)kg./metre^2$$

$$=4798\ kg./metre^2.$$

Problem 225. *A dam section is 8 metres high, the maximum depth of water impounded being 7·5 metres. The top width of section is 1 metre. The weight of masonry is 2240 kg./cu. metre while the weight of water is 1000 kg./cu. metre. Find the minimum bottom width required. Coefficient of friction between masonry and masonry is 0·6. The water face of the dam is vertical.*

Solution. Let the width of the base be b.

Consider one *metre* run of the dam.

Weight of structure

$$=W=\left(\frac{a+b}{2}\right)H\rho$$

$$\therefore \quad W=\frac{(1+b)}{2}\times 8\times 2240 \text{ kg.}$$

$$\therefore \quad W=8960\,(b+1) \text{ kg.}$$

$$\bar{x}=\frac{a^2+ab+b^2}{3\,(a+b)}$$

$$\therefore \quad \bar{x}=\frac{1+b+b^2}{3\,(b+1)}$$

Maximum water pressure

$$=P=\frac{w\,h^2}{2}$$

$$=\frac{1000\times 7{\cdot}5^2}{2} \text{ kg.}$$

$$P=28125 \text{ kg.}$$

Minimum width to avoid tension at the base

For this condition

$$z=x+\frac{h}{3}\cdot\frac{P}{W}=\frac{2}{3}b$$

$$\therefore \quad \frac{b^2+b+1}{3\,(b+1)}+\frac{7{\cdot}5}{3}\times\frac{28125}{8960(b+1)}=\frac{2}{3}b$$

$$\therefore \quad b^2+b+1+\frac{7{\cdot}5\times 28125}{8960}=2b(b+1)$$

$$\therefore \quad b^2+b+1+23{\cdot}55=2b^2+2b$$

$$\therefore \quad b^2+b=24{\cdot}55$$

$$\therefore \quad (b+0{\cdot}5)^2=24{\cdot}55+0{\cdot}25=24{\cdot}80$$

$$\therefore \quad b+0{\cdot}5=5{\cdot}0$$

$$\therefore \quad b=4{\cdot}5 \text{ metres}$$

Minimum width to avoid sliding

For this condition

$$\mu w>P$$

$$\therefore \quad 0{\cdot}6\times 8960\,(b+1)>28125$$

$\therefore \quad b+1 > \dfrac{28125}{0.6 \times 8 \times 60}$

$\therefore \quad b+1 > 5.23$

$\therefore \quad b > 4.23$ *metres.*

Hence the minimum bottom width may be made 4·5 metres.

Problem 226. *A masonry dam of trapezoidal section has a vertical water face and a height of 30 metres. Determine the widths at the top and bottom if the normal pressure on the base varies from zero pressure at one side to 90 tonnes per square metre at the other side. The depth of water impounded is 29 metres. Take the weight of water and masonry as 1000 kg./cu. metre and 2300 kg./cu. metre respectively.*

Solution. Let the top and bottom width of the dam section be a and b respectively.

Consider one *metre* run of the dam.

Weight of the structure

$= W = \dfrac{a+b}{2} H\rho$

$= \dfrac{(a+b)}{2} \times 30 \times 2300$ *kg.*

$\therefore \quad W = 34500\,(a+b)$ ...(*i*)

Maximum water pressure

$P = \dfrac{wh^2}{2}$

$= \dfrac{1000 \times 29^2}{2}$ *kg.*

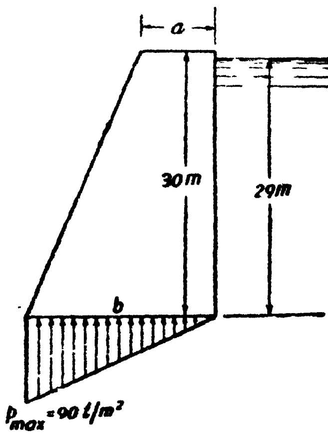


Fig. 368

$= 420500$ *kg*

Since tension at the base has just been avoided

eccentricity $\quad e = \dfrac{b}{6}$

$p_{max} = \dfrac{W}{b}\left(1 + \dfrac{6e}{b}\right)$

$= \dfrac{2W}{b} = 90 \times 1000$ *kg./metre*^{2}

$\therefore \quad \dfrac{W}{b} = 45000$ .. (*ii*)

But from Eq. (*i*)

$W = 34500(a+b)$

$\therefore \quad 34500\,\dfrac{(a+b)}{b} = 45000$

$$\therefore \quad \frac{(a+b)}{b}=\frac{45000}{34500}=1{\cdot}304$$

$$\therefore \quad a+b=1{\cdot}304\,b$$

$$b=1{\cdot}304\,b$$

$$\bar{x}=\frac{a^2+ab+b^2}{3(a+b)}$$

$$=\frac{(0{\cdot}304\,b)^2+0{\cdot}304\,b^2+b^2}{3\times 1{\cdot}304\,b}$$

$$\therefore \quad \bar{x}=\frac{1{\cdot}396}{3{\cdot}912}b$$

$$\therefore \quad \bar{x}=0{\cdot}3569\,b$$

Further, since tension has been just avoided

$$z=\frac{2}{3}b$$

$$\therefore \quad \bar{x}+\frac{h}{3}\cdot\frac{P}{W}=\frac{2}{3}b$$

$$\therefore \quad 0{\cdot}3569\,b+\frac{29}{3}\times\frac{420500}{45000\,b}=\frac{2}{3}b$$

$$\therefore \quad \frac{29\times 420500}{3\times 45000\,b}=0{\cdot}3098b$$

$$\therefore \quad b^2=\frac{29\times 420500}{3\times 45000\times 0{\cdot}3098}$$

$$\therefore \quad b=17 \textit{ metres}$$

$$\therefore \quad a=0{\cdot}304\times 17 \textit{ metres}=5{\cdot}20 \textit{ metres.}$$

§61. Trapezoidal dam section with battered water face

Fig. 369 shows a trapezoidal dam section $ABCD$. Let the top and bottom width of the section be $AB=a$ and $CD=b$. Let the water face BD be at θ with the vertical.

Let h be the depth of the section. Let the sloping length $BC=l$

$$\therefore \quad h=l\cos\theta$$

The pressure intensity on the water face changes from zero at B to wh at C, the direction of the pressure intensity being normal to the water face. Consider unit length of the dam.

Total water pressure on the dam

$=P=\frac{wh}{2}l$ acting normally to the water face at F so that $CF=\frac{1}{3}CB=\frac{l}{3}$. Obviously the line of action of P will be at θ with the horizontal.

Let G be the centre of gravity of the dam section.

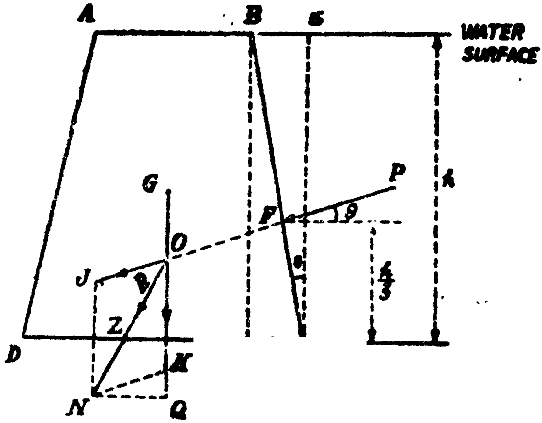


Fig. 369

Weight of dam per unit length of dam

$$=W=\rho\frac{a+b}{2}h$$

Let R be the resultant of W and P. Let the resultant R meet the base at Z.

In the figur OJ represents P

OK represents W.

The diagonal ON of the parallelogram $JOKN$ represents the resultant R. We can obtain the vertical and horizontal components of R by drawing NQ perpendicular to OK

$\therefore$ Vertical component of $R=V=OQ$

Horizontal component of $R=H=NQ$

The vertical component V will be resisted by the normal reactions at the base, and the horizontal component H will be resisted by the frictional resistance.

Let the distance $CZ=z$

Now the eccentricity on the base $=e=z-\frac{b}{2}$. The extreme pressure intensities on the base are given by

$$p=\frac{V}{b}\left[1\pm\frac{6e}{b}\right]$$

$$\therefore \quad p_{max}=\frac{V}{b}\left[1+\frac{6e}{b}\right] \text{ at } D$$

and

$$p_{min}=\frac{V}{b}\left[1-\frac{6e}{b}\right] \text{ at } C$$

As before if tension at the base should be avoided e should be less than $\frac{b}{6}$.

Now consider the total water pressure

$P=\frac{whl}{2}$ acting on the sloping face BC

Horizontal pressure on the vertical face $CE=P_h=\frac{wh^2}{2}$

Weight of water (wedge portion) $BEC=\frac{wl \sin\theta \cdot l\cos\theta}{2}$

$$=\frac{wl\sin\theta}{2}h$$

Resultant water pressure on the water face BC

$$=P=\sqrt{\left(\frac{wh^2}{2}\right)^2+\left(\frac{wl\sin\theta}{2}h\right)^2}$$

$$=\frac{wh}{2}\sqrt{h^2+l^2\sin^2\theta}$$

$$=\frac{wh}{2}\sqrt{l^2\cos^2\theta+l^2\sin^2\theta}$$

$$=\frac{wh}{2}l$$

It will therefore be convenient to consider the horizontal pressure P_h and the weight of wedge BEC of water instead of considering the total pressure P in the calculation.

Obviously P_h acts at $\frac{h}{3}$ above the base and the weight of the wedge of water acts at the centre of gravity of the wedge.

Problem 227. *A masonry dam of trapezoidal section is 10 metres high. It has a top width of 1·5 metres and a bottom width of 6·5 metres. The water face of the dam has a batter of 1 in 10. If the water level is at the top of the dam, find the maximum and the minimum normal stresses at the base. Masonry weighs 2300 kg. per cubic metre and water weighs 1000 kg. per cubic metre.*

Solution. Fig. 370 shows the section of the dam. Consider 1 *metre* run of the dam. We will now find out the various loads acting on the dam Let the resultant of all these loads act at a point Z distant z from the edge C of the base.

$$z=\frac{\text{Total moment about } C}{\text{Total } vertical \text{ load}}$$

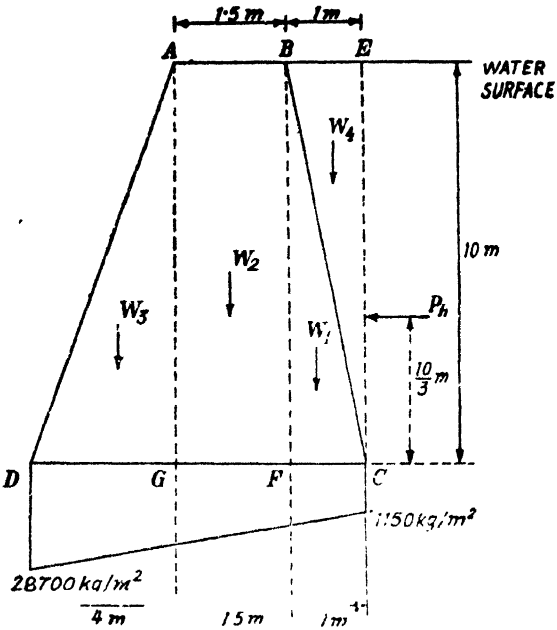


Fig. 370

The relevant calculations are shown in the following table.

Stability calculations for 1 *metre* run of the dam.

Load due to	*Magnitude of load (kg)*	*Distance from c (m)*	*Moment about c (kgm)*
W_1 Triangle of masonry : $\frac{1}{2} \times 1 \times 10 \times 2300$	11500	$\frac{2}{3}$	7667
W_2 Rectangle of masonry : $1{\cdot}5 \times 10 \times 2300$	34500	1·75	60375
W_3 Triangle of masonry : $\frac{1}{2} \times 4 \times 10 \times 2300$	46000	$\frac{23}{6}$	176333
W_4 Triangle of water : $\frac{1}{2} \times 1 \times 10 \times 1000$	5000	$\frac{1}{3}$	1667
Moment of horizontal water pressure : $= \frac{wH^3}{6} = \frac{1000 \times 10^3}{6}$			166667
Total	97000		412709

Distance of the point of application of the resultant force on the base from the edge C

$$z=\frac{\text{Total moment about } C}{\text{Total } vertical \text{ load}}$$

$$=\frac{412709}{97000}=4{\cdot}25\ m$$

eccentricity $\quad e=z-\frac{b}{2}$

$$=4{\cdot}25-\frac{6{\cdot}5}{2}=1\ m$$

But $\quad \frac{b}{6}=\frac{6{\cdot}5}{6}=1{\cdot}08\ m$

$$e \text{ is less than } \frac{b}{6}$$

extreme pressure intensity at the base

$$=\frac{\text{Total vertical load}}{\text{area}}\left[1\pm\frac{6e}{b}\right]$$

$$=\frac{97000}{6{\cdot}5\times1}\left[1\pm\frac{6\times1}{6{\cdot}5}\right] kg./m^2$$

$$=\frac{97000}{6{\cdot}5}\,[1\pm0{\cdot}923]\ kg/m^2$$

$\therefore$ $$p_{max}=\frac{97000}{6{\cdot}5}\times1{\cdot}923\ kg./m^2$$

$$=28700\ kg./m^2$$

$$p_{min}=\frac{97000}{6{\cdot}5}\times0{\cdot}077\ kg./m^2$$

$$=1150\ kg./m^2.$$

Problem 228. *A masonry dam of trapezoidal section is 12 metres high with a top width of 2 metres. The water face has a batter of 1 in 12. Find the minimum bottom width necessary so that tensile stresses are not induced on the base section. Masonry weighs 2300 kg. per cubic metre and water weighs 1000 kg. per cubic metre.*

Solution. Fig. 371 shows the section of the dam. Consider 1 *metre* run of the dam.

Let the bottom width of the dam be b *metres*. Distance of the point of application of the resultant load on the base from the edge C

$$=z=\frac{\text{Total moment about } C}{\text{Total } vertical \text{ load}}$$

If tension is to be just avoided

$$z=\frac{2}{3}\,b$$

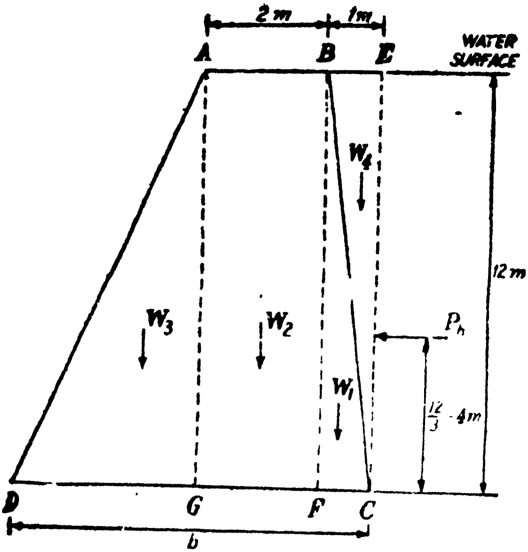


Fig. 371

The various loads on the dam per *metre* run and their moments about *c* are tabulated below.

Load due to	*Magnitude of load (kg)*	*Distance from c (m)*	*Moment about e (kgm)*
W_1 Triangle of masonry : $\frac{1}{2}\times1\times12\times2300$	13800	$\frac{2}{3}$	9200
W_2 Rectangle of masonry : $2\times12\times2300$	55200	2	110400
W_3 Triangle of masonry : $\frac{1}{2}(b\ \ 3)\times12\times2300$	$13800(b-3)$	$\frac{b+6}{3}$	$4600(b-3)(b+6)$
W_4 Triangle of water : $\frac{1}{2}\times1\times12\times1000$	6000	$\frac{1}{3}$	2000
Moment of horizontal water pressure : $=\frac{wH^3}{6}=\frac{1000\times12^3}{6}$			288000
Total	$75000+13800(b-3)$		$409600+4600(b-3)(b+6)$

$$\therefore \qquad z=\frac{409600+4600\,(b-3)\,(b+6)}{75000+13800\,(b-3)}=\frac{2}{3}\,b$$

$$\therefore \quad \frac{4096+46(b-3)(b+6)}{750+138\,(b-3)}=\frac{2}{3}b$$

$$1500b+276b(b-3)=12288+138(b-3)(b+6)$$

$$1500b+276b^2-828b=12288+138b^2+414b-2484$$

$$138b^2+261b-9804=0$$

$$b^2+1{\cdot}891b-71{\cdot}04=0$$

Solving we get $\qquad b=7{\cdot}52\ m.$

§62. Resultant thrust meeting the base at a point outside the middle third.

In a masonry dam we know that the extreme stresses on any horizontal section are given by

$$p=\frac{V}{b}\left[1\pm\frac{6e}{b}\right]$$

where b is the width of the section, V is the total vertical load per unit run of the dam on the section and e is the eccentricity.

In order tension may not be produced at the section we know that $e<\dfrac{b}{6}$.

i.e., the resultant force on the section must meet the section within the *middle third*. This condition should be satisfied at every horizontal section of the dam in order to avoid tensile stresses.

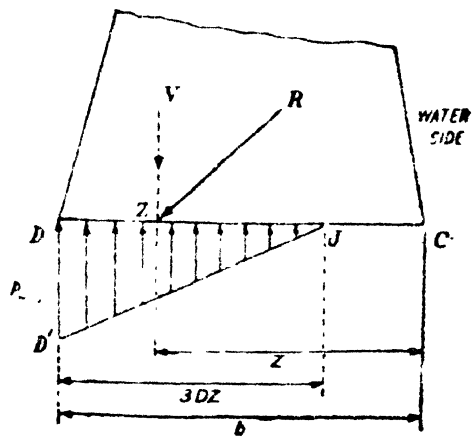


Fig. 372

Now consider the base section resting on the ground. Let for this section the resultant fall outside the middle third, at a point Z at a distance $CZ=z>\frac{2}{3}b$ (See Fig. 372). It should be realized that the nature of stress between the base and the supporting ground cannot be tensile. Hence there will be a pressure distribution following a different law different from what we had adopted earlier.

Let us assume a linear law of pressure distribution. Let the pressure intensity vary uniformly from p_{max} at the edge D to zero at a certain point J.

Total upward reaction

$$=\text{area of the pressure diagram } D'DJ$$

$$=\frac{1}{2}p_{max}DJ.$$

Total downward load

$$=V$$

Since these two should be equal,

$$\frac{1}{2}p_{max}\,DJ=V$$

$$\therefore \qquad p_{max}=\frac{2V}{DJ}$$

It should also be noted that the line of action of V through Z must also pass through the centroid of the pressure diagram.

To satisfy this condition

$$DJ=3DZ$$

$$\therefore \qquad DJ=3(b-z)$$

$$p_{max}=\frac{2V}{3(b-z)}$$

8

Deflection of Beams

In chapter 5 we studied the stresses produced by a bending moment. In this chapter we shall study the deformation produced by a bending moment. If a member is subjected to a uniform bending moment M, the radius of curvature of the deflected form of the member is given by $\frac{M}{I}=\frac{E}{R}$. If the member be subjected to a bending moment which is not the same at all sections the radius of curvature at any point of the centre line is given again by the above relation.

§63. Member bending into a circular arc.

Fig 373 shows a member AB of span l subjected to a uniform bending moment M so that the member is bent to a circular shape. Let R be the radius of the bent form of the member.

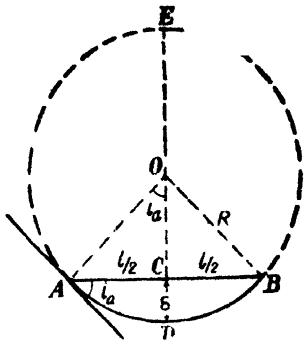


Fig. 373

Let the deflection at the centre of the span be $=CD=\delta$

But $DC.CE=AC.CB$

$$\therefore \quad \delta(2R-\delta)=\left(\frac{l}{2}\right)^2=\frac{l^2}{4}$$

$$\therefore \quad 2R\delta-\delta^2=\frac{l^2}{4}$$

For a practical beam the deflection δ being a small quantity, δ^2 can be ignored.

$$\therefore \quad 2R\delta=\frac{l^2}{4}$$

$$\therefore \quad \delta=\frac{l^2}{8R}$$

But $\quad \frac{M}{I}=\frac{E}{R}$

$$\therefore \quad \frac{1}{R}=\frac{M}{EI}$$

$$\delta=\frac{Ml^2}{8EI}$$

Let i_a be the slope at the end A. We find the angle AOC is also equal to i_a.

$$\therefore \quad \sin i_a = \frac{l}{2R} = \frac{Ml}{2EI}$$

Since i_a is also a small quantity, $\sin i_a = i_a$ *radians*

$$\therefore \quad i_a = \frac{Ml}{2EI} \text{ radians}$$

§64. Slope, deflection and radius of curvature

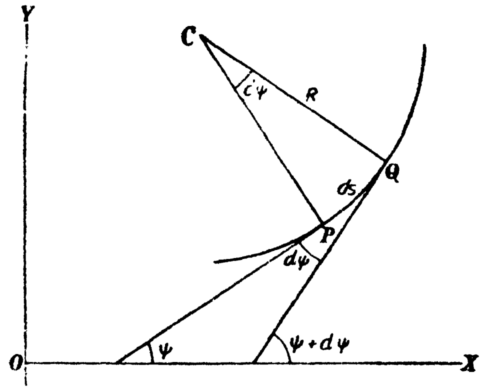


Fig. 374

Consider an elemental length $PQ = ds$ of a curve. Let the tangents at P and Q make angles ψ and $\psi + d\psi$ with the x-axis. Let the normals at P and Q meet at C. Then C is called ths centre of curvature of the curve at any point between P and Q, on the curve. The distance $CP = CQ = R$ is called the radius of curvature at any point between P and Q on the curve.

Obviously $ds = R d\psi$

or $$R = \frac{ds}{d\psi}$$

But we know that if (x, y) be the coordinates of P,

$$\frac{dy}{dx} = \tan\psi, \; \frac{dy}{ds} = \sin\psi \text{ and } \frac{dx}{ds} = \cos\psi$$

$$\therefore \quad R = \frac{ds}{d\psi} = \frac{\dfrac{ds}{dx}}{\dfrac{d\psi}{dx}} = \frac{\sec\psi}{\dfrac{d\psi}{dx}}$$

$$\tan\psi = \frac{dy}{dx}$$

Differentiating with respect to x, we have.

$$\sec^2\psi \cdot \frac{d\psi}{dx} = \frac{d^2y}{dx^2}$$

$$\therefore \quad \frac{d\psi}{dx} = \frac{\frac{d^2y}{dx^2}}{\sec^2 \psi} \qquad \ldots(ii)$$

Substituting Eq. (i), we get,

$$R = \frac{\sec^3 \psi}{\frac{d^2y}{dx^2}}$$

$$\therefore \quad \frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\sec^3 \psi}$$

$$\therefore \quad \frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{(\sec^2 \psi)^{3/2}}$$

$$\therefore \quad \frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{(1+\tan^2 \psi)^{3/2}}$$

For a practical member bent due to the bending moment, the slope $\tan \psi$ at any point is a small quantity. Hence $\tan^2 \psi$ can be ignored.

$$\therefore \quad \frac{1}{R} = \frac{d^2y}{dx^2}$$

If M be the bending moment which has produced the radius of curvature R we have

$$\frac{M}{I} = \frac{E}{R}$$

$$\therefore \quad \frac{1}{R} = \frac{M}{EI}$$

$$\therefore \quad \frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$M = EI\frac{d^2y}{dx^2}$$

§65. Cantilevers

(i) *Cantilever of length l carrying a point load at the free end*

Fig. 375 shows a cantilever AB of uniform section and of length l fixed at the end A and free at B. Let a concentrated load W be applied at the free end B.

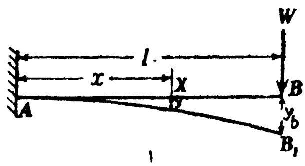


Fig. 375

Let the moment of inertia of the section of the cantilever about the neutral axis be I.

Consider any section X of the cantilever distant x from the fixed end A.

The bending moment at the section is given by

$$EI\frac{d^2y}{dx^2}=-W(l-x)$$

Integrating, we get

$$EI\frac{dy}{dx}=-W\left(lx-\frac{x^2}{2}\right)+C_1$$

where C_1 is a constant of integration

At A the fixed end the slope being zero,

we have at $x=0,\ \frac{dy}{dx}=0$

$\therefore\quad C_1=0$

$$\therefore\quad EI\frac{dy}{dx}=-W\left(lx-\frac{x^2}{2}\right) \qquad \ldots(i) \text{ Slope equation}$$

Integrating again, we get,

$$EI\,y=-W\left(\frac{lx^2}{2}-\frac{x^3}{6}\right)+C_2$$

where C_2 is a constant of integration.

At A the deflection being zero, we have

At $x=0,\ y=0$

$\therefore\quad C_2=0$

$$\therefore\quad EIy=-W\left(\frac{lx^2}{2}-\frac{x^3}{6}\right) \qquad \ldots(ii) \text{ Deflection equation}$$

Hence the slope and deflection at any section can be determined by equations (*i*) and (*ii*). The slope and deflection at the free end can be determined by putting $x=l$ in these equations. Let the slope and deflection at B be i_b and y_b respectively.

We have

$$EIi_b=-W\left(l\,.\,l-\frac{l^2}{2}\right)$$

$$\therefore\quad EIi_b=-\frac{Wl^2}{2}$$

$$\therefore\quad i_b=-\frac{Wl^2}{2EI}$$

Also,

$$EIy_b=-W\left(l\,.\,\frac{l^2}{2}-\frac{l^3}{6}\right)=-\frac{Wl^3}{3}$$

$$\therefore\quad y_b=-\frac{Wl^3}{3EI}$$

$\therefore$ **Downward deflection of B**

$$=\frac{Wl^3}{3EI}$$

(ii) Cantilever of length l carrying a concentrated load W at a distance a from the fixed end.

Fig. 376 shows a cantilever ABC of length l fixed at A and free at C and carrying a concentrated load at B distant a from the fixed end A.

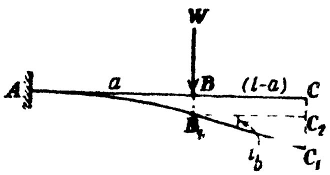


Fig. 376

Let i_b be the slope at B and y_b be the deflection at B

We have $i_b = \frac{Wa^2}{2EI}$ and $y_b = \frac{Wa^3}{3EI}$

The beam will bend only between A and B ; but from B to C it will remain straight since the B.M. between B and C is zero.

Let the end C deflect to C_1 and let B deflect to B_1

We have $$BB_1 = y_b = \frac{Wa^3}{3EI}$$

Let B_1C_2 be perpendicular to CC_1

$$CC_2 = BB_1 = y_b = \frac{Wa^3}{3EI}$$

But $$C_2C_1 = (l-a)i_b$$

$$y_c = CC_2 + C_2C_1 = \frac{Wa^3}{3EI} + (l-a)\frac{Wa^2}{2EI}$$

(iii) Cantilever of length l carrying a uniformly distributed load w per unit run over the whole length.

Fig. 377 shows a cantilever AB of length l fixed at A and free at B carrying a uniformly distributed load of w per unit run over the whole length.

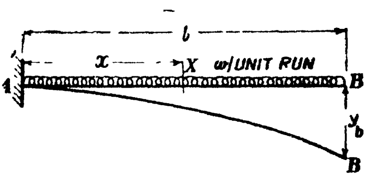


Fig. 377

The B.M. at any section X distant x from the fixed end is given by,

$$EI\frac{d^2y}{dx^2} = -\frac{w}{2}\left(l-x\right)^2$$

Integrating, we get,

$$EI\frac{dy}{dx} = +\frac{w}{6}\left(l-x\right)^3 + C_1$$

At A the fixed end the slope being zero, we have,

At $$x=0, \quad \frac{dy}{dx}=0$$

$$\therefore \quad C_1 = -\frac{wl^3}{6}$$

$$\therefore \quad EI\frac{dy}{dx} = +\frac{w}{6}\left(l-x\right)^3 - \frac{wl^3}{6}$$

...(*i*) *Slope equation*

Integrating again, we get

$$EIy = -\frac{w}{24}\left(l-x\right)^4 - \frac{wl^3}{6}x + C_2$$

Since at A the deflection is zero, we have

At $\qquad x=0,\ y=0$

$$\therefore \qquad 0 = -\frac{wl^4}{24} + C_2$$

$$\therefore \qquad C_2 = \frac{wl^4}{24}$$

$$\therefore \qquad EIy = -\frac{w}{24}\left(l-x\right)^4 - \frac{wl^3}{6}x + \frac{wl^4}{24}$$

...(*ii*) *Deflection equation*

From equations (*i*) and (*ii*) the slope and deflection at any section can be determined.

To find the slope i_b at B, putting $x=l$ in the slope equation, we get

$$EIi_b = -\frac{wl^3}{6}$$

$$\therefore \qquad i_b = -\frac{wl^3}{6EI}$$

To find the deflection y_b at B, putting $x=l$ in the deflection equation, we get,

$$EIy_b = -\frac{wl^4}{6} + \frac{wl^4}{24} = -\frac{wl^4}{8}$$

$$\therefore \qquad y_b = -\frac{wl^4}{8EI}$$

i.e., downward deflection of $B = \frac{wl^4}{8EI}$.

(*iv*) *Cantilever of length l carrying a uniformly distributed load of w per unit run for a distance a from the fixed end.*

Fig. 378 shows a cantilever ABC of length l fixed at A and free at C and carrying a uniformly distributed load w per unit run for a distance a from the fixed end A

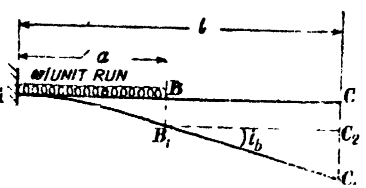


Fig. 378

Let AB_1C_1 be the deflected form of the cantilever.

Deflection at B

$$= y_b = BB_1 = \frac{wa^4}{8EI}$$

Let B_1C_2 be perpendicular to CC_1

$$\therefore \qquad CC_2 = BB_1 = y_b = \frac{\quad}{8EI}$$

$$\text{Slope at } B = i_b = \frac{wa^3}{6EI}$$

$$C_2C_1 = B_1C_2 i_b = (l-a)\,\frac{wa^3}{6EI}$$

$$\text{Deflection at } C = y_c = CC_1 = CC_2 + C_2C_1$$

$$= \frac{wa^4}{8EI} + \left(l-a\right)\frac{wa^3}{6EI}$$

The portion of the cantilever from C to B will remain straight having a slope of $\frac{wa^3}{6EI}$.

(*v*) *Cantilever of length l carrying a uniformly distributed load of w per unit run for a distance a from the free end.*

Fig. 379 shows a cantilever ABC of length l fixed at A and free at C and carrying a uniformly distributed load w per unit run for the portion BC of length a.

The deflection at C may be determined as due to the independent effect of the following load systems :

(*i*) when the whole span is loaded from A to C.

(*ii*) when an upward uniformly distributed load of w per unit run is acting for the portion AB.

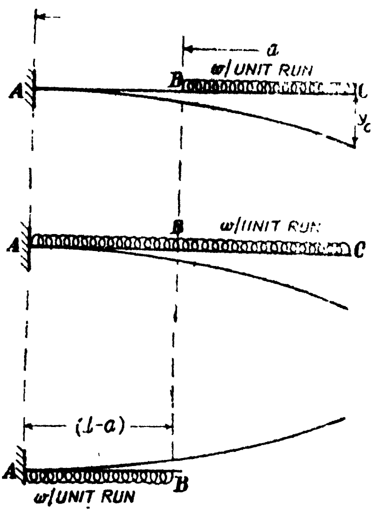


Fig 379

(*i*) Downward deflection of C when the whole length is loaded

$$= \frac{wl^4}{8EI}$$

(*ii*) Upward deflection of C due to the upward uniformly distributed load acting for the portion AB

= upward deflection of B + slope at $B \times BC$

$$= \frac{w(l-a)^4}{8EI} + \frac{w(l-a)^3}{6EI}\,a$$

Net downward deflection of the free end C

$$\frac{wl^4}{8EI} - \left\{ \frac{w(l-a)^4}{8EI} + \frac{w(l-a)^3}{6EI}\,a \right\}$$

Problem 229. *A cantilever 3 metres long carries a uniformly distributed load over the entire length. If the slope at the free end is 1° find the deflection at the free end.*

Solution.

Slope at free end $= \dfrac{wl^3}{6EI} = \dfrac{\pi}{180}$

$\therefore \quad \dfrac{wl^3}{EI} = \dfrac{\pi}{30}$

Deflection at the free end $= \dfrac{wl^4}{8EI}$

$$= \frac{wl^3}{EI} \cdot \frac{l}{8}$$

$$= \frac{\pi}{30} \times \frac{300}{8} = 3{\cdot}927 \text{ cm.}$$

Problem 230. *A horizontal cantilever of uniform section of length l carries two point loads, W at the free end and 2W at a distance of a from the free end. Find the maximum deflection due to this loading.*

If the cantilever is a steel tube of circular section 10 cms external diameter and 0·60 cm. thick and l=1·50 metres and a=0·60 metre determine the value at W in tonnes so that the maximum bending stress is 1400 kg/cm.² and calculate the maximum deflection for the loading. Take E=2×10⁶ kg./cm.²

Solution. Deflection at the free end due to the load W alone

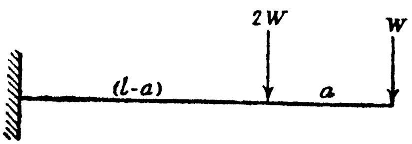


Fig. 380

$$= \frac{Wl^3}{3EI}$$

Deflection at the free end due to the load $2W$ alone

$$= \frac{2W(l-a)^3}{3EI} + \frac{2W(l-a)^2}{2EI}a$$

$\therefore$ Total deflection at the free end

$$\delta = \frac{Wl^3}{3EI} + \frac{2W(l-a)^3}{3EI} + \frac{W(l-a)^2}{EI}a$$

$$= \frac{W}{3EI}\left[l^3 + 2(l-a)^3 + 3(l-a)^2 a\right]$$

$$= \frac{W}{3EI}\left[l^2 + 2l^3 - 6l^2a + 6a^2 - 2a^3 + 3l^2a - 6a^2 + 3a^3\right]$$

$$= \frac{W}{3EI}\left[3l^2 - 3l^3a + a^3\right]$$

$$\therefore \quad \delta = \frac{W}{3EI}\left[3l^2(l-a) + a^3\right]$$

For the numerical

Max. B.M. $= M = Wl + 2W(l-a)$

$$=W(3l-2a)=W(3\times 1\cdot 5-2\times 0\cdot 6)\ kg.\ m.$$
$$=3\cdot 3\ W\ kg.\ m.$$
$$I=\frac{\pi}{64}(10^4-8\cdot 8^4)\ cm.^4$$
$$=196\cdot 5\ cm.^4$$

Stress due to max. B.M.

$$=f=\frac{M}{I}y=1400\ kg./cm^2$$

$$\therefore \quad \frac{3\cdot 3W\times 100}{196\cdot 5}\times 5=1400$$

$$\therefore \quad W=\frac{1400\times 196\cdot 5}{3\cdot 3\times 5\times 100}\ kg.$$
$$=166\cdot 70\ kg.$$
$$=0\cdot 1667\ tonne$$

Max. deflection

$$=\delta=\frac{W}{3EI}\left[3l^2(l-a)+a^3\right]$$
$$=\frac{166\cdot 7}{3\times 2\times 10^6\times 196\cdot 5}\left[3\times 2\cdot 25\times 0\cdot 9+0\cdot 216\right]\times 100^3\ cm.$$
$$=0\cdot 8894\ cm.$$

Problem 231. *A cantilever of length 2 metres carries a uniformly distributed load of 250 kg per metre for a length of 1·25 metres from the fixed end and a point load of 100 kg. at the free end. If the section B is rectangular 12 cms. wide and 24 cms. deep find the deflection at the free end. Take E=100,000 kg./cm.²*

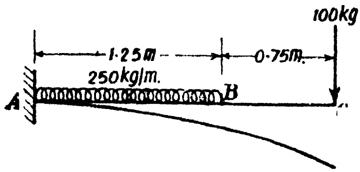


Fig. 381

Solution. Moment of inertia of the section

$$=I=\frac{bd^3}{12}$$
$$=\frac{12\times 24^3}{12}\ cm.^4$$
$$=24^3\ cm.^4$$

Deflection at C due to uniformly distributed load on AB

$$=\frac{wa^4}{8EI}+\frac{wa^3}{6EI}(l-a)$$

$$=\frac{250\times 1\cdot 25^4}{8EI}+\frac{250\times 1\cdot 25^3\times 0\cdot 75}{6EI}=\frac{137\cdot 3}{EI}$$

Deflection at C due to the point load at C

$$=\frac{Wl^3}{3EI}=\frac{100\times 2^3}{3EI}=\frac{266\cdot 7}{EI}$$

∴ Net deflection at C

$$=\frac{137 \cdot 3}{EI}+\frac{266 \cdot 7}{EI}=\frac{404}{EI}$$

$$=\frac{404\times(100)^3}{10^5\times 24^3}\ cm. = 0 \cdot 2922\ cm.$$

(*vi*) *Cantilever of length l carrying a distributed load whose intensity varies uniformly from zero at the free end to w per unit run at the fixed end.*

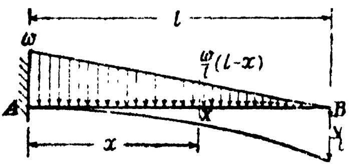


Fig. 382

Fig. 382 shows a cantilever AB of length l carrying the loading mentioned above.

Consider section X at a distance x from the fixed end A.

Intensity of loading at X

$$=\left(\frac{l-x}{l}\right) w \text{ per unit run.}$$

The B.M. at the section X is given by,

$$EI\frac{d^2y}{dx^2}=-\frac{1}{2}(l-x)\frac{w}{l}(l-x).\frac{(l-x)}{3}$$

$$=-\frac{w(l-x)^3}{6l}$$

Integrating, we get,

$$EI\frac{dy}{dx}=\frac{w(l-x)^4}{24l}+C.$$

At A the slope is zero

∴ At $x=0$,

$$\frac{dy}{dx}=0$$

$$\therefore \quad 0=\frac{wl^3}{24}+C_1$$

$$\therefore \quad C_1=-\frac{wl^3}{24}$$

$$\therefore \quad EI\frac{dy}{dx}=\frac{w(l-x)^4}{24l}-\frac{wl^3}{24} \quad \text{...(i) Slope equation}$$

Integrating again, we get,

$$EIy=-\frac{w(l-x)^5}{120l}-\frac{wl^3}{24}x+C_2$$

The deflection at A is 0

∴ $x=0$,

$y=0$

$$\therefore \quad 0=-\frac{wl^4}{120}+C_2$$

$$\therefore \quad C_2=\frac{wl^4}{120}$$

$$\therefore \quad EIy=-\frac{w(l-x)^5}{120l}-\frac{wl^3}{24}x+\frac{wl^3}{120}$$

...(*ii*) *Deflection equation*

To find the slope at B the free end, putting $x=l$ in the slope equation, we get, $EIi_b=-\frac{wl^3}{24}$

$$\therefore \quad i_b=-\frac{wl^3}{24EI}$$

To find the deflection at B putting $x=l$, in the deflection equation, we get.

$$EIy_b=-\frac{wl^4}{24}+\frac{wl^4}{120}$$

$$=-\frac{wl^4}{120}(5-1)=-\frac{wl^4}{30}$$

$$\therefore \quad y_b=-\frac{wl^4}{30EI}$$

$\therefore$ Downward deflection of $B=\frac{wl^4}{30EI}$

(*vii*) *Cantilever of length l carrying a distributed load whose intensity varies uniformly from zero at the fixed end to w per unit run at the free end.*

Fig. 383 shows a cantilever AB fixed at A and free at B and carrying the loading mentioned above.

It is easily seen that the deflection at B

= Deflection at B due to a uniformly distributed load of w per unit run over the whole length.

– Deflection at B due to a distributed load whose intensity varies uniformly from zero at the free end to w per unit run at the fixed end.

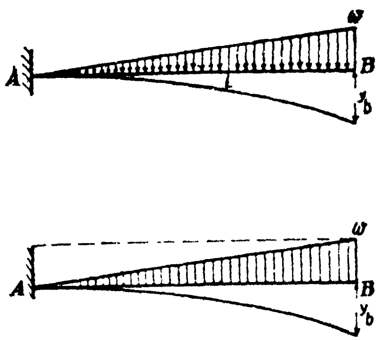


Fig. 383

$$\text{Deflection at } B=y_b=\frac{wl^4}{8EI}-\frac{wl^4}{30EI}$$

$$\frac{11}{120}\cdot\frac{wl^4}{EI}.$$

Problem 232. *.. cantilever of length l carries a concentrated load W at its mid span. If the free end be supported on a rigid prop find the reaction at the prop. Draw also S.F. and B.M. diagrams for the cantilever.*

Solution. Fig. 384 shows a cantilever AB fixed at A the free end B being supported on a rigid prop.

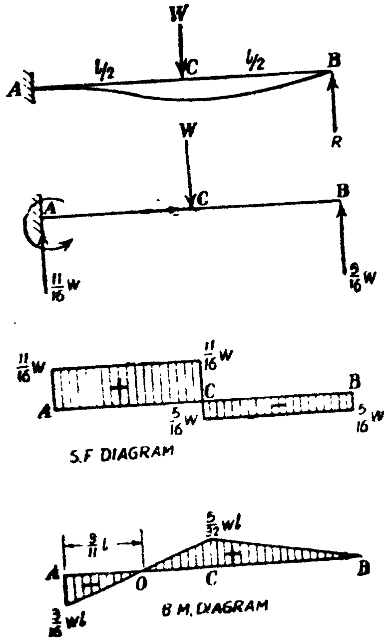


Fig. 384

Let the load W be applied at the middle point C of the span.

Let the prop reaction be R.

The cantilever can be looked upon as being subjected to the following loadings :

(*i*) Downward point load W at C.

(*ii*) Unward point load R at B.

Since the deflection at B is zero, we have

Downward deflection of B due to the load W - upward deflection B due to R

$$\therefore \quad \frac{W\left(\frac{l}{2}\right)^3}{3EI} + \frac{W\left(\frac{l}{2}\right)^2}{2EI}\cdot\frac{l}{2} = \frac{Rl^3}{3EI}$$

$$\therefore \quad R = \frac{5}{16} W$$

$$\therefore \quad \text{Reaction at } A = W - \frac{5}{16}W = \frac{11}{16}W$$

S.F diagram

S.F. at any section between A and $C = +\frac{11}{16}W$

S.F at any section between C and $B = -\frac{5}{16}W$

B.M. diagram

B.M. at $B=0$

B.M. at $C \qquad = +\frac{5}{16} W.\frac{l}{2} = \frac{5}{35}Wl$

B.M. at $A \qquad = +\frac{5}{16}Wl - \frac{Wl}{2} = -\frac{3}{16}Wl$

There will be a point of contraflexture between A and C. Let the B.M. be zero at a distance x from B. Equating the B.M. to zero, we have

$$\frac{5}{16}Wx - W\left(x-\frac{l}{2}\right)=0$$

$$\therefore \qquad \frac{5}{16}x - x + \frac{l}{2} = 0$$

$$\frac{11}{16}x = \frac{l}{2} \qquad x = \frac{8}{11}l.$$

Hence the point of contraflexure is at $\frac{8}{11}l$ from B or $\frac{3}{11}l$ from A.

Problem 233. *A cantilever of length l carries a uniformly distributed load of w per unit run over the whole length. If the free end be supported over a rigid prop, find the reaction of the prop and draw S.F. and B.M. diagrams for the cantilever. Find also the maximum deflection.*

Solution. Fig. 385 shows a cantilever AB fixed at A the end B being supported on a prop. Let the prop reaction be R.

If the prop had not been provided the dowaward deflection of B would be $\frac{wl^4}{8\,EI}$

Upward de flection at B due to R alone equals $\frac{Rl^3}{3EI}$.

Since the net deflection at B is zero, we have,

$$\frac{Rl^3}{3EI} = \frac{wl^4}{8EI}$$

$$\therefore \qquad R = \frac{3}{8}wl$$

$\therefore$ Reaction at $B = \frac{3}{8}wl$

and reaction at $A = \frac{5}{8}wl$

S.F. diagram

At any section distant x from B the S.F. is given by

$$S = wx - \frac{3}{8}wl$$

At $\quad x = 0,$

i.e., at B, $\quad S = -\frac{3}{8}wl$

At $\quad x = l,$

i.e., at A, $S = wl - \frac{3}{8}wl$

$$= +\frac{5}{8}wl.$$

Let the S.F. be zero at a distance x from B.

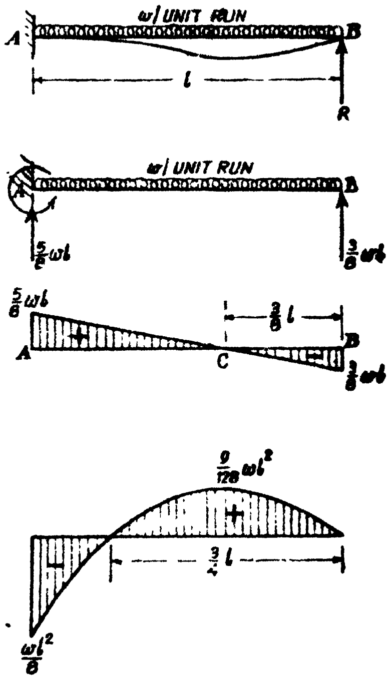


Fig. 385

Equating the S.F. to zero, we have

$$wx-\frac{3}{8}wl=0$$

$$\therefore \qquad x=\frac{3}{8}l$$

B.M. diagram

The B.M. at any section distance x from B is given by

$$M=\frac{3}{8}wlx-\frac{wx^2}{2}$$

At $x=0$. *i.e.*, at B, $M=0$

At $x=l$, *i.e.*, at A,

$$M=\frac{3}{8}wl^2-\frac{wl^2}{2}=-\frac{wl^2}{8}$$

At $x=\frac{3}{8}l$,

$$M=\frac{3}{8}wl.\frac{3}{8}l-\frac{w}{2}\cdot\left(\frac{3}{8}l\right)^2$$

$$=\frac{9}{64}wl^2-\frac{9}{128}wl^2 \qquad =+\frac{9}{128}wl^2$$

Point of contraflexure

Let the B.M. be zero at a distance x from B. Equating the B.M. to zero, we get

$$\frac{3}{8}wlx-\frac{wx^2}{2}=0$$

$$\therefore \frac{w}{16}x\left(6l-8x\right)=0$$

$$\therefore \qquad x=0 \text{ and } x=\frac{3}{4}l.$$

Deflection

At any section distant x from B, the B.M. is given by

$$EI\frac{d^2y}{dx^2}=\frac{3}{8}wlx-\frac{wx^2}{2}$$

Integrating we get,

$$EI\frac{dy}{dx}=\frac{3}{16}wlx^2-\frac{wx^3}{6}+C_1$$

At A the slope is zero

i.e., at $x=l$, $\frac{dy}{dx}=0$

$$\therefore \qquad 0=\frac{3}{16}wl^3-\frac{wl^3}{6}+C_1$$

$$\therefore \quad C_1 = -\frac{wl^3}{48}$$

$$\therefore \quad EI\frac{dy}{dx} = \frac{3}{16}wlx^2 - \frac{wx^3}{6} - \frac{wl^3}{48}$$

...(*i*) ***Slope equation***

Integrating again, we get,

$$EIy = \frac{wlx^3}{16} - \frac{wx^4}{24} - \frac{wl^3}{48}x + C_2$$

At A the deflection is zero

i.e., at $x = l,\ y = 0$

$$\therefore \quad 0 = \frac{wl^4}{16} - \frac{wl^4}{24} - \frac{wl^4}{48} + C_2$$

$$\therefore \quad C_2 = 0$$

$$\therefore \quad EIy = \frac{wlx^3}{16} - \frac{wx^4}{24} - \frac{wl^3}{48}x$$

...(*ii*) ***Deflection equation***

To find the maximum deflection

Maximum deflection will occur where the slope is zero.

Equating the slope to zero, we get

$$0 = \frac{3}{16}wlx^2 - \frac{wx^3}{6} - \frac{wl^2}{48}$$

$$9lx^2 - 8x^3 - l^3 = 0$$

Let $x = Kl$

$$\therefore \quad 9K^2l^3 - 8K^3l^3 - l^3 = 0$$

$$\therefore \quad 9K^2 - 8K^3 = 1$$

$$K^2(9 - 8K) = 1$$

Solving this equation by trial and error
we get $K = 0{\cdot}422$

Hence the maximum deflection will occur at $0{\cdot}422l$ **from the prop end** B.

Putting $x = 0{\cdot}422l$ in the deflection equation,

We get

$$EIy_{max} = \frac{wl}{16}\left(0{\cdot}422l\right)^3 - \frac{w}{24}\left(0{\cdot}422l\right)^4 - \frac{wl^3}{48}\left(0{\cdot}422l\right)$$

$$= -0{\cdot}005415\,wl^4$$

$$\therefore \quad y_{max} = -0{\cdot}005415\frac{wl^4}{EI}$$

$\therefore$ **Maximum downward deflection** $= 0{\cdot}005415\,\dfrac{wl^4}{EI}$.

Problem 234. *A cantilever of length l carries a uniformly distributed load w per unit run over the whole length. The free end of the cantilever is supported on a prop. If the prop sinks by δ find the prop reaction.*

Solution. Let the prop reaction be R

If the prop had not been present the downward deflection of the free end would be $\frac{wl^4}{8EI}$

Upward deflection of the free end due to R alone would be

$$\frac{Rl^3}{3EI}$$

Since the prop sinks by δ the net downward deflection

$$=\frac{wl^4}{8EI}-\frac{Rl^3}{3EI}=\delta$$

$$\therefore \quad \frac{Rl^3}{3EI}=\frac{wl^4}{8EI}-\delta$$

$$\therefore \quad R=\frac{3EI}{l}\left\{\frac{wl^4}{8EI}-\delta\right\}$$

Simply supported beams

(*i*) *Simply supported beam of span l carrying a point load at mid span*

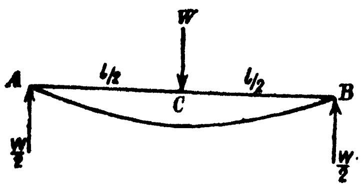


Fig. 386

Fig. 386 shows a simply supported beam AB of span l carrying a point load W at mid span C.

Since the load is symmetrically applied the maximum deflection will occur at mid span.

Each vertical reaction

$$=\frac{W}{2}.$$

Consider the left half AC of the span.

The B.M. at any section in AC distant x from A is given by

$$EI\frac{d^2y}{dx^2}=+\frac{W}{2}x$$

Integrating, we get,

$$EI\frac{dy}{dx}=\frac{Wx^2}{4}+C_1$$

Since the maximum deflection occurs at mid span C the slope at C is zero

i.e., at $\quad x=\frac{l}{2}$

$$\frac{dy}{dx}=0$$

$$\therefore \quad 0=\frac{W}{4}\cdot\left(\frac{l}{2}\right)^2+C_1$$

$$\therefore \quad C_1=-\frac{Wl^2}{16}$$

$$\therefore \quad EI\frac{dy}{dx}=\frac{Wx^2}{4}-\frac{Wl^2}{16} \qquad \text{...(i) Slope equation}$$

To find the slope at A, put $x=0$

$$\therefore \quad EIi_a=-\frac{Wl^2}{16}$$

$$\therefore \quad i_a=-\frac{Wl^2}{16EI}$$

Integrating the slope equation, we get,

$$EIy=\frac{Wx^3}{12}-\frac{Wl^2x}{16}+C_2$$

Since at A the deflection is zero,

we have at $x=0$,

$y=0$

$$\therefore \quad C_2=0$$

$$\therefore \quad EIy=\frac{Wx^3}{12}-\frac{Wl^2}{16}x$$

...(ii) *Deflection equation*

To find the deflection at C,

put $x=\frac{l}{2}$

$$\therefore \quad EIy_c=\frac{W}{12}\left(\frac{l}{2}\right)^3-\frac{Wl^2}{16}\cdot\frac{l}{2}$$

$$=-\frac{Wl^3}{48}$$

$$\therefore \quad y_c=-\frac{Wl^3}{48EI}$$

$\therefore$ Downward deflection of C

$$=\frac{Wl^3}{48EI}$$

(ii) Simply supported beam of span l carrying a uniformly dsitributed load of w per unit run over the whole span.

Fig. 387 shows a simply supported beam AB of span l carrying a uniformly distributed load w per unit run over the whole span.

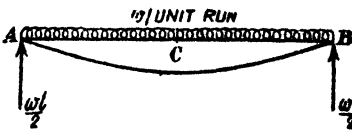


Fig. 387

Each vertical reaction

$$\frac{wl}{2}$$

The B.M. at any section distant x from the end A is given by,

$$EI\frac{d^2y}{dx^2} = \frac{wl}{2}x - \frac{wx^2}{2}$$

Integrating, we get

$$EI\frac{dy}{dx} = \frac{wl}{4}x^2 - \frac{wx^3}{6} + C_1$$

The loading being symmetrical, the maximum deflection will occur at mid span and hence the slope at mid span equals zero.

i.e., at $x = \frac{l}{2}$

$$\frac{dy}{dx} = 0$$

$$\therefore \quad 0 = \frac{wl}{4}\left(\frac{l}{2}\right)^2 - \frac{w}{6}\left(\frac{l}{2}\right)^3 + C_1$$

$$\therefore \quad C_1 = -\frac{wl^3}{24}$$

$$\therefore \quad EI\frac{dy}{dx} = \frac{wl}{4}x^2 - \frac{wx^3}{6} - \frac{wl^3}{24} \quad \text{...(i) Slope equatian}$$

Integrating again, we get,

$$EIy = \frac{wl}{12}x^3 - \frac{wx^4}{24} - \frac{wl^3}{24}x + C_2$$

At A the deflection being zero we have at $x=0$, $y=0$

$$\therefore \quad C_2 = 0$$

$$\therefore \quad EIy = \frac{wl}{12}x^3 - \frac{w}{24}x^4 - \frac{wl^3}{24}x$$

...(ii) *Deflection equation*

To find the maximum deflection which occurs at mid span C.

Putting $x = \frac{l}{2}$ in the deflection equation, we get,

$$EIy_c = \frac{wl}{12}\left(\frac{l}{2}\right)^3 - \frac{w}{24}\left(\frac{l}{2}\right)^4 - \frac{wl^3}{24}\cdot\frac{l}{2}$$

$$\frac{5}{384}wl^4$$

$$\therefore \qquad y_c=-\frac{5}{384}\frac{wl^4}{EI}$$

To find the slope at A, put $x=0$ in the slope equation.

$$\therefore \qquad EIi_a=-\frac{wl^3}{24}$$

$$\therefore \qquad i_a=-\frac{wl^3}{24EI}$$

Problem 235. *A simply supported beam of span l carries a unifomly distributed load for a distance l/2 from one support. Find the deflection at the centre.*

Solution. Let δ be the deflection at the centre when the left half of the span is loaded [see Fig. 388 (*i*)].

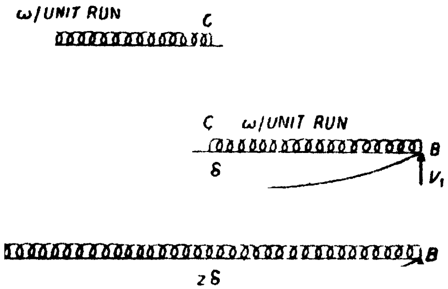


Fig. 388

If the right half of the span is loaded then also the same deflection δ will occur at the centre [see Fig 388 (*ii*)].

Hence if the left half as well as the right half are loaded, *i.e.*, if the entire span is loaded the deflection at the centre,

$$=2\delta=\ \frac{\ }{384}\ \frac{wl^4}{EI}$$

When one half of the span is loaded, the deflection at the centre

$$=\delta=\frac{5}{768}\frac{wl^4}{EI}.$$

Problem 236. *A horizontal beam of uniform section is pinned at its ends which are at the same level and is loaded at the left hand pin with an anticlockwise moment M and at the right hand pin with a clockwise moment 2M both in the same vertical plane. The length between the pins is l. Find the angles of slope at each end and the deflection of the mid point of the span in terms of M, l, E and I.*

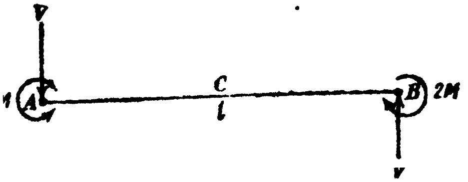


Fig. 389

Solution. Let the vertical reaction at each end be V.

For the equilibrium of the member,

$$Vl=(2M-M)$$

$$\therefore \quad V=\frac{M}{l}$$

At any section distant x from A bending moment is given by

$$EI\frac{d^2y}{dx^2}=-Vx-M$$

$$=-\frac{M}{l}x-M$$

Integrating, we get

$$EI\frac{dy}{dx}=-\frac{Mx^2}{2l}-Mx+C_1$$

Again integrating, we get,

$$EIy=-\frac{Mx^3}{6l}-\frac{Mx^2}{2}+C_1x+C_2$$

At A the deflection is zero.

$\therefore$ At $x=0$,

$y=0$

$\therefore$ $C_2=0$

At B the deflection is zero.

$\therefore$ At $x=l$,

$y=0$

$$\therefore \quad 0=-\frac{Ml^2}{6}-\frac{Ml^2}{2}+C_1l$$

$$\therefore \quad C_1=\frac{2}{3}Ml$$

$\therefore$ The slope and deflection at any section are given by

$$EI\frac{dy}{dx}=-\frac{Mx^2}{2l}-Mx+\frac{2}{3}Ml$$

and $$EIy=-\frac{Mx^3}{6l}-\frac{Mx^2}{2}+\frac{2}{3}Mlx$$

Slope at A

Putting $x=0$, in the slope equation, we get,

$$EIi_a=\frac{2}{3}Ml$$

$$\therefore \qquad i_a = \frac{2}{3}\,\frac{Ml}{EI}$$

Slope at B

Putting $x=l$, in the slope equation, we get,

$$EIi_b = -\frac{Ml}{2} - Ml + \frac{2}{3}Ml$$

$$= -\frac{5}{6}Ml$$

$$\therefore \qquad i_b = -\frac{5}{6}\,\frac{Ml}{EI}$$

Deflection at mid span

Putting $x=\frac{l}{2}$ in the deflection equation, we get,

$$EIy_c = -\frac{M}{6l}\cdot\frac{l^3}{8} - \frac{M}{2}\cdot\frac{l^2}{4} + \frac{2}{3}Ml\cdot\frac{l}{2}$$

$$\therefore \qquad = \frac{3}{16}Ml^2$$

$$y_c = \frac{3}{16}\,\frac{Ml^2}{EI}$$

Problem 237. *A horizontal cantilever of length l supports a uniformly distributed load of w per unit run along its length. The cantilever is propped to the level of the fixed end at a distance $\frac{3}{4}$ l from the fixed end. Find the reaction of the prop.*

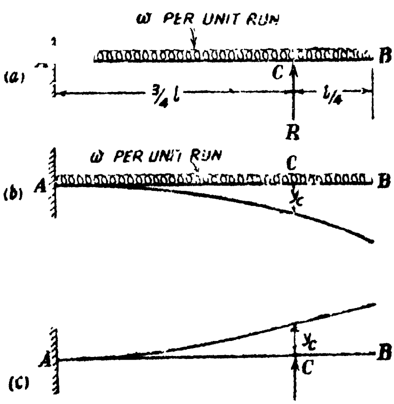


Fig. 390

Solution. Fig. 390 shows the cantilever ACB fixed at A and propped at C. Let the prop reaction be R.

We know that the deflection at C must be zero.

Hence the downward deflection at C when the prop is absent must be equal to the upward deflection of C due to an upward force R applied at C. Let us first consider the case when the prop is absent. See Fig. 390(*b*). For this case, at any section distant x from the fixed end the bending moment is given by,

$$EI\frac{d^2y}{dx^2} = -\frac{w}{2}(l-x)^2$$

Integrating, we get,

$$EI\frac{dy}{dx}=\frac{w(l-x)^3}{6}+C_1$$

At A the slope is zero.

i.e., at $x=0$

$$\frac{dy}{dx}=0$$

$$\therefore \quad 0=\frac{wl^3}{6}+C_1$$

$$\therefore \quad C_1=-\frac{wl^3}{6}$$

$$EI\frac{dy}{dx}=\frac{w(l-x)^3}{6}-\frac{wl^3}{6}$$

Integrating again, we get,

$$EIy=-\frac{w(l-x)^4}{24}-\frac{wl^3}{6}x+C_2$$

At A the deflection is zero.

i.e., at $x=0$,

$$y=0$$

$$0=-\frac{wl^4}{24}+C_2$$

$$\therefore \quad C_2=\frac{wl^4}{24}$$

$$\therefore \quad EIy=-\frac{w(l-x)^4}{24}-\frac{wl^3}{6}x+\frac{wl^4}{24}$$

Let the deflection at C be y_c

i.e., at $x=\frac{3}{4}l$

$$y=y_c$$

$$\therefore \quad EIy_c=-\frac{w}{24}\left(\frac{l}{4}\right)^4-\frac{wl^3}{6}\times\frac{3}{4}l+\frac{wl^4}{24}$$

$$=-\frac{171}{2048}wl^4$$

$$\therefore \quad y_c=-\frac{171}{2048}\frac{wl^4}{EI}$$

Hence the downward deflection at C when the prop is absent

$$=y_c=\frac{171}{2048}\frac{wl^4}{EI}$$

But upward deflection at C due to R alone

$$\frac{R\left(\frac{3}{4}l\right)^3}{3EI}=\frac{9}{64}\frac{Rl^3}{EI}$$

Since the net deflection at C is zero, we have

$$\frac{9}{64}\frac{Rl^3}{EI}=\frac{171}{2048}\frac{wl^4}{EI}$$

$$R=\frac{19}{32}wl.$$

Problem 238. *A horizontal cantilever of uniform section and length l carries two vertical point loads W_1 and W_2. W_1 acts upwards at the free end and W_2 acts downwards at a distance a from the fixed end. Find the deflection at the free end.*

Solution. If the load W_2 had been absent, the upward deflection of the free end would be

$$\frac{W_1l^3}{3EI}.$$

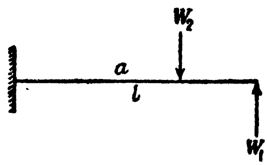


Fig. 391

If W_1 had been absent the downward deflection of the free end due to W_2 alone

$$=\frac{W_2a^3}{3EI}+\frac{W_2a^2}{2EI}(l-a)$$

∴ Net deflection of the free end

$$=\frac{W_1l^3}{3EI}-\frac{W_2a^3}{3EI}-\frac{W_2a^2}{2EI}(l-a)$$

$$=\frac{1}{6EI}\left\{2W_1l^3-2W_2a^2-3W_2a^2(l-a)\right\}$$

$$=\frac{1}{6EI}\left\{2W_1l^3-W_2a^2(3l-a)\right\}.$$

Problem 239. *A canilever of uniform section has a length $AB=l$, A is the free end and carries a point load W, while B is the fixed end. Find the deflection at a point C distant $\frac{l}{4}$ from the free end A.*

If the cantilever is propped at C find the reaction of the prop assuming that there is no deflection at C.

Draw also the B.M. and S.F. diagrams for the propped cantilever.

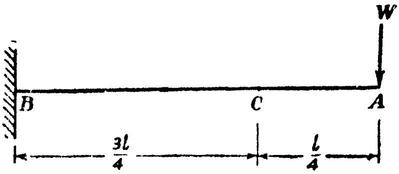


Fig.392

Solution. For the cantilever, the bending moment at any section distant x from the fixed end is given by

$$EI\frac{d^2y}{dx^2}=W(l-x)$$

Integrating, we get,

$$EI\frac{dy}{dx}=W\left(lx-\frac{x^2}{2}\right)+C_1$$

where C_1 is a constant of integration

At B the slope is zero,

i.e., at $\qquad x=0, \frac{dy}{dx}=0$

$\therefore \qquad C_1=0$

$$\therefore \qquad EI\frac{dy}{dx}=W\left(lx-\frac{x^2}{2}\right)$$

Integrating again, we get,

$$EIy=W\left(\frac{lx^2}{2}-\frac{x^3}{6}\right)+C_2$$

where C_2 is a constant of integration,

At B the deflection is zero,

i.e., at $\qquad x=0, \qquad y=0$

$\therefore \qquad C_2=0$

$$\therefore \qquad EIy=W\left(\frac{lx^2}{2}-\frac{x^3}{6}\right)$$

To find the deflection at C,

putting $\qquad x=\frac{3}{4}\,l$ in the above equation, we get

$$EIy_c=W\left(\frac{l}{2}\cdot\frac{9}{16}\,l^2-\frac{1}{6}\cdot\frac{27}{64}\,l^3\right)$$

$$=\frac{27}{128}\,Wl^3$$

$$\therefore \qquad y_c=\frac{27}{128}\,\frac{Wl^3}{EI}$$

If now the prop be provided, let the prop reaction be R.

Upward deflection at C due to the independent effect of R alone

$$= \frac{R\left(\frac{3}{4}\, l\right)^3}{3EI}$$

$$= \frac{9}{64} \frac{Rl^3}{EI}$$

Hence, if the net deflection at C is zero, we have

$$\frac{9}{64} \frac{Rl^3}{EI} = \frac{27}{128} \cdot \frac{Wl^3}{EI}$$

$$\therefore \quad R = \frac{27}{128} \times \frac{64}{9} W$$

$$\therefore \quad R = \frac{3}{2} W$$

S.F. Calculations

S.F. between C and A

$= +W$

S.F. between B and C

$= +W - \frac{3}{2} W$

$= -\frac{W}{2}$

B.M. Calculations

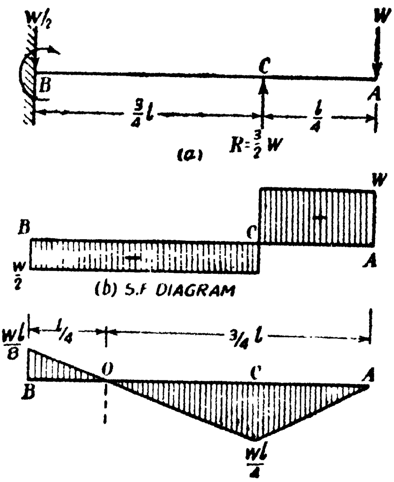
B.M. at $A = 0$

B.M. at $C = -W\frac{l}{4}$

B.M. at B

$= \frac{3}{2} W \cdot \frac{3}{4} l - Wl$

$= +\frac{Wl}{8}$



(C) B.M. DIAGRAM

Fig. 393

Point of Contraflexure

Let the B.M. be zero at O at a distance x from A $\left(x > \frac{l}{4}\right)$

Equating the B.M. to zero, we get

$$\frac{3}{2} W\left(x - \frac{l}{4}\right) - Wx = 0$$

$$\therefore \quad x = \frac{3}{4}\, l.$$

Problem 240. *A cantilever of length l carrying a uniformly distributed load of w per unit run is propped at the free end. If the prop holds the end at the level of the fixed end find the reaction of the prop. Draw S.F. and B.M. diagrams.*

Solution Fig. 394 (*a*) shows the propped cantilever *AB* the end *A* being fixed and the end *B* being propped. Let *R* be the prop-reaction. If the prop had not been present the *downward* deflection of the end *B* would have been $\frac{wl^4}{8EI}$.

If the cantilever had been subjected to an upward force *R* alone at the free end, the *upward* deflection of the end *B* would be $\frac{Rl^3}{3EI}$

Since the deflection at *B* should be zero, we have,

$$\frac{Rl^3}{3EI}=\frac{wl^4}{8EI}$$

$$\therefore \quad R=\frac{3}{8}\ wl$$

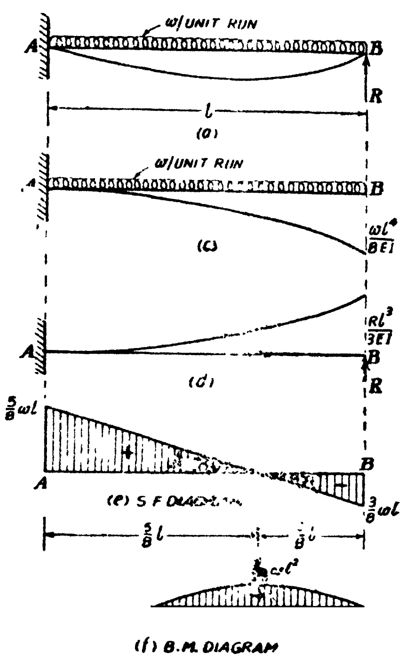


Fig. 394

S.F. diagram

The shear force S_x at any section distant *x* from the end *B* is given by

$$S_x=wx-R$$

$$\therefore \quad S_x=wx-\frac{3}{8}\ wl$$

Section at which the S.F. equals zero

Equating the general expression for S.F. to zero we have

$$wx-\frac{3}{8}\ wl=0,$$

$$\therefore \quad x=\frac{3}{8}\ l$$

At $x=0$.

$$S_x=-\frac{3}{8}\ wl$$

At $x=l$,

$$S_a=wl-\frac{3}{8}\ wl=+\frac{5}{8}\ wl.$$

B.M. diagram

At any section distant x from the end B the bending moment is given by

$$M_x = Rx - \frac{wx^2}{2}$$

$$= \frac{3}{8}wlx - \frac{wx^2}{2}$$

$$\therefore \quad M_x = \frac{wx}{8}(3l - 4x)$$

At $\quad x=0$, *i.e.*, at B,

$$M_x = 0$$

At $\quad x = l$, *i.e.*, at A,

$$M_x = \frac{w}{8}(-l^2)$$

$$= -\frac{wl^2}{8}$$

At $\quad x = \frac{3}{4}l$,

$$M_x = 0$$

At $\quad x = \frac{3}{8}l$,

$$M_x = \frac{w}{8} \cdot \frac{3}{8}l\left(3l - \frac{3}{2}l\right)$$

$$= \frac{9}{128}wl^2.$$

Problem 241. *A propped cantilever 6 metres long carries a uniformly distributed load of 1 t per metre for a distance of 4 metres from the fixed end. Calculate the reaction at the prop.*

(*A.M.I.E.*, *Nov. 1970*)

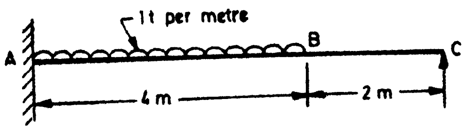


Fig. 395

Solution. Let R be the reaction at the prop C. Since the net deflection at $C=0$, we have,

$$\frac{1 \times 4^4}{8EI} + \frac{1 \times 4^3}{6EI} \times 2 - \frac{R \times 6^3}{3EI} = 0$$

$$\therefore \quad R = 0{\cdot}741 \text{ t}.$$

Problem 242. *A cantilever of length l is propped at its free end. The cantilever carries a uniformly distributed load of w per unit run over the whole span. If the prop sinks by δ, find the reaction of the prop.*

Solution. Let the prop reaction be R.

If the prop had not been present the *downward* deflection of the free end would have been $\frac{wl^4}{8EI}$. Upward deflection of the free end due to an upward force R alone acting at the free end would be

$$\frac{Rl^3}{3EI}$$

Since the net downward deflection of the free end is δ, we have

$$\frac{wl^4}{8EI}-\frac{Rl^3}{3EI}=\delta$$

$$\therefore \quad \frac{Rl^3}{3EI}=\frac{wl^4}{8EI}-\delta$$

$$\therefore \quad R=\frac{3EI}{l^3}\left(\frac{wl^4}{8EI}-\delta\right)$$

Problem 243. *A cantilever of length l is propped at its free end. The cantilever carries a uniformly distributed load of w per unit run. Find by how much above the level of the fixed end the level of the prop must be fixed so that the load may be equally shared by the supports.*

Solution. Prop reaction $=R=\frac{wl}{2}$

Level of the propped end above the level of the fixed end

=net upward deflection of the free end

$$=\frac{Rl^3}{3EI}-\frac{wl^4}{8EI}$$

$$=\left(\frac{wl}{2}\right)\frac{l^3}{3EI}-\frac{wl^4}{8EI}$$

$$=\frac{wl^4}{6EI}-\frac{wl^4}{8EI}=\frac{wl^4}{24EI}$$

Hence the propped end must be *above* the level of the fixed end by $\frac{wl^4}{24EI}$.

Problem 244. *A horizontal cantilever of uniform section and length l carries a uniformly distributed load w per unit length through-*

out its length. The cantilever is supported by a rigid prop at a distance Kl from the fixed end, the level of the beam at the prop being the same as that at the fixed end. Determine the value of K which will make the bending moment at the prop the same as that at the fixed end.

Sketch the S.F. and B.M. diagrams for the cantilever and show on them all maximum values. (*London University*)

Solution. Let AB be the cantilever fixed at A and free at B and propped at C at a distance Kl from A.

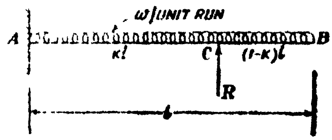


Fig.396

Let the reaction of the prop be R.

B.M. at the prop

$$= \frac{w(l-K)^2 l^2}{2}$$

B.M. at the fixed end

$$RKl - \frac{wl^2}{2}$$

Since the bending moment at fixed end equals the bending moment at the prop, we have,

$$RKl - \frac{wl^2}{2} = \frac{w(1-k)^2 l^2}{2}$$

$$\therefore \quad RK = \frac{wl}{2} - \frac{wl}{2}(1-k)^2$$

$$= \frac{wl}{2}\left[1-1+2K-K^2\right]$$

$$\therefore \quad RK = \frac{wl}{2}K(2-K)$$

$$\therefore \quad K = \frac{wl}{2}(2-k)$$

Since at C the deflection is zero it follows that the downward deflection at C when the prop is absent must be equal to the upward deflection at C due to an upward force R alone applied at C.

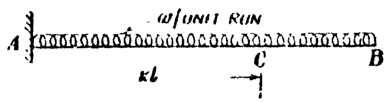


Fig. 397

Let us first find the downward deflection at C when the prop is absent. For this condition of the cantilever the bending moment at any section distant x from the fixed end is given by

$$EI\frac{d^2y}{dx^2} = \frac{w}{2}(l-x)^2 = \frac{w}{2}\left(l^2 - 2lx + x^2\right)$$

Integrating, we get,

$$EI\frac{dy}{dx} = \frac{w}{2}\left(l^2x - lx + \frac{x^3}{3}\right) + C_1$$

where C_1 is a constant of integration.

At A the slope being zero, we have,

at $x=0$

$$\frac{dy}{dx}=0$$

$\therefore$ $C_1=0$

Integrating again, we get,

$$EIy=-\frac{w}{2}\left(\frac{l^2x^2}{2}-\frac{lx^3}{3}+\frac{x^4}{12}\right)+C_2$$

At A the deflection being zero, we have,

at $x=0$

$y=0$

$\therefore$ $C_2=0$

$$\therefore \quad EIy=-\frac{w}{2}\left(\frac{l^2x^2}{2}-\frac{lx^3}{3}+\frac{x^4}{12}\right)$$

Hence to find the deflection at C, putting

$x=Kl$ in the above equation, we get,

$$EIy_c=-\frac{w}{2}\left(\frac{l^2K^2l^2}{2}-\frac{lK^3l^3}{3}+\frac{K^4l^4}{12}\right)$$

$$=-\frac{w}{2}\,\frac{l^4K^2}{12}(6-4K+K^2)$$

$$\therefore \quad y_c=-\frac{wl^4}{24EI}K^2(K^2-4K+6)$$

or the *downward* deflection at C

$$=y_c=\frac{wl^4}{24EI}K^2(K^2-4K+6)$$

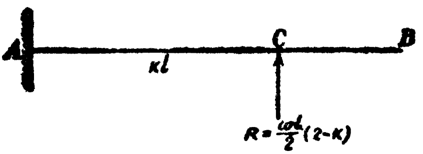


Fig. 398

Upward deflection at C due to R alone

$$=y_c=\frac{R(Kl)^3}{3EI}$$

$$=\frac{wl}{2}\,\frac{(2-K)K^3l^3}{3EI}$$

$$=\frac{wl^4(2-K)K^3}{6EI}$$

Since the net deflection at C is zero, we have,

$$\frac{wl^4}{24EI}K^2(K^2-4K+6)=\frac{wl^4}{6EI}(2-K)K^3$$

$$\therefore \quad K^2-4K+6=4K(2-K)$$

$$\therefore \quad 5K^2-12K+6=0$$

Solving as a quadratic in K,

$$K=\frac{12-\sqrt{144-120}}{10}$$

$$=\frac{12-\sqrt{24}}{10}$$

$$=\frac{12-2\sqrt{6}}{10}$$

$$=\frac{6-\sqrt{6}}{5}$$

$$=\frac{6-\sqrt{6}}{5}\left(\frac{6+\sqrt{6}}{6+\sqrt{6}}\right)$$

$$=\frac{36-6}{5(6+\sqrt{6})}$$

$$=\frac{6}{6+\sqrt{6}}$$

$$=\frac{\sqrt{6}}{\sqrt{6}+1}$$

$$=0{\cdot}71$$

S.F. and B M. diagrams

Fig. 399 shows the propped cantilever

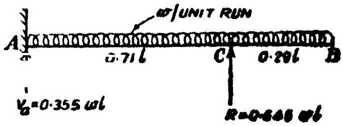


Fig. 399

$$R=\frac{wl}{2}(2-K)$$

$$=\frac{wl}{2}(2-0{\cdot}71)$$

$$=0{\cdot}645\ wl$$

∴ Vertical reaction at A

$$=V_a=wl-0{\cdot}645\ wl$$

$$=0{\cdot}355\ wl$$

S.F. Calculations :

S.F. at $A=+0{\cdot}355\ wl$

S.F. just on the left side of C

$$=+0{\cdot}355\ wl-0{\cdot}71\ wl$$
$$=-0{\cdot}355\ wl$$

S.F. just on the right side of C

$$=+0{\cdot}29\ wl$$

S.F. at $B=0$

Obviously S.F. at the middle point of $AC=0$.

B.M. Calculations :

B.M. at $B=0$

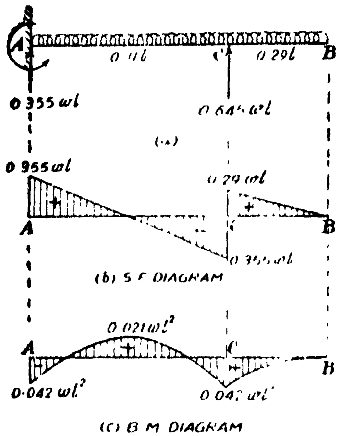


Fig. 400

$$\text{B.M. at } C = -\frac{w(0.29\ l)^2}{2}$$

$$= -0.042\ wl^2$$

$$= \text{B.M. at } A$$

B.M. at the middle point of *AC*, *i.e.*, B.M. at 0.645 *l* from *B*

$$= 0.645\ wl \times 0.355\ l - w \times \frac{(0.645\ l)^2}{2}$$

$$= +0.021\ wl^2$$

Problem 245. *A rolled steel joist 600 mm × 210 mm. is simply supported at its ends on a span of 10 metres and carries a uniformly distributed load of 1.25 tonnes per metre run including its own weight. If the maximum deflection is not to exceed 2 cms. and the maximum stress due to bending is not to exceed 1400 kg/cm.², find the greatest value of an additional concentrated load which may be added to the joist to the middle of the span. For the steel joist take* I_{xx} = 72867.6 *cm.⁴ and* $E = 2 \times 10^6$ *kg./cm.²*

Solution. (*a*) *Deflection criterion*

Let the point load be *W kg*.

Central deflection

$$= \frac{Wl^3}{48EI} + \frac{5wl^4}{384EI} = \delta \text{ cm.}$$

$$\therefore \frac{l^3}{384}\left[8W + 5wl\right] = EI\ \delta$$

$$\therefore \frac{10^3}{384}\left[8W + 5 \times 1.25 \times 1000 \times 10\right] \times 100^3$$

$$= 2 \times 10^6 \times 72867.6 \times 2$$

$$\therefore \quad 8W + 6.25 \times 10^4 = \frac{2 \times 10^6 \times 72867.6 \times 2}{10^9} \times 384$$

$$\therefore \quad 8W + 62500 = 111900$$

$$W = 6175 \text{ kg.}$$

$$= 6.175 \text{ tonnes.}$$

(*b*) *Max. stress criterion*

$$\text{Max. B.M.} = M = \frac{Wl}{4} + \frac{wl^2}{8}$$

$$= \frac{W \times 10}{4} + \frac{1.25 \times 1000 \times 10^2}{8} \text{ kg. m.}$$

$$=2{\cdot}5\ W+1{\cdot}25\times125\times100\ kg.\ m.$$

Max. stress $\quad = f = \dfrac{M}{Z} = 1400\ kg./cm.^2$

$$\therefore\quad \frac{(2{\cdot}5W+1{\cdot}25\times125\times100)\ 100}{\left(\dfrac{72867{\cdot}6}{30}\right)}=1400$$

$$\therefore\quad 2{\cdot}5\ W+1{\cdot}25\times125\times100=\frac{1400\times72867{\cdot}6}{30\times100}$$

$$\therefore\quad 2{\cdot}5\ W+1{\cdot}25\times125\times100=34000$$

$$W=13538\ kg.$$

$$=13{\cdot}538\ tonnes$$

$\therefore$ Maximum permissible value of W

$$=6{\cdot}175\ tonnes$$

Problem 246 *A simply supported beam of span l carries a uniformly distributed load w per unit run over the whole span. If now the beam be provided with a prop at the centre of the span so that the prop holds the beam to the level of the end supports, find the reaction of the prop. Draw S.F. and B.M. diagrams.*

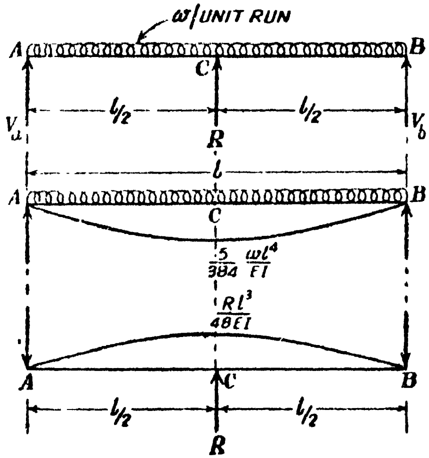


Fig. 401

Solution. Let AB be the beam of length l supported at the ends and propped at mid span C. Let the reaction of the prop be R.

If the prop had not been present the downward deflection at C would be

$$\frac{5}{384}\frac{wl^4}{EI}$$

If the beam had been subjected to an upward force R alone at C the upward deflection at C would be

$$\frac{Rl^3}{48\ EI}$$

Since the deflection at C is zero, we have,

$$\frac{Rl^3}{48\ EI}=\frac{5}{384}\frac{wl^4}{EI}$$

$$\therefore\quad R=\frac{5}{8}\ wl$$

$\therefore$ Vertical reaction at each end support

$$=V_a=V_b=\frac{\left(wl-\dfrac{5}{8}wl\right)}{2}$$

$$=\frac{3}{16}wl$$

S.F. Calculations

At any section in AC distant x from A the S.F. is given by

$$S=\frac{3}{16}\,wl-wx$$

At A *i.e.*, at $x=0$,

$$S=+\frac{3}{16}\,wl$$

At C *i.e.*, at $x=\frac{l}{2}$

$$S=\frac{3}{16}\,wl-\frac{wl}{2}$$

$$=-\frac{5}{16}wl.$$

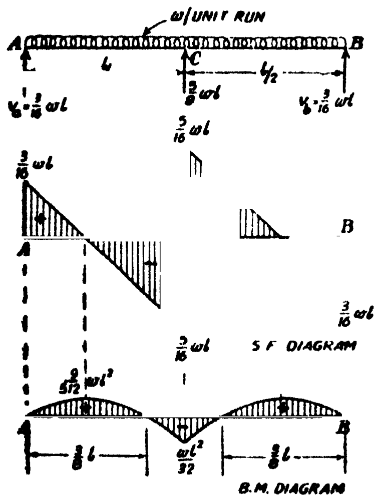


Fig. 402

Hence for the span AC the S.F. changes uniformly from $+\frac{3}{16}\,wl$ at A to $-\frac{5}{16}\,wl$ at C. Similarly for the span CB the S.F. changes uniformly from $+\frac{5}{16}\,wl$ at C to $-\frac{3}{16}\,wl$ at B.

Let at a distance x from A on the span AC the S.F. be zero. Equating the S.F. to zero, we get,

$$\frac{3}{16}\,wl-wx=0$$

$$\therefore \qquad x=\frac{3}{16}\,l$$

Consider the span AC

At any section distant x from A the B.M. is given by

$$M=\frac{3}{16}\,wl\,x-\frac{wx^2}{2}$$

$$M=\frac{w}{16}x(3l-8x)$$

At $\qquad x=0$ *i.e.*, at A and also at

$$x=\frac{3}{8}\,l$$

$$M=0$$

At $x = \frac{3}{16}l$

$$M = \frac{w}{16}\cdot\frac{3}{16}l\left(3l - 8\times\frac{3}{16}l\right)$$

$$= +\frac{9}{512}wl^3$$

At $x = \frac{l}{2}$ *i.e.*, at C

$$M = \frac{w}{16}\,\frac{l}{2}\left(3l - \frac{8\times l}{2}\right)$$

$$= -\frac{wl^3}{32}.$$

Problem 247. *A uniform girder of length 8 m is subjected to a total load of 20t uniformly distributed over the entire length. The girder is freely supported at its ends. Calculate the B.M. and the deflection at the centre.*

If a prop is introduced at the centre of the beam so as to nullify this deflection, find the net B.M. at the centre. (*AMIE, May 1972*)

Case (*i*) *When the girder is supported only at the ends*

Solution.

B.M. at centre $= \frac{wl^2}{8} = \frac{Wl}{8} = \frac{20\times 8}{8}$ *tm*

$= 20$ *tm*

Deflection at the centre $= \delta = \frac{5}{384}\frac{wl^4}{EI} = \frac{5}{384}\frac{Wl^3}{EI}$

$$= \frac{5}{384}\times\frac{20\times(800)^3}{EI}$$

$$= \frac{400000000}{3EI}$$

Case (*ii*). *When a prop is provided at the Centre.*

For this case, B.M. at centre

$$= -\frac{wl^2}{32} = -\frac{Wl}{32}$$

$$= -\frac{20\times 8}{32} = -5 \text{ tm}.$$

§66. Macaulay's Method

This is a convenient method for determining the deflections of a beam subjected to point loads or in general discontinuous loads. The method mainly consists in the special manner in which the bending moment at any section is expressed and in the manner in which the integrations are carried out.

Fig. 403 shows a simply supported beam AB supported at A and B, having a span l and carrying the loads W_1 and W_2 at C and D at distances a and b from the end A. Let V_a and V_b be the vertical reactions at A and B.

At any section between A and C distant x from A the bending moment is given by

$$M_x = V_a x$$

This expression for the bending moment holds good for all values of x between $x = 0$ and $x = a$.

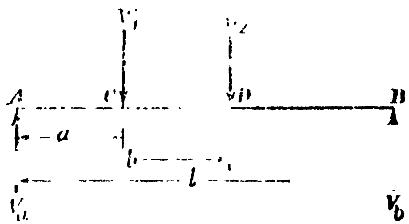


Fig. 403

At any section between C and D and distant x from A, the bending moment is given by

$$M_x = V_a x - W_1 (x-a)$$

This expression holds good for all values of x between $x=a$ and $x = b$.

At any section between D and B and distant x from the end A, the bending moment is given by

$$M_x = V_a x - W_1 (x-a) - W_2(x-b)$$

This expression holds good for all values of x between $x=b$ and $x = l$

In general at any section the bending moment is given by

$$M_x = EI \frac{d^2y}{dx^2} = V_a x \vdots - W_1 (x-a) \vdots - W_2 (x-b) \quad ...(i)$$

The manner in which the above expression is written should be noted. As the magnitude of x goes on increasing so that the law of loading changes, *additional* expressions appear.

For values of x between $x = 0$ and $x = a$, only the first term of the above expression should be considered

For values of x between $x = a$ and $x = b$ only the first two terms of the above expression should be considered.

For values of x between $x = b$ and $x = l$, all the terms of the above expression should be considered.

Integrating Eq. (i) we get, the general expression for slope

$$EI \frac{dy}{dx} = V_a \frac{x^2}{2} + C_1 \vdots - \frac{W_1 (x-a)^2}{2} - \frac{W_2 (x-b)^2}{2} \quad ...(i)$$

It is very important to note the following two points :

(a) The constant of integration C_1 should be written after the first term of the above expression.

(*b*) The quantity $(x - a)$ should be integrated as $\frac{(x-a)^2}{2}$ and not as $\frac{x^2}{2} - ax$.

Similarly the quantity $(x - b)$ should be integrated as a whole *i.e.*, as $\frac{(x-b)^2}{2}$

(*c*) The constant C_1 is valid for all values of x.

Integrating equation (*ii*) we get the deflection equation.

$$EIy = V_a \frac{x^3}{6} + C_1 x + C_2 - \frac{W_1(x-a)^3}{6} - \frac{W_2(x-b)^3}{6}$$

Again it may be noted that $(x - a)^2$ has been integrated to $\frac{(x-a)^3}{3}$ and $(x - b)^2$ has been integrated to $\frac{(x-b)^3}{3}$. The constant C_2 is written after $C_1 x$. The constant C_2 is valid for all values of x.

The constants C_1 and C_2 can be evaluated if the end conditions are known.

For instance when the beam is simply supported the deflection is zero at A and B, *i.e.*, at $x = 0$ and at $x = l$, $y = 0$.

Putting $x = 0$ and $y = 0$ in the deflection equation, we get $C_2 = 0$.

Putting $x = l$ and $y = 0$ in the deflection equation, the constant C_1 can be evaluated. Once the constants C_1 and C_2 are known, the slope and deflection at any section can be determined. The following problems show the application of Macaulay' method.

Problem 248 *A beam of length l simply supported at the ends carries a point load W at a distance a from the left end. Find the deflection under the load and the maximum deflection*

Solution Let AB be the beam of span l carrying the load W at C.

Let $AC = a$ and

$CB = b$

Let $a > b$

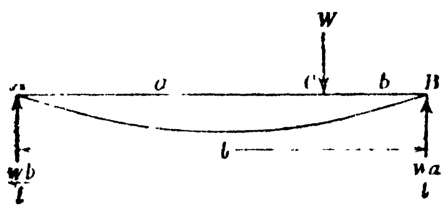


Fig. 404

It is easily seen that the vertical reactions V_a and V_b at A and B are given by,

$$V_a = \frac{Wb}{l}$$

and $$V_b = \frac{Wa}{l}$$

Following Macaulay's method, the b[illegible] moment at any section is given b

$$EI\frac{d^2y}{dx^2}=\frac{Wb}{l}x-W(x-a)$$

Integrating again, we get

$$EI\frac{dy}{dx}=\frac{Wbx^2}{2l}+C_1-\frac{W(x-a)^2}{2} \quad \text{(slope equation)}$$

Integrating again, we get

$$EI\,y=\frac{Wbx^3}{6l}+C_1x+C_2-\frac{W(x-a)^3}{6}$$

(deflection equation)

At A the deflection is zero,

i.e., at $x=0$,

$y=0$

$\therefore$ $C_2=0$

At B the deflection is zero,

$\therefore$ at $x=l$,

$y=0$

$$\therefore \quad 0=\frac{Wbl^2}{6}+C_1l-\frac{W(l-a)^3}{6}$$

$$\therefore \quad C_1l=\frac{W(l-a)^3}{6}-\frac{Wbl^2}{6}$$

Since $l-a=b$,

$$C_1l=\frac{Wb^3}{6}-\frac{Wbl^2}{6}$$

$$=-\frac{Wb}{6}(l^2-b^2)$$

$$C_1=-\frac{Wb}{6l}(l^2-b^2)$$

Hence the slope and deflection at any section are given by

$$EI\frac{dy}{dx}=\frac{Wbx^2}{2l}-\frac{Wb}{6l}(l^2-b^2)-\frac{W(x-a)^2}{2} \text{ and,}$$

$$EIy=\frac{Wbx^3}{6l}-\frac{Wb}{6l}(l^2-b^2)x-W\frac{(x-a)^3}{6}$$

To find the deflection y_c under load, putting $x=a$ in the deflection equation, we get

$$EIy_c=\frac{Wba^3}{6l}-\frac{Wb}{6l}(l^2-b^2)a$$

$$=-\frac{Wb}{6l}a(l^2-b^2-a^2)$$

But $l=a+b$

$$EIy_c=-\frac{Wba}{6l}(a^2+b^2+2ab-b^2-a^2)$$

$$=-\frac{Wba}{6l}(2ab)$$

$$=-\frac{Wa^2b^2}{3l}$$

$$\therefore \quad y_c=-\frac{Wa^2b^2}{3EI\,l}$$

To find the maximum deflection

The maximum deflection will occur on the large segment AC. Further, at the point of maximum deflection the slope is zero.

Hence equating the slope at a section in AC to zero, we have

$$0=\frac{Wbx^2}{2l}-\frac{Wb}{6l}(l^2-b^2)$$

$$\therefore \quad x^2=\frac{l^2-b^2}{3}$$

$$\therefore \quad x=\sqrt{\frac{l^2-b^2}{3}} \text{ or } \sqrt{\frac{a^2+2ab}{3}}$$

The maximum deflection y_{max} can be determined by putting

$x=\sqrt{\frac{l^2-b^2}{3}}$ in the expression for deflection.

$$EI\,y_{max}=\frac{Wb}{6l}\left(\frac{l^2-b^2}{3}\right)^{3/2}-\frac{Wb}{6l}(l^2-b^2)\left(\frac{l^2-b^2}{3}\right)^{\frac{1}{2}}$$

$$=-\frac{Wb}{6l}\left(l^2-b^2\right)^{3/2}\left[\frac{1}{\sqrt{3}}-\frac{1}{3^{3/2}}\right]$$

$$=-\frac{Wb}{6l}\left(l^2-b^2\right)^{3/2}\left[\frac{1}{\sqrt{3}}-\frac{1}{3\sqrt{3}}\right]$$

$$=-\frac{Wb}{6l}\left(l^2-b^2\right)^{3/2}\frac{2\sqrt{3}}{9}$$

$$=-\frac{Wb\,(l^2-b^2)^{3/2}}{9\sqrt{3}\,l}$$

$$\therefore \quad y_{max}=-\frac{Wb\,(l^2-b^2)^{3/2}}{9\sqrt{3}EI\,l}$$

or putting $\quad l=(a+b)$

$$y_{max}=-\frac{Wb\,(a^2+2ab)^{3/2}}{9\sqrt{3}EI\,l}$$

Problem 249. *A rolled steel beam having a span of 6 metres carries a point load of 4 tonnes at 4 metres from the left support. Find* *the deflection under the load and the position and amount of maximum deflection.* I_{xx} *for the section=7330 cm⁴. Take E=2×10³ tonnes/cm.²*

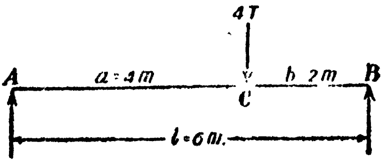


Fig 405

Solution. Fig 405 shows the beam carrying the given loading.

Bigger segment

$=AC=a=4m.$

Smaller segment

$=BC=b=2m$

Deflection under the load

$$=y_c=\frac{Wa^2b^2}{3EI\,l}$$

$$=\frac{4\times4^2\times2^2}{3\times2\times10^3\times7330\times6}(100)^3\ cm.$$

$=0.97\ cm.$

Maximum deflection occurs at

$$\sqrt{\frac{l^2-b^2}{3}}=\sqrt{\frac{36-4}{3}}=3.27\ \textit{metres from the left end.}$$

Max. deflection

$$=y_{max}=\frac{wb(l^2-b^2)^{3/2}}{9\sqrt{3}\ EI\ l}$$

$$=\frac{4\times2\ (32)^{3/2}\times(100)^3}{9\sqrt{3}\times2\times10^3\times7330\times6}\ cm.$$

$=1.056\ cm.$

Problem 250. *A simply supported beam of length l carries a load W at a distance a from one end and b from the other (a>b). Find the position and magnitude of the maximum deflection and show that the position is always within* $\frac{l}{\sqrt{3}}$ *approximately from the centre.*

(London University)

Solution. Fig 406 shows the beam AB carrying the load W at C so that $AC=a$, $BC=b$ and $a>b$.

The maximum deflection occurs at a distance

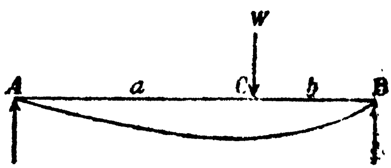


Fig. 406

$$x=\sqrt{\frac{l^2-b^2}{3}}\ \text{from } A \text{ (see problem 248)}$$

$$\text{Max. deflection}=\frac{Wb\,(l^2-b^2)^{3/2}}{9\sqrt{3}EI\,l}.$$

Hence for all possible positions that the load can take on the span, the greatest value of $x = \sqrt{\dfrac{l^2}{3}} = \dfrac{l}{\sqrt{3}}$

$\therefore$ Maximum distance of the point of maximum deflection from the mid span

$$= \frac{l}{\sqrt{3}} - \frac{l}{2}$$

$$= \left(\frac{2-\sqrt{3}}{2\sqrt{3}}\right) l$$

$$\approx \frac{l}{13}$$

Problem 251. *A horizontal girder of steel having uniform section is 14 metres long and is simply supported at its ends. It carries concentrated load of 12 tonnes and 8 tonnes at two points 3 metres and 4·5 metres from the two ends respectively. I for the section of the girder is 16×10^4 cm^4 and $E = 2·1 \times 10^6$ kg./cm^2. Calculate the deflection of the girder at points under the two loads.*

Find also the maximum deflection.

Solution. Let V_a and V_b be the vertical reactions at the support A and B.

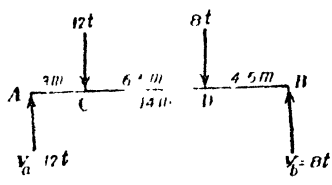


Fig. 407

Taking moment about A, we have

$V_b \times 14 = 12 \times 3 + 8 \times 9·5$

$\therefore$ $V_b = 8$ *tonnes*

$\therefore$ $V_a = 20 - 8 = 12$ *tonnes.*

The B.M. at any section distant x from A is given by

$$EI\frac{d^2y}{dx^2} = 12x \,\vdots\, -12(x-3) - 8(x-9·5)$$

Integrating, we get

$$EI\frac{dy}{dx} = 6x^2 + C_1 - 6(x-3)^2 - 4(x-9·5)^2$$

Integrating again, we get

$$EI\,y = 2x^3 + C_1x + C_2 - 2(x-3)^3 - \frac{4}{3}(x-9·5)^3$$

At $x = 0$, $y = 0$

$\therefore$ $C_2 = 0$

At $x = 14$, $y = 0$

$\therefore$ $0 = 2(14)^3 + 14C_1 - 2(14-3)^3 - \frac{4}{3}(14-9·5)^3$

$\therefore \quad 14\, C_1 = -2704{\cdot}5$

$\therefore \quad C_1 = -193{\cdot}18$

Hence the deflection at any section is given by

$$EI\, y = 2x^3 - 193{\cdot}18x \;\vdots -2\,(x-3)^3 \;\vdots -\frac{4}{3}(x \quad 9{\cdot}5)^3$$

To find the deflection at C, put $x=3m$ in the deflection equation

$$EI\, y_c = 2\times 3^3 - 193{\cdot}18\times 3$$
$$= -525{\cdot}54$$

$$\therefore \quad y_c = -\frac{525{\cdot}54}{EI}$$
$$= -\frac{525{\cdot}54\times 10^6}{2{\cdot}1\times 10^3\times 16\times 10^4}\ cm.$$
$$= -1{\cdot}564\ cm.$$

$$\therefore \quad y_c = -1{\cdot}564\ cm.\ (downward\ deflection)$$

To find the deflection at D, put $x=9{\cdot}5\ m$ in the deflection equation

$$El\, y_d = 2\,(9{\cdot}5)^3 - 193{\cdot}18\times 9{\cdot}5 - 2(9{\cdot}5-3)^3$$
$$= -669{\cdot}71$$

$$y_d = \frac{669{\cdot}71\times 10^6}{2{\cdot}1\times 10^3\times 16\times 10^4}\ cm.$$
$$= -1{\cdot}994\ cm.$$

$$y_d = -1{\cdot}994\ cm.\ (downward\ deflection).$$

Max. deflection. Let us assume that the deflection will be maximum at a section between C and D. Equating the slope at th section to zero, we have,

$$EI\frac{dy}{dx} = 6x^2 - 193{\cdot}18 - 6\,(x-3)^2 = 0$$

Solving, we get

$$x = 6{\cdot}87\ m$$

Substituting in the deflection equation, we get,

$$EI\, y_{max} = 2(6{\cdot}87)^3 - 193{\cdot}18\times 6{\cdot}87 - 2(6{\cdot}87-3)^3$$
$$= -794{\cdot}3$$

$$\therefore \quad y_{max} = -\frac{794{\cdot}3\times 10^6}{2{\cdot}1\times 10^3\times 16\times 10^4}\ cm.$$
$$= -2{\cdot}36\ cm.\ (downward\ deflection)$$

Problem 252. *A beam of uniform section is 10 m. long and is simply supported at the ends. It carries concentrated loads of 10t and 6t at distances of 2m and 5m respectively from the left end. Calculate the deflection under each load. Find also the maximum deflection. Take* $I = 18\times 10^4\ cm.^4$ *and* $E = 2\times 10^6\ kg./cm^2$.

Solution. Let V_a and V_b be the reactions at the left and right supports. Taking moments about A,

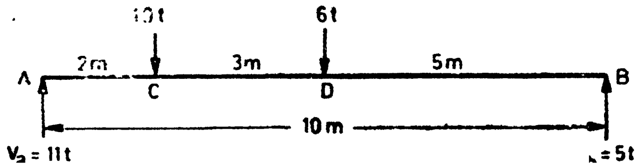


Fig. 408

$$V_b \times 10 = (10 \times 2) + (6 \times 5)$$

$$\therefore \quad V_b = 5t$$

$$\therefore \quad V_a = 16 - 5 = 11t$$

The B.M. at any section distant x from A is given by,

$$EI\frac{d^2y}{dx^2} = 11x \;\; -10\,(x-2 \;\; -6\,(x-5)$$

Integrating we get,

$$EI\frac{dy}{dx} = 5{\cdot}5x^2 + C_1 \Big| -5\,(x-2)^2 - 3\,(x-5)^2$$

Integrating again, we get,

$$EI\,y = \frac{5{\cdot}5}{3}x^3 + C_1\,x + C_2 \Big| -\frac{5}{3}(x-2)^3 \Big| -(x-5)^3$$

At $\quad x=0,$
$\quad\quad y=0$

$$\therefore \quad C_2 = 0$$

At $\quad x=10,$
$\quad\quad y=0$

$$\therefore \quad 0 = \frac{5{\cdot}5}{3}(10)^3 + 10C_1 - \frac{5}{3} \times 8^3 - 5^3$$

$$\therefore \quad C_1 = -85{\cdot}5$$

Deflection at C : putting $x = 2m$ in the deflection equation

$$EI\,y_c = \frac{5{\cdot}5}{3}(2)^3 - 85{\cdot}5 \times 2 = -156{\cdot}67$$

$$\therefore \quad y_c = -\frac{156{\cdot}67}{EI} = -\frac{156{\cdot}67 \times 10^6}{2 \times 10^3 \times 18 \times 10^4}$$

$$= -0{\cdot}435 \text{ cm.}$$

Deflection at D : putting $x = 5m$ in the deflection equation,

$$EI\,y_d = \frac{5{\cdot}5}{3}(5)^3 - 85{\cdot}5 \times 5 - \frac{5}{3}(3)^3 = -243{\cdot}33$$

$$\therefore \quad y_d = -\frac{243{\cdot}33}{EI} = -\frac{243{\cdot}33 \times 10^6}{2 \times 10^3 \times 18 \times 10^4} = -0{\cdot}676 \text{ cm.}$$

Max. deflection. The deflection is likely to be maximum at a section between C and D. Equating the slope to zero, we get,

$$5.5\,x^2-85.5-5\,(x-2)^2=0$$

$$\therefore \qquad x^2+40x-211=0$$

$$\therefore \qquad x=4.72\ m$$

This value of x justifies the correctness of our assumption about the position of maximum deflection.

Putting $x=4.72\ m$ in the deflection equation, we get,

$$EI\,y_{max}=\frac{5.5}{3}(4.72)^3-85.5\times 4.72-\frac{5}{3}\times 2.72^3$$

$$=-244.25$$

$$\therefore \qquad y_{max}=-\frac{244.25}{EI}$$

$$=-\frac{244.25\times 10^6}{2\times 10^3\times 18\times 10^4}=-0.678\ cm.$$

Problem 253. *A simply supported beam of uniform section has a span l and carries two equal loads W each symmetrically placed at a distance $\frac{l}{3}$ on either side of mid span. Find the deflection at the midspan.*

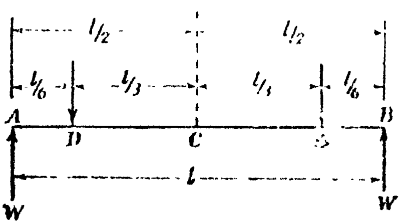


Fig 409

Solution. Fig. 409 shows the beam carrying the loading mentioned in the problem.

The problem will be solved by *Macaulay's* method.

At any section between A and mid-span and distant x from A, the bending moment is given by

$$EI\frac{d^2y}{dx^2}=Wx-W\left(x-\frac{l}{6}\right)$$

Integrating, we get,

$$EI\frac{dy}{dx}=\frac{Wx^2}{2}+C_1\ \Big|-\frac{W}{2}\left(x-\frac{l}{6}\right)^2$$

At $\qquad x=\frac{l}{2}$ the slope *i.e.*, $\frac{dy}{dx}=0$

$$\therefore \qquad 0=\frac{W}{2}\cdot\frac{l^2}{4}+C_1-\frac{W}{2}\cdot\frac{l^2}{9}$$

$$\therefore \qquad C_1=-\frac{5}{72}Wl^2$$

$$\therefore \qquad EI\frac{dy}{dx}=\frac{Wx^2}{2}-\frac{5}{72}Wl^2\ \Big|-\frac{W}{2}\left(x-\frac{l}{6}\right)^2$$

Integrating again, we have,

$$EIy=\frac{Wx^3}{6}-\frac{5}{72}Wl^2x+C_2-\frac{W}{6}\left(x-\frac{l}{6}\right)^3$$

At A the deflection is zero.

i.e., at $x=0,\ y=0$

$\therefore$ $C_2=0$

$$EI\,y=\frac{Wx^3}{6}-\frac{5}{72}Wl^2x-\frac{W}{6}\left(x-\frac{l}{6}\right)^3$$

To find the deflection y_c at mid span,

putting $x=\frac{l}{2}$ in the above equation, we get

$$EI\,y_c=\frac{W}{6}\cdot\frac{l^3}{8}-\frac{5}{72}Wl^2.\frac{l}{2}-\frac{W}{6}\cdot\frac{l^3}{27}$$

$$=Wl^3\left(\frac{1}{48}-\frac{5}{144}-\frac{1}{162}\right)$$

$$y_c=-\frac{13}{648}\ \frac{Wl^3}{EI}$$

Problem 254. *A uniform beam ($I=7800\ cm^4$) is 6m long and carries a central point load of 5t. Taking $E=2{\cdot}1\times10^6$ kg./cm.^{2} calculate the deflection under the load if (a) the beam is simply supported at its end, and (b) the beam is built in at one end and simply supported to the same level as the other.*

(AMIE November 1969)

Solution.

(*a*) *When the beam is simply supported at the ends.*

$$\delta=\frac{Wl^3}{48EI}$$

$$=\frac{5000\times600^3}{48\times2{\cdot}1\times10^6\times7800}$$

$$=1{\cdot}374\ cm.$$

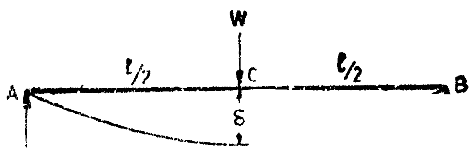


Fig. 410

(*b*) *When the beam is built-in at A and simply supported at B.*

Since the deflection at $B=0$, we have

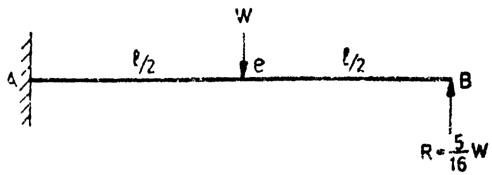


Fig. 411

$$\frac{W\left(\frac{l}{2}\right)^3}{3EI}+\frac{W\left(\frac{l}{2}\right)^2}{2EI}=\frac{Rl^3}{3EI}$$

$$\therefore \qquad R=\frac{5}{16}W$$

With B as origin

$$EI\frac{d^2y}{dx^2}=\frac{5}{16}Wx-W\left(x-\frac{l}{2}\right)$$

$$EI\frac{dy}{dx}=\frac{5}{32}Wx^2+C_1-\frac{W}{2}\left(x-\frac{l}{2}\right)^2$$

At $\quad x=l, \frac{dy}{dx}=0$

$$\therefore \qquad C_1=-\frac{Wl^2}{32}$$

$$EIy=\frac{5}{96}Wx^3-\frac{Wl^2}{32}x+C_2-\frac{W}{6}\left(x-\frac{l}{2}\right)^3$$

At $\quad x=0, y=0$

$\therefore \quad C_2=0$

At $\quad x=l/2, \quad y=y_c$

$$\therefore \qquad EIy_c=\frac{5}{96}W\frac{l^3}{8}-\frac{Wl^2}{32}\cdot\frac{l}{2}=-\frac{7}{768}Wl^3$$

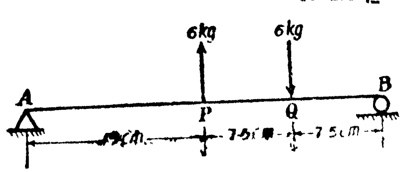
$$y_c=-\frac{7}{768}\frac{Wl^3}{EI}=\frac{7}{768}\cdot\frac{5000\times600^3}{2.1\times10^6\times7800}=0.6011 \text{ cm.}$$

Problem 255. *A simply supported beam AB of span l and uniform flexural rigidity EI is subjected to a point load W at a section C distant a from A and b from B. Prove by any standard method that deflection at a section in the part AC at distance x from A is given by the expression*

$$\frac{Wbx}{6l\,EI}\left[al+ab-x^2\right]$$

Hence calculate the net deflections at P and Q for the beam shown in Fig 412 (a) if the value of EI is 8400 kg. cm.[2]

(*A.M.I.E., November 1966*)



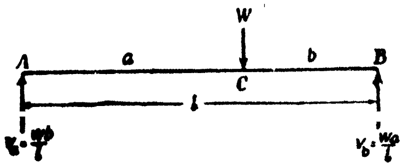


Fig. 412 (*a*)

Solution. Fig. 412 (*a*) shows a beam AB of span l carrying a point load W at C so that $AC=a$ and $BC=b$.

Let V_a and V_b be the vertical reactions at A and B.

Taking moments about the end A, we have,

$$V_b l = Wa$$

$$\therefore \quad V_b = \frac{Wa}{l}$$

$$\therefore \quad V_a = W - \frac{Wa}{l}$$

$$= \frac{W(l-a)}{l} = \frac{Wb}{l}$$

The B.M. at any section distant x from A can be written in the form

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{l}x \quad - W(x-a)$$

Integrating, we get,

$$EI\frac{dy}{dx} = \frac{Wb}{2l}x^2 + C_1 - \frac{W}{2}(x-a)^2 \quad ...(Slope\ equation)$$

Integrating again, we get

$$EIy = \frac{Wb}{6l}x^3 + C_1x + C_2 - \frac{W}{6}(x-a)^3$$

...(*Deflection equation*)

At A the deflection is zero.

$\therefore$ At $x = 0$,

$$y = 0$$

$$\therefore \quad C_2 = 0$$

At B the deflection is zero.

$\therefore$ At $x = l$,

$$y = 0$$

$$\therefore \quad 0 = \frac{Wb}{6}l^2 + C_1l - \frac{W}{6}(l-a)^3$$

$$\therefore \quad C_1l = \frac{W}{6}(l-a)^3 - \frac{Wb}{6}l^2$$

But $l - a = b$

$$\therefore \quad C_1l = \frac{Wb^3}{6} - \frac{Wbl^2}{6}$$

$$\therefore \quad C_1 = -\frac{Wb}{6l}(l^2 - b^2)$$

The deflection at any section in the part AC is given by

$$EIy = \frac{Wb}{6l}x^3 + C_1x + C_2$$

$$\therefore \quad EIy = \frac{Wb}{6l}x^3 - \frac{Wb}{6l}(l^2 - b^2)x$$

$$= -\frac{Wbx}{6l}(l^2 - b^2 - x^2)$$

$$= -\frac{Wb}{6l} x \left\{(l+b)(l-b) - x^2\right\}$$

$$= -\frac{Wb}{6l} x \left\{a(l+b) - x^2\right\}$$

$$= -\frac{Wb}{6l} x(al+ab-x^2)$$

$$\therefore \qquad y = -\frac{Wb}{6lEI} x(al+ab-x^2)$$

Or the downward deflection at any section in AC distant x from A is given by

$$y = \frac{Wb}{6lEI} x(al+ab-x^2)$$

Deflection under the load is given by putting $x=a$ in the above expression

$$y_c = \frac{Wb}{6lEI} a(al+ab-a^2)$$

$$= \frac{Wab}{6lEI} \left\{al - a^2 + ab\right\}$$

$$= \frac{Wab}{6lEI} \left\{a(l-a) + ab\right\}$$

$$= \frac{Wab}{6lEI} (ab+ab)$$

$$y_c = \frac{Wa^2b^2}{3lEI}$$

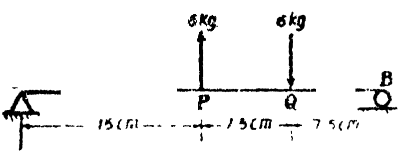


Fig. 412 (*b*)

The above formulae may be used to find the deflections at P and Q due to the given loading

Deflection at P

Deflection at P due to the upward load at P

$$y_1 = \frac{Wl^3}{48EI}$$

$$= \frac{6 \times 30^3}{48 \times 8400} \text{ cm.}$$

$$= 0{\cdot}4018 \text{ cm. (upwards)}$$

Deflection at P due to the load of 6 kg. at Q

$$y_2 = \frac{Wb}{6lEI} x(al+ab-x^2)$$

$W = 6$ *kg.*

$a = 22{\cdot}5$ cm.
$b = 7{\cdot}5$ cm.
$x = 1{\cdot}5$ cm.
and $EI = 8400$ kg. cm.^{2}

$$\therefore \quad y_2 = \frac{6\times7{\cdot}5\times15}{6\times30\times8400}(22{\cdot}5\times30+22{\cdot}5\times7{\cdot}5-15^2)\ cm.$$

$= 0{\cdot}2763$ *cm. (downwards)*

Hence net deflection of $P = (y_1 - y_2)$

$= (0{\cdot}4018 - 0{\cdot}2763)$ *cm. (upwards)*

$= 0{\cdot}1255$ *cm. (upwards)*

Deflection at Q

Deflection at Q due to the load of 6 kg. at Q

$$Y_1 = \frac{Wa^2b^2}{3lEI}$$

$$= \frac{6\times22{\cdot}5^2\times7{\cdot}5^2}{3\times30\times8400}\ cm.$$

$= 0{\cdot}2261$ *cm. (downwards).*

Deflection at Q due to the upward load of 6 kg. at P

$$Y_2 = \frac{Wb}{6lEI}x(al+ab-x^2)$$

where $W = 6$ kg.
$a = PB = 15$ cm.
$b = PA = 15$ cm.
$x = QB = 7{\cdot}5$ cm.

$$\therefore \quad Y_2 = \frac{6\times15\times7{\cdot}5}{6\times30\times8400}(15\times30+15\times15-7{\cdot}5^2)cm.$$

$\therefore \quad Y_2 = 0{\cdot}2763$ *cm. (upwards)*

$\therefore$ Net deflection of $Q = Y_2 - Y_1$

$= 0{\cdot}2763 - 0{\cdot}2261$ cm.

$= 0{\cdot}0502$ *cm. (upwards).*

Problem 256. *A beam of length l is simply supported at the ends and carries a concentrated load W at a distant a from each end. Find the deflection under each load and the deflectian at the centre.*

Solution. The loading on the beam being symmetrical, the maximum deflection occurs at mid span. Hence the slope at mid span equals zero. Hence it is enough if we discuss one half of the span.

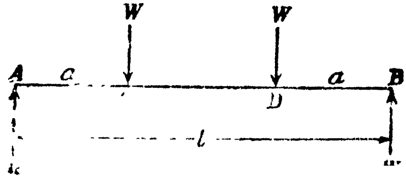


Fig. 413

The bending moment at any section between A and the midspan (Fig. 41·) distant x from A is given by

$$EI\frac{d^2y}{dx^2} = -Wx - W(x-a) \text{ following Macaulay's}$$

conventions.

Integrating, we get,

$$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + C_1 - \frac{W(x-a)^2}{2}$$

At $\quad x = \frac{l}{2}, \quad \frac{dy}{dx} = 0$

$$\therefore \quad 0 = -\frac{W}{2}\cdot\frac{l^2}{4} + C_1 - \frac{W}{2}\left(\frac{l}{2} - a\right)^2$$

$$\therefore \quad C_1 = \frac{Wl^2}{8} + \frac{W}{2}\left(\frac{l}{2} - a\right)^2$$

$$= \frac{Wl^2}{8} + \frac{W}{2}\left(\frac{l^2}{4} - la + a^2\right)$$

$$= \frac{Wl^2}{8} + \frac{Wl^2}{8} - \frac{W}{2}la + \frac{Wa^2}{2}$$

$$\therefore \quad C_1 = -\frac{Wa}{2}(l - a)$$

$$\therefore \quad EI\frac{dy}{dx} = \frac{Wx^2}{2} - \frac{Wa}{2}(l-a) - \frac{W(x-a)^2}{2}$$

Integrating again, we get,

$$EIy = \frac{Wx^3}{6} - \frac{Wa(l-a)}{2}x + C_2 - \frac{W(x-a)^3}{6}$$

At $\quad x = 0, \; y = 0$

$\therefore \quad C_2 = 0$

$\therefore$ The deflection at any section is given by

$$EIy = \frac{Wx^3}{6} - \frac{Wa(l-a)}{2}x - \frac{W(x-a)^3}{6}$$

To find the deflection under the load, *i.e.*, at C putting $x = a$ in the deflection equation, we have.

$$EIy_c = \frac{Wa^3}{6} - \frac{Wa^2(l-a)}{2}$$

$$= -\frac{Wa^2}{6}(3l - 3a - a)$$

$$= -\frac{Wa^2}{6}(-al)4$$

$$\therefore \quad y_c = -\frac{Wa^2(3l-4a)}{6EI}$$

To find the deflection at the centre, putting $x = \frac{l}{2}$ in the deflection equation, we get,

$$EI\,y_{max}=\frac{W}{6}\cdot\frac{l^3}{8}-\frac{Wa(l-a)}{2}\cdot\frac{l}{2}-\frac{W}{6}\left(\frac{l}{2}-a\right)^3$$

$$=\frac{Wl^3}{48}-\frac{Wl}{4}(la-a^2)-\frac{W}{6\times 8}(l-2a)^3$$

$$=\frac{W}{48}\left[l^3-12l^2a+12la^2-l^3+3l^2(2a)-3l(2a)^2+8a^3\right]$$

$$=\frac{W}{48}\left[l^3-12l^2a+12la^2-l^3+6l^2a+12la^2++8a^3\right]$$

$$=\frac{W}{48}\left[-6l^2a+8a^3\right]$$

$$=-\frac{Wa}{24}\left(3l^2-4a^2\right)$$

$$\therefore\; y_{max}=-\frac{Wa(3l^2-4a^2)}{24\,EI}$$

Problem 257. *A simply supported beam of uniform flexural rigidity of span l carries a uniformly distributed load of w per unit run for a distance a from the right end. Calculate the value of a for which the maximum deflection will occur at the left end of the uniformly distributed load.*

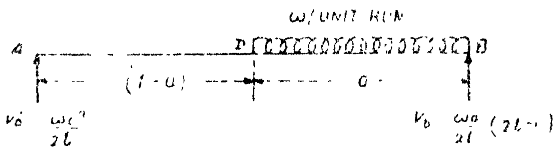


Fig. 414

Solution. Fig. 414 shows the beam AB of span l supported at A and B and carrying a uniformly distributed load of w per unit run on BD of length a. Taking moments about B, $V_a l=\frac{wa^2}{2}$

$$\therefore\quad V_a=\frac{wa^2}{2l}$$

$$\therefore\quad V_b=wa-\frac{wa^2}{2l}=\frac{wa}{2l}(2l-a)$$

The bending moment at any section distant x from the end A, is given by,

$$EI\frac{d^2y}{dx^2}=\frac{wa^2}{2l}x-\frac{w}{2}[x-(l-a)]^2$$

Integrating, we get,

$$EI\frac{dy}{dx}=\frac{wa^2}{4l}x^2+C_1-\frac{w}{6}[x-(l-a)]^3 \qquad \textit{(Slope equation)}$$

Integrating again, we get,

$$EI\ y = \frac{wa^2x^3}{12l} + C_1x + C_2 - \frac{w}{24}[x-(l-a)]^4$$

(*Deflection equation*)

At A, the deflection is zero

$\therefore$ At $x=0,\ y=0$ $\therefore$ $C_2=0$

At B, the deflection is zero

$\therefore$ At $x=l,\ y=0$

$$\therefore \quad 0 = \frac{wa^2l^2}{12} + C_1l - \frac{wa^4}{24}$$

$$\therefore \quad C_1 = -\frac{wa^2}{24l}[2l^2-a^2]$$

For the condition, that the maximum deflection should occur at D, we have

at $x=(l-a),\ \frac{dy}{dx}=0$

$$\therefore \quad 0 = \frac{wa^2}{4l}(l-a)^2 - \frac{wa^2}{24l}[2l^2-a^2] = 0$$

$$\therefore \quad \frac{wa^2}{24l}[6(l-a)^2-(2l^2-a^2)]=0$$

$\therefore$ $7a^2-12la+4l^2=0$ solving, we get $a=0.453l$

With the above value of a, $C_1 = -\frac{w(0.453l)^2}{24l}[2l^2-(0.453l)^2]$

$$\therefore \quad C_1 = -0.0153461\ wl^3$$

Substituting in the deflection equation,

$$EI\ y_{max} = \frac{w(0.453l)^2}{12l}(l-0.453l)^3 - 0.0153461wl^3(l-0.453l)$$

$$= -0.0055955\ wl^4$$

$$\therefore \quad y_{max} = -\frac{0.0055955}{EI}wl^4$$

Problem 258. *A beam of uniform section and length l is simply supported at its ends and carries a symmetrical triangular loading the intensity varying from zero at each end to w at the centre. Find the slope at each end and the deflection at the centre.*

Solution. Fig. 415 shows the beam AB of span l simply supported at A and B and carrying the symmetrical triangular load.

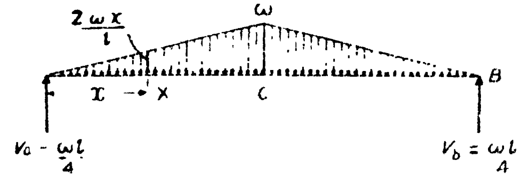


Fig. 415

Total load on the beam = Area of the load diagram

$$=\frac{1}{2}\cdot l\cdot w=\frac{wl}{2}$$

$\therefore\qquad V_a=V_b=\frac{1}{2}$ the total load $=\frac{wl}{4}$

Consider any section X in AC distant x from A.

Load intensity at $X=\frac{2wx}{l}$

The bending moment at the section is given by,

$$EI\frac{d^2y}{dx^2}=\frac{wl}{4}x-\frac{1}{2}\cdot x\cdot\frac{2wx}{l}\cdot\frac{x}{3}=\frac{wl}{4}x-\frac{wx^3}{3l}$$

Integrating, we get,

$$EI\frac{dy}{dy}=\frac{wl}{8}x^2-\frac{wx^4}{12l}+C_1$$

At $\qquad x=\frac{l}{2},\ \frac{dy}{dx}=0$ since the deflection is maximum at the centre.

$$\therefore\qquad 0=\frac{wl}{8}\cdot\frac{l^2}{4}-\frac{w}{12l}\cdot\frac{l^4}{16}+C_1$$

$$\therefore\qquad C_1=-\frac{5}{192}wl^3$$

$$\therefore\quad EI\frac{dy}{dx}=\frac{wl}{8}x^2-\frac{wx^4}{12l}-\frac{5}{192}wl^3\qquad \textit{(Slope equation)}$$

Slope at A. Put $x=0$ in the slope equation.

$$EI\ i_a=-\frac{5}{192}wl^3$$

$$\therefore\qquad i_a=-\frac{5}{192}\frac{wl^3}{EI}$$

Integrating the slope equation, we get,

$$EI\,y=\frac{wl}{24}x^3-\frac{wx^5}{60l}-\frac{5}{192}wl^3x+C_2 \quad (\textit{Deflection equation})$$

At $\quad x=0, \quad y=0 \qquad \therefore\ C_2=0$

Deflection at the centre. Put $x=\frac{l}{2}$ in the deflection equation

$$EI\,y_c=\frac{wl}{24}\cdot\frac{l^3}{8}-\frac{w}{60l}\,\frac{l^5}{32}-\frac{5}{192}wl^3\cdot\frac{l}{2}$$

$$\therefore\quad EI\,y_c=-\frac{wl^4}{120}$$

$$\therefore\quad y_c=-\frac{wl^4}{120EI}$$

or Downward deflection of $C=\frac{wl^4}{120EI}$.

Problem 259. *A beam of uniform section and length is simply supported at its ends and carries a distributed load which varies uniformly from zero at each end to a maximum intensity of w per unit run at a section* $\frac{l}{3}$ *from the right-hand end. Show that the maximum deflection occurs at a distance approximately 0·01 l from mid-span and find the maximum deflection in terms of w, l, E and I.*

(*London University*)

Solution. Fig. 416 (*a*) shows the beam carrying the loading mentioned in the problem.

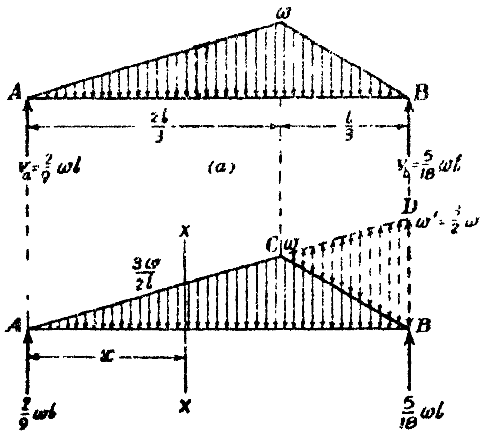
Total load on the span

=area of the load diagram

$$=\frac{wl}{2}$$

Distance of the centroid of the load diagram from the left end A

$$=\frac{l+\frac{2l}{3}}{3}=\frac{5}{9}\,l$$



(*b*)

Fig 416

Let V_a and V_b be the reactions at A and B respectively.

Taking moments about the end A, we have,

$$V_b l=\frac{wl}{2}\cdot\frac{5}{9}\,l$$

$$\therefore \quad V_b=\frac{5}{18}\,wl$$

$$\therefore \quad V_a=\frac{wl}{2}-\frac{5}{18}\,wl$$

$$\therefore \quad V_a=\frac{2}{9}\,wl$$

In Fig. 416 (*b*) the load diagram is slightly amended.

Let the line *AB* of the load diagram be produced and the point *D* be located.

The ordinate $BD=w'=\left(\dfrac{l}{\frac{2l}{3}}\right)w=\dfrac{3}{2}\,w.$

Hence we shall consider that the loading on the beam consists of (*i*) a downward triangular loading whose intensity varies from zero at the left end to $\frac{3w}{2}$ per unit run at the right end, and (*ii*) an upward triangular loading acting for a distance $\frac{l}{3}$ from the right end whose intensity varies from zero at $\frac{l}{3}$ from the right end to $\frac{3}{2}w$ at the right end.

Now following Macaulay's method, the bending moment at any section is given by

$$EI\frac{d^2y}{dx^2}=\frac{2}{9}wlx-\frac{1}{2}x\cdot\frac{3w}{2l}x\cdot\frac{x}{3}$$
$$+\frac{1}{2}\left(x-\frac{2l}{3}\right)\left(x-\frac{2l}{3}\right)\frac{\frac{3}{2}w}{\frac{l}{3}}\cdot\frac{\left(x-\frac{2l}{3}\right)}{3}$$

$$\therefore \quad EI\frac{d^2y}{dx^2}=\frac{2}{9}\,wlx-\frac{wx^3}{4l}+\frac{3w}{4l}\left(x-\frac{2l}{3}\right)^3$$

Integrating, we get,

$$EI\frac{dy}{dx}=\frac{wlx^2}{9}-\frac{wx^4}{16l}+C_1+3w\frac{\left(x-\frac{2l}{3}\right)^4}{16l}$$

Integrating again, we get,

$$EI\,y=\frac{wlx^3}{27}-\frac{wx^5}{80l}+C_1x+C_2+\frac{3w\left(x-\frac{2l}{3}\right)^5}{80l}$$

At $\quad x=0,$

$\qquad y=0$

$\therefore \quad C_2=0$

At $\quad x=l, \quad y=0$

$$\therefore \quad 0=\frac{wl^4}{27}-\frac{wl^4}{80}+C_1 l+\frac{3wl^5}{80\times 81\times 3l}$$

$$\therefore \quad 0=\frac{wl^4}{27}-\frac{wl^4}{80}+C_1 l+\frac{wl^4}{6480}$$

$$\therefore \quad C_1=-\frac{2}{81}\,wl^3$$

For maximum deflection which will occur between $x=0$ and $x=\frac{2l}{3}$, equating the expression for slope to zero, we have,

$$0=\frac{lw}{9}x^2-\frac{wx^4}{16l}-\frac{2}{81}\,wl^3$$

Putting $x=Kl$, we have,

$$\frac{wl}{9}K^2l^2-\frac{w}{16l}K^4l^4-\frac{2}{81}\,wl^3=0$$

$$\therefore \quad \frac{K^2}{9}-\frac{K^4}{16}-\frac{2}{81}=0$$

$$\therefore \quad K^4-\frac{16}{9}K^2+\frac{32}{81}=0$$

Solving as a quadratic in K^2, we get

$$K^2=\frac{\frac{16}{9}-\sqrt{\frac{256}{81}-\frac{128}{81}}}{2}$$

$$\therefore \quad K^2=\frac{4{\cdot}69}{81}$$

$$\therefore \quad K=0{\cdot}5103$$

$\therefore$ Maximum deflection occurs at a distance of $0{\cdot}5103l$ from the left end A.

Distance of the point of maximum deflection from the middle point

$$=0{\cdot}5103\,l-0{\cdot}5\,l$$
$$=0{\cdot}0103\,l$$
$$=0{\cdot}01\,l \text{ . (approximately)}$$

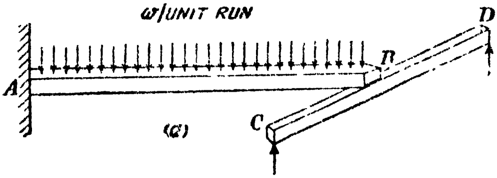
To find the maximum deflection, putting $x=0{\cdot}5103\,l$ in the expression for deflection, we have,

$$EI\,y_{max}=\frac{wl}{27}\left(0{\cdot}5103\,l\right)^3-\frac{w}{80l}(0{\cdot}5103l)^5-\frac{2}{81}\,wl^2\,(0{\cdot}5103l)$$

$$=-0{\cdot}008108\,\frac{wl^4}{EI}$$

Problem 260. *The free end of a cantilever of length l rests on the middle of a simply supported beam of the same span and having the same section. Find the reaction of the beam on the free end of the cantilever, when the cantilever carries a uniformly distributed load of w per unit run over its whole length*

Solution.



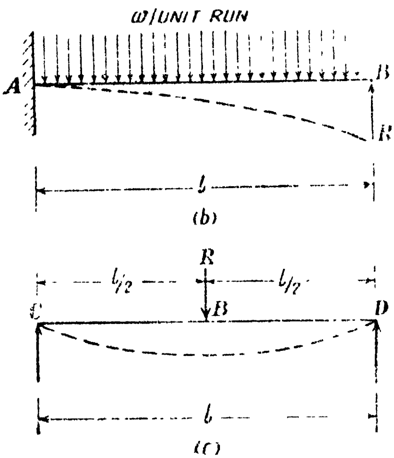


Fig. 417

Fig. 417 (*a*) shows the cantilever *AB* with its free end *B* resting on the middle of the simply supported beam *CD*. Let the common reaction between the beam and the cantilever be *R*. Now consider the cantilever *AB*. This is subjected to a uniformly distributed load of *w* per unit run on its length and an upward point load *R* at *B*

∴ Net downward deflection of the free end *B* of the cantilever

$$= \frac{wl^4}{8EI} - \frac{Rl^3}{3EI}$$

Now consider the simply supported beam *CD*. This beam is subjected to a downward point load *R* at its middle point *B*.

∴ Downward deflection of *B*

$$= \frac{Rl^3}{48\,EI}$$

Since the free end of the cantilever is always resting on the middle point of the beam, the deflection of the free end of the cantilever is equal to the deflection of the middle point of the beam.

$$\therefore \quad \frac{wl^4}{8EI} - \frac{Rl^3}{3EI} = \frac{Rl^3}{48EI}$$

$$\therefore \quad \frac{17}{48}R = \frac{wl}{8}$$

$$R = \frac{6}{17}\,wl$$

Problem 261. *A beam AB of span 5 metres is simply supported at A and B. A cantilever DC of length 3 metres which is fixed at D meets the beam AB at the midpoint C, thereby forming a rigid joint C.*

A vertical load of 20t is applied at the common joint C. Find out the reaction at the ends of the simply supported beam. Assume $I_{ab}=I_{cd}$. *(A.M.I.E May 1971)*

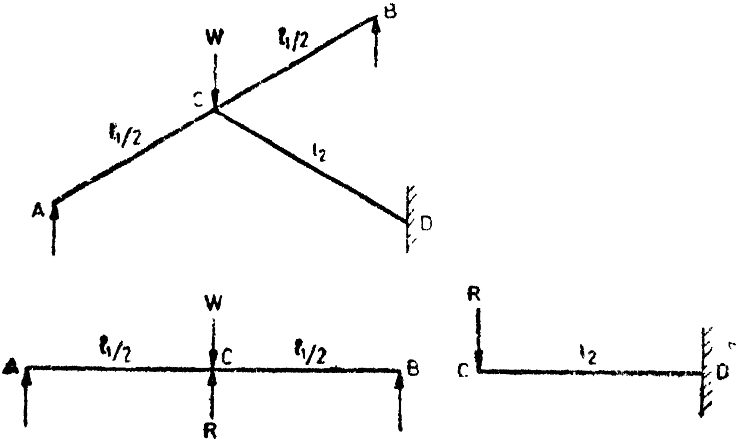


FIG. 418

Solution. Let l_1 be the length of the beam and l_2 the length of the cantilever. Let W be the load applied at C. Let R be the common reaction at C. Deflection at C is the same for the beam and the cantilever.

$$\frac{(W-R)l_1^3}{48EI}=\frac{Rl_2^3}{3EI}$$

$$\therefore \quad \frac{W-R}{R}=16\left(\frac{l_2}{l_1}\right)^3=16\times\left(\frac{3}{5}\right)^3=3.456$$

$$\therefore \quad W-R=4{\cdot}456\ R$$

$$\therefore \quad R=0{\cdot}2244W$$

$\therefore$ Reaction at each support for the beam,

$$=\frac{W-R}{2}=\frac{W-0{\cdot}2244W}{2}$$

$$=0{\cdot}3878\ W$$

$$=0{\cdot}3878\times 20$$

$$=7{\cdot}756\ t.$$

Problem 262. *A cantilever of length l carries a concentrated load W at its midspan. If the free end be supported on a rigid prop, find the maximum deflection.*

Solution.

Let R be the reaction at the propped end. Since the deflection at the propped end is zero,

We have,

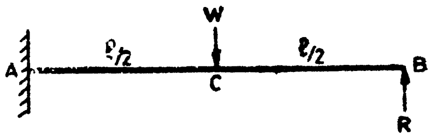


Fig. 4.19

$$\frac{Rl^3}{3EI}=\frac{W\left(\frac{l}{2}\right)^3}{3EI}+\frac{W\left(\frac{l}{2}\right)^2}{2EI}\cdot\frac{l}{2}$$

$$\therefore \qquad R=\frac{5}{16}\,W$$

The B.M. at any section distant x from the end B is given by,

$$EI\frac{d^2y}{dx^2}=Rx-W\left(x-\frac{l}{2}\right)$$

$$=\frac{5}{16}Wx-W\left(x-\frac{l}{2}\right)$$

Integrating, we get,

$$EI\frac{dy}{dx}=\frac{5}{32}\,Wx^2+C_1-\frac{W}{2}\left(x-\frac{l}{2}\right)^2$$

At $\qquad x=l,\frac{dy}{dx}=0$

$$\therefore \qquad 0=\frac{5}{32}\,Wl^2+C_1-\frac{Wl^2}{8}$$

$$\therefore \qquad C_1=-\frac{Wl^2}{32}$$

Integrating again, we get

$$EIy=\frac{5}{96}Wx^3-\frac{Wl^2}{32}x+C_2-\frac{W\left(x-\frac{l}{2}\right)^3}{6}$$

At $\qquad x=0,$

$y=0$

$\therefore \qquad C_2=0$

Maximum Deflection. Assuming the maximum deflection to occur in BC, equating the general expression for slope to zero,

$$\frac{5}{32}\,Wx^2-\frac{Wl^2}{32}=0$$

$$\therefore \quad x = \frac{l}{\sqrt{5}} \text{ from } B$$

Since the value of x is less than $\frac{l}{2}$ our assumption about the position on maximum deflection is correct.

Substituting $x = \frac{l}{\sqrt{5}}$ in the general expression for deflection, we get,

$$EI\, y_{max} = \frac{5}{96} W \left(\frac{l}{\sqrt{5}}\right)^3 - \frac{Wl^2}{32}\left(\frac{l}{\sqrt{5}}\right)$$

$$\therefore \quad y_{max} = -\frac{Wl^3}{48\sqrt{5}\, EI}$$

Problem 263. *Fig. 420 shows a simply supported beam of uniform section whose moment of inertia is 43000 cm.⁴ For the loading shown, find the position and magnitude of the maximum deflection. Take $E = 2 \times 10^3$ tonnes per cm.²*

Solution.

Taking moments about the end A, we have

$$V_b \times 8 = 4 \times 4 \times 3$$

$$\therefore \quad V_b = 6 \text{ tonnes}$$

$$\therefore \quad V_a = 4 \times 4 - 6$$

$$= 10 \text{ tonnes}$$

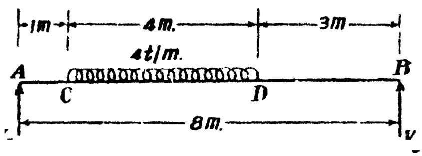


Fig. 420

In order the general expression for the bending moment at any section may be expressed in the form suitable for application of Macaulay's method the loading on the beam is arranged as shown in Fig. 421.

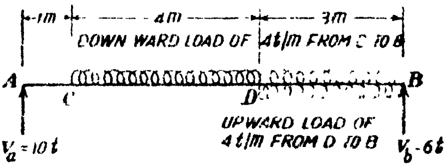


Fig. 421

Now, following Macaulay's method, the B.M. at any section distant x from A is given by,

$$EI\frac{d^2y}{dx^2} = 10x - \frac{4(x-1)^2}{2} \;\Big|\; + \frac{4(x-5)^2}{2}$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = 10x \;\Big|\; - 2(x-1)^2 \;\Big|\; + 2(x-5)^2$$

Integrating, we get,

$$EI\frac{dy}{dx} = 5x^2 + C_1 \;\Big|\; -\frac{2}{3}(x-1)^3 \;\Big|\; + \frac{2}{3}(x-5)^3$$

Integrating again, we get,

$$EI\, y = \frac{5}{3}x^3 + C_1 x + C_2 \;\Big|\; -\frac{1}{6}(x-1)^4 \;\Big|\; + \frac{1}{6}(x-5)^4$$

At A, the deflection is zero

At $x=0$, $y=0$

$\therefore$ $C_2=0$

At B also the deflection is zero,

$\therefore$ At $x=8$, $y=0$

$$\therefore \quad 0=\frac{5\times 8^3}{3}+8C_1-\frac{1}{6}\times 7^4+\frac{1}{6}\times 3^4$$

$\therefore$ $C_1=-58{\cdot}33$

Position of maximum deflection

Let us assume that the deflection will be maximum between C and D.

Equating the slope to zero, we have,

$$5x^2-58{\cdot}33-\frac{2}{3}(x-8)^3=0$$

$$\therefore \quad \frac{2}{3}(x-8)^3=5x^2-58{\cdot}33$$

Solving the above equation by trial and error we get $x=3{\cdot}82$ *metres*.

The value of x obtained confirms that our assumption about the position of maximum deflection is correct.

Substituting $x=3{\cdot}82$ *metres* in the deflection equation, we get

$$EI\,y_{max}=\frac{5}{3}(3{\cdot}82)^3-58{\cdot}33\times 3{\cdot}82-\frac{1}{6}\times 2{\cdot}82^4$$

$$=-139{\cdot}46.$$

$$\therefore \quad y_{max}=-\frac{139{\cdot}46\,(100)^3}{2\times 10^3\times 43000}\ cm.$$

$$=-1{\cdot}62\ cm.$$

Problem 264. *A beam ABC of length 3l has one support at the left end and the other support at a distance 2l from the left end. The beam carries a point load W at the right end. Find the slopes over each support and at the right end. Find also the maximum upward deflection between the supports and the deflection at the right end.*

Solution.

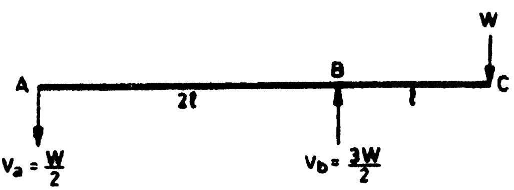


Fig. 422

Fig. 422 shows the beam ABC.

Let the reactions at A and B be V_a and V_b respectively.

Taking moments about B,

$$V_a\,(2l) = Wl \quad \therefore \quad V_a = \frac{W}{2}\downarrow$$

$$\therefore \quad V_b = W + \frac{W}{2} = \frac{3W}{2}\uparrow$$

At any section distant x from A the bending moment is given by

$$EI\,\frac{d^2y}{dx^2} = -\frac{W}{2}x \;\Big|\; +\frac{3W}{2}(x-2l)$$

Integrating, we get,

$$EI\,\frac{dy}{dx} = -\frac{Wx^2}{4} + C_1 \;\Big|\; +\frac{3W}{4}(x-2l)^2$$

Integrating again, we get,

$$EI\,y = -\frac{Wx^3}{12} + C_1x + C_2 \;\Big|\; +\frac{W(x-2l)^3}{4}$$

At $x=0,\ y=0 \quad \therefore\ C_2=0$

At $x=2l,\ y=0$

$$\therefore\ 0 = -\frac{W}{12}(8l^3) + C_1\,(2l) \qquad \therefore\ C_1 = \frac{Wl^2}{3}$$

Slope at A. Putting $x=0$ in the slope equation,

$$EI\,i_a = \frac{Wl^2}{3} \qquad \therefore\ i_a = \frac{Wl^2}{3EI}$$

Slope at B. Putting $x=2l$ in the slope equation,

$$EI\,i_b = -\frac{W(2l)^2}{4} + \frac{Wl^2}{3} = -\frac{2}{3}Wl^2$$

$$\therefore \qquad i_b = -\frac{2}{3}\,\frac{Wl^2}{EI}$$

Slope at C. Putting $x=3l$ in the slope equation,

$$EI\,i_c = -\frac{W(3l)^2}{4} + \frac{Wl^2}{3} + \frac{3W}{4}(3l-2l)^2 = -\frac{7}{6}Wl^2$$

$$\therefore \qquad i_c = -\frac{7}{6}\,\frac{Wl^2}{EI}$$

Maximum upward deflection between A and B

Equating the slope to zero, we get,

$$-\frac{Wx^2}{4} + \frac{Wl^2}{3} = 0 \quad \therefore\ x^2 = \frac{4}{3}l^2$$

$$\therefore \qquad x = \frac{2l}{\sqrt{3}}$$

Putting $x = \dfrac{2l}{\sqrt{3}}$ in the deflection equation,

$$EI\,y_{max}=-\frac{W}{12}\left(\frac{2l}{\sqrt{3}}\right)^3+\frac{Wl^2}{3}\left(\frac{2l}{\sqrt{3}}\right)$$

$$=\frac{4}{9\sqrt{3}}\,Wl^3$$

$$y_{max}=\frac{4}{9\sqrt{3}}\,\frac{Wl^3}{EI}$$

Deflection at C. Putting $x=3l$ in the deflection equation,

$$EI\,y_c=-\frac{W}{12}(3l)^3+\frac{Wl^2}{3}(3l)+\frac{W}{4}(3l-2l)^3$$

$$=-\frac{9}{4}Wl^3+Wl^3+\frac{Wl^3}{4}$$

$$=-Wl^3$$

$$\therefore \quad y_c=-\frac{Wl^3}{EI}$$

Problem 265. *A beam of length L has supports l apart with equal overhangs. The beam carries a point load W at midspan. Find the ratio $\frac{L}{l}$ in order the downward deflection at the centre is equal to the upward deflection at either end.*

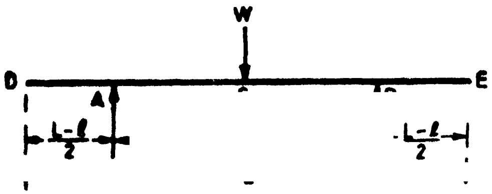


Fig. 423

Solution. Fig. 423 shows the beam. Downward deflection at the centre $=\frac{Wl^3}{48\,EI}$

Slope at A or $B=\frac{Wl^2}{16\,EI}$

$\therefore$ upward deflection at $D=$Slope at $A\times AD$

$$=\frac{Wl^2}{16\,EI}\,\frac{L-l}{2}$$

Equating the deflection at the centre to the deflection at each end,

$$\frac{Wl^3}{48\,EI}=\frac{Wl^2}{16\,EI}\cdot\frac{L-l}{2}$$

$$2l^3=3l^2(L-l)$$

$$2l^3=3l^2L-3l^3$$

$$5l^3=3l^2L$$

$$\therefore \quad \frac{L}{l}=\frac{5}{3}$$

§67. Beam subjected to couples

Fig. 424 shows a beam AB of span l supported at A and B and subjected to couples M_1 and M_2. Let M_1 be an anticlockwise couple and M_2 be a clockwise couple.

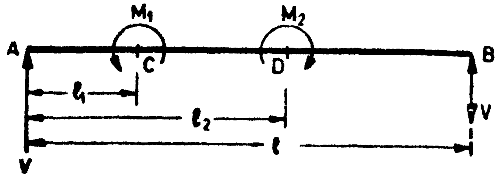


Fig. 424

Let $M_1 > M_2$

Reaction at each end $= V = \dfrac{M_1 - M_2}{l}$

The B.M. at any section distant x from A is given by

$$EI\frac{d^2y}{dx^2} = Vx \; - M_1 \; + M_2 \text{ following Macaulay's rule.}$$

But it must be noted that

When $x < l_1$ only the first expression should be considered.

When $x > l_1$ and $< l_2$ the first two expressions should be considered

When $x > l_2$ and $< l$ all the three expressions should be considered.

The above expression should be rearranged as follows.

$$EI\frac{d^2y}{dx^2} = Vx \; \Big| - M_1(x-l_1)^\circ \; \Big| + M_2(x-l_2)^\circ$$

Integrating, we get,

$$EI\frac{dy}{dx} = \frac{Vx^2}{2} + C_1 \; \Big| - M_1(x-l_1) \; \Big| + M_2(x-l_2) \quad \textit{(Slope equation)}$$

Integrating again, we get,

$$EI\; y = \frac{Vx^3}{6} + C_1x + C_2 - \frac{M_1(x-l_1)^2}{2} + \frac{M_2(x-l_2)^2}{2}$$

(Deflection equation)

At $x=0$, $y=0$ $\quad\therefore\quad C_2 = 0$

At $x=l$ $y=0$. From this condition, we can evaluate C_1. Now, we can determine the slope and deflection at any point.

Problem 266. *A beam of length l with supports at the ends is subjected to a couple M at a distance a from the left end. Find the slope at each end and the deflection at the point of application of the couple.*

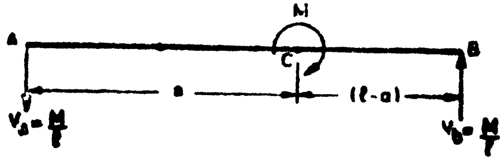


Fig. 425

Solution. Fig. 425 shows the beam

Taking moments about A,

$V_b \times l = M$

$\therefore \quad V_b = \frac{M}{l} \uparrow$

$\therefore \quad V_a = \frac{M}{l} \downarrow$

At any section distant x from A. the B.M. is given by,

$$EI \frac{d^2y}{dx^2} = -\frac{M}{l} x \, \Big| + M$$

$$= -\frac{M}{l} x \, \Big| + M(x-a)^\circ$$

(*Note the rearranged form of the expression*)

Integrating,

$$EI \frac{dy}{dx} = -\frac{Mx^2}{2l} + C_1 \; + M(x-a)$$

Integrating again,

$$EI \, y = -\frac{Mx^3}{6l} + C_1 x + C_2 \; + \frac{M(x-a)^2}{2}$$

At $\quad x = 0, \; y = 0$

$\therefore \quad C_2 = 0$

At $\quad x = l, \; y = 0$

$\therefore \quad 0 = -\frac{Ml^2}{6} + C_1 l + \frac{M(l-a)^2}{2}$

$\therefore \quad C_1 = -\frac{M}{6l}(2l^2 - 6la + 3a^2)$

Slope at A. Put $x=0$ in the slope equation

$$EI \, i_a = -\frac{M}{6l}(2l^2 - 6la + 3a^2)$$

$\therefore \quad i_a = -\frac{M}{6EIl}(2l^2 - 6la + 3a^2)$

Slope at B. Put $x = l$ in the slope equation

$$EI \, i_b = -\frac{Ml}{2} - \frac{M}{6l}(2l^2 - 6la + 3a^2)$$

$$= -\frac{M}{6l}[3l^2 + 2l^2 - 6la + 3a^2]$$

$$= -\frac{M(5l^2 - 6la + 3a^2)}{6l}$$

$\therefore \quad i_b = -\frac{M(5l^2 - 6la + 3a^2)}{6EIl}$

Deflection at C. Put $x=a$ in the deflection equation.

$$\therefore \quad EIy_c=-\frac{Ma^3}{6l}-\frac{Ma}{6l}(2l^2-6la+3a^2)$$

$$=-\frac{Ma}{6l}\left[a^2+2l^2-6la+3a^2\right]$$

$$=-\frac{Ma}{3l}(l-a)(l-2a)$$

$$\therefore \quad y_c=-\frac{Ma(l-a)(l-2a)}{3\,EIl}$$

Problem 267. *A beam of span 6 m and of uniform flexural rigidity $EI=4000$ tm^2 is subjected to a clockwise couple of 30 tm at a distance of 4 m from the left end. Find the deflection at the point of application of the couple. Find also the maximum deflection.*

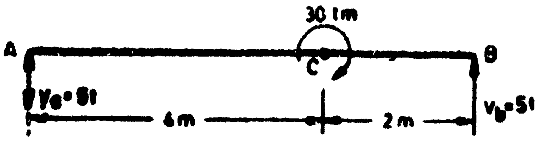


Fig. 426

Solution. Fig. 426 shows the beam AB with the couple applied at C.

Taking moments about A,

$$V_b\times 6=30$$

$$V_b=5t\uparrow$$

$$\therefore \quad V_a=5t\downarrow$$

The B.M. at any section distant x from A is given by,

$$EI\frac{d^2y}{dx^2}=-5x \;\vdots\; +30$$

$$=-5x \;\vdots\; +30(x-4)^0$$

Integrating,

$$EI\frac{dy}{dx}=-\frac{5x^2}{2}+C_1 \;\Big|\; +30(x-4)$$

Integrating again,

$$EI\,y=-\frac{5x^3}{6}+C_1x+C_2 \;\vdots\; +15(x-4)^2$$

At $x=0,\ y=0$

$\therefore \quad C_2=0$

At $x=6,\ y=0$

$$\therefore \quad 0=-5\times 6^2+6C_1+15(2)^2$$

$$\therefore \quad C_1=20$$

Deflection at C. Putting $x=4\ m$, in the deflection equation,

$$EIy_c=-\frac{5}{6}(4)^3+20\times4=26{\cdot}67$$

$$y_c=\frac{26{\cdot}67}{EI}=\frac{26{\cdot}67}{4000}\times100=0{\cdot}667\ cm.$$

Max. Deflection. This will occur in the large segment.

Equating the slope to zero, we get,

$$EI\frac{dy}{dx}=0=-\frac{5x^2}{2}+20$$

$\therefore$ $x=2\sqrt{2}\ m$

Putting $x=2\sqrt{2}\ m$ in the deflection equation

$$EIy_{max}=-\frac{5}{6}\left(2\sqrt{2}\right)^3+20\times2\sqrt{2}=\frac{80\sqrt{2}}{3}$$

$\therefore$ $$y_{max}=\frac{80\sqrt{2}\times100}{3\times4000}=0{\cdot}943\ cm.$$

Problem 268. *A beam 6 m long is subjected to two couples as follows :*

(*i*) *A clockwise couple of 20 tm at a distance of 2 m from the left end.*

(*ii*) *An anticlockwise couple of 8 tm at a distance of 4 m from the left end.*

Find the deflection at the points of application of the couples. For the beam take $EI=4150\ tm^2$.

Solution.

Reaction at each support

$$=V=\frac{20-8}{6}=2t$$

The B.M. at any section distant x from the left end A is given by

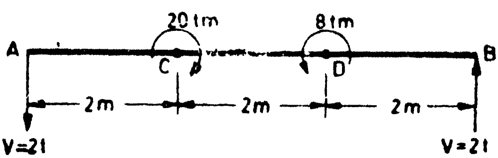


Fig. 427

$$EI\frac{d^2y}{dx^2}=-2x\ \Big|\ +20\ \vdots\ -8$$

The above equation should be rearranged as follows :

$$EI\frac{d^2y}{dx^2}=-2x\ \vdots\ +20(x-2)^0\ \vdots\ -8(x-4)^0$$

Integrating, we get,

$$EI\frac{dy}{dx}=-x^2+C_1 \;\vdots +20(x-2) \;\vdots -8(x-4)$$

Integrating again,

$$EIy=-\frac{x^3}{3}+C_1x+C_2 \;\vdots +10(x-2)^2 \;\vdots -4(x-4)^2$$

At $x=0,\ y=0$

$\therefore$ $C_2=0$

At $x=6,\ y=0$

$\therefore$ $0=-\frac{6^3}{3}+6C_1+160-16$

$\therefore$ $C_1=-12$

Deflection at C. Putting $x=2\ m$ in the deflection equation,

$$EIy_c=-\frac{2^3}{3}-12\times2=-26{\cdot}67$$

$$y_c=-\frac{26{\cdot}67}{4150}\times100\ cm.=-0{\cdot}643\ cm.$$

Deflection at D. Putting $x=4\ m$ in the deflection equation,

$$EIy_d=-\frac{4^3}{3}-12\times4+10(4-2)^2=-29{\cdot}33$$

$\therefore$ $$y_d=-\frac{29{\cdot}33}{4150}\times100\ cm.=-0{\cdot}707\ cm.$$

Problem 269. *Find the deflection at C for the beam loaded as shown in Fig. 428 (a). Take $EI=4000\ tm^2$.*

Solution.

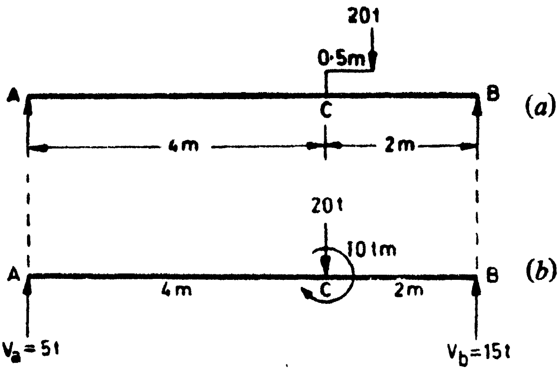


Fig. 428

The load 20*t* is acting on the bracket. This load will be transmitted to the beam at *C* along with a couple $20\times0{\cdot}5=10\ tm$ ↻ as shown in Fig. 428 (*b*).

Taking moments about A

$$V_b \times 6 = (20 \times 4) + 10$$

$\therefore \quad V_b = 15t$

$\therefore \quad V_a = 20 - 15 = 5t$

The B.M. at any section distant x from A is given by

$$EI\frac{d^2y}{dx^2} = 5x \quad -20(x-4) + 10$$

The above equation is rearranged as follows :

$$EI\frac{d^2y}{dx^2} = 5x \; -20(x-4) + 10(x-4)^0$$

Integrating, we get,

$$EI\frac{dy}{dx} = \frac{5x^2}{2} + C_1 \; -10(x-4)^2 + 10(x-4)$$

Integrating again,

$$EIy = \frac{5x^3}{6} + C_1x + C_2 \; - \frac{10}{3}(x-4)^3 + 5(x-4)^2$$

At $\quad x=0, y=0$

$\therefore \quad C_2 = 0$

At $\quad x=6, y=0$

$\therefore \quad 0 = 5 \times 6^2 + 6C_1 - \frac{10}{3}(2)^3 + 5(2)^2$

$\therefore \quad C_1 = -28{\cdot}89$

To find the deflection at C putting $x=4\ m$ in the deflection equation, we get,

$$EIy_c = \frac{5}{6}(4)^3 - 28{\cdot}89 \times 4$$

$$= -62{\cdot}23$$

$\therefore \quad y_c = -\frac{62{\cdot}33}{4000} \times 100\ cm.$

$$= -1{\cdot}56\ cm.$$

Problem 270. *Find the slope and deflection at B, C, D for the cantilever shown in Fig. 429. Take for the cantilever $EI = 5000\ tm.^2$*

Solution.

Total couple applied on the cantilever

$$= 20 + 5 - 15$$

$$= 10\ tm. \curvearrowright$$

Reacting moment at A 10 $tm.$ $\curvearrowleft$

$\therefore$ The B.M. at any section distant x from A is given by

$$EI\frac{d^2y}{dx^2} = -10 + 20 - 15$$

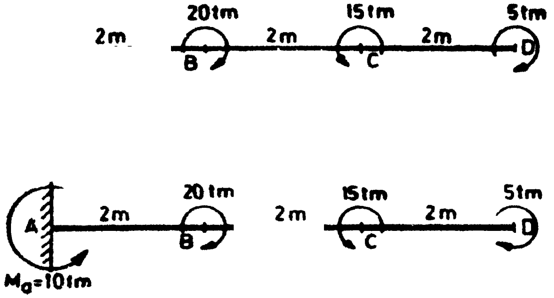


Fig. 429

The above equation is rearranged as follows :

$$EI\frac{d^2y}{dx^2}=-10 \quad +20(x-2)^\circ \quad -15(x-4)^\circ$$

Integrating, we get,

$$EI\frac{dy}{dx}=-10x+C_1 \quad +20(x-2) \quad -15(x-4)$$

At $\quad x=0, \quad \frac{dy}{dx}=0$

$\therefore \quad C_1=0$

Integrating again,

$$EIy=-5x^2+C_2+10(x-2)^2 \quad -\frac{15}{2}(x-4)^2$$

At $\quad x=0, y=0$

$\therefore \quad C_2=0$

Slope at B. Put $x=2$ *m* in the slope equation.

$$EI\, i_b=-10\times 2=-20$$

$$\therefore \quad i_b=-\frac{20}{5000}=-0{\cdot}004 \textit{ radian}$$

Slope at C. Put $x=4$ *m* in the slope equation

$$EI\, i_c=-10\times 4+20\times 2=0$$

$$\therefore \quad i_c=0$$

Slope at D. Put $x=2$ *m* in the slope equation.

$$EI\, i_d=-10\times 6+20\times 4-15\times 2=-10$$

$$\therefore \quad i_d=-\frac{10}{EI}=-\frac{10}{5000}=-0{\cdot}002 \textit{ radian}$$

Deflection at B. Put $x=2$ *m* in the deflection equation.

$$EI\, y_b=-5\times 4=-20$$

$$\therefore \quad y_b=\frac{-20}{5000}\times 100=-0{\cdot}4 \textit{ cms.}$$

Deflection at C. Put $x=4$ *m* in the deflection equation

$$EI\, y_c=-5\times 16+10\times 4=-40$$

$$\therefore \quad y_c=-\frac{40}{EI}=-\frac{40\times 100}{5000}=-0{\cdot}8 \textit{ cms.}$$

Deflection at D. Put $x=6$ *m* in the deflection equation.

$$EI\,y_d = -5\times 36+10\times 16-\frac{15}{2}\times 4 = -50$$

$$y_d = -\frac{50}{5000}\times 100 = -1 \text{ cm.}$$

§68. Moment area method—Mohr's theorems

Let AB represent part of the deflected form of a beam of uniform section.

Let A be a point of zero slope and zero deflection.

Let P and Q be two points on the deflection curve whose horizontal distances from B are x and $x+dx$ respectively.

Let the angle between the tangents at P and Q be $d\theta$. Obviously the angle between the normals at P and Q will also be equal to $d\theta$.

Let R be the radius of curvature of the elemental part PQ.

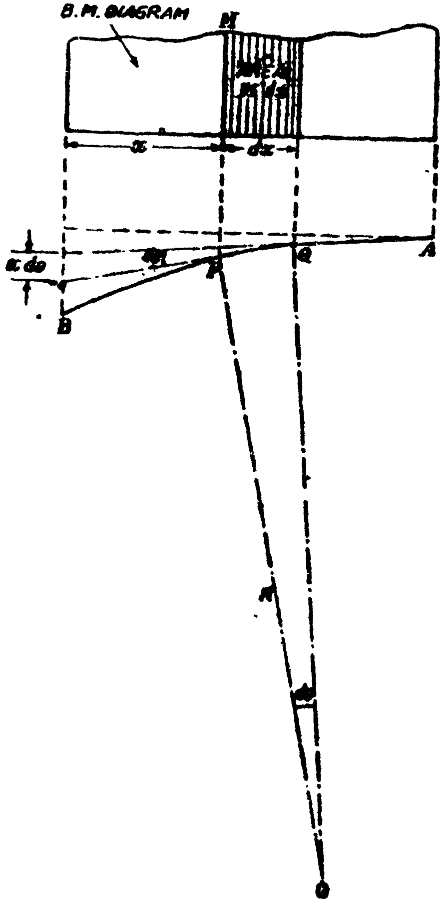


Fig. 430

$$d\theta = \frac{PQ}{R}$$

But $\quad PQ \approx dx$

$$d\theta = \frac{dx}{R}$$

But $\quad \frac{1}{R} = \frac{M}{EI}$

where M is the bending moment at any section between P and Q.

$$d\theta = \frac{M}{EI}dx \quad \ldots(i)$$

Since A is point of zero slope, the total slope at B is given by

$$\frac{1}{EI}\sum_{x=0}^{x=BA} M\,dx$$

$$\frac{1}{EI} \text{ (area of the B.M. diagram between } A \text{ and } B)$$

In case, the slope at A is not zero, we have

Total change in slope between B and A equals the area of B.M. diagram between B and A divided by the flexural rigidity EI

Deflection, due to the bending of the portion PQ

$$dy = xd\theta$$

Substituting from Eq. (i)

$$dy = \frac{M.xdx}{EI} \qquad ...(ii)$$

$\therefore$ Total deflection at B due to bending of all elemental portions like PQ

$$= y = \frac{1}{EI} \sum_{x=0}^{x=BA} \frac{Mx.dx}{EI}$$

$= \frac{1}{EI}$ [The first moment of the area of the B.M. diagram between B and A about B]

In case the point A is not a point of zero slope and deflection, *the deflection of B with respect to the tangent at A equals the first moment about B of the area of the B.M. diagram between B and A.*

The above two results are known as *Mohr's theorems.*

Though in general, problems on deflections can be solved by the above principle, it is convenient to use these principles with great advantage in the following types of problems :

(a) Cantilevers (slope at the fixed end is zero)

(b) Simply supported beams carrying symmetrical loading (slope at mid span is zero)

(c) Beams fixed at both ends (slope at each end is zero)

While dealing with members carrying uniformly distributed loading the bending moment diagrams being parabolic the following properties about areas and centroids may be of great advantage.

Fig. 431 shows part of a parabola tangential to the base AB.

Let $AB = b$

and $BC = d$

Area $DBC = A_1$

$$= \frac{2}{3} bd$$

$$\bar{x}_1 = \frac{5}{8} b$$

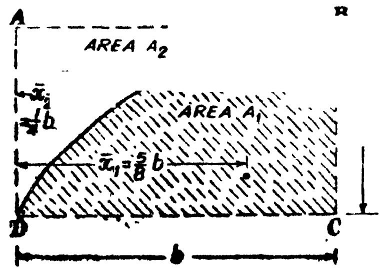


Fig. 431

Area ABD $\quad A_2 = \frac{1}{3} bd$

$$\bar{x}_2 = \frac{1}{4} b$$

We shall now apply Mohr's theorems to some problems for which the theorems may have their best application.

(*i*) *Cantilever carrying point load at the free end*

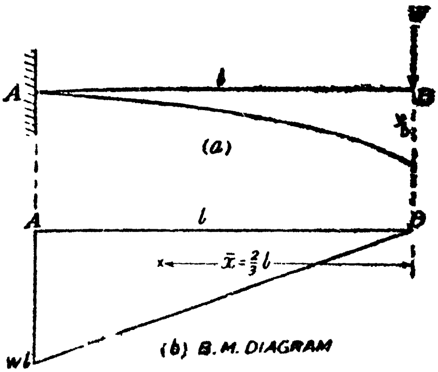


Fig. 432

Fig. 432 (*a*) shows a cantilever AB fixed at A and free at B.

Let l be the length of the cantilever. Let a point load W be applied at B.

Fig. 432 (*b*) shows the B.M. diagram for the cantilever.

Let the deflection of B with respect to A (point of zero slope) be y_b.

Let the slope at B be θ_b.

$$\theta_b = \frac{\text{Area of B.M. diagram between } A \text{ and } B}{EI}$$

Area of B.M. diagram

$$= \frac{1}{2} l . \ Wl = \frac{Wl^2}{2}$$

$$\therefore \qquad \theta_b = \frac{Wl^2}{2EI}$$

$$y_b = \frac{A\bar{x}}{EI}$$

$$\bar{x} = \frac{2}{3} l$$

$$\therefore \qquad y_b = \frac{Wl^2}{2EI} \cdot \frac{2}{3} l$$

$$= \frac{Wl^3}{3EI}$$

(*ii*) *Cantilever carrying a uniformly distributed load.*

Fig. 433 shows a cantilever of length l carrying a uniformly distributed load of w per unit run.

Area of the B.M. diagram

$$= A = \frac{1}{3} l . \ \frac{wl^2}{2}$$

$$= \frac{wl^3}{6}$$

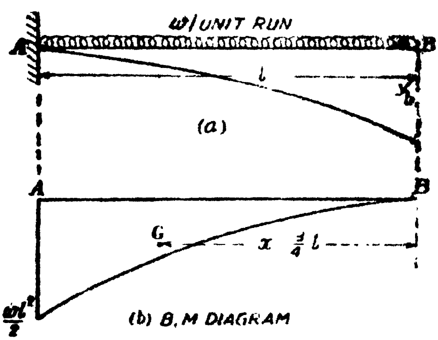


Fig. 4..

Since A is the point of zero slope,
The slope at B

$$=\frac{A}{EI}$$

$$=\frac{wl^3}{6EI}$$

Since A is the point of zero deflection,
The deflection at B

$$=y_b=\frac{A\bar{x}}{EI}$$

But $\bar{x}=\frac{3}{4}l$

$\therefore$ $y_b=\frac{wl^3}{6EI}\cdot\frac{3}{4}l$

$$=\frac{wl^4}{8EI}$$

(iii) Simply supported beam carrying a point load at mid span.

Fig. 434(*a*) shows a simply supported beam AB of span l carrying a point load W at mid span C.

Fig. 434 (*b*) shows the B.M. diagram for the beam.

Since the loading is symmetrical on the span the maximum deflection occurs at the mid span C.

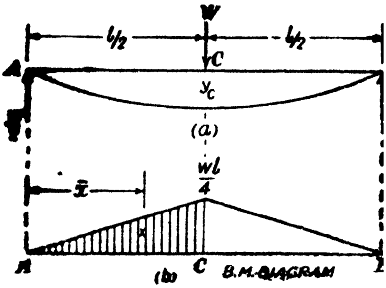


Fig. 434.

Slope at $A=\theta_a=\frac{\text{Area of B.M. diagram between } A \text{ and } C}{EI}$

Area of B.M. diagram between A and C

$$=A=\frac{1}{2}\cdot\frac{l}{2}\cdot\frac{Wl}{4}$$

$$=\frac{Wl^2}{16}$$

$\therefore$ $\theta_a=\frac{Wl^2}{16EI}$

Distance of the centroid of the B.M. diagram between A and C from A

$$=\bar{x}=\frac{2}{3}\cdot\frac{l}{2}=\frac{l}{3}$$

$\therefore$ Deflection of A with respect to C

=deflection of C with respect to A

$$\therefore y=\frac{A\bar{x}}{EI}$$

$$=\frac{\frac{Wl^2}{16}\cdot\frac{l}{3}}{EI}$$

$$=\frac{Wl^3}{48EI}$$

(*iv*) *Simply supported beam carrying a uniformly distributed load.*

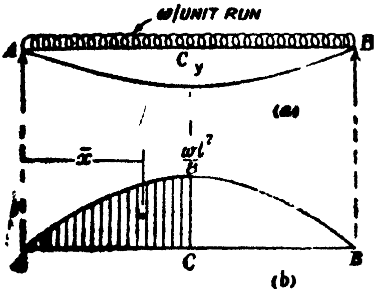


Fig. 435.

Fig. 435 (*a*) shows a simply supported beam *AB* of span *l* carrying a uniformly distributed load of *w* per unit run over the whole span.

Fig. 435 (*b*) shows the B.M. diagram for the beam.

Since the loading on the beam is symmetrical the maximum deflection occurs at *C*.
∴ At *C* the slope is zero.

Area of B.M. diagram between *A* and *C*

$$=A=\frac{2}{3}\cdot\frac{l}{2}\cdot\frac{wl^2}{8}$$

$$=\frac{wl^3}{24}$$

∴ Slope at *A*

$$=\frac{A}{EI}$$

$$=\frac{wl^3}{24EI}$$

$$\bar{x}=\frac{5}{8}\cdot\frac{l}{2}$$

$$=\frac{5}{16}l$$

∴ Deflection of *A* with respect to *C*
= Deflection of *C* with respect to *A*

$$=y=\frac{A\bar{x}}{EI}$$

$$=\frac{wl^3}{24EI}\cdot\frac{5}{16}l$$

$$=\frac{5}{384}\frac{wl^4}{EI}$$

Problem 271. *A horizontal cantilever ABC 5 metres long is built in at A and supported at B, 4 metres from A, by a rigid prop so that AB is horizontal. If AB and BC carry uniformly distributed loads of 3 t/m. and 1·5 t/m. respectively, find the load taken by the prop.*

Solution. Let the prop reaction be R.

The loading on the cantilever can be split up into the following loads :

(*i*) Upward force R at B

(*ii*) Distributed load of 1·5 *t/m* on *BC*

(*iii*) Distributed load of 3 *t/m* on *AB*

The B.M. diagrams for the separate effects of the above loading are shown in Fig. 436.

