Ravi P. Agarwal · Donal O'Regan
Samir H. Saker

Oscillation and Stability of Delay Models in Biology

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Ravi P. Agarwal: To Sadhna, Sheba, and Danah
Donal O'Regan: To Alice, Aoife, Lorna, Daniel, and Niamh
Samir H. Saker: To Mona, Meran, Maryam, Menah, and Ahmed

Preface

In mathematics you do not understand things. You just get used to them.

John L. Von Neumann (1903–1957).

We are servants rather than masters in mathematics.

Charles Hermit (1822–1901).

A mathematical model is an equation or a system of equations used to describe a natural phenomenon. Many researchers study the qualitative behavior of nonlinear delay mathematical models in a single species and also species with interactions. The qualitative analysis of delay models with constant coefficients (autonomous models) has been studied extensively. We know that the variation of the environment plays an important role in many biological and ecological dynamical systems. For example, physical environment conditions such as temperature and humidity and availability of food, water, and other resources usually vary in time with seasonal or daily variation. Therefore, more realistic models would be nonautonomous systems. One of the purposes of our book is to study oscillation and global stability of specific types of nonautonomous delay models in biology. In particular, our book presents recent research results on the qualitative behavior of mathematical models in biology.

The book consists of six chapters and is organized as follows:

In Chap. 1, we discuss the derivation and extensions of logistic models and some of their applications. This chapter also contains some useful results from mathematical analysis which are needed throughout the book.

In Chap. 2, we are concerned with oscillation and nonoscillation of different types of delay logistic models and their modified forms. In particular, we study the oscillation of models of Hutchinson type, models with delayed feedback, α-delay logistic models, α-delay models with several delays, models with nonlinear delays, hyperlogistic models, delay models with harvesting, and models with varying capacity.

In Chap. 3, we discuss the local and global stability of different types of delay logistic models. In particular, we are concerned with the local and global stability of autonomous logistic models and the uniform and 3/2 global stability of nonautonomous delay logistic models. Also we discuss a generalized logistic model and models with impulses.

In Chap. 4, we discuss autonomous and nonautonomous logistic models with piecewise arguments.

In Chap. 5, we discuss the oscillation of autonomous and nonautonomous "food-limited" population models with delay times and impulsive effects as well as the existence of periodic solutions. Also we study the 3/2 global stability of the classical model and the 3/2 uniform stability of a model with a parameter l. In addition, we discuss the global stability of models with impulses and more generalized models, "food-limited" population models with periodic coefficients, and the existence of periodic solutions.

In Chap. 6, we are concerned with oscillation, global stability, and periodicity of some diffusive logistic models. In particular, we present oscillation results of a diffusive Malthus model with several delays, oscillation results of an autonomous diffusive logistic model with a Neumann boundary condition (flux conditions), oscillation results of a nonautonomous diffusive logistic model with several delays and a Neumann boundary condition (flux conditions), global stability of the delay logistic diffusion model with a Neumann boundary condition, and periodicity and stability of a periodic diffusive logistic model of Volterra-type with instantaneous and delay effects.

We wish to express our thanks to our families and friends.

Kingsville, TX, USA — Ravi P. Agarwal
Galway, Ireland — Donal O'Regan
Mansoura, Egypt — Samir H. Saker

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Chapter 1
Logistic Models

In so far as the theorems of mathematics relate to reality, they are not certain, and in so far as they are certain they do not relate to reality.
Every thing should be made as simple as possible but not simpler.

Albert Einstein (1879–1955).

Biology is moving from being a descriptive science to being a quantitative science.

John Whitmarsh, National Inst. of Health, 2005 Joint AMS.

All processes in organisms, from the interaction of molecules to complex functions of the brain and other organs, obey physical laws. Mathematical modeling is an important step towards uncovering the organizational principles and dynamic behavior of biological systems. In general mathematical models can take many forms depending on the time scale and the space structure of the problem. For example, in population dynamics, if there is a complete overlap between generations, then the population changes in a continuous manner and studies of such systems involve the use of differential equations. For example, the equation

$$N^{'}(t) = rN(t)\left(1 - \frac{N(t)}{K}\right) \tag{1.1}$$

is used to model the changes in population dynamics and is called the logistic equation.

If there is no overlap between generations, then the appropriate models are discrete and the changes are described by difference equations relating the population in a generation $n + 1$ with size $N(n + 1)$ to that in the generation n with size $N(n)$. In this case the dynamics of the population can be written by the difference equation

$$N(n + 1) = f(N(n)), \text{ for } n \geq 0, \tag{1.2}$$

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where $f : \mathbf{R} \to \mathbf{R}$ is a continuous function representing the density. For example, in equation (1.2) if

$$f(N(n)) := \frac{(r+1)N(n)}{1 + rN(n)/K}, \ r > 0, \ K > 0,$$

we obtain the discrete analogy of (1.1) which is called the Beverton–Holt equation. If $f(N(n)) = N(n)e^{r(1-N(n)}$, then we obtain the Ricker equation which can be considered as the discrete analogy of (1.1). For other models and applications we refer the reader to [1, 4, 6–9, 11, 18–20, 23, 35–37, 40, 43, 44, 50–52, 55–63, 65, 72, 80, 83].

1.1 The Logistic Models

Motivated by Malthus' Essay on the Principle of Population [45], Verhulst [76] proposed the first-order differential equation

$$\frac{dN}{dt} = rN \tag{1.3}$$

as the geometric growth of a population in the absence of environmental constraints, where N is the density of the population and $r = b - c$ is a constant net per capita growth rate or the intrinsic growth rate, where the birth rate is b and c is the death rate. The solution of (1.3) is given by

$$N(t) = N_0 e^{rt},$$

where N_0 is the size of the population at $t = 0$. The assumption that $r > 0$ implies a generation of the population, while $r < 0$ implies that the generation of the population do not contribute in a significant manner to the future, that is, generations are not capable of replacing each other, and the assumption that $r = 0$ implies that there is no change in the population.

To develop equation (1.3) and remove the restrictions imposed on the growth in Eq. (1.3), Verhulst [76] assumed that a stable population would have a saturation level characteristic of the environment. To achieve this the exponential model was augmented by a multiplicative factor, $1 - f(N/K)$, which represents the fractional deficiency of the current size from the saturation level K. He then argued that this unbounded growth must be restrained by the Malthusian "struggle for existence" and proposed the model

$$\frac{1}{N}\frac{dN}{dt} = R = r\left(1 - f(\frac{N}{K})\right). \tag{1.4}$$

Here R is the realized *per capita* rate of growth, r is the maximum per capita rate of growth in a given environment, f is an unspecified function of population density, and the constant K is the carrying capacity of the environment. Assuming a simple linear functional relationship yields what Verhulst later called the "logistique" equation (to differentiate it from the Malthusian "logarithmique")

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right). \tag{1.5}$$

1.2 Extended Logistic Models

Turner et al. [74] suggested a generalization of the logistic growth and they termed their equation the generic logistic equation. They proposed the model

$$\frac{dN(t)}{dt} = r\,(N(t))^{1+\beta(1-\gamma)}\left[1 - \left(\frac{N(t)}{K}\right)^{\beta}\right]^{\gamma}, \tag{1.6}$$

where $\beta,\ \gamma$ are positive exponents and $\gamma < 1 + \frac{1}{\beta}$. Blumberg [16] introduced another growth equation based on a modification of the Verhulst logistic growth equation to model population dynamics or organ size evolution. Blumberg observed that the major limitation of the logistic equation was the inflexibility of the inflection point. He further observed that attempts to modify the constant intrinsic growth rate term, r, treating this as a time-dependent polynomial to overcome this limitation, often lead to an underestimation of future values. Blumberg proposed the model

$$\frac{dN(t)}{dt} = rN^{\alpha}(t)\left[1 - \frac{N(t)}{K}\right]^{\gamma}, \tag{1.7}$$

which is consistent with the generic equation when $\alpha = 2 - \gamma, \beta = 1$, and $\gamma < 2$. Von Bertalanffy [15] introduced his growth equation to model fish weight growth. Here the Verhulst logistic growth equation was modified to accommodate crude "metabolic types" based upon physiological reasoning. He proposed the form

$$\frac{dN(t)}{dt} = rN^{\frac{2}{3}}(t)\left[1 - \left(\frac{N(t)}{K}\right)^{\frac{1}{3}}\right], \tag{1.8}$$

which is a special case of the Bernoulli differential equation. Richards [54] extended the growth equation developed by Von Bertalanffy to fit empirical plant data and used the equation

$$\frac{dN(t)}{dt} = rN(t)\left[1 - \left(\frac{N(t)}{K}\right)^{\beta}\right]. \tag{1.9}$$

1.3 Delay Logistic Models

Retarded functional differential equations or delay differential equations form a class of mathematical models which allow the systems rate of change to depend on its past history. A cut forest, after replanting, will take at least 20 years before reaching any kind of maturity. Hence, any mathematical model of forest harvesting and regeneration clearly must have time delays built into it. Usually any model of species dynamics without delays is an approximation at best.

As a very simple, but typical, example consider the equation

$$N^{'}(t) = -\mu N(t) + rN(t-\tau).$$

This equation is used to model the time evolution of the population $N(t)$ of adult individuals, with per capita mortality rate $\mu > 0$ and per capita reproduction rate $r > 0$. The delay $\tau > 0$ expresses the fact that newborns take some time to become adults.

Delay time introduced in any system may lead to instability and there seems to be a common belief that incorporating delays can destabilize almost any system. However, the effects of delays may be rather complicated. A rough way of incorporating time delays is to write Eq. (1.5) as

$$N^{'}(t) = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right], \tag{1.10}$$

where $N(t)$ is the population at time t, r is the growth rate of the species, and $K > 0$ is called the carrying capacity of the habitat (note that here there is no immigration or emigration). The per capita growth rate in (1.10) is a linear function of the population N and the term $[K - N(t-\tau)]/K$ denotes the feedback mechanism which takes τ units of time to respond to change in the population size. Equation (1.10) was first introduced into ecology by Hutchinson [32].

1.4 Some Results from Analysis

In this section, we present some definitions and results from mathematical analysis which will be needed throughout this book.

We say that the subset $\mathbf{S} \subset C([a,b], \mathbf{R})$ is equicontinuous if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $|f(t_1) - f(t_2)| < \epsilon$ for all $t_1, t_2 \in [a,b]$ with $|t_1 - t_2| < \delta$ and for all $f \in \mathbf{S}$.

The set $\mathbf{S}$ is called uniformly bounded if there exits a positive number B such that $|f(t)| \leq B$ for all $t \in [a,b]$ and for all $f \in \mathbf{S}$.

Theorem 1.4.1 (Arzela–Ascoli Theorem). *A subset* $\mathbf{S}$ *in* $C([a,b], \mathbf{R})$ *is relatively compact if and only if it is uniformly bounded and equicontinuous on* $[a,b]$.

A function $g : \mathbf{R}^n \to \mathbf{R}^n$ is said to be a Hölder continuous function if there exists positive constants C and $0 \le E \le 1$ such that

$$|g(u) - g(v)| \le C\,|u - v|^E \text{ for all } u, v \in \mathbf{R}^n.$$

We now present some fixed point theorems that we will use throughout this book.

Theorem 1.4.2 (Schauder Fixed Point Theorem). *Let* $\mathbf{S}$ *be a closed, convex, and nonempty subset of a Banach space* $\mathbf{X}$*. Let* $F : \mathbf{S} \to \mathbf{S}$ *be a continuous mapping with* $F(\mathbf{S})$ *a relatively compact subset of* $\mathbf{X}$*. Then* F *has at least one fixed point in* $\mathbf{S}$.

Theorem 1.4.3 (Tychonov–Schauder Fixed Point Theorem). *Let* $\mathbf{X}$ *be a locally convex linear space, let* $\mathbf{S}$ *be a closed convex subset of* $\mathbf{X}$*, and let* $F : \mathbf{S} \to \mathbf{S}$ *be a continuous mapping with* $F(\mathbf{S})$ *compact. Then* F *has a fixed point in* $\mathbf{S}$.

Theorem 1.4.4 (Knaster's Fixed Point Theorem). *Let* $\mathbf{X}$ *be a partially ordered Banach space with ordering* $\le$*. Let* M *be a subset of* $\mathbf{X}$ *with the following properties: the infimum of* M *belongs to* M *and every nonempty subset of* M *has a supremum which belongs to* M*. Let* $F : M \to M$ *be an increasing mapping, i.e.,* $x \le y$ *implies* $Fx \le Fy$*. Then* F *has a fixed point in* M.

Next in this section we present some inequality results.

Theorem 1.4.5 (Gronwall Inequality). *Suppose* $u : [0, \beta] \to \mathbf{R}$*,* $\beta > 0$ *is a continuous function and there exist* c *and* $k \ge 0$ *such that*

$$u(t) \le c + k\int_0^t u(s)ds, \;\; for\ t \in [0, \beta].$$

Then $u(t) \le ce^{kt}$ *for* $t \in [0, \beta]$.

Theorem 1.4.6 (Gronwall–Bellman Inequality). *Let* f *and* g *be nonnegative continuous functions on* $0 \le t \le T$ *and there exist* c *such that*

$$f(t) \le c + \int_0^t g(s)f(s)ds, \;\; for\ t \in [0,\ T].$$

Then $f(t) \le c \exp\left(\int_0^t g(s)ds\right)$ *for* $t \in [0, T]$.

Theorem 1.4.7 (Halanay Lemma). *Let* t_0 *be a real number and* τ *be a nonnegative number. If* $m : [t_0 - \tau, \infty) \to [0, \infty)$ *satisfies*

$$m'(t) \le -\rho m(t) + \varrho\left(\sup_{s\in[t-\tau,t]} m(s)\right), \; for\ t \ge t_0,$$

and if $\rho > \varrho$, *then there exist positive numbers* ζ *and* η *such that*

$$m(t) \le \zeta e^{-\eta(t-t_0)}, \text{ for } t \ge t_0.$$

Theorem 1.4.8 (Barbalat [3]). *Let* $f : (0,\infty) \to \mathbf{R}$ *be Riemann integrable and uniformly continuous. Then* $\lim_{t\to\infty} f(t) = 0$.

Theorem 1.4.9 (Green's formula). *If* ϕ *and* ψ *are both twice continuously differentiable on* U *in* $\mathbb{R}^n$ *for* $n \ge 1$, *then*

$$\int\limits_U (\psi\Delta\phi - \phi\Delta\psi)\,dx = \int\limits_{\partial U} \left(\psi\frac{\partial\phi}{\partial N} - \phi\frac{\partial\psi}{\partial N}\right) dS,$$

where ∂U *is the boundary of the region* U *and* N *is the outward pointing unit normal of surface element* dS.

Theorem 1.4.10. *The set* $M \subset L_\infty[a,b]$ *is compact if and only if for every* $\epsilon > 0$, *there exists a dilatation of the interval* $[a,b]$ *to a finite number of measurable subsets* $E_i \subset [a,b]$ *such that for every* E_i *we have* $\sup_{t,s\in E_i} |f(t) - f(s)| < \epsilon$ *for all* $f \in M$.

Also we will use some results in degree theory in this book. In the following, we present some results of Mawhin [25, Theorem 7.2].

Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces and let $L : DomL \subset \mathbb{X} \to \mathbb{Y}$ be a linear operator. A linear mapping $L : DomL \subset \mathbb{X} \to \mathbb{Y}$ (with $KerL = L^{-1}(0)$ and $\operatorname{Im} L = L(DomL)$) is called a Fredholm mapping if $KerL$ has finite dimension and $\operatorname{Im} L$ is closed in $\mathbb{Y}$ and has finite codimension. The codimension of $\operatorname{Im} L$ is the dimension of $\mathbb{Y}/\operatorname{Im} L$, i.e., the dimension of the cokernel of L. When L is a Fredholm mapping, its index is the integer $IndL = \dim KerL - co\dim ImL$. If L is a Fredholm mapping of index zero then there exists continuous projections

$$P : \mathbb{X} \to KerL \quad \text{and} \quad Q : \mathbb{Y} \to \mathbb{Y}/\operatorname{Im} L.$$

Let $K_P : \operatorname{Im} L \to DomL \cap KerP$ be the inverse of the restriction L_P of L to $DomL \cap KerP$, so that $LK_P = I$ and $K_P L = I - P$.

Let Ω be a nonempty, open, and bounded subset of $\mathbb{X}$ and let $N : \mathbb{X} \to \mathbb{Y}$. The mapping N is said to be L-compact on Ω if the mapping $QN : \overline{\Omega} \to \mathbb{Y}$ is continuous, $QN(\overline{\Omega})$ is bounded, and $K_p(I - Q)N : \overline{\Omega} \to \mathbb{X}$ is compact (i.e., it is continuous and $K_P(I - Q)N(\overline{\Omega})$ is relatively compact).

Let $T : \overline{\Omega} \to \mathbb{R}^n$. The degree of T at x relative to Ω is written $\deg\{T, \Omega, x\}$. For more details about the degree theory, we refer the reader to the book [34].

Theorem 1.4.11. *Suppose L is a linear Fredholm mapping of index zero and N is L-compact in $\overline{\Omega}$. Assume*

(1) $Lx \neq \lambda Nx$ *for every* $x \in \partial\Omega \cap DomL$ *and* $\lambda \in (0, 1)$,
(2) $QNx \neq 0$ *for every* $x \in \partial\Omega \cap KerL$, *and*
(3) $\deg\{JQN| KerL, \Omega \cap KerL, 0\} \neq 0$, *where* $J : \operatorname{Im} Q \to KerL$ *is any isomorphism and* deg *denotes the Brouwer degree.*

Then $Lx = Nx$ *has at least one solution in* $DomL \cap \overline{\Omega}$.

Chapter 2
Oscillation of Delay Logistic Models

On earth there is nothing great but man, in man there is nothing great but mind.

William R. Hamilton (1805–1865).

Every problem in the calculus of variations has a solution, provided the word solution is suitably understood.

David Hilbert (1862–1943).

The qualitative study of mathematical models is important in applied mathematics, physics, meteorology, engineering, and population dynamics. In this chapter, we are concerned with the oscillation of solutions of different types of delay logistic models about their positive steady states. One of the main techniques that we will use in the proofs is the so-called linearized oscillation technique. This technique compares the oscillation of a nonlinear delay differential equation with its associated linear equation with a known oscillatory behavior.

In this chapter we establish oscillation results for a variety of autonomous and nonautonomous delay models. It is possible to extend the theory in this chapter to other models, for example, models with impulses and models with distributed delays. Results for other models (which are based on the ideas in this chapter) can be found in the reference list. Chapter 2 presents the current approach in the literature on oscillation of delay equations.

2.1 Models of Hutchinson Type

In this section, we are concerned with the oscillation of an equation of Hutchinson type about the positive equilibrium point. First, we consider the equation

$$N'(t) = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right], \tag{2.1}$$

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where $N(t)$ is the population at time t, r is the growth rate of the species, and $K > 0$ is called the carrying capacity of the habitat (note that here there is no immigration or emigration). The solution $N(t)$ of (2.1) is said to be oscillatory about the positive steady state K if $N(t_n) - K = 0$, for $n = 0, 1, 2, ..$ and $\lim_{n\to\infty} t_n = \infty$. The solution $N(t)$ of (2.1) is said to be nonoscillatory about K if there exits $t_0 \geq 0$ such that $|N(t) - K| > 0$ for $t \geq t_0$. A solution $N(t)$ is said to be oscillatory (here we mean oscillatory about zero) if there exists a sequence $\{t_n\}$ such that $N(t_n) = 0$, for $n = 0, 1, 2, ...$ and $\lim_{n\to\infty} t_n = \infty$. A solution $N(t)$ is said to be nonoscillatory if there exits $t_0 \geq 0$ such that $|N(t)| > 0$ for $t \geq t_0$.

Together with (2.1), we consider solutions of (2.1) which correspond to the initial condition

$$\begin{cases} N(t) = \phi(t) \text{ for } -\tau \leq t \leq 0, \\ \phi \in C([-\tau, 0], [0, \infty)), \text{ and } \phi(0) > 0. \end{cases} \tag{2.2}$$

Clearly the initial value problem (2.1), (2.2) has a unique positive solution for all $t \geq 0$. This follows by the method of steps. We begin with the usual result in any book on oscillation and we quote here the linearized oscillation theorem taken from [30].

Theorem 2.1.1. *Consider the nonlinear delay differential equation*

$$x^{'}(t) + \sum_{i=1}^{n} p_i f_i(x(t - \tau_i)) = 0, \tag{2.3}$$

where for $i = 1, \ldots, n$,

$$p_i \in (0, \infty),\ \tau_i \in [0, \infty),\ f_i \in C[\mathbf{R}, \mathbf{R}], \tag{2.4}$$

$$u f_i(u) > 0 \ for \ u \neq 0 \ and \ \lim_{u\to 0} \frac{f_i(u)}{u} = 1, \tag{2.5}$$

and there exits a positive constant δ *such that*

$$\left.\begin{array}{ll} either & f_i(u) \leq u \ \ for\ 0 \leq u \leq \delta \ \ and\ i = 1, 2, \ldots, n, \\ or & f_i(u) \geq u \ \ for\ -\delta \leq u \leq 0 \ \ and\ i = 1, 2, \ldots, n. \end{array}\right\} \tag{2.6}$$

Then every solution of (2.3) oscillates if and only if every solution of the linearized equation

$$y^{'}(t) + \sum_{i=1}^{n} p_i y(t - \tau_i) = 0 \tag{2.7}$$

oscillates.

Corollary 2.1.1 ([30]). *Assume that (2.4)–(2.6) hold. Then each one of the following two conditions is sufficient for the oscillation of all solutions of (2.3):*

(*a*) $\sum_{i=1}^{n} p_i \tau_i > \frac{1}{e}$;

(*b*) $\left(\prod_{i=1}^{n} p_i\right)^{\frac{1}{n}} \left(\sum_{i=1}^{n} \tau_i\right) > \frac{1}{e}$;

and when n = 1 the condition $p\tau > 1/e$ *is necessary and sufficient for oscillation.*

Now, we establish necessary and sufficient condition for the oscillation of all positive solutions of the delay logistic model (2.1) about the positive steady state K.

Theorem 2.1.2. *Every solution of (2.1) oscillates about K if and only if* $r\tau > 1/e$.

Proof. The change of variables

$$N(t) := Ke^{x(t)} \tag{2.8}$$

reduces Eq. (2.1) to the nonlinear delay equation

$$x'(t) + rf(x(t-\tau)) = 0, \tag{2.9}$$

where

$$f(u) = e^u - 1. \tag{2.10}$$

Clearly $f(u)$ satisfies the conditions (2.4)–(2.6). Corollary 2.1.1 completes the proof. ■

We now consider a generalization of the delay logistic equation (2.1) with several delays of the form

$$N'(t) = N(t)\left[\alpha - \sum_{i=1}^{n} \beta_i N(t-\tau_i)\right], \tag{2.11}$$

where

$$\alpha,\ \beta_1,\ \beta_2, \ldots, \beta_n \in (0,\infty) \text{ and } 0 \le \tau_1 < \tau_2 < \ \tau_3 \ldots < \tau_n \equiv \tau. \tag{2.12}$$

Again with (2.11), we associate the initial condition (2.2) and then it follows by the method of steps that (2.2), (2.11) has a unique solution $N(t)$ and remains positive for all $t \ge 0$.

Theorem 2.1.3. *Assume that (2.12) holds. Then each one of the following conditions implies that every solution of (2.11) oscillates about* $N^* = \alpha / \sum_{i=1}^{n} \beta_i$:

$(i)\ \ \alpha e\left(\sum_{i=1}^{n}\beta_i\tau_i\right) > \left(\sum_{i=1}^{n}\beta_i\right);$

$(ii)\ \ \alpha e\left(\prod_{i=1}^{n}\beta_i\right)^{\frac{1}{n}}\left(\sum_{i=1}^{n}\tau_i\right) > \left(\sum_{i=1}^{n}\beta_i\right).$

Proof. Set

$$N(t) = N^* e^{x(t)}.$$

Then $x(t)$ satisfies Eq. (2.3), where

$$p_i = \beta_i N^*, \quad \text{for } i = 1, 2, \ldots, n \text{ and } f_i(u) = e^u - 1. \tag{2.13}$$

Clearly $f_i(u)$ for $i = 1, 2, .., n$ satisfy the conditions (2.4)–(2.6). The proof follows from Corollary 2.1.1. ■

2.2 Models with Delayed Feedback

In order to observe the influence of a feedback mechanism on fluctuations of a population density $N(t)$ around an equilibrium K via a constant λ, Olach [53] considered a modified nonlinear delay logistic model of the form

$$N^{'}(t) = rN(t)\left|1 - \frac{N(\tau(t))}{K}\right|^{\lambda} sgn\left[\ln\frac{K}{N(\tau(t))}\right], \quad t \geq 0, \tag{2.14}$$

where $r, K, \lambda \in (0, \infty)$ and the term $1 - N(\tau(t))/K$ denotes a feedback mechanism.

We consider those solutions of (2.14) which correspond to the initial condition

$$\begin{cases} N(t) = \phi(t), \text{ for } \tau(0) \leq t \leq 0, \\ \phi \in C([\tau(0),\ 0], [0, \infty)),\ \phi(0) > 0. \end{cases} \tag{2.15}$$

It follows by the method of steps that (2.14), (2.15) has a unique positive solution $N(t)$ for all $t > 0$.

We discuss in this section the nonoscillation of positive solutions of (2.14) around the positive equilibrium point K. We begin with the following lemma.

Lemma 2.2.1. *Consider the nonlinear retarded differential equation*

$$x^{'}(t) + p(t)f(x(\tau(t))) = 0, \quad t \geq t_0 \geq 0, \tag{2.16}$$

such that for $t \geq t_0$,

$$p \in C([t_0, \infty), \mathbf{R}^+), \ \tau \in C([t_0, \infty), \mathbf{R}^+), \ \tau(t) < t, \ \lim_{t \to \infty} \tau(t) = \infty, \tag{2.17}$$

$$f \in C(\mathbf{R}, \mathbf{R}), \ \ uf(u) > 0 \ \textit{for } u \neq 0, \tag{2.18}$$

and

$$\int_{t_0}^{\infty} p(t) = \infty. \tag{2.19}$$

Then every nonoscillatory solution $x(t)$ *of (2.16) satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that (2.16) has a nonoscillatory solution $x(t)$ which we shall assume to be eventually positive (if $x(t)$ is eventually negative the proof is similar). Since $uf(u) > 0$, we note that $x^{'}(t) < 0$ eventually for $t \geq t_1 \geq t_0$. Thus

$$\lim_{t \to \infty} x(t) = L \geq 0, \text{ exists.}$$

We claim $L = 0$. If $L > 0$, we have

$$x(t_1) \geq L + \int_{t_1}^{\infty} p(s) f(x(\tau(s))) ds,$$

which with (2.19) gives a contradiction. Thus $\lim_{t\to\infty} x(t) = 0$. The proof is complete. ∎

To prove the main oscillation results for Eq. (*2.14)* we need some oscillation results for the equation

$$x^{'}(t) + p(t) \left| x(\tau(t)) \right|^{\lambda} sgn x(\tau(t)) = 0, \ \ t \geq t_0 \geq 0. \tag{2.20}$$

Let $C_{loc}([t_0, \infty), \mathbf{R})$ denote the space of continuous functions $x : [t_0, \infty) \to \mathbf{R}$ endowed with the topology of local uniform convergence.