We have

$$A_1 = +\frac{1}{2} \times 4 \times 4R$$

$$= +8R$$

The trapezium between *A* and *B* can be split into two triangles

$$A_2 = -\frac{1}{2} \times 4 \times 0{\cdot}75$$

$$= -1{\cdot}50$$

$$A_3 = -\frac{1}{2} \times 6{\cdot}75 \times 4$$

$$= -13{\cdot}50$$

$$A_4 = -\frac{1}{3} \times 4 \times 24$$

$$= -32$$

We know that the slope is zero at *A* and the deflection is zero at *B*

$$\Sigma \frac{A\bar{x}}{EI} = 0 \text{ for the}$$

portion *AB*

$$\therefore \quad A_1\bar{x}_1 = A_2\bar{x}_2 + A_3\bar{x}_3 + A_4\bar{x}_4$$

(*numerically*)

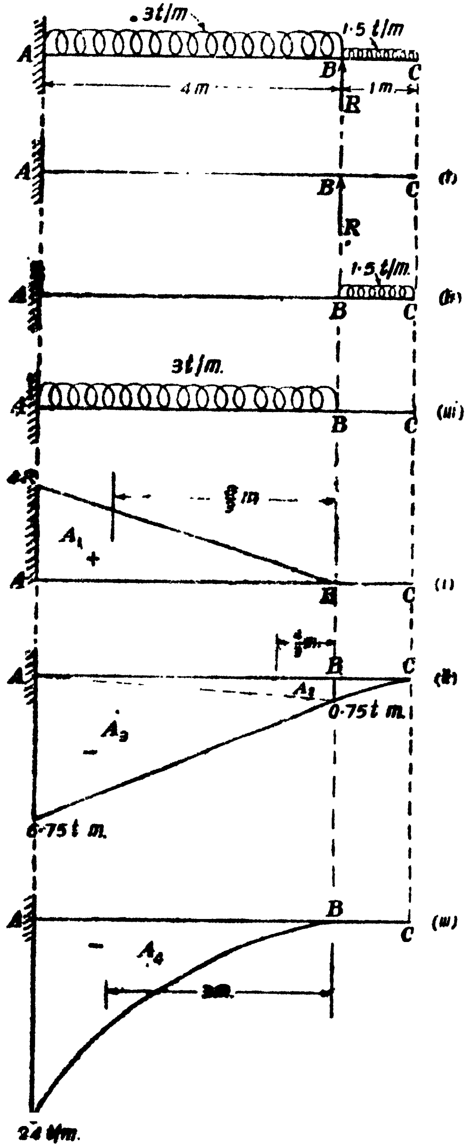


Fig. 436

$$8R \times \frac{8}{3} = 1{\cdot}5 \times \frac{4}{3} + 13{\cdot}5 \times \frac{8}{3} + 32 \times 3$$

$$\therefore \qquad R=\frac{201}{32} \textit{ tonnes}$$
$$=6{\cdot}28 \textit{ tonnes}$$

Problem 272. *A horizontal beam rests on two supports at the same level and carries a uniformly distributed load. If the supports are symmetrically placed, find their positions when the greatest downward deflection has its least value.* (London University)

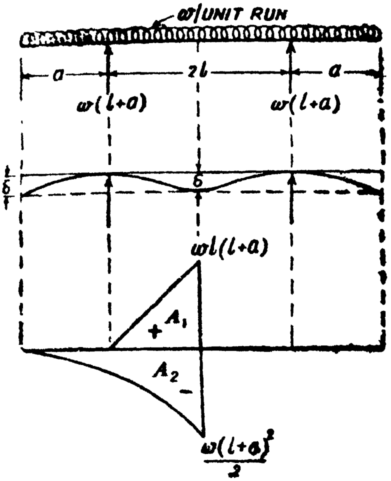


Fig. 437

Solution. Let $2l$ be the distance between the two supports and let a be the overhanging distance on each side.

Let the distributed load on the beam be w per unit run.

Each vertical reaction
$$=w(l+a)$$

In order the greatest downward deflection may have its least value the deflections at the centre and at the end must be equal.

Since the slope is zero at the centre we have $\Sigma A\bar{x}=0$ for one half the beam, about one end

Splitting the B.M. diagram into components A_1 due to support reaction and A_2 due to load, we have

$$A_1\bar{x}_1=A_2\bar{x}_2 \text{ (numerically)}$$

$$\left\{\frac{1}{2}w(l+a)\,l.l\right\}\left(a+\frac{2}{3}l\right)=\left(\frac{1}{3}\frac{w(l+a)^2}{2}(l+a)\right)\frac{3}{4}(l+a)$$

$\therefore \quad 5l^3+3l^2a-9la^2-3a^3=0$

Let $\qquad a=Kl$

$\therefore \quad 5l^3+3Kl^3-9lK^2l^2-3K^3l^3=0$

$\therefore \quad 5+3K-9K^2-3K^3=0$

Solving, by trial and error

$$K=0{\cdot}807$$

$\therefore \qquad a=0{\cdot}807\,l$

But distance between the two supports

$$=2l$$

$$\therefore \qquad a=\left(\frac{0{\cdot}807}{2}\right)(2l)$$

$$a=0{\cdot}403\times\textit{distance between the two supports.}$$

§69. Relation between maximum bending stress and maximum deflection

Case 1. Simply supported beam carrying a point load at mid span.

Let the span of the beam be l. Let W be the point load at mid span.

Maximum bending moment

$$=M=\frac{Wl}{4}$$

Let the section be symmetrical about the neutral axis

Section modulus $=Z=\left(\frac{I}{d/2}=\frac{2I}{d}\right)$

where I= Moment of inertia of beam section about the neutral axis

d = depth of the beam section.

∴ Maximum bending stress

$$=f=\frac{M}{Z}=\frac{Wl}{4}\cdot\frac{d}{2I}$$

$$\therefore \quad f=\frac{Wdl}{8I} \quad \ldots(i)$$

Maximum deflection

$$=\delta=\frac{Wl^3}{48EI} \quad \ldots(ii)$$

$$\therefore \quad \frac{f}{\delta}=\frac{6dE}{l^2}$$

$$\therefore \quad \delta=\frac{fl^2}{6Ed}$$

$$\frac{\delta}{l}\cdot\frac{d}{l}=\frac{1}{6}\frac{f}{E} \quad \ldots(a)$$

Case 2. Simply supported beam of span l carrying a uniformly distributed load w per unit run over the whole span.

For this case, maximum bending moment

$$=M=\frac{wl^2}{8}$$

Section modulus $=Z=\frac{2I}{d}$ as before.

∴ Maximum bending stress

$$f=\frac{M}{Z}=\frac{wl^2}{8}\cdot\frac{d}{2I}$$

$$\therefore \quad f=\frac{wl^2}{16}\cdot\frac{d}{I} \quad \ldots(i)$$

Maximum deflection

$$=\delta=\frac{5}{384}\cdot\frac{wl^4}{EI} \qquad \ldots(ii)$$

$$\therefore \quad \frac{f}{\delta}=\frac{24}{5}\,\frac{dE}{l^2}$$

$$\therefore \quad \delta=\frac{5}{24}\,\frac{fl^2}{Ed}$$

$$\therefore \quad \frac{\delta}{l}\cdot\frac{d}{l}=\frac{5}{24}\cdot\frac{f}{E} \qquad \ldots(b)$$

Problem 273. *A beam is rectangular in section. The beam is freely supported at its ends and is subjected to a uniformly distributed load. If the maximum bending stress is 80 kg./cm.², find the ratio of the depth of the section to span in order that the deflection may not exceed $\frac{1}{480}$ of the span. Take $E=0{\cdot}12\times10^6$ kg./cm.²*

Solution. Let l be the span, d the depth of the section and δ the central deflection.

We have

$$\frac{\delta}{l}\cdot\frac{d}{l}=\frac{5}{24}\cdot\frac{f}{E}$$

$$\therefore \quad \frac{d}{l}=\frac{5}{24}\cdot\frac{f}{E}\cdot\frac{l}{\delta}$$

$$=\frac{5}{24}\cdot\frac{80}{0{\cdot}12\times10^6}\times480$$

$$=\frac{1}{15}$$

Problem 274. *A beam consists of a symmetrical rolled steel joist. The beam is simply supported at its ends and carries a point load at the centre of the span. If the maximum stress due to bending is 1400 kg./cm²., find the ratio of the depth of the beam section to span in order the central deflection may not exceed $\frac{1}{480}$ of the span. Take $E=2\times10^6$ kg./cm.²*

Solution. Let the span of the beam be l.

Let δ be the central deflection. Let the depth of the beam section be d.

We have,

$$\frac{\delta}{l}\cdot\frac{d}{l}=\frac{1}{6}\cdot\frac{f}{E}$$

$$\therefore \quad \frac{1}{480}\cdot\frac{d}{l}=\frac{1}{6}\times\frac{1400}{2\times10^6}$$

$$\therefore \quad \frac{d}{l}=\frac{1}{6}\times\frac{1400}{2\times10^6}\times480$$

$$=\frac{7}{125}$$

Problem 275. *The stress in a steel beam is limited to 1400 kg/cm.*^{2} *and its central deflection is* $\frac{1}{600}$ *of the span. The maximum moment of inertia of a beam section 300 mm×140 mm. rolled steel joist is 8604 cm.*^{4} *Calculate the span for this beam when both the above conditions are to be satisfied if the loading is uniformly spread on the span. Find also the intensity of loading. Take* $E=2\times10^6$ *kg./cm*^{2}.

Solution. Let the uniformly distributed loading be w per unit run.

$$\therefore \text{ Max. } \quad \text{B.M.}=M=\frac{wl^2}{8}$$

$$Z=\frac{2I}{d}$$

$\therefore$ Max. bending stress f

$$=\frac{M}{Z}=\frac{wl^2}{8}\cdot\frac{d}{2I}$$

$$\therefore \quad f=\frac{wl^2}{16}\cdot\frac{d}{I} \quad \ldots(i)$$

Centre deflection $$\delta=\frac{5}{384}\cdot\frac{wl^4}{EI} \quad \ldots(ii)$$

$$\therefore \quad \frac{f}{\delta}=\frac{24}{5}\cdot\frac{dE}{l^2}$$

$$\therefore \quad \frac{\delta}{l}\cdot\frac{d}{l}=\frac{5}{24}\cdot\frac{f}{E}$$

$$\frac{1}{600}\cdot\frac{d}{l}=\frac{5}{24}\times\frac{1400}{2\times10^6}$$

$$\therefore \quad \frac{d}{l}=\frac{7}{80}$$

$$\therefore \quad l=\frac{80}{7}d$$

$$=\frac{80}{7}\times20 \text{ cm}$$

$$=344{\cdot}3 \text{ cm}$$

$$=3{\cdot}443 \text{ metres.}$$

From (*i*)

$$f=\frac{wl^2}{16}\frac{d}{I}$$

$$w=\frac{16\,If}{dl^2}$$

$$=\frac{16\times8604\times1400}{30\times(344{\cdot}3)^2} \text{ kg./cm.}$$

$$=54{\cdot}18 \text{ kg./cm.}$$

$$=5418 \text{ kg./metre run.}$$

§70. Beams of varying sections

If the section of a beam varies along the span the slopes and deflections can be determined by rearranging the relation

$$EI\frac{d^2y}{dx^2}=M \text{ to the form,}$$

$$E\frac{d^2y}{dx^2}=\frac{M}{I}$$

Now, by successive integrations the slopes and deflections can be computed.

Problem 276. *A cantilever of length l carries a point W load at the free end. If the moment of inertia of the section increases uniformly from I at the free end to 2I at the fixed end, calculate the deflection at the free end.*

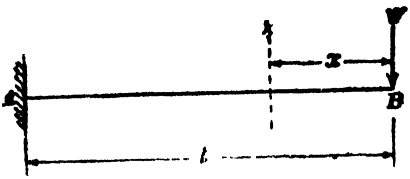


Fig. 438

Solution. At any section X, distant x from B the moment of inertia of the section

$$=I_x=I+\frac{x}{l}I$$

$$\therefore\ I_x=I\left(1+\frac{x}{l}\right)$$

B.M. at the section

$$=M=-Wx$$

$$\therefore\quad E\frac{d^2y}{dx^2}=\frac{M}{I_x}=-\frac{Wx}{I\left(1+\frac{x}{l}\right)}$$

$$\therefore\quad E\frac{d^2y}{dx^2}=-\frac{W}{I}\left\{\frac{x}{1+\frac{x}{l}}\right\}$$

$$=-\frac{Wl}{I}\frac{\left\{\frac{x}{l}\right\}}{\left\{1+\frac{x}{l}\right\}}$$

$$=-\frac{Wl}{I}\left\{1-\frac{1}{1+\frac{x}{l}}\right\}$$

$$\therefore\quad E\frac{d^2y}{dx^2}=-\frac{Wl}{I}+\frac{Wl}{I}\left(\frac{1}{1+\frac{x}{l}}\right)$$

Integrating, we get,

$$E\frac{dy}{dx}=-\frac{Wl}{I}x+\frac{Wl^2}{I}\log_e\left(1+\frac{x}{l}\right)+C_1$$

where C_1 is a constant of integration.

At A the slope is zero.

i.e., at $x=l$,

$$\frac{dy}{dx}=0$$

$$\therefore \quad 0=-\frac{Wl^2}{I}+\frac{Wl^2}{I}\log_e 2+C_1$$

$$\therefore \quad C_1=\frac{Wl^2}{I}(1-\log_e 2)$$

$$\therefore \quad E\frac{dy}{dx}=-\frac{Wl}{I}x+\frac{Wl^2}{I}\log_e\left(1+\frac{x}{l}\right)+\frac{Wl^2}{I}(1-\log_e 2)$$

Integrating again, we have,

$$Ey=-\frac{Wlx^2}{2I}+\frac{Wl^2}{2I}\left\{x\log_e\left(1+\frac{x}{l}\right)-\int\frac{x}{l}\cdot\frac{1}{1+\frac{x}{l}}dx\right\}$$
$$+\frac{Wl^2}{I}(1-\log_e 2)$$

$$\therefore \quad Ey=-\frac{Wlx^2}{2I}+\frac{Wl^2}{I}\left\{x\log_e\left(1+\frac{x}{l}\right)-x+l\log_e\left(1+\frac{x}{l}\right)\right\}$$
$$+\frac{Wl^2}{I}(1-\log_e 2)\,.\,x+C_2$$

At A the deflection is zero.

i.e., at $x=l$,

$$y=0$$

$$\therefore \quad 0=-\frac{Wl^3}{2I}+\frac{Wl^2}{I}\{l\log_e 2-l+l\log_e 2\}$$
$$+\frac{Wl^3}{I}(1-\log_e 2)+C_2$$

$$\therefore \quad 0=-\frac{Wl^3}{2I}+\frac{Wl^3}{I}\log_e 2+C_2$$

$$C_2=\frac{Wl^3}{2I}-\frac{Wl^3}{I}\log_e 2$$

$$C_2=-\frac{Wl^3}{I}(\log_e 2-0{\cdot}5)$$

$$\therefore \quad Ey=-\frac{Wlx^2}{2I}+\frac{Wl^2}{I}\left\{x\log_e\left(1+\frac{x}{l}\right)-x+l\log_e\left(1+\frac{x}{l}\right)\right\}$$
$$+\frac{Wl^2}{I}x(1-\log_e 2)-\frac{Wl^3}{I}(\log_e 2-0{\cdot}5)$$

To find the deflection at B, putting $x=0$, in the above relation, we have,

$$Ey_b=-\frac{Wl^3}{I}(\log_e 2-0{\cdot}5)$$

$$\therefore \qquad y_b = -\frac{Wl^3}{EI}(\log_e 2 - 0{\cdot}5)$$

§71. Strain energy stored due to bending

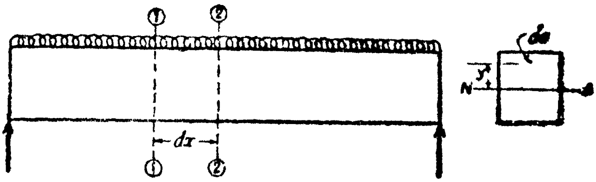


Fig. 439

In chapter 2, we have studied about strain energy stored by a member due to axial loading. If a member of length l and sectional area A is subjected to a total axial force S on the section we know the strain energy stored

$$= W_i = \left(\frac{p^2}{2E}\right) \text{volume}$$

$$= \frac{1}{2E}\left(\frac{S}{A}\right)^2 Al$$

$$= \frac{S^2 l}{2AE}$$

In Fig 439 is shown a beam subjected to an external loading. Consider two sections 1 – 1 and 2—2, dx apart Let us assume that the bending moment is *practically* constant between the two sections. Let this bending moment be M.

The part of the beam between these two sections can be taken to consist of an infinite number of elemental cylinders of area da and length dx.

Consider one such elemental cylinder at a distance y from the neutral layer.

The stress in this elemental cylinder

$$-f = \frac{M}{I} \cdot y$$

where I is the moment of inertia of the beam section about the neutral axis.

Strain energy stored by the elemental cylinder

$$= \frac{f^2}{2E} \cdot \text{volume of the elemental cylinder.}$$

$$= \frac{f^2}{2E} da.dx$$

$$= \left(\frac{M}{I} \cdot y\right)^2 \frac{da.dx}{2E}$$

$$=\frac{M^2dx}{2EI^2}.(da.y^2)$$

∴ Strain energy stored by dx length of the beam

=Strain energy stored by all the elemental cylinders between the two sections

$$=\frac{M^2dx}{2EI^2}\sum_{y=y_t}^{y=y_c} da.y^2$$

$$=\frac{M^2dx}{2\ EI^2}(I)$$

$$=\frac{M^2dx}{2EI}$$

This is the energy stored by dx length of the beam.

∴ Energy stored by the whole beam

$$=W_i=\int\frac{M^2dx}{2EI}$$

Problem 277. *A cantilever of uniform section carries a point load at the free end. Find the strain energy stored by the cantilever and hence calculate the deflection at the free end.*

Solution. Fig. 440 shows cantilever AB of length l, fixed at A and free at B and carrying a point load W at the free end B.

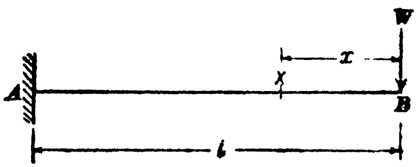


Fig. 440

At any section X distant x from B the bending moment is given by

$$M=Wx$$

∴ Strain energy stored by the cantilever

$$=W_i=\int\frac{M^2dx}{2EI}$$

$$=\int_0^l\frac{W^2x^2dx}{2EI}$$

$$=\frac{W^2}{2EI}\cdot\frac{l^3}{3}$$

$$W_i=\frac{W^2l^3}{6EI}$$

Let δ be the deflection of the free end.

∴ Work done =Average load × deflection

$$=\tfrac{1}{2}W\delta.$$

Equating the work done to energy stored, we have,

$$\tfrac{1}{2}W\delta = \frac{W^2l^3}{6EI}$$

$$\therefore \qquad \delta = \frac{Wl^3}{3EI}$$

Problem 278. *A cantilever of length l carries a uniformly distributed load of w per unit run. Find the strain energy stored by the cantilever.*

Solution. Fig. 441 shows the cantilever AB fixed at A and free at B and carrying uniformly distributed load of w per unit run over the whole length.

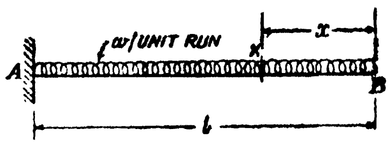


Fig. 441

The B.M. at any section X distant x from B

$$= M = \frac{wx^2}{2}$$

$\therefore$ Strain energy stored by the cantilever

$$= W_i = \int \frac{M^2 dx}{2EI}$$

$$\therefore \qquad W_i = \int_0^l \left(\frac{wx^2}{2}\right)^2 \frac{dx}{2EI}$$

$$\frac{w^2}{8EI}\int_0^l x^4\, dx = \frac{w^2}{8EI}\,\frac{l^5}{5}$$

$$W_i = \frac{w^2l^5}{40EI}.$$

Problem 279. *A simply supported beam of span l carries a point load W at mid-span. Find the strain energy stored by the beam and hence calculate the central deflection.*

Solution. Fig. 442 shows the simply supported beam AB of span l carrying a point load W at the mid span C

Each vertical reaction

$$= \frac{W}{2}$$

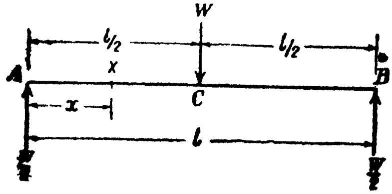


Fig. 442

At any section in AC, distant x from A, the B.M. is given by,

$$M = \frac{W}{2}x$$

$\therefore$ Strain energy stored by the part AC

$$=\int \frac{M^2dx}{2EI}$$

$$=\int_0^{l/2} \frac{W^2}{4} \frac{x^2dx}{2EI}$$

$$=\frac{W^2}{8EI}\int_0^{l/2} x^2dx$$

$\therefore$ Strain energy stored by the whole beam

$$=W_i=2\times\text{energy stored by } AC$$

$$=2\times\frac{W^2}{8EI}\int_0^{l/2} x^2dx$$

$$=\frac{W^2}{4EI}\cdot\frac{1}{3}\cdot\frac{l^3}{8}$$

$$\therefore \quad W_i=\frac{W^2l^3}{96EI}$$

Let the central deflection be δ.

$\therefore$ Work done $=\frac{1}{2}\,W\delta$

Equating the work done to strain energy stored, we have,

$$\frac{1}{2}\,W\delta=\frac{W^2l^3}{96EI}$$

$$\therefore \quad \delta=\frac{Wl^3}{48EI}$$

Problem 280. *A beam of length l is simply supported at its ends. The beam carries a uniformly distributed load of w per unit run over the whole span. Find the strain energy stored by the beam.*

Solution Fig. 443 shows a simply supported beam AB carrying a uniformly distributed load of w per unit run.

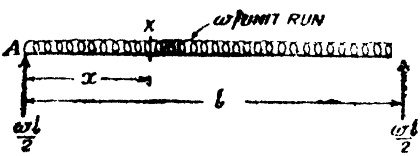


Fig. 443

Each vertical reaction

$$=\frac{wl}{2}$$

The B.M. at any section X distant x from A is given by,

$$M=\frac{wl}{2}\,x-\frac{wx^2}{2}$$

$$=\frac{w}{2}\,x(l-x)$$

$\therefore$ Strain energy stored by the beam

$$= W_i = \int \frac{M^2 dx}{2EI}$$

$$= \int_0^l \frac{w^2}{4} x^2 \frac{(l-x)^2\, dx}{2EI}$$

$$= \frac{w^2}{8EI} \int_0^l (l^2x^2 - 2lx^3 + x^4)dx$$

$$= \frac{w^2}{8EI}\left(l^2 . \frac{l^3}{3} - 2l . \frac{l^4}{4} + \frac{l^5}{5} \right)$$

$$= \frac{w^2 l^5}{240EI}$$

Problem 281. *Find the strain energy stored by the structure shown in Fig. 444 and hence compute the vertical deflection of the end A. Assume the section of the member is uniform.*

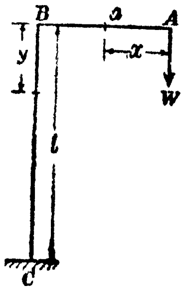


Fig. 444

Solution. At any section in AB distant x from A, the B.M. is given by,

$$M = Wx$$

$\therefore$ Strain energy stored by AB

$$= \int \frac{M^2 dx}{2EI}$$

$$= \int_0^a \frac{W^2 x^2 dx}{2EI}$$

$$= \frac{W^2 a^3}{6EI}$$

At any section in BC distant y from B, the B.M. is given by,

$$M = Wa$$

$\therefore$ Strain energy stored by BC

$$= \int \frac{M^2 dx}{2EI}$$

$$= \int_0^l \frac{W^2 a^2 dy}{2EI}$$

$$= \frac{W^2 a^2 l}{2EI}$$

∴ Total energy stored

$$=W_i=\frac{W^2a^3}{6EI}+\frac{W^2a^2l}{2EI}$$

$$=\frac{W^2a^2}{6EI}(a+3l)$$

Let δ be the vertical deflection at the end A,

Equating the work done to energy stored, we have,

$$\tfrac{1}{2}W\delta=\frac{W^2a^2}{6EI}(a+3l)$$

$$\therefore \qquad \delta=\frac{Wa^2(a+3l)}{3EI}$$

Problem 282. *Find the strain energy stored by the quadrantal ring shown in Fig. 445 of radius R. Hence calculate the vertical deflection of the end A.*

Solution. At any section X, whose radius vector OX makes an angle θ with the vertical, the B.M. is given by

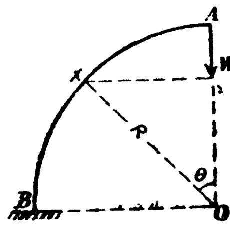


Fig. 445

$$M=WR\sin\theta$$

∴ Strain energy stored

$$=W_i=\int\frac{M^2ds}{2EI}$$

$$=\int_0^{\pi/2}\frac{(WR\sin\theta)^2(Rd\theta)}{2EI}$$

$$=\frac{W^2R^3}{2EI}\int_0^{\pi/2}\sin^2\theta\, d\theta$$

$$=\frac{W^2R^3}{2EI}\cdot\frac{\pi}{4}$$

$$=\frac{W^2\pi R^3}{8EI}$$

Let the vertical deflection at A be δ.

Equating the work done to strain energy stored, we have,

$$\tfrac{1}{2}W\delta=\frac{W^2\pi R^3}{8EI}$$

$$\delta=\frac{W\pi R^3}{4EI}$$

Problem 283. *A cantilever of length l carries a point load W at it's free end. The member is circular in section having a diameter*

D for a distance $\frac{l}{2}$ from the fixed end and a diameter $\frac{D}{2}$ for the remaining length. Show that the deflection at the free end is $\frac{23}{384}\frac{Wl^3}{EI}$, where I is the moment of inertia of the smaller section.

Solution. Fig. 446 shows the cantilever AB of length l carrying the point load W at the free end.

The part AC is of diameter D

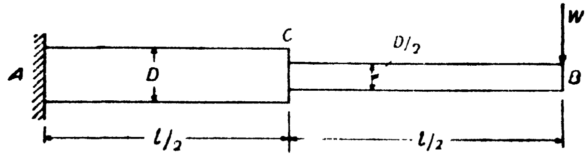


Fig. 446

The part CB is of diameter $\frac{D}{2}$.

Moment of inertia of the section for the part $AC = \frac{\pi D^4}{64}$

Moment of inertia of the section for the part

$$CD = \frac{\pi}{64}\left(\frac{D}{2}\right)^4 = \frac{\pi D^4}{1024}$$

Let the moment of inertia of the section for the part $CD = I$

$\therefore$ Moment of inertia of the section for the part $AC = 16\,I$

Strain energy stored

$$= W_i = \int \frac{(Bending\ moment)^2 dx}{2(Flexural\ rigidity)}$$

$$\therefore \quad W_i = \int_0^{l/2} \frac{(Wx)^2 x\, d}{2EI} + \int_{l/2}^{l} \frac{(Wx)^2 dx}{2E(16I)}$$

$$= \frac{W^2}{2EI}\cdot\frac{1}{3}\cdot\frac{l^3}{8} + \frac{W^2}{32EI}\cdot\frac{1}{3}\left[l^3 - \frac{l^3}{8}\right]$$

$$= \frac{W^2 l^3}{48EI} + \frac{7W^2 l^3}{768EI} = \frac{23}{768}\frac{W^2 l^3}{EI}$$

$\therefore$ Equating work done to strain energy stored,

$$\tfrac{1}{2}W\delta = \frac{23}{768}\frac{W^2 l^3}{EI}$$

$$\therefore \quad \delta = \frac{23}{384}\frac{Wl^3}{EI}$$

Problem 284. *Find the strain energy stored by the frame shown in Fig. 447 and find the horizontal deflection of the roller end D.*

Solution. Obviously there will be a horizontal reaction P at A.

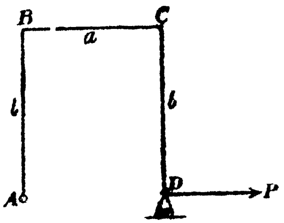


Fig. 447

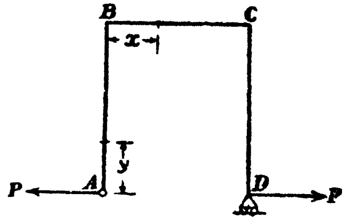


Fig. 448

Strain energy stored by the frame

$$=2\times\text{strain energy stored by } AB + \text{strain energy stored by } BC.$$

At any section AB distant y from A the B.M. is given by,

$$M=Py$$

$\therefore$ Strain energy stored by the column AB

$$=\int\frac{M^2dy}{2EI}$$

$$=\int_0^l \frac{P^2y^2dy}{2EI}$$

$\therefore$ Strain energy stored by both the columns

$$=2\int_0^l \frac{P^2y^2dy}{2EI}$$

$$=\frac{P^2}{EI}\cdot\frac{l^3}{3}$$

$$=\frac{P^2l^3}{3EI}$$

At any section in BC distant x from B, the B.M. is given by,

$$M=Pl$$

$\therefore$ Strain energy stored by BC

$$=\int\frac{M^2dx}{2EI}$$

$$=\int_0^a \frac{P^2l^2dx}{2EI}$$

$$=\frac{P^2l^2a}{2EI}$$

∴ Total strain energy stored by the frame

$$=W_i=\frac{P^2l^3}{3\,EI}+\frac{P^2l^2a}{2\,EI}$$

$$=\frac{P^2l^2}{6\,EI}(2l+3a)$$

Let δ be the horizontal movement of D

Equating the work done to strain energy stored, we have,

$$\tfrac{1}{2}\,P\delta=\frac{P^2\,l^2}{6\,EI}(2l+3a)$$

$$\delta=\frac{P\,l^2}{3\,EI}(2l+3a)$$

Problem 285. *For the semi-circular arch of radius R shown in Fig. 449, find the strain energy stored by the arch and hence find the horizontal deflection of the roller end B.*

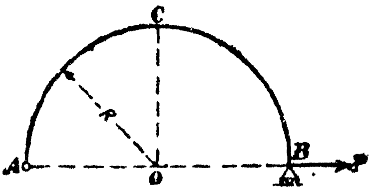


Fig. 449

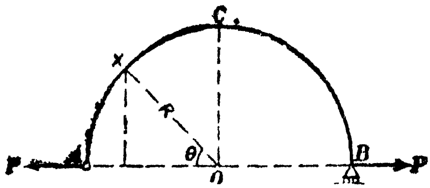


Fig. 450

Solution. See Fig. 450.

Obviously the horizontal reaction at the hinged end $=P$.

At any section X whose radius vector OX makes an angle θ with the horizontal the B.M. is given by

$$M=P\quad R\sin\theta$$

∴ Strain energy stored

$$=W_i=\int\frac{M^2\,ds}{2\,EI}$$

$$=2\int_0^{\pi/2}\frac{(P\,R\sin\theta)^2\,.\,(R\,d\theta)}{2\,EI}$$

$$=\frac{P^2\,R^3}{EI}\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$=\frac{P^2\,R^3}{EI}\cdot\frac{\pi}{4}$$

Let the horizontal deflection of the roller end be δ.

Equating the work done to strain energy stored, we have,

$$\tfrac{1}{2}P\,\delta=\frac{P^2R^3\pi}{4\,EI}$$

$$\therefore \qquad \delta=\frac{PR^3\pi}{2\,EI}$$

Problem 286. *A beam of rectangular section has a uniform breadth B and depth which varies uniformly from D at each end to 3D at the middle of its length. A second beam, made of the same material as the first beam, has the same length and breadth but a uniform depth 3D. Find the ratio of the strain energy of the first beam to that of the second beam when each is simply supported at its ends and carries a central point load W.* (*London University*)

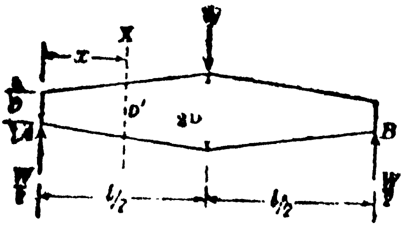


Fig. 451

Solution. (*i*) *Beam of varying depth.*

Fig. 451 shows the beam whose depth uniformly increases from D at each end to $3D$ at the centre.

Let the span of the beam be l.

At any section X distant x from $A\left(x<\dfrac{l}{2}\right)$ the depth of the section $=D'=D+\dfrac{x}{\left(\dfrac{l}{2}\right)}2D$

$$\therefore \qquad D'=D\left(1+\frac{4x}{l}\right)$$

$\therefore$ Moment of inertia at the section X

$$=I=\frac{BD'^3}{12}=\frac{BD^3}{12}\left(1+\frac{4x}{l}\right)^3$$

B.M. at X $\quad =M=\dfrac{W}{2}x$

$\therefore$ Strain energy stored by the beam

$$=U_1=\int\frac{M^2ds}{2EI}$$

$$=2\int_0^{l/2}\frac{\frac{W^2}{4}x^2dx}{2E\frac{BD^3}{12}\left(1+\frac{4x}{l}\right)^3}$$

$$=\frac{3W^2}{EBD^3}\int_0^{l/2}\frac{x^2}{\left(1+\frac{4x}{l}\right)^3}dx$$

Let $1+\frac{4x}{l}=u$

$\therefore \quad x=\frac{l}{4}(u-1)$

$\therefore \quad dx=\frac{l}{4}\,du$

When $x=0$,
$u=1$

and when $x=\frac{l}{2}$

$u=3$

$$\therefore \quad U_1=\frac{3W^2}{EBD^3}\int_1^3 \frac{l^2}{16}\,\frac{u^2-2u+1}{u^3}\cdot\frac{l}{4}\,du$$

$$=\frac{3}{64}\,\frac{W^2l^3}{EBD^3}\int_1^3\left(\frac{1}{u}-\frac{2}{u^2}+\frac{1}{u^2}\right)du$$

$$=\frac{3}{64}\,\frac{W^2l^3}{EBD^3}\left[\log_e u+\frac{2}{u}-\frac{1}{2u^2}\right]_1^3$$

$$=\frac{3}{64}\,\frac{W^2l^3}{EBD^3}\left[\left(\log_e 3+\frac{2}{3}-\frac{1}{18}\right)-\left(2-\frac{1}{2}\right)\right]$$

$$=\frac{3}{64}\,\frac{W^2l^3}{EBD^3}\left[\log_e 3-\frac{8}{9}\right]$$

(*ii*) *Beam of constant depth*

Strain energy stored

$$=U_2=\frac{W^2l^3}{96\,EI}=\frac{W^2l^3}{96\,E\,\frac{B(3D)^3}{12}}$$

$$\therefore \quad U_2=\frac{W^2l^3}{216\,EBD^3}$$

$$\therefore \quad \frac{U_1}{U_2}=\frac{3}{64}\cdot\frac{W^2l^3}{EBD^3}\left[\log_e 3-\frac{8}{9}\right]\frac{216\,EBD^3}{W^2l^3}$$

$$\therefore \quad \frac{U_1}{U_2}=\frac{9}{8}(9\log_e 3-8)$$

§72. Work done by a force on a member

It will be very convenient to realize the expression for the work done by a force on a member. Let a member be subjected to a force W (applied gradually). See Fig. 452.

Let δ be the deflection of the member in the line of action of the force. In this case the deflection δ is caused by the force W.

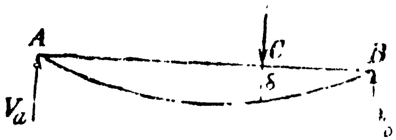


Fig. 452

Work done by the force

$$= \frac{1}{2} W \cdot \delta$$

Fig. 453 shows a member carrying a load K. This load by virtue of its own direct action will produce a deflection.

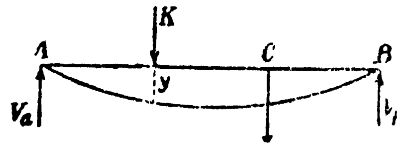


Fig. 453

Let us, for a moment, consider the load K as a very small load Hence the deflection produced by K is also very small and the work done by K may be ignored.

Suppose now due to the action of some other external agency the member is subjected to a deformation. Let y be the deflection in the line of action of the load K.

Now the work done by the load $K = K.y$

Thus we find that

(*i*) If a load acts on a member and produces a deflection δ in its line of action, by virtue of its own direct action, the work done by the load $W = \frac{1}{2} W\delta$.

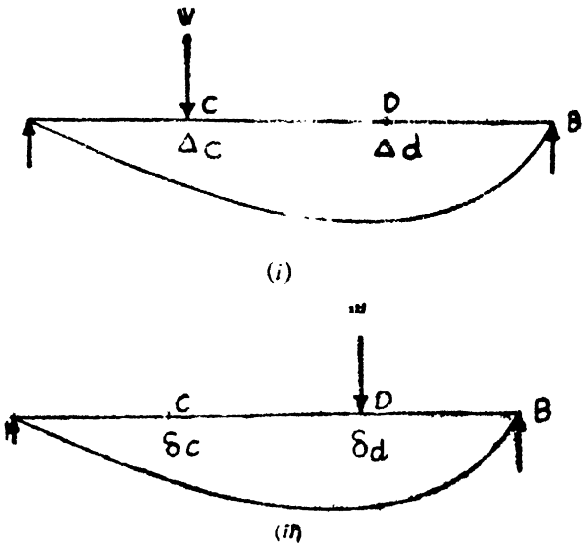
(*ii*) If a member subjected to a load K is given a deformation y in the line of action of K by virtue of some other external agency, the work done by the load $K = Ky$

§73. Law of reciprocal deflection or Maxwell's reciprocal theorem

In any beam or truss the deflection at any point D due to a load W at any other point C is the same as the deflection at C due to the same load W applied at D.

Fig. 454 (*i*) shows a structure AB carrying a load W applied at any point C. Let the deflection at C be Δc. Let the deflection at any other point D be Δd.

Fig. 454 (*ii*) shows the same structure AB carrying the same load W at D. Let the deflections at C and D be δ_c and δ_d respectively.



(ii)

Fig. 454

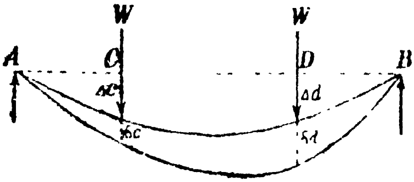


Fig. 455

Let the structure be loaded as shown in Fig. 454 (i). Work done on the structure $= \frac{1}{2} W \Delta c$, as the structure is loaded with the load W at C. Let an other equal load W be applied at D. There will be furtl [illegible] and δd at C and D as shown in Fig. 455

Total work done in this position

$$= \frac{1}{2} W \Delta c + \frac{1}{2} W \delta d + W \delta c$$

Let now the order of loading be changed.

Let the structure be first loaded as shown in Fig. 453 (ii). For

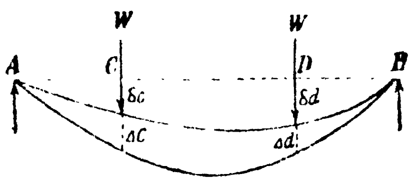


Fig. 456

this position, the work done $= \frac{1}{2} W\delta_d$. As the structure is loaded with the load W at D, let an equal load W be applied at C. Further deflections of Δc and Δd will occur at C and D respectively. See Fig. 456.

Total work done in this position

$$= \frac{1}{2} W\delta d + \frac{1}{2} W\Delta c + W\Delta d$$

Equating the two expressions obtained for the total work done when both the loads are present on the structure, we have

$$= \frac{1}{2} W\Delta c + \frac{1}{2} W\delta d + W\delta c$$
$$= \frac{1}{2} W\delta d + \frac{1}{2} W\Delta c + W\Delta d$$

$\therefore \quad \delta c = \Delta d$

i e, The deflection at C due to the load W at D

= the deflection at D due to the same load W at C.

§74. Bette's Law

In any structure the material of which is elastic and follows Hooke's law and in which the supports are unyielding and the temperature is constant, the virtual work done by a system of forces P_1, P_2, P_3, ..during the distortion caused by a system of forces W_1, W_2, W_3,...is equal to the virtual work done by the system of forces W_1, W_2, W_3 ..during the distortion caused by the system of forces P_1, P_2, P_3...

Fig. 457 shows the structure subjected separately to the two systems of forces.

Let W_e = External work done on the structure when the system of forces P_1, P_2, P_3 be applied.

Let W_e' = External work done on the structure when the system of forces W_1, W_2, W_3 be applied.

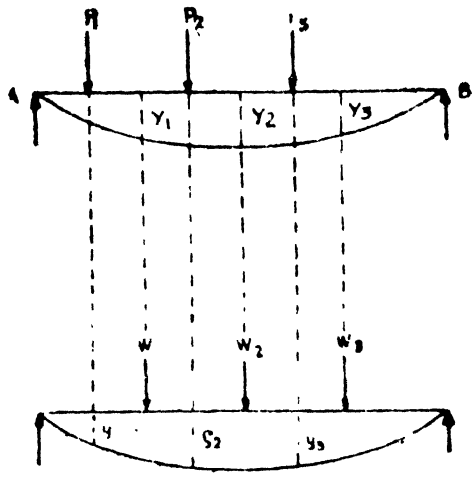


Fig. 457

Let Y_1, Y_2, Y_3 be the deflections caused by the system of forces P_1, P_2, P_3 at the points of application of the forces W_1, W_2, W_3... respectively.

Let y_1, y_2, y_3 be the deflections caused by the system of forces W_1, W_2, W_3 at the points of application of the forces P_1, P_2, P_3 respectively

Let the structure be first loaded with the system of forces P_1, P_2, P_3

Work done on the structure $= W_e$

As the structure is carrying this system of forces, let now the system of forces W_1, W_2, W_3 be applied.

Total work done

$$= W_e + W'_e + P_1y_1 + P_2y_2 + P_3y_3$$

Let now the order of loading be changed.

Let the structure be first loaded with the system of forces W_1, W_2, W_3.

Work done on the structure $= W'_e$.

As the structure is carrying this system, let the system of forces P_1, P_2, P_3 be applied.

Total work done $= W'_e + W_e + W_1Y_1 + W_2Y_2 + W_3Y_3$

Equating the expressions for the total work done when both the systems of forces are present on the structure,

We have

$$W_e + W'_e + P_1y_1 + P_2y_2 + P_3y_3 = W'_e + W_e + W_1Y_1 + W_2Y_2 + W_3Y_3$$

$$\therefore \quad P_1y_1 + P_2y_2 + P_3y_3 = W_1Y_1 + W_2Y_2 + W_3Y_3$$

$\therefore$ Virtual work done by the system of forces P_1, P_2, P_3 due to the deflections caused by the system of forces W_1, W_2, W_3 equals virtual work done by the system of forces W_1, W_2, W_3 due to the deflections caused by the system of forces P_1, P_2, P_3.

§75. The first theorem of Castigliano

In any beam or truss subjected to any load system, the deflection at any point r is given by the partial differential coefficient of the total strain energy stored with respect to a force P_r acting at the point r in the direction in which the deflection is desired.

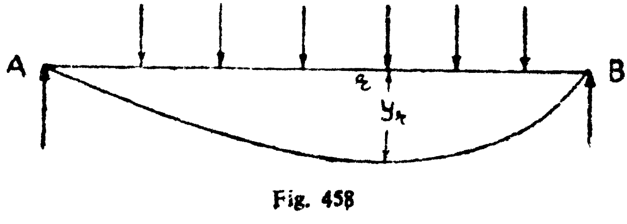


Fig. 458

Fig. 458 shows a structure AB carrying a load system P_1, P_2, $P_3 \ldots P_r \ldots P_n$.

Let the deflection at the point r be y_r.

Let W_e = External work done by the given load system

W_i = Corresponding strain energy stored.

$\therefore \quad W_e = W_i$ ...(i)

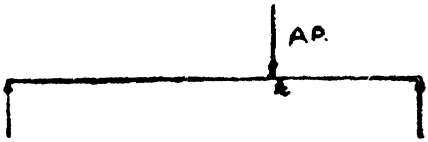


Fig. 459

Suppose the load P_r acting at the point r increases by a small amount $\triangle P_r$. The effect of such an increase in the magnitude of the load P_r can be studied as follows

Let a load $\triangle P_r$ alone be applied at r

Let the deflection at r due to the load $\triangle P_r$ be $\triangle y_r$.

$\therefore$ Work done $= \frac{1}{2} \triangle P_r . \triangle y_r$.

This being the product of two infinitely small quantities, can be ignored.

As the structure is carrying the load $\triangle P_r$ at r,

Let the given load system P_1, P_2, P_3, $\ldots P_r \ldots P_n$ be applied on the structure.

Total external work done

$$= W_e + \triangle P_r . y_r$$

Let the corresponding strain energy stored by the structure be $W_i + \triangle W_i$

$$\therefore \quad W_e + \triangle P_r . y_r = W_i + \triangle W_i \quad \ldots(ii)$$

From equations (i) and (ii), we have

$$\triangle P_r \, y_r = \triangle W_i$$

$$\therefore \quad y_r = \frac{\triangle W_i}{\triangle P_r}$$

The above is more justified when $\triangle P_r$ is an infinitely small quantity.

$$\therefore \quad y_r = \lim_{\triangle P_r \to 0} . \frac{\triangle W_i}{\triangle P_r}$$

$$\therefore \quad y_r = \frac{\partial W_i}{\partial P_r}$$

= differential coefficient of the total strain energy stored with respect to P_r.

Application of the first theorem of Castigliano to problems on deflections.

Problem 287. *Find the deflection at the free end of a cantilever carrying a concentrated load at the free end. Assume uniform flexural rigidity.*

Solution. Fig 460 shows a cantilever carrying a point load P at the free end A. The bending moment at any section distant x from the free end is given by

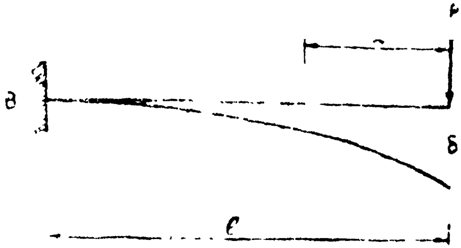


Fig 460

$$M = -Px$$

Strain energy stored by the cantilever

$$W = \int \frac{M^2 dx}{2EI}$$

$$= \int_0^l \frac{P^2x^2\, dx}{2EI}$$

$$= \frac{P^2}{2EI} \cdot \frac{l^3}{3}$$

$$W_i = \frac{P^2l^3}{6EI}.$$

$\therefore$ By the the first theorem of Castigliano,

the deflection in the line of action of the force P,

$$= \delta = \frac{\partial W_i}{\partial P} = \frac{(2P)\, l^3}{6EI} = \frac{Pl^3}{3EI}$$

Problem 288. *Find the central deflection of a simply supported beam carrying a concentrated load at mid span. Assume uniform flexural rigidity.*

Solution. Fig. 461 shows a beam AB simply supported at A and B and carrying a central load P. Each reaction

$$= \frac{P}{2}$$

The bending moment at any section in AC, distant x from the end A is given by

$$M=\frac{P}{2}x$$

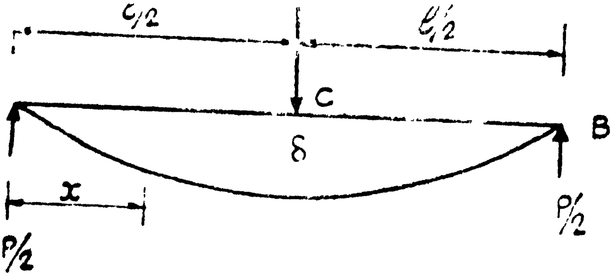


Fig. 461

∴ Strain energy stored by the beam

$$W_i=\int\frac{M^2dx}{2EI}$$

$$=2\int_0^{l/2}\frac{P^2x^2}{4}\frac{dx}{2EI}$$

$$=\frac{P^2}{4EI}\cdot\frac{1}{3}\cdot\frac{l^3}{8}$$

$$=\frac{P^2l^3}{96EI}$$

∴ $$W_i=\frac{P^2l^3}{96EI}$$

∴ The deflection in the line of action of P is given by

$$\delta=\frac{\partial W_i}{\partial P}=\frac{(2P)\,l^3}{96EI}$$

∴ $$\delta=\frac{Pl^3}{48EI}$$

Problem 289. *A simply supported beam carries a point load P eccentrically on the span. Find the deflection under the load. Assume uniform flexural rigidity.*

Solution. Fig 462 shows a beam AB of span l which carries a load P at C.

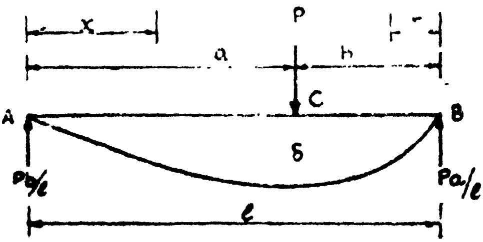


Fig. 462

Let $AC=a$ and $BC=b$.

Reaction at $A=\frac{Pb}{l}$

Reaction at $B=\frac{Pa}{l}$

The strain energy stored by the beam AB

$$W_i = \text{strain energy stored by } AC + \text{strain energy stored by } BC$$

$$=\int_0^a \left(\frac{Pb}{l}x\right)^2 \frac{dx}{2EI}+\int_0^b \left(\frac{Pa}{l}x\right)^2 \frac{dx}{2EI}$$

$$=\frac{P^2b^2a^3}{6EIl^2}+\frac{P^2a^2b^3}{6EIl^2}$$

$$=\frac{P^2a^2b^2}{6EIl^2}(a+b)$$

Since $a+b=l$

$$W_i=\frac{P^2a^2b^2}{6EIl}$$

$\therefore$ Deflection under load P is given by

$$\delta=\frac{\partial W_i}{\partial P}=\frac{(2P)a^2b^2}{6EIl}=\frac{Pa^2b^2}{3EIl}$$

Problem 290. *The semicircular arch shown in Fig. 463 has one of its ends hinged while its other end is on rollers. The roller end is pulled with a horizontal force P. Determine the horizontal movement of the roller end. Assume uniform flexural rigidity.*

Solution. There will be a horizontal reaction at A.

The bending moment at any section X is given by

$$M=Pr\sin\theta.$$

Fig. 463

$\therefore$ Strain energy stored by the arch

$$W_i=\int\frac{M^2ds}{2EI}$$

$$=2\int_0^{\pi/2}\frac{P^2r^2\sin^2\theta . r d\theta}{2EI}$$

$$= \frac{P^2 r^3}{EI} \int_0^{\pi/2} \sin^2 \theta \, d\theta$$

$$= \frac{P^2 r^3 \pi}{4EI}$$

∴ Horizontal movement of the roller end is given by

$$\delta = \frac{\partial W_i}{\partial P} = \frac{(2P) r^3 \pi}{4EI}$$

$$= \frac{P r^3 \pi}{2EI}$$

In the above examples the deflection was determined in the line of action of a force. But if it is required to find the deflection at point where no force is actually acting, then an imaginary force Q should be applied at the point and the total energy stored by the structure should be determined. This expression for the total energy should be differentiated with respect to Q. In the resulting expression Q should be put equal to zero. The following examples will further explain this point.

Problem 291. *Find the deflection at the centre of a beam of span l carrying a uniformly distributed load of w per unit run over the whole span. Assume uniform flexural rigidity.*

Solution. Introduce an imaginary concentrated load Q at the middle point.

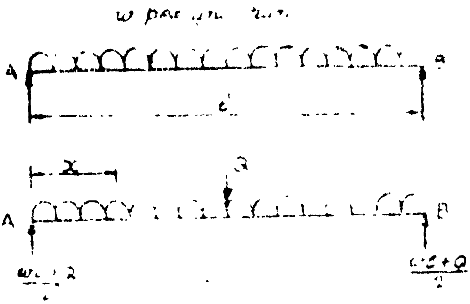


Fig. 164

Each vertical reaction

$$= \frac{wl + Q}{2}$$

Bending moment at a section distant x from one end

$$= \frac{wl + Q}{2} x - \frac{wx^2}{2}$$

∴ Total strain energy stored by the beam

$$= \int \frac{M^2 \, dx}{2EI}$$

$$\therefore \quad W_i = 2\int_0^{l/2}\left[\frac{wl+Q}{2}x - \frac{wx^2}{2}\right]^2 \frac{dx}{2EI}$$

$$\therefore \quad W_i = \frac{1}{EI}\int_0^{l/2}\left(\frac{wl+Q}{2}x - \frac{wx^2}{2}\right)^2 dx$$

$\therefore$ To find the central deflection, differentiating the total strain energy with respect to Q, we have

$$\delta = \frac{\delta W_i}{\delta Q} = \frac{1}{EI}\int_0^{l/2} 2\left[\frac{wl+Q}{2}x - \frac{wx^2}{2}\right]\frac{x}{2}\,dx$$

$$= \frac{1}{EI}\left[\int_0^{l/2}\frac{wl+Q}{2}x^2\,dx - \int_0^{l/2}\frac{wx^3}{2}\,dx\right]$$

$$= \frac{1}{EI}\left[\frac{wl+Q}{2}\cdot\frac{1}{3}\cdot\frac{l^3}{8} - \frac{w}{2}\cdot\frac{1}{4}\cdot\frac{l^4}{16}\right]$$

$$= \frac{1}{EI}\left[\frac{wl^4}{48} - \frac{wl^4}{128} + \frac{Ql^3}{48}\right]$$

$$= \frac{1}{EI}\left[\frac{5}{384}wl^4 + \frac{Ql^3}{48}\right]$$

Putting $Q = 0$, we have

$$\delta = \frac{5}{384}\frac{wl^4}{EI}$$

Problem 292. *Find the deflection at the free end of a cantilever of length l carrying a uniformly distributed load of w per unit run over the whole span. Assume uniform flexural rigidity.*

Solution. Introduce an imaginary point load Q at the free end.

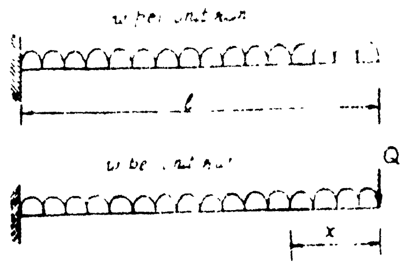


Fig. 465

The bending moment at any section distant x from the free end

$$= M = -\left(Qx + \frac{wx^2}{2}\right)$$

$\therefore$ Strain energy stored by the cantilever

$$= W_t = \frac{M^2 dx}{2EI}$$

$$= \int_0^l \left(Qx + \frac{wx^2}{2} \right)^2 \frac{dx}{2EI}$$

$\therefore$ To find the deflection at the free end, differentiating the total energy with respect to Q, we have,

$$\delta = \frac{\partial W_t}{\partial Q} = \int_0^l 2\left(Qx + \frac{wx^2}{2} \right) \frac{xdx}{2EI}$$

$$= \frac{1}{EI} \int_0^l \left(Qx^2 + \frac{wx^3}{2} \right) dx$$

$$= \frac{1}{EI} \left[\frac{Ql^3}{3} + \frac{wl^4}{8} \right]$$

Putting $Q=0$, we have,

$$\delta = \frac{wl^4}{8EI}$$

Problem 293. *The quadrantal ring AB shown in Fig. 466 is of radius r. It supports a concentrated load P at the free end A. Find the vertical and horizontal deflection of A. Assume uniform flexural rigidity.*

Solution. *Vertical deflection of A.*

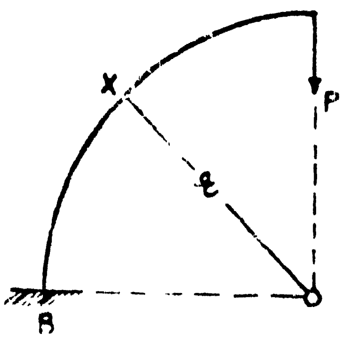


Fig. 466

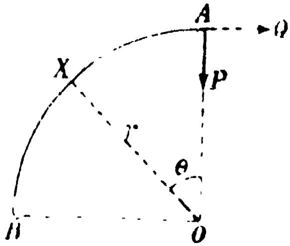


Fig. 467

The bending moment at any section X the radius vector OX making an angle θ with the vertical is given by

$$M = -Pr \sin \theta$$

∴ Strain energy stored

$$= W_i = \int \frac{M^2 ds}{2EI}$$

$$= \int_0^{\pi/2} \frac{P^2 r^2 \sin^2 \theta \cdot r d\theta}{2EI}$$

$$= \frac{P^2 r^3}{2EI} \cdot \frac{\pi}{4}$$

$$= \frac{P^2 \pi r^3}{8EI}$$

∴ Vertical deflection of A

$$\delta_v = \frac{\partial W_i}{\partial P}$$

$$= \frac{2P\pi r^3}{8EI}$$

$$= \frac{P\pi r^3}{4EI}$$

Horizontal movement of A

Introduce an imaginary horizontal force Q at A.

The bending moment at any section X is now given by

$$M = -[Pr \sin \theta + Qr(1 - \cos \theta)]$$

∴ Strain energy stored by the structure

$$= W_i = \int \frac{M^2 ds}{2EI}$$

$$= \int_0^{\pi/2} \left[Pr \sin \theta + Qr(1 - \cos \theta) \right]^2 \frac{r d\theta}{2EI}$$

∴ The horizontal movement of A is given by

$$\delta_h = \frac{\partial W_i}{\partial Q}$$

$$= \int_0^{\pi/2} 2 \left[Pr \sin \theta + Qr(1 - \cos \theta) \right] r(1 - \cos \theta) \frac{r d\theta}{2EI}$$

$$= \frac{Pr^3}{EI} \int_0^{\pi/2} \sin \theta (1 - \cos \theta) \, d\theta + \frac{Qr^3}{EI} \int_0^{\pi/2} (1 - \cos \theta)^2 \, d\theta$$

Putting $Q = 0$

$$\delta_h = \frac{Pr^3}{EI} \int_0^{\pi/2} \sin \theta (1 - \cos \theta) \, d\theta$$

$$= \frac{Pr^3}{EI} \left[(1 - \cos\theta)^2 \right]^{\pi/2}$$

$$= \frac{Pr^3}{2EI}$$

Problem 294. *The bend ABC shown in Fig. 468 carries a concentrated vertical load P at A. Find the vertical and horizontal deflection of A. Assume uniform flexural rigidity.*

Solution. *Vertical deflection of A.* Strain energy stored by the frame

W_t = strain energy stored by AB + strain energy stored by BC.

$$= \sum \int \frac{M^2 ds}{2EI}$$

$$= \int_0^a (Px)^2 \frac{dx}{2EI} + \int_0^h (Pa)^2 \frac{dy}{2EI}$$

$$= \frac{P^2a^3}{6EI} + \frac{P^2a^2h}{2EI}$$

$$\therefore \qquad W_t = \frac{P^2a^2}{6EI}(a+3h)$$

To find the vertical deflection of A differentiating the total energy stored with respect to P, we have,

$$\delta_v = \frac{\partial W_t}{\partial P} = \frac{(2P)a^2(a+3h)}{6EI}$$

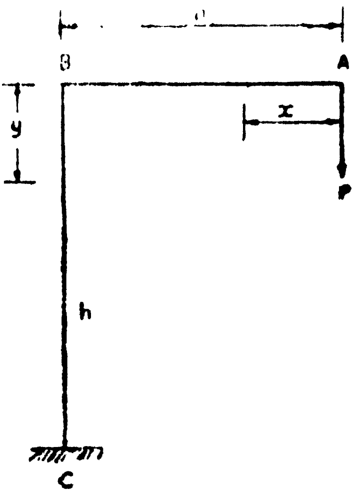


Fig. 468

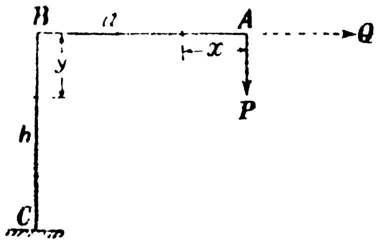


Fig. 469

$$= \frac{Pa^2(a+3h)}{3EI}$$

Particular case. When $h=0$, *i.e.*, if BA had been a cantilever fixed at B

$$\delta_v = \frac{Pa^3}{3EI}$$

Horizontal deflection of A

To find the horizontal deflection of A introduce an imaginary force Q horizontally at A.

Strain energy stored by the structure is now given by

$$W = \int_0^a \frac{(Px)^2 dx}{2EI} + \int_0^h (Pa+Qy)^2 \frac{dy}{2EI}$$

Differentiating the strain energy with respect to Q we have,
Horizontal movement of A

$$\delta_h = \frac{\partial W}{\partial Q} = \int_0^h 2(Pa+Qy)\,y \frac{dy}{2EI}$$

$$= \frac{1}{EI}\int_0^h (Pa+Qy)\,y\,dy$$

$$= \frac{1}{EI}\left[\frac{Pah^2}{2} + \frac{Qh^3}{3}\right]$$

Putting $Q=0$, we have

$$\delta_h = \frac{Pah^2}{2EI}$$

Problem 295. *Find the central deflection of the uniform bend ABCDEFG shown in Fig. 470.*

Solution. Strain energy stored by the whole structure
$W = 2$ [Energy stored by AB, BC and CD]

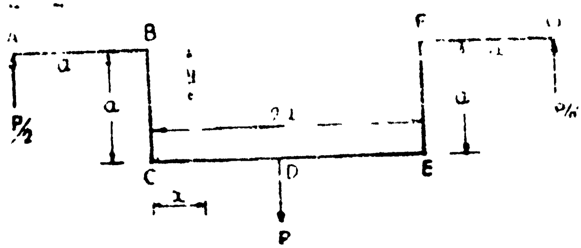


Fig. 470

$$=\sum\int \frac{M^2 ds}{2EI}$$

$$=2\left[\int_0^a \frac{\left(\frac{P}{2}x\right)^2 dx}{2EI}+\int_0^a \frac{\left(\frac{P}{2}a\right)^2 dy}{2EI}+\int_0^a\left\{\frac{P}{2}(x+a)\right\}^2\frac{dx}{2EI}\right]$$

$$=2\left[\frac{P^2}{4}\cdot\frac{a^3}{3}\cdot\frac{1}{2EI}+\frac{P^2}{4}\,\frac{a^2a}{2EI}+\frac{P^2}{4}\cdot\frac{1}{2EI}\,\frac{1}{3}\left\{(x+a)^3\right\}_0^a\right]$$

$$=2\left[\frac{P^2a^3}{24EI}+\frac{P^2a^3}{8EI}+\frac{P^2}{24EI}(8a^3-a^3)\right]$$

$$=2\times\frac{11}{24}\,\frac{P^2a^3}{EI}=\frac{11}{12}\,\frac{P^2a^3}{EI}$$

∴ Vertical deflection of D

$$=\delta_v=\frac{\partial W_i}{\partial P}=\frac{11}{12}\times\frac{(2P)a^3}{EI}$$

$$=\frac{11}{6}\,\frac{Pa^3}{EI}$$

Problem 296 *A mild steel bar 10 cm. diameter is bent as shown in Fig. 471. It is fixed horizontally at A and a load of 50 kg. hangs at D. Draw the bending moment diagram for the parts AB, BC and CD indicating the maximum values. Find the maximum bending stress Find also the deflection at D. Take $E=2\times10^6$ kg./cm.²*

Solution. *Conventions for bending moments.*

For horizontal members, sagging moments will be regarded as positive.

For the vertical member, bending moment producing concavity on the right hand side of the member will be regarded as positive.

B.M. at $D=0$

B.M. at C as a part of $CD=-50\times2=-100$ *kgm.*

B.M at C as a part of $CB=+100$ *kgm.*

B.M. at B as a part of $BC=+100$ *kgm.*

B.M. at B as a part of $AB=+100$ *kgm.*

B.M. at $A=-50\times1=-50$ *kgm.*

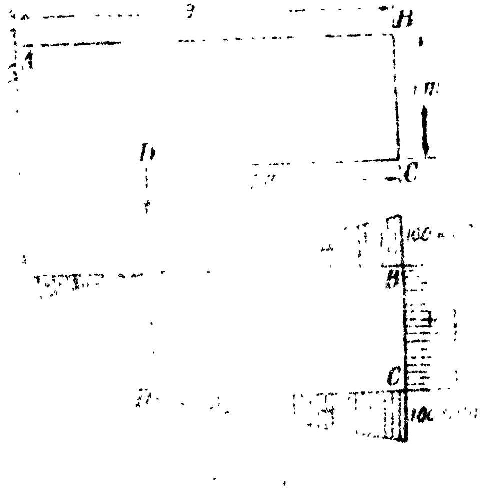
Maximum bending moment

$=100$ *kgm.*

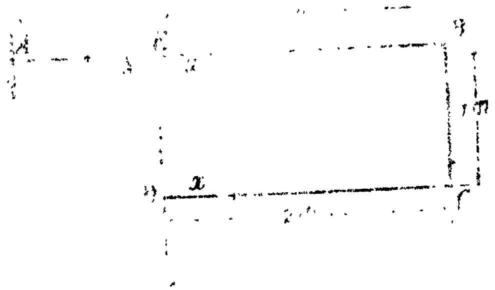
∴ Maximum bending stress

$$=\frac{M}{Z}=\frac{100\times100}{\frac{\pi(10)^3}{32}}\ \text{kg./cm.}^2$$

$$=102\ \text{kg./cm.}^2$$



[illegible] load at D [illegible]



[illegible] 72

[illegible] energy stored by the [illegible]

[illegible] strain energy stored by AB
[illegible] strain energy stored by LB
[illegible] strain energy stored by DC
[illegible] strain energy stored by BC

$$= \sum \int \frac{M^2 dx}{2EI}$$

$$= \int_0^1 \frac{P^2 x^2 dx}{2EI} + 2 \int_0^2 \frac{P^2 x^2 dx}{2EI} + \int_0^1 \frac{4P^2 dy}{2EI}$$

$$=\frac{P^2}{6EI}+\frac{2}{2EI}\cdot P^2\left(\frac{8}{3}\right)+\frac{4P^2}{2EI}(1)$$

$$=\frac{29}{6}\cdot\frac{P^2}{EI}$$

∴ Vertical deflection at D

$$=\delta_v=\frac{\partial W_i}{\partial P}=\frac{29}{6}\frac{2P}{EI}=\frac{29}{3}\cdot\frac{P}{EI}$$

$P=50$ kg.

$E=2\times10^6$ kg./cm.^{2}

$I=\frac{\pi(10)^4}{64}$ cm.^{4}

∴ $$\delta_v=\frac{29}{3}\ \frac{50\times64}{2\times10^6\times\pi(10)^4}\times(100)^3 \text{ cm.}$$

$=0{\cdot}49$ cm.

§76. Impact loading on beam

Simply supported beam

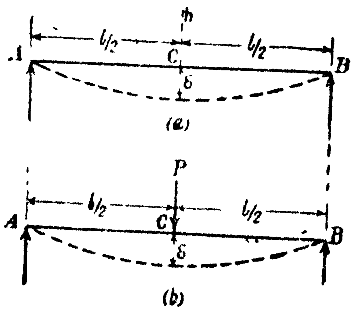


Fig. 473

Fig. 473 (*a*) shows a simply supported beam AB of span l.

Let a load W be dropped on the centre of the span from a height h.

Let δ be the instantaneous maximum deflection.

∴ Work done on the beam $=W(h+\delta)$.

Fig. 473 (*b*) shows the same beam carrying a gradually applied load P so as to produce the same deflection δ at the centre. For this case the work done$=\frac{1}{2}P\delta$.

But $$\delta=\frac{Pl^3}{48EI}$$

$$=\text{Work done}=\frac{1}{2}P\cdot\frac{Pl^3}{48EI}$$

$$=\frac{P^2l^3}{96EI}$$

Equating the two expressions obtained for the work done, we have

$$\frac{P^2l^3}{96EI}=W(h+\delta)$$

$$\therefore \quad \frac{P^2l^3}{96EI} = W\left(h + \frac{Pl^3}{48EI}\right)$$

The above equation is a quadratic equation in P. By solving the equation the equivalent gradually applied load P can be determined. Once P is known, the maximum deflection is determined from the relation $\delta = \frac{Pl^3}{48EI}$ or from the relation,

$$\frac{1}{2}P.\delta = W(h+\delta)$$

The maximum bending moment to which the beam is subjected to is give by

$$M_{max} = \frac{Pl}{4}$$

If Z be the section modulus for the section, the maximum instantaneous stress

$$= f = \frac{M}{Z}.$$

Problem 297. *A 500 mm. × 180 mm. rolled steel beam is simply supported on a span of 6 metres. A load of 2 tonnes is dropped on to the middle of the beam from a height of 1·25 cm. Find the maximum instantaneous deflection and the maximum stress induced. Take for the beam section I_{xx} = 45218·3 cm⁴ and $E = 2 \times 10^3$ tonnes/cm².*

Solution.

$$W = 2 \text{ tonnes.}$$

Let δ be the maximum deflection.

Let P *tonnes* be the equivalent gradually applied load on the middle of the beam so as to produce the same maximum deflection.

$$\therefore \quad W(h+\delta) = \tfrac{1}{2}P\delta$$

$$\therefore \quad 2(1{\cdot}25+\delta) = \tfrac{1}{2}P\delta$$

$$2{\cdot}50 + 2\delta = \tfrac{1}{2}P\delta$$

$$\therefore \quad 5 + 4\delta = P\delta$$

$$\therefore \quad \delta(P-4) = 5$$

But

$$\delta = \frac{Pl^3}{48EI} = \frac{P(600)^3}{48EI}$$

$$\therefore \quad \frac{P \times 600^3}{48EI}(P-4) = 5$$

$$\therefore \quad P(P-4) = \frac{5 \times 48 \times 2 \times 10^3 \times 45218{\cdot}3}{(600)^3}$$

$$\therefore \quad P(P-4) = 100{\cdot}4$$

$$P^2 - 4P = 100{\cdot}4$$

$\therefore \quad (P-2)^2=104{\cdot}4$

$(P-2)=10{\cdot}22$

$\therefore \quad P=12{\cdot}22$ *tonnes.*

Max. B.M.

$$=M_{max}=\frac{Pl}{4}$$

$$=\frac{12{\cdot}22\times600}{4} \text{ tonne cm.}$$

$$=12{\cdot}22\times150 \text{ tonne cm.}$$

Section modulus $=Z=\frac{I}{y}$

$$=\frac{45218{\cdot}3}{25} \text{ cm.}^3$$

$$=1808{\cdot}7 \text{ cm.}^3$$

$\therefore$ Maximum bending stress

$$=f=\frac{M}{Z}$$

$$=\frac{12{\cdot}22\times150}{1808{\cdot}7} \text{ t/cm.}^2$$

$$=1{\cdot}014 \text{ t/cm.}^2$$

$$=1014 \text{ kg./cm.}^2$$

$$\delta=\frac{Pl^3}{48EI}$$

$$=\frac{1222\times600^3}{48\times2\times10^3\times45218{\cdot}3} \text{ cm}$$

$$=0{\cdot}6084 \text{ cm.}$$

§77. Laminated Springs or Leaf Springs

This is a commonly occurring type of spring used in carriages like cars, lorries or wagons. This spring consists of a number of leaves or plates of equal width and thickness but of varying length, placed one below the other.

These springs get loaded at the ends and are supported in the middle.

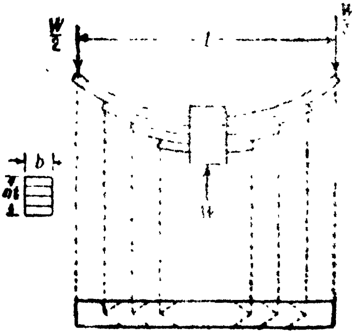


Fig. 474

In this type of spring, since friction between the plates is negligible, each plate can be considered as free to slide over its neighbouring plate. Further all the plates may be assumed to maintain the same radius. Each plate section hence has its own

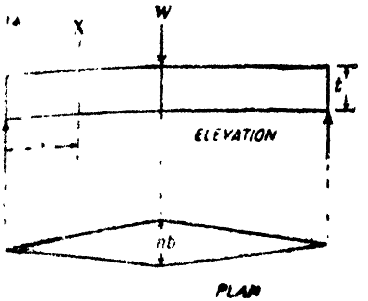


Fig. 475

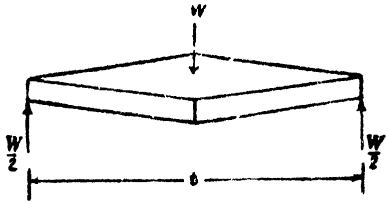


Fig. 476

neutral axis. Hence the plates may be looked upon as being arranged side by side. Hence for analysis purposes the carriage spring can be considered as equivalent to a beam whose depth is uniform and equal to the thickness of one plate and whose width uniformly increases from zero at each end to nb at the centre where n is the number of plates and b is the width of each plate.

The equivalent beam is shown in Fig. 475.

Consider the equivalent beam. Let the span of the beam be l.

Width of this beam at a section X at a distance x $\left(\text{less than } \frac{l}{2}\right)$ from the left end

$$=\frac{x}{\left(\frac{l}{2}\right)}nb$$

$$=\frac{2nb}{l}\cdot x$$

$\therefore$ Moment of inertia at the section X

$$=I=\frac{2nb}{l}x\cdot\frac{t^3}{12}$$

$$=\frac{nbt^3}{6l}\cdot x$$

B.M at the section

$$=M=\frac{W}{2}x$$

But $$\frac{M}{I}=\frac{E}{R}$$

$\therefore$ $$\frac{1}{R}=\frac{M}{EI}$$

$$=\frac{1}{E}\frac{W}{2}x\frac{6l}{nbt^3x}$$

$$\therefore \quad \frac{1}{R}=\frac{3Wl}{Enbt^3}=\text{constant}$$

Hence the beam will bend to a constant radius.

∴ Central deflection

$$=\delta=\frac{l^2}{8R}$$

$$=\frac{l^2}{8}\cdot\frac{3Wl}{Enbt^3}$$

$$\therefore \quad \delta=\frac{3}{8}\frac{Wl^3}{Enbt^3} \qquad ...(i)$$

B.M. at the centre

$$M=\frac{Wl}{4}$$

Section modulus at mid span

$$Z=n\cdot\frac{bt^2}{6}$$

∴ Bending stress

$$f=\frac{M}{Z}=\frac{Wl}{4}\cdot\frac{6}{nbt^2}$$

$$\therefore \quad f=\frac{3}{2}\cdot\frac{Wl}{nbt^2} \qquad ...(ii)$$

From equations (*i*) and (*ii*)

$$\delta=\frac{fl^2}{4Et} \qquad ...(iii)$$

It may be noted that the spring has been analysed as a *beam of uniform strength.*

Work done by the load on the spring = strain energy stored by the spring $=\frac{1}{2}W\delta$

Stiffness of the spring = load required to produce unit deflection

$$=\frac{W}{\delta}$$

Problem 298. *A leaf spring 75 cm. long is required to carry a central point load of 800 kg. If the central deflection is not to exceed 2 cm. and the bending stress is not to exceed 2 t/cm.² determine the thickness, width and number of plates.*

Also compute the radius to which the plates should be curved.

Assume width of the plates = 12 times the thickness and $E=2\times10^6$ *kg./cm.²* (*AMIE, November 1971*)

Solution. $l=75$ cm. $\quad b=12t$

$W=800$ kg. $\quad E=2\times10^6$ kg./cm.²

$\delta=2$ cm.

$f=2000$ kg./cm.²

We know the following relations :

$$f=\frac{3}{2}\cdot\frac{Wl}{nbt^2} \qquad ...(i)$$

$$=\frac{3}{8}\cdot\frac{Wl^3}{Enbt^3} \qquad ...(ii)$$

$$\delta=\frac{fl^2}{4Et} \qquad ...(iii)$$

$$\delta=\frac{l^2}{8R} \qquad ...(iv)$$

From (*iii*) $\quad t=\frac{fl^2}{4E\delta}=\frac{2000\times 75^2}{4\times 2\times 10^6\times 2}=0{\cdot}7\ cm.$

$\therefore \quad b=12\times 0{\cdot}7=8{\cdot}4\ cm.$

From (*i*) $\quad n=\frac{3}{2}\cdot\frac{Wl}{fbt^2}=\frac{3}{2}\times\frac{800\times 75}{2000\times 12(0{\cdot}7)^2}$
$=8$ *plates.*

From (*iv*) $\quad R=\frac{l^2}{8\delta}=\frac{75\times 75}{8\times 2}=351{\cdot}6\ cm.$

Problem 299. *A leaf spring is to be made of seven steel plates 6·5 cm. wide and 6·3 mm. thick. Calculate the length of the spring so that it may carry a central load of 275 kg., the stress being limited to 1600 kg./cm.² Calculate also the deflection at the centre of the spring.* $E=2{\cdot}1\times 10^6$ *kg./cm.²* (*AMIE, May 1970*)

Solution.

$n=7 \qquad l=?$
$b=6{\cdot}5\ cm. \qquad \delta=?$
$t=0{\cdot}63\ cm.$
$W=275\ kg.$
$f=1600\ kg/cm.^2$
$E=2{\cdot}1\times 10^6\ kg./cm.$

$$f=\frac{3}{2}\cdot\frac{Wl}{nbt^2}$$

$$\therefore \quad l=\frac{2}{3}\cdot\frac{nbt^2f}{W}$$

$$\therefore \quad l=\frac{2}{3}\times\frac{7\times 6{\cdot}5\times 0{\cdot}63^2\times 1600}{275}=70\ cm.$$

$$\delta=\frac{3}{8}\cdot\frac{Wl^3}{Enbt^3}$$

$$=\frac{3}{8}\cdot\frac{275\times 70^3}{(2{\cdot}1\times 10^6)\times 7\times 6{\cdot}5(0{\cdot}63)^3}$$

$$=1{\cdot}481\ cm.$$

Problem 300. *A laminated spring 1 m long is made up of plates each 5 cm. wide and 1 cm. thick. If the bending stress in the plates* *is limited to 1 t/cm.² how many plates would be required to enable the spring to carry a central point load of 200 kg.*

If E = 2·1 × 10⁶ kg./cm.², what is the deflection under the given load of 200 kg ?

Solution.

$l = 100$ cm.

$b = 5$ cm.

$t = 1$ cm.

$f = 1000$ kg./cm.²

$W = 200$ kg.

$E = 2 \cdot 1 \times 10^6$ kg./cm.²

$$f = \frac{3}{2} \cdot \frac{Wl}{nbt^2}$$

$$\therefore \quad n = \frac{3}{2} \cdot \frac{Wl}{fbt^2}$$

$$\therefore \quad n = \frac{3}{2} \cdot \frac{200 \times 100}{1000 \times 5 \times 1^2} = 6$$

$$\delta = \frac{fl^2}{4Et}$$

$$= \frac{1000 \times 100 \times 100}{4 \times 2 \cdot 1 \times 10^6 \times 1}$$

$$= 1 \cdot 19 \text{ cm.}$$

Problem 301. *A carriage spring is 1·25 metres long and built up of plates 8 cm. wide and 1 cm. thick. Find the number of plates required for the spring if a central point load of 600 kg. is to be carried and if the bending stress is not to exceed 1400 kg./cm.² Find also the central deflection. Take E = 2 × 10⁶ kg./cm.²*

Solution. Bending stress

$$= f = \frac{3}{2} \cdot \frac{Wl}{nbt^2}$$

$$\therefore \quad n = \frac{3}{2} \cdot \frac{Wl}{fbt^2}$$

$$\therefore \quad n = \frac{3}{2} \cdot \frac{600 \times 125}{1400 \times 8 \times 1^2}$$

$$= 11 \text{ plates.}$$

Central deflection

$$= \delta = \frac{3}{8} \cdot \frac{Wl^3}{Enbt^3}$$

$$= \frac{3}{8} \cdot \frac{600 \times 125^3}{2 \times 10^6 \times 11 \times 8 \times 1^3} \text{ cm.}$$

$$= \mathbf{2 \cdot 496 \text{ cm.}}$$

Problem 302. *A laminated carriage spring is 80 cms. long and made of twelve leaves of the same thickness and 4 cm. wide. Find thickness of the leaves if the bending stress is to be limited to 2000 cm.² when the spring is subjected to a point load of 600 kg. at the centre. Find also the central deflection. Take E = 2 × 10⁶ kg./cm.²*

Solution. *Maximum stress*

$$=f=\frac{3}{2}\frac{Wl}{nbt^2}$$

$$\therefore \quad t^2=\frac{3}{2}\frac{Wl}{nbf}$$

$$\therefore \quad t^2=\frac{3}{2}\frac{600\times 80}{12\times 4\times 2000}$$

$$\therefore \quad t=0{\cdot}866 \text{ cm. say } 0{\cdot}9 \text{ cm.}$$

Central deflection

$$=\delta=\frac{3}{8}\frac{Wl^3}{Enbt^3}$$

$$=\frac{3}{8}\frac{600\times 80^3}{2\times 10^6\times 12\times 4(0{\cdot}9)^3} \text{ cm.}$$

$$\therefore \quad \delta=1{\cdot}65 \text{ cm.}$$

78. Conjugate Beam Method

We have earlier discussed the slopes and deflections of beams subjected to an external loading by various methods like moment area method, Macaulay's method*, *etc.* The methods discussed earlier were convenient for cases where the beam is of uniform flexural rigidity. If the flexural rigidity is not uniform throughout the length of the beam, the methods discussed earlier are very

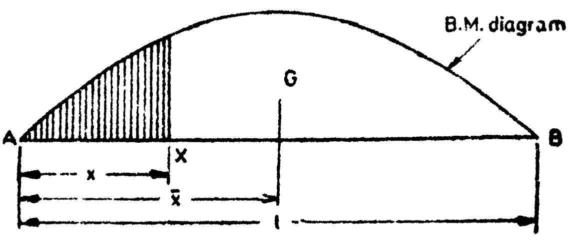


Fig. 477

laborious. The *conjugate beam method* presents a very easy approach and the student will find its application very interesting. Before this ingenious method is introduced, it will be worthy to go through the following discussion.

Fig. 477 shows the bending moment diagram for a beam *AB* with supports at *A* and *B*.

Taking *A* as origin, consider any section *X* distant *x* from *A*. Let *M* be the bending moment at *X*.

*See Chapter 8.

We have, with the usual notations,

$$EI\frac{d^2y}{dx^2}=M$$

Integrating from 0 to x, we have,

$$EI\left[\frac{dy}{dx}\right]_0^x=EI(i_x-i_a)=\int_0^x Mdx=a_x$$

where a_x is the area of the B.M. diagram from A to X.

$$\therefore \qquad i_x=i_a+\frac{a_x}{EI} \qquad \ldots(i)$$

Again, $\qquad EI\frac{d^2y}{dx^2}=M$

Multiplying by x,

$$EIx\frac{d^2y}{dx^2}=Mx.$$

Integrating for the whole range from A to B, we get

$$EI\int_0^L x\frac{d^2y}{dx^2}dx=\int_0^L Mxdx$$

$$\therefore \qquad EI\left\{x\frac{dy}{dx}-y\right\}_0^L = \text{Moment of the whole B.M. diagram about } A.$$

$$=a\bar{x}$$

where a = area of the B.M. diagram from A to B

$\bar{x}$ = centroidal distance of this diagram from A.

We know,

At $\qquad x=0, \qquad y=0$

At $\qquad x=l, \qquad y=0,$

and $\qquad \frac{dy}{dx}=i_b$

Substituting the limits, we get

$$EIl\, i_b=a\bar{x}$$

$$\therefore \qquad i_b=\frac{a\bar{x}}{EIl} \qquad \ldots(ii)$$

Similarly it can be shown,

$$i_a=-\frac{a(l-\bar{x})}{EIl} \qquad \ldots(iii)$$

Again,

$$EI\frac{d^2y}{dx^2}=M$$

$$\therefore \qquad EI\,x\frac{d^2y}{dx^2}=Mx$$

Integrating from 0 to x,

$$EI\int_0^x x\frac{d^2y}{dx^2}dx=\int_0^x Mx\,dx$$

$$\therefore \quad EI\left[x\frac{dy}{dx}-y\right]_0^x=a_x\bar{x}_x$$

= Moment of the area of B.M. diagram between A and X about A

where a_x = area of B.M. diagram between A and X

$\bar{x}_x$ = centroidal distance of this diagram from A.

Substituting the limits, we get

$$EI(xi_x-y)=a_x\bar{x}_x$$

$$\therefore \qquad y=xi_x-\frac{a_x\bar{x}_x}{EI}$$

But from equation (i) $i_x=i_a+\dfrac{a_x}{EI}$

$$\therefore \qquad y=x\left(i_a+\frac{a_x}{EI}\right)-\frac{a_x\bar{x}_x}{EI}$$

$$=xi_a+a_x\frac{(x-\bar{x}_x)}{EI}$$

It can be realized that $a_x(x-\bar{x}_x)$ is the moment of the area a_x about X.

$$y-xi_a=\frac{\text{moment of } a_x \text{ about } X}{EI} \qquad ...(iv)$$

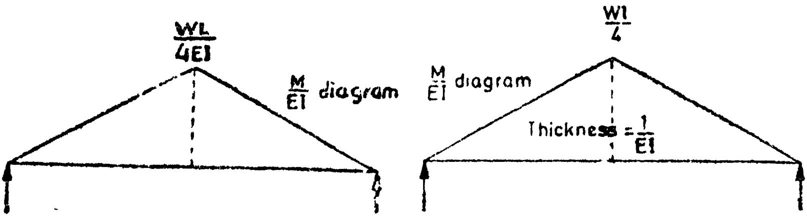
Summary of results,

$$i_a=-\frac{a(l-\bar{x})}{EIL}=-\left(\frac{a}{EI}\right)\left(\frac{l-\bar{x}}{l}\right) \qquad ...(1)$$

$$i_b=\frac{a\bar{x}}{EIL}=\left(\frac{a}{EI}\right)\frac{\bar{x}}{L} \qquad ...(2)$$

$$i_x=i_a+\frac{}{EI} \qquad ...(3)$$

and $$y=xi_x+\text{Moment of }\left(\frac{a_x}{EI}\right)\text{ about } X \qquad ...(4)$$



(a) Conjugate beam Fig. 480 (b) Conjugate beam

diagram is $\frac{Wl}{4}$ while the thickness of the diagram is $\frac{1}{EI}$. If this method is adopted the total load on the conjugate beam = volume of the load diagram = area of the load diagram × thickness.

It will be found convenient to introduce the $\frac{M}{EI}$ diagram in this way. This method will be adopted in the examples that follow.

Problem 303. *A beam of length l is simply supported at the ends and carries a concentrated load W at a distance a from each end. Find the slope at each end and under each load. Find also the deflection under each load and at the centre.*

Solution. Fig. 481 (*a*) shows a beam *AB* of span *l* carrying point loads *W* each at *D* and *E* distant *a* from the supports.

Fig. 481 (*b*) shows the B.M. diagram for the beam.

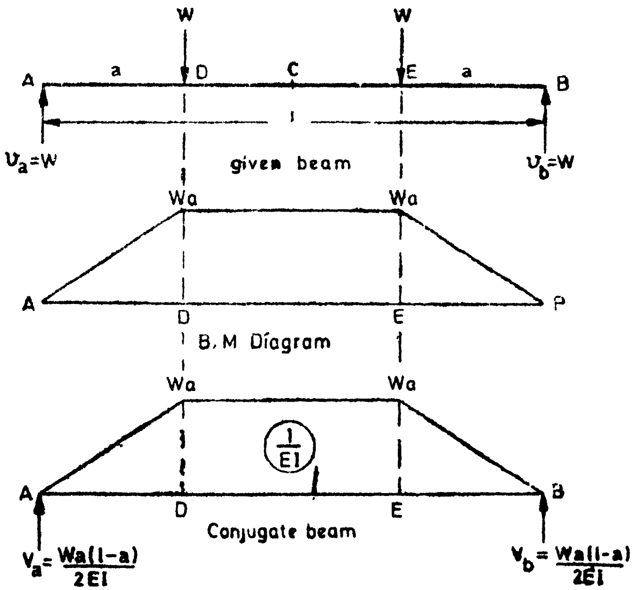


Fig. 481

Fig. 481 (c) shows the conjugate beam carrying the loading corresponding to the $\frac{M}{EI}$ *diagram*. This $\frac{M}{EI}$ *diagram* is practically the same as the *M diagram* except in the case of the $\frac{M}{EI}$ diagram, we should imagine a thickness of $\frac{1}{EI}$.

Total load on the conjugate beam

$$=\text{Volume of the load diagram}$$

$$=2\times\left\{\frac{1}{2}\,a.Wa\times\frac{1}{EI}\right\}+(l-2a)\,Wa\times\frac{1}{EI}$$

$$=\frac{Wa^2}{EI}+\frac{Wa(l-2a)}{EI}$$

$$=\frac{Wa(l-a)}{EI}$$

$\therefore$ Reaction at each support for the conjugate beam

$$=\frac{1}{2}\text{ the total load}$$

$$=\frac{Wa(l-a)}{2EI}$$

Slope at each end of the *given beam*

=Shear force at each end of the *conjugate beam*

$$=\frac{Wa(l-a)}{2EI}$$

Slope under each load of the *given beam*

= Shear force under each load of *conjugate beam*

$$=\frac{Wa(l-a)}{2EI}-\frac{1}{2}\,a\,Wa\,\frac{1}{EI}$$

$$=\frac{Wa(l-a)}{2EI}-\frac{Wa^2}{2EI}$$

$$=\frac{Wa(l-2a)}{2EI}$$

Deflection under each load of the *given beam*

=B.M. under each load of the *conjugate beam*

$$=\frac{Wa(l-a)}{2EI}\,a-\frac{1}{2}\,a.Wa.\,\frac{1}{EI}\,\frac{a}{3}$$

$$=\frac{Wa^2(l-a)}{2EI}-\frac{Wa^3}{6EI}$$

$$=\frac{Wa^2}{6EI}\,(3l-3a-a)$$

$$=\frac{Wa^2(3l-4a)}{6EI}$$

Deflection at the centre of the *given beam*

= B.M. at the centre of the *conjugate beam*

$$= \frac{Wa(l-a)}{2EI} \cdot \frac{l}{2} - \frac{1}{2}\, a\, Wa\, \frac{1}{EI}\left(\frac{l}{2} - \frac{2}{3}\, a\right) - \frac{(l-2a)}{2}\, \frac{Wa}{EI} \cdot \frac{l-2a}{4}$$

$$= \frac{Wal(l-a)}{4EI} - \frac{Wa^2}{12EI}(3l-4a) - \frac{Wa(l-2a)^2}{8EI}$$

$$\frac{Wa}{24EI}\,[6l(l-a) - 2a(3l-4a) - 3(l-2a)^2]$$

$$= \frac{Wa}{24EI}(3l^2 - 4a^2)$$

Problem 304. *A beam AB of span l is simply supported at A and B and carries a point load W at the centre C of the span. The moment of inertia of the beam section is I for the left half and 2I for the right half. Calculate the slope at each end and the centre and the deflection at the centre.*

Solution. Fig. 482 (*a*) shows the given beam.

Fig. 482 (*b*) shows the bending moment diagram for the given beam (*M*-diagram)

Fig. 482 (*c*) shows the conjugate beam.

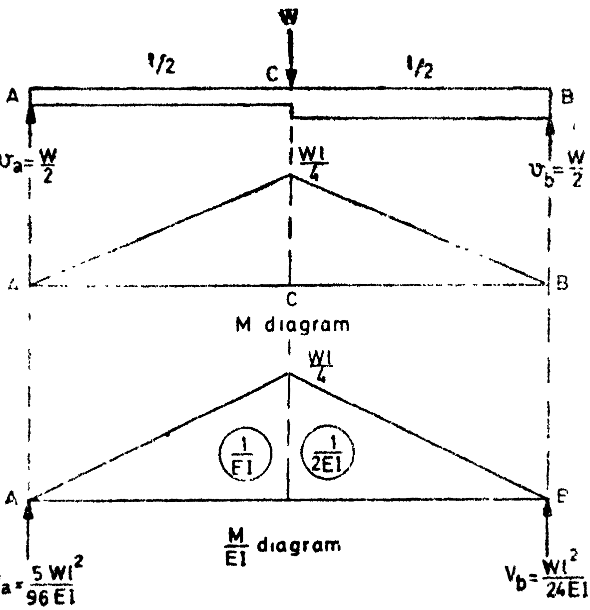


Fig. 482

The load diagram for the conjugate beam is given by the $\frac{M}{EI}$ diagram. The thickness of this diagram is $\left(\frac{1}{EI}\right)$ for the left half and $\left(\frac{1}{2EI}\right)$ for the right half.

Properties of loads on the conjugate beam are shown below.

(*Note : Volume of load diagram = load*)

Load component	*Magnitude of load*	*Distance from A*	*Moment about A*
$\frac{1}{2}\cdot\frac{l}{2}\cdot\frac{Wl}{4}\cdot\frac{1}{EI}$	$\frac{Wl^2}{16EI}$	$\frac{2}{3}\cdot\frac{l}{2}=\frac{l}{3}$	$\frac{Wl^3}{48EI}$
$\frac{1}{2}\cdot\frac{l}{2}\cdot\frac{Wl}{4}\cdot\frac{1}{2EI}$	$\frac{Wl^2}{32EI}$	$\frac{l}{2}+\frac{1}{3}\cdot\frac{l}{2}=\frac{2}{3}l$	$\frac{Wl^3}{48EI}$
Total	$\frac{3}{32}\frac{Wl^2}{EI}$		$\frac{Wl^3}{24EI}$

Let V_a and V_b be the reactions at A and B for the conjugate beam.

Taking moments about A, we have,

$$V_b \cdot l=\frac{Wl^3}{24EI}$$

$$\therefore \quad V_b=\frac{Wl^2}{24EI}$$

But total load on the conjugate beam $=\frac{3}{32}\frac{Wl^2}{EI}$

$$\therefore \quad V_a=\frac{3}{32}\frac{Wl^2}{EI}-\frac{Wl^2}{24EI}=\frac{5}{96}\frac{Wl^2}{EI}$$

Now we can easily determine the slope and deflection for the given beam.

Slope at A for the given beam = Shear force at A for the conjugate beam $=\frac{5}{96}\frac{Wl^2}{EI}$

Slope at B for the given beam = Shear force at B for the conjugate beam $=\frac{Wl^2}{24EI}$

Slope at C for the given beam = Shear force at C for the conjugate beam $=\frac{5}{96}\frac{Wl^2}{EI}-\frac{Wl^2}{16EI}$

$$=-\frac{Wl^2}{96EI}$$

$$\frac{Wl^2}{96EI} \text{ (numerically)}$$

Deflection at C for the given beam = B.M. at C for the conjugate beam

$$= \frac{5}{96}\frac{Wl^2}{EI}\cdot\frac{l}{2} - \frac{Wl^2}{16EI}\cdot\left(\frac{1}{3}\cdot\frac{l}{2}\right)$$

$$= \frac{Wl^3}{64EI}$$

Problem 305. *A beam ABCD is simply supported at its ends A and D over a span of 30 metres. It is made up of three portions AB, BC and CD each 10 metres in length. The moment of inertia of the section of these portions are I, 3I and 2I respectively, where $I=2\times10^6$ cm^4. The beam carries a point load of 15t at B and a point load of 30t at C. Neglecting the weight of the beam calculate the slopes and deflections at A, B, C and D. Take $E=2\times10^3$ t/cm^2.*

Solution. Fig. 483 (*a*) shows the given beam. Let v_a and v_d be the reactions at the supports. Taking moments about A, we have

$$v_d\times30=15\times10+30\times20$$

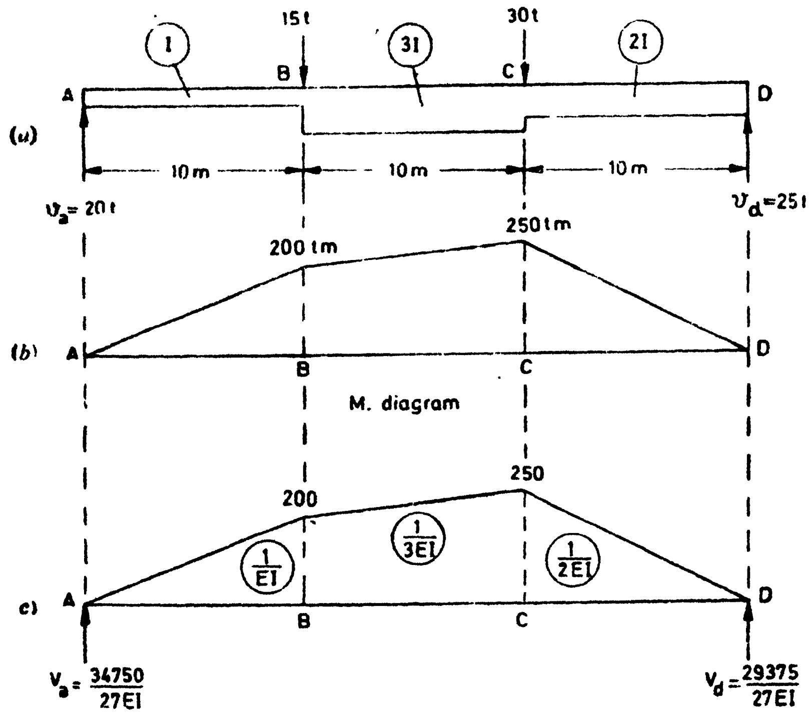


Fig. 483

$\therefore$ $v_d=25t$ and $v_a=45-25=20t$

B.M. at B $=20\times10=200$ *tm*.

B.M. at C $=25\times10=250$ *tm*.

Fig. 483 (b) shows the B.M. diagram for the given beam.

Fig. 483 (c) shows the $\frac{M}{EI}$ diagram which is the loading on the conjugate beam. The thickness on the diagram is $\frac{1}{EI}$ for the portion AB, $\frac{1}{3EI}$ for the portion BC and $\frac{1}{2EI}$ for the portion CD The properties of the loads on the conjugate beam are given below :

(*Note : Volume of load diagram = load*)

Load component	*Magnitude of load*	*Distance from A*	*Moment about A*
Load on AB			
$\frac{1}{2}\times10\times200\times\frac{1}{EI}$	$\frac{1000}{EI}$	$\frac{20}{3}$	$\frac{20000}{3EI}$
Load on BC			
$200\times10\times\frac{1}{3EI}$	$\frac{2000}{3EI}$	15	$\frac{10000}{EI}$
$\frac{1}{2}\times10\times50\times\frac{1}{3EI}$	$\frac{250}{3EI}$	$\frac{50}{3}$	$\frac{12500}{9EI}$
Load on CD			
$\frac{1}{2}\times10\times250\times\frac{1}{2EI}$	$\frac{625}{EI}$	$\frac{70}{3}$	$\frac{43750}{3EI}$
Total	$\frac{7125}{3EI}$		$\frac{293750}{9EI}$

Let V_a and V_d be the reactions at A and D for the conjugate beam.

Taking moments about A, we have,

$$V_d\times30=\frac{293750}{9EI}$$

$$\therefore \quad V_d=\frac{29375}{27EI}$$

$$\therefore \quad V_a=\frac{7125}{3EI}-\frac{29375}{27EI}=\frac{34750}{27EI}$$

Now we can easily determine the slopes and deflections at A, B, C, D for the given beam.

Slope at A for the given beam

= S.F. at A for the conjugate beam

$$=\frac{34750}{27EI}$$

$$=\frac{34750\times(100)^2}{27\times2\times10^3\times2\times10^6}$$

$$=0{\cdot}003218$$

Slope at B for the given beam

$=$S.F. at B for the conjugate beam

$$=\frac{34750}{27EI}-\frac{1000}{EI}$$

$$=\frac{7750}{27EI}$$

$$=\frac{7750\times(100)^2}{27\times2\times10^3\times2\times10^6}$$

$$=0{\cdot}0007176$$

Slope at C for the given beam

$=$S.F. at C for the conjugate beam

$$=\frac{29375}{27EI}-\frac{625}{EI}$$

$$=\frac{12500}{27EI}$$

$$=\frac{12500\times(100)^2}{27\times2\times10^3\times2\times10^6}$$

$$=0{\cdot}001157$$

Slope at D for the given beam

$=$S.F. at D for the conjugate beam

$$=\frac{29375}{27EI}$$

$$=\frac{29375\times(100)^2}{27\times2\times10^3\times2\times10^6}$$

$$=0{\cdot}00272$$

Deflection at A for the given beam

$=0$

Deflection at B for the given beam

$=$B.M. at B for the conjugate beam

$$=\frac{34750}{27EI}\times10-\frac{1000}{EI}\times\frac{10}{3}$$

$$=\frac{257500}{27EI}$$

$$=\frac{257500\times(100)^3}{27\times2\times10^3\times2\times10^6}\ cm.$$

$$=2{\cdot}384\ cm.$$

Deflection at C for the given beam

$$=\text{B.M. at } C \text{ for the conjugate beam}$$

$$=\frac{29375}{27EI}\times 10-\frac{625}{EI}\times\frac{10}{3}$$

$$=\frac{237500}{27EI}$$

$$=\frac{237500\times(100)^3}{27\times 2\times 10^3\times 2\times 10^6}\ cm.$$

$$=2{\cdot}199\ cm.$$

Relation between the given beam and conjugate beam

The relations between the given beam and the corresponding conjugate beam for different conditions are shown in the table on page 518.

We will now analyse beams with different end conditions.

Problem 306. *A cantilever of length l carries a point load W at the free end. Calculate the slope and deflection at the free end.*

Solution. Fig. 484 (a) shows the cantilever AB carrying the load W at B.

Fig. 484 (b) shows the M-diagram for the cantilever.

Fig. 484 (c) shows the corresponding conjugate beam carrying the $\frac{M}{EI}$ loading.

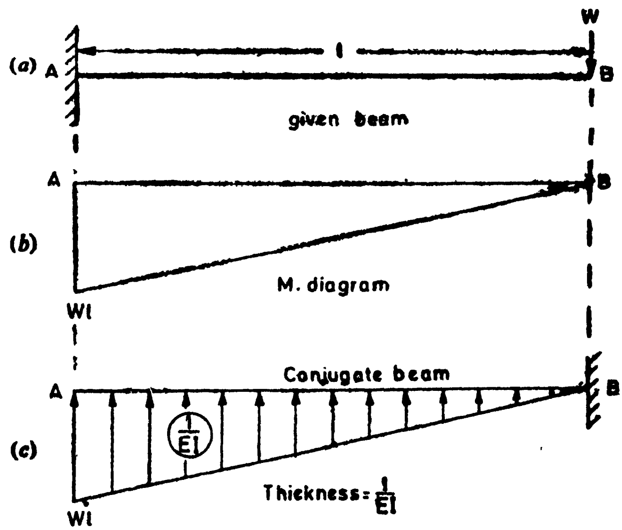


Fig. 484

Slope at B for the given beam

$$=\text{S.F. at } B \text{ for the conjugate beam}$$

$$=\frac{1}{2}l\,.\,Wl\,.\,\frac{1}{EI}=\frac{Wl^2}{2EI}$$

	Given beam	*Conjugate beam*
1	Slope at any section	S.F. at the corresponding section
2	Deflection at any section	B.M. at the corresponding section
3.	Given system of loading	The loading diagram becomes the $\frac{M}{EI}$ diagram
4	Roller Support  Slope exists, but deflection=0	 S.F. exists and B.M =0
5	Hinged support  Slope exists, but deflection=0	 S F exists and B.M.=0
6.	Free end  Slope exists and deflection exists	 S.F. exists and B.M. exists
7	Fixed end  Slope=0 Deflection=0	 Free S.F.=0 B.M.=0
8	(*a*) B M diagram positive (sagging) (*b*) B.M. diagram negative (hogging)	$\frac{M}{EI}$, load diagram is positive that is, the loading is downward $\frac{M}{EI}$, load diagram is negative that is the loading is upward

Deflection at B for the given beam

$$= \text{B.M. at } B \text{ for the conjugate beam}$$

$$= \frac{Wl^2}{2EI} \cdot \frac{2}{3} l$$

$$= \frac{Wl^3}{3EI}$$

Problem 307. *A cantilever of length l carries a point load W at a distance l_1 from the fixed end. Calculate the slope and deflection at the free end.*

Solution. Fig. 485 (a) shows a cantilever AB of length l fixed at A and carrying a point load W at a distance l_1 from A.

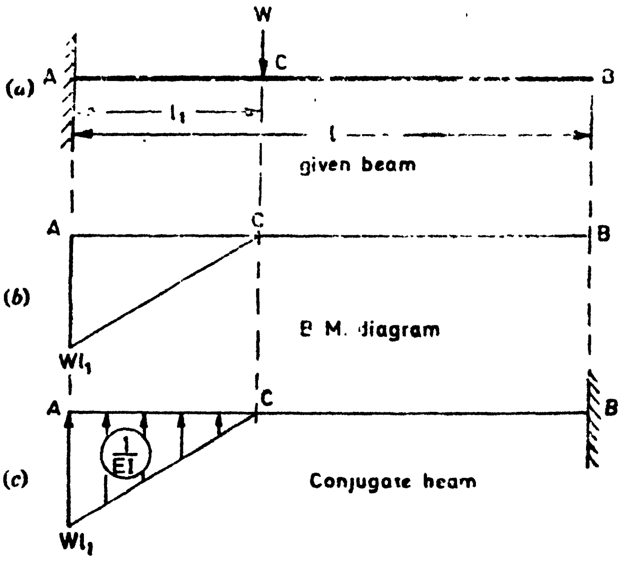


Fig. 485

Fig. 485 (b) shows the B.M. diagram for the cantilever.

Fig. 485 (c) shows the corresponding conjugate beam whose load diagram is the $\frac{M}{EI}$ diagram.

Slope at B for the given beam

$$= \text{S.F. at } B \text{ for the conjugate beam}$$

$$= \frac{1}{2} l_1 \, Wl_1 \frac{1}{EI} = \frac{Wl_1^2}{2EI}$$

Deflection at B for the given beam

$$= \text{B.M. at } B \text{ for the conjugate beam}$$

$$= \frac{Wl_1^2}{2EI}\left(l - \frac{l_1}{3}\right)$$

Problem 308. *A cantilever of length l carries a uniformly distributed load of w per unit run over the whole length. Calculate the slope and deflection at the free end.*

Solution. Fig. 486 (*a*) shows a cantilever *AB* of length *l* carrying a uniformly distributed load of *w* per unit run over the whole length.

Fig. 486 (*b*) shows the B.M. diagram for the cantilever.

Fig. 486 (*c*) shows the corresponding conjugate beam whose load diagram is the $\frac{M}{EI}$ diagram.

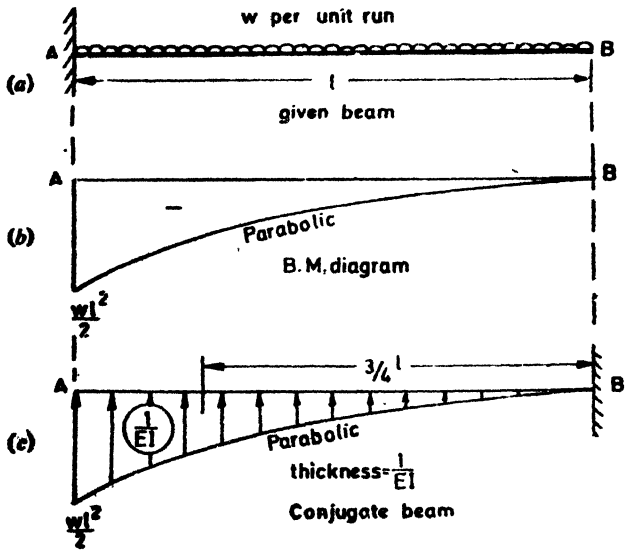


Fig. 486

Slope at *B* for the given beam

=S.F. at *B* for the conjugate beam

=Volume of load diagram

=Area × thickness

=⅓ base × altitude × thickness

$$=\frac{1}{3}\,l\,\frac{wl^2}{2}\cdot\frac{1}{EI}=\frac{wl^3}{6EI}$$

Deflection at *B* for the given beam

=B.M. at *B* for the conjugate beam

$$=\frac{wl^3}{6EI}\cdot\frac{3}{4}\,l=\frac{wl^4}{8EI}.$$

Problem 309. *A cantilever of length l is subjected to a couple M_0 at the free end. Calculate the slope and deflection at the free end.*

Solution. Fig. 487 (*a*) shows a cantilever *AB* fixed at *A*. The free end *B* is subjected to a couple M_o.

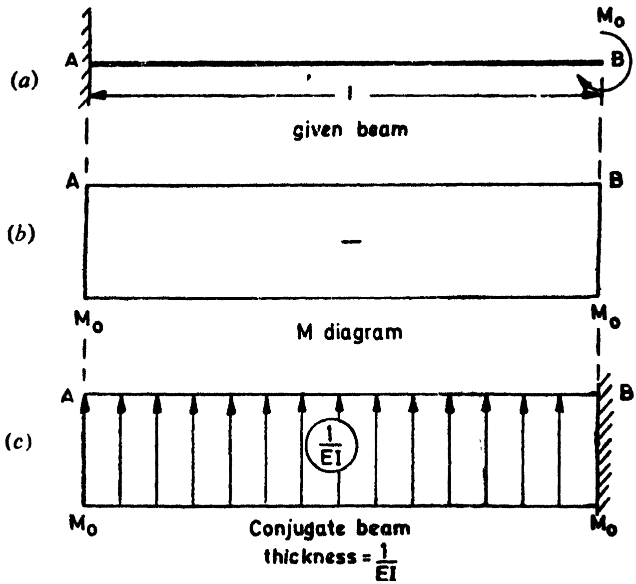


Fig. 487

Fig. 487 (*b*) shows the B.M. diagram for the cantilever.

Fig. 487 (*c*) shows the corresponding conjugate beam carrying the $\frac{M}{EI}$ loading.

Slope at *B* for the given beam = S.F. at *B* for this conjugate beam

$$= M_o l \times \frac{1}{EI} = \frac{M_o l}{EI}$$

Deflection at *B* for the given beam = B.M. at *B* for the conjugate beam

$$= \frac{M_o l}{EI} \cdot \frac{l}{2}$$

$$= \frac{M_o l^2}{2EI}.$$

Problem 310. *A cantilever of length l is subjected to a couple M_o at a distance l_1 from the fixed end. Find the slope and deflection at the free end.*

Solution. Fig 488 (*a*) shows a cantilever *AB* of length *l* fixed at *A*. Let a couple M_o be applied at *C* distant l_1 from *A*.

Fig. 488 (*b*) shows the B.M. diagram for the cantilever.

Fig. 488 (*c*) shows the corresponding conjugate beam whose load diagram is the $\frac{M}{EI}$ diagram.

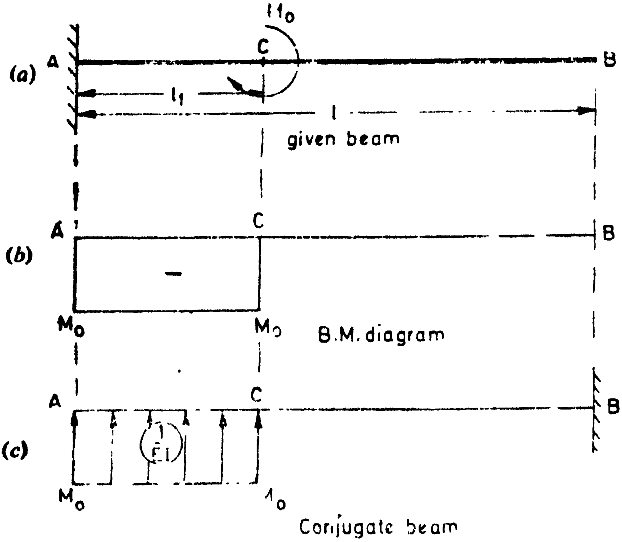


Fig. 488

Slope at B for the given beam = S.F. at B for the conjugate beam

$$=\frac{M_0 l_1}{EI}$$

Deflection at B for the given beam = B.M. at B for the conjugate beam

$$=\frac{M_0 l_1}{EI}\left(l-\frac{l_1}{2}\right)$$

Problem 311. *A propped cantilever AB of length l fixed at A carries a point load W at midspan. Find the reaction at the prop.*

Solution. Fig 489 (*a*) shows the propped cantilever carrying the point load at midspan.

Let the reactions at the fixed end A and the propped end B be v_a and v_b respectively.

Fig. 489 (*b*) is the B M. diagram when the prop is absent.

Fig. 489 (*c*) is the B.M. diagram due to prop reaction alone. The above two B.M. diagrams together constitute the final B M. diagram.

Fig. 489 (*d*) shows the corresponding conjugate beam (note the end conditions for the conjugate beam) whose load diagram is the $\frac{M}{EI}$ diagram. For the conjugate beam, since the B.M. at B

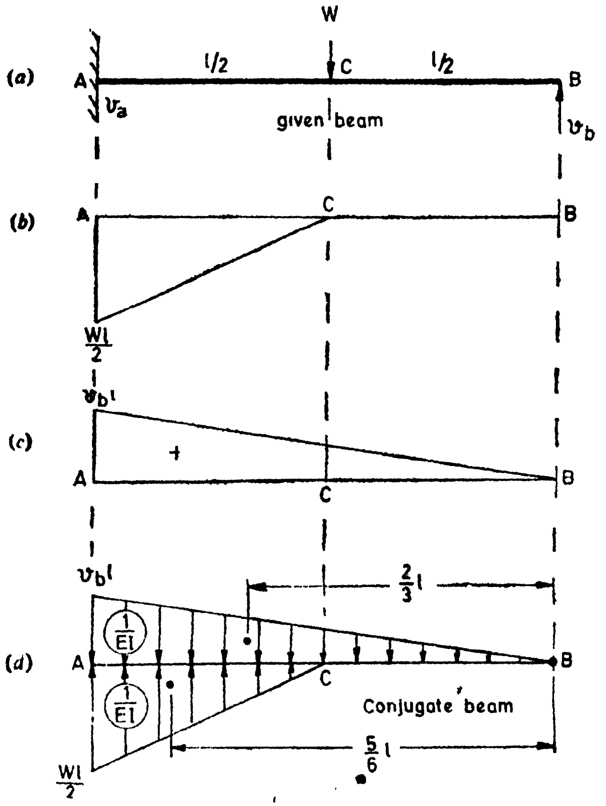


Fig. 489

equals zero, taking moments about B, we have

$$\left(\frac{1}{2}\ l v_b l\ \frac{1}{EI}\right)\frac{2}{3}\ l=\left(\frac{1}{2}\ \frac{l}{2}\ \frac{Wl}{2}\ \frac{1}{EI}\right)\frac{5}{6}\ l$$

$$\therefore \qquad v_b=\frac{5}{16}\ W$$

Problem 312. *A beam AB of span l is fixed at A and freely supported at B. For the left half AC the moment of inertia of the section is 2I, while for the right half it is I. A clockwise couple M is applied at the end B. Find the prop reaction and the slope at this end. Find also the B.M. at the fixed end.*

Solution. Fig. 490 (a) shows the given beam AB. Let v_b be the reaction at B.

Figs. 490 (b) and (c) show the B.M. diagrams due to the separate effects of M_o and v_b.

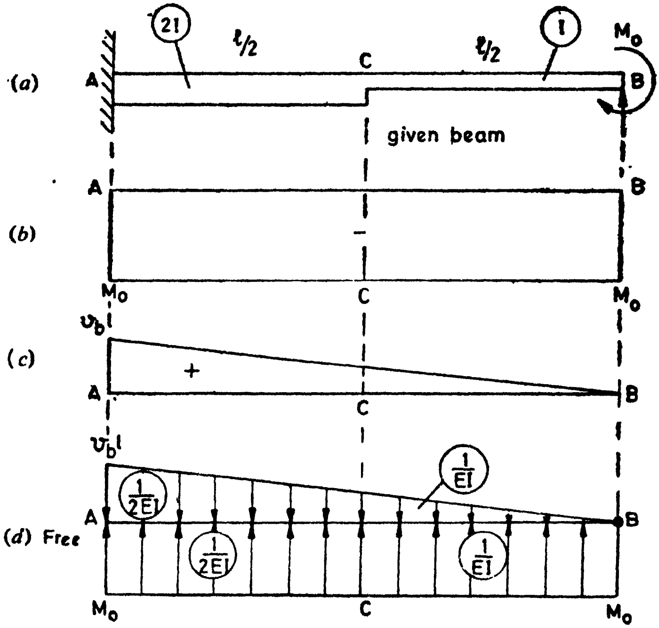


Fig. 490

Fig. 490 (*d*) shows the corresponding conjugate beam, the load diagram for which is the $\frac{M}{EI}$ diagram.

The thickness of the diagram is $\frac{1}{2EI}$ for the left half and $\frac{1}{EI}$ for the right half.

Note the end conditions for the conjugate beam.

For the conjugate beam, taking moments about B, we have,

$$M_o l \frac{1}{2EI}\,\frac{l}{2}+M_o\,\frac{l}{2}\,\frac{1}{2EI}\,\frac{l}{4}$$

$$=\frac{1}{2}\,lv_b l.\,\frac{1}{2EI}\cdot\frac{2}{3}\,l$$

$$+\frac{1}{2}\cdot\frac{l}{2}\,\frac{v_b l}{2}\cdot\frac{1}{2EI}\cdot\frac{2}{3}\cdot\frac{l}{2}$$

$$\therefore \qquad v_b=\frac{5}{3}\,\frac{M_o}{l}$$

Slope at B for the given beam = S.F. at B for the conjugat beam

$$=M_o l\,\frac{1}{2EI}+M_o\frac{l}{2}\,\frac{1}{2EI}-\frac{1}{2}\,lv_b l\,\frac{1}{2EI}-\frac{1}{2}\cdot\frac{l}{2}\,\frac{v_b l}{2}\cdot\frac{1}{2EI}$$

$$=\frac{3}{4}\frac{M_o l}{16}-\frac{5}{16}\frac{v_b l^2}{EI}=\frac{3}{4}\frac{M_o l}{EI}-\frac{5}{16}\cdot\frac{5}{3}\frac{M_o l}{EI}$$

$$=\frac{11}{48}\frac{M_o l}{EI}$$

Now for the given beam, B.M. at A

$$=v_b l-M_o=\frac{5}{3}M_o-M_o=\frac{2}{3}M_o$$

Examples in Chapter 8

1. A cast iron beam 4 *cm.* wide and 8 *cm.* deep is placed od supports 1·25 *metres* apart and is subjected to a central point load 3000 *kg.* If the central deflection is found to be 6·5 *mm.*, find the value of the Young's Modulus for the material.

($1\cdot 1\times 10^6$ *kg./cm.*^{2})

2. Find the uniform bending moment which is to be applied to a steel rod 18 *mm.* diameter so as to bend it into a circular arc of 18 *metres* radius.

If the bar is 3·25 *metres* long, find the central deflection. Take $E=2\times 10^6$ *kg./cm.*^{2}

3. A cantilever of uniform section has a length l. It is propped at the free end and carries a point load W at a distance a from the fixed end.

(*a*) If the prop holds the free end at the level of the fixed end, find the prop recti on.

(*b*) If now the prop is removed what will be the deflection at the free end ?

$$\left[\frac{Wa^2}{2l^3}(3l-a);\ \frac{Wa^2}{6EI}(3l-a)\right]$$

4. A horizontal cantilever of uniform section and length l carries two point loads, W at the free end and $2W$ at a distance a from the free end. Find the deflection at the free end.

$$\left[\frac{W}{3EI}\{3l^2(l-a)+a^3\}\right]$$

5. A beam of span $2l$ simply supported at the ends carries two equal loads W symmetrically placed at a distance of $\frac{l}{3}$ on either side of midspan Find the maximum deflection.

$$\left[\frac{23}{81}\frac{Wl^3}{EI}\right]$$

6. A horizontal cantilever of uniform section of length l carries two vertical point loads W_1 and W_2. W_1 acts upwards at the free end and W_2 acts downwards at a distance a from the fixed end. Find the end deflection.

$$\left[\frac{1}{6EI}\{2W_1l^3-W_2a^2(3l-a)\}\right]$$

7. A beam simply supported at the ends has a span l. It is subjected to equal and opposite end couples M_o. Find the slope at the ends and the central deflection.

$$\left[\frac{Ml}{2EI};\ \frac{Ml^2}{8EI}\right]$$

8. A beam simply supported at the ends having a span l carries three point loads W each symmetrically placed on the span at intervals of $\frac{l}{4}$. Find the central deflection and the slopes at the ends.

$$\left[\frac{19}{384}\frac{Wl^3}{EI};\ \frac{5}{32}\frac{Wl^2}{EI}\right]$$

9. A cantilever of length l carrying a point load W at the free end is propped at a distance a from the fixed end to the same level as the fixed end. Find the load on the prop.

Show that there is always a real point of inflexion and find its distance from the fixed end.

$$\left[\frac{W}{2a}(3l-a);\ \frac{a}{3}\right]$$

10. A cantilever of length l carries a uniformly distributed load of w per unit run for a distance $\frac{3}{4}l$ from the fixed end. Find the slope and the deflection at the free end.

$$\left[\frac{9}{128}\frac{wl^3}{EI};\ \frac{117}{2048}\frac{wl^4}{EI}\right]$$

11. A beam of uniform section of length l is loaded by its own weight only and is supported at two points with equal overhangs. Find the distance between the two supports.

(*a*) so that, with the supports at the same level the two ends of the beam remain horizontal.

(*b*) so that, the deviation from the straight is as small as possible. [0·5774 l; 0·554 l]

12. A beam of length $2l$ has one support at the left end where it is encasted while it is freely supported at a distance l from the left end. If the beam carries a concentrated load W midway between the two supports find the upward deflection of the right end.

$$\left(\frac{Wl^3}{32EI}\right).$$

13. A horizontal cantilever ACB l units long, is fixed at A and freely supported at C. It carries a uniformly distributed load of w per unit run over the entire length. Find the position of the supports so that the reactions at the two supports are equal.

(0·8375 l from the fixed end)

14. A cantilever of length l carries a total triangular load W whose intensity varies uniformly from zero at the free end to a maximum at the fixed end. Show that the deflection at any point distant x from the free end is given by

$$y=\frac{W}{60EIl^2}(4l^5-5l^4x+x^5)$$

15. A bridge across a river has a span $2l$ and is constructed with beams resting on the banks and supported at the middle on a pontoon. When the bridge is unloaded, the three supports are all at the same level, and the pontoon is such that the vertical displacement is equal to the load on it multiplied by a constant λ. Show that the load on the pontoon, due to a concentrated load W placed one-quarter of the way along the bridge, is given by

$$\frac{11W}{16\left(1+\frac{6EI\lambda}{l^3}\right)}$$

where I is the second moment of area of the section of the beams.

16. Two equal steel beams are built in at one end and connected by a steel rod as shown.

Show that the pull in the tie rod is given by

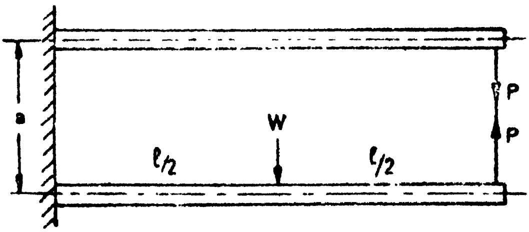


Fig. 491

$$P=\frac{5Wl^3}{32\left(\frac{6aI}{\pi d^2}+l^3\right)}$$

where d is the diameter of the rod, and I is the second moment of area of the section of each beam about its neutral axis.

17. The free end of a horizontal cantilever of length l is supported by a short vertical strut of area A and height h and of the same material as that of the cantilever. If the cantilever carries a total uniformly distributed load W over the whole length, find the load transmitted to the strut. Assume that the lower end of the strut does not sink

$$\left(\frac{3AWl^3}{8Al^3+24\ h}\right).$$

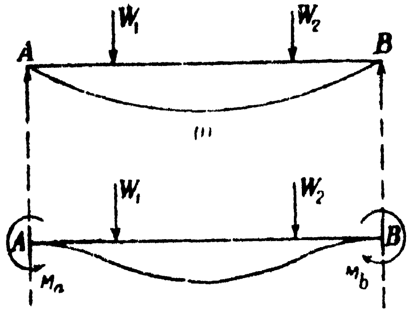
9

Fixed and Continuous Beams

§79. Fixed beams

A fixed beam is a beam whose end supports are such that the end slopes remain zero (or unaltered). Such a beam is also called a *built-in* or *encaster beam.*

Fig. 492 (*i*) shows a simply supported beam *AB* carrying an external load system. Obviously as the beam bends slopes i_a and i_b will occur at the ends *A* and *B*. If these slopes should be prevented it is necessary to apply end couples of certain definite magnitudes in the appropriate order. When the ends of the beam are bulit-in such end moments are automatically developed. Such end moments are called *Fixed End Moments*. If an end support is not able to provide sufficient *restraining* or *reacting* moment some slope will be produced at that support. But if there is absolute fixity at a support, the slope at the support will remain zero.



(*ii*)
Fig 492

§80. B.M. diagram for a fixed beam

Fig. 493 shows a fixed beam *AB* carrying an external load system. Let V_a and V_b be the vertical reactions at the supports *A* and *B*.

Let M_a and M_b be the fixed end moments. The beam may be analysed in the following stages.

(*i*) Let us first consider the beam as simply supported.

Let v_a and v_b be the reactions *A* and *B* for this condition. Fig. 493(*i*) *b* shows the bending moment diagram for this condition. At any section the bending moment M_x is a sagging moment.

(*ii*) Now let us consider the effect of end couples M_a and M_b alone. Let *V* be the reaction at each end due to this condition. Suppose $M_b > M_a$.

Then $V = \frac{M_b - M_a}{l}$. If $M_b > M_a$ the reaction V is upwards at B and downwards at A.

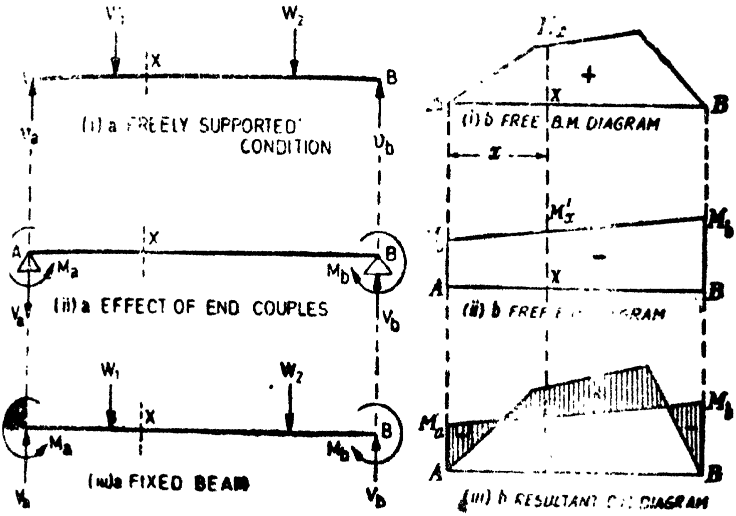


Fig. 493

Fig. 493 (*ii*) (*b*) shows the bending moment diagram for this condition. At any section the bending moment M_x' is a hogging moment.

Now the final bending moment diagram can be drawn by combining the above two B.M. diagrams as shown in Fig, 493 (*iii*) *b*.

The final reaction $V_a = v_a - V$

and $V_b = v_b + V$

The actual bending moment at any section X, distant x from the end A is given by,

$$EI\frac{d^2y}{dx^2} = M_x - M_x'$$

Integrating, we get,

$$EI\left[\frac{dy}{dx}\right]_0^l = \int_0^l M_x dx - \int_0^l M_x' dx$$

But at $x = 0, \frac{dy}{dx} = 0$

and at $x = l, \frac{dy}{dx} = 0$

Further $\int_0^l M_x dx$ = area of the Free B.M. diagram

$= a$

$\int_0^l M'_x dx$ = area of the Fixed B.M. diagram

$= a'$

Substituting in the above equation, we get,

$$0 = a - a'$$

$$\therefore \quad a = a'$$

∴ Area of the free B.M. diagram = Area of the fixed B.M. diagram.

Again consider the relation

$$EI\frac{d^2y}{dx^2} = M_x - M'_x.$$

Multiplying by x,

$$EIx\frac{d^2y}{dx^2} = M_x x - M_x'.x$$

Integrating, we get,

$$\int_0^l EIx\frac{d^2y}{dx^2}dx = \int_0^l M_x x\,dx - \int_0^l M'_x x dx$$

$$\therefore \quad EI\left[x\frac{dy}{dx} - y\right]_0^l = a\bar{x} - a'\bar{x}'$$

where $\bar{x}$ = distance of the centroid of the free B.M. diagram from A.

and $\bar{x}'$ = distance of the centroid of the fixed B.M. diagram from A.

Further, at $x = 0,\ y = 0$ and $\frac{dy}{dx} = 0$

and, at $x = l,\ y = 0$ and $\frac{dy}{dx} = 0$

Substituting in the above relation, we have,

$$0 = \bar{x}a - \bar{x}'a'$$

or $$a\bar{x} = a'\bar{x}'$$

or $$\bar{x} = \bar{x}'$$

∴ The distance of the centroid of the free B.M. diagram from A

= the distance of the centroid of the fixed B.M. diagram from A.

By using the conditions

$$a = a, \qquad \ldots(i)$$

and

$$\bar{x} = \bar{x}' \qquad \ldots(ii)$$

The unknowns M_a and M_b can be determined.

Let us now consider some standard cases.

Case (i). Fixed beam carrying a point load at midspan

Fig. 494 shows a fixed beam AB of span l carrying a point load W at midspan.

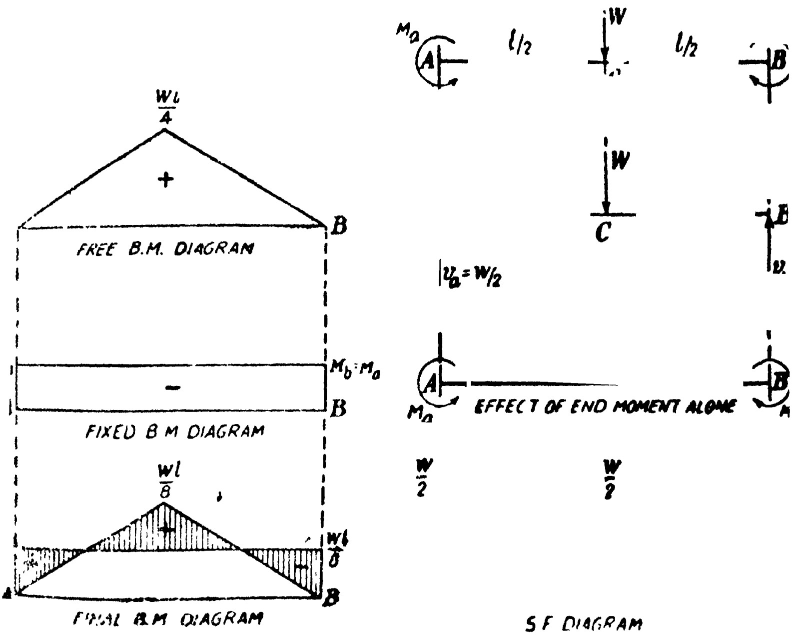


Fig. 494

Due to symmetry the end moments M_a and M_b are equal. Fig. 494 shows the free and fixed B.M. diagrams.

Equating the areas of the free and fixed B.M. diagrams, we have,

$$a' = a$$

$$M_a l = \frac{1}{2} l \cdot \frac{Wl}{4}$$

$$\therefore \quad M_a = \frac{Wl}{8}$$

$$\therefore \quad M_b = \frac{Wl}{8}$$

B.M. at midspan $=M_x-M'_x$

$$=\frac{Wl}{4}-\frac{Wl}{8}=+\frac{Wl}{8}$$

Due to symmetry the reactions V_a and V_b are equal

$$\therefore \quad V_a=V_b=\frac{W}{2}.$$

Now the S.F. and B.M. diagrams for the beam can be easily drawn.

Obviously two points of contraflexure occur at $\frac{l}{4}$ from the ends.

Slope and deflection

At any section in AC distant x from the end A, the B.M. is given by,

$$EI\frac{d^2y}{dx^2}=M_x-M'_x=\frac{W}{2}x-\frac{Wl}{8}$$

Integrating, we get,

$$EI\frac{dy}{dx}=\frac{Wx^2}{4}-\frac{Wl}{8}x+C_1 \quad (\textit{Slope equation})$$

at $x=0 \; \frac{dy}{dx}=0$

$\therefore \quad C_1=0$

Integrating again, we get,

$$EIy=\frac{Wx^3}{12}-\frac{Wlx^2}{16}+C_2 \quad (\textit{Deflection equation})$$

At $x=0,\ y=0$

$\therefore \quad C_2=0$

Maximum deflection occurs at midspan,

i.e., at $x=\frac{l}{2}.$

$$\therefore \quad EIy_c=\frac{W}{12}\left(\frac{l}{2}\right)^3-\frac{Wl}{16}\left(\frac{l}{2}\right)^2$$

$$=\frac{Wl^3}{96}-\frac{Wl^3}{64}=-\frac{Wl^3}{192}$$

$$\therefore \quad y_c=-\frac{Wl^3}{192\,EI}=\frac{1}{4} \text{ the deflection for a simply supported beam*}$$

Case (ii). Fixed beam carrying a uniformly distributed load of w per unit run over the whole span.

*It may be noted that for a simply supported beam carrying a point load W at midspan; the central deflection is $\frac{Wl^3}{48EI}$.

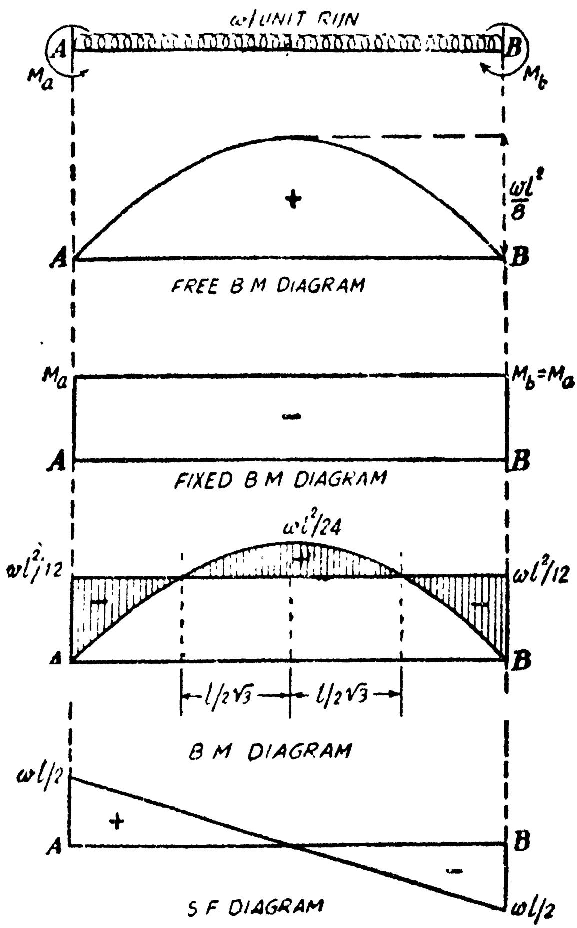


Fig. 495

Fig. 495 shows a fixed beam AB of span l carrying a uniformly distributed load w per unit run over the whole span.

By symmetry, the end moments M_a and M_b are equal.

The free B.M. diagram is a parabola whose central ordinate is $\frac{wl^2}{8}$.

Equating the areas of the fixed and free B.M. diagrams, we have,

$$a' = a$$

$$\therefore \quad M_a l = \frac{2}{3} \cdot l \cdot \frac{wl^2}{8}$$

$$\therefore \quad M_a = \frac{wl^2}{12} \text{ and } M_b = \frac{wl^2}{12}$$

Hence at any section distant x from the left end A the actual bending moment is given by

$$M = \text{Free B M.} - \text{Fixed B.M.}$$

$$M = \frac{wl}{2}x - \frac{wx^2}{2} - \frac{wl^2}{12}$$

For the points of contraflexure,

$$\frac{wl}{2}x - \frac{wx^2}{2} - \frac{wl^2}{12} = 0$$

$$\therefore \quad x^2 - lx + \frac{l^2}{6} = 0$$

Solving this quadratic we get,

$$x = \frac{l}{2} \pm \frac{l}{2\sqrt{3}}$$

Hence two points of contraflexure occur. These are equidistant from the centre of the span. Each point of contraflexure is at a distance of $\frac{l}{2\sqrt{3}}$ from the centre of the span.

Slope and deflection

The bending moment at any section is given by,

$$M = EI\frac{d^2y}{dx^2} = \frac{wl}{2}x - \frac{wx^2}{2} - \frac{wl^2}{12}$$

Integrating, we get,

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^2}{12}x + C_1 \quad (\textit{Slope equation})$$

At $\quad x = 0, \frac{dy}{dx} = 0$

$\therefore \quad C_1 = 0$

Integrating again, we get,

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^2x^2}{24} + C_2 \quad (\textit{Deflection equation})$$

At $\quad x = 0, y = 0$

$\therefore \quad C_2 = 0$

For the deflection at the centre, putting $x = \frac{l}{2}$ in the deflection equation, we get,

$$EIy_c = \frac{wl}{12}\left(\frac{l}{2}\right)^3 - \frac{w}{24}\left(\frac{l}{2}\right)^4 - \frac{wl^2}{24}\left(\frac{l}{2}\right)^2$$

$$= -\frac{wl^4}{384}$$

$$\therefore \qquad y_c = -\frac{wl^4}{384\,EI}$$

It may be remembered that a simply supported beam carrying the uniformly distributed load for over the whole span the central deflection is

$$\frac{5}{384}\,\frac{wl^4}{EI}$$

Hence the central deflection for the fixed beam

$= \frac{1}{5}$ of the central deflection of the simply supported beam.

Problem 313. *A fixed beam of 6 metres span supports two point*

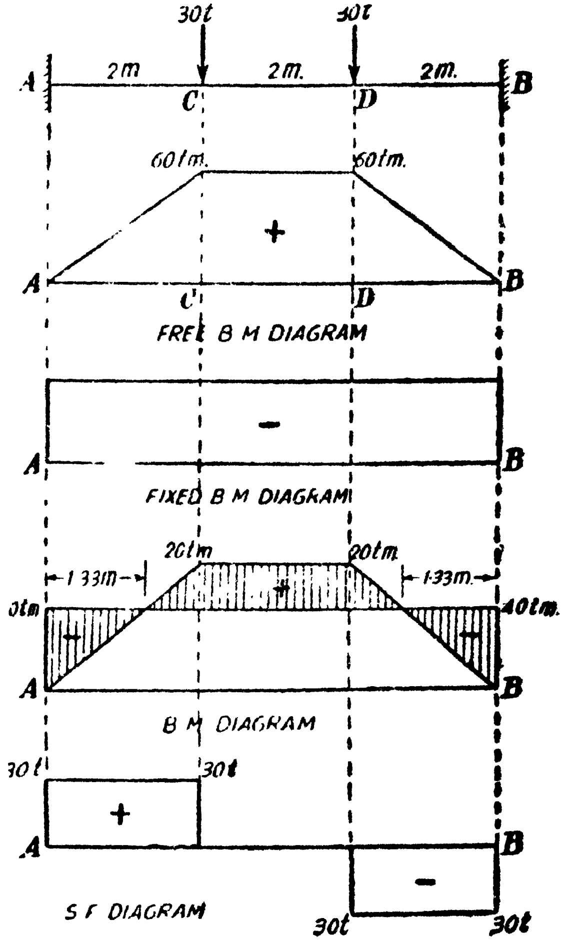


Fig. 496

loads of 30 tonnes each at 2 metres from each end. Find the fixing moments at the ends and draw the B.M. and S.F. diagrams. Find also the central deflection. Take $I=90,000$ cm.^{4} and $E=2\times10^6$ kg./cm^2.

Solution. Fig. 496 shows the fixed beam AB carrying the two point loads.

Due to symmetry the fixed moments M_a and M_b are equal. Fig. 496 shows the free and fixed B.M. diagrams.

Equating the areas of these diagrams, we have,

$$M_a\times6=\frac{1}{2}\left(60\right)\left(2+6\right)$$

$\therefore$ $$M_a=40 \ tm.$$

$\therefore$ B.M. at the centre

$$=60-40=20 \ tm.$$

Points of Contraflexure

Actual B.M. at any section in AC distant x from A is given by

$$M=\text{Free moment}-\text{Fixed moment}$$

$\therefore$ $$M=30x-40$$

For the point of contraflexure, $30x-40=0$

$\therefore$ $$x=\frac{4}{3} \ m \text{ from either end.}$$

Slope and deflection.

For any section between A and D distant x from A the bending moment is given by

$$EI\frac{d^2y}{dx^2}=30x-40 \ \vdots -30\ (x-2)$$

Integrating, we get,

$$EI\frac{dy}{dx}=15x^2-40x+C_1 \ \vdots -15\ (x-2)^2 \qquad \textit{(Slope equation)}$$

At $$x=0,\ \frac{dy}{dx}=0$$

$\therefore$ $$C_1=0$$

Integrating again, we get,

$$EIy=5x^3-20x^2+C_2 \ \vdots -5\ (x-2)^3 \qquad \textit{(Deflection equation)}$$

At $$x=0,\ y=0$$

$\therefore$ $$C_2=0$$

For the maximum deflection which occurs at the centre, putting $x=3\ m$ in the deflection, we get,

$$EI\,y_{max} = 5(3)^3 \quad 20(3)^2 - 5(3)^3 \quad -50$$

$$\therefore \qquad y_{max} = -\frac{50}{EI}$$

$$= -\frac{50}{2\times10^3\times90{,}000}\times(100)^3\ cm.$$

$$= -0{\cdot}278\ cm.$$

Case (iii). Fixed beam carrying a concentrated load eccentrically placed on the span.

Fig. 497 shows a fixed beam AB of span l carrying a point load W at C eccentrically on the span so that $AC=a$ and $BC=b$.

Obviously the free B.M. diagram is a triangle whose altitude is

$$\frac{Wab}{l}$$

Let M_a and M_b be the fixing moments at the ends.

Hence, the fixed B.M. diagram is trapezoidal.

Since the areas of the fixed and free B.M. diagrams are equal, we have,

$$\left(\frac{M_a+M_b}{2}\right) l=\frac{1}{2}\cdot l\cdot\frac{Wab}{l}$$

$$\therefore \qquad M_a+M_b=\frac{Wab}{l} \qquad \ldots(i)$$

We also know that the centroidal distances $\bar{x}'$ and $\bar{x}$ of the fixed and free B.M. diagrams from the end A should alse be equal.

We know, for the fixed B.M. diagram

$$\bar{x}'=\frac{M_a+2M_b}{M_a+M_b}\cdot\frac{l}{3} \text{ from } A$$

and, for the free B.M. diagram,

$$\bar{x}=\frac{l+a}{3} \text{ from } A$$

$$\therefore \qquad \frac{M_a+2M_b}{M_a+M_b}\cdot\frac{l}{3}=\frac{l+a}{3}$$

$$\therefore \qquad M_a+2M_b=(l+a)\frac{(M_a+M_b)}{l}$$

But $\qquad M_a+M_b=\dfrac{Wab}{l}$ from equation (i)

$$M_a+2M_b=\frac{Wab}{l^2}\,(l+a) \qquad \ldots(ii)$$

Subtracting equation (i) from equation (ii)

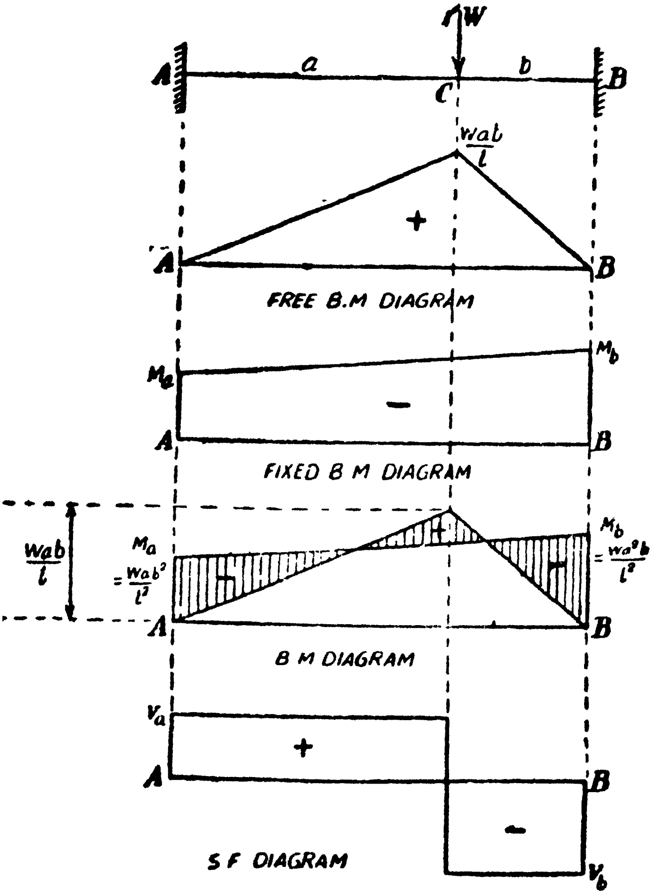


Fig. 497

$$M_b=\frac{Wab}{l^2}(l+a)-\frac{Wab}{l}$$

$$=\frac{Wab}{l^2}(l+a-l)$$

$$\therefore \quad M_b=\frac{Wa^2b}{l^2}$$

Putting $M_b=\dfrac{Wa^2b}{l^2}$ in equation (*i*)

We get,

$$M_a+\frac{Wa^2b}{l^2}=\frac{Wab}{l}$$

$$\therefore \quad M_a=\frac{Wab}{l}-\frac{Wa^2b}{l^2}$$

$$=\frac{Wab}{l^2}(l-a)$$

But $l-a=b$

$$\therefore \quad M_a=\frac{Wab^2}{l^2}$$

Thus the fixed end moments at A and B are,

$$M_a=\frac{Wab^2}{l^2}$$

and

$$M_b=\frac{Wa^2b}{l^2}$$

If $a>b$ then $M_b>M_a$

Slope and deflection

At any section distant x from the end A, the bending mome is given by,

$$EI\frac{d^2y}{dx^2}=\text{Free B.M.}-\text{Fixed B.M.}$$

$$=\frac{Wb}{l}x-\left(M_a+\frac{M_b-M_a}{l}x\right)-W(x-a)$$

But $M_a+\dfrac{M_b-M_a}{l}x$

$$=\frac{Wab^2}{l^2}+\frac{\frac{Wa^2b}{l^2}-\frac{Wab^2}{l^2}}{l}x$$

$$=\frac{Wab^2}{l^2}+\frac{Wab}{l^3}(a-b)x$$

$$\therefore \quad EI\frac{d^2y}{dx^2}=\frac{Wb}{l}x-\frac{Wab^2}{l^2}-\frac{Wab}{l^3}(a-b)x\ \Big|-W(x-a)$$

$$\therefore \quad EI\frac{d^2y}{dx^2}=\frac{Wb}{l^3}(l^2-a^2+ab)x-\frac{Wab^2}{l^2}\ \Big|-W(x-a)$$

$$=\frac{Wb}{l^3}(3ab+b^2)x-\frac{Wab^2}{l^2}\ \Big|-W(x-a)$$

$$\therefore \quad EI\frac{d^2y}{dx^2}=\frac{Wb^2}{l^3}(3a+b)x-\frac{Wab^2}{l^2}\ \Big|-W(x-a)$$

Integrating,

$$EI\frac{dy}{dx}=\frac{Wb^2(3a+b)x^2}{2l^3}-\frac{Wab^2}{l^2}x+C_1\left[-\frac{W(x-a)^2}{2}\right.$$

(*Slope equation*)

But at $x=0, \frac{dy}{dx}=0$

$\therefore \quad C_1=0$

Integrating again,

$$EIy=\frac{Wb^2(3a+b)x^3}{6l^3}-\frac{Wab^2x^2}{2l^2}+C_2\left[-\frac{W(x-a)^3}{6}\right.$$

(*Deflection equation*)

At $\quad x=0, y=0$

$\therefore \quad C_2=0$

Deflection under the load

Putting $x=a$ in the deflection equation, we have,

$$EIy_c=\frac{Wb^2(3a+b)a^3}{6l^3}-\frac{Wab^2(a^2)}{2l^2}$$

$$=\frac{Wa^3b^2}{6l^3}(-3a-b+3l)$$

$$=-\frac{Wa^3b^3}{3l^3}$$

$$\therefore \quad y_c=-\frac{Wa^3b^3}{3EI\,l^3}$$

Maximum deflection

Let $\quad a>b$

Maximum deflection will occur between A and C. For this condition equating the slope to zero, we have,

$$0=\frac{Wb^2(3a+b)x^2}{2l^3}-\frac{Wab^2}{l^2}x$$

$$\therefore \quad x=\frac{2al}{3a+b}$$

Substituting in the deflection equation, we get,

$$EIy_{max}=\frac{Wb^2(3a+b)}{6l^3}\left(\frac{2al}{3a+b}\right)^3-\frac{Wab^2}{2l^2}\left(\frac{2al}{3a+b}\right)^2$$

$$=-\frac{Wb^2}{6l^3}\left(\frac{2al}{3a+b}\right)^2\left[3al-\frac{(3a+b)(2al)}{3a+b}\right]$$

$$=-\frac{Wb^2}{6l^3}\cdot\frac{4a^2l^2}{(3a+b)^2}\cdot(al)$$

$$= -\frac{2}{3}\,\frac{Wa^3b^2}{(3a+b)^2}$$

$$\therefore \qquad y_{max} = -\frac{2}{3}\,\frac{Wa^3b^2}{(3a+b)^2\,EI}$$

Points of contraflexure

For the point of contraflexure in AC,

$$M = \frac{Wb^2}{l^3}(3a+b)\,x - \frac{Wab^2}{l^2} = 0$$

$$\therefore \qquad x = \frac{al}{3a+b}$$

For the point of contraflexure in BC,

$$M = \frac{Wb^2}{l^3}(3a+b)\,x - \frac{Wab^2}{l^2} - W(x-a) = 0$$

Solving we get $x = l - \dfrac{bl}{3b+a}$

Fixed beam carrying a uniformly distributed load for a given distance from one end.

Fig. 498 shows a fixed beam AB of span l carrying a uniformly distributed load w per unit run for a distance a from the end A.

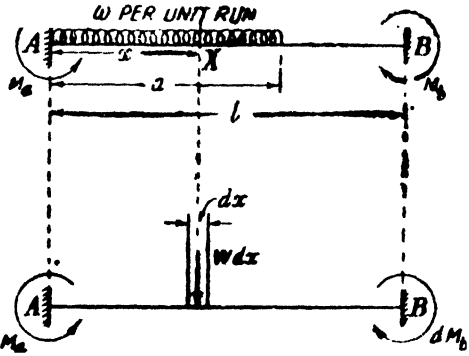


Fig. 498

Consider any section X distant x from the end A.

Load acting for an elemental distance $dx = w\,dx$.

Due to the elemental load $(w\,dx)$ the fixed end moments will be as follows.

$$dM_a = (w\,dx)\frac{x(l-x)^2}{l^2}$$

and $$dM_b = \frac{(w\,dx)x^2(l-x)}{l^2}$$

Total fixing moment at A

$$=M_a=\int_0^a \frac{wx(l-x)^2}{l^2}\,dx$$

$$=\frac{w}{l^2}\left[l^2\frac{a^2}{2}-2l\frac{a^3}{3}+\frac{a^4}{4}\right]$$

$$=\frac{w}{l^2}\,\frac{a^2}{12}\left(6l^2-8la+3a^2\right)$$

$$=\frac{wa^2}{12\,l^2}\left(6l^2-8la+3a^2\right)$$

Similarly total fixing moment at B

$$=M_b=\int_0^a \frac{wx^2(l-x)dx}{l^2}$$

$$=\frac{w}{l^2}\left[\frac{la^3}{3}-\frac{a^4}{4}\right]$$

$$M_b=\frac{wa^3}{12l^2}(4l-3a)$$

For the particular case when the distributed load covers the whole span, putting $a=l$, in the expressions for M_a and M_b we have

$$M_a=\frac{wl^2}{12\,l^2}\left(6l^2-8l^2+3l^2\right)=\frac{wl^2}{12}$$

and

$$M_b=\frac{wl^3}{12\,l^2}\left(4l-3l\right)=\frac{wl^2}{12}$$

Fixed beam carrying a triangular load whose intensity varies from zero at one end to w per unit run at the other end.

Fig 499 shows a fixed beam AB of span l carrying a triangular load whose intensity varies from zero at the end A to w per unit run at the other end B.

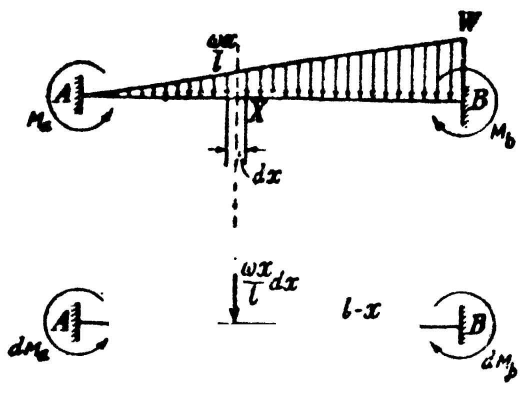


Fig. 499

Consider any section X distant x from the end A.

Intensity of loading at $X=\frac{wx}{l}$

Hence load acting for an elemental distance

$$dx=\frac{w}{l}xdx.$$

Due to this elemental load the fixed moments are as follows :

$$dM_a=\left(\frac{wx}{l}dx\right)\frac{x(l-x)^2}{l^2}$$

$$=\frac{wx^2(l-x)^2\,dx}{l^3}$$

and $$dM_b=\left(\frac{wx}{l}\,dx\right)\frac{x^2(l-x)}{l^2}$$

$$=\frac{wx^3(l-x)dx}{l^3}$$

∴ Total fixing moment at A

$$=M_a=\int_0^l \frac{wx^2(l-x)^2dx}{l^3}$$

$$=\frac{w}{l^3}\left[\,l^2\frac{l^3}{3}-2l.\,\frac{l^4}{4}+\frac{l^5}{5}\right]$$

$$=\frac{wl^2}{20}$$

Similarly,

$$M_b=\int_0^l \frac{wx^3(l-x)dx}{l^3}$$

$$=\frac{w}{l^3}\left[\,l\,\frac{l^4}{4}-\frac{l^5}{5}\right]$$

$$=\frac{wl^2}{20}$$

Thus, $$M_a=\frac{wl^2}{30}$$

and $$M_b=\frac{wl^2}{20}$$

Fixed beam carrying a triangular load for a given distance from one end.

Fig. 500 shows a fixed beam AB of span l carrying a triangular loading covering a distance a from the left end A. Let the intensity of the load vary from zero to w per unit run.

Consider any section X in the loaded range, at a distance x from A.

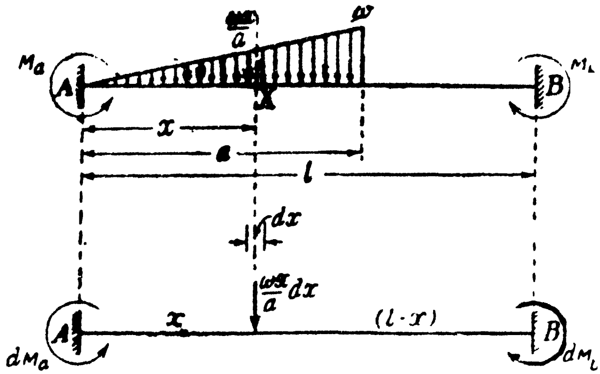


Fig. 500

Intensity of loading at

$$X=\frac{w}{a}x$$

$\therefore$ Load acting for an elemental distance

$$dx=\frac{wx}{a}dx$$

The fixed end moments due to the above elemental load are given by,

$$dM_a=\left(\frac{wx}{a}dx\right)\cdot\frac{x(l-x)^2}{l^2}$$

$$=\frac{wx^2(l-x)^2}{al^2}dx$$

and $$dM_b=\left(\frac{wx}{a}dx\right)\frac{x^2(l-x)}{l^2}$$

$$=\frac{wx^3(l-x)}{al^2}dx$$

$\therefore$ Total fixing moment at A,

$$=M_a=\int_0^a\frac{wx^2(l-x)^2}{al^2}dx$$

$$=\frac{w}{al^2}\left\{l^2\frac{a^3}{3}-2l\frac{a^4}{4}+\frac{a^5}{5}\right\}$$

$$=\frac{w}{al^2}\cdot\frac{a^3}{30}\left(10l^2-15la+6a^2\right)$$

$$=\frac{wa^2}{30l^2}\left(10l^2-15la+6a^2\right)$$

Similarly the total fixing moment at *B*

$$=M_b=\int_0^a \frac{wx^3(l-x)}{al^2}\,dx$$

$$=\frac{w}{al^2}\left[\frac{la^4}{4}-\frac{a^5}{5}\right]$$

$$=\frac{w}{al^2}\cdot\frac{a^4}{20}\left(5l-4a\right)$$

$$=\frac{wa^3}{20l^2}\left(5l-4a\right)$$

For the particular case, when the triangular loading covers the whole span, putting $a=l$, in the above expression, for M_a and M_b, we have,

$$M_a=\frac{wl^2}{30l^2}\left(10l^2-15l^2+6l^2\right)=\frac{wl^2}{30}$$

and $$M_b=\frac{wl^3}{20l^2}\left(5l-4l\right)=\frac{wl^2}{20}$$

Problem 314. *A fixed beam of span 6 metres carries point loads 20t and 15t at distances 2 m and 4 m from the left end. Find the fixed end moments and the reactions at the supports. Draw B.M. and S.F. diagrams.*

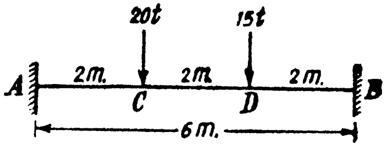


Fig. 501

Solution. Fixing moment at the left end A

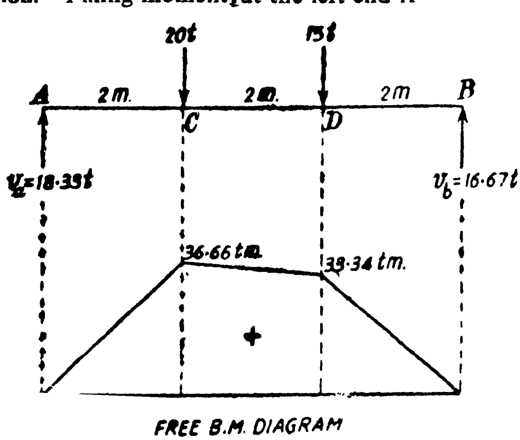


Fig. 502

$$=M_a=\sum\frac{Wab^2}{l^2}$$

$$=\frac{20\times2\times4^2}{6^2}+\frac{15\times4\times2^2}{6^2}tm$$

$$=24{\cdot}44\ tm.$$

Fixing moment at the right end B

$$=M_b=\sum\frac{Wa^2b}{l^2}$$

$$=\frac{20\times2^2\times4}{6^2}+\frac{15\times4^2\times2}{6^2}\ tm$$

$$=22{\cdot}22\ tm.$$

Free B.M. diagram

Considering the span as a simply supported beam let v_a and v_b be the reactions at the supports A and B. Taking moments about A, we have,

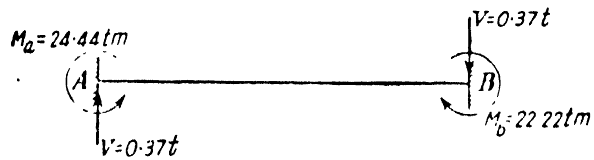


Fig. 503

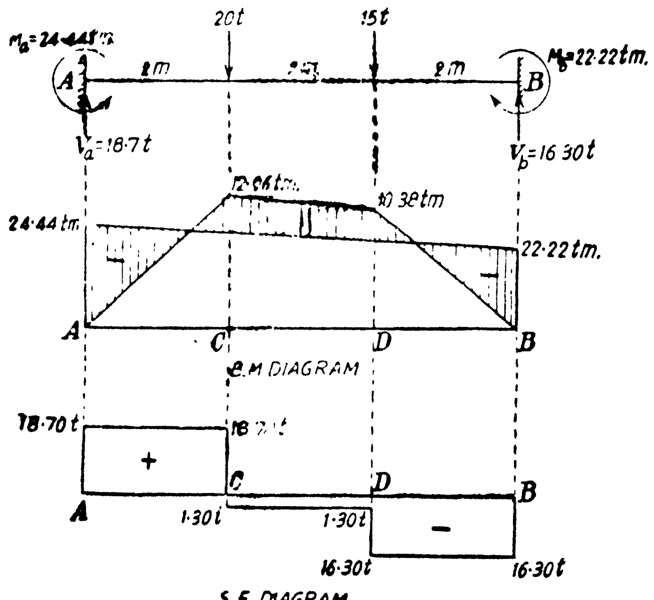


Fig. 504

$$v_b \times 6 = 20 \times 2 + 15 \times 4$$

$\therefore$ $v_b = 16{\cdot}67\ t$ and $v_a = 35 - 16{\cdot}67 = 18{\cdot}33\ t$

Free B.M. at $C = 18{\cdot}33 \times 2 = 36{\cdot}66\ tm.$

Free B.M. at $D = 16{\cdot}67 \times 2 = 33{\cdot}34\ tm.$

Reaction V at each support due to end moment

alone $\quad = V = \dfrac{24{\cdot}44 - 22{\cdot}22}{6} = 0{\cdot}37\ t.$

Since $M_a > M_b$ the reaction V at A is upwards and the reaction V at B is downwards.

$\therefore$ Final reaction at $A = V_a = v_a + V = 18{\cdot}33 + 0{\cdot}37 = 18{\cdot}70\ t$

Final reaction at $B = V_b = v_b - V = 16{\cdot}67 - 0{\cdot}37 = 16{\cdot}30\ t$

By combining the free and fixed bending moment diagrams, the final B.M. diagram can be drawn. The S.F. diagram is also drawn as usual.

Problem 315 (SI). *A fixed beam of span 6 metres carries point loads 160 KN and 120 KN at distances 2 m and 4 m from the left end.*

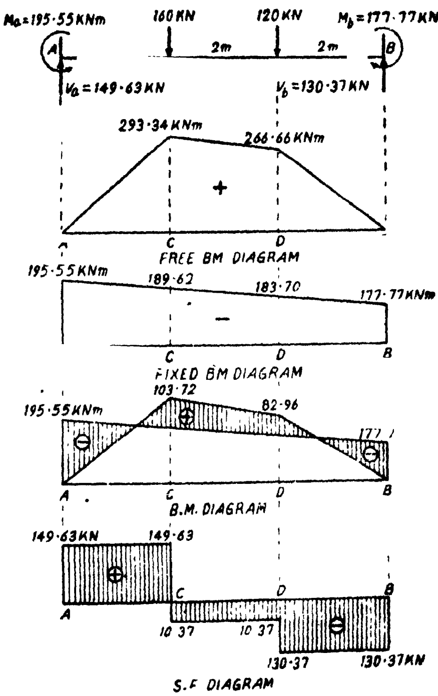


Fig. 505

***Find** the fixed end moments and the reaction at the supports. Draw **B.M.** and S.F. diagrams.*

Solution. Fig. 505 shows the fixed beam AB carrying the given loads.

Fixing moment at the end $A = M_a = \sum \frac{Wab^2}{l^2}$

$$= \frac{160 \times 2 \times 4^2}{6^2} + \frac{120 \times 4 \times 2^2}{6^2}$$

$$= 142{\cdot}22 + 53{\cdot}33 = 195{\cdot}55 \ KNm$$

Fixing moment at the end $B = M_b = \sum \frac{Wa^2b}{l^2}$

$$= \frac{160 \times 2^2 \times 4}{6^2} + \frac{120 \times 4^2 \times 2}{6^2}$$

$$= 71{\cdot}11 + 106{\cdot}66 = 177{\cdot}77 \ KNm$$

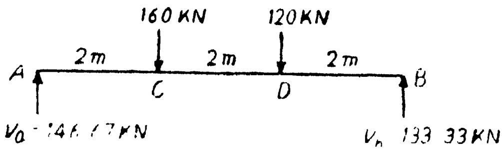


Fig. 506

Free B.M. diagram

Taking the beam as simply supported at the end, and taking moments above A,

$$v_b \times 6 = (160 \times 2) + (120 \times 4)$$

$$v_b = 133{\cdot}33 \ KN$$

$$\therefore \ v_a = 160 + 120 - 133{\cdot}33 = 146{\cdot}67 \ KN.$$

Free B.M. at $A = 0$

Free B.M. at $C = 146{\cdot}67 \times 2 = 293{\cdot}34 \ KNm$

Free B.M. at $D = 133{\cdot}33 \times 2 = 266{\cdot}66 \ KNm$

Free B.M. at $B = 0$

Reaction V at each end due to end moments

$$= V = \frac{195{\cdot}55 - 177{\cdot}77}{6} = 2{\cdot}96$$

Since $M_a > M_b$, the reaction V at A is upwards and the re V at B is downwards.

$\therefore$ Final reaction at $A = V_a = v_a + V = 146{\cdot}67 + 2{\cdot}96 = 149{\cdot}63$ *KN*

and final reaction at $B = V_b = v_b - V = 133{\cdot}33 - 2{\cdot}96 = 130{\cdot}37$ *KN*

By combining the free and the fixed bending moment diagrams, the final bending moment diagram can be drawn. The S.F. diagram is also drawn as usual.

§81. Fixed beam with ends at different levels (Effect of sinking of supports).

Consider a fixed beam AB of span l whose ends A and B are fixed at different levels. Let δ be the difference of level between the ends. Let the end A be at a higher level than the end B. Let M_a and M_b be the fixing moments at the ends. Obviously for this case M_a is negative (hogging) and M_b is positive. But numerically M_a and M_b are equal.

Let V be the reaction at each support.

Consider any section distant x from the end A.

Since the rate of loading is zero,

we have, with the usual notation

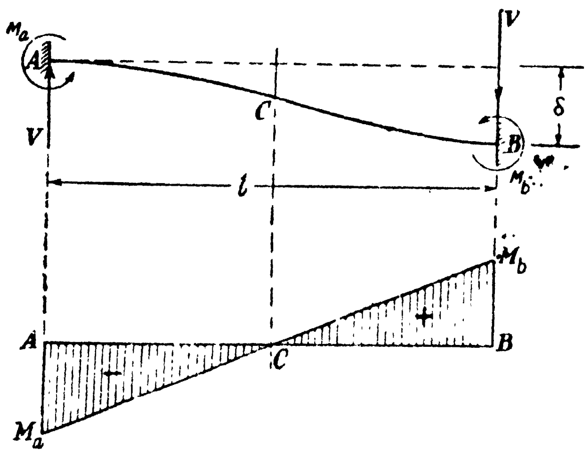


Fig. 507

$$EI\frac{d^4y}{dx^4} = 0$$

Integrating, we get,

Shear force $= EI\dfrac{d^3y}{dx^3} = C_1$

where C_1 is a constant

At $x=0$, S.F. $= +V$

$\therefore$ $C_1 = V$

Integrating again, we get

B.M. at any section $= EI\dfrac{d^2y}{dx^2} = Vx + C_2$

At $x=0$, B.M. $= -M_a$

$\therefore$ $C_2 = -M_a$

$\therefore$ $EI\dfrac{d^2y}{dx^2} = Vx - M_a$

Integrating again,

$$EI\frac{dy}{dx} = \frac{V}{2}x^2 - M_a x + C_3 \qquad (\textit{Slope equation})$$

But at $x=0$, $\dfrac{dy}{dx} = 0$

$\therefore$ $C_3 = 0$

Integrating again,

$$EIy = \frac{Vx^3}{6} - \frac{M_a x^2}{2} + C_4 \qquad (\textit{Deflection equation})$$

But at $x=0$, $y=0$

$\therefore$ $C_4 = 0$

At $x = l$ $y = -\delta$

$$\therefore \quad -EI\delta = \frac{Vl^3}{6} - \frac{M_a l^2}{2} \qquad ...(i)$$

But we also know that at B, $x = l$ and $\dfrac{dy}{dx} = 0$

$$\therefore \quad O = \frac{Vl}{2} - M_a l$$

$$\therefore \quad V = \frac{2M_a}{l} \qquad ...(ii)$$

Substituting in Eq. (i), we have,

$$-EI\delta = \frac{2M_a}{l} \cdot \frac{l^3}{6} - \frac{M_a \cdot l^2}{2}$$

$$EI\delta = \frac{M_a l^2}{6}$$

$$\therefore \qquad M_a = \frac{6EI\delta}{l^2}$$

Hence the law for the bending moment at any section distant x from A is given by

$$M = EI\frac{d^2y}{dx^2} = Vx - M_a$$

$$\therefore \qquad M = \frac{2M_a}{l}x - \frac{6EI\delta}{l^2}$$

For the B.M. at B, put $x = l$

$$\therefore \qquad M_b = \frac{2M_a}{l}.\, l - \frac{6EI\delta}{l^2}$$

$$= \frac{12EI\delta}{l^2} - \frac{6EI\delta}{2} = \frac{6EI\delta}{l^2}$$

Hence when the ends of a fixed beam are at different levels, the fixing moment at each end $= \frac{6EI\delta}{l^2}$ numerically. At the higher end this moment is a hogging moment and at the lower end this moment is a sagging moment.

Alternative approach

Fig. 508 (*i*) shows the fixed beam AB whose ends A and B are at different levels.

Let δ be the difference in level between A and B.

Let M_a and M_b be the end moments. Obviously M_a and M_b are numerically equal, but at the higher end A, the end moment M_a is a hogging moment and the moment M_b is a sagging moment. Fig. 508 (*ii*) shows the bending moment diagram for the beam. Obviously the point of contraflexure occurs at the midspan.

Fig. 508 (*iii*) shows a cantilever AC of length $\frac{l}{2}$ having the same uniform flexural rigidity as that of given beam.

Let this cantilever be subjected to a point load P at C so that the deflection

$$C_1C = \frac{\;}{2}$$

Now the deflection curve for this cantilever will be exactly the same as the deflection curve for the left half of the given fixed beam. The B.M. diagram for the cantilever should be identical with the B.M. diagram for left half of the fixed beam.

$$\therefore \qquad M_a = \frac{Pl}{2}$$

But for the cantilever, the deflection at the free end

$$= \frac{\delta}{2} = \frac{P\left(\frac{l}{2}\right)^3}{3EI} = \frac{Pl^3}{24EI}$$

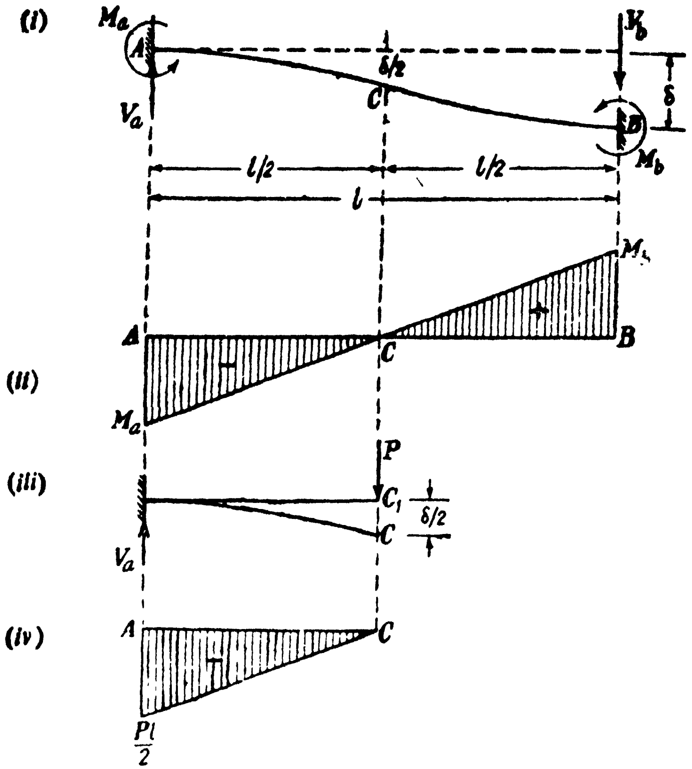


Fig. 508

$$\therefore \qquad P=\frac{12EI\delta}{l^3}$$

$$\therefore \qquad M_a=P\cdot\frac{l}{2}=\frac{12EI\delta}{l^3}\cdot\frac{l}{2}=\frac{6EI\delta}{l^2}$$

Hence each end moment for the fixed beam

$$=\frac{6EI\delta}{l^2}$$

Reaction at each support

$$=\frac{\text{Total couple on the beam}}{\text{span}}$$

$$=\frac{\frac{6EI\delta}{l^2}+\frac{6EI\delta}{l^2}}{l}$$

$$=\frac{12EI\delta}{l^3}$$

or alternatively,

reaction V_a=**Reaction at the fixed end of the cantilever**

$$=P=\frac{12EI\delta}{l^3}$$

Hence the S.F. at any section of the beam

$$=\frac{12EI\delta}{l^3}$$

Problem 316. *A fixed beam of span 5 metres carries a concentrated load of 20 t at 3 metres from the left end. If the right end sinks by 1 cm, find the fixing moments at the supports. For the beam section take $I=30,000\ cm^4$ and $E=2\times 10^3\ t/cm^2$. Find also the reaction at the supports.*

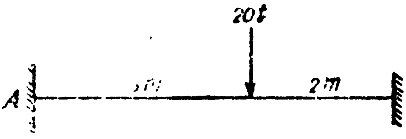


Fig. 509

Solution. Due to loading the fixing moment at each support will be a hogging moment. But due to sinking of supports the moment at each end will be $\frac{6EI\delta}{l^2}$ the nature of moment being hogging at the higher end and sagging at lower end.