Theorem 2.2.1. *Suppose that (2.17) holds,* $\lambda > 1$ *and for some* $\alpha \in (0, \lambda)$

$$\lim_{t \to \infty} \sup t \left[\tau(t)\right]^{-\alpha} \left[p(t)\right]^{(\lambda - \alpha)/\lambda} < \infty. \tag{2.21}$$

Then (2.20) has a nonoscillatory solution.

Proof. According to (2.21) there is a $c > 0$ such that

$$t \left[\tau(t)\right]^{-\alpha} \left[p(t)\right]^{(\lambda - \alpha)/\lambda} < c, \ \ \text{for} \ \ t \geq t_0.$$

Set

$$v(t) = c_0 t^{\alpha/(\alpha-\lambda)}, \text{ for } t \geq t_0, \text{ where } c_0 = \left[\frac{\alpha}{\lambda-\alpha} c^{(\lambda-\alpha)/\lambda}\right]^{1/(\lambda-1)}.$$

Let $\mathbf{S} \subset C_{loc}([t_0,\infty),\mathbf{R})$ be the set of functions which satisfy

$$0 \leq x(t) \leq v(t), \text{ for } \quad t \geq t_0$$

and define the operator

$$F : \mathbf{S} \to C_{loc}([t_0,\infty),\mathbf{R})$$

by

$$F(x)(t) = \begin{cases} \int_t^\infty p(s)[x(\tau(s))]^\lambda ds, & \text{for } t \geq t_1, \\ v(t) - v(t_1) + F(x)(t_1) & \text{for } t \in [t_0, t_1), \end{cases}$$

where $t_1 > t_0$ is such that $\tau(t) \geq t_0$ for all $t \geq t_1$. Note $F(\mathbf{S}) \subset \mathbf{S}$; to see this note if $x \in \mathbf{S}$ and $t \geq t_1$ then

$$\begin{aligned} F(x(t)) &\leq \int_t^\infty p(s)[v(\tau(s))]^\lambda ds = \int_t^\infty p(s)\, c_0^\lambda\, [\tau(s)]^{\frac{\alpha\lambda}{\alpha-\lambda}}\, ds \\ &\leq c_0^\lambda\, c^{\frac{\lambda}{\lambda-\alpha}} \int_t^\infty s^{\frac{\lambda}{\alpha-\lambda}}\, ds = v(t). \end{aligned}$$

We note that $\mathbf{S}$ is a nonempty closed convex subset of $C_{loc}([t_0,\infty),\mathbf{R})$ and the operator F is continuous. The functions belonging to the set $F(\mathbf{S})$ are equicontinuous on compact subintervals of $[t_0,\infty)$. The Tychonov–Schauder Fixed Point Theorem guarantees that the operator F has an element $y \in \mathbf{S}$ such that $y = F(y)$. The proof is complete. ■

Theorem 2.2.2. *Suppose that (2.17)–(2.19) hold, and*

$$\lim_{u\to 0} \frac{f(u)}{|u|^\lambda\, sgn\, u} = 1, \quad \lambda > 1. \tag{2.22}$$

If (2.20) has a nonoscillatory solution then (2.16) also has a nonoscillatory solution.

Proof. Assume that $v(t)$ is a nonoscillatory solution of (2.20) such that $v(\tau(t)) > 0$ for $t \geq t_0$. According to (2.22) there is a $c_1 > 1$ and $\delta > 0$ such that $f(u) \leq c_1 u^\lambda$ for $u \in [0,\delta]$. From Lemma 2.2.1 we have

$$v(t) = \int_t^\infty p(s)[v(\tau(s))]^\lambda ds, \quad t \geq t_0.$$

Now choose $T_0 > t_0$ such that $v(t) < \delta$ for $t \geq T_0$. Let $\mathbf{S} \subset C_{loc}([t_0, \infty), \mathbf{R})$ be the set of functions satisfying

$$0 \leq x(t) \leq c_2 v(t), \text{ for } \quad t \geq T_0,$$

where $c_1 c_2^{\lambda} < c_2 < 1$, and define the operator

$$F : \mathbf{S} \to C_{loc}([t_0, \infty), \mathbf{R})$$

by

$$F(x)(t) = \begin{cases} \int_t^{\infty} p(s)\, f(x(\tau(s)))\, ds, & \text{for } \ t \geq t_1, \\ c_2[v(t) - v(t_1)] + F(x)(t_1), & \text{for } \ t \in [T_0, t_1), \end{cases}$$

where $t_1 > T_0$ is such that $\tau(t) \geq T_0$ for all $t \geq t_1$. Note $F(\mathbf{S}) \subset \mathbf{S}$; to see this note if $x \in \mathbf{S}$ and $t \geq t_1$ then

$$F(x(t)) \leq \int_t^{\infty} p(s)\, c_1\, [x(\tau(s))]^{\lambda}\, ds \leq c_1 c_2^{\lambda} \int_t^{\infty} p(s)\, [v(\tau(s))]^{\lambda}\, ds \leq c_2\, v(t).$$

The remainder of the proof is similar to that of Theorem 2.2.1. ■

Consider (2.14) about the positive steady state K. The transformation $N(t) = Ke^{x(t)}$ transforms Eq. (2.14) to Eq. (2.16) with

$$f(u) = |1 - e^u|\, sgn\, u.$$

Clearly the function $f(u)$ satisfies the hypothesis (2.18) and (2.22) so the above results apply to (2.14).

2.3 α-Delay Models

Aiello [2] considered the nonautonomous delay logistic model

$$N^{'}(t) = r(t)N(t)\left[1 - \frac{N(\tau(t))}{K}\right]\left|1 - \frac{N(\tau(t))}{K}\right|^{\alpha-1}, \quad t > 0, \tag{2.23}$$

where K, α are positive constants, $\alpha \neq 1$, $r(t)$ and $\tau(t)$ are positive continuous functions defined on $[0, \infty)$ such that

$$\tau(t) \leq t, \text{ and } \lim_{t\to\infty} \tau(t) = \infty. \tag{2.24}$$

Our aim in this section is to study the oscillation and nonoscillation of all positive solutions of (2.23) about the positive steady state K. We consider (2.23) with an initial condition

$$\begin{cases} N(t) = \phi(t), \text{ for } \tau(0) \le t \le 0, \\ \phi \in C([\tau(0), 0], [0, \infty)), \ \phi(0) > 0. \end{cases} \tag{2.25}$$

The change of variables

$$y(t) = \frac{N(t)}{K} - 1 \tag{2.26}$$

in (2.23) gives us the nonlinear delay equation $y'(t) = -r(t)y(\tau(t))[1 + y(t)]\,|y(\tau(t))|^{\alpha-1}$. Since $N(t) > 0$ in (2.23) then $y(t) > -1$.

In this section we consider

$$y'(t) = -r(t)y(\tau(t))[1 + y(t)]\,|y(\tau(t))|^{\alpha-1}, \ t \ge t_0. \tag{2.27}$$

Assume that

$$\int_{t_0}^{\infty} r(s)ds < \infty \tag{2.28}$$

or

$$\int_{t_0}^{\infty} r(s)ds = \infty. \tag{2.29}$$

From the change of variables (2.26), we see that the oscillation or nonoscillation of (2.23) about K is equivalent to the oscillation or nonoscillation of (2.27) about zero. In the following, we are concerned with the existence of a nonoscillatory solution of (2.27) and the results in this section are adapted from [2].

First, we consider the case when (2.28) holds. Note for $t \ge t_0$ the function $r(t)$ is positive and

$$\int_{t_0}^{\infty} r(s)ds = R, \text{ where } 0 < R < \infty. \tag{2.30}$$

Theorem 2.3.1. *Assume that (2.24), (2.28), and (2.30) hold. Then (2.27) has a positive, nonoscillatory solution bounded away from zero.*

Proof. Note if y is a positive solution of (2.27) then

$$y'(t) = -r(t)[1 + y(t)](y(\tau(t)))^{\alpha}. \tag{2.31}$$

Let $\boldsymbol{\varphi}$ denote the locally convex space of continuous functions on $[t_0, \infty)$ with the topology of uniform convergence on compact sets of **R.** Define the set $\mathbf{S} \subset \boldsymbol{\varphi}$ as

$$\mathbf{S} := \left\{ y \in \boldsymbol{\varphi} : \begin{array}{ll} y \text{ is nonincreasing} & \\ y(t) = C_\alpha, & t_0 \le t < T \\ C_\alpha \ge y(t) \ge C_\alpha \exp\left(-\int_T^t r(s)ds\right), & t \ge T \\ \frac{y(\tau(t))}{y(t)} \le \exp\left(\int_{t_0}^t r(s)ds\right), & t \ge T; \end{array} \right.$$

here $C_\alpha > 0$ is defined so that

$$[C_\alpha + 1]C_\alpha^{\alpha-1} \le \exp\left(-\int_T^t r(s)ds\right),$$

and T is sufficiently large so that $\tau(t) \ge t_0$ for all $t \ge T$. Such a constant C_α exists since the function

$$h(u) := (u + 1)u^{\alpha-1}$$

is monotone increasing and

$$h(0) = 0 \text{ and } \quad h(1) = 2.$$

Since $0 < e^{-R} < 1$ (here R is as in (2.30)) there is a u_0 such that $h(u_0) = e^{-R}$. Then let C_α be any constant satisfying the inequality $0 < C_\alpha < u_0$, and

$$[C_\alpha + 1]C_\alpha^{\alpha-1} \le e^{-R}$$

necessarily follows. Let $R(t) = \int_{t_0}^t r(s)ds$. Note that, since $r(t) \ge 0$ and $T \ge t_0$, we have

$$\int_T^t r(s)ds \le R(t).$$

We can easily see that $\mathbf{S} \subset \boldsymbol{\varphi}$ is nonempty, since $y(t) = C_\alpha$ is in **S.** In addition, **S** is a closed convex subset of $\boldsymbol{\varphi}$. Let $y \in \mathbf{S}$ and define the map

$$F\ y(t) = \begin{cases} C_\alpha, & \text{for } t_0 \le t < T, \\ C_\alpha \exp\left(-\int_T^t \frac{r(s)(1+y(s))(y(\tau(s)))^\alpha}{y(s)} ds\right), & \text{for } \quad t \ge T. \end{cases}$$

Clearly $F\ y(t)$ is continuous, nonincreasing and satisfies

$$F\ y(t) \begin{cases} = C_\alpha, & \text{for } t_0 \le t < T, \\ \le C_\alpha, & \text{for } \quad t \ge T, \end{cases}$$

and since $y(t) \le C_\alpha$, we have by definition that

$$(1 + y(s))(y(\tau(s)))^{\alpha-1} \le e^{-R} < 1, \text{ and } \frac{y(\tau(t))}{y(t)} \le e^{R(t)} \le e^{R}.$$

Then

$$\begin{aligned}\int_T^t \frac{r(s)(1 + y(s))(y(\tau(s)))^\alpha ds}{y(s)} &\le \int_T^t e^{-R} \frac{r(s)y(\tau(s))ds}{y(s)} \\ &\le \int_T^t e^{-R} e^{R(s)} r(s)ds \le \int_T^t r(s)ds,\end{aligned}$$

so,

$$F\,y(t) \ge C_\alpha \exp\left(-\int_T^t r(s)ds\right), \text{ for } t \ge T.$$

Also for $t \ge T$

$$\begin{aligned}\frac{F\,y(\tau(t))}{F\,y(t)} &= \exp\left(\int_{\tau(t)}^t \frac{r(s)(1 + y(s))(y(\tau(s)))^\alpha}{y(s)} ds\right) \\ &\le \exp\left(\int_{\tau(t)}^t e^{-R} \frac{r(s)y(\tau(s))}{y(s)} ds\right) \\ &\le \exp\left(\int_{\tau(t)}^t e^{-R} e^{R(s)} r(s)ds\right) \\ &\le \exp\left(\int_{\tau(t)}^t r(s)ds\right) \le \exp\left(\int_{t_0}^t r(s)ds\right),\end{aligned}$$

so,

$$\frac{F\,y(\tau(t))}{F\,y(t)} \le e^{R(t)}, \text{ for } t \ge T.$$

Thus, $F(\mathbf{S}) \subset \mathbf{S}$. Note $\mathbf{S}$ is bounded above by C_α and bounded below by $C_\alpha e^{-R}$. We now prove that $\{F\,y : y \in \mathbf{S}\}$ is equicontinuous on compact sets of $[t_0, \infty)$. Let T_1 and T_2 be elements in $\mathbf{R}$ and let $T_i^* = \max\{T, T_i\}$ for $i = 1, 2$. Then

$$\begin{aligned}|F\,y(T_1) - F\,y(T_2)| &= |F\,y(T_1^*) - F\,y(T_2^*)| \\ = C_\alpha &\left|\exp\left(-\int_T^{T_1^*} \frac{r(s)(1 + y(s))(y(\tau(s)))^\alpha}{y(s)} ds\right)\right.\end{aligned}$$

$$- \exp\left(\int_T^{T_2^*} \frac{-r(s)(1+y(s))(y(\tau(s)))^\alpha}{y(s)} ds\right)\Bigg|$$

$$\leq C_\alpha \left|1 - \exp\left(\int_{T_1^*}^{T_2^*} \frac{-r(s)(1+y(s))(y(\tau(s)))^\alpha}{y(s)} ds\right)\right|$$

$$\leq C_\alpha \left|1 - \exp\left(\int_{T_1^*}^{T_2^*} -r(s)ds\right)\right| \to 0, \quad \text{as } T_1 \to T_2,$$

uniformly so $\{Fy : y \in \mathbf{S}\}$ is equicontinuous on every compact set in $[t_0, \infty)$. Apply the Arzela–Ascoli Theorem to conclude that $\overline{F\mathbf{S}}$ is compact in $\mathbf{S}$. The Tychonov-Schauder Fixed Point Theorem guarantees a fixed point y^* of F. This y^* solves (2.31) from the definition of F. The proof is complete. ∎

Now, we consider the case when (2.29) holds. First, we prove that every nonoscillatory solution of (2.27) tends to zero as t tends to infinity.

Theorem 2.3.2. *Assume that the conditions of Theorem 2.3.1 hold, except that condition (2.28) is replaced by (2.29) and (2.30) is removed. Then every nonoscillatory solution of (2.27) will satisfy* $\lim_{t\to\infty} y(t) = 0$.

Proof. First, we consider the case when $y(t) > 0$ for all $t > t_1 > 0$. Let

$$v^*(t) = \sup\{s : \tau(s) = t\},$$

and since $\lim_{t\to\infty} \tau(t) = \infty$ there exists $T = v^*(t_1)$ such that $y(t) > 0$ and $y(\tau(t)) > 0$ for all $t \geq T$. From (2.27) we have

$$y'(t) = -r(t)[1 + y(t)](y(\tau(t)))^\alpha \leq 0. \tag{2.32}$$

Thus,

$$\lim_{t\to\infty} y(t) = \gamma \geq 0 \text{ exists.}$$

Suppose $\gamma > 0$. For all $t \geq T$, $y(t) \geq \gamma$ and $y(\tau(t)) \geq \gamma$ and so (2.32) implies that

$$y'(t) \leq -r(t)[1+\gamma]\gamma^\alpha,$$

so integration and (2.29) implies that $y(t)$ is negative, and this is a contradiction. Thus $\gamma = 0$. Next, we consider the case when $y(t)$ is negative. Let $y(t)$ be an eventually negative solution of (2.27), such that

$$-1 < y(t) < 0 \text{ and } \quad y(\tau(t)) < 0,$$

for $t \geq T_0$ sufficiently large. Let $T_1 > T_0$ be such that $\tau(t) \geq T_0$ for all $t \geq T_1$. Now, since $y(\tau(t)) < 0$ for $t \geq T_1$, we have from (2.27) that

$$y^{'}(t) = -r(t)[1 + y(t)]y(\tau(t))\,|y(\tau(t))|^{\alpha-1} > 0, \quad t \geq T_1. \tag{2.33}$$

Then

$$\lim_{t\to\infty} y(t) = -\beta \ \text{ exists, where } 0 \leq \beta < 1.$$

Suppose that $\beta \neq 0$. Since $y^{'}(t) > 0$ and

$$y(\tau(t)) \leq -\beta, \quad t \geq T_1,$$

we have

$$y^{'}(t) \geq -r(t)[1 + y(t)]\beta^{\alpha}, \quad t \geq T_1. \tag{2.34}$$

Now, since $y(t)$ is nonincreasing and $\lim_{t\to\infty} y(t) = -\beta$ then there exists $T_{\varepsilon} \geq T_1$ such that

$$[1 + y(t)] \geq 1 - \beta - \varepsilon > 0,$$

so with (2.34) we have

$$y^{'}(t) \geq -r(t)[1 - \beta - \varepsilon]\beta^{\alpha}, \quad t \geq T_{\varepsilon},$$

which by integration gives a contradiction. Then $\beta = 0$ and this completes the proof. ∎

Now, we give sufficient conditions for the existence of nonoscillatory solutions of (2.27) when (2.29) holds and $\alpha \neq 1$.

Theorem 2.3.3. *Assume that (2.24) and (2.29) hold and $\alpha \neq 1$. Furthermore suppose that*

$$\lim_{t\to\infty} \sup \int_{\tau(t)}^{t} r(s)ds < \hbar, \quad \text{where } \ 0 < \hbar < \infty.$$

Then (2.27) has a nonoscillatory solution.

Proof. Let $\boldsymbol{\varphi}$ denote the locally convex space of continuous functions on $[t_0, \infty)$ with the topology of uniform convergence on compact sets of **R.** Define the set $\mathbf{S} \subset \boldsymbol{\varphi}$ as

$$\mathbf{S} = \left\{ y \in \boldsymbol{\varphi} : \begin{array}{ll} y \text{ is nonincreasing} & \\ y(t) = C_\alpha, & t_0 \le t < t_1 \\ C_\alpha \ge y(t) \ge C_\alpha \exp\left(-\int_{t_1}^{t} r(s)ds\right), & t_1 \le t < \infty \\ \frac{y(\tau(t))}{y(t)} \le e^{\hbar}, & t \ge t_1 \end{array} \right.$$

where $0 < C_\alpha < 1$ is defined so that

$$[C_\alpha + 1]C_\alpha^{\alpha-1} \le 1/e^{\hbar},$$

and t_1 is sufficiently large so that

$$\int_{\tau(t)}^{t} r(s)ds < \hbar, \text{ for } t \ge t_1.$$

The remainder of the proof is similar to that of Theorem 2.3.1 and hence is omitted. ∎

From the change of variables $y(t) = N(t)/K - 1$ and Theorems 2.3.1–2.3.3 we have the following results on the delay logistic Eq. (2.23).

Theorem 2.3.4. *Assume that (2.24), (2.28), and (2.30) hold. Then (2.23) has a positive, nonoscillatory solution bounded away from K.*

Theorem 2.3.5. *Assume that (2.24) and (2.29) hold. Then every nonoscillatory solution of (2.23) will satisfy* $\lim_{t\to\infty} N(t) = K$.

Theorem 2.3.6. *Assume that (2.24) and (2.29) hold and* $\alpha \ne 1$. *Furthermore suppose that*

$$\lim_{t\to\infty} \sup \int_{\tau(t)}^{t} r(s)ds < \hbar, \ \ where \ \ 0 < \hbar < \infty.$$

Then (2.23) has a nonoscillatory solution.

The following examples illustrate the theory.

Example 1. Consider the nonlinear delay logistic equation

$$N'(t) = \frac{1}{t^2} N(t)(1 - N(t-\tau)/K)\left|1 - N(t-\tau)/K\right|^2, \ \ t > t_0,$$

where K is a positive constant. Here $r(t) = 1/t^2$, and for $t_0 > 0$,

$$\int_{t_0}^{\infty} (1/s^2)ds = 1/t_0 < \infty.$$

The conditions of Theorem 2.3.4 are satisfied, so there exists a nonoscillatory solution to this equation which is bounded away from K.

Example 2. Consider the nonlinear delay logistic equation

$$N^{'}(t) = rN(t)(1 - N(t-\tau)/K)\left|1 - N(t-\tau)/K\right|^{2}, \;\; t > t_0,$$

where K is a positive constant. Here $r(t) = r > 0$ satisfies

$$\int_{t_0}^{\infty} r ds = \infty.$$

The conditions of Theorem 2.3.5 are satisfied, so there exists a nonoscillatory solution to this equation for any $\tau > 0$ and by Theorem 2.3.5 it tends to K when t tends to infinity.

It is important to establish necessary conditions for the existence of nonoscillatory solutions to (2.23). Li [38] considered this problem and established these conditions by analyzing the generalized characteristic equation corresponding to (2.27). These conditions are equivalent to the sufficient and necessary conditions for the existence of positive solutions of (2.23).

We begin with the following theorem which gives the characteristic equation of (2.27).

Theorem 2.3.7. *A necessary and sufficient condition for the existence of a nonoscillatory solution of (2.27) is that there exist a constant C_α, a function $\lambda(t)$, and t_1 such that*

$$\lambda(t) = |C_\alpha|^{\alpha-1}\left(1 + C_\alpha \exp\left(-\int_{t_1}^{t} r(s)\lambda(s)ds\right)\right)$$
$$\times \exp\left(\int_{\tau(t)}^{t} r(s)\lambda(s)ds + (1-\alpha)\int_{t_1}^{\tau(t)} r(s)\lambda(s)ds\right). \qquad (2.35)$$

Theorem 2.3.8. *Assume that $\alpha \in (0,1)$. Then (2.29) is a necessary and sufficient condition for every solution of (2.27) to be oscillatory.*

Proof. (i) Necessity. If (2.29) does not hold, we can assume that there exists a constant

$$k =: \frac{1}{(2-\alpha)(1+C_\alpha)C_\alpha^{\alpha-1}},$$

where C_α is a positive number, such that

$$\int_{t_0}^{\infty} r(s)ds \leq k.$$

Let $T_0 = \inf_{t \geq t_0} \tau(t)$ and let $C([T_0, \infty), \mathbf{R})$ denote the locally convex space of continuous functions on $[T_0, \infty)$ with the topology of uniform convergence on compact sets of $[T_0, \infty)$**.** Define the subset Ω of $C([T_0, \infty), \mathbf{R})$ by

$$\Omega = \{x \in C([T_0, \infty), \mathbf{R}) : x(t) \geq 0, \quad |x(t)| \leq e(1 + C_\alpha)C_\alpha^{\alpha-1}, \ t \geq T_0\}.$$

Let $x \in \Omega$ and define a mapping F on Ω by

$$(F\,x)(t) = \begin{cases} |C_\alpha|^{\alpha-1}\left(1 + C_\alpha \exp\left(-\int_{T_0}^t r(s)x(s)ds\right)\right) \\ \times \exp\left(\int_{\tau(t)}^t r(s)x(s)ds + (1-\alpha)\int_{T_0}^{\tau(t)} r(s)x(s)ds\right), & t \geq t_0, \\ (F\,x)(t_0), & t_0 \geq t \geq T_0. \end{cases}$$

Then as in the proof of Theorem 2.3.1 we have $F\,x(t)$ is continuous and $F(\Omega) \subset \Omega$. Also $\{F\,x : x \in \Omega\}$ is equicontinuous and uniformly bounded. Apply the Arzela–Ascoli Theorem to conclude that $\overline{F\,\Omega}$ is compact in Ω. Now, by using the Tychonov–Schauder Fixed Point Theorem, we see that there exists a $\lambda \in \Omega$ such that for $t \geq t_0$ we have

$$\lambda(t) = |C_\alpha|^{\alpha-1}\left(1 + C_\alpha e^{-\int_{T_0}^t r(s)\lambda(s)ds}\right) \times \exp\left(\int_{\tau(t)}^t r(s)\lambda(s)ds + (1-\alpha)\int_{T_0}^{\tau(t)} r(s)\lambda(s)ds\right). \quad (2.36)$$

By Theorem 2.3.7, (2.27) has a nonoscillatory solution.

(ii) Sufficiency. If (2.27) has an eventually positive solution, by Theorem 2.3.7 there exit C_α, t_1, and a continuous function $\lambda(t)$ satisfying

$$\begin{aligned} \lambda(t) &= \left(1 + C_\alpha e^{-\int_{t_1}^t r(s)\lambda(s)ds}\right) \\ &\quad \times |C_\alpha|^{\alpha-1}\exp\left(\int_{\tau(t)}^t r(s)\lambda(s)ds + (1-\alpha)\int_{t_1}^{\tau(t)} r(s)\lambda(s)ds\right) \\ &\geq |C_\alpha|^{\alpha-1}\exp\left(\int_{\tau(t)}^t r(s)\lambda(s)ds + (1-\alpha)\int_{t_1}^{\tau(t)} r(s)\lambda(s)ds\right) \\ &\geq |C_\alpha|^{\alpha-1}\exp\left((1-\alpha)\int_{t_1}^t r(s)\lambda(s)ds\right). \end{aligned}$$

Set

$$z(t) = \exp(-(1-\alpha)\int_{t_1}^t r(s)\lambda(s)ds)$$

and note

$$z^{'}(t) \leq -|C_\alpha|^{\alpha-1}(1-\alpha)r(t)z(t_1).$$

Integrate and we have by (2.29) that

$$\lim_{t\to\infty} z(t) = -\infty,$$

a contradiction. Similarly, we can show that (2.27) has no eventually negative solution $y(t)$ with $1 + y(t) > 0$. The proof is complete. ■

Now, we consider the case when $\alpha > 1$.

Theorem 2.3.9. *Assume that $\alpha > 1$. Then a necessary and sufficient condition for the existence of a nonoscillatory solution of (2.27) is that there exists a positive continuous function $\lambda(t)$ such that for $t \geq T$*

$$\exp\left(\int_{\tau(t)}^{t} r(s)\lambda(s)ds + (1-\alpha)\int_{T}^{\tau(t)} r(s)\lambda(s)ds\right) \leq m\lambda(t), \tag{2.37}$$

where m and T are some positive constants.