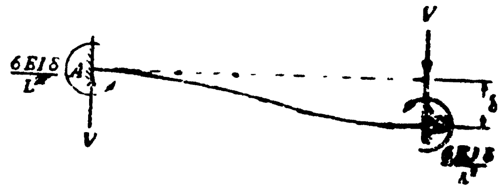


Fig. 510

Fixing moment at A

$$=-\frac{Wab^2}{l^2}-\frac{6EI\delta}{l^2}$$

$$=-\left[\frac{20\times 3\times 2^2}{5^2}+\frac{6\times 2\times 10^3\times 30,000\times 1}{5^2\times(100)^3}\right]tm$$

$$=-(9{\cdot}60+0{\cdot}48)\ tm$$

$$=-10{\cdot}08\ tm\ (hogging)$$

Fixing moment at B

$$=-\frac{Wa^2b}{l^2}+\frac{6EI\delta}{l^2}$$

$$=-\frac{20\times 3^2\times 2}{5^2}+0{\cdot}48$$

$$=-14{\cdot}40+0{\cdot}48\ tm.$$

$$=-13{\cdot}92\ tm.\ (hogging)$$

Reaction at A

$=V_a=$ Reaction due to load with simply supported condtiion + Reaction due to end moments

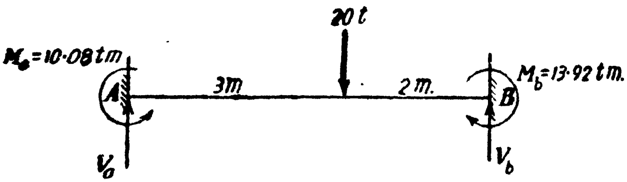


Fig. 511

$$= \frac{20\times 2}{5} - \frac{(13\ 92-10\text{'}08)}{5} = 7\text{'}232\ t.$$

V_b= Reaction due to simply supported condition + Reaction due to end moments.

$$= \frac{20\times 3}{5} + \frac{(13\text{'}92-10.08)}{5} = 12\text{'}768\ t.$$

§82. Fixed beam subjected to a couple M_o applied eccentrically on the span

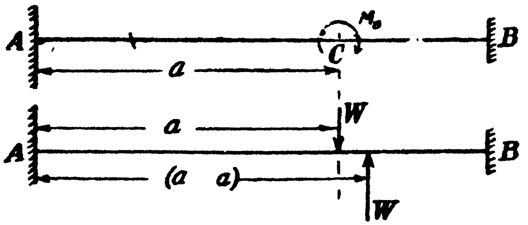


Fig. 512

Fig. 512 shows a fixed beam AB of span l subjected to a concentrated couple M_o applied at C at a distance a from the end A.

The couple M_o may be taken to consist of two equal and opposite loads W at a small distance δa apart. Now the fixing moment at A

$$= M_a = -\frac{Wa(l-a)^2}{l} + \frac{W(a+\delta a)(l-a-\delta a)^2}{l^2}$$

Ignoring $(\delta a)^2$ in the expansion of the above expression

$$M_a = -\frac{Wa(l-a)^2}{l^2} + \frac{W}{l^2}\Big[a(l-a)^2 - 2a(l-a)\delta a + \delta a(l-a)^2\Big]$$

$$= -\frac{Wa(l-a)^2}{l^2} + \frac{W}{l^2}\Big[a(l-a)^2 + \delta a(l-a)(l-3a)\Big]$$

$$= \frac{W\delta a}{l^2}(l-a)(l-3a)$$

But when δa is small M_o is equal to the couple $W\delta a$

$$\therefore \qquad M_a=\frac{M_o}{l^2}(l-a)(l-3a)$$

Similarly, it can be shown,

$$M_b=\frac{M_o}{l^2}a(2l-3a)$$

Problem 317. *A fixed beam AB of span 6 metres is subjected to a concentrated couple of 30 tm. applied at a section C 4 metres from the end A. Find the end moments from first principles and draw the B.M. and S.F. diagram.*

Solution. Taking A as the origin, the unknowns are reaction V_a and the end moment M_a. Assume these in the directions shown in Fig. 513 (If these reacting elements work out to be negative then it means they act in directions opposite to those assumed).

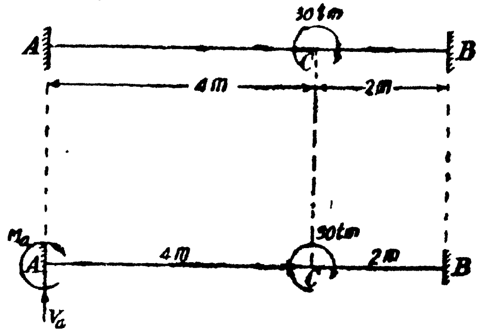


Fig. 513

At any section distant x from A the bending moment is given by

$$EI\frac{d^2y}{dx^2}=V_ax+M_a+30$$

To facilitate application of Macaulay's method the above expression will be rearranged as follows.

$$EI\frac{d^2y}{dx^2}=V_ax+M_a+30(x-4)^\circ$$

Integrating, we get,

$$EI\frac{dy}{dx}=\frac{V_ax^2}{2}+M_ax+C_1+30(x-4)\ (\textit{slope equation})$$

At $x=0$, $\frac{dy}{dx}=0$ $\qquad\therefore\ C_1=0$

Integrating again, we get,

$$EIy=\frac{V_ax^3}{6}+\frac{M_ax^2}{2}+C_2+15(x-4)^2\ (\textit{deflection equation})$$

At $x=0$, $\quad y=0$, $\quad\therefore\ C_2=0$

At B, the slope is zero,

$\therefore$ at $x=6$, $\frac{dy}{dx}=0$

$$\therefore \qquad 0=V_a\frac{6^2}{2}+M_a(6)+30(6-4)$$

$\therefore \quad 3V_a + M_a = -10$ ...(*i*)

At B, the deflection is zero,

$\therefore$ at $x=6, \quad y=0$

$$\therefore \quad 0 = V_a \frac{(6)^3}{6} + \frac{M_a(6)^2}{2} + 15(6-4)^2$$

$\therefore \quad 6V_a + 3M_a = -10$ ...(*ii*)

Solving equations (*i*) and (*ii*), we get

$$V_a = -\frac{20}{3}\ t \qquad \text{and} \qquad M_a = 10\ tm$$

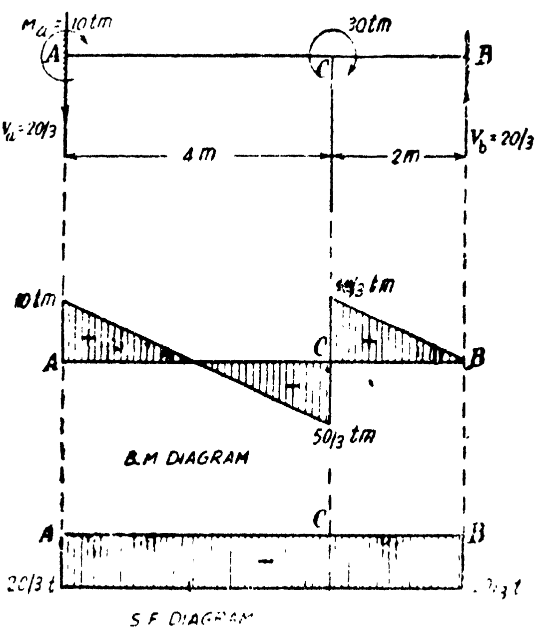


Fig. 514

The reacting moment M_a and the reacting force V_a are shown in Fig. 514

B.M. calculations

B.M. at $A = +10$ *tm.* (*sagging*)

$$M_{ca} = +10 - \frac{20}{3} \times 4 = -\frac{50}{3}\ tm. \ (hogging)$$

$$M_{cb} = -\frac{50}{3} + 30 = +\frac{40}{3}\ tm. \ (sagging)$$

$$M_b = 10 - \frac{20}{3} \times 6 + 30 = 0$$

Fig. 514 shows the B.M. diagram for the beam

S.F. diagram

Obviously the shear force at any section of the beam

$$= -\frac{20}{3} t$$

Fig. 514 shows the shear force diagram for the beam.

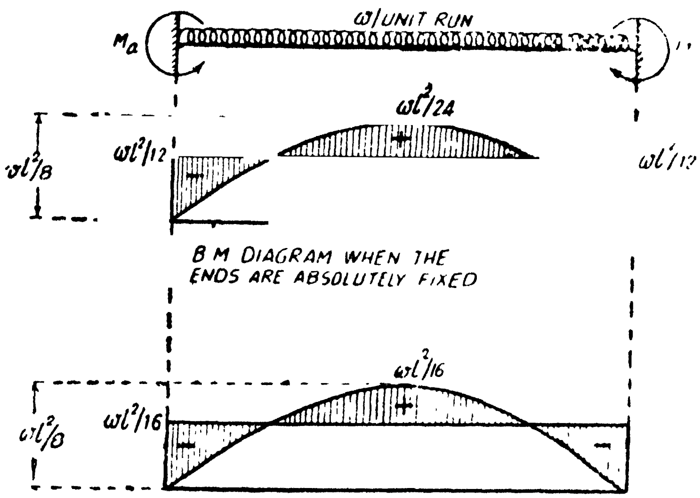
§83. Degree of fixity at supports in order the maximum bending moment is as small as poissible

Consider a fixed beam of span l carrying a point load W at mid span. For this case we know that the hogging moment at each end and the sagging moment at mid span are both equal to $\frac{Wl}{8}$.

But let us now consider the fixed beam carrying a uniformly distributed load of w per unit run over the whole span. When the ends are absolutely fixed the maximum sagging moment at the mid span is $\frac{wl^2}{24}$, while the maximum hogging moment at each end is $\frac{wl^2}{12}$

If the degree of fixity at the supports is lessened, the maximum hogging moment at the ends will be decreased. This will result in a corresponding increase in the maximum sagging moment at midspan.

Hence for the maximum bending moment for the beam to be as small as possible the condition to be satisfied is that the maximum sagging moment and the maximum hogging moment should be equal.



B.M. diagram when the degree of end fixity is such that maximum sagging moment equals maximum hogging moment.

Fig. 515

Hence for the case under discussion,

Hogging, B.M. at each support

$$=\text{Sagging moment at mid span}=\frac{wl^2}{16}$$

For this condition, obviously the end slope will not be zero, and can be determined as shown below. Further the central deflection will be greater than the corresponding deflection with absolute fixed ends.

The bending moment at any section distant x from the left support is given by,

$$EI\frac{d^2y}{dx^2}=\frac{wl}{2}x-\frac{wx^2}{2}-\frac{wl^2}{16}$$

Integrating, we get,

$$EI\frac{dy}{dx}=\frac{wlx^2}{4}-\frac{wx^3}{6}-\frac{wl^2}{16}x+C_1$$

At $\quad x=\frac{l}{2}, \quad \frac{dy}{dx}=0$

$\therefore \quad 0=\frac{wl}{4}\cdot\frac{l^2}{4}-\frac{w}{6}\cdot\frac{l^3}{8}-\frac{wl^2}{16}\cdot\frac{l}{2}+C_1$

$\therefore \quad C_1=-\frac{wl^3}{96}$

$\therefore \quad EI\frac{dy}{dx}=\frac{wlx^2}{4}-\frac{wx^3}{6}-\frac{wl^2}{\cdot}\cdots\frac{\cdot^3}{96}$

(Slope equation)

At $\quad x=0. \quad$ let $\frac{dy}{dx}=i_a$ *(slope at A)*

$\therefore \quad EI\, i_a=-\frac{wl}{96}$

$\therefore \quad i_a=-\frac{wl^3}{96\,EI}$

(This slope is just one-fourth of the end slope for a simply supported beam carrying the same loading).

Integrating the slope equation again, we have,

$$EIy=\frac{wl\,x^3}{12}-\frac{wx^4}{24}-\frac{wl^2x^2}{32}-\frac{wl^3}{96}x+C_2$$

But at $\quad x=0, \quad y=0$

$\therefore \quad C_2=0$

Let the central deflection be y_c

$\therefore$ At $\quad x=\frac{l}{2}\cdot \quad y=y_c$

$$\therefore \quad EIy_c = \frac{wl}{12}\left(\frac{l}{2}\right)^3 - \frac{w}{24}\left(\frac{l}{2}\right)^4 - \frac{wl^2}{32}\left(\frac{l}{2}\right)^2 - \frac{wl^3}{96}\left(\frac{l}{2}\right)$$

$$\therefore \quad EIy_c = -\frac{wl^4}{192}$$

$$\therefore \quad y_c = -\frac{wl^4}{192\,EI}$$

For the points of contraflexure, equating the general expression for bending moment to zero, we have,

$$\frac{wl}{2}x - \frac{wx^2}{2} - \frac{wl^2}{16} = 0$$

$$\therefore \quad x^2 - lx + \frac{l^2}{8} = 0$$

$$x^2 - lx = -\frac{l^2}{8}$$

$$\therefore \quad \left(x - \frac{l}{2}\right)^2 = \frac{l^2}{4} - \frac{l^2}{8} = \frac{l^2}{8}$$

$$\therefore \quad x - \frac{l}{2} = \pm \frac{l}{2\sqrt{2}}$$

$$\therefore \quad x = \frac{l}{2}\left[1 \pm \frac{1}{\sqrt{2}}\right]$$

$$= 0{\cdot}8535\,l \quad \text{or} \quad 0{\cdot}1465l$$

§84. Advantages and disadvantages of fixed beams

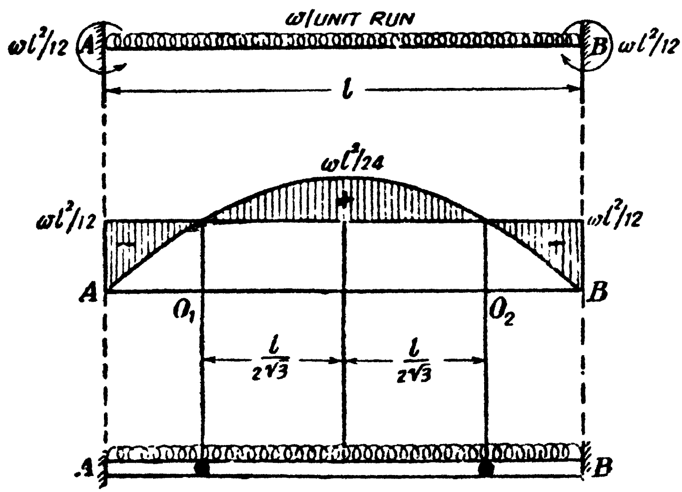


Fig. 516

Theoretically viewing the fixed beam has the following advantages :

(*i*) The fixed beam is subjected to a lesser maximum bending moment than the simply supported beam carrying the same loading.

(*ii*) For the same loading the maximum deflection for a fixed beam is less than that of the simply supported beam.

Practically the fixed beam has certain disadvantages.

(*i*) It is practically difficult to maintain the two ends of the beam at *exactly* the same level. Any subsidence of one of the supports, however small it may be, will set up considerable stresses. This is therefore a serious disadvantage. During erection of the beam, the supports therefore must be aligned with greatest accuracy.

(*ii*) Temperature variations also produce large stresses in a fixed beam.

(*iii*) When the beam is subjected to live loads (such as wheel loads passing over bridges) frequent variations of bending moment and corresponding vibrations would soon affect the degree of fixity at the ends

The above objections against fixed beams can be obviated by adopting the double cantilever construction. In this method, at the points of contraflexure for the fixed beam, hinged joints are introduced. Now the beam will therefore consist of a central simply supported girder, supported on the ends of two end cantilevers.

The bending moment diagram and the elastic curve for this beam will be the same as for the fixed beam. After introducing the hinges at the points of contraflexure of the fixed beam, temperature changes and sinking of end supports will not affect the bending moments.

§85. Continuous beams

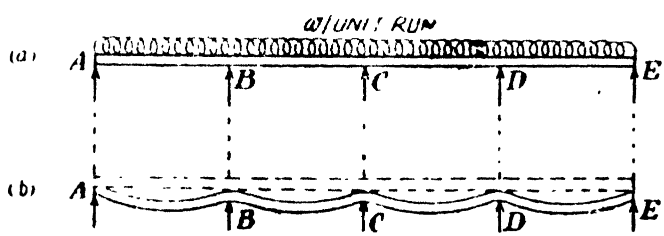


Fig. 517

Fig. 517 shows a beam continuous over a number of supports. Fig. 517 (*b*) shows the deflection curve for the beam when it is subjected to an external load system.

For the usual loadings on the beam, sagging moments occur at the mid section of the spans and hogging moments occur over the supports. If the support moments are known, the bending moment diagram can be drawn easily. The support moments can be determined by the application of *Clapeyron's theorem of three moments.*

§86. Clapeyron's theorem of three moments

If AB and BC are any two consecutive spans of a continuous beam subjected to an external loading, the support moments M_a, M_b *and* M_c *at the supports A, B and C are given by the relation,*

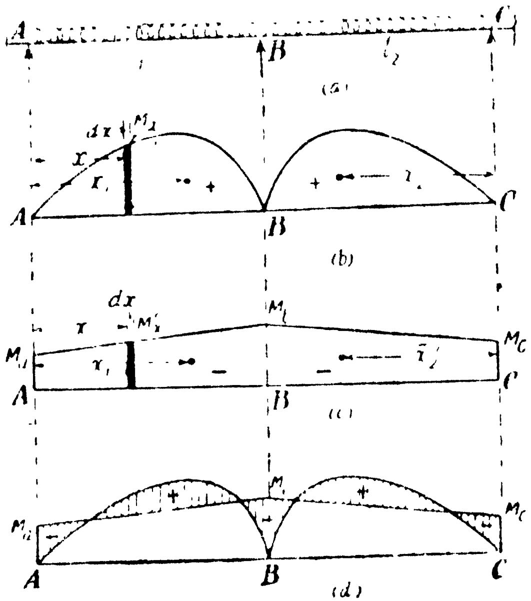


Fig. 518

$$M_a l_1 + 2M_b(l_1+l_2) + M_c l_2 = \frac{6a_1\bar{x}_1}{l_1} \quad 6a_2\bar{x}_2$$

where a_1 = area of the free B.M. diagram for the span AB

a_2 = area of the free B.M. diagram for the span BC

$\bar{x}_1$ = centroidal distance of the free B.M. diagram on AB from A

$\bar{x}_2$ = centroidal distance of the free B.M. diagram on BC from C

l_1 = span length AB

l_2 = span length CD

Let a_1' = area of the fixed B.M. diagram for span AB

a_2' = area of the fixed B.M. diagram for the span BC

Fig. 518 (*a*) shows the given beam.

Figs. 518 (*b*) and (*c*) show the free and fixed bending moment diagrams for the span *AB* and *BC*.

Consider the span *AB*.

Let at any section in AB distant x from A the free and fixed bending moments be M_x and M_x' respectively.

Hence the net bending moment at the section is given by,

$$EI\frac{d^2y}{dx^2}=M_x-M_x'$$

Multiplying by x, we get

$$EIx\frac{d^2y}{dx^2}=M_xx-M_x'x$$

Integrating from x to l_1, we get,

$$EI\int_0^{l_1} x\frac{d^2y}{dx^2}\,dx=\int_0^{l_1} M_xx\,dx-\int_0^{l_1} M_x'x\,dx$$

$$\therefore\quad EI\left[x\frac{dy}{dx}-y\right]_0^{l_1}=\int_0^{l_1} M_xx\,dx-\int_0^{l_1} M_x'x\,dx$$

But it may be seen, that

(i) At $x=0$, $y=0$

(ii) At $x=l$, $y=0$, and $\frac{dy}{dx}=i_{ba}$ (slope at B for AB)

(iii) $\int_0^{l_1} M_xx\,dx=a_1\bar{x}_1=$ Moment of the free B.M. diagram on AB about A

(iv) $\int_0^{l_1} M'_xx\,dx=a'_1\bar{x}'_1=$ Moment of the fixed B.M. diagram on AB about A

Hence,

$$EI\,l_1\,i_{ba}=a_1\bar{x}_1-a_1'\bar{x}_1'$$

But $a_1'=$ area of the fixed B.M. diagram on AB

$$=(M_a+M_b)\frac{l_1}{2}$$

$\bar{x}_1'=$ Distance of the centroid of the fixed B.M. diagram from A

$$=\frac{M_a+2M_b}{M_a+M_b}\cdot\frac{l_1}{3}$$

$$\therefore\quad a_1'\bar{x}_1'=\frac{M_a+M_b}{2}\cdot l_1\cdot\frac{M_a+2M_b}{M_a+M_b}\cdot\frac{l_1}{3}$$

$$=(M_a+2M_b)\frac{l_1^2}{6}$$

$$\therefore\quad EI\,l_1 i_{ba}=a_1\bar{x}_1-(M_a+2M_b)\frac{l_1^2}{6}$$

$$\therefore \qquad 6EI\, i_{ba} = \frac{6a_1\bar{x}_1}{l_1} - \left(M_a + 2M_b\right)l_1 \qquad \ldots(i)$$

Similarly considering the span *BC* and taking *C* as origin it can be shown that,

$$6EI\, i_{bc} = \frac{6a_2\bar{x}_2}{l_2} - \left(M_c + 2M_b\right)l_2 \qquad \ldots(ii)$$

where i_{bc} = slope for the span *CB* at *B*.

But $i_{ba} = -i_{bc}$ as the direction of x from *A* for the span *AB*, and from *C* for the span *CB* are in opposite direction.

$$\therefore \qquad i_{ba} + i_{bc} = 0 \qquad \ldots(iii)$$

Adding equation (i) and (ii), we get

$$6EI\, i_{ba} + 6EI\, i_{bc} = 6EI(i_{ba} + i_{bc}) = 0$$

$$= \frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2} - \left[M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2 \right]$$

$$\therefore \quad M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2 = \frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2}$$

This relation is called *Clapeyron's relation.*

Particular case

Suppose on the span *AB* there is a uniformly distributed load w_1 per unit run. The free B.M. diagram is a parabola having an altitude of $\frac{w_1 l_1^2}{8}$

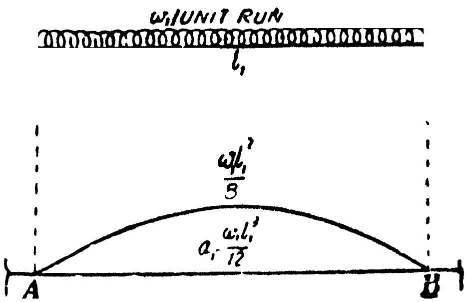


Fig. 519

$\therefore$ Area of the free B.M. diagram

$$= a_1 = \frac{2}{3} \text{ base} \times \text{altitude}$$

$$= \frac{2}{3} \cdot l_1 \times \frac{w_1 l_1^2}{8} = \frac{w_1 l_1^3}{12}$$

and $\qquad \bar{x}_1 = \frac{l_1}{2}$

$$\therefore \qquad \frac{6a_1\bar{x}_1}{l_1} = \frac{6}{l_1} \cdot \frac{w_1 l_1^3}{12} \cdot \frac{l_1}{2} = \frac{w_1 l_1^3}{4}$$

Hence when the span AB and BC carry uniformly distributed loads of w_1 per unit run and w_2 per unit run,

$$M_a l_1 + 2M_b (l_1 + l_2) + M_c l_2 = \frac{w_1 l_1^3}{4} + \frac{w_2 l_2^3}{4}$$

It will be found convenient to use the above relation in case of distributed loads on continuous beams.

Problem 318. *A continuous beam ABC covers two consecutive spans AB and BC of lengths 4m and 6m, carrying uniformly distributed loads of 6 t/m and 10 t/m respectively. If the ends A and C are simply supported find the support moments at A, B and C. Draw also B.M. and S.F. diagrams.*

Solution. Since the end A and C are simply supported the support moments at A and C are zero.

$$\therefore \quad M_a = M_c = 0$$

Applying the theorem of three moments for the spans AB and BC, we get,

$$0 \times 4 + 2M_b(4+6) + 0 \times 6 = \frac{6 \times 4^3}{4} + \frac{10 \times 6^3}{4}$$

$$\therefore \quad 20M_b = 636$$

$$\therefore \quad M_b = 31{\cdot}8 \ tm.$$

Max. Free B.M. for the span $AB = \frac{6 \times 4^2}{8} = 12 \ tm.$

$\therefore$ B.M. at the centre of span AB

$$= -\frac{31{\cdot}8}{2} + 12 = -3{\cdot}9 \ tm.$$

Max. free B.M. for this span $BC = \frac{10 \times 6^2}{8} = 45 \ tm.$

$\therefore$ B.M. at the centre of span BC

$$= -\frac{31{\cdot}8}{2} + 45 = +29{\cdot}10 \ tm.$$

Reactions

Consider the span AB,

$$\text{B.M. at } B = V_a \times 4 - \frac{6 \times 4^2}{2} = -31{\cdot}8$$

$$\therefore \quad V_a = 4{\cdot}05 \ t$$

Similarly considering the span BC,

$$\text{B.M. at } B = V_c \times 6 - \frac{10 \times 6^2}{2} = -31{\cdot}8$$

$$\therefore \quad V_c = 24{\cdot}7 \ t$$

$\therefore$ the reaction $V_b = \text{Total load} - (V_a + V_c)$

$$= (6 \times 4 + 10 \times 6) - (4{\cdot}05 + 24{\cdot}7)$$

$$= 55{\cdot}25 \ t$$

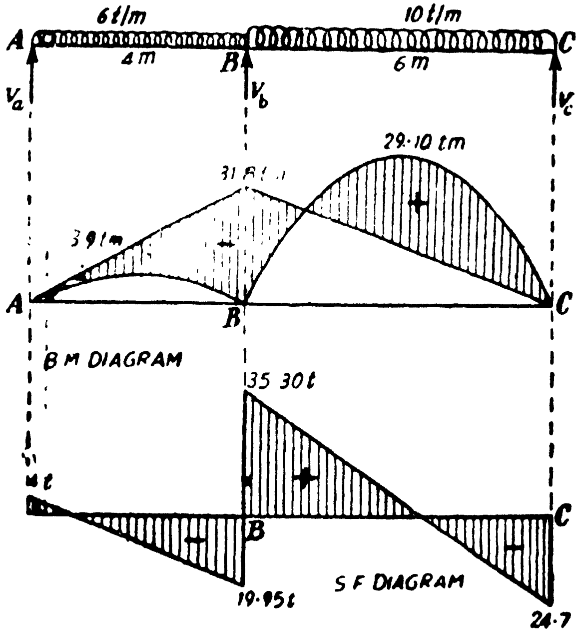


Fig. 520

Problem 319. *A continuous beam consists of three successive spans of 8 metres, 10 metres and 6 metres and carries loads of 6 t per metre, 4t per metre and 8t per metre respectively on the spans. Determine the bending moments and reactions at the supports.*

Solution. Applying the theorem of three moments for the spans *AB* and *BC*, we have,

$$M_a \times 8 + 2M_b(8+10) + M_c \times 10 = \frac{6 \times 8^3}{4} + \frac{4 \times 10^3}{4}$$

But since *A* is the simply supported end of the girder,

$$M_a = 0$$

$\therefore\ 36M_b + 10M_c = 1768$

$\therefore\ 18M_b + \ 5M_c = 884$ ...(*i*)

Now consider the spans *BC* and *CD*

Applying the theorem of three moments for these spans, we have,

$$M_b \times 10 + 2M_c(10+6) + M_d \times 6 = \frac{4 \times 10^3}{4} + \frac{8 \times 6^3}{4}$$

But, since *D* is the simply supported end of the girder,

$$M_d = 0$$

$\therefore\ 10\ M_b + 32\ M_c = 1432$

$\therefore\ \ 5\ M_b + 16\ M_c = 716$ ...(*ii*)

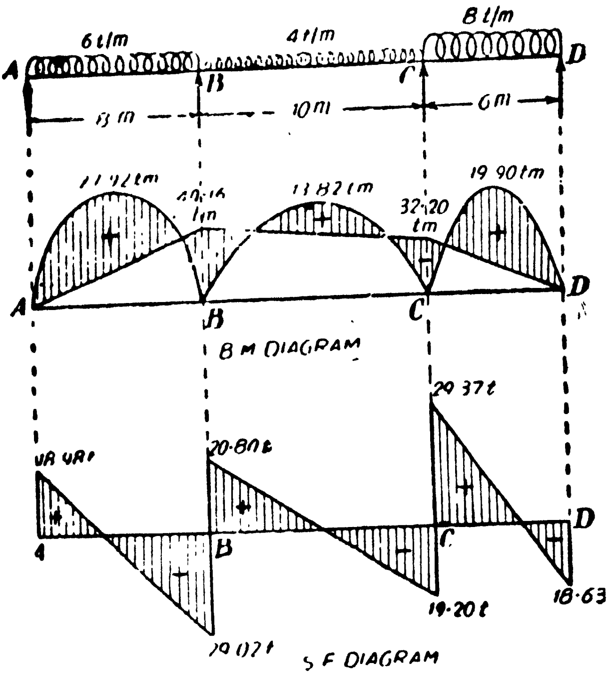


Fig. 521

Solving equations (*i*) and (*ii*) we get,

$$M_b = 40{\cdot}16 \; tm. \; (hogging)$$

and $$M_c = 32{\cdot}20 \; tm. \; (hogging)$$

Max. free bending moment for span $AB = \dfrac{6 \times 8^2}{8} = 48 \; tm.$

Max. free bending moment for span $BC = \dfrac{4 \times 10^2}{8} = 50 \; tm.$

Max. free bending moment for span $CD = \dfrac{8 \times 6^2}{8} = 36 \; tm.$

Now the free and the fixed bending moment diagram can be drawn.

Reactions

$$\text{B.M. at } B = V_a \times 8 - \frac{6 \times 8^2}{2} = -40{\cdot}16$$

$$\therefore \quad V_a = 18{\cdot}98 \; t$$

$$\text{B.M. at } C = 18{\cdot}98 \times 18 + V_b \times 10 - 6 \times 8 \times 1' - 4 \times 10 \times 5 = -32{\cdot}20$$

$$\therefore \quad V_b = 49{\cdot}82 \; t$$

Again B.M. at $C=V_d\times 6-\frac{8\times 6^2}{2}=-32{\cdot}20$

$\therefore \quad V_d=18{\cdot}63\ t$

$\therefore \quad V_c=\text{Total load}-(V_a+V_b+V_d)$

$=(6\times 8+4\times 10+8\times 6)$

$-(18{\cdot}98+49{\cdot}82+18{\cdot}63)$

$=48{\cdot}57\ t$

Now it is easy to draw the S.F. diagram.

Problem 320. *A continuous beam ABC consists of two spans AB and BC of lengths 6 metres and 8 metres. The span AB carries a point load of 12 t at 4 metres from A, while the span BC carries a point load of 16 t at 5 m from C. Find the moments and reactions at the supports.*

Solution. In this case, it is necessary to draw the free bending moment diagrams for the two spans.

Max. free B.M. for the span AB

$=\frac{12\times 4\times 2}{6}=16\ tm.$

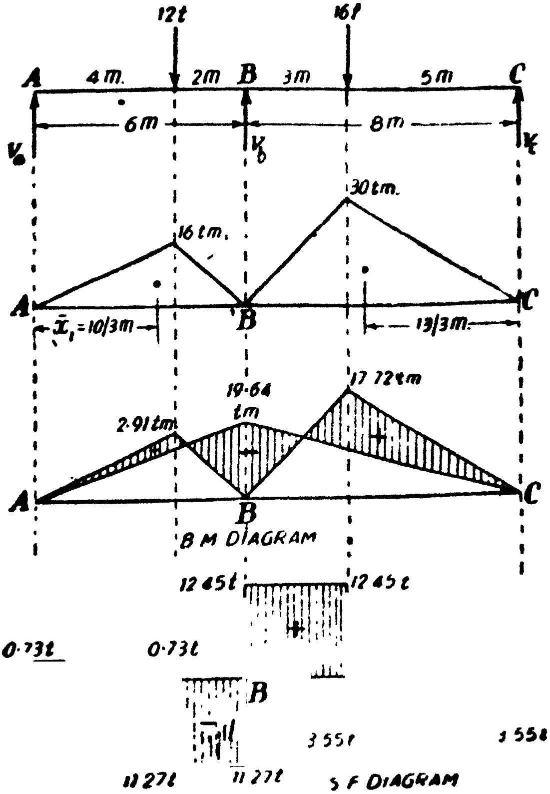


Fig. 522

Max. free B.M. for the span BC

$$=\frac{16\times3\times5}{8}=30\ tm.$$

Area of the free B.M. diagram of span AB

$$=a_1=\frac{1}{2}\times6\times16=48\ units$$

Area of the free B.M. diagram of span BC

$$=a_2=\frac{1}{2}\times8\times30=120\ units$$

Centroidal distance of free B.M. diagram on span AB from A

$$=\bar{x}_1=\frac{6+4}{3}=\frac{10}{3}\ m$$

Centroidal distance of free B.M. diagram on span BC from C

$$=\bar{x}_2=\frac{8+5}{3}=\frac{13}{3}\ m$$

Since each end of the beam is simply supported,

$$M_a=M_c=0$$

Applying the theorem of three moments for the spans AB and BC, we have,

$$M_a l_1+2M_b(l_1+l_2)+M_c l_2$$

$$=\frac{6a_1\bar{x}}{l_1}+\frac{6a_2\bar{x}_2}{l_2}$$

$$\therefore\quad O+2M_b(6+8)+O$$

$$=\frac{6\times48}{6}\times\frac{10}{3}+\frac{6\times120}{8}\times\frac{13}{3}$$

$$\therefore\quad 28M_b=550$$

$$\therefore\quad M_b=19\cdot64\ tm.$$

Reactions

B.M at $B=V_A\times6-12\times2=-19\cdot04$

$$\therefore\quad 6V_A=4\cdot36$$

$$\therefore\quad V_A=0\cdot73\ t$$

Again,

B.M. at $B=V_C\times8-16\times3=-19\cdot64$

$$\therefore\quad V_C=3\cdot55\ t$$

$$\therefore\quad V_b=(12+16)-(0\cdot73+3\cdot55)=23\cdot72\ t$$

§87. Fixed ends of Continuous beams

Consider the continuous beam shown in Fig. 523. Let the end A of the beam be fixed.

With the usual notation, the bending moment at any section distant x from B is given by

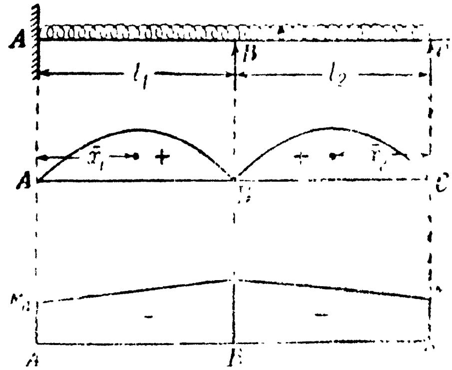


Fig. 523

$$EI\frac{d^2y}{dx^2}=M_x-M'_x$$

Multiplying by x, we get,

$$EIx\frac{d^2y}{dx^2}=M_x x-M'_x x$$

Integrating from $x=0$, to $x=l_1$ we have,

$$EI\left[x\frac{dy}{dx}-y\right]_0^{l_1}=\int_0^{l_1}M_x x\,dx-\int_0^{l_1}M'_x x\,dx$$

At $x=0$, $y=0$

and at $x=l_1$

$y=0$

$\therefore$ 0 = Moment of the free B.M. diagram on AB about B.

− Moment of the fixed B.M. diagram on AB about B.

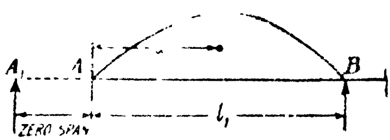


Fig. 524

$$\therefore\quad 0=a_1(l_1-\bar{x}_1)-\frac{l_1^2}{6}\left(M_b+2M_a\right)$$

Rearranging the terms,

$$2M_a l_1+M_b l_1=\frac{6a_1(l_1-\bar{x}_1)}{l_1}$$

The above relation can be obtained by introducing an imaginary zero span A_1A.

Applying the theorem of three moments for the spans A_1A and AB, we get,

$$0+2M_a(0+l_1)+M_bl_1=0+\frac{6a_1(l_1-\bar{x}_1)}{l_1}$$

$$\therefore \quad 2M_al_1+M_bl_1=\frac{6a_1(l_1-\bar{x}_1)}{l_1}$$

If the span AB carries a uniformly distributed load of w_1 per unit run over the whole span then the R.H.S. of the above equation will be equal to $\frac{w_1l_1^3}{4}$.

The above method can be applied to a fixed beam also to find the end moments.

For instance, consider the fixed beam AB of span l carrying a uniformly distributed load of w per unit run over the whole span.

Obviously the end moments M_a and M_b are equal. Introducing an imaginary zero span A_1A and applying the theorem of three moments for the spans A_1A and AB we have

$$0+2M_a(0+l)+M_bl=0+\frac{wl^3}{4} \qquad \textit{(See Fig. 525)}$$

Since, $M_a=M_b$,

$$3M_al=\frac{wl^3}{4}$$

$$\therefore \quad M_a=\frac{wl^2}{12}=M_b$$

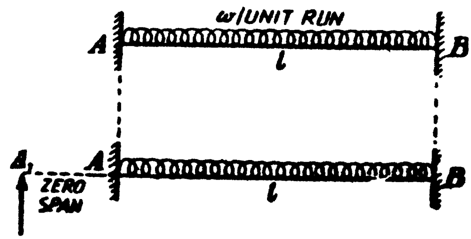


Fig. 525

§88. Propped cantilever

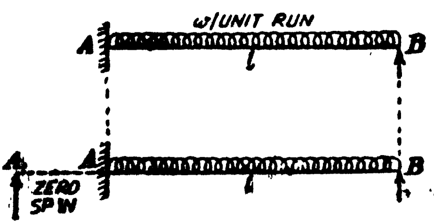


Fig. 526

Consider the propped cantilever AB, fixed at A and simply supported at B.

Introduce an imaginary zero span A_1A. Applying the theorem of three moments, we have,

$$0+2M_a(0+l)+M_bl=0+\frac{wl^3}{4}$$

But $M_b=0$,

since B is the simply supported end.

$$\therefore \qquad M_a = \frac{wl^2}{8}.$$

Problem 321. *A continuous beam ABC consists of two consecutive spans AB and BC 4 metres each and carrying a distributed load of 6 t per metre run. The end A is fixed and the end C is simply supported. Find the support moments and the reactions.*

Solution. Introduce an imaginary zero span A_1A.

Applying the theorem of three moments for the spans AB and BC, we have,

$$0+2M_a(0+4)+M_b\times4=0+6\times\frac{4^3}{4}$$

$$\therefore \qquad 8M_a+4M_b=96$$

$$\therefore \qquad 2M_a+M_b=24 \qquad \ldots(i)$$

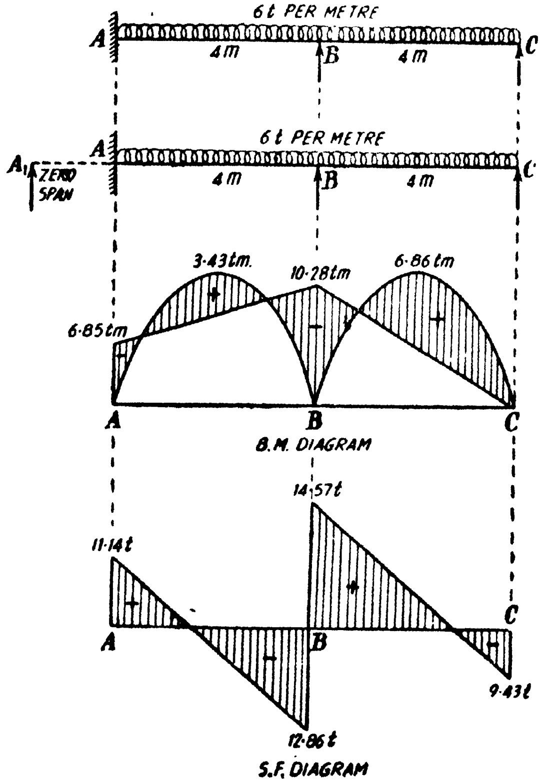


Fig. 527

Applying the theorem of three moments for the spans AB and BC, we have,

$$M_a \times 4 + 2M_b(4+4) + M_c \times 4 = 6 \times \frac{4^3}{4} + 6 \times \frac{4^3}{4}$$

But since C is the simply supported end,

$$M_c = 0$$

$\therefore \quad 4M_a + 16M_b = -192$

$\therefore \quad M_a + 4M_b = -48 \qquad \ldots(ii)$

Solving equations (i) and (ii), we get,

$$M_a = -\frac{48}{7} = -6.85 \ tm. \ (hogging)$$

and $\quad M_b = -\frac{72}{7} = -10.28 \ tm. \ (hogging)$

Max. free B.M. for span $AB = \frac{6 \times 4^2}{8} = 12 \ tm.$

Max. free B.M. for span $BC = 12 \ tm.$

Reactions

B.M. at $B = -6.85 + V_a \times 4 - 6 \times \frac{4^2}{2} = -10.28$

$\therefore \quad V_a = 11.14 \ t$

Again B.M. at $\quad B = V_c \times 4 - 6 \times \frac{4^2}{2} = -10.28$

$\therefore \quad V_c = 9.43 \ t$

$\therefore \quad V_b = 6 \times 8 - 11.14 - 9.43$

$\quad = 27.43 \ t$

§89. Continuous beam with supports at different levels

Consider the continuous beam shown in Fig. 518. Let the support B be δ_1 *below* A and δ_2 *below* C.

Proceeding as in §86, we have for the span AB, with A as origin

$$EI\left[x \frac{dy}{dx} - y\right]_0^{l_1} = \int_0^{l_1} M_x \, x \, dx + \int_0^{l_1} M_x' x \, dx$$

At $\quad x = l_1, \ y = -\delta$

$\therefore \quad EI(l_1 i_{ba} + \delta_1) = a_1 \bar{x}_1 - a_1' \bar{x}_1'$

$\therefore \quad EI(l_1 i_{ba} + \delta_1) = a_1 \bar{x}_1 + (M_a + 2M_b) \frac{l_1^2}{6}$

$$6EI i_{ba} = -\frac{6a_1 \bar{x}_1}{l_1} - \frac{6EI \delta_1}{l_1} - (M_a + 2M_b) l_1 \qquad \ldots(i)$$

Similarly, for the span BC taking C as origin it can be shown that

$$6EI\, i_{bc} = \frac{6a_2\bar{x}_2}{l_2} - \frac{6EI\,\delta_2}{l_2} - (M_c + 2M_b)l_2 \qquad (ii)$$

But $\qquad i_{ba} + i_{bc} = 0$

$\therefore$ Adding equations (i) and (ii) we get,

$$6EI\, i_{ba} + 6EI\, i_{bc} = 0 = \frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2} - \frac{6EI\,\delta_1}{l_1} - \frac{6EI\,\delta_2}{l_2}$$

$$-\left[M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2\right]$$

$$\therefore \quad M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2$$

$$= \frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2} - 6EI\left(\frac{\delta_1}{l_1} + \frac{\delta_2}{l_2}\right)$$

If the span AB and BC carry distributed loads of intensity w_1 and w_2 per unit run, then the above equation simplifies to,

$$M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2$$

$$= \frac{w_1 l_1^3}{4} + \frac{w_2 l_2^3}{4} - 6EI\left(\frac{\delta_1}{l_1} + \frac{\delta_2}{l_2}\right)$$

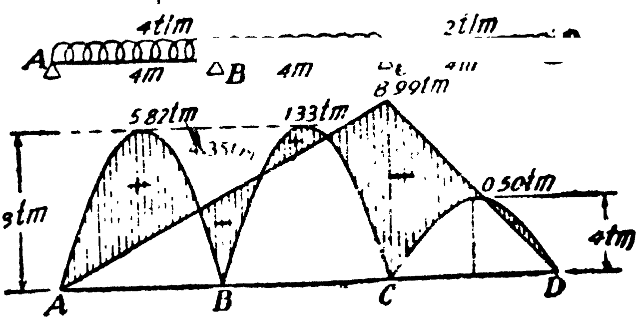
Problem 322. *Fig. 528 shows a continuous beam carrying an external loading. If the support B sinks by 0.25 cm. below the level of the other support find the support moments. Take I for section = 15000 cm⁴ and E = 2 × 10³ t/cm².*

Solution. Consider the spans AB and BC

$$M_a = 0$$

$$\delta_1 = -0.25 \text{ cm.}$$

$$\delta_2 = +0.25 \text{ cm.}$$



B M DIAGRAM

Fig. 528

$$\therefore \quad 0 + 2M_b(4+4) + 4M_c$$

$$= \frac{4 \times 4^3}{4} + \frac{4 \times 4^3}{4} - \frac{6 \times 2 \times 10^3 \times 15000}{(100)^2}\left\{\frac{0.25}{400} + \frac{0.25}{400}\right\}$$

$\therefore \quad 16M_b+4M_c=64+64-22{\cdot}5$

$\therefore \quad 16M_b+4M_c=105{\cdot}5$

$\therefore \quad 4M_b+M_c=26{\cdot}375 \qquad \ldots(i)$

Now consider the spans BC and CD

$$M_d=0$$

$$\delta_1=-0{\cdot}25\ cm.$$

$$\delta_2=0$$

$\therefore \quad 4M_b+2M_c(4+4)+0$

$$=\frac{4\times4^3}{4}+\frac{2\times4^3}{4}-\frac{6\times2\times10^3\times15000}{(100)^2}\left\{-\frac{0{\cdot}25}{400}\right\}$$

$\therefore \quad 4M_b+16M_c=64+32+11{\cdot}25$

$\therefore \quad 4M_b+16M_c=107{\cdot}25 \qquad \ldots(ii)$

Solving equations (i) and (ii), we get,

$$M_b=4{\cdot}97\ tm.\ (hogging)$$

and $$M_c=6{\cdot}49\ tm.\ (hogging)$$

Max. free B.M. for span $AB=\frac{4\times4^2}{8}=8\ tm.$

Max. free B.M. for span $BC=8\ tm.$

Max. free B.M. for span $CD=\frac{24\times^2}{8}=4\ tm.$

Fig. 528 shows the B.M. diagram for the girder.

The reactions at the supports may be determined as in the previous examples.

§90. Continuous beam with overhang

In this case the support moment over the support beyond which the beam overhangs is known. Hence the beam can be analysed as usual by applying the theorem of three moments. The following example explains this case.

Problem 323. *Find the support moments for the continuous girder shown in Fig. 529 and draw B.M. and S.F. diagrams.*

Solution. Consider the overhang $A.E$. The support moment at A is obviously

$$M_a=2\times1{\cdot}50=3\ tm.\ (hogging)$$

Consider the spans AB and BC. Applying the theorem of three moments, we have,

$$M_a(3)+2M_b(3+3)+M_c(3)=\frac{4\times3^3}{4}+\frac{3\times3^3}{4}$$

$$3(3)+12M_b+3M_c=27+\frac{81}{4}$$

$\therefore \quad 12M_b + 3M_c = 38{\cdot}25$

$\therefore \quad 4\,M_b + M_c = 12{\cdot}75 \qquad \ldots(i)$

Now consider the spans BC and CD

Obviously $\qquad M_d = 0$

Applying the theorem of three moments, we have,

$$M_b(3) + 2M_c(3+3) + 0 = \frac{3 \times 3^3}{4} + \frac{5 \times 3^3}{4}$$

$\therefore \quad 3M_b + 12M_c = 54$

$\therefore \quad M_b + 4M_c = 18 \qquad \ldots(ii)$

Solving equations (i) and (ii), we get,

$M_c = 3{\cdot}95$ *tm* and $M_b = 2{\cdot}20$ *tm*

Max. free B.M. at the centre of span $AB = \dfrac{4 \times 3^2}{8} = 4{\cdot}50$ *tm.*

Max. free B.M. at the centre of span $BC = \dfrac{3 \times 3^2}{8} = 3{\cdot}375$ *tm.*

Max. free B.M. for the span $CD = \dfrac{5 \times 3^2}{8} = 5{\cdot}625$ *tm.*

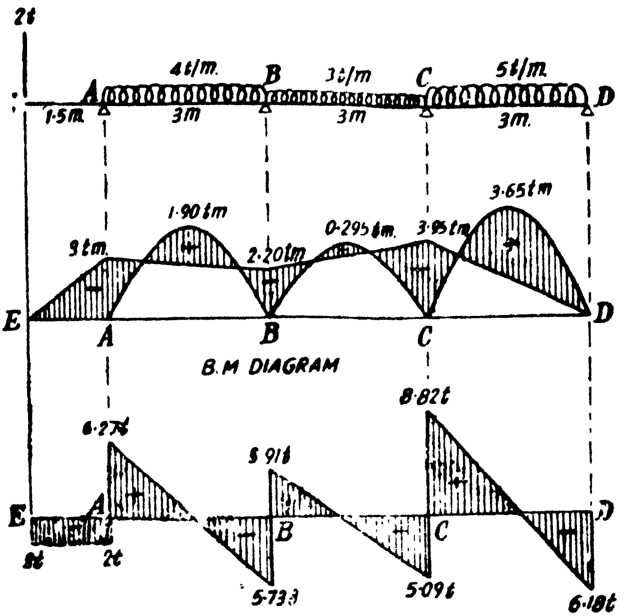


Fig. 529

Reactions

B.M. at $B = V_a \times 3 - \frac{4\times 3^2}{2} - 2\times 4\cdot 5 = -2\cdot 20$

$\therefore$ $V_a = 8\cdot 27\ t$

B.M at $C = 8\cdot 27 \times 6 + V_b \times 3 - \frac{3\times 3^2}{2}$

$-4\times 3\times 4\cdot 5 - 2\times 7\cdot 5 = -3\cdot 95$

$\therefore$ $V_b = 9\cdot 64\ t$

Again B.M. at $C = V_d \times 3 - \frac{5\times 3^2}{2} = -3\cdot 95$

$\therefore$ $V_d = 6\cdot 18\ t$

Total applied load on the girder

$= 2 + (4\times 3) + 3(3) + 5(3) = 38\ t$

$\therefore$ $V_c = 38 - (8\cdot 27 + 9\cdot 64 + 6\cdot 18)$

$\therefore$ $V_c = 13\cdot 91\ t$

§91. Continuous beam with different moment of inertia for different spans

Let AB and BC be two successive spans of a continuous beam Let l_1, l_2 be the span lengths AB and BC respectively.

Let M_a, M_b, M_c be the support moments at A, B and C respectively.

Let I_1 and I_2 be the moments of inertia of the beam section for the spans AB and CD. With the usual notations, the bending moment at any section in AB is given by

$$EI_1 \frac{d^2y}{dx^2} = M_x - M'_x$$

Multiplying by x,

$$EI_1\, x \frac{d^2y}{dx^2} = M_x x - M'_x x$$

Integrating from $x = 0$ to $x = l_1$

$$EI_1 \left[x\frac{dy}{dx} - y \right]_0^{l_1} = \int_0^{l_1} M_x x\, dx - \int_0^{l_1} M'_x x dx$$

We know that

(*i*) At $x = 0$, $y = 0$

(*ii*) At $x = l_1$, $y = 0$ and $\frac{dy}{dx} = i_{ba}$

(*iii*) $\int^{l_1} M_x x\, dx = a_1 x_1$

(iv) $\int_0^{l_1} M'_x x\, dx = a_1'\bar{x}_1'$

$\therefore \quad EI_1 l_1 \quad i_{ba} = a_1\bar{x}_1 - a_1'\bar{x}_1'$

But $$a_1'x_1' = \frac{M_a+M_b}{2}\, l_1\, \frac{M_a+2M_b}{M_a+M_b} \cdot \frac{l_1}{3}$$

$$= (M_a+2M_b)\frac{l_1^2}{6}$$

$\therefore \quad EI_1 l_1 \quad i_{ba} = a_1\bar{x}_1 - (M_a+2M_b)\frac{l_1^2}{6}$

$$\therefore \quad 6E\, i_{ba} = \frac{6a_1\bar{x}_1}{I_1 l_1} - \frac{(M_a+2M_b)\, l_1}{I_1} \quad \ldots(i)$$

Similarly considering the span BC it can be shown that,

$$6E\, i_{bc} = \frac{6a_2\bar{x}_2}{I_2 l_2} - \frac{(M_c+2M_b)\, l_2}{I_2} \quad \ldots(ii)$$

But $\quad i_{ba} + i_{bc} = 0$

$\therefore$ Adding equations (i) and (ii)

$$6E(i_{ba}+i_{bc}) = \frac{6a_1\bar{x}_1}{I_1 l_1} + \frac{6a_2\bar{x}_2}{I_2 l_2} - \left\{ M_a\frac{l_1}{I_1} + 2M_b\left(\frac{l_1}{I_1}+\frac{l_2}{I_2}\right) + M_c\frac{l_2}{I_2}\right\} = 0$$

$$\therefore \quad M_a\left(\frac{l_1}{I_1}\right) + 2M_b\left(\frac{l_1}{I_1}+\frac{l_2}{I_2}\right) + M_c\left(\frac{l_2}{I_2}\right) = \frac{6a_1\bar{x}_1}{I_1 l_1} + \frac{6a_2\bar{x}_2}{I_2 l_2}$$

For the particular case when $I_1 = I_2 = I$ the above relation simplifies to,

$$M_a l_1 + 2M_b(l_1+l_2) + M_c l_2 = \frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2}$$

Problem 324. *A continuous beam ABC consists of spans AB and BC of lengths 3 m and 4 m respectively the ends A and C being simply supported. If the spans AB and BC carry uniformly distributed loads of 5 t per metre and 8 t per metre respectively determine the support moments at A, B and C. Draw S.F. and B.M. diagrams. The moments of inertia for the spans AB and BC are I and 2I respectively.*

Solution. Since the ends A and C are simply supported

We have, $\quad M_a = M_c = 0$

Applying the theorem of three moments for the spans AB and BC

$$M_a\left(\frac{l_1}{I_1}\right) + 2M_b\left(\frac{l_1}{I_1}+\frac{l_2}{I_2}\right) + M_c\left(\frac{l_2}{I_2}\right) = \frac{6a_1\bar{x}_1}{I_1 l_1} + \frac{6a_2\bar{x}_2}{I_2 l_2}$$

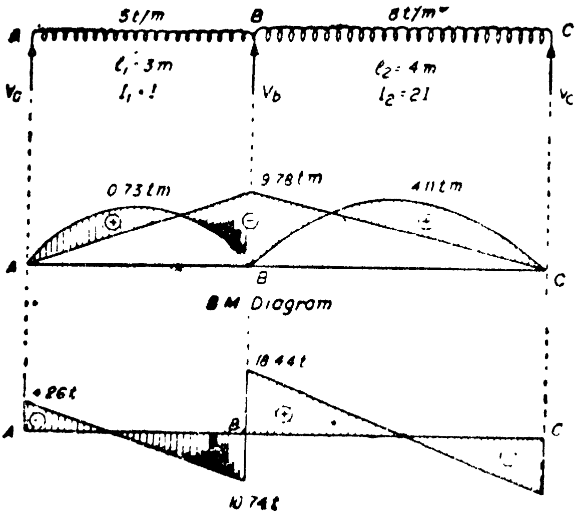


Fig. 530

$$=\frac{w_1l_1^3}{4I_1}+\frac{w_2l_2^3}{4I_2}$$

$$\therefore\quad 0+2M_b\left(\frac{3}{I}+\frac{4}{2I}\right)+0=\frac{5\times3^3}{4I}+\frac{8\times4^3}{4(2I)}$$

$\therefore$ $10\ M_b=97\cdot75$ $\quad\therefore$ $M_b=9\cdot775$ *tm.* *say 9·78 tm.*

Maximum free B.M. for span $AB=5\times\frac{3^2}{8}=5\cdot62$ *tm.*

Maximum free B.M. for span $BC=8\times\frac{3^2}{8}=9$ *tm.*

$\therefore$ Actual B.M. at centre of $AB=5\cdot62-\frac{1}{2}\,(9\cdot78)=0\cdot73$ *tm.*
Actual B.M. at centre of $BC=9\cdot00-\frac{1}{2}\,(9\cdot78)=4\cdot11$ *tm.*

Reactions

Consider the span *AB*,

B.M. at $B=V_a\times3-5\times\frac{3^2}{2}=-9\cdot78$

$\therefore$ $V_a=4\cdot26\ t$

Similarly considering the span *BC*,

B.M. at $B=V_c\times4-8\times\frac{4^2}{2}=-9\cdot78$

$\therefore$ $V_c=13\cdot56\ t$

$\therefore$ $V_b=(5\times3)+(8\times4)-(4\cdot26+13\cdot56)$

$=29\cdot18\ t$

Problems for Exercise

1. A fixed beam of span l carries two points loads W each placed symmetrically at a distance a from each support. Find the fixing moment at the supports and the bending moments at the centre. Find also the deflections under either load and at the centre.

$$\left[\frac{Wa\,(l-a)}{l}\,;\ \frac{Wa^2}{l}\,;\ \frac{Wa^3(2l-3a)}{6\,EI\,l}\,;\ \frac{Wa^2\,(3l-4a)}{24EI}\right]$$

2. A beam of span l is fixed at both the ends. It is subjected to a couple M applied at the middle point of the beam about a horizontal axis normal to the beam. Show that the fixing couple at each support is $\frac{M}{4}$ in the same direction as M and that the slope at the centre is $\frac{Ml}{16\,EI}$.

3. A fixed beam of span 4 *metres* carries a uniformly distributed load of 3 *t* per *metre*. Find the fixing moment at either support and the bending moment at the centre. (4 *tm.* ; 2 *tm.*)

4. A fixed beam of span 6 *metres* carries two point loads, 15 *t* and 24 *t* at distances 2*m* and 4*m* from the left end. Find the end moments Find also the maximum sagging moment. Draw S.F. and B.M. diagrams.

(24 *tm.* ; 28 *tm.* ; 15·33 *tm.* Reaction at left Support = 17·33 *t* ; Reaction at the right support = 21·67 *t*)

5. A fixed beam of span 5 *metres* carries a point load of 10 *t* at the centre and point loads of 4 *t* at 2 *m* from each end. Find the maximum positive and negative moments.

6. A beam of uniform section and span l firmly encastered at the ends carries two point loads, each W symmetrically situated at $\frac{l}{3}$ from the ends. The beam is propped at mid span by a force of such magnitude that the greatest bending moment to which the beam is subjected is as small as possible. (*a*) Determine magnitude of the supporting force at mid span. (*b*) Sketch the bending moment diagram for the beam showing on it values required for drawing it to cale. (*c*) Determine the resultant deflection at midspan in terms of W, s, E and I, and state whether it is upwards or downwards. Neglect the weight of the beam.

$$\left[(a)\ \frac{4}{3}\,W\,;\ (b)\ \text{Fixing moment} = \frac{Wl}{18}\,;\ (c)\ \frac{Wl}{1296EI}\ \text{downwards}\right]$$

(*London University*)

7. A beam of span l carries a central load W. It is so constrained at the ends that when the end slope is i, the restraining couple at the supports is μi. Prove that the magnitude of the restraining

couple at each end is $\frac{\mu w l^2}{8} \div (\mu l + 2EI)$ and that the magnitude of the central deflection is $\frac{Wl^3}{192EI}\left(\frac{\mu l+8EI}{\mu l+2EI}\right)$

(*London University*)

8. A beam of span 12 *m* carries two point loads 10 *t* and 15 *t* at distances 4 *m* and 8 *m* respectively from the left end. Find the fixed end moments and draw the B.M. diagram.

(31·11 *tm.* and 35·56 *tm.* ; 2·75 *m* from left end and 2·6 *m* from right end)

9. A beam of uniform section and span *l* is firmly built-in at the ends and carries a load whose intensity varies from zero at the left end to *w* at the right end. Determine the fixing moments and the reactions. Find also the position and magnitude of the maximum sagging moment.

10. A uniform beam simply supported at the ends carries a uniformly distributed load which produces a maximum deflection of 2·5 *cm.* and a maximum bending stress of 300 *kg./cm.*^{2} An equal beam, built in at the ends carries a uniformly distributed load of a different magnitude. For the built-in beam determine (*a*) the maximum bending stress if the maximum deflections of the two beams are equal, and (*b*) the maximum deflection if the maximum bending stresses for the two beams are equal.

(1000 *kg./cm.*^{2} ; 0·75 *cm.*)

11. A beam of uniform section simply supported at its ends carries a concentrated load of 8 *t* at mid span. Find the concentrated load which the same beam will carry at mid span when its ends are built-in and (*a*) the maximum deflection remains unchanged (*b*) the maximum bending moment remains the same.

(32 *t* ; 16 *t*)

12. A fixed beam *AB* of span 6 *m* carries a uniformly distributed load of 2 *t* per *metre* run over the left half and 3 *t* per *metre* run over the right half and a concentrated load of 4 *t* at the centre of the span. Calculate the fixed end moments. Assume uniform flexural rigdity.

$\left(9\frac{15}{16}\ tm.;\ 11\frac{1}{16}\ tm.\right)$

13. A beam *AB* 6 *metres* long is fixed at *A* and simply supported at *B* and carries a point load of 20 *t* at 4 *metres* from *A*. Find the fixing moment at *A* and the reactions at the two supports. Draw S.F. and B.M. diagrams.

(M_a=17·78 *tm.* Max. Sagging moment=20·74 *tm.* V_a=9·63 *t* and V_b=10·37 *t*)

14. A beam *AB* 6 *metres* long is fixed at *A* and simply supported at *B* and carries a uniformly distributed load of 4 *t* per *metre* over the whole span and a point load of 10 *t* at mid span. Find

the fixing moment at A and the reactions at the two supports. Draw S.F. and B.M. diagrams.

[$M_a = 29{\cdot}25$ *tm*. Max. Sagging moment $= 18{\cdot}375$ *tm*. $V_a = 21{\cdot}875$ *t*, $V_b = 12{\cdot}125$ *t*]

15. A continuous beam ABC 8 *metres* long consists of two spans $AB=3$ *m* and $BC=5$ *m*. The span AB carries a load of 5 *t/m* while the span BC carries a load of 3 *t/m*. Find the support moment at B and the reactions at the supports.

[$M_b = 10{\cdot}625$ *tm*. ; $V_a = 3{\cdot}96$ *t* ; $V_b = 20{\cdot}66$ *t* ; $V_c = 5{\cdot}38$ *t*]

16. A continuous beam $ABCDE$ 12 *metres* long consists of four spans of 3 *metres* each. The beam carries a distributed load of 4 *t/m* over the whole span. Find the moments and reactions at the supports

$$M_a = M_e = 0 \quad M_b = M_d = \frac{27}{7}\ tm. \ ; \ M_c = \frac{18}{7}\ tm.$$

$$V_a = V_e = \frac{33}{7}\ t \ ; \ V_b = V_d = \frac{96}{7}\ t \ ; \ V_c = \frac{78}{7}\ t \ \Big]$$

17. A continuous beam ABC has two spans $AB = l_1$ and $BC = l_2$. The beam simply rests on the end supports and it carries a uniformly distributed load of w per unit length on its whole length. If the support B sinks an amount δ below the level of the supports A and C, show that the reaction at B is

$$R_b = \frac{w\,(l_1+l_2)}{2} + \frac{w\,(l_1^3 - l_2^3)}{8 l_1 l_2} \quad \frac{3EI\,\delta(l_1+l_2)}{l_1^2\, l_2^2}$$

(*London University*)

18. A continuous beam ABC consists of two spans AB of length 4 *m* and BC of length 3 *m*. The span AB carries a point load of 10 *t* at its middle point. The span BC carries a point load of 12 *t* at 1 *m* from C. Find the moments at the support, and the maximum positive moment for each span. Find also the reactions at the supports.

$$\Big[\ M_a = M_c = 0 \ ; \ M_b = \frac{46}{7}\ tm.$$

$$\text{Max.} + ve \text{ Moment for } AB = \frac{47}{7}\ tm.$$

$$\text{Max.} + ve \text{ Moment for } BC = \frac{122}{21}\ tm.$$

$$V_a = \frac{47}{14}\ t \ ; \ V_b = \frac{100}{7}\ t \ ; \ V_c = \frac{61}{14}\ t \ \Big]$$

10

Torsion of Shafts

§92. Pure torsion

A shaft of circular section is said to be in pure torsion when it is subjected to equal and opposite end couples whose axes coincide with the axis of the shaft. While a beam *bends* as an effect of bending moment, a shaft *twists* as an effect of torsion. At any point in the section of the shaft, a shear stress is induced or more exactly, the state of stress at any point in the cross-section of the shaft is one of *pure-shear*. By the principle of complementary shear stresses, we know that in a state of simple shear there are two planes carrying the shear stress of the same intensity. These planes must be perpendicular to each other.

In the case of the shaft in torsion, the planes of shear at a point are (*i*) the cross-section itself and (*ii*) the plane containing the point and the axis of the shaft.

§93. Theory of pure torsion

Fig. 531 shows a solid cylindrical shaft of radius R and length l subjected to a couple or a twisting moment T at one end, while its other end is held or fixed by the application of a balancing couple of the same magnitude.

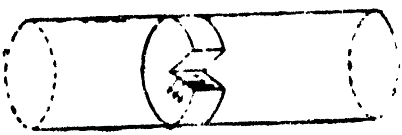


Fig. 531

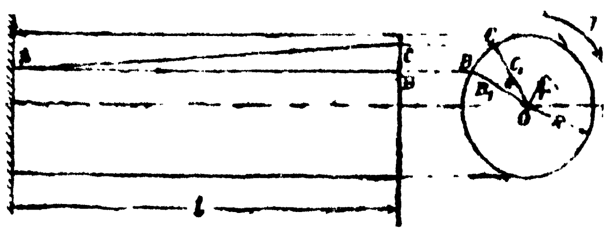


Fig. 532

Let AB be a line on the surface of the shaft and parallel to the axis of the shaft before the deformation of the shaft. As an effect of torsion this line, after the deformation of the shaft, takes the form AC.

The angle $CAB=\phi$ represents the shear strain of the shaft material at the surface. This angle being small, we have,

$$BC=l\phi$$

$$\therefore \quad \phi=\frac{BC}{l} \quad ...(i)$$

Let the angle BOC be the angular movement of the radius OB due to the strain in the length of the shaft. Let $BOC=\theta$. Let f_s be the shear stress intensity at the surface of the shaft.

We know, $f_s=\phi C$ ...(ii)

where C = Modulus of rigidity of the shaft material.

$$\therefore \quad f_s=\left(\frac{BC}{l}\right)C$$

But $BC=R\theta$

$$\therefore \quad f_s=\frac{R\theta}{l}.C$$

$$\therefore \quad \frac{f_s}{R}=\frac{C\theta}{l} \quad ...(iii)$$

The shaft may be taken to consist of infinite number of elemental hollow shafts, one surrounding the other.

If the deformation of a line on the surface of any such interior cylinder, at a radius r be considered it can be similarly visualized that the shear stress intensity q at the radius r is given by the relation,

$$\frac{q}{r}=\frac{C\theta}{l}$$

$$\therefore \quad \frac{f_s}{R}=\frac{q}{r}=\frac{C\theta}{l}$$

Since C, θ and l are constants, it follows that at any section of the shaft, the shear stress intensity at any point is proportional to the distance of the point from the axis of the shaft. Hence the shear stress is maximum at the surface and the shear stress is zero at the axis of the shaft.

§94. Moment of resistance

Fig. 533 shows the section of a shaft of radius R subjected to pure torsion. Let f_s be the maximum shear stress which occurs at the surface.

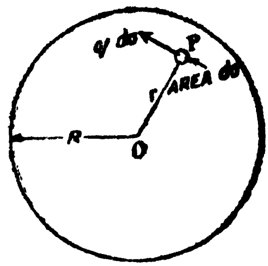


Fig. 533

Consider an elemental area da at a distance r from the axis of the shaft.

Shear stress offered by the elemental area $=q=\frac{r}{R}f_s$.

∴ Shear resistance offered by the elemental area

$$q.da = \frac{r}{R} f_s \,.\, da$$

∴ Moment of resistance offered by the elemental area

$$= \frac{r}{R} f_s da.\, r$$

$$= \frac{f_s}{R} .\, da \,.\, r^2$$

∴ Total moment of resistance offered by the cross-section of the shaft

$$= T = \frac{f_s}{R} \Sigma da \,.\, r^2$$

But $\Sigma da.r^2$ represents the moment of inertia of the section of the shaft about the axis of the shaft, *i.e.*, the quantity $\Sigma da.r^2$ is the polar moment of inertia I_p of the section of the shaft.

$$\therefore \quad T = \frac{f_s}{R} . I_p$$

$$\therefore \quad \frac{T}{I_p} = \frac{f_s}{R} \qquad \ldots(iv)$$

But from eq. (*iii*),

$$\frac{f_s}{R} = \frac{C\theta}{l}$$

$$\therefore \quad \frac{T}{I_p} = \frac{f_s}{R} = \frac{C\theta}{l}$$

§95. Assumptions in the theory of pure torsion

The theory of pure torsion is based on the following assumptions :

(*i*) The material of the shaft is uniform throughout.

(*ii*) The twist along the shaft is uniform.

(*iii*) The shaft is of uniform circular section throughout.

(*iv*) Cross-sections of the shaft, which are plane before twist remain plane after twist.

(*v*) All radii which are straight before twist remain straight after twist.

§96. Polar modulus

Let T be the moment of torsional resistance of the section of a shaft of radius R and I_p the polar moment of inertia of the shaft section.

The shear stress intensity q at any point on the section distant r from the axis of the shaft is given by

$$q = \frac{T}{I_p} .\, r$$

The maximum shear stress f_s occurs at the greatest radius R

$$\therefore \quad f_s = \frac{T}{I_p} . R$$

or $$T = f_s . \frac{I_p}{R}$$

or $$T = f_s . Z_p$$

where $$Z_p = \frac{I_p}{R}$$

$$= \frac{\text{Polar moment of inertia of the shaft section}}{\text{maximum radius}}$$

This ratio is called *polar modulus* of the shaft section. The greatest twisting moment which a given shaft section can resist

= maximum permissible shear stress × polar modulus.

Hence for a shaft of a given material the magnitude of the polar modulus is a measure of its strength in resisting torsion.

Given a number of shafts of the same length and material, the shaft which can resist the greatest twisting moment is the one whose polar modulus is greatest. Shafts of the same material and length having the same polar modulus have the same strength.

For a solid circular shaft of diameter d

$$I_p = \frac{\pi d^4}{32}$$

$$R = \frac{d}{2}$$

$$\therefore \quad Z_p = \frac{\pi d^3}{16}$$

$\therefore$ Moment of resistance

$$= f_s . Z_p$$

$\therefore$ Moment of resistance

$$= f_s . \frac{\pi d^3}{16}$$

For a hollow shaft whose external and internal diameters are d_1 and d_2,

$$= \frac{\pi}{32} (d_1^4 - d_2^4)$$

$$R = \frac{d_1}{2}$$

$$Z_p = \frac{\pi}{16 d_1} (d_1^4 - d_2^4)$$

Moment of resistance

$$=f_s \cdot Z_p$$

$$=f_s \cdot \frac{\pi}{16d_1}(d_1^4-d_2^4)$$

Torsional rigidity

Let a twisting moment T produce a twist of θ radians in a length l.

We know the relation,

$$\frac{T}{I_p}=\frac{C\theta}{l}$$

$$\therefore \qquad \theta=\frac{Tl}{CI_p}$$

For a given shaft the twist is therefore proportional to the twisting moment T. In a beam the bending moment produces a *bend* or *deflection*; in the same manner a torque produces a twist in a shaft. The expression CI_p corresponds to a similar quantity EI* in the expression for deflection of beams. The quantity CI_p is called *Torsional rigidity*. Obviously the quantity CI_p stands for the torque required to produce a twist of 1 radian per unit length of the shaft.

§97. Horse power transmitted by a shaft

Let a shaft turning at N rpm transmit P horse power. Let the mean torque to which the shaft is subjected be T *kg. m.*

$\therefore$ Work done per second

$$=\text{Mean Torque} \times \text{angle turned per second}$$

$$=T\frac{N}{60}2\pi \text{ kg. m. per second.}$$

$\therefore$ H.P. Transmitted

$$=\frac{\left(\frac{2\pi NT}{60}\right)}{75}$$

$$=\frac{2\pi NT}{4500}$$

Problem 325. *In a tensile test, a test piece 25 mm. in diameter, 200 mm. gauge length stretched 0·0975 mm. under a pull of 5000 kg. In a torsion test, the same rod twisted 0·025 radian over a length of 200 mm., when a torque of 4000 kg. cm. was applied. Evaluate the Poisson's ratio and the three elastic moduli for the material.*

(*A.M.I.E.*, *Winter 1975*)

*The quantity EI in expressions for beam deflections is called *flexural rigidity*.

Solution.

$$d=2{\cdot}5 \text{ cm.}, \quad l=20 \text{ cm.}, \quad P=5000 \text{ kg.}$$

Tensile stress $=f=\dfrac{5000}{\frac{\pi}{4}(2{\cdot}5)^2}=1018{\cdot}59 \text{ kg./cm.}^2$

Tensile strain $=e=\dfrac{0{\cdot}0975}{200}$

∴ Elastic Modulus

$$=E=\frac{f}{e}=\frac{1018{\cdot}59\times 200}{0{\cdot}0975}=2{\cdot}089\times 10^6 \text{ kg./cm.}^2$$

$$\frac{T}{I_p}=\frac{C\theta}{l}$$

$$\therefore \quad C=\frac{Tl}{I_p\theta}=\frac{4000\times 20}{\frac{\pi(2{\cdot}5)^4}{32}\times 0{\cdot}025}=0{\cdot}834\times 10^6 \text{ kg./cm.}^2$$

We know,

$$E=2C\left(1+\frac{1}{m}\right)$$

$$\therefore \quad 1+\frac{1}{m}=\frac{E}{2C}=\frac{2{\cdot}089\times 10^6}{2\times 0{\cdot}834\times 10^6}$$

$$\therefore \quad 1+\frac{1}{m}=1{\cdot}252$$

$$\therefore \quad \frac{1}{m}=0{\cdot}252$$

∴ *Poisson's Ratio* $=0{\cdot}252$

We know,

$$E=3K\left(1-\frac{2}{m}\right)$$

$$K=\frac{E}{3\left(1-\frac{2}{m}\right)}$$

$$=\frac{2{\cdot}089\times 10^6}{3(1-2\times 0{\cdot}252)} \text{ kg./cm.}^2$$

$$=1{\cdot}404\times 10^6 \text{ kg./cm.}^2$$

Problem 326. *A steel shaft transmits 140 horse power at 160 rpm. If the shaft is 100 mm. diameter, find the Torque on the shaft and the maximum shear stress induced. Find also the twist of the shaft in a length of 6 metres. Take* $C=8\times 10^5$ *kg./cm.*²

Solution.

$$\text{H.P.}=\frac{2\pi NT}{4500}$$

$$\therefore \quad 140=\frac{2\pi\times160\times T}{4500}$$

$$T=\frac{140\times4500}{2\pi\times160}\ kg.\ m.$$

$$=626{\cdot}6\ kg.\ m.$$

$$=62660\ kg.\ cm.$$

$$T=f_s\,.\,\frac{\pi D^3}{16}$$

$$\therefore \quad f_s=\frac{16T}{\pi D^3}=\frac{16\times62660}{\pi\times10^3}\ kg./cm.^2$$

$$=319{\cdot}1\ kg./cm.^2$$

$$\frac{T}{I_p}=\frac{C\theta}{l}$$

$$\therefore \quad \theta=\frac{T}{I_p}\,.\,\frac{l}{C}\ \text{radians}$$

$$=\frac{62660}{\left(\frac{\pi\times10^4}{32}\right)}\,.\,\frac{600}{8\times10^5}\ \text{radian}$$

$$=0{\cdot}04786\ \text{radian}$$

$$=2°\,45'$$

Problem 327. *Find the H.P. that can be transmitted by a shaft 60 mm. diameter, at 180 r.p.m. if the permissible shear stress is 850 kg./cm.²*

Solution.

$$T=f_s\,\frac{\pi}{16}D^3=\frac{850\times\pi(6)^3}{16}\ kg./cm.^2$$

$$=36060\ kg.\ cm.$$

$$=360{\cdot}60\ kg.\ m.$$

$$\therefore \text{ H.P. transmitted}=\frac{2\pi NT}{4500}=\frac{2\pi\times180\times360{\cdot}6}{4500}$$

$$=90{\cdot}63\ H.P.$$

Problem 328. *A solid circular shaft transmits 100 hp at 200 rpm. Calculate the shaft diameter if the twist in the shaft is not to exceed 1° in 2 metres length of shaft and the shearing stress is limited to 500 kg./cm². Take C= [illegible] m.²* *(A.M.I.E., Summer 1979)*

Solution.

$$\text{H.P. of shaft}\qquad \frac{2\pi NT}{4500}=100\ hp.$$

$$T=\frac{4500\times100}{2\pi\times200}=358{\cdot}10\ kg.m.=35810\ kg.\ cm.$$

(*i*) *Twist consideration*

$$\frac{T}{I_p}=\frac{C\theta}{l}$$

$$\therefore \quad I_p=\frac{Tl}{C\theta} \qquad \frac{\pi d^4}{32}=\frac{Tl}{C\theta}$$

$$\therefore \quad d^4=\frac{2Tl}{\pi C\theta}$$

$$\therefore \quad d^4=\frac{32\times 35810\times 200\times 180}{\pi\times 1\times 10^6\times\pi}=4179{\cdot}8$$

$$\therefore \quad d=8{\cdot}04 \text{ cm.}$$

(*ii*) *Shear stress consideration*

$$T=f_s\frac{\pi d^3}{16}=35810$$

$$d^3=\frac{16\times 35810}{500\pi}=364{\cdot}76$$

$$\therefore \quad d=7{\cdot}14 \text{ cm}$$

Hence we should provide at least a diameter of 8·04 cm.

Problem 329. *A shaft has to transmit 140 H.P. at 160 r.p.m. If the shear stress is not to exceed 650 kg./cm.² and the twist in a length of 350 cms. must not exceed 1°, find a suitable diameter. Take* $C=8\times 10^5$ *kg./cm.²*

Solution.

$$HP=\frac{2\pi NT}{4500}$$

$$\therefore \quad 140=\frac{2\pi\times 160\times T}{4500}$$

$$\therefore \quad T=\frac{140\times 4500}{2\pi\times 160} \text{ kg. m}$$

$$=626{\cdot}6 \text{ kg. m.}$$

$$=62{,}660 \text{ kg. cm.}$$

From shear stress consideration,

$$T=f_s\cdot\frac{\pi D^3}{16}$$

$$\therefore \quad D^3=\frac{16T}{f_s\pi}$$

$$=\frac{16\times 62660}{650\times\pi}$$

$$D=7{\cdot}89 \text{ cm.}$$

From stiffness consideration,

$$\frac{T}{I} = \frac{C\theta}{l}$$

$$\therefore \quad I_p = \frac{Tl}{C\theta}$$

$$\therefore \quad \frac{\pi D^4}{32} = \frac{Tl}{C\theta}$$

$$\therefore \quad D^4 = \frac{32}{\pi} \cdot \frac{Tl}{C\theta}$$

$$\therefore \quad D^4 = \frac{32}{\pi} \cdot \frac{62660 \times 350 \times 180}{8 \times 10^5 \times \pi}$$

$$\therefore \quad D = 11 \cdot 25 \text{ cm.}$$

Hence the required diameter

$$= 11 \cdot 25 \text{ cm.}$$

Problem 330. *A solid shaft is of 100 mm. diameter. It transmits 160 HP at 200 rpm. Find the maximum intensity of shear stress induced and the angle of twist for a length of 6 metres. Take* $C = 8 \times 10^5$ *kg/cm.*^{2}

Solution.

$$\text{H.P. transmitted} = \frac{2\pi NT}{60 \times 75}$$

$$\therefore \quad 160 = \frac{2\pi \times 200T}{60 \times 75}$$

$$\therefore \quad T = \frac{160 \times 60 \times 75}{2\pi \times 200} \text{ kg. m.}$$

$$= 572 \cdot 8 \text{ kg. m.}$$

$$= 57280 \text{ kg. cm.}$$

Polar moment of inertia

$$= I_p = \frac{\pi d^4}{32}$$

$$R = \frac{d}{2}$$

$$\frac{T}{I_p} = \frac{f_s}{R}$$

$$\therefore \quad f_s = \frac{T}{I_p} \cdot R = \frac{16T}{\pi d^3}$$

$$= \frac{16 \times 57280}{\pi \times 10^3} \text{ kg./cm.}^2$$

$$= 291 \cdot 7 \text{ kg./cm.}^2$$

$$\frac{T}{I_p} = \frac{C\theta}{l}$$

$$\therefore \quad \theta = \frac{T}{I_p} \cdot \frac{l}{C}$$

$$= \frac{57280 \times 32}{\pi \times 10^4} \cdot \frac{600}{8 \times 10^5} \textit{ radians.}$$

$$= 0{\cdot}04375 \textit{ radians.}$$

$$= 2° \; 30'.$$

Problem 331. *Find the diameter of the shaft required to transmit 80 horsepower at 150 r.p.m. if the maximum torque is likely to exceed the mean torque by 25% for a maximum permissible shear stress of 600 kg./cm² Find also the angle of twist for a length of 2 5 metres.*

Take $C = 8 \times 10^5$ *kg./cm.²*

Solution

$$\text{H.P.} = \frac{2\pi NT}{75 \times 60}$$

$$80 = \frac{2\pi \times 150 \times T}{4500}$$

$$\therefore \quad T = \frac{80 \times 4500}{2\pi \times 150} \textit{ kg. m.}$$

$$= 381{\cdot}9 \textit{ kg. m.}$$

$$= 38190 \textit{ kg. cm.}$$

The torque calculated above is the mean torque.

$\therefore$ Max. torque $= T_{max} = 1{\cdot}25 \times$ mean torque.

$$= 1{\cdot}25 \times 38190 \textit{ kg. cm.}$$

$$= 47737{\cdot}5 \textit{ kg. cm.}$$

Polar modulus $= \dfrac{I_p}{r_{max}} = \dfrac{\pi d^4 \times 2}{32 \times d} = \dfrac{\pi d^3}{16}$

$$T_{max} = f_s \times \text{Polar modulus}$$

$$\therefore \quad 47737{\cdot}5 = 600 \times \frac{\pi d^3}{16}$$

$$\therefore \quad d^3 = \frac{47737{\cdot}5 \times 16}{600\pi}$$

$$\therefore \quad d = 7{\cdot}40 \textit{ cms.}$$

$$\frac{T}{I_p} = \frac{C\theta}{l}$$

$$\therefore \quad \theta = \frac{T}{I_p} \cdot \frac{l}{C}$$

$$=\frac{47737{\cdot}5\times32}{\pi\times7{\cdot}4^4}\times\frac{250}{8\times10^5}\ radian$$

$$=0{\cdot}0507\ radian$$

$$=2^\circ\ 54'.$$

Problem 332. *A hollow shaft is to transmit 400 hp at 80 rpm. If the shear stress is not to exceed 600 kg./cm.² and the internal diameter is 0·6 of the external diameter, find the external and internal diameters, assuming that the maximum torque is 1·4 times the mean torque.* *(A.M.I.E. Winter 1976)*

Solution.

H.P. of the shaft $=\frac{2\pi NT_{mean}}{4500}=400$

$$\therefore\quad T_{mean}=\frac{4500\times400}{2\pi\times80}=3580{\cdot}98\ kg.\ m.$$

$$\therefore\quad T_{max}=1{\cdot}4\ T_{mean}=1{\cdot}4\times3580{\cdot}98=5013{\cdot}37\ kg.\ m.$$

$$=501337\ kg.\ cm.$$

Polar modulus $Z_p=\dfrac{\frac{\pi}{32}(D^4-d^4)}{\frac{D}{2}}=\dfrac{\pi}{16D}(D^4-d^4)$

$$=\frac{\pi}{16D}[D^4-0{\cdot}6^4D^4]=0{\cdot}1709\ D^3$$

$$T_{max}=f_sZ_p=501337$$

$$\therefore\quad 600\times0{\cdot}1709\ D^3=501337$$

$$\therefore\quad D^3=\frac{501337}{600\times0{\cdot}1709}=4889{\cdot}18$$

$$\therefore\quad D=16{\cdot}97\ cm.$$

$$\therefore\quad d=0{\cdot}6\times16{\cdot}97=10{\cdot}18\ cm.$$

Problem 333. *A hollow shaft with diameter ratio $\frac{3}{5}$ is required to transmit 600 horse power at 120 rpm with a uniform twisting moment. The shearing stress in the shaft must not exceed 600 kg./cm.² and the twist in a length of 2·5 m must not exceed 1°. Calculate the minimum external diameter of the shaft satisfying these conditions. Take the modulus of rigidity C = 8 × 10⁵ kg./cm.².* *(London University)*

Solution. Let the internal and external diameters of the shaft be d and D respectively.

$$\therefore\quad \frac{d}{D}=\frac{3}{5}$$

$$H.P.=\frac{2\pi NT}{4500}$$

$$600=\frac{2\pi\times120\times T}{4500}$$

$$\therefore \qquad T = \frac{600 \times 4500}{2\pi \times 120} \; kgm.$$

$$= 3581 \; kgm.$$

$$= 358100 \; kg.\ cm.$$

Polar moment of inertia

$$= I_p = \frac{\pi}{32}(D^4 - d^4)$$

$$\therefore \text{ Polar modulus } = \frac{I_p}{\left(\frac{D}{2}\right)} = \frac{2I_p}{D}$$

$$= \frac{2\pi}{32\,D}(D^4 - d^4)$$

$$= \frac{\pi}{16\,D}(D^4 - d^4)$$

$$= \frac{\pi}{16\,D}\left(D^4 - \frac{81}{625}D^4\right)$$

$$= \frac{\pi D^3}{16} \times \frac{554}{625}$$

$$= 0.1709\, D^3\ cm^3$$

Since the maximum shear stress is 600 kg/cm²

$$T = f_s \times \text{polar modulus}$$

$$\text{Polar modulus} = \frac{T}{f_s}$$

$$\therefore \qquad 0.1709\, D^3 = \frac{358100}{600}$$

$$\therefore \qquad D = 15.17\ cm$$

When the twist in a length of 2.5 m is 1°

$$I = \frac{\pi}{32}(D^4 - d^4) = \frac{\pi}{32}\left(D^4 - \frac{81}{625}D^4\right)$$

$$= \frac{544}{32 \times 625}\pi D^4$$

$$\frac{T}{I_p} = \frac{C\theta}{l}$$

$$\theta = \frac{l}{C} \cdot \frac{T}{I_p}$$

$$\frac{\pi}{180} = \frac{250}{8 \times 10^5} \times \frac{358100 \times 35 \times 625}{544\,\pi\,D^4}$$

$$\therefore \qquad D^4 = 75000$$

$$\therefore \qquad D = 16.55\ cm.$$

In order that the shear stress should not exceed 600 kg/cm^2 and that the twist in a length of 2·5 *m* should not exceed 1° the outside diameter of the shaft must be at least 16·55 *cm*.

Problem 334. *A solid aluminium shaft 100 cm. long and of 5 cm. diameter is to be replaced by a tubular steel shaft of the same length and the same outside diameter (i.e., 5 cm.), such that each of the two shafts could have the same angle of twist per unit torsional moment over the total length. What must the inner diameter of the tubular steel shaft be ? Modulus of rigidity of steel is three times that of aluminium.* *(A.M.I.E., Nov. 1966)*

Solution. If a torsional moment T be applied to a shaft of length l, the twist θ for the length l is given by

$$\frac{T}{I_p}=\frac{C\theta}{l}$$

where I_p = polar moment of inertia

and C = Modulus of rigidity

$\therefore$ Angle of twist per unit torsional moment

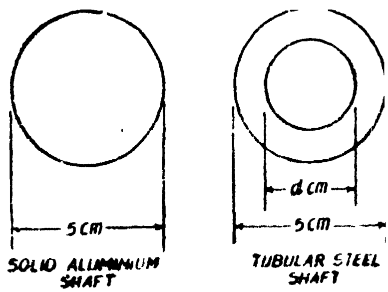


Fig. 534

$$=\frac{\theta}{T}=\frac{l}{I_pC}$$

Since the angle of twist per unit torsional moment is the same for the two shafts, we have $\left(\frac{l}{I_pC}\right)$ should have the same value for the two shafts. Since the two shafts have the same length, (I_pC) should be the same for the two shafts.

Let I_a and I_s be the polar moments of inertia of the aluminium and steel shafts.

Let C_a and C_s be the modulus of rigidity of aluminium and steel. Let d be the internal diameter of the steel shaft.

Hence we have

$$I_a\,C_a=I_sC_s$$

But $$C_s=3C_a$$

$\therefore$ $$I_aC_a=I_s.3I_a$$

$\therefore$ $$I_a=3I_s$$

$$\frac{\pi\times5^4}{32}=3\times\frac{\pi}{32}\left[5^4-d^4\right]$$

$\therefore$ $$5^4-d^4=\frac{5^4}{3}$$

$\therefore$ $$d^4=5^4-\frac{5^4}{3}$$

$$=625-\frac{625}{3}$$

$$d^4=\frac{1250}{3}$$

$$d=4{\cdot}518 \text{ cm.}$$

Problem 335. *The shaft shown in Fig. 535 rotates at 200 r.p.m. with 40 hp. and 20 hp taken off at A and B respectively and 60 hp. applied at C. Find the maximum shear stress developed in the shaft and the angle of twist (degree) of the gear A relative to C. Assume $G=0{\cdot}85\times10^6$ kg./cm².*

Solution.

Shaft between B and C

H.P. of the shaft $=60$ *hp.*

Let the torque in this part of the shaft be T_{bc}.

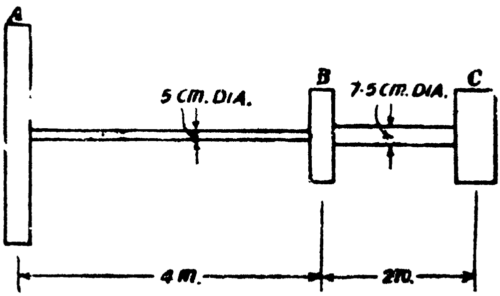


Fig. 535

$$\text{H.P.}=\frac{2\pi NT}{75\times60}$$

$$60=\frac{2\pi\times200}{4500}\times T_{bc}$$

$$\therefore \quad T_{bc}=\frac{60\times4500}{400\pi} \text{ kg. m.}$$

$$=214{\cdot}8 \text{ kg. m.}$$

$$=21480 \text{ kg. cm.}$$

Let f_s be the maximum shear stress in this part of the shaft.

$$f_s.\frac{\pi d^3}{16}=T_{bc}$$

$$f_s=\frac{16T_{bc}}{\pi d^3}$$

$$=\frac{16\times21480}{\pi\times7{\cdot}5^3} \text{ kg./cm}^2.$$

$$=259{\cdot}2 \text{ kg./cm.}^2$$

Shaft between B and A

H.P. of the shaft $=60-20=40$ *hp.*

Let the torque in this part of the shaft be T_{ab}

$$\therefore \quad 40=\frac{2\pi\times200\times T_{ab}}{4500}$$

$$\therefore \quad T_{ab}=\frac{40\times4500}{400\pi} \text{ kg. m.}$$

=14[illegible]·2 *kg. m.*

=[illegible] *kg. cm.*

Let f_s' be the maximum shear stress in this part of the shaft

$$\therefore \quad f_s' \frac{\pi d^3}{16} = T_{ab}$$

$$\therefore \quad f_s' = \frac{16 T_{ab}}{\pi d'^3}$$

$$= \frac{16 \times 14320}{\pi \times 5^3} \ kg./cm^2$$

$$= 583{\cdot}3 \ kg./cm^2.$$

Hence the greatest shear stress occurs in the 5 *cm* diameter shaft.

$\therefore$ *Max. shear stress* $= 583{\cdot}3 \ kg./cm.^2$

Twist of the shaft

Let θ_{bc} be the twist of the shaft *BC*.

$$\therefore \quad \theta_{bc} = \frac{l}{C} \cdot \frac{T_{bc}}{I_p}$$

$$= \frac{200}{0{\cdot}85 \times 10^6} \times \frac{21480 \times 32}{\pi \times 7{\cdot}5^4} \ radian$$

$$= 0{\cdot}01626 \ radian.$$

Let θ_{ab} be twist of the shaft *AB*

$$\theta_{ab} = \frac{l'}{C} \cdot \frac{T_{ab}}{I'_p}$$

$$= \frac{400}{0{\cdot}85 \times 10^6} \times \frac{14320 \times 32}{\pi \times 5^4} \ radian.$$

$$= 0{\cdot}1098 \ radian.$$

$\therefore$ Angle of twist of *A* with respect to *C*

$$= \theta_{ab} + \theta_{bc}$$

$$= 0{\cdot}1098 + 0{\cdot}01626$$

$$= 0{\cdot}12606 \ radian.$$

$$= 7^\circ \ 13'$$

Problem 336. *Show that for a given maximum shear stress the minimum diameter required for a solid circular shaft to transmit P horse power at N r.p.m. can be expresed as*

$$d = constant \times \sqrt[3]{\frac{P}{N}}$$

What value of the maximum shear stress has been used if the constant equals 7·70, d being in centimetres.

Solution.

$$\text{H.P.} = P = \frac{2\pi NT}{4500}$$

$$\therefore \quad T=\frac{4500P}{2\pi N}\ kg.\ m.$$

$$=\frac{450{,}000P}{2\pi N}\ kg.\ cm.$$

$$T=\frac{f_s.\ \pi d^3}{16}$$

$$\therefore \quad d^3=\frac{16T}{f_s.\pi}$$

$$=16\times\frac{450{,}000P}{2\pi N f_s.\pi}$$

$$=\frac{16\times 450{,}000}{2\pi^2 f_s}\cdot\frac{P}{N}$$

$$\therefore \quad d=K\sqrt[3]{\frac{P}{N}}$$

where $$K=\sqrt[3]{\frac{16\times 450{,}000}{2\pi^2 f_s}}$$

$$K=7\cdot 7=\sqrt[3]{\frac{16\times 450{,}000}{2\pi^2 f_s}}$$

$$(7\cdot 7)^3=\frac{16\times 450{,}000}{2\pi^2 f_s}$$

$$f_s=\frac{16\times 450{,}000}{2\pi^2\times(7\cdot 7)^3}$$

$$=800\ kg./cm^2.$$

Problem 337. *A hollow marine propeller shaft turning at 110 rpm is required to propel a vessel at 12 metres per sec. for the expenditure of 8450 shaft horse power, the efficiency of the propeller being 68 percent. The diameter ratio of the shaft is to be ½ and the direct stress due to thrust is not to exceed 80 kg./cm². Calculate (a) the shaft diameters (b) the maximum shearing stress due to torque.*

(*London University* ; **A.M.I.E**)

Solution.

Let the thrust exerted be F kg.

$\therefore$ Useful work done per second

= thrust × distance moved per second

$= F\times 12\ kg.\ m.\ per\ sec.$

$$\text{Efficiency} = \frac{\text{Useful work done}}{\text{Energy of the shaft}}=0\cdot 68$$

$$\therefore \quad \frac{F\times 12}{1450\times 75}=0\cdot 68$$

$\therefore \quad F = \frac{0·68 \times 8450 \times 75}{12}$ kg.

$= 35910$ kg.

Let the external diameter be D cm.

$\therefore$ Internal diameter

$= d = \frac{2}{3} D$ cm.

$\therefore$ Area of the section

$= A = \frac{\pi}{4}\left(D^2 - \frac{4}{9} D^2\right)$

$= \frac{\pi}{4} \cdot \frac{5}{9} D^2$

$= \frac{5\pi}{36} D^2$

Stress due to direct load

$= \frac{\text{thrust}}{\text{Area of shaft}} = 80$ kg./cm².

$\therefore \quad \frac{35910}{\frac{5\pi}{36} D^2} = 80$

$\therefore \quad D = 32·07$ cm.

$\therefore$ internal diameter $d = \frac{2}{3} \times 32·07$ cm.

$= 21·38$ cm.

Polar moment of inertia

$= I_p = \frac{\pi}{32}\left[32·07^4 - 21·38^4\right]$ cm.^{4}

$= 83300$ cm^4.

H.P. $\quad = P = \frac{2\pi NT}{4500}$

$\therefore \quad T = \frac{4500 \times P}{2\pi N}$

$= \frac{4500 \times 8450}{2\pi \times 110} \times 100$ kg. cm.

$= 5501000$ kg. cm.

$\frac{T}{I_p} = \frac{f_s}{R}$

$f_s = \frac{T}{I_p} \cdot R$

$$=\frac{5501000}{83300}\times\frac{32{\cdot}07}{2}\ kg./cm^2.$$

$$=1060\ kg./cm^2.$$

Problem 338. *Two shafts of the same meterial are subjected to the same torque. If the first shaft is of solid circular section and the second shaft is of hollow section whose internal diameter is $\frac{2}{3}$ of the outside diameter, compare the weights of the two shafts.*

(*AMIE, May 1974*)

Solution. The maximum torque a shaft section can safely resist is given by

$$T=f_s\ .\ Z_p$$

where f_s = permissible shear stress

Z_p = polar modulus of shaft section

In order the two shafts may have the same strength to resist torque, the polar moduli of the shafts must be equal.

Let D be the dimeter of the solid shaft.

$\therefore$ Polar modulus of the solid shaft section

$$=\frac{\pi D^3}{16}$$

Let D_1 be the external diameter of the hollow shaft.

$\therefore$ Internal diameter $=\frac{2}{3}D_1$

$\therefore$ Polar modulus of the hollow shaft section

$$=\frac{\pi\left(D_1^4-\frac{16}{81}D_1^4\right)}{16D_1}$$

$$=\frac{65}{81}\ \frac{\pi D_1^3}{16}$$

Equating the polar moduli of the two shaft sections, we have,

$$\frac{65}{81}\ \frac{\pi D_1^3}{16}=\frac{\pi D^3}{16}$$

$$\therefore\qquad \frac{D_1}{D}=\sqrt[3]{\frac{81}{65}}=1{\cdot}075$$

Since the two shafts are of the same material and are of the same length.

$$\frac{\text{Weight of hollow shaft}}{\text{Weight of solid shaft}}$$

$$=\frac{\text{Area of hollow shaft section}}{\text{Area of solid shaft section}}$$

$$=\frac{\frac{\pi}{4}\left(D_1^2-\frac{4}{9}D_1^2\right)}{\frac{\pi}{4}D^2}$$

$$=\frac{5}{9}\left(\frac{D_1}{D}\right)^2$$

$$=\frac{5}{9}(1.075)^2$$

$$=0.642$$

Problem [illegible]. *A solid circular shaft is to transmit 400 hp at 100 r.p.m. If the shear stress is not to exceed 800 kg/cm², find the diameter of the shaft.*

What percentage saving in weight would be obtained if the shaft is replaced by a hollow one whose internal diameter equals 0.6 of the external diameter, the length, the material and the maximum shear stress being the same. (A.M.I.E., Summer 197[illegible])

Solution.

Horse power transmitted

$$=\frac{2\pi NT}{4500}=400$$

$$\therefore \quad T=\frac{4500\times 400}{2\pi\times 100}=2864.78 \text{ kg. m.}$$

$$=286478 \text{ kg. cm.}$$

When a solid shaft is provided

$$f\cdot\frac{\pi D^3}{16}=T$$

$$D^3=\frac{16T}{f\pi}=\frac{16\times 286478}{800\pi}=[illegible]$$

$$\therefore \quad D=12.[illegible] \text{ cm.}$$

When a hollow shaft is provided

External diameter $= D_1$

Internal diameter $= 0.6\,D_1$

Polar modulus of hollow shaft $\Big\} = Z_{hollow} = \frac{\pi}{16D_1}\left[D_1^4-0.6^4D_1^4\right]$

$$=0.8704\,\frac{\pi D_1^3}{16}$$

Polar modulus of solid shaft $\Big\} = Z_{solid} = \frac{\pi D^3}{16}$

$$Z_{hollow}=Z_{solid}$$

$$0.8704\,\frac{\pi D_1^3}{16}=\frac{\pi D^3}{16}$$

$$\left(\frac{D_1}{D}\right)^3 = 1{\cdot}1489$$

$$\therefore \quad \frac{D_1}{D} = 1{\cdot}047$$

Percentage saving in weight

$$= \left(\frac{A_{solid} - A_{hollow}}{A_{solid}}\right) 100\%$$

$$= \left[1 - \frac{A_{hollow}}{A_{solid}}\right] 100\%$$

$$= \left[1 - \frac{D_1^2 - 0{\cdot}36 D_1^2}{D^2}\right] 100\%$$

$$= (1 - 0{\cdot}64 \times 1{\cdot}047^2)\ 100\%$$

$$= 29{\cdot}84\%$$

Problem 340. *A solid steel shaft has to transmit 100 h.p at 200 r.p.m. Taking allowable shear stress as 700 kg/cm² find the suitable diameter of the shaft, if the maximum torque transmitted on each revolution exceeds the mean by 30%. Also find the outer diameter of a hollow shaft to replace the solid shaft if the diameter ratio is 0·7.*

(AMIE, Winter 1977)

Solution.

$$\text{H.P. of shaft} = \frac{2\pi N T_{mean}}{4500} = 100$$

$$\therefore \quad T_{mean} = \frac{4500 \times 100}{2\pi \times 200} = 358{\cdot}10 \text{ kg m}$$

$$= 35810 \text{ kg. cm.}$$

$$\therefore \quad T_{max} = 1{\cdot}3\ T_{mean} = 1{\cdot}3 \times 35810 = 46553 \text{ kg. cm.}$$

Case (i). *When a solid shaft is provided*

$$T_{max} = f_s \frac{\pi d^3}{16} = 46553$$

$$\therefore \quad d^3 = \frac{46553 \times 16}{700\pi} = 338{\cdot}7$$

$$\therefore \quad d = 6{\cdot}97 \text{ cm.}$$

Case (ii). *When a hollow shaft of diameter ratio 0·7 is provided*

External diameter $= D$

Internal diameter $= 0{\cdot}7D$

$$\text{Polar modulus} = \frac{\pi}{16D}[D^4 - 0{\cdot}7^4 D^4] = 0{\cdot}7599 \frac{\pi D^3}{16}$$

$$Z_{hollow} = Z_{solid}$$

$$0{\cdot}7599 \frac{\pi D^3}{16} = \frac{\pi}{16}(6{\cdot}97)^3$$

$$\therefore \quad D = 7{\cdot}638 \text{ cm.}$$

Internal diameter $= 0{\cdot}7D = 0.7 \times 7{\cdot}638 = 5{\cdot}347$ *cm.*

Problem 341. *A hollow shaft having an inside diameter 60% of its outer diameter is to replace a solid shaft transmitting the same power at the same speed. Calculate the percentage saving in material, if the material to be used is also the same.*

(AMIE, Summer 1975)

Solution.

Let the diameter of the solid shaft be D

Let the outer diameter of the hollow shaft be D_1

$\therefore$ Inner diameter of the hollow shaft $= 0.6D_1$

Polar modulus of the solid shaft

$$Z_{solid} = \frac{\pi D^3}{16}$$

Polar modulus of the hollow shaft

$$Z_{hollow} = \frac{\pi}{16 D_1}[D_1^4 - 0.6^4 D_1^4]$$

$$= 0.8704 \frac{\pi D_1^3}{16}$$

Since both the shafts should have the same polar modulus,

$$Z_{hollow} = Z_{solid}$$

$$0.8704 \frac{\pi D_1^3}{16} = \frac{\pi D^3}{16}$$

$$\left(\frac{D_1}{D}\right)^3 = \frac{1}{0.8704} = 1.1489$$

$$\therefore \quad \frac{D_1}{D} = 1.047$$

Percentage saving in weight,

$$\left[\frac{A_{solid} - A_{hollow}}{A_{solid}}\right] \times 100\%$$

$$= \left[1 - \frac{A_{hollow}}{A_{solid}}\right] \times 100\%$$

$$= \left[1 - \frac{D_1^2 - 0.36 D_1^2}{D^2}\right] \times 100\%$$

$$= \left[1 - 0.64 \frac{D_1^2}{D^2}\right] \times 100\%$$

$$= (1 - 0.64 \times 1.047^2) \times 100\%$$

$$= 29.84\%.$$

Problem 342. *A solid shaft is to tranmit 450 H.P. at 120 r.p.m. If the shear stress of the material must not exceed 800 kg./cm². find the diameter required.*

What percentage saving in weight would be obtained if this shaft is replaced by a hollow one whose internal diameter equals 0·6 × exter-

nal diameter the length, material and maximum shearing stress remaining unchanged.

Solution.

$$H.P. = \frac{2\pi NT}{4500}$$

$$450 = \frac{2\pi \times 120 \times T}{4500}$$

$$\therefore \quad T = \frac{450 \times 4500}{2\pi \times 120} \text{ kg m.}$$

$$= 2685 \text{ kg. m.}$$

$$= 2,68\,500 \text{ kg. cm.}$$

But
$$T = f_s \frac{\pi D^3}{16}$$

$$\therefore \quad D^3 = \frac{16T}{f_s \pi}$$

$$D^3 = \frac{16 \times 2,68,500}{800 \times \pi}$$

$$\therefore \quad D = 11{\cdot}95 \text{ cm.}$$

Let D_1 be the external diameter of hollow shaft

$\therefore$ Internal diameter of the hollow shaft $= 0{\cdot}6 D_1$

Since the solid and the hollow shafts have to transmit the same torque at the same maximum shear stress their polar moduli must be equal.

Polar modulus of the solid shaft section

$$= \frac{\pi D^3}{16}$$

Polar modulus of the hollow shaft section

$$= \frac{\frac{\pi}{32}\left[D_1^4 - (\cdot 6 D_1)^4\right]}{\frac{D_1}{2}}$$

$$= 0{\cdot}8704 \frac{\pi D_1^3}{16}$$

Equating the polar moduli of the two shafts, we have,

$$0{\cdot}8704 \frac{\pi D_1^3}{16} = \frac{\pi D^3}{16}$$

$$\therefore \quad D_1^3 = \frac{D^3}{0{\cdot}8704}$$

Problem 341. *A hollow shaft having an inside diameter 60% of its outer diameter is to replace a solid shaft transmitting the same power at the same speed. Calculate the percentage saving in material, if the material to be used is also the same.*

(AMIE, Summer 1975)

Solution.

Let the diameter of the solid shaft be D

Let the outer diameter of the hollow shaft be D_1

$\therefore$ Inner diameter of the hollow shaft $=0.6D_1$

Polar modulus of the solid shaft

$$Z_{solid}=\frac{\pi D^3}{16}$$

Polar modulus of the hollow shaft

$$Z_{hollow}=\frac{\pi}{16D_1}[D_1^4-0.6^4D_1^4]$$

$$=0.8704\frac{\pi D_1^3}{16}$$

Since both the shafts should have the same polar modulus,

$$Z_{hollow}=Z_{solid}$$

$$0.8704\frac{\pi D_1^3}{16}=\frac{\pi D^3}{16}$$

$$\left(\frac{D_1}{D}\right)^3=\frac{1}{0.8704}=1.1489$$

$$\therefore\quad \frac{D_1}{D}=1.047$$

Percentage saving in weight,

$$\left[\frac{A_{solid}-A_{hollow}}{A_{solid}}\right]\times 100\%$$

$$=\left[1-\frac{A_{hollow}}{A_{solid}}\right]\times 100\%$$

$$=\left[1-\frac{D_1^2-0.36D_1^2}{D^2}\right]\times 100\%$$

$$=\left[1-0.64\frac{D_1^2}{D^2}\right]\times 100\%$$

$$=(1-0.64\times 1.047^2)\times 100\%$$

$$=29.84\%.$$

Problem 342. *A solid shaft is to tranmit 450 H.P. at 120 r.p.m. If the shear stress of the material must not exceed 800 kg./cm²: find the diameter required.*

What percentage saving in weight would be obtained if this shaft is replaced by a hollow one whose internal diameter equals 0.6 × exter-

nal diameter the length, material and maximum shearing stress remaining unchanged.

Solution.

$$\text{H.P.} = \frac{2\pi NT}{4500}$$

$$450 = \frac{2\pi \times 120 \times T}{4500}$$

$\therefore$ $$T = \frac{450 \times 4500}{\pi \times 120} \; kg. \; m.$$

$$= 2685 \; kg \; m.$$

$$= 2,68,500 \; kg. \; cm.$$

But $$T = f_s \frac{\pi D^3}{16}$$

$\therefore$ $$D^3 = \frac{16T}{f_s \pi}$$

$$D^3 = \frac{16 \times 2,68,500}{800 \times \pi}$$

$\therefore$ $$D = 11{\cdot}95 \; cm.$$

Let D_1 be the external diameter of hollow shaft.

$\therefore$ Internal diameter of the hollow shaft $= 0{\cdot}6 D_1$

Since the solid and the hollow shafts have to transmit the same torque at the same maximum shear stress their polar moduli must be equal.

Polar modulus of the solid shaft section

$$= \frac{\pi D^3}{16}$$

Polar modulus of the hollow shaft section

$$= \frac{\frac{\pi}{32}\left[D_1^4 - (\cdot 6 D_1)^4\right]}{\frac{D_1}{2}}$$

$$= 0{\cdot}8704 \; \frac{\pi D_1^3}{16}$$

Equating the polar moduli of the two shafts, we have,

$$0{\cdot}8704 \frac{\pi D_1^3}{16} = \frac{\pi D^3}{16}$$

$\therefore$ $$D_1^3 = \frac{D^3}{0{\cdot}8704}$$

$$D_1^3 = \frac{(11 \cdot 95)^3}{0.8704}$$

$$\therefore \quad D_1 = 12 \cdot 51 \text{ cm.}$$

Since both the shafts are of the same material and length, percentage saving in weight

$$= \left\{ \frac{\text{Area of solid shaft} - \text{area of hollow shaft}}{\text{Area of solid shaft}} \right\} \times 100$$

$$= \left\{ 1 - \frac{\text{area of hollow shaft}}{\text{area of solid shaft}} \right\} \times 100$$

$$= \left\{ 1 - \frac{\frac{\pi}{4}\left[12 \cdot 51^2 - (0 \cdot 6 \times 12 \cdot 51)^2\right]}{\frac{\pi}{4} 11 \cdot 95^2 \times} \right\} \times 100$$

$$= 29 \cdot 84\%$$

Problem 343 *The propeller shaft of a steam ship has to transmit 10,000 H.P. at 240 rpm. The shaft has an internal diameter of 15 cm. Calculate the minimum external diameter if the shearing stress in the shaft is to be limited to 1575 kg/cm².* (London University)

Solution

$$\text{H.P} = \frac{2\pi NT}{4500}$$

$$10{,}000 = \frac{2\pi \times 240T}{4500}$$

$$T = \frac{10{,}000 \times 4500}{2\pi \times 240} \text{ kg. m.}$$

$$= 29840 \text{ kg. m.}$$

$$= 2984000 \text{ kg cm.}$$

Let the external diameter be d_1 cm.

Internal diameter $= d_2 = 15$ cm.

Polar moment of inertia

$$= I_p = \frac{\pi}{32}\left(d_1^4 - 15^4\right) \text{ cm.}^4$$

$$= \frac{\pi}{32}\left(d_1^4 - 50625\right) \text{ cm.}^4$$

$\therefore$ Polar modulus

$$= Z_p = \frac{\pi}{32} \frac{(d_1^4 - 50625)}{\left(\frac{d_1}{2}\right)} \text{ cm.}^3$$

$$\therefore \quad Z_p = \frac{\pi}{16} \frac{(d_1^4 - 50625)}{d_1} \text{ cm.}^3$$

$$T = f_s Z_p$$

$$\therefore \qquad 2984000 = 1570 \times \frac{\pi}{16}\left(\frac{d_1^4 - 50625}{d_1}\right)$$

Rearranging,

$$d_1^4 - 50625 = \frac{2984000 \times 16}{1570\pi}$$

$$\therefore\ d_1^4 - 9679 d_1 - 50625 = 0$$

Solving by trial and error, we get

$$d_1 = 25 \text{ cm.}$$

Problem 344. *A hollow steel shaft 24 cm external and 16 cm. internal diameter is to be replaced by a solid alloy shaft. If both the shafts should have the same polar modulus, find the diameter of the latter and the ratio of the torsional rigidities. Take C for steel $= 2{\cdot}4 \times C$ for alloy.*

If alternatively, the two shafts should have the same torsional rigidity, find the ratio of their polar moduli.

Solution. *Case* (*i*). When the polar moduli of the two shafts are equal.

i.e. $\qquad Z_{steel} = Z_{alloy}$

Polar modulus of the steel shaft

$$= Z_{steel} = \frac{\pi}{16 D_1}\left[D_1^4 - D_2^4\right]$$

$$= \frac{\pi}{16 \times 24}\left(24^4 - 16^4\right) \text{ cm.}^3$$

$$= \frac{2080\pi}{3} \text{ cm.}^3$$

Let the diameter of the solid alloy shaft be *d cm.*

$\therefore$ Polar modulus of the alloy shaft

$$= Z_{alloy} = \frac{\pi d^3}{16}$$

$\therefore$ Equating the polar moduli of the two shafts, we get

$$\frac{\pi d^3}{16} = \frac{2080}{3}\pi$$

$$\therefore \qquad d^3 = \frac{2080 \times 16}{3}$$

$$\therefore \qquad d = 22{\cdot}30 \text{ cm.}$$

Ratio of torsional rigidities

$$= \frac{\text{Torsional rigidity of steel shaft}}{\text{Torsional rigidity of alloy shaft}}$$

$$= \frac{C_s I_s}{C_a . I_a}$$

But since the polar moduli are equal

$$\frac{I_s}{r_s} = \frac{I_a}{r_a}$$

$$\therefore \quad \frac{I_s}{I_a} = \frac{r_s}{r_a}$$

$\therefore$ Ratio of torsional rigidities

$$= \frac{C_s}{C_a} \cdot \frac{r_s}{r_a}$$

$$= 2{\cdot}4 \times \frac{12}{11{\cdot}15}$$

$$= 2{\cdot}586$$

Case (*ii*). When the torsional rigidities of the two shafts are equal

$$C_s I_s = C_a I_a$$

$$C_s \cdot \frac{\pi}{32}(24^4 - 16^4) = C_a \frac{\pi}{32} d^4$$

$$\therefore \quad d^4 = \frac{C_s}{C_a}(24^4 - 16^4)$$

$$= 2{\cdot}4\,(24^4 - 16^4)$$

$$\therefore \quad d = 28{\cdot}28 \text{ cm.}$$

$\therefore$ Ratio of Polar moduli

$$= \frac{Z_s}{Z_a} = \frac{I_s}{I_s} \cdot \frac{r_a}{I_a}$$

$$= \frac{I_s}{I_a} \cdot \frac{r_a}{r_s}$$

Since the torsional rigidities are equal

$$C_s I_s = C_a I_a$$

$$\frac{I_s}{I_a} = \frac{C_a}{C_s} = \frac{1}{2{\cdot}4}$$

$$\frac{Z_s}{Z_a} = \frac{1}{2{\cdot}4} \times \frac{14{\cdot}14}{12}$$

$$= 0{\cdot}4909.$$

Problem 345. *The stepped steel shaft shown in Fig. 536 is subjected to a torque T at the free end and a torque 2T in the opposite direction at the junction of the two sizes. What is the total angle of twist at the free end if the maximum shear stress in the shaft is limited to 700 kg. per cm² ?*

Assume the modulus of rigidity to be $0{\cdot}84 \times 10^6$ kg./cm²

(*A.M.I.E., May 1967*)

Solution. Torque on the shaft *BC*

$$= T \text{ kg. cm.}$$

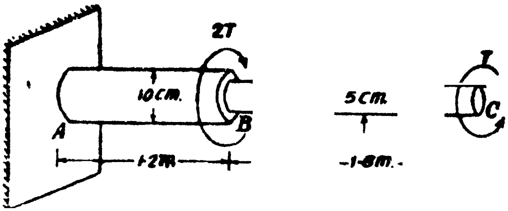


Fig. 536

∴ Torque on the shaft AB

$$=T-2T=-T \ kg. \ cm.$$

Hence the two shafts are subjected to a torque of same magnitude but of opposite *sense*.

Hence the shaft BC which is of smaller diameter will be subjected to a bigger shear stress than shaft AB.

Equating the torsional strength of the shaft BC to the external torque, we have

$$f_s \frac{\pi d^3}{16}=T$$

$$\therefore \quad T=700\times\frac{\pi\times 5^3}{16} \ kg. \ cm.$$

$$=17180 \ kg. \ cm.$$

Let the twist of the shaft BC be θ_{bc} radians,

$$\therefore \quad \frac{T}{I_{bc}}=\frac{C\theta_{bc}}{l_{bc}}$$

$$\theta_{bc}=\frac{17180}{\left(\frac{\pi\times 5^4}{32}\right)}\times\frac{180}{0.84\times 10^6} \ radian$$

$$=0.06 \ radian.$$

Similarly the twist θ_{ab} of the shaft AB is given by,

$$\theta_{ab}=\frac{17180}{\left(\frac{\pi\times 10^4}{32}\right)}\times\frac{120}{0.84\times 10^4} \ radian$$

$$=0.0025 \ radian.$$

Since the directions of twists θ_{bc} and θ_{ab} are opposite to each other.

Net angle of twist of the free end

$$=\theta_{bc}-\theta_{ab}$$

$$=0.06-0.0025 \ radian$$

$$=0.0575 \ radian$$

$$=3^\circ \cdot 295$$

$$say \ 3^\circ \ 18'$$

Problem 346. *A solid shaft 6.50 metres long is securely fixed at each end. A torque of 900 kg. cm. is applied to the shaft at a section 2.50 metres from one end. Find the fixing torques set up at the ends of the shaft.*

If the shaft is 3.50 cms. diameter find the maximum shear stresses in the two portions. Find also the angle of twist for the section where the torque is applied.

Take $C = 0.84 \times 10^6$ *kg./cm*^{2}.

Solution

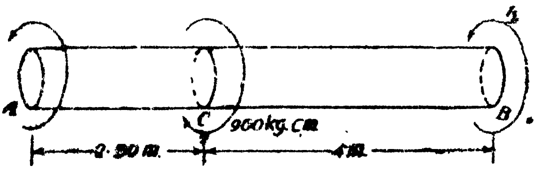


Fig. 537

Fig. 537 shows the shaft AB. Let the fixing couples at the ends A and B be T_a and T_b respectively.

Since the fixing couples must oppose the applied couple,

$$T_a + T_b = 900 \text{ kg. cm.} \qquad \ldots(i)$$

Further, the ends being fixed the angle of twist in the length AC of the shaft must be equal to the angle of twist in the length CB.

The twist θ for a length l is given by

$$\theta = \frac{Tl}{CI_p}$$

Since C and I_p have the same value for the two portions

We have $T_a . AC = T_b . CB$

$\therefore \quad T_a . (2.5)100 = T_b(4) 100$

$$\therefore \quad T_a = 1.6T_b \qquad \ldots(ii)$$

Substituting in equation (i), we get

$$1.6T_b + T_b = 900$$

$$\therefore \quad T_b = \frac{900}{2.6} \text{ kg. cm.} = 346.2 \text{ kg. cm.}$$

$$\therefore \quad T_a = 900 - 346.2 \text{ kg. cm.}$$

$$\therefore \quad T_a = 553.8 \text{ kg. cm.}$$

$$\text{Polar modulus} = Z_p = \frac{\pi d^3}{16} = \frac{\pi \times 3.5^3}{16} \text{ cm}^3$$

Maximum shear stress in the portion AC

$$= q_{ao} = \frac{T_a}{Z_p}$$

$$= \frac{553.8 \times 16}{\pi (3.5)^3} \ kg./cm.^2$$

$$= 65.75 \ kg./cm.^2$$

Maximum shear stress in the portion *CB*

$$= q_{cb} = \frac{T_b}{Z_p}$$

$$= \frac{346.2 \times 16}{\pi \times (3.5)^3} kg./cm.^2$$

$$= 41.1 \ kg./cm.^2$$

The angle of twist can be determined considering either the part *AC* or *CB*.

$$\theta = \frac{Tl}{CI_p}$$

$$= \frac{T_a . AC}{CI_p}$$

$$= \frac{553.8 \times 250 \times 32}{0.84 \times 10^6 \times \pi \times (3.5)^4} \ radian$$

$$= 0.01118 \ radian$$

$$= 0\cdot64^\circ$$

$$= 0^\circ \ 38' \ 24''$$

Problem 347. *A 3 cm. diameter circular steel shaft is provided with enlarged portions A and B as shown in Fig. 538. On to this enlarged portion a steel tube 0·20 cm thick is shrunk. While the shrinking process is going on the 3 cm. diameter shaft is held twisted by a couple of magnitude 800 kg. cm. When the tube is firmly set on the shaft, this twisting couple is removed. Calculate what twisting couple is left on the shaft, the shaft and the tube being made of the same material.*

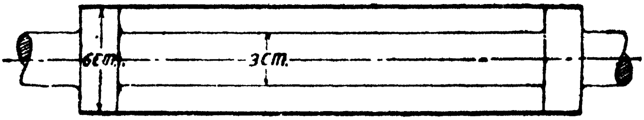


Fig. 538

Solution. When the shaft is subjected to a couple of 800 *kg. cm.* let the angle of twist of the shaft be θ

Polar moment of inertia of the shaft section

$$= I_p = \frac{\pi}{32} (3)^4 \ cm^4 = \frac{81}{32} \pi \ cm^4$$

$$\frac{Tl}{CI_p} = \frac{800 \ l \times 32}{C \times 81\pi} \ radian.$$

$$= \frac{800 \times 32}{81\pi} \frac{l}{C} \text{ radian.}$$

Now consider the sleeve

External diameter $= d_1 = 6 + 0{\cdot}4 = 6{\cdot}4$ *cm.*

Internal diameter $= d_2 = 6$ *cm.*

Moment of inertia

$$I' = \frac{\pi}{32}(6{\cdot}4^4 - 6^4) \text{ cm.}^4$$

$$= \frac{\pi}{32} \times 76{\cdot}96 \times 4{\cdot}96 \text{ cm.}^4$$

Let T_s be the couple left on the shaft

∴ Corresponding twist of the shaft

$$= \theta_s = \frac{T_s l}{C I_p}$$

$$= \frac{T_s l \times 32}{C\, 81\, \pi} \text{ radian.}$$

$$= \frac{32}{81\,\pi} \cdot \frac{T_s l}{C} \text{ radian.}$$

Corresponding twist in the sleeve

$$= \theta_s = \frac{T_s l}{C I'}$$

$$= \frac{T_s l \times 32}{C \times \pi \times 76{\cdot}96 \times 4{\cdot}96} \text{ radian.}$$

But $\theta_s + \theta_s = \theta$

$$\therefore \frac{32}{81\pi} \frac{T_s l}{C} + \frac{32}{\pi \times 76{\cdot}96 \times 4{\cdot}96} \frac{T_s l}{C} = \frac{32 \times 800}{81\,\pi} \frac{l}{C}$$

$$\frac{T_s}{81} + \frac{T_s}{76{\cdot}96 \times 4{\cdot}96} = \frac{800}{81}$$

$$T_s\left[\frac{1}{81} + \frac{1}{381{\cdot}7}\right] = \frac{800}{81}$$

$$\frac{462{\cdot}7}{81 \times 381{\cdot}7} T_s = \frac{800}{81}$$

$$\therefore \quad T_s = \frac{800 \times 381{\cdot}7}{462{\cdot}7} \text{ kg. cm.}$$

$$= 660 \text{ kg cm.}$$

Problem 348. *A composite shaft consists of a steel rod 6 cm. diameter surrounded by a closely fitting tube of brass fixed to it. Find the outside diameter of the tube so that when a torque is applied to the composite shaft, it will be shared equally by the two materials. Take, C for steel* $= 0{\cdot}84 \times 10^6$ *kg./cm*^{2}. *and C for brass* $= 0{\cdot}42 \times 10^6$ *kg./cm*^{2}.

If the torque is 100,000 kg. cm. find the maximum shearing stress in each material and the angle of twist in a length of 4 metres.

Solution. Let the twist of each shaft be θ

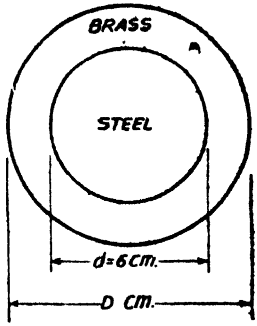


Fig. 539

Polar moment of inertia of steel shaft

$$= \frac{\pi \times 6^4}{32} \; cm.^4$$

Polar moment of inertia of the brass shaft

$$= I_b = \frac{\pi}{32}\left(D^4 - 6^4 \right) cm.^4$$

If T_s and T_b be the torques in the steel and brass shafts,

We have

$$\frac{T_s}{I_s} = \frac{C_s \theta}{l} \text{ and}$$

$$\frac{T_b}{I_b} = \frac{C_b \theta}{l}$$

But $$T_s = T_b$$

$\therefore$ $$\frac{I_b}{I_s} = \frac{C_s}{C_b} = \frac{0{\cdot}84}{0{\cdot}42} = 2$$

$\therefore$ $$I_b = 2 I_s$$

$\therefore$ $$\frac{\pi}{32}(D^4 - d^4) = 2 \times \frac{\pi}{32} d^4$$

$\therefore$ $$D^4 - d^4 = 2 \times d^4$$

$\therefore$ $$D^4 = 3 \times d^4 = 3 \times 6^4$$

$\therefore$ $$D = 7{\cdot}898 \; cm.$$

$$T_s \quad T_b = \frac{100{,}000}{\ } = 50{,}000 \; kg. \; cm.$$

For the steel shaft, maximum shear stress

$$= q_s = \frac{T_s}{Z_s}$$

$$= \frac{50{,}000 \times 16}{\pi \times 6^3} \; kg./cm.^2$$

$$= 1179 \; kg./cm.^2$$

For the brass shaft, maximum shearing stress

$$= q_b = \frac{T_b}{Z_b}$$

$$= \frac{50{,}000 \times 16 D}{\pi (D^4 - d^4)}$$

Since $D^4-d^4=2d^4$

$$q_b=\frac{50{,}000\times16\times7\cdot898}{\pi\times2\times(6)^4}\ kg./cm.^2$$

$$=776\ kg./cm.^2$$

The common angle of twist

$$=\theta=\frac{T_s l}{C_s I_s}$$

$$=\frac{50{,}000\times400\times32}{0\cdot84\times10^6\times\pi\times6^4}\ radian$$

$$=0\cdot1871\ radian$$

$$=10^\circ\ 42'$$

Problem 349. *A solid alloy shaft 5 cm diameter is to be coupled in series with a hollow steel shaft of the same external diameter. Find the internal diameter of the steel shaft if the angle of twist per unit length of the steel shaft is to be 75% of that of the alloy shaft. Determine the speed at which the shafts are to be driven to transmit 250 H.P. if the limits of shearing stress are to be 560 kg./cm.² and 800 kg./cm.² in alloy and steel respectively. Take $C_{steel}=2\cdot2\ C_{alloy}$.*

(*London University*)

Solution. Angle of twist per unit length of a shaft is given by

$$\frac{\theta}{l}=\frac{T}{CI_p}$$

$$\therefore\quad \frac{T}{C_s I_s}=0\cdot75\frac{T}{C_a I_a}$$

$$\therefore\quad \frac{I_a}{I_s}=0\cdot75\frac{C_s}{C_a}$$

$$\therefore\quad \frac{\frac{\pi}{32}\times5^4}{\frac{\pi}{32}\left(5^4-d^4\right)}=0\cdot75\times2\cdot2$$

$$\therefore\quad 5^4-d^4=\frac{5^4}{0\cdot75\times2\cdot2}=378\cdot7$$

$$\therefore\quad d^4=625-378\cdot8=246\cdot3$$

$$\therefore\quad d=3\cdot962\ cm.$$

Also for a shaft, we know the relation

$$\frac{f_s}{r}=\frac{C\theta}{l}$$

$$\therefore\quad \frac{f_{steel}}{r_{steel}}=C_s\left(\frac{\theta}{l}\right)_{steel}\ \text{and}\ \frac{f_{alloy}}{r_{alloy}}=C_a\left(\frac{\theta}{l}\right)_{alloy}$$

$$\frac{f_{steel}}{f_{alloy}}=\frac{C_s}{C_a}\cdot\frac{r_s}{r_a}\times0\cdot75=2\cdot2\times0\cdot75=1\cdot65$$

$\therefore$ $f_{steel} = 1{\cdot}65\, f_{alloy}$

when $f_{steel} = 800\ kg./cm.^2$

$f_{alloy} = \frac{800}{1{\cdot}65} = 484{\cdot}9\ kg./cm.^2$ (this is less than the permissible stress of 560 $kg./cm^2$.)

Now the torque can be determined by considering any one of the shafts.

$$T = f_{alloy} \frac{\pi D^3}{16}$$

$$= \frac{484{\cdot}3 \times \pi(5^3)}{16}\ kg.\ cm.$$

$$= 11890\ kg.\ cm. = 118{\cdot}90\ kgm.$$

$$\therefore \quad HP = \frac{2\pi NT}{4500}$$

$$\therefore \quad N = \frac{4500(H.P.)}{2\pi T}$$

$$= \frac{4500 \times 250}{2\pi \times 1189}$$

$$\therefore \quad N = 1505\ rpm.$$

Problem 350. *A steel shaft ABCD having a total length of 240 cm. consists of three lengths having different sections as follows. AB is hollow having outside and inside diameters of 8 cms. and 5 cms. respectively and BC and CD are solid, BC having a diameter of 8 cms. and CD a diameter of 7 cms. If the angle of twist is the same for each section, determine the length of each section and the total angle of twist if the maximum shear stress is 500 kg./cm.² Take C = 0·82 × 10⁶ kg./cm.²*

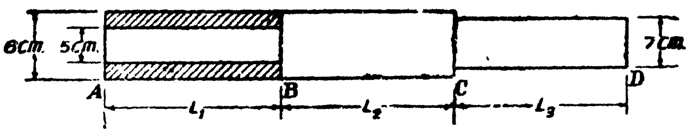


Fig. 540

Solution. Polar moment of inertia of the shaft sections are,

For shaft *AB*, $I_1 = \frac{\pi}{32}\left(8^4 - 5^4\right)\ cm.^4$

$$= 340{\cdot}9\ cm.^4$$

For shaft *BC*, $I_2 = \frac{\pi}{32} \cdot \left(8\right)^4\ cm.^4$

$$= 402{\cdot}4\ cm.^4$$

For shaft CD, $I_3 = \frac{\pi}{32}\left(7\right)^4$ $cm.^4$

$$=235{\cdot}8 \; cm.^4$$

Since the angle of twist for each length is the same $\theta = \left(\frac{Tl}{CI_p}\right)$ must be the same for each length.

Since T and C are the same throughout the shaft,

$$\frac{l_1}{I_1} = \frac{l_2}{I_2} = \frac{l_3}{I_3}$$

$$\therefore \quad \frac{l_1}{340{\cdot}9} = \frac{l_2}{402{\cdot}4} = \frac{l_3}{235{\cdot}8}$$

$$\therefore \quad l_1 = 1{\cdot}44 \; l_3$$

and $\quad l_2 = 1{\cdot}71 \; l_3$

But $\quad l_1 + l_2 + l_3 = 240 \; cm.$

$$\therefore \quad 1{\cdot}44 l_3 + 1{\cdot}71 l_3 + l_3 = 240$$

$$\therefore \quad 4{\cdot}15 \; l_3 = 240$$

$$\therefore \quad l_3 = 57{\cdot}8 \; cm.$$

$$\therefore \quad l_1 = 1{\cdot}44 \times 57{\cdot}8 \; cm.$$

$$\therefore \quad l_1 = 83{\cdot}2 \; cm.$$

$$\therefore \quad l_2 = 240 - 57{\cdot}8 - 83{\cdot}2 = 99 \; cm.$$

Twist of the shaft AB

$$= \theta_1 = \frac{f_s l_1}{Cr}$$

$$= \frac{500 \times 83{\cdot}2}{0{\cdot}82 \times 10^6 \times 4} \; radian$$

$$= 0{\cdot}01269 \; radian$$

$\therefore$ Total angle of twist of the whole shaft

$$= 3 \times 0{\cdot}01269 \; radian$$

$$= 0{\cdot}03807 \; radian$$

$$= 2^\circ \; 11'.$$

Problem 351. *A steel shaft ABCD has a total length of 127·5 cm.. made up as follows. AB = 30 cm. BC = 37·5 cm., and CD = 60 cm. AB is hollow its outside diameter being 10 cm. and inside diameter d_1 cm. BC and CD are solid having diameters of 10 cm. and 8·75 cm. respectively. If equal opposite torques are applied at the ends of the shaft find the maximum permissible value of d_1 for the maximum shearing stress in AB not to exceed that in CD. If the torque applied to the shaft is 90 t cm. what is the total angle of twist ? Take $C = 0{\cdot}8 \times 10^6$ kg./cm.^{2}*

(London University)

Solution. Let the common torque on the shaft be T *kg. cm.*

Let d_1 = internal diameter of the shaft AB,

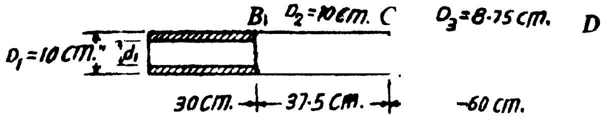


Fig. 541

For the shaft AB,

$$T=q_1 Z_1$$

$$=q_1 \cdot \frac{\pi}{16}\left(\frac{D_1^4-d_1^4}{D_1}\right)$$

$\therefore$ $$T=q_1 \frac{\pi}{16}\left(\frac{10^4-d_1^4}{10}\right)$$

$\therefore$ $$q_1=\frac{160T}{\pi(10^4-d_1^4)}$$

For the shaft CD,

$$T=q_3 Z_3$$

$$=q_3 \frac{\pi D_3^3}{16}$$

$$=q_3 \frac{\pi\,(8{\cdot}75)^3}{16}$$

$\therefore$ $$q_3=\frac{16T}{\pi(8{\cdot}75)^3}$$

Since $q_1=q_3$

$$\frac{160T}{\pi(10^4-d_1^4)}=\frac{16}{\pi(8{\cdot}75)^3}$$

$\therefore$ $$\frac{10}{10^4-d_1^4}=\frac{1}{(8{\cdot}75)^3}$$

$\therefore$ $$10^4-d_1^4=10\,(8{\cdot}75)^3$$

$\therefore$ $$d_1^4=3301$$

$\therefore$ $$d_1=7{\cdot}58 \text{ cm.}$$

We know the twist of a shaft is given by

$$\theta=\frac{Tl}{CI_p}$$

In our case,

$$\theta=\Sigma\frac{Tl}{CI_p}$$

$$=\frac{T}{C}\Sigma\frac{l}{I_p}$$

$$=\frac{30{,}00{,}000}{0{\cdot}8\times10^6}\left\{\frac{30}{\frac{\pi}{32}(10^4-7{\cdot}58^4)}+\frac{37{\cdot}5}{\frac{\pi}{32}(10^4)}+\frac{60}{\frac{\pi}{32}(8{\cdot}75)^4}\right\}$$

$=0{\cdot}0214$ *radian*

$=1^\circ\ 12'$.

§98. Keys

A key is a wedge-like piece inserted between two machine parts so as to prevent relative movement between them. For instance a key is a necessity for connecting a shaft and the surrounding hub (Fig. 542).

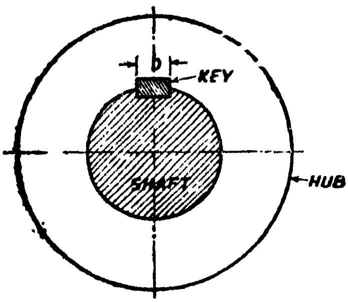


Fig. 542

Let l and b be the length and width of the key.

Let f_k be the safe shearing stress in the key. The resistance set up by the key is $f_k\ lb$. If the diameter of the shaft be d the moment that can be transmitted by the key

$$=f_k\ lb\ \frac{d}{2}.$$

If f_s be the maximum shearing stress in the shaft the maximum torsion on the shaft

$$=T=f_s\cdot\frac{\pi d^3}{16}.$$

Equating the torsion on the shaft to the moment transmitted by the key, we have,

$$T=f_s\frac{\pi d^3}{16}=f_k\ lb\ \frac{d}{2}.$$

§99. Coupling

A coupling is used to connect two shafts so that the rotary motion of one can be transmitted to the other.

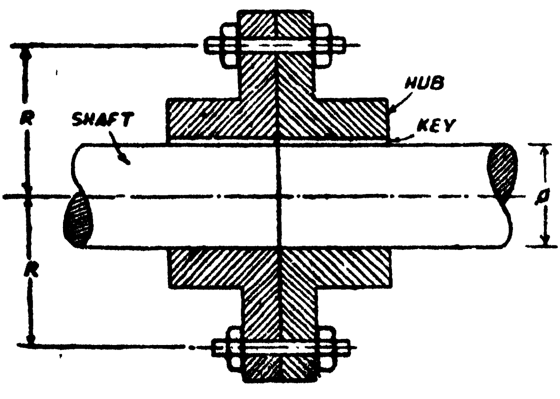


Fig. 543

Fig. 543 shows an arrangement for such a coupling.

The coupling surrounds the two shafts to be connected. Connection between each shaft and coupling is provided by the key. The two parts of the coupling are held together by bolts. The bolts are arranged along a circle called the bolt circle.

Let the diameter of the bolt be d_b. Let n bolts be provided on a bolt circle of radius R.

Let f_b be safe shearing stress in the bolt.

Maximum load that can be resisted by one bolt

$$=f_b.\ \frac{\pi d_b^2}{4}$$

∴ Moment that can be transmitted by all the bolts

$$=n\times f_b\ \frac{\pi d_b^2}{4}\times R.$$

Equating the maximum torsion on the shaft to the moment transmitted by the bolts,

$$T=f_s\frac{\pi d^3}{16}=nf_b\ \frac{\pi d_b^2}{4}R.$$

Problem 352. *A 10 cm. diameter shaft transmits 140 HP at 120 rpm. A flanged coupling is keyed to the shaft, the key being 2·5 cm. wide and 14 cm. long. Six bolts of 2 cm. diameter are symmetrically arranged along a bolt circle of 28 cm. diameter. Find the shear stresses induced in the shaft, the key and the bolts.*

Solution.

$$H.P.=\frac{2\pi NT}{4500}$$

∴ $$T=\frac{4500\times 140}{2\pi\times 120}\times 100\ kg.\ cm.$$

$$=83540\ kg.\ cm.$$

We know $$T=f_s\frac{\pi d^3}{16}=f_k lb\frac{d}{2}=nf_b\ \frac{\pi d_b^2}{4}.\ R$$

∴ $$83540=f_s\frac{\pi(10)^3}{16}=f_k\ 14\times 2{\cdot}5\times 5$$

$$=6f_b\ \frac{\pi}{4}(2)^2\times 14$$

∴ $$f_s=425\ kg./cm.^2$$

$$f_k=477\ kg./cm.^2$$

and $$f_b=317\ kg./cm.^2$$

§100. Shear and Torsional resilience

Shear resilience

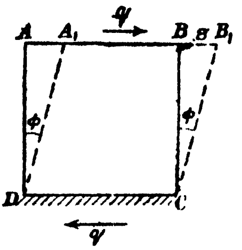


Fig. 544

Fig. 544 shows a square block *ABCD* of side *l* whose thickness perpendicular to the plane of the drawing is unity.

Let the faces *AB* and *CD* be subjected to shear stresses of intensity *q*.

With respect to the face *CD* let the face *AB* be deformed to the position A_1B_1. Let the angle $BCB_1=\phi$. Obviously ϕ represents the shear strain. Let $BB_1=\delta$.

Shear force on the face $AB=P=q\ AB\times 1=ql$

∴ Work done by *P* (if gradually applied)

$$=\tfrac{1}{2}\ P.\delta$$

$$=\tfrac{1}{2}ql.\delta$$

But $$\delta=l\phi=l\times\frac{q}{C}$$

Work done $$=\tfrac{1}{2}(ql)\left(\frac{lq}{C}\right)=\frac{q^2l^2}{2C}.$$

This is also the energy stored by the block.

Volume of the block $=l^2\times 1$

∴ Strain energy stored per unit volume

$$2C$$

Torsional resilience

When the cross-section of a member is subjected to a constant, *i.e.*, uniform shear stress of intensity *q*, we know that the energy stored by the member is $\frac{q^2}{2C}\times$volume. But in case of a cylindrical shaft the shear stress due to torsion varies uniformly from zero at the axis to a maximum value f_s at the surface.

Let the shaft be a solid shaft of diameter *D* and length *l*. The shaft may be taken to consist of an infinite number of elemental concentric hollow shafts.

Consider one such elemental hollow shaft of radius *r* and thickness *dr*. The shear stress *q* at the radius *r* is given by

$$q=\frac{r}{R}f_s=\frac{2r}{D}f_s$$

∴ Strain energy stored by the elemental cylinder

$$=\frac{q^2}{2C}\times\text{volume}$$

$$\frac{\left(\frac{2r}{D} f_s\right)^2}{2C} 2\pi r dr.\ l$$

$$= \frac{4\pi l}{CD^2} f_s^2 r^3 dr$$

$\therefore$ Strain energy stored by the whole shaft

$$= \frac{4\pi l f_s^2}{CD^2} \int_0^{D/2} r^3 dr$$

$$= \frac{4\pi l f_s^2}{CD^2} \cdot \frac{1}{4} \cdot \frac{D^4}{16}$$

$$= \frac{\pi D^2 l f_s^2}{16\ C}$$

$$= \frac{\pi D^2}{4} . l . \frac{f_s^2}{4C}$$

$$= \frac{f_s^2}{4C} \times \text{volume of the shaf}$$

Similarly for a hollow shaft of internal diameter d and external diameter D.

$$\text{Strain energy stored} = \frac{4\pi l f_s^2}{CD^2} \int_{d/2}^{D/2} r^3 dr$$

$$= \frac{4\pi l}{CD^2} f_s^2 \ \frac{1}{4} \left(\frac{D^4}{16} - \frac{d^4}{16} \right)$$

$$= \frac{f_s^2}{4C} \left\{ \frac{D^2+d^2}{D^2} \right\} \frac{\pi}{4} (D^2-d^2) l$$

$$= \frac{f_s^2}{4C} \left\{ \frac{D^2+d^2}{D^2} \right\} \times \text{volume of the shaft.}$$

§101. Torsion of shafts of non-circular sections

The theory of pure torsion described in the preceding sections is correct only for shafts of circular sections. (See article on assumptions in the theory of pure torsion) Estimation of twisting moment on a shaft of non-circular sections is a highly complicated problem.*

This is due to warping of the cross-section during the twist. Let on a rectangular bar of rubber, a system of small squares be traced. If now the bar be twisted, it will be seen that the lines which were originally perpendicular to the axis of the bar will now be curved. The distortion of the small squares varies along the sides

*Shafts of noncircular sections have been analysed by St. Venant. An account of this work can be seen in Todhunter and Pearson's *History of the Theory of Elasticity*, Cambridge, Vol 2, page 311—1893.

of the cross-section, the distortion being a maximum at the middle of the side and zero at the corners. St. Venant's investigations lead to the following results.

(*i*) *Rectangular section.*

Torsional resistance $= T = \dfrac{x^2y^2}{3y+1{\cdot}8x} f_s$

where x = short side.

y = long side

The maximum shearing stress f_s occurs at the middle of longer side.

(*ii*) *Square section.*

Torsional resistance $= T = 0{\cdot}208\ x^3 f_s$

where x = side of square

The maximum shearing stress occurs at the middle point of a side.

(*iii*) Twist for rectangular and square sections is given by

$$\theta = \frac{Tl}{C} \cdot \frac{42 I_p}{A^4}$$

when the ratio $\dfrac{y}{x}$ is less than 3

A = Area of shaft section.

§102. Close Coiled helical Springs

Fig. 545 shows a closely coiled helical spring carrying an axial load W. Let the spring consist of n coils. Let d be the diameter of the rod of the spring and R be the mean radius of the coil. Every section of the rod is subjected to a torsion WR.

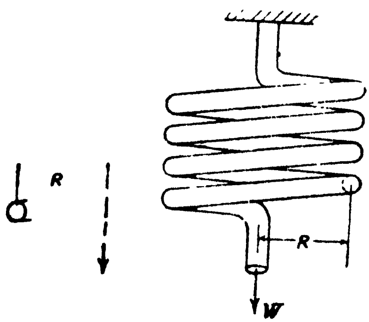


Fig 545

$\therefore$ Maximum shear stress at any section of the rod

$$= f_s = \frac{T}{Z_p}$$

$$= \frac{WR}{\frac{\pi d^3}{16}}$$

$$= \frac{16\,WR}{\pi d^3}$$

Length of the rod $= l = 2\pi\, Rn$

∴ Strain energy stored by the spring

$$= \frac{f_s^2}{4C} \times \text{volume}$$

$$= \left(\frac{16\,WR}{\pi d^3}\right)^2 \frac{1}{4C}\,\frac{\pi d^2}{4}\,2\pi Rn$$

$$= 32\,\frac{W^2R^3n}{Cd^4}$$

If δ be the deflection of the spring, *i.e.*, the downward movement of the load,

Work done on the spring $= \frac{1}{2}\,W\delta$

Equating the work done to energy stored, we have,

$$\tfrac{1}{2}W\delta = 32\,\frac{W^2R^3n}{Cd^4}$$

$$\therefore \qquad \delta = \frac{64\,WR^3n}{Cd^4}$$

Strain energy stored by the spring

= work done on the spring

$= \frac{1}{2}\,W\delta$

Stiffness of the spring

s = load required to produce unit deflection

$$\therefore \qquad s = \frac{W}{\delta} = \frac{Cd^4}{64R^3n}$$

Problem 353. *A closely coiled helical spring is made out of 10 mm. diameter steel rod, the coil consisting of 10 complete turns with a mean diameter of 12 cms. The spring carries an axial pull of 20 kg. Find the maximum shear stress induced in the section of the rod. If $C = 0{\cdot}8 \times 10^6$ kg./cm.², find the deflection of the spring, the stiffness and the strain energy stored by the spring.*

Solution.

$$f_s = \frac{16WR}{\pi d^3}$$

$$= \frac{16 \times 20 \times 6}{\pi \times (1)^3}\ kg./cm.^2$$

$$= 611\ kg./cm.^2$$

$$\delta=\frac{64WR^3n}{Cd^4}$$

$$=\frac{64\times 20\times (6)^3\times 10}{0{\cdot}8\times 10^6\times (1)^4}\ cm.$$

$$=3{\cdot}45\ cm.$$

Stiffness $\quad s=\frac{W}{\delta}$

$$=\frac{20}{3{\cdot}45}$$

$=5{\cdot}8$ *kg. per cm. of deflection*

Energy stored $\quad =\frac{1}{2}W\delta$

$$=\frac{1}{2}\times 20\times 3{\cdot}45\ kg.\ cm.$$

$$=34{\cdot}5\ kg.\ cm.$$

Problem 354. *A close coiled helical spring is to carry a load of 12 kg. and the mean coil diameter is to be 9 times the wire diameter. Calculate these diameters if the maximum shear stress is 1000 kg./cm.*^{2}

Solution.

$$D=2R=9d$$

$\therefore \quad R=4{\cdot}5\ d$

$$W=12\ kg.$$

$$f_s=\frac{16WR}{\pi d^3}$$

$\therefore \quad 1000=\frac{16\times 12\times 4{\cdot}5d}{\pi d^3}$

$\therefore \quad d^2=\frac{12\times 12\times 4{\cdot}5}{1000\pi}$

$$d=0{\cdot}45\ cm.$$

$\therefore \quad D=0{\cdot}45\times 9\ cm.$

$$=4{\cdot}05\ cm.$$

Problem 355. *A close coiled helical spring is to have a stiffness of 1 kg./cm of compression under a maximum load of 4·5 kg. and a maximum shearing stress of 1200 kg./cm.*^{2} *The solid length of the spring (when the coils are touching) is to be 4·5 cm. Find the diameter of the wire, the mean diameter of the coils and the number of coils required. Modulus of rigidity C = 4·2 × 10*^{5} *kg./cm*^{2}.

(*A.M.I.E., May 1965*)

Solution.

Stiffness $\quad \frac{W}{\delta}=\frac{Cd^4}{64R^3n}$

$$1=\frac{42\times10^4\times d^4}{64R^3n}$$

$$\therefore \quad d^4=\left(\frac{64}{42\times10^4}\right)R^3n \qquad ...(i)$$

$$f_s=\frac{16WR}{\pi d^3}$$

$$\therefore \quad 1260=\frac{16\times4{\cdot}5R}{\pi d^3}$$

$$\therefore \quad R=\frac{1260\times\pi d^3}{16\times4{\cdot}5}$$

$$\therefore \quad R=55\ d^3 \qquad ...(ii)$$

Solid length of the spring, when the coils are touching

$$=nd=4{\cdot}5\ cm.$$

$$\therefore \quad n=\frac{4{\cdot}5}{d} \qquad ..(iii)$$

Substituting the values R and n in equation (i) we get,

$$d^4=\frac{64}{42\times10^4}\cdot(55)^3\ d^4\times\frac{4{\cdot}5}{d}$$

$$\therefore \quad d^4=\frac{42\times10^4}{64\times55^3\times4{\cdot}5}$$

$$\therefore \quad d=0{\cdot}3059\ cm.$$

$$\therefore \quad R=55\ (0{\cdot}3059)^3$$

$$=1{\cdot}575\ cm.$$

$$\therefore \quad D=3{\cdot}150\ cm.$$

$$n=\frac{4{\cdot}5}{0{\cdot}3059}$$

$$=14{\cdot}7$$

Problem 356. *A weight of 260 kg. is dropped on a closely coiled helical spring consisting of 16 coils. Find the height by which the weight is dropped before striking the spring so that the spring may be compressed by 22 cm. The coils have a mean radius of 12 cm and the diameter of the rod of the spring is 3 cm. Take $C=0{\cdot}2\times10^6$ kg./cm.^{2}.*

Solution. Let P be the gradually applied load producing the same compression of 22 cm.

$$\therefore \quad \frac{64PR^3n}{Cd^4}=22$$

$$\therefore \quad P=\frac{22\times0{\cdot}9\times10^6\times(3)^4}{64(12)^3\times16}\ kg.$$

$$=906{\cdot}3\ kg.$$

Equating the energy supplied by the falling load to the energy stored by the spring,

$$W(h+\delta)=\frac{1}{2}P\delta.$$

$$260(h+22)=\frac{1}{2}\times 906{\cdot}3\times 22$$

$$\therefore \quad h+22=38{\cdot}33 \text{ cm.}$$

$$\therefore \quad h=16{\cdot}33 \text{ cm.}$$

Problem 357. *A close coiled helical spring has a stiffness of 10 kg./cm. Its length when fully compressed, with adjacent coils touching each other is 40 cm. The modulus of rigidity of the material of the spring is* $0{\cdot}8\times 10^6$ *kg./cm.*^{2}.

(*i*) *Determine the wire diameter and the mean coil diameter if their ratio* $=\frac{1}{10}$.

(*ii*) *If the gap between any two adjacent coils is* 0·2 *cm., what maximum load can be applied before the spring becomes solid, i.e., adjacent coils touch ?*

(*iii*) *What is the corresponding maximum shear stress in the spring ?* (*AMIE May, 1976*)

Solution.

$$\frac{W}{\delta}=10 \text{ kg./cm.}$$

$$nd=40 \text{ cm.}$$

$$C=0{\cdot}8\times 10^6 \text{ kg./cm.}^2$$

$$\frac{d}{D}=0{\cdot}1$$

Gap between adjacent coils

$$=0{\cdot}2 \text{ cm.}$$

$$\delta=\frac{64WR^3n}{Cd^4}$$

$$\therefore \quad \frac{W}{\delta}=\frac{Cd^4}{64R^3n}=10$$

$$d^4=\frac{64\times 10\cdot R^3n}{0{\cdot}8\times 10^6}$$

$$\therefore \quad d^4=\left[\frac{8}{10^4}\right]R^3n \qquad ..(i)$$

But $\quad nd=40 \quad \therefore \quad n=\frac{40}{d}$

and $\quad d=0{\cdot}1\,D=0{\cdot}2\,R$

$$\therefore \qquad R=5\,d \qquad ...(ii)$$

From (*i*) and (*ii*),

$$d^4=\frac{8}{10^4}\times 125\,d^3\times\frac{40}{d}$$

$$\therefore \qquad d^2=4 \qquad \therefore \quad d=2\ cm.$$

$$n=\frac{40}{d}=\frac{40}{2}=20\ turns$$

$$R=5\,d=5\times 2=10\ cm.$$

$$D=20\ cm.$$

Gap between adjacent coils

$$=0{\cdot}2\ cm.$$

Max. deflection $=\delta=0{\cdot}2\times 20=4\ cm.$

But $\dfrac{W}{\delta}=10 \qquad \therefore \quad W=10\times 4=40\ kg.$

$$T=WR=\frac{f_s\pi d^3}{16} \quad f_s=\frac{16\times 40\times 10}{\pi\times 8}$$

$$=254{\cdot}65\ kg./cm.^2$$

Problem 358. *It is required to design a close coiled helical spring which shall deflect 1 cm. under an axial load of 10 kg. at a shear stress of 900 kg./cm.². The spring is to be made out of round wire having a modulus of rigidity of 8 × 10⁵ kg./cm.². The mean diameter of the coils is to be 10 times the diameter of the wire. Find the diameter and length of the wire necesssary to form the spring.*
(*AMIE Summer, 1978*)

Solution.

$$\delta=1\ cm.$$
$$W=10\ kg.$$
$$f_s=900\ kg./cm.^2$$
$$C=8\times 10^5\ kg./cm.^2$$
$$D=10\,d$$
$$R=5\,d$$

$$T=WR=\frac{f_s\pi d^3}{16}$$

$$f_s=\frac{16WR}{\pi d^3}$$

$$900=\frac{16\times 10\times 5d}{\pi d^3}$$

$$d^2=0{\cdot}2829$$

$$\therefore \qquad d=0{\cdot}532\ cm.$$

$$\therefore \qquad R=5\times 0{\cdot}532=2{\cdot}66\ cm.$$

$$\delta=\frac{64WR^3n}{Cd^4}$$

$$n=\frac{Cd^4\delta}{64WR^3}=\frac{8\times10^5(0\cdot532)^4\times1}{64\times10\times2\cdot66^3}=5\cdot32$$

Length of wire required

$$=2\pi Rn$$
$$=2\pi\times2\cdot66\times5\cdot32$$
$$=88\cdot91\ cm.$$

Problem 359. *A vehicle weighing 2500 kg. and running at 2 metres per second has to be brought to rest by a buffer spring. Find the number of springs of 15 coils each required to absorb the energy of motion during a compression of 25 cm. Each spring is made of 25 mm. diameter rod forming a coil of 20 cm. mean diameter. Take* $C=9\cdot45\times10^5$ *kg./cm.*^{2} *and* $g=9\cdot8$ *m./sec.*^{2}.

Solution. The kinetic energy of the truck

$$=\frac{Wv^2}{2g}\ kg.\ m.$$
$$=\frac{2500\times2^2}{2\times9\cdot8}\ kg.\ m.$$
$$=510\cdot2\ kg.\ m.$$
$$=51020\ kg.\ cm.$$

Let P *kg.* be the gradually applied load on one spring so as to compress it by 25 *cm.*

$$\therefore\quad 25=\frac{64PR^3n}{Cd^4}$$

$$\therefore\quad P=\frac{25\times9\cdot45\times10^5\times(2\cdot5)^4}{64(10)^3\times15}\ kg.$$
$$=960\cdot7\ kg.$$

$\therefore$ Strain energy stored by one spring

$$=\frac{1}{2}P\delta$$
$$=\frac{1}{2}\times960\cdot7\times25\ kg.\ cm.$$
$$=12000\ kg.\ cm.$$

$\therefore$ Number of springs required

$$=\frac{51020}{12000}=4\cdot25\ say\ 5\ springs.$$

Problem 360. *The following data apply to two close-coiled helical springs.*

Spring	*Number of turns* n	*Diameter of coil* D *(cm.)*	*Diameter of rod* d *(cm.)*	*Uncompressed axial length (cm.)*
A	8	10	0·60	7·00
B	10	8	0·54	7·50

Spring B is placed inside A and both are compressed between a pair of parallel plates until the distance between the two plates measures 6·00 cm. Calculate (i) the load applied to the plates and (ii) the maximum shear stress in each spring. Take $C=0{\cdot}82\times10^6$ *kg./cm* ^{2}.

Solution. *Spring A.* Let the load on the spring be W_1 *kg.* Compression of the spring

$$=\delta_1=7{\cdot}00-6{\cdot}00=1 \; cm.$$

$$\delta_1=\frac{64\,W_1R_1{}^3n_1}{Cd_1{}^4}$$

$$\therefore \quad W_1=\frac{\delta_1\;Cd_1{}^4}{64\;R_1{}^3n_1}$$

$$=\frac{1\times0{\cdot}82\times10^6\,(0{\cdot}6^4)}{64\times5^3\times8}\;kg.$$

$$=1{\cdot}662\;kg.$$

Spring B. Let the load on this spring be W_2 *kg.*

Compression of the spring

$$=\delta_2=7{\cdot}50-6{\cdot}00=1{\cdot}50\;cm.$$

$$\therefore \quad W_2=\frac{\delta_2\;Cd_2{}^4}{64\;R_2{}^3n_2}$$

$$\therefore \quad W_2=\frac{1{\cdot}50\times0{\cdot}82\times10^6(0{\cdot}54)^4}{64(4)^3\times10}\;kg.$$

$$=2{\cdot}554\;kg.$$

∴ Total load applied on the plates

$$=W_1+W_2$$

$$=1{\cdot}662+2{\cdot}554\;kg.$$

$$=4{\cdot}216\;kg.$$

Shear stress for the spring *A*

$$=f_{s1}=\frac{W_1\;R_1}{\left(\frac{\pi d_1{}^3}{16}\right)}$$

$$=\frac{1{\cdot}662\times5\times16}{\pi(0{\cdot}6)^3}\ kg./cm.^2$$

$$=195{\cdot}8\ kg/cm.^2$$

Shear stress for the spring B

$$=f_{s2}=\frac{W_2R_2}{\left(\frac{\pi d_2^3}{16}\right)}$$

$$=\frac{2{\cdot}554\times4\times16}{\pi(0{\cdot}54)^3}\ kg./cm.^2$$

$$=330{\cdot}4\ kg./cm.^2$$

Problem 361. *Find the maximum permissible load for a closely coiled spring made out of 8 mm × 8 mm. square rod with 10 coils of 8 cm. mean diameter, if the maximum shearing stress is limited to 700 kg. per cm.*^{2} *Find also the deflection of the load. Take* $C=0{\cdot}9\times10^6\ kg./cm.^2$

Solution. Since the rod is of square section

$$T=0{\cdot}208\ x^3 f_s$$

$$=0{\cdot}208\ (0{\cdot}8)^3\times700\ kg.\ cm.$$

$$=74{\cdot}56\ kg.\ cm.$$

But $T=WR$

$\therefore$ $W\times4=74{\cdot}56\ kg.\ cm.$

$\therefore$ $W=18{\cdot}64\ kg.\ cm.$

Twist of a rod of square section

$$=\theta=\frac{7{\cdot}11\ Tl}{Cx^4}$$

$$\frac{7{\cdot}11\ T2\pi Rn}{Cx^4}$$

$$=\frac{7{\cdot}11\times74{\cdot}56\times2\pi\times4\times10}{0{\cdot}9\times10^6\times(0{\cdot}8)^4}\ radian$$

$$=0{\cdot}3616\ radian$$

$\therefore$ $\delta=R.\ \theta$

$$=4\times0{\cdot}3616\ cm.$$

$$=1{\cdot}4464\ cm.$$

§103. Torsion of a tapering shaft

Let a twisting moment T be applied to a tapering shaft of length l whose radius change uniformly from r_1 at the left end to r_2 at the right end

Let f_{s1} = **maximum shear stress at the left end**

f_{s2} = **maximum shear stress at the right end**

and f_s = **maximum shear stress at a section X distant x from the left end.**

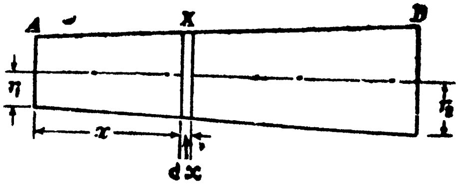


Fig. 546

Let the radius at the section X be r.

$$\therefore \quad T=f_{s1}\frac{\pi}{2}r_1^3=f_{s2}\cdot\frac{\pi}{2}r_2^3=f_s\frac{\pi r^3}{2}$$

$$\therefore \quad f_{s1}r_1^3=f_{s2}r_2^3=f_sr^3$$

Consider a short length dx of the shaft.

For this short length the shaft may be considered as having a uniform radius r.

$\therefore$ Angle of twist of the small length dx of the shaft

$$=d\theta=\frac{T}{CI_p}dx$$

$$=\frac{2T}{C\pi}\frac{dx}{r^4}$$

But, $\qquad r=r_1+\frac{r_2-r_1}{l}x=r_1+kx$

where $\qquad k=\frac{r_2-r_1}{l}$

$$\therefore \quad d\theta=\frac{2T}{C\pi}\frac{dx}{(r_1+kx)^4}$$

$\therefore$ Total angle of twist for the whole length l of the shaft

$$=\theta=\int d\theta$$

$$=\int_0^l \frac{2T}{C\pi}\frac{dx}{(r_1+kx)^4}$$

$$=-\frac{2T}{C\pi}\frac{1}{3k}\left[\frac{1}{(r_1+kx)^3}\right]_0^l$$

$$=-\frac{2}{3k}\frac{T}{C\pi}\left[\frac{1}{(r_1+kl)^3}-\frac{1}{r_1^3}\right]$$

Put $\qquad k=\frac{r_2-r_1}{l}$

$$\therefore \quad kl=r_2-r_1$$

$$\therefore \quad \theta = -\frac{2}{3k}\frac{T}{C\pi}\left[\frac{1}{r_2^3}-\frac{1}{r_1^3}\right]$$

$$\therefore \quad \theta = \frac{2}{3k}\frac{T}{C\pi}\left[\frac{1}{r_1^3}-\frac{1}{r_2^3}\right]$$

$$\therefore \quad \theta = \frac{2}{3}\cdot\frac{Tl}{C\pi}\left(\frac{l}{r_2-r_1}\right)\left(\frac{r_2^3-r_1^3}{r_1^3r_2^3}\right)$$

$$\therefore \quad \theta = \frac{2Tl}{3C\pi}\left[\frac{r_1^2+r_1r_2+r_2^2}{r_1^3r_2^3}\right]$$

For the particular case of the shaft of uniform radius,

$$r_1 = r_2 = r$$

and,
$$\theta = \frac{2}{3}\frac{Tl}{C\pi}\cdot\frac{3r^2}{r^6}$$

$$= \frac{2Tl}{C\pi r^4}$$

$$= \frac{Tl}{CI_0}$$

Examples in Chapter 10

1. A shaft running at 140 rpm is required to transmit 50 H P. If the maximum torque is likely to exceed the mean torque by 25%, find the diameter of the shaft, if the maximum shear stress is 600 $kg./cm^2$. Find also the angle of twist for a length of 2·25 *metres*. Take $C = 0{\cdot}8\times10^6$ kg/cm^2. *(6·474 cm ; 2° 59')*

2. Find the H.P. that can be transmitted by a 60 *mm.* diameter shaft at 160 rpm if the permissible shear stress is 800 $kg./cm^2$ the maximum torque being 30% greater than the mean torque *(58·32 H P.)*

3. Find the size of a square shaft to transmit 100 *h p* at 120 rpm if shear stress is not to exceed 500 $kg./cm^2$ *(83 mm. × 83 mm.)*

4. A shaft is 2 *metres* long, 6 *cm.* diameter at one end, and tapers at a uniform rate to 8 *cm* diameter at the other end. The larger end is firmly fixed and a torque of 350 *kg. m.* is applied to the smaller end. Find the maximum shear stress and the total angle of twist. Take $C = 0{\cdot}8\times10^6$ $kg./cm^2$ *(825 kg/cm.² ; 2° 16')*

5. A hollow shaft of diameter ratio 3 : 5, is required to transmit 800 H P. at 110 rpm the maximum torque being 12% greater than the mean. The shearing stress is not to exceed 4 *tons per sq. in.* and the twist in a length of 10 *ft.* is not to exceed one degree. Find the minimum external diameter of the shaft satisfying these conditions *(London University)* *(7·68 in.)*

6. A hollow shaft of circular section is to have an inside diameter one half the outside diameter. It is to be designed to

transmit 40 H.P. at a speed of 480 rpm and the shear stress is not to exceed 850 $kg./cm^2$

Calculate :—

(*a*) the external diameter of the hollow shaft ;

(*b*) the angle of relative twist in degrees between two sections 2 *metres* apart ;

(*c*) percentage difference in the weight of the hollow shaft compared with a solid circular shaft designed for the same conditions. Take $C=0.84\times10^6$ $kg./cm^2$ (*3.38 cm ; 6° .8 ; 21.7%*)

7. A steel shaft *ABCD* having a total length of 120 *cm* is made up of three lengths *AB*, *BC* and *CD* each 40 *cm* long. *AB* and *BC* are solid having diameters of 4.5 *cm*. and 5.5 *cm*. respectively and *CD* is hollow, having outside and inside diameter of 5.5 *cm* and 3.5 *cm*. respectively. When an axial torque of 16000 *kg cm*. is transmitted from one end of the shaft to the other, the total angle of twist from *A* to *D* is 2 *degrees*.

Determine :—

(*a*) the maximum shearing stress in the shaft and state where this occurs ;

(*b*) the angle of twist for each of the three lengths *AB*, *BC* and *CD* ;

(*c*) the modulus of rigidity of the material

($f_{s(max)}$ occurs in the shaft *AB*. f_{smax} = 895.8 kg/cm^2 ;
1° .01 ; 0° 45 ; 0° .54 9.036 × 10^5 $kg./cm^2$.)

8. A hollow shaft subjected to pure torque, attains a maximum shearing stress f_s. Given that the strain energy stored per unit volume is $\frac{f_s^2}{3C}$, where *C* is the modulus of rigidity calculate the ratio of shaft diameters ($\sqrt{3}$).

9. A hollow steel shaft 8 *in*. external diameter and 5 *in*. internal diameter transmits 1800 H.P. at a speed of 150 R.P.M. Calculate the shearing stress at the inner and outer surfaces of the shaft and the strain energy per foot length. Take $C=12\times10^6$ $lbs./in^2$ (*London University*) (*5550 lbs/in.² ; 8890 lbs./in.² ; 840.6 in. lbs.*)

10. Compare the weight of a solid shaft with that of a hollow one to transmit a given horse-power at a given speed with a given maximum shearing stress, the outside diameter of the hollow shaft being 1½ times the internal diameter. (*0.644*)

11. A shaft tapers uniformly from a radius $(r+a)$ at one end to $(r-a)$ at the other. If it is under the action of an axial torque *T* and $a=0.1r$, find the percentage error in the angle of twist for a given length when calculated on the assumption of a constant radius *r*. (*London University*) (*3.25%*)

12. A shaft AB of length l is fixed at both ends. A torque T is applied at a section X. If $AX=a$ and $BX=b$ and $a>b$, find the diameter of shaft in order the shear stress may not exceed f_s

$$\left(d=\sqrt[3]{\frac{16aT}{(a+b)\pi f_s}} \right)$$

13. A shaft ABC has pulleys at A, B and C. If through pulley A 500 H.P. is transmitted to the shaft, and through pulleys B and C 200 H.P. and 300 H P. are drawn off, find the ratio of the diameters of the shafts AB and BC. Find also the ratio of the angles of twist of these two parts if $AB=l_1$ and $BC=l_2$

$$\left(\sqrt[3]{\frac{5}{3}}; \ \frac{l_1}{l_2} \sqrt[3]{\frac{3}{5}} \right)$$

14. A shaft AB of length l is fixed at both ends. Two like twisting moments M_1 and M_2 are applied at sections C and D. If $AC=a$, $CD=b$ and $DB=c$, find the torque in each portion of the shaft.

$$\left(\frac{M_1(b+c)+M_2c}{l}; \ \frac{M_1a-M_2c}{l}; \ \frac{M_1a+M_2(a+b)}{l} \right)$$

15. Determine the diameter at which the angle of twist and not the maximum stress, is the controlling factor in the design, if $C=0{\cdot}84\times10^6$ *kg./cm.*^{2}, and $f_s=210$ *kg./cm.*^{2} and the maximum allowable twist is $\frac{1}{4}°$ per *metre*.

(*When the diameter is less than 11·46 cm., the angle of twist is the controlling factor*).

16. A close coiled helical spring is to have a stiffness of 400 *lbs. per inch* and is subjected to a maximum load of 500 *lbs.* If the mean diameter of the coils is to be 3·5 *in.* and the working stress is 14 *tons/in.*^{2}, find the number of coils and the diameter of the steel rod from which the spring should be made. Take the modulus of rigidity as 12×10^6 *lbs./in*^{2}. (*London University*) (*5·93 ; 0·51 in.*)

17. Close-coiled helical springs having n turns are made of round wire such that the mean diameter of the coils D (*in*) is 10 times the diameter of the wire. Show that the 'stiffness' in *lbs per in.* for any such spring is $\frac{D}{n}\times a$ constant and determine the constant if the modulus of rigidity of the material is 12×10^6 *lbs./in*^{2}.

Such a spring is required to support a load of 200 *lbs.* with an extension of 4 *in.* and a maximum shea ring stress of 50,000 *lbs./in*^{2} Calculate (*i*) its weight, (*ii*) the mean diameter of the coils and. (*iii*) the number of turns. The material weighs 0·28 *lb. per cubic inch* (*London University*) (*2·15 lbs. ; 3·2 in. ; 9·6.*)

11

Principal Stresses and Strains

§104. Normal and tangential or shear stresses

Case 1. Member subjected to axial load

Fig. 547 shows a rectangular bar *ABCD* of uniform sectional area *A*. Let the member be subjected to an axial tensile load *W* producing a tensile stress $p = \frac{W}{A}$ on sections normal to the axis of loading. On such section normal to the axis of loading only normal stresses are induced and no tangential stresses are induced.

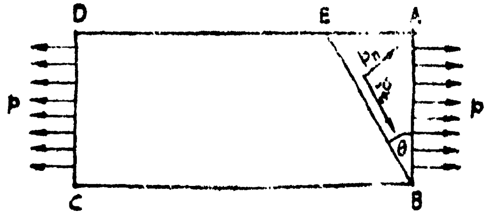


Fig. 547

Consider any oblique plane *BE* at an angle θ to the cross-section. On the plane *BE* normal and tangential stresses are induced.

The sectional area of the member along the plane $BE = A \sec \theta$.

Normal stress on the plane *BE*

$$= p_n = \frac{\text{Total force normal to the plane } BE}{\text{Sectional area along the plane } BE}$$

$$\therefore \quad p_n = \frac{pA \cos \theta}{A \sec \theta}$$

$$\therefore \quad p_n = p \cos^2 \theta$$

Tangential stress on the plane *BE*

$$= p_t = \frac{\text{Tangential force along the plane } BE}{\text{Sectional area along the plane } BE}$$

$$\therefore \quad p_t = \frac{pA \sin \theta}{A \sec \theta}$$

$$= p \sin \theta \cos \theta$$

$$\therefore \quad p_t = \frac{p}{2} \sin 2\theta$$

From the expressions obtained for p_n and p_t it follows that p_n is maximum when $\theta=0$ and the maximum value of $p_n = p \cos^2 0 = p$

p_n is a minimum when $\theta = 90°$

Minimum value of $p_n = p \cos^2 90° = 0$

Also, p_t is a maximum when $\sin 2\theta$ is a maximum.

$$i.e. \quad \text{when } \sin 2\theta = 1$$

$$2\theta = 90°$$

$$\therefore \quad \theta = 45°$$

$$\text{Maximum value of } p_t = \frac{p}{2} \sin 2\theta$$

$$= \frac{p}{2} \sin 90° = \frac{p}{2}$$

$$\text{when} \quad \theta = 0 \text{ or } 90°$$

$$p_t = 0$$

Hence, we find that, if a member be subjected to axial tensile load, the plane normal to the axis of loading carries the greatest normal stress and the plane inclined at 45° to the plane carrying the greatest normal stress, carries the maximum shear stress and the intensity of the greatest shear stress is one half the intensity of the greatest normal stress.

Suppose the tensile strength of a member is greater than double the shear strength. If such a member is subjected to axial load up to failure, the failure of the member will occur by shear. But if the tensile strength is less than twice the shear strength, the failure of the member will occur due to maximum normal stress.

For the planes corresponding to $\theta = 0$ and $\theta = 90°$, *i.e.*, for the sectional planes *BA* and *BC*, we find there are no shear or tangential stresses. Such a plane, on which no tangential stress occurs is called a *principal plane*. If at all there is a stress on a principal plane, it should be normal to the plane. The only normal stresses occurring on the principal planes are called *principal stresses*. In our case the principal stress on the principal plane *BA* equals p. But the principal stress on the principal plane *BC* equals zero.

Case 2. Member subjected to like principal stresses

Consider a rectangular block *ABCD* whose thickness perpendicular to the plane of the paper is unity.

Let the block be subjected to principal stresses p_1 and p_2 as shown.

Consider any oblique sectional plane *BE* at an angle θ with the principal plane *BA*.

Fig. 548

Normal stress on the plane BE

$$p_n = \frac{\text{Total force normal to the plane } BE}{\text{Sectional area along the plane}}$$

$$p_n = \frac{p_1\, BA \cos\theta + p_2\, EA \sin\theta}{BE}$$

$\therefore$ $$p_n = p_1 \cos^2\theta + p_2 \sin^2\theta$$

$$= \frac{p_1}{2}\left(1 + \cos 2\theta\right) + \frac{p_2}{2}\left(1 - \cos 2\theta\right)$$

$\therefore$ $$p_n = \frac{p_1 + p_2}{2} + \frac{p_1 - p_2}{2} \cos 2\theta$$

Tangential stress on the plane BE

$$= p_t = \frac{\text{Total force tangential or parallel to the plane } BE}{\text{Sectional area along the plane } BE}$$

$\therefore$ $$p_t = \frac{p_1\, BA \sin\theta - p_2\, EA \cos\theta}{BE}$$

$\therefore$ $$p_t = p_1 \cos\theta \sin\theta - p_2 \sin\theta \cos\theta$$
$$= (p_1 - p_2) \sin\theta \cos\theta$$

$\therefore$ $$p_t = \frac{p_1 - p_2}{2} \sin 2\theta$$

Hence on the plane BE the resultant stress p is the resultant of p_n and p_t

$\therefore$ $$p = \sqrt{(p_1 \cos^2\theta + p_2 \sin^2\theta)^2 + (p_1 - p_2)^2 \sin^2\theta \cos^2\theta}$$

This simplifies to,

$$p = \sqrt{p_1^2 \cos^2\theta + p_2^2 \sin^2\theta}$$

The angle that the line of action of the resultant stress makes with the normal to the plane is called the *obliquity*.

If the obliquity be ϕ, we have,

$$\tan\phi = \frac{p_t}{p_n}$$

Let p_x be the force on unit area of the plane BE parallel to the direction of the principal stress p_1.

Let p_y be the force on unit area of the plane BE parallel to the direction of the principal stress p_2

We have $\quad p_x = p_1 \cos\theta$

and $\quad p_y = p_2 \sin\theta$

$\therefore$ Resultant stress

$$p = \sqrt{p_x^2 + p_y^2}$$

or $$p = \sqrt{p_1^2 \cos^2\theta + p_2^2 \sin^2\theta}$$

a result obtained earlier.

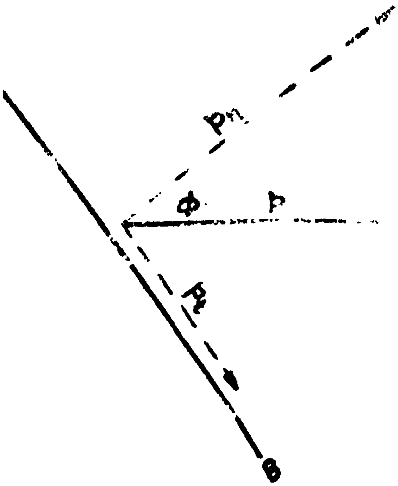


Fig. 549

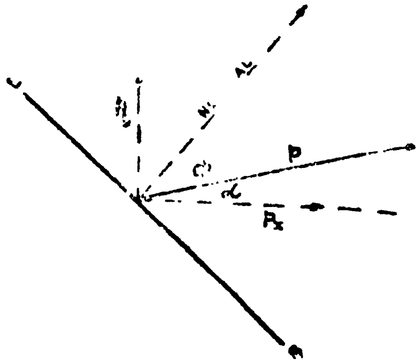


Fig. 550

Let α be the inclination of the resultant stress with the principal stress p_1.

We have

$$\tan\alpha=\frac{p_y}{p_x}=\frac{p_2}{p_1}\tan\theta$$

Now consider the expressions obtained earlier *viz.*

$$p_n=\frac{p_1+p_2}{2}+\frac{p_1-p_2}{2}\cos 2\theta$$

$$p_t=\frac{p_1-p_2}{2}\sin 2\theta$$

For the principal planes, p_t should be equal to zero,

i.e. $\frac{p_1-p_2}{2}\sin 2\theta=0$

$2\theta=0$ or $180°$

Or $\theta=0$ or $90°$

when $\theta=0$, $p_n=\frac{p_1+p_2}{2}+\frac{p_1-p_2}{2}(1)=p_1$

when $\theta=90°$, $p_n=\frac{p_1+p_2}{2}+\frac{p_1-p_2}{2}(-1)=p_2$

Further p_t is maximum when

$\sin 2\theta=1$

$\therefore$ $2\theta=90°$ or $270°$

or $\theta=45°$ or $135°$

$\therefore$ **Max. shear stress**

$$=p_{t(max)}=\frac{p_1-p_2}{2}$$

Hence there are two mutually perpendicular planes along which the greatest shear stress occurs. These planes are at angles of 45° and 135° with the principal plane carrying the principal stress p_1.

Problem 362. *The principal tensile stresses at a point across two perpendicular planes are 800 kg/cm² and 400 kg./cm². Find the normal, tangential stresses and the resultant stress and its obliquity on a plane at 20° with the major principal plane. Find also the intensity of stress which acting alone can produce the same maximum strain. Take Poisson's ratio = ¼.*

Solution.

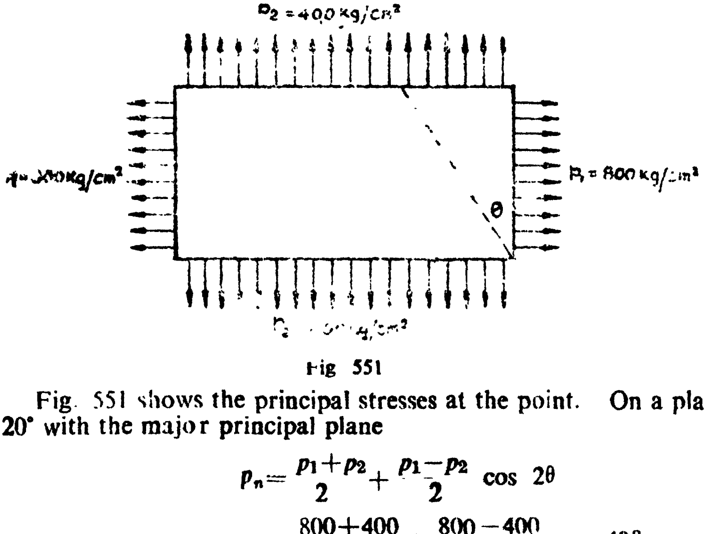


Fig 551

Fig. 551 shows the principal stresses at the point. On a plane at 20° with the major principal plane

$$p_n = \frac{p_1+p_2}{2} + \frac{p_1-p_2}{2} \cos 2\theta$$

$$= \frac{800+400}{2} + \frac{800-400}{2} \cos 40°$$

$$= 600+200 \cos 40°$$

$$\therefore \quad p_n = 753{\cdot}2 \text{ kg./cm.}^2 \text{ (tensile)}$$

$$p_t = \frac{p_1-p_2}{2} \sin 2\theta = \frac{800-400}{2} \sin 40°$$

$$= 200 \sin 40°$$

$$p_t = 128{\cdot}56 \text{ kg./cm.}^2$$

$\therefore$ Resultant stress

$$= \sqrt{p_n^2+p_t^2} = \sqrt{753{\cdot}2^2+128{\cdot}56^2}$$

$$= 764 \text{ kg./cm.}^2$$

Obliquity $= \phi = \tan^{-1} \dfrac{p_t}{p_n} = \tan^{-1} \dfrac{128{\cdot}56}{753{\cdot}2}$

$$\therefore \quad \phi = 9° \ 41'$$

Maximum strain $= \dfrac{p_1}{E} - \dfrac{p_2}{mE}$

of the major and minor axes of an ellipse formed as a result of the deformation of the circle marked.

Take $E=2\times10^6$ *kg./cm.*^{2} and $\frac{1}{m}=\frac{1}{4}$

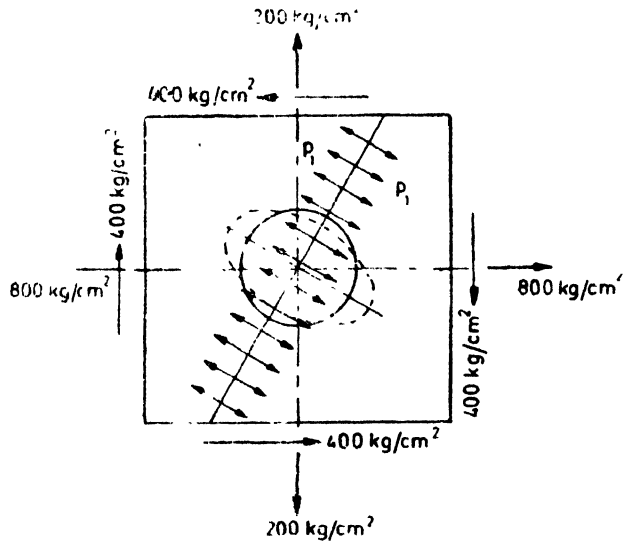


Fig. 552.

Solution.

$$p=800 \textit{ kg./cm.}^2$$
$$p'=200 \textit{ kg./cm.}^2$$
$$p=400 \textit{ kg./cm.}^2$$

Hence the principal stresses are given by

$$\frac{p+p'}{2}\pm\sqrt{\left(\frac{p-p'}{2}\right)^2+q^2}$$

$$\frac{p+p'}{2}=\frac{800+200}{2}=500 \textit{ kg./cm.}^2$$

and $$\frac{p-p'}{2}=\frac{800-200}{2}=300 \textit{ kg./cm.}^2$$

$\therefore$ **The principal stresses are** $500\pm\sqrt{300^2+400^2}$ *kg./cm.*^{2}

$$=500\pm500 \textit{ kg./cm.}^2$$

$\therefore$ $p_1=1000$ *kg./cm.*^{2} and $p_2=0$

$$e_1=\frac{p_1}{E}-\frac{p_2}{mE}=\frac{1000}{2\times10^6}=\frac{1}{2000}$$

∴ Increase in length of diameter of the circle,

$$=e_1\ d=\frac{1}{2000}\times 10=\frac{1}{200}\ cm.$$

$$e_2=\frac{p_2}{E}-\frac{p_1}{mE}=-\frac{1000}{4\times 2\times 10^6}=-\frac{1}{8000}$$

∴ Decrease in length of diameter $=e_2 d=\frac{1}{8000}\times 10=\frac{1}{800}$ *cm.*

Thus the circle will become an ellipse whose major axis $=10+\frac{1}{200}$

$=10{\cdot}005$ *cm.* and minor axis

$=9{\cdot}99875$ *cm.*

Graphical Methods

First Method

Let p_1 and p_2 be the two principal stresses. Let p_1 be the major principal stress. Let it be required to find the normal and tangential stresses on a plane at an angle θ with the major principal plane.

Draw two concentric circles with centre O and radii $OA=p_1$ and $OB=p_2$. Mark the plane XOX at an angle θ with major principal plane. Let the normal OCD to the plane XOX intersect the two circles at C and D. Draw DE pernendicular to OA and CP perpendicular to DE. Join OP. OP represents the resultant stress on the plane XOX. Draw PG perpendicular to OD. OG represents the normal stress and GP the tangential stress on the plane XOX.

The angle $G\hat{O}P=\phi=$obliquity.

The inclination of the resultant stress with the direction of the major principal stress $=\alpha=POE$.

Proof : Let CF be perpendicular to OA

Now $OE=OD\cos\theta=p_1\cos\theta=p_x$

$EP=FC=OC\sin\theta=p_2\sin\theta=p_y$

The resultant stress p being the resultant of p_x and p_y

we have $OP=p$

Hence OP represents the resultant stress.

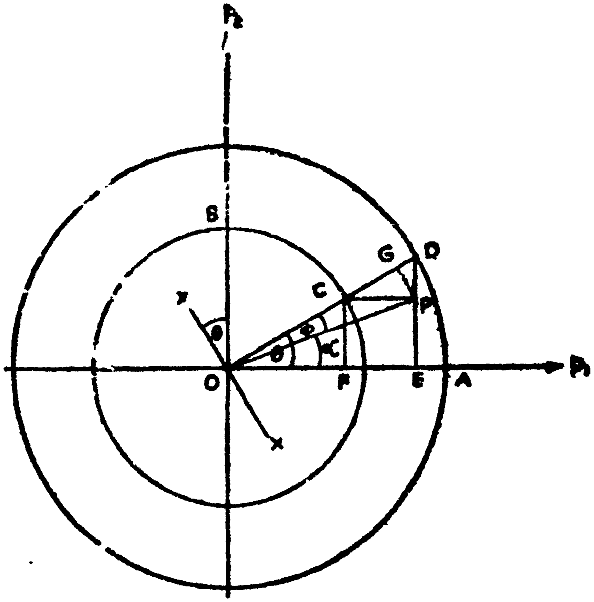


Fig. 553

The Ellipse of Stress

In the above graphical construction the co-ordinates of P with O as origin, are

$$x = OE = p_x = p_1 \cos\theta$$

$$y = EP = p_y = p_2 \sin\theta$$

Hence the coordinates of P for any plane XOX are given by the above relation.

$$\therefore \qquad \frac{x^2}{p_1^2} + \frac{y^2}{p_2^2} = \cos^2\theta + \sin^2\theta = 1$$

$$\therefore \qquad \frac{x^2}{p_1^2} + \frac{y^2}{p_2^2} = 1$$

The above is the equation to an ellipse.

Hence the locus of P is an ellipse whose major axis is $2\,p_1$ and minor axis is $2\,p_2$. This ellipse is called the *Ellipse of Stress.*

Two like and equal principal stresses.

From the relations

$$p_n = \frac{p_1 + p_2}{2} + \frac{p_1 - p_2}{2} \cos 2\theta$$

and $$p_t = \frac{p_1 - p_2}{2} \sin 2\theta$$

we find that when $p_1 = p_2$

$$p_n = p_1 \qquad \text{and } p_t = 0$$

Hence all planes are principal planes without any tangential stresses.

Following the above graphical construction we find that the two concentric circles coincide. Points *P, C, G, D* also coincide.

Hence the resultant stress which is represented by *OP* is normal to the plane *XOX*

Two equal and unlike principal stresses

Let the major principal stress be p_1 and let the other principal stress be $p_2=-p_1$

we have

$$p_n=\frac{p_1+p_2}{2}+\frac{p_1-p_2}{2}\cos 2\theta$$

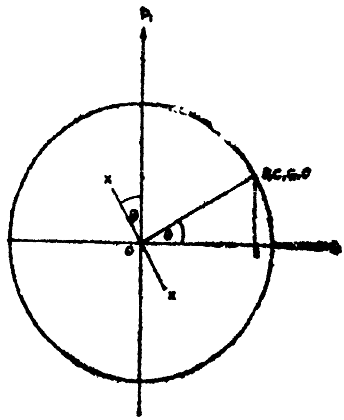


Fig. 554

$$p_n=\frac{p_1-p_1}{2}+\frac{p_1+p_1}{2}\cos 2\theta=p_1\cos 2\theta$$

$$p_t=\frac{p_1-p_2}{2}\sin 2\theta$$

$$=\frac{p_1+p_1}{2}\sin 2\theta$$

$$\therefore \quad p_t=p_1\sin 2\theta.$$

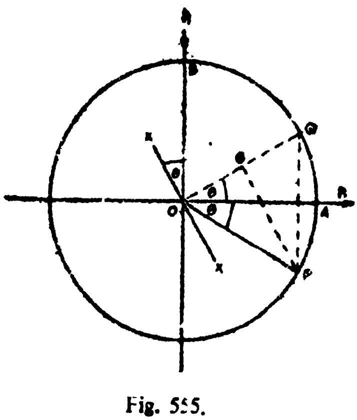


Fig. 555.

Since $p_n=p_1\cos 2\theta$ and $p_t=p_1\sin 2\theta$ it follows that the resultant stress is at an obliquity of 2θ.

In this case the graphical solution may be obtained as follows.

Draw a circle with centre *O* and radius p_1.

Mark the direction of the two principal stresses. Mark the plane *XOX* at an angle θ with the major principal plane. Draw the normal *OQ* to the plane. Draw *QP* perpendicular to *OA* and obtain the point *P* (see Fig. 555).

The resultant stress is given by *OP*. Draw *PG* perpendicular to *OQ*.

Now $OG=OP\cos 2\theta=p_1\cos 2\theta=p_n$

and $GP=OP\sin 2\theta=p_1\sin 2\theta=p_t$.

It is worth remembering that

(*i*) *When the two principal stresses are equal and like, the resultant stress on any plane is normal to the plane and equal in magnitude to either of the principal stresses.*

(*ii*) *When the two principal stresses are equal and unlike, the resultant stress on any plane equals in magnitude to either of the principal stresses, but at an obliquity of 2θ where θ is the angle between the plane and the major principal plane.*

Second Method

Let p_1 and p_2 be two unequal like principal stresses. Let it be required to find the resultant stress on a plane inclined at angle θ with the major principal plane. The stresses p_1 and p_2 can be written as

$$p_1=\frac{p_1+p_2}{2}+\frac{p_1-p_2}{2}$$

and
$$p_2=\frac{p_1+p_2}{2}-\frac{p_1-p_2}{2}$$

Hence the principal stresses p_1 and p_2 may be split up into the following systems of principal stresses.

(*i*) Two equal like principal stresses of intensity $\frac{p_1+p_2}{2}$

(*ii*) Two equal unlike principal stresses of intensity $\frac{p_1-p_2}{2}$

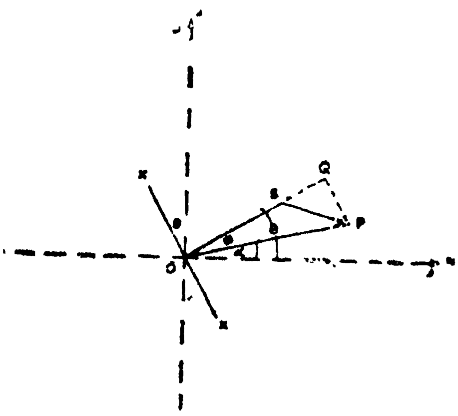


Fig. 556

Let XOX be the plane at θ with the major principal plane.

Draw OS perpendicular to XOX and equal to $\frac{p_1+p_2}{2}$ · OS represents the resultant of the first system of stresses.

Now draw SP making an angle 2θ with the direction of OS.

Let SP be equal to $\frac{p_1-p_2}{2}$

Now SP represents the resultant of the second system of stress. Hence the resultant stress on the plane XOX is the vectorial sum of OS and SP and is given by OP.

Draw PQ perpendicular to OS

Now OQ represents the normal stress and QP represents the tangential stress.

Angle $\hat{QOP}=\phi=$obliquity.

and $\hat{POA}=\alpha=\theta-\phi$

Even by studying the geometry of Fig. 556
We have $OQ=OS+SQ=OS+SP\cos 2\theta$

$$=\frac{p_1+p_2}{2}+\frac{p_1-p_2}{2}\cos 2\theta=p_n$$

and $$QP=SP\sin 2\theta=\frac{p_1-p_2}{2}\sin 2\theta=p_t$$

Third Method

By Mohr's Circle

Let p_1 and p_2 be two unequal like principal stresses. Let it be required to find the resultant stress on a plane inclined at an angle θ with the major principal plane.

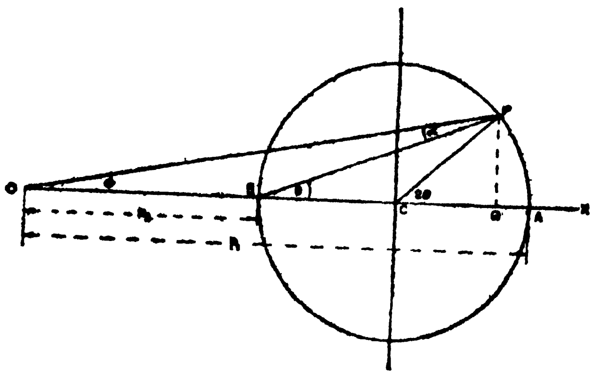


Fig. 557

On any axis OX set off $OA=p_1$ and $OC=p_2$. Describe a circle with AB as diameter. Let C be the centre of this circle. This circle is called *Mohr's circle.* Set off CP at 2θ with the line OX. Join OP, Draw PQ perpendicular to OX.

Now, $$OC=\frac{OA+OB}{2}$$

$$=\frac{p_1+p_2}{2}$$

$$AB=OA-OB$$

$$=p_1-p_2$$

$\therefore$ Radius of the Mohr's circle

$$=CP=BC$$

$$=\frac{p_1-p_2}{2}$$

$$\therefore \quad OQ=OC+CQ=OC+CP\cos 2\theta$$

$$=\frac{p_1+p_2}{2}+\frac{p_1-p_2}{2}\cos 2\theta$$

$$=p_n$$

$$QP=CP\sin 2\theta$$

$$=\frac{p_1-p_2}{2}\sin 2\theta$$

Hence OQ and QP represent the normal and tangential stresses. OP therefore represents the resultant stress and the angle POQ represents the obliquity.

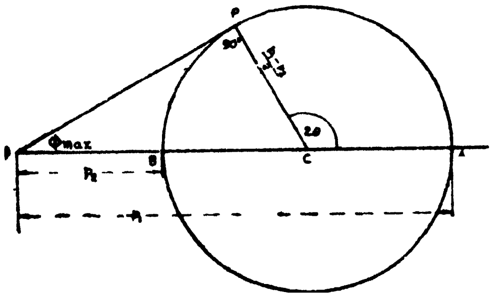


Fig. 558

We find from the *Mohr's circle diagram* as the inclination θ of the plane with the major principal plane goes on increasing the obliquity also increases. But there is a certain plane on which the obliquity is a maximum.

For this condition, the line OP representing the resultant stress should be tangential to the Mohr's circle.

The plane on which the obliquity is greatest is given by

$$2\theta=\frac{\pi}{2}+\phi_{max}$$

or

$$\theta=\frac{\pi}{4}+\frac{\phi_{max}}{2}$$

Further,

$$\sin\phi_{max}=\frac{CP}{OC}=\frac{\frac{p_1-p_2}{2}}{\frac{p_1+p_2}{2}}$$

$$\therefore \quad \sin\phi_{max}=\frac{p_1-p_2}{p_1+p_2}$$

Mohr's circle of stress for two unequal unlike principal stresses

Let the major principal stress be p_1 (*tensile*) and the other principal stress be p_2 (*compressive*).

Taking p_1 as positive and p_2 as negative, the normal and tangential stress on any plane at an angle θ with the major principal plane are therefore given by

$$p_n = \frac{p_1 - p_2}{2} + \frac{p_1 + p_2}{2} \cos 2\theta$$

and $$p_t = \frac{p_1 + p_2}{2} \sin 2\theta.$$

Maximum shear stress equals $\frac{p_1 + p_2}{2}$

Hence the radius of the Mohr's circle represents the maximum shear stress.

Along any axis set off OA and OB along opposite directions to represent p_1 and p_2.

Describe a circle on AB as diameter. Let C be the centre of this circle.

Draw CP at an angle 2θ with CA. Join OP and draw PQ perpendicular to OA.

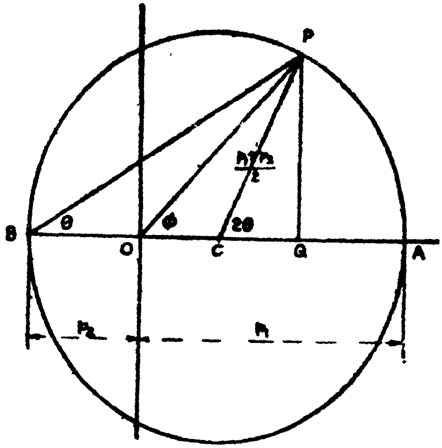


Fig. 559

Radius of the Mohr's circle

$$= BC = PC = AC$$

$$= \frac{p_1 + p_2}{2}$$

$$OC = BC - BO$$

$$= \frac{p_1 + p_2}{2} - p_2 = \frac{p_1 - p_2}{2}$$

$$OQ = OC + CQ = OC + CP \cos 2\theta$$

$$= \frac{p_1 - p_2}{2} + \frac{p_1 + p_2}{2} \cos 2\theta$$

$$= p_n$$

And, $$QP = CP \sin 2\theta = \frac{p_1 + p_2}{2} \sin 2\theta$$

$$= p_t$$

Hence QO and QP represent the normal and tangential stresses. OP therefore represents the resultant stress and the angle POQ represents the obliquity. Further the greatest shear stress

$$= \frac{p_1 + p_2}{2} = \text{radius of the } \textit{Mohr's circle}$$

An important note :

When the principal stresses are unlike, in the Mohr's circle, the normal and resultant stresses will be negative for points on the circle to the left of the vertical axis through O.

Determination of principal planes and principal stresses

Case I. At a point the complementary shear stresses of intensity q are induced. To determine the principal planes and the principal stresses.

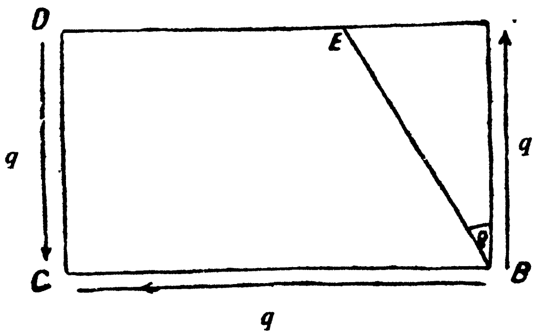


Fig. 560

Fig. 560 shows a rectangular block $ABCD$ in a state of simple shear.

Let the thickness of the block normal to the plane of the paper be unity. Consider any sectional plane BE at an angle θ with the plane AB.

Let the normal and tangential stresses on the plane BE be p_n and p_t.

$$p_n=\frac{qAB\sin\theta+qEA\cos\theta}{BE}$$

$\therefore$ $$p_n=q\sin\theta\cos\theta+q\sin\cos\theta$$

$\therefore$ $$p_n=q\sin 2\theta$$

$$p_t=\frac{qAB\cos\theta-pEA\sin\theta}{BE}$$

$$=q\cos^2\theta-q\sin^2\theta$$

$$p_t=q\cos 2\theta$$

For the principal planes, equating the tangential stress to zero we have

$$q\cos 2\theta=0$$

$\therefore$ $$2\theta=90^\circ \quad \text{or} \quad 270^\circ$$

or $$\theta=45^\circ \quad \text{or} \quad 135^\circ$$

Hence the two principal planes are at 45° and 135° with the plane *AB*. The principal planes are also at right angles to each other.

The two principal stresses are

$$p_1 = q \sin 2\theta = q \sin 90^\circ = +q$$

and $$p_2 = q \sin 2\theta = \sin 270^\circ = -q$$

Thus on one principal plane there will be tensile stress and the other principal plane will carry a compressive stress. These principal stresses are called diagonal tensile and diagonal compressive stresses.

Further for p_t to be maximum

$$p_t = q \cos 2\theta \text{ should be maximum.}$$

when $2\theta = 0$, *i.e.*, when $\theta = 0$, $p_t = q$

when $2\theta = 180^\circ$ *i.e.*, when $\theta = 90^\circ$, $p_t = -q$

This is so, since the planes *AB* and *AD* carry the shear stress of intensity q.

Fig. 561 shows the planes carrying the maximum shear stress and the principal planes. Planes *AB* and *AD* carry the maximum shear stresses. Planes *LM* and *MN* carry the principal stresses.

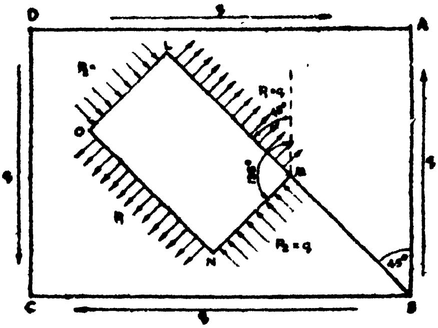


Fig. 561

It is worthy to note that

(*i*) *The two principal planes are normal to each other.*

(*ii*) *The planes carrying the maximum shear stress are normal to each other.*

(*iii*) *The planes carrying the maximum shear stress are at 45° with the principal planes.*

Case II. At a point in a strained material the normal and tangential stresses are given. To locate the principal planes and to determine the principal stresses.

Fig. 562 shows a rectangular block *ABCD* whose thickness normal to the plane of the paper is unity. Let it be subjected to the normal stresses p and p' and the tangential stresses q as shown.

Consider a sectional plane *BE* at an angle θ with the plane *AB*

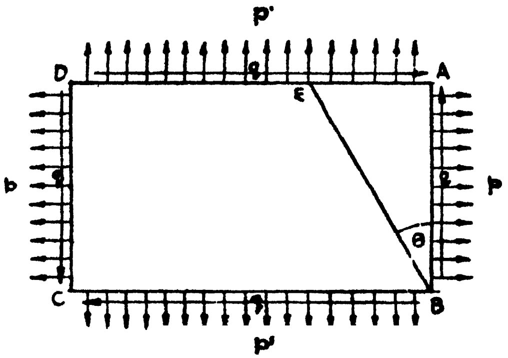


Fig. 562

Normal stress on the plane BE

$$=p_n=\frac{pAB\cos\theta+p'AE\sin\theta+qAE\cos\theta+qAB\sin}{BE}$$

$\therefore\ P_n=p\cos^2\theta+p'\sin^2\theta+q\sin\theta\cos\theta+q\cos\theta\sin\theta$

$\therefore\ P_n=p\cos^2\theta+p'\sin^2\theta+2\,q\sin\theta\cos\theta$

$$=\frac{p}{2}(1+\cos\ 2\theta)+\frac{p'}{2}(1-\cos 2\theta)+q\sin 2\theta$$

$$\boxed{\therefore\ P_n=\frac{p+p'}{2}+\frac{p-p'}{2}\cos\ 2\theta+q\sin\ 2\theta}$$

Tengential stress on the plane *BE*

$$=P_t=\frac{pAB\sin\theta-p'AE\cos\theta+qAE\sin\theta-qAB\cos\theta}{BE}$$

$\therefore\ p_t=p\cos\theta\sin\theta-p'\sin\theta\cos\theta+q\sin^2\theta-q\cos^2\theta$

$$\boxed{\therefore\ p_t=\frac{p-p'}{2}\sin 2\theta-q\cos 2\theta}$$

In order a plane may be a principal plane, the tangential stress on the plane must be zero. Equating the tangential stress p_t to zero, we have

$$\frac{p-p'}{2}\sin 2\theta-q\cos\ 2\theta=0$$

$$\therefore\qquad \tan 2\theta=\frac{2q}{p-p'}$$

There are thus two values of 2θ differing by 180° satisfying the above relation. Let $2\theta_1$ and $2\theta_2$ be the solutions to the equation

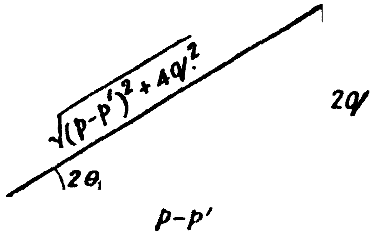


Fig. 563

$$\tan 2\theta=\frac{2q}{p-p'}$$

we have,

$$\sin 2\theta_1=\frac{2q}{\sqrt{(p-p')^2+4q^2}}$$

$$\cos 2\theta_1=\frac{p-p'}{\sqrt{(p-p')^2+4q^2}}$$

and
$$\sin 2\theta_2=\frac{-2q}{\sqrt{(p-p')^2+4q^2}}$$

and
$$\cos 2\theta_2=-\frac{p-p'}{\sqrt{(p-p')^2+4q^2}}$$

Obviously θ_1 and θ_2 differ by 90°. Hence the principal planes which are at θ_1 and θ_2 with the plane AB are normal to each other.

To determine the principal stresses p_1 and p_2 we should substitute the values of $2\theta_1$ and $2\theta_2$ for 2θ in the expression for the normal stress p_n.

$$\therefore \quad p_1=\frac{p+p}{2}+\frac{p-p'}{2}\cos 2\theta_1+q\sin 2\theta_1$$

$$=\frac{p+p'}{2}+\frac{p-p'}{2}\,\frac{(p-p')}{\sqrt{(p-p)^2+4q^2}}+\frac{2q^2}{\sqrt{(p-p')^2+4q^2}}$$

$$=\frac{p+p'}{2}+\frac{1}{2\sqrt{(p-p')^2+4q^2}}\left[(p-p')^2+4q^2\right]$$

$$=\frac{p+p'}{2}+\frac{1}{2}\sqrt{(p-p')^2+4q^2}$$

$$\boxed{\therefore \quad p_1=\frac{p+p'}{2}+\sqrt{\left(\frac{p-p'}{2}\right)^2+q^2}}$$

$$p_2=\frac{p+p'}{2}+\frac{p-p'}{2}\cos 2\theta_2+q\sin 2\theta_2$$

$$=\frac{p+p'}{2}-\frac{p-p'}{2}\,\frac{(p-p')}{\sqrt{(p-p')^2+4q^2}}-\frac{2q^2}{\sqrt{(p-p')^2+4q^2}}$$

$$=\frac{p+p'}{2}-\frac{1}{2\sqrt{(p-p')^2+4q^2}}\left[(p-p')^2+4q^2\right]$$

$$\therefore \qquad p_2 = \frac{p+p'}{2} - \frac{1}{2}\sqrt{(p-p')^2+4q^2}$$

$$\therefore \qquad \boxed{p_2 = \frac{p+p'}{2} - \sqrt{\left(\frac{p-p'}{2}\right)^2+q^2}}$$

These principal stresses may be like or unlike. Once the principal stresses and the principal planes are known, the planes of maximum shear stress are easily determined. The planes of maximum shear stress will be at $\theta_1+45°$ and $\theta_1+135°$ with the plane *AB*.

Fig. 564 shows the principal planes.

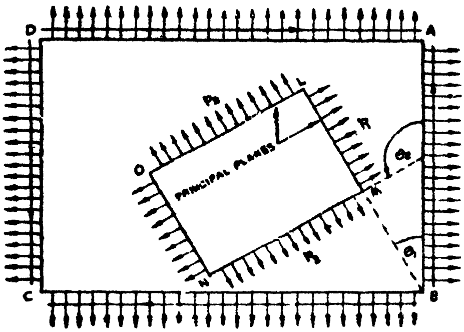


Fig. 564

Greatest shear stress $= q_{max}$

$$= \frac{p_1-p_2}{2}.$$

Problem 368. *A rectangular block of material is subjected to a tensile stress of 1100 kg./cm.² on one plane and a tensile stress of 470 kg./cm.² on a plane at right angles, together with shear stresses of 630 kg./cm.² on the same planes. Find :*

(*i*) *The direction of the principal planes.*

(*ii*) *The magnitudes of the principal stresses.*

(*iii*) *The magnitude of the greatest shear stress.*

Solution. Let p_1 and p_2 be the principal stresses.

The inclination of the principal planes with the plane *AB* carrying the tensile stress of $p=1100$ *kg./cm.²* is given by

$$\tan 2\theta = \frac{2q}{p-p'}$$

$$= \frac{2\times 630}{1100-470} = 2$$

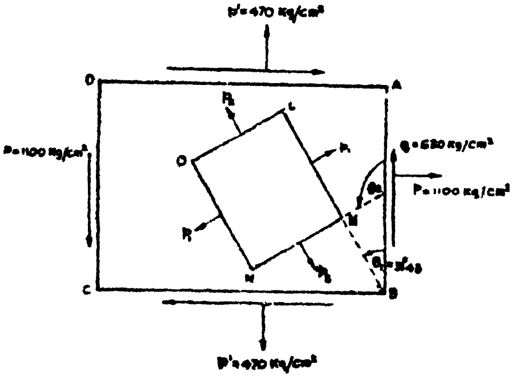


Fig 565

$\therefore \quad 2\theta = 63^\circ\ 26'$ or $243^\circ\ 26'$

$\therefore \quad \theta = 31^\circ\ 43'$ or $121^\circ\ 43'$

Major principal stress

$$= p_1 = \frac{p+p'}{2} + \sqrt{\left(\frac{p-p'}{2}\right)^2 + q^2}$$

$$= \frac{1100+470}{2} + \sqrt{\left(\frac{1100-470}{2}\right)^2 + 630^2} \ kg./cm.^2$$

$$= 785 + 704 \ kg./cm.^2$$

$$= 1489 \ kg./cm.^2 \ (tensile)$$

Minor principal stress

$$= p_2 = \frac{p+p'}{2} - \sqrt{\left(\frac{p-p'}{2}\right)^2 + q^2}$$

$$= 785 - 704 \ kg./cm.^2$$

$$= 81 \ kg./cm.^2 \ (tensile)$$

Maximum shear stress

$$= q_{max} = \frac{p_1 - p_2}{2} = \frac{1489 - 81}{2} \ kg./cm.^2$$

$$= 704 \ kg./cm.^2$$

This will occur at planes at $31^\circ\ 43' + 45^\circ = 76^\circ\ 43'$ and $76^\circ\ 43' + 90^\circ = 166^\circ\ 43'$ with the plane *AB* carrying the normal stress of 1100 $kg./cm.^2$

Problem 369 (SI). *A rectangular block of material is subjected to a tensile stress of 100 MN/m² on one plane and a tensile stress of 50 MN/m² on a plane at right angles, together with shear stresses of 60 MN/m² on the same planes. Find :*

(*i*) *The direction of the principal planes.*

(*ii*) *The magnitudes of the principal stresses.*

(*iii*) *The magnitude of the greatest shear stress.*

Solution. Let p_1 and p_2 be the principal stresses.

Let AB represent the plane carrying the normal tensile stress of 100 $MN/m.^2$

Let θ be the inclination of a principal plane with the plane AB.

$$\tan 2\theta = \frac{2q}{p-p'} = \frac{2\times 60}{100-50} = 2\cdot 40$$

$\therefore$ $2\theta = 67° \, 22'$ or $247° \, 22'$

$\therefore$ $\theta = 33° \, 41'$ or $123° \, 41'$

Major principal stresses

$$= p_1 = \frac{p+p'}{2} + \sqrt{\left(\frac{p-p'}{2}\right)^2 + q^2}$$

$$= \frac{100+50}{2} + \sqrt{\left(\frac{100-50}{2}\right)^2 + 60^2}$$

$$= 75 + 65 = 140 \ MN/m^2 \ (tensile)$$

Minor principal stress

$$= p_2 = \frac{p+p'}{2} - \sqrt{\left(\frac{p-p'}{2}\right)^2 + q^2}$$

$$= 75 - 65 = 10 \ MN/m^2 \ (tensile)$$

Maximum shear stress

$$= q_{max} = \frac{p_1 - p_2}{2}$$

$$= \frac{140-10}{2}$$

$$= 65 \ MN/m^2.$$

The maximum shear stress will occur on planes at $33° \, 41' + 45° = 78° \, 41'$, and $78° \, 41' + 90° = 168° \, 41'$ with the plane AB carrying the normal stress of 100 MN/m^2.

Problem 370. *When a certain thin-walled tube is subjected to internal pressure and torque the stresses in the tube wall are*

(*a*) *600 kg. per cm.² tensile.*

(*b*) *300 kg. per cm.² tensile in a direction at right angles to* (*a*).

(*c*) *Complementary shear stress of 450 kg./cm.² in the directions of* (*a*) *and* (*b*).

Calculate the normal and tangential stresses on the two planes which are equally inclined to (*a*) *and* (*b*).

What are the results if due to an end thrust, (*b*) *is compressive,* (*a*) *and* (*c*) *being unchanged ?*

Solution. The normal and tangential stresses on any plane are given by

$$p_n = \frac{p+p'}{2} + \frac{p-p'}{2} \cos 2\theta + q \sin 2\theta \qquad ...(i)$$

and $$p_t=\frac{p-p'}{2}\sin 2\theta-q\cos 2\theta \qquad ...(ii)$$

In our case $p=600\ kg./cm.^2$,

$p'=300\ kg./cm.^2$

and $q=450\ kg./cm.^2$

The planes equally inclined to the directions of (*a*) and (*b*) are corresponding to

$\theta=45°$

and $\theta=135°$

when $\theta=45°$,

$$p_n=\frac{600+300}{2}+\frac{600-300}{2}\cos 90°$$

$$+450\sin 90°=450+450$$

$$=900\ kg./cm.^2\ (tensile)$$

and $$p_t=\frac{600-300}{2}\sin 90°-450\cos 90°$$

$$=150\ kg./cm.^2$$

Similarly,

when $\theta=135°$,

$$p_n=450+150\cos 270°+450\sin 270°$$

$$=450-450=0$$

and $$p_t=150\sin 270-450\cos 270$$

$$=-150\ kg./cm.^2$$

In the case with end thrust,

$p=600\ kg./cm.^2$

$p'=-300\ kg./cm.^2$

$q=450\ kg./cm.^2$

when $\theta=45°$,

$$p_n=\frac{600-300}{2}+\frac{600+300}{2}\cos 90°+450\sin 90°$$

$$=150+450$$

$$=600\ kg./cm.^2\ (tensile)$$

and $$p_t=\frac{600+300}{2}\sin 90°-450\cos 90°$$

$$=+450\ kg./cm.^2$$

when $\theta=135°$,

$$p_n=150+450\cos 270°+450\sin 270$$

$$=-300\ kg./cm.^2\ (compressive)$$

and $$p_t=450\sin 270°-450\cos 270$$

$$=-450\ kg./cm.^2$$

Problem 371 (SI). *In a stress element, the normal stresses in two mutually perpenaicular directions are 600 MN/m² and 300 MN/m² both tensile. The complementary shear stresses in these directions are of intensity 450 MN/m². Find the normal and tangential stresses on the two planes which are equally inclined to the planes carrying the normal stresses mentioned above.*

Solution. $p=600\ MN/m^2$: $p'=300\ MN/m^2$; $q=450\ MN/m^2$ corresponding to the planes equally inclined to the planes which carry the above normal stresses,

$$\theta=45^\circ \quad \text{and} \quad 135^\circ$$

Normal stress $=p_n=\frac{p+p'}{2}+\frac{p-p'}{2}\cos 2\theta+q\sin 2\theta$

When $\theta=45^\circ$ $\quad p_n=\frac{600+300}{2}+\frac{600-300}{2}\cos 90^\circ+450\sin 90$

$=450+450=900\ MN/m^2$

When $\theta=135^\circ$ $\quad p_n=\frac{600+300}{2}+\frac{600-300}{2}\cos 270^\circ$

$+450\sin 270^\circ$

$=450-450=0$

Tangential stress $p_t=\frac{p-p'}{2}\sin 2\theta-q\cos 2\theta$

When $\theta=45^\circ$, $\quad p_n=\frac{600-300}{2}\sin 90^\circ-450\cos 90^\circ$

$=150\ MN/m^2$

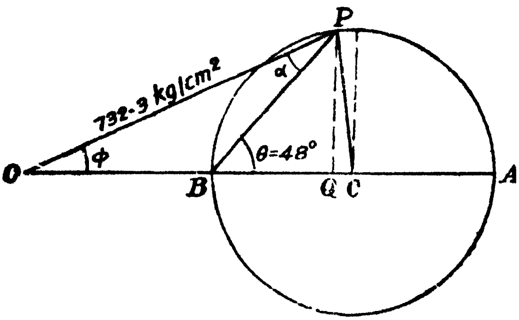
When $\theta=135^\circ$, $\quad p_t=\frac{600-300}{2}\sin 270^\circ-450\cos 270^\circ$

$=-150\ MN/m^2$.

Problem 372. *At a certain point in a strained material the principal stresses are 1000 kg./cm.² and 400 kg. per cm.² both tensile. Find the normal tangential and resultant stresses across a plane through the point at 48° to the major principal plane, using Mohr's circle of stress.*

Solution. Draw OA and OB to represent the two principal stresses.

$$OA=p_1=1000\ kg./cm^2$$
$$OB=p_2=400\ kg./cm.^2$$



$OA\ \ p_1=1000\ kg./cm.^2$

$OB=p_2=400\ kg./cm.^2$

$p_n=OQ=668{\cdot}6\ kg./cm.^2$

$p_t=QP=298{\cdot}3\ kg./cm.^2$

$\phi=P\hat{O}B=24^\circ\ 2'$

$\alpha=OPB=23^\circ\ 58'$

Fig. 566

On AB as diameter describe a circle.

Draw BP so that $PBA = \theta = 48°$

Draw PQ perpendicular to OA.

Now OQ represents normal stress on the given plane. By measurement, $OQ = 668{\cdot}6$ $kg./cm.^2$ QP represents the tangential stress on the given plane. By measurement, $QP = 298{\cdot}3$ $kg./cm^2$.

Obliquity $= \phi = POQ = 24°\ 2'$ by measurement.

Resultant stress is given by OP.

By measurement $OP = 732{\cdot}3$ $kg./cm.^2$ (*tensile*)

Inclination of the resultant stress with the direction of the major principal stress

$$\alpha = O\hat{P}B$$
$$= \theta - \phi$$
$$= 48° - 24°\ 2'$$
$$= 23°\ 58'.$$

Problem 373. *Draw "Mohr's stress circle" for principal stresses of 800 kg./cm.² tensile and 500 kg./cm.² compressive, and find the resultant stresses on planes making 22° and 64° with the major principal plane. Find also the normal and tangential stresses on these planes.*

Solution. Fig. 567 shows the Mohr's circle corresponding to

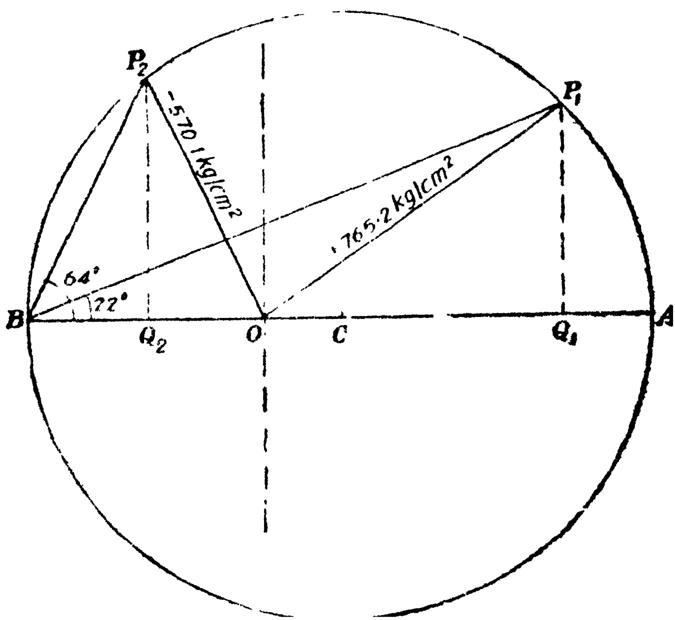


FIG. 567

the given principal stresses. Since p_1 and p_2 are unlike OA and OB have been

$$OA = p_1 = 800 \text{ kg./cm.}^2 \text{ (tensile)}$$
$$OB = p_2 = 500 \text{ kg./cm.}^2 \text{ (compressive)}$$
$$OQ_1 = p_{n1} = 617\cdot5 \text{ kg./cm.}^2 \text{ (tensile)}$$
$$P_1Q_1 = p_{t1} = 451\cdot5 \text{ kg./cm.}^2$$
$$OQ_2 = p_{n2} = 250\cdot2 \text{ kg./cm.}^2 \text{ (compressive)}$$
$$P_2Q_2 = p_{t2} = 512\cdot2 \text{ kg./cm.}^2$$
$$BP_1O = \alpha_1 = 14^\circ\ 11'$$
$$BP_2O = \alpha_2 = 52^\circ\ 2'$$

drawn on opposite sides of O. P_1 and P_2 correspond to the 22° and 64° planes. The results are given in the figure.

Problem 374. *At a point in a bracket the stresses on two mutually perpendicular planes are 1200 kg./cm.² tensile and 600 kg./cm.² tensile. The shear stress across these planes is 300 kg./cm.² Find, using the Mohr's stress circle; the principal stresses and maximum shear stress at the point.*

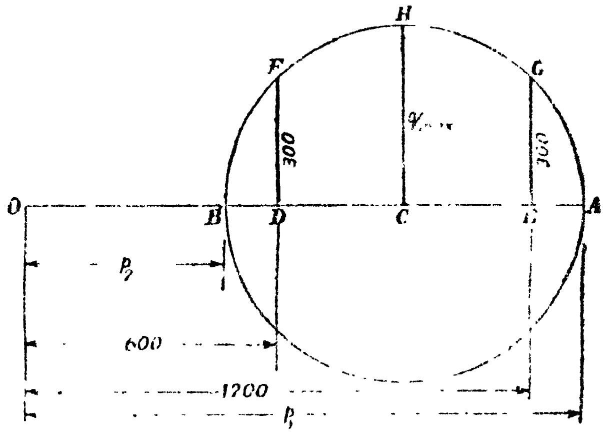


Fig. 568

Solution. Mark OE and OD to represent the normal stresses of 1200 kg./cm.² (*tensile*) and 600 kg./cm.² (*tensile*) respectively. Draw $DF = EG$ = shear stress of 300 kg./cm.² With C, the middle point of DE as centre, describe a circle with CF or CG as radius and obtain the points A and B.

Major principal stress

$$= p_1 = OA = 1324 \text{ kg./cm.}^2, \text{ tensile (by measurement)}$$
$$p_2 = OB = 424 \text{ kg./cm.}^2, \text{ tensile (measurement)}$$

Problem 375. *At a point the principal stresses are 1200 kg./cm.² and 800 kg./cm.² both tensile. Find by the ellipse of stress the resultant stress on a plane inclined at 30° to the major principal stress.*

Solution. The given plane is at 30° to the major principal stress or at 60° to the major principal plane.

See Fig. 569. With centre O, draw circles of radii $OA=p_1=$ 1200 $kg./cm.^2$ and $OB=p_2=800$ $kg./cm.^2$

Mark the plane CD at 60° with the major principal plane. Set off the perpendicular offset OEF intersecting the two circles at E and F. Draw EP and FP respectively parallel to OA and OB and obtain the point P. Resultant stress $=p=OP=916{\cdot}6$ $kg./cm.^2$ by measurement. Obliquity $\phi=EOP=10°\ 54'$; $\alpha=POA=49°\ 6'$.

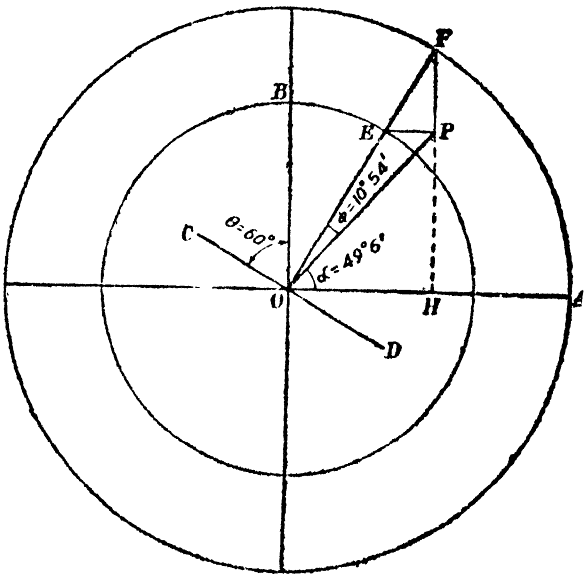


Fig. 569

Problem 376. *A rectangular block of material is subjected to stresses on perpendicular faces as shown. Using Mohr's circle of stress find*

(*i*) *the normal and shear stresses on a plane for which* $\theta=30°$

(*ii*) *the magnitudes of the principal stresses and the inclination of the plane on which each principal stress acts.*

Solution. Mark off $OD=p'=900$ $kg./cm.^2$ and $OE=p=$ 1400 $kg./cm.^2$ Step of perpendicular offsets $EF=DG=q=$ 500 $kg./cm.^2$ From C the middle point of DE, draw a circle of radius CF or CG, and obtain the points A and B. Principal stress $p_1=OA=1709$ $kg./cm.^2$ by measurement. Principal stress $p_2=OB=$ 591 $kg./cm.^2$ by measurement. Position of the first principal plane is given by $\theta_1=FBA=31°\ 48'$ by measurement

For the plane on which the normal and tangential stresses are required, draw CJ so that $J\hat{C}F = 2 \times 30 = 60°$. Draw JK perpendicular to OC.

Now the normal stress on the plane

$$= OK = 850 \text{ kg./cm.}^2 \text{ (by measurement).}$$

Tangential stress on the plane

$$= KJ = 472 \text{ kg./cm.}^2 \text{ (by measurement).}$$

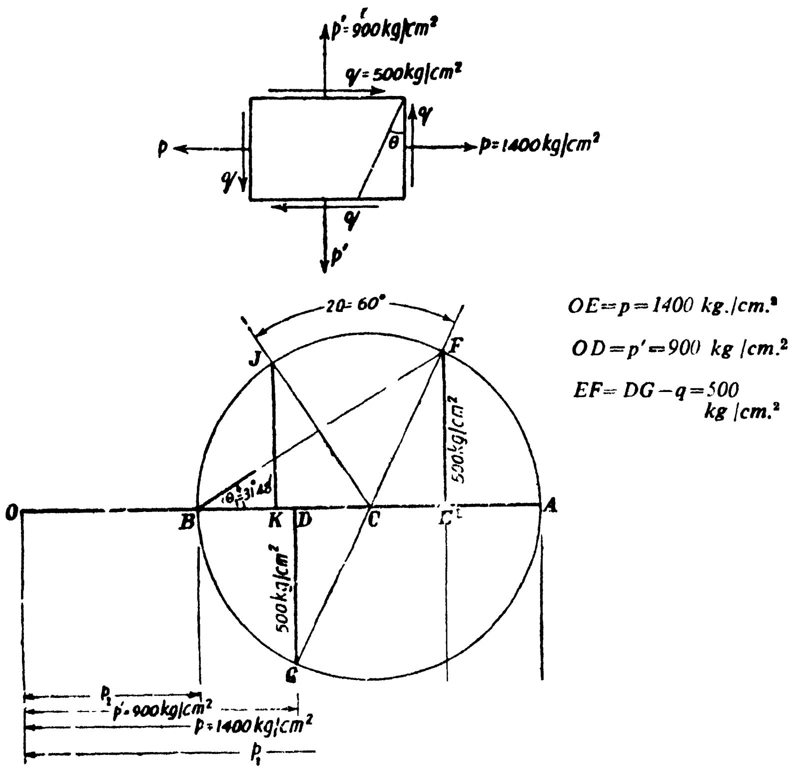


Fig. 570

§105. Combined bending and Torsion

Let a shaft of diameter d be subjected to a bending moment M and a twisting moment T at a section.

At any point in the section at a radius r and at a distance y from the neutral axis the bending stress is given by

$$p = \frac{M}{I} y,$$

I being the moment of inertia of the section about the neutral axis

The shear stress at the point is given by

$$q=\frac{T}{I_p}\,r$$

I_p being the polar moment of inertia.

The location of the principal planes through the point is given by,

$$\tan 2\theta=\frac{2q}{p},$$

and the principal stresses are given by,

$$p_1=\frac{p}{2}+\sqrt{\frac{p^2}{4}+q^2}$$

and $$p_2=\frac{p}{2}-\sqrt{\frac{p^2}{4}+q^2}$$

The effect of bending moment and torsion will be most predominant at the points A and B

At these points, the maximum bending stress is given by

$$p=\frac{M}{Z}=\frac{M}{\frac{\pi d^3}{32}}$$

$$=\frac{32M}{\pi d^3}\text{ (Compressive}$$

at A and tensile at B)

At these points the shear stress is given by

$$q=\frac{T}{Z_p}$$

$$=\frac{T}{\frac{\pi d^3}{16}}=\frac{16T}{\pi d^3}$$

Hence the position of the principal planes through any of these two points is given by

$$\tan 2\theta=\frac{2q}{p}$$

$$\tan 2\theta=\frac{T}{M}$$

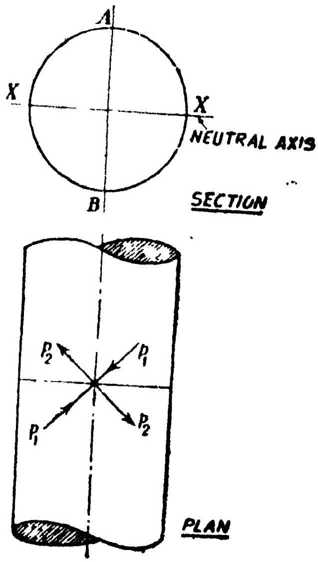


Fig. 571

The principal stresses are given by

$$p_1=\frac{p}{2}+\sqrt{\frac{p^2}{4}+q^2}$$

$$=\frac{16}{\pi d^3}\left\{M+\sqrt{M^2+T^2}\right\}$$

and $$p_2=\frac{p}{2}-\sqrt{\frac{p^2}{4}+q^2}$$

$$=\frac{16}{\pi d^3}\left\{M-\sqrt{M^2+T^2}\right\}$$

The maximum shear stress is given by

$$q_{max}=\frac{p_1-p_2}{2}$$

$$\therefore \quad q_{max}=\frac{16}{\pi d^3}\sqrt{M^2+T^2}$$

Problem 377. *At a certain section of a shaft 80 mm. in diameter there is a bending moment of 35 tonne cm. and a twsiting moment of 50 tonne cm. Find the maximum direct stress induced in the section and specify the position of the plane on which it acts. Taking Poission's ratio as 0·28, find what stress acting alone can produce the same maximum strain.*

Solution. $M=35000$ *kg. cm.*

$T=50000$ *kg. cm.*

$d=80$ *mm.* $=8$ *cm.*

The principal stresses are given by

$$p_1=\frac{16}{\pi d^3}\left\{M+\sqrt{M^2+T^2}\right\}$$

and $$p_2=\frac{16}{\pi d^3}\left\{M-\sqrt{M^2+T^2}\right\}$$

$$\therefore \quad p_1=\frac{16}{\pi\times 8^3}\left\{35000+\sqrt{35{,}000^2+50{,}000^2}\right\} \text{ kg./cm.}^2$$

$$=955 \text{ kg./cm.}^2$$

$$p_2=\frac{16}{\pi\times 8^2}\left\{35000-\sqrt{35000^2+50000^2}\right\} \text{ kg./cm.}^2$$

$$=-260 \text{ kg./cm.}^2$$

The position of the principal planes is given by

$$\tan 2\theta=\frac{T}{M}=\frac{50{,}000}{35{,}000}=1{\cdot}4286$$

$$\therefore \quad 2\theta=55^\circ$$

$$\therefore \quad \theta=27^\circ\ 30'$$

Maximum strain

$$e=\frac{p_1}{E}-\frac{p_2}{mE}=\frac{1}{E}\left(955+0{\cdot}28\times 260\right)$$

$$=\frac{1027 \cdot 8}{E}$$

If f be the stress producing the same maximum strain,

$$\frac{f}{E}=\frac{1027 \cdot 8}{E}$$

$$\therefore \quad f=1027 \cdot 8 \text{ kg./cm.}^2$$

Problem 378 (S.I.). *A shaft section 100 mm in diameter is subjected to a bending moment of 4000 Nm and a torque of 6000 Nm. Find the maximum direct stress induced on the section, and specify the position of the plane on which it acts. Find also, what stress acting alone can produce the same maximum strain. Take Poisson's ratio 0·25.*

Solution.

$$M=4000 \text{ Nm}=400{,}000 \text{ N cm.}$$

$$T=6000 \text{ Nm}=600{,}000 \text{ N cm.}$$

$$d=100 \text{ mm}=10 \text{ cm.}$$

The principal stresses are given by

$$p_1=\frac{16}{\pi d^3}\left(M+\sqrt{M^2+T^2} \right)$$

and

$$p_2=\frac{16}{\pi d^3}\left(M-\sqrt{M^2+T^2} \right)$$

$$p_1=\frac{16}{\pi \times 10^3}\left(400{,}000+\sqrt{400{,}000^2+600{,}000^2} \right)$$

$$=\frac{16}{\pi \times 10^3}\left(40 \times 10^4+10^4\sqrt{40^2+60^2} \right)$$

$$=5710 \text{ N/cm}^2.$$

$$p_2=\frac{16}{\pi \times 10^3}\left(40 \times 10^4-10^4\sqrt{40^2+60^2} \right)$$

$$=-1635 \text{ N/cm}^2.$$

Position of the major principal planes

$$\tan 2\theta=\frac{T}{M}=\frac{6000}{4000}=1 \cdot 5$$

$$\therefore \quad 2\theta=56° \, 18'$$

$$\therefore \quad \theta=28° \, 9'$$

Maximum strain

$$=e_1=\frac{p_1}{E}-\frac{p_2}{mE}=\frac{1}{E}(5710+0 \cdot 25 \times 1635)$$

Let f be the stress acting alone to produce the same maximum strain.

$$\therefore \quad \frac{f}{E}=\frac{6118 \cdot 75}{E}$$

$$\therefore \quad f=6118 \cdot 75 \ N/cm.^2$$

Problem 379. *A crankshaft 20 cm. in diameter is subjected to a bending moment of 185000 kg. cm. and a twisting moment of 277500 kg. cm. Find the greatest shear stress. Find also the plane of the maximum shear stress with respect to the axis of the shaft.*

Solution. $M=185000 \ kg. \ cm.$

$T=277500 \ kg. \ cm.$

Maximum shear stress

$$=q_{max}=\frac{16}{\pi d^3}\sqrt{M^2+T^2}$$

$$=\frac{16}{\pi \times 20^3}\sqrt{(185000)^2+(277500)^2} \ kg./cm.^2$$

$$=212 \cdot 3 \ kg./cm.^2$$

The position of the principal planes is given by

$$\tan 2\theta=\frac{T}{M}=\frac{277500}{185000}=1 \cdot 5$$

$$\therefore \quad 2\theta=56^\circ \ 18'$$

$$\therefore \quad \theta=28^\circ \ 9'$$

$\therefore$ Angle made by the principal plane with respect to the axis of the shaft $=90-28^\circ \ 9'=60^\circ \ 51'$

$\therefore$ Angle made by the planes of maximum stress

$=61^\circ 51' \pm 45^\circ$

$=106^\circ \ 51'$ with the axis of the shaft and $16^\circ \ 51'$ with the axis of the shaft

§106. Strain Energy in terms of the principal stresses

Let a strained element be subjected to the principal stress p_1 and p_2.

The strains in the directions of the principal stresses are

$$e_1=\frac{p_1}{E}=\frac{p_2}{mE}$$

and

$$e_2=\frac{p_2}{E}-\frac{p_1}{mE}$$

where $\frac{1}{m}$ is Poisson's ratio

Per unit volume the strain energy stored

$$=\frac{1}{2}p_1 e_1+\frac{1}{2}p_2 e_2$$

$$=\frac{p_1}{2}\left(\frac{p_1}{E}-\frac{p_2}{mE}\right)+\frac{p_2}{2}\left(\frac{p_2}{E}-\frac{p_1}{mE}\right)$$

$$=\frac{1}{2E}\left(p_1^2+p_2^2-\frac{2p_1 p_2}{m}\right)$$

Problem 380. *A cross-section of a shaft 90 mm. diameter is subjected to a bending moment of 60 tonne cm. and a twisting moment of 90 tonne cm. Find the direct stress which acting alone will make the shaft store the same maximum strain energy per unit volume. Take*

$$\frac{1}{m}=0{\cdot}3.$$

Solution. $M=60000$ *kg. cm.*

$T=90000$ *kg. cm.*

$d=90$ *mm.*$=9$ *cm.*

The principal stresses are given by

$$p_1=\frac{16}{\pi d^3}\left(M+\sqrt{M^2+T^2}\right)$$

and $$p_2=\frac{16}{\pi d^3}\left(M-\sqrt{M^2+T^2}\right)$$

$\therefore$ $$p_1=\frac{16}{\pi\times 9^3}\left(60000+\sqrt{60000^2+90000^2}\right)$$

$$=1174 \text{ kg./cm.}^2$$

and $$p_2=\frac{16}{\pi\times 9^3}\left(60000-\sqrt{60000^2+90000^2}\right)$$

$$=-134 \text{ kg./cm.}^2$$

Maximum strain energy per unit volume

$$=\frac{1}{2E}\left(p_1^2+p_2^2-\frac{2\,p_1 p_2}{m}\right)$$

Let f be the direct stress acting alone to store the same energy per unit volume.

$\therefore$ $$\frac{f^2}{2E}=\frac{1}{2E}\left(p_1^2+p_2^2-\frac{2\,p_1 p_2}{m}\right)$$

$\therefore$ $$f^2=p_1^2+p_2^2-\frac{2p_1p_2}{m}$$

$\therefore$ $$f^2=1174^2+134^2+2\times 1174(-134)0{\cdot}3$$

$\therefore$ $$f=1221 \text{ kg./cm.}^2$$

Problem 381. *In a circular shaft subjected to an axial twisting moment T and a bending moment M show that when M = 1·2 T, the ratio of maximum shearing stress to the greater principal stress is approximately 0·566.*

Solution. Let T_e be the equivalent twisting moment to produce the maximum shearing stress q_{max}

$$= T_e = \sqrt{M^2 + T^2} = q_{max}\, Z_0$$

where Z_0 = polar modulus of the shaft section

$$\therefore \quad q_{max} = \frac{\sqrt{M^2+T^2}}{Z_0}$$

Let M_e be the equivalent bending moment to produce the maximum bending stress equal to the greater principal stress p_1

$$M_e = \tfrac{1}{2}\left[M + \sqrt{M^2+T^2} = p_1 \frac{Z_0}{2} \right]$$

$$\therefore \quad p_1 = \frac{M+\sqrt{M^2+T^2}}{Z_0}$$

$$\therefore \quad \frac{q_{max}}{p_1} = \frac{\sqrt{M^2+T^2}}{M+\sqrt{M^2+T^2}} = \frac{\sqrt{(1·2T)^2+T^2}}{1·2T+\sqrt{(1·2T)^2+T^2}}$$

$$= \frac{\sqrt{2·44}}{1·2+\sqrt{2·44}}$$

$$= \frac{1·56}{2·76} = 0·566$$

Problem 382. *A beam 3 metres long, of I section is freely supported at its ends with the web vertical. It carries concentrated loads of 10 tonnes at 0·6 metre from each end. The flanges are each 15 cm. wide and 2·5 cm. thick, the overall depth being 40 cm. The thickness of the web is 1·25 cm. Calculate the principal stresses and the maximum shearing stress in a section of the beam where the bending moment and shearing force, both have maximum values.*

Solution. Moment of Inertia of the section about the neutral axis

$$I = \frac{15 \times 40^3}{12} - \frac{13·75 \times 35^3}{12}$$

$$= 30870 \text{ cm.}^4$$

Max. B.M. $= M = 10 \times 0·6 = 6$ *tm.*

$= 600$ *t cm.*

Max. S.F. $= S = 10\ t$

Max. bending stress in the

$$\text{web} = p = \frac{M}{I} \cdot y$$

$$=\frac{600\times17{\cdot}5}{30870}=0{\cdot}34\ t/cm.^2\ (compressive)$$

Max. shearing stress in the web

$$=q=\frac{Sa\bar{y}}{Ib}=\frac{10\times15\times2{\cdot}5\times16{\cdot}25}{30870\times1{\cdot}25}$$

$$=0{\cdot}16\ t/cm.^2$$

The principal stresses are given by $\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q^2}$

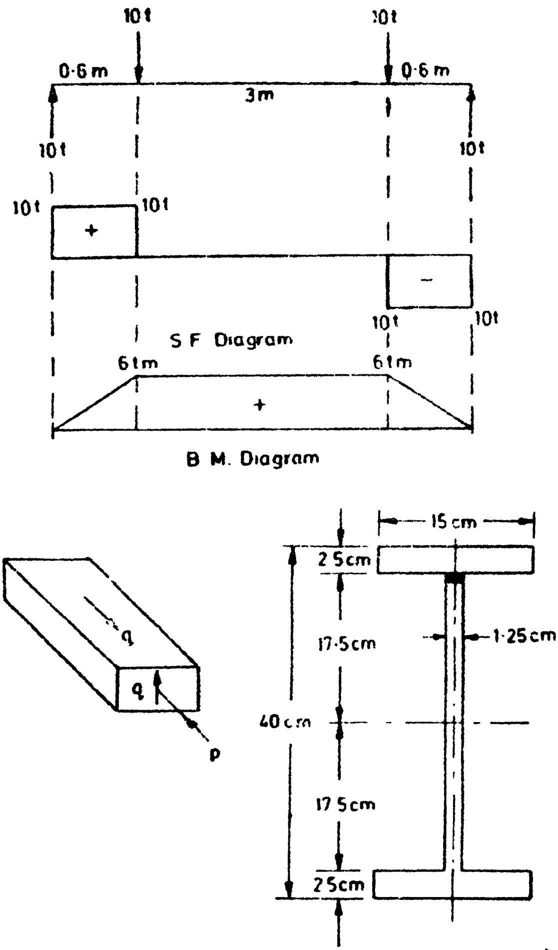


Fig. 572

$$=-0.17\pm\sqrt{0.17^2+0.16^2}$$

$\therefore$ $p_1=+0.06\ t/cm.^2$

and $p_2=-0.40\ t/cm.^2$

Problem 383. *A steel shaft ABCD of circular section is 180 cm. long and is supported in bearings at the ends A and D. AB =67·5 cm., BC=45 cm. and CD=67·5 cm. The shaft is horizontal and two horizontal arms, rigidly connected to the shaft at B and C project from it at right angles on opposite sides. The arm B carries a vertical load of 1600 kg at 30 cm. from the shaft axis and the arm at C carries a vertical balancing load at 37·5 cm. from the axis.*

If the shearing stress is not to exceed 800 kg /cm.² determine the minimum permissible diameter of the shaft. Assume the bearings give simple point support to the shaft.

Solution. Fig. 573 shows the centre line of the shaft and the loads applied on it.

For rotational equilibrium,

$$W\times37.5=1600\times30$$

$$\therefore\quad W=1280\ kg.$$

The shaft is subjected to uniform torque between B and C. This torque

$$=1600\times30$$

$$=48000\ kg.\ cm.$$

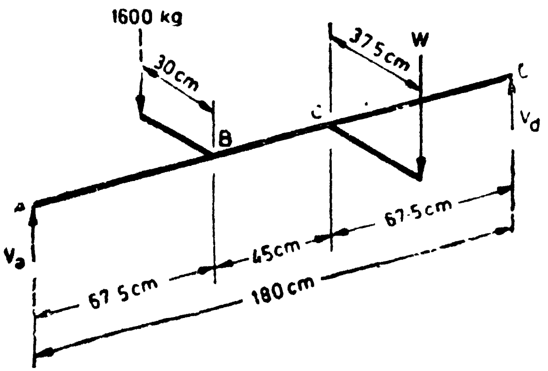


Fig. 573

Taking moments about A,

$$V_d\times180=(1600\times67.5)+(1280\times112.5)$$

$$\therefore\quad V_d=1400\ kg.$$

$$\therefore\quad V_a=1600+1280-1400$$

$$=1480\ kg.$$

Max. B.M. for the shaft occurs at *B*

B.M. at *B* $= M = 1480 \times 67{\cdot}5 = 99900$ *kg. cm.*

$\therefore$ Just on RHS of the section *B*,

We have,

B.M. $= M = 99900$ *kg. cm.*

Torque $= T = 48000$ *kg. cm.*

Let T_e be the equivalent torque to produce the same max. shear stress

$$T_e = \sqrt{M^2 + T^2} = \sqrt{(99900)^2 + (48000)^2}$$

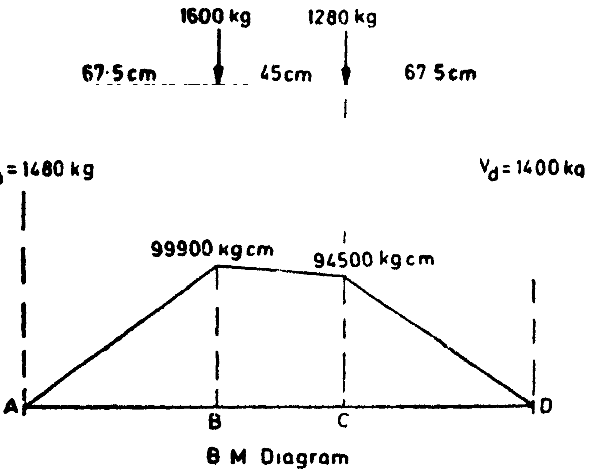


Fig. 574

$= 110830$ *kg. cm.*

Permissible shear stress $f_s = 800$ *kg./cm.*^{2}

$$= f_s \frac{\pi d^3}{16}$$

$$110830 = 800 \frac{\pi d^3}{16}$$

$$d = \sqrt[3]{\frac{110830 \times 16}{800\pi}} \text{ cm.}$$

$= 8{\cdot}90$ *cm.*

Problem 384. *A flywheel weighing 600 kg. is mounted on a shaft 8 cm. in diameter and midway between bearings 60 cm. apart in which the shaft may be assumed to be directionally free. If the shaft is transmitting 40 horse power at 360 rpm, calculate the principal stresses and the maximum shearing stresses in the shaft at the* *ends of a vertical and horizontal diameter in a plane close to that of the flywheel.*

Solution. H.P. $=\dfrac{2\pi NT}{4500}$

$$\therefore \quad T=\frac{40\times 4500}{2\pi\times 360}\ kg.\ m.$$
$$=79{\cdot}58\ kg\ m.$$
$$=7958\ kg.\ cm.$$
$$=7{\cdot}958\ t\ cm.$$

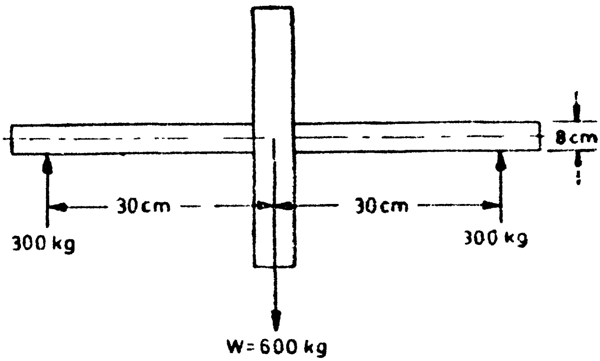


Fig. 575

Max. shear stress due to torque

$$=q=\frac{T}{\frac{\pi d^3}{16}}=\frac{16+7{\cdot}958}{\pi\times 8^3}=0{\cdot}079\ t/cm^2.$$

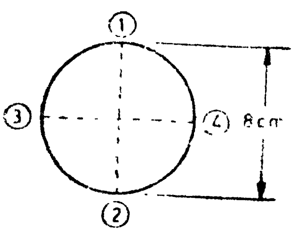
Max. B.M. $=M=300\times 30=9000\ kg.\ cm.$

$=9\ t\ cm.$

Max. bending stress at the extremities of the vertical diameter, (*i.e.*, at the points 1 and 2). (See Fig. 576)

$$=p=\frac{M}{\frac{\pi d^3}{32}}=\frac{32\times 9}{\pi\times 8^3}\ t/cm.^2$$

$$=0{\cdot}179\ t/cm.^2$$



Section of Shaft

Fig 576

At the points 1 or 2

$$p_1=\frac{p}{2}+\sqrt{\frac{p^2}{4}+q^2}$$
$$=\frac{0{\cdot}179}{2}+\sqrt{\left(\frac{0{\cdot}179}{2}\right)^2+0{\cdot}079^2}$$

$= 0.209\ t/cm^2$ *compressive at 1 and tensile at 2*

$$p_2 = \frac{p}{2} - \sqrt{\frac{p^2}{4} + q^2} = 0.03 \quad t/cm.^2 \text{ tensile}$$

at 1 and compressive at 2

At these points

$$q_{max} = \sqrt{\frac{p^2}{4} + q^2}$$

$$= \sqrt{\left(\frac{0.179}{2}\right)^2 + 0.079^2}$$

$$= 0.119\ t/cm^2.$$

Now consider the points 3 and 4, *i.e.* at the ends of the horizontal diameter.

At these points,

Stress due to bending is zero,

$\therefore \qquad p = 0$

Shear stress due to torsion $= q' = 0.079\ t/cm^2$.

Shear force at the section $= S = 300\ kg$

Shear stress due to shear force

$$= q'' = \frac{4}{3}\ q_{avg}$$

$$= \frac{4}{3} \times \frac{300}{\left(\frac{\pi \times 8^2}{4}\right)}\ kg./cm^2 = 7.96\ kg./cm.^2$$

say $0.007\ t/cm.^2$

Total shear stress

$$= q = q' + q''$$

$$= 0.079 + 0.007$$

$$= 0.086\ t/cm.^2$$

Since at these points only these shear stresses exist the principal stresses at these points are also $0.086\ t/cm.^2$ (*compressive as well as tensile*).

Problem 385. *A solid shaft of 15 cm. diameter transmits 2400 h.p. at 600 rpm and is also subjected to an axial thrust of 25000 kg. If the maximum principal stress is not to exceed 800 kg./cm², find what additional bending moment may be safely carried. What will be the direction of the maximum principal stress ?*

Solution. Let the shaft be subjected to a torque T

$$HP = \frac{2\pi NT}{4500}$$

$$T=\frac{4500\times 2400}{2\pi\times 600}\ kgm.$$

$=2864\ kgm.=286400\ kg.\ cm.$

$=286{\cdot}4\ t\ cm.$

Shear stress due to torque

$$=q=\frac{16T}{\pi d^3}$$

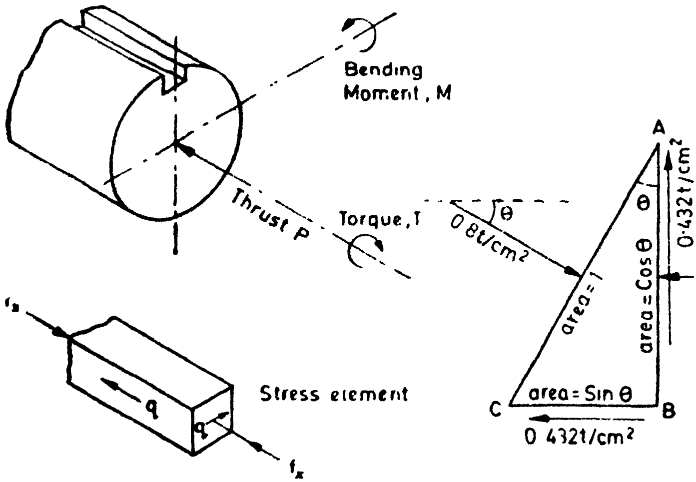


Fig. 577

$$=\frac{16\times 286{\cdot}4}{\pi\times 15^3}\ t/cm^2$$

$=0{\cdot}432\ t/cm^2$

Compressive stress due to thrust

$$=\frac{P}{A}=\frac{2500}{\left(\frac{\pi\times 15^2}{4}\right)}=142{\cdot}0\ kg./cm.^2$$

$=0{\cdot}142\ t/cm.^2$

Consider a rectangular stress element at the upper end of the vertical diameter. Let f_x be the total compressive stress due to direct stress caused by the thrust and due to bending moment.

$\therefore$ f_x=direct stress due to thrust + bending stress.

We will now determine the value of f_x in combination with the shear stress of 0·432 $t/cm.^2$ so as to produce a principal stress of 800 $kg./cm.^2$ (0·8 t/cm^2).

Consider the wedge ACB of the stress element taken such that the face AC is in the principal plane. Let the area of this principal plane AC be *unity*.

Area of face $BC = 1 \times \sin\theta = \sin\theta$

Area of face $AB = 1 \times \cos\theta = \cos\theta$

where θ is the inclination of the *principal stress with the axis of the shaft*.

Resolving vertically, $\quad 0·8 \sin\theta = 0·432 \cos\theta$

$\therefore \quad \tan\theta = 0·54$

$\therefore \quad \theta = 28° \; 22'$

Resolving horizontally,

$$0·8 \cos\theta = f_x \cos\theta + 0·432 \sin\theta$$

$$f_x \cos\theta = 0·8 \cos\theta - 0·432 \sin\theta$$

$$\therefore \quad f_x = 0·8 - 0·432 \tan\theta$$

$$= 0·8 - 0·432 \times 0·54$$

$$= 0·567 \ t/cm.^2$$

But the compressive stress due to thrust alone $= 0·142 \ t/cm.^2$

$\therefore$ Compressive stress due to bending moment

$$= f = 0·567 - 0·142 = 0·425 \ t/cm^2$$

Let the B.M. producing the above bending stress f be M t cm.

$$M = \frac{fI}{y}$$

$$= \frac{\pi d^3 f}{32}$$

$$= \frac{\pi (15)^3 \times 0·425}{32} \ t \ cm$$

$$= 140·9 \ t \ cm.$$

§107. Equivalent bending moment and equivalent torque

Let a shaft of diameter d be subjected to a maximum bending moment M and a torque T. We know principal stresses are :

$$p_1 = \frac{16}{\pi d^3}\left\{M + \sqrt{M^2 + T^2}\right\}$$

and $$p_2 = \frac{16}{\pi d^3}\left\{M - \sqrt{M^2 + T^2}\right\}$$

and the maximum shearing stress

$$= q_{max} = \frac{16}{\pi d^3}\sqrt{M^2 + T^2}$$

Let M_e be the equivalent bending moment which acting alone produces the maximum tensile stress equal to p_1

then $$p_1 = \frac{M_e}{\frac{\pi d^3}{32}} = \frac{32}{\pi d^3} M_e$$

$$\frac{32}{\pi d^3} M_e = \frac{16}{\pi d^3}\left(M+\sqrt{M^2+T^2}\right)$$

$$M_e = \left(M+\sqrt{M^2+T^2}\right)$$

Similarly let T_e be the equivalent torque which acting alone produces the maximum shearing stress q_{max}

$$\therefore \qquad q_{max} = \frac{T_e}{\frac{\pi d^3}{16}} = \frac{16\,T_e}{\pi d^3}$$

$$\therefore \qquad \frac{16\,T_e}{\pi d^3} = \frac{16}{\pi d^3}\sqrt{M^2+T^2}$$

$$\therefore \qquad T_e = \sqrt{M^2+T^2}$$

Problem 386. *Two views of an overhung crank are shown in Fig. 578 a force of 2000 kg. being applied to the crank pin in the direction shown and at a distance of 15 cm from the centre of the adjacent bearing. The crankshaft is of solid section of 7·5 cm. diameter. Calculate the maximum principal stress and maximum shear stress in the section of the shaft at the centre of the bearing*

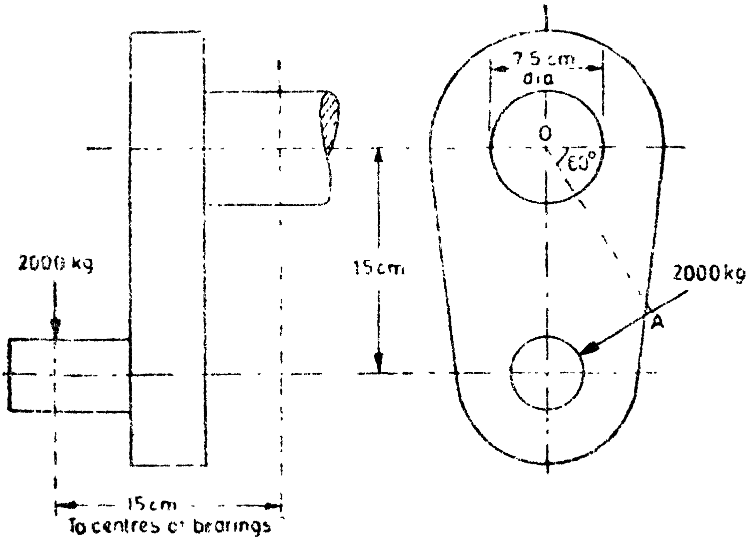


Fig. 578

Solution.

Torque on the shaft $T = 2000 \times OA$

$= 2000 \times 15 \sin 60° = 25980$ *kg cm.*

$= 25·98$ *t cm.*

Max. B.M. $= M = 2000 \times 15 = 30000$ *kg. cm.*

$= 30$ *t cm.*

Equivalent bending moment M

$$=\frac{1}{2}\left\{M+\sqrt{M^2+T^2}\right\}$$

$$=\frac{1}{2}\left\{30+\sqrt{30^2+25\cdot98^2}\right\} t\ cm.$$

$$=34\cdot85\ t\ cm.$$

∴ Maximum principal stress

$$=p_1=\frac{Me}{\frac{\pi d^3}{32}}=\frac{32\ M_e}{\pi d^3}$$

$$\frac{32\times34\cdot85}{\pi\times7\cdot5^3}=0\cdot841\ t/cm^2.$$

$$=841\ kg/cm.^2$$

Equivalent torque $T_e=\sqrt{M^2+T^2}$

$$=\sqrt{30^2+25\cdot98^2}$$

$$=39\cdot70\ t\ cm.$$

Maximum shear stress

$$=q_{max}=\frac{T_e}{\frac{\pi d^3}{16}}\quad\frac{16\ T_e}{\pi d^3}$$

$$=\frac{16\times39\cdot70}{\pi\times7\cdot5^3}=0\ 479\ t/cm.^2$$

$$=479\ kg./cm.^2$$

§108. Principal Strains

These are the greatest and least direct strains in a material subjected to complex stresses. These strains are produced in the direction of the principal stresses.

In general, if there exist at a point three like principal stresses p_1, p_2 and p_3 acting on three principal planes, the principal strains are given by

$$\frac{p_1}{E}\quad\frac{p_2+p_3}{mE}$$

$$\frac{p_2}{E}\quad\frac{p_3+p_1}{mE}$$

and $$e_3=\frac{p_3}{E}-\frac{p_1+p_2}{mE}$$

where = Poisson's ratio

Problem 387. *Tests on a brass plate subjected to principal stresses gave the following results :*

Principal strain $e_1 = 7{\cdot}1 \times 10^{-4}$

Principal strain $e_2 = 1{\cdot}67 \times 10^{-4}$

Find the principal stresses. Find also the plane on which the normal stress is $0{\cdot}6$ *t/cm.*^{2} *Take* $E = 0{\cdot}8 \times 10^3$ *t/cm.*^{2} *and* $\frac{1}{m} = \frac{1}{3}$

Solution.
$$e_1 = \frac{1}{E}\left(p_1 - \frac{p_2}{m}\right)$$

$$\therefore \quad 7{\cdot}1 \times 10^{-4} = \frac{1}{0{\cdot}8 \times 10^3}\left(p_1 - \frac{p_2}{3}\right)$$

$$\therefore \quad p_1 - \frac{p_2}{3} = 0{\cdot}568 \qquad \text{.. (i)}$$

$$e_2 = \frac{1}{E}\left(p_2 - \frac{p_1}{m}\right)$$

$$\therefore \quad 1{\cdot}67 \times 10^{-4} = \frac{1}{0{\cdot}8 \times 10^3}\left(p_2 - \frac{p_1}{3}\right)$$

$$\therefore \quad p_2 - \frac{p_1}{3} = 0{\cdot}134 \qquad \text{...(ii)}$$

Solving equations (*i*) and (*ii*) we get $p_1 = 0{\cdot}70$ *t/cm.*^{2} *and* $p_2 = 0{\cdot}35$ *t/cm.*^{2}

Let θ be the inclination of the plane on which the normal stress is $0{\cdot}6$ *t/cm.*^{2}

$$\therefore \quad p_n = \frac{p_1 + p_2}{2} + \frac{p_1 - p_2}{2}\cos 2\theta = 0{\cdot}6$$

But
$$\frac{p_1 + p_2}{2} = \frac{0{\cdot}70 + 0{\cdot}35}{2} = 0{\cdot}525 \text{ t/cm.}^2$$

and
$$\frac{p_1 - p_2}{2} = \frac{0{\cdot}70 - 0{\cdot}35}{2} = 0{\cdot}175 \text{ t/cm.}^2$$

$$\therefore \quad p_n = 0{\cdot}525 + 0{\cdot}175 \cos 2\theta = 0{\cdot}6$$

$$\therefore \quad \cos 2\theta = \frac{0{\cdot}075}{0{\cdot}175}$$

$$\therefore \quad 2\theta = 64^\circ\, 36'$$

$$\therefore \quad \theta = 32^\circ\, 18' \text{ with the major principal plane.}$$

§109. Strain energy in terms of principal stresses

Let p_1 and p_2 be the principal stresses at a point in a strained material.

$\therefore$ The principal strains are given by

$$e_1 = \frac{p_1}{E} - \frac{p_2}{mE}$$

and $$e_2 = \frac{p_2}{E} - \frac{p_1}{mE}$$

Hence strain energy stored per unit volume

$$= \tfrac{1}{2} p_1 e_1 + \tfrac{1}{2} p_2 e_2$$

$$= \frac{1}{2} p_1 \left(\frac{p_1}{E} - \frac{p_2}{mE} \right) + \frac{1}{2} p_2 \left(\frac{p_2}{E} - \frac{p_1}{mE} \right)$$

$$= \frac{1}{2E} \left(p_1^2 + p_2^2 - \frac{2 p_1 p_2}{m} \right)$$

Similarly it can be shown, if three principal stresses p_1, p_2, p_3 exist then the strain energy stored per unit volume

$$= \frac{1}{2E} \left\{ p_1^2 + p_2^2 + p_3^2 - \frac{2p_1p_2 + 2p_2p_3 + 2p_3p_1}{m} \right\}$$

§110. Criterion for failure

While designing a member various theories are adopted. These are given below :

(*i*) *The maximum principal stress theory of Rankine.* According to this theory the maximum principal stress shall not exceed the safe stress for the material.

(*ii*) *The maximum strain theory of St. Venant.* According to this theory the equivalent direct stress acting alone which can produce a strain equal to the greater principal strain shall not exceed the safe limit.

Let p_1 and p_2 be the principal stresses. Greater principal strain

$$= e_1 = \left(\frac{p_1}{E} - \frac{p_2}{mE} \right)$$

Let f be the stress acting alone producing the strain e_1

$$\therefore \quad e_1 = \frac{f}{E} = \frac{p_1}{E} - \frac{p_2}{mE}$$

$$\therefore \quad f = \left(p_1 - \frac{p_2}{m} \right)$$

This stress f should not exceed the safe limit.

(*iii*) *The maximum shear stress theory due to Sir J.J. Guest.* Let p_1 and p_2 be the principal strains.

$$\therefore \text{ Max. shear stress} = q_{max} = \frac{p_1 - p_2}{2}$$

This shear stress should not exceed the safe shear stress for the material.

(iv) Maximum Strain energy theory due to Beltrami and Haigh. Let p_1 and p_2 be the principal stresses.

∴ Strain energy stored per unit volume

$$= \frac{1}{2E}\left(p_1^2+p_2^2-\frac{2p_1 p_2}{m}\right)$$

Let f be the stress acting alone to store the same energy per unit volume

$$\therefore \quad \frac{f^2}{2E}=\frac{1}{2E}\left(p_1^2+p_2^2-\frac{2p_1 p_2}{m}\right)$$

$$\therefore \quad f=\sqrt{p_1^2+p_2^2-\frac{2p_1 p_2}{m}}$$

this stress should not exceed the safe limit.

§111. To determine the strain in any direction in terms of the principal strains

Fig. 579 shows a rectangular plate *ABCD* subjected to a stress system so that the principal strains are e_1 and e_2. After the deformation the rectangle *ABCD* takes the shape $A_1B_1C_1D_1$.

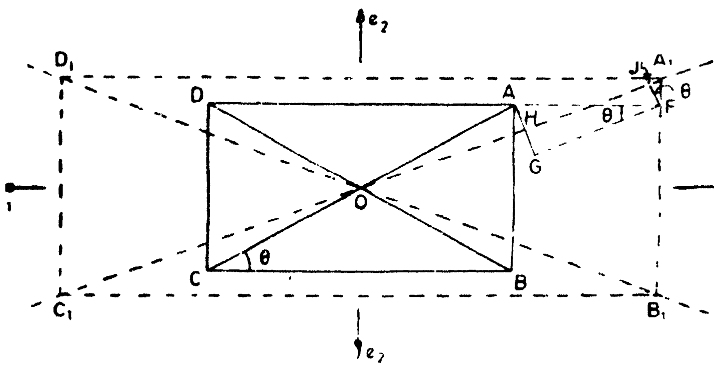


Fig. 579

The strain in *OA* due to the principal strains e_1 and e_2

$$=e_\theta=\frac{HA_1}{OA}=\frac{HJ+JA_1}{OA}=\frac{GF+JA_1}{OA}$$

$$=\frac{AF\cos\theta+A_1F\sin\theta}{OA}$$

$$=\frac{\frac{1}{2}DA\,e_1\cos\theta+\frac{1}{2}AB\,e_2\sin\theta}{\frac{1}{2}AC}$$

$$\therefore \quad e_\theta=e_1\cos^2\theta+e_2\sin^2\theta$$

The strain in a perpendicular direction (shear strain) is given by $\frac{AH}{OA}$

$$e'_\theta=\frac{AG-HG}{OA}=\frac{AG-JF}{OA}$$

$$=\frac{AF\sin\theta-A_1F\cos\theta}{OA}$$

$$=\frac{\frac{1}{2}DAe_1\sin\theta-\frac{1}{2}ABe_2\cos\theta}{\frac{1}{2}AC}$$

$$=\frac{e_1-e_2}{2}\sin 2\theta$$

§112. Ellipse of strain

If e_1 and e_2 are the principal strains, the strain in the direction making an angle θ with e_1

$$=e_\theta=e_1\cos^2\theta+e_2\sin^2\theta \qquad ...(i)$$

The strain in a direction perpendicular to the above direction

$$=e'_\theta=\frac{e_1-e_2}{2}\sin 2\theta$$

In Fig 580 oh represents the strain in the direction at θ with the direction of e_1 and hd represents the strain at right angles to the above direction.

$\therefore$ Resultant strain is given by od

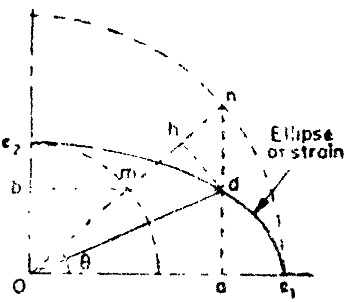


Fig. 580

Let $oa=x$ and $ad=y$

$\therefore$ $oa=x=e_1\cos\theta$

and $ad=y=e_2\sin\theta$

$\frac{x^2}{e_1^2}+\frac{y^2}{e_2^2}$ 1 this is the equation to an ellipse having a major axis of length $2e_1$ and a minor axis of length $2e_2$. This ellipse is the locus of the point d. This ellipse is called the ellipse of strain

Problem 388. *A rosette of three main gauges on the surface of a metal plate under stress gave the following readings :*

No. 1 at 0° : +0·000592

No. 2 at 45° : +0·000308

No. 3 at 90° : +0·000432

the angles being measured anticlockwise from gauge No. 1. Determine the magnitude of the principal strains and their directions relative to the axis of gauge No. 1.

If $E=2{\cdot}1\times10^6$ kg./cm.^{2} and $m=3$ find the principal stresses.

Solution Let the principal strains be e_1 and e_2. Let the axis of gauge No. 1 be at α with e_1

$\therefore$ For gauge No 1, $\theta=\alpha$
For gauge No 2, $\theta=45^\circ+\alpha$
For gauge No 3, $\theta=90^\circ+\alpha$

If e_θ is the strain in the direction inclined at θ with the direction of e_1

Then $$e_\theta=\frac{e_1+e_2}{2}+\frac{e_1-e_2}{2}\cos 2\theta$$

when $\theta=\alpha$, $e_\theta=0.000592$

$$=\frac{e_1+e_2}{2}+\frac{e_1-e_2}{2}\cos 2\alpha \quad \ldots(i)$$

when $\theta=45^\circ+\alpha$, $e_\theta=0.000308$

$$=\frac{e_1+e_2}{2}+\frac{e_1-e_2}{2}\cos (90^\circ+2\alpha) \quad \ldots(ii)$$

when $\theta=90^\circ+\alpha$, $e_\theta=-0.000432$

$$=\frac{e_1+e_2}{2}+\frac{e_1-e_2}{2}\cos (180^\circ+2\alpha) \quad \ldots(iii)$$

Adding equations (i) and (iii)

$$0.000160=e_1+e_2$$

Substituting in equation (ii)

$$0.000308=0.000080-\frac{e_1-e_2}{2}\sin 2\alpha$$

$$\therefore \quad \frac{e_1-e_2}{2}\sin 2\alpha=-0.000228 \quad \ldots(iv)$$

Similarly substituting for (e_1+e_2) in equation (i)

$$0.000592=0.000080+\frac{e_1-e_2}{2}\cos 2\alpha$$

$$\therefore \quad \frac{e_1-e_2}{2}\cos 2\alpha=0.000512 \quad \ldots(v)$$

Dividing equation (iv) by equation (v)

$$\tan 2\alpha=-\frac{0.000228}{0.000512}=-0.446$$

$$\therefore \quad 2\alpha=156^\circ$$

$$\therefore \quad \alpha=78^\circ$$

Substituting in equation (iv)

$$\frac{e_1-e_2}{2}\sin 156^\circ=-0.000228$$

$\therefore \qquad e_1-e_2=-0{\cdot}00112$

but $\qquad e_1+e_2 = 0{\cdot}00016$

Solving the above equations we get $e_1=-0{\cdot}00048$

and $\qquad e_2=0{\cdot}00064$

Thus the principal strains are $-0{\cdot}00048$ and $+0{\cdot}00064$ the first being at 78° measured clockwise from the direction of gauge No. 1.

Let p_1 and p_2 be the principal stresses

$$e_1=\frac{1}{E}\left[p_1-\frac{p_2}{m}\right]$$

$$\therefore \qquad -0{\cdot}00048=\frac{1}{2{\cdot}1\times10^6}\left[p_1-\frac{p_2}{3}\right] \qquad \ldots(A)$$

$$e_2=\frac{1}{E}\left[p_2-\frac{p_1}{m}\right]$$

$$\therefore \qquad 0{\cdot}00064=\frac{1}{2{\cdot}1\times10^6}\left[p_2-\frac{p_1}{3}\right] \qquad \ldots(B)$$

From equations (A) and (B) we get

$p_1=-630$ *kg./cm.*^{2} *(compressive)*

and $\qquad p_2=1134$ *kg./cm.*^{2} *(tensile)*

Examples in Chapter 11

1. The plate of a boiler is subjected to principal stresses of 1200 *kg./cm.*^{2} and 600 *kg./cm.*^{2} both tensile. Find the intensity of stress which acting alone will produce the same maximum strain. Take Poisson's ratio $=0{\cdot}3$ (1020 *kg./cm.*^{2})

2. The principal stresses at a certain point in a strained material are 1200 *kg./cm.*^{2} and 480 *kg./cm.*^{2} both tensile. Find the normal and tangential stresses on a plane inclined at 20° with the major principal plane. ($P_n=1115{\cdot}7$ *kg./cm.*^{2} ; $P_t=231{\cdot}4$ *kg./cm.*^{2})

3. At a point the principal stresses are 1400 *kg./cm.*^{2} and 750 *kg./cm.*^{2} both tensile. Find the normal and tangential stresses on a plane inclined at 60° to the axis of the major principal stress. ($p_n=1237{\cdot}5$ *kg./cm.*^{2} ; $P_t=281{\cdot}5$ *kg./cm.*^{2})

4. A rectangular steel bar is subjected to a tensile stress of 800 *kg./cm.*^{2}, as well as a shear stress of 300 *kg./cm.*^{2}. Determine the principal stresses and the principal planes. Find also what stress acting alone can produce the same maximum strain. Take $\frac{1}{m}=0{\cdot}3$.

(900 *kg./cm.*^{2} tensile. 100 *kg./cm.*^{2} compressive
$\theta_1=18°\ 26'$; $\theta_2=108°\ 26'$; 930 *kg./cm.*^{2})

5. At a point in a strained material the principal stresses are 600 *kg./cm.*^{2} and 400 *kg./cm.*^{2}. Find the position of the plane across which the resultant stress is most inclined to the normal and determine the value of this stress.

(50° 46′ with the major principal plane ; 490 *kg./cm.*^{2})

6. At a point in a beam section there is a longitudinal bending stress of 1200 $kg/cm.^2$ tensile and a transverse shear stress of 500 $kg/cm.^2$. Find the resultant stress on a plane inclined at 30° to the longitudinal axis. (1060 $kg./cm.^2$ at 13° 40' to longitudinal axis)

7. In a circular shaft subjected to an axial twisting moment T and a bending moment M, show that when $M=1{\cdot}2\ T$ the ratio of the maximum shearing stress to the greater principal stress is nearly 0·566. (*London University*)

8. At a certain point in a piece of elastic material there are normal stresses of 480 $kg./cm.^2$ tensile and 320 $kg/cm.^2$ compressive on two planes at right angles to one another, together with shearing stresses of 240 $kg/cm.^2$ on the same planes. If the loading on the material is increased so that the stresses reach K times those given, find the maximum value of K if the maximum direct stress in the material is not to exceed 1280 $kg/cm.^2$ and the maximum shearing stress is not to exceed 800 $kg./cm.^2$ (*London University*)

($K=2{\cdot}342$ for direct stress condition ;
$K=1{\cdot}715$ for shear stress condition)

9. A solid circular shaft is subjected to an axial torque T and to a bending moment M. If $M=KT$, determine in terms of K the ratio of the maximum principal stress to the maximum shear stress.

$$\left(\frac{K+\sqrt{1+K^2}}{\sqrt{1+K^2}}\right)$$

10. At a certain section of a shaft 100 $mm.$ diameter there is a bending moment of 40000 $kg.\ cm.$ and a twisting moment of 60,000 $kg.\ cm.$ Find the maximum direct stress induced in the section. If Poisson's ratio $=0{\cdot}25$, find what stress acting alone can produce the same maximum strain. (570·9 $kg/cm.^2$; $\theta=28°9$; 611·3 $kg./cm.^2$)

11. A horizontal circular shaft of diameter d and diametral moment of inertia I is subjected to a bending moment $M\cos\theta$ in a vertical plane and to an axial twisting moment $M\sin\theta$. Show that the principal stresses at ends of a vertical diameter are

$$\tfrac{1}{2}\,MK(\cos\theta\pm 1),$$

where $$K=\frac{d}{2I}$$

If strain energy is t e criterion of failure, show that,

$$S=\frac{S_0\sqrt{2}}{\sqrt{\cos^2\theta\,(1-\sigma)+(1+\sigma)}}$$

where $S=$ maximum shearing stress
$S_0=$ maximum shearing stress in the special case where $\theta=0$
$\sigma=$ Poisson's ratio (*London University*)

12. A circular shaft of external radius r and diametral moment of inertia I is subjected to a bending moment $M\cos\alpha$ and a twisting moment $M\sin\alpha$. Show that the magnitude of the maximum shearing stress has a constant value $\tfrac{1}{2}MK$ for all values of α where $K=\dfrac{r}{I}$. Show clearly on a diagram the planes on which the maximum shearing stresses occur when $\alpha=45°$.

12

Thin Cylinders and Spheres

§113. Thin Cylinders

Fig. 581 shows a thin cylindrical shell whose internal diameter is d, the thickness of the shell being t. Let the length of the shell be l. Let the shell be subjected to an internal pressure of intensity p.

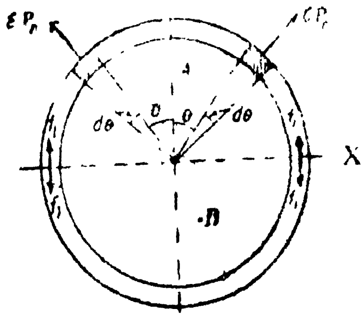


Fig. 581

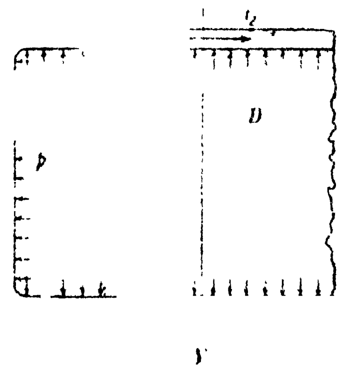


Fig. 582

Let us consider a longitudinal section XX through the axis, dividing the shell into two halves A and B. Now let us consider two elementary strips subtending an angle $\delta\theta$ at the centre at an angle θ on either side of the vertical through the centre.

Normal force on each strip

$$\delta P_n = pr\,\delta\theta\,l,$$

where r = radius of the shell.

The resultant of the two normal forces on the two elemental strips $= \delta P = 2prl\,\delta\theta \cos\theta$ acting vertically, *i.e.*, normal to XX.

∴ Total force normal to XX on one side of XX

= total bursting force

$$= P = \int_0^{\pi/2} 2prl \cos\theta \, d\theta$$

$= 2prl = pdl$

= Intensity of radial pressure × projected area

Let f_1 be the intensity of tensile stress induced in the metal across the section XX.

Resisting force offered by the section XX

$$=f_1\, 2lt$$

Equating the resisting force to the bursting force

$$\therefore \quad f_1\, 2lt = pdl$$

$$\therefore \quad f_1 = \frac{pd}{2t}$$

Just like the section XX, if we had considered any other longitudinal section, the intensity of tensile stress would be found to be the same. Hence the direction of this tensile stress is along the circumference of the shell. A stress so induced is called a *hoop* or *circumferential* stress.

Let us now consider a section YY normal to the axis of the shell. Let the section divide the shell into two parts C and D.

Force acting on the end of the shell

$$=P = p\frac{\pi d^2}{2}$$

In order the shell may not be split up at the section YY, this section will offer a resistance. Let f_2 be the tensile stress on the section YY.

Equating the resistance offered by the section YY to the total force on one of the shell

$$f_2\, \pi dt = p\frac{\pi d^2}{4}$$

$$\therefore \quad f_2 = \frac{pd}{4t}$$

This stress is called the *longitudinal* stress.

Hence at any point in the metal of the shell there are two principal stresses, namely a hoop stress of $\frac{pd}{2t}$ acting circumferentially and a longitudinal stress of $\frac{pd}{4t}$ acting parallel to the axis of the shell.

Greatest shear stress

$$=q_{max} = \frac{f_1 - f_2}{2} = \frac{pd}{8t}$$

Circumferential strain

$$=e_1 = \frac{f_1}{E} - \frac{f_2}{mE}$$

where E is Young's modulus and $\frac{1}{m}$ is Poisson's ratio

$$\therefore \quad e_1 = \frac{pd}{2tE} - \frac{pd}{4tEm}$$

$$\therefore \quad e_1 = \frac{pd}{2tE}\left(1 - \frac{1}{2m}\right)$$

Longitudinal strain

$$= e_2 = \frac{f_2}{E} - \frac{f_1}{mE} = \frac{pd}{4tE} - \frac{pd}{2tEm}$$

$$\therefore \quad e_2 = \frac{pd}{2tE}\left(\frac{1}{2} - \frac{1}{m}\right)$$

Circumferential strain

$$= \frac{\text{change of circumference}}{\text{original circumference}}$$

$$= \frac{\pi \delta d}{\pi d} = e_1$$

$$\therefore \quad \frac{\delta d}{d} = e_1$$

But $\frac{\delta d}{d}$ = strain of the diameter

∴ Change in diameter $= e_1 \times$ original diameter.

Longitudinal strain

$$= \frac{\text{change in length}}{\text{original length}} = e_2$$

$$\therefore \quad \frac{\delta l}{l} = e_2$$

∴ Change in length $= e_2 \times$ original length

Capacity of the shell

$$= V = \frac{\pi}{4} d^2 l$$

∴ Change in capacity of the shell

$$= \delta V = \frac{\pi}{4} d^2 \, \delta l + \frac{\pi}{4} \cdot 2d \, \delta d l$$

$$\therefore \quad \frac{\delta V}{V} = \frac{\delta l}{l} + 2 \frac{\delta d}{d}$$

$$\therefore \quad \frac{\delta V}{V} = e_2 + 2e_1$$

From this relation it is possible to find the change in the capacity of the shell under the radial pressure p.

Let f be the permissible tensile stress for the shell material. Hence for the shell to be safe the major principal stress f_1 shall not exceed f

i.e., $$\frac{pd}{2t} \leq f$$

$$\therefore \quad t > \frac{pd}{2f}$$

From the above relation, for a given radial pressure, the thickness of the shell may be determined for a given diameter for a given permissible stress.

Problem 389. *A seamless pipe 80 cm. diameter contains a fluid under a pressure of 20 kg./cm.² If the permissible tensile stress be 1000 kg./cm.², find the minimum thickness of the pipe.*

Solution $p = 20$ *kg./cm.*² ; $d = 80$ *cm.* diameter

and $$f = 1000 \text{ kg./cm.}^2$$

$$t = \frac{pd}{2f} = \frac{20 \times 80}{2 \times 1000} \text{ cm.}$$
$$= 0{\cdot}8 \text{ cm.}$$

Problem 390 (SI). *A 90 cm. diameter pipe contains a fluid at a pressure of 250 N/cm². If the safe stress in tension is 10,000 N/cm.², find the minimum thickness of the pipe.*

Solution. $p = 250$ *N/cm*² ; $d = 90$ *cm.*, $f_1 = 10{,}000$ *N/cm.*²

$$f_1 = \frac{pd}{2t}$$

$$t = \frac{pd}{2f_1} = \frac{250 \times 90}{2 \times 10{,}000} = 1{\cdot}125 \text{ cm.}$$

Problem 391. *A cylindrical thin drum 80 cm. in diameter and 3 metres long has a shell thickness of 1 cm. If the drum is subjected to an internal pressure of 25 kg./cm.², determine (i) the change in diameter (ii) change in length, and (iii) change in volume. Take $E = 2 \times 10^6$ kg./cm²; Poisson's ratio $= \frac{1}{4}$.* *(A.M.I.E., Summer 1978)*

Solution.

Hoop stress $$= f_1 = \frac{pd}{2t} = \frac{25 \times 80}{2 \times 1} = 1000 \text{ kg./cm.}^2$$

Longitudinal stress

$$= f_2 = \frac{f_1}{2} = \frac{1000}{2} = 500 \text{ kg./cm.}^2$$

Circumferential strain

$$e_1 = \frac{f_1}{E} - \frac{f_2}{mE}$$
$$= \frac{1}{E}\left(f_1 - \frac{f_2}{m}\right)$$

$$=\frac{1}{2\times 10^6}(1000-0\cdot25\times 500)$$

$$=0\cdot0004375$$

Longitudinal strain

$$=e_2=\frac{f_2}{E}-\frac{f_1}{mE}$$

$$=\frac{1}{E}\left(f_2-\frac{f_1}{m}\right)$$

$$=\frac{1}{2\times 10^6}(500-0\cdot25\times 1000)$$

$$=0\cdot0001250$$

$$\therefore \quad e_v=2e_1+e_2=2(0\cdot0004375)+0\cdot0001250$$

$$=0\cdot001$$

$\therefore$ *Increase in length* $=\delta l=e_2 l=0\cdot0001250\times 300$

$=$ ***0·0375 cm.***

Increase in diameter

$$=\delta d=e_1 d=0\cdot0004375\times 80$$

$=$ ***0·035 cm.***

Increase in the volume

$$=\delta V=e_v V=0\cdot001\times\frac{\pi}{4}(80)^2\,300$$

$=$ ***1507·968 cc.***

Problem 392. *The air vessel of a torpedo is 53 cm. external diameter and 1 cm. thick, the length being 183 cm. Find the change in the external diameter and the length when charged to 105 kg./cm.² internal pressure. Take $E=2\cdot1\times10^6$ kg./cm.² and Poisson's ratio $=0\cdot3$.*

(A.M.I.E. November 1965)

Solution. Internal diameter $=53-2=51$ cm.

Circumferential stress

$$=f_1=\frac{pd}{2t}=\frac{105\times 51}{2\times 1}\ \text{kg./cm.}^2$$

$$=2578\ \text{kg./cm.}^2$$

Longitudinal stress $=f_2$

$$=\frac{f_1}{2}=1339\ \text{kg./cm.}^2$$

Circumferential strain

$$=e_1=\frac{f_1}{E}-\frac{f_2}{mE}$$

$$=\frac{1}{E}\left(f_1-\frac{f_2}{m}\right)$$

$$= \frac{1}{2{\cdot}1 \times 10^6}(2678 - 0{\cdot}3 \times 1339) = \frac{2276{\cdot}3}{2{\cdot}1 \times 10^6}$$

$$= 0{\cdot}001084$$

$\therefore$ Change in external diameter

$$= e_1 \times \text{original external diameter}$$

$$= 0{\cdot}001084 \times 53 \text{ cm.}$$

$$= 0{\cdot}05742 \text{ cm.}$$

Longitudinal strain $= e_2$

$$= \frac{f_2}{E} - \frac{f_1}{mE}$$

$$= \frac{1}{E}\left(f_2 - \frac{f_1}{m}\right)$$

$$= \frac{1}{2{\cdot}1 \times 10^6}(1339 - 0{\cdot}3 \times 2678)$$

$$= 0{\cdot}0002551$$

$\therefore$ Change in length $= e_2 \times$ original length

$$= 0{\cdot}0002551 \times 183 \text{ cm.}$$

$$= 0{\cdot}04669 \text{ cm.}$$

Problem 393. *A shell 3·25 metres long, 1 metre in diameter is subjected to an internal pressure of 10 kg/cm.² If the thickness of the shell is 10 mm., find the circumferential and longitudinal stresses. Find also the maximum shear stress and the changes in the dimensions of the shell. Take* $E = 2 \times 10^6$ *kg/cm.² and* $\frac{1}{m} = 0{\cdot}3$.

Solution. $p = 10$ kg./cm.²

$l = 325$ cm.

$d = 100$ cm.

$t = 1$ cm.

Circumferential stress

$$= f_1 = \frac{pd}{2t}$$

$$= \frac{10 \times 100}{2 \times 1} \text{ kg/cm.}^2$$

$$= 500 \text{ kg/cm.}^2$$

Longitudinal stress $= f_2$

$$= \frac{pd}{4t} = \frac{f_1}{2}$$

$$= 250 \text{ kg./cm.}^2$$

Maximum shear stress

$$_{max} = \frac{f_1 - f_2}{2} = \frac{500 - 250}{2} \ kg./cm.^2$$

$$= 125 \ kg./cm.^2$$

Circumferential strain

$$= e_1 = \frac{f_1}{E} - \frac{f_2}{mE} \quad \frac{1}{E}\left(f_1 - \frac{f_2}{m}\right)$$

$$= \frac{1}{E}\ (500 - 0{\cdot}3 \times 250)$$

$$= \frac{425}{E}$$

Longitudinal strain

$$= e_2 = \frac{f_2}{E} - \frac{f_1}{mE}$$

$$= \frac{1}{E}\left(f_2 - \frac{f_1}{m}\right)$$

$$= \frac{1}{E}\ (250 - 0{\cdot}3 \times 500)$$

$$= \frac{100}{E}$$

Strain of the capacity

$$= e_v = \frac{\delta V}{V} = e_2 + 2e_1$$

$$= \frac{100}{E} + \frac{850}{E}$$

$$= \frac{950}{E}$$

Change in diameter $= e_1 \times$ original diameter

$$= \frac{425}{E}\ d\ cm. = \frac{425}{2 \times 10^6} \times 100\ cm.$$

$$= 0{\cdot}02125\ cm.$$

Change in length $= e_2 \times$ original length

$$= \frac{100}{E}\ l\ cm.$$

$$= \frac{100}{2 \times 10^6} \times 325\ cm.$$

$$= 0{\cdot}01625\ cm.$$

Change in the capacity

$$= e_v \times \text{original capacity}$$

$$=e_v \cdot \frac{\pi d^2}{4} \, l$$

$$=\frac{950}{2\times10^6}\cdot\frac{\pi}{4}\cdot(100)^2\times325$$

$$=1213 \text{ cm.}^3$$

Problem 394. *A water main 80 cm. diameter contains water at a pressure head of 115 metres. If the weight of water is 1000 kg. per cubic metre, find the thickness of the metal required if the permissible stress is 250 kg./cm.*^{2}

Solution. Pressure intensity inside the pipe

$$=wH$$

$$=1000\times115 \text{ kg./metre}^2$$

$$=\frac{1000\times115}{(100)^2} \text{ kg./cm.}^2$$

$$=11{\cdot}5 \text{ kg./cm.}^2$$

∴ Thickness required for the pipe

$$=t=\frac{pd}{2f}$$

$$=\frac{11{\cdot}5\times80}{2\times250} \text{ cm.}$$

$$=1{\cdot}84 \text{ cm.}$$

Problem 395. *A cylindrical shell 90 cm. long, 15 cm. internal diameter, having a thickness of metal=8 mm., is filled with a fluid at atmospheric pressure. If an additional 20 cm.*^{3} *of fluid is pumped into the cylinder find (i) the pressure exerted by the fluid on the cylinder, and (ii) the hoop stress induced. Take* $E=2\times10^6$ *kg./cm.*^{2} *and* $\frac{1}{m}=0{\cdot}3$ (*A.M.I.E., Summer 1977*)

Solution. Let the internal pressure be p kg./cm.^{2}

Hoop stress $=f_1=\frac{pd}{2t}=\frac{p\times15}{2\times0{\cdot}8}=9{\cdot}375\, p$ kg./cm.^{2}

Longitudinal stress

$$=f_2=\frac{f_1}{2}=4{\cdot}6875\, p \text{ kg./cm.}^2$$

Circumferential strain

$$=e_1=\frac{1}{E}\left(f_1-\frac{f_2}{m}\right)$$

$$=\frac{p}{E}\,(9{\cdot}375-0{\cdot}3\times4{\cdot}6875)$$

$$=7{\cdot}96875\,\frac{p}{E}$$

Longitudinal strain

$$=e_2=\frac{1}{E}\left(f_2-\frac{f_1}{m}\right)$$

$$=\frac{p}{E}(4{\cdot}6875-0{\cdot}3\times9{\cdot}375)$$

$$=1{\cdot}875\frac{p}{E}$$

Volumetric strain

$$=e_v=2e_1+e_2$$

$$=(2\times7{\cdot}96875+1{\cdot}875)\ \frac{p}{E}$$

$$=17{\cdot}8125\ \frac{p}{E}$$

Increase in volume

$$=\delta V=e_vV=20\ cm.^3$$

$$\therefore\quad 17{\cdot}8125\ \frac{p}{2\times10^6}\times\frac{\pi}{4}(15)^2\ 90=20$$

$$\therefore\quad p=141{\cdot}2\ kg./cm.^2$$

$$\therefore\quad f_1=\frac{pd}{2t}=\frac{141{\cdot}2\times15}{2\times0{\cdot}8}$$

$$=1323{\cdot}75\ kg./cm.^2$$

Problem 396. *A cylindrical shell 100 cm. long, 18 cm. internal diameter, thickness of metal 8 mm. is filled with a fluid at atmospheric pressure. If an additional 20 cm.^{3} of the fluid is pumped into the cylinder find the pressure exerted by the fluid on the wall of the cylinder. Find also the hoop stress induced. Take $E=2\times10^6$ kg./cm.^{2} and $\frac{1}{m}=0{\cdot}3$.*

Solution. Let the pressure on the walls of the cylinder be p kg./cm.^{2}

Circumferential stress$=f_1$

$$=\frac{pd}{2t}$$

$$=\frac{p\times18}{2\times0{\cdot}8}\ kg./cm.^2$$

$$=11{\cdot}25\,p\ \ kg./cm.^2$$

Longitudinal stress $=f_2$

$$=\frac{f_1}{2}=5{\cdot}625\,p\ kg./cm.^2$$

Circumferential strain

$$=e_1=\frac{f_1}{E}-\frac{f_2}{mE}$$

$$=\frac{1}{E}\left(f_1-\frac{f_2}{m}\right)$$

$$=\frac{1}{E}(11{\cdot}25\,p-0{\cdot}3\times5{\cdot}625)\,p$$

$$=\frac{9{\cdot}5625\,p}{E}$$

Longitudinal strain

$$=e_2=\frac{f_2}{E}-\frac{f_1}{mE}$$

$$=\frac{1}{E}\left(f_2-\frac{f_1}{mE}\right)$$

$$=\frac{1}{E}(5{\cdot}625\,p-0{\cdot}3\times11{\cdot}25\,p)$$

$$=\frac{2{\cdot}25\,p}{E}$$

∴ Strain of the capacity

$$=e_v=e_2+2e_1$$

$$=\frac{2{\cdot}25\,p}{E}+\frac{2\times9{\cdot}5625\,p}{E}$$

$$=\frac{21{\cdot}375\,p}{E}$$

But $e_v=\frac{\delta V}{V}$

∴ $\frac{\delta V}{V}=\frac{21{\cdot}375\,p}{E}$

But $\delta V=20\ cm.^3$

∴ $p=\frac{\delta V}{V}\cdot\frac{E}{21{\cdot}375}$

$$=\frac{20\times2\times10^6}{\pi\times\frac{18^2}{4}\times100\times21{\cdot}375}\ kg./cm.^2$$

$$=73{\cdot}52\ kg./cm.^2$$

Hoop stress $=f_1$

$$=\frac{pd}{2t}$$

$$=\frac{73{\cdot}52\times18}{2\times0{\cdot}8}\ kg/.cm.^2$$

$$=827{\cdot}1\ kg./cm.^2$$

§114. Riveted cylindrical boilers

A boiler of the desired capacity can be made by bending plates to the required diameter and connecting them, usually by a butt joint. The desired length of the boiler can be obtained by connecting individually fabricated shells by usually a lap joint. These joints are shown in Figs. 583 and 584.

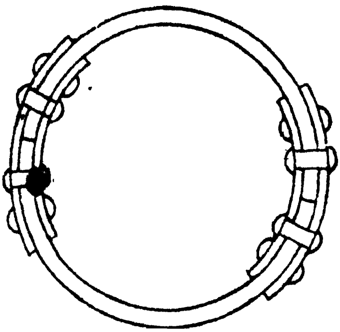


Fig. 583

In the case of riveted shells the circumferential and longitudinal stresses are greater than what are given in the previous articles. This is due to weakening of the plates due to rivet holes.

If η_1 is the efficiency of the longitudinal joints, hoop stress $= f_1 = \frac{p\,d}{2t\eta_1}$ on the section through rivet holes. Similarly if η_2 is the efficiency of the circumferential joint, the longitudinal stress

$$= f_2 = \frac{p\,d}{4t\eta_2} \text{ on the section through rivet holes.}$$

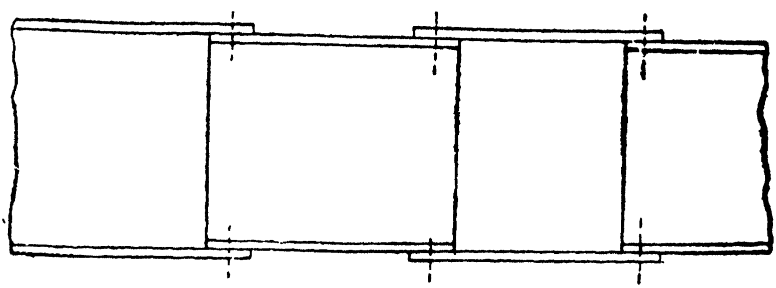


Fig. 584

The thickness of the shell in order the hoop stress may not exceed the permissible stress is given by

$$t = \frac{p\,d}{2f\eta_1}$$

Problem 397. *A riveted boiler is made out of 2 cm. thick plates, to a diameter of 200 cms. If the efficiency of the longitudinal and circumferential joints be 75% and 60% find the safe pressure in the boiler if the maximum tensile stress on the plate section through rivets may not exceed 1200 kg./cm.² Find also the longitudinal stress on the section through the rivets.*

Solution. Circumferential stress

$$=f_1=\frac{p\,d}{2t\eta_1}$$

$$p=\frac{f_1 2t\,\eta_1}{d}$$

$$=\frac{1200\times2\times2\times0{\cdot}75}{200}\,kg./cm.^2$$

$$=18\ kg./cm.^2$$

Longitudinal stress

$$f_2=\frac{p\,d}{4t\eta_2}=\frac{18\times200}{4\times2\times0{\cdot}6}\ kg/cm.^2$$

$$=750\ kg./cm.^2$$

Problem 398. *A boiler is subjected to an internal pressure of 20 kg./cm². The thickness of the boiler plate is 2 cm and the permissible tensile stress is 1200 kg./cm². Find the maximum permissible diameter when the efficiency to the longitudinal joint is 90% and that of the circumferential joint is 40%.* *(AMIE, Winter 1976)*

Solution. Limiting the hoop stress to 1200 $kg./cm.^2$

$$f_1=\frac{p\,d}{2t\eta}=1200$$

$$d=\frac{1200\times2\times2\times0{\cdot}9}{20}=216\ cm.$$

Limiting the longitudinal stress to 1200 $kg./cm.^2$

$$f_2=\frac{p\,d}{4t\eta}=1200$$

$$\therefore\quad d=\frac{1200\times4\times2\times0{\cdot}4}{20}=192\ cm.$$

$\therefore$ *Maximum permissible diameter* $=192$ *cm.*

Problem 399. *A riveted boiler 225 cm. in diameter has to sustain an internal pressure of 5·6 kg./cm². The efficiency of the riveted joints is 70% and a safe stress of 600 kg/cm.² is allowed in the material. Find the thickness of the shell and the necessary pitch of rivets for the longitudinal joints, which is a single rivetted butt joint. Take diameter of rivet d=1·9√t where t=thickness of plate in cm.*

(AMIE, May 1976)

Solution.

Hoop stress $=f=\frac{p\,d}{2t\eta}$

$$\therefore\quad t=\frac{p\,d}{2f\eta}=\frac{5{\cdot}6\times225}{2\times660\times0{\cdot}7}=1{\cdot}5\ cm.$$

Diameter of rivets $=1{\cdot}9\sqrt{t}=1{\cdot}9\sqrt{1{\cdot}5}=2{\cdot}33\ cm.$ say 24 mm.

Efficiency of the joint

$$=\frac{\text{pitch}-\text{diameter of rivet}}{\text{pitch}}=0\cdot7$$

$$\text{pitch}-2\cdot4=0\cdot7\times\text{pitch}$$

$$0\cdot3\times\text{pitch}=2\cdot4$$

$$\textit{pitch}=8\ cm.$$

§115. **Wire bound thin pipes**

Suppose a wire under tension is wound round a pipe at a close pitch, compressive stresses will be initially developed in the pipe section. If now a fluid under pressure be admitted into the pipe, the bursting force will be resisted by the pipe as well as the wires, offering tensile stresses. A pipe closely bound by wires can therefore withstand a higher fluid pressure than an unbound pipe. The following problem explains the analysis of such a pipe.

Problem 400. (SI). *A 20 cm. diameter cast iron pipe has a thickness of 12 mm. and is closely wound with a layer of 5 mm. diameter steel wire under a tensile stress of 60 N/mm.² If now water under a pressure of 3·5 N/mm² is admitted into the pipe, find the stresses induced in the pipe and the steel wire. For cast iron take* $E_c=1\times10^5$ *N/mm.² and Poisson's ratio = 0·3. For steel take* $E_s=2\times10^5$ *N/mm.²*

Solution. Consider one *centimetre* run of the pipe. For this length number of sections of steel wire

$$=2\times\frac{1}{0\cdot5}=4 \text{ (two on each side)}$$

Crushing force for 1 *cm.* run of the pipe

$$=P_c=4\times\frac{\pi}{4}\times5^2\times60=4712\ N.$$

Let f_c be the intensity of compressive stress induced in the pipe section.

$\therefore$ Resisting force $=f_c 2t\times10\ N$

$\therefore$ $f_c 2t\times10=4712$

$\therefore$ $f_c=\dfrac{4712}{2\times12\times10}=19\cdot63\ N/mm.^2$

Hence before the water is admitted into the pipe, the stresses are as follows :

In the wires : 60 $N/mm.^2$ (*tensile*)

In the pipe : 19·63 $N/mm.^2$ (*compressive*)

Stresses due to fluid pressure alone

Let f_p and f_w be the stresses in the pipe and wires due to the independent effect of fluid pressure of 3·5 $N/mm.^2$

Consider 1 *cm.* length of the pipe.

Bursting force $=pdl=3{\cdot}5\times200\times10=7000\ N$

Resisting force $=f_p\ 2lt+f_w\times4\times\dfrac{\pi}{4}(5)^2$

$=f_p\times2\times10\times12+f_w\ \pi(5)^2$

$=240\ f_p+78{\cdot}54\ f_w$

Equating the resisting force to the bursting force

$240\ f_p+78{\cdot}54\ f_w=7000$ ...(*i*)

Circumferential strain in the pipe

$$=\frac{f_p}{E_c}-\frac{1}{m}\ \frac{pd}{4tE_c}$$

$$=\frac{1}{E_c}\left\{f_p-\frac{1}{m}\ \frac{pd}{4t}\right\}$$

$$=\frac{1}{E_c}\left\{f_p-0{\cdot}3\times\frac{3{\cdot}5\times200}{4\times12}\right\}$$

$$=\frac{f_p-4{\cdot}375}{E_c}$$

Circumferential strain in the steel wire

$$=\frac{f_w}{E_s}$$

Equating the circumferential strains of the wire and pipe

$$\frac{f_w}{E_s}=\frac{f_p-4{\cdot}375}{E_c}$$

$\therefore\quad f_w=2(f_p-4{\cdot}375)$ ...(*ii*)

Substituting in equation (*i*),

$240\ f_p+78{\cdot}54\times2\ (f_p-4{\cdot}375)=7000$

$\therefore\quad 397{\cdot}08 f_p=7687{\cdot}225$

$\therefore\quad f_p=19\ 36\ N/mm.^2$

$\therefore\quad f_w=2(19\ 36-4{\cdot}375)=29{\cdot}97\ N/mm.^2$

The stresses in the pipe and wire are as follows :

	Pipe	*Steel wire*
Initial stresses	19·63 N/mm^2 (*compressive*)	60 $N/mm.^2$ (*tensile*)
Stresses due to fluid pressure alone	19·36 $N/mm.^2$ (*tensile*)	29·97 $N/mm.^2$ (*tensile*)
Final stresses	0·27 N/mm^2 (*compressive*)	89·97 $N/mm.^2$ (*tensile*)

Problem 401. *A 20 cm diameter cast iron pipe has a thickness of 12 mm. and is closely wound with a layer 5 mm. diameter steel wire under a tensile stress of 600 kg./cm². If now water under a pressure of 35 kg./cm². is admitted into the pipe, find the stresses induced in the pipe and steel wire. For cast iron, take* $E_c = 1\times10^6$ *kg./cm². and Poisson's ratio=0·3. For steel take* $E_s = 2\times10^6$ *kg./cm².*

Solution. *Initial stresses.*

Consider one *centimetre* run of the pipe. For this length number of sections of steel wire

$$= 2\times\frac{1}{0{\cdot}5} = 4 \text{ (two on each side)}$$

∴ Crushing force for 1 *cm.* run of the pipe

$$= P_c = 4\times\frac{\pi}{4}(0{\cdot}5)^2\times600$$

$$= 471{\cdot}2 \; kg.$$

Let f_c be the intensity of compressive stress induced in the pipe section.

∴ Resisting force $= f_c\, 2t$

∴ $f_c\, 2t = 471{\cdot}2$

∴ $$f_c = \frac{471{\cdot}2}{2\times1{\cdot}2}\; kg./cm.^2$$

$$= 196{\cdot}3 \; kg./cm.^2$$

Hence before the water is admitted into the pipe the stresses are as follows.

In the wires : 600 *kg./cm.*² (*tensile*)

In the pipe : 196·3 *kg./cm.*² (*compressive*)

Stresses due to fluid pressure alone

Let f_p and f_w be the stresses in the pipe and wires due to the independent effect of fluid pressure of 35 *kg./cm.*²

Consider one *cm.* length of the pipe.

Bursting force $= pdl$

$= 35\times20\times1$ *kg.*

$= 700$ *kg.*

Resisting force $= f_p 2\,lt + f_w\times4\times\frac{\pi}{4}(0{\cdot}5)^2$

$= f_p\times2\times1{\cdot}2 + f_w\pi\times0{\cdot}25$

$= 2{\cdot}4 f_p + 0{\cdot}7854 f_w$

Equating the resisting force to the bursting force

$$2{\cdot}4 f_p + 0{\cdot}7854 f_w = 700 \qquad (i)$$

Circumferential strain in the pipe

$$= \frac{f_p}{E_c} - \frac{1}{m}\left(\frac{pd}{4tE_c}\right)$$

$$=\frac{1}{E_c}\left(f_p-\frac{1}{m}\frac{pd}{4t}\right)$$

$$=\frac{1}{E_c}\left(f_p-0{\cdot}3\times\frac{35\times 20}{4\times 1{\cdot}2}\right)$$

$$=\frac{f_p-43{\cdot}75}{E_c}$$

Circumferential strain in the steel wire

$$=\frac{f_w}{E_s}$$

Equating the circumferential strains of the wire and pipe

$$\frac{f_w}{E_s}=\frac{f_p-43{\cdot}75}{E_s}$$

$$\therefore \quad f_w=(\tfrac{2}{1})\,(f_p-43{\cdot}75)$$

$$\therefore \quad f_w=2(f_p-43{\cdot}75) \qquad (ii)$$

Substituting in eq. (*i*),

$$2{\cdot}4\,f_p+0{\cdot}7854\times 2(f_p-43{\cdot}75)=700$$

$$3{\cdot}9708\,f_p=768{\cdot}7$$

$$\therefore \quad f_p=193{\cdot}6 \; kg/cm.^2 \; (tensile)$$

$$\therefore \quad f_w=2(193{\cdot}6-43{\cdot}75)\; kg./cm.^2$$

$$\therefore \quad f_w=299{\cdot}7\; kg./cm.^2 \; (tensile)$$

The stresses in the pipe and wire are as follows :

	Pipe	*Steel wire*
Initial stresses	196·3 *kg./cm.*² (*compressive*)	600 *kg./cm.*² (*tensile*)
Stresses due to fluid pressure alone	193·6 *hg /cm.*² tensile	299·7 *kg./cm.*² (*tensile*)
Final stresses	.·7 *kg /cm.*² (*compressive*)	899·7 *kg./cm.*² (*tensile*)

§116. Thin spherical shells

Fig. 585 shows a seamless spherical shell of internal diameter d and thickness t and subjected to an internal pressure of intensity p.

Let us consider a section XX through the centre of the shell.

Bursting force $P=p\times$ projected area

$$=p\times\frac{\pi d^2}{4}$$

Let f_1 be the tensile stress induced on the section of the metal at the section XX.

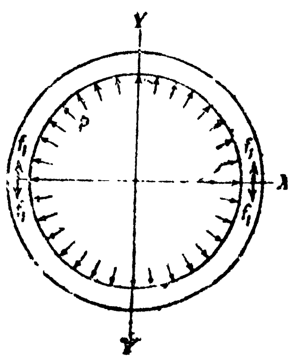


Fig. 585

$\therefore$ *Resisting force* $= f_1 \pi dt$

Equating the resisting force to the bursting force

$$f_1 \pi dt = p \frac{\pi d^2}{4}$$

$$\therefore \qquad f_1 = \frac{pd}{4t}$$

Similarly if any other section through the centre of the shell had been considered, it can be similarly shown that the tensile stress for the section would be $\frac{pd}{4t}$.

In the case of the thin spherical shell, the principal stresses f_1 and f_2 at any point are equal and like. Hence, every plane through the point is a principal plane with an intensity of stress equal to $\frac{pd}{4t}$. Obviously, no shear stress exists anywhere in the shell.

The strain in any direction

$$= e = \frac{f_1}{E} - \frac{f_2}{mE}$$

$$= \frac{f_1}{E} - \frac{f_1}{mE}$$

$$= \frac{f_1}{E}\left(1 - \frac{1}{m}\right)$$

$$= \frac{pd}{4tE}\left(1 - \frac{1}{m}\right)$$

The increase in diameter due to internal pressure is given by the relation

$$\frac{\delta d}{d} = e = \frac{pd}{4tE}\left(1 - \frac{1}{m}\right)$$

Original volume $= V = \frac{4}{3} \pi r^3$

$$\therefore \qquad V = \frac{\pi d^3}{6}$$

$$\therefore \qquad \delta V = \frac{3\pi d^2}{6} \delta d$$

$$\therefore \qquad \frac{\delta V}{V} = 3 \frac{\delta d}{d} = 3e$$

$$\therefore \qquad \frac{\delta V}{V} = \frac{3pd}{4tE}\left(1 - \frac{1}{m}\right)$$

From the above relation, the change in volume can be determined.

If the shell had been riveted, the stress in the plate would given by

$$f=\frac{pd}{4t\eta}$$

where η is the efficiency of joint.

Problem 402. *A vessel in the shape of a spherical shell 40 cm in diameter and 1 cm. shell thickness is completely filled with a fluid at atmospheric pressure. Additional fluid is then pumped in till the pressure increases by 50 kg./cm². Find the volume of this additional fluid given that $\mu=0{\cdot}25$ and $F=1\times10^6$ kg./cm.² for the shell material.* (*AMIE, Winter 1978*)

Solution.

$$f_1=f_2=\frac{pd}{4t}=\frac{50\times80}{4\times1}=1000\ kg/cm.^2$$

$$e_1=\frac{f_1}{E}-\frac{f_2}{mE}=\frac{1000}{E}-\frac{0{\cdot}25\times1000}{E}$$

$$\therefore\quad e_v=3e_1=\frac{3\times750}{2\times10^6}=\frac{750}{2\times10^6}$$

$$\therefore \text{ Increase in volume } \delta V=e_vV=\frac{750}{2\times10^6}\times\frac{4}{3}\pi(40)^3$$

$$=603{\cdot}19\ cc.$$

$\therefore$ *Volume of additional fluid=603·19 cc.*

Problem 403. *A thin seamless spherical shell of 1·5 metres diameter is 8 mm. thick. It is filled with a liquid so that the internal pressure is 15 kg./cm². Determine the increase in diameter and capacity of the shell. Take $\frac{1}{m}=0{\cdot}3$ and $E=2\times10^6$ kg./cm².*

Solution.

$$f_1=f_2=\frac{pd}{4t}$$

$$=\frac{15\times150}{4\times0{\cdot}8}\ kg./cm.^2$$

$$=703\ kg./cm.^2$$

$\therefore$ Strain of the diameter

$$=e=\frac{f_1}{E}-\frac{f_1}{mE}$$

$$=\frac{f_1}{E}\left(1-\frac{1}{m}\right)$$

$$=\frac{703}{2\times10^6}\left(1-0{\cdot}3\right)$$

$$=0{\cdot}00246$$

$\therefore$ **Increase in diameter**

$$=e\times\text{original diameter}$$

$=0{\cdot}000246\times150$ *cm.*

$=0{\cdot}0369$ *cm.*

Strain of capacity

$=e_v=3e$

$=3\times0{\cdot}000246$

$=0{\cdot}000738$

∴ Change in volume

$=e_v\times$ original volume

$=0{\cdot}000738\times\dfrac{\pi\times(150)^3}{6}$ $cm.^3$

$=1304$ cm^3.

Problem 404. *A thin spherical shell 140 cm. diameter is subjected to an internal pressure of 18 kg./cm². If the permissible stress in the plate material is 1400 kg./cm.², and the joint efficiency is 75%, find the minimum thickness.*

Solution. $p=18$ $kg./cm.^2$

$f=1400$ $kg./cm.^2$

$\eta=75\%$

$t=\dfrac{pd}{4f\eta}$

$=\dfrac{18\times140}{4\times1400\times0{\cdot}75}$ cm.

$=0{\cdot}6$ *cm.*

§117. Biaxial stresses in doubly curved walls of pressure vessels

Consider a closed thin walled vessel whose form is a surface of revolution. Let the vessel be subjected to an internal pressure of intensity p. Let t be the thickness of the wall of the vessel. This thickness is so small compared with the principal radii of curvature and hence the wall may be taken to be incapable of offering any resistance against bending. Hence the wall acts as a thin membrane. The stresses in the membrane are taken to be uniform and acting tangential to the middle surface of the wall. These stresses are called the *membrane stresses.*

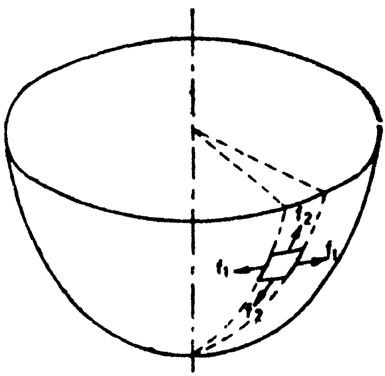


Fig. 586

These can be determined by the conditions of equilibrium.

Consider an elemental part of wall bounded by two meridian and two circumferential arcs.

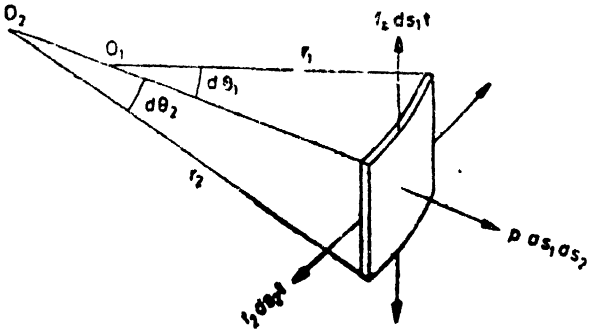


Fig. 587

Let f_1 be the stress intensity in the circumferential direction. This is the *hoop stress*. Let f_2 be the stress intensity along the meridian. This stress is called the *meridional* stress. Let r_1 be the radius of curvature of the circumferential arc. Let r_2 be the radius of the meridional arc.

Let $d\theta_1$ = angle subtended by the circumferential elemental arc.

$d\theta_2$ = angle subtended by the meridional elemental arc.

ds_1 = length of the circumferential elemental arc.

ds_2 = length of the meridional elemental arc.

The forces acting on the edges of the elemental part of the wall are, $f_1\ ds_2t$ in the circumferential direction and f_2ds_1t in the meridional direction.

Resolving the forces on the elemental part along the normal to the elemental part,

$$f_1\ ds_2td\theta_1+f_2\ ds_1td\theta_2=pds_1ds_2$$

But $$d\theta_1=\frac{ds_1}{r_1} \text{ and } d\theta_2=\frac{ds_2}{r_2}$$

$$\therefore \quad f_1\ \frac{ds_1ds_2t}{r_1}+\frac{f_2\ ds_1ds_2t}{r_2}=pds_1ds_2$$

$$\therefore \quad \frac{f_1}{r_1}+\frac{f_2}{r_2}=\frac{p}{t}$$

This is the most general equation giving the relation between the circumferential and hoop stresses and the internal pressure in the vessel.

Particular cases

(*i*) If the vessel is spherical

$$f_1=f_2=f \text{ and } r_1=r_2=r$$

$$\frac{f}{r}+\frac{f}{r}=\frac{p}{t}$$

$$\therefore \qquad f=\frac{pr}{2t}$$

(*ii*) If the vessel is cylindrical $r_2=\infty$

$$\frac{f_1}{r_1}+\frac{f_2}{\infty}=\frac{p}{t}$$

$$\therefore \qquad f_1=\frac{pr_1}{t} \text{ (Hoop stress)}$$

In this case the longitudinal stress is due to the pressure on the ends of the cylinder

$$\text{Longitudinal stress}=f_2=\frac{p\,\pi r_1^2}{2\pi r_1\,t}$$

$$=\frac{pr_1}{2t}$$

Stresses in a conical water tank

Fig. 588 shows an open conical tank uniformly suspended around its upper rim. Let h be the depth of water. Consider any level XX distant y from the apex O.

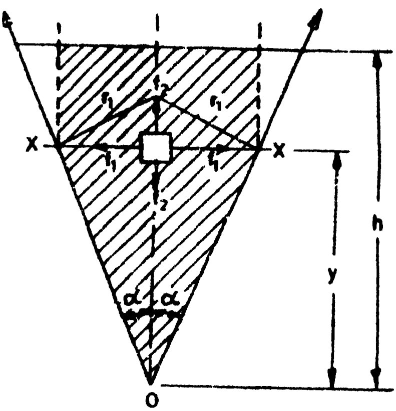


Fig. 588

At this level $\qquad r_2=\infty$

$$\therefore \quad \frac{f_1}{r_1}+\frac{f_2}{\infty}=\frac{p}{t}$$

$$\therefore \quad f_1=\frac{pr_1}{t}$$

where r_1=radius of the circumferential curvature.

But $\qquad p=w\,(h-y)$ and $r_1=\dfrac{y\tan\alpha}{\cos\alpha}$

where $\qquad w$=weight per unit volume of water.

$$\therefore \qquad f_1=\frac{w\,(h-y)}{t}\;\frac{y\tan\alpha}{\cos\alpha}$$

This is the hoop stress at the level XX.

Maximum hoop stress

For the hoop stress to be maximum,

$$\frac{df_1}{dy}\quad\frac{w\tan\alpha}{t\cos\alpha}\;(h-2y)=0$$

$$\therefore \qquad y=\frac{h}{2}$$

Putting $y=\frac{h}{2}$ in the general expression for the hoop stress, we get

$$f_{1(max)}=\frac{wh^2 \tan \alpha}{4t \cos \alpha}$$

We can also calculate the stress f_2 in the meridional direction. The total weight of the shaded volume of water must be resisted by the vertical component of the induced meridional tension on the circumference XX.

∴ For this condition

$$f_2\, 2\pi y \tan \alpha\; t \cos \alpha = w(h-y)\, \pi y^2 \tan^2 \alpha + w\pi y^2 \tan^2 \alpha \frac{y}{3}$$

$$\therefore \qquad f_2=\frac{w \tan \alpha}{2t \cos \alpha}\left(hy-\frac{2}{3}y^2 \right)$$

Maximum meridional stress

For the meridional stress to be maximum,

$$\frac{df_2}{dy}=\frac{w \tan \alpha}{2t \cos \alpha}\left(h-\frac{4}{3}y \right)=0$$

$$\therefore \qquad y=\frac{3}{4}h$$

Putting $y=\frac{3}{4}h$ in the general expression for the meridional stress we get,

$$f_{2(max)}=\frac{3wh^2 \tan \alpha}{16\, t \cos \alpha}$$

Examples in Chapter 12

1. A seamless pipe 100 *cm.* diameter contains a fluid under a pressure of 15 *kg per cm.*^{2} If the permissible tensile stress be 1000 *kg cm.*^{2} find the minimum thickness of the pipe. *(0·75 cm)*

2. A cylindrical shell is 40 *cm.* internal diameter and 0·8 *cm.* thick and 100 *cm.* long. Find the change in the internal diameter and the length, when the cylinder is charged with an internal pressure of 80 *kg./cm*^{2}.
Take $E=2\times 10^6$ *kg./cm.*^{2} and Poisson's ratio=0·3.
(0·034 *cm.*, 0·02 *cm.*)

3. A cylindrical shell is 3 *metres* long, 1 *metre* in diameter and is subjected to an internal pressure of 10 *kg,/cm.*^{2} If the thickness of the shell is 12 *mm.* find the circumferential and longitudinal stresses. Find also the max. shear stress and the changes in the dimensions of the shell.

Take $E=2\times10^6$ *kg./cm.*^{2} and $\frac{1}{m}=0.3$

(416.7 *kg./cm.*^{2} ; 208.4 *kg./cm.*^{2}
104.4 *kg./cm.*^{2}
$\delta d=0.018$ *cm*
$\delta l=0.012$ *cm.*
$\delta v=931$ *cm*^{3}).

4. A water main 60 *cm.* diameter contains water at a pressure head of 100 *metres.* If the weight of water is 1000 *kg. per cubic metre* find the thickness of the metal required if the permissible stress is 300 *kg./cm*^{2}. (1 *cm.*)

5. A cylindrical shell 120 *cm.* long, 20 *cm.* internal diameter and 10 *mm.* thick is filled with a fluid at atmosphere pressure. If an additional 30 *cm.*^{3} of the fluid be pumped into the cylinder, find the pressure exerted by the fluid on the wall of the cylinder. Find also the hoop stress induced

Take $E=2\times10^6$ *kg./cm*^{2} and $\frac{1}{m}=0.3$

(83.75 *kg./cm.*^{2} ; 837.5 *kg./cm.*^{2})

6. A riveted boiler is made out of 15 *mm.* thick plates to a diameter of 180 *cm.* If the efficiency of the longitudinal and circumferential joints be 75% and 60% respectively, find the safe pressure in the boiler if the maximum tensile stress on the plate section through rivets may not exceed 1200 *kg./cm*^{2}. Find also the longitudinal stress on the section through the rivets.

(15 *kg./cm*^{2} ; 750 *kg./cm.*^{2})

7. A thin spherical shell of 1.80 *metres* diameter is 10 *mm.* thick. It is filled with a liquid so that the internal pressure is 10 *kg./cm.*^{2} Find the increase in diameter and capacity of the shell.

Take $E=2\times10^6$ *kg./cm.*^{2} and $\frac{1}{m}=0.3$

(0.028 *cm.* ; 1443 *cm.*^{3})

8. A thin special shell 150 *cm.* diameter is subjected to an internal pressure of 20 *kg./cm.*^{2} If the permissible stress in the plate material is 1400 *kg./cm.*^{2} and the joint efficiency is 80% find the maximum thickness. (0.7 *cm.*)

13

Thick Cylinders and Spheres

§118. Thick Cylinders

In the case of thin cylinders, the hoop stress was determined assuming it to be uniform across the thickness of the cylinder. But actually, the hoop stress will not be uniform across the thickness and it will be seen that the hoop stress will vary from a maximum value at the inner circumference to a minimum value at the outer circumference.

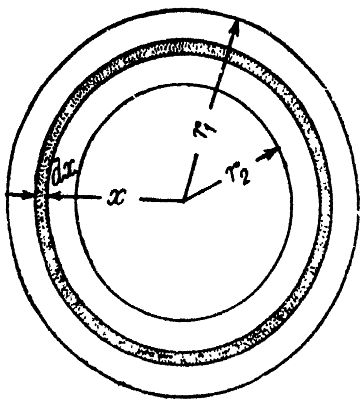


Fig. 589

Fig. 589 shows a thick cylinder of external radius r_1 and internal radius r_2 and length l. Let the cylinder be subjected to an internal pressure of intensity p_0. The thick cylinder may be taken to consist of a number of concentric elemental rings. Consider one such elemental ring of radius x and thickness dx. Let the radial pressure intensities be p_x and (p_x+dp_x) at the inner and outer circumference of the elemental ring. Let the hoop stress intensity induced in the elemental ring be f_x. Consider a longitudinal section XX.

Bursting force $=p_x(2xl)-(p_x+dp_x)2(x+dx)\,l$

Resisting force $=f_x\,2dx\,l$

Equating the resisting force to the bursting force, we have,

$$f_x 2dxl=2p_x xl-2(p_x+dp_x)(x+dx)l$$

$$\therefore \quad f_x dx=-p_x dx-x dp_x-dp_x dx$$

Ignoring products of very small quantities, we have,

$$f_x=-p_x-x\frac{dp_x}{dx} \qquad \ldots(i)$$

Now we will obtain another relation between the radial pressure and hoop stress by using the condition that the longitudinal strain at any point in the section is the same.

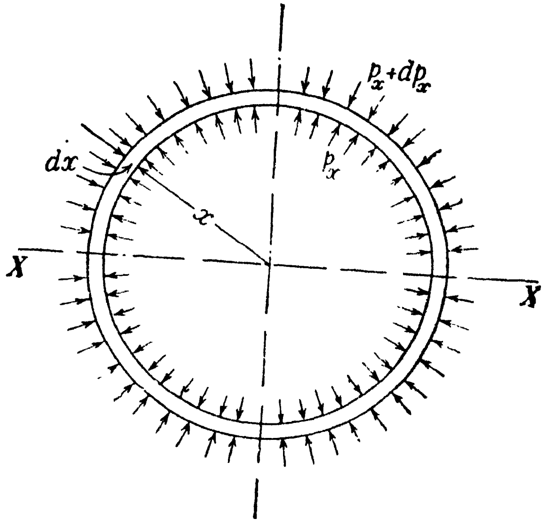


Fig. 590

The longitudinal stress

$$p_o = \frac{p\pi r_2^2}{x(r_1^2 - r_2^2)} = \frac{pr_2^2}{r_1^2 - r_2^2}$$

Hence at any point in the section of the elemental ring considered above, the following three principle stresses exist :

(*i*) The radial pressure p_x

(*ii*) The hoop stress f_x

(*iii*) The longitudinal tensile stress p_o

Since the longitudinal strain is constant, we have,

$$\frac{p_o}{E} - \frac{f_x}{mE} + \frac{p_x}{mE} = \text{constant}$$

where $\frac{1}{m}$ is the Poisson's ratio.

But since p_o, m and E are constants

$$f_x - p_x = \text{constant}$$

$\therefore$ Let $\qquad f_x - p_x = 2a \qquad$ ...(*ii*)

Putting $\qquad f_x = (p_x + 2a)$ in equation (*i*), we get

$$(p_x + 2a) = -p_x - x\frac{dp_x}{dx}$$

$$\therefore \quad \frac{dp_x}{dx} = -\frac{2(p_x + a)}{x}$$

$$\therefore \quad \frac{dp_x}{p_x + a} = -\frac{2dx}{x}$$

Integrating, we get,

$$\log_e (p_x+a) = -2 \log_e x + \log_e b$$

where, $\log_e b$ is a constant

$$\therefore \quad \log_e (p_x+a) = \log_e \frac{b}{x^2}$$

$$\therefore \quad p_x + a = \frac{b}{x^2}$$

$$\therefore \quad p_x = \frac{b}{x^2} - a$$

Substituting, in equation (*ii*) we have

$$f_x = \frac{b}{x^2} + a$$

Then, the radial pressure p_x and the hoop stress f_x at any radius x are given by,

$$p_x = \frac{b}{x^2} - a \quad \ldots(iii)$$

and

$$f_x = \frac{b}{x^2} + a \quad \ldots(iv)$$

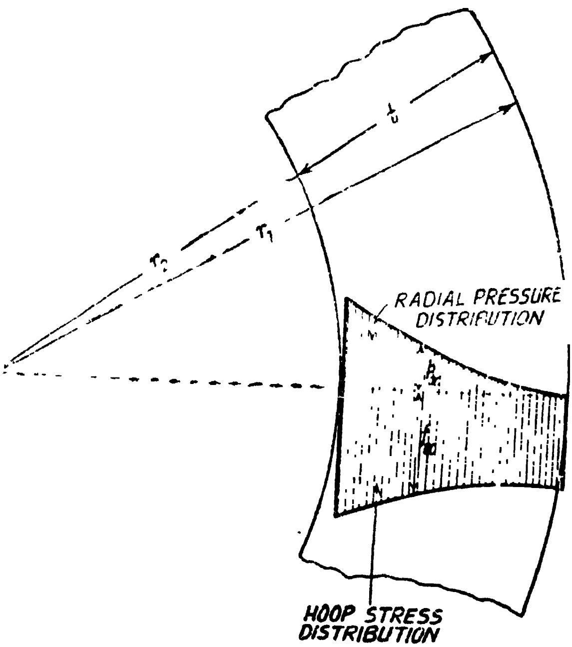


Fig. 591

The above two equations are called *Lame's equations*. From the known conditions, namely at $x=r_2$, $p_x=p_o$ and at $x=r$, $p_x=0$ the constants a and b may be evaluated.

After finding the constants a and b the hoop stress at any radius can be easily determined.

Fig. 592 shows the radial pressure distribution and hoop stress distribution across the thickness of the shell.

Fig. 592 also shows the radial pressure and hoop-stress variation.

In this representation the two curves are parallel distant $2a$ apart vertically.

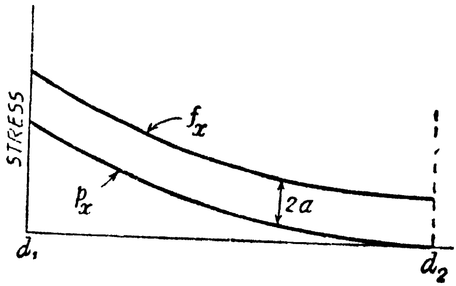


Fig. 592

Problem 405. *A pipe of 40 cm. internal diameter and 10 cm. thickness contains a fluid at a pressure of 80 kg./cm². Find the maximum and minimum hoop stress across the section. Also, sketch the radial pressure distribution and hoop stress distribution across the section.*

Solution. Let the radial pressure intensity and the hoop stress at any radius x be given by

$$p_x = \frac{b}{x^2} - a$$

and $$f_x = \frac{b}{x^2} + a$$

At $x = 20$ cm., $p_x = 80$ kg./cm² and at $x = 30$ cm. $p_x = 0$.

$$\therefore \frac{b}{400} - a = 80$$

and $$\frac{b}{900} - a = 0$$

Solving the above equations we get $b = 57600$

and $a = 64$

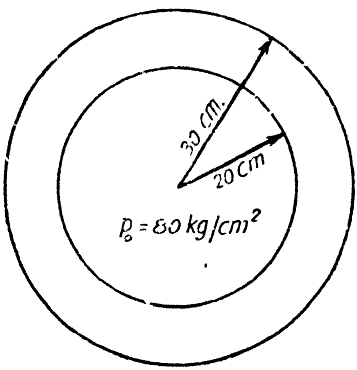


Fig. 593

Hence the hoop stress at any radius x is given by

$$f_x = \frac{57600}{x^2} + 64$$

At $x = 20$ cm., $$f_{20\,cm} = \frac{57600}{400} + 64 = 208 \text{ kg/cm.}^2$$

At $x = 30$ cm., $$f_{30\,cm} = \frac{57600}{900} + 64 = 128 \text{ kg./cm.}^2$$

Fig. 594 (*a*) and 594 (*b*) show the distribution of radial pressure and hoop stress across the section.

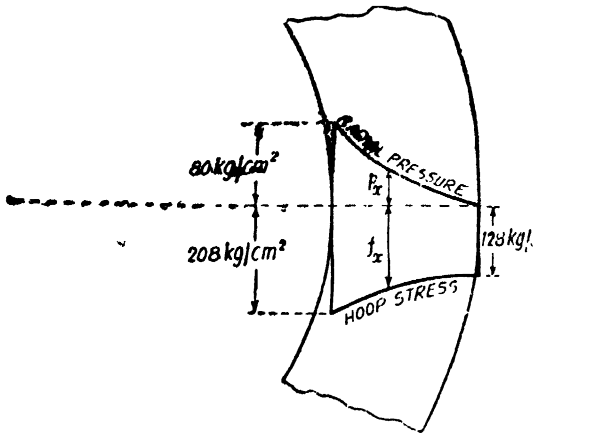


Fig. 594 (*a*)

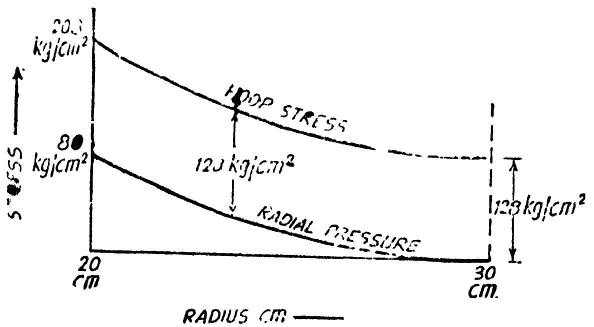


Fig. 594 (*b*)

Problem 406 (SI). *A pipe of 400 mm. internal diameter and 100 mm. thickness contains a fluid at a pressure of 8 N/mm². Find the maximum and the minimum hoop stress across the section.*

Solution. Let the radial pressure intensity and the hoop stress at any radius x be given by,

$$p_x = \frac{b}{x^2} - a$$

and $$f_x = \frac{b}{x^2} + a$$

At $x = 200$ *mm.*, $p_x = 8$ $N/mm.^2$

and At $x = 300$ *mm.*, $p_x = 0$

$$\frac{b}{40000} - a = 8$$

and $$\frac{b}{90000} - a = 0$$

Solving the above equations, we get $b = 576{,}000$

and $a = 6.4$

Hence the hoop stress at any radius x is given by

$$f_x = \frac{576{,}000}{x^2} + 6.4$$

At $x = 200$ *mm.*, $$f_{200} = \frac{576000}{200 \times 200} + 6.4 = 20.8 \ N/mm^2$$

At $x = 300$ *mm.*, $$f_{300} = \frac{576000}{300 \times 300} + 6.4 = 1.28 \ N/mm^2$$

Problem 407. *Find the thickness of metal necessary for a steel cylindrical shell of internal diameter 15 cm. to withstand an internal pressure of 500 kg./cm². The maximum hoop stress in the section is not to exceed 1500 kg./cm².*

Solution. Let the external radius be r_1 *cm.*

Let the radial pressure and hoop stress at any radius x be given by

$$p_x = \frac{b}{x^2} - a$$

and $$f_x = \frac{b}{x^2} + a$$

At $x = 7.5$ *cm.*

$p_x = 500$ *kg./cm.*²

$$\therefore \quad 500 = \frac{b}{56.25} - a \quad \ldots(i)$$

At $x = 7.5$ *cm.*

$f_x = 1500$ *kg./cm.*²

$$\therefore \quad 1500 = \frac{b}{56.25} + a \quad \ldots(ii)$$

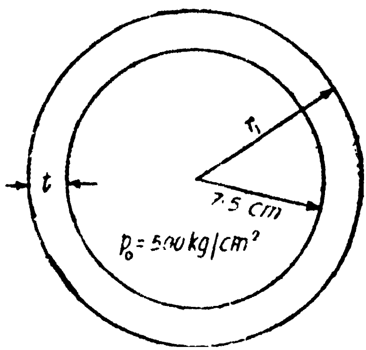


Fig. 595

Subtracting equation (i) from equation (ii)

$$1000 = 2a$$

$$\therefore \quad a = 500$$

Substituting in equation (i)

$$500 = \frac{b}{56.25} - 500$$

$\therefore \qquad \frac{b}{56{\cdot}25}=1000$

$\therefore \qquad b=56250$

We also know that at $x=r_1$, $p_x=0$

$\therefore \qquad 0=\frac{b}{r_1^2}-a$

$\therefore \qquad r_1^2=\frac{b}{a}=\frac{56250}{500}=112{\cdot}5$

$\therefore \qquad r_1=10{\cdot}60$ *cm.*

$\therefore$ Thickness of metal $=t=10{\cdot}60-7{\cdot}50=3{\cdot}1$ *cm.*

Problem 408. (S.I.) *Find the thickness of metal necessary for a steel cylindrical shell of internal diameter 150 mm. to withstand an internal pressure of 50 N/mm². The maximum hoop stress in the section is not to exceed 150 N/mm².*

Solution. Let the external radius be r_1 *mm.* Let the radial pressure and hoop stress at any radius x be given by,

$$p_x=\frac{b}{x^2}-a$$

and
$$f_x=\frac{b}{x^2}+a$$

At $x=75$ *mm.*, $p_x=50$ *N/mm.²*

$$50=\frac{b}{5625}-a \qquad ...(i)$$

At $x=75$ *mm.*, $f_x=150$ *N/mm.²*

$$150=\frac{b}{5625}+a \qquad ...(ii)$$

Solving equations (*i*) and (*ii*), we get,

$$a=50 \text{ and } b=562{,}500$$

We also know that at

$$x=r_1,\ p_x=0$$

$\therefore \qquad 0=\frac{b}{r_1^2}-a$

$\therefore \qquad r_1^2=\frac{b}{a}=\frac{562{,}500}{50}=11250$

$\therefore \qquad r_1=106$ *mm.*

$\therefore$ *Thickness of metal*

$$=t=r_1-r_2$$
$$=106-75=31 \text{ mm.}$$

§119. Compound Cylinders

In the examples shown above we find that when the cylindrical shell is subjected to internal pressure the hoop stress across the section is not uniform. The maximum hoop stress occurs at the inner circumference and the hoop stress decreases towards the outer circumference. Hence the maximum pressure inside the shell is limited corresponding to the condition that the hoop stress at the inner circumference reaches the permissible value.

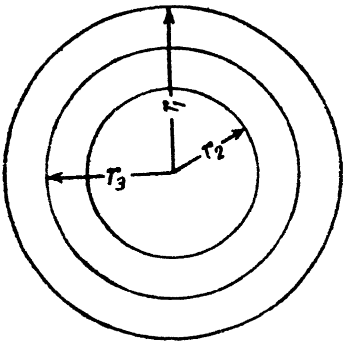


Fig. 596

But, suppose the shell is made by shrinking one tube over the other. This will initially introduce hoop compressive stresses in the inner tube and hoop tensile stresses in the outer tube.

If now the compound tube is subjected to internal perssure, both the inner and outer tubes will be subjected to hoop tensile stress, due to the internal pressure alone. Adding the internal stresses caused while shrinking and the stresses due to internal pressure alone, the final hoop stresses in both the tubes can be determined. By this arrangement the hoop stresses throughout the metal will be more or less uniform.

Let r_1 and r_2 be the outer and inner radii of the compound tube. Let the radius at the junction of the two tubes be r_3.

Let p_s be the radial pressure intensity at the junction of the two tubes after shrinking the outer tube over the inner tube.

Let Lamme's relation for the outer tube be given by,

$$p_x = \frac{b_1}{x^2} - a_1$$

and

$$f_x = \frac{b_1}{x^2} + a_1$$

At $x = r_1$, $p_x = 0$

$\therefore$
$$0 = \frac{b_1}{r_1^2} - a_1 \qquad \ldots(i)$$

and at $x = r_3$,

$\therefore$
$$p_s = \frac{b_1}{r_3^2} - a_1 \qquad \ldots(ii)$$

The constants a_1 and b_1 can be determined from equations (*i*) and (*ii*)

Let Lamme's relations for the inner tube be given by

$$p_x = \frac{b_2}{x^2} - a_2$$

and $$f_x = \frac{b_2}{x^2} + a_2$$

At $x = r_2$, $p_x = 0$

$$\therefore \quad 0 = \frac{b}{r_2^2} - a_2 \quad \ldots(iii)$$

and at $x = r_3$, $p_x = p_s$

$$\therefore \quad p_s = \frac{b_2}{r_3^2} - a_2 \quad \ldots(iv)$$

The constants a_2 and b_2 can be determined from equations (*iii*) and (*iv*).

Now the hoop stresses for the outer and the inner tube can be easily determined.

Suppose the compound tube is subjected to an internal fluid pressure p_o. For this analysis, the inner and the outer tubes will together be considered as one thick shell. The stresses due to internal fluid pressure alone can now be determined. For this condition let Lamme's relations be,

$$p_x = \frac{B}{x^2} - A$$

and, $$f_x = \frac{B}{x^2} + A$$

At $x = r_1$, $p_x = 0$

$$\therefore \quad 0 = \frac{B}{r_1^2} - A \quad \ldots(v)$$

At $x = r_2$, $p_x = p_o$

$$\therefore \quad p_o = \frac{B}{r_2^2} - A \quad \ldots(vi)$$

The constants A and B can now be evaluated. The hoop stresses across the section can now be easily determined.

By algebraically adding, the hoop stresses caused due to shrinking to the hoop stresses caused by internal fluid pressure, the final hoop stresses may be determined.

Problem 409. *A compound tube is composed of a tube 25 cm. internal diameter and 2 5 cm. thick shrunk on a tube of 25 cm. external diameter and 2·5 cm. thick. The radial pressure at the junction is 80 kg./cm². The compound tube is subjected to an interal fluid pressure of 845 kg./cm². Find the variation of the hoop stress over the wall of the compound tube.*

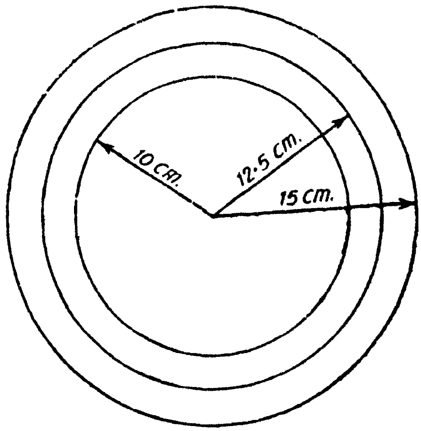


Fig. 597

Solution. Stress due to shrinking the outer tube on the inner tube.

Outer tube

Let Lamme's relations for the outer tube be given by

$$p_x = \frac{b_1}{x^2} - a_1$$

and $$f_x = \frac{b_1}{x^2} + a_1$$

At $x = 15$ cm., $p_x = 0$

$\therefore$ $$0 = \frac{b_1}{225} - a_1 \quad \ldots(i)$$

At $x = 12·5$ cm., $p_x = 80$ kg./cm.

$\therefore$ $$80 = \frac{b_1}{156·25} - a_1 \quad \ldots(ii)$$

Solving equations (*i*) and (*ii*), we get

$$a_1 = 181·8 \text{ and } b_1 = 40910$$

Hoop stresses for the outer tube are given by,

$$f_{12\,5} = \frac{40910}{156·25} + 181·8 = 443·6 \text{ kg/cm.}^2 \text{ (tensile)}$$

$$f_{15} = \frac{40910}{225} + 181·8 = 363·6 \text{ kg./cm.}^2 \text{ (tensile)}$$

Inner tube

Let Lamme's relations for the inner tube be given by

$$p_x = \frac{b_2}{x^2} - a_2$$

and $$f_x = \frac{b_2}{x^2} + a_2$$

At $x = 12.5$ cm , $p_x = 80$ kg./cm.²

$$80 = \frac{b_2}{156.25} + a_2 \quad \ldots(iii)$$

At $x = 10$ cm., $p_x = 0$

$$0 = \frac{b_2}{100} - a_2 \quad \ldots(iv)$$

Solving equations (*iii*) and (*iv*) we get $a_2 = -222$ and

$b_2 = -22220$

Hence the hoop stresses for the inner **tube are given by,**

$$f_{12.5} = -\frac{22220}{156.25} - 222$$

$$= -364.2 \text{ kg./cm.}^2 \text{ (compressive)}$$

$$f_{10} = -\frac{22220}{100} - 222$$

$$= -444.2 \text{ kg./cm.}^2 \text{ (compressive)}$$

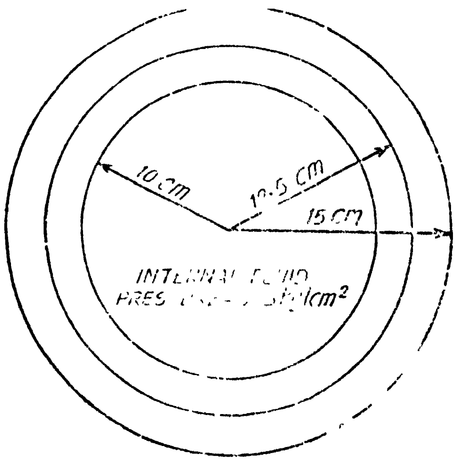


Fig. 598

Stresses due to internal fluid pressure alone

For this condition both the tubes together will be considered as acting as one cylinder.

Let Lamme's relations for this condition be

$$p_x = \frac{B}{x^2} - A$$

and

$$f_x = \frac{B}{x^2} + A$$

At $x = 10$ cm., $p_x = 845$ kg./cm.²

$$\therefore \qquad 845 = \frac{B}{100} - A \qquad \ldots(v)$$

At $\qquad x = 15\ cm,\ p_x = 0$

$$\therefore \qquad 0 = \frac{B}{225} - A \qquad \ldots(vi)$$

Solving equations (*v*) and (*vi*), we get,

$$A = 676{\cdot}1 \text{ and } B = 152200$$

Hence the hoop stresses due to internal fluid pressure alone are given by,

$$f_{10} = \frac{152200}{100} + 676{\cdot}1 = 2198{\cdot}1\ kg./cm.^2\ (tensile)$$

$$f_{12{\cdot}5} = \frac{152200}{156{\cdot}25} + 676{\cdot}1 = 1650{\cdot}1\ kg./cm.^2\ (tensile)$$

$$f_{15} = \frac{152200}{225} + 676{\cdot}1 = 1352{\cdot}2\ kg./cm.^2\ (tensile)$$

Hence due to the combined effect of shrinking the outer tube on the inner tube and internal fluid pressure the final hoop stresses will be as follows :

Outer tube

$$F_{12{\cdot}5} = 443{\cdot}6 + 1650{\cdot}1 = 2093{\cdot}7\ kg./cm.^2\ (tensile)$$
$$F_{15} = 363{\cdot}6 + 1352{\cdot}2 = 1715{\cdot}8\ kg./cm.^2\ (tensile)$$

Inner tube

$$F_{12{\cdot}5} = -364{\cdot}2 + 1650{\cdot}1 = 1285{\cdot}9\ kg./cm.^2\ (tensile)$$
$$F_{10} = -444{\cdot}2 + 2198{\cdot}1 = 1753{\cdot}9\ kg./cm.^2\ (tensile)$$

§120. Initial difference in radii at the junction of a compound cylinder

In order to shrink on the outer tube over the inner tube, it is necessary that the radius of the inner face of the outer tube should be slightly less than the radius of the outer face of the inner tube, so that after the tubes are fitted the required radial pressure at the junction of the tubes may be reached.

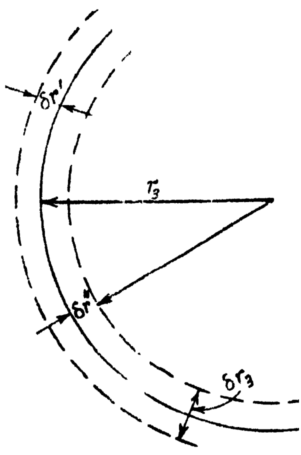


Fig. 599

Let r_3 = radius of junction after shrinking on.

$\delta r'$ = difference between the radius of the outer face of the inner tube and r_3.

$\delta r''$ = difference between the radius of the inner face of the outer tube and r_3.