Proof. (i) Sufficiency. We only consider the case (since the other case is similar) when

$$\int_{t_0}^{\infty} r(s)\lambda(s)ds < \infty.$$

Then there exist ϱ, T and $C_\alpha > 0$ such that

$$\int_{T}^{\infty} r(s)\lambda(s)ds < \varrho, \ \ (1 + C_\alpha)C_\alpha^{\alpha-1} < \frac{1}{m\varrho}.$$

Let $T_0 = \inf_{t\geq t_0} \tau(t)$. Define a mapping F on $C([T_0,\infty),\mathbf{R}^+)$ as follows

$$(F\,y)(t) := \begin{cases} \int_t^{\infty} r(s)(1+y(s))y^{\alpha}(\tau(s))ds, & t \geq T \\ (F\,y)(T) + C_\alpha \exp(-\int_{T_0}^{t} r(s)\lambda(s)ds) & \\ -C_\alpha \exp(-\int_{T_0}^{T} r(s)\lambda(s)ds), & T_0 \leq t \leq T. \end{cases}$$

Clearly F is an increasing operator. Set

$$y_0 := C_\alpha \exp(-\int_{T}^{t} r(s)\lambda(s)ds), \ \ y_{n+1} = F\,y_n, \quad n = 1, 2, \ldots .$$

Then we have that

$$y_0(t) \geq y_1(t) \geq \ldots \geq y_n(t) \geq \ldots \geq 0, \text{ for } t \geq T_0. \tag{2.38}$$

In fact

$$\begin{aligned} y_1(t) = (F\,y_0)(t) &\leq \int_t^\infty r(s)\left(1 + C_\alpha \exp(-\int_T^s r(u)\lambda(u)du)\right) \\ &\quad \times \left(C_\alpha^\alpha \exp(-\alpha \int_T^{\tau(s)} r(u)\lambda(u)du\right) ds \\ &\leq C_\alpha^\alpha (1 + C_\alpha) m \int_t^\infty r(s)\lambda(s)ds \exp\left(-\int_T^t r(s)\lambda(s)ds\right) \\ &\leq C_\alpha \exp\left(-\int_T^t r(s)\lambda(s)ds\right) = y_0(t), \quad t \geq T. \end{aligned}$$

Continue to obtain (2.38). Then $\lim_{n\to\infty} y_n(t) = y(t) \geq 0$, $t \geq T_0$, exists. From the Lebesgue's Dominated Convergence Theorem

$$y(t) := \begin{cases} \int_t^\infty r(s)(1 + y(s))y^\alpha(\tau(s))ds, & t \geq T \\ \\ (F\,y)(T) + C_\alpha \exp(-\int_{T_0}^t r(s)\lambda(s)ds) \\ -C_\alpha \exp(-\int_{T_0}^T r(s)\lambda(s)ds), & T_0 \leq t \leq T. \end{cases}$$

It is easy to see that $y(t) > 0$ on $[T_0, T]$ and hence $y(t) > 0$ for all $t \geq T_0$. Therefore, $y(t)$ is a positive solution of (2.27) on $[T, \infty)$.

(ii) Necessity. If (2.27) has an eventually positive solution then from Theorem 2.3.7 there exists a continuous positive function $\lambda(t)$ such that

$$\begin{aligned} \lambda(t) &= \left(1 + C_\alpha \exp\left(-\int_{t_1}^t r(s)\lambda(s)ds\right)\right) \\ &\quad \times C_\alpha^{\alpha-1} \exp\left(\int_{\tau(t)}^t r(s)\lambda(s)ds + (1-\alpha)\int_{t_1}^{\tau(t)} r(s)\lambda(s)ds\right) \\ &\geq C_\alpha^{\alpha-1} \exp\left(\int_{\tau(t)}^t r(s)\lambda(s)ds + (1-\alpha)\int_{t_1}^{\tau(t)} r(s)\lambda(s)ds\right). \end{aligned} \tag{2.39}$$

Let $m = 1/C_\alpha^{\alpha-1}$. Then (2.39) implies (2.37). If (2.27) has an eventually negative solution, then

$$\lambda(t) \geq (1-|C_\alpha|) \left|C_\alpha^{\alpha-1}\right| \exp\left(\int_{\tau(t)}^{t} r(s)\lambda(s)ds + (1-\alpha)\int_{t_1}^{\tau(t)} r(s)\lambda(s)ds\right),$$

where $|C_\alpha| < 1$. Thus (2.37) is also true. The proof is complete. ■

From Theorems 2.3.8 and 2.3.9 one can immediately derive some explicit necessary and sufficient conditions for the oscillation and the existence of nonoscillatory solutions of (2.23) about the positive steady state K.

2.4 α-Models with Several Delays

In this section, we consider the nonlinear delay logistic equation with several delays of the form

$$N^{'}(t) = \sum_{k=1}^{m} r_k(t)N(t)\left[1 - \frac{N(h_k(t))}{K}\right]\left|1 - \frac{N(h_k(t))}{K}\right|^{\alpha_k - 1}, \quad t > 0, \tag{2.40}$$

where $\alpha_k < 1$, $k = 1, \ldots, m$ or $\alpha_k > 1$, $k = 1, \ldots, m$ under the conditions:

(b_1) r_k, $k = 1, 2, \ldots, m$, are Lebesgue measurable functions essentially bounded in each finite interval $[0, b]$, $r_k \geq 0$,

(b_2) $h_k : [0, \infty) \to \mathbf{R}$ are Lebesgue measurable functions, $h_k(t) \leq t$, $\lim_{t\to\infty} h_k(t) = \infty$, $k = 1, 2, \ldots, m$.

The case $\alpha_k = 1$, $k = 1, \ldots, m$, will be considered in detail in Sect. 2.6.

We consider positive solutions of (2.40) with an initial condition

$$\begin{cases} N(t) = \phi(t), \text{ for } \tau_* \leq t \leq 0, \\ \phi \in C([\tau_*, 0], [0, \infty)), \ \ \phi(0) > 0, \end{cases} \tag{2.41}$$

where

$$\tau_* = \min_{1\leq k\leq m}\left(\inf_{t\geq 0}\{h_k(t)\}\right).$$

Clearly the initial value problem (2.40), (2.41) has a unique positive solution for all $t \geq 0$. This follows from the method of steps. In this section we consider

$$x^{'}(t) = -[x(t) + 1]\sum_{k=1}^{m} r_k(t)x(h_k(t))\left|x(h_k(t))\right|^{\alpha_k - 1}, \ t \geq 0, \tag{2.42}$$

and it is also possible to consider

$$x^{'}(t) = -[x(t)+1]\sum_{k=1}^{m} r_k(t)x(h_k(t))\,|x(h_k(t))|^{\alpha_k-1}\,,\ t \ge t_0,$$
$$x(t) = \varphi(t),\quad t < t_0,\quad \text{and}\quad x(t_0) = x_0 > -1,$$

where

(b_3) $\varphi : (-\infty, t_0) \to$ is a Borel measurable bounded function.

We also consider the delay differential inequalities

$$x^{'}(t) \le -[x(t)+1]\sum_{k=1}^{m} r_k(t)x(h_k(t))\,|x(h_k(t))|^{\alpha_k-1}\,,\ t \ge 0, \tag{2.43}$$

$$x^{'}(t) \ge -[x(t)+1]\sum_{k=1}^{m} r_k(t)x(h_k(t))\,|x(h_k(t))|^{\alpha_k-1}\,,\ t \ge 0. \tag{2.44}$$

In the following we discuss the nonoscillation of solutions of (2.42) which is equivalent to the nonoscillation of positive solutions of (2.40) about K. The results in this section are adapted from [5].

In the following we assume $\alpha_k < 1$, $k = 1, 2, \ldots, m$, and that $(b_1) - (b_2)$ hold and we consider solutions of (2.42), (2.43), and (2.44) for which $1 + x(t) > 0$.

We prove the following comparison theorem.

Theorem 2.4.1. *The following statements are equivalent:*

(*1*) *Either inequality (2.43) has an eventually positive solution or inequality (2.44) has an eventually negative solutions.*

(*2*) *There exist* $t_0 \ge 0$, $\varphi : (-\infty, t_0) \to \mathbf{R}$, *with either* $\varphi(t) \ge 0$, $C > 0$, *or* $\varphi(t) \le 0$, $-1 < C < 0$, *such that the inequality*

$$u(t) \ge \left(1 + C\exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right)\sum_{k=1}^{m}(F_k u)(t), \tag{2.45}$$

where

$$(F_k u)(t) = \begin{cases} |C|^{\alpha_k-1}\, r_k(t) \times \exp\{\int_{h_k(t)}^{t} u(s)ds\} \\ \times \exp\{(1-\alpha_k)\int_{t_0}^{h_k(t)} u(s)ds\}, & \text{if } h_k(t) \ge t_0 \\ \frac{r_k(t)}{|C|}\exp\{\int_{t_0}^{t} u(s)ds\}\,|\varphi(h_k(t))|^{\alpha_k}\,, & \text{if } h_k(t) < t_0 \end{cases}$$

has a nonnegative locally integrable solution on $[t_0, \infty)$.

(*3*) *Equation (2.42) has a nonoscillatory solution.*

Proof. 1) $\Rightarrow$ (2) Let x be a solution of (2.43) and $x(t) > 0$ for $t \geq t_1$. Then there exists $t_0 > t_1$ such that $h_k(t) \geq t_1$ for $t \geq t_0$, $k = 1, \ldots, m$. Denote $\varphi(t) = x(t)$, $t < t_0$, and $C = x(t_0)$. Let

$$u(t) = \frac{-x^{'}(t)}{x(t)}, \quad t \geq t_0.$$

Then $u(t) \geq 0$ and

$$x(t) = \begin{cases} C \exp\{-\int_{t_0}^{t} u(s)ds\}, & t \geq t_0, \\ \varphi(t), & t < t_0. \end{cases} \tag{2.46}$$

Then by substituting x in (2.43) we obtain inequality (2.45). Similarly (2.45) can be obtained, if $x(t) < 0$ is a solution of (2.44).

2) $\Rightarrow$ 3). Let u_0 be a nonnegative solution of inequality (2.45) with

$$\varphi(t) \leq 0, \ -1 < C < 0.$$

Denote a sequence

$$u_n(t) = \left(1 + C \exp\left\{-\int_{t_0}^{t} u_{n-1}(s)ds\right\}\right) \sum_{k=1}^{m} (F_k u_{n-1})(t). \tag{2.47}$$

Inequality (2.45) implies $u_1(t) \leq u_0(t)$. By induction, we have

$$0 \leq u_n(t) \leq u_{n-1}(t) \leq u_0(t).$$

Then there exits a pointwise limit of the nonincreasing nonnegative limit $u_n(t)$. Let

$$\lim_{n \to \infty} u_n(t) = u(t).$$

Then by the Lebesgue Convergence Theorem

$$\lim_{n \to \infty} (F_k u_n)(t) = (F_k u)(t), \quad k = 1, 2, \ldots, m.$$

Thus (2.47) implies that

$$u(t) = \left(1 + C \exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right) \sum_{k=1}^{m} (F_k u)(t).$$

Hence the function $x(t)$ defined by (2.46) is an eventually negative solution of (2.42). Now let u_0 be a nonnegative solution of inequality (2.45) with $\varphi(t) \geq 0$, $C > 0$. Let $C_1 = -C$, $\varphi_1(t) = -\varphi(t)$. Then u is also a solution of (2.45) with

C_1 (respectively $\varphi_1(t)$) instead of C (respectively, $\varphi(t)$). As in the previous case it follows that there exists an eventually negative solution of (2.42). Implication 3) ⇒ 1) is evident. The proof is complete. ■

Corollary 2.4.1. *Suppose there exist* t_0 *and* $A > 1$ *such that the inequality*

$$u(t) \geq A \sum_{k=1}^{m} r_k(t) \exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\} \times \exp\left\{(1-\alpha_k)\int_{t_0}^{h_k(t)} u(s)ds\right\} \tag{2.48}$$

has a nonnegative, locally integrable solution, where the sum contains only such terms for which $h_k(t) \geq t_0$. *Then (2.42) has a nonoscillatory solution.*

In the following we give some necessary and sufficient conditions for the existence of nonoscillatory solutions of (2.42).

Theorem 2.4.2. *There exists a nonoscillatory solution of (2.42) if and only if*

$$\int_0^\infty r_k(t)dt < \infty, \quad k = 1, 2, \ldots, m. \tag{2.49}$$

Proof. First, suppose that (2.49) holds. Then there exist t_0 and $A > 1$ such that

$$A \exp\left\{2\int_{t_0}^{\infty} \sum_{k=1}^{m} r_k(t)dt\right\} < 2.$$

For any nonnegative u

$$\begin{aligned} &A \sum_{k=1}^{m} r_k(t) \exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\} \times \exp\left\{(1-\alpha_k)\int_{t_0}^{h_k(t)} u(s)ds\right\} \\ &\leq A \sum_{k=1}^{m} r_k(t) \exp\left\{\int_{t_0}^{t} u(s)ds\right\}. \end{aligned}$$

Let

$$u(t) = 2\sum_{k=1}^{m} r_k(t).$$

From the above inequalities we see that u is a solution of inequality (2.48). Corollary 2.4.1 implies that (2.42) has an eventually positive solution.

Suppose now that for some i, $1 \leq i \leq m$, we have $\int_0^\infty r_i(t)dt = \infty$. Let x be a positive or negative solution of (2.42) for $t \geq t_1$. There exists $t_0 > t_1$ such that $h_k(t) \geq t_1$, for $t \geq t_0$ and $k = 1, 2, \ldots m$. Let

$$u(t) = \frac{-x'(t)}{x(t)}, \quad t \geq t_0.$$

Then $u(t) \geq 0$ and $x(t)$ satisfies (2.46) where $C = x(t_0)$. Substituting x in (2.42) we obtain for $t \geq t_0$

$$u(t) = \begin{cases} \sum_{k=1}^{m} |C|^{\alpha_k - 1} r_k(t)(1 + C\exp\{-\int_{t_0}^{t} u(s)ds\}) \\ \quad \times \exp\{-\alpha_k \int_{t_0}^{h_k(t)} u(s)ds\} \exp\{\int_{t_0}^{t} u(s)ds\}. \end{cases}$$

Then

$$u(t) \geq \min\{1, 1 + C\} |C|^{\alpha_k - 1} r_i(t) \exp\{(1 - \alpha_i) \int_{t_0}^{t} u(s)ds\}.$$

Hence

$$r_i(t) \leq \frac{|C|^{1-\alpha_i}}{\min\{1, 1 + C\} |C|} u(t) \exp\{-(1 - \alpha_i) \int_{t_0}^{t} u(s)ds\}$$

and so

$$\begin{aligned} \int_{t_0}^{t} r_i(s)ds &\leq \frac{|C|^{1-\alpha_i}}{\min\{1, 1 + C\} |C|} \int_{t_0}^{t} u(s) \exp\{-(1 - \alpha_i) \int_{t_0}^{s} u(\tau)d\tau\}ds \\ &= \frac{|C|^{1-\alpha_i}}{\min\{1, 1 + C\} |C|} \left(1 - \exp\{-(1 - \alpha_i) \int_{t_0}^{t} u(s)ds\}\right) \\ &\leq \frac{|C|^{1-\alpha_i}}{\min\{1, 1 + C\} |C|}. \end{aligned}$$

Hence

$$\int_{t_0}^{\infty} r_i(s)ds < \infty,$$

which gives a contradiction. The proof is complete. ∎

It is also possible to establish results when $\alpha_k = 1$ for $k = 1, 2, \ldots, m$ (see Sect. 2.6 where a more general situation is considered).

Next, we consider the case when $\alpha_k > 1$ for $k = 1, 2, \ldots, m$.

Lemma 2.4.1. *If $h \in L_\infty[a, b]$, then the linear integral operator*

$$(Hx)(t) = \begin{cases} \int_a^{h(t)} x(s)ds, & \text{if } h(t) \in [a, b] \\ 0, & \text{if } h(t) \notin [a, b] \end{cases}$$

is a completely continuous operator in $L_\infty[a, b]$.

Proof. Let $\epsilon > 0$ be given. Divide $H([a,b]) \cap [a,b]$ into a finite number of subsets F_i, $i = 1, \ldots, n$, such that for every $s_1, s_2 \in F_i$ we have $|s_1 - s_2| < \epsilon$. Let

$$E_i = h^{-1}(F_i),\ i = 1, \ldots, n,\ \ E_0 = \{t \in [a,b] : h(t) \notin [a,b]\},$$

$$S = \{x \in L_\infty[a,b] : ||x|| = 1\} \ \text{ and } \ M = H(S).$$

For dilatation E_i, $i = 1, 2, \ldots$, we have

$$\sup_{t,s \in E_i} |(Hx)(t) - (Hx)(s)| = \sup_{t,s \in E_i} |\int_{h(t)}^{h(s)} x(w)dw| \leq \sup_{t,s \in E_i} |h(t) - h(s)| < \epsilon.$$

If $i = 0$ then $\sup_{t,s \in E_0} |(Hx)(t) - (Hx)(s)| = 0$. Now Theorem 1.4.10 implies $M = H(S)$ is a compact set. ■

Theorem 2.4.3. *Suppose for some $\varepsilon > 0$, there exists a nonoscillatory solution of the linear delay differential equation*

$$x^{'}(t) = -\varepsilon \sum_{k=1}^{m} r_k(t) x(h_k(t)). \tag{2.50}$$

Then there exists a nonoscillatory solution of (2.42).

Proof. Let $t_0 > 0$, C, and $\varphi : (-\infty, t_0) \to \mathbf{R}$ be such that

$$-1 < C < 0, \ \ \varphi(t) \leq 0, \ \ |\varphi(t)| < |C| < \varepsilon^{1/(\alpha_k - 1)},$$

and hence $C \leq \varphi(t) \leq 0$. Now (2.50) with $x(t) = \varphi(t)$, $t < t_0$, and $x(t_0) = x_0$ with $x_0 = C$ has a negative solution $x_0(t) < 0$. Let

$$w_0 = -\frac{x_0^{'}(t)}{x_0(t)}.$$

Then $w_0(t) > 0$ and

$$x_0(t) = C \exp\{-\int_{t_0}^{t} w_0(s)ds\}, \ \ t \geq t_0.$$

By substituting x_0 in (2.50), we have

$$w_0(t) = \varepsilon \sum_{k=1}^{m} r_k(t) \times \begin{cases} \exp\{\int_{h_k(t)}^{t} w_0(s)ds\}, & \text{if } h_k(t) \geq t_0, \\ \exp\{\int_{t_0}^{t} w_0(s)ds\} \frac{\varphi(h_k(t))}{C}, & \text{if } h_k(t) < t_0. \end{cases}$$

Consider now two sequences

$$w_n(t) = \left(1 + C \exp\left\{-\int_{t_0}^{t} w_{n-1}(s)ds\}\right\}\right)\sum_{k=1}^{m} r_k(t)$$
$$\times \begin{cases} |C|^{\alpha_k-1} \exp\left\{\int_{h_k(t)}^{t} w_{n-1}(s)ds\right\} \\ \times \exp\left\{-(\alpha_k-1)\int_{t_0}^{h_k(t)} v_{n-1}(s)ds\right\}, & \text{if } h_k(t) \ge t_0, \\ \exp\left\{\int_{t_0}^{t} w_{n-1}(s)ds\right\} \frac{|\varphi(h_k(t))|^{\alpha_k}}{|C|}, & \text{if } h_k(t) < t_0, \end{cases}$$

$$v_n(t) = \left(1 + C \exp\left\{-\int_{t_0}^{t} v_{n-1}(s)ds\}\right\}\right)\sum_{k=1}^{m} r_k(t)$$
$$\times \begin{cases} |C|^{\alpha_k-1} \exp\left\{\int_{h_k(t)}^{t} v_{n-1}(s)ds\right\} \\ \times \exp\left\{-(\alpha_k-1)\int_{t_0}^{h_k(t)} w_{n-1}(s)ds\right\}, & if\ h_k(t) \ge t_0, \\ \exp\left\{\int_{t_0}^{t} v_{n-1}(s)ds\right\} \frac{|\varphi(h_k(t))|^{\alpha_k}}{|C|}, & if\ h_k(t) < t_0, \end{cases}$$

where $v_0 = 0$. We have

$$|\varphi(h_k(t))|^{\alpha_k-1} < |C|^{\alpha_k-1} < \varepsilon.$$

Then

$$w_0(t) \ge w_1(t), v_1(t) \ge v_0(t) = 0, \text{ and } w_0(t) \ge v_0(t).$$

Hence by induction

$$0 \le w_n(t) \le w_{n-1}(t) \le \ldots \le w_0(t),\ v_n(t) \ge v_{n-1}(t) \ge \ldots \ge v_0(t) = 0,$$

and $w_n(t) \ge v_n(t)$. There exist pointwise limits of the nonincreasing nonnegative sequence $w_n(t)$ and of the nondecreasing sequence $v_n(t)$. If we denote

$$w(t) = \lim_{n\to\infty} w_n(t) \text{ and } v(t) = \lim_{n\to\infty} v_n(t),$$

then by the Lebesgue Convergence Theorem, we conclude

$$w(t) = \left(1 + C \exp\left\{-\int_{t_0}^{t} w(s)ds\}\right\}\right)\sum_{k=1}^{m} r_k(t)$$
$$\times \begin{cases} |C|^{\alpha_k-1} \exp\left\{\int_{h_k(t)}^{t} w(s)ds\right\} \\ \times \exp\left\{-(\alpha_k-1)\int_{t_0}^{h_k(t)} v(s)ds\right\}, & \text{if } h_k(t) \ge t_0, \\ \exp\left\{\int_{t_0}^{t} w(s)ds\right\} \frac{|\varphi(h_k(t))|^{\alpha_k}}{|C|}, & \text{if } h_k(t) < t_0, \end{cases} \tag{2.51}$$

$$v(t) = \left(1 + C \exp\left\{-\int_{t_0}^{t} v(s)ds\right\}\right) \sum_{k=1}^{m} r_k(t)$$
$$\times \begin{cases} |C|^{\alpha_k - 1} \exp\left\{\int_{h_k(t)}^{t} v(s)ds\right\} \\ \times \exp\left\{-(\alpha_k - 1)\int_{t_0}^{h_k(t)} w(s)ds\right\}, & \text{if } h_k(t) \geq t_0, \\ \exp\left\{\int_{t_0}^{t} v(s)ds\right\} \frac{|\varphi(h_k(t))|^{\alpha_k}}{|C|}, & \text{if } h_k(t) < t_0. \end{cases} \tag{2.52}$$

Fix $b \geq t_0$ and denote the operator $F : L_\infty[t_0, b] \to L_\infty[t_0, b]$ by

$$(Fu)(t) = \left(1 + C \exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right) \sum_{k=1}^{m} r_k(t)$$
$$\times \begin{cases} |C|^{\alpha_k - 1} \exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\} \\ \times \exp\left\{-(\alpha_k - 1)\int_{t_0}^{h_k(t)} u(s)ds\right\}, & \text{if } h_k(t) \geq t_0, \\ \exp\left\{\int_{t_0}^{t} u(s)ds\right\} \frac{|\varphi(h_k(t))|^{\alpha_k}}{|C|}, & \text{if } h_k(t) < t_0. \end{cases}$$

Note for every function u from the interval $v \leq u \leq w$, we have $v \leq Fu \leq w$. Lemma 2.4.1 implies that the operator F is completely continuous on the space $L_\infty[t_0, b]$ (for every $b \geq t_0$). Then by the Schauder Fixed Point Theorem there exists a nonnegative solution of equation $u = Fu$. Let

$$x(t) = \begin{cases} C \exp\{-\int_{t_0}^{t} u(s)ds\}, & t \geq t_0, \\ \varphi(t), & t < t_0. \end{cases}$$

Then $x(t)$ is a negative solution of (2.42), which completes the proof. ■

2.5 Models with Harvesting

In this section we study the dynamics of a population affected by harvesting, i.e.,

$$\frac{dN}{dt} = r(N(t), t)N(t) - E(N(t), t), \tag{2.53}$$

where $E(N, t)$ is a harvesting strategy for the population.

We consider the delay model

$$N'(t) = r(t)N(t)\left[a - \sum_{k=1}^{m} b_k N(h_k(t))\right] - \sum_{l=1}^{n} c_l(t)N(g_l(t)), \quad t \geq 0, \tag{2.54}$$

with

$$N(t) = \varphi(t),\ t < 0,\ N(0) = N_0, \tag{2.55}$$

under the following conditions:

(a_1) $a > 0$, $b_k > 0$;
(a_2) $r(t) \geq 0$, $c_l(t) \geq 0$ are Lebesgue measurable and locally essentially bounded functions;
(a_3) $h_k(t)$, $g_l(t)$ are Lebesgue measurable functions, $h_k(t) \leq t$, $g_l(t) \leq t$, $\lim_{t\to\infty} h_k(t) = \infty$, $\lim_{t\to\infty} g_l(t) = \infty$;
(a_4) $\varphi : (-\infty, 0) \to \mathbf{R}$ is a Borel measurable bounded function, $\varphi(t) \geq 0$, $N_0 > 0$.

In this section we obtain sufficient conditions for positiveness, boundedness, and extinction of solutions of equation (2.54). The results in this section are adapted from [14]. An absolutely continuous function N (: $\mathbf{R} \to \mathbf{R}$) on each interval $[0, b]$ is called a solution of problem (2.54), (2.55), if it satisfies equation (2.54) for almost all $t \in [0, \infty)$ and equality (2.55) for $t \leq 0$.

First, we present some lemmas (the proofs can be found in [12, 13], and [30]) which will be used in the proof of the main results. Consider the linear delay differential equation

$$x'(t) + \sum_{l=1}^{n} c_l(t) x(g_l(t)) = 0,\ \ t \geq 0, \tag{2.56}$$

and a corresponding differential inequality

$$y'(t) + \sum_{l=1}^{n} c_l(t) y(g_l(t)) \leq 0,\ \ t \geq 0. \tag{2.57}$$

Lemma 2.5.1. *Suppose that for the functions c_l, g_l, hypotheses $(a_2) - (a_3)$ hold. Then*

(1) *If $y(t)$ is a positive solution of (2.57) for $t \geq t_0$, then $y(t) \leq x(t)$, $t \geq t_0$, where $x(t)$ is a solution of (2.56) and $x(t) = y(t)$, $t \leq t_0$.*
(2) *For every nonoscillatory solution $x(t)$ of (2.56), we have $\lim_{t\to\infty} x(t) = 0$.*
(3) *If*

$$\sup_{t\geq 0} \sum_{l=1}^{n} \int_{\min_k g_k(t)} c_l(s)ds \leq \frac{1}{e}, \tag{2.58}$$

then equation (2.56) has a nonoscillatory solution.

If in addition, $0 \leq \varphi(t) \leq N_0$, then the solution of the initial value problem (2.56)–(2.55), where $N(t)$ in (2.55) is replaced by $x(t)$, is positive.

Consider also the linear delay equation

$$x'(t) + \sum_{l=1}^{n} c_l(t)x(g_l(t)) - a(t)x(t) = 0, \quad t \geq 0. \tag{2.59}$$

A solution $X(t,s)$ of the problem

$$x'(t) + \sum_{l=1}^{n} c_l(t)x(g_l(t)) - a(t)x(t) = 0, \quad t \geq s,$$

$$x(t) = 0, \quad t < s, \quad x(s) = 1,$$

is called a fundamental function of (2.59).

Lemma 2.5.2. *Suppose for the functions c_l, g_l, hypotheses $(a_2)-(a_3)$ hold, a is a locally bounded function such that $a(t) \geq 0$,*

$$\sum_{l=1}^{n} c_l(t) \geq a(t), \quad \int_0^\infty \left[\sum_{l=1}^{n} c_l(t) - a(t)\right] = \infty, \tag{2.60}$$

and

$$\lim_{t\to\infty} \sup \left[a(t)(t - G(t)) + \sum_{l=1}^{n} c_l(t)(G(t) - g_l(t))\right] < 1, \tag{2.61}$$

where $G(t) = \max_l g_l(t)$. Then

(1) *If there exists a nonoscillatory solution of (2.59), then for some t_0 and $t \geq t_0$ we have $X(t,s) > 0$ for $t \geq s \geq t_0$, where $X(t,s)$ is a fundamental function of (2.59).*
(2) *For every nonoscillatory solution $x(t)$ of (2.59) we have $\lim_{t\to\infty} x(t) = 0$.*

Let

$$h(t) = \min_k \{h_k(t)\}, \quad g(t) = \min_l \{g_l(t)\}.$$

In addition to $(a_1)-(a_4)$ consider the following hypothesis:
(a_5). $h(t)$ is a nondecreasing continuous function.

If in (2.54) we neglect harvesting terms, i.e., assume $c_l \equiv 0$, then the positive equilibrium becomes $a/\sum_{k=1}^m b_k$.

Theorem 2.5.1. *Suppose $(a_1)-(a_5)$ hold,*

$$\varphi(t) \leq N_0 < \frac{a}{\sum_{k=1}^m b_k} \quad \text{for } t < 0, \tag{2.62}$$

and

$$\sup_{t>0} \sum_{l=1}^{n} \int_{g(t)}^{t} c_l(s) \exp\left\{ \varkappa(t) \int_{g_l(t)}^{t} r(\tau) d\tau \right\} ds \le \frac{1}{e}, \tag{2.63}$$

where

$$\varkappa(t) = a \left[\exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(\xi) d\xi \right\} - 1 \right].$$

Then for any solution of (2.54)–(2.55), we have

$$0 < N(t) \le \frac{a}{\sum_{k=1}^{m} b_k} \exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(s) ds \right\}. \tag{2.64}$$

Proof. Suppose (2.64) is not valid. Then either there exists a $\bar{t} > 0$ such that

$$0 < N(t) \le \frac{a}{\sum_{k=1}^{m} b_k} \exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(s) ds \right\},\ 0 \le t < \bar{t},$$

$$N(\bar{t}) = \frac{a}{\sum_{k=1}^{m} b_k} \exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(s) ds \right\},\ N'(\bar{t}) > 0, \tag{2.65}$$

or there exists a $\bar{t} > 0$ such that

$$0 < N(t) \le \frac{a}{\sum_{k=1}^{m} b_k} \exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(s) ds \right\},\ 0 \le t < \bar{t},\ N(\bar{t}) = 0. \tag{2.66}$$

Suppose we have the first possibility for a solution $N(t)$ of (2.54)–(2.55). Denote by

$$t_1 < t_2 < \cdots < t_k < \ldots,$$

a sequence of all points t_k, such that

$$N(h(t_k)) = \frac{a}{\sum_{i=1}^{m} b_i},\ \ N'(h(t_k)) > 0.$$

Now

$$N(0) = N_0 < \frac{a}{\sum_{k=1}^{m} b_k},\ N(\bar{t}) > \frac{a}{\sum_{k=1}^{m} b_k},$$

and (a_5) imply that the set $\{t_k\}$ is not empty. Suppose t^* is a point where we have a local maximum for $N(t)$. We prove that if

$$N(t^*) > \frac{a}{\sum_{i=1}^{m} b_i}, \text{ then } t^* \in \bigcup_k [h(t_k), t_k].$$

Let t_k be the greatest among all points of the sequence $\{t_k\}$ satisfying $h(t_k) < t^*$.

Suppose first

$$N(t) \le \frac{a}{\sum_{i=1}^{m} b_i},$$

for some t and $h(t_k) < t \le t_k$. The definition of t_k and t^* imply $t^* < t$ and hence $t^* \in [h(t_k), t_k]$.

Now suppose

$$N(t) > \frac{a}{\sum_{i=1}^{m} b_i}, \text{ for } h(t_k) < t \le t_k.$$

Suppose there exists a smallest point t' such that

$$N(t') = \frac{a}{\sum_{i=1}^{m} b_i}.$$

Then (2.54) implies $N^{'}(t) < 0,\ t_k \le t < t'$. Hence in this interval $N(t)$ has no maximal points. Thus $h(t_k) < t^* < t_k$.

If such a t' does not exist then $N^{'}(t) \le 0$ for $t > t_k$ and so once again $h(t_k) < t^* < t_k$.

Equation (2.54) implies now that

$$N^{'}(t) \le ar(t)N(t),\ \ h(t_k) \le t \le t^*,\ \ N(h(t_k)) = \frac{a}{\sum_{i=1}^{m} b_i}.$$

Then

$$\begin{aligned} N(t^*) &\le \frac{a}{\sum_{i=1}^{m} b_i} \exp\left\{a \int_{h(t_k)}^{t^*} r(s)ds\right\} \\ &\le \frac{a}{\sum_{i=1}^{m} b_i} \exp\left\{a \int_{h(t_k)}^{t_k} r(s)ds\right\} \\ &\le \frac{a}{\sum_{l=1}^{m} b_i} \exp\left\{a \sup_{t>0} \int_{h(t)}^{t} r(s)ds\right\}, \end{aligned}$$

which contradicts our assumption (2.65).

Suppose now there exists a $\bar{t} > 0$ such that (2.66) holds. After substituting

$$N(t) = \exp\left\{\int_0^t r(s)\left[a - \sum_{k=1}^m b_k N(h_k(s))\right] ds\right\} x(t), \tag{2.67}$$

in (2.54)–(2.55), we have the system

$$x'(t) = -\sum_{l=1}^n c_l(t) \exp\left\{-\int_{g_l(t)}^t r(s)\left[a - \sum_{k=1}^m b_k N(h_k(s))\right] ds\right\} x(g_l(t)), \tag{2.68}$$

for $t > 0$, and (we assume $r(t) = 0,\ t < 0$)

$$x(t) = \varphi(t),\ \text{for}\ \ t < 0,\ x(0) = N_0. \tag{2.69}$$

Consider now the initial value problem

$$y'(t) = -\sum_{l=1}^n p_l(t) y(g_l(t)), t > 0, \tag{2.70}$$

$$y(t) = \psi(t),\ t < 0,\ y(0) = y_0, \tag{2.71}$$

where

$$p_l(t) = c_l(t) \exp\left\{-\int_{g_l(t)}^t r(s)\left[a - \sum_{k=1}^m b_k N(h_k(s))\right] ds\right\}.$$

It is evident that if $\psi(t) = \varphi(t),\ y_0 = N_0$, then the solutions of (2.68)–(2.69) and (2.70)–(2.71) coincide. Inequalities (2.64) and (2.63) imply that

$$\begin{aligned}
&\sum_{l=1}^n \int_{g(t)}^t p_l(s) ds \\
&= \sum_{l=1}^n \int_{g(t)}^t c_l(s) \exp\left\{\int_{g_l(s)}^s r(\tau)\left[\sum_{k=1}^m b_k N(h_k(\tau)) - a\right] d\tau\right\} ds \\
&\le \sup_{t>0} \sum_{l=1}^n \int_{g(t)}^t c_l(s) \exp\left\{\varkappa(t) \int_{g_l(s)}^s r(\tau) d\tau\right\} ds \le \frac{1}{e},
\end{aligned}$$

where

$$\varkappa(t) = a\left[\exp\left\{a \sup_{t>0} \int_{h(t)}^t r(\xi) d\xi\right\} - 1\right].$$

Note (2.62) which say $\varphi(t) \leq N_0$. Thus Lemma 2.5.1 yields that if $\psi(t) = \varphi(t)$, $y_0 = N_0$, then $y(t) > 0, t > 0$. Hence $x(t) > 0,\ t > 0$. Consequently by (2.67) we have $N(t) > 0,\ t > 0$, which contradicts assumption (2.66). The proof is complete. ∎

Theorem 2.5.2. . *Suppose* $(a1) - (a5)$ *hold, then for every eventually positive solution of (2.54)–(2.55) there exists* $t_0 \geq 0$ *such that (2.64) holds for* $t \geq t_0$.

Proof. Suppose $N(t)$ is an eventually positive solution of (2.54)–(2.55). If

$$N(t) \leq \frac{a}{\sum_{k=1}^{n} b_k},$$

for some $t_0 \geq 0$ and $t \geq t_0$, then the statement of the theorem is true.

Suppose now that

$$N(t) > \frac{a}{\sum_{k=1}^{n} b_k},$$

for some $t_1 \geq 0$ and $t \geq t_1$. Now (2.54) implies that

$$N^{'}(t) \leq -\sum_{l=1}^{n} c_l(t)N(g_l(t)),\quad t \geq t_2,$$

for some $t_2 \geq t_1$. Lemma 2.5.1 implies that $0 < N(t) \leq x(t),\ t \geq t_2$, where $x(t)$ is a solution of the equation

$$x^{'}(t) + \sum_{l=1}^{n} c_l(t)x(g_l(t)) = 0, t \geq t_1,\quad x(t) = N(t),\ t \leq t_2,$$

and $\lim_{t\to\infty} x(t) = 0$. Then $\lim_{t\to\infty} N(t) = 0$. We have a contradiction with our assumption.

Hence there exists a sequence $\{t_n\},\ \lim_{n\to\infty} t_n = \infty$, such that

$$N(h(t_n)) = \frac{a}{\sum_{k=1}^{n} b_k}.$$

The end of the proof is similar to the corresponding part of the proof of Theorem 2.5.1. ∎

Consider now

$$N^{'}(t) = r(t)N(t)\left[a - b_0 N(t) - \sum_{k=1}^{m} b_k N(h_k(t))\right] - \sum_{l=1}^{n} c_l(t)N(g_l(t)). \quad (2.72)$$

Theorem 2.5.3. *Suppose $b_0 > 0$, hypotheses $(a_1) - (a_4)$ hold,*

$$\varphi(t) \leq N_0 < \frac{a}{b_0}, \tag{2.73}$$

and

$$\sup_{t>0} \sum_{l=1}^{n} \int_{g(t)}^{t} c_l(s) \exp\left\{ \left[\frac{a \sum_{k=1}^{m} b_k}{b_0} \right] \int_{g_l(s)}^{s} r(u) du \right\} ds \leq \frac{1}{e}. \tag{2.74}$$

Then for any solution of (2.72)–(2.73) we have

$$0 < N(t) \leq \frac{a}{b_0}. \tag{2.75}$$

Proof. We follow the scheme of the proof in Theorem 2.5.1. Suppose (2.75) is not true. Then either there exists $\bar{t} > 0$ such that

$$0 < N(t) \leq \frac{a}{b_0},\ 0 \leq t < \bar{t},\ N(\bar{t}) = \frac{a}{b_0}, N'(\bar{t}) > 0, \tag{2.76}$$

or there exists $\bar{t} > 0$ such that

$$0 < N(t) \leq \frac{a}{b_0},\ \ 0 \leq t < \bar{t},\ \ N(\bar{t}) = 0. \tag{2.77}$$

Suppose the first possibility (2.76) holds. Then for $0 < t < \bar{t}$ we have

$$N'(t) \leq r(t)N(t)[a - b_0 N(t)],\ N(0) = N_0.$$

Denote by x a solution of the problem

$$x'(t) = r(t)x(t)[a - b_0 x(t)],\ \ x(0) = N_0. \tag{2.78}$$

Then

$$N(t) \leq x(t) < \frac{a}{b_0},\ 0 \leq t \leq \bar{t},$$

since the solution of (2.78) tends to a/b_0 and is always less than a/b_0. We have a contradiction with assumption (2.76).

Suppose now that for $\bar{t} > t_0$ (2.77) holds. Substituting in (2.72),

$$N(t) = \exp\left\{ \int_0^t r(s) \left[a - b_0 N(s) - \sum_{k=1}^{m} b_k N(h_k(s)) \right] ds \right\} x(t), \tag{2.79}$$

we have the system

$$x'(t) = -\sum_{l=1}^{n} p_l(t)x(g_l(t)),\ t > 0, \tag{2.80}$$

$$x(t) = \varphi(t),\ t < 0,\ x(0) = N_0,$$

where

$$p_l(t) = c_l(t)\exp\left\{-\int_{g_l(t)}^{t} r(s)\left[a - b_0N(s) - \sum_{k=1}^{m} b_kN(h_k(s))\right]ds\right\}.$$

Inequalities (2.75) and (2.74) imply that

$$\begin{aligned}
&\sum_{l=1}^{n}\int_{g(t)}^{t} p_l(s)ds \\
&\leq \sum_{l=1}^{n}\int_{g(t)}^{t} c_l(s) \\
&\quad\times\exp\left\{\int_{g_l(s)}^{s} r(\tau)\left[\sum_{k=1}^{m} b_kN(h_k(\tau)) + b_0N(\tau) - a\right]d\tau\right\} ds \\
&\leq \sup_{t>0}\sum_{l=1}^{n}\int_{g(t)}^{t} c_l(s)\exp\left\{\left[\frac{a\sum_{k=1}^{m} b_k}{b_0}\right]\int_{g_l(s)}^{s} r(\tau)d\tau\right\} ds \leq \frac{1}{e}.
\end{aligned}$$

As in the proof of Theorem 2.5.1, Lemma 2.5.1 implies $N(t) > 0,\ 0 \leq t \leq \bar{t}$. This contradiction proves the theorem. ■

Similar reasoning to that in Theorem 2.5.2 yields the next result.

Theorem 2.5.4. *Suppose $b_0 > 0$, $(a_1) - (a_4)$ hold. Then for every eventually positive solution of (2.72)–(2.55) there exists a $t_0 \geq 0$ such that (2.75) holds for $t \geq t_0$.*

Now we obtain sufficient extinction conditions for solutions of the logistic equation with harvesting. To this end consider the following equation which is more general than (2.54):

$$N'(t) = N(t)\left[a(t) - \sum_{k=1}^{m} b_k(t)N(h_k(t))\right] - \sum_{l=1}^{n} c_l(t)N(g_l(t)),\ t \geq 0. \tag{2.81}$$

Theorem 2.5.5. *Suppose $a(t) \geq 0$, $b_k \geq 0$ are locally essentially bounded functions and for c_l, h_k, g_l conditions (a_2), (a_3) hold. Suppose in addition (2.60)–(2.61) hold. Then for any solution of (2.81)–(2.55) either*

$$\lim_{t \to \infty} N(t) = 0$$

or there exists $\bar{t} > 0$ such that $N(\bar{t}) < 0$.

Proof. It is sufficient to prove that for every positive solution $N(t)$ of (2.81)–(2.55) we have $\lim_{t\to\infty} N(t) = 0$.

Suppose $N(t) > 0$ is a solution of (2.81)–(2.55). Equation (2.81) implies

$$N^{'}(t) + \sum_{l=1}^{n} c_l(t) N(g_l(t)) - a(t) N(t) \leq 0.$$

Lemma 2.5.2 guarantees that there exists $t_0 \geq 0$, such that the fundamental function $X(t,s)$ of the equation

$$x^{'}(t) + \sum_{l=1}^{n} c_l(t) x(g_l(t)) - a(t) x(t) = 0 \tag{2.82}$$

is positive for $t \geq s \geq t_0$. Then the variation of constant formula [30] implies

$$N(t) = x(t) + \int_{t_0}^{t} X(t,s) f(s) ds,$$

where $x(t)$ is a solution of (2.82) with the initial condition $x(t) = N(t)$, $t \leq t_0$, and $f(t)$ is a nonpositive function. Hence $0 < N(t) \leq x(t)$. Lemma 2.5.2 implies that

$$\lim_{t \to \infty} x(t) = 0.$$

Thus $\lim_{t\to\infty} N(t) = 0$. The proof is complete. ∎

2.6 Models with Nonlinear Delays

We return now to Sect. 2.4 when $\alpha_k = 1$, $k = 1, \ldots, m$. Consider the delay logistic model with several delays

$$N^{'}(t) = N(t) \sum_{k=1}^{m} r_k(t) \left[1 - \frac{N(h_k(t))}{K} \right], \quad h_k(t) \leq t. \tag{2.83}$$

Motivated by (2.83) in this section we consider first the scalar delay differential equation

$$x'(t) = -\sum_{k=1}^{m} r_k(t)x(h_k(t))\,[x(t)+1] \tag{2.84}$$

under the following conditions

(c_1) r_k, $k = 1, 2, \ldots, m$, are Lebesgue measurable functions essentially bounded in each finite interval $[0, b]$, $r_k \geq 0$,

$$\int_{t_0}^{\infty} \sum_{k=1}^{m} r_k(t)dt = \infty, \quad \lim_{t\to\infty} \inf \sum_{k=1}^{m} \int_{\max_k h_k(t)}^{t} r_k(s)ds > 0;$$

(c_2) $h_k : [0, \infty) \to \mathbf{R}$ are Lebesgue measurable functions, $h_k(t) \leq t$, $\lim_{t\to\infty} h_k(t) = \infty$, $k = 1, 2, \ldots, m$.

Together with (2.84), we consider for each $t_0 \geq 0$ an initial value problem

$$x'(t) = -\sum_{k=1}^{m} r_k(t)x(h_k(t))\,[x(t)+1], \quad t \geq t_0, \tag{2.85}$$

$$x(t) = \varphi(t), \quad t < t_0, \quad \text{and } x(t_0) = x_0 > -1, \tag{2.86}$$

where

(c_3) $\varphi : (-\infty, t_0) \to \mathbf{R}$ is a Borel measurable bounded function.

Consider the linear delay differential equation

$$x'(t) + \sum_{k=1}^{m} r_k(t)x(h_k(t)) = 0 \tag{2.87}$$

and the delay differential inequalities

$$x'(t) + \sum_{k=1}^{m} r_k(t)x(h_k(t)) \leq 0, \quad t \geq 0, \tag{2.88}$$

$$x'(t) + \sum_{k=1}^{m} r_k(t)x(h_k(t)) \geq 0, \quad t \geq 0. \tag{2.89}$$

The following Lemma follows a standard argument (see the proof of Theorem 2.4.1).

Lemma 2.6.1. *Assume that* $(c_1) - (c_3)$ *hold. Then the following statements are equivalent:*

(1) There exits a nonoscillatory solution of (2.87).
(2) There exists an eventually positive solution of t inequality (2.88).
(3) There exists an eventually negative solution of (2.89).
(4) There exists $t_0 \geq 0$ such that the inequality

$$u(t) \geq \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u(s)ds\right), \quad t \geq t_0, \ u(t) = 0, \ t < t_0, \tag{2.90}$$

has a nonnegative locally integrable solution.

If $x(t)$, $y(t)$, $z(t)$, $t \geq 0$, are positive solutions of (2.87), (2.88), (2.89), respectively, $x(t) = y(t) = z(t)$, $t < 0$, then $y(t) \leq x(t) \leq z(t)$ for $t \geq 0$.

Lemma 2.6.2. *Assume that for the equation*

$$x'(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t)) = 0, \quad t \geq 0, \tag{2.91}$$

assumptions $(c_1) - (c_2)$ hold.

(*i*) *If $a_k(t) \leq r_k(t), g_k(t) \geq h_k(t)$, and (2.87) has a nonoscillatory solution, then (2.91) has a nonoscillatory solution.*
(*ii*) *If $a_k(t) \geq r_k(t), g_k(t) \leq h_k(t)$, and all solutions of (2.87) are oscillatory, then all solutions of (2.91) are oscillatory.*

Theorem 2.6.1. *Assume that $(c_1) - (c_3)$ hold. Suppose that for every sufficiently small $\varepsilon \geq 0$ all solutions of the linear delay differential equation*

$$x'(t) + (1-\varepsilon)\sum_{k=1}^{m} r_k(t)x(h_k(t)) = 0, \quad t \geq t_0, \tag{2.92}$$

are oscillatory. Then all solutions of (2.85) are oscillatory.

Proof. Suppose (2.85) has a nonoscillatory solution. Then by the condition $x(t) + 1 > 0$ either there exists a positive solution $x(t) > 0$ for all $t \geq T \geq t_0$ or there exists a solution $x(t)$ such that

$$-1 < x(t) < 0, \quad \text{for } t \geq T.$$

We can assume $h_k(t) \geq t_0$ for all $t \geq T$, since $\lim_{t\to\infty} h_k(t) = \infty$.

First, we suppose that $x(t) > 0$ for $t \geq T$. From (2.85), we have

$$x'(t) + \sum_{k=1}^{m} r_k(t)x(h_k(t)) \leq 0, \quad t \geq t_0.$$

Lemma 2.6.1 implies for $\varepsilon = 0$ that (2.92) has a nonoscillatory solution, which gives a contradiction.

Suppose now

$$-1 < x(t) < 0, \text{ for } \ t \geq T.$$

Let us introduce the function u as a solution of

$$x^{'}(t) = -u(t)x(t)\left[x(t)+1\right], \ \ x(T) = x_0 < 0. \tag{2.93}$$

Now, since $x(t)+1 > 0$, we have $x^{'}(t) > 0$ and this implies that $u(t) \geq 0$. From (2.93) we obtain

$$x(t) = -\frac{\exp\left(-\int_T^t u(s)ds + c\right)}{1+\exp\left(-\int_T^t u(s)ds + c\right)},$$

where $c = \ln\left[|x_0|/(1+x_0)\right]$. Substituting in (2.85) we have

$$u(t)\frac{\exp\left(-\int_T^t u(s)ds + c\right)}{1+\exp\left(-\int_T^t u(s)ds + c\right)} = \sum_{k=1}^{m} r_k(t)\frac{\exp\left(-\int_T^{h_k(t)} u(s)ds + c\right)}{1+\exp\left(-\int_T^{h_k(t)} u(s)ds + c\right)}.$$

Hence

$$u(t) = \sum_{k=1}^{m} r_k(t)\exp\left(\int_{h_k(t)}^{t} u(s)ds\right)\frac{1+\exp\left(-\int_T^t u(s)ds + c\right)}{1+\exp\left(-\int_T^{h_k(t)} u(s)ds + c\right)}. \tag{2.94}$$

Equality (2.94) implies that $u(t) \geq \sum_{k=1}^{m} r_k(t)$ and from (c_1) we have

$$\int_T^{\infty} u(t)dt = \infty.$$

Consequently there exists $T_1 \geq T$ such that

$$\max_{1\leq k\leq m}\frac{1+\exp\left(-\int_T^t u(s)ds + c\right)}{1+\exp\left(-\int_T^{h_k(t)} u(s)ds + c\right)} \geq (1-\varepsilon), \text{ for } t \geq T_1.$$

Then,

$$u(t) \geq (1-\varepsilon)\sum_{k=1}^{m} r_k(t)\exp\left(\int_{h_k(t)}^{t} u(s)ds\right).$$

From Lemma 2.6.1, (2.92) has a nonoscillatory solution, which is a contradiction. The proof is complete. ∎

From Lemma 2.6.2 and Theorem 2.6.1 we have the following oscillation comparison theorem.

Theorem 2.6.2. *Suppose $a_k(t) \geq r_k(t)$, $g_k(t) \leq h_k(t)$, and the assumptions of Theorem 2.6.1 hold. Then all the solutions of the equation*

$$x^{'}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t))[1 + x(t)] = 0, \quad t \geq 0, \tag{2.95}$$

are oscillatory.

Theorem 2.6.3. *Assume that $(c_1) - (c_3)$ hold. Suppose for every sufficiently small $\varepsilon \geq 0$ there exists a nonoscillatory solution of the linear delay differential equation*

$$x^{'}(t) + (1 + \varepsilon) \sum_{k=1}^{m} r_k(t)x(h_k(t)) = 0, \quad t \geq t_0. \tag{2.96}$$

Then (2.85) has a nonoscillatory solution.