$\therefore$ Original difference of the radii of the two tubes at the junction

$$=\delta r_3=\delta r'+\delta r''$$

Now, for the outer tube let Lamme's relations be

$$p_x=\frac{b_1}{x^2}-a_1$$

and $$f_x=\frac{b_1}{x^2}+a_1$$

$\therefore$ Circumferential tensile strain for the outer tube at the junction

$$=\frac{\delta r''}{r_3}=\frac{1}{E}\left[\left(\frac{b_1}{r_3^2}+a_1\right)+\frac{p'}{m}\right] \qquad ...(i)$$

where p'=radial pressure at the junction.

and $\frac{1}{m}$=Poisson's ratio.

Similarly, for the inner tube, let Lamme's relations be

$$p_x=\frac{b_2}{x^2}-a_2$$

and $$f_x=\frac{b_2}{x^2}+a_2$$

$\therefore$ Circumferential compressive strain for the inner tube at the junction

$$=-\frac{\delta r'}{r_3}=\frac{1}{E}\left[\left(\frac{b_2}{r_3^2}+a_2\right)+\frac{p'}{m}\right] \qquad ...(ii)$$

Subtracting equation (*ii*) from equation (*i*), we get

$$\frac{\delta r'+\delta r''}{r_3}=\frac{\delta r_3}{r_3}=\frac{1}{E}\left[\left(\frac{b_1}{r_3^2}+a_1\right)-\left(\frac{b_2}{r_3^2}+a_2\right)\right] \qquad ...(iii)$$

$$\therefore \quad \frac{\delta r_3}{r_3}=\frac{\text{Algebraic difference between the hoop stresses in the tubes at the junction}}{E}$$

Problem 410. *For the compound tube of problem 409 find the original difference in diameter of the two tubes before shrinking-on, so that after shrinking on the radial pressure at the junction may be 80 kg./cm². Find also the minimum temperature to which the outer tube should be heated in order it can be slipped on the inner tube. Take* $E=2\times10^6$ *kg./cm.²*

Solution. Due to the radial pressure of 80 *kg./cm.*² at junction the hoop stresses for the two tubes at the junction are as follows:

Outer tube: hoop stress at the junction

$$=+443{\cdot}6 \textit{ kg./cm.}^2 \textit{ (tensile)}$$

Inner tube : hoop stress at the junction

$$= -364{\cdot}2 \; kg./cm.^2 \; (compressive)$$

$$\therefore \quad \frac{\delta r_3}{r_3} = \frac{443{\cdot}6 + 364{\cdot}2}{2 \times 10^6} = \frac{807{\cdot}8}{2 \times 10^6}$$

$$= 0{\cdot}0004039$$

$$\therefore \quad \delta r_3 = 0{\cdot}0004039 \times 12{\cdot}5 = 0{\cdot}005 \; cm.$$

$\therefore$ original difference in diameter $= 2 \times 0{\cdot}005 = 0{\cdot}010$ *cm.*

Let T be the temperature by which the outer tube is to be heated in order to slip it on the inner tube.

$$\therefore \quad \alpha T(\pi d) = \pi \delta d$$

$$T = \frac{\delta d}{\alpha d}$$

$$= \frac{0{\cdot}010}{(12 \times 10^{-6})25}$$

$$= 33^\circ{\cdot}34C.$$

Problem 411. *A steel tube of 20 cm. external diameter is to be shrunk on to another steel tube of 6 cm. internal diameter. After shrinking the diameter at the junction is 12 cm. Before shrinking on the difference of diameter at the junction is 0·008 cm. Find the hoop stresses developed in the two tubes after shrinking-on, and the radial pressure at the junction. Take* $E = 2 \times 10^6$ *kg./cm².*

Solution. Let Lamme's relations for the outer tube be given by

$$p_x = \frac{b_1}{x^2} - a_1$$

and $$f_x = \frac{b_1}{x^2} + a_1$$

Let p_j be the radial pressure at the junction.

At $x = 10$ *cm.*, $p_x = 0$

$$\therefore \quad 0 = \frac{b}{100} - a_1 \qquad ...(i)$$

At $x = 6$ *cm.*, $p_x = p_j$

$$\therefore \quad p_j = \frac{b^2}{36} - a_1 \qquad ..(ii)$$

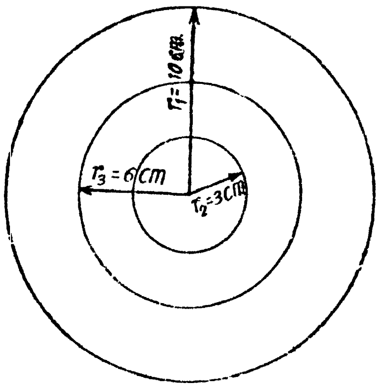


Fig. 600

Now let Lamme's relation for the inner tube be given by,

$$p_r = \frac{b_2}{x^2} - a_2$$

and $$f_x = \frac{b_2}{x^2} + a_2$$

At $x=3$ cm., $p_x=0$ $\quad\therefore\quad 0=\frac{b_2}{9}-a_2$ ...(iii)

At $x=6$ cm., $p_x=p_j$ $\quad\therefore\quad p_j=\frac{b_2}{36}-a_2$ ...(iv)

From equations (ii) and (iv), we get,

$$p_j=\frac{b_1}{36}-a_1=\frac{b_2}{36}-a_2 \quad ...(v)$$

Finally,

$$\frac{\delta r_3}{r_3}=\frac{0{\cdot}008}{6}=\frac{1}{E}\left[\left(\frac{b_1}{r_3^2}+a_1\right)-\left(\frac{b_2}{r_3^2}\right)+a_2\right] \quad ...(vi)$$

From equations (i) and (iii), we have.

$$b_1=100a_1$$

and $$b_2=9a_2$$

Substituting for b_1 and b_2 in equations (v) and (vi)

$$\frac{100a_1}{63}-a_1=\frac{9a_2}{36}-a_2$$

$$\therefore \quad a_1=-\frac{27}{64}a_2 \quad ...(vii)$$

and $$\left(\frac{100a_1}{36}+a_1\right)-\left(\frac{9a_2}{36}+a_2\right)=\frac{0{\cdot}008}{6}\times 2\times 10^6$$

$$\frac{34}{9}a_1-\frac{5}{4}a_2=\frac{0{\cdot}008}{3}\times 10^6$$

$$\frac{34}{9}a_1-\frac{5}{4}a_2=\frac{8000}{3} \quad ...(viii)$$

Solving equations (vii) and (viii), we get

$$a_2=-937{\cdot}8 \text{ and } a_1=395{\cdot}7$$

$$\therefore \quad b_2=-937{\cdot}8\times 9$$

$$=-8440{\cdot}2$$

and $$b_1=395{\cdot}7\times 100$$

$$=39570$$

Hence, Lamme's relations for the two tubes are as follows.

Outer tube

$$p_x=\frac{39570}{x^2}-395{\cdot}7$$

and $$f_x=\frac{39570}{x^2}+395{\cdot}7$$

Inner tube

$$p_x=-\frac{8440{\cdot}2}{x^2}+937{\cdot}8$$

and $$f_x = -\frac{8440 \cdot 2}{x^2} - 937 \cdot 8$$

At the junction, *i.e.*, at $x=6$ *cm.* be radial pressure

$$= p_1 = \frac{39570}{36} - 395 \cdot 7$$

or alternatively $$-\frac{8440 \cdot 2}{36} + 937 \cdot 8$$

$$= 703 \cdot 4 \text{ kg./cm.}^2$$

Now the hoop stresses can be calculated as follows

Outer tube

$$f_{10} = \frac{39570}{100} + 395 \cdot 7 = 791 \cdot 4 \text{ kg./cm.}^2 \text{ (tensile)}$$

$$f_6 = \frac{39570}{36} + 395 \cdot 7 = 1494 \cdot 9 \text{ kg./cm.}^2 \text{ (tensile)}$$

Inner tube

$$f_6 = -\frac{8440 \cdot 2}{36} - 937 \cdot 8 = -1172 \cdot 2 \text{ kg./cm.}^2 \text{ (compressive)}$$

$$f_3 = -\frac{8440 \cdot 2}{9} - 937 \cdot 8 = -1877 \cdot 6 \text{ kg./cm.}^2 \text{ (compressive)}$$

§121. Thick Spherical Shells

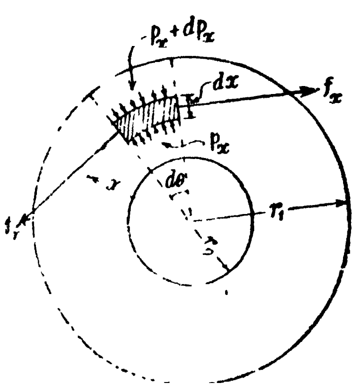


Fig. 601

Consider a spherical shell of external radius r_1 and internal radius r_2 subjected to an internal fluid pressure p.

Consider an elemental disc of the spherical shell at a radius x, having a thickness dx. Let this elemental part subtend an angle $d\theta$ at the centre.

Consequent to the internal pressure, let the radius increase from x to $(x+u)$

Circumferential strain

$$e_\theta = \frac{(x+u)d\theta - xd\theta}{xd\theta}$$

$$\therefore \quad e_\theta = \frac{u}{x} \qquad \text{...(i)}$$

The radial strain $= e_x = \dfrac{d(x+u) - dx}{dx}$

$$\therefore \qquad e_s = \frac{du}{dx} \qquad ...(ii)$$

The above strains are tensile if positive.

Now consider an element spherical shell at the radius x having a thickness dx Let the radial pressure at the radii x and $(x+dx)$ be P_x and (P_x+dP_x) respectively.

Let f_x be the circumferential stress which is equal in all directions since this is a spherical shell.

The bursting force on any diametral plane on the section of the elemental shell

$$= P_x \pi x^2 - (P_x + dP_x)\pi(x+dx)^2$$

Resisting force

$$= f_x \times 2\pi x\, dx$$

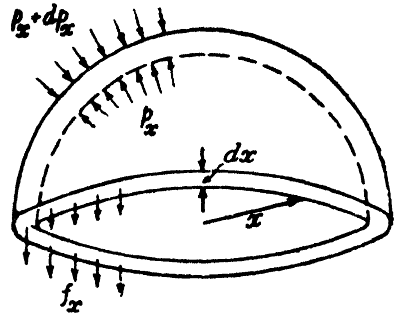


Fig. 602

Equating the resisting force and bursting force, we have,

$$f_x(2\pi x dx) = P_x \pi x^2 - (P_x + dP_x)\pi(x+dx)^2$$

Neglecting squares and products of very small quantities,

$$2f_x = -2p_x - x\frac{dP_x}{dx} \qquad ..(iii)$$

At any point at a radius x the three principal stresses are

(*i*) The radial pressure P_x (*compressive*)

(*ii*) The hoop stress f_x (*tensile*)

(*iii*) The hoop stress f_x on a plane at right angles (*tensile*)

$\therefore$ Radial strain

$$e_x = \frac{P_x}{E} + \frac{2f_x}{mE}$$

(*compressive*)

$$\therefore \quad e_x = -\frac{1}{E}\left(p_x + \frac{2f_x}{m}\right)$$

(*tensile*) ...(*iv*)

Circumferential strain

$$= e_y = \frac{f_x}{E} - \frac{f_x}{mE} + \frac{p_x}{mE}$$

(*tensile*)

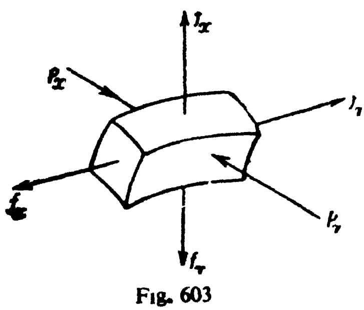


Fig. 603

$$\therefore \qquad e_y = \frac{1}{E}\left(\frac{m-1}{m}f_x + \frac{P_x}{m}\right) \text{ (tensile)}$$

From equations (*i*) and (*ii*)

$$\frac{du}{dx} = \frac{d}{dx}\left(x\, e_y\right)$$

$$\therefore \qquad e_x = e_y + x\frac{de_y}{dx} \qquad \text{...(vi)}$$

Substituting for e_x and e_y from equations (*iv*) and (*v*) in equation (*vi*), we have,

$$-\frac{1}{E}\left(p_x+\frac{2f_x}{m}\right)=\frac{1}{E}\left(\frac{m-1}{m}f_x+\frac{p_x}{m}\right)+\frac{x}{E}\left[\left(\frac{m-1}{m}\right)\frac{df_x}{dx}+\frac{1}{m}\frac{dp_x}{dx}\right]$$

on simplification,

$$\left(p_x+f_x\right)\left(m+1\right)-\left(m-1\right)x\frac{pf_x}{dx}+x\frac{dp_x}{dx}=0 \qquad \text{...(vii)}$$

But from equation (*iii*)

$$f_x=-p_x-\tfrac{1}{2}\,\frac{dp_x}{dx}$$

Differentiating the above relation,

$$\frac{df_x}{dx}\quad-\frac{dp_x}{dx}-\tfrac{1}{2}\left[x\,\frac{d^2p_x}{dx^2}+\frac{dp_x}{dx}\right]$$

Substituting for f_x and $\frac{df_x}{dx}$ in equation (*vii*) and on simplification

we get,

$$x\,\frac{d^2p_x}{dx^2}+4\,\frac{dp_x}{dx}=0 \qquad \text{.(viii)}$$

putting $\qquad \frac{dp_x}{dx}=v,$

we get,

$$x\,.\,\frac{dv}{dx}+4v=0$$

$$\therefore \qquad \frac{dv}{v}=-4\frac{dx}{x}$$

Integrating, we get

$$\log_e v=-4\log_e x+\log_e C_1$$

where $\log_e C_1$ is a constant of integration.

$$\therefore \qquad \log_e v=\log_e\left(\frac{C_1}{x^4}\right)$$

$$\therefore \qquad v=\frac{C_1}{x^4}$$

$$\therefore \qquad \frac{dp_x}{dx}=\frac{C_1}{x^4}$$

$$\therefore \qquad dp_x = C_1 \frac{dx}{x^4}$$

Integrating we get,

$$p_x = -\frac{C_1}{3x^3} + C_2$$

where C_2 is a constant of integration.

But we know

$$f_x = p_x - \frac{1}{2}\, x \frac{dp_x}{dx}$$

$$\therefore \qquad f_x = \frac{C_1}{3x^3} - C_2 - \frac{C_1}{2x^3}$$

$$\therefore \qquad f_x = -\frac{C_1}{6x^3} - C_2$$

Now consider the two expressions obtained for p_x and f_x namely

$$p_x = -\frac{C_1}{3x^3} + C_2$$

and

$$f_x = -\frac{C_1}{6x^3} - C_2$$

putting $C_1 = -6b$ and $C_2 = -a$

we get,

$$p_x = \frac{2b}{x^3} - a \qquad \ldots(ix)$$

and

$$f_x = \frac{b}{x^3} + a \qquad \ldots(x)$$

The above two relations will be found very useful. From the known initial conditions the constants a and b can be evaluated.

For instance corresponding to the external radius.

$$x = r_1 \quad \text{and} \quad p_x = 0$$

$$0 = \frac{2b}{r_1^3} - a \qquad \ldots(A)$$

But at $x = r_2$ (internal radius) $p_x = p$

$$\therefore \qquad p = \frac{2b}{r_2^3} - a \qquad \ldots(B)$$

Solving equations (A) and (B)

we get,

$$a = \frac{pr_2^3}{r_1^3 - r_2^3}$$

and

$$b = \frac{pr_1^3 r_2^3}{2(r_1^3 - r_2^3)}$$

Problem 412. *A thick spherical shell of 10 cm. internal diameter is subjected to an internal fluid pressure of 300 kg./cm². If the permissible tensile stress is 800 kg./cm², find the thickness of the shell.*

Solution. Let the radial pressure and the hoop stress at any radius be given by,

$$p_x=\frac{2b}{x^3}-a$$

$$\therefore \quad f_x=\frac{b}{x^3}+a$$

At $x=5$ *cm.*, $p_x=300$ *kg./cm.*^{2} ...(*i*)

$$\therefore \quad 300=\frac{2b}{125}-a$$

At $x=5$ *cm.*, $f_x=800$ *kg /cm.*^{2}

$$\therefore \quad 800=\frac{b}{125}+a \qquad ...(ii)$$

Solving equations (*i*) and (*ii*),

we get,

$$a=\frac{1300}{3}$$

and
$$b=\frac{137500}{3}$$

Let the external radius be r_1

$\therefore$ At $x=r_1$, $p_x=0$

$$\therefore \quad 0=\frac{2\times 137500}{3r_1^3}-\frac{1300}{3}$$

$$\therefore \quad r_1=5{\cdot}96 \text{ cm.}$$

$\therefore$ *Thickness of the shell*$=5{\cdot}96-5=0{\cdot}96$ *cm.*

Problem 413 (SI). *A thick spherical shell of 100 mm. internal diameter is subjected to an internal fluid pressure of 30 N/mm². If the permissible tensile stress is 80 N/mm.², find* the *thickness of the shell.*

Solution. Let the radial pressure and the hoop stress at any radius be given by,

$$p_x=\frac{2b}{x^3}-a$$

and
$$f_x=\frac{b}{x^3}+a$$

At $x=50$ *mm.*, $p_x=30$ *N/mm*^{2}

$$\therefore \quad 30=\frac{2b}{125{,}000}-a \qquad ...(i)$$

At $x=50$ *mm.*, $f_x=80$ *N/mm.*^{2}

$$80=\frac{b}{125{,}000}+a \qquad ...(ii)$$

Solving equations (*i*) and (*ii*), we get,

$$a=\frac{130}{3}$$

and $$b=\frac{13750000}{3}$$

Let the external radius be r_1

At $x=r_1$, $p_x=0$

$$\therefore \quad 0=\frac{2\times 13750000}{3r_1^3}-\frac{130}{3}$$

$$r_1^3=\frac{2\times 13750000}{130}$$

$$\therefore \quad r_1^3=211538{\cdot}46$$

$$\therefore \quad r_1=59{\cdot}6 \text{ mm.}$$

$\therefore$ *Thickness of the shell*

$$=r_1-r_2$$
$$=59{\cdot}6-50$$
$$=9{\cdot}6 \text{ mm.}$$

Examples in chapter 13

1. A pipe of 50 *cm*. internal diameter and 10 *cm*. thickness contains a fluid at a pressure of 60 $kg./cm^2$. Find the maximum and minimum hoop stresses across the section. Also sketch the radial pressure distribution across this section.
(Hoop stresses : 152·4 $kg./cm.^2$, 59·8 $kg/cm.^2$)

2. Find the thickness of metal necessary for a steel cylindrical shell of internal diameter 20 *cm* to withstand an internal pressure of 400 $kg./cm^2$. The maximum hoop stress in the section is not to exceed 1500 $kg./cm^2$. (3·15 *cm.*)

3. A compound tube is composed of a tube 20 *cm*. internal diameter and 2 *cm*. thick shrunk on a tube of 20 *cm*. external diameter and 2 *cm*. thick. The radial pressure at the junction is 60 $kg./cm^2$. The compound tube is subjected to an internal pressure of 800 $kg./cm^2$. Find the variation of the hoop stress over the wall of the compound tube.

Initial stresses : *Inner tube* $f_8=333{\cdot}3$ $kg/cm.^2$ (*compressive*)
$f_{10}=273{\cdot}3$ $kg./cm.^2$ (*compressive*)
outer tube $f_{10}=332{\cdot}7$ $kg./cm.^2$ (*tensile*)
$f_{12}=222{\cdot}3$ $kg/cm.^2$ (*tensile*)

Final stresses : *Inner tube* $f_8=1721{\cdot}7$ $kg./cm.^2$ *tensile*)
$f_{10}=1287{\cdot}7$ $kg./cm.^2$ (*tensile*)
outer tube $f_{10}=1893{\cdot}7$ $kg./cm^2$ *tensile*)
$f_{12}=1602{\cdot}3$ $kg./cm.^2$ (*tensile*)

4. A thick spherical shell of 16 *cm*. internal diameter is subjected to an internal fluid pressure of 400 $kg./cm^2$. If the permissible tensile stress is 800 $kg./cm.^2$ find the thickness of the shell. (2·1 *cm.*)

14

Columns and Struts

Introduction

We come across various instances of members subjected to compressive loads. These members are given different names depending on the particular situation in which they are placed.

Columns and *stanchions* are vertical members used in building frames.

A *post* is a general term applied to a compression member.

A *strut* is a compression member of a truss.

A *boom* is the principal compression member in a crane.

§122. Axially loaded compression members

Fig. 604 shows a short column of uniform sectional area A, subjected to an axial load P. The stress intensity p induced on the section of the column is obviously

$$p = \frac{P}{A}$$

Let the load on the member be gradually increased till the member fails by crushing. Let P_c be the crushing load. Let f_c be the ultimate crushing stress.

$\therefore$ Crushing load $= P_c = f_c A$.

A very short column will fail at the above load by crushing. By dividing the crushing load by a suitable factor of safety the safe load for the member can be computed.

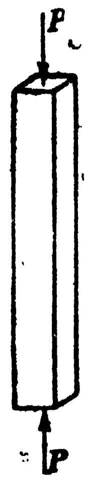


Fig. 604

But the compression members which we come across do not fail entirely by crushing. These members are considerably long in comparison with their lateral dimensions. Hence, these members start bending, *i.e.*, buckling when the axial load reaches a certain critical value. Once a member shows signs of buckling it will lead to the failure of the member. This load at which the member just buckles is called the *buckling load* or *critical load* or *crippling load*. The buckling load is less than the crushing load. The value of the buckling load is low for long members and relatively high for short members. The value of the buckling load for a given member depends upon the length of the member and the least lateral dimension. When an axially loaded compression member just buckles, it is said to develop an elastic instability.

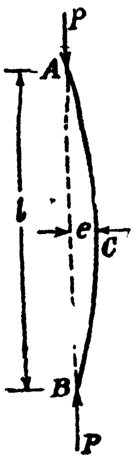


Fig. 605

Fig. 605 shows a column loaded axially, which has just buckled, due to the applied load P. Let l be the length of the column and A the sectional area. The maximum lateral deflection e has occurred at the middle point of the column.

The extreme stresses on the mid section are given by

$$\frac{P}{A} \pm \frac{Pe}{Z} = p_o \pm p_b$$

where Z = section modulus of the section about the axis of bending

p_o = stress due to direct load

p_b = extreme stress due to bending.

$\therefore \quad p_{max} = p_o + p_b$

and $p_{min} = p_o - p_b$

When p_{max} reaches the crushing stress f_c for the column material the member will fail. Thus it should be realized that the failure of the member is not caused by p_o alone but due to the combined effect of p_o and p_b.

§123. Euler's Theory of long columns

We will now discuss the case of very long columns having low crippling loads. In this case the stress p_o due to direct load is very small in comparison with the stress p_b due to buckling. We will therefore regard that the failures of these members are entirely due to bending. The following four cases arise :

Case 1. *When both ends of the member are pinned.*
Case 2. *When one end is fixed and the other is free.*
Case 3. *When both ends are fixed.*
Case 4. *When one end fixed and the other is pinned.*

Sign conventions for bending moments.

A bending moment which bends the column so as to present convexity towards the initial centre line of the member will be regarded as positive.

In Fig. 606, BA represents the initial centre line of a member. Whether the member bends taking the shape BA' or BA'' the bending moment producing this type of curvature is positive.

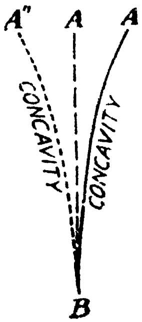


Fig. 606

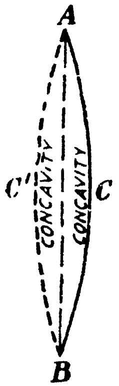


Fig. 607

In Fig. 607, BA represents the initial centre line of a column. If the member bends taking the shape ACB or $AC'B$ presenting concavity towards the initial centre line of the member, the bending moment producing this type of curvature is regarded as negative.

Case 1. *When both ends of the column are pinned or hinged.*

Fig. 608 shows a column AB of length l and uniform sectional area A, hinged at both the ends A and B. Let P be the crippling load at which the column has just buckled.

Consider any section at a distance x from the end B. Let y be the deflection (lateral displacement) at the section.

The bending moment at the section is given by,

$$EI\frac{d^2y}{dx^2} = -Py$$

$$\therefore \quad EI\frac{d^2y}{dy^2}+Py=0$$

$$\therefore \quad \frac{d^2y}{dx^2}+\frac{P}{EI}y=0$$

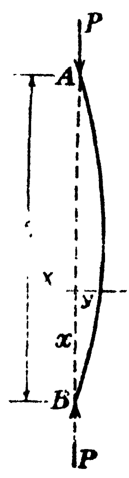


Fig. 608

The solution to the above differential equation is

$$y=C_1 \cos\left(x\sqrt{\frac{P}{EI}}\right)+C_2 \sin\left(x\sqrt{\frac{P}{EI}}\right)$$

where C_1 and C_2 are constants of integration.

At B, the deflection is zero.

$\therefore$ At $x=0, \quad y=0$

$\therefore \quad C_1=0$

At A also, the deflection is zero.

i.e. at $x=l, \quad y=0$

$$\therefore \quad 0=C_2 \sin\left(l\sqrt{\frac{P}{EI}}\right)$$

Since $C_1=0$, we conclude that C_2 cannot be zero.

This is because if both C_1 and C_2 are zero the column will not bend at all.

$$\text{Hence} \quad \sin\left(l\sqrt{\frac{P}{EI}}\right)=0$$

$$\therefore \quad l\sqrt{\frac{P}{EI}}=0, \pi, 2\pi, 3\pi, 4\pi, \ldots \ldots$$

Considering the least practical value,

$$l\sqrt{\frac{P}{EI}}=\pi$$

$$\therefore \quad P=\frac{\pi^2 EI}{l^2}$$

Case 2. *When one end is fixed and the other is free.*

Fig. 609 shows a column AB of length l whose lower end B is fixed, the upper end A being free. Let due to the crippling load P the column just buckle. Let a be the deflection at the top end.

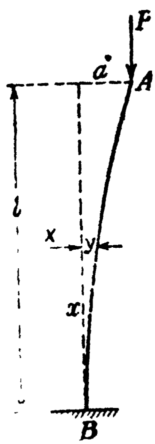


Fig. 609

At any section distant x from the fixed end B, the bending moment is given by

$$EI\frac{d^2y}{dx^2}=P(a-y)$$

where y is the deflection at X.

$$\therefore \quad EI\frac{d^2y}{dx^2}+Py=Pa$$

$$\therefore \quad \frac{d^2y}{dx^2}+\frac{P}{EI}y=\frac{Pa}{EI}$$

The solution to the above differential equation is

$$y=C_1 \cos\left(x\sqrt{\frac{P}{EI}}\right)+C_2 \sin\left(x\sqrt{\frac{P}{EI}}\right)+a$$

where C_1 and C_2 are constants of integration.

At B, the deflection is zero.

$\therefore$ At $x=0$, $y=0$

$\therefore$ $0=C_1+a$ $\quad\therefore\quad C_1=-a$

The slope at any section is given by

$$\frac{dy}{dx}=-C_1\sqrt{\frac{P}{EI}}\sin\left(x\sqrt{\frac{P}{EI}}\right)+C_2\sqrt{\frac{P}{EI}}\cos\left(x\sqrt{\frac{P}{EI}}\right)$$

At B the slope is zero,

$\therefore$ At $x=0$, $\frac{dy}{dx}=0$

$$\therefore\quad 0=C_2\sqrt{\frac{P}{EI}}$$

$\therefore$ $C_2=0$

At A the deflection is a

$\therefore$ At $x=l$, $y=a$

$$\therefore\quad a=-a\cos\left(l\sqrt{\frac{P}{EI}}\right)+a$$

$$\therefore\quad \cos\left(l\sqrt{\frac{P}{EI}}\right)=0$$

$$\therefore\quad l\sqrt{\frac{P}{EI}}=\frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2},\ldots\ldots$$

Considering the first practical value,

$$l\sqrt{\frac{P}{EI}}=\frac{\pi}{2}$$

$$\therefore\quad P=\frac{\pi^2 EI}{4l^2}$$

Case 3. *When both ends of the column are fixed.*

Fig. 610 shows a column AB of length l whose ends A and B are both fixed. Obviously there will be a restraint moment say M_o at each end. Let P be the crippling load.

Consider any section X distant x from the lower end B. The bending moment at the section X, is given by,

$$EI\frac{d^2y}{dx^2}=M_0-Py$$

$$\therefore \quad EI\frac{d^2y}{dx^2}+Py=M_0$$

$$\therefore \quad \frac{d^2y}{dx^2}+\frac{P}{EI}\,y=\frac{M_0}{EI}$$

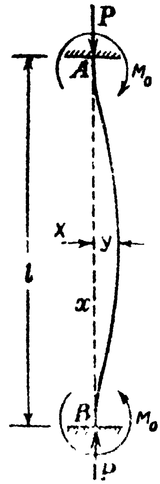


Fig. 610

The solution to the above differential equation is,

$$y=C_1\cos\left(x\sqrt{\frac{P}{EI}}\right)+C_2\sin\left(x\sqrt{\frac{P}{EI}}\right)+\frac{M_0}{P} \quad \ldots(i)$$

where, C_1 and C_2 are constants of integration. The slope at any section is given by

$$\frac{dy}{dx}=-C_1\sqrt{\frac{P}{EI}}\sin\left(x\sqrt{\frac{P}{EI}}\right)+C_2\sqrt{\frac{P}{EI}}\cos\left(x\sqrt{\frac{P}{EI}}\right) \quad \ldots(ii)$$

At B, the deflection is zero.

$\therefore$ At $x=0$, $y=0$

$$\therefore \quad 0=C_1+\frac{M_0}{P} \qquad \therefore \quad C_1=-\frac{M_0}{P}$$

At B, the slope is zero.

$\therefore$ At $x=0$, $\frac{dy}{dx}=0$

$$\therefore \quad 0=C_2\sqrt{\frac{P}{EI}} \qquad \therefore \quad C_2=0$$

At A, the deflection is zero.

$\therefore$ At $x=l$, $y=0$

$$\therefore \quad 0=-\frac{M_0}{P}\cos\left(l\sqrt{\frac{P}{EI}}\right)+\frac{M_0}{P}$$

$$\therefore \quad \frac{M_0}{P}\left[1-\cos\left(l\sqrt{\frac{P}{EI}}\right)\right]=0$$

$$\therefore \quad \cos\left(l\sqrt{\frac{P}{EI}}\right)=1$$

$$\therefore \quad l\sqrt{\frac{P}{EI}}=0,\ 2\pi,\ 4\pi,\ 6\pi,\ldots\ldots$$

Considering the first practical value,

$$l\sqrt{\frac{P}{EI}}=2\pi$$

$$\therefore \qquad P=\frac{4\pi^2 EI}{l^2}$$

Case 4. *When one end of the column is fixed and the other end pinned or hinged*

Fig. 611 shows a column *AB* of length *l*, whose upper end *A* is hinged while its lower end *B* is fixed.

Let *P* be the crippling load. Studying the nature of bending we realize that there will be a restraint moment M_0 at the lower fixed end.

The existence of the restraint moment therefore justifies the need for a horizontal force also at the top end *A* without which no bending moment can occur at *B*. Hence the hinge at *A* must exert a horizontal force *H* at *A*.

Consider any section *X* at a distance *x* from the lower fixed end *B*.

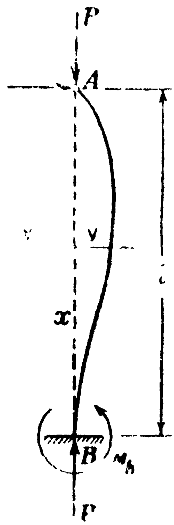


Fig. 611

The bending moment at the section is given by,

$$EI\frac{d^2y}{dx^2}=-Py+H(l-x)$$

$$\therefore \qquad EI\frac{d^2y}{dx^2}+Py=H(l-x)$$

$$\therefore \qquad \frac{d^2y}{dx^2}+\frac{P}{EI}\,y=\frac{H}{EI}(l-x)$$

The solution to the above differential equation is,

$$y=C_1\cos\left(x\sqrt{\frac{P}{EI}}\right)+C_2\sin\left(x\sqrt{\frac{P}{EI}}\right)+\frac{H}{P}(l-x) \qquad \ldots(i)$$

where C_1 and C_2 are constants of integration. The slope at any section is given by,

$$\frac{dy}{dx}=-C_1\sqrt{\frac{P}{EI}}\sin\left(x\sqrt{\frac{P}{EI}}\right)+C_2\sqrt{\frac{P}{EI}}\cos\left(x\sqrt{\frac{P}{EI}}\right)-\frac{H}{P} \qquad \ldots(ii)$$

At *B*, the deflection is zero.

$\therefore$ At $x=0, y=0$

$$\therefore \quad 0=C_1+\frac{H}{P}l \quad \therefore \quad C_1=-\frac{H}{P}l$$

At B, the slope is zero.

$$\therefore \quad \text{At } x=0, \frac{dy}{dx}=0$$

$$\therefore \quad 0=C_2\sqrt{\frac{P}{EI}}-\frac{H}{P} \quad \therefore \quad C_2=\frac{H}{P}\sqrt{\frac{EI}{P}}$$

At A, the deflection is zero.

$\therefore$ At $x=l, y=0$

$$\therefore \quad 0=-\frac{H}{P}l\cos\left(l\sqrt{\frac{P}{EI}}\right)+\frac{H}{P}\sqrt{\frac{EI}{P}}\sin\left(l\sqrt{\frac{P}{EI}}\right)$$

Simplifying, we get

$$\tan\left(l\sqrt{\frac{P}{EI}}\right)=\left(l\sqrt{\frac{P}{EI}}\right)$$

The solution to this equation is

$$l\sqrt{\frac{P}{EI}}=4\cdot5 \textit{ radians}$$

$$\therefore \quad \frac{l^2P}{EI}=(4\cdot5)^2=20\cdot25$$

$$\therefore \quad P=\frac{20\cdot25\,EI}{l^2}$$

Approximately $20\cdot25=2\pi^2$

$$\therefore \quad P=\frac{2\pi^2 EI}{l^2}$$

Summary of results :

(*i*) When both ends of column are pinned (hinged)

Crippling load $=P=\dfrac{\pi^2 EI}{l^2}$

(*ii*) When one end of the column is fixed and the other end is free,

Crippling load $=P=\dfrac{\pi^2 EI}{4l^2}$

(*iii*) When both ends of the column are fixed,

Crippling load $=P=4\dfrac{\pi^2 EI}{l^2}$

(iv) When one end of the column is fixed, and the other end is pinned (hinged),

$$\text{Crippling load} = P = \frac{2\pi^2 EI}{l^2}$$

§124. Effective length of a column

The effective length of a given column with given end conditions is the length of an equivalent column of the same material and section with hinged ends having the value of the crippling load equal to that of the given column.

Actual length of column = l, effective length = L

End conditions of column	*Crippling load*	*Relation between equivalent length and actual length*
Case 1 : Both ends hinged	$\frac{\pi^2 EI}{l^2} = \frac{\pi^2 EI}{L^2}$	$L = l$
Case 2 : One end fixed one end free	$\frac{\pi^2 EI}{4l^2} = \frac{\pi^2 EI}{L^2}$	$L = 2l$
Case 3 : Both ends fixed	$\frac{4\pi^2 EI}{l^2} = \frac{\pi^2 EI}{L^2}$	$L = \frac{l}{2}$
Case 4 : One end fixed one end hinged	$\frac{2\pi^2 EI}{l^2} = \frac{\pi^2 EI}{L^2}$	$L = \frac{l}{\sqrt{2}}$

If L be effective length of a column, the crippling load,

$$= \frac{\pi^2 EI}{L^2}$$

The effective length of columns corresponding to different end conditions are given in the above table.

In the above formula I should be taken as the moment of inertia of the section about the axis of least resistance. Hence I should be taken as the least moment of inertia of the section.

Putting $I = AK^2$

where K = least radius of gyration of the column section.

We have, crippling load $= P = \frac{\pi^2 EAK^2}{L^2}$

or, the stress intensity corresponding to the crippling load

$$= \frac{P}{A} = \pi^2 E \frac{K^2}{L^2}$$

$$\frac{P}{A} = \frac{\pi^2 E}{\left(\frac{L}{K}\right)^2}$$

The ratio $\frac{L}{K} = \frac{\text{effective length}}{\text{least radius of gyration}}$ is called the *slenderness ratio* of the column.

§125. Assumptions made in Euler's theory

The Euler's formula for the crippling load is based on the following assumptions :

(*i*) The column is initially perfectly straight and is axially loaded.

(*ii*) The section of the column is uniform.

(*iii*) The column material is perfectly elastic, homogeneous and isotropic and obeys Hooke's law.

(*iv*) The length of the column is very large compared to the lateral dimensions.

(*v*) The direct stress is very small compared with the bending stress corresponding to the buckling condition

(*vi*) The self-weight of the column is ignorable.

(*vii*) The column will fail by buckling alone.

§126. Limitation of Euler's formula

The validity of Euler's formula is subject to the satisfaction of the assumptions mentioned in the above article. Further accepting the formula for the cripping load,

$$P = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 E\, A K^2}{L^2}$$

The stress at failure

$$= \frac{P}{A} = \frac{\pi^2 E}{\left(\frac{L}{K}\right)^2}$$

From the above it can be realized that the stress at failure $\frac{P}{A}$ according to the formula will be high when the slenderness ratio is small But the stress at failure cannot be greater than the crushing stress for the column material. Hence when the slenderness ratio is less than a certain limit, Euler's formula gives a value of the crippling load even greater than the crushing load.

For instance, consider a mild steel column. The crushing stress for mild steel $= F_c = 3300$ kg/cm^2. Young's Modulus for mild steel $= E = 2{\cdot}1 \times 10^6$ $kg./cm^2$. Equating the crippling stress to the crushing stress corresponding to the minimum value of slenderness ratio.

We have,

$$\frac{\pi^2 E}{\left(\frac{L}{K}\right)^2} = F_c$$

$$\left(\frac{L}{K}\right)^2=\frac{\pi^2 E}{F_c}\quad\frac{\pi^2\times 2{\cdot}1\times 10^6}{3300}=6282$$

$$\frac{L}{K}=79{\cdot}27 \text{ say } 80.$$

Hence when the slenderness ratio is less than this limit for mild steel columns, Euler's formula will not be valid.

Problem 414. *A mild steel tube 4 metres long 3 cm. internal diameter and 4 mm. thick is used as a strut with both ends hinged. Find the collapsing load. Take $E=2{\cdot}1\times 10^6$ kg./cm.²*

Solution. Moment of inertia of section

$$=I=\frac{\pi}{64}\,(D^4-d^4)$$

$$=\frac{\pi}{64}\,[3{\cdot}8^4-3^4]\ cm.^4$$

$$=6{\cdot}259\ cm.^4$$

Since both ends of the column are hinged

Effective length $=L=4\ m=400\ cm.$

$\therefore$ Crippling load $=P=\dfrac{\pi^2 EI}{L^2}$

$$=\frac{\pi^2\times 2{\cdot}1\times 10^6\times 6{\cdot}259}{(400)^2}\ kg.$$

$$=811\ kg.$$

Problem 415. *A strut 2·50 metres long is 6 cm. in diameter. One end of the strut is fixed while its other end is hinged. Find the safe compressive load for the member using Euler's formula, allowing a factor of safety of 3·5. Take $E=2{\cdot}1\times 10^6$ kg./cm².*

Solution. Since one end of the strut is fixed and the other end is hinged, effective length of strut

$$=L=\frac{l}{\sqrt{2}}$$

$$=\frac{250}{\sqrt{2}}=176{\cdot}8\ cm.$$

Moment of inertia of the section

$$=I=\frac{\pi d^4}{64}=\frac{\pi\times 6^4}{64}\ cm.^4$$

$$=63{\cdot}62\ cm.^4$$

$\therefore$ Crippling load $=P=\dfrac{\pi^2 EI}{L^2}$

$$=\frac{\pi^2\times 2{\cdot}1\times 10^6\times 63{\cdot}62}{(176{\cdot}8)^2}\ kg.$$

$$=42190\ kg.$$

$$\therefore \quad \text{Sate load} = \frac{\text{Crippling load}}{\text{Factor of safety}}$$

$$= \frac{42190}{3 \cdot 5} = 12{,}050 \ kg.$$

Problem 416. *Calculate the critical load for a strut which is made of a bar circular in section and 5 metres long and which is pin-jointed at both ends. The same bar when freely supported gives a mid span deflection of 1 cm under a load of 8 kg. at the centre.*

(A.M.I.E., Summer 1979)

Solution. *Anlaysis as a beam*

$$l = 500 \ cm, \ W = 8 \ kg, \ \delta = 1 \ cm., \ \delta = \frac{Wl^3}{48EI}$$

$$\therefore \quad EI = \frac{Wl^3}{48\delta} = \frac{8 \times 500^3}{48 \times 1} = \frac{125}{6} \times 10^6 \ kg./cm.^2$$

Analysis as a strut

Euler's critical load

$$P = \frac{\pi^2 EI}{l^2}$$

$$= \pi^2 \times \frac{125}{5} \times 10^6 \ \frac{1}{500^2}$$

$$= 822 \cdot 48 \ kg.$$

Problem 417. *A bar of length 4 metres when used as a simply supported beam and subiected to a uniformly distributed load of 3 t/m over the whole span deflects 1·5 cm at the centre. Determine the crippling loads when it is used as a column with the following conditons.*

(*i*) *Both ends pinjointed*

(*ii*) *One end fixed and the other hinged,*

(*iii*) *Both ends fixed* *(AMIE. Summer 1977)*

Solution.

Analysis as a beam

$$w = 3 \ t/m = \frac{3000}{100} \ kg/cm. = 30 \ kg./cm.$$

$$l = 400 \ cm, \ \delta = 1 \cdot 5 \ cm.$$

$$= \frac{5}{384} \ \frac{wl^4}{EI}$$

$$\therefore \quad EI = \frac{5}{384} \cdot \frac{wl^4}{\delta} = \frac{5}{384} \times \frac{30(400)^4}{1 \cdot 5} = \frac{2}{3} \times 10^{10} \ kg. \ cm.^2$$

Analysis as a column

(*i*) *When both ends are hinged*

Crippling load $P=\frac{\pi^2 EI}{l^2}$

$=\pi^2\times\frac{2}{3}\times 10^{10}\ \frac{1}{(400)^2}=411235{\cdot}4\ kg.$

(*ii*) *When one end is fixed and the other end is hinged*

Crippling load $P=\frac{2\pi^2 EI}{l^2}$

$=2\times 411235{\cdot}4=822470{\cdot}8\ kg.$

(*iii*) *When both ends are fixed*

Crippling load $P=\frac{4\pi^2 EI}{l^2}$

$=4\times 411235{\cdot}4=1644941{\cdot}6\ kg.$

Problem 418. *A round steel rod of diameter 1·5 cm. and length 200 cm is subjected to a gradually increasing axial compressive load. Using Euler's formula find the buckling load. Find also the maximum lateral deflection corresponding to the buckling condition. Both ends of the rod may be taken as hinged. Take $E=2{\cdot}1\times 10^6$ kg./cm.² and the yield stress of steel = 2400 kg./cm²*

Solution. Area of the rod

$$=A=\frac{\pi d^2}{4}=\frac{\pi}{4}\times 1{\cdot}5^2=1{\cdot}767\ cm^2.$$

Moment of inertia of the section

$$=I=\frac{\pi d^4}{64}=\frac{\pi\times 1{\cdot}5^4}{64}=0{\cdot}2485\ cm.^4$$

Since both ends of the member are hinged, the effective length

$=L=200\ cm.$

∴ Buckling load (using Euler's formula)

$$=P=\frac{\pi^2 EI}{L^2}$$

$$=\frac{\pi^2\times 2{\cdot}1\times 10^6\times 0{\cdot}2485}{(200)^2}\ kg.$$

$=121{\cdot}8\ kg.$

Direct compressive stress

$$=P_0=\frac{P}{A}$$

$$=\frac{128{\cdot}8}{1{\cdot}767}\ kg./cm.^2$$

$$=72{\cdot}90\ kg./cm.^2$$

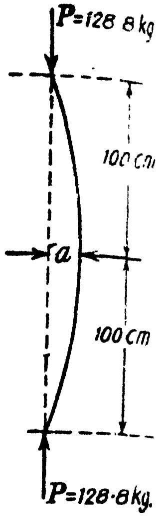


Fig. 612

Let the maximum bending stress corresponding to buckling condition be p_b.

$$\therefore \quad p_b+p_o=\text{yield stress}$$

$$\therefore \quad p_b=2400-72{\cdot}90$$

$$=2327{\cdot}1\ kg./cm.^2$$

Let M be the maximum bending moment which occurs at the centre.

$$\frac{M}{\ } = \frac{p_b}{\left(\frac{d}{2}\right)}$$

$$M=\frac{237{\cdot}1}{0{\cdot}75}\times 0{\cdot}2485\ kg.\ cm.$$

$$=771{\cdot}1\ kg.\ cm.$$

Let the maximum central deflection be a *cm.*

$$\therefore \quad M=Pa=128{\cdot}8\ a=771{\cdot}1$$

$$\therefore \quad a=\frac{771{\cdot}1}{128{\cdot}8}=5{\cdot}99\ cm.$$

§127. Empirical formulae

Rankine's formula. In the case of a short column which fails by curshing the load at failure equals $P_c=F_cA$ where F_c is the crushing stress for the column material and A is the sectional area But, for a long column which fails by buckling, the load at failure, *i e*, the buckling load equals $P_e=\frac{\pi^2.EI}{L^2}$ where L is the effective length of the column. The struts and columns which we come across are neither too short nor long. The failure of the member will be due to the combined effect of direct and bending (buckling) stresses. Rankine devised an empirical formula for the collapse load which should cover all columns whether they are short or long.

Let P be the actual crippling load. Rankine stated his empirical formula

$$\frac{1}{P}=\frac{1}{P_c}+\frac{1}{P_e}$$

where $P_c=F_cA=$crushing load

$P_e=$Buckling load according to Euler's formula

$$=\text{Eulerian load}=\frac{\pi^2EI}{L^2}$$

It should be noted that for a given column material the crushing stress F_c is a constant and hence the crushing load $F_c A$ is also a constant for a given sectional area of a column. But the Eulerian load depends upon the effective length of the column.

If the column is short P_e will be large and $\frac{1}{P_e}$ will be small enough to be ignored compared with $\frac{1}{P_c}$. Hence as the length of the column is decreased,

$$\frac{1}{P} \to \frac{1}{P}$$

$$\therefore \qquad P \to P_c$$

Similarly if the column is long P_e will be small and $\frac{1}{P_e}$ will be large enough compared with $\frac{1}{P_c}$ and we may ignore $\frac{1}{P_c}$.

Hence as the length of the column is increased

$$\frac{1}{P} \to \frac{1}{P_e}$$

$$\therefore \qquad P \to P_e$$

Hence the formula

$$\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}$$

gives satisfactory results for the extreme cases of long as well as short columns. Hence Rankine's formula is taken to be valid for all lengths of columns.

Hence,
$$\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}$$

$$\therefore \qquad \frac{1}{P} = \frac{P_e + P_c}{P_c \,.\, P_e}$$

$$\therefore \qquad P = \frac{P_c \,.\, P_e}{P_c + P_e}$$

$$\therefore \qquad P = \frac{P_c}{1 + \frac{P_c}{P_e}}$$

$$\therefore \qquad P = \frac{F_c A}{1 + \frac{F_c A}{\frac{\pi^2 EI}{L^2}}}$$

Putting $I = AK^2$ where K = least radius of gyration of the column section,

$$P = \frac{F_c C}{1 + \frac{F_c A L^2}{\pi^2 E A K^2}}$$

$$\therefore \quad P = \frac{F_c A}{1 + \frac{F_c}{\pi^2 E}\left(\frac{L}{K}\right)^2}$$

$$\therefore \quad P = \frac{F_c A}{1 + \alpha\left(\frac{L}{K}\right)^2}$$

where $\alpha = \frac{F_c}{\pi^2 E}$

For a given material of column, the quantity

$$\alpha = \frac{F_c}{\pi^2 E}$$

is a constant. Hence both F_c and α are constants for given column **material.**

The following table shows the values of F_c and α for different **column materials.**

Material	F_c kg./cm.^{2}	F_c N/mm^2	$\frac{F_c}{\pi^2 E}$
Wrought Iron	2500	250	$\frac{1}{9000}$
Cast Iron	5500	550	$\frac{1}{1600}$
Mild Steel	3200	320	$\frac{1}{7500}$
Strong Timber	500	50	$\frac{1}{750}$

Studying Rankine's formula

$$P = \frac{F_c A}{1 + \alpha\left(\frac{L}{K}\right)^2}$$

we find $$\frac{\text{Crushing load}}{1 + \alpha\left(\frac{L}{K}\right)^2}$$

The factor $1+\alpha\left(\frac{L}{K}\right)^2$ has thus been introduced to take into account the buckling effect.

Problem 419. *An ISMB 250 R.S.J. is to be used as a column 4 metres long with one end fixed and the other end hinged. Find the safe axial load on the column allowing a factor of safety of 3. Take $F_c=3200\ kg/cm^2$ and $\alpha=\frac{1}{7500}$. Properties of column section are as follows :*

Area $=47{\cdot}55\ cm.^3$, $I_{xx}=5131{\cdot}6\ cm^4$, $I_{yy}=334{\cdot}5\ cm.^4$

Solution. Effective length of the column

$$=L=\frac{l}{\sqrt{2}}=\frac{400}{\sqrt{2}}\ cm.$$

$$K^2=\frac{I_{yy}}{A}=\frac{334{\cdot}5}{47{\cdot}55}=7{\cdot}033$$

Crippling load $=P=\dfrac{F_c A}{1+\alpha\left(\frac{l}{K}\right)^2}$

$$=\frac{3200\times47{\cdot}55}{1+\frac{1}{7500}\times\frac{400\times400}{2\times7{\cdot}033}}\ kg.$$

$$=60450\ kg.$$

Allowing a factor of a safety of 3,

Safe load on the column

$$=\frac{60450}{3}=20150\ kg.$$

Problem 420. *A hollow cylindrical cast iron column is 4 metres long, both ends being fixed. Design the column to carry an axial load of 25 tonnes. Use Rankine's formula and adopt a factor of safety of 5. The internal diameter may be taken as 0·80 times the external diameter. Take $F_c=5500\ kg/cm.^2$ and $\alpha=\frac{1}{1600}$.*

Solution. Let the external diameter be D cm.

Internal diameter $=d=0{\cdot}8\ D\ cm.$

Area of the section

$$=A=\frac{\pi}{4}(D^2-d^2)\ cm.^2$$

$$=0{\cdot}09\pi\ D^2\ cm.^2$$

Moment of inertia of the section

$$=\frac{\pi}{64}(D^4-d^4)\ cm.^4$$

$$\therefore \quad K^2 = \frac{I}{A} = \frac{\frac{\pi}{64}(D^4 - d^4)}{\frac{\pi}{4}(D^2 - d^2)} \; cm.^2$$

$$\therefore \quad K^2 = \frac{D^2 + d^2}{16} \; cm.^2$$

$$= \frac{1.64}{16} D^2 \; cm.^2$$

$$= 0.1025 \; D^2 \; cm.^2$$

Safe load on the column = 25 *tonnes*

$\therefore$ Crippling load = safe load × factor of safety

$= 25 \times 5 = 125$ *tonnes*

Effective length of the column

$$= L = \frac{l}{2} = \frac{400}{2} = 200 \; cm.$$

Applying Rankine's formula

$$P = \frac{F_c A}{1 + \alpha \frac{L^2}{K^2}}$$

$$\therefore \quad 125000 = \frac{5500 \times 0.09 \pi D^2}{1 + \frac{1}{1600} \times \frac{200 \times 200}{0.1025 \; D^2}}$$

$$\therefore \quad 125000 = \frac{5500 \times 0.09 \; \pi D^4}{D^2 + \frac{200 \times 200}{1600 \times 0.1025}}$$

$$\therefore \quad 125000 = \frac{1555 \; D^4}{D^2 + 243.9}$$

$$\therefore \quad 1555 \; D^4 = 125000 \; D^2 + 30487500$$

$$\therefore \quad D^4 - 80.40 \; D^2 = 19610$$

$$(D^2 - 40.2)^2 = (40.2)^2 + 19610 = 21226$$

$$\therefore \quad D^2 - 40.2 = 145.7$$

$$\therefore \quad D^2 = 185.9$$

$$\therefore \quad D = 13.63 \; cm. \text{ (say 14 cm.)}$$

Internal diameter $= 0.8 \times 13.63$ *cm.*

$= 10.904$ *cm.* (*say 11 cm.*)

Problem 421. *A hollow C.I. column whose outside diameter is 200 mm. has a thickness of 20 mm. It is 4·5 metres long and is fixed at both ends. Calculate the safe load by Rankine's formula using a factor of safety of 4. Calculate the slenderness ratio and the ratio of Euler's and Rankine's critical loads.* (*A.M.I.E., Winter 1979*)

Solution. D = Outer diameter = 20 $cm.$

d = Inner diameter = 20 − 4 = 16 $cm.$

Area $= A = \frac{\pi}{4}(D^2 - d^2)$

$= \frac{\pi}{4}(26^2 - 16^2) = 113{\cdot}10\ cm.^4$

Moment of inertia of section

$$I = \frac{\pi}{64}(D^4 - d^4)$$

$$= \frac{\pi}{64}(20^4 - 16^4) = 4637\ cm.^4$$

Let K = radius of gyration

$$K^2 = \frac{I}{A} = \frac{4637}{113{\cdot}1} = 41\ cm.^2$$

$\therefore$ $K = \sqrt{41} = 6{\cdot}40\ cm.$

Effective length $= \frac{450}{2} = 225\ cm.$

$\therefore$ *Slenderness ratio* $= \frac{225}{6{\cdot}4} = 35{\cdot}15$

Rankine's critical load

$$= P_r = \frac{f_c A}{1 + \alpha \frac{l^2}{K^2}}$$

For cast iron, $f_c = 5500\ kg./cm.^2$ and $\alpha = \frac{1}{1600}$

$\therefore$ $P_r = \dfrac{5500 \times 113{\cdot}1}{1 + \frac{1}{1600} \times \frac{225 \times 225}{41}} = 351098\ kg.$

Safe load $= \frac{351098}{4} = 87774{\cdot}5\ kg$

Euler's critical load

$$P_e = \frac{\pi^2 EI}{l^2}$$

Taking $E = 940\ t/cm.^2$, $P_e = \dfrac{\pi^2 \times 940{,}000 \times 4637}{225 \times 225}$

$= 849770\ kg.$

$\dfrac{\textit{Euler's critical load}}{\textit{Rankine's critical load}} = \dfrac{849770}{351098} = 2{\cdot}42.$

Problem 422. *A 1·5 metre long column has a circular cross section of 5 cm. diameter. One end of the column is fixed in direction and position and the other end is free. Taking a factor of safety of 3 calculate the safe load using* (*i*) *Rankine's formula : Take* $f_c = 5600$ *kg./cm.*^{2}

$$\alpha = \frac{1}{1600} \text{ for pinned ends}$$

(*ii*) *Euler's formula : Young's modulus for* $CI = 1{\cdot}2 \times 10^6$ *kg./cm.*^{2}

(*A.M.I.E., May 1976*)

Solution. (*i*) *By Rankine's formula*

$d = 5$ *cm.* Effective length $L = 2l$

$= 2 \times 1{\cdot}5 = 3$ $m = 300$ *cm.*

$$A = \frac{\pi}{4}(5)^2 = 19{\cdot}635 \text{ cm.}^2$$

$$K^2 = \frac{\pi d^4/64}{\pi d^2/4} = \frac{d^2}{16} = \frac{5 \times 5}{16} = \frac{25}{16}$$

$$P = \frac{f_c A}{1 + \alpha \frac{L^2}{K^2}} = \frac{5600 \times 19{\cdot}635}{1 + \frac{1}{1600} \times \frac{300 \times 300}{\left(\frac{25}{16}\right)}} = 2971{\cdot}78 \text{ kg.}$$

$$\therefore \text{ Safe load} = \frac{2971{\cdot}78}{3} = 990{\cdot}59 \text{ kg.}$$

(*ii*) *By Euler's formula*

$$P = \frac{\pi^2 EI}{L^2}$$

$$= \frac{\pi^2}{300 \times 300} \times 1{\cdot}2 \times 10^6 \times \frac{\pi \times 5^4}{64} = 4012{\cdot}22 \text{ kg.}$$

$$\therefore \text{ Safe load} = \frac{4012{\cdot}22}{3} = 1337{\cdot}4 \text{ kg.}$$

Problem 423. *A column 9 metres long has a cross-section shown in Fig. 613. The column is pinned at both ends. If the column is subjected to an axial load equal in value 1/4 of the Euler critical load for the column, determine the factor of safety on the Rankine ultimate stress value. Take* $F_c = 3{\cdot}26$ *t/cm.*^{2} *Rankine's constant*

$$\alpha = \frac{1}{7500}, \; E = 2000 \text{ t/cm.}^2$$

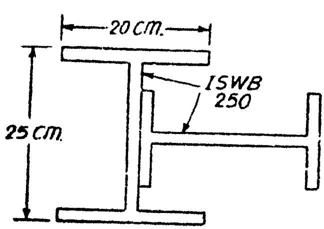


Fig. 613

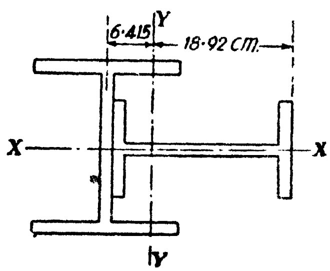


Fig. 614

Properties of one R.S.J.

$$Area = 52{\cdot}05\ cm.^2$$
$$I_{xx} = 5943{\cdot}1\ cm.^4$$
$$I_{yy} = 857{\cdot}5\ cm.^4$$

Thickness of web $= 6{\cdot}7\ mm.$

Solution. *Properties of the combined section*

Area of the combined section

$$= A = 52{\cdot}05 \times 2\ cm^2$$
$$= 104{\cdot}10\ cm.^2$$

Moment of inertia about the xx axis

$$I_{xx} = 5943{\cdot}1 + 857{\cdot}5\ cm.^4$$
$$= 6800{\cdot}6\ cm.^4$$
$$I_{yy} > I_{xx}$$

$$\therefore \quad K^2 = \frac{I}{A} = \frac{6800{\cdot}6}{2 \times 52{\cdot}05}\ cm^2$$
$$= 65{\cdot}34\ cm.^2$$

Euler critical load $= \dfrac{\pi^2 EI}{l^2}$

$$= \frac{\pi^2 \times 2000 \times 6800{\cdot}6}{900 \times 900}\ tonnes$$
$$= 165{\cdot}8\ tonnes$$

Safe load $= \dfrac{1}{4} \times 165{\cdot}8 = 41{\cdot}45\ tonnes$

Crippling load as per Rankine's formula

$$= \frac{f_c A}{1 + \alpha \dfrac{l^2}{K^2}}$$

$$= \frac{3{\cdot}26 \times 104{\cdot}1}{1 + \dfrac{1}{7500} \cdot \dfrac{900 \times 900}{65{\cdot}34}}$$

$$= 205{\cdot}4\ tonnes$$

Safe load $= 41{\cdot}45\ tonnes$

$\therefore$ Factor of safety on Rankine's ultimate stress value

$$= \frac{205{\cdot}4}{41{\cdot}45} = 4{\cdot}96$$

Problem 424 *A uniform bar of cross-sectional area A and flexural stiffness EI is heated so that its temperature varies linearly from ½ t at one end to t at the other end One end is pinned to a rigid foundation ; the other end is pin-jointed so that it can slide in the direction of the length of the bar, the thermal expansion of which is* *resisted by a compression spring of stiffness k. If there is no load in the spring when t = 0, obtain an expression for the stress in the bar when it is heated and show that it buckles in flexure when*

$$t=\frac{4\pi^2 I}{3\alpha l^2 A}\left(1+\frac{EA}{Kl}\right)$$

where α = *coefficient of linear thermal expansion.*

Solution. Average temperature of the bar

$$=\frac{\frac{1}{2}t+t}{2}=\frac{3}{4}t.$$

Free expansion of the bar $=\frac{3}{4}\alpha t l$

Let P be the force exerted on the bar by the spring

Decrease in length of bar due to this force

$$=\frac{Pl}{AE}$$

$\therefore$ Net expansion of the bar

$$=\frac{3}{4}\alpha t l-\frac{Pl}{AE}$$

Decrease in axial length of spring

$$=\frac{P}{K}$$

But net expansion of the bar = decrease in axial length of spring

$$\therefore \quad \frac{3}{4}\alpha t l-\frac{Pl}{AK}=\frac{P}{K}$$

$$\therefore \quad P=\frac{\frac{3}{4}\alpha t l}{\frac{l}{AE}+\frac{1}{K}}$$

$\therefore$ Stress in the bar

$$\therefore \quad =\frac{P}{A}=\frac{\frac{3}{4}\alpha t l}{\frac{l}{E}+\frac{A}{K}}$$

Corresponding to the buckling condition of the bar

$$P=\frac{\pi^2 EI}{l^2}=\frac{\frac{3}{4}\alpha t l}{\frac{l}{AE}+\frac{1}{K}}$$

$$\therefore \quad t=\frac{4\pi^2 I}{3\alpha l^2 A}\left(1+\frac{AE}{Kl}\right)$$

§128 The straight line formula

It may be noted that Euler's formula and Rankine's formula are not exact compared with actual results by experiments, due to the following reasons :

(*i*) The effect of direct compression has been neglected in the case of Euler's formula

(*ii*) The loading is not exactly applied as desired.

(*iii*) The pin-joints are not practically frictionless.

(*iv*) Absolute fixation of ends is not possible.

(*v*) The members are not perfectly straight, uniform and homogeneous.

The formulae therefore lead to results which are approximate and may be taken to agree within 5 to 10 percent. Hence approximate empirical formulae are often used in practical designing. Some of them are given below :

(*i*) Stress at critical load for structural steel

$$\frac{P}{A}=3675-20\left(\frac{l}{K}\right)\ kg./cm.^2$$

(*ii*) Stress at critical load for cast iron

$$=238-6\frac{l}{K}\ kg./cm.^2$$

(*iii*) Safe working stress for mild steel

$$1500\left[1-0\cdot0038\frac{l}{K}\right]kg./cm.^2$$

Johnson's parabolic formula

According to Prof. Johnson the stress at critical load is given by

$$\frac{P}{A}=f_c-g\left(\frac{l}{K}\right)^2 \text{ for pin-jointed}$$

struts, where,

f_c = compressive yield stress,

g = a constant depending on the column material and is

taken as $\frac{f_c^2}{4\pi^2 E}$, where

E = Young's Modulus

§129. Factor of safety

This is the ratio of the critical load to the safe load on the column.

Where, otherwise mentioned, the factors of safety may be taken as follows.

Column material	*Factor of Safety*
Timber	6
Wrought Iron, Mild steel Medium steel	3
Cast Iron	5

§130. Choice of a column formula

The suitability of a column formula will depend upon the slenderness ratio $\left(\frac{l}{K}\right)$. The following table indicates the suitability of column formula.

Column formula	*Slenderness Ratio Range*	
	Wrought Iron or steel	*Cast Iron, Timber*
Straight line formula	0—140	0—140
Johnson's parabolic formula	0—100	0—60
Rankine's formula	any value	any value
Euler's formula	over 90	over 50

§131. Formula given by the I.S. code for Mild steel

The direct stress in compression on the gross area of the section of an axially loaded compression member shall not exceed the values of P_c calculated as follows and given in the table below.

$$P_c = P_c' = \frac{\frac{f_y}{m}}{1 + 0.20 \sec\left[\frac{l}{K}\sqrt{\frac{mP_c'}{4E}}\right]}$$

for $\frac{l}{K}$ between 0 and 160

where P_c = the allowable average axial compression stress
P_c' = a value obtained from the above secant formula
f_y = the guaranteed minimum yield stress

m=factor of safety taken as 1·68

$\frac{l}{K}$=slenderness ratio

E=modulus of Elasticity = 2,047,000 $kg/cm.^2$

For values of $\frac{l}{K}=160$ and above,

$$P_c=P_c'\left(1\cdot2-\frac{l}{800K}\right)$$

where

$$P_c'=\frac{\frac{f_y}{m}}{1+0\cdot20\sec\left[\frac{l}{K}\sqrt{\frac{mP_c'}{4E}}\right]}$$

Safe stresses in axial compression in accordance with the formula suggested by the I.S code are tabulated below for various values of slenderness ratio.

Slenderness ratio	*Safe stress in kg/cm*
0	1250
10	1246
20	1236
30	1224
40	1203
50	1172
60	1130
70	1075
80	1007
90	928
100	840
110	753
120	671
130	597
140	531
150	474
160	423
170	377
180	366
190	300
200	270
210	243
220	219
230	199
240	181
250	166
300	109
350	76

Problem 425. *Fig 615 shows a compound stanchion made up of two channels ISJC 200 weighing 13·9 kg. per channel and two 250 mm. × 10 mm. plates riveted one to each flange. Calculate the safe load that can be carried by the column The column is 6 metres long and both its ends are fixed. Allow a factor of safety of 3·5.*

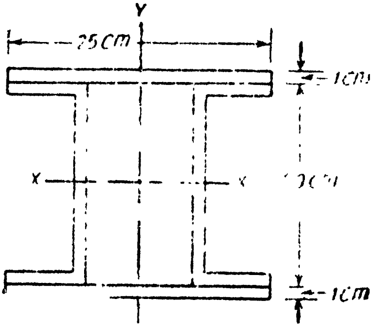


Fig. 615

Properties of one channel are given below :

$$A = 17{\cdot}77 \ cm.^2$$
$$I_{xx} = 1161{\cdot}2 \ cm.^4$$
$$I_{yy} = 84{\cdot}2 \ cm.^4$$

Distance of centroid from back of web = 1·97 cm.

Take $F_c = 3{\cdot}2\ t\ cm^{-2}$ *and* $\alpha = \dfrac{1}{7500}$

Solution. Properties of the composite section :

$$\text{Area} = A = 2[17{\cdot}77 + 25 \times 1]\ cm.^2$$
$$= 85{\cdot}54\ cm.^2$$

$$I_{xx} = 2 \times 1161{\cdot}2 + 2\left(\frac{25 \times 1^3}{12} + 25 \times 1 \times 10{\cdot}5^2\right) cm.^4$$
$$= 2322{\cdot}4 + 4516{\cdot}6 = 6839\ cm.^4$$

$$I_{yy} = 2\left[\frac{1 \times 25^3}{12} + 84{\cdot}2 + 17{\cdot}77 \times 6{\cdot}97^2\right] cm.^4$$
$$= 4499\ cm.^4$$

$$\therefore \qquad \text{Least } K^2 = \frac{4499}{85{\cdot}54}\ cm.^2$$
$$= 52{\cdot}59\ cm.^2$$

Effective length of the member $= l = \dfrac{6}{2} = 3m = 300\ cm.$

$$\text{Crippling load} = \frac{F_c A}{1 + \alpha \dfrac{L^2}{K^2}}$$

$$= \frac{3{\cdot}2 \times 85{\cdot}54}{1 + \dfrac{1}{7500} \times \dfrac{300 \times 300}{52{\cdot}59}} \text{ tonnes}$$

$$= 222{\cdot}9 \text{ tonnes}$$

$$\therefore \quad \text{Safe axial load} = \frac{\text{Crippling load}}{\text{Factor of safety}}$$

$$= \frac{222{\cdot}9}{3{\cdot}5} \text{ tonnes}$$

$$= 63{\cdot}7 \text{ tonnes.}$$

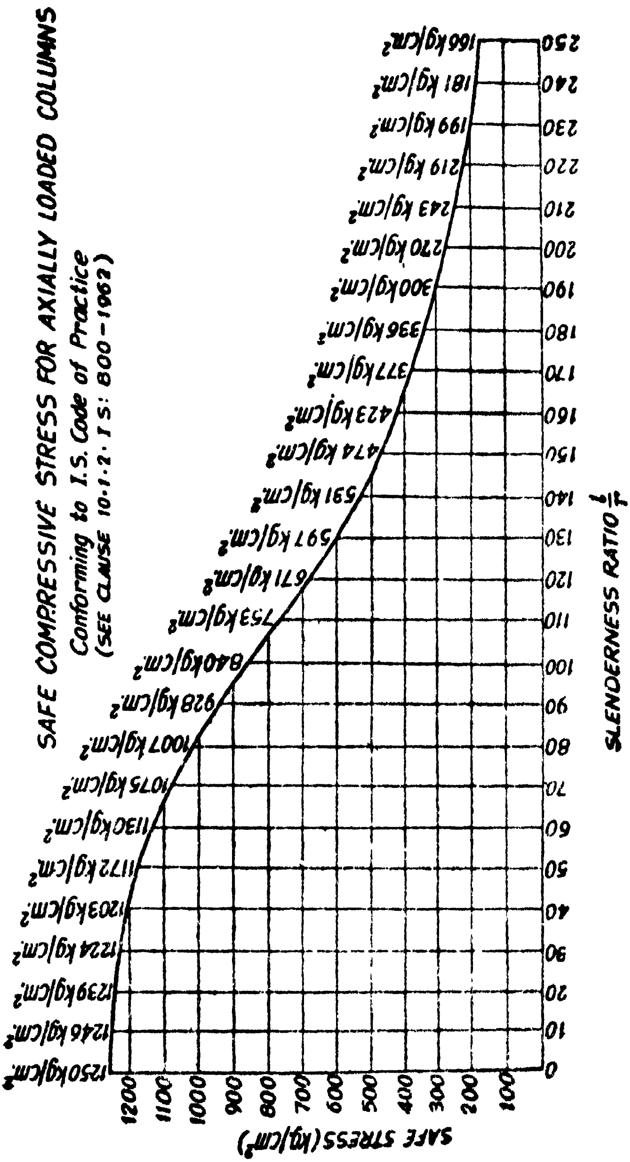

SAFE COMPRESSIVE STRESS FOR AXIALLY LOADED COLUMNS
Conforming to I.S. Code of Practice
(SEE CLAUSE 10.1.2. I S: 800–1962)
1250 kg/cm²
1246 kg/cm²
1239 kg/cm²
1224 kg/cm²
1203 kg/cm²
1172 kg/cm²
1130 kg/cm²
1075 kg/cm²
1007 kg/cm²
928 kg/cm²
840 kg/cm²
753 kg/cm²
671 kg/cm²
597 kg/cm²
531 kg/cm²
474 kg/cm²
423 kg/cm²
377 kg/cm²
336 kg/cm²
300 kg/cm²
270 kg/cm²
243 kg/cm²
219 kg/cm²
199 kg/cm²
181 kg/cm²
166 kg/cm²
0
10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
160
170
180
190
200
210
220
230
240
250
SLENDERNESS RATIO $\frac{l}{r}$
1200
1100
1000
900
800
700
600
500
400
300
200
100
SAFE STRESS (kg/cm²)

Fig. 616

§132. Column subjected to eccentric loading

(*i*) *Rankine's method.* Consider a short column subjected to an eccentric load P. Let e be the eccentricity from the geometric axis. Let A be the sectional area of the member.

∴ Maximum compressive stress

$$=p_{max}=\frac{P}{A}+\frac{P.e}{I}y_c$$

$$=\frac{P}{A}+\frac{P.e}{AK^2}y_c$$

$$=\frac{P}{A}\left[1+\frac{ey_c}{K^2}\right]$$

$$\therefore \quad P=\frac{p_{max}A}{1+\frac{ey_c}{K^2}}$$

Let f be the safe stress for the column material.

∴ Safe load for the column at the eccentricity e is given by

$$P=\frac{fA}{\left(1+\frac{ey_c}{K^2}\right)}$$

If the effect of buckling be also included, the safe eccentric load

$$P=\frac{fA}{\left(1+\frac{ey_c}{K^2}\right)\left(1+\alpha\frac{L^2}{K^2}\right)}$$

where L=effective length of the column.

(*ii*) *Euler's method*

Consider a column AB of length l subjected to an eccentric load P at eccentricity e. Let the top of the column be free and the bottom of the column be fixed. Let y be the deflection at any section X distant x from the fixed end B. Let a be the deflection at A.

The bending moment at the section X is given by,

$$EI\frac{d^2y}{dx^2}=P(a+e-y)$$

$$\therefore \quad \frac{d^2y}{dx^2}+\frac{P}{EI}y=\frac{P(a+e)}{EI}$$

The solution to the above differential equation is given by,

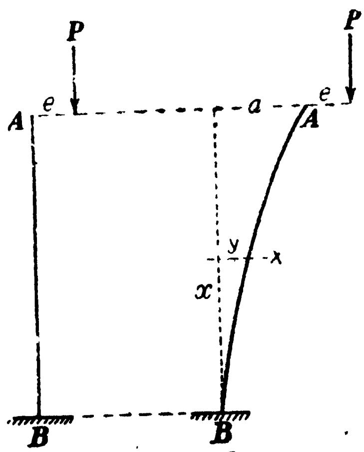


Fig. 617

$$y = C_1 \cos x \sqrt{\frac{P}{EI}} + C_2 \sin x \sqrt{\frac{P}{EI}} + (a+e) \qquad \ldots(i)$$

The slope at any section is given by,

$$\frac{dy}{dx} = -C \sqrt{\frac{P}{EI}} \sin x \sqrt{\frac{P}{EI}} + C_2 \sqrt{\frac{P}{EI}} \cos x \sqrt{\frac{P}{EI}}$$

At B, $x=0$ and $y=0$, and $\frac{dy}{dx}=0$

$\therefore$ $0 = C_1 + (a+e)$

and $0 = C_2 \sqrt{\frac{P}{EI}}$

$\therefore$ $C_2 = 0$ and $C_1 = -(a+e)$

At A, $x = l$ and $y = a$

$\therefore$ $a = -(a+e) \cos l \sqrt{\frac{P}{EI}} + (a+e)$

$\therefore$ $a = (a+e)\left[1 - \cos l \sqrt{\frac{P}{EI}}\right]$

$\therefore$ $(a+e) \cos l \sqrt{\frac{P}{EI}} = e$

$$a + e = e \sec l \sqrt{\frac{P}{EI}}$$

The maximum bending moment for the column occurs at B and is equal to $P(a+e)$

$\therefore$ Max. B.M. $= M = Pe \sec l \sqrt{\frac{P}{EI}}$

Hence the maximum compressive stress for the column section at B

$$= \frac{P}{A} + \frac{Pe \sec l \sqrt{\frac{P}{EI}}}{Z}$$

If both ends of the column had been hinged, it can be shown that the maximum bending moment

$$= M = Pe \sec \frac{L}{2} \sqrt{\frac{P}{EI}}$$

For all cases we will remember the above expressions and take L as the effective length of the column.

Problem 426. *A column of circular section made of cast iron 20 cm. external diameter and 2 cm. thick is used as a column 4 metres long. Both ends of the column are fixed. The column carries a load of 15 tonnes at an eccentricity of 2·5 cm. from the axis of the column. Find the extreme stresses on the column section.*

Find also the maximum eccentricity in order there may be no tension anywhere on the section. Take $E=940\ t/cm^2$.

Solution. Area of the column

$$=A=\frac{\pi}{4}(20^2-16^2)\ cm.^2$$

$$=113{\cdot}1\ cm.^2$$

Moment of inertia of the section about a diameter

$$=I=\frac{\pi}{64}(20^4-16^4)\ cm.^4$$

$$=4637\ cm.^4$$

$\therefore$ Section modulus

$$Z=\frac{I}{y_{max}}=\frac{4637}{10}=463{\cdot}7\ cm.^3$$

Effective length of the column

$$=L=\frac{l}{2}=\frac{4}{2}=2\ metres=200\ cm$$

Maximum bending moment

$$M=Pe\sec\frac{L}{2}\sqrt{\frac{P}{EI}}$$

Let us determine the angle $\frac{L}{2}\sqrt{\frac{P}{EI}}$

$$\frac{L}{2}\cdot\sqrt{\frac{P}{EI}}=100\sqrt{\frac{15}{940\times4637}}\ radian$$

$$=0{\cdot}1856\ radian$$

$$=10^\circ{\cdot}64=10^\circ\ 38'$$

say $10^\circ\ 40'$

$$\sec 10^\circ\ 40'=1{\cdot}017$$

$\therefore$ Maximum bending moment

$$M=15\times2{\cdot}5\times1{\cdot}017\ t\ cm.$$

$$=38{\cdot}14\ t\ cm.$$

$\therefore$ Maximum compressive stress

$$=p_{max}=\frac{P}{A}+\frac{M}{Z}$$

$$=\frac{15}{113{\cdot}1}+\frac{38{\cdot}14}{463{\cdot}7}\ t/cm.^2$$

$$=0{\cdot}215\ t/cm.^2$$

$$=215\ kg/cm.^2$$

If tension is just to be avoided corresponding to the maximum eccentricity.

$$\frac{P}{A}=\frac{M}{Z}$$

$$\frac{P}{A}=\frac{P.e.\sec\frac{L}{2}\sqrt{\frac{P}{EI}}}{Z}$$

$$\frac{15}{113{\cdot}1}=\frac{14\times e\times 1{\cdot}017}{463{\cdot}7}$$

$$e=\frac{463{\cdot}7}{113{\cdot}1\times 1{\cdot}017}=4{\cdot}432\ cm.$$

Problem 427. *Fig. 618 shows a compound stanchion made up of two channels ISJC 200 weighing 13 9 kg. per metre per channel and two 250 mm × 10 mm plates riveted one to each flange. If the maximum permissible compressive stress is 800 kg./cm.² find, the maximum eccentricity of a 40 t load from the YY axis of the column. The load line lies in the vertical plane through the XX axis. Take $E=2\times 10^3$ t/cm², the effective length of the column being 3 metres.*

Solution. Properties of the column section (see problem 425)

Area of the section

$$=85{\cdot}54\ cm.^2$$

$$I_{yy}=4499\ cm.^4$$

Stress due to direct load

$$=p_o=\frac{40{,}000}{85{\cdot}54}=468\ kg./cm.^2$$

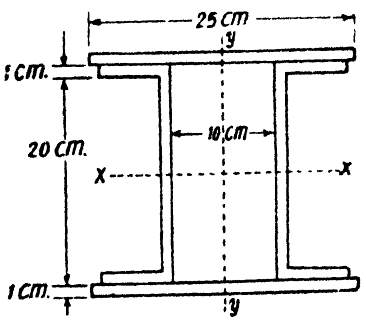


Fig. 618

Maximum compressive stress

$$=800\ kg./cm.^2$$

∴ Maximum bending stress

$$=800-468=332\ kg/cm.^2$$

Section modulus about the YY axis

$$=Z_{yy}=\frac{4499}{12\cdot5}=360 \quad ^3$$

$\therefore$ Max. B.M. $=M=332\times360$ *kg. cm.*

$$\therefore \quad M=Pe\sec\frac{L}{2}\sqrt{\frac{P}{EI}}=332\times360 \text{ kg. cm.}$$

Now,
$$\frac{L}{2}\sqrt{\frac{P}{EI}}$$

$$=150\sqrt{\frac{40}{10^3\times4499}} \text{ radian}$$

$$=0\cdot3160 \text{ radian}$$
$$=18^\circ\ 7'$$

$$\therefore \sec\frac{L}{2}\sqrt{\frac{P}{EI}}=\sec 18^\circ\ 7'=1\cdot052$$

$\therefore$ Max. B.M.

$$=P.e.\sec\frac{L}{2}\sqrt{\frac{P}{EI}}=40{,}000\times e\times1\cdot052=332\times360$$

$$\therefore \quad e=\frac{332\times360}{40{,}000\times1\cdot052} \text{ cm.}$$
$$=2\cdot84 \text{ cm.}$$

§133. Prof. Perry's formula

This is a formula which is found useful for cases where we have to determine the safe load that can be applied on a column at a given eccentricity.

Let L = effective length of the column.

p_{max} = maximum permissible compressive stress

p_o = stress due to direct load $=\frac{P}{A}$

p_b = maximum compressive stress due to bending moment

$$=\frac{M}{Z}=\frac{My_c}{AK^2}$$

$$=\frac{Pe\sec\frac{L}{2}\sqrt{\frac{P}{EI}}\cdot y_c}{AK^2}$$

$$=\frac{Pey_c}{AK^2}\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}$$

where $$P_e=\frac{\pi^2EI}{L^2}$$

$$\therefore \quad p_{max}=\frac{P}{A}+\frac{Pe\,y_c}{AK^2}\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}$$

But $$\frac{P}{A}=p_o$$

$$\therefore \quad p_{max}=p_o\left[1+\frac{ey_c}{K^2}\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}\right]$$

According to Prof. Perry $\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}$

is approximately equal to $\frac{1{\cdot}2\,P_e}{P_e-P}$

Let $$p_e=\frac{P_e}{P}$$

$$\therefore \sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}=\frac{1{\cdot}2\,P_e}{P_e-P}=\frac{1{\cdot}2\,p_e}{p_e-p_o}$$

$$\therefore \quad p_{max}=\left[1+\frac{ey_c}{K^2}\cdot\frac{1{\cdot}2\,p_e}{p_e-p_o}\right]$$

$$\therefore \quad \frac{p_{max}}{p_o}=\left[1+\frac{ey_c}{K^2}\cdot\frac{1{\cdot}2p_e}{p_e-p_o}\right]$$

$$\therefore \quad \left(\frac{p_{max}}{p_o}-1\right)=\frac{ey_c}{K^2}\,\frac{1{\cdot}2p_e}{p_e-p_o}$$

$$\therefore \quad \left(\frac{p_{max}}{p_o}-1\right)\left(1-\frac{p_o}{p_e}\right)=\frac{1{\cdot}2\,ey_c}{K^2}$$

...*(Prof. Perry's formula)*

Problem 428. *For the column in problem 427 find the maximum load that can be applied at an eccentricity of 2 cm. from the axis yy The maximum permissible compressive stress is limited to 800 kg./cm^2 Take $E=2\times 10^3$ t/cm^2.*

Solution. $$K^2=\frac{I}{A}=\frac{4499}{85{\cdot}54}=52{\cdot}59\ cm.^2$$

Eularian load $$P_e=\frac{\pi^2 EI}{L^2}$$

$$=\frac{\pi^2\times 2\times 10^3\times 4499}{(300)^2}\ tonnes$$

$$=987\ tonnes$$

$\therefore \quad p_e = \dfrac{987}{85{\cdot}54}\ t/cm.^2$

$= 11{\cdot}53\ t/cm.^2$

$p_{max} = 800\ kg.cm.^2 = t/cm.^2$

Applying Perry's formula

$$\left(\frac{0{\cdot}8}{p_o} - 1\right)\left(1 - \frac{p_o}{11{\cdot}53}\right) = \frac{1{\cdot}2 \times 2 \times 12{\cdot}5}{52{\cdot}59}$$

$$\left(\frac{0{\cdot}8 - p_o}{p_o}\right)\frac{11{\cdot}53 - p_o}{11{\cdot}53} = \frac{1{\cdot}2 \times 2 \times 12{\cdot}5}{52{\cdot}59}$$

$$\therefore \quad (0{\cdot}8 - p_o)(11{\cdot}53 - p_o) = \frac{1{\cdot}2 \times 2 \times 12{\cdot}5 \times 11{\cdot}53}{52{\cdot}59} p_o$$

$$p_o^2 - 12{\cdot}33\ p_o + 9{\cdot}224 = 6{\cdot}577\ p_o$$

$$\therefore \quad p_o^2 - 18{\cdot}90\ p_o + 9{\cdot}224 = 0$$

$$(p_o - 9{\cdot}45)^2 = -9{\cdot}224 + (9{\cdot}45)^2$$

$$(p_o - 9{\cdot}45)^2 = 80{\cdot}08$$

$$\therefore \quad p_o - 9{\cdot}45 = -8{\cdot}95$$

$$\therefore \quad p_o = 0{\cdot}50\ t/cm.^2$$

$\therefore$ Safe load $= P = p_oA = 0{\cdot}50 \times 85{\cdot}54 = 42{\cdot}77$ *tonnes*

Problem 429. *A strut of length l is encastered at its lower end; its upper end is elastically supported against lateral deflection so that the resisting force is K times the end deflection. Show that the crippling load P is given by*

$$\frac{\tan \alpha l}{\alpha l} = 1 - \frac{P}{Kl} \quad \text{where } \alpha^2 = \frac{P}{EI}$$

Solution. At any section distant x from the lower fixed end B, the bending moment is given by

$$EI\frac{d^2y}{dx^2} = P(a-y) - Ka(l-x)$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{Pa - Ka(l-x)}{EI}$$

The solution to this differential equation is,

$$y=C_1 \cos x\sqrt{\frac{P}{EI}}+C_2 \sin x\sqrt{\frac{P}{EI}}+\frac{Pa-Ka(l-x)}{P}$$

$$\therefore\ y=C_1 \cos \alpha x+C_2 \sin \alpha x+\frac{Pa-Ka(l-x)}{P}$$

$$\therefore\ \frac{dy}{dx}=-C_1\alpha \sin \alpha x+C_2\alpha \cos \alpha x+\frac{Ka}{P}$$

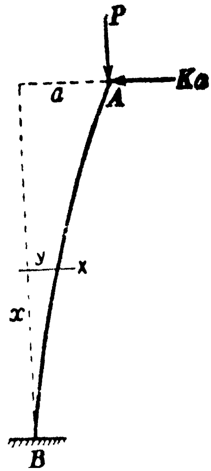


Fig. 619

At $x=0, \quad y=0,$

$$\therefore\ 0=C_1+\frac{Pa-Kal}{P}$$

$$\therefore\ C_1=-\frac{a(P-Kl)}{P}$$

At $x=0, \quad \frac{dy}{dx}=0$

$$\therefore\ 0=C_2\alpha+\frac{Ka}{P}$$

$$\therefore\ C_2=-\frac{Ka}{P\alpha}$$

At $x=l, \quad y=a$

$$a=-\frac{a(P-Kl)}{P}\cos \alpha l-\frac{Ka}{P\alpha}\sin \alpha l+a$$

$$\therefore\ \frac{KA}{P\alpha}\sin \alpha l=-\frac{a(P-Kl)}{P}\cos \alpha l$$

$$\tan \alpha l=-\left(\frac{P-Kl}{PK}\right)P\alpha$$

$$\tan \alpha l=-\frac{P\alpha}{K}+\alpha l$$

$$\therefore\ \frac{\tan \alpha l}{\alpha l}=1-\frac{P}{Kl}$$

§134. Column with initial curvature (axial loading)

Fig. 620 shows a column AB of length l with both its ends pinned. The column has an initial curvature having a central deflection a'.

Let at a distance x from the end B the initial deflection be y'. For purposes of analysis let us assume a sine curve for

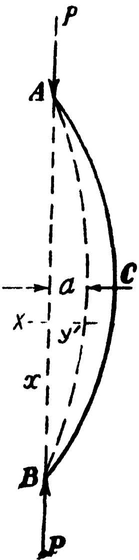


Fig. 620

the initial profile of the centre line of the column, so that

$$y'=a'\sin\frac{\pi x}{l}$$

$$\therefore\quad \frac{dy'}{dx}=\frac{\pi a}{l}\cos\frac{\pi x}{l}$$

$$\therefore\quad \frac{d^2y'}{dx^2}=-\frac{\pi^2a'}{l^2}\sin\frac{\pi x}{l} \qquad \text{...(i)}$$

When the loading on the column reaches the critical value P, the column will deflect to the form ACB, so that the deflection at x changes from y' to y. This happens due to the bending moment Py.

$$\therefore\quad EI\frac{d^2(y-y')}{dx^2}=-Py$$

$$\therefore\quad \frac{d^2(y-y')}{dx^2}=-\frac{P}{EI}y$$

$$\therefore\quad \frac{d^2y}{dx^2}+\frac{P}{EI}y=\frac{d^2y'}{dx^2}=-\frac{\pi^2a'}{l}\sin\frac{\pi x}{l} \qquad \text{...(ii)}$$

Let the solution to the above differential equation be given by

$$y=Ca'\sin\frac{\pi x}{l}$$

where C is a constant of integration.

$$\therefore\quad \frac{dy}{dx}=Ca'\frac{\pi}{l}\cos\frac{\pi x}{l}$$

and
$$\frac{d^2y}{dx^2}=-Ca'\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}$$

Substituting the expressions for y and $\frac{d^2y}{dx^2}$ in equation (*ii*), we have,

$$-Ca'\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}+\frac{P}{EI}Ca'\sin\frac{\pi x}{l}=-\frac{\pi^2a'}{l^2}\sin\frac{\pi x}{l}$$

$$\therefore\quad C\left\{\frac{\pi^2}{l^2}-\frac{P}{EI}\right\}=\frac{\pi^2}{l^2}$$

$$C=\frac{\dfrac{\pi^2}{l^2}}{\dfrac{\pi^2}{l^2}-\dfrac{P}{EI}}=\frac{1}{1-\dfrac{Pl^2}{\pi^2EI}}=\frac{1}{1-\dfrac{P}{P_e}}$$

$$C=\frac{P_e}{P_e-P}$$

Hence the equation to the deflected form of the column is given by

$$y=\frac{P_e}{P_e-P}a' \sin\frac{\pi x}{l} \qquad ..(iii)$$

The deflection will be a maximum at the mid-section C.

Let a be the central deflection

$\therefore$ At $\qquad x=\frac{l}{2}, \quad y=a$

$$a=\frac{P_e}{P_e-P}a'$$

Maximum $\qquad$ B.M. $=M$

$=$B.M. at the mid section

$=Pa$

$$=\frac{PP_e}{P_e-P}a'$$

Maximum compressive stress

$$=p_{max}=p_o+p_b=\frac{P}{A}+\frac{M}{Z}=\frac{P}{A}+\frac{My_c}{AK^2}$$

$$=\frac{P}{A}+\frac{PP_e}{P_e-P}a'\frac{y_c}{AK^2}$$

$$=\frac{P}{A}\left[1+\frac{P_e}{P_e-P}\cdot\frac{a'y_c}{K^2}\right]$$

$$=p_o\left[1+\frac{P_e}{P_e-P}\cdot\frac{a'y_c}{K^2}\right]$$

$$=p_o\left[1+\frac{p_e}{p_e-p}\cdot\frac{a'y_c}{K^2}\right]$$

or rearranging,

$$\left[\frac{p_{max}}{p_o}-1\right]\left[1-\frac{p_o}{p_e}\right]=\frac{a'y_c}{K^2}$$

§135. Laterally loaded struts

Case (i). *Strut pinned at both ends and subjected to an axial thrust P and a transverse point load W at the centre.*

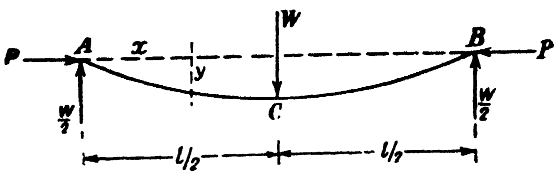


FIG. 621

Fig. 621 shows the laterally loaded strut.

Consider any section in AC distant x from the end A. The bending moment at the section is given by,

$$EI\frac{d^2y}{dx^2}=-Py-\frac{W}{2}x$$

$$\frac{d^2y}{dx^2}+\frac{P}{EI}y=-\frac{Wx}{2\,EI}$$

The solution to the above differential equation is,

$$y=C_1\cos x\sqrt{\frac{P}{EI}}+$$

$$C_2\sin x\sqrt{\frac{P}{EI}}-\frac{Wx}{2P} \qquad ...(i)$$

The slope at any section in AC is given by,

$$\frac{dy}{dx}=-C_1\sqrt{\frac{P}{EI}}\sin x\sqrt{\frac{P}{EI}}+$$

$$C_2\sqrt{\frac{P}{EI}}\cos x\sqrt{\frac{P}{EI}}-\frac{W}{2P} \qquad ...(ii)$$

At $\quad x=0,\ y=0$

$$C_1=0$$

At $\quad x=\frac{l}{2},\ \frac{dy}{dx}=0$

$$\therefore \quad 0=C_2\sqrt{\frac{P}{EI}}\cdot\cos\frac{l}{2}\sqrt{\frac{P}{EI}}-\frac{W}{2P}$$

$$\therefore \quad C_2=\frac{W}{2P}\cdot\sqrt{\frac{EI}{P}}\sec\frac{l}{2}\sqrt{\frac{P}{EI}}$$

Hence the deflection at any section in AC, is given by,

$$y=\frac{W}{2P}\sqrt{\frac{EI}{P}}\sec\frac{l}{2}\sqrt{\frac{P}{EI}}$$

$$\sin x\sqrt{\frac{P}{EI}}-\frac{Wx}{2P}$$

The maximum deflection will occur at the centre,

$\therefore$ At $\quad x=\frac{l}{2},\ y=y_{max}$

$$\therefore \quad y_{max}=\frac{W}{2P}\sqrt{\frac{EI}{P}}\sec\frac{l}{2}\sqrt{\frac{P}{EI}}$$

$$\sin\frac{l}{2}\sqrt{\frac{P}{EI}}-\frac{Wl}{4P}$$

$$\therefore \quad y_{max}=\frac{W}{2P}\sqrt{\frac{EI}{P}}\tan\frac{l}{2}\sqrt{\frac{P}{EI}}-\frac{Wl}{4P}$$

∴ Maximum bending moment

$$M_{max}=-Py_{max}-\frac{W}{2}\cdot\frac{l}{2}$$

$$=-\frac{W}{2}\sqrt{\frac{EI}{P}}\tan\frac{l}{2}\sqrt{\frac{P}{EI}}$$

We know, the expansion,

$$\tan\theta=\theta+\frac{\theta^3}{3}+\ldots\ldots$$

When θ is small,

$$\tan\theta=\theta+\frac{\theta^3}{3}$$

∴

$$M_{max}=-\frac{W}{2}\sqrt{\frac{EI}{P}}\left[\frac{l}{2}\sqrt{\frac{P}{EI}}+\frac{1}{3}\,\frac{l^3}{8}\,\frac{P}{EI}\sqrt{\frac{EI}{P}}\right]$$

$$=-\left[\frac{Wl}{4}+\frac{Wl^3}{48EI}\cdot P\right]$$

Case (*ii*). *Strut pinned at both ends and subjected to an axial thrust P and a lateral uniformly distributed load of intensity w per unit run.*

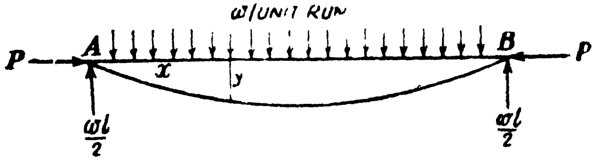


Fig. 622

Fig. 622 shows the laterally loaded strut. Consider any section distant x from the end A. The bending moment at the section is given by,

$$EI\,\frac{d^2y}{dx^2}=-Py+\frac{wx^2}{2}-\frac{wl}{2}x$$

∴

$$\frac{d^2y}{dx^2}+\frac{P}{EI}y=-\frac{wx\,(l-x)}{2\,EI}$$

The solution to the above differential equation is,

$$y=C_1\cos x\sqrt{\frac{P}{EI}}+C_2\sin x\sqrt{\frac{P}{EI}}-\frac{wx(l-x)}{2P}-\frac{wEI}{P^2}$$

∴

$$\frac{dy}{dx}=-C_1\sqrt{\frac{P}{EI}}\sin x\sqrt{\frac{P}{EI}}+$$

$$C_2\sqrt{\frac{P}{EI}}\cos x\sqrt{\frac{P}{EI}}+\frac{wx}{P}-\frac{wl}{2P}$$

At A, $x=0$,

$y=0$

$\therefore$ $$0=C_1-\frac{wEI}{P^2}$$

$\therefore$ $$C_1=\frac{wEI}{P^2}$$

At $x=\frac{l}{2}$, $\frac{dy}{dx}=0$

$\therefore$ $$0=-\frac{wEI}{P^2}\sqrt{\frac{P}{EI}}\sin\frac{l}{2}\sqrt{\frac{P}{EI}}$$
$$+C_2\sqrt{\frac{P}{EI}}\cos\frac{l}{2}\sqrt{\frac{P}{EI}}$$

$\therefore$ $$C_2=C_1\tan\frac{l}{2}\sqrt{\frac{P}{EI}}$$

Substituting the values of C_1 and C_2 in the deflection equation, we have

$$y=\frac{wEI}{P^2}\left[\cos x\sqrt{\frac{P}{EI}}+\tan\frac{l}{2}\sqrt{\frac{P}{EI}}\sin x\sqrt{\frac{P}{EI}}\right]-\frac{wx(l-x)}{2P}-\frac{wEI}{P^2}$$

Let y_c be the deflection at the centre.

$\therefore$ At $x=\frac{l}{2}$, $y=y_c$

$\therefore$ $$y_c=\frac{wEI}{P^2}\left[\cos\frac{l}{2}\sqrt{\frac{P}{EI}}+\tan\frac{l}{2}\sqrt{\frac{P}{EI}}\sin\frac{l}{2}\sqrt{\frac{P}{EI}}\right]-\frac{wl^2}{8P}-\frac{wEI}{P^2}$$

$\therefore$ $$y_c=\frac{wEI}{P^2}\left[\sec\frac{l}{2}\sqrt{\frac{P}{EI}}-1\right]-\frac{wl^2}{8P}$$

Maximum bending moment which will occur at C is given by,

$$M_c=-\frac{wl^2}{8}-Py_c$$

$$= -\frac{wl^2}{8} - P\left[\frac{wEI}{P^2}\left(\sec\frac{l}{2}\sqrt{\frac{P}{EI}} - 1\right) - \frac{wl^2}{8P}\right]$$

$$= -\frac{wEI}{P}\left[\sec\frac{l}{2}\sqrt{\frac{P}{EI}} \quad 1\right]$$

We know the expansion

$$\sec\theta = 1 + \quad + \frac{5\theta^4}{\lfloor 4} + \frac{61\theta^6}{\lfloor 6} + .$$

When θ is small,

$$\sec\theta = 1 + \quad + \frac{5\theta^4}{\lfloor 4}$$

$$= 1 + \frac{\theta^2}{2} \quad \frac{5}{24}$$

$$M_c = -\frac{wEI}{P}\left[\frac{1}{2} \quad \frac{l^2}{4} \quad \frac{P}{EI} + \frac{5}{24} \quad \frac{l^4}{16} \cdot \frac{P^2}{E^2I^2}\right]$$

$$= -\left[\frac{wl^2}{} \quad \frac{5}{384}\frac{wl^4}{EI}P\right]$$

Examples in Chapter 14

1. A strut 3 *metres* long is 6 *cm* in diameter. One end of the strut is fixed while its other end is hinged. Allowing factor of safety of 3 find the safe compressive load. Use Euler's formula. Take $E = 2 \times 10^6$ *kg/cm.*^{2} (*9303 kg.*)

2. A mild steel tube 8 *metres* long, 3 *cm* internal diameter and 4 *mm.* thick is used as a strut with both ends fixed. Find the collapsing load by Euler's formula. Take $E = 2\ 1 \times 10^6$ *kg./cm.*^{2} (*811 kg.*)

3. A column of timber section 15 *cm.* × 20 *cm.* is 6 *metres* long both ends being fixed. Find the safe load for the column. Use Euler's formula and allow a factor of safety of 3. Take $E = 175$ *t/cm* ^{2} (*36 tonnes*)

4. A steel column consists of two channels ISLC, 350 × 38·8 *kg./m.* per channel the clear spacing between them being 22 *cm.* Find the crippling load for the column if it has an effective length of 10 *metres.* Take $E = 2 \times 10^3$ *t/cm.*^{2}

For one channel, $A = 41{\cdot}47$ *cm.*^{2} Flange width = 10 *cm.* overall depth = 35 *cm.*

$I_{xx} = 9312{\cdot}2$ *cm.*^{4}
$I_{yy} = 394{\cdot}6$ *cm.*^{4}
$C_{yy} = 2{\cdot}41$ *cm.*
$t_w = 0{\cdot}74$ *cm.*
$t_f = 1{\cdot}25$ *cm.* (*336·8 t*)

5. An I section 300 *mm* × 150 *mm* is provided with a flange plate 200 *mm.* × 12 *mm.* for each flange. The composite member is used as a column with one end fixed and the other end hinged. Calculate the length of the member for which the crippling load by Rankine's formula and Euler's formula will be the same.

Take $E=2100\ t/cm.^2$, $f_c=3{\cdot}3\ t/cm.^2$ and $\alpha=\frac{1}{7500}$

(*12·60 m*)

6. A hollow circular column 2 *metres* long has one of its ends fixed and the other end free and has to support an axial load of 50,000 *kg*. The internal diameter is 0·8 times the external diameter. Allowing a factor of safety of 4 calculate the external diameter and the thickness of metal. Use Rankine's formula

Take $f_c=3300\ kg./cm.^2$ and $\alpha=\frac{1}{7500}$

(*17·7 cm., 1·8 cm.*)

7. A steel bar 2 *cm.* × 3 *cm.*, 2 *metres* long is subjected to a gradually increasing axial compressive load. Find the buckling load using Euler's formula. Find also the maximum lateral deflection corresponding to the buckling condition. Both ends of the rod may be taken as hinged. Take $E=2{\cdot}1\times10^6\ kg./cm.^2$ and the yield stress of steel $=2400\ kg./cm.^2$ (*1036 kg.* ; *4·3 cm.*)

8. In a compression test on a short length of a tube of 6 *cm.* external diameter and with thickness 0·5 *cm.* it failed at a load of 37 *t.* When the same is tested as a strut with both ends hinged, 2 *metres* long, it failed at a load of 18 *t*. Find the value of α in Rankine's formula.

$\left(\frac{1}{10{,}000}\right)$

9. A strut 3 *metres* long with both ends hinged consists of two equal angles 100 × 100 × 10 *mm.* the spacing between the angles being 1 *cm.* Find the safe compressive load for the strut allowing a factor of safety of 4. Use Rankine's formula. Take $f_c=3200\ kg./cm.^2$ and $\alpha=\frac{1}{7500}$. Properties of one angle are given below

$A=19{\cdot}03\ cm.^2$

$I_{xx}=I_{yy}=177\ cm.^4$ (*13298 kg.*)

10. Find the minimum value of the slenderness ratio of a mild steel column for which Euler's formula is vaild. Take $f_c=3300\ kg./cm.^2$ and $E=2{\cdot}1\times10^6\ kg./cm.^2$ (*79·27*)

15

Riveted Joints

Rivets are used to connect together permanently two or more plates. Rivets are permanent fastenings. Rivets have their greatest application in boiler work, connections of truss members at joints, built up columns, plate-girders, *etc.*

Riveted joints are mainly of two types,

viz., Lap joints and Butt joints. Two plates are said to be connected by a Lap joint when the connected ends of the plates lie in parallel planes. In a butt joint the connected end of the plates lie in the same plane. The abutting ends of the plates are covered by one or two cover plates or strap plates.

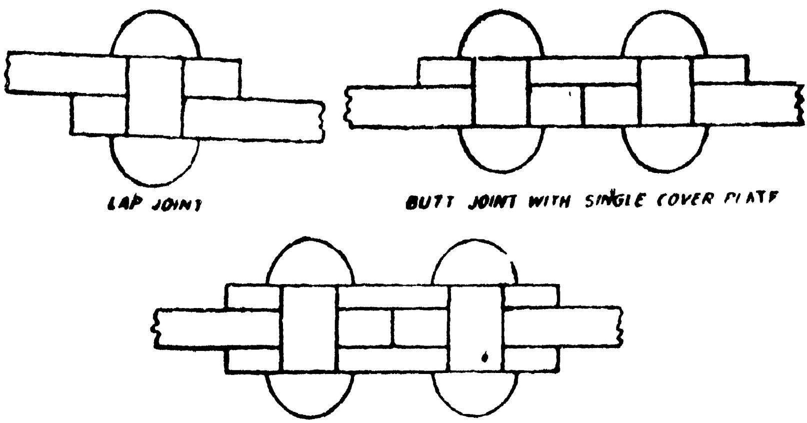


Fig. 623

Lap joints may be further classified into single riveted, double riveted, treble riveted, *etc.*, joints depending on whether one, two, three, *etc.*, rows of rivets are used for the connection. Very often single and double riveted joints are used. Fig. 623 shows single and double riveted lap joints.

Butt joints may be classified into single riveted, double riveted and treble riveted butt joints depending upon whether one, two or three rows of rivets are used *on each side of the joint.*

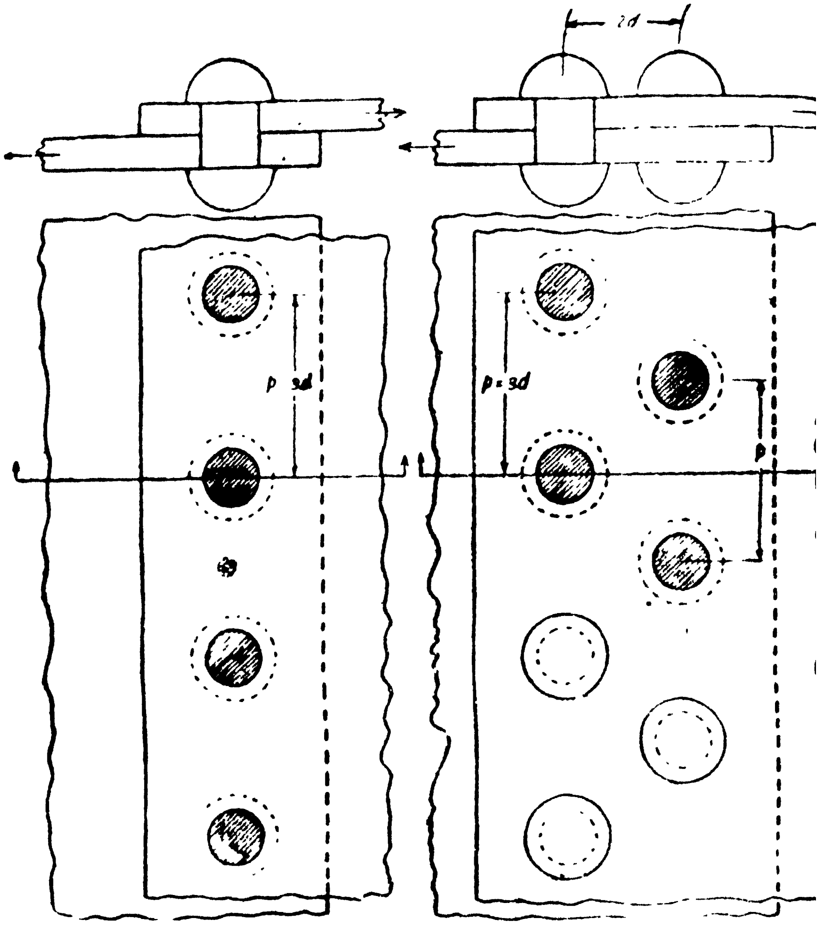


Fig. 624

§136. Failure of a riveted joint

A riveted joint may fail in any of the following manners :

(*i*) *By tearing of the plate between the rivet hole and the edge of the plate.*

Such a failure is due to insufficient margin. If d be the diameter of the rivet, then the effective margin, *i.e.*, the distance between the centre of the rivet and the nearest edge of the plate should be at least $1{\cdot}5d$, in order a failure may not occur.

(*ii*) *By tearing of plates between rivets.*

This failure is due to excessive tensile stress in the plates on the section corresponding to the line of rivets. Let p be the pitch of rivets. Consider one pitch length. Fig. 628.

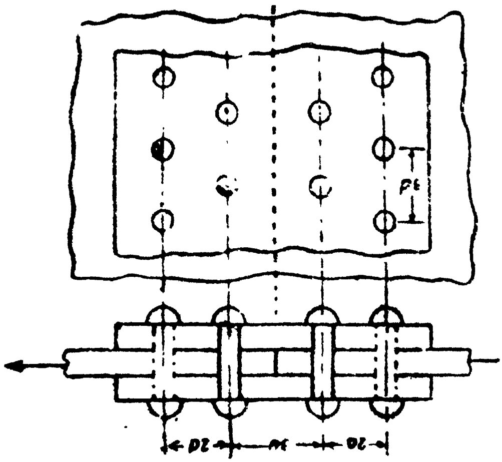


Fig. 625

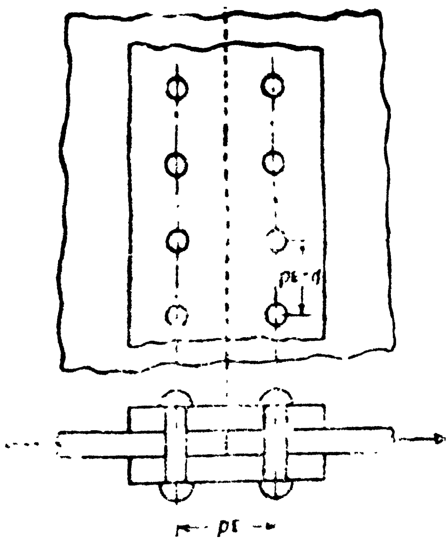


Fig. 626

The safe tensile load that the plates can withstand for one pitch length is called the tearing strength.

Let f_t = safe tensile stress in the plates.

and t = thickness of the plate.

Hence tearing strength per pitch length

$= p_t = f_t \times$ net area of plate

$= (p-d)t$

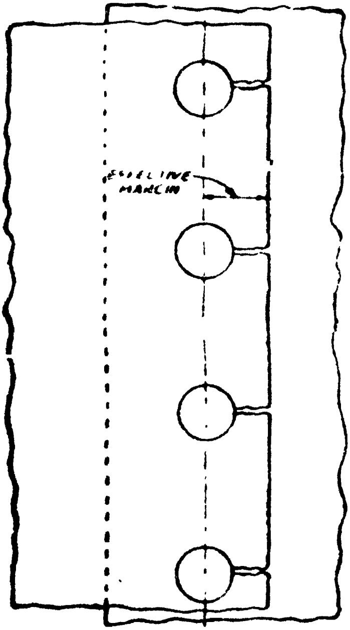


Fig. 627

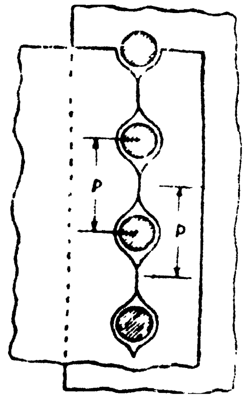


Fig. 628

(iii) Failure due to shearing of rivet

Fig. 629 shows a single riveted lap joint. Consider one pitch length of the rivet. When the load per pitch length is large it is possible that the rivet may shear off.

If f_s = safe shearing stress for the rivet then the safe load per pitch length to prevent failure

$$= P_s = f_s \, \pi \, d^2$$

For the case cited above, only one rivet is covered by a pitch length. If the joint had been a double-riveted joint, two rivets would be covered in one pitch length, and for this case the shearing strength per pitch length

$$= P_s = 2 \times f_s \frac{\pi d^2}{4}$$

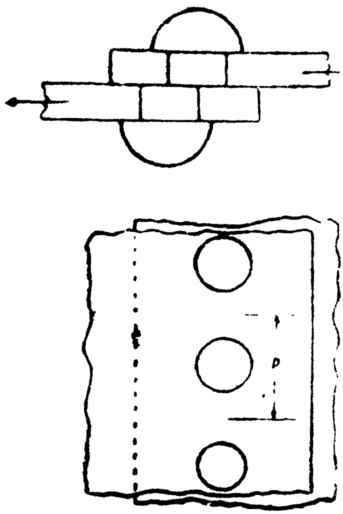


Fig. 629

In general, in a lap joint if n rivets are covered per pitch length, the shearing strength per pitch length would be $n \times \left(f_s \frac{\pi d^2}{4} \right)$

In a lap joint when a rivet is liable to fail by shear, there is only one plane along which the rivet can fail by shear. Hence rivets of a lap joint are said to be in single shear.

The strength of one rivet in shear is $f_s \frac{\pi d^2}{4}$. This is called the shear value of one rivet.

But in a butt joint when a rivet will fail by shear, it will simultaneously fail along two planes Hence rivets used in such a joint are said to be in double shear.

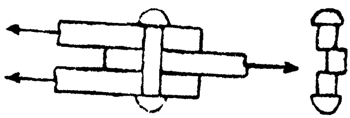


Fig. 630

Safe load which a rivet can withstand in double shear

$$=2 \times \frac{f_s \pi d^2}{4}.$$

Strength of the joint per pitch length

$$=n \times \left(\frac{2 f_s \pi d^2}{4} \right)$$

where n = number of rivets covered per pitch length,
$n=1$ for a single riveted butt joint.
$n=2$ for a double riveted butt joint.
$n=3$ for a treble riveted butt joint.

(*iv*) *Failure by bearing or crushing of rivet or plate.*

Suppose in the lap joint shown, the top plate is weaker than the bottom plate.

If the top plate be pulled bearing stress is induced between the plate at A and the rivet. If these stresses are high it is quite possible that the plate or rivet may be crushed. Fig 631 shows a case in which the top plate has failed due to excessive bearing stress.

If f_b = allowable bearing stress, then for design purposes the safe load on the rivet $= P_b = f_b dt$.

where d = diameter of rivet
and t = thickness of plate

P_b is called the *bearing value* of the rivet.

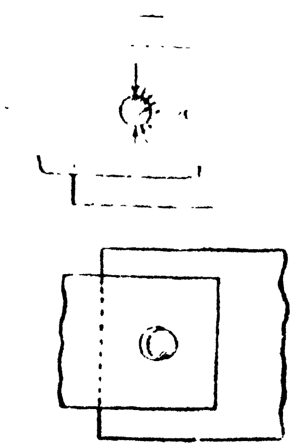


Fig. 631

Hence safe load per pitch length of the joint

$= n f_b dt$, where n is the number of rivets covered per pitch length.

§137. Efficiency of a joint

Consider one pitch length of a joint. Let P_t, P_s and P_b be the safe loads per pitch length from tearing, shearing and bearing considerations. Let p be the pitch of rivets and t the thickness of the plate.

Safe pull on a solid plate for a length p would be $P = f_t pt$

$$\text{Efficiency} = \eta = \text{least of } P_t, P_s \text{ and } P_b$$

§138. Diameter of rivet

The diameter of rivet to suit the thickness of a plate may be determined from the following empirical formulae.

1. Unwin's formula

$d =$ dia. of rivet (*mm.*)

$t =$ thickness of plate (*mm.*)

$$d = 6{\cdot}05 \sqrt{t}$$

2. The French formula

$d =$ dia. of rivet (*mm.*)

$t =$ thickness of plate (*mm.*) not exceeding 15 *mm.*

$$d = 1{\cdot}5t + 4$$

3 The German formula

$d =$ dia. of rivet (*cm.*)

$t =$ thickness of plate (*cm.*)

$$d = \sqrt{5t} - 0{\cdot}2$$

Problem 430. *Find the efficiencies of the following riveted joints :*

(*i*) *Single riveted lap joint for 8 mm. thick plates with 16 mm. diameter rivets at a pitch of 5 cm. centres.*

(*ii*) *Double riveted lap joint for 8 mm. thick plates with 16 mm. diameter rivets at a pitch of 7·5 cm. centres.*

For each case, take the finished diameter of rivets to be 15·5 mm. Adopt the following working stresses :

Permissible tensile stress in steel plates	$= 1500$ *kg./cm*^{2}.
Permissible bearing stress in rivets	$= 1575$ *kg./cm*^{2}.
Permissible shearing stress in rivets	$= 785$ *kg./cm*^{2}.

Solution. ( ) *Single riveted lap joint for 8 mm. thick plates at a pitch of 5 cm. centres.*

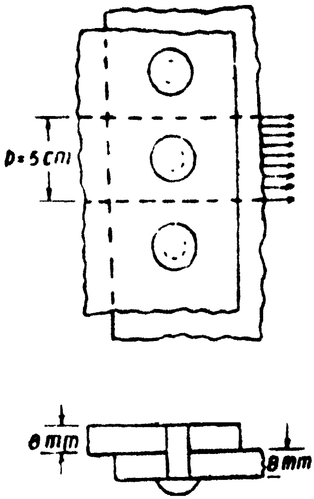


Fig. 632

Fig. 632 shows the single riveted lap joint.

Diameter of rivets

$=17{\cdot}5$ *mm.*

$=1{\cdot}75$ *cm.*

Consider one pitch length of the joint.

Number of rivets covered per pitch length = 1

Rivets are in single shear.

(*a*) Tearing strength per pitch length $=P_t=(p-d)\,t\,f_t$

$(5-1{\cdot}75)0{\cdot}8\times1500$ *kg.*

$=3900$ *kg.*

(*b*) Bearing strength per pitch length

$=P_b=1\times f_b\,dt$

$=1\times1575\times1{\cdot}75\times0{\cdot}8$ *kg.*

$=2205$ *kg.*

(*c*) Shearing strength per pitch length

$$=P_s=1\times f_s\times\frac{\pi d^2}{4}$$

$$=1\times785\times\frac{\pi}{4}(1{\cdot}75)^2\ kg.$$

$$=1888\ kg.$$

$\therefore$ Least strength per pitch length

$=1888$ *kg.*

If the plate had been solid, without any joint, the strength of the solid plate would be,

$$P=p\,t\,f_t$$

$$=5\times0{\cdot}1\times1500\ kg.$$

$$=6000\ kg.$$

$$\therefore\ \text{efficiency of the joint}=\frac{\text{Least strength per pitch length}}{\text{Strength of solid plate}}\times100$$

$$=\frac{1888}{6000}\times100\%$$

$$=31{\cdot}47\%$$

(*ii*) *Double riveted lap joint for 8 mm. thick plates at a pitch 7·50 cm. centres.*

Fig. 633 shows the double riveted lap joint. Consider one pitch length of the joint. Number of rivets covered per pitch length=2. Rivets are in single shear.

(*a*) Tearing strength per pitch length

$=P_t=(p-d)\,t\,f_t$

$=(7{\cdot}5-1{\cdot}75)0{\cdot}8\times1500$ *kg*.

$=6900$ *kg*.

(*b*) Bearing strength per pitch length

$=P_b=2\,(f_b\,dt)$

$=2\times(1575\times1{\cdot}75\times0{\cdot}8)$ *kg*.

$=4410$ *kg*

(*c*) Shearing strength per pitch length

$=P_s=2\left(f_s\,\frac{\pi d^2}{4}\right)$

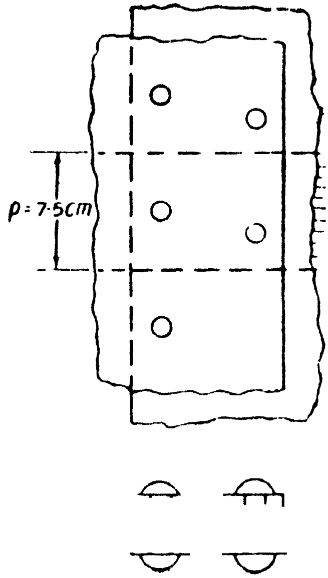


Fig. 633

$=2\times785\times\frac{\pi}{4}\,(1{\cdot}75)^2$ *kg*.

$=3776$ *kg*.

Least strength $=3776$ *kg*.

Strength of solid plate

$=p\,t\,f_t$

$=7{\cdot}5\times0{\cdot}8\times1500$ *kg*.

$=9000$ *kg*.

$$\text{Efficiency}=\frac{\text{Least strength}}{\text{Strength of solid plate}}$$

$=\frac{3776}{9000}\times1{\cdot}5\%$

$=41{\cdot}95\%$

Problem 431. *A single riveted double cover butt joint in plates 14 mm. thick is made with 22 mm. diameter rivets at a pitch of 9 cms. If the allowable tensile, shear and bearing stresses are 1400 kg./cm.², 800 kg./cm.² and 1600 kg./cm.² respectively, find the safe load per pitch length of the joint. Find also the efficiency of the joint.*

Solution. Since this is a butt joint with two cover plates, the rivets are in double shear. Since the joint is single riveted, number of rivets on one side of the joint, covered in one pitch length equals 1.

Consider one pitch length of the joint.

Tearing strength $=P_t=(p-d)\,t\,f_t$

$=(9-2{\cdot}2)\ 1{\cdot}4\times1400\ kg.$

$=13330\ kg.$

Shearing strength $=P_s=1\times\left(2f_s\,\frac{\pi d^2}{4}\right)$

$=1\times2\times800\times\frac{\pi}{4}\ (2{\cdot}2)^2\ kg.$

$=6082\ kg.$

Bearing strength $=P_b=1\times(f_b d t)$

$=1\times1600\times2{\cdot}2\times1{\cdot}4$

$=4928\ kg.$

∴ Safe load per pitch length

$=4928\ kg.$

Strength of solid plate

$=P=p t f_t$

$=9\times1{\cdot}4\times1400\ kg.$

$=17640\ kg,$

∴ Efficiency of the joint

$=\frac{\text{Least strength of joint}}{\text{Strength of solid plate}}\times100$

$=\frac{4928}{17640}\times100$

$=27{\cdot}94\%$

Problem 432 (SI). *A single riveted butt joint is used to connect two plates 12 mm thick. The rivets are 25 mm in diameter and are provided at a pitch of 10 cm. The permissible stresses in tension, shear and bearing are 120 N/mm.², 94·5 N/mm.² and 212·5 N/mm.². Find the efficiency.*

Solution. Single riveted butt joint,

$t=12\ mm.=1{\cdot}2\ cm.$

$d=25\ mm.=2{\cdot}5\ cm.$

$p=10\ cm.$

$f_s=94{\cdot}5\ N/mm.^2=9450\ N/cm.^2$

$f_b=212{\cdot}5\ N/mm.^2=21250\ N/cm.^2$

$f_t=120\ N/mm.^2=12000\ N/cm.^2$

$\eta=?$

The rivets are in double shear.

Number of rivets covered per pitch length $=1$

Consider one pitch length of the joint.

Shearing strength per pitch length

$$=P_s=[1]\left(2f_s\frac{\pi d^2}{4}\right)$$

$$=1\times2\times9450\times\pi\times\frac{2{\cdot}5^2}{4}\ kg.$$

$$=92775\ Newton$$

Bearing strength per pitch length

$$=P_b=[1]\,(f_b dt)$$

$$=1\times21250\times2{\cdot}5\times1{\cdot}2\ kg.$$

$$=63750\ Newton$$

Tearing strength per pitch length

$$=P_t=(p-d)\,t\,f_t$$

$$=(10-2{\cdot}5)\ 1{\cdot}2\times12000\ Newton$$

$$=108000\ Newton$$

∴ **Safe load per pitch length**

$$=63750\ Newton$$

∴ $$Efficiency=\frac{\text{Safe load per pitch length}}{p\,t\,f_t}\times100\%$$

$$=\frac{63750}{10\times1{\cdot}2\times1200}\times100\%$$

$$=44{\cdot}27\%$$

Problem 433. *Two plates of 12 mm. thickness are to be connected in a double riveted double cover butt joint using 18 mm. diameter rivets at a pitch of 8 cms. If the ultimate tensile, shearing and bearing stresses are 4600 kg./cm.², 3200 kg./cm.² and 6400 kg./cm.² respectively. find the pull per pitch length at which the joint will fail. Find also the efficiency of the joint.* (*AMIE, May 1974*)

Solution. Since this is a butt joint with two cover plates, the rivets are in double shear. Further, since the joint is double riveted, the number of rivets on one side of the joint covered in one pitch length equals 2.

Consider one pitch length of the joint.

Tearing strength $=P_t=(p-d)\,t\,f_t$

$$=(8-1{\cdot}8)1{\cdot}2\times4600\ kg.$$

$$=34{,}220\ kg.$$

Shearing strength

$$=P_s=2\times\left(\frac{2f_s\pi d^2}{4}\right)$$

$$=2\times2\times3200\times\frac{\pi\times1{\cdot}8^2}{4}\ kg.$$

$$=32570\ kg.$$

Bearing strength

$$=P_b=2\times(f_b\,dt)$$
$$=2\times6400\times1.8\times1.2\ kg.$$
$$=27650\ kg.$$

$\therefore$ The joint will fail at a pull of 27650 *kg*.

Ultimate strength of solid plate

$$=P=pt\,f_t$$
$$=8\times1.2\times4600\ kg.$$
$$=44160\ kg.$$

$$\therefore\quad \text{Efficiency}=\frac{27650}{44160}\times100$$
$$=62.62\%$$

Problem 434. *A single riveted double cover butt joint in a structure is used for connecting two plates 12 mm. thick. The diameter of rivets is 24 mm. The permissible stresses are 1200 kg./cm.² in tension, 1000 kg./cm.² in shear and 2000 kg./cm.² in bearing. Calculate the necessary pitch and the efficiency of the joint.* (*AMIE, May 1967*)

Solution. Since this is a butt joint with two cover plates, the rivets are in double shear. Further, since the joint is single riveted, number of rivets on one side of the joint covered per pitch length equals 1.

Let the pitch length of the joint be *p cm*

Consider one pitch length of the joint.

Tearing strength

$$=P_t=(p-d)\,t\,f_t$$
$$=(p-2.4)\ 1.2\times1200\ kg.$$
$$=1440\ (p-2.4)\ kg$$

Shearing strength

$$=P_s=1\times\left(\frac{2f_s\pi d^2}{4}\right)$$
$$=1\times2\times1000\times\frac{\pi\times2.4^2}{4}$$
$$=9050\ kg$$

Bearing strength

$$=P_b=1\times(f_b\,dt)$$
$$=1\times2000\times2.4\times1.2$$
$$=5760\ kg.$$

Equating P_t to lesser of P_s and P_b,

$$1440\ (p-2.4)=5760$$
$$\therefore\quad p=6.4\ cm.$$

Strength of solid plate

$$=P=ptf_t$$
$$=6{\cdot}4\times1{\cdot}2\times1200$$
$$=9216\ kg.$$

$\therefore$ Efficiency $=\dfrac{5760}{9216}\times100$

$$=62{\cdot}5\%$$

Problem 435. *A double riveted double cover butt joint is used for connecting plates 1·2 cm. thick. The diameter of the rivets is 2·2 cm. The permissible stresses are 1000 kg./cm.² in tension, 800 kg./cm.² in shear and 1600 kg./cm.² in bearing. Draw a neat sketch of the joint and calculate the necessary pitch and the efficiency of the joint.*

(AMIE, November 1966)

Solution. Since the joint is a butt joint the rivets are in double shear. Further, since the joint is double riveted, the number of rivets on one side of the joint covered in one pitch length equals 2.

Let the pitch be p *cm.*

Consider one pitch length of the joint.

Tearing strength

$$=P_t=(p-d)\,t\,f_t$$
$$=(p-2{\cdot}2)\,1{\cdot}2\times1000\ kg.$$
$$=1200\,(p-2{\cdot}2)\ kg.$$

Shearing strength

$$=P_s=2\times\left(\frac{2f_s\pi d^2}{4}\right)$$
$$=2\times2\times800\times\pi\times\frac{2{\cdot}2^2}{4}\ kg.$$
$$=12160\ kg.$$

Bearing strength

$$=P_b=2\times f_b\,dt$$
$$=2\times1600\times2{\cdot}2\times1{\cdot}2$$
$$=8448\ kg.$$

Equating P_t to the lesser of P_s and P_b,

$$1200\,(p-2{\cdot}2)=8448$$

$\therefore$ $p=9{\cdot}24$ *cm.*

say $9{\cdot}25$ *cm.*

Strength of solid plate

$$=P=pt\,f_t$$
$$=9{\cdot}25\times1{\cdot}2\times1000\ kg.$$
$$=11{,}100\ kg.$$

$$\therefore \qquad \eta = \frac{8448}{11{,}100} \times 100$$

$$= 76{\cdot}1\%$$

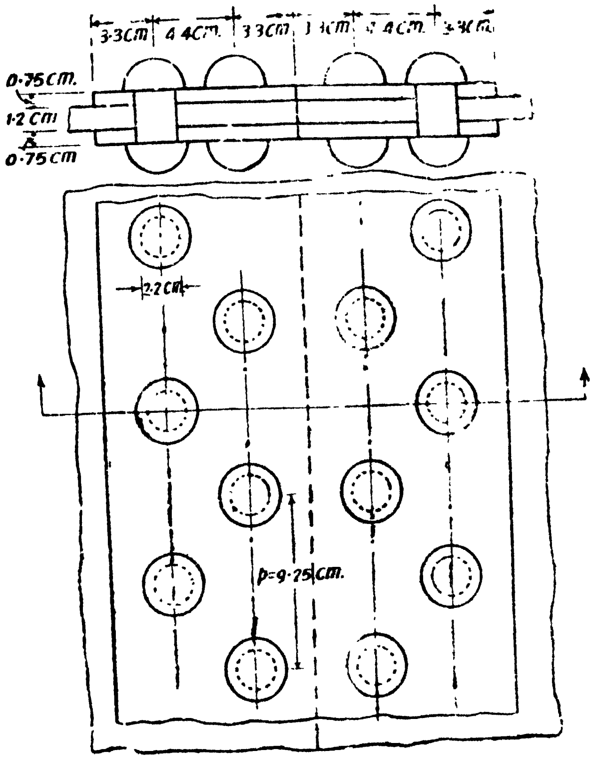


Fig. 634

Problem 436. *A thin cylindrical shell 1500 mm. in diameter is made of 12 mm. plates. The circumferential joint is a single riveted lap joint with 22 mm. diameter rivets at a pitch of 50 mm. If the ultimate tensile stress in the plate is 4500 kg./cm.² and the ultimate shearing and bearing stresses for the rivets are 3000 kg./cm.² and 6000 kg./cm.² respectively, calculate the efficiency of the joint.*

(*AMIE, Winter 1974*)

Solution.

$t = 1{\cdot}2$ *cm.*

$d = 2{\cdot}2$ *cm.*

$p = 5$ *cm.*

$f_t = 4500$ *kg./cm.² ultimate*

$f_s = 3000$ *kg./cm.² ultimate*

$f_b = 6000$ *kg./cm.² ultimate*

Single riveted lap joint.

The rivets are in single shear.

Number of rivets covered per pitch length = 1

Consider one pitch length of the joint.

Shearing strength per pitch length

$$= P_s = [1] f_s \frac{\pi d^2}{4}$$

$$= 1 \times 3000 \times \pi \times \frac{2{\cdot}2^2}{4}$$

$$= 11404 \ kg.$$

Bearing strength per pitch length

$$= P_b = [1] f_b \ dt$$

$$= 1 \times 6000 \times 2{\cdot}2 \times 1{\cdot}2$$

$$= 15840 \ kg.$$

Tearing strength per pitch length

$$= P_t = (p-d) \ t \ f_t$$

$$= (5 - 2{\cdot}2) \ 1{\cdot}2 \times 4500$$

$$= 15120 \ kg.$$

Least strength per pitch length

$$= 11404 \ kg.$$

$$\text{Efficiency} = \frac{\text{Least strength per pitch length}}{p t f_t} \times 100\%$$

$$= \frac{11404}{5 \times 1{\cdot}2 \times 4500} \times 100$$

$$= 42{\cdot}24\%$$

Problem 437. *Find the efficiency of a double riveted two strap butt joint, if the main plates are 20 mm. thick, diameter of rivet is 22 mm. and rivet pitch is 8·4 cm. The safe stresses in tension, bearing and shear are respectively 800, 1400 and 600 kg./cm.².*

(AMIE, Winter 1976)

Solution. Double riveted butt joint.

$t = 2 \ cm.$

$d = 2{\cdot}2 \ cm.$

$p = 8{\cdot}4 \ cm.$

$f_t = 800 \ kg./cm.^2$

$f_b = 1400 \ kg./cm.^2$

$f_s = 600 \ kg./cm.^2$

The rivets are in double shear.

Number of rivets covered per pitch length = 2

Consider one pitch length of the joint.

Shearing strength per pitch length

$$=P_s=[2]\ 2\ f_s\ \frac{\pi d^2}{4}$$

$$=2\times 2\times 600\times\pi\times\frac{2{\cdot}2^2}{4}$$

$$=9123{\cdot}2\ kg.$$

Bearing strength per pitch length

$$=P_b=[2]\ f_c\ dt$$

$$=2\times 1400\times 2{\cdot}2\times 2$$

$$=12320\ kg.$$

Tearing strength per pitch length

$$=P_t=(p-d)\ t\ f_t$$

$$=(8{\cdot}4-2{\cdot}2)2\times 800$$

$$=9920\ kg.$$

∴ Safe load per pitch length

$$=9123{\cdot}2\ kg.$$

∴ Efficiency $= \dfrac{\text{Safe load per pitch length}}{\text{Strength of the solid plate per pitch length}}$

$$=\frac{9123{\cdot}2}{8{\cdot}4\times 2\times 800}\times 100$$

$$=67{\cdot}88\%$$

Problem 438 *Find the suitable pitch for a riveted lap joint for plates 1 cm. thick if safe working stresses in tension in the plates and crushing and shearing of the rivet material are respectively 1500 kg./cm.², 2125 kg./cm.² and 945 kg./cm.² in the following types of joints :*

(i) Single riveted, and (ii) Double riveted. Find also the efficiency of the joint in the above two cases. Take $d=1{\cdot}9\sqrt{t}$.

(*AMIE, Summer 1977*)

Solution Diameter of rivet

$$d=1{\cdot}9\sqrt{t}=1{\cdot}9\sqrt{1}=1{\cdot}9\ cm.\ \text{say}\ 2\ cm.$$

(i) Single riveted lap joint Rivets are in single shear.

Number of rivets covered per pitch length = 1.

Consider one pitch length of the joint.

Shearing strength per pitch length

$$=P_s=[1]\ f_s\ \pi\ \frac{d^2}{4}$$

$$=1\times 945\times\pi\times\frac{2^2}{4}$$

$$=2968{\cdot}8\ kg.$$

Bearing strength per pitch length

$$=P_b=[1]\,f_b\,dt$$
$$=1\times2125\times2\times1$$
$$=4250\ kg.$$

Tearing strength per pitch length

$$=P_t=(p-d)\,t\,f_t$$
$$=(p-2)1\times1500$$
$$=1500\,(p-2)$$

Equating the tearing strength to the lesser of bearing and shearing strengths

$$1500\,(p-2)=2968{\cdot}8$$

$$\therefore\qquad p=3{\cdot}98\ cm.$$

Generally pitch should not be less than

$$3d=3\times2$$
$$=6\ cm.$$

Hence provide a pitch of 6 *cm*

$$\text{Efficiency}=\eta=\frac{\text{least of } P_s.P_b.P_t}{pt\,f_t}\times100$$
$$=\frac{2968{\cdot}8}{6\times1\times1500}\times100\%$$
$$=33\%.$$

(*ii*) *Double riveted lap joint.* Rivets are in single shear.

Number of rivets covered per pitch length = 2.

Consider one pitch length of the joint

Shearing strength per pitch length

$$=P_s=[2]\,f_s\,\frac{\pi d^2}{4}$$
$$=2\times945\times\frac{\pi\times2^2}{4}$$
$$=5937{\cdot}6\ kg.$$

Bearing strength per pitch length

$$=P_b=[2]\,f_b\,dt$$
$$=2\times2125\times2\times1$$
$$=8500\ kg.$$

Tearing strength per pitch length

$$=P_t=(p-d)\,t\,f_t$$
$$=(p-2)1\times1500$$
$$=1500\,(p-2)$$

Equating the tearing strength to the lesser of bearing and shearing strengths,

$$1500\,(p-2)=5937{\cdot}6$$

$$\therefore \qquad p=5{\cdot}96\ cm.\ \text{say } 6\ cm.$$

$$\text{Efficiency}=\frac{5937{\cdot}6}{6\times1\times1500}\times100\%$$

$$=65{\cdot}97\%.$$

§139. Riveted joints in Structural steel werk

A truss or bridge is fabricated by connecting a number of members. A joint for connecting a number of members is made by riveting the members to a common plate called the gusset plate.

The members meeting at the joint *A* for instance may be connected as shown in Fig. 635.

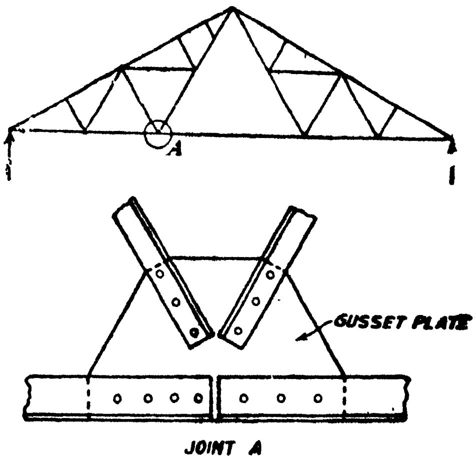


Fig. 635

The number of rivets required to connect a member to the gusset plate

$$=\frac{\text{Force in the member}}{\text{Strength of one rivet}}$$

The minimum number of rivets for the connection of a member to a gusset plate is 2.

Problem 439 *Four members OA, OB, OC and OD are to be connected at a joint O. Details of the members and the loads carried by them are shown in Fig 636. If the connection to the gusset plate be made by 20 mm. diameter rivets find the number of rivets required to connect each member. Thickness of the gusset plate is 8 mm. Sketch the joint. Take* $f_s=700$ *kg./cm.*^{2} *and* $f_b=1400$ *kg /cm*^{2}.

Solution.

Member OA

Load in *OA*

$=12700\ kg.$

Rivets connecting this member are in double shear.

Rivet value in double shear

$$=2f_s\frac{\pi d^2}{4}$$

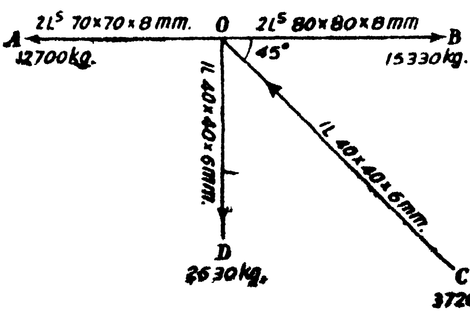


Fig. 636

$=2\times700\times\frac{\pi}{4}\times2^2$ *kg.*

$=4000$ *kg.*

Rivet value in bearing $=f_b\ dt.$

$=1400\times2\times0.8$ *kg.*

$=2240$ *kg.*

$\therefore$ Number of rivets required

$$=\frac{12700}{2240}$$

$=6$ rivets

Member OB

Rivets connecting this member are also in double shear.

$\therefore$ Number of rivets required

$$=\frac{15330}{2240}$$

$=7$ rivets

Member OD

Rivets connecting this member are in single shear.

Rivet value in single shear

$$=f_s\frac{\pi d^2}{4}$$

$$=700\times\frac{\pi}{4}\times2^2$$

$=2200$ *kg.*

Rivet value in bearing $=f_b\ dt$

$=1400\times2\times0.6$ *kg.*

$=1680$ *kg.*

$\therefore$ Least rivet value $=1680$ *kg.*

$\therefore$ Number of rivets required

$$=\frac{2630}{1680}$$

This is less than 2. Hence let us provide two rivets.

Member OC

Rivets connecting this member are in single shear.

$\therefore$ Number of rivets required

$$=\frac{3720}{1680}$$

$=3$ rivets

Fig. 637 shows details of the joint O.

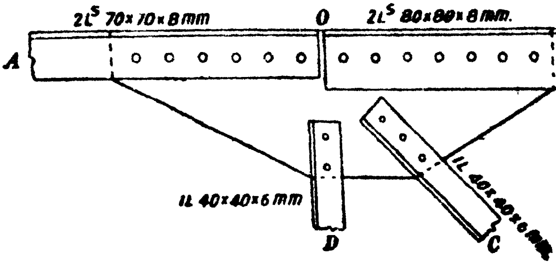
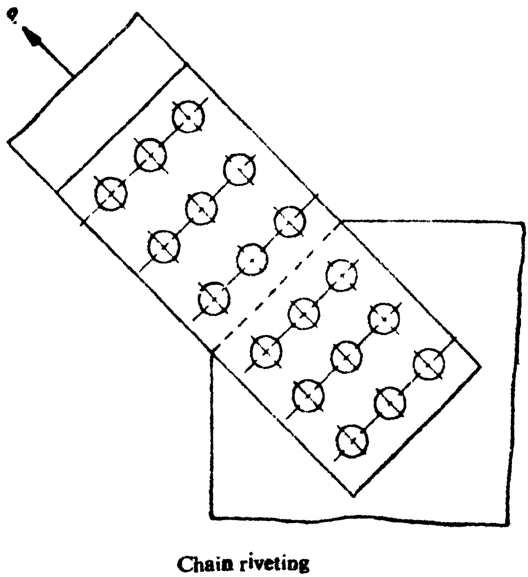


Fig. 637

§140. Chain riveting and diamond riveting

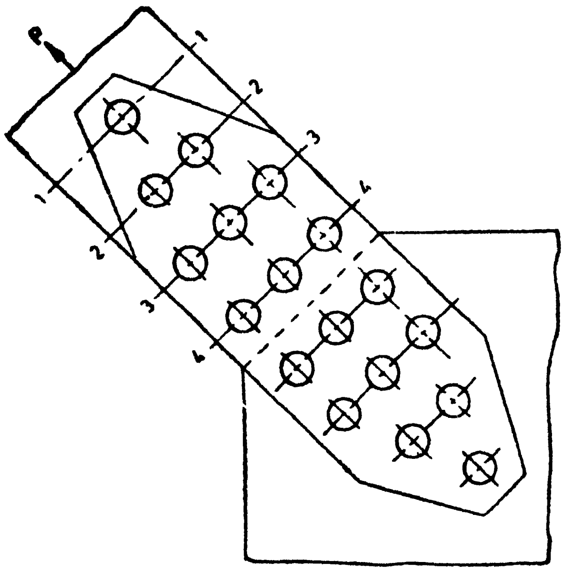
In the case of bridge truss members it will become necessary to provide a large number of rivets to connect a member to the gusset plate. The strength of the joint will depend upon the actual arrangement of rivets. Suppose a flat plate is used as a member of a truss. Let 9 rivets be required to transmit the load in the member to the gusset plate. Let a butt joint be provided. Two arrangements are possible as shown in Figs. 638 and 639.

In the arrangement shown in Fig. 638 the nine rivets have been arranged in three rows with three rivets in each row. This arrangement is called chain riveting.



Chain riveting
Fig. 638

The width of the flat needed to transmit the tension P must be determined such that the tensile stress on the weakest section does not exceed the permissible stress f_t. In the chain riveted arrangement



Diamond riveting
Fig. 639

shown in Fig. 638 the section is weakened by three rivet holes. Let d be the diameter of rivet holes. The minimum width b of the plate shall therefore be such that

$$P=(b-3d)\, t f_t$$

or

$$b=\frac{P}{t f_t}+3d$$

In the diamond-riveted arrangement the nine rivets have been arranged in four rows with one rivet in the first row, two rivets in the second row, three rivets in the third row and three rivets in the fourth row.

Suppose we assume that the section 1—1 passing through the first rivet hole is the weakest section. The width of the flat plate required is given by

$$P=(b-d)t f_t$$

or

$$b=\frac{P}{t f_t}+d$$

The width of plate required in this arrangement is less than what is required in chain riveting by $2d$. This reduction in the requirement of width of the flat plate is a considerable saving in the case of long members of bridges. Hence diamond riveting is usually preferred to chain riveting.

Generally in diamond riveting the weakest section is the section 1—1 passing through one rivet hole of the first row. The successive sections 2—2, 3—3 and 4—4 are stronger.

If f_t is the limiting tensile stress for the plate the pull required to tear the plate at section 1—1

$$=P_1=(b-d)\,t\,f_t$$

If the plate section 2—2 must fail, the rivet in the first row should also fail. Hence the pull required to tear the plate at the section 2—2

$$=P_2=(b-2d)t\,f_t+\text{strength of 1 rivet in the row 1.}$$

Similarly if the plate section 3—3 must fail, the rivets in the first and second rows should also fail, *i.e.*, three rivets must fail. Hence the pull required to tear the plate at section 3—3

$$=P_3=(b-3d)\,t\,f_t+\text{strength of three rivets.}$$

Similarly, the pull required to tear the plate at section 4—4

$$=P_4=(b-3d)\,t\,f_t+\text{strength of six rivets}$$

Comparing P_1. P_2, P_3 and P_4, we find generally P_1 is the least. Hence in diamond riveting the section 1—1 is generally the weakest section. The efficiency of the joint $=\eta$

$$=\frac{P_1}{P}=\frac{(b-d)\,t\,f_t}{b\,t\,f_t}=\frac{b-d}{b}$$

Problem 440. *In a bridge truss, a tie bar consists of a flat 24 cm. wide and 2 cm. thick and is connected to a gusset plate of the same thickness by a double cover butt joint with 2 cm. diameter rivets. If the permissible stresses in tension, shear and bearing are 1400 kg./cm.² 900kg./cm.² and 1800 kg./cm.² respectively, design the joint.*

Solution. It is usual to adopt diamond grouping of rivets (Fig. 640).

Rivet value in double shear

$$=2f_s\frac{\pi d^2}{4}$$

$$=2\times900\times\frac{\pi}{4}\times2^2$$

$$=5660\ kg.$$

Rivet value in bearing

$$=f_b\,dt$$

$$=1800\times2\times2$$

$$=7200\ kg.$$

$\therefore$ **Lesser rivet value $=5660\ kg$.**

Safe pull on the plate at section 1—1

$=P_1=f_t\,(b-d)\,t$

$=1400\,(24-2)2\ kg.$

$=61600\ kg.$

Safe pull at the section 2—2

$=P_2=f_t(b-2d)\,t+$ Strength of one rivet in front of section

$=1400\,(24-4)2+5660\ kg.$

$=61660\ kg.$

Safe pull at the section 3—3

$=P_3=f_t\,(b-3d)\,t+$ Strength of three rivets in front of the section

$=1400\,(24-6)2+3\times 5660\ kg.$

$=67380\ kg.$

Safe pull at section 4—4

$P_4=f_t(b-4d)\,t+$ strength of six rivets in front of the section

$=1400\,(24-8)2+6\times 5660\ kg.$

$=78760\ kg.$

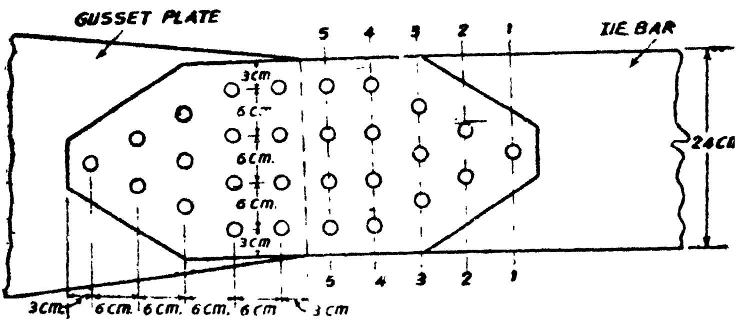


Fig. 640

Hence we find that the sections closer to the joint are stronger than those away from it. The weakest section is therefore 1—1 and the maximum pull that can be applied is 61600 kg.

∴ Minimum number of rivets required on each side of the joint

$$=\frac{P}{\text{lesser rivet value}}$$

$$\frac{61600}{5660}=11$$

Let us provide 14 rivets on each side of the joint.

§141. Eccentric riveted connections

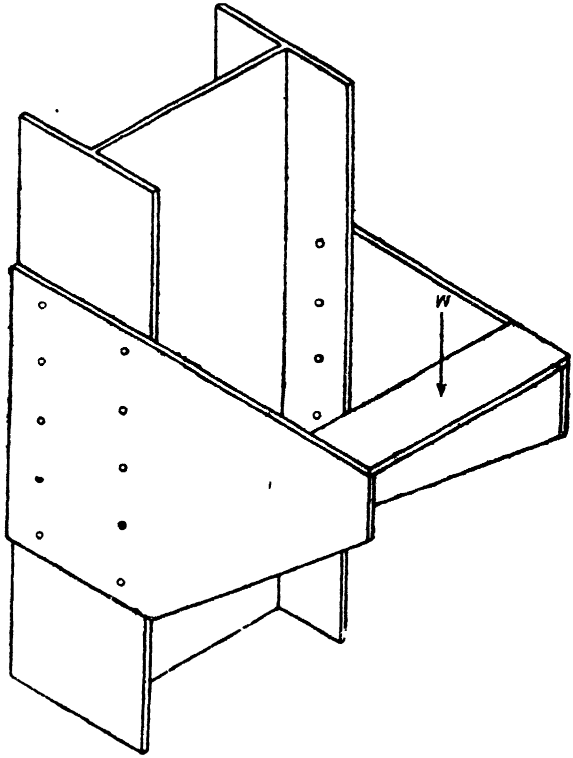


Fig. 641

In the discussions we had so far, the resistance offered by a rivet was entirely to prevent a linear or translatory displacement of the plate or member connected.

The are some circumstances in which the rivets used for a connection have to offer not only resistances to prevent translatory dispalcements but also resistances to prevent rotary displacements. A bracket connection is one example of this type of connection. Fig. 641 shows an eccentric riveted connection for a bracket. It consists of two bracket plates riveted to the flanges of a rolled steel column. If a load W be applied to the bracket, a load of $P=\frac{W}{2}$ is transferred to each bracket plate. The line of action of the load P on a bracket plate does not pass through the centroid of the rivet group.

The perpendicular distance between the line of action of the load P on the bracket plate and the centroid of the rivet group is called the eccentricity.

The rivets connecting the bracket plate and the flange of the column have to offer the following resistances.

(*i*) ***Resistance against translation***

This resistance is assumed to be uniform for all the rivets.

If P be the load on the bracket plate, resistance against translation per rivet

$$=\frac{P}{n}$$

where n=number of rivets on one bracket plate.

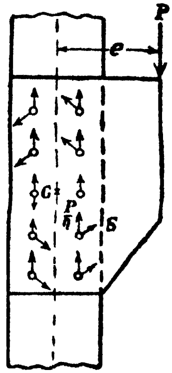


Fig. 642

(*ii*) ***Resistance against rotation***

The load being eccentric, there is a tendency for the bracket plate to rotate about G the centroid of the rivet group. The rivets therefore have to offer a resistance to prevent such a rotation. Such a resistance offered by a rivet is called the torsional shear in the rivet.

It will be assumed that the torsional shear in any rivet is directly proportional to the distance of the rivet from the centroid of the rivet group.

Let S be the torsional shear in a rivet distant r from the centroid of the rivet group. The direction of the resistance S is at right angles to the line joining G and the rivet.

Hence $S=K\,r$ where K is a constant.

Hence the resisting moment offered by the rivet against the rotation of the bracket plate is $Sr=Kr^2$.

$\therefore$ Total resisting moment offered by all the rivets

$$=\Sigma\,Kr^2$$
$$=K.\,\Sigma r^2$$

But the external moment applied $=P.e$.

$$\therefore \quad K\,\Sigma r^2=P.e$$

$$\therefore \quad K=\frac{P\,e}{\Sigma r^2}.$$

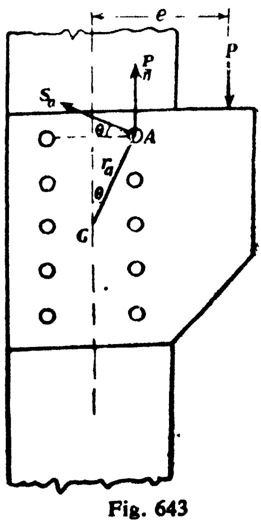


Fig. 643

If (x, y) be the coordinates of rivet distant r from G,

$$x^2+y^2=r^2$$

$$\therefore \quad K=\frac{P.e}{\Sigma x^2+\Sigma y^2}$$

Now consider the rivet A (Fig. 644) which is most distant from G. Torsional shear in the rivet

$$=S_a=Kr_a$$

Resistance against translation $=\dfrac{P}{n}$

These resistances are shown in Fig. 644.

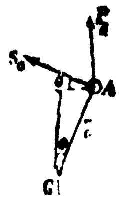


Fig. 644

The greatest resistance offered by the rivet is the resultant of the above two resistances offered by the rivet A.

For the design to be safe, this resultant resistant must be less than the lesser rivet value.

The resultant resistance may be calculated as follows.

Total vertical component on the rivet A

$$=V=\frac{P}{n}+S_a \sin\theta$$

Total Horizontal component on rivet A

$$=H=S_a \cos\theta$$

$\therefore$ Resultant resistance $=R=\sqrt{V^2+H^2}$

Problem 441. *A line shaft transmits a load of 2500 kg. at an eccentricity of 50 cms. across a bracket plate riveted to a stanchion. Two rows of rivets 10 cms. apart are provided with five rivets per row. The pitch of rivets in each row is 6 cm. Find the greatest force induced in any rivet.*

Solution. Resistance against translation per rivet

$$=\frac{P}{n}$$

$$=\frac{2500}{10}\ kg$$

$$=250\ kg.$$

Torsional shear in any rivet distant r from the centroid G of rivet group

$$=S=Kr$$

where $$K=\frac{Pe}{\Sigma x^2+\Sigma y^2}$$

In our case,

$$\Sigma x^2+\Sigma y^2=10\,(5)^2+4(12)^2+4(6)^2\ cm.^2$$

$$=970\ cm.^2$$

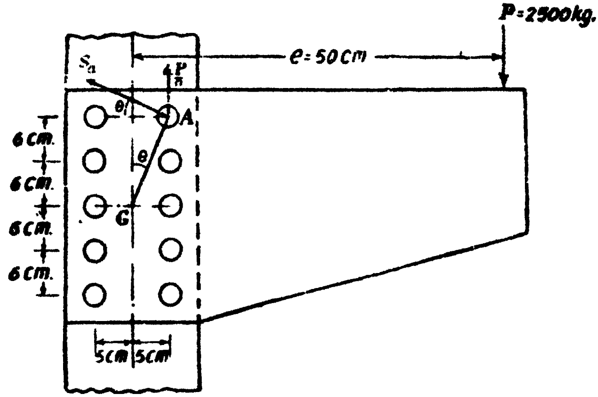


Fig. 645

$$\therefore \quad K=\frac{2500\times 50}{970}$$

$$=129$$

Now consider the rivet A distant r_a from G.

Torsional shear on the rivet $A=S_a=Kr_a=129\ r_a$

S_a acts normal to the line GA.

Total vertical component on the rivet A

$$=V=\frac{P}{n}+S_a \sin\theta$$

$$=250+129\ r_a \sin\theta \ kg.$$

$$=250+129\times 5\ kg.$$

$$=895\ kg.$$

Total horizontal component on the rivet A

$$=H=S_a \cos\theta$$

$$=129\ r_a \cos\theta\ kg.$$

$$=129\times 12\ kg.$$

$$=1548\ kg.$$

$\therefore$ Resultant force on the rivet A

$$=R=\sqrt{V^2+H^2}$$

$$=\sqrt{895^2+1548^2}\ kg.$$

$$=1787\ kg.$$

Problem 442. *A load of 15000 kg. is applied to a bracket plate at an eccentricity of 30 cms. Sixteen rivets of 2 cm. diameter are arrange.l in two rows with eight rivets per row. The rows are 20 cm. apart. The pitch of rivets in each vertical row is 8 cm. If the permissible stresses in shear and bearing be 800 kg./cm.² and 1600 kg./cm.²*

respectively, investigate the safety of the connection. The bracket plate in 1·25 cm. thick.

Solution. Resistance against translation per rivet

$$= \frac{P}{n} = \frac{15000}{16} \; kg.$$

$$= 937{\cdot}5 \; kg.$$

Torsional shear in any rivet distant r from the centroid G of the rivet group is given by

$$S = Kr$$

where $$K = \frac{P.e}{\Sigma x^2 + \Sigma y^2}$$

In our case, $\Sigma x^2 + \Sigma y^2 = 16(10)^2 + 4(28)^2 + 4(20)^2 + 4(12)^2$

$$+4(4)^2 \; cm.^2 = 6976 \; cm.^2$$

$$K = \frac{15000 \times 30}{6976} = 64{\cdot}51$$

Now consider the rivet A

Torsional shear on the rivet A

$$= S_a = Kr_a$$

$$= 64{\cdot}51 \; r_a$$

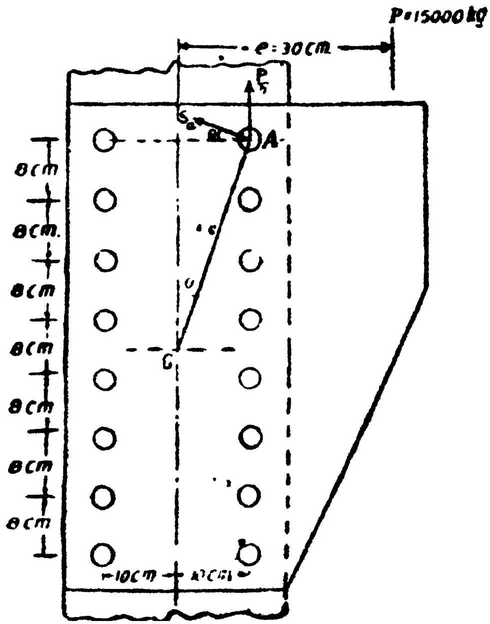


Fig. 646

S_a acts normal to the line joining G and the rivet A.

Total vertical component on the rivet A

$$=V=\frac{P}{\cdot}+S_a \sin\theta$$

$$=937{\cdot}3+64{\cdot}51\ r_a \sin\theta\ kg.$$
$$=937{\cdot}5+64{\cdot}51\times 10\ kg.$$
$$=1582{\cdot}6\ kg.$$

Total horizontal component on the rivet A

$$=H=S_a\cos\theta\ kg.$$
$$=64{\cdot}51\ r_a\cos\theta\ kg.$$
$$=64{\cdot}51\times 28\ kg.$$
$$=1806{\cdot}3\ kg.$$

∴ Resultant force on the rivet A

$$R=\sqrt{V^2+H^2}$$
$$=\sqrt{1582{\cdot}6^2+1806{\cdot}3^2}\ kg.$$
$$=2402\ kg.$$

The rivets are in single shear.

Rivet value in single shear

$$=f_s\frac{\pi d^2}{4}$$

$$=800\times\frac{\pi}{4}\ 2^2\ kg.$$

$$=2514\ kg.$$

Rivet value in bearing $f_b\,dt$

$$=1600\times 2\times 1{\cdot}25\ kg=4000\ kg.$$

∴ Lesser rivet value = 2514 kg.

Since the maximum force on the rivet A is less than the lesser rivet value, the design is safe.

Examples in Chapter 15

1. A single riveted lap joint is provided to connect 10 *mm.* plates with 20 *mm.* rivets at a pitch of 80 *mm.* State how the joint will fail. Calculate also the efficiency of the joint. Take $f_s=800$ $kg./cm.^2$, $f_b=1600$ $kg./cm.^2$ and $f_t=1200$ $kg./cm^2$.

(By shearing of rivets ; *26·2%*)

2. A double riveted lap joint is provided to connect 8 *mm.* plates with 16 *mm.* rivets at a pitch of 6 *cm.* Calculate the strength of the joint per pitch length. Find also the efficiency of the joint. Take $f_s=800$ $kg./cm.^2$, $f_b=1600$ $kg./cm.^2$ and $f_t=1200$ $kg./cm^2$.

(*3216 kg. 73.3%*)

3. A single riveted double cover butt joint in plates 12 *mm.* thick is made with 20 *mm.* diameter rivets at a pitch of 9 *cm.* Find the safe load per pitch length of the joint. Find also the efficiency of the joint. Take $f_s = 800$ *kg./cm.*^{2}, $f_b = 1600$ *kg./cm.*^{2} and $f_t = 1400$ *kg./cm*^{2}. (5024 *kg.* ; 33·2%)

4. Two plates of 10 *mm.* thickness are to be connected in a double riveted double cover butt joint with 20 *mm.* rivets at a pitch of 7 5 *cm.* If the ultimate tensile, shearing and bearing stresses are 4600 *kg./cm.*^{2}, 3 00 *kg./cm* ^{2} and 6400 *kg./cm* ^{2} respectively, find the pull per pitch length at which the joint will fail! Find also the efficiency of the joint. (25300 *kg* ; 73·3%)

5. A single riveted butt joint is to be provided for connecting two plates 10 *mm.* thick with 20 *mm.* diameter rivets. Calculate the necessary pitch and the efficiency of the joint. Take $f_s = 1000$ *kg./cm.*^{2}, $f_b = 2000$ *kg./cm.*^{2} and $f_t = 1200$ *kg /cm*^{2} ($p = 5·3$ *cm* ; 62·3%)

6. A double riveted double cover butt joint is to be provided for connecting 10 *mm.* thick plates with 20 *mm.* diameter rivets. Calculate the necessary pitch and the efficiency of the joint. Take $f_s = 1600$ *kg /cm.*^{2}, $f_b = 1600$ *kg./cm* ^{2} and $f_t = 1200$ *kg /cm*^{2}.
($p = 10·4$ *cm* ; 82·4%)

7. A bridge diagonal consists of a flat of thickness 20 *mm.* and has to transmit a tension of 45000 *kg.* to a gusset plate by a double cover butt joint, using 20 *mm.* rivets. Find the number of rivets required with diamond riveting. Find also the width of flat required. Also calculate for the arrangement suggested, the actual stresses in shear and bearing. The permissible stresses are 1200 *kg./cm.*^{2} in tension, 1000 *kg /cm.*^{2} in shear and 2000 *kg /cm* ^{2} in bearing.
(*21 cm* ; 9 rivets on each side for a convenient arrangement ; *796 kg./cm.*^{2} ; *1250 kg./cm.*^{2})

8. A vertical load of 2*t* is applied to a bracket plate at an eccentricity of 20 *cm.* as shown in Fig 647.

Find the maximum resistance offered by any rivet (*1.39t*)

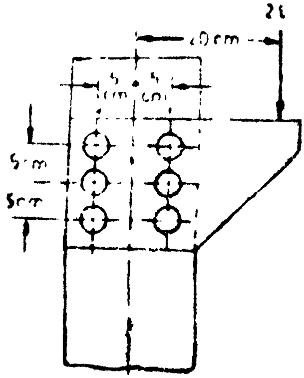


Fig. 647

9. A structural member is connected to a gusset plate by a lap joint by five rivets of 20 *mm.* diameter as shown in Fig. 648. The

member has to transmit a pull of 10*t* and a clockwise moment of 30 *t cm*. Find the greatest shear stress produced in any rivet.

(*1·03 t/cm.²*)

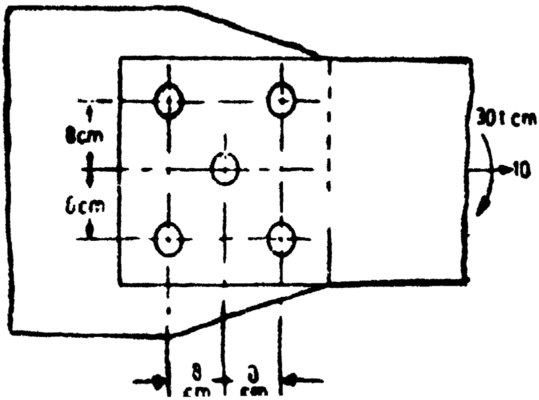


Fig. 648

10. A vertical load of 12 *t* is applied to a bracket plate at an eccentricity of 25 *cm*. Two vertical rows of rivets are provided 12 *cm*. apart. The spacing of rivets in each vertical row is 10 *cm* The rivets are 20 *mm*. diameter. Investigate the safety of the design. Permissible shear stress=945 *kg./cm²*. Permissible bearing stress=2125 *kg./cm²*. Strength of rivets shall be calculated on the basis of finished diameter which shall be taken as 1·5 *mm*. in excess of the nominal diameter. Thickness of bracket plate 10 *mm*.

(Lesser rivet value=*3430 kg*.
Max. load on any rivet=*2402 kg*.
∴ The design is safe)

11. A heavy riveted bracket connection has to resist a girder reaction of 70*t*. at an eccentricity of 50 *cm*. This load is transmitted to the two bracket plates. Four rows of 20 *mm*. diameter rivets are provided in each bracket plate of 12 *mm* thickness. Each row contains 10 rivets. The rivets in each row are provided at a pitch of 8 *cm*. The distance between consecutive vertical rows is 6 *cm*. Find the maximum load an any rivet. Also investigate the safety of the design. The strength of rivets shall be based on a finished diameter of 21·5 *mm*. Take f_s=945 *kg./cm.²*, f_b=2125 *kg/cm²*)

(Lesser rivet value=*3430 kg*.
Max. load an any rivet=*2402 kg*.
∴ Design is safe)

16

Welded Connections

§142. The welding process

Welding is a process of connecting metal parts by fusion. Arc welding and oxy-acetylene welding are the two usual methods adopted. Molten or fused metal is deposited between the metal parts which are to be connected. The metal parts are also fused to a specified depth. When the deposited fused metal is cooled, the metal parts get joined by the new metal. The ends of metal parts to be connected and the tip of the weld rod are fused by arc which causes a high temperature of about 3300°C. In the oxy-acetylene method a jet of burning oxygen and acetylene is used as a source of heat. The weld rod has a coating which also melts during the welding process and forms a shield preventing combination of the heated metal with the freely available oxygen and nitrogen of the atmosphere.

§143. Advantages of welded connection

(*i*) Since the process does not involve driving holes, the gross sectional area of the welding member is effective. In the case of riveted tension member deductions have to be made for the area lost due to punching holes.

(*ii*) Welded structures are comparatively lighter than corresponding riveted structures.

(*iii*) A welded joint has a great strength. Often a welded joint has the strength of the parent metal itself.

(*iv*) Repairs and further new connections can be done more easily than in riveting.

(*v*) Welded joints provide rigidity. Hence welded members, for the same loading, are subjected to smaller bending moments than corresponding riveted members.

(*vi*) Often welded joints are economical to riveted joints. For a welded structure maintenance and painting costs are less than for the riveted structure.

(*vii*) Members of such shapes that afford difficulty for riveting can be more easily welded.

(*viii*) A welded structure has a better finish and appearance than the corresponding riveted structure.

(*ix*) Connecting angles, gusset plates, splicing plates can be minimized and in many cases, can be avoided in welded structures

(*x*) Steel bars in reinforced concrete structures may be welded easily. Lapping of bars may be avoided if welding is resorted to.

(*xi*) It is possible to weld at any point at any part of a structure. But riveting will always require enough clearance.

(*xii*) The process of welding takes less time than riveting.

(*xiii*) The process of welding does not involve great noise compared to the noise produced in the riveting process.

§144. Disadvantages of welded connection

(*i*) Welding requires skilled labour and supervision.

(*ii*) Testing a weld joint is difficult. An X-ray examination alone can enable us to study the quality of the connection.

(*iii*) Due to uneven heating and cooling the welded members are likely to get warped at the welded surfaces.

(*iv*) Internal stresses in the welded zones are likely to be set up.

§145. Types of Welds

Welds may be classified into two main types namely fillet-weld and butt-weld.

Fillet weld. This type of weld is used when the members to be connected overlap each other. Fig. 649 shows a fillet weld. The section of the fillet weld for design purposes will be taken as an isosceles right-angled triangle. The length of either of the equal sides of the triangle is called the size of the weld. The perpendicular distance between the hypotenuse of the triangle and the opposite apex is called the throat thickness. If t be the throat thickness and s be the size of the fillet weld,

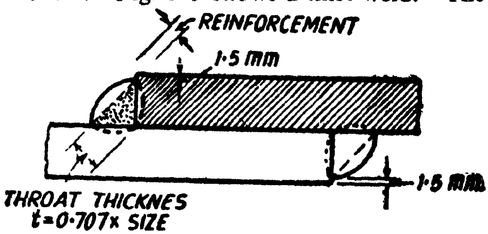


Fig. 649

$$t=\frac{\ }{\sqrt{2}}=0{\cdot}707\,s. \quad say\ 0{\cdot}75.$$

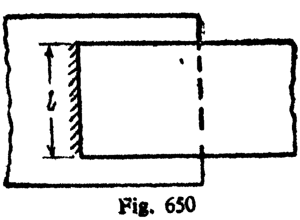


Fig. 650

If l be the length of a weld and t the throat thickness, the product lt will be regarded as the effective or resisting area resisting shear.

Safe load on a weld

= Length of weld × throat thickness × permissible shear stress in weld.

The permissible shear stress is usually taken as 1025 $kg./cm^2$.

Butt weld. This type of weld is used when the members to be connected butt each other.

The following types of butt welds are in practice :

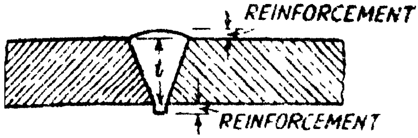


Fig. 651

(*i*) *Single V-butt weld.* Fig. 651 shows this type of weld. The effective throat thickness is taken as the thickness of the thinner part. The permissible stresses for this type of weld are 1420 $kg/cm.^2$ in axial tension or compression,

and

$\begin{cases} 1575\ kg./cm.^2 \text{ in bending} \\ 1025\ kg/cm^2 \text{ in shear} \end{cases}$

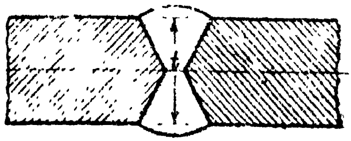


Fig. 652

(*ii*) *Double V-butt weld.* Fig. 652 shows this type of butt weld.

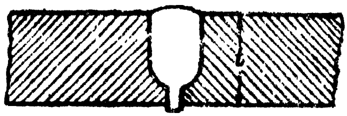


Fig. 653

(*iii*) *Single U-butt weld.* (Fig 653)

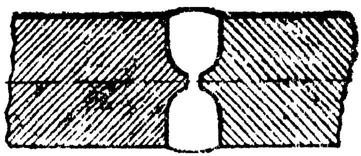


Fig. 654

(*iv*) *Double U-butt weld.* (Fig. 654)

Abstract of IS Specifications [*IS* 816]

§146. Minimum sizes of weld

The I.S. code has recommended the following :

Thickness of thinner part	*Minimum size*
Up-to and including 9·5 *mm.*	3 *mm.*
Over 9·5 *mm.*, up to and including 19 *mm.*	5 *mm.*
Over 19 *mm.*, up to and including 32 *mm.*	6 *mm.*
Over 32 *mm.*	8 *mm.*
	9·5 *mm.* minus minimum size of fillet

§147. Effective length

The effective length of a fillet weld shall be taken as that length only which is of the specified size and required thickness. For practical purposes, the effective length may be taken as the actual length minus twice the weld size.

§148. Minimum length

The effective length of a fillet weld designed to transmit loading shall not be less than four times the size of weld.

§149. Fillet weld applied to the edge of a part

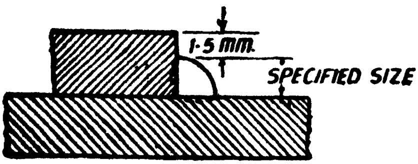


Fig. 655.

When a fillet weld is applied to the square edge of a part, the specified size of the weld should generally be at least 1·5 *mm*. less than the edge thickness.

§150. Angle between fusion faces

Fillet welds should not be used for connecting parts whose fusion faces form an angle more than 120° or less than 60° unless such welds are demonstrated by practical tests to develop the strength required.

§151. Throat thickness

Fillet weld. For the purpose of stress calculation the effective throat thickness of a fillet weld should be taken as $K \times$ fillet size, where K is a constant. The value of K for different angles between fusion faces shall be as in the following table :

ANGLE BETWEEN FUSION FACES	60°-90°	91°-100°	101°-106°	107°-113°	114°-120°
CONSTANT K	0·70	0·65	0·60	0·55	0·50

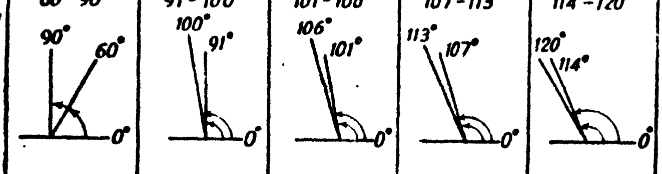


Fig. 656

Butt weld

(*a*) *Complete penetration butt weld.* The effective throat thickness of a complete penetration butt weld shall be taken as the thickness of the thinner part joined.

(*b*) *Incomplete penetration or unsealed single but welds.* The effective throat thickness of an incomplete penetration butt weld shall be taken as the minimum thickness of the weld metal common to the parts joined excluding reinforcement.

Unsealed single butt of V and U and incomplete butt welds welded from one side only, should have a throat thickness of atleast $\frac{7}{8}$ of the thickness of the thinner parts joined. If required, evidence should be produced by the fabricator to show that the effective throat thickness has been obtained. For the purpose of stress calculation, a reduced effective throat thickness not exceeding $\frac{5}{8}$ of the thickness of the thinner part joined should be used.

Note. (*i*) An incomplete penetration butt weld means a butt weld in which the weld metal is intentionally not deposited through the full thickness of the joint.

(*ii*) The nature of evidence to be produced by the fabricator to ensure the necessary effective thickness should be decided by agreement between the designer/purchaser and the fabricator, and may, for example, comprise :

(*a*) Tests carried out before welding to show that the welding procedure is capable of providing the required penetration, and inspection during welding to establish that the correct procedure has been followed.

(*b*) Test pieces made as continuations of the seams during welding ; and

(*c*) Examination, after welding, by radiographic or other suitable non-destructive methods.

The unwelded portion in incomplete penetration butt welds, welded from both sides, shall not be greater than $\frac{1}{4}$ of the thickness of the thinner part joined and should be central on the depth of the weld. For the purpose of stress calculation a reduced effective throat thickness not exceeding $\frac{5}{8}$ of the thickness of the thinner part joined should be used.