Proof. From Lemma 2.6.1 for some $T \geq t_0$ and for $t \geq T$ there exists a nonnegative solution u_0 of

$$u(t) \geq (1 + \varepsilon) \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u(s)ds \right), \quad t \geq T. \tag{2.97}$$

This inequality implies that $u_0(t) \geq \sum_{k=1}^{m} r_k(t)$, and hence by (c_1) we have that

$$\int_T^{\infty} u_0(s)ds = \infty.$$

Let c be some negative number. Then there exists $T_1 \geq T$ such that

$$\max_{1 \leq k \leq m} \frac{1 - \exp\left(-\int_{T_1}^{t} u_0(s)ds + c \right)}{1 - \exp\left(-\int_{T_1}^{h_k(t)} \sum_{k=1}^{m} r_k(s)ds + c \right)} < (1 + \varepsilon), \text{ for } t \geq T_1, \tag{2.98}$$

and by (c_1) for $t \geq T_1$, we have

$$\min_{1 \leq k \leq m} \exp\left[\int_{h_k(t)}^{t} \sum_{k=1}^{m} r_k(s)ds \right] \frac{1 - \exp\left(-\int_{T_1}^{h_k(t)} \sum_{k=1}^{m} r_k(s)ds + c \right)}{1 - \exp\left(-\int_{T_1}^{t} u_0(s)ds + c \right)} > 1.$$

From (2.97) and (2.98), we have

$$u_0(t) \geq \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u_0(s)ds\right) \frac{1 - \exp\left(-\int_{T_1}^{t} u_0(s)ds + c\right)}{1 - \exp\left(-\int_{T_1}^{h_k(t)} \sum_{k=1}^{m} r_k(s)ds + c\right)}. \tag{2.99}$$

Let us fix $t_1 > T_1$ and consider the nonlinear operator

$$(F_1 u)(t) = \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u(s)ds\right) \times \frac{1 - \exp\left(-\int_{T_1}^{t} u(s)ds + c\right)}{1 - \exp\left(-\int_{T_1}^{h_k(t)} \sum_{k=1}^{m} r_k(s)ds + c\right)}$$

in the Banach space $L_\infty[T_1, t_1]$. We have

$$(F_1 u)(t) = \sum_{k=1}^{m} r_k(t) \frac{\exp\left(\int_{T_1}^{t} u(s)ds\right)}{\exp\left(\int_{T_1}^{t} \zeta_k(t,s)u(s)ds\right)} \times \frac{1 - \exp\left(-\int_{T_1}^{t} u(s)ds + c\right)}{1 - \exp\left(-\int_{T_1}^{h_k(t)} \sum_{k=1}^{m} r_k(s)ds + c\right)}, \tag{2.100}$$

where $\zeta_k(t,s) = 1$, if $s < h_k(t) < t$, and $\zeta_k(t,s) = 0$, if $h_k(t) < s$. The operator F_1 is continuous. Consider all functions $v \in L_\infty[T_1, t_1]$ such that

$$\sum_{k=1}^{m} r_k(t) \leq v(t) \leq u_0(t).$$

We have $(F_1 v)(t) \geq \sum_{k=1}^{m} r_k(t)$. Inequality (2.98) implies that

$$(F_1 v)(t) \leq \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u_0(s)ds\right) \times \frac{1 - \exp\left(-\int_{T_1}^{t} u_0(s)ds + c\right)}{1 - \exp\left(-\int_{T_1}^{h_k(t)} \sum_{k=1}^{m} r_k(s)ds + c\right)}$$

$$\leq (1+\varepsilon) \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u_0(s)ds\right) \leq u_0(t).$$

Hence for each v such that

$$\sum_{k=1}^{m} r_k(t) \le v(t) \le u_0(t)$$

we have

$$\sum_{k=1}^{m} r_k(t) \le (F_1 v)(t) \le u_0(t).$$

Then by Knaster's Fixed Point Theorem (see Sect. 1.4), there exists u_1 such that

$$\sum_{k=1}^{m} r_k(t) \le u_1(t) \le u_0(t) \quad \text{and} \quad u_1 = F\, u_1.$$

This means that

$$u_1(t) = \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u_1(s)ds\right) \frac{1-\exp\left(-\int_{T_1}^{t} u_1(s)ds + c\right)}{1-\exp\left(-\int_{T_1}^{h_k(t)} \sum_{k=1}^{m} r_k(s)ds + c\right)}. \tag{2.101}$$

Consider the operator

$$(F_2 u)(t) = \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u(s)ds\right) \frac{1-\exp\left(-\int_{T_1}^{t} u(s)ds + c\right)}{1-\exp\left(-\int_{T_1}^{h_k(t)} u_1(s)ds + c\right)}.$$

If

$$\sum_{k=1}^{m} r_k(t) \le v(t) \le u_1(t),$$

then (2.101) and (2.98) imply

$$\begin{aligned}(F_2 v)(t) &\le \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u_1(s)ds\right) \frac{1-\exp\left(-\int_{T_1}^{t} u_1(s)ds + c\right)}{1-\exp\left(-\int_{T_1}^{h_k(t)} u_1(s)ds + c\right)} \\ &\le u_1(t),\end{aligned}$$

and

$$
\begin{aligned}
&(F_2 v)(t) \\
&\geq \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} \sum_{k=1}^{m} r_k(s) ds \right) \frac{1 - \exp\left(- \int_{T_1}^{t} \sum_{k=1}^{m} r_k(s) ds + c \right)}{1 - \exp\left(- \int_{T_1}^{h_k(t)} u_0(s) ds + c \right)} \\
&\geq \sum_{k=1}^{m} r_k(t).
\end{aligned}
$$

Hence

$$
\sum_{k=1}^{m} r_k(t) \leq (F_2 v)(t) \leq u_1(t)
$$

and as in the previous case we obtain that there exists a solution u_2 of the equation $u = F_2 u$ such that

$$
\sum_{k=1}^{m} r_k(t) \leq u_2(t) \leq u_1(t).
$$

By induction we prove that there exists a solution u_n of the equation $u = F_n\, u$ which satisfies

$$
\sum_{k=1}^{m} r_k(t) \leq u_n(t) \leq u_{n-1}(t),
$$

where

$$
(F_n u)(t) = \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u(s) ds \right) \frac{1 - \exp\left(- \int_{T_1}^{t} u(s) ds + c \right)}{1 - \exp\left(- \int_{T_1}^{h_k(t)} u_{n-1}(s) ds + c \right)}.
$$

A monotone bounded sequence $\{u_n\}$ has a limit $u = \lim_{n \to \infty} u_n(t)$ and this limit is a solution of the equation

$$
u(t) = \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u(s) ds \right) \frac{1 - \exp\left(- \int_{T_1}^{t} u(s) ds + c \right)}{1 - \exp\left(- \int_{T_1}^{h_k(t)} u(s) ds + c \right)}.
$$

From this, we have that

$$x(t) = -\frac{\exp\left(-\int_T^t u(s)ds + c\right)}{1 + \exp\left(-\int_T^t u(s)ds + c\right)}$$

(where $c = \ln[|x(T_1)|/(1 + x(T_1)])$ is a positive solution of (2.85) for $T_1 \le t \le t_1$. Since t_1 is an arbitrary number, we have a positive solution for all $t \ge T_1$. The proof is complete. ■

For the remainder of this section we consider

$$x'(t) + \sum_{k=1}^{m} r_k(t) f_k[x(h_k(t))] = 0 \tag{2.102}$$

under the following assumptions:

($a1$) $r_k(t) \ge 0$, $k = 1, \ldots, m$, are Lebesgue measurable locally essentially bounded functions;
($a2$) $h_k : [0, \infty) \to \mathbf{R}$, for $k = 1, \ldots, m$, are Lebesgue measurable functions $h_k(t) \le t$, $\lim t_{t\to\infty} h_k(t) = \infty$;
($a3$) $f_k : \mathbf{R} \to \mathbf{R}$, $k = 1, \ldots, m$, are continuous functions, $x f_k(x) > 0$ for $x \ne 0$.

Together with (2.102), we consider for each $t_0 \ge 0$ an initial value problem

$$x'(t) + \sum_{k=1}^{m} r_k(t) f_k[x(h_k(t))] = 0, \ t \ge t_0, \tag{2.103}$$

$$x(t) = \phi(t), \ \ t < t_0, \ \ x(t_0) = x_0. \tag{2.104}$$

We also assume that the following hypothesis holds:

($a4$) $\phi : (-\infty, t_0) \to \mathbf{R}$ is a Borel measurable bounded function.

We will also use the following lemma (whose proof is standard) which can be found in [33].

Lemma 2.6.3. *Suppose there exists an index k such that*

$$\int_0^\infty r_k(t)dt = \infty \tag{2.105}$$

and $x(t)$ is a nonoscillatory solution of (2.103). Then $\lim_{t\to\infty} x(t) = 0$.

Theorem 2.6.4. *Assume that $(a_1) - (a_4)$ and (2.105) hold. Furthermore assume that*

$$\lim_{u\to\infty} \frac{f_k(u)}{u} = 1, \ \ k = 1, 2, \ldots, m. \tag{2.106}$$

If for some $\varepsilon > 0$ all solutions of the linear equation

$$x'(t) + (1-\varepsilon)\sum_{k=1}^{m} r_k(t)\, x(h_k(t)) = 0, \;\; t \ge t_0, \tag{2.107}$$

are oscillatory, then all solutions of (2.103) are also oscillatory.

Proof. Assume (2.103) has a nonoscillatory solution $x(t)$. Then, by Lemma 2.6.3 we have that $\lim_{t\to\infty} x(t) = 0$.

Assume that there exists $t_1 \ge t_0$ sufficiently large such that $x(t) > 0$ for $t \ge t_1$ and $h_k(t) \ge t_1$ for $t \ge t_2$. From condition (2.106) there exists $t_3 \ge t_2$ such that

$$f_k(x(h_k(t))) \ge (1-\varepsilon)x(h_k(t)), \;\; t \ge t_3.$$

Hence

$$x^{'}(t) + (1-\varepsilon)\sum_{k=1}^{m} r_k(t)x(h_k(t)) \le 0, \quad t \ge t_3.$$

Now Lemma 2.6.1 implies that (2.107) has a nonoscillatory solution. This is a contradiction.

Suppose now, $x(t) < 0$ for $t \ge t_1$ for some t_1 sufficiently large such that $h_k(t) \ge t_1$ for $t \ge t_2$. Let

$$y(t) := -x(t), \quad g_k(y) = -f_k(-y)$$

and the functions g_k satisfy all the assumptions for f_k, and $y(t)$ is an eventually positive solution of the equation

$$y^{'}(t) + \sum_{k=1}^{m} r_k(t)g_k(y(h_k(t))) = 0.$$

As was shown above, we have

$$y^{'}(t) + (1-\varepsilon)\sum_{k=1}^{m} r_k(t)y(h_k(t)) \le 0,$$

for $t_2 \ge t_1$. Now Lemma 2.6.1 implies that (2.107) has a nonoscillatory solution. This contradiction proves the theorem. ∎

Theorem 2.6.5. *Assume that $(a_1)-(a_4)$ hold. Suppose for all $k = 1,\dots,m$, either*

$$f_k(x) \le x \;\; \text{for} \;\; x > 0 \;\; \text{or} \;\; f_k(x) \ge x \;\; \text{for} \; x < 0, \tag{2.108}$$

and there exists a nonoscillatory solution of the linear delay differential equation (2.87).

Then there exists a nonoscillatory solution of (2.103).

Proof. Suppose $f_k(x) \le x$ for $x > 0$, $k = 1, \dots, m$. By Lemma 2.6.1 there exist $t_0 > 0$ and $u_0(t) \ge 0$, $t \ge t_0$, $u_0(t) = 0$, $t < t_0$, such that

$$u_0(t) \ge \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u_0(s)ds\right), \quad t \ge t_0.$$

Let us fix $b > t_0$ and consider the nonlinear operator $F : L_\infty[t_0, b] \to L_\infty[t_0, b]$ given by

$$(Fu)(t) = \sum_{k=1}^{m} r_k(t) f_k\left(\exp\left(-\int_{t_0}^{h_k(t)} u(s)ds\right)\right) \exp\left(\int_{t_0}^{t} u(s)ds\right).$$

For any function u from the interval $0 \le u \le u_0$ we have

$$0 \le (Fu)(t) \le \sum_{k=1}^{m} r_k(t) \exp\left(-\int_{t_0}^{h_k(t)} u(s)ds\right) \exp\left(\int_{t_0}^{t} u(s)ds\right)$$
$$\le \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u_0(s)ds\right) \le u_0(t).$$

Hence $0 \le Fu \le u_0$. Lemma 2.4.1 implies that the operator F is completely continuous in $L_\infty[t_0, b]$. Then by the Schauder Fixed Point Theorem, there exists a nonnegative solution of the equation $u = Fu$. Let

$$x(t) = \begin{cases} \exp\left(-\int_{t_0}^{t} u(s)ds\right), & t \ge t_0, \\ 0, & t < t_0. \end{cases}$$

Then $x(t)$ is an eventually positive solution of (2.87).

If $f_k(x) \ge x$, $x \le 0$, $k = 1, \dots, m$, then (2.87) has an eventually negative solution, which completes the proof of the theorem. ■

Consider (2.83). Let $N(t) = Ke^{x(t)}$. Then x is a solution of (2.102) with

$$f_k(x) = f(x) = e^x - 1.$$

Note $f_k(u) \ge u$ for $u \le 0$ and $u f_k(u) > 0$ for $u \ne 0$.

2.7 Hyperlogistic Models

In this section, we are concerned with the oscillation of the delay hyperlogistic models. First, we consider an autonomous delay hyperlogistic model of the form

$$N^{'}(t) = rN(t)\prod_{j=1}^{m}\left[1-\frac{N(t-\tau_j)}{K}\right]^{\alpha_j}, \quad t \geq 0, \tag{2.109}$$

where $r, K, \tau_j \in (0,\infty)$, and $\alpha_j = p_j/q_j$ are rational numbers with q_j odd, p_j and q_j are co-prime, $1 \leq j \leq m$, and

$$\prod_{j=1}^{m}(-1)^{\alpha_j} = -1.$$

By making a change of variables

$$x(t) = \frac{N(t)}{K} - 1,$$

Eq. (2.109) becomes

$$x^{'}(t) + r\,[1+x(t)]\prod_{j=1}^{m}x^{\alpha_j}\left(t-\tau_j\right) = 0. \tag{2.110}$$

We are interested in those solutions $x(t)$ of (2.110) satisfying $x(t) \geq -1$ which correspond to solutions $N(t)$ of (2.109) satisfying $N(t) \geq 0$. Thus we consider the initial condition

$$\begin{cases} x(t) = \phi(t) \geq -1, & t \in [t_0-\tau, t_0]\,, \\ \phi \in C\left([t_0-\tau, t_0]\,, [-1,\infty)\right) \text{ and } & \phi(t_0) > -1, \end{cases} \tag{2.111}$$

where $\tau = \max\{\tau_1,\ldots,\tau_m\}$. Now (2.110), (2.111) has a unique solution $x(t;t_0,\phi)$ on $[t_0-\tau,\infty)$ and $x(t) > -1$ for $t \geq t_0$. We will show that all solutions of (2.110) and (2.111) are oscillatory when $\sum_{j=1}^{m}\alpha_j < 1$, but at least one nonoscillatory solution exists when $\sum_{j=1}^{m}\alpha_j > 1$. For the case where $\sum_{j=1}^{m}\alpha_j = 1$, we will establish an equivalence, as far as oscillation is concerned, between (2.110) and its so-called quasilinearized equation

$$y^{'}(t) + r\prod_{j=1}^{m}y^{\alpha_j}(t-\tau_j) = 0. \tag{2.112}$$

The results in this section are adapted from [84].

The case $\sum_{j=1}^{m} \alpha_j < 1.$

Theorem 2.7.1. *If* $\alpha = \sum_{j=1}^{m} \alpha_j < 1$, *then every solution of (2.110)–(2.111) oscillates.*

Proof. Assume that (2.110)–(2.111) has a nonoscillatory solution $x(t)$. We first suppose that $x(t)$ is eventually positive. Then, by (2.110), we eventually have

$$x^{'}(t) = -r(1 + x(t)) \prod_{j=1}^{m} x^{\alpha_j}(t - \tau_j) < 0,$$

which implies that $x(t)$ is eventually decreasing. Thus

$$x(t - \tau_j) \geq x(t), \quad \text{eventually}, \quad \text{for } j = 1, \ldots, m,$$

and hence (note $\alpha = \sum_{j=1}^{m} \alpha_j$)

$$x^{'}(t) + r(1 + x(t))x^{\alpha}(t) \leq x^{'}(t) + r(1 + x(t)) \prod_{j=1}^{m} x^{\alpha_j}(t - \tau_j) = 0.$$

Thus

$$\frac{d}{dt} x^{1-\alpha}(t) \leq -(1 - \alpha)r\,[1 + x(t)] \leq -(1 - \alpha)r,$$

which implies that

$$x^{1-\alpha}(t) \to -\infty, \quad \text{as } t \to \infty.$$

This is impossible since $x(t) > 0$ eventually and $1 - \alpha > 0$.

We next suppose that $x(t)$ is eventually negative. Noting that $x(t) > -1$ for $t \geq 0$, we have eventually

$$\begin{aligned} x^{'}(t) &= -r(1 + x(t)) \prod_{j=1}^{m} x^{\alpha_j}(t - \tau_j) \\ &= r(1 + x(t)) \prod_{j=1}^{m} \left[-x(t - \tau_j)\right]^{\alpha_j} > 0, \end{aligned}$$

which implies that $x(t)$ is eventually increasing, so there exists $T_1 > 0$ such that $x(t-\tau_j) \le x(t) < 0$ for $j = 1, \ldots, m$ and

$$1 + x(t) > 1 + x(T_1) > 0, \quad \text{for all } t > T_1.$$

Therefore

$$\begin{aligned} & x^{'}(t) + r(1+x(t))x^{\alpha}(t) \\ \ge\ & x^{'}(t) + r(1+x(t))\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) = 0, \quad t > T_1, \end{aligned}$$

and hence

$$\begin{aligned} \frac{d}{dt}x^{1-\alpha}(t) &\le -r(1-\alpha)(1+x(t)) \\ &< -r(1-\alpha)(1+x(T_1)) < 0, \;\; t > T_1. \end{aligned}$$

Integrating the above inequality from T_1 to $t > 0$ and letting $t \to \infty$, we get $x^{1-\alpha}(t) \to -\infty$, as $t \to \infty$. This is a contradiction to the fact that $x(t) > -1$ for $t \ge 0$ and completes the proof. ■

The case $\sum_{j=1}^{m} \alpha_j > 1$.

We now recall the following well-known result.

Lemma 2.7.1. *Every solution of (2.112) with* $\sum_{j=1}^{m} \alpha_j = 1$ *oscillates if and only if*

$$r\sum_{j=1}^{m} \alpha_j \tau_j > \frac{1}{e}.$$

Moreover, the above inequality holds if and only if

$$y^{'}(t) + r\prod_{j=1}^{m} y^{\alpha_j}(t-\tau_j) \le 0, \; \textit{has no eventually positive solution,}$$

$$y^{'}(t) + r\prod_{j=1}^{m} y^{\alpha_j}(t-\tau_j) \ge 0, \; \textit{has no eventually negative solution.}$$

Theorem 2.7.2. *If* $\alpha = \sum_{j=1}^{m} \alpha_j > 1$, *then (2.110) has a nonoscillatory solution.*

Proof. Choose rational numbers $\beta_j = \frac{r_j}{s_j} \in [0, \infty)$ with s_j odd, $1 \le j \le m$, such that

$$\beta_j \le \alpha_j, \text{ for } j = 1, \ldots, m, \ \sum_{j=1}^{m} \beta_j = 1, \ \prod_{j=1}^{m} (-1)^{\beta_j} = -1.$$

Let $\epsilon > 0$ satisfy

$$r\epsilon \sum_{j=1}^{m} \beta_j \tau_j \le \frac{1}{e}.$$

Then, by Lemma 2.7.1, the equation

$$x'(t) + r\epsilon \prod_{j=1}^{m} x^{\beta_j}(t - \tau_j) = 0 \tag{2.113}$$

has a positive solution $x(t)$ defined on $[t_0, \infty)$ for some $t_0 \ge 0$. It is clear that $x(t) \to 0$ as $t \to \infty$. Since $\beta_j \le \alpha_j$ and

$$\sum_{j=1}^{m} \beta_j < \sum_{j=1}^{m} \alpha_j,$$

we have

$$\lim_{t \to \infty} (1 + x(t)) \frac{\prod_{j=1}^{m} x^{\alpha_j}(t - \tau_j)}{\prod_{j=1}^{m} x^{\beta_j}(t - \tau_j)} = 0.$$

Thus, there exists $t_1 > t_0$ such that

$$(1 + x(t)) \prod_{j=1}^{m} x^{\alpha_j}(t - \tau_j) < \epsilon \prod_{j=1}^{m} x^{\beta_j}(t - \tau_j), \ \text{ for } t \ge t_1,$$

and hence for $t \ge t_1$, we see that

$$x'(t) + r(1 + x(t)) \prod_{j=1}^{m} x^{\alpha_j}(t - \tau_j) < x'(t) + \epsilon r \prod_{j=1}^{m} x^{\beta_j}(t - \tau_j) = 0. \tag{2.114}$$

Set $y(t) = \ln(1 + x(t))$. Then, from (2.114), we have

$$y'(t) + r \prod_{j=1}^{m} \left[e^{y(t-\tau_j)} - 1\right]^{\alpha_j} < 0, \text{ for } t \geq t_1,$$

which yields

$$y(t) > r \int_t^\infty \prod_{j=1}^{m} \left[e^{y(s-\tau_j)} - 1\right]^{\alpha_j} ds, \quad \text{for } t \geq t_1. \tag{2.115}$$

Define X to be the set of piecewise continuous functions $z : [t_1 - \tau, \infty) \to [0, 1]$ and endow X with the usual pointwise ordering $\leq$, that is,

$$z_1 \leq z_2 \Leftrightarrow z_1(t) \leq z_2(t), \text{ for } t \geq t_1 - \tau.$$

Then $(X; \leq)$ becomes an ordered set. It is obvious that for any nonempty subset M of X, $\inf(M)$ and $\sup(M)$ exist. Thus $(X; \leq)$ is a complete lattice. Define a mapping Ψ on X as follows:

$$(\Psi z)(t) = \begin{cases} \frac{r}{y(t)} \int_t^\infty \prod_{j=1}^{m} \left[e^{y(s-\tau_j)z(s-\tau_j)} - 1\right]^{\alpha_j} ds, & t \geq t_1, \\ \frac{t}{t_1} (\Psi z)(t) + \left(1 - \frac{t}{t_1}\right), & t_1 - \tau \leq t \leq t_1. \end{cases}$$

For each $z \in X$, we see that

$$0 \leq (\Psi z)(t) \leq \frac{r}{y(t)} \int_t^\infty \prod_{j=1}^{m} \left[e^{y(s-\tau_j)} - 1\right] ds < 1, \text{ for } t \geq t_1,$$

and

$$0 \leq (\Psi z)(t) \leq 1, \quad \text{for} \quad t \in [t_1 - \tau, t_1].$$

This shows that $\Psi X \subseteq X$. Moreover, it can be easily verified that Ψ is a monotone increasing mapping. Therefore, by the Knaster–Tarski Fixed Point Theorem (see Sect. 1.4), we have that there exists a $z \in X$ such that $\Psi z = z$, that is,

$$z(t) = \begin{cases} \frac{r}{y(t)} \int_t^\infty \prod_{j=1}^{m} \left[e^{y(s-\tau_j)z(s-\tau_j)} - 1\right]^{\alpha_j} ds, & t \geq t_1, \\ \frac{t}{t_j} (\Psi z)(t_1) + (1 - \frac{t}{t_1}), & t_1 - \tau \leq t \leq t_1. \end{cases} \tag{2.116}$$

By (2.116), $z(t)$ is continuous on $[t_1 - \tau, \infty)$. Moreover, since $z(t) > 0$ for $t \in [t_1 - \tau, t_1)$, we must have $z(t) > 0$, for all $t \geq t_1$. Set $w(t) = y(t)z(t)$. Then $w(t)$ is positive, continuous on $[t_1 - \tau, \infty)$, and satisfies

$$w(t) = r \int_t^\infty \prod_{j=1}^m \left[e^{w(s-\tau_j)} - 1\right]^{\alpha_j} ds, \quad \text{for } t \geq t_1. \tag{2.117}$$

Differentiating (2.117) yields

$$\frac{d}{dt} w(t) + r \int_t^\infty \prod_{j=1}^m \left[e^{w(s-\tau_j)} - 1\right]^{\alpha_j} = 0, \quad \text{for } t \geq t_1,$$

which shows that $e^{w(t)} - 1$ is a positive solution of (2.110) on $[t_1, \infty)$. This completes the proof. ■

The case $\sum_{j=1}^m \alpha_j = 1.$

The following theorem establishes an equivalence between the oscillation of (2.110)–(2.111) and the oscillation of (2.112).

Theorem 2.7.3. *When* $\sum_{j=1}^m \alpha_j = 1$, *every solution of (2.110)–(2.111) oscillates if and only if every solution of (2.112) oscillates.*

Proof. $\Rightarrow$: Assume that (2.112) has a nonoscillatory solution $y(t)$. Since $-y(t)$ is also a solution of (2.112), we may assume that $y(t)$ is eventually positive. We, will prove that (2.110)–(2.111) has a nonoscillatory solution for some t_0. To this end, we only need to prove that the equation

$$z'(t) + r \prod_{j=1}^m (1 - e^{-z(t-\tau_j)})^{\alpha_j} = 0 \tag{2.118}$$

has an eventually positive solution. Let t_0 be such that $y(t - \tau) > 0$ for $t \geq t_0$. Using the inequality $1 - e^{-x} \leq x$ for $x \geq 0$, we have for $t \geq t_0$ that

$$y'(t) + r \prod_{j=1}^m (1 - e^{-y(t-\tau_j)})^{\alpha_j} \leq y'(t) + r \prod_{j=1}^m y^{\alpha_j}(t - \tau_j) = 0. \tag{2.119}$$

It can be easily shown that $y(t) \to 0$, as $t \to \infty$. Integrating the above inequality from t to ∞, we obtain

$$y(t) \geq r \int_t^\infty \prod_{j=1}^m (1 - e^{-y(t-\tau_j)})^{\alpha_j}, \quad \text{for } t \geq t_0.$$

Now an argument similar to the proof of Theorem 2.7.2 shows that (2.119) would have an eventually positive solution $z(t)$ on $[t_0, \infty)$ satisfying $z(t) > 0$ for all $t \geq t_0$.

$\Leftarrow$: Assume, for the sake of contradiction, that (2.110)–(2.111) has a nonoscillatory solution $x(t)$ for every t_0. Then $1 + x(t) > 0$, for $t \geq t_0$. We now distinguish two cases:

Case (i): $x(t)$ is eventually positive.

Then there exists $T \geq t_0$ such that $x(t) > 0$, for $t \geq T$. From (2.110) it follows that

$$x'(t) + r\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) \leq x'(t) + r(1+x(t))\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) = 0. \quad (2.120)$$

This, together with Lemma 2.7.1, implies that (2.112) has a nonoscillatory solution, contrary to the assumption that every solution of (2.112) oscillates.

Case (ii): $x(t)$ is eventually negative.

Since $1 + x(t) > 0$ for $t \geq t_0$ and $x(t) < 0$ for $t \geq T$ for some $T \geq t_0$, we have

$$x'(t) = r(1+x(t))\prod_{j=1}^{m}[-x(t-\tau_j)]^{\alpha_j} > 0, \quad \text{for } t \geq T,$$

from which we can easily see that $x(t) \to 0$ as $t \to \infty$. On the other hand, in view of Lemma 2.7.1, we can choose $\epsilon \in (0, 1)$ such that

$$r(1-\epsilon)\sum_{j=1}^{m}\alpha_j\tau_j > \frac{1}{e}. \quad (2.121)$$

Now, let $T_1 > T$ be sufficiently large such that $1 > 1 + x(t) > 1 - \epsilon$, for $t \geq T$. Then by (2.110), we have for $t \geq T + \tau$ that

$$\begin{aligned} & x'(t) + r(1-\epsilon)\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) \\ & \geq x'(t) + r(1+x(t))\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) = 0, \end{aligned} \quad (2.122)$$

which is also a contradiction since, by Lemma 2.7.1, (2.122) implies that the inequality

$$x'(t) + r(1-\epsilon)\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) \geq 0$$

cannot have an eventually negative solution. This completes the proof. ■

The following corollary is an immediate result from Theorem 2.7.3 and Lemma 2.7.1.

Corollary 2.7.1. *If* $\sum_{j=1}^{m} \alpha_j = 1$, *then every solution of (2.110)–(2.111) oscillates (or every positive solution of (2.111) oscillates about the steady state* K*) if and only if*

$$r \sum_{j=1}^{m} \alpha_j \tau > \frac{1}{e}.$$

Next, in the following we consider the nonautonomous hyperlogistic delay model

$$N'(t) = r(t)N(t) \prod_{j=1}^{m} \left[1 - \frac{N(t-\tau_j)}{K}\right]^{\beta_j}, \text{ for } t \geq 0, \tag{2.123}$$

where $0 < \tau_1 \leq \tau_2 \leq \ldots \leq \tau_m$, $\beta_1, \ldots, \beta_m$ are rational numbers with denominators that are positive odd integers, and

$$r \in C([t_0, \infty), [0, \infty)), \; K > 0.$$

We will establish some sufficient conditions for the oscillation of all positive solutions of (2.123) about K. The results are adapted from [71]. To prove the main results we study the oscillation of the equation

$$x'(t) + p(t) \prod_{j=1}^{m} \left|x(t-\tau_j)\right|^{\alpha_j} sign[x(t-\tau_j)] = 0, \quad t \geq t_0, \tag{2.124}$$

where

$$p \in C([t_0, \infty), [0, \infty)), \; 0 < \tau_1 \leq \tau_2 \leq \ldots \leq \tau_m, \; \alpha_j > 0, \; j = 1, 2, \ldots, m,$$

and then apply the obtained results on the hyperlogistic model (2.123).

We will consider the equation

$$x'(t) + p(t) f(x(t-\tau_1), \ldots, x(t-\tau_m)) = 0, \text{ for } t \geq t_0, \tag{2.125}$$

where the function f satisfies the following condition (H):

(H). $f \in C(\mathbf{R}^m, \mathbf{R})$, $f(x_1, \ldots, x_m)$ is nondecreasing on each x_i, $i = 1, \ldots, m$, and

$$x_i > 0, \text{ for } i = 1, \ldots, m \Rightarrow f(x_1, \ldots, x_m) > 0,$$

$$x_i < 0, \text{ for } i = 1, \ldots, m \Rightarrow f(x_1, \ldots, x_m) < 0,$$

and

$$\lim_{(x_1,\ldots,x_m)\to(0,\ldots,0)} \frac{|f(x_1,\ldots,x_m)|}{\prod_{j=1}^{m} \left|x_j\right|^{\alpha_j}} = M > 0.$$

We will apply the results on the equation

$$x^{'}(t) + \sum_{j=1}^{m} p_j(t)\, x^{\beta_j}\left(t-\tau_j\right) = 0, \text{ for } t \geq t_0, \tag{2.126}$$

where $\beta_1, \ldots, \beta_m$ are rational numbers with denominators that are positive odd integers and

$$p_j \in C([t_0,\infty),[0,\infty)), \text{ for } \; j = 1,2,\ldots,m.$$

In the following, we consider the case when

$$\sum_{j=1}^{m} \alpha_j > 1 \tag{2.127}$$

and study the oscillatory behavior of (2.124) in terms of $p(t)$ and the delays $\tau_1, \ldots, \tau_m$.

The following lemma whose proof is standard (see [21]) will be needed to prove the main results.

Lemma 2.7.2. *Assume that* (H) *holds, and for large* t,

$$p(s) \neq 0, \text{ for } s \in [t, t+\tau], \tag{2.128}$$

where $\tau = \max\{\tau_1,\tau_2,\ldots,\tau_m\}$. *Then (2.125) has an eventually positive solution if and only if the corresponding inequality,*

$$x^{'}(t) + p(t) f(x(t-\tau_1),\ldots,x(t-\tau_m)) \leq 0, \quad t \geq t_0, \tag{2.129}$$

has an eventually positive solution.

Associated with (2.125), we consider the equation

$$x^{'}(t) + q(t) f(x(t-\tau_1),\ldots,x(t-\tau_m)) = 0, \text{ for } t \geq t_0, \tag{2.130}$$

where $q \in C([t_0,\infty),[0,\infty))$. Applying Lemma 2.7.2, we have the following lemma.

Lemma 2.7.3. *Assume that* (H) *and (2.128) hold, and that for large* t

$$p(t) \le q(t). \tag{2.131}$$

If every solution of (2.125) oscillates, then every solution of (2.130) oscillates.

Theorem 2.7.4. *Assume that (2.127) holds. Then the following conclusions hold:*

(i) *If there exists* $\lambda > 0$ *such that*

$$\sum_{j=1}^{m} \alpha_j e^{-\lambda \tau_j} < 1, \tag{2.132}$$

and

$$\lim_{t\to\infty} \inf \left[p(t) \exp\left(-e^{\lambda\tau}\right) \right] > 0, \tag{2.133}$$

then every solution of (2.124) oscillates.

(ii) *If (2.128) holds and there exists* $\mu > 0$ *such that*

$$\sum_{j=1}^{m} \alpha_j e^{-\mu \tau_j} > 1, \tag{2.134}$$

and

$$\lim_{t\to\infty} \sup \left[p(t) \exp\left(-e^{\mu\tau}\right) \right] < \infty, \tag{2.135}$$

then (2.124) has an eventually positive solution.

Proof. (i) From (2.132) and (2.133), we may choose $\lambda_2 < \lambda_1 < \lambda$ and $T > t_0$ such that

$$\sum_{j=1}^{m} \alpha_j e^{-\lambda \tau_j} < \sum_{j=1}^{m} \alpha_j e^{-\lambda_1 \tau_j} < \sum_{j=1}^{m} \alpha_j e^{-\lambda_2 \tau_j} < 1, \tag{2.136}$$

and

$$p(t) \ge \lambda_1 e^{\lambda_1 t} \exp\left[\frac{1}{2} \left(\sum_{j=1}^{m} \alpha_j - 1 \right) e^{\lambda_1 t} \right], \quad t \ge T. \tag{2.137}$$

Set

$$q(t) = \lambda_1 e^{\lambda_1 t} \exp\left[\frac{1}{2} \left(\sum_{j=1}^{m} \alpha_j - 1 \right) e^{\lambda_1 t} \right]. \tag{2.138}$$

By Lemma 2.7.3, it suffices to prove that every solution of the equation

$$x^{'}(t) + q(t) \prod_{j=1}^{m} \left|x(t-\tau_j)\right|^{\alpha_j} sign\,[x(t-\tau_1)] = 0, \quad t \geq t_0, \tag{2.139}$$

oscillates. Assume the contrary, and let $x(t)$ be an eventually positive solution of (2.139). Then there exists a $T_1 > T$ such that

$$1 > x(t-\tau_m) > 0 \text{ and } x^{'}(t) \leq 0, \text{ for } t \geq T_1.$$

Let $y(t) = -\ln x(t)$ for $t \geq T_1 - \tau_m$. Then $y(t) > 0$ for $t \geq T_1 - \tau_m$, and from (2.139) we have

$$y'(t) = q(t) \exp\left[y(t) - \sum_{j=1}^{m} \alpha_j y(t-\tau_j) \right], \text{ for } t \geq T_1. \tag{2.140}$$

Set $l = \sum_{j=1}^{m} \alpha_j e^{-\lambda_2 \tau_j}$. Then $0 < l < 1$. We consider the following three possible cases.

Case (1): Consider the case when $y(t) \leq \sum_{j=1}^{m} \alpha_j e^{(\lambda_1 - \lambda_2)\tau_j} y(t-\tau_j)$ eventually holds.

Choose $T_2 > T_1$ such that

$$y(t) \leq \sum_{j=1}^{m} \alpha_j e^{(\lambda_1-\lambda_2)\tau_j} y(t-\tau_j), \text{ for } t \geq T_2.$$

Consequently, we have for $t \geq T_2$ that

$$\frac{y(t)}{e^{\lambda_1 t}} \leq \sum_{j=1}^{m} \frac{\alpha_j e^{\lambda_1 t - \lambda_2 \tau_j}}{e^{\lambda_1 t}} \frac{y(t-\tau_j)}{e^{\lambda_1 (t-\tau_j)}} = \sum_{j=1}^{m} \alpha_j e^{-\lambda_2 \tau_j} \frac{y(t-\tau_j)}{e^{\lambda_1 (t-\tau_j)}}.$$

Set $z(t) = y(t)e^{-\lambda_1 t}$. Then

$$z(t) \leq \sum_{j=1}^{m} \alpha_j e^{-\lambda_2 \tau_j} z(t-\tau_j), \quad \text{for } t \geq T_2. \tag{2.141}$$

This implies that

$$\lim_{t \to \infty} z(t) = 0. \tag{2.142}$$

From (2.142), it follows that there exists a $T_3 > T_2$ such that

$$y(t) < \frac{1}{2}e^{\lambda_1 t}, \quad t \geq T_3, \tag{2.143}$$

which, together with (2.140), implies for $t \geq T_3$ that

$$\begin{aligned} y'(t) &\geq q(t)\exp\left[\left(1-\sum_{j=1}^{m}\alpha_j\right)y(t)\right] \\ &\geq q(t)\exp\left[\frac{1}{2}\left(1-\sum_{j=1}^{m}\alpha_j\right)e^{\lambda_1 t}\right] = \lambda_1 e^{\lambda_1 t}. \end{aligned}$$

It follows that

$$y(t) \geq y(T_3) + e^{\lambda_1 t} - e^{\lambda_1 T_3}, \quad t \geq T_3,$$

which contradicts (2.143).

Case (2): Consider the case when $y(t) - \sum_{j=1}^{m} \alpha_j e^{(\lambda_1-\lambda_2)\tau_j} y(t-\tau_j)$ is oscillatory.

In this case, there exists an increasing infinite sequence $\{t_n\}$ of real numbers with $T_3 < t_1 < t_2 < \ldots$ such that

$$y(t_n) = \sum_{j=1}^{m} \alpha_j e^{(\lambda_1-\lambda_2)\tau_j} y(t_n-\tau_j), \quad n = 1, 2, \ldots, \tag{2.144}$$

and

$$y(t) > \sum_{j=1}^{m} \alpha_j e^{(\lambda_1-\lambda_2)\tau_j} y(t-\tau_j), \quad t \in (t_{2n-1}, t_{2n}), \ n = 1, 2, \ldots \tag{2.145}$$

Set

$$u(t) = y(t) - \sum_{j=1}^{m} \alpha_j e^{(\lambda_1-\lambda_2)\tau_j} y(t-\tau_j).$$

Then $u(t)$ is oscillatory and there exists an increasing infinite sequence $\{\xi_n\}$ of real numbers such that

$$u(\xi_n) = \max\{u(t) \ : t_{2n-1} \leq t \leq t_{2n}\},$$

and $u'(\xi_n) = 0, n = 1, 2, \ldots$. Note

$$u'(\xi_n) = y'(\xi_n) - \sum_{j=1}^{m} \alpha_j e^{(\lambda_1-\lambda_2)\tau_j} y'(\xi_n - \tau_j),$$

and for $t \geq T_1$

$$y'(t) = q(t) \exp\left[u(t) + \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1) y(t - \tau_j)\right]. \tag{2.146}$$

It follows that

$$\begin{aligned}
&q(\xi_n) \exp\left[u(\xi_n) + \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1) y(\xi_n - \tau_j)\right] \\
&= \sum_{i=1}^{m} \alpha_i e^{(\lambda_1-\lambda_2)\tau_i} q(\xi_n - \tau_i) \\
&\quad \times \exp\left[u(\xi_n - \tau_i) + \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1) y(\xi_n - \tau_i - \tau_j)\right] \\
&< \lambda_1 e^{\lambda_1 \xi_n} \exp\left[\frac{1}{2}\left(\sum_{j=1}^{m} \alpha_j - 1\right) e^{\lambda_1(\xi_n - \tau_1)}\right] \\
&\quad \times \exp\left[\max_{1 \leq i \leq m} \{u(\xi_n - \tau_i)\} + \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1) y(\xi_n - \tau_1 - \tau_j)\right].
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&u(\xi_n) + \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1) y(\xi_n - \tau_j) \\
&< \max_{1 \leq i \leq m} \{u(\xi_n - \tau_i)\} + \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1) y(\xi_n - \tau_1 - \tau_j) \\
&- \frac{1}{2}\left(\sum_{j=1}^{m} \alpha_j - 1\right) (1 - e^{-\lambda_1 \tau_1}) e^{\lambda_1 \xi_n}, \qquad n = 1, 2, 3, \ldots .
\end{aligned} \tag{2.147}$$

If

$$\limsup_{t\to\infty} u(t) = \limsup_{n\to\infty} u(\xi_n) = \infty,$$

then there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that

$$u\left(\xi_{n_k}\right) = \max\{u(t) : T_2 \le t \le \xi_{n_k}\},\ \ k = 1, 2, \dots .$$

Hence, from (2.147), we have

$$0 < \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1)\left[y(\xi_{n_k} - \tau_j) - y(\xi_{n_k} - \tau_1 - \tau_j)\right]$$
$$< -\frac{1}{2}\left(\sum_{j=1}^{m} \alpha_j - 1\right)(1 - e^{-\lambda_1\tau_1})e^{\lambda_1\xi_{n_k}} < 0,\ k = 1, 2, \dots .$$

This is a contradiction. If

$$\limsup_{t\to\infty} u(t) = \limsup_{n\to\infty} u(\xi_n) < \infty,$$

then from (2.147),

$$0 < \limsup_{n\to\infty} \Big(u(\xi_n)$$
$$+ \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1)\left[y(\xi_{n_k} - \tau_j) - y(\xi_{n_k} - \tau_1 - \tau_j)\right]\Big)$$
$$\le \limsup_{n\to\infty} \left\{ \max_{1\le i\le m} \{u(\xi_n - \tau_i)\} \right.$$
$$\left. -\frac{1}{2}\left(\sum_{j=1}^{m} \alpha_j - 1\right)(1 - e^{-\lambda_1\tau_1})e^{\lambda_1\xi_n} \right\} = -\infty.$$

This is also a contradiction.

Case (3): Consider the case when $y(t) \ge \sum_{j=1}^{m} \alpha_j e^{(\lambda_1-\lambda_2)\tau_j} y(t - \tau_j)$ eventually holds.

Let $T_4 > T_3$ be such that

$$y(t) \ge \sum_{j=1}^{m} \alpha_j e^{(\lambda_1-\lambda_2)\tau_j} y(t - \tau_j),\ t \ge T_4.$$

It follows from (2.140) that

$$y'(t) = q(t)\exp\left[y(t) - \sum_{j=1}^{m}\alpha_j y(t-\tau_j)\right]$$
$$\geq q(t)\exp\left[\left(1 - e^{(\lambda_2-\lambda_1)\tau_1}\right) r(t)\right], \text{ for } t \geq T_4.$$

Set $c = 1 - e^{(\lambda_2-\lambda_1)\tau_1}$. Then $0 < c < 1$, and the above inequality reduces to

$$y'(t)e^{-cy(t)} \geq q(t), \text{ for } t \geq T_4.$$

Integrating the above inequality from T_4 to ∞, we obtain

$$\int_{T_4}^{\infty} q(t)dt \leq \int_{T_4}^{\infty} y'(t)e^{-cy(t)}dt \leq \frac{1}{c}e^{-cy(T_4)} < \infty,$$

which contradicts the definition of $q(t)$.

Cases 1, 2, and 3 complete the proof of (i).

(ii) By (2.134) and (2.135), we may choose $\mu_1 > \mu$ and $T > t_0$ such that

$$\sum_{j=1}^{m}\alpha_j e^{-\mu\tau_j} > \sum_{j=1}^{m}\alpha_j e^{-\mu_1\tau_j} > 1, \tag{2.148}$$

and

$$p(t) \leq \mu_1 e^{\mu_1 t}\exp\left[\left(\sum_{j=1}^{m}\alpha_j e^{-\mu_1\tau_j} - 1\right)e^{\mu_1 t}\right], \ t \geq T. \tag{2.149}$$

Set $\varphi(t) = e^{\mu_1 t}$ and $x(t) = e^{-\varphi(t)}$. Then for $t \geq T$,

$$x'(t) + p(t)\prod_{j=1}^{m}\left|x(t-\tau_j)\right|^{\alpha_j} sign\left[x(t-\tau_1)\right]$$

$$= -\varphi(t)e^{-\varphi(t)} + p(t)\prod_{j=1}^{m} e^{-\alpha_j\varphi(t-\tau_j)}$$

$$= \prod_{j=1}^{m} e^{-\alpha_j\varphi(t-\tau_j)}\left\{p(t) - \mu_1 e^{\mu_1 t}\exp\left[\left(\sum_{j=1}^{m}\alpha_j e^{-\mu_1\tau_j} - 1\right)e^{\mu_1 t}\right]\right\} \leq 0.$$

This shows that the inequality

$$x'(t) + p(t)\prod_{j=1}^{m}\left|x(t-\tau_j)\right|^{\alpha_j} sign\left[x(t-\tau_1)\right] \le 0, \;\; t \ge t_0,$$

has an eventually positive solution. In view of Lemma 2.7.2, the corresponding equation (2.124) also has an eventually positive solution. The proof is complete. ∎

Applying Theorem 2.7.4 on the special form

$$x^{'}(t) + p(t)\left|x(t-\tau_j)\right|^{\alpha} sign\left[x(t-\tau)\right] = 0, \qquad t \ge t_0, \tag{2.150}$$

where

$$p \in C([t_0,\infty),[0,\infty)),\; \tau > 0,\; \alpha > 0,$$

we have immediately the following result.

Corollary 2.7.2. *Assume that $\alpha > 1$. Then the following conclusions hold:*

(i) If there exists $\lambda > \tau^{-1}\ln\alpha$ such that (2.133) holds, then every solution of (2.150) oscillates.
(ii) If $p(t) \not\equiv 0$ on any interval of length τ, and there exists $\mu < \tau^{-1}\ln\alpha$ such that (2.135) holds, then (2.150) has an eventually positive solution.

Note that if $\sum_{j=1}^{m}\alpha_j > 1$, then it follows that there exists a unique $\lambda_0 > 0$ such that

$$\sum_{j=1}^{m}\alpha_j e^{-\lambda_0\tau_j} = 1.$$

Therefore, applying Theorem 2.7.4 to the following equation which is a special form of (2.124)

$$x'(t) + C\exp(e^{\lambda t})\prod_{j=1}^{m}\left|x(t-\tau_j)\right|^{\alpha_j} sign\left[x(t-\tau_1)\right] = 0, \;\; t \ge t_0, \tag{2.151}$$

where $C > 0$, we have that every solution of (2.151) oscillates if $\lambda > \lambda_0$ and (2.151) has an eventually positive solution in $\lambda < \lambda_0$.

In the following, we apply Theorem 2.7.4 to (2.125), (2.126), and (2.123).

Theorem 2.7.5. *Assume that* (H) *holds and* $\sum_{j=1}^{m} \alpha_j > 1$. *Then the following conclusions hold:*

(i) If there exists $\lambda > 0$ *such that (2.132) and (2.133) hold, then every solution of (2.125) oscillates.*
(ii) If (2.128) and

$$\int_{t_0}^{\infty} p(t)dt = \infty \tag{2.152}$$

hold and there exists $\mu > 0$ *such that (2.134) and (2.135) hold, then (2.125) has an eventually positive solution.*

Proof. (i) Assume the contrary, and let $x(t)$ be an eventually positive solution of (2.125). Then from (2.125) and (2.133), we easily see that $\lim_{t\to\infty} x(t) = 0$. Then from (2.125) and (H) there exists a $T_1 > t_0$ such that

$$1 > x(t-\tau_m) > 0, \text{ and } x^{'}(t) \le 0, \text{ for } \quad t \ge T_1,$$

and

$$f(x(t-\tau_1), \ldots, x(t-\tau_m)) \ge \frac{1}{2} M \prod_{j=1}^{m} \left[x(t-\tau_j)\right]^{\alpha_j}, \quad t \ge T_1. \tag{2.153}$$

Substituting (2.153) into (2.125), we have

$$x'(t) + \frac{1}{2} M p(t) \prod_{j=1}^{m} \left[x(t-\tau_j)\right]^{\alpha_j} \le 0, \quad \text{for } t \ge T_1. \tag{2.154}$$

This shows that the inequality (2.154) has an eventually positive solution. In view of Lemma 2.7.2, the corresponding equation,

$$x'(t) + \frac{1}{2} M p(t) \prod_{j=1}^{m} |x(t-\tau_j)|^{\alpha_j} sign\,[x(t-\tau_1)] = 0, \quad t \ge t_0, \tag{2.155}$$

also has an eventually positive solution. But, by Theorem 2.7.4, (2.132) and (2.133) imply that every solution of (2.155) oscillates, and this contradiction completes the proof of (i).

(ii) In view of Theorem 2.7.4, (2.128), (2.134), (2.135), and (2.152) imply that the equation

$$x'(t) + 2Mp(t)\prod_{j=1}^{m}|x(t-\tau_j)|^{\alpha_j} sign\,[x(t-\tau_1)] = 0, \quad t \geq t_0, \tag{2.156}$$

has an eventually positive solution $x(t)$ with $\lim_{t\to\infty} x(t) = 0$. From this, (H), and (2.156), there exists a $T_2 > t_0$ such that

$$x(t-\tau_m) > 0, \text{ and } \ x^{'}(t) \leq 0 \text{ for } \ t \geq T_2,$$

and

$$f(x(t-\tau_1),\ldots,x(t-\tau_m)) \leq 2M\prod_{j=1}^{m}|x(t-\tau_j)|^{\alpha_j}, \qquad t \geq T_2. \tag{2.157}$$

Substituting (2.157) into (2.156), we have

$$x^{'}(t) + p(t)f(x(t-\tau_1),\ldots,x(t-\tau_m)) \leq 0, \quad t \geq T_2. \tag{2.158}$$

This shows that inequality (2.158) has an eventually positive solution. In view of Lemma 2.7.2, the corresponding equation (2.125) also has an eventually positive solution. The proof is complete. ∎

Theorem 2.7.6. *Assume that* $\sum_{j=1}^{m}\beta_j > m$, *and that there exists* $\lambda > 0$ *such that*

$$\sum_{j=1}^{m}\beta_j e^{-\lambda\tau_j} < m, \tag{2.159}$$

and

$$\liminf_{t\to\infty}\left\{\left[\prod_{j=1}^{m}p_j(t)\right]\exp\left(-me^{\lambda t}\right)\right\} > 0. \tag{2.160}$$

Then every solution of (2.126) oscillates.

Proof. Assume the contrary, and let $x(t)$ be an eventually positive solution of (2.126). It follows from (2.126) that there exists a $T > t_0$ such that

$$x(t-\tau_m) > 0, \text{ and } \ x^{'}(t) \leq 0, \quad \text{for} \quad t \geq T.$$

From (2.126), we have

$$x^{'}(t) + m\left[\prod_{j=1}^{m} p_j(t)\right]^{\frac{1}{m}} \prod_{j=1}^{m} \left[x(t-\tau_j)\right]^{\frac{\beta_j}{m}} \le 0, \quad t \ge T. \tag{2.161}$$

This shows that inequality (2.161) has an eventually positive solution. In view of Lemma 2.7.2, the corresponding equation,

$$x^{'}(t) + m\left[\prod_{j=1}^{m} p_j(t)\right]^{\frac{1}{m}} \prod_{j=1}^{m} |x(t-\tau_j)|^{\frac{\beta_j}{m}} sign\left[x\left(t-\tau_1\right)\right] = 0, \; t > t_0, \tag{2.162}$$

also has an eventually positive solution. But Theorem 2.7.4, (2.159), and (2.160) imply that every solution oscillates. This contradiction completes the proof. ■

Now, we consider equation (2.123). Note that if

$$\prod_{j=1}^{m} (-1)^{\beta_j} = -1,$$

then by making a change of variables,

$$x(t) = \ln\left[\frac{N(t)}{K}\right],$$

one can write (2.123) as

$$x^{'}(t) + r(t)\prod_{j=1}^{m} \left[e^{x(t-\tau_j)} - 1\right]^{\beta_j} = 0, \quad \text{for } t \ge 0. \tag{2.163}$$

Set

$$f(x_1, \ldots, x_m) = \prod_{j=1}^{m} (e^{x_j} - 1)^{\beta_j}.$$

Then f satisfies condition (H) for $\beta_1, \ldots, \beta_m$.

Hence, in view of Theorem 2.7.5, we have immediately the following result.

Theorem 2.7.7. *Assume that*

$$\prod_{j=1}^{m} (-1)^{\beta_j} = -1 \text{ and } \sum_{j=1}^{m} \beta_j > 1.$$

Then the following conclusions hold:

(*i*) *If there exists* $\lambda > 0$ *such that*

$$\sum_{j=1}^{m} \beta_j e^{-\lambda \tau_j} < 1, \tag{2.164}$$

and

$$\lim_{t \to \infty} \inf \left[r(t) \exp\left(-e^{\lambda t}\right) \right] > 0, \tag{2.165}$$

then every positive solution of (2.123) oscillates about K.

(*ii*) *If* $r(t) \neq 0$ *for any interval of length* τ, *where* $\tau = \max\{\tau_1, \ldots, \tau_m\}$,

$$\int_0^{\infty} r(s)ds = \infty, \tag{2.166}$$

and there exists $\mu > 0$ *such that*

$$\sum_{j=1}^{m} \beta_j e^{-\mu \tau_j} > 1, \tag{2.167}$$

and

$$\lim_{t \to \infty} \sup \left[r(t) \exp\left(-e^{\mu t}\right) \right] < \infty, \tag{2.168}$$

then (2.123) has a solution greater than K *eventually.*

2.8 Models with a Varying Capacity

In the delay logistic equations we assumed that the carrying capacity $K > 0$ is a constant. The variation of the environment plays an important role in many biological and ecological dynamical systems. It is realistic to assume that the parameters in the models are positive periodic functions of period ω.

Consider the nonautonomous delay logistic model

$$N'(t) = r(t)N(t)\left[1 - \frac{N(t - m\omega)}{K(t)}\right], \tag{2.169}$$

where m is a positive integer and $\omega > 0$. Assume r and K are positive periodic functions of period ω. We consider solutions of (2.169) corresponding to the initial condition

$$\begin{cases} N(t) = \varphi(t), \text{ for } m\omega < t < 0, \\ \varphi \in C[[-m\omega, 0], \mathbf{R}^+], \ \varphi(0) > 0. \end{cases} \tag{2.170}$$

It is easy to see that there exist a unique positive periodic solution $N^*(t)$ of (2.169).

Theorem 2.8.1. *If*

$$\int_0^\infty \frac{r(t)N^*(t)}{K(t)} dt = \infty, \tag{2.171}$$

then every nonoscillatory solution $N(t)$ of (2.169) satisfies

$$\lim_{t\to\infty} N(t) = N^*(t). \tag{2.172}$$

Proof. Assume that $N(t) > N^*(t)$ for t sufficiently large (the proof when $N(t) < N^*(t)$ is similar and will be omitted). Set

$$N(t) = N^*(t)e^{z(t)}. \tag{2.173}$$

Then $z(t) > 0$ for t sufficiently large, and for t large

$$z'(t) + \frac{r(t)N^*(t)}{K(t)} \left(e^{z(t-m\omega)} - 1\right) = 0, \tag{2.174}$$

so

$$z'(t) = -\frac{r(t)N^*(t)}{K(t)} \left(e^{z(t-m\omega)} - 1\right) < 0.$$

Thus, $z(t)$ is decreasing, and therefore

$$\lim_{t\to\infty} z(t) = \alpha \in [0, \infty).$$

We claim $\alpha = 0$. If $\alpha > 0$, then there exist $\varepsilon > 0$ and $T_\varepsilon > 0$ such that for $t \geq T_\varepsilon$,

$$0 < \alpha - \varepsilon < z(t) < \alpha + \varepsilon.$$

However, then from (2.174), we find

$$z'(t) + \frac{r(t)N^*(t)}{K(t)}(e^{\alpha-\varepsilon} - 1) \leq 0, \quad t \geq T_\varepsilon,$$

By integrating from T_ε to ∞ and using (2.171) we immediately get a contradiction. Hence $\alpha = 0$. Thus

$$\lim_{t\to\infty}(N(t) - N^*(t)) = \lim_{t\to\infty} N^*(t)(e^{z(t)} - 1) = 0.$$

This completes the proof. ∎

Theorem 2.8.2. *Assume that r and K are positive periodic functions of period ω such that (2.171) holds. Suppose for every sufficiently small $\varepsilon \geq 0$ all solutions of the linear delay differential equation*

$$x'(t) + (1-\varepsilon)\frac{r(t)N^*(t)}{K(t)}x(t - m\omega) = 0, \quad t \geq t_0, \tag{2.175}$$

are oscillatory. Then all solutions of (2.169) are oscillatory about $N^*(t)$.