§152. Intermittent Fillet welds

Intermittent fillet welds may be used to transfer calculated stress across a joint when the strength required is less than that developed by a continuous fillet weld of the smallest allowable size for the thickness of the parts joined. Any section of intermittent fillet welding shall have an effective length of not less than four times the weld size with a minimum of 38 *mm*.

The clear spacing between the effective lengths of intermittent fillet welds carrying calculated stresses shall not exceed the following number of times the thickness of the thinner part joined and shall in no case be more than 30 *cm*.

16 times for compression members

24 times for tension members.

Longitudinal fillet welds at the ends of built up members shall have an effective length of not less than the width of the component part joined unless end transverse welds are used, in which case, the sum of the end longitudinal and end transverse welds shall be not less than twice the width of the component part.

Chain intermittent welding is to be preferred to staggered intermittent welding. Where staggered intermittent welding is used, the ends of the component parts shall be welded on both sides

In a line of intermittent fillet welds, the welding should extend to the ends of parts connected ; for welds staggered about two edges, this applies generally to both edges, but need not apply to subsidiary fittings or components such as intermediate web stiffeners

§153. Lap joints

The overlap of parts at stress carrying lap joints shall be not less than five times the thickness of the thinner part unless lateral deflection of parts is prevented, they shall be connected by at least two transverse lines of fillet, plug or slot welds or by two or more longitudinal fillet or slot joints.

If the longitudinal fillet welds are used alone in end connections, the length of each fillet weld shall be not less than the perpendicular distance between them. The transverse spacing of longitudinal fillet welds used in end connections shall not exceed 16 times the thickness of the thinner part connected, unless end transverse welds or intermediate plug or slot welds are used to prevent butting or separation of the parts.

§154. Fillet welds in slots or holes

Where fillet welds are used in slots or holes through one or more of the parts being joined, the dimensions of the slot or hole should comply with the following limits in terms of the thickness of the part in which the slot or hole is formed.

(*a*) The width or diameter should be not less than 3 times the thickness.

(*b*) Corners at the enclosed ends of slots should be rounded with a radius not less than 1·5 times the thickness.

and (*c*) The distance between the edge of the part and the edge of the slot or hole, or between adjacent slots or holes should be not less than twice the thickness.

When welding inside a slot or a hole, in a plate or other part, in order to join the same to an underlying part, fillet welding may be used along the wall or walls of the slot or the hole, but the latter shall not be filled with weld metal or partially filled in such a manner as to form a direct weld metal connection between opposite walls, except that fillet welds along opposite walls may overlap each other for a distance of ¼th of their size.

§155. End returns

Fillet welds terminating at the ends or sides of members should, whenever practicable, be returned continuously around the corners for distances not less than twice the size of the weld. This provision should in particular apply to side and top fillet welds in tension which connect brackets, beam seatings and similar parts.

§156. Bending about a single fillet

A single fillet weld should not be subjected to a bending moment about the longitudinal axis of the fillet.

§157. Permissible stresses in welds

Welded joints shall be proportioned so that the stress therein shall not exceed the values given in the following table.

Kind of Stress	*Permissible stress (kg./cm.²)*
(*i*) Axial tensile stress on throat section of butt weld	1420
(*ii*) Axial compressive stress on throat section of butt weld	1420
(*iii*) Maximum bending stress	1575
(*iv*) Maximum shear stress	1025

Stress in fillet welds shall be considerd as shear on throat for any direction of the applied load. Welds in plugs and slots shall not be considered as having any value in resistance to stress other than shear.

§158. Combined stresses in welds

Fillet welds. When the fillet welds in a connection are subjected to tension or compression bending forces, combined with a direct shear force, the maximum resultant stress may be calculated as the vector sum and should not exceed the permissible shear stress of 1025 $kg./cm^2$.

Butt welds. In butt welds subjected to tensile or compressive stress (axial and/or bending) in combination with direct shear stress, the weld shall be so proportioned that the quantity

$$\left(\frac{p_s}{P_s}\right)^2+\left(\frac{p_t}{P_t}\right)^2 \text{ shall not exceed unity.}$$

where,

p_s = Actual shear stress in the weld

P_s = Permissible shear stress in the weld

p_t = Actual tensile or compressive stress in the weld

P_t = Permissible tensile or compressive stress in the weld.

Problem 443. *A tie bar 100 mm. × 16 mm. thick is to be welded to another plate as shown in Fig. 657. Find the minimum overlap required if 8 mm. fillet welds are used. Adopt the following working stresses :*

Tensile stress in plates = 1500 kg./cm.²

Shear stress in weld = 1025 kg./cm.²

Solution. Maximum tension in the tie bar

$=10\times1{\cdot}6\times1500$ *kg.*

$=24{,}000$ *kg.*

Total length of weld

$=2\times10+2x$ *cm.*

$=(20+2x)$ *cm.*

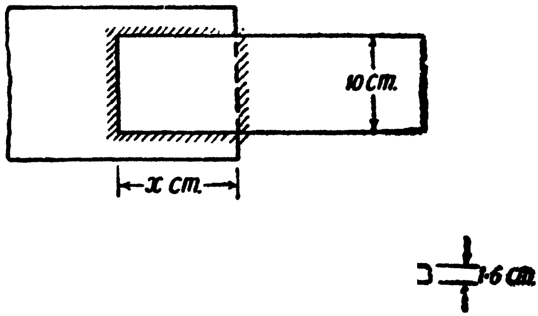


Fig. 657

Throat thickness

$=t=0{\cdot}7\times$ size of weld

$=0{\cdot}7\times0{\cdot}8$ *cm.*

$=0{\cdot}56$ *cm.*

Effective area of weld

$=0{\cdot}56\ (20+2x)$ $cm.^2$

Total resistance of the weld

= Effective area provided by weld × permissible shear stress

$=0{\cdot}56(20+2x)\ 1025$ *kg.*

$=574\ (20+2x)$ *kg.*

Equating the resistance of the weld to the load on the tie bar, we have,

$$574\ (20+2x)=24000 \text{ kg.}$$

$$20+2x=41{\cdot}81$$

$$x=10{\cdot}905 \text{ cm. say } 11 \text{ cm.}$$

Problem 444. *Find the minimum lap length required for the joint shown in Fig. 658 if 6 mm. fillet welds are used. Permissible shear stress in the weld may be taken as 1025 kg./cm².*

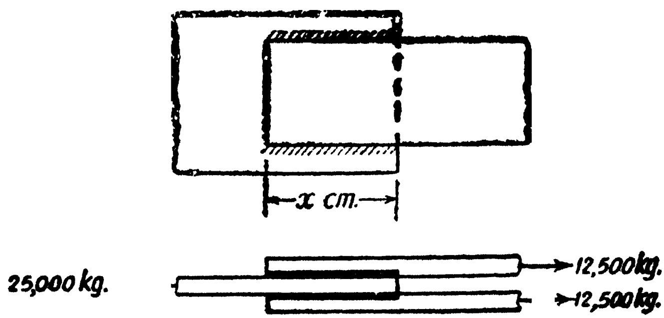


Fig. 658

Solution. Let the length of lap required be x *cm.*

Total length of the weld

$$=2x+2x$$
$$=4x \text{ cm.}$$

Resistance of weld $=0{\cdot}7\times0{\cdot}6\times4x\times1025$ *kg.*

$$=1722\,x \text{ kg.}$$

Equating the resistance of the weld to the load on the joint, we have,

$$1722\,x=25000 \text{ kg.}$$

$$x=\frac{25000}{1722} \text{ cm.}$$

$$=14{\cdot}25 \text{ cm. say } 15 \text{ cm.}$$

Problem 445. *A welded lap joint is to be provided to connect two tie bars 150 mm. × 10 mm. The working stress in the tie bar is 1500 kg./cm². Investigate the design if the size of the fillet welds be 8 mm. Safe stress for the weld may be taken as 1025 kg./cm².*

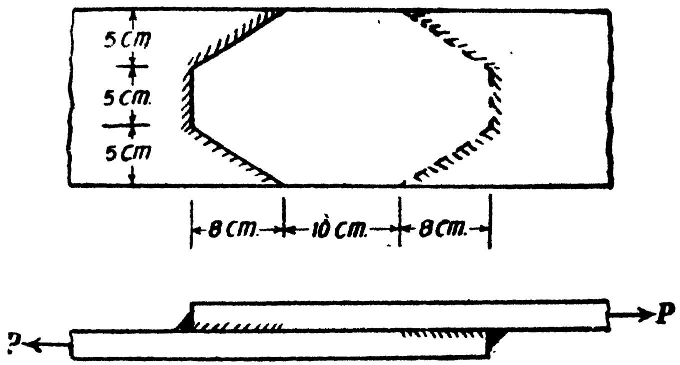


Fig. 659

Solution. Safe load in the tie bar

$$=15\times1\times1500 \text{ kg.}$$
$$=22500 \text{ kg.}$$

Total length of weld $=2\times5+4\sqrt{5^2+8^2}$ *cm.*

$$=47{\cdot}74 \text{ cm.}$$

Strength of weld $=0{\cdot}7\times0{\cdot}8\times47{\cdot}74\times1025$ *kg.*

$$=27400 \text{ kg.}$$

Since the strength of the weld is greater than even the maximum tension in the tie bars the design is safe.

Problem 446. *A 150 mm. × 115 mm. × 8 mm. angle carrying a tensile load of 20000 kg. is to be connected to a gusset plate by 6 mm. fillet welds at the extremities of the longer leg as show in Fig. 660. Design the joint allowing a shear stress of 1025 kg./cm.² in the welds.*

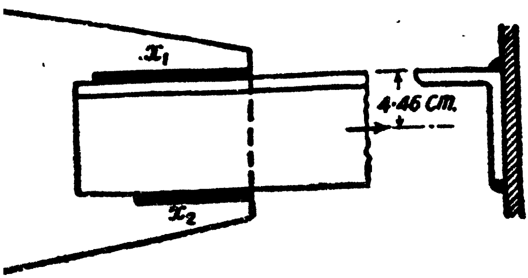


Fig. 660

Solution. Let the length of weld at the top extremity be x_1 *cm*.

Let the length of weld at the bottom extremity be x_2 *cm*.

Distance of longitudinal centroidal axis of angle from the top edge $=4\text{·}46$ *cm*.

Safe load on weld per *cm*. length

$$=0\text{·}7\times0\text{·}6\times1025 \text{ kg. per cm.}$$
$$=430\text{·}5 \text{ kg. per cm.}$$

Equating the total resistance of the weld to the tension in the member, we have,

$$430\text{·}5(x_1+x_2)=20{,}000$$
$$x_1+x_2=43\text{·}46 \text{ cm.} \qquad ...(i)$$

Taking moments about the upper weld line,

$$430\text{·}5x_2\times15=20{,}000\times4\text{·}46$$
$$\therefore \quad x_2=13\text{·}82 \text{ cm.}$$
$$\therefore \quad x_1=46\text{·}46-13\text{·}82$$
$$=32\text{·}64 \text{ cm.}$$ Let us provide $x_1=33$ *cm*. and $x_2=14$ *cm*.

Problem 447. *A circular plate 15 cm diameter is welded to*

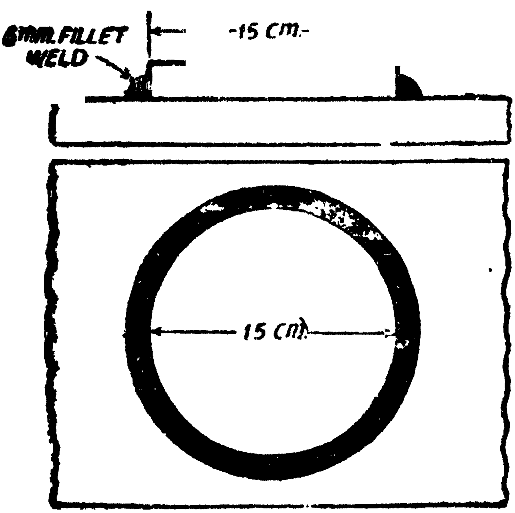


Fig. 661

another plate by means of 6 mm. fillet weld. Calculate the greatest twisting moment that can be resisted by the weld if the permissible shearing stress in the weld is 1025 kg./cm².

Solution. Safe load that can be resisted per *cm.* length of weld

$$=0{\cdot}7\times0{\cdot}6\times1025\ kg.\ per\ cm.$$

$$=430{\cdot}5\ kg.\ per\ cm.\ length\ of\ weld.$$

Greatest twisting moment

$$=430{\cdot}5\times\pi\times15\times\frac{15}{2}\ kg\ cm.$$

$$=152{,}200\ kg.\ cm.$$

Problem 448. *Design a lap joint for connecting two plates of sizes 15 cm. × 1 cm. and 20 cm × 1 cm. allowing a safe shear stress of 1025 kg./cm². in the weld. Permissible tensile stress in the plate = 1500 kg./cm².*

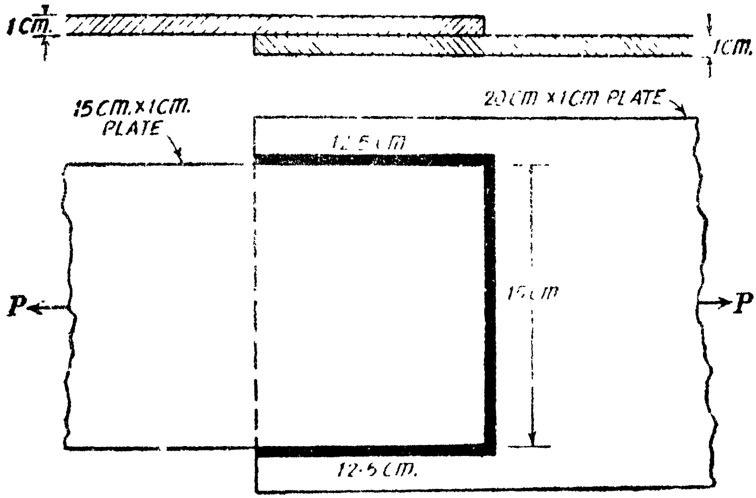


Fig. 662

Solution. We will design the joint for maximum strength of the smaller plate.

∴ Maximum strength of the smaller plate

$$=15\times1\times1500\ kg.$$

$$=22500\ kg.$$

Let 8 mm. welds be used.

Strength of weld per *cm.* length

$$=0{\cdot}7\times0{\cdot}8\times1\times1025\ kg.\ per\ cm.$$

$$=574\ kg.\ per\ cm.\ of\ weld\ length.$$

∴ Total length of weld required

$$= \frac{22500}{574} \ cm.$$

$$= 39{\cdot}20 \ cm.$$

Providing the weld at the end of the tie and at the sides symmetrically, length of weld at each side

$$= \frac{1}{2}(39{\cdot}20 - 15)$$

$$= 12{\cdot}1 \ cm. \text{ say } 12{\cdot}5 \ cm.$$

Problem 449. *A tie bar 100 mm. × 10 mm. is connected to another by fillet welds around the end of the bar and also inside a machined slot as shown in Fig. 663. Allowing a tensile stress of 1500 kg./cm.² in tie bar and a shearing stress of 1025 kg./cm². in the fillet weld. find the size of the fillet weld.*

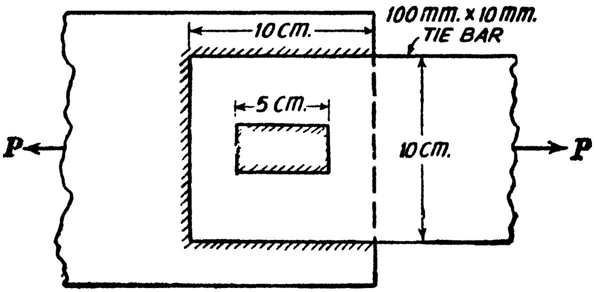


Fig. 663.

Solution. Pull in the tie bar

$$= 10 \times 1 \times 1500 = 15000 \ kg.$$

Strength of weld per *cm.* length

$$= 1025 \times 0{\cdot}7 \ s \ kg.$$

$$= 717{\cdot}5 \ s \ kg.$$

where s is the size of the weld in *cm.*

Total length of weld

$$= 3 \times 10 + 2 \times 5 = 40 \ cm.$$

Equating the strength of the weld to the tension in the tie bar we have,

$$40 \times 717{\cdot}5 \ s = 15000 \ kg.$$

$$t = 0{\cdot}52 \ cm.$$

Let us provide 6 *mm.* fillet weld.

Problem 450. *For the single V unsealed butt welded joint shown in Fig. 664, find the permissible load. Safe stress in weld may be taken as 1420 kg./cm².*

Solution. Since the joint is an unsealed *V*-butt joint the effective thickness of the throat of weld

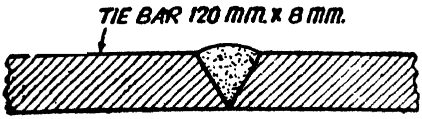


Fig. 664

$$= \frac{5}{8} \times \text{thickness of plate}$$

$$= \frac{5}{8} \times 0.8$$

$$= 0.5 \ cm.$$

Safe load = length of weld × Effective throat thickness × permissible tensile stress in the weld

$$= 12 \times 0.5 \times 1420 \ kg.$$

$$= 8720 \ kg.$$

Problem 451. *The tension member of a truss consists of two angles 80 mm. × 80 mm. × 8 mm. If the two angles are welded on either side of a gusset plate at the joint, design the joint. Axial tension in the member is 22000 kg. Permissible shear in the welds = 1025 kg./cm². Use 6 mm. fillet welds.*

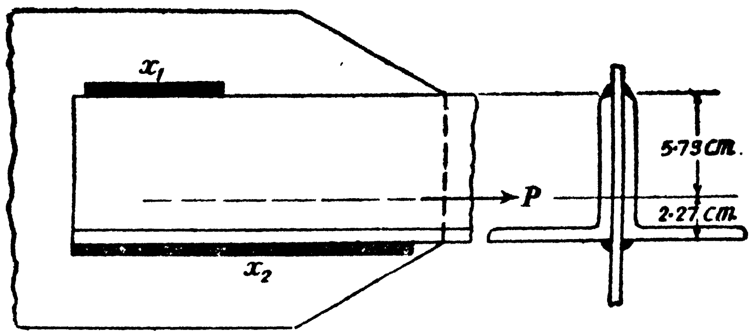


Fig. 665

Solution. Fig. 665 shows the arrangement for the connection.

Let the length of weld at top and bottom edges be x_1 and x_2 for each angle.

Total length of weld

$$= 2(x_1 + x_2) \ cm.$$

Strength of weld per *cm.* length

$$= 0.7 \times 0.6 \times 1025 \ kg$$

$$= 430.5 \ kg.$$

Equating the resistance of the weld to the tension in the member, we have,

$$430.5 \times 2(x_1 + x_2) = 22000 \ kg.$$

$$\therefore \quad x_1 + x_2 = 25.55 \quad \ldots(i)$$

Taking moments about the bottom welds,

$$430{\cdot}5 \times 2x_1 \times 8 = 22000 \times 2{\cdot}27$$

$$x_1 = 7{\cdot}25 \ cm.$$

$$x_2 = 25{\cdot}55 - 7{\cdot}25 \ cm.$$

$$= 18{\cdot}30 \ cm$$

Problem 452. *A tie in a truss, consisting of a double angle section 100×65×10 mm. thick carrying a tensile load of 25000 kg. is to be welded to a gusset plate as shown in Fig. 666. Design the joint with 8 mm. fillet welds allowing a shearing stress of 1025 kg / cm.² in the weld.*

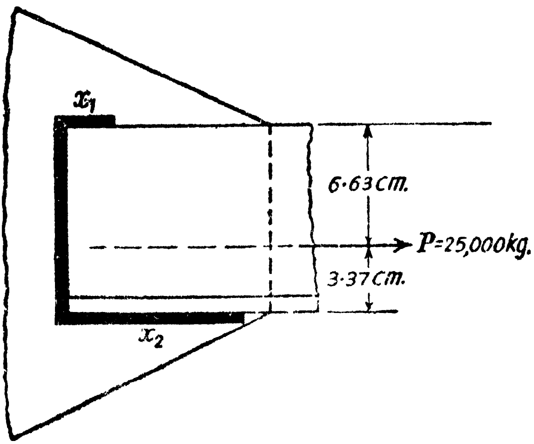


Fig. 666

Solution. Safe load per *cm.* length of weld

$$= 0{\cdot}7 \times 0{\cdot}8 \times 1 \times 1025 \ kg. \ per \ cm.$$

$$= 574 \ kg. \ per \ cm. \ length \ of \ weld$$

Total length of weld

$$= 2[x_1 + x_2 + 10] \ cm.$$

Equating the resistance of the weld to the load in the member, we have,

$$574 \times 2(x_1 + x_2 + 10) = 25000$$

$$\therefore \quad x_1 + x_2 + 10 = \frac{25000}{574 \times 2}.$$

$$= 21{\cdot}78$$

$$\therefore \quad x_1 + x_2 = 11{\cdot}78 \ cm. \qquad ...(i)$$

Taking moments about the top weld line,

$$2\left[574 \times 10 \times \frac{10}{2} + 574\, x_2 \times 10\right] = 25000 \times 6{\cdot}63 \text{ (see Fig. 666)}$$

$$x_2 = 9{\cdot}4 \ cm.$$

$\therefore \quad x_1 = 11{\cdot}78 - 9{\cdot}4 = 2{\cdot}38 \; cm$

Let us provide the following lengths of weld

$x_1 = 2{\cdot}5 \; cm$

$x_2 = 6{\cdot}5 \; cm.$

Vertical length of weld

$= 10 \; cm.$

Problem 453. *A welded plate girder consists of flange plates 200 mm. × 10 mm. one at top and the other at the bottom and a vertical web plate 600 mm × 10 mm. If the allowable shear stress in the weld is 1025 kg./cm.² and the mean shear stress in the web is 945 kg./cm.², determine the permissible shear force on the section of the beam if 8 mm fillet welds are used for the connection.*

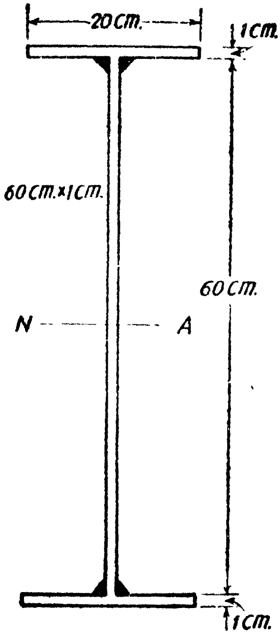


Fig. 667

Solution Moment of inertia of the beam section about the neutral axis

$$= I = \frac{20 \times 62^3}{12} - \frac{19 \times 60^3}{12} \; cm^4$$

$$= 55210 \; cm^4$$

(i) Shear force based on the strength of weld. Let S be the shear force which produces a shear stress of 1025 $kg./cm^2$. in the weld.

$$\therefore \text{ shear stress} = q = \frac{S \times \text{moment of flange area about netural axis}}{I(2t)}$$

where t = throat thickness

$$\therefore \; q = 1025 = \frac{S(20 \times 1 \times 30{\cdot}5)}{55210 \times 2 \times 0{\cdot}7 \times 0{\cdot}8} \; kg./cm.^2$$

$$\therefore \qquad S = 103{,}900 \; kg.$$

(ii) Shear force based on the average shear stress in the web.

Maximum shear force

= Average shear stress in the web × area of web

$= 945 \times 60 \times 1 \; kg.$

$56{,}700 \; kg.$

Hence, allowable shear force on the section of the girder

$= 56{,}700 \; kg.$

Problem 454. *A plate girder simply supported at ends having a span of 15 metres consists of a web plate 700 mm. × 12 mm and a flange plate 300 mm. × 18 mm. for each flange. The girder carries an all inclusive load of 4500 kg. per metre run. Find the size of the* *weld required for connecting the flange plates to the web plates, near the supports. Use 10 mm. fillet welds. Permissible shear stress in the weld equals 1025 kg./cm².*

Solution. Maximum shear force for the girder

$$=S=\frac{4500\times 15}{2}\ kg.$$

$$=33750\ kg.$$

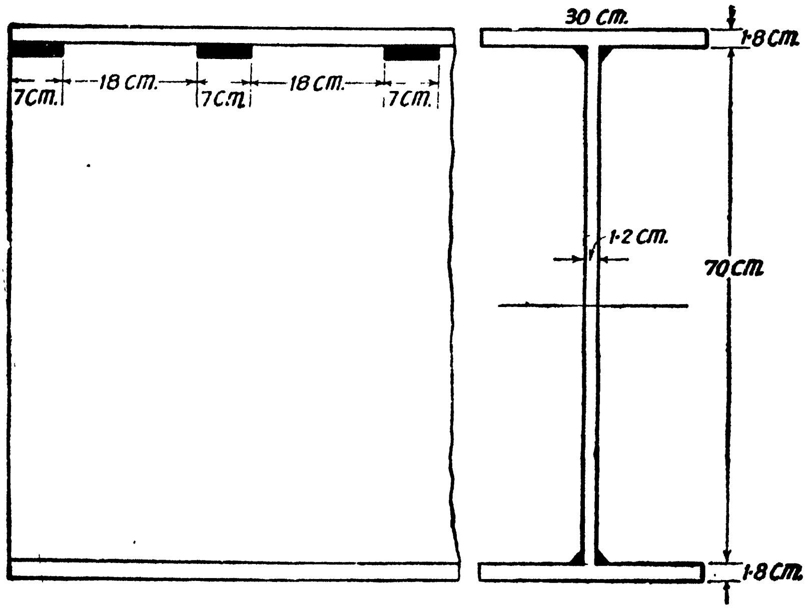


Fig. 668

Moment of inertia of the section of the grider

$$=I=\tfrac{1}{12}\,(30\times 73{\cdot}6^3-28{\cdot}8\times 70^3)\ cm^4.$$

$$=173520\ cm.^4$$

Maximum shear stress in the web

$$=q=\frac{SA\bar{y}}{Ib}$$

$$=\frac{33750\times 30\times 1{\cdot}8\times 35{\cdot}9}{173520\times 1{\cdot}2}\ kg./cm.^2$$

$$=314{\cdot}2\ kg./cm.^2$$

Horizontal shear per *metre* length which the weld has to resist

$$=F=q\times\text{thickness of web}\times 100\ kg.$$

$$=314{\cdot}2\times 1{\cdot}2\times 100\ kg.$$

$$=37{,}700\ kg.$$

Using 10 *mm.* fillet weld and allowing a shear stress of 1025 *kg./cm.*^{2} the total length of weld required in 1 *metre* length of girder (for one flange) to resist the shear force of 37,700 *kg.* is given by

$$\text{Length of weld} = l_w = \frac{37{,}700}{0{\cdot}7 \times 1 \times 1025} \text{ cm.}$$

$$= 52{\cdot}5 \text{ cm.}$$

Since the welding will be done on either side of the web length of weld required per *metre* length, on each side

$$\frac{52{\cdot}5}{2} = 26{\cdot}25 \text{ cm.}$$

Suppose we provide 4 welds of 7 *cm.* length each in a length of 100 *cm.*

$$\text{Spacing of weld} = \frac{100 - 4 \times 7}{4} \text{ cm.}$$

$$= 18 \text{ cm.}$$

§159. Eccentric welded connection

In the discussions we had so far, the resistance offered by a weld was entirely to prevent a linear or translatory displacement of the plate or member connected. There are also circumstances in which the welds provided for a connection may have to offer not only a resistance to prevent translatory displacements but also resistances to prevent rotary displacements. A bracket connection is one example of this type of connection.

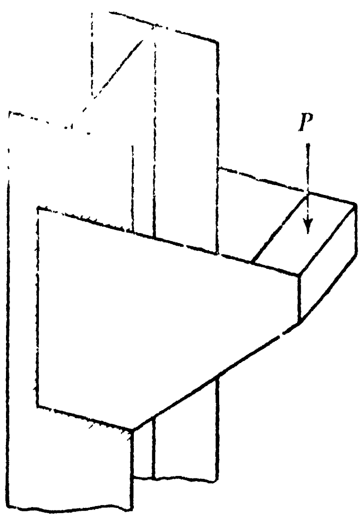


Fig. 609

Fig. 669 shows an eccentric welded connection, for a bracket. It consists of two bracket plates welded to the flanges of a column. If a load W be applied to the bracket, a load $P=\frac{W}{2}$ is transmitted to each bracket plate. The line of action of the load P does not pass through the centroid of the weld group Hence the connection is called an eccentric welded connection

There are two types of bracket connections, *viz.*,

(*i*) Welded connection subjected to moment in a plane of the weld.

(*ii*) Welded connection subjected to moment in a plane normal to the plane of the weld.

Case 1. *Welded connection subjected to moment in the plane of the weld.*

Consider the bracket connection shown in Fig. 670.

Let load on one bracket plate be P.

Let G be the centroid of the weld lengths.

Let e = eccentricity of the load

= distance between the centroid G of the weld group and the load line

$=\bar{x}+a$

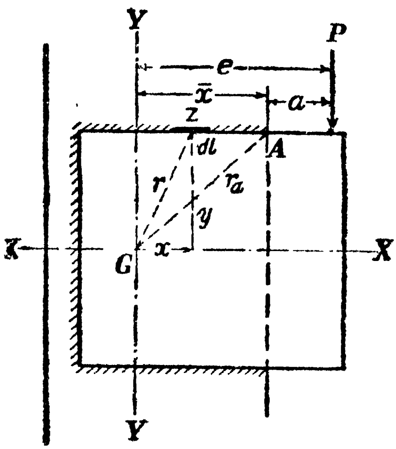


Fig. 670

The weld has to offer the following resistances :

(*i*) a resistance against translation

(*ii*) a resistance against rotation.

(*i*) *Resistance against translation.* This resistance is assumed to be uniform in the weld.

$\therefore$ Resistance per unit length

$$=\frac{\text{Load on bracket plate}}{\text{Total length of weld}}$$

$$=\frac{P}{L}$$

(*ii*) *Resistance against rotation.* The value of this resistance per unit length at any point is assumed to be proportional to the

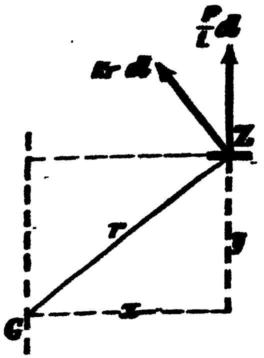


Fig. 671

distance of the point from the centroid G of the weld group.

Consider an elemental length dl of weld at any point Z distant r from G.

∴ Resistance against rotation per unit run at $Z = Kr$ where K is a constant.

∴ Resistance offered by the elemental length dl of weld at Z.

$$= Kr\ dl.$$

This resistance acts normal to GZ.

∴ Moment of resistance offered by the elemental length of weld

$$= Kr\ dl\ .\ r$$
$$= K\ dl\ .\ r^2$$

∴ Total moment of resistance offered against rotation

$$= \Sigma\ K\ dl\ r^2$$
$$= K\ \Sigma\ dl\ r^2$$

But $\Sigma dl\ r^2 = I_p =$ Polar moment of inertia of the weld lengths about an axis through G normal to the plan of the weld

But $I_p = I_{xx} + I_{yy}$

where I_{xx} = Moment of inertia of the weld length about any (say horizontal) axis XX, in the plane of the weld, through G.

I_{yy} = Moment of inertia of the weld lengths about an axis in the plane of the weld, normal to the axis XX and passing through G.

Total t of resistance offered by the weld lengths against

$$= K\,[I_{xx} + I_{yy}]$$

t on the connection

$$= P\ .\ e.$$

of resistance offered by the weld against rotation to the external moment, we have,

$$K\,[I_{xx} + I_{yy}] = P\ .\ e$$

$$\therefore \qquad K = \frac{P\ .\ e}{I_{xx} + I_{yy}}$$

From the above relation the constant K can be computed for any given arrangement of weld length.

The maximum resistance against rotation is offered by the weld at the most distant point of the weld from G.

For design purposes consider the point A (See Fig. 672).

The forces acting at A per unit length of weld are shown in Fig. 672.

Let θ be the inclination of GA with YY axis.

Total vertical force per unit length of weld at A

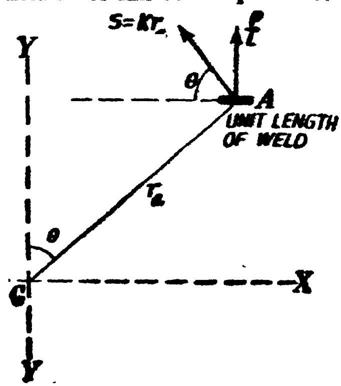


Fig. 672

$$=V=\frac{P}{l}+S\sin\theta$$

Total horizontal force per unit length of weld at A

$$=H=S\cos\theta$$

∴ Resultant forces per unit length of weld at A

$$=\sqrt{V^2+H^2}$$

For the design to be safe this resultant force should not exceed the safe load per unit length of the weld.

The positions of the centroid G of the weld group for the two usual arrangements of weld are given below :

(i) When the rectangle $ABCD$ is the weld length (Fig. 673). For this case,

$$\bar{x}=\frac{b}{2}$$

$$e=\bar{x}+a=\frac{b}{2}+a$$

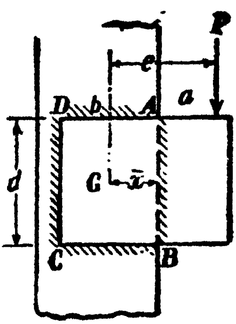


Fig. 673

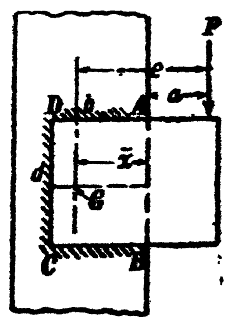


Fig. 674

(*ii*) When the length *ADCB* alone is welded. (Fig. 674). For this case,

$$\bar{x}=\frac{b(b+d)}{b+(b+d)}$$

Moment of inertia of a weld length

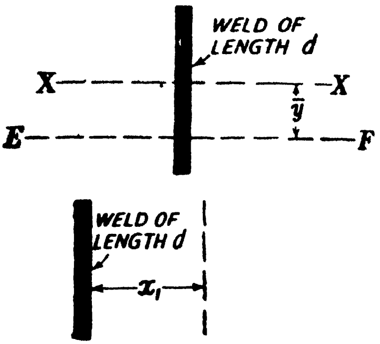


Fig. 675

Moment of inertia of the weld of length d (Fig. 675) about the XX axis,

$$I_{xx}=\frac{d^3}{12}$$

Moment of inertia of the weld length about the parallel axis EF,

$$=I_{ef}=I_{xx}+d(\bar{y})^2$$

Moment of inertia of a weld of length d about the longitudinal axis $=0$.

Moment of inertia of a weld of length d about an axis parallel to the weld length, at a distance x_1 from the weld $=d(x_1)^2$

Problem 455. *Fig. 676 shows an arrangement to support a bracket plate. The load applied to the bracket plate is 10,000 kg. Find the greatest resistance offered by the weld per cm. length. If 6 mm. fillet welds are used find the greatest stress intensity in the weld.*

Solution. $\bar{x}=\frac{10}{2}$

$= 5$ *cm.*

Eccentricity $=e=5+5$

$= 10$ *cm.*

Moment of inertia of weld lengths

$$I_{xx}=2\left[\frac{20^3}{12}+10\times 10^2\right]$$

$$=3333\ cm.^3$$

$$I_{yy}=2\left[\frac{10^3}{12}+20\times 5^2\right]$$

$$=1167\ cm.^3$$

$$I_{xx}+I_{yy}=3333+1167$$

$$=4500\ cm^3$$

Resistance against translation per *cm.* length of weld,

$$=\frac{P}{L}$$

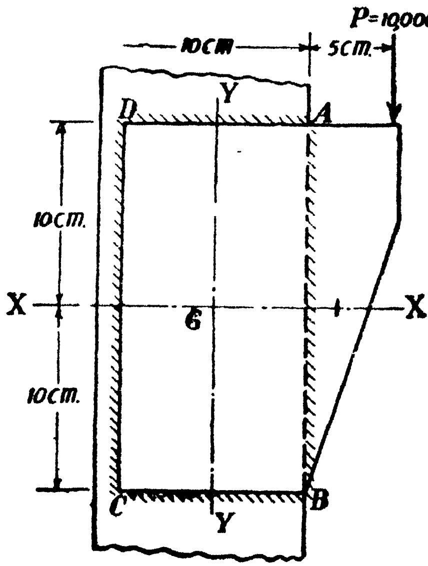


Fig. 676

$$=\frac{10{,}00}{60}=166{\cdot}7 \ kg./cm.$$

Resistance against rotation per *cm.* length of weld at a point distant r from the centroid G

$$=S=Kr$$

where
$$K=\frac{Pe}{I_{xx}+I_{yy}}$$
$$=\frac{10{,}000\times 10}{4500}$$
$$=\frac{200}{9}$$

Resistance against rotation at A per *cm.* length of weld

$$=S_a=Kr_a$$

$$=\frac{200}{9}r_a \ kg./cm.$$

Total vertical component at A per *cm.* length of weld

$$=V=\frac{P}{L}+S_a \sin\theta$$
$$=166{\cdot}7+\frac{200}{9}r_a \sin\theta \ kg./cm.$$
$$=166{\cdot}7+\frac{200}{9}\times 5 \ kg/cm$$
$$=277{\cdot}8 \ kg./cm.$$

Total horizontal component at A per *cm.* length of weld

$$=H=S_a\cos\theta$$

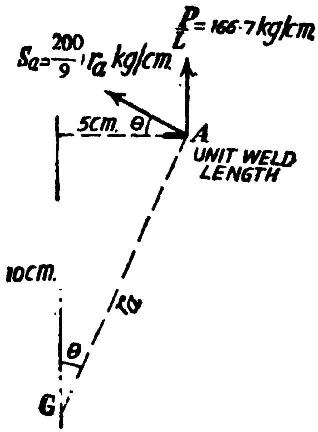


Fig. 679

$$=\frac{200}{9}r_a\cos\theta \ kg./cm.$$
$$=\frac{200}{9}\times 10 \ kg./cm.$$
$$=222{\cdot}2 \ kg./cm.$$

$\therefore$ Resultant resistance per *cm.* length at A

$$=\sqrt{(222{\cdot}2)^2+(277{\cdot}8)^2}$$
$$=355{\cdot}8 \ kg./cm.$$

Let the maximum shear stress intensity in the weld be q *kg./cm*^{2}.

$\therefore \quad 0{\cdot}7\times 0{\cdot}6\times 1\times q=355{\cdot}8$

$\therefore \quad q=847{\cdot}1 \ kg./cm^2.$

Problem 456. *Find the maximum load per centimetre run on the weld for the arrangement shown in Fig. 678. Suggest also a suitable size of weld.*

Solution.

$$\bar{x}=\frac{b\,(b+d)}{b+(b+d)}$$

$$=\frac{15\times 40}{15+40}\ cm.$$

$$=10{\cdot}91\ cm.$$

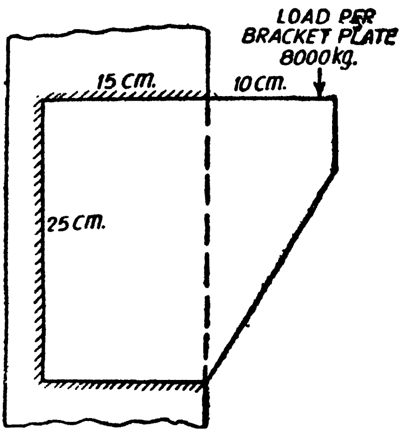


Fig. 678

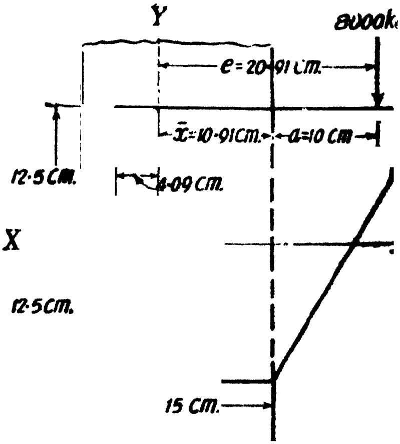


Fig. 679

Resistance offered by the weld per *cm.* length against translation

$$\frac{P}{L}=\frac{8000}{30+25}\ kg./cm.$$

$$=145{\cdot}4\ kg./cm.$$

At any point of the weld distant r from the centroid G of the weld group, the resistance of the weld per *cm.* length against rotation

$$=S=K\,r$$

where

$$K=\frac{P\,.\,e}{I_{xx}+I_{yy}}$$

$$I_{xx}=2\times 15(12{\cdot}5)^2+\frac{25^3}{12}$$

$$=5989{\cdot}5\ cm^3.$$

$$I_{yy}=2\left[\frac{15^3}{12}+15\,(10{\cdot}91-7{\cdot}50)^2\right]$$

$$+25\times(4{\cdot}09)^2=1329{\cdot}5\ cm^3.$$

$\therefore \quad I_{xx}+I_{yy}=5989{\cdot}5+1329{\cdot}5$

$=7319 \; cm.^3$

$\therefore \quad K=\dfrac{8000\times 20{\cdot}91}{7319}$

$=22{\cdot}86$

Now consider 1 *cm.* length of weld at *A*.

Resistance against rotation at *A* per *cm.* length

$=S_a=Kr_a$

$=22{\cdot}86 \; r_a$

Total vertical component per *cm.* length of weld at *A*

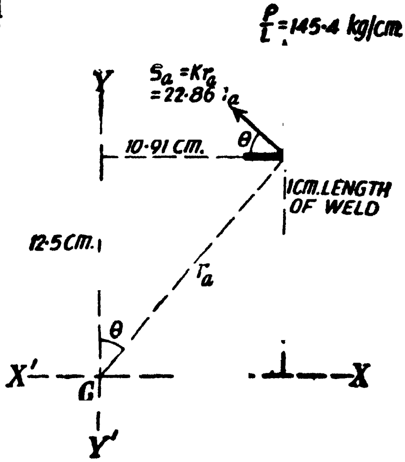


Fig. 680

$=V=\dfrac{P}{L}+S_a \sin\theta$

$=145{\cdot}4+22.86 \; r_a \sin\theta$

$=145{\cdot}4+22{\cdot}86\times 10{\cdot}91 \; kg./cm.$

$=396{\cdot}4 \; kg./cm.$

Total horizontal component per *cm.* length of weld at *A*

$=H=S_a\cos\theta$

$=22{\cdot}86 \; r_a \cos\theta$

$=22{\cdot}86\times 12{\cdot}5 \; kg./cm.$

$=286 \; kg./cm.$

$\therefore$ Resultant resistance per *cm.* length

$=\sqrt{(396{\cdot}4)^2+286^2} \; kg./cm.$

$=489 \; kg./cm.$

Let the size of weld be *s cm.*

Equating the strength of the weld per *cm.* length to the maximum resistance of the weld per *cm.* length, we have

$0{\cdot}7\times s\times 1025=489$

$s=\dfrac{489}{0{\cdot}7\times 1025} \; cm.$

$=0{\cdot}68 \; cm.$

$=6{\cdot}8 \; mm.$

Provide 7 *mm.* fillet weld.

Problem 457. *Fig. 681 shows an eccentric welded connection, with 8 mm. fillet welds.*

Determine the greatest load P per bracket plate which can be applied on the connection. Shear stress in the weld is not to exceed 1025 kg./cm².

Solution.

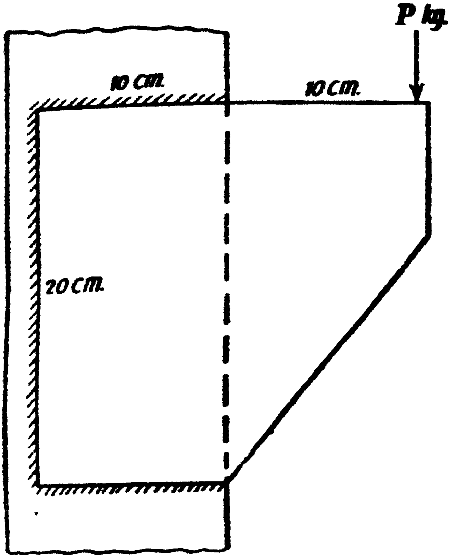


Fig. 681

$$\frac{b(b+d)}{b+(b+d)}$$

$$\frac{10\times 30}{10+30}\ cm$$

$$\frac{300}{40}\ cm.$$

$$=7{\cdot}5\ cm.$$

Eccentricity

$$e=\bar{x}+10$$

$$=7{\cdot}5+10$$

$$=17{\cdot}5\ cm.$$

Moment of inertia of weld length.

$$I_{xx}=2\times 10\times 10^2+\frac{20^3}{12}\ cm.^3$$

$$=2667\ cm.^3$$

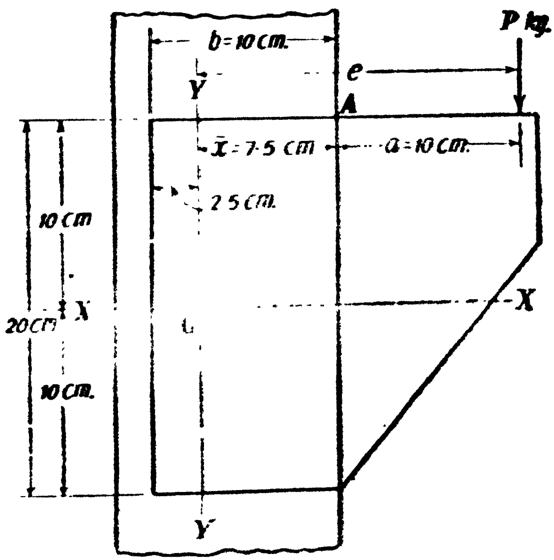


Fig. 682

$$I_{yy}=2\left[\frac{10^3}{12}+10(7\cdot5-5)^2\right]+20(2\cdot5)^2 \text{ cm.}^3$$

$$=417 \text{ cm.}^3$$

$$\therefore \quad I_{xx}+I_{yy}=2667+417$$

$$=3084 \text{ cm.}^3$$

Resistance against translation per *cm.* length

$$=\frac{P}{40} \text{ kg./cm.}$$

Resistance per *cm.* length at any point of weld distant r from the centroid G of the weld group is given by

$$S=Kr$$

where $$K=\frac{P\cdot e}{I_{xx}+I_{yy}}$$

$$\frac{P\times 17\cdot5}{3084}$$

Consider 1 *cm.* length of weld at the point A (Fig. 683).

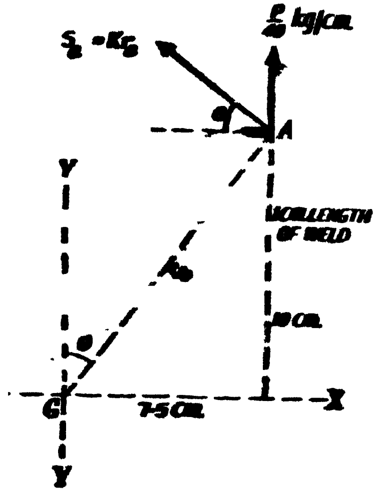


Fig. 683

Resistance against rotation per *cm.* length at A

$$=S_a=K\,r_a$$

$$=\frac{P\times17\cdot5}{3084}\,r_a$$

$$=\left[\frac{17\cdot5}{30\cdot84}\right]P\,r_a$$

Total vertical component per *cm.* length of weld at A

$$=V=\frac{P}{40}+S_a\sin\theta$$

$$=\frac{r}{40}+\frac{17\cdot5}{3084}P\,r_a\sin\theta$$

$$=\frac{P}{40}+\frac{17\cdot5}{3084}P(7\cdot5)\text{ kg/cm.}$$

$$=0\cdot0675\,P \text{ kg./cm.}$$

Horizontal component per *cm.* length of weld at A

$$=H=S_a\cos\theta$$

$$A=\frac{17\cdot5}{3084}P\,r_a\cos\theta$$

$$=\frac{17\cdot5}{3084}P(10)\text{ kg./cm.}$$

$$=0\cdot0567\,P \text{ kg./cm.}$$

∴ Resultant resistance per *cm*. length of weld at A,

$$=\sqrt{V^2+H^2}$$
$$=\sqrt{(0{\cdot}0675\,P)^2+(0{\cdot}0567\,P)^2}\ kg/cm.$$
$$=0{\cdot}08815\,P\ kg./cm.$$

Equating the maximum resistance per *cm*. length of weld to the strength to the weld per *cm*. length of weld, we have,

$$0{\cdot}08815\,P=0{\cdot}7\times0{\cdot}8\times1025\ kg./cm.$$
$$P=6512\ kg.$$

Case 2. *Welded connection subjected to moment in a plane normal to the plane of connection.*

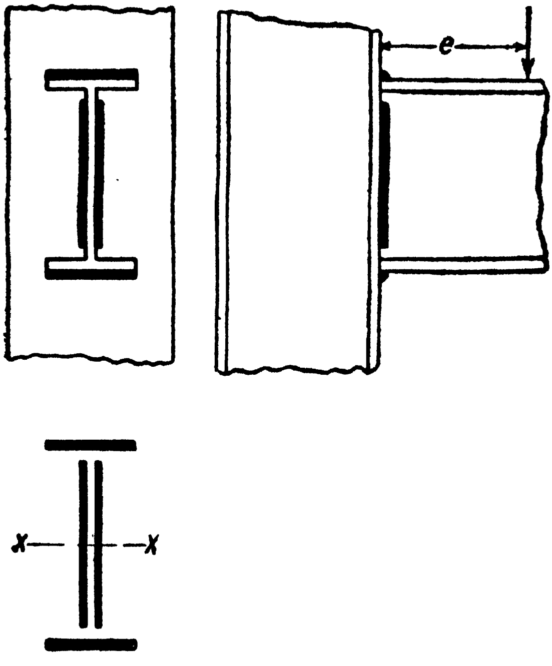


Fig. 684

Fig. 684 shows an eccentric welded connection. Let all the welds be of the same size. In this case also the welds have to offer resistances against translation and rotation.

Resistance per *cm*. length of weld against translation is assumed to be uniform.

∴ Resistance per *cm*. length against translation

$$=V=\frac{r}{L}\ kg./cm.$$

where L=total length of weld.

Resistance per *cm.* length at any point of the weld distant *y* from the *xx* axis of the weld line (Fig. 684) is given by

$$H=\left[\frac{P.e}{I_{xx}}\right]y \ kg./cm.$$

∴ Resultant resistance per *cm.* length of weld at any point

$$=\sqrt{V^2+H^2} \ kg./cm.$$

Problem 458. *Find the size of weld required for the bracket connection shown in Fig. 685. The stress in the weld is not to exceed 1025 kg /cm².*

Solution. Moment of inertia of the weld lengths about the xx-axis

$$=I=2\times\frac{15^3}{12}+2\times 14\times 10^2 \ cm^{\cdot 3}$$

$$=3363 \ cm^2$$

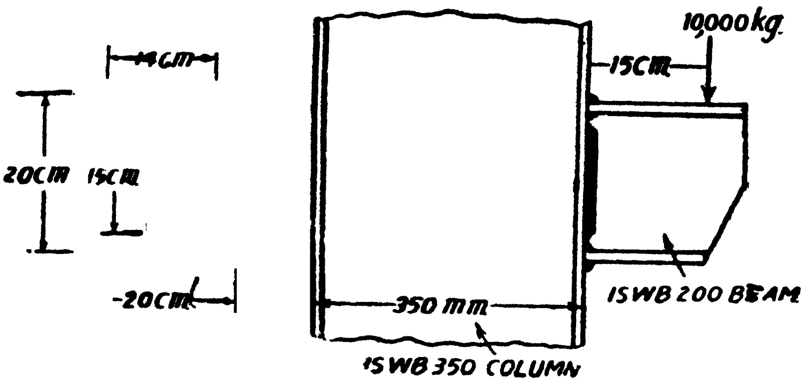


Fig. 685

Resistance against translation per *cm*. length of weld

$$=V=\frac{P}{L}=\frac{10000}{28+30} \ kg./cm.$$

$$=172 \ kg./cm.$$

∴ Resistance against rotation per *cm* length of weld

$$=H=\frac{M}{I}y$$

$$=\frac{10{,}000\times 15}{3363}\times 10 \ kg./cm.$$

$$=446 \ kg./cm.$$

∴ Resultant resistance per *cm.* length of weld

$$=\sqrt{(172)^2+(446)^2}=478 \ kg/cm.$$

Let the size of weld be s *cm*. Equating the strength of weld per *cm*. length to the maximum resistance per *cm*. length, we have,

$$0.7 \times s \times 1025 = 478$$

$$s = 0.67 \text{ cm.} \qquad \text{Use 7 mm. weld.}$$

Problem 459. *Find the minimum size of the fillet weld required to connect the bracket plate to the column as shown in Fig 686. Stress in the weld is not to exceed 1025 kg./cm².*

Solution. Moment of inertia of the weld lengths about the xx-axis

$$I = 2 \times \frac{30^3}{12} \text{ cm}^3.$$

$$= 4500 \text{ cm}^3.$$

Resistance against translation per *cm*. length

$$= V = \frac{P}{L}$$

$$= \frac{6000}{60}$$

$$= 100 \text{ kg./cm.}$$

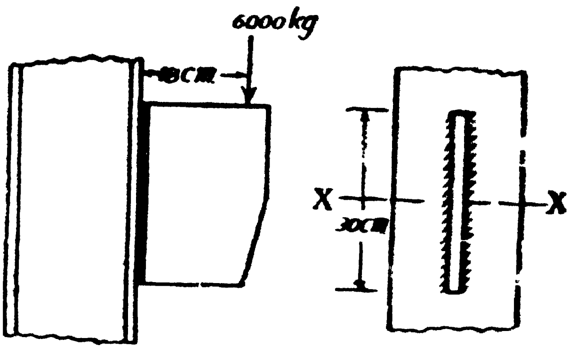


Fig. 686

Maximum resistance against rotation per *cm*. length

$$= H = \frac{M}{I} \cdot y_{max}$$

$$= \frac{6000 \times 18}{4500} \times \frac{30}{2} \text{ kg./cm.}$$

$$= 360 \text{ kg./cm.}$$

∴ Resultant resistance per *cm*. length of weld

$$= \sqrt{V^2 + H^2}$$

$$= \sqrt{100^2 + 360^2}$$

$$= 374 \text{ kg./cm.}$$

Let the size of the weld be s *cm*. Equating the strength of weld per *cm*. length to the maximum resistance per *cm*. length,

we get, $0.7 \times s \times 1025 = 374$

$$\therefore \qquad s = 0.52 \text{ cm.}$$

Provide a fillet weld 6 *mm*. size.

Problem 460. *A load of 10,000 kg. is applied to a bracket fillet-welded to a stanchion, as shown in Fig. ... Find the greatest resistance offered by the weld per cm. ... weld is 15 cm. long.*

Solution. Moment of inertia of weld length about the XX axis

$$=2\times 15\times 10^2 \text{ cm}^3$$

$$=3000 \text{ cm}^3.$$

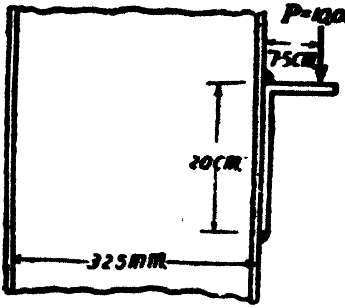


Fig. 687

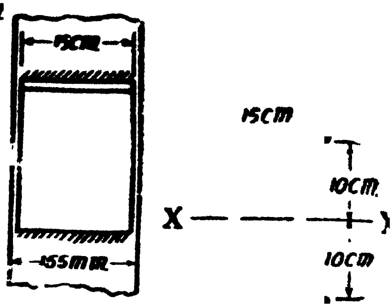


Fig. 688

Resistance against translation per *cm.* length of weld

$$=V=\frac{P}{L}$$

$$=\frac{10{,}000}{30} \text{ kg./cm.}$$

$$=333.3 \text{ kg./cm.}$$

Resistance against rotation per *cm.* length of weld,

$$=H=\frac{M}{I}\cdot y$$

$$=\frac{10{,}000\times 7.5}{3000}\times\frac{20}{2} \text{ kg./cm.}$$

$$=250 \text{ kg./cm.}$$

∴ Resultant resistance per *cm.* length

$$=\sqrt{(333.3)^2+250^2} \text{ kg./cm.}$$

$$=416.6 \text{ kg./cm.}$$

Problem 461. *Fig. 689 shows a bracket welded to a column. Find the maximum resistance offered by the weld per cm. length.*

Solution. Moment of inertia of the welded length about the xx-axis

$$I=2\times\frac{20^3}{12} \text{ cm.}^3$$

$$=1333 \text{ cm}^3$$

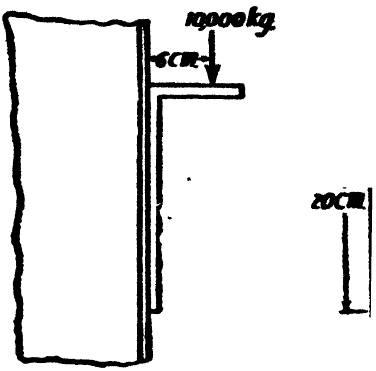


Fig. 689

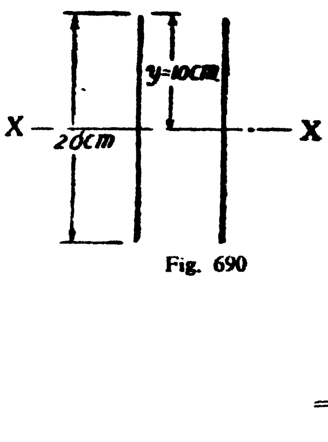


Fig. 690

Resistance against translation, per *cm* length of weld

$$=V=\frac{P}{L}$$

$$=\frac{10000}{40}=250\ kg./cm.$$

Resistance against rotation per *cm*. length of weld

$$=H=\frac{M}{}$$

$$=\frac{10000\times 6}{1333}\times 10\ kg./cm.$$

$$=450\ kg./cm.$$

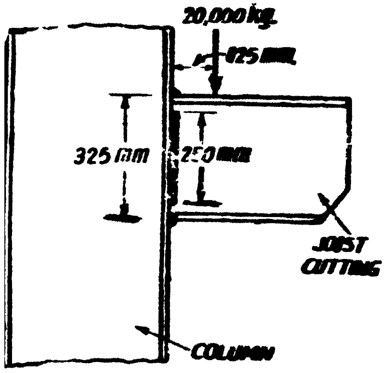
$\therefore$ Resultant resistance per *cm*. length of weld

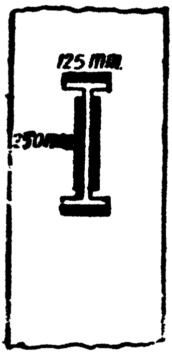
$$=\sqrt{250^2+450^2}$$

$$=515\ kg./cm.$$

Problem 462. *A joist cutting is used as a bracket as shown in Fig. 691. The bracket carries a load of 20 tonnes at a distance of 125 mm from the flange of the column. Determine the load in the welds per cm. length if the flange welds have double the throat thickness as compared to the welds in the web.*

Solution. Let the thickness of the throat of each of the flange welds be unity. Since the flange welds have double the throat thickness as compared to the web welds the two vertical web weld lengths can be replaced by an equivalent weld of unit throat thickness for purposes of analysis. (See Fig. 692).





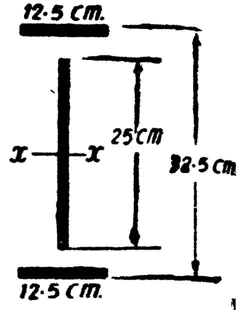


Fig 691 Fig. 692

Analysis of flange welds

Moment of the inertia of the equivalent weld lengths about the x-x axis.

$$=I=\frac{25^3}{12}+2\times 12{\cdot}5\times(16{\cdot}25)^2\ cm.^3$$

$$=7903\ cm^3.$$

Resistance against translation per *cm*. length of weld

$$=V=\frac{20{,}000}{50}$$

$$=400\ kg./cm.$$

Resistance against rotation per *cm*. length of weld

$$=H=\frac{M}{I}.y$$

$$=\frac{20{,}000\times 12{\cdot}5}{7903}\times 16{\cdot}25\ kg./cm.$$

$$=514\ kg./cm.$$

Resultant resistance per *cm*. length of weld

$$=\sqrt{400^2+514^2}$$

$$=651\ kg./cm.$$

Analysis of web welds

Note. The web is provided with two weld lengths.

Resistance against rotation per *cm*. length

$$=\tfrac{1}{2}\ (400)$$

$$=200\ kg./cm.$$

Maximum resistance against rotation per *cm*. length.

$$=\frac{1}{2}.\left(\frac{M}{I}.y\right)$$

$$=\frac{1}{2}\ \frac{(20{,}000\times 12{\cdot}5)\times 12{\cdot}5}{7903}\ kg./cm.$$

$$=198\ kg./cm.$$

∴ Resultant resistance per *cm*. length of weld

$$=\sqrt{200^2+198^2}$$

$$=280\ kg./cm.$$

17

Analysis of Framed Structures

160. Perfect Frame

A framed structure consists of a number of members connected to each other so as to form a frame to support an external load system. In the present discussion only pin-jointed connections between members will be assumed.

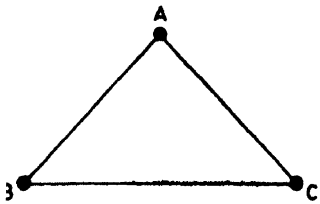


Fig. 693 (*a*). Basic perfect frame

The simplest frame is a triangle, Fig. 693 (*a*), consisting of three members pin-jointed to each other. This can be easily analysed by the conditions of equilibrium. This frame is called the basic perfect frame. It has three members AB, BC and CA and three joints A, B and C.

Suppose we add to this basic perfect frame two members AD and CD and a joint D, we get a frame [Fig. 693 (*b*)] which can also be analysed by the conditions of equilibrium. This frame is called a perfect frame. Suppose we add to this frame again a set of two members and a joint as shown in Fig. 693 (*c*). We again get a perfect frame.

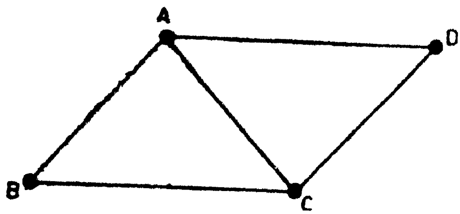


Fig 693 (*b*). Perfect frame

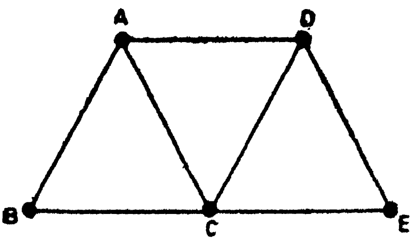


Fig. 693 (*c*). Perfect frame

In this way, we can go on adding any number of sets (each set consists of two members and a joint) and obtain a perfect frame.

Relation between the number of joints and the number of members in a perfect frame.

Let there be n members and j joints in a perfect frame. See Fig. 694 (a).

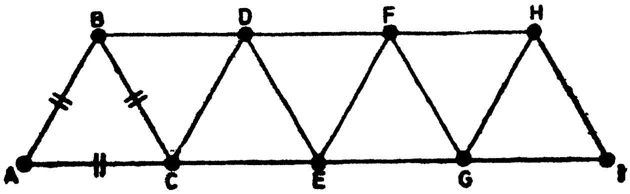


Fig. 694 (a)

Suppose we remove three members AB, BC and CA and the three joints A, B and C We are now left with $(n-3)$ members and $(j-3)$ joints.

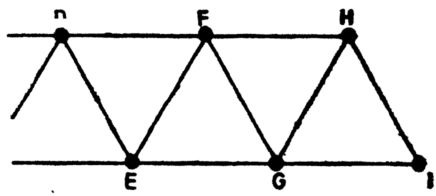


Fig. 694 (b)

Studying this remaining part of the frame [Fig. 694 (b)] we find that the number of members is such that, for each joint, there are two members.

Hence for the $(j-3)$ joints we have $2(j-3)$ members.

$$\therefore \quad n-3=2(j-3)$$

$$\therefore \quad n=2j-3$$

Hence for a stable frame the minimum number of members required = twice the number of joints minus three.

If the number of members provided is less than the above requirement the frame will not be stable.

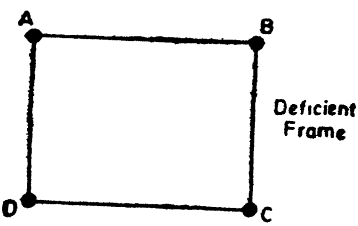


Fig. 695 (a)

For instance the frame $ABCD$ shown in Fig. 695 (a) has four members and four joints. Since there are four joints, we need at least $2\times4-3=5$ members to provide a stability to the frame. The four member frame shown in Fig. 695 (a) is therefore unstable.

Such a frame is called a *deficient frame.*

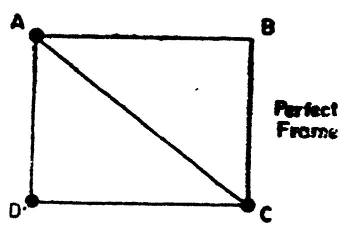
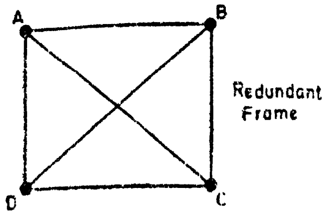


Fig. 695 (b)

Suppose we add one more member—say the member AC [See Fig. 695 (b)]. The frame becomes stable and perfect. Hence, a deficient frame has less number of members than what is required for a perfect frame.



g. 696

Suppose, to the rectangular frame *ABCD* we provide the diagonal *AC* as well as *BD*. This frame would remain stable even if one of the members *AC* or *BD* is removed. In other words, the frame has more members than what is required for a perfect frame. Such a frame is called a *redundant* frame.

In general let a frame have *j* joints and *n* members

If $n=2j-3$, the frame is a *perfect frame*

If $n<2j-3$, the frame is a *deficient frame*

If $n>2j-3$, the frame is a *redundant frame*.

A perfect frame can always be analysed by the conditions of equilibrium. A redundant frame cannot be fully analysed by the conditions of equilibrium. In the present chapter we will discuss the analysis of perfect frames only.

§161 Reactions at Supports

Frames are usually provided with either

(*i*) Roller or free supports or (*ii*) hinged support.

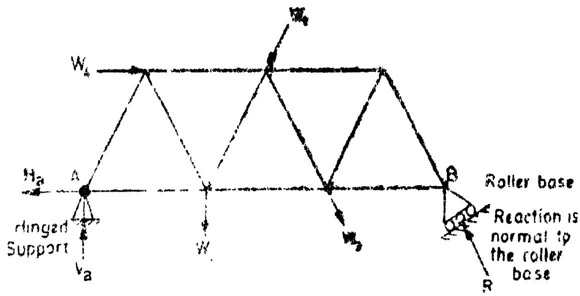


Fig. 697

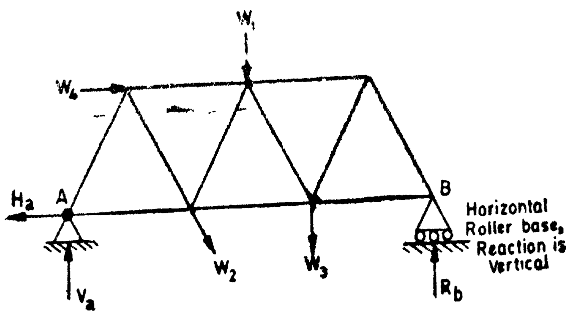


Fig. 698

At a roller support the line of action of reaction will be at right angles to the roller base. The reaction R_b at the roller support B of the truss shown in Fig. 697 will act normal to the roller base.

For the particular case, when the roller base is horizontal the reaction at such a support will be vertical.

In the cantilever frame shown in Fig. 699, the roller base at B is vertical and hence the reaction at this support is horizontal.

At a hinged support, the direction and the line of action of reaction will depend upon the load system on the structure.

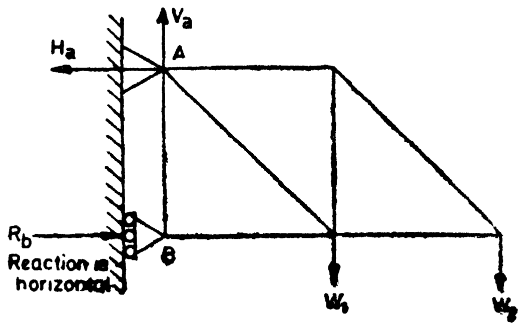


Fig. 699

To determine the reactions.

Reaction at the supports of a structure can be determined by the conditions of equilibrium. The external load system applied on the structure and the reactions at the supports must form a system in equilibrium.

Consider the cantilever truss shown in Fig. 700. The truss is provided with a hinged support at A and a roller support at E.

The roller base at E being vertical the reaction at E is horizontal. Hence there will be no vertical reaction at E.

Taking moments about A,

$$H_e \times 4=(6+4)3+(3\times 6)$$

$$H_e=12t\rightarrow$$

Total applied vertical force

$$=6+4+3=13t\downarrow$$

$\therefore$ Vertical reaction at $A=V_a=13t\uparrow$

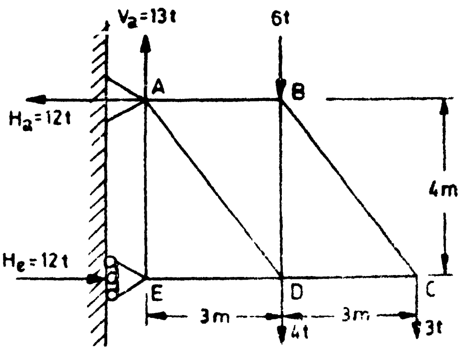
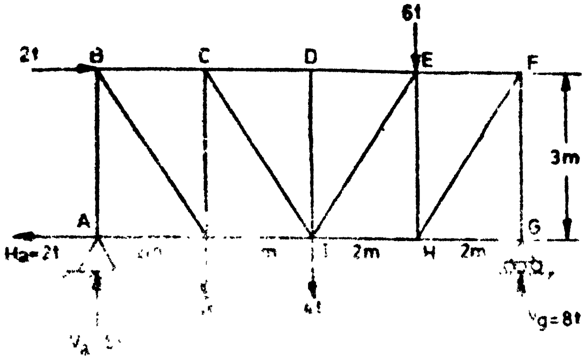


Fig. 700

Resolving the forces horizontally we get $H_a=12t\leftarrow$

Thus the reaction at A consists of a vertical component $V_a=13t\uparrow$ and a horizontal component $H_a=12t\leftarrow$

Now consider the truss shown in Fig. 701 provided with a hinged support at A and a roller support at G. The roller base at G is horizontal and hence the reaction at G is entirely vertical. There will be no horizontal reaction at G.

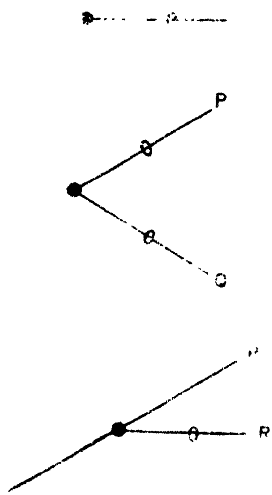


Take ...
F × ... 6 × ...

Total ...
... 3
... ?t

... truss ... forces

In a ... some ... may not carry ... the following principles



(b)

(c)

(a) A single force cannot form a system in equilibrium. Hence if there is only one force acting at a joint, then for the equilibrium of the joint, this force equals zero.

In Fig. 702 (a) $P = 0$

(b) If two forces act at a joint, then for the equilibrium of the joint these two forces should act along the same straight line. The two forces will ... equal and opposite. If the two forces acting at a joint are not along the same straight line, then for the equilibrium of the joint each force ... 0.

In Fig. 702 (b) $P = 0$ and $Q = 0$

(c) If three forces act at a joint and two of them are along ... the third force should be equal to zero.

In Fig. 702 (c) $R = 0$

It will be found very convenient to apply the above principles in finding which of the members will not carry forces.

For instance in the truss shown in Fig 703 consider the joint H. Forces at this joint are

P_{ha} in the member HA

P_{hg} in the member HG.

and P_{hb} in the member HB

Since P_{ha} and P_{hg} are in the same straight line, $P_{hb}=0$. Similarly $P_{cg}=0$ and $P_{fd}=0$.

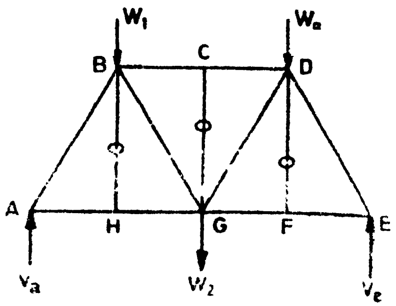


Fig. 703

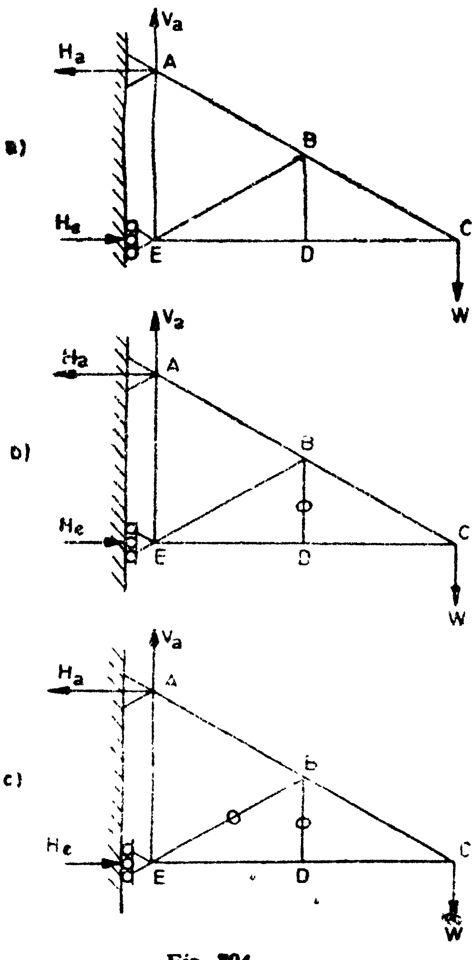


Fig. 704

Now consider the cantilever truss shown in Fig. 704 at the joint *D*. There are three forces namely

P_{de}, P_{dc} and P_{db}.

The two forces P_{de} and P_{dc} are along the same straight line

$\therefore \quad P_{db} = 0$

Now consider the joint *B*

Since $\quad P_{bd} = 0$, the forces at this joint are

P_{ba}, P_{bc} and P_{be}

Of these forces, P_{ba} and P_{bc} are along the same straight line.

$\therefore \quad P_{be} = 0$

Now consider the truss shown in Fig. 705.

Obviously the reaction at *H* is vertical and $\quad V_h = W$

Reaction at $\quad A = 0$

By using the principles explained above we find that there will be no force in the members other than *EF*, *FG* and *GH*.

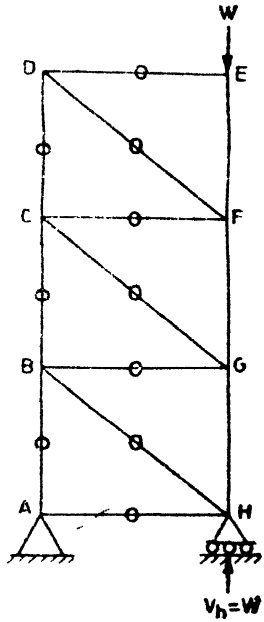


Fig. 705

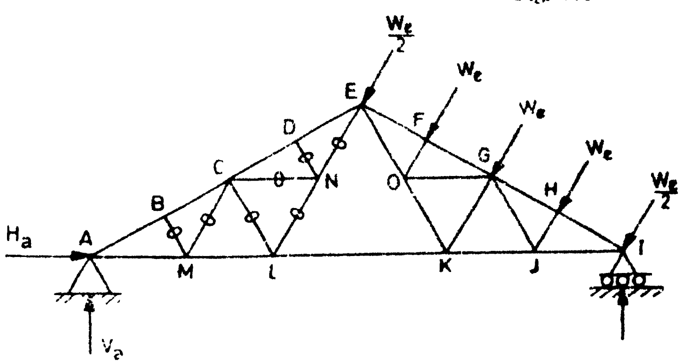


Fig. 706

Similarly for the roof truss shown in Fig. [illegible] it can be realised that there will be no forces in the members BM, [illegible], NE, NL.

§162. Analysis of a truss

The analysis of a truss consists of [illegible]

(*i*) Determination of the reactions [illegible] supports.

(*ii*) Determination of the forces [illegible] of the truss.

The reactions are determined by [illegible] *load system and the induced reactions* [illegible] *equilibrium.*

The forces in the members of the [illegible] condition that *every joint should be* [illegible] *acting at every joint should form a system* [illegible]

A structure can be analysed by the [illegible]

(*i*) Method of Resolution or method of joints

(*ii*) Method of sections

(*iii*) Graphical analysis.

(*i*) **Method of joints**. In this method, after determining the reactions at the supports, we will consider the equilibrium of every joint. The following examples explain this method. Note that a compression member will *push* the joint to which it is connected; and a tension member will *pull* the joint to which it is connected.

Problem 463 *Analyse the truss shown in Fig 707.*

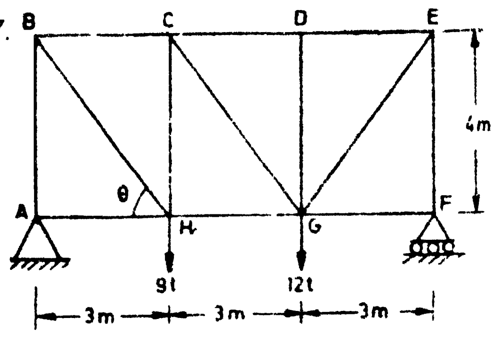


Fig. 707

Solution. Let V_a and V_f be the reactions at supports A and F.

Taking moments about A,

$$V_f \times 9 = (9 \times 3) + (12 \times 6)$$

$$\therefore \quad V_f = 11\ t \uparrow$$

and $\quad V_a = 21 - 11$

$$= 10\ t \uparrow$$

Members which do not carry forces

At the joint A there are three forces, two of which are along the same straight line. Hence the third force namely

P_{ab} [illegible]

Similarly considering the joint F,

we know $P_{fg}=0$

Similarly considering the joint D,

we know $P_{dg}=0$

See Fig. 708.

Now let as consider the equilibrium of the various joints.

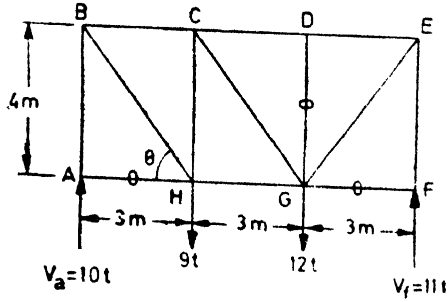


Fig. 708

Joint A

Resolving vertically,

$$P_{ab}=10\ t\ (compressive)$$

See Fig. 709.

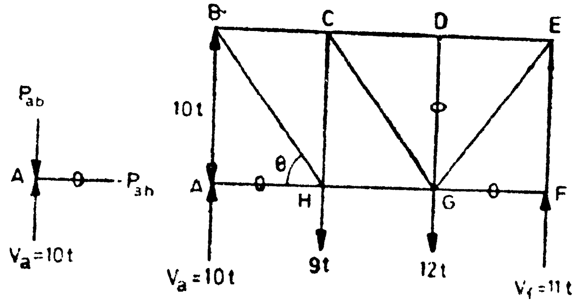


Fig. 709

Joint B

Resolving vertically,

$$P_{bh}\sin\theta=10\ t$$

$$\therefore \qquad P_{bh}=\frac{10}{\sin\theta}$$

But, $$\tan\theta=\frac{4}{3}$$

$$\therefore \qquad \sin\theta=\frac{4}{5} \text{ and } \cos\theta=\frac{3}{5}$$

$$\therefore \qquad P_{bh}=\frac{10\times5}{4}=\frac{25}{2}\ t\ (tensile)$$

Resolving horizontally,

$$P_{bc}=\frac{25}{3}\cos\theta$$

$$=\frac{25}{2}\times\frac{3}{5}=\frac{15}{2}\ t\ (compressive)$$

See Fig. 710.

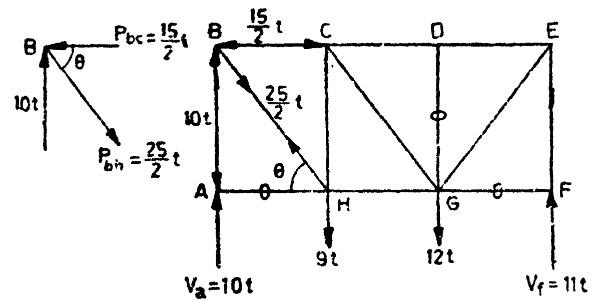


Fig. 710

Joint H

Resolving vertically we have the following components :

(*i*) $9\ t\downarrow$

(*ii*) $$\frac{25}{2}\sin\theta=\frac{25}{2}\cdot\frac{4}{5}=10\ t\uparrow$$

$\therefore$ Balancing vertical force needed $=1\ t\downarrow$

$\therefore$ $P_{hc}=1\ t$ (*compressive*)

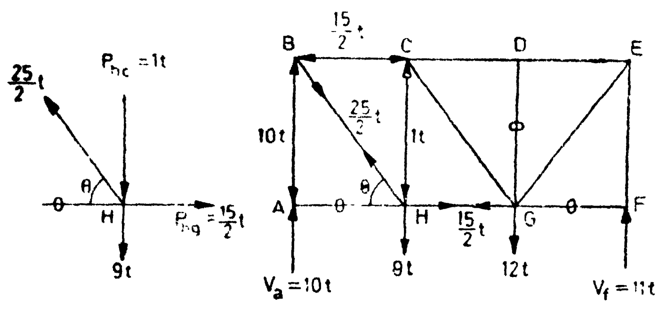


Fig. 711

Resolving horizontally,

$$P_{hg}=\frac{25}{2}\cos\theta$$

$$=\frac{25}{2}\times\frac{3}{5}=\frac{15}{2}\ t\ (tensile)$$

Joint C

Resolving vertically

$$P_{cg}\sin\theta=1$$

$$\therefore\quad P_{cg}=\frac{1}{\sin\theta}=\frac{5}{4}\ t\ (tensile)$$

Resolving horizontally, we have the following components :

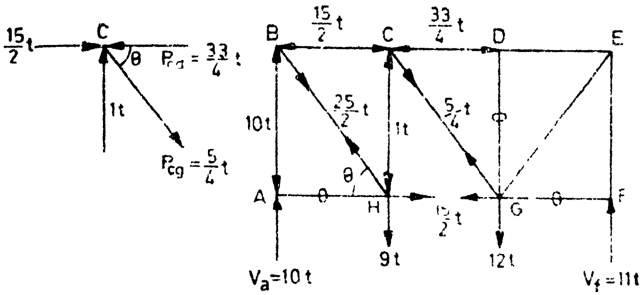


Fig. 712

(*i*) $\frac{15}{2} t \rightarrow$

(*ii*) $\frac{5}{4} \cos \theta = \frac{5}{4} \cdot \frac{3}{5} = \frac{3}{4} t \rightarrow$

∴ Balancing horizontal force needed

$$= \frac{33}{4} t \leftarrow$$

∴ $$P_{cd} = \frac{33}{4} t \text{ (compressive)}$$

See Fig. 712.

Joint D

Resolving horizontally at *D*

$$P_{de} = \frac{33}{4} t. \text{ (compressive)}$$

See Fig. 713.

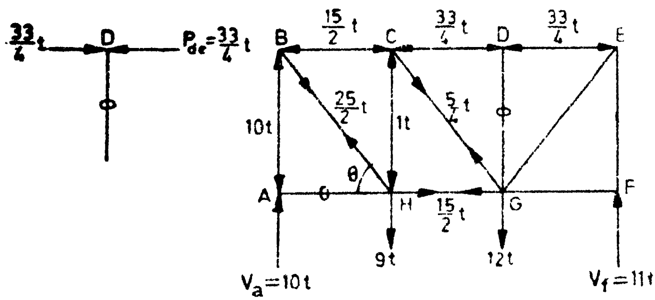


Fig. 713

Joint G

Resolving vertically we have the following components :

(*i*) 12*t* ↓

(*ii*) $\frac{5}{4} \sin \theta$

$$= \frac{5}{4} \cdot \frac{4}{5} = 1\ t \uparrow$$

$\therefore$ Balancing vertical force needed

$$= 11\ t \uparrow$$

$$\therefore \quad P_{ge} \sin \theta = 11\ t$$

$$\therefore \quad P_{ge} = 11 \times \frac{5}{4} = \frac{55}{4}\ t$$ See Fig. 713

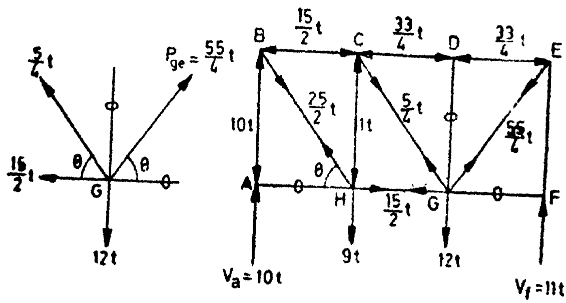


Fig. 714

Arithmetical check :

Resolving horizontally we have in following components :

(*i*) $\frac{15}{2}\ t \leftarrow$

(*ii*) $\frac{5}{4} \cos \theta = \frac{5}{4} \cdot \frac{3}{5} = \frac{3}{4}\ t \leftarrow$

(*iii*) $\frac{55}{4} \cos \theta = \frac{55}{4} \cdot \frac{3}{5} = \frac{33}{4}$

These forces balance.

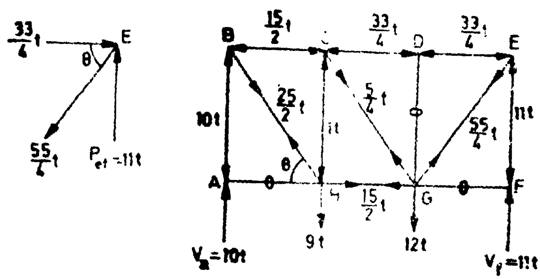


Fig. 715

Joint E

Resolving vertically at E

$$P_{ef} = \frac{55}{4} \sin \theta$$

$$= \frac{55}{4} \cdot \frac{4}{5} = 11 \ t \ (compressive)$$

Arithmetical check.

Resolving horizontally the components are :

(*i*) $\frac{33}{4} \ t \rightarrow$

(*ii*) $\frac{55}{4} \cos \theta = \frac{55}{4} \cdot \frac{3}{5} = \frac{33}{4} \ t \leftarrow$

The forces at the joint F obviously are in equilibrium. Fig. 7.16 shows the forces in all the members of the truss. These are tabulated below

Abstract of forces

Member	*Force in the member*	
	Compression	*Tension*
AB	10 t	
BC	$\frac{15}{2}$ t	
CD	$\frac{33}{4}$ t	
DE	$\frac{33}{4}$ t	
EF	11 t	
FG	0	0
GH		$\frac{15}{2}$ t
HA	0	0
BH		25/[illegible] t
HC	1 t	
CG		$\frac{5}{4}$ t
DG	0	0
GE		$\frac{55}{4}$ t

Problem 464. *Determine the forces in the various members of the truss shown in Fig. 716.*

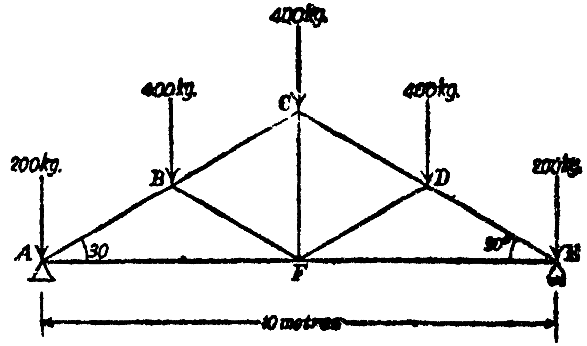


Fig. 716

Solution. Each vertical reaction

= Half the total load = 800 *kg.*

Now we will discuss the equilibrium of all the joints of the structure.

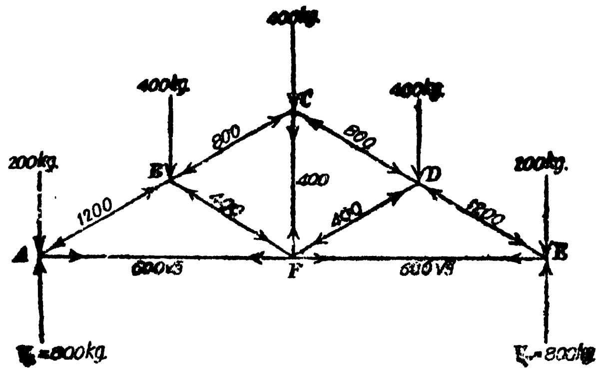


Fig. 717

The forces keeping the joint *A* in equilibrium are the following :

(*i*) Downward load 200 *kg*.

(*ii*) Upward reaction V_a=800 *kg*.

(*iii*) Force in the member *AB i.e.*, P_{ab}.

(*iv*) Force in the member *AF i.e.*, P_{af}.

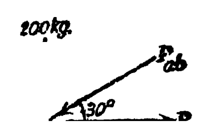


Fig. 718

Resolving the forces at *A* vertically we find the following vertical components, *viz.*, the downward load of 200 *kg*, and an upward force of 800 *kg*.

The resultant of the above two forces is 600 *kg*. ↑ acting upwards. Hence, to keep the joint *A* in equilibrium, we require a downward force of 600 *kg.* ↓ at *A*. The member *AF* being horizontal, the force in this member will not have a vertical component. Hence only the vertical component of the force in the member *AB* should provide a vertical force of 600 *kg.* ↓ downwards at *A*. To satisfy this condition the force in *AB* should be a compressive force, *i.e.*, a force pushing the joint *A*. If P_{ab} is the force in the member *AB*, its vertical component should be equal to 600 *kg.* ↓ downwards at *A*.

i.e. $P_{ab} \sin 30° = 600$ *kg.*

$\therefore$ $P_{ab} = 1200$ *kg.* (*compressive*)

Now resolve the forces at the joint *A* horizontally. The external load of 200 *kg.* and the reaction of 800 *kg* being vertical forces, do not have horizontal components. But the force P_{ab} has a horizontal component of $P_{ab} \cos 30°$, *i.e.*, $1200 \cos 30°$, *i.e.*, $600\sqrt{3}$ *kg.* This horizontal component of P_{ab}, at *A*, acts in a direction towards the left ←. Hence to keep the joint *A* in equilibrium we require a horizontal force of $600\sqrt{3}$ *kg.* towards the right →.

This necessary force is provided by the force in the member *AF*. Hence if P_{af} be the force in the member *AF*, this force should *pull* the joint *A* towards the right. Hence P_{af} is a tensile force

$\therefore$ $P_{af} = 600\sqrt{3}$ *kg.* (*tensile*)

Joint B

Now consider the joint *B*.

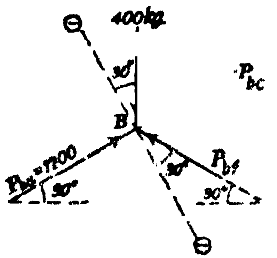


Fig. 719

The forces keeping this joint in equilibrium are the following:

(*i*) Downward external load 400 *kg* ↓

(*ii*) Force in the member *BA* = P_{ba} = 1200 *kg.*

(*iii*) Force in the member *BC* = P_{bc}.

(*iv*) Force in the member *BF* = P_{bf}.

The unknown forces are P_{bc} and P_{bf}.

Hence a direction for resolution will now be chosen at right angles to the line of action of one of the unknowns. For example, consider a line 1—1 normal to the force P_{bc}. If now the forces at *B* are resolved along the line 1—1, the force in *BC* will not have any component along 1—1. The only forces which can have components along 1—1 are the external force of 400 *kg.* and the force P_{bf}.

Hence, resolving these forces along 1—1, we get

$400 \cos 30° = P_{bf} \cos 30°$

$\therefore$ $P_{bf} = 400$ *kg*

The component of 400 *kg.* at *B* along 1—1 acts down the line 1—1. Hence the component of P_{bf} along 1—1 should act up the line 1—1. Hence the force P_{bf} should be a force pushing the joint *B*.

$\therefore$ $P_{bf} = 400$ *kg.* (*compressive*)

Now let us resolve the forces at B along the principal rafter. We have the following components along the principal rafter.

(i) Force P_{ba} of 1200 $kg.$ acting up the line of rafter.

(ii) Component of P_{bf}, *i.e.*, P_{bf} cos 60°, *i.e.*, 400 cos 60° $=200$ $kg.$ acting down the line of rafter.

(iii) Component of 400 the $kg.$ load $=400$ cos 60° $=200$ $kg.$ acting down the line of the rafter.

Hence these forces along the line of rafter have a resultant of 800 $kg.$ acting *up* the line of rafter at B. Hence we require a balancing force of 800 $kg.$ ***down*** the line of rafter at B. This is supplied by the force P_{bc} in the member BC. This force P_{bc} should therefore ***push*** the joint B down the rafter.

$\therefore$ $P_{bc}=800$ $kg.$ (*compressive*)

By symmetry

Force in $CD=P_{cd}=800$ $kg.$ (***compressive***)

Force in $DE=P_{de}=1200$ $kg.$ (***compressive***)

Force in $FD=P_{fd}=400$ $kg.$ (***compressive***)

Force in $FE=P_{fe}=600\sqrt{3}$ $kg.$ (***tensile***)

Members	*Force (kg.)*	
	Compressive	*Tensile*
AB	1200	
BC	800	
CD	800	
DE	1200	
[illegible]		600√3
[illegible]		600√3
[illegible]	400	
DF	400	
CF		400

Joint F

Resolving the forces at this joint vertically, we have the following components.

(*i*) Vertical component of $P_{fb}=400 \sin 30° = 200$ *kg*. ↘

(*ii*) Vertical component of $P_{fa}=400 \sin 30° = 200$ *kg*.

The resultant of the above two vertical components is 400 *kg*. ↓

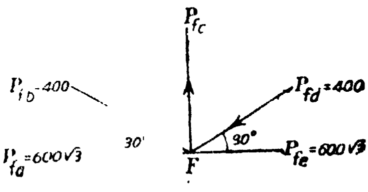


Fig. 720

Hence we require a balancing upward force of 400 *kg*. ↑ at *F*. This is supplied by the tension in the member *FC*

∴ P_{fc} = 400 *kg* (*tensile*)

Problem 465 *Determine the forces in all the members of the truss shown in Fig. 721.*

Solution. See graphical solution at the end of the chapter.

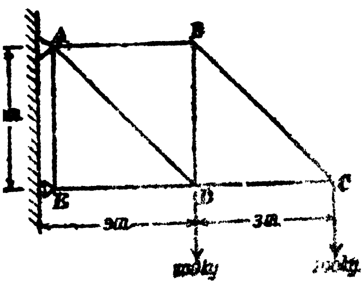


Fig. 721

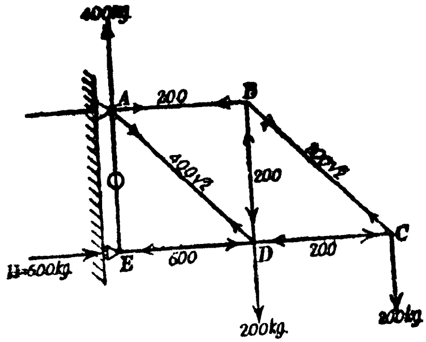


Fig. 722

Analytical solution

Since the end *E* has been placed on rollers, the reaction at this support must be at right [illegible] to the roller base.

Hence this reaction [illegible] be horizontal

Let [illegible] at *E*

Taking moments [illegible] we have

[illegible] 3 = 200 × 6

∴ *H* = [illegible]

Horizontal reaction at *A* = 600 *kg*. ←

Since these are the only horizontal forces on the whole truss.

Total external vertical load

= 200 + 200 = 400 *kg*. ↓

∴ Vertical reaction at *A*

= 400 *kg*. ↑

Joint E

At this joint the following forces are acting :

(*i*) Horizontal reaction

$H = 600$ *kg.*

(*ii*) Horizontal force P_{ed}

(*iii*) Vertical force P_{ea}

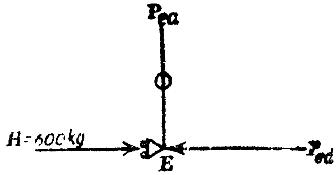


Fig. 723

Out of these three forces, two of them are along the same straight line.

Hence, the third force, *viz*., P_{ea} must be zero.

Hence $\quad P_{ed} = -600$ *kg.* (*compressive*)

Joint C

The forces keeping the joint *C* in equilibrium are the following

(*i*) Downward load of 200 *kg.*

(*ii*) P_{cb}

(*iii*) P_{cd}

Resolving these forces vertically, we have

$$P_{cb} \sin 45° = 200$$

$$\therefore \quad P_{cb} = 200\sqrt{2} \text{ kg. (tensile)}$$

Now resolving the forces horizontally

$$P_{cd} = -P_{cb} \cos 45°$$

$$\therefore \quad P_{cd} = -200\sqrt{2} \cos 45° = -200 \text{ kg. (compressive)}$$

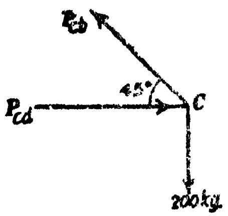


Fig. 724

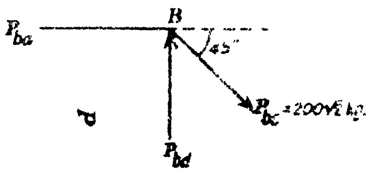


Fig. 725

Joint B

This joint is in equilibrium under the action of the following forces :

(*i*) Tensile force $P_{bc} = 200\sqrt{2}$ *kg.*

(*ii*) Horizontal force P_{ba}.

(*iii*) Vertical force P_{bd}.

Resolving these forces vertically,

We have

$$P_{bd} = -200\sqrt{2} \sin 45°$$

$$\therefore \quad P_{bd} = 200 \text{ kg. (compressive)}$$

Now, resolving the forces horizontally,

We have

$$P_{ba} = 200\sqrt{2}\cos 45°$$

$$\therefore \quad P_{ba} = 200 \text{ kg. } (\textit{tensile})$$

Joint D

The forces acting at this joint are shown in Fig. 726.

Resolving these forces vertically, we have

$$P_{da}\sin 45° = 200 + 200$$

$$= 400$$

$$\therefore \quad P_{da} = 400\sqrt{2} \text{ kg. } (\textit{tensile})$$

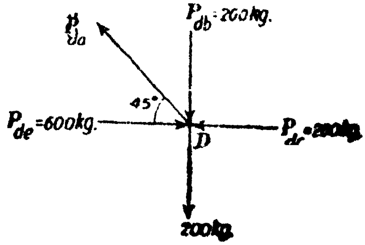


Fig. 726

Arithmetical Check

Resolving horizontally, the horizontal components are :

(*i*) $P_{de} = 600$ *kg.* ⟶

(*ii*) $P_{da}\cos 45° = 400\sqrt{2}\cos 45° = 400$ *kg.* ⟵

(*iii*) $P_{dc} = 200$ *kg.* ⟵

These components are balancing.

The forces in the various members have been tabulated below :

Members	*Force (kg.)*	
	Compressive	*Tensile*
AB		200
BC		$200\sqrt{2}$
CD	200	
DE	600	
EA	0	0
DA		$400\sqrt{2}$
DB	200	

Problem 466. *Determine the forces in the various members of the structure shown in Fig. 727.*

Solution. See graphical solution at the end of the chapter.

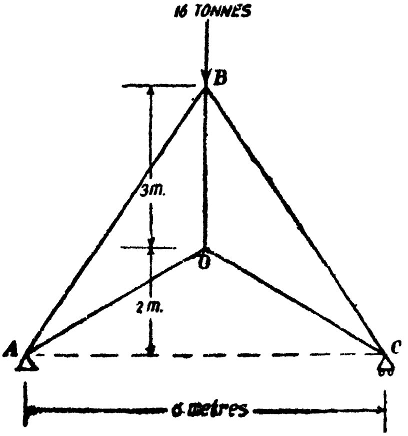


Fig. 727

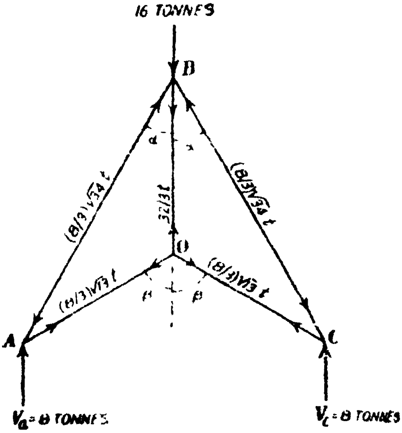


Fig 728

Analytical Solution

Each vertical reaction = $\frac{10}{2}$ 8 *tonnes*

It can be easily concluded that the members AB and BC are compression members and that AO and OC are tension members. The above can be determined as follows :

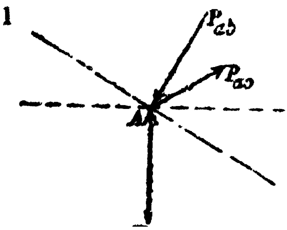


Fig. 729

For instance, consider the joint A which is in equilibrium under the action of three forces :

(*i*) 8 *tonnes* acting vertically upwards.

(*ii*) Force P_{ab} in the member AB.

(*iii*) Force P_{ao} in the member AO.

If these forces be resolved along a direction 1 – 1 at right angles to the force P_{ab}, we find that the component of the upward force of 8 *tonnes* and the component of the force P_{ao}, along 1 – 1 should balance. To satisfy this condition P_{ao} should be a tensile force.

Having now established that P_{ao} is a tensile force, resolving the forces at A horizontally we find that the horizontal components of the forces P_{ab} and P_{ao} should balance. To satisfy this condition, P_{ab} should be a compressive force.

Similarly P_{cb} is a compressive force and P_{co} is a tensile force.

Now consider the joint O. The vertical components of the forces P_{oa} and P_{oc} act downwards.

Hence P_{ob} should act upwards at O, *i.e.*, the force P_{ob} is a tensile force.

Thus we have determined the nature of forces in all the members of the structure.

It is now proposed to solve this problem by the *principle of triangle of forces*.

Consider the joint *A*. This joint is in equilibrium under the action of the following forces :

(*i*) An upward force $V_a = 8$ *tonnes*

(*ii*) Force P_{ab} and

(*iii*) Force P_{ao}

Hence these three forces can be represented by the three sides of a triangle which shall be respectively parallel to these three forces. Then the magnitudes of these three forces remain in the same proportion as the lengths of the corresponding sides of the triangle.

Now instead of actually constructing a triangle whose sides are parallel to the three forces at *A*, let us study the *geometry* of the triangle *ABO*.

The force $V_a = 8$ *tonnes* is parallel to the side *OB* of the triangle.

The force P_{ao} is parallel to the side *AO* of the triangle.

The force P_{ba} is parallel to the side *BA* of the triangle.

Hence the three sides of the triangle *ABO* are parallel to the lines of action of the three forces acting at *A*. Hence, by the principle of triangle of forces,

we have,

$$8 : P_{ao} : P_{ab} = OB : AO : BA$$

By the geometry of the triangle *ABO*,

we have,

$$OB = 3 \text{ metres}$$

$$AO = \sqrt{3^2+2^2} = \sqrt{13} \text{ metres.}$$

$$BA = \sqrt{3^2+5^2} = \sqrt{34} \text{ metres.}$$

$$\therefore \quad 8 : P_{ao} : P_{ab} = 3 : \sqrt{13} : \sqrt{34}$$

$$= 1 : \frac{\sqrt{13}}{3} : \frac{\sqrt{34}}{3}$$

$$= 8 : \frac{8}{3}\sqrt{13} : \frac{8}{3}\sqrt{34}$$

$$\therefore \quad P_{ao} = \frac{8}{3}\sqrt{13} \text{ tonnes (tensile)}$$

$$P_{ab} = \frac{8}{3}\sqrt{34} \text{ tonnes (compressive)}$$

Now consider the joint *O*. Resolving the forces at *O*, vertically, we have,

$$P_{ob}=2\,P_{oa}\cos\beta=2\times\frac{8}{3}\sqrt{13}\times\frac{2}{\sqrt{13}}$$

$$\therefore \qquad P_{ob}=\frac{32}{3}\ \textit{tonnes (tensile)}$$

This result may also be checked by resolving the forces at the joint *B* vertically.

The forces in the various members are tabulated below.

Member	*Force (Tonnes)*	
	Compressive	*Tensile*
AB	$\frac{8}{3}\sqrt{.4}=15{\cdot}55$	
BC	15·55	
AO		$\frac{8}{3}\sqrt{13}=9{\cdot}62$
CO		9·62
BO		$\frac{32}{3}=10{\cdot}67$

Problem 467. *Determine the forces in all the members of the truss shown in Fig. 730. All the inclined members are at 45° with the horizontal.*

Solution.

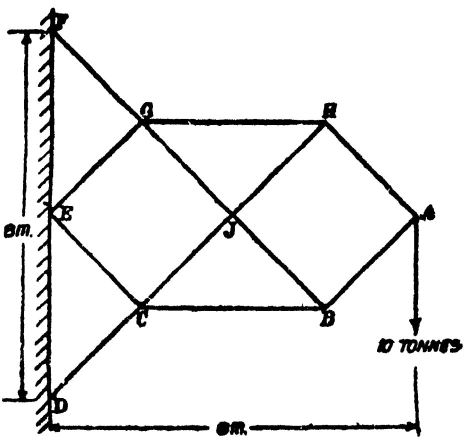


Fig. 730

Analytical Solution

Joint A. This is in equilibrium under the action of the three forces, *viz*. (*i*) 10 *tonnes*, (*ii*) P_{ah} and (*iii*) P_{ab}.

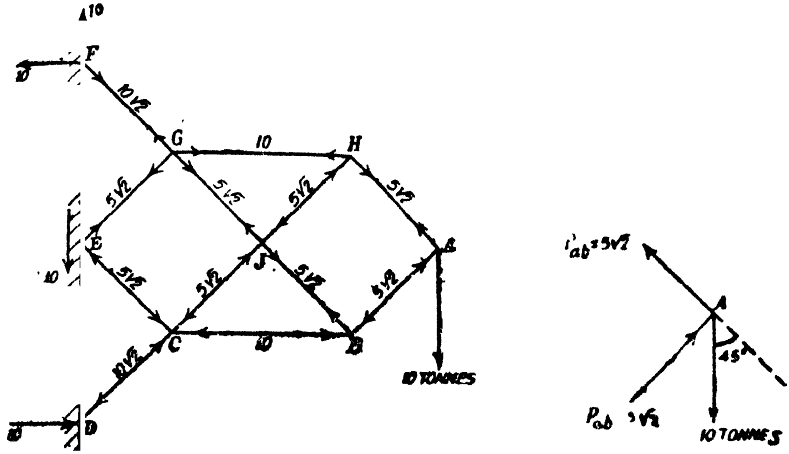


Fig. 731

Resolving the forces at A in line with AH, we have,

$$P_{ah} = 10 \cos 45^\circ$$

$$= \frac{10}{\sqrt{2}} = 5\sqrt{2} \text{ tonnes (tensile)}$$

Now since the horizontal components of the forces P_{ah} and P_{ab} should balance, we conclude

$$P_{ab} = 5\sqrt{2} \text{ tonnes (compressive)}$$

Joint H

Since the vertical components of the forces P_{ha} and P_{hj} should balance, we conclude

$$P_{hj} = 5\sqrt{2} \text{ tonnes (compressive)}$$

Resolving the forces at H horizontally, we have

$$P_{hg} = P_{ha} \cos 45^\circ + P_{hj} \cos 45^\circ$$

$$= 2 \times 5\sqrt{2} \; \frac{1}{\sqrt{2}} = 10 \text{ tonnes (tensile)}$$

Joint B

Since the vertical components of the forces P_{ba} and P_{bj} should balance, we conclude

$$P_{bj} = 5\sqrt{2} \text{ tonnes (tensile)}$$

Now resolving horizontally,

$$P_{bc} = P_{bj} \cos 45^\circ + P_{ba} \cos 45^\circ$$

$$= 2 \times 5\sqrt{2} \; \frac{1}{\sqrt{2}} = 10 \text{ tonnes (compressive)}$$

Joint J

Resolving the forces in line with *HJC*, we have,

$$P_{jc}=P_{jh}=5\sqrt{2}\ tonnes\ (compressive)$$

Now, resolving the forces in line with *GJB*, we have,

$$P_{jg}=P_{jb}=5\sqrt{2}\ tonnes\ (tensile)$$

Joint G

Resolving the forces in line with *GE*, we have,

$$P_{ge}=P_{gh}\cos 45°$$

$$\therefore \quad P_{ge}=\frac{10}{\sqrt{2}}=5\sqrt{2}\ tonnes\ (tensile)$$

Now, resolving the forces in line with *FGJ*, we have,

$$P_{gf}=P_{gj}+P_{gh}\cos 45°$$

$$\therefore \quad P_{gf}=5\sqrt{2}+10\cos 45°$$

$$=10\sqrt{2}\ tonnes\ (tensile)$$

Joint C

Resolving the forces in line with *CE*, we have,

$$P_{ce}=P_{cb}\cos 45°$$

$$=10\cos 45°$$

$$\therefore \quad P_{ce}=5\sqrt{2}\ tonnes\ (compressive)$$

Now resolving the forces in line with *DCJ*, we have,

$$P_{cd}=P_{cj}+P_{cb}\cos 45°$$

$$\therefore \quad P_{cd}=5\sqrt{2}+10\cos 45°$$

$$\therefore \quad P_{cd}=10\sqrt{2}\ tonnes\ (compressive)$$

Reactions

At F,

Horizontal reaction $=P_{fg}\cos 45°$

$=10\sqrt{2}\cos 45°=10\ tonnes\leftarrow$

Vertical reaction $=P_{fg}\sin 45°=10\ tonnes\uparrow$

At E,

Since the horizontal components of the forces P_{eg} and P_{ec} balance, there will be no horizontal reaction.

Vertical reaction $=P_{eg}\sin 45°+P_{ec}\sin 45°$

$=2\times 5\sqrt{2}\sin 45°$

$=10\ tonnes\downarrow$

At D,

Horizontal reaction $=P_{dc}\cos 45°$

$=10\sqrt{2}\cos 45°=10\ tonnes\rightarrow$

Vertical reaction $=P_{dc}\sin 45°=10\ tonnes\uparrow$

The forces in the various members computed above are tabulated below.

Member	Force (Tonnes)	
	Compressive	*Tensile*
AB	$5\sqrt{2}$	
BC	10	
CD	$10\sqrt{2}$	
FG		$10\sqrt{2}$
GH		10
HA		$5\sqrt{2}$
LG		$5\sqrt{2}$
EC	$5\sqrt{2}$	
GJ		$5\sqrt{2}$
JB		$5\sqrt{2}$
JH	$5\sqrt{2}$	
JC	$5\sqrt{2}$	

Problem 468. *Calculate the forces induced in the members of the pin joined truss shown in fig. 732. Show the values on a neat diagram of the truss. Mention clearly the nature of the forces (tension or compression) in each case.* (*A.M.I.E., November 1966*)

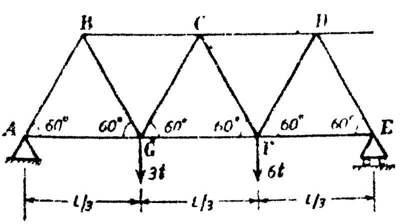


Fig. 732

Solution. We find that at *E* a roller support has been provided. Hence the reaction at *E* must be normal to the roller base, *i.e.* the reaction at *E*, in this case, should be vertical.

Let this vertical reaction at *E* be *V*.

Length of each member$=\frac{L}{3}$

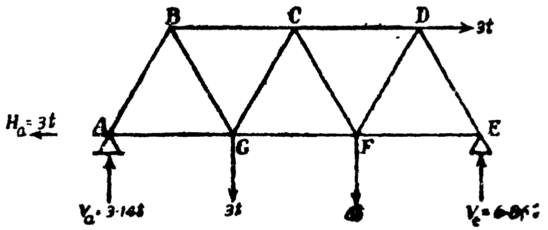


Fig. 733

Height of the truss

$$=\frac{L}{3}\sin 60$$

$$=\frac{L}{3}\cdot\frac{\sqrt{3}}{2}$$

$$=\frac{L}{6}\sqrt{3}$$

For the equilibrium of the truss taking moments of the forces on the truss about the end A, we have,

$$V_e.L=3\times\frac{L}{3}+6\times\frac{2L}{3}+3+\frac{L}{6}\sqrt{3}$$

$$\therefore \quad V_e=\frac{10+\sqrt{3}}{2}t=5{\cdot}86\ t$$

Total vertical external load$=3+6=9t$

$\therefore$ Vertical reaction at $A=V_a$

$$=9-\left(\frac{10+\sqrt{3}}{2}\right)$$

$$=\left(\frac{8-\sqrt{3}}{2}\right)t\uparrow$$

$$=3{\cdot}14\ t\uparrow$$

and Horizontal reaction at $A=3\ t\leftarrow$

Joint A

Resolving vertically, we get

$$P_{ab}\sin 60^\circ=3{\cdot}14t$$

$$\therefore \quad P_{ab}=3{\cdot}62\ t\ (\textit{compressive})$$

Resolving horizontally, we get

$$_{ag}=3+P_{ab}\cos 60^\circ$$

$$=3+3{\cdot}62\cos 60^\circ$$

$$=4{\cdot}81\ t\ (\textit{tensile})$$

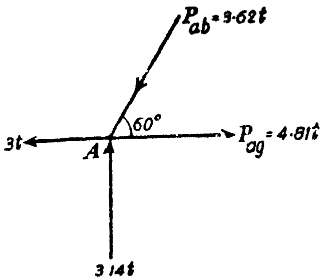


Fig. 734

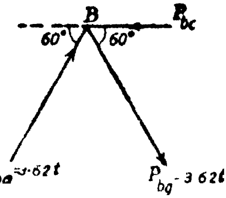


Fig. 735

Joint B

Resolving vertically, we get

$$P_{bg} \sin 60^\circ = P_{ba} \sin 60^\circ$$

$\therefore \quad P_{bg} = P_{ba}$

$\therefore \quad P_{bg} = 3{\cdot}62\ t$ (*tensile*)

Resolving horizontally, we get

$$P_{bc} = P_{ba} \cos 60^\circ + P_{bg} \cos 60^\circ$$
$$= 2 \times 3{\cdot}62 \times \tfrac{1}{2}$$
$$= 3{\cdot}62\ t \text{ (compressive)}$$

Joint G

Resolving vertically we have the following vertical components

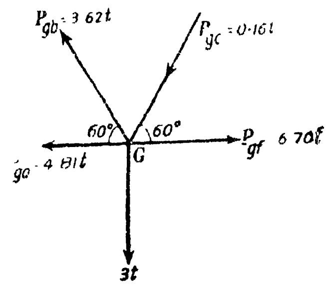


Fig. 736

(*i*) $3t \downarrow$ (*downwards*)

(*iii*) Vertical component of P_{gb}

$= 3{\cdot}62 \sin 60^\circ$

$= 3{\cdot}14\ t \uparrow$ (*upwards*)

Hence we require a balancing force of $0{\cdot}14t \downarrow$ (*downwards*)

$\therefore \quad P_{gc} \sin 60^\circ = 0{\cdot}14\ t$

$\therefore \quad P_{gc} = 0{\cdot}16\ t$ (*compressive*)

Resolving horizontally, we get,

$$P_{gf} = 4{\cdot}81 + 3{\cdot}62 \cos 60^\circ + 0{\cdot}16 \cos 60^\circ$$
$$= 6{\cdot}7\ t \text{ (tensile)}$$

Joint C

Resolving vertically, we get

$$P_{cf} \sin 60^\circ = P_{cg} \sin 60^\circ$$

$\therefore \quad P_{cf} = P_{cg}$

$\therefore \quad P_{cf} = 0{\cdot}16\ t$ (*tensile*)

Resolving horizontally, we get

$$P_{cd} = 3{\cdot}62 + 0{\cdot}16 \cos 60 + 0{\cdot}16 \cos 60$$
$$= 3{\cdot}62 + 2 \times 0{\cdot}16 \times \tfrac{1}{2}$$
$$= 3{\cdot}78\ t \text{ (compressive)}$$

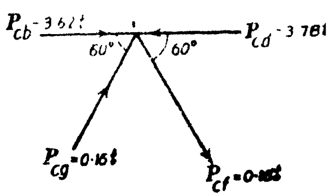


Fig. 737

Joint F

Resolving vertically, we get,

$$P_{fd} \sin 60° = 6 - 0{\cdot}16 \sin 60°$$
$$= 6 - 0{\cdot}14 = 5{\cdot}86$$

$$\therefore \quad P_{fd} = \frac{5{\cdot}86}{\sin 60°} = 6{\cdot}78\ t\ (tensile)$$

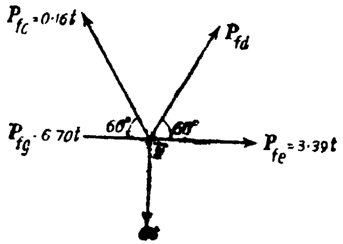


Fig. 738

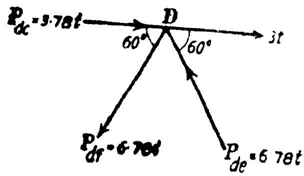


Fig. 739

Resolving horizontally, we get

$$P_{fe} = 6{\cdot}70 + 0{\cdot}16 \cos 60° - 6{\cdot}78 \cos 60°$$
$$= 3{\cdot}39\ t\ (tensile)$$

Joint D

Resolving vertically, we get

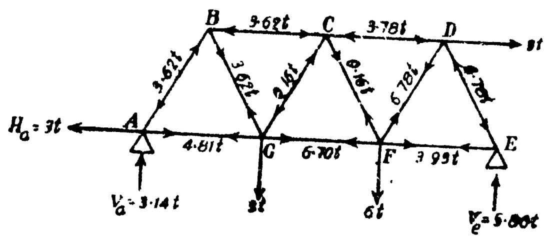


Fig. 740

$$P_{de} \sin 60° = P_{df} \sin 60°$$

$$\therefore \quad P_{de} = P_{df}$$

$$\therefore \quad P_{de} = 6{\cdot}78\ t\ (tensile)$$

Fig. 740 shows the forces in the various members of the structure.

Problem 469. *Determine analytically the forces in the members of the roof truss loaded as shown in Fig. 741 and tabulate the forces in members of the structure.* *(A.M. I E., May 1966)*

Solution. (See graphical solution at the end of the chapter)

Let θ be inclination af the member AE with the horizontal.

$$\tan \theta = \frac{\text{rise}}{\text{half span}} = \frac{4{\cdot}5}{6} = \frac{3}{4}$$

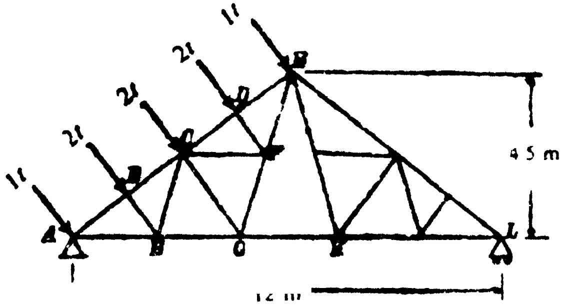


Fig. 741

$\therefore \quad \sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$

Length $\quad AE = \text{half span} \times \sec\theta$

$= 6 \times \frac{5}{4} = 7.5\ m$

Reactions

Since the end L is placed on rollers with the roller base horizontal, the reaction at L should be vertical. Hence there will be no horizontal reaction at L.

Let the vertical reaction at L be V_l.

Taking moments about A,

we have,

$$8 \times \frac{7.5}{2} = V_l \times 12$$

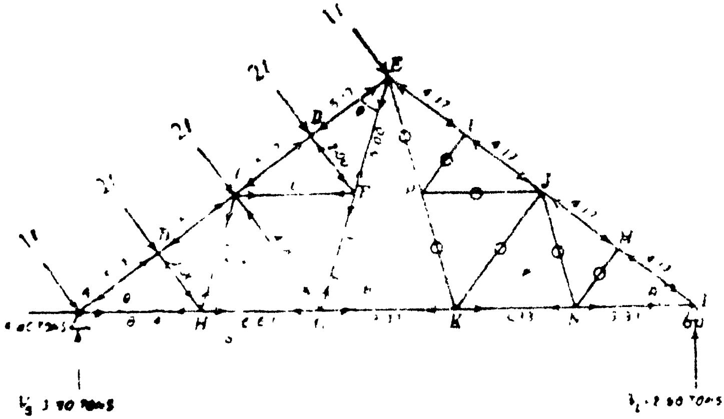


Fig. 742

$$\therefore \quad V_l = \frac{8 \times 7.5}{2 \times 12} = 2.50\ t \uparrow$$

Vertical component of the total external load $=8 \cos\theta$

$$=8\times\frac{4}{5}=6{\cdot}40\ t$$

$\therefore$ Vertical reaction at A

$$=V_a=6{\cdot}40-2{\cdot}50$$

$$=3{\cdot}90\ t$$

Horizontal component of the total external load

$$=8 \sin\theta=8\times\frac{3}{5}=4{\cdot}80\ t\rightarrow$$

$\therefore$ Horizontal reaction at A

$$=H_a=4{\cdot}80\ t\leftarrow$$

Analysis of Joints

Joint L. Resolving the forces at this joint vertically,

we have,

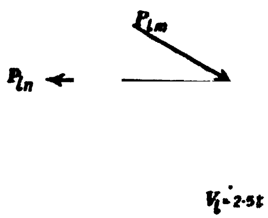


Fig. 743

$$P_{lm}\sin\theta=V_l$$

$$\therefore \qquad P_{lm}\times\frac{3}{5}=2{\cdot}5$$

$$P_{lm}=4{\cdot}167\ t\ (compressive)$$

Resolving horizontally,
we have,

$$P_{ln}=P_{lm}\cos\theta$$

$$=4{\cdot}167\times\frac{4}{5}$$

$$=3{\cdot}33\ t\ (tensile)$$

Joint M

At this joint there are three forces namely P_{ml}, P_{mj} and P_{mn}. Of these the forces P_{ml} and P_{mj} are along the same straight line. Hence, the third force $P_{mn}=0$.

$$\therefore \qquad P_{mj}=P_{ml}=4{\cdot}17\ t\ (compressive)$$

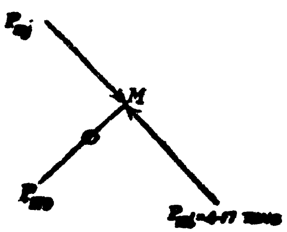


Fig. 744

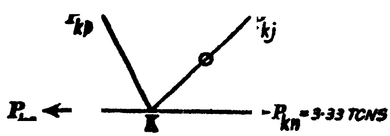


Fig. 745

Joint N

Since there is no force in the number NM, the joint N should be in equlibrium under the action of the forces P_{nk}, P_{nl} and P_{nj}.

Out of these forces, the forces P_{nk} and P_{nl} are along the same straight line. Hence, the third force P_{nj} should be zero.

$$\therefore \quad P_{nk}=P_{nl}=3{\cdot}33\ t\ (tensile)$$

Joint I

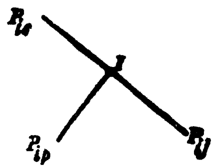


Fig. 746

This joint is in equilibrium under the action of the forces P_{ie}, P_{ij} and P_{ip}. Out of these three forces the two forces P_{ie} and P_{ij} are along the same straight line. Therefore the third force, namely

$$P_{ip}=0$$

$$\therefore \quad P_{ie}=P_{ij}.$$

Joint P

Since there is no force in the member PI, at the joint P, we therefore have the following three forces P_{pk}, P_{pe} and P_{pj}. Of these forces, the two forces P_{pe} and P_{pk} are along the same straight line.

Hence the third force

$$P_{pj}=0$$

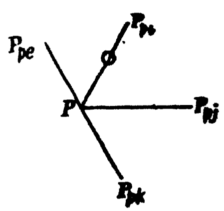


Fig. 747

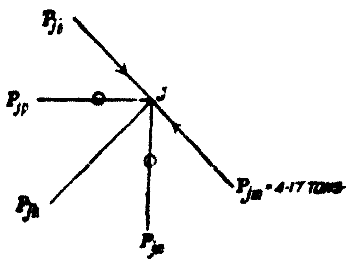


Fig. 748

Joint J

Since the members JP and JN do not carry forces, we have at the joint J the following three forces, namely,

$$P_{jm},\ P_{ji} \text{ and } P_{jk}.$$

Of these three forces the two forces P_{ji} and P_{jm} are along the same straight line.

Hence the third force $P_{jk}=0$

$$\therefore \quad P_{ji}=P_{jm}=4{\cdot}17\ t\ (compressive)$$

By studying the equilibrium of the joint I

we get

$$P_{ie}=P_{ij}=4{\cdot}17\ t\ (compressive)$$

Joint K

Since there is no force in the member KJ, we have only the following three forces at this joint, namely P_{kj}, P_{kn} and P_{kp}. Out of these forces, two forces P_{kj} and P_{kn} are along the same straight line. Hence, the third force

$$P_{kp}=0$$

$$\therefore \quad P_{kj}=P_{kn}=3{\cdot}33 \ t \ (tensile)$$

If now we consider the equilibrium of the joint P we conclude that the force in the member $PE=0$.

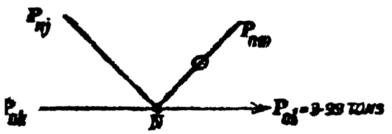


Fig. 749

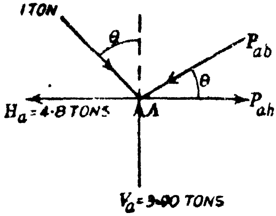


Fig. 750

Joint A

Resolving vertically,

we have

$$P_{ab}\sin\theta+1\cos\theta=3{\cdot}9$$

$$\therefore \quad \frac{3}{5}P_{ab}+\frac{4}{5}=3{\cdot}9$$

$$\therefore \quad P_{ab}=5{\cdot}17 \ t \ (compressive)$$

Resolving horizontally,

we have,

$$P_{ah}+1\sin\theta=4{\cdot}8+P_{ab}\cos\theta$$

$$\therefore \quad P_{ah}+\frac{3}{5}=4{\cdot}8+5{\cdot}17\times\frac{4}{5}$$

$$\therefore \quad P_{ah}=8{\cdot}34 \ t \ (tensile)$$

Joint B

We have

$$P_{bc}=P_{ba}=5{\cdot}17 \ t \ (compressive)$$

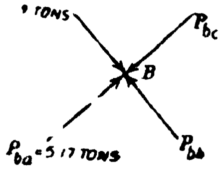


Fig. 751

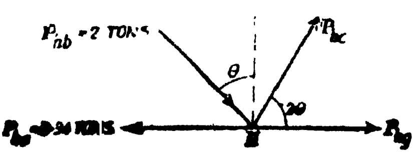


Fig. 752

Resolving normal to the principal rafter,

we have,

$$P_{bc}=2t \text{ (compressive)}$$

Joint H

Resolving the forces vertically, we get,

$$P_{hc} \sin 2\theta = P_{hb} \cos \theta$$

$$\therefore \quad P_{hc}\, 2 \sin \theta \cos \theta = 2 \cos \theta$$

$$\therefore \quad P_{hc} = \frac{1}{\sin \theta} = \frac{5}{3} = 1\cdot67\ t \text{ (tensile)}$$

Resolving horizontally, we have,

$$P_{ha} = P_{hb} \sin \theta + P_{hc} \cos 2\theta + P_{hg}$$

$$\therefore \quad 8\cdot34 = 2 \times \frac{3}{5} + 1\cdot67\left(1 - 2 \times \frac{9}{25}\right) + P_{hg}$$

$$\therefore \quad 8\cdot34 = 1\cdot20 + 0\cdot47 + P_{hg}$$

$$\therefore \quad P_{hg} = 6\cdot67\ t \text{ (tensile)}$$

Joint D

Resolving the forces normal to the principal rafter, we have

$$P_{df} = 2\ t \text{ (compressive)}$$

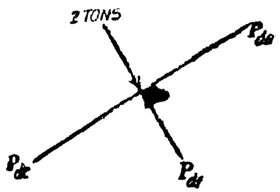


Fig. 753

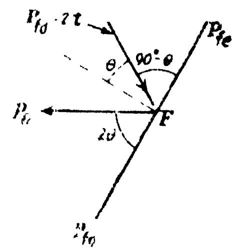


Fig. 754

Joint F

Resolving the forces normal to the line *GFE*, we have,

$$P_{fe} \sin 2\theta = 2 \cos \theta$$

$$\therefore \quad P_{fe}\, 2 \sin \theta \cos \theta = 2 \cos \theta$$

$$\therefore \quad P_{fe} = \frac{1}{\sin \theta} = \frac{5}{3}$$

$$\therefore \quad P_{fe} = 1\cdot67\ t \text{ (tensile)}$$

Joint C

Resolving the forces normal to the principal rafter, we have

$$P_{cg} = 2 + P_{ch} \sin \theta + P_{cf} \sin \theta$$

$$= 2 + 2 \times 1\cdot67 \times \frac{3}{5}$$

$$\therefore \quad P_{cg} = 4\cdot00\ t \text{ (compressive)}$$

Resolving along the principal rafter, we have

$$P_{cd} = P_{cb} = 5{\cdot}17 \ t. \ (compressive)$$

Now considering the equilibrium of the joint D, we have

$$P_{dc} = P_{de} = 5{\cdot}17 \ t. \ (compressive)$$

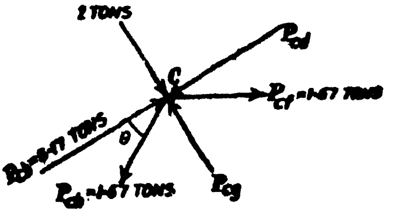


Fig. 755

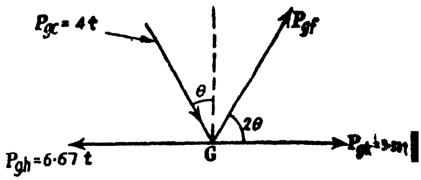


Fig. 756

Joint G

Resolving vertically, we get

$$P_{gf} \sin 2\theta = P_{ge} \cos \theta$$

$$\therefore \quad P_{gf}\, 2 \sin \theta \cos \theta = 4 \cos \theta$$

$$\therefore \quad P_{gf} = \frac{2}{\sin \theta} = 3{\cdot}33 \ t. \ (tensile)$$

Joint F

Resolving the forces along the line GFE, we have,

$$P_{fe} = P_{fg} + P_{fc} \cos 2\theta + P_{fd} \sin \theta$$

$$= 3{\cdot}33 + 1{\cdot}67 \left(1 - 2 \times \frac{9}{25}\right) + 2 \times \frac{3}{5}$$

$$\therefore \quad P_{fe} = 5{\cdot}00 \ t \ (tensile)$$

The forces in the various members are tabulated on page 865. Members EP, PI, PJ, PK, JK, JN, NM do not carry any force.

§163. Method of sections

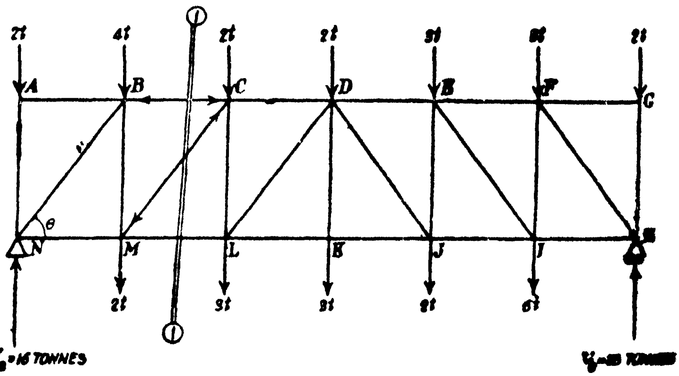


Fig 757

Fig. 757 shows a truss subjected to an external load system. Suppose, it is required to find the forces in the members BC, CM, ML and CL. The method of resolution will not be a convenient method, since it would involve the analysis of various other joints.

Member	Force (tonnes)	
	Compressive	Tensile
AB	5·17	
BC	5·17	
CD	5·17	
DE	5·17	
EI	4·17	
IJ	4·17	
JM	4·17	
ML	4·17	
AH		8·34
HG		6·67
GK		3·33
KN		3·33
NL		3·33
HB	2	
HC		1·67
CG	4	
CF		1·67
DF	2	
FG		3·33
FE		5·00

In a case like this, where it is required to find the forces in some selected intermediate members of a structure, the method of sections is adopted.

Let V_n and V_h be the reactions at the left and right supports. Taking moments about the left support, we have,

$$V_h \times 18 = (4+2)3+(2+3)6+(2+3)9+(3+2)12+(3+6)\times 15 + 2\times 18$$

$\therefore \quad V_h = 18$ *tonnes.*

$\therefore \quad V_n = 34 - 18 = 16$ *tonnes.*

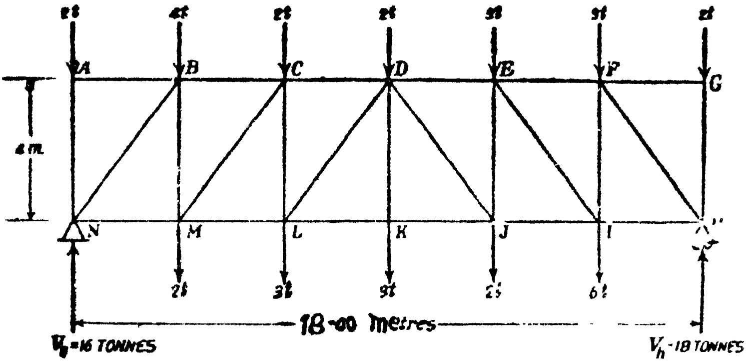


Fig. 758

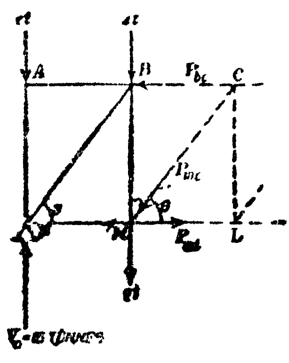


Fig. 759

Suppose the structure be split into two parts by a section 1—1. Fig 759 shows the left part of section 1—1.

If really the structure be cut or split by a plane like 1—1, the continuity being broken, each part (the left part of section 1—1 and the right part of section 1—1) will collapse.

This section 1—1 is cutting the members BC, MC and ML. Let BC and MC be compression members and let ML be a tension member. Now let us consider as to how the part of the structure on the left hand side of 1—1 was in equilibrium before the structure was split by the section 1—1.

The member BC was pushing the joint B with a force P_{bc}.
The member MC was pushing the joint M with a force P_{mc}.
The member ML was pulling the joint M with a force P_{ml}.

Now if the structure be split by the section 1—1, since the above forces P_{bc}, P_{mc} and P_{ml} are not present, the left part will not be in equilibrium and will hence collapse. But after splitting the strcuture by section 1—1, if we apply pushing forces of magnitude

P_{bc}, P_{mc} and a pulling force P_{ml} on the left part of section 1—1 along the lines BC, MC and ML the part will remain in equilibrium. Hence P_{bc}, P_{mc} and P_{ml} may be determined as the forces necessary to keep the left part in equilibrium. Hence the external forces already acting on the left part and the forces P_{bc}, P_{mc} and P_{ml} form a system in equilibrium. It is on this principle that the method of section is based.

The nature and magnitude of the forces P_{bc}, P_{mc} and P_{ml} may be determined as follows.

Consider the equilibrium of the left part of section 1—1. Resolving the forces on this part vertically, we have the following vertical components :

(*i*) $V_a = 16$ *tonnes* ↑ (upwards)

(*ii*) At *A* 2 *tonnes* ↓ (downwards)

(*iii*) At *B* 4 *tonnes* ↓ (downwards)

(*iv*) At *M* 2 *tonnes* ↓ (downwards)

The resultant of these forces = 8 *tonnes* ↑ (upwards)

Hence we require a balancing force of 8 *tonnes* ↓ (downwards)

The balancing forces is to be provided by the vertical components of the force P_{bc}, P_{mc} and P_{ml}. Out of these forces two forces namely P_{bc} and P_{ml} are horizontal forces and, therefore, they do not have vertical components.

Hence the vertical component of P_{mc} should be 8 *tonnes* ↓ (*downwards*). This is possible only if P_{mc} pushes the joint M. Hence P_{mc} is a compressive force.

∴ Vertical component of P_{mc}

$$= P_{mc} \sin\theta = 8 \text{ tonnes.}$$

But $$\tan\theta = \frac{4}{3}$$

∴ $$\sin\theta = \frac{4}{5}$$

∴ $$P_{mc} = \frac{8}{\sin\theta}$$

$$= \frac{8}{4/5} = 10 \text{ tonnes}$$

∴ $$P_{mc} = 10 \text{ tonnes (compressive)}$$

Force P_{bc}.

To find the force P_{bc}, we use the condition that the algebraic sum of the moments of the forces on the left hand side of section 1—1 about any point should be equal to zero. A convenient point is selected for taking moment. The point selected must be such that, out of the unknown forces P_{bc}, P_{mc} and P_{ml}, only the force P_{bc} should have a moment about the point and the forces P_{mc} and P_{ml} should not have moments about this point. To satisfy this condition the point of intersection of the forces P_{mc} and P_{ml} will be selected, *i.e.*, we choose the point M.

Now taking moments about M of the forces on the left hand side of section 1—1 we have the following moments about M.

(*i*) $16 \times 3 = 48$ *tonnes metres* ↻ (clockwise)

(*ii*) $2 \times 3 = 6$ *tonnes metres* ↺ (anticlockwise)

∴ Resultant moment

$= 42$ *tonnes metres* ↻ (clockwise)

Hence we require a balancing moment of 42 *tonnes metres* ↺ (anticlockwise) about M. This balancing moment should be provided by the forces P_{bc}, P_{mc} and P_{ml}.

But, P_{mc} and P_{ml} will not provide any moment about M.

Hence, P_{bc} must provide the necessary anticlockwise moment of 42 *tonnes metres*, about M.

This is possible only if P_{bc} pushes the joint B, *i.e.* P_{bc} is a compressive force.

$$\therefore \quad P_{bc} \times 4 = 42 \textit{ tonne metres}$$

$$\therefore \quad P_{bc} = \frac{42}{4} = 10{\cdot}5 \textit{ tonnes (compressive)}$$

Force P_{ml}

Now, to find the force P_{ml} we will take moments about C, the point of intersection of the forces P_{bc} and P_{mc}.

Taking moments about C of the forces on the left hand side of section 1—1, we have the following moments about C.

(*i*) $16 \times 6 = 96$ *tonne metres* ↻ (clockwise)

(*ii*) $2 \times 6 = 12$ *tonne metres* ↺ (anticlockwise)

(*iii*) $4 \times 3 = 12$ *tonne metres* ↺ (anticlockwise)

(*iv*) $2 \times 3 = 6$ *tonne metres* ↺ (anticlockwise)

The resultant moment $= 96 - 30 = 66$ *tonne metres* ↻ (clockwise)

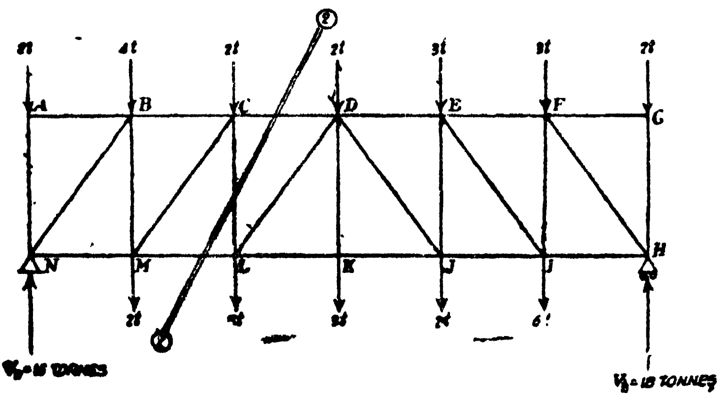


Fig. 760

Hence we require a balancing moment of 66 *tonne metres* ↶(anticlockwise) about C. This is to be provided by the forces P_{bc}, P_{mc} and P_{ml}. But the forces P_{bc} and P_{mc} do not have moments about C.

Hence the moment of P_{ml} about C should be 66 *tonne metres* (anticlockwise). To satisfy this condition P_{ml} should be a *pulling* force. Hence P_{ml} is a tensile force.

$$\therefore \qquad P_{ml} \times 4 = 66 \text{ tonne metres}$$

$$\therefore \qquad P_{ml} = 16{\cdot}5 \text{ tonnes (tensile)}$$

Force in the member CL

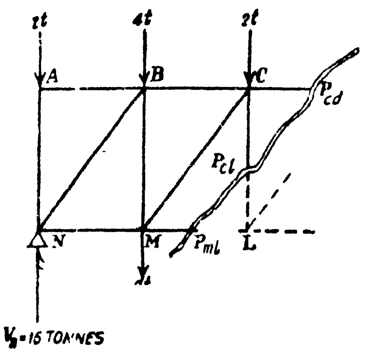


Fig. 761

Pass a section 2—2 as shown cutting the member CL. The section is passed cutting the minimum number of members including the member in which we are interested.

Now consider the equilibrium of the forces on the left hand side of section 2—2 (see Fig. 761). Resolving the forces vertically, we have the following vertical components.

(*i*) 16 *tonnes* ↑ *upwards at N*
(*ii*) 2 *tonnes* ↓ *downwards at A*
(*iii*) 2 *tonnes* ↓ *downwards at M*
(*iv*) 4 *tonnes* ↓ *downwards at B*
(*v*) 2 *tonnes* ↓ *downwards at C*

The resultant of these forces = 6 *tonnes* ↑ *upwards*

Hence we require a downward force of 6 *tonnes*. This is to be provided by the forces P_{cd}, P_{cl} and P_{ml}. But the forces P_{cd} and P_{ml} are horizontal forces and hence they do not have vertical components. Hence the force P_{cl} should provide a downward force of 6 *tonnes*, *on the left part of section 2—2*. This is possible only if P_{cl} is a pulling force. Hence P_{cl} is a tensile force.

$$\therefore \qquad P_{cl} = 6 \text{ tonnes (tensile)}$$

The above principles may be used to determine the force in any other selected member.

Problem 470. *Determine the forces in the members HC, BC, HG, and EF of the truss shown in Fig. 762 by the method of sections.*

Solution. Let θ be the inclination of the principal rafter with the horizontal.

$$\tan\theta = \frac{4{\cdot}5}{6} = \frac{3}{4}$$

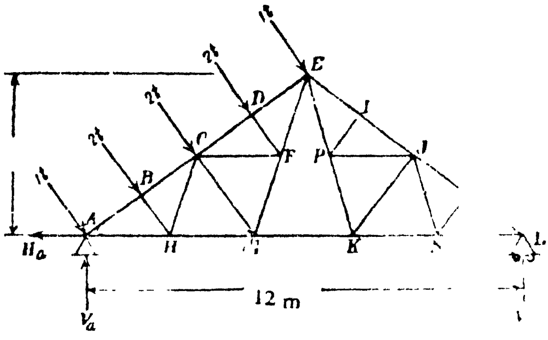


Fig. 762

$$\sin\theta=\frac{3}{5} \text{ and } \cos\theta$$

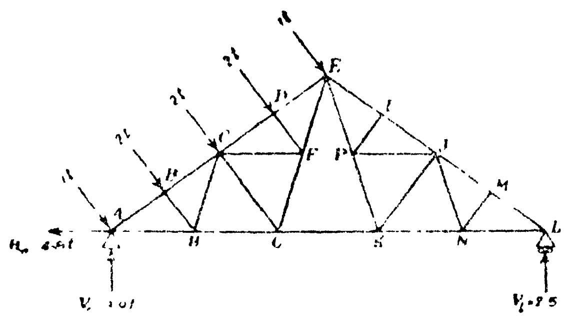


Fig. 763

Length of $AE=\sqrt{6^2+4\cdot5^2}=7\cdot5\ m$

Taking moments about A, we have

$$V_l\times12=8\times\frac{7\cdot5}{2}$$

$$\therefore \qquad V_l=2\cdot50\ t$$

Vertical component of the applied load system

$$=8\cos\theta=8\times\frac{4}{5}=6\cdot4\ t$$

$$\therefore \qquad V_a=6\cdot4-2\cdot5=3\cdot9\ t$$

Horizontal component of the applied load system

$$=8\sin\theta=8\times\frac{3}{5}=4\cdot8\ t$$

$\therefore$ Horizontal reaction at $A=4\cdot8\ t=H_a$

To find the force in the member HC

Pass section 1—1 as shown in Fig. 764 and consider the left part of section 1—1. Taking moments of the forces on this part about A **(the point of intersection of** P_{bc} **and** P_{ha}**),**

we get, $\quad P_{bc} \times AQ = 2 \times \frac{7{\cdot}5}{4}$

$\therefore \quad P_{bc} = \frac{15}{4\,AQ}$ *(tensile)*

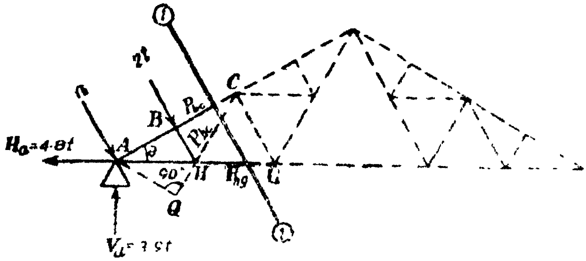


Fig. 764

But $\quad AQ = AC \sin\theta = \frac{7{\cdot}5}{2} \times \frac{3}{5} = 2{\cdot}25\ m$

$\therefore \quad P_{bc} = \frac{15}{4 \times 2{\cdot}25} = 1{\cdot}667\ t$ *(tensile)*

To find the force in BC

Consider the left part of section 1 – 1. Taking moments about *H* (the point of intersection of P_{bh} and P_{hg}), we have the following moments :

(i) $\quad 1 \times AB = 1 \times \frac{7{\cdot}5}{4} = 1{\cdot}875$ *t.m.* ↺ (*anticlockwise*)

(ii) $3{\cdot}9 \times AH = 3{\cdot}9 \times \frac{7{\cdot}5}{4} \sec\theta$

$\quad = 3{\cdot}9 \times \frac{7{\cdot}5}{4} \times \frac{5}{4} = 9{\cdot}14$ *t.m.* ↻ (*clockwise*)

$\therefore$ Total moment about $H = 7{\cdot}265$ *t.m.* ↻ (*clockwise*)

Hence we require a balancing anticlockwise moment of 7·265 *tonne metre* about *H*. This can be provided only by P_{bc}. In order the moment of P_{bc} about *H* may be anticlockwise, P_{bc} should be a compressive force.

$\therefore \quad P_{bc} \times AH \sin\theta = 7{\cdot}265$

$\therefore \quad P_{bc} \times AB \sec\theta \sin\theta = 7{\cdot}265$

$P_{bc} \times \frac{7{\cdot}5}{4} \tan\theta = 7{\cdot}265$

$P_{bc} \times \frac{7{\cdot}5}{4} \times \frac{3}{4} = 7{\cdot}265$

$P_{bc} = \frac{7{\cdot}265 \times 16}{7{\cdot}5 \times 3}$

$P_{bc} = 5{\cdot}167\ t$ *(compressive)*

To find the force in HG

Consider the left part of section 1—1. Take moments about C, the point of intersection of P_{bc} and P_{hc}.

We have the following moments about C.

$$1 \times \frac{7.5}{2} = 3.75 \ tm. \text{ (anticlockwise)}$$

$$2 \times \frac{7.5}{4} = 3.75 \ tm. \text{ (anticlockwise)}$$

$$4.8 \times \frac{7.5}{2} \sin\theta = 4.8 \times \frac{7.5}{2} \times \frac{3}{5} = 10.80 \ tm \text{ (clockwise)}$$

$$3.9 \times \frac{7.5}{2} \cos\theta = 3.9 \times \frac{7.5}{2} = 11.7 \ tm. \text{ (clockwise)}$$

Total moment about $C = 15$ *tm.* (*clockwise*).

Hence we require a *balancing anticlockwise* moment of 15 *tm.* about C. This can be provided only by P_{hg}. In order the moment of P_{hg} about C may be *anticlockwise* P_{hg} should be a *tensile* force.

$$\therefore \quad P_{hg} \times \frac{7.5}{2} \sin\theta = 15 \ tm.$$

$$\therefore \quad P_{hg} \times \frac{7.5}{2} \cdot \frac{3}{5} = 15$$

$$\therefore \quad P_{hg} = \frac{15 \times 10}{7.5 \times 3} \ t$$

$$\therefore \quad P_{hg} = 6.667 \ t \text{ (tensile)}$$

Force in EF

Pass section 2—2 as shown in Fig. 765. Consider the left part of section 2—2.

Take moments about A (the point of intersection of the forces P_{ae} and P_{gk}), we have the following moments about A.

(*i*) $2 \times \frac{7.5}{4} = 3.75$ *tm.* (*clockwise*)

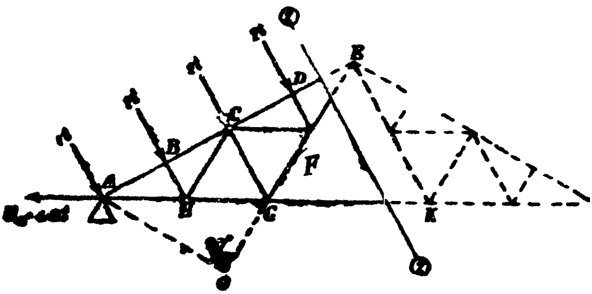


Fig. 765

(*ii*) $2\times$ $7{\cdot}5$ $7{\cdot}50$ *tm.* ↻ (*clockwise*)

(*iii*) $2\times\frac{3}{4}\times7{\cdot}5=11{\cdot}25$ *tm.* ↻ (*clockwise*)

∴ Total moment about $A=22{\cdot}50$ *tm.* ↻ (*clockwise*)

Hence, we require a balancing anticlockwise moment of 22·50 *tm.* about *A*. This can be provided only by P_{fe}. In order the moment of P_{fe} about *A* may be *anticlockwise*,

P_{fe} should be a tensile force.

∴ $P_{fe}\times AO=22{\cdot}50$

But $AO=7{\cdot}5\sin\theta$

$$=7{\cdot}5\times\frac{3}{5}=4{\cdot}5\ m$$

$$P_{fe}=\frac{22{\cdot}50}{AO}=\frac{22{\cdot}50}{4{\cdot}5}=5\ t\ (tensile)$$

Problem 471. *Fig. 766 shows a cantilever bridge truss. Find the stresses in members 1, 2, 3 and 4 of the truss.* (*A.M.I.E.*)

Fig. 766

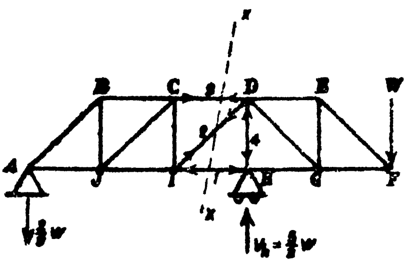


Fig. 767

Solution. The whole structure consists of two cantilevering trusses and an intermediate truss supported at the ends of the two cantilever trusses. The loading on the intermediate truss is transferred to the ends of the cantilevering trusses. Since the total load $2W$ on the intermediate truss is symmetrically applied on it, a load of W will be transferred to the end of each cantilevering truss. Now consider the cantilevering truss on the left (see Fig. 767).

Let the vertical reaction at the support *H* be V_h.

Let the length of each panel be *a*.

Taking moments about *A*, we have,

$$V_h\,3a=W\,5a$$

∴ $$V_h=\frac{5}{3}W\ (upwards)$$

∴ **Vertical reaction at** *A*

$$=\frac{2}{3}W\ (\textit{downwards})$$

Resolving the forces at H vertically,

We have $$P_{hd}=\frac{5W}{3}\ (compressive)$$

i.e. Force in member 4 is $\frac{5W}{3}$ (*compressive*)

Pass section XX as shown.

Consider the part of the truss on the left side of section XX. Resolving the forces on this part vertically, we have

$$P_{id}\sin 45^\circ=\frac{2}{3}W$$

$$P_{id}=\frac{2\sqrt{2}}{3}W\ (tensile)$$

i.e. the force in member 2 is $\frac{\sqrt{2}}{3}W$ (*tensile*)

To find the force in the member CD, taking moments of the forces on the LHS of XX about I, we have,

$$\frac{2}{3}W(2a)=P_{cd}\,a$$

$\therefore$ $$P_{cd}=\frac{4}{3}W\ (tensile)$$

i.e. The force in member 3 is $\frac{4}{3}W$ (*tensile*)

To find the force on the member IH, taking moments of the forces on the LHS of section XX about D, we have,

$$\frac{2}{3}W(3a)=P_{ih}\,h$$

$\therefore$ $$P_{ih}=2W\ (compressive)$$

i.e. The force in member 1 is $2W$ (*compressive*)

The forces in the members 1, 2, 3 and 4 are tabulated below.

Member	*Force*	*Type of Force*
1	$2W$	Compressive
2	$\frac{2\sqrt{2}}{3}W$	Tensile
3	$\frac{4}{3}W$	Tensile
4	$\frac{5}{3}W$	Compressive

Problem 472. *Find graphically or otherwise the forces in all the members of the truss shown in Fig. 768. Indicate the results in a tabular form.* (*A.M.I.E.*)

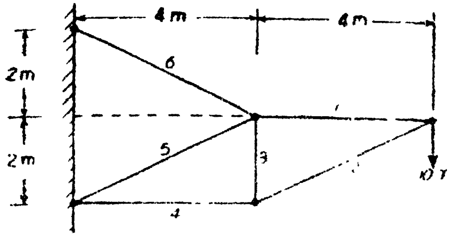


Fig. 768

Solution. See graphical solution at the end of the chapter.

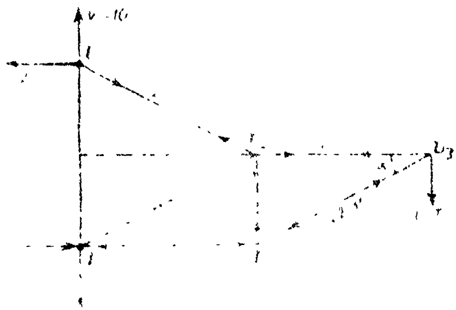
Analytical Method (*See Fig.* [illegible])

Let V and H be the vertical and horizontal components of the reaction at L_1

$$\tan \alpha = \frac{2}{4} = \frac{1}{2}$$

$$\therefore \quad \sin \alpha = \frac{1}{\sqrt{5}}$$

and $$\cos \alpha = \frac{2}{\sqrt{5}}$$



Taking moments about U_1, we have

$$H \times 4 = 10 \times 8$$

$$\therefore \quad H = 20\ t$$

Hence horizontal reaction at U_1 = [illegible]

Let the vertical reaction at U_1 be [illegible]

Resolving the forces at U_1 horizontally

$$P_{U_1} \cdot _2 \cos \alpha = 2\ [illegible]$$

[illegible]

Resolving vertically at [illegible]

[illegible]

Hence vertical reaction at [illegible]

Joint U_3

Resolving vertically

$$P_{U_3 \ldots} \sin \alpha = [illegible]$$

$$\therefore \quad P_{\ldots} = [illegible]\ 10$$

$$\therefore \quad P_{U_3 - U_2} = [illegible] = 22{\cdot}36\ t\ (compression)$$

Resolving horizontally

$$P_{U_3-U_2} = P_{L_3-U_3} \cos \alpha$$

$$=10\sqrt{5}\times\frac{2}{\sqrt{5}}=20\ t\ (\textit{tensile})$$

Joint L_2

Resolving vertically

$$P_{L2U2}=P_{L2U3}\sin\alpha$$

$$=10\sqrt{5}\,\frac{1}{\sqrt{5}}=10\cdot00\ t\ (\textit{tensile})$$

Resolving horizontally

$$P_{L2-L1}=p_{L2U3}\cos\alpha$$

$$=10\sqrt{5}\times\frac{2}{\sqrt{5}}=20\ t\ (\textit{compressive})$$

Joint L_1

Resolving vertically

$$P_{U1-U2}\sin\alpha=V=0$$

$$\therefore\quad P_{L1-U2}=0$$

The forces in the various members are entered in the following table :

Member	Forces in tonnes	
	Compressive	Tensile
U_1U_2		22·36
U_2U_3		20 00
U_3L_2	22·36	
L_1L_2	20·00	
U_2L_1	0	0
U_2L_2		10

Problem 473. *A pinjointed frame is as shown in Fig. 770. It is hinged at A and loaded at D. A horizontal chain is attached to C and pulled so that AD is horizontal. Determine the pull in the chain and also the force in each member stating whether it is in tension or compression.*

Solution. See graphical solution at the end of the chapter.

Analytical Solution

Let the tension in the chain be T (see Fig. 771).

The external load of 2 t can be resolved into its horizontal and vertical components.

Since the given load is at 45° with the vertical, this load can be replaced by a horizontal

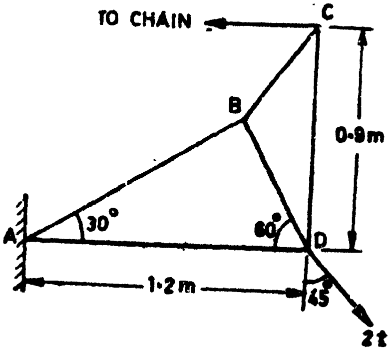


Fig. 770

load of $\sqrt{2}\ t$ and a vertical load of $\sqrt{2}\ t$ at D.

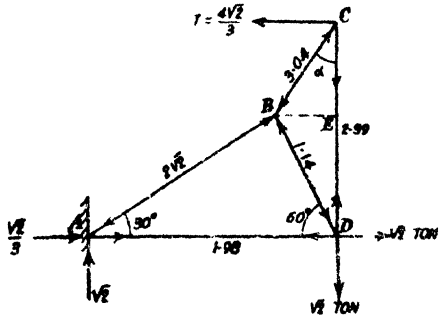


Fig. 771

Taking moments of the forces on the structure about the hinge A, we have

$$T \times 0.9 = \sqrt{2} \times 1.2$$

$$\therefore \qquad T = \frac{1.2\sqrt{2}}{0.9}\ t = 1.885\ t$$

Resolving the forces on the whole structure horizontally we have the following horizontal components :

(i) Tensile load of $\frac{4\sqrt{2}}{3}\ t$ in the chain $\leftarrow$

(ii) Horizontal force $\sqrt{2}\ t \rightarrow$

Hence the horizontal reaction at $A = \frac{\sqrt{2}}{3}\ t \rightarrow$

Resolving the forces on the whole structure vertically, we get Vertical reaction at $A = \sqrt{2}\ t \uparrow$

Joint A

Resolving vertically

$$P_{ab} \sin 30° = \sqrt{2} \qquad P_{ab} = 2\sqrt{2}t \text{ (compressive)}$$

Resolving horizontally, at A, we have the following horizontal components.

Horizontal reaction at $A = \frac{\sqrt{2}}{3}\ t \rightarrow = 0.470\ t \rightarrow$

(ii) Horizontal component of $P_{ab} = 2\sqrt{2} \cos 30° = \sqrt{6} = 2.45\ t \leftarrow$

$\therefore$ Force in $AD = P_{ad} = 2.45 - 0.47$

$= 1.98\ t$ (*tensile*)

Joint C.

Let the inclination of CB with the vertical be α.

Length $BD = 1{\cdot}2 \sin 30° = 0{\cdot}6\ m$

Length $BE = 0{\cdot}6 \sin 30° = 0{\cdot}3\ m$ and length $ED = 0{\cdot}6 \cos 30°$
$= 0{\cdot}520\ m$

$\therefore$ Length $CE = 0{\cdot}90 - 0{\cdot}52 = 0{\cdot}38\ m$

$$\tan\alpha = \frac{BE}{CE} = \frac{0{\cdot}3}{0{\cdot}38}$$

$$\alpha = 38°\ 15'$$

Resolving horizontally

$$P_{cb} \sin\alpha = T = 1.885$$

$$\therefore \quad P_{cb} = \frac{1{\cdot}885}{\sin 38°\ 15'} = 3{\cdot}04\ t \ (\textit{compressive})$$

Resolving vertically,

$$P_{cd} = P_{cb} \cos\alpha$$
$$= 3{\cdot}04 \cos 38°\ 15'$$
$$= 2{\cdot}9\ t\ (\textit{tensile})$$

Joint D

Resolving horizontally

$$P_{db} \cos 60° = P_{da} - \sqrt{2}$$
$$= 1{\cdot}98 - 1{\cdot}41 = 0{\cdot}57$$

$$\therefore \quad P_{db} = 1{\cdot}14\ t\ (\textit{compressive})$$

The forces in the various members of the structure are tabulated below :

Members	*Force in member (tonnes)*	
	Compressive	*Tensile*
AB	2·83	
BC	3·04	
CD		2·9
DA		1·98
DB	1·14	

Problem 474. *The loading and support conditions of a plane are shown in Fig 772. Find graphically or otherwise the forces members AB, AF, BC, BF, BE, and FE.* (*A.M.I.E.*)

Solution. See graphical solution at the end of the chapter.

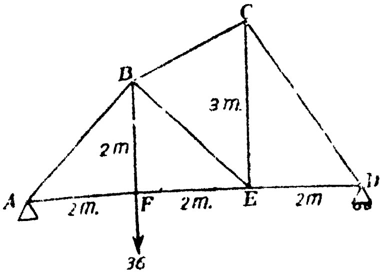


Fig. 772

Analytical Solution

Let the reactions of the supports be V_d and V_a (Fig.

Taking moments about A, we have,

$$V_d \times 6 = 36 \times 2$$

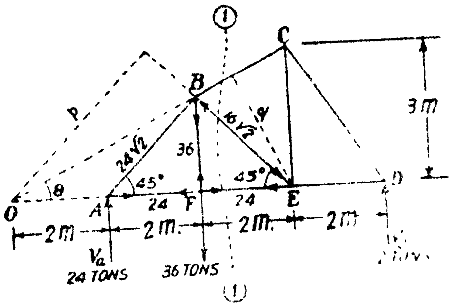


Fig. 773

$$V_d = 12\ t$$
$$V_a = 36 - 12 = 24\ t$$

Joint A

Resolving vertically, we have

$$P_{ab} \sin 45° = 24\ t$$

$\therefore$ $P_{ab} = 24\sqrt{2}\ t$ *(compressive)*

Resolving horizontally, we have,

$$P_{af} = P_{ab} \cos 45°$$
$$= 24\sqrt{2} \cos 45°$$

$\therefore$ $P_{af} = 24\ t$ *(tensile)*

Joint F

Resolving vertically, we have.

$$P_{fb} = 36\ t \text{ (tensile)}$$

Resolving horizontally, we have.

$$P_{fe} = 24\ t \text{ (tensile)}$$

Produce CB to meet FA produced at O

Let $\hat{BOA} = \theta$

$$\tan\theta = \frac{BF}{OF} = \frac{CE}{OE}$$

$$\therefore \quad \frac{2}{OF} = \frac{3}{OF+2}$$

$$\therefore \quad OF = 4\ m$$

$\therefore \qquad OA=2\ m$

To find the force in the member BE pass section $1-1$ as shown. Consider the equilibrium of the forces acting on the left hand side of section 1-1. Take moments of these forces about O. We have the following moments :

(i) $\qquad 24\times 2=48\ tm$ ↺

(ii) $\qquad 35\times 4=144\ tm$ ↻

Hence we require a balancing anticlockwise moment of

$$(144-48)=96\ tm \text{ about } O.$$

This moment is supplied by the moment of P_{be} about O.

Let the perpendicular distance between O and the member BE be p.

Hence $\qquad P_{be}\times p=96$

$$\therefore \qquad P_{be}=\frac{96}{p}$$

But $\qquad p=6 \sin 45^\circ$

$$=\frac{6}{\sqrt{2}}\ m=3\sqrt{2}m$$

$$\therefore \qquad P_{be}=\frac{96}{3\sqrt{2}}=16\sqrt{2}t \ (compressive)$$

To find the force in the member BC, consider the equilibrium of the forces acting on the left hand side of section $1-1$.

Take moments about E. We have the following moments :

(i) $\qquad 24\times 4=96\ tm.$ ↻

(ii) $\qquad 36\times 2=72\ tm$ ↺

Hence we require a balancing anticlockwise moment of

$$96-72=24\ tm \text{ about } E.$$

Let the perpendicular distance between E and the member BC be q.

Hence the balancing moment about E provided by P_{bc} is equal to $P_{bc}\ q$.

$$\therefore \qquad P_{bc}q=24$$

$$\therefore \qquad P_{bc}=\frac{24}{q}$$

But $\qquad q=6 \sin\theta$

But $\qquad \tan\theta=\frac{1}{2}$

$$\therefore \qquad \sin\theta=\frac{1}{\sqrt{5}}$$

$$\therefore \qquad q=\frac{6}{\sqrt{5}}\ m$$

$$\therefore \qquad P_{bc}=\frac{24\sqrt{5}}{6}$$

$\therefore \qquad P_{bc}=4\sqrt{5}\ t$ *(compressive)*

The forces in the members which have been computed are tabulated below :

Member	Force (tonnes)	
	Compressive	Tensile
AB	$24\sqrt{2}$	
BC	$4\sqrt{5}$	
AF		24
FG		24
BF		36
BE	$16\sqrt{2}$	

Problem 475. *Determine the forces in the various members of a pin-jointed framework shown in Fig. 774. All loads are in kg.*

Solution See graphical solution at the end of the chapter.

Analytical Solution

Let the joints be named as shown in Fig. 774.

Let H_e and V_e be the horizontal and vertical reactions at *E*.

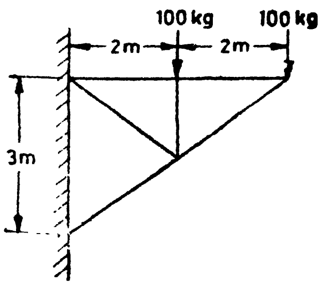


Fig. 774

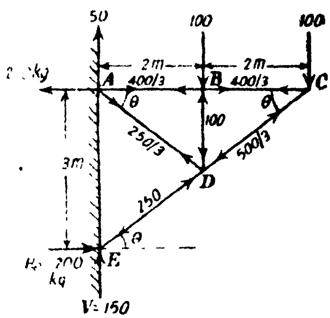


Fig. 775

Taking moments about *A*, we have

$$H_e \times 3 = 100 \times 2 + 100 \times 4 = 600$$

$$\therefore \qquad H_e = 200\ kg.$$

Resolving the forces at *E* horizontally, we have,

$$P_{ed} \cos \theta = 200$$

$$\therefore \quad P_{ed} = \frac{200}{\cos \theta}$$

But $\tan \theta = \frac{3}{4}$

$\therefore \quad \sin \theta = \frac{3}{5}$

and $\cos \theta = \frac{4}{5}$

$$\therefore \quad P_{ed} = \frac{200 \times 5}{4} = 250 \ kg. \ (compressive)$$

Resolving the forces at E vertically,

we have,

$$V_e = 250 \sin \theta$$

$$= 250 \times \frac{3}{5} = 150 \ kg. \uparrow$$

$\therefore$ Vertical reaction at $A = 200 - 150 = 50 \ kg. \uparrow$

and Horizontal reaction at $A = 200 \ kg.$ $\leftarrow$

Joint C

Resolving the forces vertically,

we have,

$$P_{cd} \sin \theta = 100$$

$$\therefore \quad P_{cd} = \frac{100}{\sin \theta} \qquad 100 \times 5$$

$$\therefore \quad P_{cd} = \frac{500}{3} kg. \ (compressive)$$

Resolving horizontally,

we have,

$$P_{cb} = P_{cd} \cos \theta$$

$$= \frac{500}{3} \times \frac{4}{5} = \frac{400}{3}$$

$$\therefore \quad P_{cb} = \frac{400}{3} kg. \ (tensile)$$

Joint B

Resolving vertically,

we have,

$$P_{bd} = 100 \ kg. \ (compressive)$$

Resolving horizontally,

we have,

$$P_{ba} = \frac{400}{3} \text{ kg. (tensile)}$$

Joint A

Resolving vertically,
we have,

$$P_{ad} \sin \theta = 50$$

$$\therefore \quad P_{ad} = \frac{50 \times 5}{3} = \frac{250}{3}$$

$$\therefore \quad P_{ad} = \frac{250}{3} \text{ kg. (tensile)}$$

The forces in the various members are tabulated below.

Members	*Force (kg)*	
	Compressive	*Tensile*
AB		$\frac{400}{3}$
BC		$\frac{400}{3}$
CD	$\frac{500}{3}$	
DE	250	
AD		$\frac{250}{3}$
DB	100	

Problem 476. *Find graphically or analytically the magnitude and nature of the stresses in all the members of the truss shown in Fig. 776.*

Solution. See graphical solution at the end of the chapter.

Analytical Solution.

Let V_a and V_d be the reactions at the supports.

Taking moments about the end *A*,

we have,

$$V_d \times 10 = 2 \times 2{\cdot}5 + 4 \times 7{\cdot}5$$

$$\therefore \quad V_d = \frac{35}{10} = \frac{7}{2} \ t$$

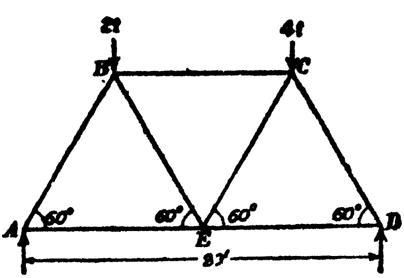


Fig. 776

$$V_a = 6 - \frac{7}{2} = \frac{5}{2}\ t$$

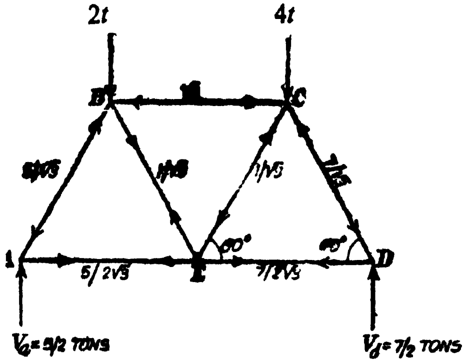


Fig. 777

Joint A

Resolving the forces at A vertically

we have,

$$P_{ab} \sin 60° = \frac{5}{2}$$

$$\therefore \qquad P_{ab} = \frac{5}{2} \cdot \frac{2}{\sqrt{3}} = \frac{5}{\sqrt{3}}\ t \ \ \textit{(compressive)}$$

Resolving horizontally,

we have,

$$P_{ae} = P_{ab} \cos 60° = \frac{5}{\sqrt{3}} \cos 60°$$

$$\therefore \qquad P_{ae} = \frac{5}{2\sqrt{3}}\ t \ \ \textit{(tensile)}$$

Joint B

Resolving the forces at B vertically we have the following vertical components :

(*i*) Vertical component of $P_{ba} = \frac{5}{\sqrt{3}} \sin 60° = \frac{5}{2}\ t \uparrow$

(*ii*) Downward load of $2\ t \downarrow$

Hence we require a downward force of $\frac{1}{2}\ t$ for equilibrium.

$\therefore$ Vertical component of P_{be} should be equal to $\frac{1}{2}\ t$.

$$\therefore \qquad P_{be} \sin 60 = \frac{1}{2}.$$

$\therefore \qquad P_{bc}=\frac{1}{2}\cdot\frac{2}{\sqrt{2}}=\frac{1}{\sqrt{3}}\ t$ *(tensile)*

Resolving the forces horizontally,

we have,

$$P_{bc}=P_{ba}\cos 60^\circ+P_{be}\cos 60^\circ$$

$$=\frac{\;}{\sqrt{3}}\cdot\frac{1}{2}+\frac{1}{\sqrt{3}}\cdot\frac{1}{2}$$

$$=\sqrt{3}\ t \text{ (compressive)}$$

Joint E

Since the vertical components of P_{eb} and P_{ec} should balance, we have,

$$P_{ec}=\frac{1}{\sqrt{3}}\ t \text{ (compressive)}$$

Resolving the forces horizontally,

$$P_{ed}=P_{ea}+P_{eb}\cos 60^\circ+P_{ec}\cos 60^\circ$$

$$=\frac{5}{2\sqrt{3}}+\frac{1}{\sqrt{3}}\cos 60^\circ+\frac{1}{\sqrt{3}}\cos 60^\circ$$

$$=\frac{5}{2\sqrt{3}}+\frac{1}{2\sqrt{3}}+\frac{1}{2\sqrt{3}}=\frac{7}{2\sqrt{3}}\ t.$$

$\therefore \qquad P_{ed}=\frac{7}{2\sqrt{3}}\ t$ *(tensile)*

Joint C

Resolving the forces vertically we have the following vertical components :

(*i*) Downward load of 4 t ↓

(*ii*) Vertical component of P_{ce}

$$\frac{1}{\sqrt{3}}\sin 60^\circ=\tfrac{1}{2}\ t\ \uparrow$$

Hence we require an upward force of $\frac{7}{2}$ t ↑

This will be provided by the vertical component of P_{cd}

$\therefore \qquad P_{cd}\sin 60^\circ=\frac{7}{2}$

$\therefore \qquad P_{cd}=\frac{1}{2}\cdot\frac{2}{\sqrt{3}}$

$\therefore \qquad P_{cd}=\frac{7}{\sqrt{3}}\ t$ *(compressive)*

The forces in the various members are tabulated below :

Members	Force (tonnes)	
	Compressive	*Tensile*
AB	$\frac{5}{\sqrt{3}}$	
BC	$\sqrt{3}$	
CD	$\frac{7}{\sqrt{3}}$	
DE		$\frac{7}{2\sqrt{3}}$
EA		$\frac{5}{2\sqrt{3}}$
BE		$\frac{1}{\sqrt{3}}$
EC	$\frac{1}{\sqrt{3}}$	

Problem 477. *Fig. 778 shows a shear leg crane lifting a 40,000 kg. load. The legs are 12 m long and 6 m apart at the base. The back stay is 14 m long. All members are pinjointed and A, E and C are at the same level on the ground. Find the force in the various members.*

Solution See graphical solution at the end of the chapter.

Analytical Solution

Consider a vertical plane containing the member *AB*. This plane will intersect the plane of the triangle *EBC* along *BD*.

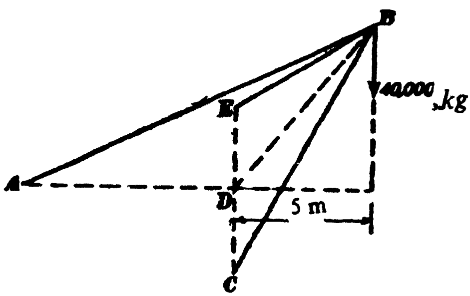


Fig. 778

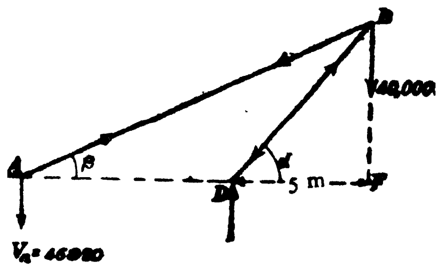


Fig. 779

Now let us imagine that the members BE and BC are removed and a member BD is provided. The frame now takes the shape shown in Fig. 779.

If now the force in the member BD be determined, then the forces in the members BE and BC can be easily determined since the force in BD represents the resultant of the forces in the members BE and BC

Now to find the length BD, consider the true shape of the triangle BEC (Fig. 780)

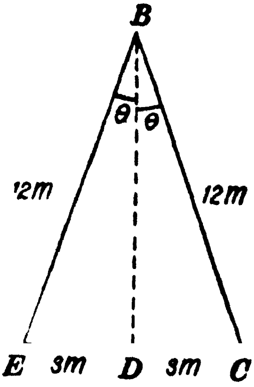


Fig. 780

$$BE = BC = 12 \ m.$$
$$ED = DC = 3 \ m.$$
$$\therefore \quad BD = \sqrt{12^2 - 3^2}$$
$$= \sqrt{135} \ m.$$
$$= 11{\cdot}62 \ m.$$

Let the imaginary member BD be inclined at α with the horizontal and let AB be inclined at β with the horizontal.

$$\therefore \quad \cos\alpha = \frac{5}{11{\cdot}62}$$
$$\therefore \quad \alpha = 64° \ 31'$$
$$\text{and} \quad BF = 5 \tan 64° \ 31'$$
$$\therefore \quad \sin\beta = \frac{5 \tan 64° \ 31'}{14}$$
$$\therefore \quad \beta = 48° \ 31'$$
$$\therefore \quad AF = 14 \cos 48° \ 31' = 9{\cdot}274 \ m$$
$$\therefore \quad AD = 9{\cdot}274 - 5 = 4{\cdot}274 \ m$$

Let the vertical component of the reaction at D be V_d. Taking moments about A, we have

$$V_d \times AD = 40{,}000 \times AF$$
$$V_d = 40{,}000 \times \frac{AF}{AD}$$
$$= \frac{40{,}000 \times 9{\cdot}274}{4{\cdot}274}$$
$$= 86820 \ kg. \uparrow$$

$\therefore$ Vertical component of the reaction at A

$$= V_a = 86820 - 40{,}000 = 46820 \ kg \downarrow$$

Resolving forces at D vertically, we have,

$$P_{db} \sin\alpha = 86820$$
$$\therefore \quad P_{db} = \frac{86820}{\sin\alpha}$$

$$=\frac{86820}{\sin 64^\circ 31'}$$

$\therefore$ $P_{ab}=96160\ kg.$ (*compressive*)

Resolving the forces at A vertically, we have

$$P_{ab}\sin\beta=46820$$

$$P_{ab}=\frac{46820}{\sin\beta}$$

$$=\frac{46820}{\sin 48^\circ 31}$$

$\therefore$ $P_{ab}=62480\ kg.$ (*tensile*)

Now to find the force in each of the members BE and BC, consider the plane containing the triangle BEC. By symmetry, forces in BE and BC are equal.

i.e., $P_{be}=P_{bc}.$

Since the resultant of P_{be} and P_{bc} is P_{bd}, we have

$$2P_{bc}\cos\theta=96160$$

But $\sin = \frac{3}{12}=\frac{1}{4}$

$$\theta=14^\circ 25'$$

$$P_{bc}=\frac{96160}{2\cos 14^\circ 25'}=49650\ kg.\ (compressive)$$

$\therefore$ Force in the back stay BA

$=62480\ kg.$ (*tensile*)

and Force in each leg, *i.e.*, in the members BE and BC

$=49650\ kg.$ (*compressive*)

Problem 478. *Determine the reactions and the forces in the members of the vertical frame shown in Fig. 781.*

Solution. See graphical solution at the end of the chapter.

Analytical Solution.

Let the vertical components of the reactions at the left and right supports be V_1 and V_2 respectively.

Taking moments about the left support, we have

$$V_2\times 10=1\times 15$$

$\therefore$ $V_2=\frac{3}{2}\ t\uparrow$

$\therefore$ $V_1=2-\frac{3}{2}=\frac{1}{2}\ t\uparrow$

Since the right support is a roller support there will be no horizontal reaction at the right support. Hence the external horizontal force of 1 t will be balanced by the horizontal reaction at the left support.

Hence horizontal reaction at the left support $=H=1\ \overleftarrow{t}$

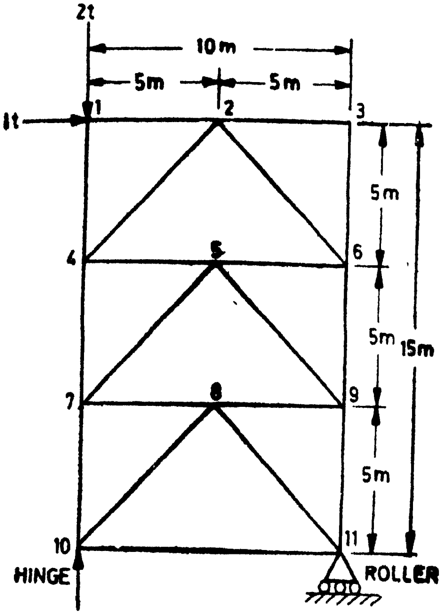


Fig. 781

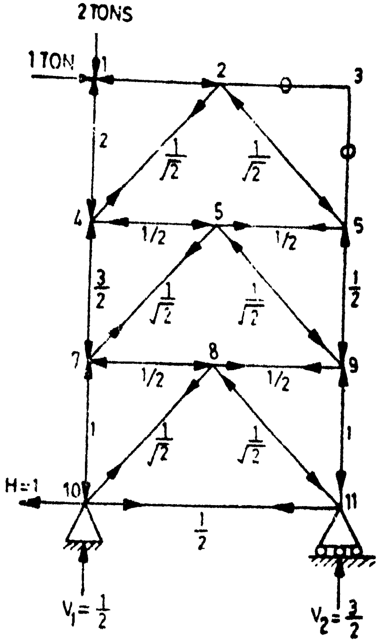


Fig. 782

Joint 3

Only two forces are present at this joint, *viz.*, P_{3-2} and P_{3-6}. Since these are the only two forces at the joint 3 and since these are not along the same straight line, we have, for the equilibrium of the joint 3,

$$P_{3-2}=P_{3-6}=0$$

Joint 1. Resolving the forces vertically, we have

$$P_{1-4}=2\ t\ (compressive)$$

Resolving the forces horizontally, we have

$$P_{1-2}=1\ t\ (compressive)$$

Joint 2. Resolving the forces along the line 2—6, we have,

$$P_{2-6}=1 \cos 45^\circ$$

$$\therefore \quad P_{2-6}=\frac{1}{\sqrt{2}}\ t\ (compressive)$$

Since the vertical components of P_{2-6} and P_{2-4} must balance, we have

$$P_{2-4}=\frac{1}{\sqrt{2}}\ t\ (tensile)$$

Joint 4. Resolving horizontally, we have,

$$P_{4-5}=\frac{1}{\sqrt{2}}\cos 45^\circ$$

$\therefore$ $$P_{4-5}=\tfrac{1}{2}\ t\ (compressive)$$

Resolving vertically, we have,

$$P_{4-7}=2-\frac{1}{\sqrt{2}}\sin 45^\circ$$

$\therefore$ $$P_{4-7}=\frac{3}{2}\ t\ (compressive)$$

Joint 6. Resolving vertically,

$$P_{6-9}=\frac{1}{\sqrt{2}}\sin 45^\circ=\tfrac{1}{2}\ t\ (compressive)$$

Resolving horizontally,

$$P_{6-5}=\frac{1}{\sqrt{2}}\cos 45^\circ=\frac{1}{2}\ t\ (tensile)$$

Joint 5. Resolving the forces along the line 5—9

$$P_{5-9}=\left(\frac{1}{2}+\frac{1}{2}\right)\cos 45^\circ$$

$\therefore$ $$P_{5-9}=\frac{1}{\sqrt{2}}\ t\ (compressive)$$

Since the vertical components of P_{5-7} and P_{5-9} should balance, we have,

$$P_{5-7}=\frac{1}{\sqrt{2}}\ t\ (tensile)$$

Joint 7 Resolving vertically, we have,

$$P_{7-10}=\frac{3}{2}-\frac{1}{\sqrt{2}}\cos 45^\circ$$

$\therefore$ $$P_{7-10}=\frac{3}{2}-\frac{1}{2}=1\ t\ (compressive)$$

Resolving horizontally, we have,

$$P_{7-8}=\frac{1}{\sqrt{2}}\cos 45^\circ=\frac{1}{2}\ t\ (compressive)$$

Joint 9. Resolving vertically,

$$P_{9-11}=\frac{1}{2}+\frac{1}{\sqrt{2}}\sin 45^\circ$$

$\therefore$ $$P_{9-11}=1\ t\ (compressive)$$

Resolving horizontally, we have,

$$P_{9-8}=\frac{1}{\sqrt{2}}\cos 45^\circ=\frac{1}{2}\ t\ (tensile)$$

Joint 8. Resolving the forces along the line 8—11, we nave,

$$P_{8-11}=\left(\frac{1}{2}+\frac{1}{2}\right)\cos 45^\circ$$

$$\frac{1}{\sqrt{2}}\ t\ (compressive)$$

Since the vertical components of P_{8-11} and P_{8-10} should balance, we have,

$$P_{8-10}=\frac{1}{\sqrt{2}}\ t\ (tensile)$$

Joint 10. Resolving horizontally, we have,

$$P_{10-11}=1-\frac{1}{\sqrt{2}}\cos 45^\circ=\frac{1}{2}\ t\ (tensile)$$

The forces in the various members are shown in Fig. 782 and tabulated on page 892.

§164. Graphic Statics. Girders and frames can also be analysed graphically. When analytical solutions are difficult a graphical solution may be found convenient. While the analytical methods provide absolutely correct results, the graphical methods provide reasonably accurate results.

Bow's Notation. A force is designated by two letters which are written on either side of the line of action of the force.

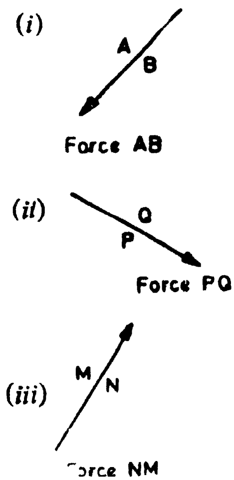


Fig. 783

Fig. 783 (*i*) shows a force with the letters *A* and *B* on either side of the line of action of the force. This force will be called as *AB*. A force has a point of action. With respect to the point of action of the force the letters used to designate the force are pronounced in the clockwise order.

Similarly the force shown in Fig. 783 (*ii*) will be called as *PQ*.

Similarly the force shown in Fig. 783 (*iii*) will be called as *MN*.

§165. To determine the resultant of a system of forces

Fig. 784 shows three parallel forces $AB=10t$, $BC=8t$ and $CD=12t$. Let it be required to find the resultant of these forces. To a convenient scale, draw $ab=10t$, $bc=8t$ and $cd=12t$. Choose any pole o. Join ao, bo, co, do.

Take any point 1 on the line of action of the force AB Through 1, draw 1—2 parallel to bo to meet the line of action of the force

Member	Force	
	Compressive (tonnes)	Tensile (tonnes)
1—2	1	
2—3	—	
3—6	—	—
6—9	½	
9—11	1	
11—10		½
10—7	1	
7—4	$\frac{3}{2}$	
4—1		
4—2		$\frac{1}{\sqrt{2}}$
2—6	$\frac{1}{\sqrt{2}}$	
4—5	½	
5—6		½
5—7		$\frac{1}{\sqrt{2}}$
5—9	$\frac{1}{\sqrt{2}}$	
7—8	½	
8—9		½
8—10		$\frac{1}{\sqrt{2}}$
8—11	$\frac{1}{\sqrt{2}}$	

BC at 2. Through 2 draw 2—3 parallel to *co* to meet the line of action of *CD* at 3. Draw 1—4 and 3—4 parallel to *ao* and *do*

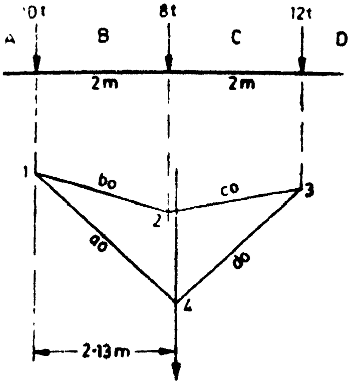


Fig. 784. Funicular polygen.

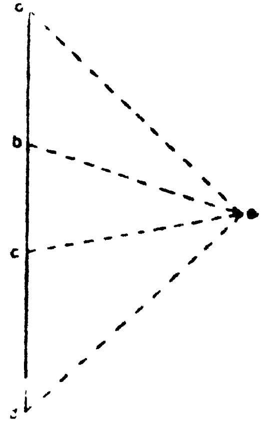


Fig. 785. Polar diagram

respectively and obtain the point 4. Through 4 draw a line parallel to the given forces. This line is the line of action of the resultant of all the forces By measurement this line is 2·13 *m* from the line of action of the force *AB* Now let us consider another system of forces. Fig 786 shows a system of forces which are not parallel to each other. The various forces are named as per Bow's notations The forces are *AB*=100 *kg*., *BC*=150 *kg*., *CD*=100 *kg*. and *DE*=80 *kg*.

First draw *ab*, *bc*, *cd* and *de* to represent the given forces and find the magnitude and the direction of the resultant. This is given by *ae*.

Take any convenient point *o* and draw *oa*, *ob*, *oc* and *od*. (*o* is called a pole).

Now from any point 1 on the line of action of the force *AB* draw a line parallel to *bo* to meet force *BC* at 2. From 2 draw a line parallel to *co* to meet the force *CD* at 3. From 3 draw a line parallel to *do* to meet the force *DE* at 4. Now from the points 1 and 4, draw lines parallel to *ao* and *eo* respectively and obtain the point of intersection 5. Through the point 5 draw a line parallel to *ae*. This line represents the line of action of the resultant of the given forces The polygon 1 2 3 4 5 is called the *funicular polygon, the string polygon* or *the link polygon* The polygon along with the forces *AB*, *BC*, *CD* and *DE* is called the *funicular diagram*

The figure *a b c d e a* is called the *force polygon*. The force polygon along with the radii *oa*, *ob*, *oc*, *od* and *oe* is called the *polar diagram*. The graphical solution is found to be convenient particularly when the given forces have different points of application and do not converge at the same point.

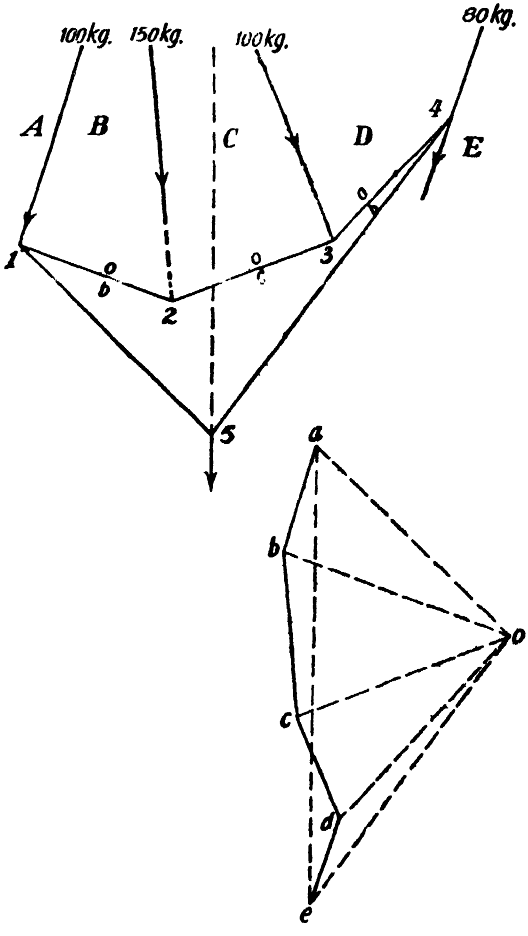


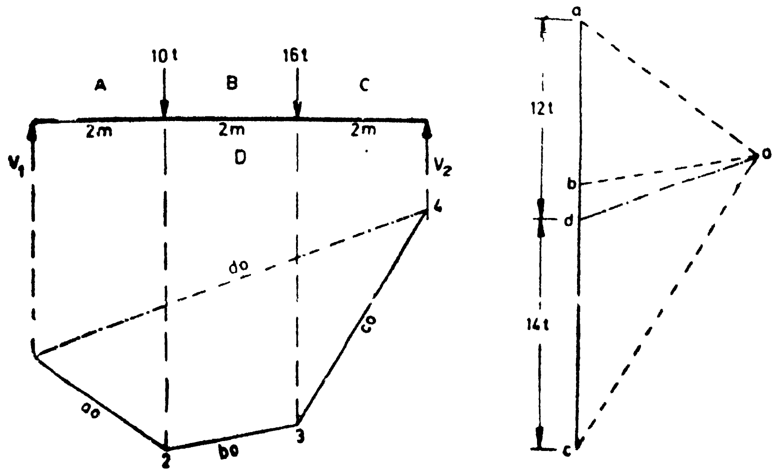
Fig. 786

§166. To determine the reactions at the supports of a simply supported beam.

Fig. 787 shows a simply supported beam 6 *metres* long with the supports at the ends. The beam carries point loads 10 *t* and 16 *t* at distances 2 *m* and 4 *m* from the left support. Let us determine the reactions V_1 and V_2 at the left and right supports.

The forces on the beam and the reactions are named following Bow's notations. Now the polar diagram and the funicular polygon are drawn as follows :

Draw *ab* to represent the force *AB* equal to 10 *t*. Draw *bc* to represent the force *BC* equal to 16 *t*. Select any convenient pole *o*. Join *oa*, *ob*, *oc*. Now through any convenient point 1 on the line



Funicular polygon Fig. 787. Polar diagram

of action of V_1 draw a line 1—2 parallel to *ao* to meet the line of action of the force *AB* at 2. Now through 2, draw a line 2—3 parallel to *bo* to meet the line of action of the force *BC* at 3. Now through 3, draw a line 3—4 parallel to *co* to meet the line of action of the reaction V_2 at 4. Join 1—4. The figure 1—2—3—4 is the *funicular polygon*. The line 1—4 is the *closing line* of the funicular polygon. Now through the pole *o* draw *od* parallel to this closing line 1—4, to intersect the load line at *d*. Now *da* represents V_1 and *cd* represents V_2. By measurement $V_1 = da = 12\ t$ and $V_2 = cd = 14\ t$.

Problem 479. *Determine the forces in all the members of the truss shown in Fig. 788. The inclined members are at 45° to the horizontal.*

Solution.

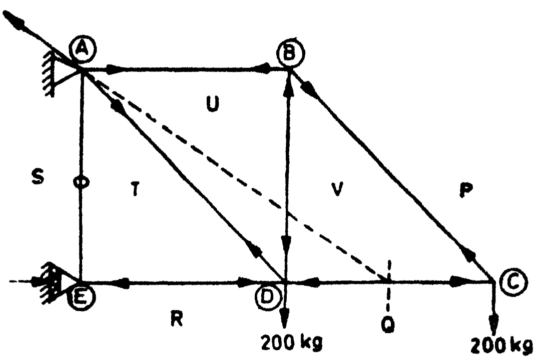


Fig. 788

Since at E a roller support has been provided the reaction at E should be normal to the roller base, *i.e.*, at E the reaction is horizontal. The resultant of the two loads of 200 *kg*. each applied at the joints D and C will act through the middle point of DC. Now the forces keeping the truss in equilibrium are the following :

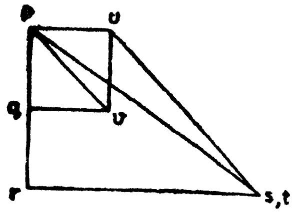


Fig. 789

(*i*) Resultant external load 200+200 $=400$ *kg*.

(*ii*) Horizontal reaction at E

(*iii*) Reaction at A.

These forces are concurrent.

Hence the reaction at A should act in line with the line joining A and the middle point of DC.

Having determined the directions of reactions, we proceed as follows :

Name the various forces adopting ***Bow's notation*** as shown. Draw pq, qr to represent the applied loads of 200 *kg* each applied at the joints C and D

Draw a horizontal line rs through r (which is parallel to the reaction at E). Draw ps parallel to the reaction at A and thus obtain the point s Since at E there are only three forces, two of which are along a straight line, the third force, *i.e.*, the force in AE, *i.e.*, the force st, is zero. Hence s and t coincide in the stress diagram, tr represents the force TR, *i.e.*, the force in ED.

Now consider the joint A. The forces at this joint are :

(*i*) Reaction at A, *i.e.*, SP

(*ii*) Force in AB, *i e.*, PU

(*iii*) Force in AD, *i.e.*, UT.

Now through the point u draw a line tu parallel to the force UT. Draw pu parallel to the force PU. Thus, obtain the point u. Obtain other points in the stress diagram similarly. Drawing uv parallel to UV and pv parallel to PV obtain the point v. Obviously vq should be horizontal since it represents the horizontal force VQ.

To find the force from the stress diagram

For instance to find the force in the member AB. consider either the joint A or B Let us consider the joint A. At any joint the forces which are named following Bow's notation should be pronounced in the clockwise order, *i e.*, the force in AB, at the joint A should be pronounced as PU and not UP. Hence at A. the force in AB is PU. Now find the direction of pu in the stress diagram. Now the direction of the force in AB at A will be the same as the direction pu. Mark an arrow along AB at A in the direction pu. The force AB at A pulls the joint A and hence AB is a tension member. The member AB will of course pull the joint B also. Mark an arrow along BA at B.

Following the same principle mark the arrows along the extremities of the various members indicating whether they are tensile or compressive.

Now the forces in the members of the structure can be scaled from the stress diagram. These are tabulated below :

MEMBER		*FORCE (kg.)*	
Designation by end joints	*Designation as per Bow's Notation*	*Compressive*	*Tensile*
AB	*PU*		200·0
BC	*PV*		282·8
CD	*VQ*	200	
DE	*TR*	600	
EA	*ST*	0	0
DA	*UT*		565·6
DB	*UV*	200	

Problem 480. *Determine the forces in the various members of the structure shown in Fig. 790.*

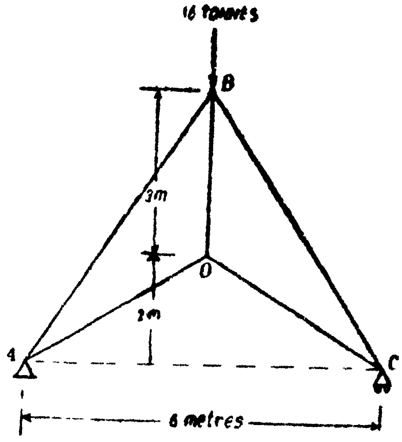


Fig. 790

Name the various forces following Bow's notation.
Draw *pq* to represent 16 *tonnes*.
Reaction at $C = QR = 8$ *tonnes*.
Reaction at $A = RP = 8$ *tonnes*.

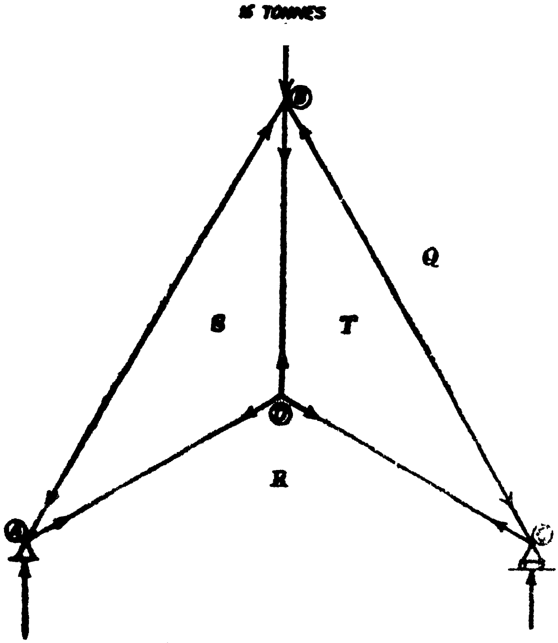


Fig. 791

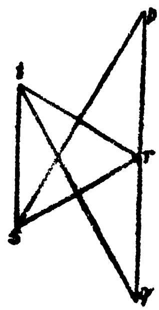


Fig. 792

Measure *qr* vertically and equal to 8 *tonnes* and obtain the point *r*.

Now obtain the various points in the stress diagram in succession in the following order :

Draw *rs* and *ps* parallel to *RS* and *PS* and obtain the point *s*.

Member		*Force (tonnes)*	
Designation by end Joint.	*Designation as per Bow's Notation*	*Compressive*	*Tensile*
AB	*PS*	15·5	
BC	*QT*	15·5	
AO	*SR*		9·6
CO	*TR*		9·6
BO	*ST*		10·6

Draw *st* and *qt* parallel to *ST* and *QT* and obtain the point *t*.

Now *tr* should remain parallel to *TR*.

The forces in the various members of the structure may be scaled from the stress diagram. These are tabulated on page 898.

Problem. 481. *Determine the forces in all the members of the truss shown in Fig. 793. All the inclined members are at 45° with the horizontal.*

Solution.

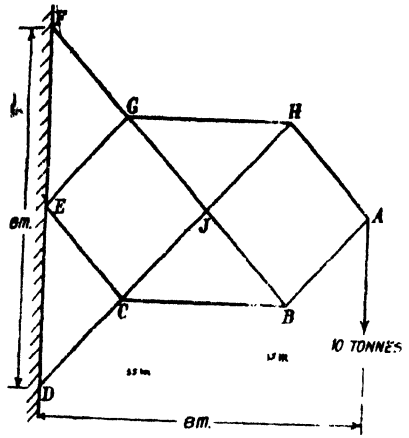


Fig. 793

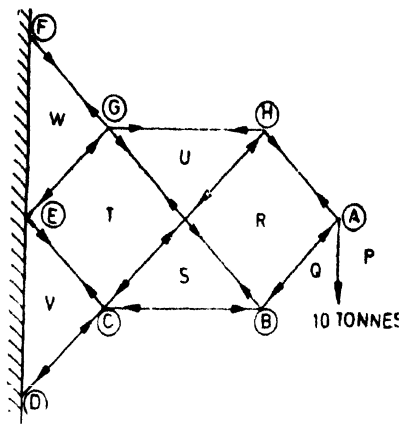


Fig. 794

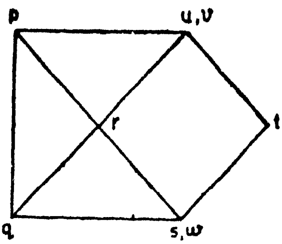


Fig. 795

Name the various forces following Bow's notation.

Darw *pq* to represent the force *PQ* = 10 *tonnes*. Draw *qr* and *pr* parallel to *QR* and *PR* and obtain the point *r*.

Draw *ru* and *pu* parallel to *RU* and *PU* and obtain the point *u*.

Draw *rs* and *qs* parallel to *RS* and *QS* and obtain the point *s*.

Draw *st* and *ut* parallel to *ST* and *UT* and obtain the point *t*.

Draw *tw* and *pw* parallel to *TW* and *PW* and obtain the point *w*.

Draw *tv* and *qv* parallel to *TV* and *QV* and obtain the point *v*.

The forces in the various members may now be scaled from the stress diagram. These are tabulated on page 900.

Member		*Force (tonnes)*	
Designation by end joints	***Designation by Bow's Notation***	***Compessive***	***Tensile***
AB	*QR*	7·07	
BC	*QS*	10·00	
CD	*QV*	14·4	
FG	*WP*		14·14
GH	*PU*		10
HA	*PR*		7·07
EG	*WT*		7·07
EC	*VT*	7·07	
GJ	*TU*		7·07
JB	*SR*		7·07
JH	*UR*	7·07	
JC	*TS*	7·07	

Problem. 482. *Determine graphically the forces in the members of the roof truss loaded as shown in Fig. 776 and tabulate the forces in the members of the structure.* (*A.M. I. E.*)

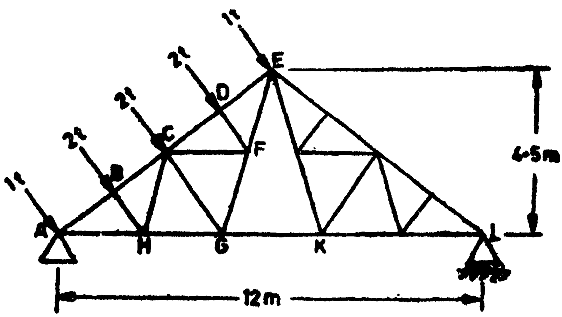


Fig. 796

Solution

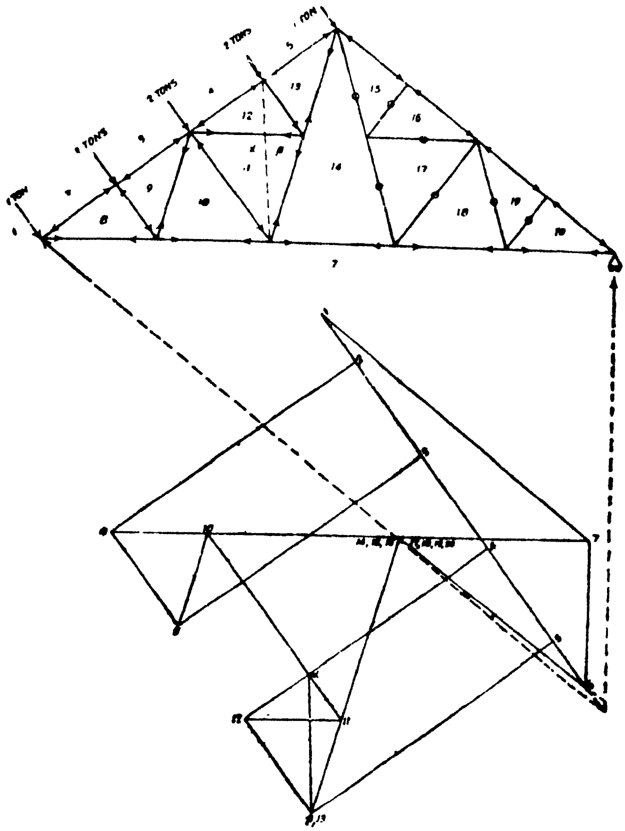


Fig. 797

The various forces are named following Bow's notation.

The resultant of the external loading will act normal to the roof through the joint *C*. The reaction at the right end will be vertical. The reaction at the left support should therefore pass through the point of intersection of resultant external loading and the vertical through the support at the right end. Thus the directions of the two reactions are known.

Draw *1—2*, *2—3*, *3—4*, *4—5* and *5—6* to represent the forces applied at the joint *A*, *B*, *C*, *D*, and *E*. Draw 6—7 vertically. Draw 1—7 parallel to the reaction at the left support and thus obtain the point 7.

Draw 7—8 horizontally and 2—8 parallel to 2—8 and obtain the point 8.

Draw 8—9 parallel to 8—9 and 3—9 parallel to 3—9 and obtain the point 9.

Draw 9—10 parallel to 9—10 and 7—10 parallel to 7—10 and obtain the point 10.

Imagine that the members *CF* and *FD* are removed and instead of these a member *GD* is provided.

Hence discard the forces 11—12 and 12—13, and introduce the force $\alpha-\beta$

Member		Force (tonnes)	
Designation by end joints	*Designation by Bow's natation*	*Compressive*	*Tensile*
AB	2—8	5·17	
BC	3—9	5·17	
CD	4—12	5·17	
DE	5—13	5·17	
EL	6—15	4·17	
AH	7—8		8·34
HG	7—10		6·67
GK	7—14		3·33
KL	18—7		3·33
HB	8—9	2·00	
HC	9—10		1·67
CG	10—11	4·00	
CF	12—11		1·67
DF	12—13	2·00	
FG	11—14		3·33
FE	13—14		5·00

Draw 10—x parallel to CD and 4—α parallel to roof and obtain the point α.

Draw α—β parallel to GD and 5—β parallel to roof and obtain the point β.

Draw β—14 parallel to GE and 7—14 horizontally and obtain the point 14.

Now the point 13 can be obtained as the point of intersection of 14—13 and 5—13.

Point 12 is the point of intersection of 13—12 and 4—12.

Point 11 is the point of intersection of 12—11 and 10—11.

Point 15 is the point of intersection of 14—15 and 6—15.

It will be seen that the points 14, 15, 16, 17, 18, 19 and 20 will all coincide.

The above method is called the method of substitution.

The forces in the various members may now be scaled from the stress diagram. They are tabulated on page 902.

Problem 483. *Find graphically or otherwise the forces in the members of the structure shown in Fig. 798.*

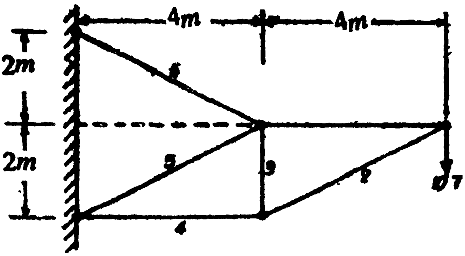
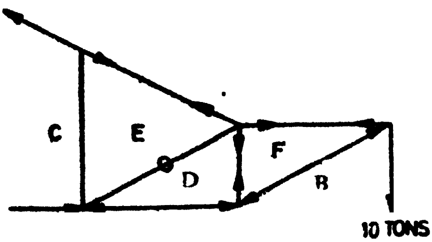


Fig 798



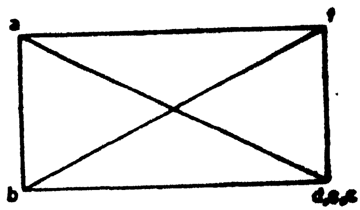


Fig. 799

Fig. 800

Graphical Solution

The various forces keeping all the joints of the structure in equilibrium are named following Bow's notation.

Draw ab to represent the force AB of 10 t Draw bf and af parallel to BF and AF and obtain the point f. Draw fd and bd parallel to FD and BD and obtain the point d. It will be seen that the points d, e and c will coincide.

Now the forces in the various members may be scaled from the stress diagram. These forces are entered in the following table :

Member		Force in the member (tonnes)	
Designation by end joints	*Designation by Bow's Notation*	*Compressive*	*Tensile*
U_1U_2	*AE*		22·36
U_2U_3	*AF*		22·00
U_3L_3	*BF*	22·36	
L_1L_3	*DB*	20·00	
U_2L_1	*ED*	0	0
U_1L_3	*DF*		10

Problem 484. *Find graphically or otherwise the tension in the chain and the forces in the members of the structure shown in Fig. 801.*

Solution.

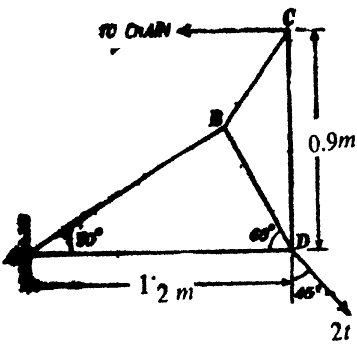


Fig. 801

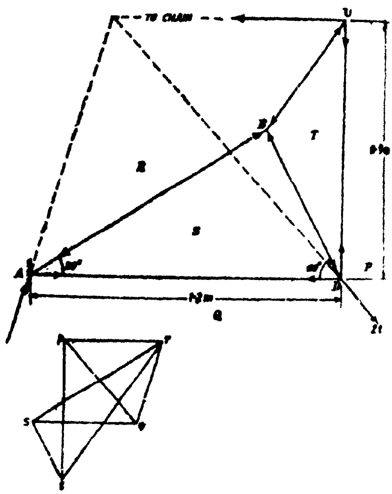


Fig. 802

The forces on the structure are named following Bow's notation.

The structure is in equilibrium under the action of

(*i*) The applied load of 2*t*

(*ii*) Tension in the chain

(*iii*) Reaction at *A*

These three forces should therefore be concurrent. Using this condition the direction of the reaction at *A* can be determined.

Draw pq to represent the force $PQ=2\ t$. Draw pr horizontally, *i.e.*, parallel to the chain and qr parallel to the reaction at A and obtain the point r.

Draw rs and qs parallel to RS and QS and obtain the point s. Draw st parallel to ST and pt parallel to PT and obtain the point t.

Now the tension in the chain is given by the force RP. By measurement $rp=1{\cdot}89\ t$. Now the forces in the various members are tabulated below :

Member		*Force (tonnes)*	
Designation by end joints	*Designation By Bow's Natation*	*Compression*	*Tension*
AB	*RS*	2·83	
BC	*RT*	3·04	
CD	*TP*		2·39
DA	*SQ*		1·98
DB	*ST*	1·14	

Problem 485. *Find the factors in the members of the structure shown in Fig. 803.* (*A.I.M.E.*)

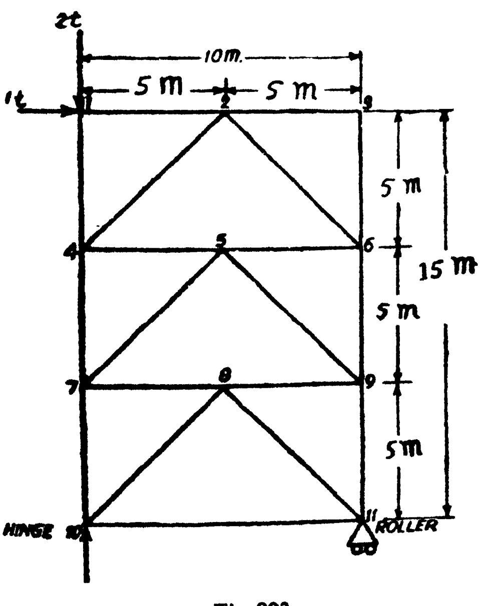


Fig 803.

Solution. The directions of the reactions may be determined as follows.

The right support (Joint 11) is a roller support. Hence the reaction is vertical. The external loads applied are the 2 *tonnes* load and the 1 *tonne* load on the joint 1 applied vertically and horizontally respectively. The resultant of these two forces, the vertical reaction at the roller support and the reaction at the hinged left support are in equilibrium, and should therefore be concurrent. Using this principle the direction at the hinged support can be determined.

The various forces may now be designated following Bow's notation (Fig 805).

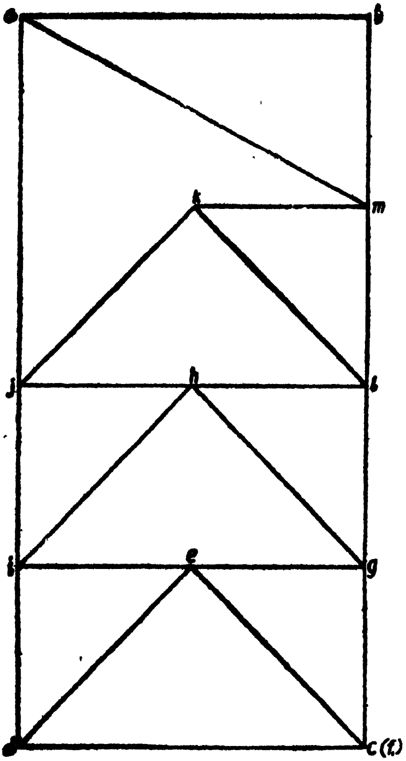


Fig. 804

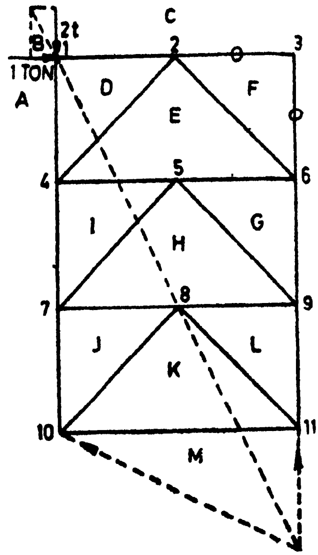


Fig. 805

Selecting a suitable scale, draw *ab* horizontally to represent force *AB* of 1 *t*. Draw *bc* vertically to represent the force *BC* of 2 *t*. Draw *cm* and *am* parallel to the vertical reaction of the roller support and the reaction of the hinged support and obtain the point *m*. It will be seen that the points *c* and *f* will coincide. Draw *cd* and *ad* horizontally and vertically respectively parallel to the forces *CD* and *AD* and obtain the point *d*. Draw *de* and *fe* parallel to *DE* and *FE* and obtain the point *e*. Draw *ei* and *aj* parallel to *EI* and *AI* and obtain the point *i*. Draw *eg* and *cg* parallel to *EG* and *CG* and obtain the point *g*. Draw *ih* and *gh* parallel to *IH* and *GH* and obtain the point *h*. Draw *hj* and *al* parallel to *HJ* and *AJ* and obtain the point *j*. Draw *hl* and *cl* parallel to *HL* and *CL* and obtain the point *l*. Draw *k* and *jk* parallel to *LK* and *JK* and obtain the point *K*.

The forces in the various members may now be scaled from the stress diagram. These are tabulated below :

Member		*Force (tonnes)*	
Designation by end joints	*Designation by Bow's Notation*	*Compressive*	*Tensile*
1—2	*CD*	1	
2—3	*CF*	0	0
3—6	*CF*	0	0
6—9	*CG*	0·50	0
9—11	*CL*	1·00	
11—10	*MK*		0·50
10—7	*AJ*	1·00	
7—4	*AI*	1·50	
4—1	*AD*	2·00	
4—2	*DE*		0·71
2—6	*EF*	0·71	
4—5	*EI*	0·50	
5—6	*EG*		0·50
7—5	*IH*		0·71
5—9	*HG*	0·71	
7—8	*HJ*	0·50	
8—9	*HL*		0·50
10—8	*JK*		0·71
8—11	*KL*	0·71	

Problem 486. *Find the forces in the members of the truss shown in Fig. 806.*

Solution.

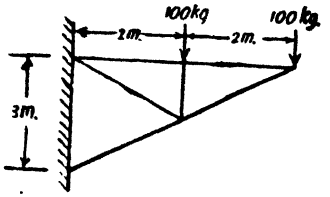


Fig. 806

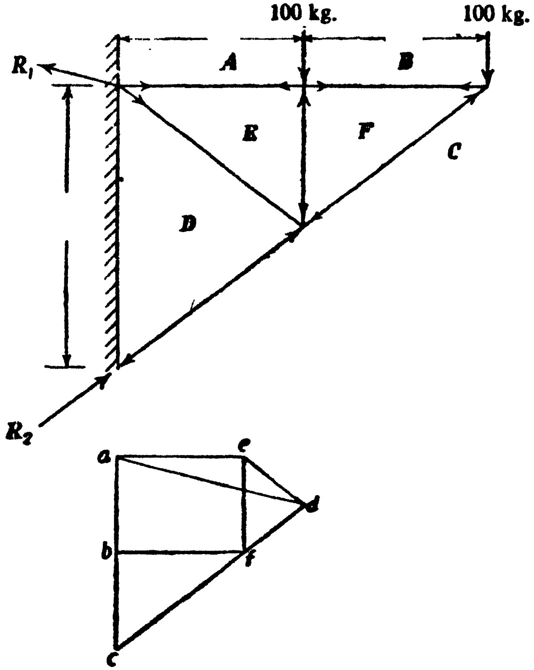


Fig. 807

Name the various forces following Bow's Notation. Adopting a convenient scale, draw *ab* and *bc* vertically representing the two loads of 100 *kg*. each. Draw *cf* and *bf* parallel to *CF* and *BF* and obtain the point *f*. Draw *fe* and *ae* parallel to *FE* and *AE* and obtain the point *e*. Draw *ed* and *cd* parallel to *ED* and *CD* and obtain the point *d*. The forces in the members may now be scaled from the stress diagram. These are tabulated below :

Member	Force (kg.)	
Designation by Bow's Notation	*Compressive*	*Tensile*
AE		133·3
BF		133·3
CF	166.7	
FE	100	
ED		83·3
DC	250	

Problem 487. *Find the forces in the members of the truss shown in Fig. 808.*

Solution

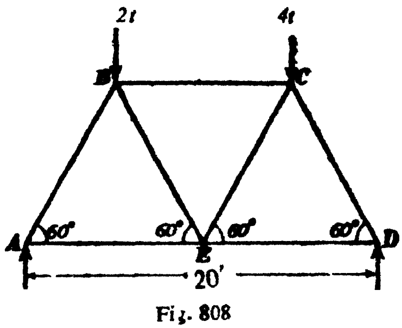


Fig. 808

Name the various forces following Bow's Notation.

Draw *pq* and *qr* to represent the vertical loads of 2 *t* and 4 *t*. Select a pole *O*. Join *po*, *qo* and *ro*. From any point on the left reaction (say from the left support itself), draw a line *po* parallel to *po* to meet the load line of 2 *t*. From the point of intersection obtained draw a ray parallel to *ro* to meet the load line of 4*t*. From the point of intersection obtained draw a ray parallel to *ro* to meet the load line of the reaction at *D* and thus obtain the funicular polygon. Through the pole *o* draw *os* parallel to the closing line of the funicular polygon to meet the line *pqr* at *s*. Now *rs* represents the reaction at the right support and *sp* represents the reaction at the left support. Now all the points in the stress diagram are obtained in succession.

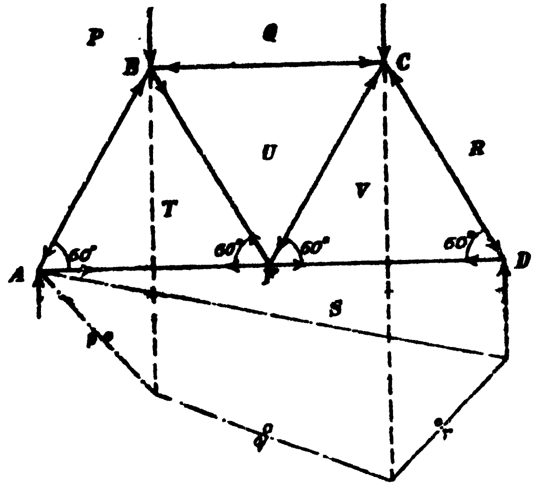


Fig. 809

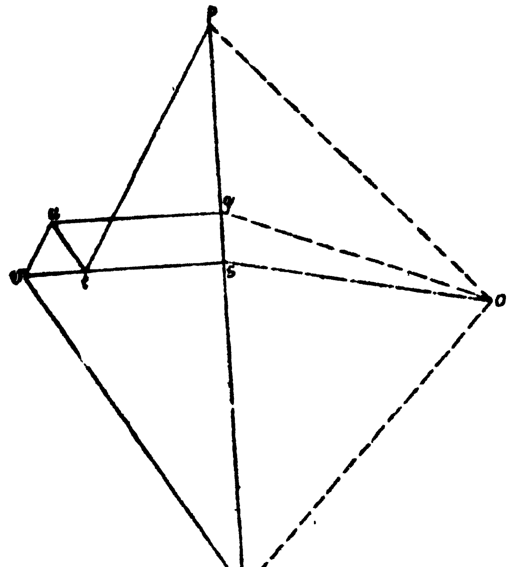


Fig. 810

Draw *pt* and *st* parallel to the forces *PT* and *ST* and obtain the point *t*. Draw *tu* and *qu* parallel to *TU* and *QU* and obtain the point *u*. Draw *uv* and *rv* parallel to *UV* and *RV* and obtain the point *v*. The forces in the members may now be scaled from the stress diagram. These are tabulated below :

Member		*Force (tonnes)*	
Designation By End Joints	*Designation by Bow's Notation*	*Compressive*	*Tensile*
AB	*PT*	2·9	
BC	*QU*	1·7	
CD	*VR*	4·0	
DE	*SV*		2·0
EA	*ST*		1·5
BE	*TU*		0·6
EC	*UV*	0·6	

Problem 488. *Find the forces in the members of the structure shown in Fig. 811.* *(A.M.I.E)*

Solution.

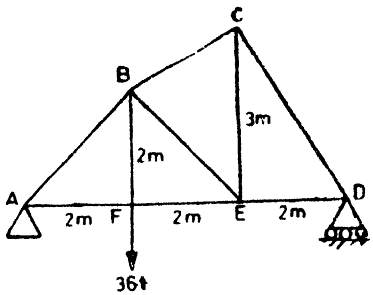


Fig. 811

Name the various forces following Bow's notation.

Draw *pq* vertically to represent the load of 36 *t*. Select a pole *O* and complete the funicular polygon as explained in the previous problem. Through the pole *O* draw *ov* parallel to the closing line of the funicular polygon to meet the load line *pq* at *v*. Now *vp* represents the reaction at the right support and *qv* represents the reaction at the left support. Now the various points in the stress diagram may be obtained in succession. These forces are tabulated on page 912.

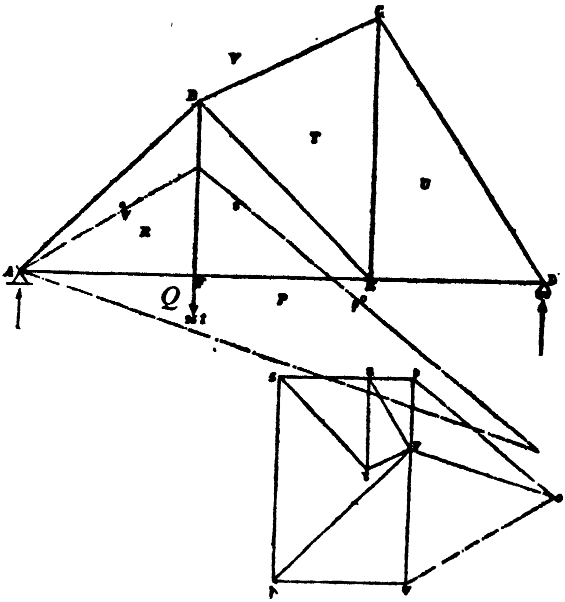


Fig. 812

Member		Force (tonnes)	
Designation by End Joints	Designation by Bow's Notation	Compressive	Tensile
AB	VR	33·9	
BC	VT	8·9	
CD	VU	14·4	
DE	PU		8
EF	PS		24
FA	QR		24
BF	RT		36
CE	TU		16
BE	ST	22·6	

Problem 489. *Fig. 813 shows a shear leg crane lifting a 40,000 kg. load. The legs are 12 m long and 6 m apart at the base. The back stay is 14 m long. All members are pinjointed and A, E and C are all at the same level on the ground. Find the forces in the various members.*

Solution.

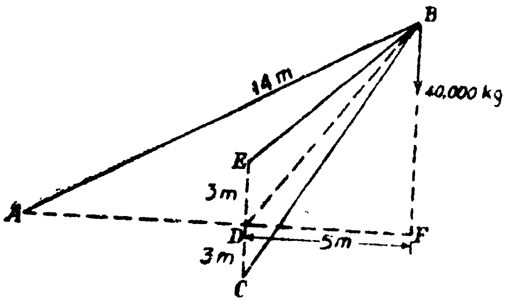


Fig. 813

In place of the members *BE* and *BC* introduce the members *BD* so that *BD*, *BA* and the load of 40,000 *kg*. are on the same vertical

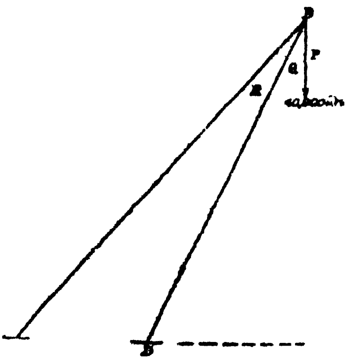


Fig. 814

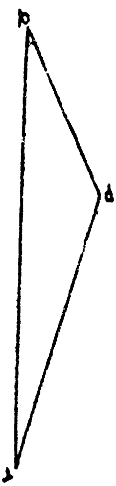


Fig. 815

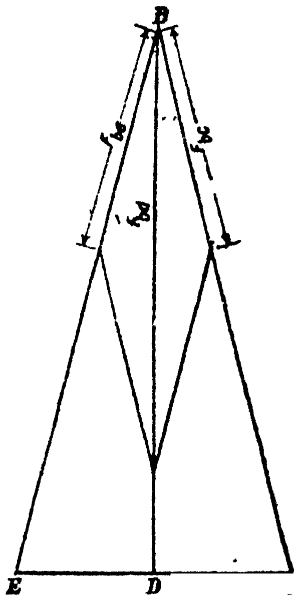


Fig. 816

plane. Solving this structure graphically the force in the member *BA* which is represented by *RP* by Bow's notation, is given by *rp* in the stress diagram. By scaling from the stress diagram.

$$rp = 62{,}500 \text{ kg. (tensile)}$$

Now *qr* represents the force in *BD*.

Now draw the real shape of the triangle *EBC*.

This is shown in Fig. 816. Plot *D* on this diagram. Along *BD* measure a distance equal to *qr*. Now resolve this force into the components along *BE* and *BC* as shown, by completing the parallelogram.

The component along *BE*

=Component along *BC*

=Force in *BE* or *BC*

=49,600 *kg*. (*compressive*) (by scaling)

18

Simple Mechanical Properties of Metals

From an engineering point of view, properties connected with metals are elasticity, plasticity, brittleness, malleability and ductility. Many of these properties are contrasting in their nature so that a given metal cannot exhibit simultaneously all these properties. For instance mild steel exhibits the property of elasticity Copper possesses the property of ductility. Wrought iron is malleable, lead is plastic and cast iron is brittle.

When a ductile metal is subjected to a tensile test it is found that the tensile stress is proportional to the strain upto a certain limit. Slightly before reaching the proportionality limit the elasticity of the material will be just broken down. On further increasing the load, the material will *yield* or *flow* so that the material is in a semi plastic stage. Further increase of load will result in extensions which increase with time The increase of load makes the strains to increase at a higher rate with the stresses until finally local stretch of the material occurs. This is marked by the formation of a *waist*. In this condition due to the considerable decrease in cross-section even a smaller load than the load at which the waste was formed, can produce further elongation. As elongation continues the sectional area continues to become smaller so that the load necessary to produce further extensions will be gradually reduced until fracture occurs

§167. Tensile stress-strain diagram for mild steel

Fig. 817 shows the stress strain diagram obtained for mild steel

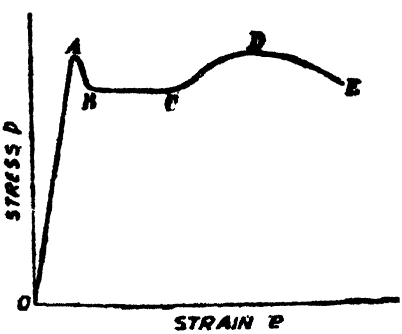


Fig. 817

If the specimen is loaded axially the stress-strain diagram is linear upto the point A The ratio of stress to strain *i.e*, Young's Modulus for the linear portion OA is usually $2{\cdot}1 \times 10^6$ kg/cm^2 If the specimen is strained beyond the point A, the stresses must be reduced almost suddenly to maintain equilibrium The reduction of stress AB occurs quickly and hence the exact nature of the curve between. A

and B is uncertain. Further straining continues at an almost constant stress along BC. From A to C the material is taken to *yield*. The point A is called the upper yield point and B is called the lower yield point. Beyond the point C, the material becomes a little hardened, and hence the stress again increases with strain. At D the stress reached is the greatest, based on the original sectional area of the specimen. The stress at D is called the ultimate stress. The stress falls from D to E and the specimen fails at the condition E. When the ultimate stress is reached the *waist* just begins.

Properties of some important engineering materials are tabulated below :

Material	*Limit of proportionality (kg./cm²)*	*Ultimate stress (kg./cm.²)*	*Elongation for a tensile fracture*	*Young's Modulus E (kg /cm.²)*	*Coefficient of linear expansion α (per °C)*
Medium strength mild steel	2820	3760	30%	$2{\cdot}1 \times 10^6$	$1{\cdot}2 \times 10$[illegible]
High strength steel	7840	15680	10%	$2{\cdot}1 \times 10^6$	$1{\cdot}3 \times 10^{-}$[illegible]
Wrought iron	1880	3140	—	$1{\cdot}69 \times 10^6$	$1{\cdot}2 \times 10$[illegible]
Cast iron					
Tension	—	1570	—	$1{\cdot}4 \times 10^6$	$1{\cdot}1 \times 10$[illegible]
Compression	—	7060	—		
Concrete					
Tension	—	450	—	$0{\cdot}14 \times 10^6$	$1{\cdot}2 \times 10$[illegible]
Compression	—	4500	—		

Working stress

This is the stress to which a material is actually subjected to in a stressed condition. For the safety of members, certain stresses are specified as the permissible stresses. A member must be so designed that the stress on it is less than the permissible stress.

Factor of safety

It is necessary that the working stress should be well below the elastic limit and to achieve this condition the ultimate stress is divided by a factor called the 'factor of safety' to obtain the working stress.

Problem 490. *A mild steel bar specimen of diameter 20 mm. is subjected to a tensile test. The bar was found to yield under a load* *of 8·25 tonnes, and the specimen attained a maximum load of 15·50 tonnes and ultimately broke at a load of 7·25 tonnes.*

Find (i) the tensile stress at the yield point
(ii) the ultimate stress
(iii) the average stress at the breaking point if the diameter of the neck is 10·75 mm.

Solution. Original sectional area of the specimen $= \frac{\pi}{4}(2)^2$

$= 3{\cdot}142\ cm.^2$

$\therefore$ Tensile stress at yielding $= \frac{8{\cdot}25 \times 1000}{3{\cdot}142}\ kg./cm.^2$

$= \mathbf{2626}\ kg./cm.^2$

Ultimate stress means the nominal stress corresponding to the maximum load

$\therefore$ Ultimate stress $= \frac{15{\cdot}5 \times 100}{3{\cdot}142}\ kg./cm.^2$

$= 4933\ kg./cm.^2$

Area of the specimen in the necked portion $= \frac{\pi}{4}(1{\cdot}075)^2\ cm.^2$

$= 0{\cdot}9076\ cm.^2$

$\therefore$ Average stress at the breaking point

$= \frac{7{\cdot}25 \times 1000}{0{\cdot}9076}\ kg./cm.^2$

$= 7987\ kg./cm.^2$

§168. Measurement of Ductility

When a mild steel test bar is subjected to a tensile test, it is observed that the elongation is practically distributed uniformly over the bar, till the maximum load is reached. After reaching the maximum load the cross-section decreases due to local yielding at the yield point.

Two methods are in use to estimate the ductility of the material. The first method is based on the total elongation produced in the specimen while the second method is based on the total reduction in sectional area.

First method

Let L = length of the test bar at fracture
l = length of the test bar before application of stress,

Percentage elongation $= \left(\frac{L-l}{l} \times 100\right)$

Since local yielding occurs before the fracture of the specimen the percentage elongation depends on the length of the specimen. Suppose the test bar is marked off in *centimetre* divisions, the percentage elongation of 1 *cm* length in the fracture zone will be very large. The effect of local yielding becomes less important if the length of the test bar is more. Hence it is really necessary to always specify the length of the test bar for which the percentage elongation has been computed.

The following table shows the percentage elongations for different values of the original length of test specimens.

Original length (*cm.*)	1·25	3·75	6·25	8.75	11·25	13·75	16·25	20
Final length (*cm.*)	2·25	5·50	8·70	11·62	14·69	17·70	20·71	25·25
Percentage elongation	80	47·7	39·25	32·84	30·61	28·70	27·42	26·25

In the above results, the percentage elongation has been determined without taking into consideration the sectional area of the specimen. For specimens of different sectional area the percentage elongation may slightly vary.

According to Prof. Unwin, for geometrically similar bars the results for percentage elongation are found to be comparable.

Let l= original length of the specimen

y= local extension

x= total extension

Then it is found

$$x=p+zl$$

where z is a coefficient.

The local extension y is found to depend on the area A of the specimen such that,

$$y=S\sqrt{A} \quad \text{where } S \text{ is a coefficient.}$$

$$\therefore \quad x=S\sqrt{A}+zl$$

If the coefficients S and z are known it is possible to reasonably determine the elongation of another bar of the same meterial having different dimensions. The following problem illustrates this method.

Problem 491. *The following results refer to a tensile test bar*

(*i*) *Diameter of test bar* = *22 mm.*

(*ii*) *gauge length* = *20 cm.*

(*iii*) *yield load* = *8·62 t.*

(*iv*) *Maximum load* = *12·90 t.*

(*v*) *Load at the instant of failure* = *10·95 t.*

(*vi*) *Total elongation* = *5·75 cm.*

(vii) *Elongation up to the point of maximum load* } $= 4.50$ *cm.*

(viii) *Area of the reduced section at fracture* } $= 1.99$ *cm.*^{2}

From the above results find the results for a tensile test bar of the same material 25 mm. diameter and 15 cm. gauge length, giving the elongation percentage and load at the instant of fracture.

Solution. Consider the 22 *mm.* diameter test bar

Total extension $= x = S\sqrt{A} + zl$

zl = elongation up to the point of *max. load* $= 4.50$ *cm.* and $x = 5.75$ *cm.*

$$\therefore \quad S\sqrt{A} = x - zl = 5.75 - 4.50 = 1.25 \text{ cm.}$$

Sectional area of the bar $= A = \frac{\pi}{4}(2.2)^2 = 3.801$ *cm.*^{2}

$$\therefore \quad S = \frac{1.25}{\sqrt{3.801}} = 0.641$$

But $\quad zl = 4.50$

$$\therefore \quad z = \frac{4.50}{20} = 0.225$$

$$\text{yield stress} = \frac{8.62}{3.801} = 2.268 \ t/cm.^2$$

Till the maximum load is reached the elongation is uniform and the sectional area at maximum load can be determined from the relation,

length × sectional area at max. load = original length × original area

(20+4.5) × sectional area at max. load $= 20 \times 3.801$

$$\therefore \quad \text{Sectional area at max. load} = \frac{20 \times 3.801}{24.5} = 3.102 \ cm^2$$

$$\therefore \quad \text{Max. stress} = \frac{\text{Max. load}}{\text{sectional area at max. load}}$$

$$= \frac{12.90}{3.102} \ t/cm.^2$$

$$= 4.159 \ t/cm.^2$$

Percentage reduction in area

$$= \frac{3.801 - 1.99}{3.801} \times 100 = 47.64\%$$

$$\text{Stress at failure} = \frac{10.95}{1.99} = 5.501 \ t/cm.^2$$

Summary of results

$S = 0.641$

$z = 0.225$

yield stress $=2{\cdot}268\ t/cm.^2$
Max. stress$=4{\cdot}159\ t/cm.^2$
Percentage reduction in area $=47{\cdot}64\%$
stress at failure$=5{\cdot}501\ t/cm.^2$

Now consider the 25 *mm.* diameter bar. Since this bar is also of the same material, the corresponding properties for this bar will be the same as given in the summary of results above.

Area of the second specimen$=A=\frac{\pi}{4}(2{\cdot}5)^2=4{\cdot}908\ cm.^2$

Total extension$=x=S\sqrt{A}+zl$

$$=0{\cdot}641\sqrt{4{\cdot}908}+0{\cdot}225\times 15$$
$$=1{\cdot}420+3{\cdot}375=4{\cdot}795\ cm.$$

$\therefore$ Percentage elongation$=\frac{L-l}{l}\times 100$

$$=\frac{4{\cdot}795}{15}\times 100$$
$$=31{\cdot}97\%$$

yield load=yield stress × sectional area

$$=22{\cdot}68\times 4{\cdot}908=11{\cdot}12\ t$$

length of rod at max. load × sectional area at max. load
=original length × original sectional area.

$\therefore$ Sectional area at max. load$=\frac{15\times 4{\cdot}908}{15+0{\cdot}225\times 15)}=4{\cdot}005\ cm.^2$

$\therefore$ Max. load=Max. stress × sectional area at max. load

$$=4{\cdot}159\times 4{\cdot}005=16{\cdot}65\ t$$

Percentage reduction in area$=47{\cdot}64\%$.

$\therefore$ Decrease in area of the section$=0{\cdot}4764\times 4{\cdot}908\ cm.^2$
$=2{\cdot}338\ cm.^2$

$\therefore$ Sectional area at fracture$=4{\cdot}908-2{\cdot}338=2{\cdot}570\ cm.^2$

$\therefore$ Load at fracture=stress at fracture × area of fracture

$$=5{\cdot}501\times 2{\cdot}570=14{\cdot}14\ t$$

Second method of determining the ductility

In this method the ductility is expressed in terms of the percentage reduction in area.

Let A=area of the section before the application of the load
A'=area after elongation.

$$\text{Percentage reduction in area}=\left(\frac{A-A'}{A}\times 100\right)\%$$

Hardness

The hardness of a meterial is the resistance which it can offer to indentation by other bodies.

Tests on hardness may be classified into (*i*) Scratch tests (*ii*) Indentation tests.

(*i*) *Scratch test*

This consists of pressing a loaded diamond into the surface to be tested, and then pulling the diamond so as to make a scratch. Depending on the load required to make a scratch of a given width or depending on the width of the scratch made with a given load the *hardness number* is determined.

(*ii*) *Indentation test*

This test consists of pressing a body of a standard shape into the material whose hardness is to be tested. The hardness number is based on the hardness number of the load and the indent produced.

Brinnell's Method

In this method a hardened steel ball of a given diameter is squeezed into the material to be tested, under a fixed load. Usually a load of 3000 *kg.* is applied and the steel ball has a diameter of 10 *mm.*

Brinnell hardness number

$$= \frac{\text{Load in } kg.}{\text{Spherical area of indent in } sq.\ mm.}$$

$$= \frac{P}{\frac{\pi D}{2}\left[D - \sqrt{D^2 - d^2}\right]}$$

where S = Standard load (*kg.*)
D = Diameter of the steel ball (*mm.*)
d = Diameter of indent (*mm.*)

§169. Impact Testing

For deciding the suitability of a material which is expected to resist repeated shocks, the ordinary static tensile test is not found satisfactory. Testing machines have been devised so that a specimen can be subjected to a single shock or a number of repeated shocks. The energy required to break the specimen is taken as a measure of the resistance of the material against shock loading.

The Izod testing machine

This testing machine consists of a heavy pendulum which is pivoted at the top two supporting A frames. The pendulum carries a pointer which moves over graduated scale graduated in *kg. metre* and fixed on the top of the machine. The specimen will be made of square section 10 *mm.* side and is notched in one face. The notch is 2 *mm.* deep and has a radius of 0·25 *mm.* at the bottom. The specimen is firmly held in a vice and fastened to the base of the machine.

The notch of the specimen should be at the top of jaw of the vice and facing the pendulum. Now the pendulum is raised so that the energy stored is 16·5 *kgm*. By using a trigger the pendulum is released so that the pendulum while swinging will knock the specimen and break it. The residual energy in the pendulum is noted on the graduated scale. The difference between the initial and the final energy represents the energy required to break the specimen.

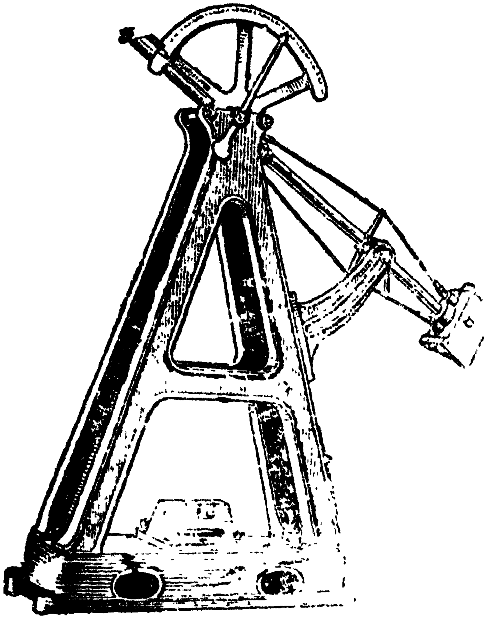


Fig. 818

§170. Fatigue of Metals

Sometimes we come across members which are subjected to loads which vary continuously from one magnitude to another. Sometimes a member may be subjected to even a reversible loading, *i.e.*, the member may be subjected to repeated tensile and compressive stresses. It is found that members subjected to such repeated stresses fail at stresses lower than the ultimate stresses determined by static tests. This property of metals is called *fatigue of metals*.

According to Wohler's experiments the following conclusions have been arrived :

(*i*) Reversed stresses of magnitude less than the static breaking stress may cause failure if repeated a sufficient number of times.

(*ii*) When subjected to fluctuating stress the resistance to fracture depends upon the range of stress within certain limits.

It is also found that within a certain range of stress the number of cycles of fluctuations of stress required to cause failure becomes infinite. This range is called the *Limiting range of stress* or the *endurance limit*.

19

Elements of Reinforced Concrete

General principles of design. Fig. 819 shows two bars made of concrete and steel having the same length L and cross-sectional area A. Let the two members be subjected to a total load W. Let W_s and W_c be the loads transferred to the steel and concrete members. Let E_s and E_c be the Young's moduli for steel and concrete respectively.

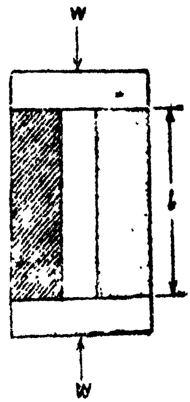


Fig. 819

Since the change in length of each member will be the same, we have,

Strain in concrete = strain in steel

$$\therefore \quad \frac{W_c}{AE_c} = \frac{W_s}{AE_s}$$

$$= \frac{W_s + W_c}{AE_c + AE_s}$$

$$= \frac{W}{A(E_c + E_s)}$$

$\therefore$ Stress in steel

$$= t = \frac{W_s}{A} = \frac{E_s}{E_c} \frac{W_c}{A}$$

Stress in concrete

$$= c = \frac{W_c}{A}$$

$$\therefore \quad t = \frac{E_s}{E_c} c$$

$$\therefore \quad t = mc \quad \text{where} \quad m = \frac{E_s}{E_c}$$

This ratio $\frac{E_s}{E_c}$ is called the *modular ratio* between steel and concrete.

Suppose the modular ratio equals 18.

We have,

Stress in steel = 18 × stress in concrete

$$\text{Load on steel} = W_s = \frac{E_s}{E_c+E_s} \cdot W$$

$$= \frac{\frac{E_s}{E_c}}{1+\frac{E_s}{E_c}} \cdot W$$

$$= \frac{m}{m+1} \cdot W$$

If $m=18$,

$$W_s = \frac{18}{19} W$$

$$\therefore \quad W_c = \frac{1}{19} W$$

Hence, we find that the steel member is subjected to a greater load than concrete. Hence, steel when provided in combination with concrete will be very useful in sharing a considerable part of the load on the member of composite section.

§171. Assumptions

The analysis and design of a reinforced concrete member subjected to bending are based on the following assumptions :

(*a*) Plane sections transverse to the centre line of a member before bending remain plane sections after bending.

(*b*) Elastic modulus for concrete has the same value within the limits of deformation of the member.

(*c*) Elastic modulus for steel has the same value within the limits of deformation of the member.

(*d*) The reinforcement does not slip from the concrete surrounding it.

(*e*) Tension is borne entirely by steel.

(*f*) The steel is free from initial stresses when embedded in concrete.

(*g*) There is no resultant thrust on any transverse section of the member.

Of the above assumptions, the assumption that plane sections transverse to the centre line of a member, before bending remain plane after bending may require further clarification.

Fig. 820 (*a*) shows a beam subjected to an external loading. Consider sections *AB*, *CD*, *EF*, *GH*, *IJ*, *etc.*, which are at right angles to the centre line of the member. After the beam bends, the various fibres are subjected to deformations of such amounts that these planes respectively occupy the positions A_1B_1, C_1D_1, E_1F_1, G_1H_1, I_1J_1, *etc.* This is illustrated in Fig. 820 (*b*).

Fig. 821 (*a*) shows a simply supported singly reinforced beam. Consider two sections *AB* and *CD* unit distance apart. Let the beam be subjected to an external loading. Let A_1C_1 be the length of the topmost fibre.

Let B_2C_2 be the length of a fibre of concrete at the level of the reinforcement.

We have strain in concrete in the top fibre

$$=AC-A_1C_1=e_c$$

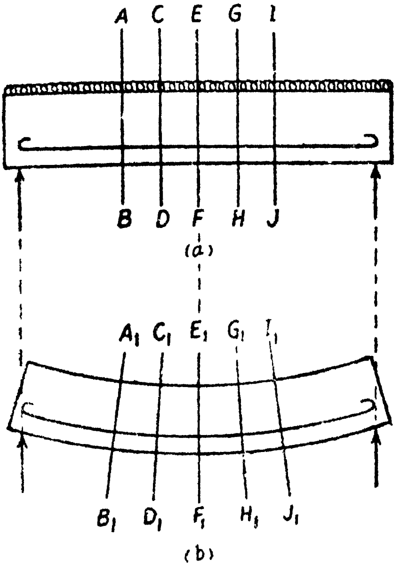


Fig. 820

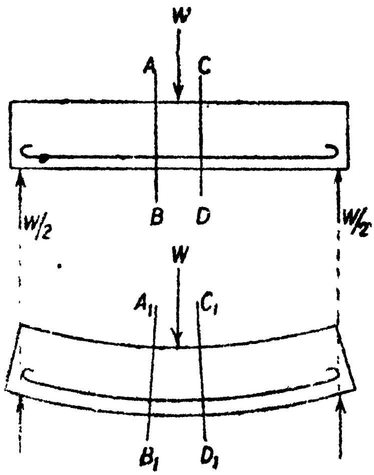


Fig. 821

Similarly, strain in concrete just surrounding the steel

$$=B_1D_1-BD=e_t$$

Since, there is no slip between steel and the concrete surrounding it, the strain in steel is also equal to e_t

∴ Stress in steel

$$t=E_s e_t=mE_c e_t$$

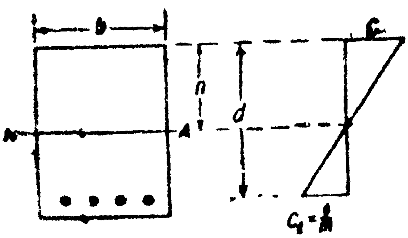


Fig 822

Stress in concrete surrounding steel

$$=C_t=E_c e_t=E_c\,\frac{t}{E_s}$$

$$\therefore \qquad C_t=\frac{t}{m}$$

Fig. 822 shows the stress distribution in concrete across the section of the beam.

§172. Neutral axis. The neutral axis for a beam section is the line of intersection of the neutral layer with the beam section.

This is a straight line dividing cross-section into tension and compression zones One of the basic assumptions made in the analysis of reinforced concrete beams is that the tension is borne completely by steel. Hence, it is important to note that in determining the neutral axis, the concrete in the tension zone should not be taken into account The tension should be considered as resisted by the steel. If the area of the reinforcement is A_t and the tensile stress in the reinforcement is t, the total tension resisted

$$=T=A_t t$$

$$=A_t mC_t$$

$$=(mA_t)C_t$$

Hence, a reinforcement of area A_t can be regarded as equivalent to an area (mA_t) of concerte. Let b and d be the breadth and effective depth of the beam section. (Effective depth is the depth from the compression edge to the centre of tensile reinforcement). Let n be the depth of neutral axis.

One of the assumptions in the analysis is that there is no resultant thrust on the section

i e., Total compression = Total tension

$\therefore$ Compression area × average compressive stress

= area of tensile reinforcement × stress in steel

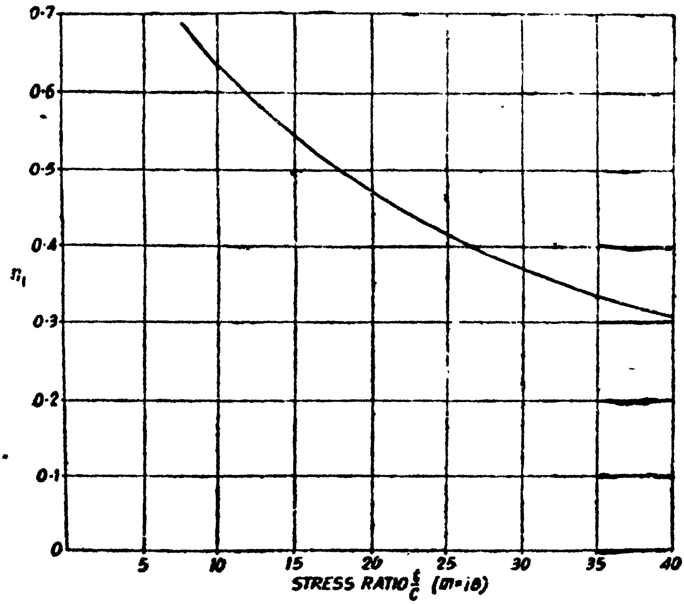


Fig. 823

$$\therefore \quad bn\frac{c}{2} = A_t t \qquad \ldots(i)$$

Further, by the geometry of the stess diagram (Fig. 822) we have,

$$\frac{c}{t/m} = \frac{n}{d-n} \qquad \ldots(ii)$$

Equating the moments of areas of compression and tension zones about the neutral axis, we have,

$$\frac{bn^2}{2} = mA_t(d-n) \qquad \ldots(iii)$$

Besides the above results, the following result will be found interesting,

we have,

$$\frac{c}{(t/m)} = \frac{n}{d-n}$$

$$\therefore \quad \frac{mc}{t} = \frac{n}{d-n}$$

Putting $\frac{t}{c} = r \quad \frac{m}{r} = \frac{n}{d-n}$

$$\therefore \quad md - mn = nr$$

$$\therefore \quad n(m+r) = md \qquad \ldots(iv)$$

$$\therefore \quad n = \left[\frac{m}{m+r}\right]d$$

Putting $n = n_1 d$,

We have, $n_1 = \frac{m}{m+r}$ ..(v)

With $m = 18$, we have

$$n_1 = \frac{18}{18+r}$$

Fig. 823 shows the values of n_1 for various values of r

Alternative expression for n

$$n_1 = \frac{m}{m+r} = \frac{m}{m+\frac{t}{c}}$$

$$\therefore \quad n_1 = \frac{mc}{mc+t}$$

§173. Lever arm

This is the distance between the point of application of the resultant compression and the point of application of the resultant tension. The point of application of the resultant compression is at the level of the centroid of the compressive stress diagram, *i.e.*, at a depth of $\frac{n}{3}$ from the compression edge. But, the resultant tension is at the level of the reinforcement since the tensile resistance of concrete is ignored.

$$\therefore \qquad \text{Lever arm}=a=d-\frac{n}{3} \qquad \ldots(vii)$$

Sometimes the lever arm is expressed as the product of a coefficient a_l and the effective depth.

$$\therefore \qquad a=a_l d$$

$$\text{But} \qquad a=d-\frac{n}{3}$$

$$\therefore \qquad a_l d=d-\frac{n}{3}$$

$$\therefore \qquad a_l=1-\frac{1}{3}\frac{n}{d}$$

$$\therefore \qquad a_l=1-\frac{n_1}{3} \qquad \ldots(viia)$$

§174. Moment of resistance

This is the resisting moment offered by a beam section to resist the bending moment at the section.

Moment of resistance

=total compression or total tension × lever arm

$$\text{M.R.}=bn\frac{c}{2}\left(d-\frac{n}{3}\right) \qquad \ldots(viii)$$

$$\text{M.R.}=A_t t\left(d-\frac{n}{3}\right) \qquad \ldots(ix)$$

Sometimes the moment of resistance is put in another form.

$$\text{M.R.}=M=bn\frac{c}{2}\left(d-\frac{n}{3}\right)$$

$$\text{Put} \qquad n=n_1 d$$

$$\therefore \qquad M=b\,n_1 d\frac{c}{2}\left(d-\frac{n_1 d}{3}\right)$$

$$\therefore \qquad M=\tfrac{1}{2}\,n_1\left(1-\frac{n_1}{3}\right)cbd^2$$

$$\therefore \qquad M=Q\,bd^2$$

$$\text{where} \qquad Q=\tfrac{1}{2}n_1\left(1-\frac{n_1}{3}\right)c$$

§175. Balanced or economic or critical sections

A section may be reinforced with such an amount of steel, that when the most distant concrete fibre in the compression zone reaches the allowable stress in compression, the tensile stress in the reinforcement reaches the allowable stress in steel. For instance, if the allowable stresses in concrete and steel are 50 $kg./cm.^2$ and 1400 $kg./cm^2$., respectively, then the section is so much reinforced that when the extreme compressive stress in concrete reaches 50 $kg./cm^2$, the tensile stress in steel reaches 1400 $kg./cm.^2$ A section reinforced to satisfy this condition is called a balanced or economic or critical section. The neutral axis corresponding to this condition is called the critical neutral axis.

Let n_c be the depth of critical neutral axis.

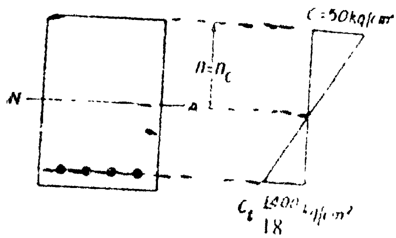


Fig. 824

Moment of resistance of the section

$$= M.R. = b\,n_c \frac{c}{2}\left(d - \frac{n_c}{3}\right)$$

or,

$$= M.R. = A_t\, t\left(d - \frac{n_c}{3}\right)$$

c and t shall be taken at the allowable values

$$n_c = \frac{m}{m+r}$$

Taking $c = 50\ kg./cm.^2$, $t = 1400\ kg/cm.^2$ and $m = 18$.

$$r = \frac{t}{c} = \frac{1400}{50} = 28$$

$$n_1 = \frac{18}{18+28} = \frac{18}{46} = \frac{9}{23}$$

Lever arm factor

$$= a_1 = 1 - \frac{n_1}{3}$$

$$= 1 - \frac{3}{23} = \frac{20}{23} = 0{\cdot}87$$

∴ Lever arm $= a = 0{\cdot}87\ d$

∴ $$M.R. = bn_c \frac{c}{2}\left(d - \frac{n_c}{3}\right)$$

$$= b \times \frac{9}{23}\ d \times \frac{50}{2} \times 0{\cdot}87\ d$$

∴ $M.R. = 8{\cdot}50\ bd^2$

If $M.R. = Q\ bd^2$, $Q = 8{\cdot}50$

Also,

Total compression = total tension

$$bn_c \frac{c}{2} = A_t\ t$$

∴ $$b \times \frac{9}{23}\ d \times \frac{50}{2} = A_t \times 1400$$

∴ $$\frac{A_t}{bd} = 0{\cdot}00699$$

Stresses in Steel
t

Stress in Concrete c kg/cm²	t=1200 kg/cm²			t=1400 kg/cm²			t=1600 kg/cm³			t=1800 kg/cm²		
	n_1	a_l	Q	n_1	a_l	Q	n_1	a_l	Q	n_1	a_l	Q
20	0·231	0·923	2·13	0·205	0·[illegible]32	1·91	0·184	0·939	1·73	0·167	0·944	1·58
25	0·273	0·909	3·10	0·243	0·919	2·79	0·205	0·932	2·39	0·200	0·933	2·33
30	0·3[illegible]0	0·897	4·17	0·278	0·907	3·79	0·252	0·919	3·47	0·231	0·933	3·20
35	0·344	0·852	5·13	0·310	0·897	4·87	0·283	0·906	4·49	0·259	0·914	4·14
40	0·375	0·875	6·56	0·340	0·887	6·03	0·310	0·897	5·56	0·286	0·[illegible]05	5·18
45	0·403	0·866	7·85	0·367	0·878	7·25	0·336	0·888	6·71	0·310	0·897	5·26
50	0·429	0·857	9·19	0·391	0·870	8·50	0·3[illegible]0	0·880	7·92	0·333	0·889	7·40
55	0·452	0·849	10·55	0·415	0·862	9·84	0·382	0·873	9·17	0·355	0·882	8·61
[illegible]0	0·474	0·842	11·98	0·435	0·855	11·15	0·403	0·866	10·47	0·375	0·875	9·84
65	0·494	0·835	13·40	0·455	0·848	12·54	0·423	0·859	11·81	0·394	0·899	11·13
70	0·512	0·829	14·86	0·474	0·842	13·97	0·440	0·853	13·14	0·412	0·863	12·45
75	0·529	0·824	16·3[illegible]	0·491	0·836	15·39	0·45[illegible]	0·847	14·55	0·429	0·857	13·79

$\therefore \qquad A = 0{\cdot}699\%$ of bd

The values of n_1, a_1 and Q for various working stresses are given in the table with modular ratio $m = 18$. (See page 930).

§176. Unbalanced section

If the steel reinforcement provided is more or less than what is required for a balanced section, the section is termed an *unbalanced section*. Unbalanced sections may be further classified into *under-reinforced* and *over-reinforced sections*.

§177. Under-reinforced section

This is a section such that the amount of steel provided is less than what is required for a balanced section. Hence if the stress in steel just reaches the allowable stress, the corresponding extreme compressive stress in concrete will be less than its allowable limit. Suppose the allowable stresses in concrete and steel are 50 $kg./cm.^2$ and 1400 $kg./cm.^2$ and $m = 18$. For the under-reinforced section, when the stress in steeel just reaches 1400 $kg./cm.^2$ the corresponding stress in concrete will be less than 50 $kg./cm.^2$ (Fig. 825). The depth n of the actual neutral axis is less than the depth n_c of the critical neutral axis The moment of resistance of the under-reinforced section will be less than that of the balanced section.

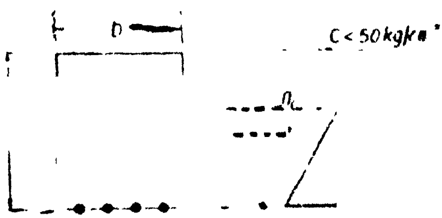


Fig 825

The moment of resistance is given by

$$\text{M.R.} = At\left(d - \frac{n}{3}\right)$$

the stress t being taken at the allowable stress for steel.

§178. Over-reinforced section

This is a section such that the amount of steel provided is more than what is required for a balanced section Hence, when the stress in concrete reaches its allowable limit, the corresponding stress in steel will be less than its allowable limit Suppose the allowable stresses in concrete and steel are 50 $kg./cm.^2$ and 1400 $kg./cm.^2$ and $m = 18$. For the over-reinforced section, when the stress in concrete reaches 50 $kg./cm.^2$, the corresponding stress in steel will be less

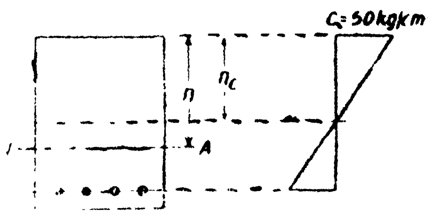


Fig. 826

than 1400 $kg./cm.^2$ (Fig. 826). The depth n of the actual neutral axis in this case will be greater than the depth n_c of the critical neutral axis. The moment of resistance of the over-reinforced section will be greater than that of the balanced section. The moment of resistance is given by

$$M.R.=bn\frac{c}{2}\left(d-\frac{n}{3}\right)$$

the stress c being taken at the allowable stress for concrete.

§179. Types of Problems

In singly reinforced sections we come across the following types of problems :

Type A. *Data* : Dimensions of the section, permissible stress in concrete and steel and modular ratio.

Required : Moment of resistance of the section.

This type of problem may be solved as follows :

(*i*) First determine the position of actual neutral axis by equating the moment of concrete area in compression about the neutral axis to the moment of equivalent tension area about the neutral axis

i.e., use the relation,

$$\frac{bn^2}{2}=mA_t(d-n) \text{ and find } n.$$

(*ii*) Find the position of critical neutral axis, corresponding to the given safe stresses in concrete and steel

(*iii*) Ascertain whether the section is under-reinforced or over-reinforced. If the actual neutral axis lies above the critical neutral axis, the section is under-reinforced. But, if the actual neutral axis is below the critical neutral axis, the section is over-reinforced.

(*iv*) If the section is over-reinforced concrete attains the maximum stress earlier than steel, and the moment of resistance is given by

$$M.R.=bn\frac{c}{2}\left(d-\frac{n}{3}\right)$$

Taking c = permissible stress in concrete

and n = depth of actual neutral axis.

If the section is under-reinforced, steel attains the maximum stress earlier than concrete, and the moment of resistance is given by

$$M.R.=A_t t\left(d-\frac{n}{3}\right)$$

Taking t = permissible stress in steel

and n = depth of actual neutral axis

Type B. *Data* : Dimensions of the section, Area of reinforcement, Bending moment M and modular ratio.

Required : Stresses in concrete and steel.

This type of problem may be solved as follows :

(*i*) Determine the position of the actual neutral axis.

(*ii*) Find the stress in concrete by equating the moment or resistance to the given bending moment

i.e., use the relation,

$bn \frac{c}{2}\left(d-\frac{n}{3} \right)=M$ and find c.

(*iii*) Find the stress in steel from the relation,

$$\frac{mc}{t}=\frac{n}{d-n}$$

Type C. *Data* : Permissible stresses is concrete and steel Bending moment M and modular ratio.

Required : To design the section.

This type of problem may be solved as follows ·

(*i*) Determine the depth of critical neutral axis in terms of the effective depth d

i.e., use the relation

$$\frac{c}{t/m}=\frac{n_c}{d-n_c}$$ and find n_c in terms of d.

(*ii*) Choose a convenient width b. By equating the moment of resistance to the given bending moment find the effective depth

i.e., use the relation, $bn_c \frac{c}{2}\left(d-\frac{n_c}{3} \right)=M$ and find d.

(*iii*) Find the area of steel by equating the total compression on the beam section to the total tension on the beam section.

i e., use the relation,

$$bn_c \frac{c}{2}=A_t t$$ and find A_t.

The following problems illustrate the above types of problems.

Problem 492. *A reinforced concrete beam section of width b and effective depth d is reinforced on the tension side only. If the allowable stresses in concrete and steel are 50 kg./cm.² and 1400 kg./cm.² and m=18, determine for a balanced section (i) the depth of neutral axis, (ii) the moment of resistance and (iii) the percentage of steel reinforcement.*

Solution. Steel ratio

$$=r=\frac{t}{c}=\frac{1400}{50}=28$$

$$n_c=\frac{m}{m+r}\,d$$

$$= \frac{18}{18+28} d$$

$$\therefore \quad n_c = \frac{9}{23} d$$

$$\text{M.R.} = bn_c \frac{c}{2}\left(d - \frac{n_c}{3}\right)$$

$$= b \times \frac{9}{23} d \times \frac{50}{2}\left(d - \frac{3}{23} d\right)$$

$$\therefore \quad \text{M.R.} = 8{\cdot}50\ bd^2$$

Total tension = Total compression

$$A_t t = bn_c \frac{c}{2}$$

$$A_t \times 1400 = b \times \frac{9}{23} d \times \frac{50}{2}$$

$$\therefore \quad \frac{A_t}{bd} = 0{\cdot}00699$$

∴ Percentage of steel reinforcement

$$= 0{\cdot}699\%.$$

Problem 493. *The cross-section of a singly reinforced concrete beam is 30 cm. wide and 40 cm. deep to the centre of the reinforcement which consists of four bars of 14 mm. dia. If the stresses in concrete and steel are not to exceed 50 kg/cm.² and 1400 kg/cm², determine the moment of resistance of the section. Take m = 18.*

Solution. See Fig. 827.

$A_t = 4 \times 1{\cdot}54 = 6{\cdot}16$ *cm.*²

Position of actual neutral axis Equating moments of compression and equivalent tension zones about the actual neutral axis, we have

$$\frac{30n^2}{2} = 18 \times 6{\cdot}16\,(40 - n)$$

$$\therefore \quad n^2 + 6{\cdot}16n - 246{\cdot}4 = 0$$

$$n = 13{\cdot}89 \text{ cms.}$$

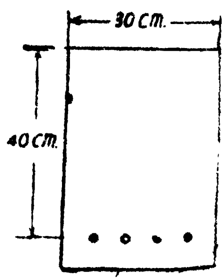


Fig. 827

For a balanced section,

$$\text{Steel ratio} = r = \frac{t}{c} = \frac{1400}{50} = 28$$

Depth of critical neutral axis

$$= n_c = \frac{m}{m+r} d$$

$$= \frac{18}{18+28} \times 40 \text{ cm.}$$

$=15{\cdot}65\ cm.$

But $\quad n=13{\cdot}89\ cm.$

Hence $\quad n<n_c$

$\therefore$ The section is under-reinforced and steel will attain the maximum stress earlier to concrete

$$\therefore \quad \text{M.R.}=A_t\,t\left(d-\frac{n}{3}\right)$$

$$=6{\cdot}16\times1400\left(40-\frac{13{\cdot}89}{3}\right)\ kg.\ cm.$$

$$=305030\ kg.\ cm.$$

Problem 494 (S.I). *The cross-section of a singly reinforced concrete beam is 30 cm. wide and 40 cm. deep to the centre of the reinforcement which consists of 4 bars of 14 mm. diameter. If the stresses in concrete and steel are not to exceed 5 N/mm.² and 140 N/mm² determine the moment of resistance of the section. Take m=18.*

Solution.

$$A_t=4\times\frac{\pi}{4}(1{\cdot}4)^2=6{\cdot}16\ cm^2$$

Position of actual neutral axis. Equating moments of compression and equivalent tension zones about the neutral axis, we have,

$$\frac{30\,n^2}{2}=18\times6{\cdot}16\,(40-n)$$

$$\therefore \quad n^2+7{\cdot}392\,n-295{\cdot}7=0$$

$$\therefore \quad n=13{\cdot}89\ cm.$$

For a balanced section,

$$\text{Steel ratio}=r=\frac{t}{c}=\frac{140}{5}=28$$

Depth of critical neutral axis,

$$=n_c=\left(\frac{m}{m+r}\right)d$$

$$=\left(\frac{18}{18+28}\right)\times40=15{\cdot}65\ cm.$$

Hence $\quad n<n_c$

$\therefore$ The section is *under reinforced* and steel attains its maximum stress earlier to concrete.

$$\text{M.R.}=A_t\,t\left(d-\frac{n}{3}\right)$$

$$=6{\cdot}16\times14000\left(40-\frac{13{\cdot}89}{3}\right)N\ cm.$$

$$=3050310\ N\ cm.$$

Problem 495. *A singly reinforced concrete beam 25 cm. wide and 40 cm. deep to the centre of tensile reinforcement has a span of 5 metres and carries a total uniformly distributed load of 1250 kg. per metre including its weight. The stresses in concrete and steel are not to exceed 50 kg./cm.² and 1400 kg./cm.² respectively. Find the steel reinforcement necessary. Take m = 18.*

Solution. Total load per *metre*

$$=1000 \ kg./m.$$

Span $= 5$ *metres*

∴ Maximum bending moment

$$M=\frac{1250\times 5^2}{8}\times 100 \ kg. \ cm.$$

$$=390625 \ kg. \ cm.$$

$$\text{M.R.}=Qbd^2=Q\times 25\times 40^2=390625 \ kg. \ cm.$$

$$\therefore \quad Q=\frac{390652}{25\times 1600}=9{\cdot}77$$

$$\therefore \quad M=9{\cdot}77 \ bd^2$$

If the section had been a balanced section,

$$n_c=\frac{m}{m+r}=\frac{18}{18+\frac{1400}{50}}=0{\cdot}39$$

Balanced M.R.

$$=b\times 0{\cdot}39 \ d\times \frac{50}{2}\left(d-\frac{0{\cdot}39 \ d}{3}\right)$$

$$=8{\cdot}50 \ bd^2$$

Since the moment of resistance of the beam has to be greater than that of the balanced section, the beam is to be *over-reinforced*. Concrete attains its maximum stress earlier to steel.

$$\therefore \quad c=50 \ kg./cm.^2$$

M.R. of the beam

$$=b\,n\frac{c}{2}\left(d-\frac{n}{3}\right)$$

$$=25 \ n\times\frac{50}{2}\left(40-\frac{n}{3}\right)=390625 \ kg \ cm.$$

$$\therefore \quad n\left(40-\frac{n}{3}\right)$$

$$=\frac{390625}{625}=625$$

$$40 \ n-\frac{n^2}{3}=625$$

$$n^2-120 \ n+1875=0$$

Solving,

we get $n = 18.46\ cm.$

Stress in steel is given by

$$\frac{mc}{t} = \frac{n}{d-n}$$

$$\frac{18 \times 50}{t} = \frac{18.46}{40 - 18.46}$$

$$\therefore \quad t = \frac{18 \times 50 \times 21.54}{18.46}\ kg./cm^2$$

$$= 1050\ kg/cm^2$$

Total compression = Total tension

$$bn\frac{c}{2} = A_t \cdot t$$

$$25 \times 18.46 \times \frac{50}{2} = A_t \times 1050$$

$$\therefore \quad A_t = \frac{25 \times 18.46 \times 50}{2 \times 1050}\ cm.^2$$

$$= 10.98\ cm.^2$$

Problem 496 (SI). *A singly reinforced concrete beam 25 cm. wide and 40 cm deep to the centre of the tensile reinforcement has a span of 5 metres and carries a total uniformly distributed load of 12500 Newtons/metre including its weight. The stresses in concrete and steel are not to exceed 5 N/mm² and 140 N/mm² respectively. Find the steel reinforcement necessary. Take m = 18.*

Solution. Total load per *metre* = 12500 *Newtons/metre*

Span = 5 *metres*

$$\text{Maximum bending moment} = M = \frac{12500 \times 5^2}{8}\ Nm.$$

$$= 39062.5\ Nm.$$

$$= 3906250\ Ncm.$$

$$M.R. = Qbd^2 = Q \times 25 \times 40^2 = 3906250\ N\ cm.$$

$$\therefore \quad Q = 97.7$$

If the section had been a balanced section,

$$n_c = \frac{m}{m + \frac{t}{c}} = \frac{18}{18 + \frac{140}{5}} = \frac{9}{23}d$$

$$\text{Balanced M.R.} = b \times \frac{9}{23}d \times \frac{500}{2}\left(d - \frac{3}{23}d\right) = 85\ bd^2$$

But the M.R. of the given beam section = $97.7\ bd^2$

Since the moment of resistance of the beam has to be greater than that of the balanced section, the beam is to be *over reinforced.* Hence concrete attains its maximum stress earlier to steel.

Hence $c = 5\ N/cm^2 = 500\ N/mm.^2$

M R. of the beam section

$$= bn\,\frac{c}{2}\left(d - \frac{n}{3}\right)$$

$$= 25n \times \frac{500}{2}\left(40 - \frac{n}{3}\right) = 3906250\ N\ cm.$$

$\therefore\quad n^2 - 120n + 1875 = 0$

Solving, we get $\quad n = 18{\cdot}46\ cm.$

Stress in steel is given by,

$$\frac{mc}{t} = \frac{n}{d-n}$$

$$\therefore\quad \frac{18 \times 500}{t} = \frac{18{\cdot}46}{40 - 18{\cdot}46}$$

$$\therefore\quad t = 10500\ N/cm.^2$$

Total compression = Total tension

$$bn\,\frac{c}{2} = A_t\, t$$

$$25 \times 18{\cdot}46 \times \frac{500}{2} = A_t \times 10500$$

$$\therefore\quad A_t = \frac{25 \times 18{\cdot}46 \times 500}{2 \times 10500}$$

$$= 10{\cdot}98\ cm.^2$$

Problem 497. *A reinforced concrete beam 30 cm. wide by 60 cm. total depth has a span of 6 metres Find the necessary tension reinforcement at the mid span to enable the beam to carry a load of 800 kg./m. in addition to its own weight.*

Concrete cover below the steel centre

$= 3{\cdot}5\ cm.$

Weight of R.C C.

$= 2400\ kg/cm.^3$

Allowable stress in steel

$= 1400\ kg/cm^2$

Allowable stress in concrete

$= 50\ kg./cm.^2$

Modular ratio $m = 18$

Solution. Dead load per *metre* length

$$= 0{\cdot}3 \times 0{\cdot}6 \times 1 \times 2400 = 432\ kg./m$$

Superimposed load on the beam $=800$ *kg./m.*

$\therefore$ Total load per *metre* $=1232$ *kg./m.*

Maximum bending moment

$$=M=\frac{1232\times 6^2}{8}\times 100 \text{ kg. cm.}$$

$$=554{,}400 \text{ kg. cm.}$$

Let the M.R. of the beam be Qbd^2

$\therefore$ $Qbd^2=Q\times 30\times 56\cdot 5^2=554400$

$\therefore$ $Q=\frac{554400}{30\times 56\cdot 5^2}$

$\therefore$ $Q=5\cdot 79$

$\therefore$ $M=5\cdot 79\ bd^2$

For a balanced section,

with $c=50$ *kg./cm.*^{2}

$t=1400$ *kg /cm.*^{2}

and $m=18$

M.R. of the balanced section $=8\cdot 5\ bd^2$

But the M.R. of the given beam $=5\cdot 79\ bd^2$

Since the M.R. of the beam is less than the M.R. of the balanced section, the beam should be designed as an *under-reinforced* beam. Steel attains its maximum stress earlier to concrete

$\therefore$ $t=1400$ *kg/cm.*^{2}

Corresponding stress in concrete is given by

$$\frac{mc}{t}=\frac{n}{d-n}$$

$$c=\frac{t}{m}\cdot\frac{n}{d-n}$$

$$c=\frac{1400}{18}\cdot\frac{n}{d-n}$$

Effective depth $=56\cdot 5$ *cm.*

$$\therefore\quad c=\frac{1400}{18}\cdot\frac{n}{56\cdot 5-n}$$

$$\text{M.R.}=bn\frac{c}{2}\left(d-\frac{n}{3}\right)$$

$$=30\ n\times\frac{1400}{2\times 18}\cdot\frac{n}{56\cdot 5-n}\left(56\cdot 5-\frac{n}{3}\right)=554{,}400 \text{ kg cm.}$$

$$\therefore\quad \frac{n^2\left(56\cdot 5-\frac{n}{3}\right)}{56\cdot 5-n}=\frac{554{,}400\times 36}{30\times 1400}$$

$$\therefore \quad \frac{n^2\left(56{\cdot}5-\frac{n}{3}\right)}{56{\cdot}5-n}=475{\cdot}2$$

Solving the above equation by trial and error,

we get $\quad n=18{\cdot}88\ cm$

$$\therefore \quad c=\frac{1400}{18}\times\frac{18{\cdot}88}{56{\cdot}5-18{\cdot}88}\ kg./cm.^2$$

$$=39\ kg\ /cm.^2$$

Total compression $=$ Total tension

$$bn\frac{c}{2}=A_t\ t$$

$$30\times18{\cdot}88\times\frac{39}{2}=A_t\times1400$$

$$\therefore \quad A_t=\frac{30\times18{\cdot}88}{1400}\times\frac{39}{2}\ cm.^2$$

$$=7{\cdot}89\ cm.^2$$

Problem 498 (SI). *A reinforced concrete beam 30 cm. wide and 60 cm. deep has a span of 6 metres Find the necessary tension reinforcement at mid span to enable the beam to carry a load of 8000 Newtons per metre in addition to its own weight.*

Concrete cover below the steel centre	$=3{\cdot}5$ *cm.*
Weight of R.C.C.	$=25000\ N/metre^3$
Allowable stress in steel	$=140\ N/mm.^2$
Allowable stress in concrete	$=5\ N/mm.^2$
Modular ratio	$=m=18$

Solution.

Dead load of beam $=0{\cdot}3\times0{\cdot}6\times1\times25000=4500\ N/m$

Superimposed load $=8000\ N/m$

Total $=12500\ N/m$

Maximum bending moment

$$=M=\frac{12500\times6^2}{8}\times100\ N\ cm.$$

$$=5625000\ N\ cm.$$

Let the M.R. of the beam section be $Q\ b.l^2$

$\therefore \quad Q\times30\times56{\cdot}5^2=5625000$

$\therefore \quad Q=58{\cdot}7$

$\therefore \quad$ M.R. $=58{\cdot}7\ bd^2\ N\ cm.$

For a balanced section,

with $c=5\ N/mm.^2=500\ N/cm.^2$

$t=140\ N/mm.^2=14000\ N/cm.^2$

and $m=18$

M.R. of the balanced section

$$=85\ bd^2\ N\ cm.$$

Since the M.R. of the given beam section is less than the M R. of the balanced section, the beam should be designed as an *under reinforced beam.* Steel attains its maximum stress earlier to concrete.

$\therefore$ $t=14000\ N/cm.^2$

$$\frac{mc}{t}=\frac{n}{d-n}$$

$$\therefore \quad c=\frac{t}{m}\cdot\frac{n}{d-n}=\frac{14000}{18}\times\frac{n}{56\cdot5-n}$$

$$\therefore \quad c=\frac{14000}{18}\times\frac{n}{56\cdot5-n}$$

$$\text{M.R.}=bn\frac{c}{2}\left(d-\frac{n}{3}\right)$$

$$=30\,n\times\frac{14000}{2\times18}\times\frac{n}{56\cdot5-n}\left(56\cdot5-\frac{n}{3}\right)$$

$$=5625000$$

$$\therefore \quad \frac{n^2\left(56\cdot5-\frac{n}{3}\right)}{56\cdot5-n}=482\cdot14$$

Solving by trial and errors, we get

$$n=18\cdot99\ cm.$$

$$\therefore \quad c=\frac{14000}{18}\times\frac{18\cdot99}{(56\cdot5-18\cdot99)}$$

$$393\cdot8\ N/cm.^2$$

Total compression = Total tension

$$bn\frac{c}{2}=A_t\ t$$

$$\therefore\ 30\times18\cdot99\times\frac{393\cdot8}{2}=A_t\times14000$$

$$\therefore \quad A_t=\frac{30\times18\cdot99}{14000}\times\frac{393\cdot8}{2}\ cm.^2$$

$$=8\cdot01\ cm^2.$$

Problem 499. *The moment of resistance of a rectangular singly reinforced beam of width b cm. and effective depth d cm. is 10 bd^2 kg. cm. If the stressses in the outside fibre of the concrete and in steel* *do not exceed 50 kg./cm.² and 1400 kg./cm.² respectively and the modular ratio equals 18, determine the ratio of the depth of neutral axis from the extreme compression fibres to the effective depth of the beam and the ratio of the area of the tensile steel to the effective area of the beam.*

Solution.

For a balanced section, with $c=50$ *kg./cm.*², $t=1400$ *kg./cm.*² and $m=18$, the M.R. of the section $=M=8{\cdot}50\ bd^2$.

But the M R. of the given beam $=10\ bd^2$. Since the M.R. of the given beam is greater than the M.R. of the balanced section, the beam should be *over-reinforced* Hence concrete attains its maximum stress earlier to steel.

Let n = actual depth of neutral axis

$$\frac{n}{d}=n_1$$

$$\text{M.R.}=bn\frac{c}{2}\left(d-\frac{n}{3}\right)=10\ bd^2$$

$$\therefore\ bn_1d\times\frac{50}{2}\left(d-\frac{n_1d}{3}\right)=10\ bd^2$$

$$\therefore\quad n_1\left(1-\frac{n_1}{3}\right)\frac{10}{25}=0{\cdot}4$$

$$n_1-\frac{n_1^2}{3}=0{\cdot}4$$

$$n_1^2-3n_1+1{\cdot}2=0$$

Solving, we get $n_1=0{\cdot}476$

Stress in steel is given by,

$$\frac{mc}{t}=\frac{n}{d-n}=\frac{n_1d}{d-n_1d}=\frac{n_1}{1-n_1}$$

$$\therefore\quad t=mc\,\frac{1-n_1}{n_1}=18\times50\times\frac{(1-0{\cdot}476)}{0{\cdot}476}\ kg./cm.^2$$

$$=769\ kg./cm.^2$$

Total compression = Total tension

$$bn\frac{c}{2}=A_t.t$$

$$b\times0{\cdot}476d\times\frac{50}{2}=A_t\times769$$

$$\therefore\quad\frac{A_t}{bd}=\frac{0{\cdot}476\times25}{769}=0{\cdot}0155.$$

Problem 500 (SI). *The moment of resistance of a rectangular singly reinforced concrete beam section of width b cm. and effective depth d cm. is 100 bd² Newton centimetre. If the stresses in the outside fibres of concrete and steel do not exceed 5 N/mm.² and 140*

N/mm.^{2} respectively and the modular ratio equals 18, determine the ratio of the depth of neutral axis from the extreme compression fibres to the effective depth of the beam and the ratio of the area of the tensile steel to the effective area of the beam.

Solution. For a balanced section with $c=5$ N/mm^2=500 N/cm.^{2}, $t=140$ N/mm.^{2}=14000 N/cm^2 and $m=18$, the M R. of the beam section = $M=85\ bd^2$ N cm.

But, the M.R. of the given beam section is $100\ bd^2$ N cm. Since the M.R of the given beam section is greater than the M.R. of the balanced section, the beam should be *over-reinforced* Hence, concrete attains its maximum stress earlier to steel.

Let n=depth of actual neutral axis

Let $\frac{n}{d}=n_1$

$$\text{M.R.}=bn\frac{c}{2}\left(d-\frac{n}{3}\right)$$

$$=100\ bd^2$$

$$b\ n_1 d\times\frac{500}{2}\left(d-\frac{n_1 d}{3}\right)=100\ bd^2$$

$$\therefore \quad n_1\left(1-\frac{n_1}{3}\right)=0{\cdot}4$$

$$\therefore \quad n_1^2-3n_1+1{\cdot}2=0$$

Solving we get $n_1=0{\cdot}476.$

Stress in steel is given by,

$$\frac{mc}{t}=\frac{n}{d-n}=\frac{n_1 d}{d-n_1 d}=\frac{n_1}{1-n_1}$$

$$t=mc\ \frac{1-n_1}{n_1}$$

$$=18\times500\times\left[\frac{1-0{\cdot}476}{0{\cdot}476}\right]$$

$$=9907\ N/cm.^2$$

Total compression=Total tension

$$bn\frac{c}{2}=A_t\ t$$

$$b\times0{\cdot}476d\times\frac{500}{2}=A_t\times9907$$

$$\therefore \quad \frac{A_t}{bd}=\frac{0{\cdot}476\times250}{9907}$$

$$=0{\cdot}012$$

Problem 501. *The moment of resistance of a singly reinforced rectangular reinforced concrete beam of breadth b cm. and effective* *depth d cm. 6 bd^2 kg. cm. If the stresses in the extreme fibre of concrete and in steel do not exceed 50 kg/cm.^{2} and 1400 kg./cm.^{2} respectively and the modular ratio equals 18, determine the ratio of the depth of neutral axis from the outside compression fibres to the effective depth of the beam and the ratio of the area of the tensile steel to the effective area of the beam.*

Solution For a balanced section, with $c=50$ *kg./cm.*^{2}, $t=1400$ *kg./cm.*^{2} and $m=18$, the moment of resistance of the balanced section

$$=8{\cdot}5\ bd^2$$

But the M.R. of the given beam is only 6 bd^2.

Since the M.R of the given beam is less than the M.R. of the balanced section, the beam section should be *under-reinforced.* Hence steel attains the maximum stress earlier to concrete.

Hence $t=1400\ kg/cm.^2$

Corresponding maximum stress in concrete is given by,

$$\frac{18c}{1400}=\frac{n}{d-n}$$

$$\therefore \quad c=\frac{1400}{18}\cdot\frac{n}{d-n}$$

Let $n=n_1 d$

$$\therefore \quad c=\frac{1400}{18}\cdot\frac{n_1 d}{d-n_1 d}$$

$$\therefore \quad c=\frac{1400}{18}\left(\frac{n_1}{1-n_1}\right)$$

$$\text{M.R.}=bn\frac{c}{2}\left(d-\frac{n}{3}\right)=6\ bd^2$$

$$\therefore\ b\ n_1 d\,\frac{1400}{2\times18}\left(\frac{n_1}{1-n_1}\right)\left(d-\frac{n_1 d}{3}\right)=6\ bd^2$$

$$\frac{n_1^2\left(1-\frac{n_1}{3}\right)}{1-n_1}=\frac{6\times2\times18}{1400}$$

$$\therefore \quad \frac{n_1^2\left(1-\frac{n_1}{3}\right)}{1-n_1}=0{\cdot}154$$

Solving the above equation by trial and error, we get

$$n_1=0{\cdot}34$$

$$\therefore \quad n=0{\cdot}34\ d$$

$$\therefore \quad c=\frac{1400}{18}\cdot\frac{0{\cdot}34}{1-0{\cdot}34}\ kg/cm.^2$$

$$=40\ kg./cm.$$

Total compression =Total tension

$$\therefore \quad bn\frac{c}{2}=A_t\ t$$

$$\therefore \quad b\times 0{\cdot}34d\times\frac{40}{2}=A_t\times 1400$$

$$\therefore \quad \frac{A_t}{bd}=\frac{0{\cdot}34\times 20}{1400}=0{\cdot}00486$$

Problem 502. *A reinforced concrete slab has an overall depth of 10 cm., the effective cover to the centre of the reinforcement being 2 cm. If the stresses in concrete and steel are not to exceed 50 kg /cm.² and 1400 kg /cm.², find the safe uniformly distributed load which can be placed on the slab. The slab is supported on beams spaced at 3 metres centres. Find also the spacing of 10 mm. diameter bars to resist the maximum bending moment. The maximum bending moment for a one metre wide strip of the slab be taken as* $\frac{wl^2}{12}$ *kg. m. where w is the load on the slab in kg. per metre² and l is the spacing of the beams in metres. Take m=18.*

Solution. Corresponding to

$$c=50\ kg./cm.^2$$

$$t=1400\ kg\ /cm.^2$$

and $$m=18$$

For the balanced section,

$$\text{M.R.}=8{\cdot}5\ bd^2$$

and the lever arm $a=0{\cdot}87\ d.$

Hence the moment of resistance offered by the balanced section per *metre* width

$$=8{\cdot}5\times 100\times(10-2)^2\ kg.\ cm.$$
$$=54{,}400\ kg.\ cm.$$

Let the safe distributed load on the slab be w *kg./metre²*.

Maximum bending moment for a 1 *metre* wide strip

$$=\frac{wl^2}{12}=\frac{w\times 3^2}{12}\times 100\ kg.\ cm.$$

Equating the maximum bending moment to the moment of resistance, we get,

$$w\times\frac{3^2}{12}\times 100=54400$$

$$\therefore \quad w=\frac{54400\times 12}{9\times 100}\ kg./metre^2$$

$$=725{\cdot}3\ kg./metre^2.$$

The above load is the total load on the slab including the weight of the slab also,

Dead load of the slab per *square metre*

$$=1\times\frac{10}{100}\times 2400=240\ kg./metre^2$$

$\therefore$ Net external load which the slab can support

$$=725{\cdot}3-240=485{\cdot}3\ kg./metre^2$$

Area of steel $\quad A_t=\dfrac{M}{ta}$

$$=\frac{54400}{1400\times 0{\cdot}87\times 8}\ cm.^2$$

$$=5{\cdot}58\ cm.^2$$

$\therefore$ Spacing of 10 *mm.* diameter bars

$$=\frac{\text{area of 1 bar}\times 100}{\text{Total area of steel per } metre \text{ width}}$$

$$=\frac{0{\cdot}79\times 100}{5{\cdot}58}\ cm.$$

$$=14{\cdot}2\ cm.$$

Hence, let us provide 10 *mm. diameter bars at 14 cm. centres.*

180. Effect of varying the steel ratio on the depth of neutral axis and the moment of resistance

For this discussion let us consider M 150 grade of concrete.

Permissible compressive stress in concrete

$$=50\ kg./cm.^2$$

Permissible tensile stress in steel

$$=1400\ kg./cm.^2$$

Modular ratio $\quad =18$

For a balanced section depth of the neutral axis is given by

$$\frac{mc}{t}=\frac{n}{d-n}$$

$$\therefore\quad \frac{18\times 50}{1400}=\frac{n}{d-n}$$

$$\therefore\quad n=0{\cdot}39\ d$$

Lever arm $\quad =a=d- \quad =d-0{\cdot}39\ d$

$$=0{\cdot}87\ d$$

Moment of resistance $=$ M.R. $=b\ n\ \dfrac{c}{2}\left(d-\dfrac{n}{3}\right)$

$$=b\times 0{\cdot}39\ d\times\frac{50}{2}\times 0{\cdot}87\ d$$

$$=8{\cdot}5\ bd^2$$

$$=Q\,bd^2 \text{ where } Q=8\ 50$$

Total tension $=$ Total compression

$$A_t \times t = bn\frac{c}{2}$$

$$A_t \times 1400 = b \times 0{\cdot}39d \times \frac{50}{2}$$

$\therefore$ Steel ratic $= p = \dfrac{A_t}{bd} = \dfrac{0{\cdot}39 \times 25}{1400}$

$$=0{\cdot}00696$$

$\therefore$ Percentage of steel $= 100\,p = 0{\cdot}696\%$

Now for all values of n less than 0·39 d the corresponding values of the lever arm a, 100 p and the M.R. are tabulated in the following table (page 948). For this range in the value of n the section will be an *under-reinforced section*. The moment of resistance should therefore be estimated corresponding to the maximum tensile stress in steel.

Now let us consider a further range of values of n from 0·39 d to 0 7 d. These sections are *over-reinforced sections* and hence for these section the moment of resistance should be computed corresponding to the condition of maximum stress in concrete. The values of a, 100 p and the moment of resistance for various values of n are tabulated.

It is worthy to note :

(*i*) As the percentage of steel increases the depth of neutral axis n also increases.

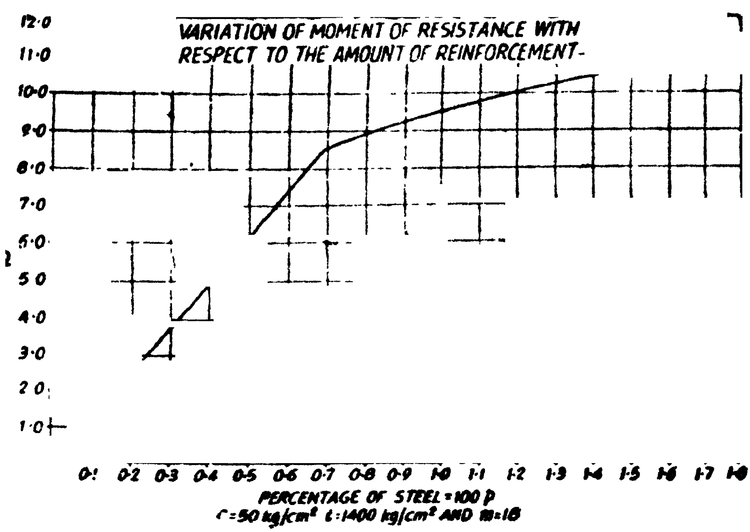


Fig. 828

Properties of Under-reinforced Sections

m = 18 Stress in Steel = 1400 kg./cm.^{2}

n	Lever arm *a*	Percentage of steel 100 *p*	Safe Moment of Resistance $M = p\ bd\ ta$	$Q = \frac{M}{bd^2}$
0·10 *d*	0·967 *d*	0·03	0·41 bd^2	0·41
0·15 *d*	0·950 *d*	0·07	0·93 bd^2	0·93
0·20 *d*	0·933 *d*	0·14	1·83 bd^2	1·83
0·25 *d*	0·917 *d*	0·23	2·96 bd^2	2·96
0·30 *d*	0·900 *d*	0·36	4·54 bd^2	4·54
0·35 *d*	0·883 *d*	0·52	6·43 bd^2	6·43
0·39 *d*	0·870 *d*	0·69	8·50 bd^2	8·50

Properties of Over-reinforced Sections

m = 18. Stress in Concrete = 50 kg./cm.^{2}

n	Lever arm *a*	Percentage of steel 100 *p*	Safe Moment of Resistance $M = bn\frac{c}{2}a$	$Q = \frac{M}{bd^2}$
0·39 *d*	0·870 *d*	0·69	8·50 bd^2	8·50
0·45 *d*	0·850 *d*	1·02	9·56 bd^2	9·56
0·50 *d*	0·833 *d*	1·39	10·41 bd^2	10·41
0·55 *d*	0·817 *d*	1·87	11·23 ba^2	11·23
0·60 *d*	0·800 *d*	2·50	12·00 bd^2	12·00
0·65 *d*	0·783 *d*	3·35	12·72 bd^2	12·72
0·70 *d*	0·767 *d*	4·54	13·42 bd^2	13·42

(*ii*) As the percentage of steel increases the lever arm decreases.

(*iii*) The safe moment of resistance increases rapidly and somewhat uniformly as 100 p (percentage of steel) varies from 0 to 0·69. For further increase in 100 p the safe moment of resistance increases at an *appreciably slower rate*. The critical point at which the change in rate of increase of moment of resistance occurs represents the condition for a balanced section. Hence it is important to note that by providing more steel than the requirement of the balanced section though it is possible to increase the moment of resistance, it is uneconomical due to appreciably slower rate of increase of the moment of resistance. (See Fig. 828).

§181. Doubly Reinforced Beams

Beams reinforced with steel in compression and tension zones are called doubly reinforced beams. This type of beam will be found necessary when due to head room, appearance considerations the size of a beam is limited. The beam with its limited dimensions, if reinforced on the tension side, may not have enough moment of resistance to resist the bending moment. By increasing the steel only on the tension zone the moment of resistance cannot indefinitely be increased. Usually the moment of resistance can be increased by not more than 25% over the balanced moment of resistance, by making the section over-reinforced on the tension side Hence in order to further increase the moment of resistance of a section of limited dimensions, a doubly reinforced beam is provided. Besides this a doubly reinforced beam is also used in the following circumstances :

(*i*) The external live loads may alternate, *i.e.*, may occur on either face of the member.

Ex : A pile which is lifted in such a manner that the tension and compression zones may alternate.

(*ii*) The loading may be eccentric and the eccentricity of the load may change from one side of the axis to another side.

(*iii*) The member may be subjected to a shock or impact or accidental lateral thrust.

Analysis of a doubly reinforced section

Neutral axis. Fig 829 shows a doubly reinforced beam b *cm* wide and d *cm* deep to the centre of tensile reinforcement.

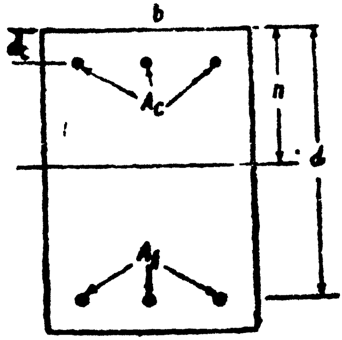


Fig. 829

Let A_c and A_t be the areas of reinforcement in the compression and tension zones.

Let n be the depth of neutral axis.

Equating moments of areas on either side of the neutral axis, about the neutral axis, we have,

$$\frac{bn^2}{2} \times mA_c(n-d_c) - A_c(n-d_c) = mA_t(d-n)$$

$$\therefore \quad \frac{bn^2}{2} + (m-1)A_c(n-d_c) = mA_t(d-n) \qquad ...(1)$$

If the stresses c and t, *i.e.*, the stresses in concrete and tension reinforcement are known, we have

$$\frac{mc}{t} = \frac{n}{d-n} \qquad ...(2)$$

It is very important to note in the case of a singly reinforced beam of chosen dimensions, in order that the actual neutral axis may coincide with the critical neutral axis, there is a certain definite amount of steel required.

But, in a doubly reinforced beam of chosen dimensions, the reinforcements A_c and A_t may be adjusted in an **infinite** *number of ways so that the actual neutral axis and the critical neutral axis may coincide.*

Since total compression = Total tension, we have,

$$C = T$$

$$\therefore \quad bn\frac{c}{2} + (m-1)A_c c' = A_t\, t \qquad ...(3)$$

Stress in compression steel. If c' is the stress in concrete at the level of the compression steel the stress in compression steel $= mc'$

But $\quad c' = \frac{n-d_c}{n} \cdot c$

$\therefore$ Stress in compression steel $= m.$ [illegible]

Moment of resistance

This is computed by taking moments of the compressive forces about the centre of gravity of the tension reinforcement.

$$\therefore \quad M.R. = bn\frac{c}{2}\left(d - \frac{n}{3}\right) + (m-1)A_c\, c'(d-d_c) \qquad ...(4)$$

§182. Types of Problems

The following are the types of problems we come across in the analysis of doubly reinforced sections.

Type 1 Data : Overall section, A_c, A_t and the working stresses c and t.

Required : Moment of resistance.

Solution. The position of the actual neutral axis first determined (*see* Eq. 1). The position of the critical neutral axis is found from Eq (2) with the given values of the working stresses c and t. If the depth of actual neutral axis is greater than the depth of ciritical neutral axis, concrete will attain its maximum stress earlier. Hence the *M.R.* is worked out from Eq. (4) taking c at the given working stress. c should be taken as $\frac{n-d_c}{n}c$.

If the depth of neutral axis is less than the depth of critical axis, the steel in the tension zone reaches its maximum stress earlier. The stress in concrete corresponding to the working stress of t in the tension reinforcement is found from Eq. (2). With this value of c the *M R.* is found from Eq. (4).

The following example illustrates the above type.

Problem 503. *A beam of reinforced concrete is 25 cm wide and 48 cm deep to the centre of tensile steel. It is reinforced with four bars of 20 mm diameter as compressive steel at an effective cover of 5 cm. and with four bars of 26 mm. diameter as tensile steel. If the stresses in concrete and steel are not to exceed 50 kg./cm.² and 1400 kg./cm.² respectively, determine the moment of resistance of the section. The modular ratio = 18.*

Solution $A_c = 12{\cdot}57\ cm.^2$

$A_t = 21{\cdot}24\ cm.^2$

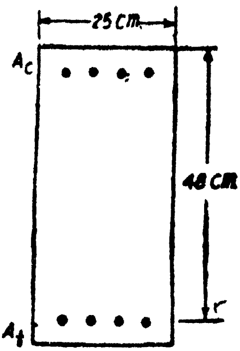


Fig. 830

Depth of actual neutral axis

Taking moments about the neutral axis, we have,

$$25\frac{n^2}{2}+(18-1)12{\cdot}57\,(n-5)$$

$$=18\times 21{\cdot}24\,(48-n)$$

Solving, we get $n=22{\cdot}17\ cm.$

The depth of critical neutral axis is given by

$$\frac{18\times 50}{1400}=\frac{n_c}{48-n_c}$$

$$\therefore \qquad n_c=18{\cdot}78\ cm.$$

Since the depth of actual neutral axis is greater than the depth of critical neutral axis, concrete attains the maximum stress earlier to steel.

$\therefore$ Stress in concrete will be allowed to reach 50 *kg./cm.²*

$\therefore \quad c=50\ kg./cm.^2$

$$\therefore\ c'=\frac{n-d_c}{n}c=\frac{22{\cdot}17-5}{22{\cdot}17}\times 50=38{\cdot}73\ kg./cm.^2$$

$$\therefore \quad M\,R.=25\times 22{\cdot}17\times\frac{50}{2}\left[48-\frac{22{\cdot}17}{3}\right]+(18-1)12{\cdot}57\times 38{\cdot}73\,[48-5]\ kg.\ cm.$$

$$=562{,}600+355{,}800=918{,}400\ kg.\ cm.$$

Type 2 Data : Overall section, A_c and A_t, Max. *B.M.*

Required : Stresses c and t in concrete and tensile steel and also in compression steel.

Solution. The position of the actual neutral axis is first determined from Eq. (1). Express the moment of resistance in terms of the stress c in concrete and equate to the given bending moment. From this eqation the stress c in concrete is determined.

The stress in compression steel $= mc' = \frac{(n-d_c)}{n} c.$

The stress in tension steel is given by the relation

$$\frac{mc}{t} = \frac{n}{d-n}$$

The following example illustrates the above type.

Problem 504. *A doubly reinforced concrete beam is 25 cm. wide and 50 cm. deep to the centre of tension reinforcement. The areas of the compression and tension steel are 12·9 cm.² each. The centre of compression steel is 5 cm. from the compression edge. If $m=18$ and the bending moment at the section is 700,000 kg. cm.. calculate the stresses in concrete and steel.*

Solution. *See* Fig. 831

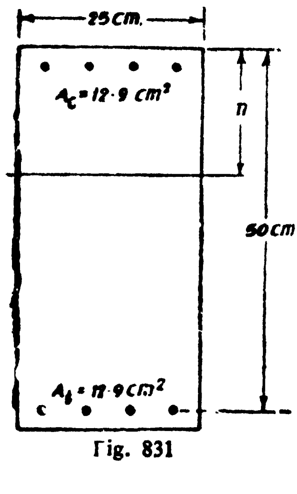


Fig. 831

Taking moments about the neutral axis, we have,

$$\frac{25n^2}{2} + (18-1)12{\cdot}9(n-5)$$
$$= 18 \times 12{\cdot}9(50-n)$$

Solving, we get $n = 16{\cdot}40$ *cm.*

Let the maximum stress in concrete be c *kg./cm.²*

$\therefore$ Stress in concrete at the level of the compression steel

$$= c' = \frac{n-d_c}{n} c.$$
$$= \frac{16{\cdot}40-5}{16{\cdot}4} c.$$
$$= 0{\cdot}6952\, c$$

$$M.R. = bn\frac{c}{2}\left(d-\frac{n}{3}\right) + (m-1)A_c c' \;(d-d_c)$$

$$= 25 \times 16{\cdot}4 \frac{c}{2}\left[50 - \frac{16{\cdot}40}{3}\right] + (18-1)12{\cdot}0 \times 0{\cdot}6952c$$
$$(50-5) = 700{,}000 \; kg.\ cm.$$

$\therefore$ $9128\, c + 6860\, c = 700{,}000$ *kg. cm.*

$\therefore$ $15988\, c = 700{,}000$ *kg. cm.*

$\therefore$ $c = 43{\cdot}77$ *kg. cm.²*

Stress in compression steel $= 18 \times 30{\cdot}43 = 517{\cdot}74$ *kg./cm.*²

Stress in tension steel $= \left(\frac{d-n}{n}\right) mc$

$$= \frac{(70 - 16{\cdot}4)}{16{\cdot}4} \times 18 \times 43{\cdot}77 \; kg./cm.^2$$

$$= 1614 \; kg./cm.^2$$

Type 3. Data : Overall section, the maximum bending moment, safe stresses c and t in concrete and steel

Required : A_c and A_t.

Solution. With the working stresses c and t determine the depth of critical neutral axis. Find the *M.R.* in terms of the stress in concrete and equate to the given bending moment. This equation will involve only A_c as the unknown which can be determined.

By equating total compression to total tension, find A_t. The following example illustrates this type.

Problem 505. *A rectangular beam reinforced on both sides is 30 cm wide and 75 cm. deep. The centres of steel are 5 cm. from the respective edges. If the limiting stresses in concrete and steel are 50 kg./cm.² and 1400 kg./cm.² respectively, determine the steel area for a bending moment of 1,400,000 kg. cm. Take m = 18.*

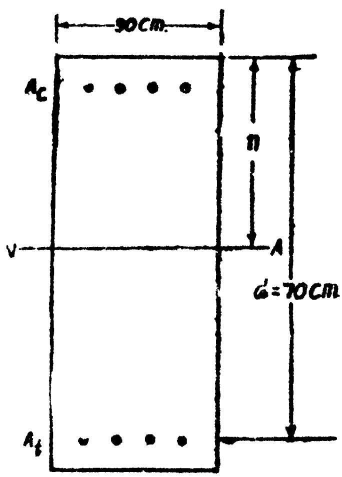


Fig. 832

Solution. The section will be designed as a balanced section.

The depth of critical neutral axis is given by,

$$\frac{18 \times 50}{1400} = \frac{n_c}{70 - n_c}$$

$\therefore \quad n_c = 27{\cdot}39$ *cm.*

Stress in concrete $= c = 50$ *kg./cm*².

Stress in concrete at the level of compression steel

$$= c' = \frac{\quad}{27{\cdot}39} \times 50 \; kg./cm.^2$$

$$= 40{\cdot}87 \; kg./cm.^2$$

$$\therefore \quad M.R. = 30 \times 27{\cdot}39 \times \frac{50}{2} \left[70 - \frac{27{\cdot}39}{3}\right] + (18-1)\, A_c \times 40{\cdot}87 (70-5) = 1{,}400{,}000 \; kg.\ cm.$$

$\therefore \quad 1{,}250{,}000 + 45160 \; A_c = 1{,}400{,}000$

$$\therefore \quad A_c = 3{\cdot}32 \; cm.^2$$

Total compression = Total tension

$$30\times 27.39\times \frac{50}{2}+(18-1)\times 3.32\times 40.87=A_t\times 1400$$

$$20540+2307=1400\,A_t$$

$$\therefore\qquad A_t=16.32\ cm.^2$$

§183. The steel beam theory

This is a method of designing doubly reinforced beams. The theory assumes the following :

(*i*) compression is resisted only by compression steel ;

(*ii*) tension is resisted only by tension steel ;

(*iii*) stress in compression steel = stress in tension steel ;

(*iv*) concrete serves only as a web of an I beam whose flanges are represented by the compression and tension reinforcement.

In this method areas of compression and tension steel are equal, *i.e.*, $A_c=A_t$

The *M.R.* of the beam is, therefore, given by,

$$A_c\,t(d-d_c)\ \text{or}\ A_t\,t(d-d_c)$$

$$\therefore\quad A_c=A_t=\frac{M}{t(d-d_c)}\qquad \ldots(4)$$

A stress of $t=1400\ kg/cm.^2$ may be assumed in determining the reinforcement.

Problem 506. *A rectangular beam reinforced on both sides is 30 cm. wide and 75 cm. deep. The centres of steel are 5 cm. from the respective edges. Find the reinforcement required by the steel beam theory, for a bending moment of 1,400,000 kg. cm.*

Solution. Distance between compression and tension reinforcement

$$=75-5-5=65\ cm$$

Allowing, $t=1400\ kg./cm.^2$

$$A_c=A_t=\frac{1,400,000}{1400\times 65}\ cm.^2=15.4\ cm.^2$$

§184. Shear stresses in homogeneous sections

Fig. 833 shows a simply supported beam subjected to concentrated load W at the centre.

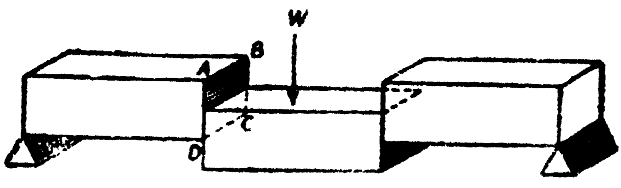


Fig. 833

We find that the shear force at the section $ABCD = \frac{W}{2}$.

Hence, the cross-section $ABCD$ has to offer an equal and opposite resistance. If really the intensity of the shear resistance is uniform, the shear stresses at the section would have been $\frac{W/2}{\text{area } ABCD}$

But, actually, the intensity of shear resistance is not uniform.

Now consider the beam shown in Fig. 834 subjected to a load system. Consider two sections 1-1 and 2-2, dx apart. Let the bending moments at sections 1-1 and 2-2 be M and $(M+dM)$ respectively.

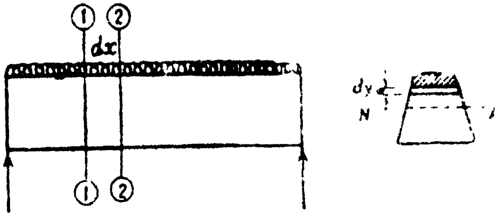


Fig. 834

Now consider an elemental part of the beam, of width b and thickness dy and length dx, situated at a height of y from the neutral layer. This part is separately shown in Fig. 835.

Let I be the moment of inertia of the cross-section of the beam about the neutral axis.

Bending stress on the elemental part at section (1-1)

$$=f=\frac{M}{I}y$$

Bending stresses on the elemental part at section (2-2)

$$=f+df=\frac{M+dM}{I}y$$

∴ Net force at the elemental part

$=df\times$ area of cross-section of the elemental part

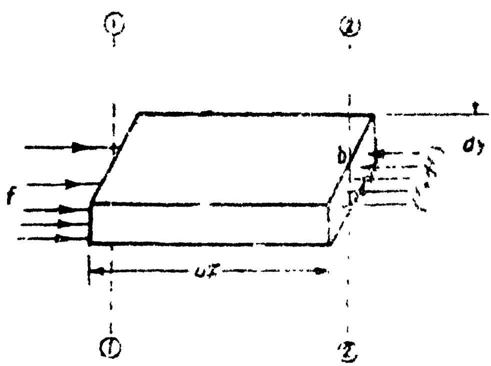


Fig. 835

$$=df.\ bdy=\frac{dM}{I}y.\ bdy$$

∴ Total force on the part of the beam of length dx and of area as shown shaded (Fig. 835)

$$=\int_{y}^{y_c} \frac{dM}{I} b.\ ydy = \frac{dM}{I}\int_{y}^{y_c} bdy.\ y$$

$$=\frac{dM}{I} \times \text{moment of shaded area about the neutral axis}$$

$$=\frac{dM}{I}.\ A\bar{y}$$

where A = area shaded

$\bar{y}$ = distance of C.G. of shaded area from the N A.

But this net force should be balanced by horizontal shear.

Let q be the intensity of horizontal shear stress.

∴ Horizontal shear resistance = force on the part of the beam of length dx and of area shown shaded

$$b\ dx\ .\ q = \frac{dM}{I}\ A\bar{y}$$

$$q = \frac{dM}{dx} . \frac{A\bar{y}}{Ib}$$

But $$\frac{dM}{dx} = \text{S.F. at the section} = S$$

∴ $$q = \frac{SA\bar{y}}{Ib} \quad ...(i)$$

This is the intensity of shear stress in a horizontal direction. But, this also represents the intensity of shear stress in a vertical direction (by the principle of complementary shear).

§185. Shear distribution in a beam of rectangular section

Consider a beam b *cm.* by d *cm.* Let the shear force at the section be S. The shear stress q at a point y *cm.* above the neutral axis is given by

$$q = \frac{S}{Ib} \times \text{ moment of area}$$

above the point considered about the N.A.

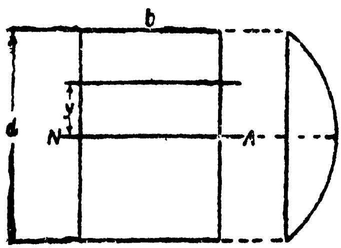


Fig. 836

$$= \frac{S}{Ib} . b\left(\frac{d}{2} - y\right)\left(\frac{\frac{d}{2}+y}{2}\right)$$

$$q = \frac{S}{2I}\left[\frac{d^2}{4} - y^2\right] \quad ...(ii)$$

Hence, the distribution of shear intensity follows a parabolic law. This is shown in Fig. 836.

We find that the shear stress

$$=0 \text{ at } y=\frac{d}{2}$$

and the shear stress is maximum at

$$y=0$$

$$\therefore \qquad q_{max}=\frac{S}{2I}\cdot\frac{d^2}{4}=\frac{Sd^2}{8I}$$

Since $\qquad I=\frac{bd^3}{12}$

$$q_{max}=\frac{Sd^2}{8}\frac{12}{bd^3}=\frac{3}{2}\cdot\frac{S}{bd}$$

But $\qquad \frac{S}{bd}=$ average shear stress.

$$\therefore \qquad q_{max}=\frac{3}{2}q_{average}$$

§186. Shear stresses in an R.C. beam

In the case of reinforced concrete beam in which it is assumed that tensile stresses are not resisted by concrete, the distribution of shear stresses will not exactly follow the law of equations (*i*) and (*ii*). However, for the compression zone of the section, the shear distribution follows the law of equations (*i*) and (*ii*). The shear stress in concrete in the tension zone is constant, as shown in Fig 837.

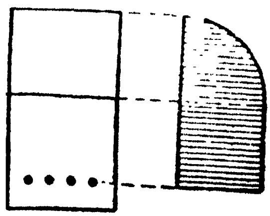


Fig. 837

The shear strsss in concrete in the tension zone therefore is the maximum shear stress. This can be determined as follows :

Consider the concrete beam subjected to a bending. Consider two sections 1-1 and 2-2 dx apart. Let the cross-section of the beam be b *cm.* and d *cm.* effective depth.

Let the bending moments at sections 1-1 and 2-2 be M and $(M+dM)$ respectively. Let the lever arm

$$=a=d-\frac{n}{3}$$

Tension in the reinforcement at section 1-1

$$=T=\frac{M}{a}$$

Tension in the reinforcement at section 2-2

$$=T+dT$$

$$=\frac{M+dM}{a}$$

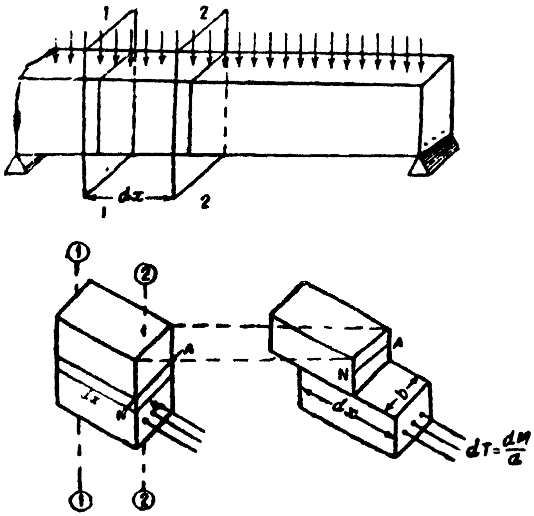


Fig. 838

$\therefore$ Net force on the reinforcements tending them to move

$$= dT = \frac{dM}{a}$$

If the reinforcement be firmly bonded with concrete so that the reinforcement will not slip out of concrete, then the net force in the reinforcement will induce shear stress equal to

$$\frac{dT}{\text{Horizontal shear area of beam between section 1-1 and 2-2}}$$

$$\therefore \quad q = \frac{dT}{bdx} = \frac{dM}{a} \cdot \frac{1}{bdx} = \frac{1}{ab} \cdot \frac{dM}{dx}$$

$$= \frac{S}{ab} \text{ since the S.F.} = S = \frac{dM}{dx}$$

$$\therefore \quad q = \frac{S}{ab} \quad \ldots(iii)$$

This is the horizontal shear stresses in concrete which will also be equal to vertical shear stress intensity.

§187. Effect of shear stresses

Consider a rectangular block *ABCD* acted upon by shear stresses q as shown in Fig. 839. The effect of the shear stresses is to deform the block to the shape shown in Fig. 839 (*c*). We find that compressive stresses are developed on the diagonal plane *AC* and tensile stresses are developed on the diagonal plane *BD*. It can be easily shown that the intensity of diagonal compression or diagonal tension is also equal to the shear stress q applied on the block. If the material of the block is weak in tension, the failure will occur along the diagonal BD and the block will be split up into two along this plane. If the material is weak in compression, then failure can occur by crushing along the plane AC.

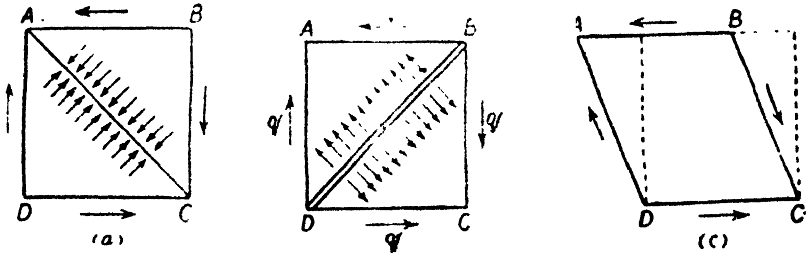


Fig. 839

Concrete is a material weak in tension, and strong in compression. It is at least ten times as strong in compression as in tension. Hence if a concrete block is subjected to shear stress failure may result by diagonal tension.

The I.S specification has specified that the safe diagonal tensile stress for concrete is 5 $kg./cm.^2$ for M 150 grade concrete. Hence if the shear stress which is also equal to the diagonal tensile stress is less than 5 $kg./cm.^2$, the concrete block is safe. If the shear stress exceeds 5 $kg./cm.^2$ then the block requires to be strengthened by diagonal or vertical reinforcement. See Figs. 840 (*a*) and (*b*).

Further the design will not be considered safe even with such reinforcements if the shear stress exceeds four times the allowable shear stress, *i.e.*, if the shear stress exceeds $4 \times 5 = 20$ $kg./cm.^2$, the size of the block should be suitably increased so that the shear stress does not exceed 20 $kg./cm^2$.

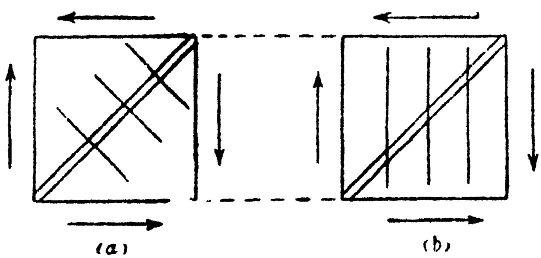


Fig. 840

Hence summarising the above, we have,

when $q < 5$ $kg./km.^2$: no shear reinforcement is required.

$q > 5$ $kg./cm.^2$ and $q < 20$ $kg./cm.^2$: shear reinforcement is provided to resist the diagonal tension.

$q > 20$ $kg./cm.^2$: size of the block is to be changed so that the shear stress will not exceed 20 $kg./cm^2$.

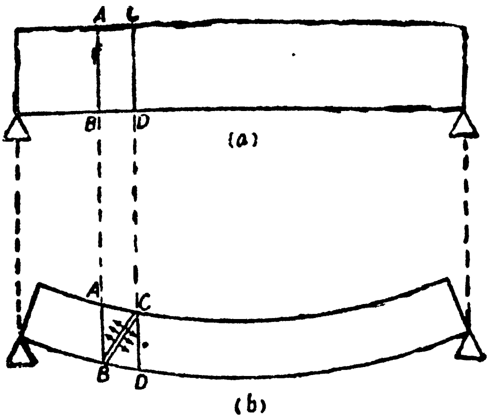


Fig. 841

Figs. 841 (*a*) and (*b*) show how such diagonal tensile stresses are developed in a beam. Suppose a failure by diagonal tension occurs so that the part *ABCD* is split up into the parts *ABC* and *DCB*. In order to safeguard the structure against such failure by diagonal tension, reinforcement connecting the two parts *ACB* and *DCB* are required. These reinforcements are called shear reinforcements. As already suggested, [in Fig. 842 (*a*) and (*b*)] the reinforcement may be provided vertically or diagonally.

When the shear reinforcements are provided vertically they are called stirrups. These consist of bars of 6 *mm*. to 10 *mm*. diameter bent round the tensile steel, as shown in Fig. 842 (*a*).

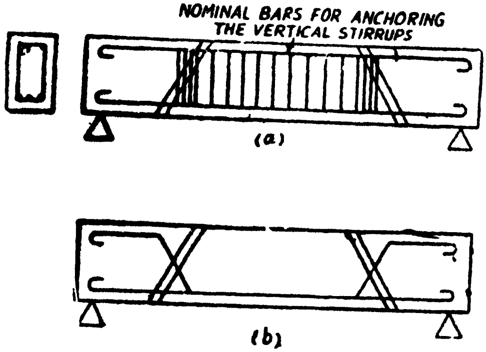


Fig. 842

It is also necessary to provide small diameter bars of 10 *mm*. to 12 *mm*. in the compression zone of the beam in order to properly anchor the stirrups. In Fig. 842 (*a*) each stirrup consists of two legs as shown in the cross-section. In such a case we say the stirrups are two-legged. In some cases in order to

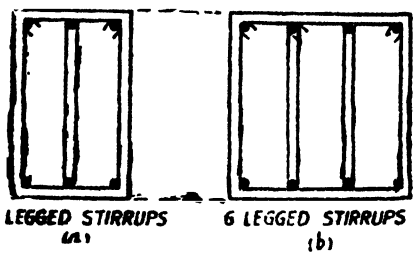


Fig. 843

resist greater shear stresses it becomes necessary to provide several legged stirrups (four-legged, six-legged, *etc.*). These are shown in Figs. 843 (*a*) and (*b*).

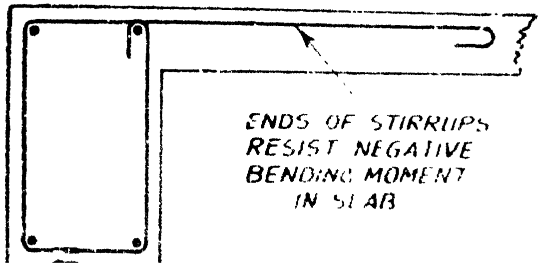


Fig. 844. Two-Legged Stirrups in *L* Beam.

Design of vertical stirrups

Let us assume that the concrete has failed by diagonal tension so that the stirrups which are all crossing the crack are subjected to a load equal to the maximum shear force *S*, which is the same as the end reaction. See Fig. 845. Let us also assume that the cracked plane is inclined at 45° to the centre line of the beam and that it extends over a horizontal distance equal to the lever arm *a* of the beam.

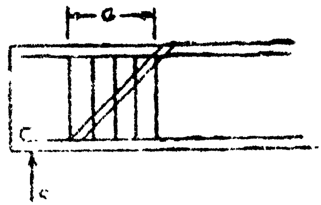


Fig. 845

Let A_w = area of one stirrup

t_w = allowable stress in tension in one stirrup

= 1400 kg/cm^2.

p = pitch of stirrups

Number of stirrups in the horizontal distance $a = \frac{a}{p}$

$\therefore$ Total load on the stirrups $= \frac{a}{p} \cdot A_w \cdot t_w = S$

$$p = A_w t_w a \qquad \ldots(iv)$$

A_w should be determined taking into account the number of legs provided in each stirrup

Since the shear force decreases from the ends towards the centre, the spacing of these stirrups will be close at the ends and can be increased towards the centre. As per our code of practice, the spacing of these stirrups shall not exceed the lever arm distance of the beam

Strictly at sections where the shear stress $q = \frac{S}{ab}$ is less than the allowable limit of 5 $kg./cm.^2$, shear reinforcement is not theoretically required. But it is a practice to always provide stirrups even if the induced shear stress is less than 5 $kg./cm^2$.

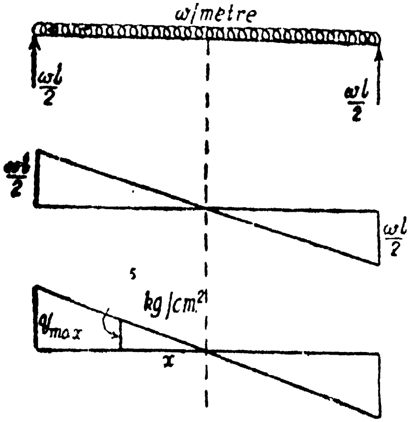


Fig. 846

Fig. 846 shows a simply supported beam carrying a distributed load of w $kg./metre$. Let the span of the beam be l $metres$.

We have

$$q_{max}=\frac{S}{ab}=\frac{\frac{wl}{2}}{ab}$$

This occurs at the end.

Let the section where the shear stress is 5 $kg./cm.^2$ be at a distance of x $metres$ from the centre.

$$\therefore \frac{x}{l/2}=\frac{5}{q_{max}}$$

From the above relation, x can be determined.

Hence for a distance of x on each side of the middle point, shear reinforcement is not theoretically required, but practically provided to a nominal extent at a spacing not exceeding the lever arm distance of the beam.

For the distance of $\left(\frac{l}{2}\quad x\right)$ from each end stirrups are provided in accordance with the requirement of Eq. (*iv*). The distance $\left(\frac{l}{2}-x\right)$ from each end may be divided into a number of sections and the shear forces at these sections can be determined. The pitch of stirrups corresponding to these sections can now be calculated. This will assist in deciding about changing the pitch at suitable sections.

Problem 507. *A beam 25 cm. wide, 50 cm. effective depth and 6 metres span supports a total load of 19050 kg. including its weight. Find the maximum shear stress and determine the spacing of 10 mm. stirrups.*

Solution. Assume $c=50$ $kg./cm.^2$ and $t=1400$ $kg./cm.^2$ $m=18$

$$a=0{\cdot}87\,d$$

$$S=\frac{19050}{50}=9525\ kg.$$

$$q=\frac{S}{ab}=\frac{9525}{(0{\cdot}87\times 50\times 25)}=8{\cdot}75\ kg./cm.^2$$

This is greater than 5 $kg./cm.^2$

∴ Shear reinforcement is necessary. Further the shear stress is less than $4 \times 5 = 20$ *kg./cm²*. Hence, the dimensions of the beam need not be changed.

Suppose two-legged 10 *mm.* bars are suggested for stirrups, their spacing will be

$$p = \frac{A_w t_w a}{S}$$

$$= \frac{2 \times 0.79 \times 1400 \times 0.87 \times 50}{9525} = 10.3 \text{ cm.}$$

Let us, therefore, suggest 10 *mm.* diameter two-legged stirrups at 10 *cm.* centres.

Let the point where the shear stress equals the allowable safe shear stress of 5 *kg./cm²* be *x metres* from the centre.

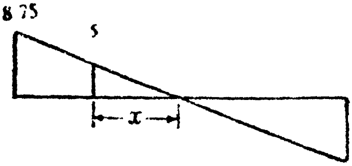


Fig. 847

$$\therefore \quad \frac{x}{3} = \frac{5}{8.75}$$

$$\therefore \quad x = \frac{15}{8.75} = 1.71 \text{ metres}$$

Hence shear reinforcement is needed for $3 - 1.71 = 1.29$ *metres* say 1.30 *metres* from the support.

For the middle 3.4 *metres* nominal stirrups at 25 *cm.* centres may be provided.

§188. **Inclined or diagonal reinforcement.** When inclined reinforcements are provided, they consist of main tension reinforcement bent up at a certain angle as shown in Fig. 848 Bars are usually bent at 45° with the horizontal Bars can be bent up only if the remaining bars are sufficient to resist the prevailing bending moments.

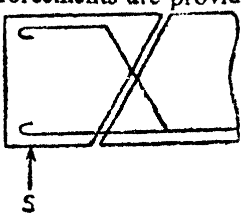


Fig. 848

Suppose A_w is the area of reinforcement bent up. If t_w is the stress in these bars the vertical component of the tension in these bars $= A_w t_w \sin 45° = 0.707\, A_w t_w$.

Usually t_w is taken at 1400 *kg./cm²*.

If this quantity exceeds the maximum shear force *S*, then the shear is safely resisted by the bent up bars. Often with the bars which can be spared to be bent up the shear value of $0.707\, A_w t_w$ may be less than the maximum shear force. In such cases for the balance shear force of $(S - 0.707\, A_w t_w)$ vertical stirrups should be provided. However, vertical stirrups are always provided with a spacing not exceeding the lever arm distance of the beam.

§189. **Lattice girder effect.** (*a*) **Single system.** It is not enough if bars are bent up just near the ends to resist the shear. In order that the beam is safe against shear failure, shear reinforcement is provided throughout the length whether with diagonal steel or with vertical steel.

When a number of bars are available to be bent up to resist shear, it is usual to assume that the beam is equivalent to a truss consisting of concrete compression members and steel tension members as shown in Fig. 849.

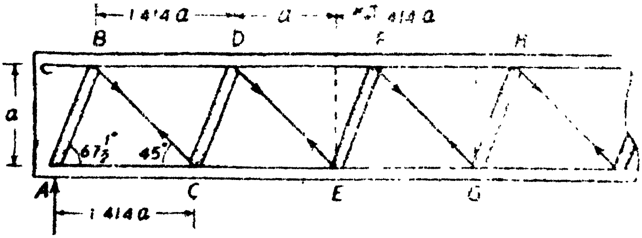


Fig. 849

The usual arrangement is to assume the first imaginary compression member AB at $67\frac{1}{2}°$ with the horizontal. The inclined tension members are at 45° with the horizontal.

With such a system, the shear resistance at any section $= A_{v} t_{v} \sin 45° = 0.707\, A_{v} t_{v}$

The tension members of this imaginary truss are provided by the bent up bars. Bending up bars are possible only when a good number of bars are present at the bottom and a number of them are no longer required to resist the bending moment. The arrangement shown in Fig. 850 is called a single system of bent up bars. The height of the imaginary truss is equal to the lever arm distance a.

From the geometry of the triangle ABC of the truss, we have

$$AC = CB = a\sqrt{2} = 1.414\, a$$

i.e., the bars bent up at C are bent from a point at a distance of $1.414\, a$ from the support.

If the inclination of the bars be at any angle θ with the horizontal, the arrangement would be as shown in Fig. 850.

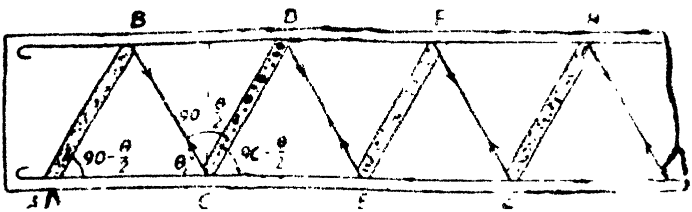


Fig. 850

It is worth noting that if the bent up bar say CB is at θ with the horizontal, the compression member CD should be bisecting the angle $B\hat{C}E$ so that $B\hat{C}D = D\hat{C}E = \frac{180-\theta}{2} = 90 - \frac{\theta}{2}$. If this arragement is not done, *i.e.*, if the compression members are not taken at

$90 - \frac{\theta}{2}$ with the horizontal, we find the design of shear reinforcement becomes uneconomical.

Sometimes the imaginary compression members are also taken at 45° with the horizontal while the bars are also bent up at 45° as shown in Fig. 851.

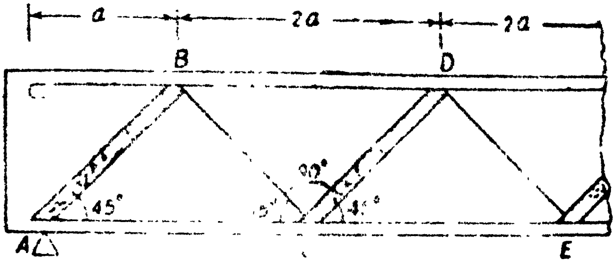


Fig. 851

In such cases while [illegible]lculating the shear resistance at any section by the expression 0·7[illegible] $A_s t_s$, care should be taken to assume a value of t_s such th[illegible] exce[illegible] [illegible]e tensile stresses are not developed in the horizontal part of the be[illegible]p bars.

A value of t_s = 0·707 [illegible]owable tensile stress in steel should be used while calculati[illegible] the she[illegible] resistance. Suppose the allowable tensile stress is 1400 $kg./cm^2$ [illegible]he value of t_s in this case should be taken at 0·707 × 1400 = 990 kg/cm^2.

(*b*) **Double system.** If in addition to the bars bent up as shown in Fig. 849, additional bars are also bent up as shown in Fig. 852, the arrangement is called a double system arrangement.

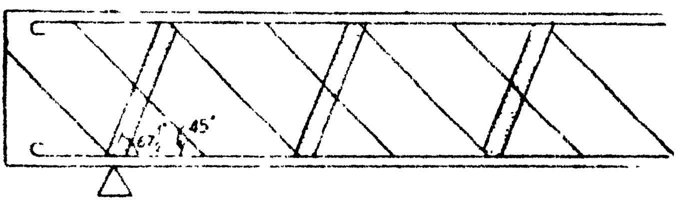


Fig. 852

In this case the shear resistance at any section $= 2 \times \cdot 707\ A_s t_s$.

§190. **Limiting the shear stress.** It was stated in an earlier paragraph that the shear stress should not be allowed to exceed four times the safe shear stress of concrete. Hence if the safe shear stress of concrete is 5 $kg./cm^2$ the shear stress in no case shall be allowed to exceed $4 \times 5 = 20\ kg/cm^2$. It may be felt as to why there should be an upper limit to the shear stress. It may also appear that after all, if the shear stress has exceeded the safe stress of 5 $kg./cm.^2$, shear reinforcement of proper amount may be provided. But it should not be forgotten that there should not also be any failure by diagonal compreisson. Suppose the shear stress is so large that the diagonal compression stress can just reach its working stress. Corresponding to this, the diagonal tension would require such large amount of steel that concreting will become difficult which will result in air pockets. The concrete in such a case would consist of a number of disjointed concrete pieces separated by steel bars.

§191. Bond

One of the main assumptions in developing the theory of reinforced concrete is that the reinforcements do not slip from the concrete surrounding it. When concrete sets and thus hardens, it will firmly grip round the reinforcements. It is because of this grip between concrete and steel, the two materials share the applied loads. Once this grip is absent the reinforcement provided would serve no purpose.

Suppose the reinforcement of the beam shown in Fig. 853 has lost the grip with concrete.

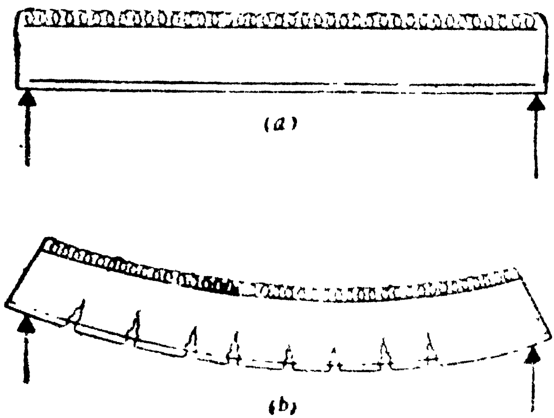


Fig. 853

Then it is just as good as loosely providing a reinforcement in ›les present earlier. When the beam is loaded the steel rods slip nd the beam fails since the reinforcement has not really shared the oading. Hence this grip between the concrete and steel is very mportant and the gripping stress, hereafter called the bond stress, should therefore be within a limit.

It may be realised that the function of this bond in reinforced concrete members is exactly the same as the function of rivets in built up plate girders which consist of web plates and flange angles nd cover plates. If for instance the flange angles are not properly iveted to the web plate the flange angles will not function. If ne number of rivets connecting the flange angles and the web is nsufficient, these rivets will fail and immediately the flange becomes separated from the web. In almost the same manner when the induced bond stresses are very large the bars get separated from the concrete or let us say the bond between the concrete and the steel is broken and this will result in the load of the beam to be resisted only by the concrete. The beam will thus fail by tension even though sufficient amount of steel from bending moment considerations has been provided. Bond stresses often are not receiving that much good attention which they deserve. While designing particularly, beams subjected to heavy loads and footings, bond stresses shall always be determined and shall be compared with the allowable bond stress.

§192. Direct bond

In Fig. 854 (*a*) is shown a bar embedded in concerete only for a short length ; while in Fig. 854 (*b*) is shown a bar embedded for a

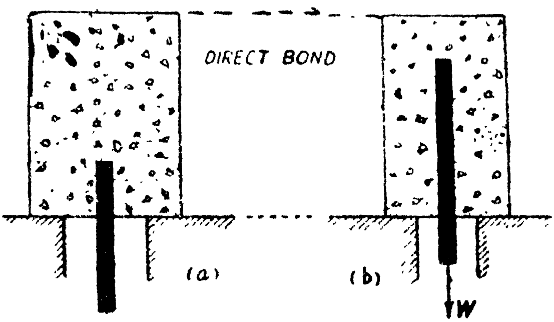


Fig 854

greater length. It is obvious that assuming the bars not to fail, it requires a greater load to pull out the bar in the arrangement of Fig. 854 (*b*) than what is required to pull out the bar in the arrangement of Fig 854 (*a*)

The intensity of bond stress

$$= \frac{\text{Load in the bars}}{\text{Contact area of the bar with concrete}}$$

Let l be the length of embedment

Bond stress $= S_b = \dfrac{W}{\pi\, dl}$

where d = diameter of bar

S_b is called the *average bond stress.* Let the bar be subjected to its maximum stress t.

$$\therefore \quad W = \frac{\pi d^2}{4}\, t.$$

$$\therefore \quad S_b = \frac{\frac{\pi d^2}{4}\, t}{\pi dl} = \frac{dt}{4l}$$

$\therefore$ Length of embedment $= l = \dfrac{dt}{4S_b}$

The I.S. specification allows a value of $S_b = 6$ $kg./cm.^2$ for M 150 grade concrete.

If $\quad t = 1400$ $kg./cm.^2$

and $\quad S_b = 6$ $kg./cm.^2$

we have $\quad l = \dfrac{d \times 1400}{4 \times 6}$

$\therefore \quad l = 58\, d$ ...(1)

The practice is thus to provide a bond length of 58 diameters.

The following are the factors that give the property of a good bond between concrete and reinforcement.

(*a*) Sufficient cover for reinforcement.

(*b*) Richness of concrete.

(*c*) Using twisted bars, welding the stirrup bars with the main bars.

(*d*) Roughness of steel.

§193. Local bond

Consider an R.C. beam subjected to a certain loading. Consider two sections 1-1 and 2-2 distant dx apart. See Fig. 855. Let the bending moments at sections 1-1 and 2-2 be M and $(M+dM)$ respectively. Let the lever arm distance be a.

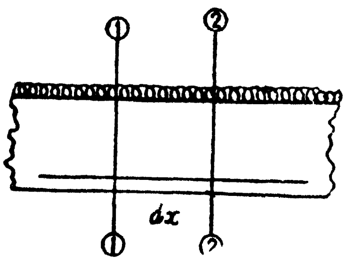


Fig. 855

Tension in steel reinforcement at section 1-1 $= \dfrac{M}{d}$ and Tension in steel reinforcement at section 2-2 $= \dfrac{M+dM}{a}$

$\therefore$ Net force in the tension reinforcement in a length $dx = \dfrac{dM}{a}$.

For, this length of reinforcement, the bond force between the steel and the concrete $= (\Sigma 0)\ dx\ S_b$ where $(\Sigma 0)$ is the total perimeter of reinforcement. Equating the force of bond to the net force on the tensile reinforcement, we have, $(\Sigma 0)\ dx.\ S_b = \dfrac{dM}{a}$

$$\therefore \quad S_b = \frac{dM}{dx} \cdot \frac{1}{a(\Sigma o)}$$

But $\quad \dfrac{dM}{dx} = \text{S.F.} = S.$

$$\therefore \quad S_b = \frac{S}{a\Sigma o} \quad ...(2)$$

This bond stress induced due to the rate of change of bending moment is called the *local bond stress.*

I.S. specification has recommended a safe local bond stress of $10\ kg./cm.^2$ for *M* 150 mix concrete.

In designs the Eq. (2) is in the form $\Sigma 0 = \dfrac{S}{aS_b}$ ...(3)

From this, the perimeter of bars required to limit the bond stress to $10\ kg./cm.^2$ can be computed.

Hence reinforcement required from bond considerations must be provided though from bending moment considerations no reinforcement may be required.

Eq. (3) shows that the required perimeter of reinforcement is proportional to the shear force. For a simply supported beam, for instance, the bending moment at the support is zero and the shear force is maximum.

Hence some bars are to be maintained in the bottom such that the perimeter of bars provided is at least $= \dfrac{S}{aS_b}$. The remaining bars may be curtailed at such places where they are no longer required for bending moment. These bars may also be bent up at suitable places to serve as shear reinforcement.

§194. End anchorage

Bars embedded in concrete are sometimes hooked so as to have proper anchorage with concrete. If bars are provided with hooks, the necessary grip or bond length can be reduced. A hook shall always confirm to the specifications shown in Fig. 856. The anchorage value of the hook alone is considered as 16*d* where *d* is [illegible]meter of the bar.

Fig. 856. A standard hook.

In this connection [illegible] specification has stated the following :

Anchorage Value of B[illegible] [illegible] in a reinforcing bar may be assumed to [illegible] equivalent to that of the length of the bar [illegible] diameter of the round bar for each 45° th[illegible], pr[illegible]

(*a*) The radius of the [illegible] be not less than twice the diameter of the round bar

(*b*) The length of the straight part of the bar beyond the end of the curve be at least four times the diameter of the round bar ; and

(*c*) Whatever the angle through which the bar is bent, the assumed anchorage value should not be taken as more than equivalent to a length of bar equal to sixteen times the diameters of the round bar.

Bars in tension. In the tensile reinforcement the length measured from any section to the end of the bar plus the equivalent anchorage value of hook shall be such that the average bond stress induced to develop the actual stress at the section shall not exceed the permissible average bond stress. In the case of the tensile reinforcement of circular section, the length measured from such section up to the beginning of the hook shall at least be equal to n times the diameters of the bar minus the anchorage value to the hook.

where $$n = \frac{\text{Actual tensile stress in the bar}}{\text{Four times the permissible average bond stress}}$$

In no case shall such value of n be less than 12.

Bars in compression In this case, the length measured from any section to the end of the bar shall be such that the average bond stress induced to develop the actual stress at the section shall not exceed 1·25 times the permissible average bond stress. In the case of bars of circular section the length measured from such section shall be at least equal to n times the diameter of the bar where

$$n = \frac{\text{Actual compressive stress in the bar}}{\text{Five times the permissible average bond stress}}$$

In no case the value of n be less than 12. Hooks are unnecessary. But when a hook is provided it shall not be accounted for anchorage purposes.

§195. Reinforcement

Reinforcement used shall conform to the requirements of *I.S. 432 specification for Mild steel and high tensile steel bars and hard drawn steel wire for concrete reinforcement (Revised).* The reinforcement shall be free from loose mill scale, loose rust, oil and grease or any such harmful matter, immediately before placing the concrete. The reinforcement shall be placed and positioned strictly following the requirements shown in the structural drawings.

196. Bending Bars Bending bars shall be done with great caution. Often this job does not receive that much attention which it deserves. Bending bars are expected to fulfil certain definite functions and hence bars must be bent so that there is good advantage of the worked out design. There are instances of failures of structures for want of correct bending, even though the designs worked out are not faulty.

§197. The necessity of bending reinforcement. Bars are bent under different circumstances. They may be bent to form hooks so as to develop proper anchorage. Sometimes bars have to be bent so as to form loops as in the case of stirrups as shear reinforcement. Bars may also be bent to resist diagonal tension They may also be bent up to form necessary rienforcement for hogging bending moments. The following are the types of bend we normally come across :

(*a*) Hooks at the end of bars in beams.
(*b*) Bars bent up at ends and hooked in beams for resisting diagonal tension.
(*c*) Bars which serve for positive bending moment which are bent up to resist negative bending moment.
(*d*) Bars bent to form loops to serve as shear reinforcement.

These are shown in Fig. 857.

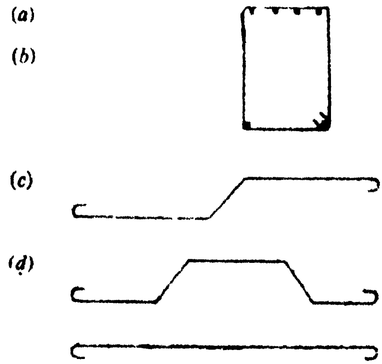


Fig. 857

The I.S. code has further recommended the following :

Splices in tensile reinforcement. Splices at point of maximum tensile stress shall be avoided wherever possible ; splices where used shall be welded, lapped or otherwise fully developed. In any case the splice shall transfer the entire computed stress from bar to bar. Lapped splices in tension shall not be used for bars of sizes larger than 36 *mm*. diameter, such splices shall preferably be welded.

For contact splices, spaced laterally closer than 12-bar diameter or located closer than 15 *cm* or 6 bar diameters from the outside edge, the lap shall be increased by 20 per cent or stirrup or closely spaced spirals shall enclose the splice for its full length.

Where more than one half of the bars are spliced within a length of 40-bar diameters or where splices are made at points of maximum stress special precaution shall be taken such as increasing the length of the lap, and/or using spirals or closely spaced stirrups around and for the length of the splice

Splices in compression reinforcement When lapped splices are used, the lap lengths shall conform to the requirements mentioned earlier Welded splices may be used instead of lapped splices. Where bar size exceeds 36 *mm* diameter welded splices shall preferably be used. In bars required for compression only the compressive stress may be transmitted by bearing of square cut ends held in concrete contact by a suitably welded sleeve or mechanical device

In columns where longitudinal bars are offset at a splice, the slope of the inclined portion of the bar with the axis of the column shall not exceed 1 in 6 and the portions of the bar above and below the offset shall be parallel to the axis of the column. Adequate horizontal support at the offset bends shall be treated as a matter of design, and shall be provided by metal ties, spirals or parts of the floor construction Metal ties or spirals so designed shall be placed near (not more than eight-bar diameters from) the point of bend.