Proof. Assume that (2.169) has a solution which does not oscillate about $N^*(t)$. Without loss of generality we assume that $N(t) > N^*(t)$, so that $z(t) > 0$; here z is defined in Theorem 2.8.1. (The case $N(t) < N^*(t)$ implies that $z(t) < 0$ and the proof is similar. In fact, we will see below that if $z(t)$ is a negative solution of (2.176) then $U(t) = -z(t)$ is positive solution of (2.176)). It is clear that $N(t)$ oscillates about $N^*(t)$ if and only if $z(t)$ oscillates about zero. Also

$$z'(t) + \frac{r(t)N^*(t)}{K(t)} f(z(t - m\omega)) = 0, \tag{2.176}$$

where

$$f(u) = (e^u - 1).$$

Note that

$$\lim_{u\to 0}\frac{f(u)}{u} = 1.$$

Then by Theorem 2.6.4, since every solution of (2.175) oscillates, then every solution of (2.176) oscillates. Thus every positive solution of (2.169) oscillates about $N^*(t)$. The proof is complete. ∎

Next we discuss the oscillation of (2.169) about the positive periodic function $K(t)$. The result is adapted from [86].

Theorem 2.8.3. *Assume the following:*

(i) K is a nonconstant positive differentiable periodic function of period ω.
(ii) r is positive and continuous for $t \geq 0$ such that

$$\lim_{t\to\infty}\inf r(t) > 0, \textit{ and } \quad \lim_{t\to\infty}\inf \int_{t-m\omega}^{t} r(s)ds > \frac{1}{e}. \tag{2.177}$$

Then every positive solution of (2.169) is oscillatory about K.

Proof. If we define $y(t) = \ln[N(t)/K(t)]$, then y is governed by

$$y^{'}(t) = r(t)\left[1 - e^{y(t-m\omega)}\right] - \frac{K^{'}(t)}{K(t)}, \tag{2.178}$$

and the oscillation of N about K is equivalent to that of y about zero and thus it is sufficient to consider the usual oscillation of y. We simplify (2.178) by letting

$$Q(t) = \ln(\frac{K(t_0)}{K(t)}) \tag{2.179}$$

and note that (2.178) becomes

$$y^{'}(t) + r(t)\left[e^{y(t-m\omega)} - 1\right] = Q^{'}(t). \tag{2.180}$$

Suppose now the conclusion of the theorem is false. Then there exists an eventually positive or eventually negative solution for (2.180).

Let us first assume that (2.180) has an eventually positive solution y. Since Q is a nonconstant periodic function, there exist two sequences $\{t_n^{'}\}$ and $\{t_n^{''}\}$ such that $\lim_{n\to\infty} t_n^{'} = \infty$, $\lim_{n\to\infty} t_n^{''}$, and

$$-\infty < q_1 \le Q(t) \le q_2 < \infty,$$

$$q_1 = Q(t_n^{'}) \text{ and } q_2 = Q(t_n^{''}), \quad n = 1, 2, \ldots . \tag{2.181}$$

Let

$$u(t) = y(t) - Q(t), \quad \text{for } t \ge T,$$

(where $y(t - m\omega) > 0$ for $t \ge T$). Note that (2.180) becomes

$$u^{'}(t) = r(t)\left[1 - e^{y(t-m\omega)}\right] < 0. \tag{2.182}$$

We claim $u(t) + q_1 > 0$. Suppose for some $t \ge T$, $u(t) + q_1 \le 0$. Since $y(t) > 0$, we have $u(t) + Q(t) = y(t) > 0$ and hence $u(t_n^{'}) + q_1 = y(t_n^{'}) > 0$ showing that $u(t) + q_1 \le 0$ is not possible. Therefore,

$$u(t) + q_1 > 0, \text{ for large } t \ge T. \tag{2.183}$$

Let $z(t) = u(t) + q_1$ and we see that

$$
\begin{aligned}
z^{'}(t) &= u^{'}(t) = y^{'}(t) - Q^{'}(t) \\
&= r(t)\left[1 - e^{y(t-m\omega)}\right] \\
&= r(t)\left[1 - e^{u(t-m\omega)+Q(t-m\omega)}\right] \\
&\leq -r(t)\left[u(t-m\omega) + Q(t-m\omega)\right] \leq -r(t)z(t-m\omega).
\end{aligned} \tag{2.184}
$$

Note that (2.184) has an eventually positive solution and this is impossible due to (2.177) (a standard argument is used here).

Let us now consider the case when $y(t)$ is an eventually negative solution of (2.169). This implies that

$$
\frac{N(t)}{K(t)} < 1, \quad \text{for large } t. \tag{2.185}
$$

The boundedness of K (due to periodicity) and (2.185) imply that $N(t)$ is bounded. It follows from (2.169) that $N^{'}(t) > 0$ eventually and this implies that

$$
\lim_{t\to\infty} N(t) = l > 0. \tag{2.186}
$$

Integrating (2.169), we have

$$
\ln \frac{l}{N(t_0)} = \int_{t_0}^{\infty} r(t)\left(1 - \frac{N(t-m\omega)}{K(t)}\right) dt < \infty. \tag{2.187}
$$

Hence

$$
\lim_{t\to\infty} \inf r(t)\left(1 - \frac{N(t-m\omega)}{K(t)}\right) = 0.
$$

But $\liminf_{t\to\infty} r(t) > 0$, so

$$
\lim_{t\to\infty} \sup \frac{N(t-m\omega)}{K(t)} = 1,
$$

i.e., there exists a sequence $\{t_k\}$ such that

$$
\lim_{k\to\infty} \frac{N(t_k - m\omega)}{K(t_k)} = 1.
$$

Since $N(t) < K$, we see that $\lim_{t\to\infty} N(t) = l = \min_{t\in[0,\omega]} K(t)$. But then

$$\begin{aligned}
&\int_{t_0}^{\infty} r(t)\left(1 - \frac{N(t-m\omega)}{K(t)}\right) dt \\
&\geq \frac{\inf r(t)}{\max_{t\in[0,\omega]} K(t)} \int_{t_0}^{\infty} (K(t) - N(t-m\omega))\, dt \\
&\geq \frac{\inf r(t)}{\max_{t\in[0,\omega]} K(t)} \int_{t_0}^{\infty} \left(K(t) - \min_{t\in[0,\omega]} K(t)\right) dt = \infty,
\end{aligned}$$

which contradicts (2.187). This completes the proof. ■

Chapter 3
Stability of Delay Logistic Models

The essence of mathematics lies in its freedom.

Georg Cantor (1845–1915).

As for everything else, so for a mathematical theory: beauty can be perceived but not explained.

Arthur Cayley (1821–1895).

The stability of the equilibrium points is important in the study of mathematical models. The equilibrium point $\overline{N}$ is locally stable if the solution of the model $N(t)$ approaches $\overline{N}$ as time increases for all the initial values, in some neighborhood of $\overline{N}$. The equilibrium point $\overline{N}$ is globally stable for a mathematical model if for all initial values the solution of the model approaches $\overline{N}$ as time increases. A model is locally or globally stable if its positive equilibrium point is locally or globally stable.

To study local asymptotic stability, we use a standard approach to analyze the stability of a linearization about the trivial solution. The stability of the trivial solution of the linearized equation depends on the location of the roots of the associated characteristic equation. If all the roots of the characteristic equation for the linearized equation have negative real parts, and if all the roots are uniformly bounded away from the imaginary axis, then the trivial solution of the linear equation is locally asymptotically stable.

In this chapter we present the current approach on stability (local, global, and uniform) for autonomous and nonautonomous delay equations. We note that the theory in Chap. 3 can be extended (using the ideas in this chapter) to cover other models, for example models with distributed delays.

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3.1 Autonomous Models of Hutchinson Type

In this section we discuss autonomous models of Hutchinson type.

3.1.1 Local Stability

First we consider the local stability of a Hutchinson type model

$$N^{'}(t) = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right], \quad \text{for } t \geq 0, \tag{3.1}$$

where $N(t)$ is the population at time t, r is the growth rate of the species, and $K > 0$ is called the carrying capacity of the habitat (note that there is no immigration or emigration). It is well known that the trivial solution of (3.1) is unstable, since the linearization of (3.1) about $N = 0$ satisfies the linear equation $dN(t)/dt \simeq rN(t)$ which shows that $N = 0$ is unstable with exponential growth. Next, we consider the perturbations about the positive steady state K. Set

$$N^*(t^*) = N(t)/K, \quad t^* = rt, \quad \tau^* = r\tau,$$

where the asterisk denotes dimensionless quantities. Then (3.1) becomes, on dropping the asterisks for algebraic convenience, but keeping in mind that we are now dealing with non-dimensional quantities,

$$N^{'}(t) = N(t)[1 - N(t-\tau)]. \tag{3.2}$$

Linearizing about the steady state, $N = 1$, by writing $N(t) = 1 + n(t)$ we have

$$\frac{dn}{dt} \simeq -n(t-\tau), \tag{3.3}$$

and its corresponding characteristic equation is given by

$$p(\lambda) = \lambda + e^{-\lambda\tau} = 0. \tag{3.4}$$

Clearly if $\tau = 0$, (3.4) has a real root $\lambda = -1$ which shows that $n = 0$ is stable with exponential decay, which leads to the local stability of the steady state K. On the other hand for $\tau > 0$ we have

$$p(-1/\tau) = -\frac{1}{\tau} + e = \frac{e}{\tau}(\tau - 1/e) \leq 0,$$

when $\tau \le 1/e$ which shows that there exists a real root in the interval $[-1/\tau, 0]$ which means that there exists a nonoscillatory solution of (3.3) which also tends to zero as t tends to infinity and if $\tau > 1/e$, Eq. (3.4) has no real roots which leads to the oscillation of all solutions (see Sect. 2.1).

Now, we wish to know whether there are any solutions of (3.4) with $\mathrm{Re}\,\lambda > 0$ which would imply that the trivial solution is unstable with exponential growth. Let $\lambda = \mu + iw$ be a root of (3.4) where μ and w are real numbers. We claim that there is a real number μ_0 such that all solutions of (3.4) satisfy $\mathrm{Re}\,\lambda < \mu_0$. To see this, note $\lambda = e^{-[\mu+iw]\tau}$ so $|\lambda| = e^{-\mu\tau}$, and so, if $|\lambda| \to \infty$ then $e^{-\mu\tau} \to \infty$ which requires that $\mu \to -\infty$. Thus there must be a number μ_0 which bounds $\mathrm{Re}\,\lambda$ from above.

Set $z = 1/\lambda$, and

$$f(z) = 1 + ze^{-\tau/z}.$$

Then $f(z)$ has an essential singularity at $z = 0$. So by Picard's Theorem, $f(z)$ has infinitely many complex roots in the neighborhood of $z = 0$. Now, from (3.4) we have

$$\mu = -e^{-\mu\tau}\cos w\tau, \quad w = e^{-\mu\tau}\sin w\tau. \tag{3.5}$$

The aim now is to determine the range of τ such that $\mu < 0$.

First, let $w = 0$. Then we have $\mu = -e^{-\mu\tau}$, and this has no positive roots since $e^{-\mu\tau} > 0$ for all $\mu\tau$. Consider the case $w \neq 0$. From (3.5) if w is a solution then $-w$ is also a solution, so we consider $w > 0$ without loss of generality. From

$$\mu = -e^{-\mu\tau}\cos w\tau,$$

$\mu < 0$ requires $w\tau < \pi/2$ since $-e^{-\mu\tau} < 0$ for all $\mu\tau$. Multiplying

$$w = e^{-\mu\tau}\sin w\tau,$$

by τ we have

$$\tau e^{-\mu\tau}\sin w\tau = \tau w < \pi/2.$$

Next we consider the generalized delay logistic equation

$$N^{'}(t) + a[N(t) - N^*] = rN(t)\left[1 - \sum_{j=1}^{\infty}\frac{N(t-\tau_j)}{K_j}\right], \quad t > 0, \tag{3.6}$$

and establish some sufficient conditions for the local asymptotic stability of the positive steady state N^*. The results are adapted from Gopalsamy [26]. By using the transformation

$$N(t) = N^* + x(t),$$

we see that the Eq. (3.6) reduces to the nonlinear delay equation

$$x'(t) + ax(t) + [N^* + x(t)] \sum_{j=1}^{\infty} b_j x(t - \tau_j) = 0, \quad t > 0, \tag{3.7}$$

where

$$\frac{r}{N^*} = \sum_{j=1}^{\infty} b_j, \quad b_j = \frac{r}{K_j}, \qquad j = 1, 2, \dots . \tag{3.8}$$

We assume that a is a nonnegative constant, N^*, b_j, τ_j for $j = 1, 2, 3, \dots$ are positive constants such that

$$\sum_{j=1}^{\infty} b_j = b < \infty, \quad 0 < \inf_j \tau_j = \tau_* \le \sup_j \tau_j = \tau^* < \infty. \tag{3.9}$$

With (3.6) we associate the initial condition

$$\begin{cases} N(t) = \phi(t), & \text{for } -\tau^* \le t \le 0, \\ \phi \in C([-\tau^*, 0], [0, \infty)), & \text{and } \phi(0) > 0. \end{cases} \tag{3.10}$$

It follows from the substitution $N(t) = N^* + x(t)$ that the asymptotic stability (local or global) of N^* of (3.6) is equivalent to that of the trivial solution of (3.7) where the relevant initial condition for (3.7) is inherited from (3.10) through the substitution $N(t) = N^* + x(t)$.

Theorem 3.1.1. *Assume that (3.9) holds, and*

$$\tau^* N^* \sum_{j=1}^{\infty} b_j < \pi/2. \tag{3.11}$$

Then the trivial solution of (3.7) is locally asymptotically stable (or equivalently, the positive steady state N^* of (3.6) is locally asymptotically stable).

Proof. The linear variational equation corresponding to the trivial solution of (3.7) is

$$z'(t) + az(t) + N^* \sum_{j=1}^{\infty} b_j z(t - \tau_j) = 0, \quad t > 0, \tag{3.12}$$

and its associated characteristic equation is given by

$$\lambda + a + N^* \sum_{j=1}^{\infty} b_j e^{-\lambda \tau_j} = 0. \tag{3.13}$$

It is well known that the trivial solution of (3.12) is asymptotically stable in the sense that every solution z of (3.12) corresponding to the initial function $\phi : [-\tau^*, 0] \to (-\infty, \infty)$,

$$z(s) = \phi(s), \; s \in [-\tau^*, 0], \; \phi \; \text{ is continuous on } \; [-\tau^*, 0],$$

is such that (i). $|z(t)|$ is nonincreasing in t for $t \geq 0$ and (ii). $\lim_{t\to\infty} |z(t)| = 0$ if and only if $\text{Re}\,\lambda \leq -\sigma < 0$ for some positive number σ where λ is any root of (3.13). The asymptotic stability of the trivial solution of (3.12) implies the local asymptotic stability of the trivial solution of (3.7).

We show (3.13) cannot have roots with nonnegative real parts. Suppose $\lambda = \mu + i\upsilon$ is a root of (3.13) where μ and υ are real numbers and suppose that $\mu \geq 0$. Then we have from (3.13) that

$$\mu + a = -N^* \sum_{j=1}^{\infty} b_j e^{-\mu \tau_j} \cos \upsilon \tau_j, \tag{3.14}$$

$$\upsilon = N^* \sum_{j=1}^{\infty} b_j e^{-\mu \tau_j} \sin \upsilon \tau_j. \tag{3.15}$$

It follows from (3.15) and $\mu \geq 0$ that

$$|\upsilon| \leq N^* \sum_{j=1}^{\infty} b_j,$$

which with (3.11) implies that

$$|\upsilon|\, \tau_j \leq N^* \tau^* \sum_{j=1}^{\infty} b_j < \pi/2. \tag{3.16}$$

Since $\mu + a > 0, \; \mu \geq 0$, we have from (3.14) and (3.16) that

$$-\,(\mu + a)/N^* \leq 0 \text{ and } -(\mu + a)/N^* = \sum_{j=1}^{\infty} b_j e^{-\mu \tau_j} \cos \upsilon \tau_j > 0, \tag{3.17}$$

which is impossible. Suppose now that (3.13) has a sequence of roots $\lambda_n = \mu_n + i\upsilon_n$ $(n = 1, 2, 3, \ldots)$ such that $\mu_n < 0$ and $\mu_n \to 0$ as $n \to \infty$. We now show that this is not possible. Since f defined by

$$f(\lambda) = \lambda + a + N^* \sum_{j=1}^{\infty} b_j e^{-\lambda \tau_j},$$

is analytic in λ, the zeros of f are isolated and hence any limit point of the roots of (3.13) cannot be in the finite part of the complex plane. Let us suppose that we have

$$\mu_n < 0, \ \ \upsilon_n > 0, \ \ \mu_n \to 0, \ \upsilon_n \to \infty \ \ as \ \ n \to \infty.$$

We have from (3.13) that

$$\mu_n + a = -N^* \sum_{j=1}^{\infty} b_j e^{-\mu_n \tau_j} \cos \upsilon_n \tau_j, \tag{3.18}$$

$$\upsilon_n = N^* \sum_{j=1}^{\infty} b_j e^{-\mu_n \tau_j} \sin \upsilon_n \tau_j, \tag{3.19}$$

for $n = 1, 2, 3, \ldots$, and this implies that

$$\upsilon_n \leq N^* e^{-\mu_n \tau^*} \sum_{j=1}^{\infty} b_j,$$

which leads to

$$1 \leq N^* \sum_{j=1}^{\infty} b_j \left(\frac{e^{-\mu_n \tau^*}}{\upsilon_n} \right) \to 0, \ \ \text{as } n \to \infty,$$

and this is a contradiction. Thus (3.13) can have only roots with negative real parts, and this completes the proof. ■

3.1.2 $\frac{3}{2}$-Global Stability

In this subsection we are interested in the 3/2 global stability of the positive steady state K of (3.1). Motivated by (3.1) (let $N(t) = K(y(t) + 1)$) in this section we examine

$$y'(t) = -\alpha y(t-1)[1 + y(t)], \quad \alpha > 0. \tag{3.20}$$

If we were considering (3.1) then $\alpha = r\tau$. The global stability result in this section is due to Wright [78].

We consider solutions of (3.20) which correspond to the initial condition

$$\begin{cases} y(t) = \phi(t), \text{ for } -1 \le t \le 0, \ \phi \in C[-1,0] \\ 1+\phi(t) \ge 0 \text{ for } t \in [-1,0] \text{ and } 1+\phi(0) > 0. \end{cases} \tag{3.21}$$

By the method of steps we see that (3.20), (3.21) has a solution y with $1+y(t) > 0$ for $t \ge 0$.

Theorem 3.1.2. *Let y be a solution of (3.20), (3.21). If $\alpha \le 3/2$, then* $\lim_{t\to\infty} y(t) = 0$.

Proof. If $y(t)$ is nonoscillatory, then $y(t) > 0$ or $y(t) < 0$ for some $t \ge t_0 \ge 0$. Note also from (3.20) that

$$1 + y(t) = (1 + y(t_0)) \exp\left(-\alpha \int_{t_0-1}^{t-1} y(u)du\right). \tag{3.22}$$

Without loss of generality we assume that $y(t) > 0$, since the case when $y(t) < 0$ is similar and will be omitted. Now, since $y(t) > 0$ for $t \ge t_0$, then (3.20) implies that

$$y^{'}(t) < 0, \quad \text{for } t \ge t_0 + 1.$$

Hence $y(t)$ is positive and strictly decreasing and there exists c such that

$$\lim_{t\to\infty} y(t) = c \ge 0.$$

Let $c > 0$. Then

$$\lim_{t\to\infty} y^{'}(t) = -\alpha c(1+c),$$

which leads to a contradiction with $y(t) > 0$. Therefore $c = 0$. This means that every nonoscillatory solution of (3.20) satisfies

$$\lim_{t\to\infty} y(t) = 0.$$

To complete the proof, we prove that every oscillatory solution of (3.20) satisfies

$$\lim_{t\to\infty} y(t) = 0.$$

First, we prove every oscillatory solution is bounded above. Let $t_2 > t_1 > 0$ be arbitrary two consecutive zeros of $y(t)$ such that $y(t) > 0$ for $t \in [t_1, t_2]$, and assume that $y(t)$ attains its maximum at t^*. Then $y^{'}(t^*) = 0$, which implies that $y(t^* - 1) = 0$. Letting $t_0 = t^* - 1$ in (3.22), we have

$$1 + y(t^*) = \exp\left(-\alpha \int_{t^*-2}^{t^*-1} y(\zeta)d\zeta\right) < e^{\alpha},$$

since $y(\zeta) > -1$. Hence $y(t^*) < e^{\alpha} - 1$. This proves that $y(t)$ is bounded above by $(e^{\alpha} - 1)$.

Define

$$u = \lim_{t\to\infty} \sup y(t) \quad \text{and } v = -\lim_{t\to\infty} \inf y(t). \tag{3.23}$$

Let ϵ be a positive constant such that, for $t \geq t_1 = t_1(\epsilon) > 0$,

$$-v - \epsilon < y(t) < u + \epsilon. \tag{3.24}$$

If $y(T)$ is a local maximum or minimum with $T > t_1 + 2$, then $y(T-1) = 0$, and

$$-\alpha(u+\epsilon) < (\ln(1 + y(T)) =) - \alpha \int_{T-2}^{T-1} y(\zeta)d\zeta < \alpha(v+\epsilon),$$

which leads to

$$-1 + \exp(-\alpha(u+\epsilon)) < y(T) < -1 + \exp(\alpha(v+\epsilon)). \tag{3.25}$$

From the definition of $u,\ v$ we see that there is a $T > 0$ such that $y(T)$ is a local maximum and $y(T) > u - \epsilon$, and a $T^{'} > 0$ such that $y(T^{'})$ is a local minimum and $y(T^{'}) < -v + \epsilon$. Hence,

$$u - \epsilon < \exp(\alpha(v+\epsilon)) - 1, \;\; v - \epsilon < 1 - \exp(-\alpha(u+\epsilon)). \tag{3.26}$$

Since (3.26) is true for all $\epsilon > 0$, this leads to

$$u \leq e^{\alpha v} - 1, \quad v \leq 1 - e^{-\alpha u}. \tag{3.27}$$

It follows that $v < 1$ and that, if one of u and v is zero, then so is the other, and so $\lim_{t\to\infty} y(t) = 0$. Therefore, we assume in the following that

$$u > 0, \qquad 0 < v < 1. \tag{3.28}$$

If $\alpha \leq 1$, then from (3.27), we have

$$1 + u \leq e^{v} \leq \exp(1 - e^{-u}),$$

and this implies that

$$1 + u - \exp(1 - e^{-u}) \leq 0. \tag{3.29}$$

However, since $u > 0$, we have

$$\begin{aligned}&1 + u - \exp(1 - e^{-u})\\ &= \int_0^u \int_0^{u_1} (1 - e^{-u_2}) \exp(1 - e^{-u_2} - u_2) du_2 du_1 > 0,\end{aligned}$$

a contradiction with (3.29). This proves the theorem when $\alpha \le 1$.