Permissible Stresses in Steel Reinforcement

Permissible stresses in kg./cm²

Sl. No	Type of stress in the steel	Mild steel conforming to grade I of I.S. 432—1960 or to I.S. 1139-1959	Medium Tensile steel conformig to I.S 432-1960 or to I.S,: 1139-1959	Cold twisted steel bars conforming to I.S.: 1786-1961
(i)	Tension : Other than in (a) helical reinforcement in a column and (b) shear reinforcement.			
	Upto and including 40 *mm*	1400	Half the guaranteed yield stress subject to a maximum of 1900	1900
	Over 40 *mm*.	1300		
(ii)	Tension in helical reinforcement in a compression member.	1000	1300	1600
(iii)	Tension in shear reinforcement.	1400	1400	1400
(iv)	Compression in column bars	1300	1300	1300
(v)	Compression in bars in a beam or slab when the compressive resistance of the concrete is taken into account	The calculated compressive stress in the surrounding concrete multiplied by the modular ratio.		
(vi)	Compression in bars in a beam or slab where the compressive resistance of the concrete is not taken into account.			
	Upto and including 40 *mm*.	1400	Half the guaranteed yield stress subject to a maximum of 1900	1900
	Over 40 *mm*.	1300		

Note 1. When mild steel conforming to grade II of *I.S.* : *432-1960* is used, the permissible stress shall be reduced by 10 per cent, or if the design details have already been worked out on the basis of mild steel conforming to Grade 1 *I.S.* : *432-1960*, the area of reinforcement shall be increased by 10 per cent of that required of Grade 1 steel.

Note 2. Yield stress of steels for which there is no clearly defined yield point should be taken to be 0·2 per cent of proof stress

The horizontal thrust to be resisted shall be assumed as $1\frac{1}{2}$ times the horizontal component of the nominal stress in the inclined portion of the bar. Offset bars shall be bent before they are placed in forms.

Joining or Lapping. The length of lap in reinforcement shall not be less than :

(*a*) For bars in tension :

$$\text{bar diameter} \times \frac{\text{actual tensile stress}}{\text{four times the permissible average bond stress}}$$

or 30-bar diameters whichever is greater.

(*b*) For bars in compression :

$$\text{bar diameter} \times \frac{\text{actual compressive stress}}{\text{five times the permissible average bond stress}}$$

or 24-bar diameter whichever is greater

Minimum spacing of reinforcement The minimum horizontal distance between parallel reinforcements shall not be less than the following :

(*i*) Diameter of bar when bars are of the same diameter. Diameter of the thickest bar when bars of more than one size are used.

(*ii*) Maximum size of coarse aggregate plus 6 *mm*. *If the* aggregate size be taken as 19 *mm*., this spacing equals 25 *mm*. *A greater distance should be provided when convenient.*

The vertical distance between two horizontal main bars shall be not less than 5 *mm*. But this does not apply at a splice or lap and when such reinforcements are transverse to each other.

Cover. All reinforcements shall have a cover of concrete and the thickness of such a cover exclusive of plaster or other decorative finish, as per I.S. code shall be as follows :

(*a*) At each end of a reinforcing bar – not less than 25 *mm*. nor less than twice the diameter of such bar.

(*b*) For longitudinal reinforcement in a column—not less than 40 *mm*. nor less than the diameter of bar. In the case of columns of minimum dimensions of 20 *cm*. or less, whose bars do not exceed 13 *mm*. dia —25 *mm*. cover may be used.

(*c*) For longitudinal reinforcement in beams—not less than 25 *mm*. nor less than diameter of such bar.

(*d*) For tensile, compressive shear or other reinforcement in a slab– not less than 13 *mm*, nor dia. of reinforcement.

(*e*) For any other reinforcement—not less than 13 *mm*. nor less than the diameter of reinforcement.

§198. T-beams

Strictly rectangular beams are uncommon in reinforced concrete since the beam carries in almost all cases a slab with which it is monolithic. Hence the structure becomes a slab which is stiffened by concrete ribs. The slab and the rib due to their monolithic nature form a T-beam. The flange of the T-beam provides the necessary resistance to compression while the vertical rib provides the depth and hence the necessary lever arm. The width of rib must be such as to accommodate the tens:le reinforcement.

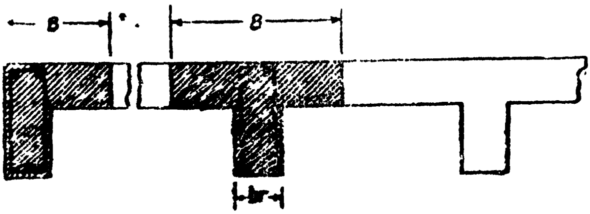


Fig. 858

§199. Width of flange of a T-beam

A certain portion of the slab on either side of the beam can be considered as forming the compression flange. The width B (*see* Fig. 858) of the flange which can be considered as acting effectively with the rib depends upon the span of the T-beam, the breadth of the rib, the overall thickness of the rib and the spacing of T-beams.

If the spporting beam happens to be an end beam, the flange of the beam is present only on one side of the beam and the beam in such a case is called on L-beam.

The width of flange of T and L beams may be determined from the following requirements recommended by I.S. 456 specification.

(*a*) For T beams, $B=\frac{l}{6}+b_r+6d_s$

(*b*) For L beams, $B=\frac{l}{12}+b_r+3d_s$

where l = effective span

b_r = width of the rib.

It is important to note that the part of the slab considered as the flange of the T-beam can function with the beam only when the flange has adequate reinforcement transverse to the beam and it shall be built integrally with the beam or effectively bonded together with the beam.

In this connection the I.S. code has stated the following :

The flanges of the T-beam or L-beam may be part of a slab which is spanning either transverse to the beam or in the same direction as the beam. In any case the flange shall have adequate reinforcement transverse to the beam and it shall be built integrally with the beam. However, where the main reinforcement of a slab, which is considered as the flange of the T-beam or L-beam is parallel to the beam, transverse reinforcement extending to the length indi-

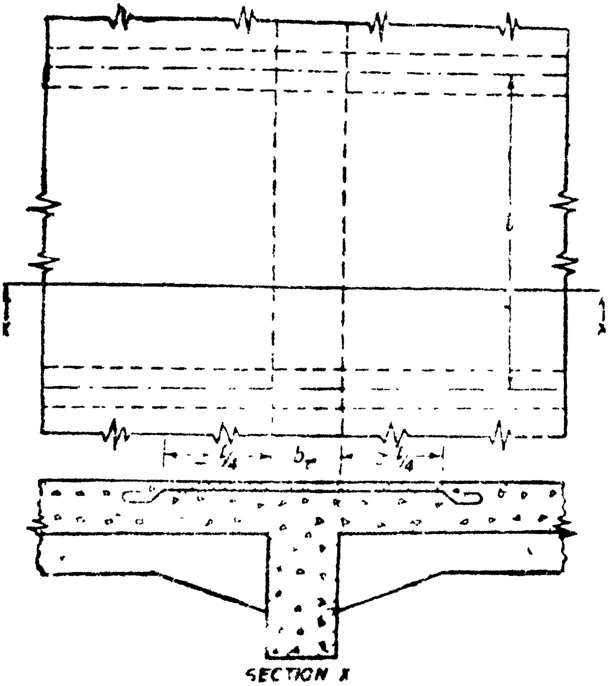


Fig. 859

cated in Fig. 859 shall be provided near the top surface of the slab. If the quantity of such reinforcement is not specifically determined by calculation, it shall be not less than 60 percent of the main reinforcement'in the centre of the span of the slab constituting the flange.

Depth of Rib. This is determined by the effective depth of the beam. The effective depth of a T or L beam is the distance between the top compression edge and the centre of the tensile reinforcement. In preliminary computations the depth can be taken as $\frac{1}{12}$ of the span for heavy loads, $\frac{1}{12}$ to $\frac{1}{15}$ of the span for medium and $\frac{1}{15}$ to $\frac{1}{20}$ for light loads. Some designers follow the following specifications, *viz.*

The ratio of effe ,tive span to overall depth of a beam shall not exceed the following :

Simply supported beams	: 20
Continuous beams	: 25
Cantilever beams	: 10

§200. Width of rib

This shall be such as to accommodate the necessary tensile reinforcement. The width shall also be such as to prevent lateral instability. This width is generally between $\frac{1}{3}$ and $\frac{2}{3}$ of the depth of the rib. More often architectural requirements fix the width which shall be the same as the width of the supporting column.

§201 Neutral axis of a T-beam

The depth of the neutral axis can be determined by equating moments of areas on either side of the neutral axis. Three cases arise, *viz.*

(*i*) The neutral axis may be situated within the flange.

(*ii*) The neutral axis may be just at the bottom edge of the slab.

(*iii*) The neutral axis may be below the slab.

Case (*i*). **Neutral axis within the flange** (Fig. 860).

Such a beam will behave as a rectangular beam of width *B*.

Taking moments about the N.A.

$$\frac{Bn^2}{2}=mA_t(d-n)\quad\ldots(1)$$

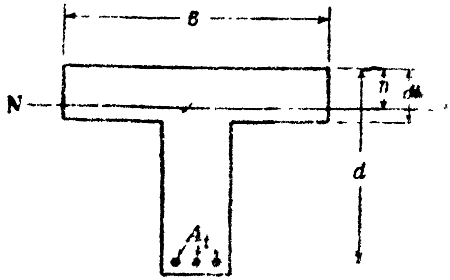


Fig. 860

Case (*ii*). In this case $n=d_s$ = thickness of the slab.

Hence Eq. (1) still holds good.

Case (*iii*). **Neutral axis below the slab** (Fig. 861).

This is the usual case. In this case, taking moments about N.A.

We have,

$$Bd_s\left(n-\frac{d_s}{2}\right)+\frac{b_r(n-d_s)^2}{2}=mA_t(d-n)\quad\ldots(2)$$

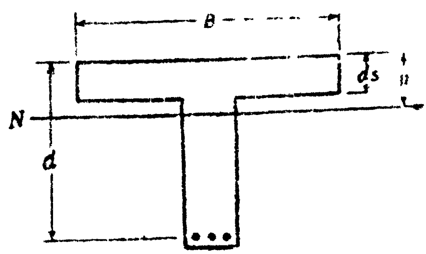


Fig. 861

In these computations it is usual to ignore the compressive force in the rib of the beam.

Hence, it is usual to discard the term $b_r\frac{(n-d_s)^2}{2}$ in Eq. (2) and hence practically, we have

$$Bd_s\left(n-\frac{d_s}{2}\right)=mA_t(d-n)\quad\ldots(3)$$

It is to be noted that the above amendment on equation (2) is made only to simplify the calculations. The compression area of the T-beam has thus been taken as Bd_s.

§202. Lever arm of the T-beam

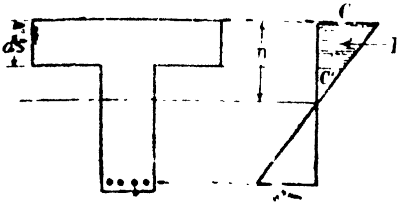


Fig. 862

This is the distance between the line of action of the resultant compression to the line of action of the resultant tension.

Let the maximum compressive stress in concrete be c $kg./cm.^2$ Let the compressive stress in concrete at the bottom edge of the flange be c' $kg./cm.^2$

The centre of gravity of the resultant compression is situated at a depth y from the compression edge ; y is the actual depth of centroid of the pressure trapezium corresponding to the flange (Fig. 862).

$$y=\frac{c+2c'}{c+c'}\cdot\frac{d_s}{3} \qquad (4)$$

∴ Lever arm $\quad a=d-y$

But $\quad c'=\left(\frac{n-d_s}{n}\right)c$

Substituting in Eq. (4), we have,

$$y=\frac{c+\frac{2(n-d_s)}{n}c}{c+\frac{n-d_s}{n}c}\cdot\frac{d_s}{3}$$

$$y=\frac{3n-2d_s}{2n-d_s}\cdot\frac{d_s}{3} \qquad ...(5)$$

Eq. (5) may be found to be more convenient than Eq (4).

Moment of Resistance of the beam

M.R. = Total compression or total tension × lever arm

$$\therefore \quad M.R.=Bd_s\frac{c+c'}{2}\,(d-y) \qquad ...(6)$$

Substituting for c'

$$M.R.=Bd_s\,\frac{c+\frac{n-d_s}{n}c}{2}\,(d-y)$$

$$M.R.=\frac{\left(n-\frac{d_s}{2}\right)}{n}\times c\,Bd_s(d-y) \qquad ...(7)$$

M R. is also equal to $A_t\,.\,t(d-y)$ ...(8)

Problem 508. *An R.C. T-beam with a flange width of 102 cm. has a tension steel of area 22·8 cm². Taking the modular ratio as 18 and the permissible stresses in concrete and steel as 50 kg./cm.² and 1440 kg./cm.² determine the moment of resistance of the beam. The thickness of the flange is 13 cm. and the effective depth of the beam is 51 cm.*

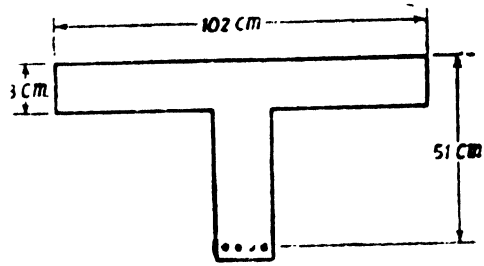


Fig. 963

Solution.

$A_t = 22·8\ cm.^2$

Taking moments about the neutral axis and assuming it to be situated below the slab,

We have,

$102 \times 13(n-6·5) = 18 \times 22·8(51-n)$

$\therefore \qquad n = 17\ cm.$

Since this value of n is greater than the thickness of the slab our assumption that the neutral axis is below the slab is correct.

Depth of critical neutral axis is given by

$$\frac{18 \times 50}{1400} = \frac{n_c}{51-n_c}$$

$\therefore \qquad n_c = 19·9\ cm.$

Since $n < n_c$ steel attains the maximum stress earlier.

Distance of $C.G.$ of the total compression from the top edge

$$= \bar{y} = \frac{3n-2d_s}{2n-d_s} \cdot \frac{d_s}{3}$$

$$= \frac{3 \times 17 - 2 \times 13}{2 \times 17 - 13} \cdot \frac{13}{3} = 5·16\ cm.$$

or alternatively,

$$\bar{y} = \frac{c+2c'}{c+c'} \cdot \frac{d_s}{3}$$

But $\qquad c' = \frac{n-d_s}{n} c = \frac{17-13}{17} c = \frac{4}{17} c$

$$= 0·235c$$

$$\therefore \qquad \bar{y} = \frac{c+2 \times 0·235c}{c+0·235c} \cdot \frac{13}{3}\ cm.$$

$$= \frac{1·47}{1·235} \times \frac{13}{3}\ cm$$

$$= 5·16\ cm.$$

$\therefore$ Lever arm $= a = d - \bar{y}$

$$= 51 - 5{\cdot}16 \text{ cm.}$$

$$= 45{\cdot}84 \text{ cm.}$$

$\therefore$
$$\text{M.R.} = A_t\, t(d - \bar{y})$$

$$= 22{\cdot}8 \times 1400 \times 45{\cdot}84 \text{ kg. cm.}$$

$$= 1{,}463{,}000 \text{ kg. cm.}$$

Problem 509. *A T-beam has a flange width of 120 cm, the flange thickness being 10 cm. The reinforcement consists of 5 bars of 24 mm diameter placed at an effective depth of 40 cm. If the stresses in concrete and steel shall not exceed 50 kg./cm.² and 1400 kg./cm.² respectively find the moment of resistance of the beam section. Modular ratio = 18.*

Solution

$$A_t = 22{\cdot}62 \text{ cm.}^2$$

Assuming that the neutral axis is below the slab, taking moments about the N.A., we have,

$$120 \times 10\,(n - 5) = 18 \times 22{\cdot}62\,(50 - n)$$

$$n = 13{\cdot}87 \text{ cm.}$$

Since $n > d_s$, our assumption about the position of neutral axis is correct.

Depth of critical neutral axis is given by

$$\frac{18 \times 50}{1400} = \frac{n_c}{40 - n_c}$$

$\therefore$
$$n_c = 15{\cdot}65 \text{ cm.}$$

Since $n < n_c$ steel will attain the maximum stress earlier

$$\bar{y} = \frac{3n - 2d_s}{2n - d_s} \times \frac{d_s}{3}$$

$$= \frac{3 \times 13{\cdot}87 - 2 \times 10}{2 \times 13{\cdot}87 - 10} \times \frac{10}{3} \text{ cm.}$$

$$= 4{\cdot}06 \text{ cm.}$$

$\therefore$ Lever arm $= d - \bar{y} = 40 - 4{\cdot}06 = 35{\cdot}94$ *cm.*

$\therefore$ *M.R.* $= A_t\, t\,(d - \bar{y})$

$$= 22{\cdot}62 \times 1400 \times 35{\cdot}94 \text{ kg. cm.}$$

$$= 1139000 \text{ kg. cm.}$$

Problem 510. *The breadth and thickness of the flange of a T-beam are 170 cm. and 15 cm. respectively. Steel reinforcement of area 45·24 cm.² is provided at an effective depth of 70 cm. If the modular ratio equals 18 find the stresses in concrete and steel if the moment of resistance of the beam section is 32,000 kgm.*

Solution. Assuming the *N.A.* to be below the flange, taking moments about the *N.A.* we have,

$$170\times 15\,(n-7\cdot5)=18\times 45\cdot24\,(70-n)$$

$$\therefore \quad n=22\cdot63 \text{ cm.}$$

$$y=\frac{3n-2d_s}{2n-d_s}\cdot\frac{d_s}{3}$$

$$=\frac{2\times 22\cdot63-2\times 15}{2\times 22\cdot63-15}\cdot\frac{15}{3}\text{ cm.}$$

$$=6\cdot26 \text{ cm.}$$

Lever arm $=(d-y)=70-6\cdot26=63\cdot74$ *cm.*

M.R. $=A_t\, t\,(d-y)=32000\times 100$

$=45\cdot24\times t\times 63\cdot74$

$=32000\times 100$

$$\therefore \quad t=\frac{32000\times 100}{45\cdot24\times 63\cdot74}\text{ kg./cm.}^2$$

$$=1{,}110 \text{ kg./cm}^2.$$

Stress in concrete is given by

$$\frac{18\times c}{1110}=\frac{22\cdot63}{70-22\cdot63}$$

$$\therefore \quad c=29\cdot45 \text{ kg./cm}^2$$

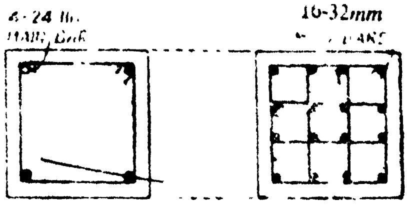
§203. Axially Loaded Columns

A column forms a very important component of a structure Columns support beams which in turn support walls and slabs. It should not be forgotten that the failure of a column results in the collapse of the structure. The design of a column should therefore receive importance

§204. **Axially loaded columns.** An axially loaded column is one in which the line of thrust of the load supported by the column coincides with the longitudinal axis of the column.

§205 **Plain concrete columns.** Columns whose unsupported lengths do not exceed four times the least lateral dimension may be made of plain concrete Further the load should be axially applied. Columns of greater lengths should be adequately reinforced.

§206. **R.C. columns.** Columns may be cast to any of the following shapes—square, circular, hexagonal, octagonal, *etc* Longitudinal reinforcement (or main steel) is provided to resist compressive loads along with with concrete. As per I.S. 456—a reinforced concrete column shall have longitudinal steel reinforcement and the cross-sectional area of such reinforcement shall be not less than 0·8% nor more than 6% of the cross-sectional area of the column required to transmit all the loading. The object of stipulating a minimum percentage of steel is to make provision to prevent buckling of the column due to an accidental eccentricity of the load on it. The object of stipulating a maximum percentage of steel is to provide reinforcement within such a limit to avoid congestion of reinforcements which would make it very difficult to place the concrete and consolidate it. This may be best realized from the following two examples. Consider two columns 45 *cm.* × 45 *cm.*



(*a*) Fig. 864 (*b*)

$$\text{Reinforcement required at } 0{\cdot}8\% \text{ of gross area} = \frac{0{\cdot}8}{100} \times 45^2$$

$$= 16{\cdot}2 \ cm.^2$$

This may be provided by four bars of 24 *mm.* diameter with an area of 18·10 *cm.*^{2} [Fig. 864 (*a*)].

$$\text{Reinforcement required at } 6\% \text{ of the gross area} = \frac{6}{100} \times 45^2$$

$$= 121{\cdot}5 \ cm^2.$$

Even if the bigger diameter bars be selected, say 32 *mm.* diameter bars, we will require 16 bars of 32 *mm.* diameter providing a total area of 8·04 × 16 = 128·64 *cm.*^{2} [Fig. 864 (*b*)]. The difficulty of placing concrete between the 16 bars of 32 *mm.* diameter with the overall size of 45 *cm* × 45 *cm.* may be quite apparent. Practically the maximum percentage of steel may be limited to 4 per cent of the gross area so as to ensure a good and sound concrete.

If the ratio of the length to least radius of gyration is less than 12, the requirement regarding minimum amount of steel will not apply.

The longitudinal reinforement should be laterally tied by transverse links to provide a restraint against outward buckling of each of the longitudinal bars. I.S. 456 code stipulates that the diameter of longitudinal bars shall not be less than 12 *mm.* and that the diameter of the transverse reinforcement shall be not less than one fourth of the diameter of the main rods and in no case less than 5 *mm.* in diameter.

In the case of pedestals and columns in which the longitudinal reinforcement is ignored for purposes of calculating the permissible load on the column the longitudinal reinforcement shall not be less than 0·15% of the gross area of the column section.

§207. Spacing of transverse links. This shall not exceed the least of the following :

(*a*) The least lateral dimension of the columns.

(*b*) Sixteen times the diameter of the smallest longitudinal reinforcing rod in the column.

(*c*) Forty-eight times the diameter of the transverse reinforcement.

§208. **Cover.** The minimum cover to a column reinforcement equals 40 *mm*. or diameter of bar whichever is greater.

§209. **Effective length of a column.** The effective length of a column is not necessarily its actual length. It depends on the degree of fixity of the ends of the columns. The following table (page 983) gives the effective lengths (corresponding to the actual length l which refers to the length of the column from floor to floor or between properly restrained supports). This table is in accordance with I.S. 456 code.

§210. **Safe loads on R.C. columns.**

Short columns. A column will be considered as a short column when the ratio of effective length to its least lateral dimension does not exceed 12*.

§211. **Permissible load on a column**

Reinforced concrete columns may be designed by the following methods :

(*i*) method based on the elastic theory.

(*ii*) I.S. code method.

(*i*) **Method based on the elastic theory**

Let A be the overall area of the column section.

Let the area of compressive reinforcement be A_c and let the modular ratio be m.

Let the compressive stress in concrete be c.

Since there is no slipping between the reinforcement and the concrete, strains in concrete and steel are equal. Hence to satisfy this condition

Stress in steel = modular ratio × stress in concrete

$$t = mc$$

Let W be the load on the column.

∴ Load on concrete + load on steel = W.

$$c(A-A_c) = A_c\, mc = W$$

$$\therefore c[A+(m-1)\, A_c] = W$$

*For more exact computations a column can be regarded as a short column when the ratio of the effective length to its least radius of gyration does not exceed 50.

Degree of end restraint of compression member	*Theoretical value of effective length*	*Recommended value of effective length*
Effectively held in position and restrained against rotation at both (*i e.* both ends are fixed)	0·5 *l*	0·65 *l*
Effectively held position at both ends, restrained against rotation at one end (*i.e.* fixed at one end and hinged at the other end)	0·7 *l*	0·80 *l*
Effectively held in position at both ends, but not restrained against rotation (*i e.,* both ends are hinged.	1·00 *l*	1·00 *l*
Effectively held in position and restrained against rotation at one end, and at the other partially restrained against rotation but not held in position		1·50 *l*
Effectively held in position at one end but not restrained against rotation, and at the other end restrained against rotation but not held in position	2·00 *l*	2·00 *l*
Effectively held in position and restrained against rotation at one end but not held in position nor restrained against rotation at the other end, *i.e.,* fixed at one end and free at the other end	2·00 *l*	2·00 *l*

Safe Compressive Stress in Concrete in Columns (IS 456)

S.I. UNITS		*M.K.S. UNITS*	
Grade of concrete	*Safe stress in Compression N/mm².*	*Grade of Concrete*	*Safe stress in Compression kg./cm.²*
M 10	2·5	M 100	25
M 15	4·0	M 150	40
M 20	5·0	M 200	50
M 25	6·0	M 250	60
M 30	8·0	M 300	80
M 35	9·0	M 350	90
M 40	10·0	M 400	100

On this theory taking c at 40 $kg./cm.^2$ (or the stipulated stress) the safe load on the column section can be determined. Until recently columns were designed on the elastic theory.

(ii) ***I.S. code method***

The safe load on a short column is given by,

$$W=\begin{bmatrix}\text{area of}\\\text{concrete}\end{bmatrix}\times\begin{bmatrix}\text{safe stress}\\\text{in concrete}\end{bmatrix}+\begin{bmatrix}\text{area of}\\\text{steel}\end{bmatrix}\times\begin{bmatrix}\text{safe stress}\\\text{in steel}\end{bmatrix}$$

Safe compressive stress in concrete shall be taken as follows :

Safe compressive stress in mild steel bars is equal to 130 N/mm^2 (1300 $kg./cm^2$).

Problem 511. *An R.C. column 30 cm.×30 cm. in section is reinforced with 8 bars of 20 mm. diameter. If the permissible stress in concrete is 40 kg./cm.² find the safe compressive load by simple elastic theory. Take m=18.*

Solution.

Area of column $=A=30\times30=900\ cm.^2$

Area of steel $=A_t=8\times\frac{\pi}{4}(2)^2=25{\cdot}12\ cm.^2$

Equivalent concrete area $=A_e=A+(m-1)A_t$

$=900+(18-1)\ 25{\cdot}12=1327{\cdot}04\ cm^2$

$$\text{Safe load}=\begin{bmatrix}\text{Safe stress}\\\text{in concrete}\end{bmatrix}\times\begin{bmatrix}\text{Equivalent}\\\text{concrete area}\end{bmatrix}$$

$=40\times1327{\cdot}04\ kg.$

$=53082\ kg.$

Problem 512. (S.I.) *A short column 30 cm.×30 cm. in section is reinforced with 8 bars of 22 cm. diameter. Find the safe load on the column by simple elastic theory. Take m=18 Safe stress in concrete is 4N/mm².*

Solution. Area of the column

$=A=30\times30=900\ cm.^2$

Area of steel $=A_t=8\times\frac{\pi}{4}(2{\cdot}2)^2=30{\cdot}4\ cm^2$

Equivalent concrete area

$=A_e=A+(m-1)A_t$

$=900+(18-1)\ 30{\cdot}4\ cm.^2$

$=1416{\cdot}8\ cm.^2$

Safe stress in concrete

$=4\ N/mm.^2=400\ N/cm^2$

$$\text{Safe load} = \begin{bmatrix}\text{Safe stress}\\ \text{in concrete}\end{bmatrix} \times \begin{bmatrix}\text{Equivalent}\\ \text{concrete area}\end{bmatrix}$$

$= 400 \times 1416.8$ *Newtons*

$= 566720$ *Newtons*

$= 566.72$ *Kilonewtons.*

Problem 513. *A short column of square section is to be designed to carry an axial load of 1,02,300 kg. Design the column by I.S. code method.*

Solution. Let us assume that *M* 150 grade concrete is used.

Hence $c = 40\ kg./cm.^2$

and $c_1 = 1300\ kg./cm.^2$

Let the sectional area of the column be $A\ cm.^2$

Let us assume 2% of area of column as steel reinforcement.

$\therefore$ **Area of steel** $= 0.02A\ cm.^2$

$\therefore$ **Area of concrete** $= A - 0.02A\ cm.^2$

$= 0.98A\ cm.^2$

$\therefore$ Safe load = concrete area × safe stress in concrete + steel area × safe stress in steel

$\therefore\ 0.98A \times 40 + 0.02A \times 1300$

$= 1,02,000\ kg$

$\therefore\ 65.2A = 1,02,300$

$A = 1570\ cm.^2$

$\therefore$ Size of column $= \sqrt{1570} = 39.62\ cm.$

Hence provide $40\ cm \times 40\ cm$

Area of the column section $= 1600\ cm.^2$

$\therefore\ 40(1600 - A_s) + A_s \times 1300$

$= 1,02,300$

$1260 A_s = 38300$

$\therefore\ A_s = 30.4\ cm^2$

Provide 8 bars of 22 *mm* diameter ($30.41\ cm.^2$).

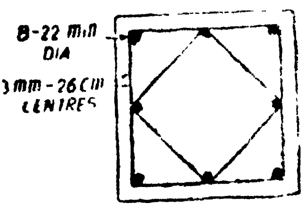


Fig. 865

Lateral ties. Diameter to be not less than

(*i*) 5 *mm*

(*ii*) One-fourth of diameter of main steel $= \frac{1}{4} \times 22 = 5.5$ *mm.*

Let us, therefore, provide 6 *mm.* diameter ties.

Spacing. This shall not exceed the following :

(*i*) Least lateral dimension = 40 *cm.*

(*ii*) 12 times the diameter of main bar $= 12 \times 2.2 = 26.4$ *cm.*

(*iii*) 48 times the diameter of transverse steel

$= 48 \times 6 = 288$ *mm.* $= 28.8$ *cm.*

Let us, therefore, provide 6 *mm*. diameter ties at 26 *cm*. centres [*See* Fig. 865].

§212. Long columns. When the ratio of effective length of a column to its least lateral dimension exceeds 12 the column will be regarded as a long column. Such columns are liable to be buckled and to include this factor in the design the working stresses in concrete and steel are taken at a lower value, by multiplying the usual working stresses by a co-efficient C_r called a *reduction co-efficient.*

Hence for a long column

Safe stress in concrete $= C_r$ corresponding safe stress for short column

and safe stress in steel $= C_r$ corresponding safe stress for short column

The co-efficient C_r is to be determined from the following relation

$$C_r = 1{\cdot}25 - \frac{l_{ef}}{48b}$$

$$\left[\text{For more exact computations } C_r = 1{\cdot}25 - \frac{l_{ef}}{160\,K_m}\right]$$

where C_r = reduction co-efficient

l_{ef} = effective length of the column

b = least lateral dimension

K_m = least radius of gyration.

Values of C_r for various values of $\frac{l_{ef}}{b}$ are given in the following table :

$\frac{l_{ef}}{b}$	C_r	$\frac{l_{ef}}{b}$	C_r	$\frac{l_{ef}}{b}$	C_r	$\frac{l_{ef}}{b}$	C_r
12	1·000	24	0·750	36	0·500	48	0·250
13	0·979	25	0·729	37	0·479	49	0·229
14	0·958	26	0·708	38	0·458	50	0·208
15	0·937	27	0·688	39	0·438	51	0·188
16	0·917	28	0·667	40	0·417	52	0·167
17	0·896	29	0·646	41	0·396	53	0·146
18	0·875	30	0·625	42	0·375	54	0·125
19	0·854	31	0·604	43	0·354	55	0·104
20	0·833	32	0·583	44	0·333	56	0·083
21	0·813	33	0·563	45	0·313	57	0·063
22	0·792	34	0·542	46	0·292	58	0·042
23	0·771	35	0·521	47	0·271	59	0·021

Problem 514. *A column 38 cm.* $\times$ *38 cm.* $\times$ *8 metres long has to support a load of 80,000 kg. Find the necessary reinforcement for the column. Take* $c=40$ *kg./cm.*^{2} *and* $c_1=1300$ *kg./cm.*^{2}

Solution b = least lateral dimension = 38 *cm.*

l_{ef} = length of column = 800 *cm.*

$$\therefore \quad \frac{l_{ef}}{b}=\frac{800}{38}=21{\cdot}05.$$

This is greater than 12. Therefore, the column is a long column.
Reduction co-efficient

$$C_r=1{\cdot}25-\frac{l_{ef}}{48b}$$

$$=1{\cdot}25-\frac{800}{48\times 38}=0{\cdot}811$$

$\therefore$ Safe load on the long column = Reduction co-efficient $\times$ safe load on short column.

Let the area of reinforcement be A_c $cm.^2$

$\therefore$ Area of concrete $=38\times 38-A_c=(1444-A_c)$ $cm.^2$

$\therefore$ Safe load $=0{\cdot}811\ [40(1444-A_c)+1300\times A_c]$
$=80{,}000$ *kg.*

$\therefore \quad A_c=33{\cdot}2$ $cm.^2$

This is between 0·8% and 6% of the area of the column section.

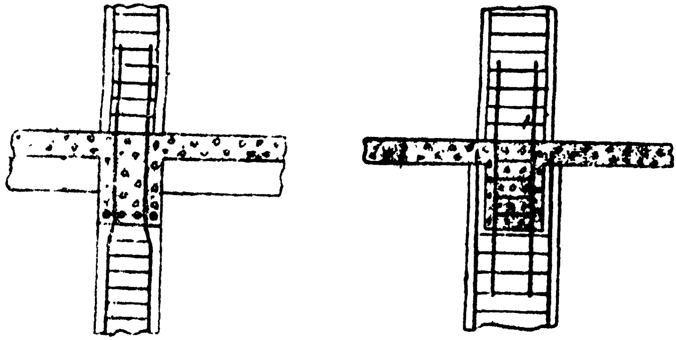
8 bars of 24 *mm.* diameter will provide 36·19 $cm.^2$

Lateral ties shall also be provided as in the previous example.

§213. **Continuous columns.** Often in multistoried structures, a column continues up through a floor from one storey to another. In such cases the main bars of the column must be first continued up either within or outside the reinforcements of the floor beam which frames into this column. When the main bars continue up outside the reinforcement of the beam, it is necessary that the width of the column should be at least 8 *cm* more than the width of the beam. Sometimes the column size in plan may be smaller above the floor than below it. In such cases the main bars of the column will have to be bent inwards at the floor level, or alternatively these main bars may be stopped just below the floor level and separate lap bars may be provided for connecting the part of the column above and below the floor.

Figs. 866 (*a*) and (*b*) show two alternative arrangements.

§214 **Spirally reinforced column.** These are circular columns, which are reinforced with closely and uniformly spaced spiral reinforcement in addition to longitudinal steel. Columns of circular section are usually spirally reinforced. Sometimes separate loops may also be provided in place of the spiral. The continuous spiral is adopted in preference to separate loops. A column with helical reinforcement shall have atleast six bars as longitudinal reinforcement.



(*a*) Column reinforcement taken within the main bars of the beam. Fig. 866 (*b*) Column reinforcement taken outside the main bars of the beam.

The safe load on a spirally reinforced column is given by the following expression

$$P = cA_k + c_1 A_c + 2t_b A_b$$

where P = Safe load on the column

A_k = The cross-sectional area of concrete in the column core excluding the area of longitudinal reinforcement (For core diameter see Fig. 867).

A_b = The equivalent (volume of helical reinforcement area of helical reinforcement per unit length of the column).

A_c = The cross-sectional area of longitudinal steel in compression.

t_b = The permissible stress in helical reinforcement and c_1 = the permissible compressive stress for column bars. The sum of the terms cA_k and $2t_bA_b$ shall not exceed 0·5 F_cA where F_c is the ultimate cube strength of the concrete req d from the work tests. The permissible tensile stress in the helical reinforcement may be taken at 1000 $kg./cm^2$.

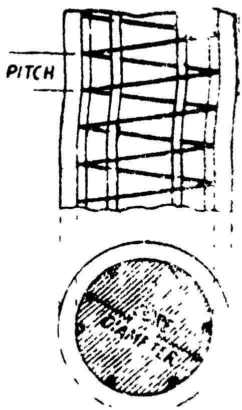


Fig. 867

Problem 515. *A column of circular section is 326 mm in diameter and is reinforced with 6 bars of 22 mm. diameter as longitudinal steel and helically reinforced with 6 mm. diameter bars at a pitch of 6 cm. Find the safe load on the column. Adopt* $c=40$ *kg./cm.*^{2}, $c_1=1300$ *kg./cm*^{2} *and* $t_b=1000$ *kg./cm.*^{2}

Solution. Diameter of column = 326 *mm.* = 32·6 *cm* Allowing 40 *mm.* cover to longitudinal steel, the diameter of the core = 32·6 − 2 × 4 = 24·6 *cm.*

Safe load on the column

$P=cA_k+c_1A_c+2t_bA_b$

$A_b=$ Volume of the spiral per *cm* run of the column length of the spiral per *cm.* run of the column × area of helical reinforcement.

$=\frac{\text{Length of the spiral per pitch length}}{\text{Pitch}}\times$ area of helical reinforcement

Length of the spiral per pitch length

$=\pi\times$ diameter of centre line of helical reinforcement

$=\pi\times(24\cdot6+0\cdot6)=79\cdot18$ *cm.*

$$\therefore \quad A_b=\frac{79\cdot18\times0\cdot28}{6}=3\cdot70 \text{ cm.}^2$$

Adopting $c=40$ *kg./cm.*^{2}, $c_1=1300$ *kg./cm.*^{2} and t_b 1000 *kg./cm*^{2}.

Safe load $=P=40\left[\frac{\pi\times25^2}{4}-22\cdot81\right]+1300\times22\cdot81+1000\times3\cdot70$ *kg*

$=55780$ *kg.*

$=55\cdot78$ *tonnes.*

218. Combined Bending and Direct Stresses

Often we come across members subjected to direct stresses accompanied by bending stresses. A very common example is that of a column subjected to an eccentric load or a column of a storeyed building. Other instances are in arches, tank walls, chimneys, silos, bins, *etc.* In some cases the direct stress may be predominant and the bending stresses may be small. While in some cases the bending stresses may be predominant as compared with the direct stresses. There will also be instances in which both types of stresses may be predominant. Let us discuss the case that we usually come across.

When the section is subjected to an axial load and a moment, the ratio of moment to the load is called the eccentricity and such a section can be analysed as though the load has been applied at the above eccentricity.

Case 1. Rectangular section subjected to compression and bending. Eccentricity is less than $\frac{D}{6}$. *See* Fig. 868.

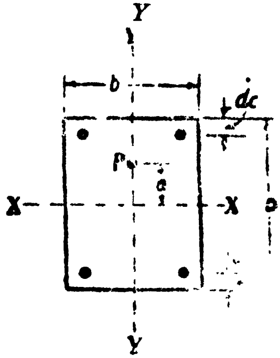


Fig. 868

In this case the resultant stresses in concrete will remain compressive and the resultant stresses are given by

$$c=\frac{P}{A_e}\pm\frac{M}{Z}$$

where P = Eccentric load

M = B.M. due to eccentric load = $P.e$.

Z = Section modulus about the axis with repect to which the load is eccentric.

A_e = Equivalent concrete area

For the section shown in the figure with symmetrical reinforcement we have the equivalent area of section

$$A_e=bD+(m-1)A_t$$

where A_t = Total area of reinforcement.

Equivalent moment of inertia about the axis XX

$$I_e=\frac{bD^3}{12}+(m-1)A_t\left[\frac{D}{2}-d_c\right]^2$$

$$Z=\frac{I_e}{\frac{D}{2}}$$

$\therefore$ The resultant stresses are

$$c=\frac{P}{A_e}\pm\frac{P.e}{I_e}\left(\frac{D}{2}\right)$$

The following examples illustrate the above case.

Problem 516. *A rectangular reinforced concrete section 70 cm. deep and 45 cm. wide is reinforced with 7 bars of 28 mm. diameter placed at an effective cover of 5 cm from the top edge and seven similar bars at the same effective cover from the bottom edge. Determine the maximum thrust on the section, which can be applied at a distance of 10 cm from the centre line if the compressive stress in concrete is not to exceed 50 kg/cm.² Take m=18.*

Solution.

$e=10$ *cm.*

$$\frac{D}{6}=\frac{70}{6}=11{\cdot}67 \text{ cm}$$

$$e<\frac{D}{6}$$

Total area of steel provided = 2 × 43·1 = 86·2 *cm.*²

Equivalent area of concrete for the given section

$= A_e = bD+(m-1)A_t$

$= 45\times 70+(18-1)\,86{\cdot}2 \text{ cm.}^2$

$= 4615{\cdot}4 \text{ cm}^2$

Equivalent moment of Inertia about the centroidal axis XX

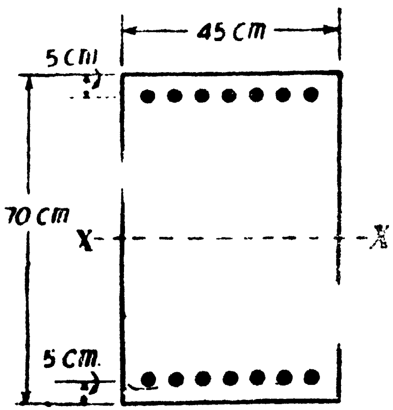


Fig. 869

$$=I_e=\frac{bD^3}{13}+(m-1)\,A_t\left(\frac{D}{2}-d_c\right)^2$$

$$=\frac{45\times70^3}{12}+17\times86{\cdot}2\,(35-5)^2\ cm.^4$$

$$=2{,}605{,}110\ cm.^4$$

Maximum compressive stress in concrete

$$=c=\frac{P}{A_e}+\frac{M}{I_e}\,y_c$$

$$\therefore\qquad 50=\frac{P}{4615{\cdot}4}+\frac{P\times10}{2605110}\times35$$

$$\therefore\qquad P=142{,}400\ kg.$$

Problem 517. *The horizontal thrust for a two-hinged arch is found to be 11,000 kg. The crown section of the arch is a square of 30 cm. side. It is reinforced with 6·03 cm.² of steel above the centroidal axis and equal amount of steel below the axis. The centres of steel are respectively 5 cm. from the upper and lower edges of the crown section. The section is also subjected to a bending moment of 100,000 kg. cm. Assuming that the concrete does not fail in tension, calculate the maximum tensile and compressive stresses in concrete at the crown section. Find also the tensile and compressive stresses in steel. Take m=18.*

Solution. Equivalent concrete area of the arch section

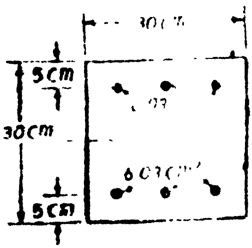


Fig. 870

$$A_e=30\times30+(18-1)\times2\times6{\cdot}03\ cm.^2$$
$$=1105\ cm.^2$$

Equivalent moment of Inertia

$$I_e=\frac{30\times30^3}{13}+(18-1)2\times6{\cdot}03\ cm.^4$$

$$=88000\ cm.^4$$

Extreme stresses in concrete

$$=\frac{11000}{1105}\pm\frac{100{,}000}{88000}\times15\ kg./cm^2.$$

$$=9{\cdot}95\pm17{\cdot}05\ kg./cm.^2$$

$\therefore$ Maximum *compressive stress* in concrete

$$=9{\cdot}95+17{\cdot}05\ kg./cm.^2$$
$$=27\ kg./cm.^2$$

Maximum *tensile stress* in concrete

$$=17{\cdot}05-9{\cdot}95=7{\cdot}10\ kg./cm.^2$$

Compressive stress in steel

$$=18\left[9.95+\frac{10}{15}\times17{\cdot}05\right]kg./cm.^2$$

$$=383{\cdot}76\ kg./cm^2$$

Tensile stress in steel $=18\left[\frac{10}{15}\times 17{\cdot}05-9{\cdot}95\right]$ *kg./cm.*²

$=25{\cdot}6$ *kg./cm.*²

Problem. 518. *An R.C. column 30 cm × 30 cm. is reinforced with four bars of 26 mm. diameter placed at an effective cover of 5 cm. Find how far from the centre, the line of thrust may pass along the YY axis without causing tension in concrete. Take m=18.*

Solution. Area of 4 bars of 26 *mm.* diameter $=21{\cdot}24$ *cm*

Let the maximum eccentricity be *e cm.*

(*See* Fig. 871)

Equivalent concrete area $=30\times30+(18-1)\,21{\cdot}24$ *cm.*²

$=1261$ *cm.*²

Equivalent moment of Inertia

$$=\frac{30\times30^3}{12}+(18-1)\,21{\cdot}24\times10^2 \text{ cm.}^4$$

$=103600$ *cm.*⁴

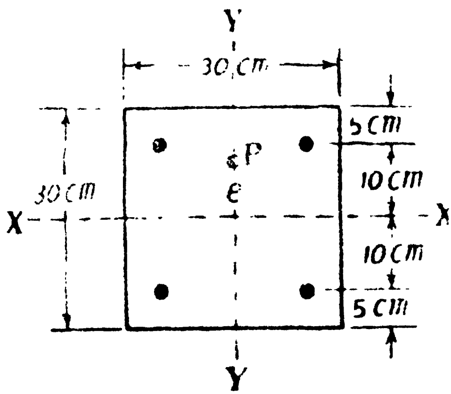


Fig. 871

If tension in concrete should be avoided the direct and bending stresses should be equal for the section.

Let P be the applied load

$$\therefore \quad \frac{P}{1261}=\frac{P.e}{103600}\times\frac{30}{2}$$

$e=5{\cdot}48$ *cm.*

Problem 519. *A two-hinged arched rib has a section 30 cm. wide and 90 cm. deep, at the crown. The section is reinforced with six bars of 22 mm. diameter at the top and an equal reinforcement at the bottom. The reinforcements are placed at an effective cover of 5 cm. from the respective edges. If the resultant thrust on the section is 100,000 kg. inclined at 3° with the tangent to the arch centre line and acting on the vertical axis and 10 cm. from the centre line of the section, determine the stresses in concrete and steel at the top and bottom of the section ; modular ratio may be taken as 15.*

Solution. *See* Fig. 872.

Since $e=10$ *cms*. which is less than $\frac{90}{6}$ the stresses will remain compressive.

Thrust normal to the cross-section

$=100,000 \cos 3^{\circ}$

$=99860$ *kg*.

Equivalent area of the section

$=30\times90+(15-1)\times12\times3{\cdot}80$

$=3338{\cdot}4\ cm.^4$

Equivalent Moment of Inertia

$$=\frac{30\times90^3}{12}+14\times12\times3{\cdot}8(45-5)^2\ cm.^4$$

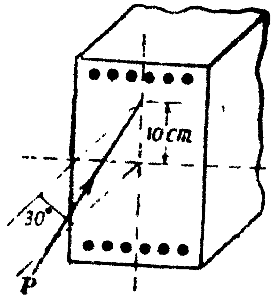


Fig. 872

$=18,22,500+14,21,440\ cm.^4$

$=28,43,940\ cm.^4$

$\therefore$ Extreme stresses in concrete

$$=\frac{99860}{3338{\cdot}4}\pm\frac{99860\times10}{28,43,340}\times\frac{90}{2}\ kg./cm.^2$$

$=29{\cdot}92\pm15{\cdot}81\ kg./cm.^2$

$\therefore$ Maximum compressive stress $=45{\cdot}73\ kg./cm.^2$

Minimum compressive stress $=14{\cdot}11\ kg./cm.^2$

Stress in steel $=15\left[29{\cdot}92\pm\frac{45-5}{45}(15{\cdot}81)\right]kg./cm.^2$

$=659{\cdot}55\ kg./cm.^2$ compressive in steel at top and $238{\cdot}95\ kg./cm.^2$ compressive in steel at bottom.

When the eccentricity is so large that excesive tensile stresses are developed. In this case the concrete on the tension zone is ignored and the position of the neutral axis found from first principles.

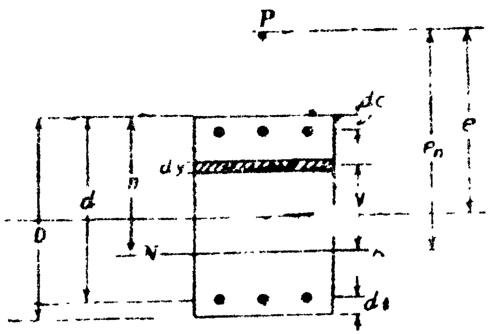


Fig. 873

Fig. 873 shows the section of an *R.C.* member subjected to an eccentric load. The stress at a point is proportional to its distance from the neutral axis.

Let the stress at any point be given by $c=ry$

where y=distance of the point from the neutral axis

and r=a co-efficient of proportionality

Total compression—Total tension=Thrust on the section

$$\therefore \quad \int_0^n b\, dy\, ry+(m-1)\, A_c r(n-d_c)-mA_t\, r(d-n)=P$$

$$\therefore \quad P=r\left[\frac{bn^2}{3}+(m-1)\, A_c(n-d_c)-m\, A_t(d-n)\right]$$

Similarly, Moment of resistance=bending moment

or Total moment of the induced compressive and tensile stresses about the neutral axis } =bending moment.

$$\therefore \quad bn\,.\,\frac{nr}{2}\,\tfrac{2}{3}n+(m-1)\, A_c r(n-d_c)^2+mA_t r(d-n)^2=Pe_n$$

$$\therefore \quad Pe_n=r\left[\frac{bn^3}{3}+(m-1)A_t(n-d_c)^2+mA_t(d-n)^2\right]$$

$$\therefore \quad e_n=\frac{Pe_n}{P}=\frac{\frac{bn^3}{3}+(m-1)A_c(n-d_c)^2+mA_t(d-n)^2}{\frac{bn^2}{2}+(m-1)A_c(n-d_c)-mA_t(d-n)}$$

But $$e_n=e+n-\frac{D}{2}$$

$$\therefore \quad e+n-\frac{D}{2}=\frac{\frac{bn^2}{3}+(m-1)A_c(n-d_c)^2+mA_t(d-n)^2}{\frac{bn^2}{2}+(m-1)A_c(n-d_c)-mA_t(d-n)}$$

Hence, for a given position of the eccentric load the depth of neutral axis can be determined from the above relation.

Problem 528. *The haunch section of a bowstring girder bridge is subjected to a bending moment of 1,931,500 kg. cm. and a normal thrust of 81,260 kg. The section is 40 cm. wide and 80 cm. deep. Eight bars of 18 mm. dia. are provided at the top and an equal reinforcement at the bottom. The reinforcements are placed at an effective cover of 6 cm. from the respective edges. Determine the maximum stress in concrete and steel. The modular ratio may be taken as 15.*

Solution. $A_c = A_t = 20{\cdot}36\ cm.^2$

See Fig. 874.

Eccentricity from the centre of the section

$$= e = \frac{M}{P} = \frac{1{,}93{,}15{,}000}{81260}\ cm.$$
$$= 23{\cdot}77\ cm$$

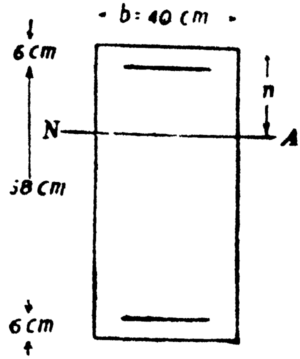


Fig. 874

The eccentricity is considerably large to produce tensile stresses.

Hence the depth of the neutral axis is given by

$$e + n - \frac{D}{2} = \frac{\frac{bn^3}{3} + (m-1)A_c(n-d_c)^2 + mA_t(d-n)^2}{\frac{bn^2}{2} + (m-1)A_c(n-d_c) - mA_t(d-n)}$$

$$\therefore\ 23{\cdot}77 + n - \frac{80}{2} = \frac{\frac{40n^3}{3} + 14\times 20{\cdot}36(n-6)^2 + 15\times 20{\cdot}36(74-n)^2}{\frac{40n^2}{2} + 14\times 20{\cdot}36(n-6) - 15\times 20{\cdot}36(74-n)}$$

Simplifying, we get

$$n^3 - 48{\cdot}69n^2 + 2209{\cdot}05n - 193212{\cdot}471 = 0$$

or $\quad n[n^2 - 48{\cdot}69n + 2209{\cdot}05] = 193212{\cdot}471$

Solving by trial and error, we have

When $n = 63$	$LHS = 195966{\cdot}54$
$n = 62$	$LHS = 188124{\cdot}74$
$n = 62{\cdot}7$	$LHS = 193593{\cdot}77$
$n = 62{\cdot}6$	$LHS = 192796{\cdot}73$
$n = 62{\cdot}66$	$LHS = 193269{\cdot}13$
$n = 62{\cdot}652$	$LHS = 193206{\cdot}24$
$n = 62{\cdot}653$	$LHS = 193213{\cdot}71$

Taking moments about the tensile steel,

$$40n\frac{c}{2}\left(74 - \frac{n}{3}\right) + (15-1)\times 20{\cdot}36\left(\frac{n-6}{n}\right)c \times 68$$
$$= 81260 \times (34 + 23{\cdot}77)$$

$$\therefore\ 20\times 62{\cdot}653\ \frac{c}{2}\left(74 - \frac{62{\cdot}653}{3}\right) + 14\times 20{\cdot}36\times \frac{56{\cdot}653}{62{\cdot}653}\times 68c$$
$$= 81260 \times 57{\cdot}77$$

$$66557{\cdot}5c + 17526{\cdot}5c = 46{,}94{,}390{\cdot}2$$

$$\therefore \qquad c = 55{\cdot}83\ kg./cm.^2$$

Corresponding stress in tensile steel is given by

$$\frac{15\times 55{\cdot}83}{t}=\frac{62{\cdot}653}{74-62{\cdot}653}$$

$$\therefore \qquad t=\frac{15\times 55{\cdot}83\times 11{\cdot}347}{62{\cdot}653}\ kg./cm.^2$$

$$t=151.7\ kg./cm.^2$$

Examples on Chapter 19

1. The cross-section of a singly reinforced concrete beam is 30 *cms*. wide and 50 *cms* deep to the centre of the tension reinforcement which consists of four bars of 16 *mm* diameter. If the stresses in concrete and steel are not to exceed 50 $kg/cm.^2$ and 1400 $kg/cm.^2$ respectively, determine the moment of resistance of the beam. Take $m=18$. (497,-00 *kg. cm.*)

2. A reinforced concrete beam 30 *cm.* wide by 60 *cm.* total depth has a span of 8 *metres*. Find the necessary tension reinforcement at the centre of the span to enable the beam to carry a load of 600 *kg./m.* in addition to its own weight.

Concrete cover below the steel centre	$=4$ *cm.*
Weight of the beam	$=2400\ kg./metre^3$
Permissible stress in concrete	$=50\ kg./cm.^2$
Permissible stress in steel	$=1400\ kg./cm.^2$
Modular ratio	$=18$ ($12{\cdot}94\ cm.^2$)

3. A singly reinforced beam is 20 *cm.* wide and 40 *cm.* deep to the centre of tension reinforcement which consists of four bars of 16 *mm.* diameter. If the stresses in concrete and steel are not to exceed 50 $kg/cm.^2$ and 1400 $kg./cm^2$; find the moment of resistance of the beam. Take $m=18$. (325,600 *kg. cm.*)

4. The moment of resistance of a rectangular concrete beam of breadth *b cm.* and effective depth *d cm.* is $9{\cdot}15\ bd^2$. If the stresses in the outside fibre of concrete and steel shall not exceed 50 kg/cm^2 and 1400 $kg/cm.^2$ respectively and the modular ratio equals 18 find the ratio of the depth of neutral axis from the outside compression fibres to the effective depth of a beam and the ratio of the area of the tensile steel to the effective area of the beam. The beam is reinforced for tension only ($0{\cdot}427$; $0{\cdot}00584$)

5. Find the percentage of tension reinforcement necessary for a singly reinforced balanced rectangular section if the permissible stresses in concrete and steel are *c* and *t* $kg./cm.^2$ respectively and the modular ratio is *m*

$$\left(50\frac{mc^2}{t(mc+t)}\%\right)$$

6. The section of a singly reinforced concrete beam is subjected to a sagging bending moment of 20,000 *kg m*. If the stresses in concrete and steel are not to exceed 50 $kg./cm.^2$ and 1400 $kg./cm.^2$ respectively, find the dimensions of the beam. The width of the beam may be made ¾ the effective depth. Take $m=18$.

(*51 cm.* × *68 cm.* effective)

7. A simply supported slab is subjected to a bending moment of 850 *kg. m.* per *metre* width. If the stresses in concrete and steel are not to exceed 50 $kg./cm.^2$ and 1400 $kg./cm.^2$, find the effective depth required and the area of steel. Take. $m=18$.

(*10 cm.* ; *6·98* $cm.^2$ per *metre* width)

8. A simply supported rectangular concrete beam is reinforced for tension only. The beam is subjected to a bending moment of 918,000 *kg. cm.* If the beam is 30 *cm.* wide and the stresses in concrete and steel are not to exceed 50 $kg./cm.^2$ and 1400 $kg./cm.^2$ respectively, find the effective depth and the area of steel.

(*60 cm.* ; *12·53* $cm.^2$)

9. A doubly reinforced rectangular concrete beam is 30 *cm.* wide and 50 *cm.* deep to the centre of tension steel. It is reinforced with four bars of 20 *mm.* diameter as compressive steel at an effective cover of 4 *cm* and with four bars of 24 *mm* diameter as tensile steel. If the stresses in concrete and steel shall not exceeed 50 $kg./cm.^2$ and 1400 $kg./cm.^2$ respectively, find the moment of resistance of the section. Take $m=18$ (*1,053,200 kg. cm.*)

10. A doubly reinforced concrete beam is 24 *cm.* wide and 55 *cm.* deep to the centre of tensile reinforcement. The areas of compression and tension steel are 15 $cm.^2$ each. The centre of the compression steel is 5 *cm* from the compression edge. If $m=18$ and the section is subjected to a maximum bending moment of 750,000 *kg. cm.*, find the stress in concrete and steel.

($c=34$ $kg./cm.^2$; $t=1020$ $kg./cm.^2$; $t_c=463·5$ $kg./cm.^2$)

11. A rectangular beam reinforced on both sides is 30 *cm.* wide and 70 *cm.* deep. The centres of steel are 4 *cm.* from the respective edges. If the limiting stresses in concrete and steel are 50 $kg./cm.^2$ and 1400 $kg./cm.^2$ respectively, find the steel areas for a bending moment of 1,300,000 *kg. cm.* Take $m=18$.

($A_c=4·22$ $cm.^2$; $A_t=15·99$ $cm.^2$)

12. An R.C. beam 30 *cm.* × 30 *cm.* in section is reinforced with four bars of 24 *mm.* diameter as compression steel and an equal amount as tension steel, the effective cover to the reinforcement being 5 *cm.* The section is subjected to a bending moment of 3500 *kg. m.* If the modular ratio is 18, find the stresses induced in concrete and steel. ($c=37·89$ $kg./cm.^2$; $t=748·2$ $kg./cm.^2$; $t_c=396·2$ $kg./cm^2$.)

13. A doubly reinforced beam 30 *cm*. wide and 64 *cm*. deep is reinforced with four bars of 24 *m*. diameter as compression steel and an equal amount as tensile steel. The respective reinforcements are provided at an effective cover of 4 *cm*. If the stresses in concrete and steel shall not exceed 50 *kg./cm.*^{2} and 1400 *kg./cm.*^{2} respectively and the modular ratio is 18, find the moment of resistance of the section.

What would be the moment of resistance based on steel beam theory? (*1,373,400 kg. cm.* ; *1,419,040 kg. cm.*)

14. A pile 30 *cm.*$\times$30 *cm*. in section is reinforced with four bars of 24 *mm*. diameter, the effective cover to the centre of reinforcement being 5 *cm* The pile section is subjected to a bending moment of 1600 *kg. m*. while hoisting it. If the modular ratio is 18, find the stresses induced in concrete and steel.

(c=*47·56 kg./cm*^{2}. ; t=*1126 kg./cm.*^{2} : t_c=*439·2 kg./cm.*^{2})

APPENDIX I

TABLE 1

Moment Coefficients for Continuous Beams

(Equal Spans) (Dead Loads)

M = Coefficient × total load on span × span

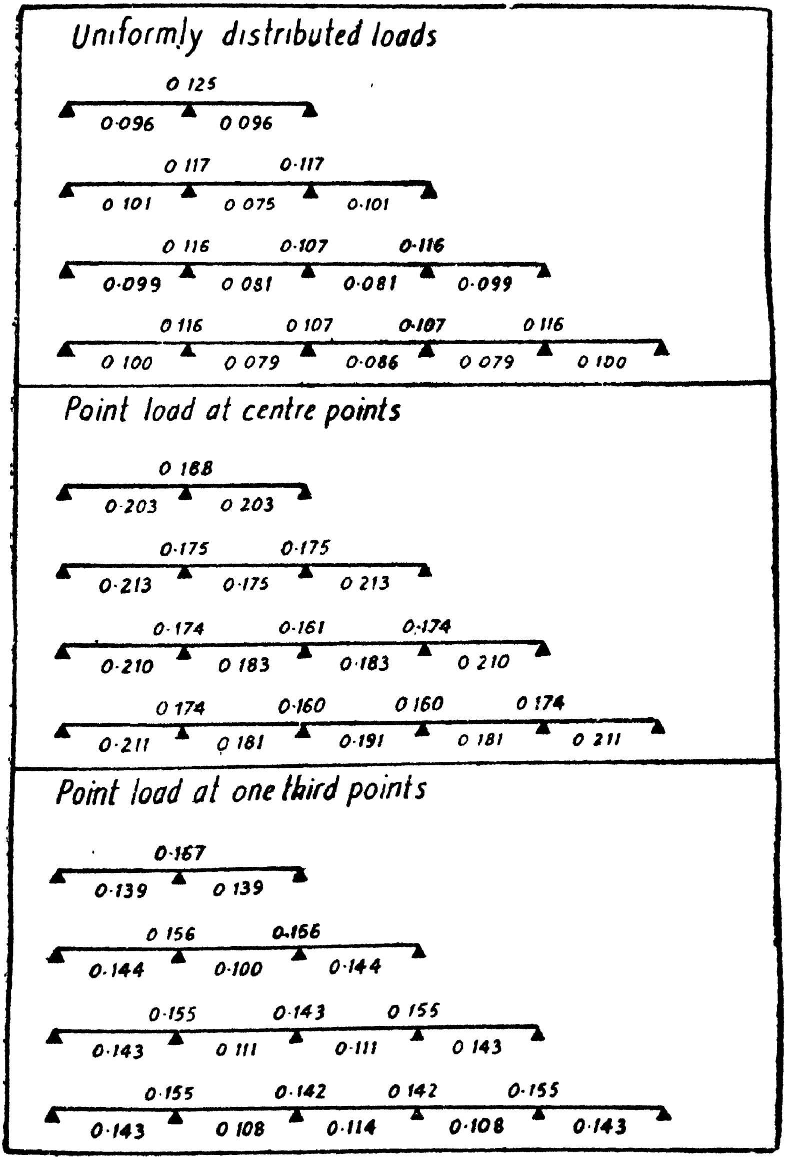


TABLE 2

Moment Coefficients for Continuous Beams

(Equal Spans) (Live Loads)

M = Coefficient × total load on span × span

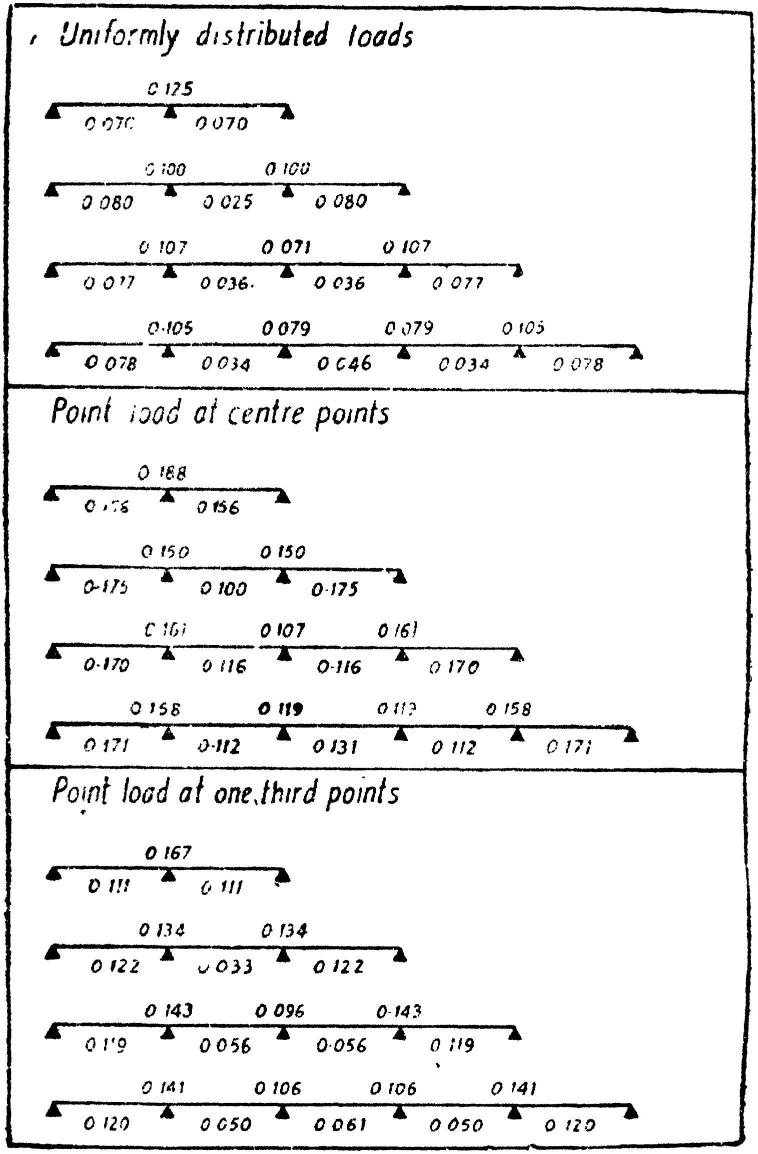
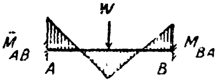
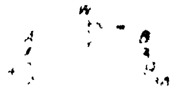


TABLE 3

Fixed end moments M_{AB} and M_{BA} and end slope angles θ_{AB} and θ_{BA}

Concentrated Loads





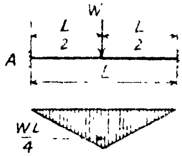
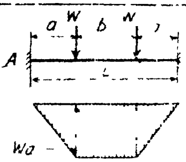
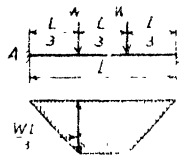
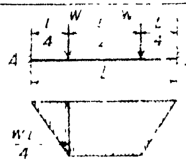
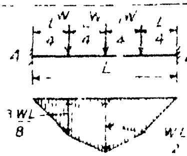
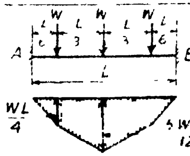
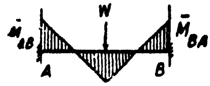
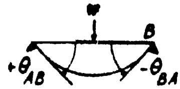
No	LOADING SYSTEMS F.cc B M DIAGRAMS	$\bar{M}_{AB}$ AND $\bar{M}_{BA}$	θ_{AB} AND θ_{BA}
1		$\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{WL}{8}$	$\theta_{AB} = -\theta_{BA} = \frac{WL^2}{16EI}$
2		$\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{Wa(L-a)}{L}$	$\theta_{AB} = -\theta_{BA} = \frac{Wa(L-a)}{2EI}$
3		$\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{2WL}{9}$	$\theta_{AB} = -\theta_{BA} = \frac{WL^2}{9EI}$
4		$\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{3WL}{16}$	$\theta_{AB} = -\theta_{BA} = \frac{3WL^2}{32EI}$
5		$\bar{M}_{Ab} = -\bar{M}_{BA} = -\frac{5WL}{16}$	$\theta_{AB} = -\theta_{BA} = \frac{5WL^2}{32EI}$
6		$\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{19WL}{12}$	$\theta_{AB} = -\theta_{BA} = \frac{19WL^2}{144EI}$

TABLE 3 (Contd.)

Fixed end moments M_{AB} and M_{BA} and end slope angles θ_{AB} and θ_{BA}

Concentrated Loads





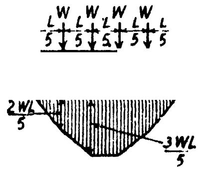
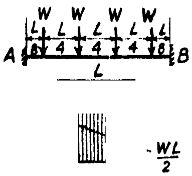
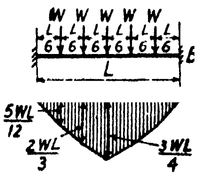
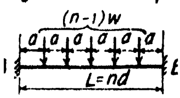
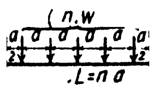
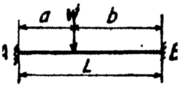
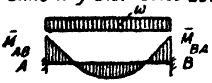
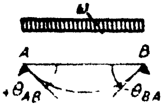
No	LOADING SYSTEMS Free B.M DIAGRAMS	$\bar{M}_{AB}$ AND M_{BA}	θ_{AB} AND θ_{BA}
		$\bar{M}_{AB} = -M_{BA} \cdot 2WL$	$\theta_{AB} = -\theta_{BA} = \frac{WL^2}{5EI}$
		$\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{11WL}{32}$	$\theta_{AB} = -\theta_{BA} = \frac{11WL^2}{64EI}$
		$\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{35WL}{72}$	$\theta_{AB} = -\theta_{BA} = \frac{35WL^2}{144EI}$
10		$\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{WL}{12} \cdot \frac{n^2-1}{n}$	$\theta_{AB} = -\theta_{BA} = \frac{WL^2}{24EI} \cdot \frac{n^2-1}{n}$
11		$\bar{M}_{AB} = -\bar{M}_{BA} = \frac{WL}{24} \cdot \frac{2n^2+1}{n}$	$\theta_{AB} = -\theta_{BA} = \frac{WL^2}{48EI} \cdot \frac{2n^2+1}{n}$
12		$\bar{M}_{AB} = wab^2$ $\bar{M}_{BA} = +\frac{wa^2b}{L^2}$	$\theta_{AB} = \frac{wab}{6LEI}(b+L)$ $-\theta_{BA} = \frac{wab}{6LEI}(a+L)$

TABLE 4

Fixed end moments M_{AB} M_{BA} and end slope angles θ_{AB} and θ_{BA}

Uniformly Distributed Loads





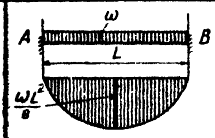
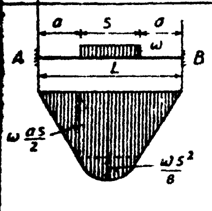
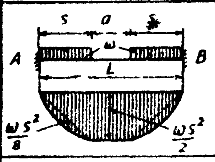
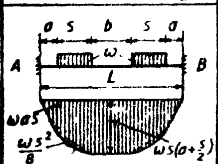
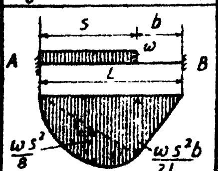
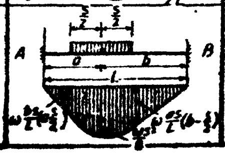
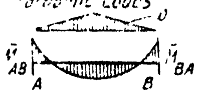
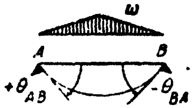
No	LOADING SYSTEMS Free B M DIAGRAM	$\bar{M}_{AB}$ AND $\bar{M}_{BA}$	θ_{AB} AND θ_{BA}
1		$\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{\omega L^2}{12}$	$\theta_{AB} = -\theta_{BA} = \frac{\omega L^3}{24EI}$
2		$\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{\omega s}{24L}(3L^2 - s^2)$ For $s = \frac{L}{2}$, $\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{11\omega L^2}{192}$ For $s = \frac{L}{3}$, $\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{13\omega L^2}{324}$ For $s = \frac{L}{4}$, $\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{47\omega L^2}{1536}$	$\theta_{AB} = -\theta_{BA} = \frac{\omega s}{48EI}(3L^2 - s^2)$ For $s = \frac{L}{2}$, $\theta_{AB} = -\theta_{BA} = \frac{11\omega L^3}{384EI}$ For $s = \frac{L}{3}$, $\theta_{AB} = -\theta_{BA} = \frac{13\omega L^2}{648EI}$ For $s = \frac{L}{4}$, $\theta_{AB} = -\theta_{BA} = \frac{47\omega L^3}{3072EI}$
3		$\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{\omega s^2}{6L}(2L + a)$ For $a = s = \frac{L}{3}$, $\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{7\omega L^2}{162}$	$\theta_{AB} = -\theta_{BA} = \frac{\omega s^2}{12EI}(2L + a)$ For $a = s = \frac{L}{3}$, $\theta_{AB} = -\theta_{BA} = \frac{7\omega l^3}{324EI}$
4		$\bar{M}_{AB} = -\bar{M}_{BA} =$ $-\frac{\omega s}{12L}[3L^2 - 3(b+s)^2 - s^2]$ For $a = s = b = \frac{L}{5}$, $\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{31\omega L^2}{750}$	$\theta_{AB} = -\theta_{BA} =$ $= \frac{\omega s}{24EI}[3L^2 - 3(b+s)^2 - s^2]$ For $a = s = b = \frac{L}{5}$, $\theta_{AB} = -\theta_{BA} = \frac{31\omega L^3}{1500EI}$
5		$\bar{M}_{AB} = -\frac{\omega s^2}{12L^2}[2L(3L - 4s) + 3s^2]$ $\bar{M}_{BA} = +\frac{\omega s^3}{12L^2}(4L - 3s)$ For $s = b = \frac{L}{2}$, $\bar{M}_{AB} = -\frac{11\omega L^2}{192}$ $\bar{M}_{BA} = \frac{5\omega L^2}{192}$	$\theta_{AB} = \frac{\omega s^2}{24EIL}(2L - s)^2$ $-\theta_{BA} = \frac{\omega s^2}{24EIL}(2L^2 - s^2)$ For $s = b = \frac{L}{2}$, $\theta_{AB} = \frac{9\omega L^3}{384EI}$ $-\theta_{BA} = \frac{7\omega L^3}{384EI}$
6		$\bar{M}_{AB} = -\frac{\omega s}{12L^2}[12ab^2 + s^2(L - 3b)]$ $\bar{M}_{BA} = +\frac{\omega s}{12L^2}[12a^2b + s^2(L - 3a)]$	$\theta_{AB} = \frac{\omega bs}{24L}[4a(b+L)s^2]\frac{1}{EI}$ $-\theta_{BA} = \frac{\omega as}{24L}[4b(a+L) - s^2]\frac{1}{EI}$

TABLE 5

Fixed end moments M_{AB} and M_{BA} and end slope angles θ_{AB} and θ_{BA}

Triangular Loads, Applied Loads, Parabolic Loads





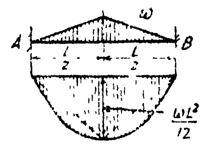
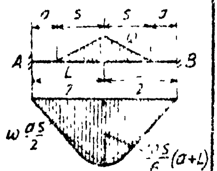
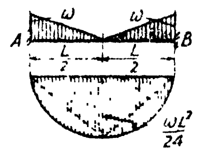
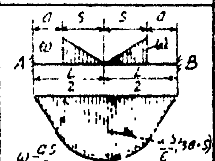
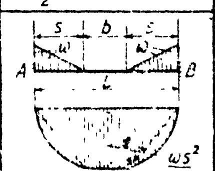
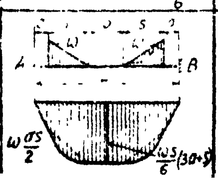
No	LOADING SYSTEMS Free B M DIAGRAM.	$\bar{M}_{AB}$ AND $\bar{M}_{BA}$	θ_{AB} AND θ_{BA}
1		$\bar{M}_{AB}=-\bar{M}_{BA}=-\frac{5\omega L^2}{96}$	$\theta_{AB}=-\theta_{BA}=\frac{5\omega L^3}{192EI}$
2		$\bar{M}_{AB}=-\bar{M}_{BA}=-\frac{\omega s}{24L}(3L^2-2s^2)$ For $a=s=\frac{L}{4}$, $\bar{M}_{AB}=-\bar{M}_{BA}=-\frac{23\omega L^2}{768}$	$\theta_{AB}=-\theta_{BA}=\frac{\omega s}{48EI}(3L^2-2s^2)$ For $a=s=\frac{L}{4}$, $\theta_{AB}=-\theta_{BA}=\frac{23\omega L^3}{1536EI}$
3		$\bar{M}_{AB}=-\bar{M}_{BA}=-\frac{\omega L^2}{32}$	$\theta_{AB}=-\theta_{BA}=\frac{\omega L^3}{64EI}$
4		$\bar{M}_{AB}=-\bar{M}_{BA}=-\frac{\omega s}{6L}(L^2-2s^2)$ For $a=s=\frac{L}{4}$, $\bar{M}_{AB}=-\bar{M}_{BA}=-\frac{7\omega L^2}{256}$	$\theta_{AB}=-\theta_{BA}=\frac{\omega s}{16EI}(L^2-2s^2)$ For $a=s=\frac{L}{4}$, $\theta_{AB}=-\theta_{BA}=\frac{7\omega L^3}{512EI}$
5		$\bar{M}_{AB}=-\bar{M}_{BA}=-\frac{\omega s^2}{12L}(2L-s)$ For $s=b=\frac{L}{3}$, $\bar{M}_{AB}=-\bar{M}_{BA}=-\frac{5\omega L^2}{324}$	$\theta_{AB}=-\theta_{BA}=\frac{\omega s^2}{24EI}(2L-s)$ For $s=b=\frac{L}{3}$, $\theta_{AB}=-\theta_{BA}=\frac{\omega L^3}{648EI}$
6		$\bar{M}_{AB}=-\frac{\omega s}{12L}[6a(L-a)+s(2b+3s)]$ For $a=b=s=\frac{L}{5}$, $\bar{M}_{AB}=-\bar{M}_{BA}=-\frac{29\omega L^2}{1500}$	$\theta_{AB}=-\theta_{BA}=\frac{\omega s}{24EI}[6a(L-a)+s(2b+3s)]$ For $a=b=s=\frac{L}{5}$, $\theta_{AB}=-\theta_{BA}=\frac{29\omega L^3}{3000EI}$

TABLE 5 (Contd.)

Fixed end moments M_{AB} and M_{BA} and end slope angles θ_{AB} and θ_{BA}

Triangular Loads
Applied Moments
Parabolic Loads

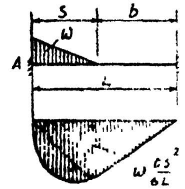
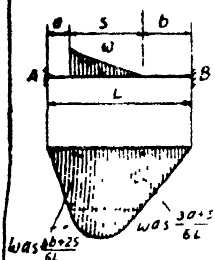
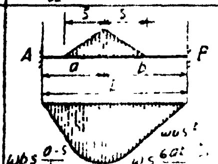
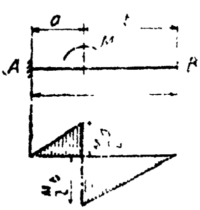
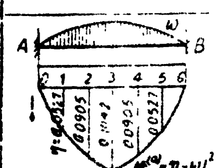
No.	LOADING SYSTEMS Free B M DIAGRAM	$\bar{M}_{AB}$ AND $\bar{M}_{BA}$	θ_{AB} AND θ_{BA}
7		$\bar{M}_{AB} = -\frac{ws^2}{60L^2}(10bL+3s^2)$ $\bar{M}_{BA} = +\frac{ws^3}{60L^2}(5b+2\ [illegible])$ For $s = b = \frac{L}{2}$, $\bar{M}_{AB} = -\frac{[illegible]\,wL^2}{560}$ $\bar{M}_{BA} = +\frac{[illegible]\,wL^2}{960}$	$\theta_{AB} = \frac{ws^2}{360L}\left[5b(4L+s)-8s^2\right]\frac{1}{EI}$ $-\theta_{BA} = \frac{ws^2}{360L}\left[10b(L+s)-7s^2\right]\frac{1}{EI}$ For $s = b = \frac{L}{2}$, $\theta_{AB} = \frac{53wL^3}{5760EI}$ $-\theta_{BA} = \frac{37wL^3}{5760EI}$
8		$\bar{M}_{AB} = -\frac{ws}{60L^2}\left[10b^2(3a+s)+\right.$ $\left.+s^2(15a+10b+3s)+40abs\right]$ $\bar{M}_{BA} = +\frac{ws}{60L^2}\left[10a^2(3b+2s)+\right.$ $\left.+s^2(10a+5b+2s)+20abs\right]$ For $a = s = b = \frac{L}{3}$, $\bar{M}_{AB} = -\frac{wL^2}{45}$, $\bar{M}_{BA} = +\frac{29wL^2}{1620}$	$\theta_{AB} = \frac{ws}{360L}\left[10a^2(3b+2s)+\right.$ $+20b^2(3a+s)+s^2(40a+$ $\left.+25b+8s)+100abs\right]\frac{1}{EI}$ $\theta_{BA} = \frac{ws}{360L}\left[20a^2(3b+2s)+\right.$ $+10b^2(3a+s)+s^2(35a+$ $\left.+20b+7s)+80abs\right]\frac{1}{EI}$ For $a = s = b = \frac{L}{3}$, $\theta_{AB} = \frac{101wL^3}{9720EI}$, $-\theta_{BA} = \frac{47wL^3}{4860EI}$
9		$\bar{M}_{AB} =$ $= -\frac{ws}{6L^2}\left[6ab^2+s^2(a-2b)\right]$ $\bar{M}_{BA} =$ $+\frac{ws}{6L^2}\left[6a^2b+s^2(b-2a)\right]$	$\theta_{AB} = \frac{wbs}{12L}\left[2a(b+L)-s^2\right]\frac{1}{EI}$ $-\theta_{BA} = \frac{was}{12L}\left[2b(a+L)-s^2\right]\frac{1}{EI}$
10		$\bar{M}_{AB} = +M\frac{b}{L}\left(2-\frac{3b}{L}\right)$ $\bar{M}_{BA} = +M\frac{a}{L}\left(2-\frac{3a}{L}\right)$ For $a = 0$, $\bar{M}_{AB} = -M$, $\bar{M}_{BA} = 0$ For $a = \frac{L}{2}$, $\bar{M}_{AB} = \bar{M}_{BA} = +\frac{M}{4}$ For $a = L$, $\bar{M}_{AB} = 0$, $\bar{M}_{BA} = +M$	$\theta_{AB} = M\frac{L}{6}\left[\frac{3b^2}{L^2}-1\right]\frac{1}{EI}$ $\theta_{BA} = M\frac{L}{6}\left[1-\frac{3a^2}{L^2}\right]\frac{1}{EI}$ For $a = 0$, $\theta_{AB} = M\frac{L}{3EI}$, $\theta_{BA} = M\frac{L}{6EI}$ For $a = \frac{L}{2}$, $\theta_{AB} = \theta_{BA} = \frac{ML}{24EI}$ For $a = L$, $\theta_{AB} = -M\frac{L}{6EI}$ $\theta_{BA} = -M\frac{L}{3EI}$
11		$\bar{M}_{AB} = -\bar{M}_{BA} = -\frac{wL^2}{15}$	$\theta_{AB} = -\theta_{BA} = \frac{wL^3}{30EI}$

TABLE 5 (Contd.)

Fixed end moments M_{AB} and M_{BA} and end slope angles θ_{AB} and θ_{BA}

Triangular Loads
Applied Moments
Parabolic Loads

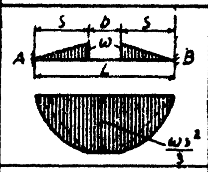
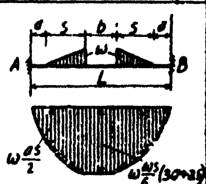
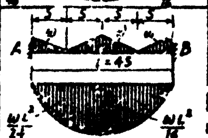
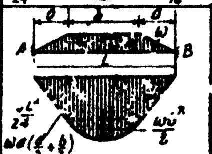
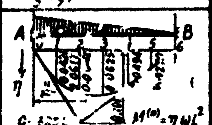
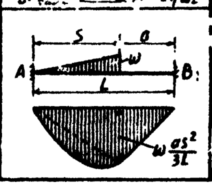
No	LOADING SYSTEMS Free B.M DIAGRAM	$\bar{M}_{AB}$ AND $\bar{M}_{BA}$	θ_{AB} AND θ_{BA}
12		$\bar{M}_{AB}=-\bar{M}_{BA}=-\frac{wS^2}{12L}(4L-3S)$ For $s=b=\frac{L}{3}$, $\bar{M}_{AB}=-\bar{M}_{BA}=-\frac{wL^2}{36}$	$\theta_{AB}=-\theta_{BA}=\frac{wS^2}{24EI}(4L-3S)$ For $s=b=\frac{L}{3}$, $\theta_{AB}=-\theta_{BA}=\frac{wL^3}{12EI}$
13		$\bar{M}_{AB}=-\bar{M}_{BA}=$ $=+\frac{wS}{12L}[6a(L-a)-S(4b+5S)]$ For $a=s=b=\frac{L}{5}$, $\bar{M}_{AB}=-\bar{M}_{BA}=-\frac{11wL^2}{900}$	$\theta_{AB}=-\theta_{BA}=$ $=\frac{wS}{24EI}[6a(L-a)+S(4b+5S)]$ For $a=b=\frac{L}{5}$, $\theta_{AB}=-\theta_{BA}=\frac{11wL^3}{1000EI}$
14		$\bar{M}_{AB}=-\bar{M}_{BA}=-\frac{17wL^2}{384}$	$\theta_{AB}=-\theta_{BA}=\frac{17wL^3}{768EI}$
15		$\bar{M}_{AB}=-\bar{M}_{BA}=-\frac{3wL^2}{128}$	$\theta_{AB}=-\theta_{BA}=\frac{5wL^3}{256EI}$
16		$\bar{M}_{AB}=-\bar{M}_{BA}=$ $=-\frac{w}{12L}[L^3-a^2(2L-a)]$ For $a=b=\frac{L}{3}$, $\bar{M}_{AB}=-\bar{M}_{BA}=-\frac{11wL^2}{162}$	$\theta_{AB}=-\theta_{BA}=\frac{w}{24EI}[L^3-a^2(2L-a)]$ For $a=b=\frac{L}{3}$, $\theta_{AB}=-\theta_{BA}=\frac{11wL^3}{324EI}$
17		$\bar{M}_{AB}=-\frac{wL^2}{20}$ $\bar{M}_{BA}=+\frac{wL^2}{30}$	$\theta_{AB}=\frac{wL^3}{45EI}$ $-\theta_{BA}=\frac{7wL^3}{360EI}$
18		$\bar{M}_{AB}=-\frac{wS^2}{30L}[10a^2+S(5a+S)]$ $\bar{M}_{BA}=+\frac{wS^3}{20L^2}(5a+S)$ For $S=a=\frac{L}{2}$, $M_{AB}=-\frac{wL^2}{30}$, $M_{BA}=+\frac{wL^2}{160}$	$\theta_{AB}=\frac{wS^2}{360L}[40a^2+7S(5a+S)]$ $\theta_{BA}=\frac{wS^2}{180L}[10a^2+4S(5a+S)]$ For $S=a=\frac{L}{2}$, $\theta_{AB}=\frac{41wL^3}{2880EI}$, $\theta_{BA}=\frac{17wL^3}{1440EI}$

TABLE 6
Fixed end moments M°_{AB} for hinged members

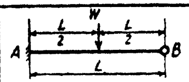
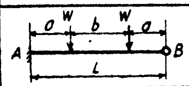
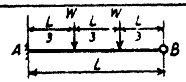
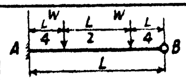
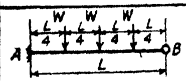
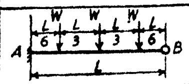
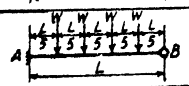
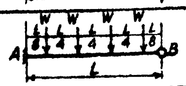
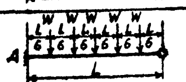
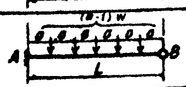
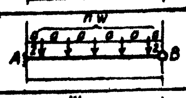
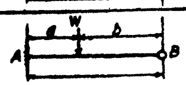
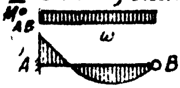
No	LOADING SYSTEMS Free B.M DIAGRAM	$\bar{M}^{o}_{AB}$
1		$\bar{M}^{o}_{AB} = -\frac{3WL}{16}$
2		$\bar{M}^{o}_{AB} = -\frac{3Wa(L-a)}{2L}$
3		$\bar{M}^{o}_{AB} = -\frac{WL}{3}$
4		$\bar{M}^{o}_{AB} = -\frac{9WL}{32}$
5		$\bar{M}^{o}_{AB} = -\frac{15WL}{32}$
6		$\bar{M}^{o}_{AB} = -\frac{19WL}{48}$
7		$\bar{M}^{o}_{AB} = -\frac{3WL}{5}$
8		$\bar{M}^{o}_{AB} = -\frac{33WL}{64}$
9		$\bar{M}^{o}_{AB} = -\frac{35WL}{48}$
10		$\bar{M}^{o}_{AB} = -\frac{WL}{8} \cdot \frac{n^2-1}{n}$
11		$\bar{M}^{o}_{AB} = \frac{WL}{16} \cdot \frac{2n^2+1}{n}$
12		$\bar{M}^{o}_{AB} = \frac{Wab}{2L^2}(b+L)$

TABLE 7

Fixed end moments M°_{AB} for hinged members

Uniformly Distributed Loads



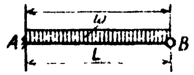
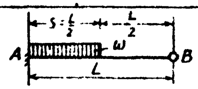
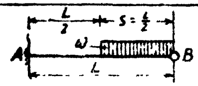
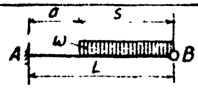
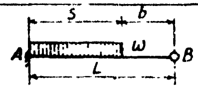
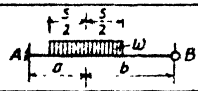
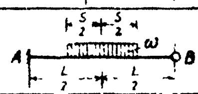
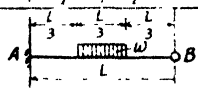
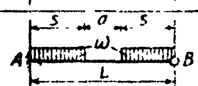
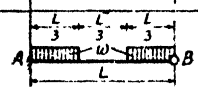
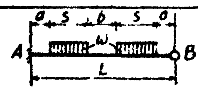
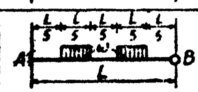
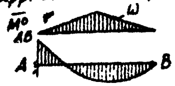
No	LOADING SYSTEMS Free B M DIAGRAM	$\overline{M^{\circ}_{AB}}$
1		$\overline{M^{\circ}_{AB}} = -\frac{\omega L^2}{8}$
2		$\overline{M^{\circ}_{AB}} = -\frac{9\omega L^2}{128}$
3		$\overline{M^{\circ}_{AB}} = -\frac{7\omega L^2}{128}$
4		$\overline{M^{\circ}_{AB}} = -\frac{\omega s^2}{8L^2}(2L^2-s^2)$
5		$\overline{M^{\circ}_{AB}} = -\frac{\omega s^2}{8L^2}(4bL+s^2)$
6		$\overline{M^{\circ}_{AB}} = -\frac{\omega bs}{8L^2}(4a^2+8ab-s^2)$
7		$\overline{M^{\circ}_{AB}} = -\frac{\omega s}{16L}(3L^2-s^2)$
8		$\overline{M^{\circ}_{AB}} = -\frac{13\omega L^2}{216}$
9		$\overline{M^{\circ}_{AB}} = -\frac{\omega s^2}{4L}(2L+a)$
10		$\overline{M^{\circ}_{AB}} = -\frac{7\omega L^2}{108}$
11		$\overline{M^{\circ}_{AB}} = -\frac{\omega s}{8L}\left[3L^2-3(b+s)^2-s^2\right]$
12		$\overline{M^{\circ}_{AB}} = -\frac{31\omega L^2}{500}$

TABLE 8
Fixed end moments M°_{AB} for hinged members

Triangular Loads, Applied Moments, Parabolic Loads



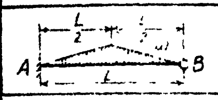
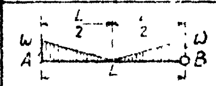
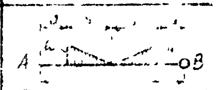
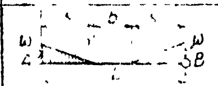
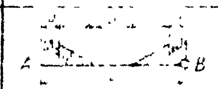
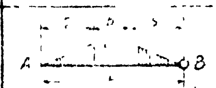
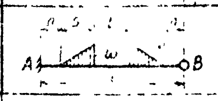
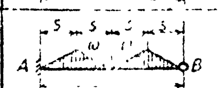
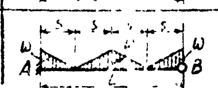
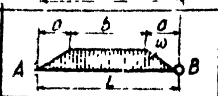
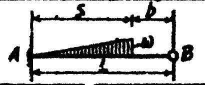
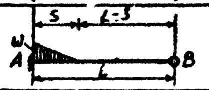
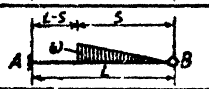
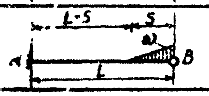
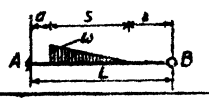
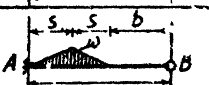
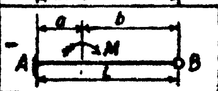
No	LOADING SYSTEMS Free B M DIAGRAM	$\overline{M}^{o}_{AB}$
1		$\overline{M}^{o}_{AB} = -\frac{5wL^2}{64}$
2		$\overline{M}^{o}_{AB} = -\frac{ws}{16L}(3L^2-2s^2)$, For $a = s = \frac{L}{4}$, $\overline{M}^{o}_{AB} = -\frac{23wL^2}{512}$
3		$\overline{M}^{o}_{AB} = -\frac{3wL^2}{64}$
4		$\overline{M}^{o}_{AB} = -\frac{3ws}{16L}(L^2-2s^2)$, For $a = s = \frac{L}{4}$, $\overline{M}^{o}_{AB} = -\frac{21wL^2}{512}$
5		For $s = b = \frac{L}{3}$, For $s = \frac{L}{4}$, $\overline{M}^{o}_{AB} = -\frac{ws^2}{8L}(4L-s)$, $\overline{M}^{o}_{AB} = -\frac{5wL^2}{216}$, $\overline{M}^{o}_{AB} = -\frac{7wL^2}{512}$
6		For $a = b = s = \frac{L}{5}$, $\overline{M}^{o}_{AB} = -\frac{ws}{8L}[6a(L-a)+s(7b+$ [illegible] $)]$, $\overline{M}^{o}_{AB} = -\frac{29wL^2}{\text{[illegible]}}$
7		$\overline{M}^{o}_{AB} = -\frac{ws^2}{8L}(4L-3s)$, For $s = b = \frac{L}{3}$, $\overline{M}^{o}_{AB} = -\frac{wL^2}{2\text{[illegible]}}$
8		For $a = b = \frac{L}{\text{[illegible]}}$, $\overline{M}^{o}_{AB} = -\frac{ws}{8L}[6a(L-a)+s(4b+5s)]$, $\overline{M}^{o}_{AB} = -\frac{33wL^2}{1000}$
9		$\overline{M}^{o}_{AB} = -\frac{17wL^2}{256}$
10		$\overline{M}^{o}_{AB} = -\frac{15wL^2}{256}$
11		For $a = b = \frac{L}{3}$, $\overline{M}^{o}_{AB} = -\frac{w}{8L}[L^3-a^2(2L-a)]$, $\overline{M}^{o}_{AB} = \frac{33wL^2}{324}$
12		$\overline{M}^{o}_{AB} = -\frac{wL^2}{15}$

TABLE 8 (Contd)

Fixed end moments $\overline{M}^o_{AB}$ for hinged members

Triangular Loads
Applied Moments
Parabolic Loads

No.	LOADING SYSTEMS Free B.M. DIAGRAM	$\overline{M}^o_{AB}$
13		$\overline{M}^o_{AB} = -\frac{7wL^2}{120}$
14		$\overline{M}^o_{AB} = -\frac{ws^2}{120L^2}(40b^2+35bs+7s^2)$, For $s=b=\frac{L}{2}$, $\overline{M}^o_{AB} = -\frac{41wL^2}{960}$
15		$\overline{M}^o_{AB} = -\frac{ws^2}{120L^2}(20L^2-15Ls+3s^2)$, For $s=\frac{L}{2}$, $\overline{M}^o_{AB} = -\frac{53wL^2}{1920}$
16		$\overline{M}^o_{AB} = -\frac{ws^2}{30L^2}(5L^2-3s^2)$. For $s=\frac{L}{2}$, $\overline{M}^o_{AB} = -\frac{17wL^2}{480}$
17		$\overline{M}^o_{AB} = -\frac{ws^2}{120L^2}(10L^2-3s^2)$, For $s=\frac{L}{2}$, $\overline{M}^o_{AB} = -\frac{37wL^2}{1920}$
18		$\overline{M}^o_{AB} = -\frac{ws}{120L^2}\left[10a^2(3b+2s)+20a(3b^2+2s^2)+5bs(4b+5s)+4s(25ab+2s^2)\right]$ For $a=s=b=\frac{L}{3}$, $\overline{M}^o_{AB} = -\frac{101wL^2}{3240}$
19		$\overline{M}^o_{AB} = -\frac{ws}{120L^2}\left[10a^2(3b+s)-20a(3b^2+s^2)+5bs(8b+7s)+s(80ab+7s^2)\right]$ For $a=s=b=\frac{L}{3}$, $\overline{M}^o_{AB} = -\frac{47wL^2}{1620}$
20		$\overline{M}^o_{AB} = -\frac{ws^2}{4L^2}(4L^2-7Ls+3s^2)$, For $b=\frac{L}{2}$, $\overline{M}^o_{AB} = -\frac{39wL^2}{1024}$
21		$\overline{M}^o_{AB} = -\frac{ws^2}{4L^2}(2L^2-3s^2)$, For $a=\frac{L}{2}$, $\overline{M}^o_{AB} = -\frac{29wL^2}{1024}$
22		$\overline{M}^o_{AB} = -\frac{w}{120L}(b+L)(7L^2-3b^2)$
23		$\overline{M}^o_{AB} = +\frac{M}{2}\left[1-\frac{3b^2}{L^2}\right]$ For $b=\frac{L}{2}$, $\overline{M}^o_{AB} = +\frac{M}{8}$ For $b=0$, $\overline{M}^o_{AB} = +\frac{M}{2}$
24		$\overline{M}^o_{AB} = -\frac{wL^2}{10}$

APPENDIX II
MATHEMATICAL TABLES

LOGARITHMS AND ANTILOGARITHMS

TABLE 9

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	14	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	3	5	8	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	17	19	21
						2672	2695	2718	2742	2765	2	4	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

TABLE 9 (Contd.)

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 6 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 1 2	3 4 4	5 6 6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1 1 2	3 4 4	5 6 6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1 1 2	3 3 4	5 6 6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 1 2	3 3 4	5 5 6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 1 2	3 3 4	5 5 6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1 1 2	3 3 4	5 5 6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 1 2	3 3 4	5 5 6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1 1 2	3 3 4	5 5 6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 1 2	3 3 4	4 5 6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1 1 2	2 3 4	4 5 6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1 1 2	2 3 4	4 5 6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1 1 2	2 3 4	4 5 5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1 1 2	2 3 4	4 5 5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1 1 2	2 3 4	4 5 5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1 1 2	2 3 4	4 5 5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1 1 2	2 3 3	4 5 5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 1 2	2 3 3	4 5 5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 1 2	2 3 3	4 4 5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 1 2	2 3 3	4 4 5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1 1 2	2 3 3	4 4 5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1 1 2	2 3 3	4 4 5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1 1 2	2 3 3	4 4 5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1 1 2	2 3 3	4 4 5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1 1 2	2 3 3	4 4 5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1 1 2	2 3 3	4 4 5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1 1 2	2 3 3	4 4 5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1 1 2	2 3 3	4 4 5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0 1 1	2 2 3	3 4 4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0 1 1	2 2 3	3 4 4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0 1 1	2 2 3	3 4 4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0 1 1	2 2 3	3 4 4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0 1 1	2 2 3	3 4 4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0 1 1	2 2 3	3 4 4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0 1 1	2 2 3	3 4 4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0 1 1	2 2 3	3 4 4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0 1 1	2 2 3	3 4 4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0 1 1	2 2 3	3 4 4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0 1 1	2 2 3	3 4 4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1 1	2 2 3	3 4 4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0 1 1	2 2 3	3 3 4

TABLE 10
ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	123	456	789
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1	1 1 1	2 2 2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 0 1	1 1 1	2 2 2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0 0 1	1 1 1	2 2 2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0 0 1	1 1 1	2 2 2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0 1 1	1 1 2	2 2 2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0 1 1	1 1 2	2 2 2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0 1 1	1 1 2	2 2 2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0 1 1	1 1 2	2 2 2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0 1 1	1 1 2	2 2 3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0 1 1	1 1 2	2 2 3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0 1 1	1 1 2	2 2 3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0 1 1	1 2 2	2 2 3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0 1 1	1 2 2	2 2 3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0 1 1	1 2 2	2 3 3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0 1 1	1 2 2	2 3 3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0 1 1	1 2 2	2 3 3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1	1 2 2	2 3 3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0 1 1	1 2 2	2 3 3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1	1 2 2	2 3 3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0 1 1	1 2 2	3 3 3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1 2 2	3 3 3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0 1 1	2 2 2	3 3 3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1	2 2 2	3 3 3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0 1 1	2 2 2	3 3 4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0 1 1	2 2 2	3 3 4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1	2 2 2	3 3 4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0 1 1	2 2 3	3 3 4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0 1 1	2 2 3	3 3 4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0 1 1	2 2 3	3 4 4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0 1 1	2 2 3	3 4 4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0 1 1	2 2 3	3 4 4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0 1 1	2 2 3	3 4 4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1 1 2	2 3 3	4 4 5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1 1 2	2 3 3	4 4 5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1 1 2	2 3 3	4 4 5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1 1 2	2 3 3	4 4 5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1 1 2	2 3 3	4 5 5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2 3 4	4 5 5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1 1 2	2 3 4	4 5 5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 5 6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	3 3 4	4 5 6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 5 6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

TABLE 10 (Contd.)

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

TABLE 11

NATURAL SINES

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	15
5	·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	12	14
11	·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	·2079	2096	2113	2130	2147	2164	2181	2198	2215	2232	3	6	9	11	14
13	·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	10	12
36	·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

TABLE 11 (Contd.)

NATURAL SINES

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	·7547	7558	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	2
85	·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	·9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000	0	0	0	0	0
90	1·000														

TABLE 12

NATURAL COSINES

(Numbers in difference columns to be subtracted, not added.)

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	1·000	1·000	1·000	1·000	1·000	1·000	·9999	9999	9999	9999	0	0	0	0	0
1	·9998	9998	9998	9997	9997	9997	9996	9996	9995	9995	0	0	0	0	0
2	·9994	9993	9993	9992	9991	9990	9990	9989	9988	9987	0	0	0	1	1
3	·9986	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	·9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	·9962	9960	9959	9957	9956	9954	9952	9951	9949	9947	0	1	1	1	2
6	·9945	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	·9925	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	·9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	·9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10	·9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	·9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	·9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	·9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	·9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	·9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	3	4
16	·9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	·9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	3	3	4
18	·9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	·9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20	·9397	9391	9385	9379	9373	9367	9361	9354	9348	9342	1	2	3	4	5
21	·9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	·9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	·9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	3	5	6
24	·9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	·9063	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	3	4	5	6
26	·8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	·8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	·8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	·8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30	·8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7
31	·8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	·8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	·8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	·8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	3	5	7	8
35	·8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	3	5	7	8
36	·8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	·7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	·7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	·7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40	·7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	4	6	8	10
41	·7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	·7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	·7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	·7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	4	6	8	10

TABLE 12 (Contd.)

NATURAL COSINES

(Numbers in difference columns to be subtracted, not added.)

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	·7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10
46	·6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2	4	6	8	11
47	·6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	·6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	·6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11
50	·6428	6414	6401	6388	6374	6361	6347	6334	6320	6307	2	4	7	9	11
51	·6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2	5	7	9	11
52	·6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2	5	7	9	12
53	·6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2	5	7	9	12
54	·5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2	5	7	9	12
55	·5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	2	5	7	10	12
56	·5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2	5	7	10	12
57	·5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2	5	7	10	12
58	·5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12
59	·5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3	5	8	10	13
60	·5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	3	5	8	10	13
61	·4848	4833	4818	4802	4787	4772	4756	4741	4726	4710	3	5	8	10	13
62	·4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	3	5	8	10	13
63	·4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	3	5	8	10	13
64	·4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	3	5	8	11	13
65	·4226	4210	4195	4179	4163	4147	4131	4115	4099	4083	3	5	8	11	13
66	·4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3	5	8	11	14
67	·3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3	5	8	11	14
68	·3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	3	5	8	11	14
69	·3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3	5	8	11	14
70	·3420	3404	3387	3371	3355	3338	3322	3305	3289	3272	3	5	8	11	14
71	·3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3	6	8	11	14
72	·3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	3	6	8	11	14
73	·2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	3	6	8	11	14
74	·2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	3	6	8	11	14
75	·2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	3	6	8	11	14
76	·2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3	6	8	11	14
77	·2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3	6	9	11	14
78	·2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3	6	9	11	14
79	·1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	3	6	9	11	14
80	·1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	3	6	9	12	14
81	·1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3	6	9	12	14
82	·1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3	6	9	12	14
83	·1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3	6	9	12	14
84	·1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	3	6	9	12	14
85	·0872	0854	0837	0819	0802	0785	0767	0750	0732	0715	3	6	9	12	15
86	·0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	3	6	9	12	15
87	·0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	3	6	9	12	15
88	·0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	·0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15
90	·0000														

TABLE 13

NATURAL TANGENTS

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20	·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	[illegible]	13	18	22
36	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	·7526	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	·8098	8127	8156	8165	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

TABLE 13 (Contd.)

NATURAL TANGENTS

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	2·0057	2·0145	2·0233	2·0323	2·0413	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	131
71	2·9042	9208	9375	9544	9714	9887	3·0061	3·0237	3·0415	3·0595	29	58	87	116	145
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	53	107	160	213	267
77	4·3315	3662	4015	4374	4737	5107	5483	5864	6252	6646	Mean differences cease to be sufficiently accurate				
78	4·7046	7453	7867	8288	8716	9152	9594	5·0045	5·0504	5·0970					
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140					
80	5·6713	7297	7894	8502	9124	9758	6·0405	6·1066	6·1742	6·2432					
81	6·3138	3859	4596	5350	6122	6912	7720	8548	9395	7·0264					
82	7·1154	2066	3002	3962	4947	5958	6996	8062	9158	8·0285					
83	8·1443	2636	3863	5126	6427	7769	9152	9·0579	9·2052	9·3572					
84	9·5144	9·677	9·845	10·02	10·20	10·39	10·58	10·78	10·99	11·20					
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95					
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					
90	∞														

TABLE 14

LOGARITHMS OF SINES

Degree	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1′	2′	3′	4′	5′
0	$\bar{0}$·0000	0000	0000	0000	0000	0000	0000	0000	0000	$\bar{1}$9999	0	0	0	0	0
1	$\bar{1}$·9999	9999	9999	9999	9999	9999	9998	9998	9998	9998	0	0	0	0	0
2	$\bar{1}$·9997	9997	9997	9996	9996	9996	9996	9995	9995	9994	0	0	0	0	0
3	$\bar{1}$·9994	9994	9993	9993	9992	9992	9991	9991	9990	9990	0	0	0	0	0
4	$\bar{1}$·9989	9989	9988	9988	9987	9987	9986	9985	9985	8984	0	0	0	0	0
5	$\bar{1}$·9983	9983	9982	9981	9981	9980	9979	9978	9978	9977	0	0	0	0	1
6	$\bar{1}$·9976	9975	9975	9974	9973	9972	9971	9970	9969	9968	0	0	0	1	1
7	$\bar{1}$·9968	9967	9966	9965	9964	9963	9962	9961	9960	9959	0	0	1	1	1
8	$\bar{1}$·9958	9956	9955	9954	9953	9952	9951	9950	9949	9947	0	0	1	1	1
9	$\bar{1}$·9946	9945	9944	9943	9941	9940	9939	9937	9936	9935	0	0	1	1	1
10	$\bar{1}$·9934	9932	9931	9929	9928	9927	9925	9924	9922	9921	0	0	1	1	1
11	$\bar{1}$·9919	9918	9916	9915	9913	9912	9910	9909	9907	9906	0	1	1	1	1
12	$\bar{1}$·9904	9902	9901	9899	9897	9896	9894	9892	9891	9889	0	1	1	1	1
13	$\bar{1}$·9887	9885	9884	9882	9880	9878	9876	9875	9873	9871	0	1	1	1	2
14	$\bar{1}$·9869	9867	9865	9863	9861	9859	9857	9855	9853	9851	0	1	1	1	2
15	$\bar{1}$·9849	9847	9845	9843	9841	9839	9837	9835	9833	9831	0	1	1	1	2
16	$\bar{1}$·9828	9826	9824	9822	9820	9817	9815	9813	9811	9808	0	1	1	2	2
17	$\bar{1}$·9806	9804	9801	9799	9797	9794	9792	9789	9787	9785	0	1	1	2	2
18	$\bar{1}$·9782	9780	9777	9775	9772	9770	9767	9764	9762	9759	0	1	1	2	2
19	$\bar{1}$·9757	9754	9751	9749	9746	9743	9741	9738	9735	9733	0	1	1	2	2
20	$\bar{1}$·9730	9727	9724	9722	9719	9716	9713	9710	9707	9704	0	1	1	2	2
21	$\bar{1}$·9702	9699	9696	9693	9690	9687	9684	9681	9678	9675	0	1	1	2	2
22	$\bar{1}$·9672	9669	9666	9662	9659	9656	9653	9650	9647	9643	1	1	2	2	3
23	$\bar{1}$·9640	9637	9634	9631	9627	9624	9621	9617	9614	9611	1	1	2	2	3
24	$\bar{1}$·9607	9604	9601	9597	9594	9590	9587	9583	9580	9576	1	1	2	2	3
25	$\bar{1}$·9573	9569	9566	9562	9558	9555	9551	9548	9544	9540	1	1	2	2	3
26	$\bar{1}$·9537	9533	9529	9525	9522	9518	9514	9510	9507	9503	1	1	2	3	3
27	$\bar{1}$·9499	9495	9491	9487	9483	9479	9475	9471	9467	9463	1	1	2	3	3
28	$\bar{1}$·9459	9455	9451	9447	9443	9439	9435	9431	9427	9422	1	1	2	3	3
29	$\bar{1}$·9418	9414	9410	9406	9401	9397	9393	9388	9384	9380	1	1	2	3	4
30	$\bar{1}$·9375	9371	9367	9362	9358	9353	9349	9344	9340	9335	1	1	2	3	4
31	$\bar{1}$·9331	9326	9322	9317	9312	9308	9303	9298	9294	9289	1	2	2	3	4
32	$\bar{1}$·9284	9279	9275	9270	9265	9260	9255	9251	9246	9241	1	2	2	3	4
33	$\bar{1}$·9236	9231	9226	9221	9216	9211	9206	9201	9196	9191	1	2	3	3	4
34	$\bar{1}$·9186	9181	9175	9170	9165	9160	9155	9149	9144	9139	1	2	3	3	4
35	$\bar{1}$·9134	9128	9123	9118	9112	9107	9101	9096	9091	9085	1	2	3	4	5
36	$\bar{1}$·9080	9074	9069	9063	9057	9052	9046	9041	9035	9029	1	2	3	4	5
37	$\bar{1}$·9023	9018	9012	9006	9000	8995	8989	8983	8977	8971	1	2	3	4	5
38	$\bar{1}$·8965	8959	8953	8947	8941	8935	8929	8923	8917	8911	1	2	3	4	5
39	$\bar{1}$·8905	8899	8893	8887	8880	8874	8868	8862	8855	8849	1	2	3	4	5
40	$\bar{1}$·8843	8836	8830	8823	8817	8810	8804	8797	8791	8784	1	2	3	4	5
41	$\bar{1}$·8778	8771	8765	8758	8751	8745	8738	8731	8724	8718	1	2	3	5	6
42	$\bar{1}$·8711	8704	8697	8690	8683	8676	8669	8662	8655	8648	1	2	3	5	6
43	$\bar{1}$·8641	8634	8627	8620	8613	8606	8598	8591	9584	8577	1	2	4	5	6
44	$\bar{1}$·8569	8562	8555	8547	8540	8532	8525	8517	8510	8502	1	2	4	5	6

TABLE 14 (Contd.)

LOGARITHMS OF SINES

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	$\bar{1}$·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46	$\bar{1}$·8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	1	2	4	5	6
47	$\bar{1}$·8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48	$\bar{1}$·8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	4	6
49	$\bar{1}$·8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50	$\bar{1}$·8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51	$\bar{1}$·8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52	$\bar{1}$·8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53	$\bar{1}$·9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54	$\bar{1}$·9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55	$\bar{1}$·9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56	$\bar{1}$·9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57	$\bar{1}$·9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58	$\bar{1}$·9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59	$\bar{1}$·9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60	$\bar{1}$·9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61	$\bar{1}$·9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62	$\bar{1}$·9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63	$\bar{1}$·9499	9503	9506	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64	$\bar{1}$·9537	9540	9544	9548	9551	9555	9558	9562	9566	[illegible]	1	1	2	2	3
65	$\bar{1}$·9573	9576	9580	9583	9587	9590	9594	9597	9601	[illegible]	1	1	2	2	3
66	$\bar{1}$·9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67	$\bar{1}$·9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68	$\bar{1}$·9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	2
69	$\bar{1}$·9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70	$\bar{1}$·9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71	$\bar{1}$·9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72	$\bar{1}$·9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73	$\bar{1}$·9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	0	1	1	2	2
74	$\bar{1}$·9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75	$\bar{1}$·9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76	$\bar{1}$·9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	0	1	1	1	2
77	$\bar{1}$·9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	0	1	1	1	1
78	$\bar{1}$·9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	0	1	1	1	1
79	$\bar{1}$·9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	0	0	1	1	1
80	$\bar{1}$·9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	0	0	1	1	1
81	$\bar{1}$·9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	0	0	1	1	1
82	$\bar{1}$·9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	0	0	1	1	1
83	$\bar{1}$·9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	0	0	0	1	1
84	$\bar{1}$·9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	0	0	0	0	1
85	$\bar{1}$·9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	0	0	0	0	0
86	$\bar{1}$·9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	0	0	0	0	0
87	$\bar{1}$·9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	0	0	0	0	0
88	$\bar{1}$·9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	0	0	0	0	0
89	$\bar{1}$·9999	9999	0·0000	0000	0000	0000	0000	0000	0000	0000					
90	0·0000														

TABLE 15
LOGARITHMS OF COSINS

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	∞	3̄·2419	3̄·5429	7190	8439	9408	3̄·0200	2̄·0870	2̄·1450	2̄·1961					
1	2̄·2419	2832	3210	3553	3880	4179	4459	4723	4971	5206					
2	2̄·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3	2̄·7188	7330	7468	7602	7731	7857	7979	8098	8213	8326					
4	2̄·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	16	32	48	64	80
5	2̄·9403	9489	9573	9655	9736	9816	9894	9970	1̄·0046	1̄·0120	13	26	39	52	65
6	1̄·0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7	1̄·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8	1̄·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9	1̄·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10	1̄·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11	1̄·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12	1̄·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13	1̄·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14	1̄·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15	1̄·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16	1̄·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17	1̄·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18	1̄·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19	1̄·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20	1̄·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21	1̄·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22	1̄·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23	1̄·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24	1̄·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25	1̄·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26	1̄·6418	6434	6449	6465	6480	6495	[illegible]	6526	6541	6556	3	5	8	10	13
27	1̄·6570	6585	6600	6615	6629	6044	[illegible]	6673	6687	6702	2	5	7	10	12
28	1̄·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29	1̄·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30	1̄·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31	1̄·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32	1̄·7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33	1̄·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34	1̄·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35	1̄·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36	1̄·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37	1̄·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38	1̄·7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39	1̄·7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40	1̄·8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41	1̄·8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42	1̄·8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43	1̄·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44	1̄·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6

TABLE 15 (Contd.)

LOGARITHMS OF COSINES

[Numbers in difference columns to be subtracted, not added.]

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	$\bar{1}$·8495	8487	8480	[illegible]	8464	8457	8440	8441	8433	8426	1	3	4	5	6
46	$\bar{1}$·8418	8410	8402	8394	8386	8378	8370	8362	8354	8346	1	3	4	5	7
47	$\bar{1}$·8338	8331	[illegible]	8313	8305	8297	8289	8280	8272	8264	1	3	4	6	7
48	$\bar{1}$·8255	8247	8238	8230	8221	8213	8204	8195	8187	8178	1	3	4	6	7
49	[illegible]	8161	[illegible]	[illegible]	8134	[illegible]	8117	8108	8099	8090	1	3	4	6	7
50	$\bar{1}$·8081	8072	8063	[illegible]	8044	8035	8026	8017	8007	7998	2	3	5	6	8
51	$\bar{1}$·7989	7979	7970	7960	7951	7941	7932	7922	7913	7903	2	3	5	6	8
52	$\bar{1}$·7893	7884	7874	7864	7854	7844	7835	7825	7815	7805	2	3	5	7	8
53	[illegible]	7785	[illegible]	7764	7754	7744	7734	7723	7713	7703	2	3	5	7	9
54	$\bar{1}$·7692	7682	7671	7660	7650	7640	7629	7618	7607	7597	2	4	5	7	9
55	$\bar{1}$·7586	7575	7564	7553	7542	7531	7520	7509	7498	7487	2	4	6	7	9
56	$\bar{1}$·7476	7464	7453	7442	7430	7419	7407	7396	7384	7373	2	4	6	8	10
57	$\bar{1}$·7361	7349	7338	7326	7314	7302	7290	7278	7266	7254	2	4	6	8	10
58	$\bar{1}$·7242	[illegible]	[illegible]	7205	7193	7181	7168	7156	7144	7131	2	4	6	8	10
59	$\bar{1}$·7118	[illegible]	[illegible]	7080	7068	7055	7042	7029	7016	7003	2	4	6	9	11
60	[illegible]	6977	6963	[illegible]	6937	6923	6910	6896	6883	6869	2	4	7	9	11
61	[illegible]	[illegible]	6828	[illegible]	6801	6787	6773	6759	6744	6730	2	5	7	9	12
62	[illegible]	[illegible]	[illegible]	6673	6659	6644	6629	6615	6600	6585	2	5	7	10	12
63	[illegible]	6556	6541	6526	6510	6495	6480	6465	6449	6434	3	5	8	10	13
64	[illegible]	6403	6387	[illegible]	6356	6340	6324	6308	6292	6276	3	5	8	11	13
65	[illegible]	6243	6227	6210	6194	6177	6161	6144	6127	6110	3	6	8	11	14
66	$\bar{1}$·6093	[illegible]	[illegible]	6042	6024	6007	599-	5972	5954	5937	3	6	9	12	15
67	$\bar{1}$·5919	5901	5883	5865	5847	5828	5810	5792	5773	5754	3	6	9	12	15
68	$\bar{1}$·5736	5717	5698	5679	5660	5641	5621	5602	5583	5563	3	6	10	13	16
69	$\bar{1}$·5543	5523	5504	5484	5463	5443	5423	5402	5382	5361	3	7	10	14	17
70	$\bar{1}$·5341	5320	5299	5278	5256	5235	5213	5192	5170	5148	4	7	11	14	18
71	$\bar{1}$·5126	5104	[illegible]	5060	5037	5015	4992	4969	4946	4923	4	8	11	15	19
72	$\bar{1}$·4900	4876	4853	4829	4805	4781	4757	4733	4709	4684	4	8	12	16	20
73	$\bar{1}$·4659	4634	4609	4584	4559	4533	4508	4482	4456	4430	[illegible]	9	13	17	21
74	$\bar{1}$·4403	4377	4350	4323	4296	4269	4242	4214	4186	4158	[illegible]	9	14	18	23
75	$\bar{1}$·4130	4102	4073	4044	4015	3986	3957	3927	3897	3867	[illegible]	10	15	20	24
76	$\bar{1}$·3837	3806	3775	3745	3713	3682	3650	3618	3586	3554	[illegible]	11	16	21	26
77	$\bar{1}$·3521	3488	3455	3421	3387	3353	3319	3284	[illegible]	[illegible]	[illegible]	11	17	[illegible]	[illegible]
78	$\bar{1}$·3179	3143	3107	3070	3034	2997	2959	2921	[illegible]	[illegible]	6	12	19	25	31
79	$\bar{1}$·2806	2767	2727	2687	2647	2606	2565	2524	[illegible]	2439	7	14	20	27	34
80	$\bar{1}$·2397	2353	2310	2266	2221	2176	2131	2085	2038	[illegible]	8	15	23	[illegible]	[illegible]
81	$\bar{1}$·1943	1895	1847	1797	1747	1697	1646	1594	1542	[illegible]	[illegible]	17	25	34	[illegible]
82	$\bar{1}$·1436	1381	1326	1271	1214	1157	1099	1040	0981	0920	10	19	29	38	48
83	$\bar{1}$·0859	0797	0734	0670	0605	0539	0472	0403	0334	0264	11	22	33	44	55
84	$\bar{1}$·0192	0120	0046	$\bar{2}$·9970	$\bar{2}$·9894	$\bar{2}$·9816	$\bar{2}$·9736	$\bar{2}$·9655	$\bar{2}$·9573	[illegible]	13	26	39	52	65
85	$\bar{2}$·9403	9315	9226	9135	9042	8946	8849	8749	8647	8543	16	32	48	64	80
86	$\bar{2}$·8436	8326	8213	8098	7979	7857	7731	7602	7468	7330					
87	$\bar{2}$·7188	7041	6889	6731	6567	6397	6220	6035	5842	564-					
88	$\bar{2}$·5428	5206	4971	4723	4459	4179	3880	3558	3210	2832					
89	$\bar{2}$·2419	[illegible]	1450	0870	[illegible]	$\bar{3}$·9408	$\bar{3}$·8439	$\bar{3}$·7190	$\bar{3}$·5429	$\bar{3}$·2419					
90															

TABLE 16

LOGARITHMS OF TANGENTS

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	$-\infty$	$\bar{3}.2419$	$\bar{3}.5429$	$\bar{3}.7190$	$\bar{3}.8439$	$\bar{3}.9409$	$\bar{2}.0200$	$\bar{2}.0870$	$\bar{2}.1450$	$\bar{2}.1962$					
1	$\bar{2}.2419$	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2	$\bar{2}.5431$	5643	5845	6038	6223	6401	6571	6736	6894	7046					
3	$\bar{2}.7194$	7337	7475	7609	7739	7865	7988	8107	8223	8336					
4	$\bar{2}.8446$	8554	8659	8762	8862	8960	9056	9150	9241	9331	16	32	48	64	81
5	$\bar{2}.9420$	9506	9591	9674	9756	9836	9915	9992	$\bar{1}.0068$	$\bar{1}.0143$	13	26	40	53	66
6	$\bar{1}.0216$	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	56
7	$\bar{1}.0891$	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8	$\bar{1}.1478$	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9	$\bar{1}.1997$	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10	$\bar{1}.2463$	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	28	35
11	$\bar{1}.2887$	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	32
12	$\bar{1}.3275$	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	30
13	$\bar{1}.3634$	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14	$\bar{1}.3968$	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15	$\bar{1}.4281$	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16	$\bar{1}.4575$	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17	$\bar{1}.4853$	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18	$\bar{1}.5118$	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19	$\bar{1}.5370$	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20	$\bar{1}.5611$	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21	$\bar{1}.5842$	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	19
22	$\bar{1}.6064$	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23	$\bar{1}.6279$	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24	$\bar{1}.6486$	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25	$\bar{1}.6687$	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	16
26	$\bar{1}.6882$	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27	$\bar{1}.7072$	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28	$\bar{1}.7257$	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29	$\bar{1}.7438$	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30	$\bar{1}.7614$	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31	$\bar{1}.7788$	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32	$\bar{1}.7958$	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33	$\bar{1}.8125$	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34	$\bar{1}.8290$	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35	$\bar{1}.8452$	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36	$\bar{1}.8613$	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37	$\bar{1}.8771$	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38	$\bar{1}.8928$	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39	$\bar{1}.9084$	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40	$\bar{1}.9238$	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41	$\bar{1}.9392$	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42	$\bar{1}.9544$	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43	$\bar{1}.9697$	9712	9727	9742	9757	9772	9788	9803	9818	9833	3	5	8	10	13
44	$\bar{1}.9848$	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13

TABLE 16 (Contd.)

LOGARITHMS OF TANGENTS

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	3	5	8	10	13
46	·0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	3	5	8	10	13
47	·0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	3	5	8	10	13
48	·0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	3	5	8	10	13
49	·0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	3	5	8	10	13
50	·0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	3	5	8	10	13
51	·0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	3	5	8	10	13
52	·1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	3	5	8	10	13
53	·1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	3	5	8	11	13
54	·1387	1403	1419	1435	1451	1467	1483	1499	1516	1532	3	5	8	11	13
55	·1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	3	5	8	11	14
56	·1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	3	5	8	11	14
57	·1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	3	6	8	11	14
58	·2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	3	6	9	11	14
59	·2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	3	6	9	12	14
60	·2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	3	6	9	12	15
61	·2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	3	6	9	12	15
62	·2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	3	6	9	12	15
63	·2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3	6	9	13	16
64	·3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3	6	10	13	16
65	·3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3	7	10	13	17
66	·3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3	7	10	14	17
67	·3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	4	7	11	14	18
68	·3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4	7	11	15	19
69	·4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	4	8	12	15	19
70	·4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4	8	12	16	20
71	·4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	4	8	13	17	21
72	·4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	4	9	13	18	22
73	·5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	5	9	14	19	23
74	·5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5	10	15	20	25
75	·5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	5	10	16	21	26
76	·6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	6	11	17	22	28
77	·6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	6	12	18	24	30
78	·6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	6	13	19	26	32
79	·7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	7	14	21	28	35
80	·7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	8	16	23	31	39
81	·8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	9	17	26	35	43
82	·8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	10	20	29	39	49
83	·9109	9172	9236	9301	9367	9433	9501	9570	9640	9711	11	22	34	45	56
84	·9784	9857	9932	1·0008	1·0085	1·0164	1·0244	1·0326	1·0409	1·0494	13	26	40	53	66
85	1·0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	16	32	48	64	81
86	1·1554	1664	1777	1893	2012	2135	2261	2391	2525	2663					
87	1·2806	2954	3106	3264	3429	3599	3777	3962	4155	4357					
88	1·4569	4792	5027	5275	5539	5819	6119	6441	6789	7167					
89	1·7581	8038	8550	9130	9800	2·0591	2·1561	2·2810	2·4571	2·7581					

TABLE 17
POWERS, ROOTS AND RECIPROCALS

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$\sqrt{10n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$\frac{1}{n}$
1	1	1	1	1	3·162	2·154	4·642	1
2	4	8	1·414	1·260	4·472	2·714	5·848	·5000
3	9	27	1·732	1·442	5·477	3·107	6·694	·3333
4	16	64	2	1·587	6·325	3·420	7·368	·2500
5	25	125	2·236	1·710	7·071	3·684	7·937	·2000
6	36	216	2·449	1·817	7·746	3·915	8·434	·1667
7	49	343	2·646	1·913	8·367	4·121	8·879	·1429
8	64	512	2·828	2·000	8·944	4·309	9·283	·1250
9	81	729	3·000	2·080	9·487	4·481	9·655	·1111
10	100	1000	3·162	2·154	10·0	4·642	10·000	·1000
11	121	1331	3·317	2·224	10·488	4·791	10·323	·09091
12	144	1728	3·464	2·289	10·954	4·932	10·627	·08333
13	169	2197	3·606	2·351	11·402	5·066	10·914	·07692
14	196	2744	3·742	2·410	11·832	5·192	11·187	·07143
15	225	3375	3·873	2·466	12·247	5·313	11·447	·06667
16	256	4096	4·000	2·520	12·649	5·429	11·696	·06250
17	289	4913	4·123	2·571	13·038	5·540	11·935	·05882
18	324	5832	4·243	2·621	13·416	5·646	12·164	·05556
19	361	6859	4·359	2·668	13·784	5·749	12·386	·05263
20	400	8000	4·472	2·714	14·142	5·848	12·599	·0500
21	441	9261	4·583	2·759	14·491	5·944	12·806	·04762
22	484	10648	4·690	2·802	14·832	6·037	13·006	·04545
23	529	12167	4·796	2·844	15·166	6·127	13·200	·04348
24	576	13824	4·899	2·884	15·492	6·214	13·389	·04167
25	625	15625	5·000	2·924	15·811	6·300	13·572	·0400
26	676	17576	5·099	2·962	16·125	6·383	13·751	·03846
27	729	19683	5·196	3·000	16·432	6·463	13·925	·03704
28	784	21952	5·292	3·037	16·733	6·542	14·095	·03571
29	841	24389	5·385	3·072	17·029	6·619	14·260	·03448
30	900	27000	5·477	3·107	17·321	6·694	14·422	·03333
31	961	29791	5·568	3·141	17·607	6·768	14·581	·03226
32	1024	32768	5·657	3·175	17·889	6·840	14·736	·03125
33	1089	35937	5·745	3·208	18·166	6·910	14·888	·03030
34	1156	39304	5·831	3·240	18·439	6·980	15·037	·02941
35	1225	42875	5·916	3·271	18·708	7·047	15·183	·02857
36	1296	46656	6·000	3·302	18·974	7·114	15·326	·02778
37	1369	50653	6·083	3·332	19·235	7·179	15·467	·02703
38	1444	54872	6·164	3·362	19·494	7·243	15·605	·02632
39	1521	59319	6·245	3·391	19·748	7·306	15·741	·02564
40	1600	64000	6·325	3·420	20·00	7·368	15·874	·0250
41	1681	68921	6·403	3·448	20·248	7·429	16·005	·02439
42	1764	74088	6·481	3·476	20·494	7·489	16·134	·02381
43	1849	79507	6·557	3·503	20·736	7·548	16·261	·02326
44	1936	85184	6·633	3·530	20·976	7·606	16·386	·02273
45	2025	91125	6·708	3·557	21·213	7·663	16·510	·02222
46	2116	97336	6·782	3·583	21·448	7·719	16·631	·02174
47	2209	103823	6·856	3·609	21·679	7·775	16·751	·02128
48	2304	110592	6·928	3·634	21·909	7·830	16·869	·02083
49	2401	117649	7·000	3·659	22·136	7·884	16·985	·02041
50	2500	125000	7·071	3·684	22·361	7·937	17·100	·020

TABLE 17 (Contd.)

POWERS, ROOTS AND RECIPROCALS

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$\sqrt{10n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$\frac{1}{n}$
51	2601	132651	7·141	3·708	22·583	7·990	17·213	·01961
52	2704	140608	7·211	3·733	22·804	8·041	17·325	·01923
53	2809	148877	7·280	3·756	23·022	8·093	17·435	·01887
54	2916	157464	7·348	3·780	23·238	8·143	17·544	·01852
55	3025	166375	7·416	3·803	23·452	8·193	17·652	·01818
56	3136	175616	7·483	3·826	23·664	8·243	17·758	·01786
57	3249	185193	7·550	3·849	23·875	8·291	17·863	·01754
58	3364	195112	7·616	3·871	24·083	8·340	17·967	·01724
59	3481	205379	7·681	3·893	24·290	8·387	18·070	·01695
60	3600	216000	7·746	3·915	24·495	8·434	18·171	·01667
61	3721	226981	7·810	3·936	24·698	8·481	18·272	·01639
62	3844	238328	7·874	3·958	24·900	8·527	18·371	·01613
63	3969	250047	7·937	3·979	25·100	8·573	18·469	·01587
64	4096	262144	8·000	4·000	25·298	8·618	18·566	·01562
65	4225	274625	8·062	4·021	25·495	8·662	18·663	·01538
66	4356	287496	8·124	4·041	25·690	8·707	18·758	·01515
67	4489	300763	8·185	4·062	25·884	8·750	18·852	·01493
68	4624	314432	8·246	4·082	26·077	8·794	18·945	·01471
69	4761	328509	8·307	4·102	26·268	8·837	19·038	·01449
70	4900	343000	8·367	4·121	26·458	8·879	19·129	·01429
71	5041	357911	8·426	4·141	26·646	8·921	19·220	·01408
72	5184	373248	8·485	4·160	26·833	8·963	19·310	·01389
73	5329	389017	8·544	4·179	27·019	9·004	19·399	·01370
74	5476	405224	8·602	4·198	27·203	9·045	19·487	·01351
75	5625	421875	8·660	4·217	27·386	9·086	19·574	·01333
76	5776	438976	8·718	4·236	27·568	9·126	19·661	·01316
77	5929	456533	8·775	4·254	27·749	9·166	19·747	·01299
78	6084	474552	8·832	4·273	27·928	9·205	19·832	·01282
79	6241	493039	8·888	4·291	28·107	9·244	19·916	·01266
80	6400	512000	8·944	4·309	28·284	9·283	20·000	·01250
81	6561	531441	9·000	4·327	28·460	9·322	20·083	·01235
82	6724	551368	9·055	4·344	28·636	9·360	20·165	·01220
83	6889	571787	9·110	4·362	28·810	9·398	20·247	·01205
84	7056	592704	9·165	4·380	28·983	9·435	20·328	·01190
85	7225	614125	9·220	4·397	29·155	9·473	20·408	·01176
86	7396	636056	9·274	4·414	29·326	9·510	20·488	·01163
87	7569	658503	9·327	4·431	29·496	9·546	20·567	·01149
88	7744	681472	9·381	4·448	29·665	9·583	20·646	·01136
89	7921	704969	9·434	4·465	29·833	9·619	20·724	·01124
90	8100	729000	9·487	4·481	30·000	9·655	20·801	·01111
91	8281	753571	9·539	4·498	30·166	9·691	20·878	·01099
92	8464	778688	9·592	4·514	30·332	9·726	20·954	·01087
93	8649	804357	9·644	4·531	30·496	9·761	21·029	·01075
94	8836	830584	9·695	4·547	30·659	9·796	21·105	·01064
95	9025	857375	9·747	4·563	30·822	9·830	21·179	·01053
96	9216	884736	9·798	4·579	30·984	9·865	21·253	·01042
97	9409	912673	9·849	4·595	31·145	9·899	21·327	·01031
98	9604	941192	9·899	4·610	31·305	9·933	21·400	·01020
99	9801	970299	9·950	4·626	31·464	9·967	21·472	·01010
100	10000	1000000	10·000	4·642	31·623	10·000	21·544	·0100

TABLE 18

SQUARE ROOTS. FROM 1 TO 10

	0	1	2	3	4	5	6	7	8	9	Mean Differences		
											1 2 3	4 5 6	7 8 9
1·0	1·000	1·005	1·010	1·015	1·020	1·025	1·030	1·034	1·039	1·044	0 1 1	2 2 3	3 4 4
1·1	1·049	1·054	1·058	1·063	1·068	1·072	1·077	1·082	1·086	1·091	0 1 1	2 2 3	3 4 4
1·2	1·095	1·100	1·105	1·109	1·114	1·118	1·122	1·127	1·131	1·136	0 1 1	2 2 3	3 4 4
1·3	1·140	1·145	1·149	1·153	1·158	1·162	1·166	1·170	1·175	1·179	0 1 1	2 2 3	3 3 4
1·4	1·183	1·187	1·192	1·196	1·200	1·204	1·208	1·212	1·217	1·221	0 1 1	2 2 2	3 3 4
1·5	1·225	1·229	1·233	1·237	1·241	1·245	1·249	1·253	1·257	1·261	0 1 1	2 2 2	3 3 4
1·6	1·265	1·269	1·273	1·277	1·281	1·285	1·288	1·292	1·296	1·300	0 1 1	2 2 2	3 3 3
1·7	1·304	1·308	1·311	1·315	1·319	1·323	1·327	1·330	1·334	1·338	0 1 1	2 2 2	3 3 3
1·8	1·342	1·345	1·349	1·353	1·356	1·360	1·364	1·367	1·371	1·375	0 1 1	1 2 2	3 3 3
1·9	1·378	1·382	1·386	1·389	1·393	1·396	1·400	1·404	1·407	1·411	0 1 1	1 2 2	3 3 3
2·0	1·414	1·418	1·421	1·425	1·428	1·432	1·435	1·439	1·442	1·446	0 1 1	1 2 2	2 3 3
2·1	1·449	1·453	1·456	1·459	1·463	1·466	1·470	1·473	1·476	1·480	0 1 1	1 2 2	2 3 3
2·2	1·483	1·487	1·490	1·493	1·497	1·500	1·503	1·507	1·510	1·513	0 1 1	1 2 2	2 3 3
2·3	1·517	1·520	1·523	1·526	1·530	1·533	1·536	1·539	1·543	1·546	0 1 1	1 2 2	2 3 3
2·4	1·549	1·552	1·556	1·559	1·562	1·565	1·568	1·572	1·575	1·578	0 1 1	1 2 2	2 3 3
2·5	1·581	1·584	1·587	1·591	1·594	1·597	1·600	1·603	1·606	1·609	0 1 1	1 2 2	2 3 3
2·6	1·612	1·616	1·619	1·622	1·625	1·628	1·631	1·634	1·637	1·640	0 1 1	1 2 2	2 2 3
2·7	1·643	1·646	1·649	1·652	1·655	1·658	1·661	1·664	1·667	1·670	0 1 1	1 2 2	2 2 3
2·8	1·673	1·676	1·679	1·682	1·685	1·688	1·691	1·694	1·697	1·700	0 1 1	1 1 2	2 2 3
2·9	1·703	1·706	1·709	1·712	1·715	1·718	1·720	1·723	1·726	1·729	0 1 1	1 1 2	2 2 3
3·0	1·732	1·735	1·738	1·741	1·744	1·746	1·749	1·752	1·755	1·758	0 1 1	1 1 2	2 2 3
3·1	1·761	1·764	1·766	1·769	1·772	1·775	1·778	1·780	1·783	1·786	0 1 1	1 1 2	2 2 3
3·2	1·789	1·792	1·794	1·797	1·800	1·803	1·806	1·808	1·811	1·814	0 1 1	1 1 2	2 2 2
3·3	1·817	1·819	1·822	1·825	1·828	1·830	1·833	1·836	1·838	1·841	0 1 1	1 1 2	2 2 2
3·4	1·844	1·847	1·849	1·852	1·855	1·857	1·860	1·863	1·865	1·868	0 1 1	1 1 2	2 2 2
3·5	1·871	1·873	1·876	1·879	1·881	1·884	1·887	1·889	1·892	1·895	0 1 1	1 1 2	2 2 2
3·6	1·897	1·900	1·903	1·905	1·908	1·910	1·913	1·916	1·918	1·921	0 1 1	1 1 2	2 2 2
3·7	1·924	1·926	1·929	1·931	1·934	1·936	1·939	1·942	1·944	1·947	0 1 1	1 1 2	2 2 2
3·8	1·949	1·952	1·954	1·957	1·960	1·962	1·965	1·967	1·970	1·972	0 1 1	1 1 2	2 2 2
3·9	1·975	1·977	1·980	1·982	1·985	1·987	1·990	1·992	1·995	1·997	0 1 1	1 1 2	2 2 2
4·0	2·000	2·002	2·005	2·007	2·010	2·012	2·015	2·017	2·020	2·022	0 0 1	1 1 1	2 2 2
4·1	2·025	2·027	2·030	2·032	2·035	2·037	2·040	2·042	2·045	2·047	0 0 1	1 1 1	2 2 2
4·2	2·049	2·052	2·054	2·057	2·059	2·062	2·064	2·066	2·069	2·071	0 0 1	1 1 1	2 2 2
4·3	2·074	2·076	2·078	2·081	2·083	2·086	2·088	2·090	2·093	2·095	0 0 1	1 1 1	2 2 2
4·4	2·098	2·100	2·102	2·105	2·107	2·110	2·112	2·114	2·117	2·119	0 0 1	1 1 1	2 2 2
4·5	2·121	2·124	2·126	2·128	2·131	2·133	2·135	2·138	2·140	2·142	0 0 1	1 1 1	2 2 2
4·6	2·145	2·147	2·149	2·152	2·154	2·156	2·159	2·161	2·163	2·166	0 0 1	1 1 1	2 2 2
4·7	2·168	2·170	2·173	2·175	2·177	2·179	2·182	2·184	2·186	2·189	0 0 1	1 1 1	2 2 2
4·8	2·101	2·193	2·195	2·198	2·200	2·202	2·205	2·207	2·209	2·211	0 0 1	1 1 1	2 2 2
4·9	2·214	2·216	2·218	2·220	2·223	2·225	2·227	2·220	2·232	2·234	0 0 1	1 1 1	2 2 2
5·0	2·236	2·238	2·241	2·243	2·245	2·247	2·249	2·252	2·254	2·256	0 0 1	1 1 1	2 2 2
5·1	2·258	2·261	2·263	2·265	2·267	2·260	2·272	2·274	2·276	2·278	0 0 1	1 1 1	2 2 2
5·2	2·280	2·283	2·285	2·287	2·289	2·201	2·293	2·296	2·298	2·300	0 0 1	1 1 1	2 2 2
5·3	2·302	2·304	2·307	2·309	2·311	2·313	2·315	2·317	2·319	2·322	0 0 1	1 1 1	2 2 2
5·4	2·324	2·326	2·328	2·330	2·332	2·335	2·337	2·339	2·341	2·343	0 0 1	1 1 1	1 2 2

TABLE 18 (Contd.)

RECIPROCALS OF NUMBERS. FROM 1 TO 10

[Numbers in difference columns to be subtracted, not added.]

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1 2 3	4 5 6	7 8 9
5·5	·1818	1815	1812	1808	1805	1802	1799	[illegible]	1792	[illegible]	[illegible] 1 1	1 2 2	2 3 3
5·6	1786	1783	1779	1776	1773	1770	1767	1764	176[illegible]	[illegible]	1 1 1	[illegible] 2 2	2 3 3
5·7	·1754	1751	1748	1745	1742	1739	1736	1733	[illegible]	[illegible]	[illegible]	1 1 2	2 2 3
5·8	1724	1721	1718	1715	[illegible]	1709	1706	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2 2 [illegible]
5·9	[illegible]	1692	1689	[illegible]	[illegible]	1681	1678	1675	[illegible]	[illegible]	[illegible]	[illegible]	[illegible] 2
6·0	·1667	[illegible]	1661	1658	1656	1653	1650	[illegible]	[illegible]	[illegible]	[illegible]	1 1 2	2 2 [illegible]
6·1	[illegible]	1637	1634	1631	1629	1626	1623	1621	1618	[illegible]	0 1 1	1 1 2	2 2 [illegible]
6·2	1613	1610	1608	1605	1603	1600	[illegible]	1595	[illegible]	1590	0 1 1	1 1 2	2 2 2
6·3	1587	1585	1582	1580	1577	1575	157	1570	1567	1565	0 0 1	1 1 1	2 2 2
6·4	·1562	1560	1558	1555	1552	1550	1548	1545	1543	1541	0 0 1	1 1 1	2 2 2
6·5	·1538	1536	1534	1531	1529	1527	1524	1522	1520	1517	0 0 1	1 1 1	2 2 2
6·6	·1515	1513	1511	1508	1506	1504	1502	1499	1497	1495	0 0 1	1 1 1	2 2 2
6·7	·1493	1490	1488	1486	1484	1481	1479	1477	1475	1473	0 0 1	1 1 1	2 2 2
6·8	·1471	1468	1466	1464	1462	1460	1458	1456	1453	1451	0 0 1	1 1 1	2 2 2
6·9	·1449	1447	1445	1443	1441	1439	[illegible]	1435	1433	1431	0 0 1	1 1 1	2 2 2
7·0	·1429	1427	1425	1422	1420	1418	[illegible]	1414	1412	1410	0 0 1	1 1 1	1 2 2
7·1	·1408	1406	1404	1403	1401	1399	1397	1395	1393	1391	0 0 1	1 1 1	1 2 2
7·2	1389	1387	1385	1383	1381	1379	1377	1376	1374	1372	0 0 1	1 1 1	[illegible] 2 2
7·3	1370	1368	1366	1364	1362	1361	1359	1357	1355	1353	0 0 1	1 1 1	1 2 2
7·4	1351	1350	1348	1346	1344	134[illegible]	1340	1339	1337	1335	0 0 1	1 1 1	1 1 2
7·5	1333	1332	1330	1328	1326	1325	1323	1321	1319	1318	0 0 1	1 1 1	1 1 2
7·6	1316	1314	1312	1311	1309	1307	1305	1304	1302	1300	0 0 1	1 1 1	1 1 2
7·7	1299	1297	1295	1294	1292	1290	1289	1287	1285	1284	0 0 0	1 1 1	1 1 1
7·8	1282	1280	1279	1277	1276	1274	1272	1271	1269	1267	0 0 0	1 1 1	1 1 1
7·9	·1266	1264	1263	1261	1259	1258	1256	1255	1253	1252	0 0 0	1 1 1	1 1 1
8·0	·1250	1248	1247	1245	1244	1242	1241	1239	1238	1236	0 0 0	1 1 1	1 1 1
8·1	1235	1233	1232	1230	1229	1227	1225	1224	1222	1221	0 0 0	1 1 1	1 1 1
8·2	1220	1218	1217	1215	1214	1212	1211	1209	1208	1206	0 0 0	1 1 1	1 1 1
8·3	·1205	1203	1202	1200	1199	1198	1196	1195	1193	1192	0 0 0	1 1 1	1 1 1
8·4	·1190	1189	1188	1186	1185	1183	1182	1181	1179	1178	0 0 0	1 1 1	1 1 1
8·5	·1176	1175	1174	1172	1171	1170	1168	1167	1166	1164	0 0 0	1 1 1	1 1 1
8·6	1163	1161	1160	1159	1157	1156	1155	1153	1152	1151	0 0 0	1 1 1	1 1 1
8·7	1149	1148	1147	1145	1144	1143	1142	1140	1139	1138	0 0 0	1 1 1	1 1 1
8·8	·1136	1135	1134	1133	1131	1130	1129	1127	1126	1125	0 0 0	1 1 1	1 1 1
8·9	·1124	1122	1121	1120	1119	1117	1116	1115	1114	1112	0 0 0	1 1 1	1 1 1
9·0	·1111	1110	1109	1107	1106	1105	1104	1103	1101	1100	0 0 0	1 1 1	1 1 1
9·1	·1099	1098	1096	1095	1094	1093	1092	1090	1089	1088	0 0 0	0 1 1	1 1 1
9·2	·1087	1086	1085	1083	1082	1081	1080	1079	1078	1076	0 0 0	0 1 1	1 1 1
9·3	·1075	1074	1073	1072	1071	1070	1068	1067	1066	1065	0 0 0	0 1 1	1 1 1
9·4	·1064	1063	1062	1060	1059	1058	1057	1056	1055	1054	0 0 0	0 1 1	1 1 1
9·5	1053	1052	1050	1049	1048	1047	1046	1045	1044	1043	0 0 0	0 1 1	1 1 1
9·6	1042	1041	1039	1038	1037	1036	1035	1034	1033	1032	0 0 0	0 1 1	1 1 1
9·7	1031	1030	1029	1028	1027	1026	1025	1024	1022	1021	0 0 0	0 1 1	1 1 1
9·8	·1020	1019	1018	1017	1016	1015	1014	1013	1012	1011	0 0 0	0 1 1	1 1 1
9·9	1010	1009	1008	1007	1006	1005	1004	1003	1002	1001	0 0 0	0 0 1	1 1 1

TABLE 19

SQUARE ROOTS. *From 10 to 100*

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
10	3·162	3·178	3·194	3·209	3·225	3·240	3·256	3·271	3·286	3·302	2	3	5	6	8	9	11	12	14
11	3·317	3·332	3·347	3·362	3·376	3·391	3·406	3·421	3·435	3·450	1	3	4	6	7	9	10	12	13
12	3·464	3·479	3·493	3·507	3·521	3·536	3·550	3·564	3·578	3·592	1	3	4	6	7	8	10	11	13
13	3·606	3·619	3·633	3·647	3·661	3·674	3·688	3·701	3·715	3·728	1	3	4	5	7	8	10	11	12
14	3·742	3·755	3·768	3·782	3·795	3·808	3·821	3·834	3·847	3·860	1	3	4	5	7	8	9	11	12
15	3·873	3·886	3·899	3·912	3·924	3·937	3·950	3·962	3·975	3·987	1	3	4	5	6	8	9	10	11
16	4·000	4·012	4·025	4·037	4·050	4·062	4·074	4·087	4·099	4·111	1	2	4	5	6	7	9	10	11
17	4·123	4·135	4·147	4·159	4·171	4·183	4·195	4·207	4·219	4·231	1	2	4	5	6	7	8	10	11
18	4·243	4·254	4·266	4·278	4·290	4·301	4·313	4·324	4·336	4·347	1	2	3	5	6	7	8	9	10
19	4·359	4·370	4·382	4·393	4·405	4·416	4·427	4·438	4·450	4·461	1	2	3	5	6	7	8	9	10
20	4·472	4·483	4·494	4·506	4·517	4·528	4·539	4·550	4·561	4·572	1	2	3	4	6	7	8	9	10
21	4·583	4·593	4·604	4·615	4·626	4·637	4·648	4·658	4·669	4·680	1	2	3	4	5	6	8	9	10
22	4·690	4·701	4·712	4·722	4·733	4·743	4·754	4·764	4·775	4·785	1	2	3	4	5	6	7	8	9
23	4·796	4·806	4·817	4·827	4·837	4·848	4·858	4·868	4·879	4·889	1	2	3	4	5	6	7	8	9
24	4·899	4·909	4·919	4·930	4·940	4·950	4·960	4·970	4·980	4·990	1	2	3	4	5	6	7	8	9
25	5·000	5·010	5·020	5·030	5·040	5·050	5·060	5·070	5·079	5·089	1	2	3	4	5	6	7	8	9
26	5·099	5·109	5·119	5·128	5·138	5·148	5·158	5·167	5·177	5·187	1	2	3	4	5	6	7	8	9
27	5·196	5·206	5·215	5·225	5·235	5·244	5·254	5·263	5·273	5·282	1	2	3	4	5	6	7	8	9
28	5·292	5·301	5·310	5·320	5·329	5·339	5·348	5·357	5·367	5·376	1	2	3	4	5	6	7	7	8
29	5·385	5·394	5·404	5·413	5·422	5·431	5·441	5·450	5·459	5·468	1	2	3	4	5	5	6	7	8
30	5·477	5·486	5·495	5·505	5·514	5·523	5·532	5·541	5·550	5·559	1	2	3	4	4	5	6	7	8
31	5·568	5·577	5·586	5·595	5·604	5·612	5·621	5·630	5·639	5·648	1	2	3	3	4	5	6	7	8
32	5·657	5·666	5·675	5·683	5·692	5·701	5·710	5·718	5·727	5·736	1	2	3	3	4	5	6	7	8
33	5·745	5·753	5·762	5·771	5·779	5·788	5·797	5·805	5·814	5·822	1	2	3	3	4	5	6	7	8
34	5·831	5·840	5·848	5·857	5·865	5·874	5·882	5·891	5·899	5·908	1	2	3	3	4	5	6	7	8
35	5·916	5·925	5·933	5·941	5·950	5·958	5·967	5·975	5·983	5·992	1	2	2	3	4	5	6	7	8
36	6·000	6·008	6·017	6·025	6·033	6·042	6·050	6·058	6·066	6·075	1	2	2	3	4	5	6	7	7
37	6·083	6·091	6·099	6·107	6·116	6·124	6·132	6·140	6·148	6·156	1	2	2	3	4	5	6	7	7
38	6·164	6·173	6·181	6·189	6·197	6·205	6·213	6·221	6·229	6·237	1	2	2	3	4	5	6	6	7
39	6·245	6·253	6·261	6·269	6·277	6·285	6·293	6·301	6·309	6·317	1	2	2	3	4	5	6	6	7
40	6·325	6·332	6·340	6·348	6·356	6·364	6·372	6·380	6·387	6·395	1	2	2	3	4	5	6	6	7
41	6·403	6·411	6·419	6·427	6·434	6·442	6·450	6·458	6·465	6·473	1	2	2	3	4	5	5	6	7
42	6·481	6·488	6·496	6·504	6·512	6·519	6·527	6·535	6·542	6·550	1	2	2	3	4	5	5	6	7
43	6·557	6·565	6·573	6·580	6·588	6·595	6·603	6·611	6·618	6·626	1	2	2	3	4	5	5	6	7
44	6·633	6·641	6·648	6·656	6·663	6·671	6·678	6·686	6·693	6·701	1	2	2	3	4	5	5	6	7
45	6·708	6·716	6·723	6·731	6·738	6·745	6·753	6·760	6·768	6·775	1	1	2	3	4	4	5	6	7
46	6·782	6·790	6·797	6·804	6·812	6·819	6·826	6·834	6·841	6·848	1	1	2	3	4	4	5	6	7
47	6·856	6·863	6·870	6·877	6·885	6·892	6·899	6·907	6·914	6·921	1	1	2	3	4	4	5	6	7
48	6·928	6·935	6·943	6·950	6·957	6·964	6·971	6·979	6·986	6·993	1	1	2	3	4	4	5	6	6
49	7·000	7·007	7·014	7·021	7·029	7·036	7·043	7·050	7·057	7·064	1	1	2	3	4	4	5	6	6
50	7·071	7·078	7·085	7·092	7·099	7·106	7·113	7·120	7·127	7·134	1	1	2	3	4	4	5	6	6
51	7·141	7·148	7·155	7·162	7·169	7·176	7·183	7·190	7·197	7·204	1	1	2	3	4	4	5	6	6
52	7·211	7·218	7·225	7·232	7·239	7·246	7·253	7·259	7·266	7·273	1	1	2	3	3	4	5	6	6
53	7·280	7·287	7·294	7·301	7·308	7·314	7·321	7·328	7·335	7·342	1	1	2	3	3	4	5	5	6
54	7·348	7·355	7·362	7·369	7·376	7·382	7·389	7·396	7·403	7·409	1	1	2	3	3	4	5	5	6

TABLE 19 (Contd.)

SQUARE ROOTS FROM 10 TO 100

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
55	7·416	7·423	7·430	7·436	7·443	7·450	7·457	7·463	7·470	7·477	1	1	2	3	3	4	5	5	6
56	7·483	7·490	7·497	7·503	7·510	7·517	7·523	7·530	7·537	7·543	1	1	2	3	3	4	5	5	6
57	7·550	7·556	7·563	7·570	7·576	7·583	7·589	7·596	7·603	7·609	1	1	2	3	3	4	5	5	6
58	7·616	7·622	7·629	7·635	7·642	7·649	7·655	7·662	7·668	7·675	1	1	2	3	3	4	5	5	6
59	7·681	7·688	7·694	7·701	7·707	7·714	7·720	7·727	7·733	7·740	1	1	2	3	3	4	4	5	6
60	7·746	7·752	7·759	7·765	7·772	7·778	7·785	7·791	7·797	7·804	1	1	2	3	3	4	4	5	6
61	7·810	7·817	7·823	7·829	7·836	7·842	7·849	7·855	7·861	7·868	1	1	2	3	3	4	4	5	6
62	7·874	7·880	7·887	7·893	7·899	7·906	7·912	7·918	7·925	7·931	1	1	2	3	3	4	4	5	6
63	7·937	7·944	7·950	7·956	7·962	7·969	7·975	7·981	7·987	7·994	1	1	2	3	3	4	4	5	6
64	8·000	8·006	8·012	8·019	8·025	8·031	8·037	8·044	8·050	8·056	1	1	2	2	3	4	4	5	6
65	8·062	8·068	8·075	8·081	8·087	8·093	8·099	8·106	8·112	8·118	1	1	2	2	3	4	4	5	6
66	8·124	8·130	8·136	8·142	8·149	8·155	8·161	8·167	8·173	8·179	1	1	2	2	3	4	4	5	5
67	8·185	8·191	8·198	8·204	8·210	8·216	8·222	8·228	8·234	8·240	1	1	2	2	3	4	4	5	5
68	8·246	8·252	8·258	8·264	8·270	8·276	8·283	8·289	8·295	8·301	1	1	2	2	3	4	4	5	5
69	8·307	8·313	8·319	8·325	8·331	8·337	8·343	8·349	8·355	8·361	1	1	2	2	3	4	4	5	5
70	8·367	8·373	8·379	8·385	8·390	8·396	8·402	8·408	8·414	8·420	1	1	2	2	3	4	4	5	5
71	8·426	8·432	8·438	8·444	8·450	8·456	8·462	8·468	8·473	8·479	1	1	2	2	3	4	4	5	5
72	8·485	8·491	8·497	8·503	8·509	8·515	8·521	8·526	8·532	8·538	1	1	2	2	3	3	4	5	5
73	8·544	8·550	8·556	8·562	8·567	8·573	8·579	8·585	8·591	8·597	1	1	2	2	3	3	4	5	5
74	8·602	8·608	8·614	8·620	8·626	8·631	8·637	8·643	8·649	8·654	1	1	2	2	3	3	4	5	5
75	8·660	8·666	8·672	8·678	8·683	8·689	8·695	8·701	8·706	8·712	1	1	2	2	3	3	4	5	5
76	8·718	8·724	8·729	8·735	8·741	8·746	8·752	8·758	8·764	8·769	1	1	2	2	3	3	4	5	5
77	8·775	8·781	8·786	8·792	8·798	8·803	8·809	8·815	8·820	8·826	1	1	2	2	3	3	4	4	5
78	8·832	8·837	8·843	8·849	8·854	8·860	8·866	8·871	8·877	8·883	1	1	2	2	3	3	4	4	5
79	8·888	8·894	8·899	8·905	8·911	8·916	8·922	8·927	8·933	8·939	1	1	2	2	3	3	4	4	5
80	8·944	8·950	8·955	8·961	8·967	8·972	8·978	8·983	8·989	8·994	1	1	2	2	3	3	4	4	5
81	9·000	9·006	9·011	9·017	9·022	9·028	9·033	9·039	9·044	9·050	1	1	2	2	3	3	4	4	5
82	9·055	9·061	9·066	9·072	9·077	9·083	9·088	9·094	9·099	9·105	1	1	2	2	3	3	4	4	5
83	9·110	9·116	9·121	9·127	9·132	9·138	9·143	9·149	9·154	9·160	1	1	2	2	3	3	4	4	5
84	9·165	9·171	9·176	9·182	9·187	9·192	9·198	9·203	9·209	9·214	1	1	2	2	3	3	4	4	5
85	9·220	9·225	9·230	9·236	9·241	9·247	9·252	9·257	9·263	9·268	1	1	2	2	3	3	4	4	5
86	9·274	9·279	9·284	9·290	9·295	9·301	9·306	9·311	9·317	9·322	1	1	2	2	3	3	4	4	5
87	9·327	9·333	9·338	9·343	9·349	9·354	9·359	9·365	9·370	9·375	1	1	2	2	3	3	4	4	5
88	9·381	9·386	9·391	9·397	9·402	9·407	9·413	9·418	9·423	9·429	1	1	2	2	3	3	4	4	5
89	9·434	9·439	9·445	9·450	9·455	9·460	9·466	9·471	9·476	9·482	1	1	2	2	3	3	4	4	5
90	9·487	9·492	9·497	9·503	9·508	9·513	9·518	9·524	9·529	9·534	1	1	2	2	3	3	4	4	5
91	9·539	9·545	9·550	9·555	9·560	9·566	9·571	9·576	9·581	9·586	1	1	2	2	3	3	4	4	5
92	9·592	9·597	9·602	9·607	9·612	9·618	9·623	9·628	9·633	9·638	1	1	2	2	3	3	4	4	5
93	9·644	9·649	9·654	9·659	9·664	9·670	9·675	9·680	9·685	9·690	1	1	2	2	3	3	4	4	5
94	9·695	9·701	9·706	9·711	9·716	9·721	9·726	9·731	9·737	9·742	1	1	2	2	3	3	4	4	5
95	9·747	9·752	9·757	9·762	9·767	9·772	9·778	9·783	9·788	9·793	1	1	2	2	3	3	4	4	5
96	9·798	9·803	9·808	9·813	9·818	9·823	9·829	9·834	9·839	9·844	1	1	2	2	3	3	4	4	5
97	9·849	9·854	9·859	9·864	9·869	9·874	9·879	9·884	9·889	9·894	1	1	1	2	3	3	4	4	5
98	9·899	9·905	9·910	9·915	9·920	9·925	9·930	9·935	9·940	9·945	0	1	1	2	2	3	3	4	4
99	9·950	9·955	9·960	9·965	9·970	9·975	9·980	9·985	9·990	9·995	0	1	1	2	2	3	3	4	4

TABLE 20

RECIPROCALS OF NUMBERS. FROM 1 TO 10

(Numbers in difference columns to be subtracted, not added.)

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
1·0	1·000	9901	9804	9709	9615	9524	9434	9346	9259	9174									
1·1	9091	9009	8929	8850	8772	8696	8621	8547	8475	8403									
1·2	8333	8264	8197	8130	8065	8000	7937	7874	7813	7752									
1·3	7692	7634	7576	7519	7463	7407	7353	7299	7246	7194									
1·4	7143	7092	7042	6993	6944	6897	6849	6803	6757	6711	5	10	14	19	24	29	33	38	43
1·5	6667	6623	6579	6536	6494	6452	6410	6369	6329	6289	4	8	13	17	21	25	29	33	38
1·6	6250	6211	6173	6135	6098	6061	6024	5988	5952	5917	4	7	11	15	18	22	26	29	33
1·7	5882	5848	5814	5780	5747	5714	5682	5650	5618	5587	3	6	10	13	16	20	23	26	29
1·8	·5556	5525	5495	5464	5435	5405	5376	5348	5319	5291	3	6	9	12	15	17	20	23	26
1·9	5263	5236	5208	5181	5155	5128	5102	5076	5051	5025	3	5	8	11	13	16	18	21	24
2·0	5000	4975	4950	4926	4902	4878	4854	4831	4808	4785	2	5	7	10	12	14	17	19	21
2·1	4762	4739	4717	4695	4673	4651	4630	4608	4587	4566	2	4	7	9	11	13	15	17	20
2·2	·4545	4525	4505	4484	4464	4444	4425	4405	4386	4367	2	4	6	8	10	12	14	16	18
2·3	4348	4329	4310	4292	4274	4255	4237	4219	4202	4184	2	4	5	7	9	11	13	14	16
2·4	4167	4149	4132	4115	4098	4082	4065	4049	4032	4016	2	3	5	7	8	10	12	13	15
2·5	·4000	3984	3968	3953	3937	3922	3906	3891	3876	3861	2	3	5	6	8	9	11	12	14
2·6	3846	3831	3817	3802	3788	3774	3759	3745	3731	3717	1	3	4	6	7	8	10	11	13
2·7	3704	3690	3676	3663	3650	3636	3623	3610	3597	3584	1	3	4	5	7	8	9	11	12
2·8	3571	3559	3546	3534	3521	3509	3497	3484	3472	3460	1	2	4	5	6	7	9	10	11
2·9	3448	3436	3425	3413	3401	3390	3378	3367	3356	3344	1	2	3	5	6	7	8	9	10
3·0	3333	3322	3311	3300	3289	3279	3268	3257	3247	3236	1	2	3	4	5	6	7	9	10
3·1	3226	3215	3205	3195	3185	3175	3165	3155	3145	3135	1	2	3	4	5	6	7	8	9
3·2	·3125	3115	3106	3096	3086	3077	3067	3058	3049	3040	1	2	3	4	5	6	7	8	9
3·3	3030	3021	3012	3003	2994	2985	2976	2967	2959	2950	1	2	3	4	4	5	6	7	8
3·4	2941	2933	2924	2915	2907	2899	2890	2882	2874	2865	1	2	3	3	4	5	6	7	8
3·5	·2857	2849	2841	2833	2825	2817	2809	2801	2793	2786	1	2	2	3	4	5	6	6	7
3·6	2778	2770	2762	2755	2747	2740	2732	2725	2717	2710	1	2	2	3	4	5	5	6	7
3·7	·2703	2695	2688	2681	2674	2667	2660	2653	2646	2639	1	1	2	3	4	4	5	6	6
3·8	2632	2625	2618	2611	2604	2597	2591	2584	2577	2571	1	1	2	3	3	4	5	5	6
3·9	2564	2558	2551	2545	2538	2532	2525	2519	2513	2506	1	1	2	3	3	4	4	5	6
4·0	·2500	2494	2488	2481	2475	2469	2463	2457	2451	2445	1	1	2	2	3	4	4	5	5
4·1	·2439	2433	2427	2421	2415	2410	2404	2398	2392	2387	1	1	2	2	3	3	4	5	5
4·2	·2381	2375	2370	2364	2358	2353	2347	2342	2336	2331	1	1	2	2	3	3	4	4	5
4·3	·2326	2320	2315	2309	2304	2299	2294	2288	2283	2278	1	1	2	2	3	3	4	4	5
4·4	2273	2268	2262	2257	2252	2247	2242	2237	2232	2227	1	1	2	2	3	3	4	4	5
4·5	2222	2217	2212	2208	2203	2198	2193	2188	2183	2179	0	1	1	2	2	3	3	4	4
4·6	2174	2169	2165	2160	2155	2151	2146	2141	2137	2132	0	1	1	2	2	3	3	4	4
4·7	2128	2123	2119	2114	2110	2105	2101	2096	2092	2088	0	1	1	2	2	3	3	4	4
4·8	2083	2079	2075	2070	2066	2062	2058	2053	2049	2045	0	1	1	2	2	3	3	3	4
4·9	·2041	2037	2033	2028	2024	2020	2016	2012	2008	2004	0	1	1	2	2	2	3	3	4
5·0	·2000	1996	1992	1988	1984	1980	1976	1972	1969	1965	0	1	1	2	2	2	3	3	4
5·1	·1961	1957	1953	1949	1946	1942	1938	1934	1931	1927	0	1	1	2	2	2	3	3	3
5·2	·1923	1919	1916	1912	1908	1905	1901	1898	1894	1890	0	1	1	1	2	2	3	3	3
5·3	·1887	1883	1880	1876	1873	1869	1866	1862	1859	1855	0	1	1	1	2	2	2	3	3
5·4	·1852	1848	1845	1842	1838	1835	1832	1828	1825	1821	0	1	1	1	2	2	2	3	3

TABLE 20 (Contd.)

SQUARE ROOTS. From 1 to 10

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
5·5	2·345	2·347	2·349	2·352	2·354	2·356	2·358	2·360	2·362	2·364	0	0	1	1	1	1	1	2	2
5·6	2·366	2·369	2·371	2·373	2·375	2·377	2·379	2·381	2·383	2·385	0	0	1	1	1	1	1	2	2
5·7	2·387	2·390	2·392	2·394	2·396	2·398	2·400	2·402	2·404	[illegible]	0	0	1	1	1	1	1	2	2
5·8	2·408	2·410	2·412	2·415	2·417	2·419	2·421	2·423	[illegible]	[illegible]	0	0	1	1	1	1	1	2	2
5·9	2·429	2·431	2·433	2·435	2·437	2·439	2·441	2·443	[illegible]	2·447	0	0	1	1	1	1	1	2	2
6·0	2·449	2·452	2·454	2·456	2·458	2·460	2·462	2·464	2·466	[illegible]	0	0	1	1	1	1	1	2	2
6·1	2·470	2·472	2·474	2·476	2·478	2·480	2·482	2·484	2·486	[illegible]	0	0	1	1	1	1	1	2	2
6·2	2·490	2·492	2·494	2·496	2·498	2·500	2·502	2·504	[illegible]	[illegible]	0	0	1	1	1	1	1	2	2
6·3	2·510	2·512	2·514	2·516	2·518	2·520	2·522	2·524	2·526	[illegible]	0	0	1	1	1	1	1	2	2
6·4	2·530	2·532	2·534	2·536	2·538	2·540	2·542	2·544	2·546	2·548	0	0	1	1	1	1	1	2	2
6·5	2·550	2·551	2·553	2·555	2·557	2·559	2·561	2·563	2·565	2·567	0	0	1	1	1	1	1	2	2
6·6	2·569	2·571	2·573	2·575	2·577	2·579	2·581	2·583	2·585	[illegible]	0	0	1	1	1	1	1	2	2
6·7	2·588	2·590	2·592	2·594	2·596	2·598	2·600	2·602	[illegible]	[illegible]	0	0	1	1	1	1	1	2	2
6·8	2·608	2·610	2·612	2·613	2·615	2·617	2·619	2·621	2·623	2·625	0	0	1	1	1	1	1	2	2
6·9	2·627	2·629	2·631	2·632	2·634	2·636	2·638	2·640	2·642	[illegible]	0	0	1	1	1	1	1	2	2
7·0	2·646	2·648	2·650	2·651	2·653	2·655	2·657	2·659	2·661	[illegible]	0	0	1	1	1	1	1	2	2
7·1	2·665	2·666	2·668	2·670	2·672	2·674	2·676	2·678	2·680	[illegible]	0	0	1	1	1	1	1	1	2
7·2	2·683	2·685	2·687	2·689	2·691	2·693	2·694	2·696	[illegible]	[illegible]	0	0	1	1	1	1	1	1	2
7·3	2·702	2·704	2·706	2·707	2·709	2·711	2·713	2·715	2·717	[illegible]	0	0	1	1	1	1	1	1	2
7·4	2·720	2·722	2·724	2·726	2·728	2·729	2·731	2·733	2·735	2·737	0	0	1	1	1	1	1	1	2
7·5	2·739	2·740	[illegible]	2·744	2·746	2·748	2·750	2·751	2·753	2·755	0	0	1	1	1	1	1	1	2
7·6	2·757	2·759	2·760	[illegible]	2·764	2·766	2·768	2·769	2·771	2·773	0	0	1	1	1	1	1	1	2
7·7	2·775	2·777	2·778	2·780	2·782	2·784	2·786	2·787	2·789	2·791	0	0	1	1	1	1	1	1	2
7·8	2·793	2·795	2·796	2·798	2·800	2·802	2·804	2·805	2·807	2·809	0	0	1	1	1	1	1	1	2
7·9	2·811	2·812	2·814	2·816	2·818	2·820	2·821	2·823	2·825	2·827	0	0	1	1	1	1	1	1	2
8·0	2·828	2·830	2·832	2·834	2·835	2·837	2·839	2·841	2·843	2·844	0	0	1	1	1	1	1	1	2
8·1	2·846	[illegible]	[illegible]	[illegible]	[illegible]	2·855	2·857	2·858	2·860	2·862	0	0	1	1	1	1	1	1	2
8·2	2·864	2·865	[illegible]	2·869	2·871	2·872	2·874	2·876	2·877	2·879	0	0	1	1	1	1	1	1	2
8·3	2·881	2·883	[illegible]	[illegible]	2·888	2·890	2·891	2·893	2·895	2·897	0	0	1	1	1	1	1	1	2
8·4	2·898	2·900	[illegible]	2·903	[illegible]	[illegible]	2·909	2·910	2·912	2·914	0	0	1	1	1	1	1	1	2
8·5	2·915	[illegible]	[illegible]	2·921	2·922	2·924	2·926	2·927	2·929	2·931	0	0	1	1	1	1	1	1	2
8·6	2·933	2·934	[illegible]	2·938	2·939	2·941	2·943	2·944	2·946	2·948	0	0	1	1	1	1	1	1	2
8·7	2·950	2·951	2·953	2·955	[illegible]	2·958	[illegible]	2·961	2·963	2·965	0	0	1	1	1	1	1	1	2
8·8	2·966	2·968	2·970	2·972	[illegible]	[illegible]	2·977	2·978	2·980	2·982	0	0	1	1	1	1	1	1	2
8·9	2·983	2·985	2·987	2·988	2·990	[illegible]	2·993	2·995	2·997	2·998	0	0	1	1	1	1	1	1	2
9·0	3·000	3·002	3·003	[illegible]	3·007	3·008	3·010	3·012	3·013	3·015	0	0	0	1	1	1	1	1	1
9·1	3·017	3·018	3·020	3·022	3·023	[illegible]	3·027	3·028	3·030	3·032	0	0	0	1	1	1	1	1	1
9·2	3·033	3·035	[illegible]	3·038	3·040	3·041	3·043	3·045	[illegible]	3·048	0	0	0	1	1	1	1	1	1
9·3	3·050	3·051	3·053	3·055	3·056	3·058	3·059	3·061	3·063	3·064	0	0	0	1	1	1	1	1	1
9·4	3·066	3·068	3·069	3·071	3·072	3·074	3·076	3·077	3·079	3·081	0	0	0	1	1	1	1	1	1
9·5	3·082	3·084	3·085	3·087	3·089	3·090	3·092	3·094	3·095	3·097	0	0	0	1	1	1	1	1	1
9·6	3·098	3·100	3·102	3·103	3·105	3·106	3·108	3·110	3·111	3·113	0	0	0	1	1	1	1	1	1
9·7	3·114	3·116	3·118	3·119	3·121	3·122	3·124	3·126	3·127	3·129	0	0	0	1	1	1	1	1	1
9·8	3·130	3·132	3·134	3·135	3·137	3·138	3·140	3·142	3·143	3·145	0	0	0	1	1	1	1	1	1
9·9	3·146	3·148	3·150	3·151	3·153	3·154	3·156	3·158	3·159	3·161	0	0	0	1	1	1	1	1	1

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