Now we assume that $\alpha > 1$. Let T be a maximum or minimum point such that $T > t_1 + 3$, so that $y(T-1) = 0$. For $t > 0$

$$\ln(1 + y(t)) = -\alpha \int_{T-2}^{t-1} y(\zeta) d\zeta = \alpha \int_{t-1}^{T-2} y(\zeta) d\zeta,$$

and so for $t_1 + 1 < t < T - 1$, by (3.24) we have

$$-\alpha(v + \epsilon)(T - t - 1) < \ln(1 + y(t)) < \alpha(u + \epsilon)(T - t - 1).$$

Hence,

$$\begin{aligned}&-1 + \exp\{-\alpha(v + \epsilon)(T - t - 1)\}\\ &< y(t)) < -1 + \exp\{\alpha(u + \epsilon)(T - t - 1)\}.\end{aligned} \tag{3.30}$$

Let $\tau \in [0, 1]$ be an arbitrary constant. The inequalities (3.24) and (3.30) yield

$$\begin{aligned}&\ln(1 + y(T))\\ &= -\alpha \int_{T-2}^{T-1} y(\zeta) d\zeta = -\alpha \int_{T-2}^{T-1-\tau} y(\zeta) d\zeta - \alpha \int_{T-1-\tau}^{T-1} y(\zeta) d\zeta\\ &\le \alpha(1 - \tau)(v + \epsilon) + \alpha \int_{T-1-\tau}^{T-1} (1 - e^{-\alpha(v+\epsilon)(T-t-1)}) d\zeta\\ &\le \alpha(1 - \tau)(v + \epsilon) + \alpha\tau - \frac{[1 - \exp\{-\alpha\tau(v + \epsilon)\}]}{v + \epsilon},\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}&\ln(1 + y(T))\\ &\ge -\alpha(1 - \tau)(u + \epsilon) + \alpha \int_{T-1-\tau}^{T-1} (e^{\alpha(u+\epsilon)(T-t-1)} - 1) d\zeta\\ &\ge -\alpha(1 - \tau)(u + \epsilon) + \alpha\tau + \frac{[1 - \exp\{-\alpha\tau(u + \epsilon)\}]}{u + \epsilon}.\end{aligned} \tag{3.32}$$

We can find (as before) T such that $y(T) > u - \epsilon$. Using this in (3.31) and letting $\epsilon \to 0$, we obtain

$$\ln(1+u) \leq \alpha(1-\tau)v + \alpha\tau - \frac{[1-\exp\{-\alpha\tau v\}]}{v}. \tag{3.33}$$

Letting $\tau = 1$, we get from (3.33) that

$$\ln(1+u) \leq \alpha - \frac{[1-\exp(-\alpha v)]}{v}. \tag{3.34}$$

If $\alpha v \geq -\ln(1-v)$, we may let $\tau = -\ln(1-v)/\alpha v$ in (3.33) and obtain

$$\ln(1+u) < \alpha v - (\frac{1-v}{v})\ln(1-v) - 1. \tag{3.35}$$

Next, we choose a minimum point T for which $y(T) < -v+\epsilon$. Using this in (3.32), letting $\epsilon \to 0$ and setting

$$\tau = \frac{\ln(1+u)}{\alpha u} \leq 1,$$

we obtain

$$-\ln(1-v) \leq \alpha u - \frac{1+u}{u}\ln(1+u) + 1. \tag{3.36}$$

We now claim if $1 < \alpha \leq 3/2$, $u > 0$ and $v > 0$ then

$$\ln(1+u) < v - \frac{v^2}{6}, \quad \text{and} \quad -\ln(1-v) < u + \frac{u^2}{6}. \tag{3.37}$$

The claim follows at once from (3.34) and (3.36) if we show that

$$\alpha v - 1 + e^{-\alpha v} \leq v^2 - \frac{v^3}{6}, \tag{3.38}$$

whenever

$$\alpha \leq \frac{3}{2}, \quad \alpha v < -\ln(1-v), \tag{3.39}$$

and that

$$\frac{3}{2}v + \left(\frac{1-v}{v}\right)\ln\left(\frac{1}{1-v}\right) - 1 < v - \frac{v^2}{6}, \tag{3.40}$$

and

$$\frac{3}{2}u - \left(\frac{1+u}{u}\right)\ln(1+u) + 1 < u + \frac{u^2}{6}. \tag{3.41}$$

Since $u > 0$, we have

$$\begin{aligned}&(1+u)\ln(1+u) - u\\ &= \int_0^u \left(\int_0^s \frac{d\theta}{1+\theta}\right) ds \geq \int_0^u \int_0^s (1-\theta) d\theta ds = \frac{u^2}{2} - \frac{u^3}{6},\end{aligned}$$

which is (3.41). Also, since $0 < v < 1$, we have

$$\begin{aligned}&v - (1-v)\ln(\frac{1}{1-v})\\ &= \int_0^v \left(\int_0^s \frac{d\theta}{1-\theta}\right) ds \geq \int_0^u \int_0^s (1+\theta) d\theta ds = \frac{v^2}{2} + \frac{v^3}{6},\end{aligned}$$

which is (3.40). It remains to be proved that (3.38) is true whenever (3.39) is true. Let $W = 1 - e^{-w}$. We have from (3.39) that

$$\begin{aligned}\alpha v - 1 + e^{-\alpha v} &= \int_0^{\alpha v} (1-e^{-w}) dw < \int_0^{-\ln(1-v)} (1-e^{-w}) dw\\ &= \int_0^v \frac{W}{W-1} dW\\ &< \frac{1}{1-v} \int_0^v W dW = \frac{v^2}{2(1-v)}.\end{aligned}$$

If $0 < v < 0.45$, we have

$$\frac{v^2}{2(1-v)} < 0.925v^2 < v^2(1-\frac{v}{6}),$$

and so we have (3.38) for such a v. Since $\alpha \leq 3/2$, we have

$$\begin{aligned}\alpha v - 1 + e^{-\alpha v} &= \int_0^{\alpha v} (1-e^{-w}) dw < \int_0^{3v/2} (1-e^{-w}) dw\\ &\leq \int_0^{3v/2} (w - \frac{1}{2}w^2 + \frac{1}{6}w^3) dw\\ &= \frac{9}{8}v^2 - \frac{9}{16}v^3 + \frac{27}{128}v^4,\end{aligned}$$

and the last expression is less than $v^2 - v^3/6$ provided that

$$81v^2 - 152v + 48 < 0,$$

which is equivalent to

$$\left(v - \frac{76}{81}\right)^2 < \frac{1888}{(81)^2} = \left(\frac{43.45}{81}\right)^2,$$

and this is true when

$$1 \geq v \geq \frac{(76 - 43.45)}{81} = 0.402,$$

and so certainly for $v > 0.45$. This proves the claim. Let $v_3 = v_3(u)$ be the smaller root of

$$\ln(1 + u) = v_3 - \frac{v_3^2}{6}. \tag{3.42}$$

Clearly, $v_3 > 0$. We also define v_4 by

$$-\ln(1 - v_4) = u + \frac{u^2}{6}. \tag{3.43}$$

Equation (3.43) yields

$$v_4 = 1 - \exp\{-u - \frac{u^2}{6}\} < u.$$

Hence by the claim above, we have

$$0 < v_3 < v < v_4 < u. \tag{3.44}$$

Thus,

$$\frac{1}{1 + u} \frac{du}{dv_3} = 1 - \frac{v_3}{3}, \quad \frac{1}{1 - v_4} \frac{dv_4}{du} = 1 + \frac{u}{3},$$

and so

$$\frac{dv_4}{dv_3} = (1 + u)(1 + \frac{u}{3})(1 - v_4)(1 - \frac{v_3}{3}).$$

Hence, by (3.42) and (3.43), we have

$$\ln(\frac{dv_4}{dv_3}) = \ln(1 + u) + \ln(1 + u/3) + \ln(1 - v_4)$$
$$+ \ln(1 - v_3/3) \leq \frac{3}{2}\{\ln(1 + u) - u\} < 0.$$

Then,

$$d(v_4 - v_3)/du < 0,$$

and so (note $v_4 \to 0$ and $v_3 \to 0$ as $u \to 0$) we have $v_4 < v_3$ for $u > 0$. This contradicts (3.44) and so for $1 < \alpha \le 3/2$ we must have

$$u = v = 0.$$

This completes the proof. ∎

3.1.3 Global Exponential Stability

In this subsection we discuss global exponential asymptotic stability. Motivated by (3.1) in this subsection we examine the problem,

$$x'(t) = -rx(t-\tau)[1+x(t)], \quad r,\tau \in (0,\infty). \tag{3.45}$$

We consider solutions of (3.45) which correspond to the initial condition

$$\begin{cases} x(t) = \phi(t), \text{ for } -\tau \le t \le 0, \ \phi \in C[-\tau, 0] \\ 1+\phi(t) \ge 0 \ \text{ for } \ t \in [-\tau, 0] \ \text{ and } \ 1+\phi(0) > 0. \end{cases} \tag{3.46}$$

By the method of steps we see that (3.45), (3.46) have a solution x with $1 + x(t) > 0$ for $t \ge 0$. The results in this section are adapted from [85].

Lemma 3.1.1. *If x is a nonoscillatory of (3.45), (3.46) then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose x is eventually positive, so $x'(t) < 0$ for t large. Thus $\lim_{t\to\infty} x(t) = l \ge 0$. Since x is bounded then $x'(t)$ is bounded and hence by Barbalat's Theorem (see Sect. 1.4) $\lim_{t\to\infty} x'(t) = 0$. From (3.45) we have $0 = -r\, l\, (1+l)$. Thus $l = 0$

Suppose x is eventually negative, so $x'(t) > 0$ for t large and so we have $\lim_{t\to\infty} x(t) = m \le 0$. Also $0 = -r\, m\, (m+l)$. Note $1+m \ge 0$. We need to only consider the case when $m = -1$, and in this case for t large $x'(t) < -r(1-\frac{1}{2})\,\frac{1}{2}$ which yields $\lim_{t\to\infty} x(t) = -\infty$, a contradiction since $1 + x(t) > 0$. Thus $m = 0$. ∎

We say that the trivial solution of (3.45) is globally exponentially asymptotically stable if for any solution $x(t)$ of (3.45) corresponding to a given initial function

$$\phi : [-\tau, 0] \to \mathbf{R}, \ \text{ where } \ \phi(0) > -1,$$

then there exist positive numbers $T,\ M,\ \delta$ such that

$$|x(t)| \le Me^{-\delta t}, \ \text{ for } \ t \ge T.$$

To prove the main results, we need an estimate on the lower and upper bounds of the oscillatory solution of (3.45).

Lemma 3.1.2. *If $x(t)$ is a solution of (3.45), (3.46), then there exists a number $T_0 > 0$ such that*

$$\exp(-r\tau(e^{-r\tau}-1)) - 1 \le x(t) \le e^{r\tau} - 1,\ \ t \ge T_0. \tag{3.47}$$

Proof. If $x(t)$ is a nonoscillatory solution of (3.45), then by Lemma 3.1.1 $1 + x(t) \to 1$ as $t \to \infty$ and there exists a $t_1 \ge 0$ such that (3.47) holds.

Suppose that x is an oscillatory solution and let $\{t_n\}$ be a sequence of zeros of x such that $\lim_{n\to\infty} t_n = \infty$. Let t^* be the point where x attains its local maximum. Then, from (3.45), we have

$$0 \le x^{'}(t^*) = -rx(t^* - \tau)[1 + x(t^*)],$$

and therefore $x(t^* - \tau) \le 0$, and as a consequence of this, there exists $\zeta \in [t^* - \tau, t^*]$ such that $x(\zeta) = 0$. An integration of (3.45) over $[\zeta, t^*]$ yields

$$\ln\left(x(t^*) + 1\right) = -r\int_{\zeta}^{t^*} x(t-\tau)dt \le r\int_{t^*-\tau}^{t^*} dt = r\tau,$$

so

$$x(t^*) + 1 \le e^{r\tau}. \tag{3.48}$$

Since $x(t^*)$ is an arbitrary local maximum, then

$$x(t) + 1 \le e^{r\tau} \quad \text{for} \ \ t \ge t_1 \tag{3.49}$$

where t_1 is the first zero of the oscillation solution. Let t^{**} be the point where x attains its local minimum. Then (like above) there exists $\mu \in [t^{**} - \tau, t^{**}]$ such that $x(\mu) = 0$. An integration of (3.45) over $[\mu, t^{**}]$ using (3.49) yields

$$\begin{aligned}\ln\left(x(t^{**}) + 1\right) &\ge -r\int_{\mu}^{t^{**}} (e^{r\tau} - 1)\, dt \\ &\ge -r(e^{r\tau} - 1)\int_{t^{**}-\tau}^{t^*} dt = -r\tau(e^{r\tau} - 1),\end{aligned}$$

implying

$$x(t^{**}) + 1 \ge \exp\{-r\tau(e^{r\tau} - 1)\}.$$

Since $x(t^{**})$ is an arbitrary local minimum then

$$x(t) \geq \exp\{-r\tau(e^{r\tau} - 1)\} - 1, \quad \text{for } t \geq t_1. \tag{3.50}$$

This completes the proof. ■

Note if we use the lower bound in (3.47) then we can obtain immediately that

$$1 + x(t) \leq \exp\left(r\,\tau\,(1 - \exp(-r\tau\,(e^{r\tau} - 1)))\right) \quad \text{for} \quad t \geq T_1 \geq T_0.$$

Now, we are ready to prove the main result for the global exponential stability of (3.45).

Theorem 3.1.3. *Assume that* $r,\ \tau \in (0, \infty)$ *and satisfy*

$$r\tau \exp(r\tau(1 - \exp(-r\tau(e^{-r\tau} - 1)))) < 1. \tag{3.51}$$

Then the trivial solution of (3.45), (3.46) is exponentially globally asymptotically stable.

Proof. We rewrite (3.45) in the form

$$x^{'}(t) = -a(t)x(t - \tau), \tag{3.52}$$

where $a(t) = r[1 + x(t)]$ and define u by

$$u = \sigma(t) = \int_{t_0}^{t} a(s)ds, \quad t \geq t_0$$

where t_0 is a nonnegative number. By Lemma 3.1.1 and Lemma 3.1.2, we note that $\sigma^{-1}(.)$ exists and $u(t) \to \infty$, as $t \to \infty$. Furthermore

$$\sigma(t - \tau) = u - \int_{\sigma^{-1}(u)-\tau}^{\sigma^{-1}(u)} a(s)ds, \;\; t - \tau = \sigma^{-1}\left(u - \int_{\sigma^{-1}(u)-\tau}^{\sigma^{-1}(u)} a(s)ds\right).$$

Let

$$x(t) = x(\sigma^{-1}(u)) = y(u),$$

and then y satisfies

$$\frac{dy(u)}{du} = -y(u - \eta(u)), \tag{3.53}$$

where

$$\eta(u) = \int_{\sigma^{-1}(u)-\tau}^{\sigma^{-1}(u)} a(s)ds = \int_{t-\tau}^{t} a(s)ds$$
$$\leq r\tau \exp(r\tau(1 - \exp(-r\tau(e^{-r\tau} - 1)))).$$

We rewrite (3.53) by using the mean value theorem in the form

$$\frac{dy(u)}{du} = -y(u) + (y(u) - y(u - \eta(u)))$$
$$= -y(u) + \eta(u)y'(\zeta), \quad \text{for } \zeta \in [u - \eta(u), u],$$

and for all u for which $y(u) \neq 0$,

$$\frac{d}{du}|y(u)| \leq -|y(u)| + \eta(u)\left(\sup_{s\in[u-2\eta(u),u]} |y(s)|\right)$$
$$\leq -|y(u)| + \eta^*\left(\sup_{s\in[u-2r\tau e^{r\tau},u]} |y(s)|\right), \tag{3.54}$$

where η^* is the left-hand side of (3.51). Now, since $\eta \leq \eta^* < 1$, it follows from (3.53) and (3.54) and Halanay's Lemma (see Sect. 1.4) that there exist positive numbers M and α such that

$$|y(u)| \leq Me^{-\alpha u}, \tag{3.55}$$

which is also true if $y(u) = 0$. Thus we have

$$|x(t)| \leq Me^{-\alpha \int_{t_0}^t a(s)ds}$$
$$\leq M \exp(-\alpha \exp(r\tau[1 - \exp(-r\tau(e^{r\tau} - 1)]) (t - t_0)).$$

The proof is complete. ■

Consider the logistic equation with variable delay of the form

$$N'(t) = rN(t)\left[1 - \frac{N(t - \tau(t))}{K}\right], \quad \text{for } t \geq 0, \tag{3.56}$$

where $r > 0$ and $\tau(t) < t$ and $\lim_{t\to\infty} \tau(t) = \tau_0 > 0$. Motivated by (3.56) in this section we consider

$$x'(t) = -r(1 + x(t))x(t - \tau(t)). \tag{3.57}$$

We consider solutions of (3.57) corresponding to the initial condition

$$x(s) = \phi(s),\ 1 + \phi(s) \geq 0,\ 1 + \phi(0) > 0,\ s \in [-\sup_{u>0} \tau(u), 0].$$

Theorem 3.1.4. *Let* $\tau : [0,\infty) \to [0,\infty)$ *be such that* $\tau(t) \to \tau_0 > 0$, *as* $t \to \infty$ *and*

$$r\tau_0 \exp(r\tau_0(1 - \exp(-r\tau_0(e^{-r\tau_0} - 1)))) < 1. \tag{3.58}$$

Then the trivial solution of (3.57) is exponentially asymptotic stable.

Proof. We consider solutions of (3.57) which satisfy $1 + x(t) > 0$ for $t \geq t_0$. We rewrite (3.57) in the form

$$x^{'}(t) = -r(1 + x(t))x(t - \tau_0) + r(1 + x(t))(x(t - \tau_0) - x(t - \tau(t))). \tag{3.59}$$

Our strategy in the proof is to compare (3.59) with

$$z^{'}(t) = -r(1 + z(t))z(t - \tau_0), \tag{3.60}$$

since we know from Theorem 3.1.3 and condition (3.58) that the trivial solution of (3.60) is exponentially globally asymptotically stable. By the non-linear variation of constant formula (see [64]) we can represent the solution of (3.59) in the form

$$x(t) = z(t) + r\int_{t_0}^{t}(T(t,s,x_s)U_0)(1 + x(s)(x(s - \tau_0) - x(s - \tau(s)))ds, \tag{3.61}$$

for $t \geq t_0 \geq 0$, where z denotes a solution of (3.60) with

$$z(s) = x(s), \quad s \in [-\sup_{t>0} \tau(t), 0],$$

and $T(t,s,x_s)U_0$ is a solution of

$$\frac{\partial(T(t,s,x_s)U_0)}{\partial t} = -r(T(t,s,x_s)U_0), \quad t \geq s \geq 0,$$
$$T(s,s,x_s)U_0 = x_s = x(s + \theta), \quad \theta \in [-\tau_0, 0],$$

associated with the linear variational system corresponding to (3.60). From the properties of (3.60) we see that there exist numbers $B_0,\ B_1,\ \beta > 0$ such that

$$|z(t)| \leq B_0 e^{-\beta(t-t_0)}, \quad t \geq t_0 \geq 0, \tag{3.62}$$

$$\|T(t,s,x_s)U_0\| \leq B_1 e^{-\beta(t-s)}, \quad t \geq s \geq t_0 \geq 0, \tag{3.63}$$

for sufficiently large t_0. We have from (3.61), (3.62), and (3.63),

$$|x(t)| \leq B_0 e^{-\beta(t-t_0)} + B_1 \int_{t_0}^{t} e^{-\beta(t-s)} \left|x^{'}(\zeta(s))\right| |\tau_0 - \tau(s)|\, ds, \ \ t \geq t_0, \tag{3.64}$$

where $\zeta(s)$ lies between $s - \tau_0$ and $s - \tau(s),\ s \geq t_0$. Using the boundedness of solutions of (3.57) (see Lemma 3.1.2), one can estimate $x^{'}(\zeta(s))$ and therefore

$$|x(t)| \leq B_0 e^{-\beta(t-t_0)} + B_2 \int_{t_0}^{t} e^{-\beta(t-s)} |\tau_0 - \tau(s)| \, ds, \ \ t \geq t_0, \tag{3.65}$$

for some constant $B_2 > 0$ representing an upper bound of $x^{'}(t)$ for $t \geq t_0$. An application of the Gronwall–Bellman inequality (see Sect. 1.4) to (3.65) leads to

$$|x(t)| \leq B_0 e^{\beta t_0} \exp\left(t\left(-\beta + B_2 \frac{1}{t} \int_{t_0}^{t} |\tau_0 - \tau(s)| \, ds\right)\right). \tag{3.66}$$

Since $\tau(t) \to \tau_0 > 0$ *as* $t \to \infty$, for every $\varepsilon > 0$ there exists a t_0 such that

$$\int_{t_0}^{t} |\tau_0 - \tau(s)| \, ds < \varepsilon t, \quad t \geq t_{0,}$$

and hence if t_0 is sufficiently large we have from (3.66) that

$$|x(t)| \leq B_0 e^{\beta t_0} \exp\left(t\left(-(\beta - B_2 \varepsilon)\right)\right). \tag{3.67}$$

Since ε is arbitrary the exponential asymptotic stability of the trivial solution of (3.57) follows. The proof is complete. ■

3.2 A Nonautonomous Hutchinson Model

In this section we examine the nonautonomous nonlinear delay logistic model of Hutchinson's type

$$N^{'}(t) = r(t)N(t)\left[1 - \frac{N(t-\tau)}{K}\right]. \tag{3.68}$$

3.2.1 $\frac{3}{2}$*–Uniform Stability*

Motivated by (3.68) (let $N(\tau\, t) = K(y(t) + 1)$ and $\alpha = r\,\tau$) in this section we examine the equation

$$y^{'}(t) = -\alpha(t)y(t-1)[1 + y(t)], \tag{3.69}$$

where α is a positive continuous function of t.

We consider solutions of (3.69) which correspond to the initial condition for any $t_0 \geq 0$

$$\begin{cases} y(t) = \phi(t), \text{ for } t_0 - 1 \leq t \leq t_0, \ \phi \in C[t_0 - 1, t_0] \\ 1 + \phi(t) \geq 0 \ \text{ for } \ t \in [t_0 - 1, t_0] \ \text{ and } \ 1 + \phi(t_0) > 0. \end{cases} \tag{3.70}$$

In this case

$$1 + y(t) = (1 + y(t_0)) \exp^{-\int_{t_0}^{t} \alpha(s) y(s-1) ds} > 0,$$

and so

$$y(t) > -1, \quad \text{for all } \ t \geq t_0.$$

The results in this section are adapted from [17, 70].

The zero solution of (3.69) is uniformly stable if, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\|\phi\| = \sup_{t \in [t_0 - 1, t_0]} |\phi(t)| < \delta$ imply $|y(t; t_0, \phi| < \varepsilon$ for all $t \geq t_0$ where $y(t; t_0, \phi)$ is a solution of (3.69) with the initial value ϕ at t_0.

We need the following lemma in the proof of the main result.

Lemma 3.2.1. *Suppose that there exists a constant* $\alpha_0 > 0$ *such that*

$$\int_{t-1}^{t} \alpha(s) ds \leq \alpha_0 \leq \frac{3}{2} \quad \text{for} \ \ t \geq 1. \tag{3.71}$$

Let $\eta \in (1, 2)$ *be a constant satisfying* $\alpha_0 \eta < 3/2$ *and let* $y(t)$ *be a solution of (3.69) on* $[t_0 - 1, \infty)$ *such that* $y(t_1) = 0$ *for some* $t_1 \geq t_0 + 1$ $(t_0 \geq 0)$. *Then, for any* $\rho < \eta - 1$, $|y(t)| \leq \rho$ *for* $t \in [t_0 - 1, t_1]$ *implies* $|y(t)| \leq \rho$ *for all* $t \geq t_1$.

Proof. Suppose that it is not true. Then there exists $t_2 > t_1$ such that

$$|y(t_2)| = \rho, \quad |y(t_2 + \tau)| > \rho$$

for a sufficiently small $\tau > 0$ and $|y(t)| \leq \rho$ for $t_1 \leq t \leq t_2$. We assume $y(t_2) = \rho > 0$ (since the proof in the case when $y(t_2) = -\rho$ is similar). Hence, there exists a $t_3 \in (t_2, t_2 + \tau)$ such that

$$y^{'}(t_3) > 0 \quad \text{and } y(t_3) > \rho. \tag{3.72}$$

From (3.69), it is easy to prove that there exists $t_4 > t_3$ such that $y^{'}(t_4) = 0$ and $y(t_4) > \rho$. Clearly $t_4 < t_2 + 1$ and $y(t_4 - 1) = 0$. Since $|y(t)| \leq \rho$ for all $t \in [t_0 - 1, t_2]$,

$$\left| y^{'}(t) \right| \leq \alpha(t) \left| y(t-1) \right| [1 + |y(t)|] \leq \rho \eta \alpha(t), \ t \in [t_4 - 1, t_2],$$

and hence

$$\begin{aligned} |y(t-1)| &= |y(t_4 - 1) - y(t-1)| \\ &\leq \int_{t-1}^{t_4 - 1} \rho \eta \alpha(u) du, \ \ t \in [t_4 - 1, t_2]. \end{aligned}$$

Consequently, for all $t \in [t_4 - 1, t_2]$

$$\left|y^{'}(t_2)\right| \le \min\left\{\rho\eta\alpha(t),\ \rho\eta^2\alpha(t)\int_{t-1}^{t_4-1}\alpha(u)du\right\}.$$

Thus

$$\begin{aligned}\rho = y(t_2) &= \int_{t_4-1}^{t_2} y^{'}(s)ds \\ &\le \int_{t_4-1}^{t_2}\min\left\{\rho\eta\alpha(s),\ \rho\eta^2\alpha(s)\int_{s-1}^{t_4-1}\alpha(u)du\right\}ds.\end{aligned}$$

If $\eta\int_{t_4-1}^{t_4}\alpha(s)ds \le 1$, then

$$y(t_2) \le \int_{t_4-1}^{t_2}\rho\eta\alpha(s)ds < \int_{t_4-1}^{t_4}\rho\eta\alpha(s)ds \le \rho,$$

which is a contradiction.

If $\eta\int_{t_4-1}^{t_4}\alpha(s)ds > 1$, choose $q \in (0,1)$ such that $\eta\int_{t_4-1-q}^{t_4}\alpha(s)ds = 1$. Then

$$\begin{aligned}&y(t_2)\\ &\le \int_{t_4-1}^{t_4-1+q}\rho\eta\alpha(s)ds + \int_{t_4-1+q}^{t_4}\rho\eta^2\alpha(s)\int_{s-1}^{t_4-1}\alpha(u)duds\\ &= \rho\eta^2\int_{t_4-1+q}^{t_4}\eta\int_{t_4-1}^{t_4-1+q}\alpha(s)\alpha(u)duds + \rho\eta^2\int_{t_4-1+q}^{t_4}\int_{s-1}^{t_4-1}\alpha(s)\alpha(u)duds\\ &\le \rho\eta\alpha_0 - \frac{1}{2}\rho\eta^2\int_{t_4-1+q}^{t_4} d\left(\int_{t_4-1+q}^{s}\alpha(u)du\right)^2\\ &= \rho(\eta\alpha_0 - \frac{1}{2}) < \rho(\frac{3}{2} - \frac{1}{2}) = \rho.\end{aligned}$$

This contradicts the assumption $y(t_2) = \rho$. The proof is complete. ■

Theorem 3.2.1. *Suppose that there exists a constant $\alpha_0 > 0$ such that (3.71) holds. Then the zero solution of (3.69) is uniformly stable.*

Proof. Let $\eta \in (1,2)$ be such that $\eta\alpha_0 < 3/2$. Then for every $\epsilon \in (0, \eta - 1)$ we choose a $\delta = \delta(\epsilon) > 0$ so small that

$$\rho := (1+\delta)e^{\alpha_0\delta}e^{\alpha_0((1+\delta)e^{\alpha\delta}-1)} - 1 < \epsilon.$$

Consider a solution $y(t) = y(t; t_0, \phi)$ of (3.69) with $t_0 \geq 0$ and with $\|\phi\| = \sup_{t\in[t_0-1,t_0]} |\phi(t)| < \delta$. Suppose that $|y(t_4)| > \rho$ for some $t_4 > t_0$. Then it follows from $\delta < \rho$ that there exist constants t_2 and t_3 such that

$$t_0 < t_2 < t_3 \leq t_4, \ \ |y(t_2)| = \rho, \ \ |y(t_3)| > \rho, \ \ |y(t)| < \rho,$$

for all $t \in [t_0 - 1, t_2)$, $|y(t)| > \rho$ for all $t \in (t_2, t_3]$, and $y^{'}(t_3)y(t_3) > 0$. Suppose that $y(t) > 0$ for $t_2 \leq t \leq t_3$ (the case when $y(t) < 0$ is similar so the proof is omitted). It is easy to see from Lemma 3.2.1 that there exists a $t_1 \in (t_3 - 1, t_3)$ such that $y(t_1) = 0$. For $t \in [t_0, t_0 + 1]$, we have

$$\left|[\ln(y(t) + 1)]^{'}\right| \leq \delta\alpha(t),$$

and hence

$$\begin{aligned} &|[\ln(y(t) + 1)]| \\ &\leq |[\ln(y(t_0) + 1)]| + |\ln(y(t) + 1) - \ln(y(t_0) + 1)| \\ &\leq \ln(1 + \delta) + \delta \int_{t_0}^{t} \alpha(t) \leq \ln(1 + \delta) + \delta\alpha_0, \end{aligned}$$

and therefore, we have

$$y(t) \leq (1 + \delta)e^{\delta\alpha_0} - 1,$$

and

$$y(t) \geq \frac{1}{(1 + \delta)} e^{-\delta\alpha_0} - 1.$$

That is

$$|y(t)| \leq (1 + \delta)e^{\delta\alpha_0} - 1 < \rho, \ \ t \in [t_0, t_0 + 1].$$

Similarly, for $t_0 + 1 < t < t_0 + 2$, we can show that

$$|y(t)| \leq (1 + \delta)e^{\delta\alpha_0} e^{\alpha_0((1+\delta)e^{\alpha\delta} - 1)} - 1 = \rho.$$

Therefore, we have $t_3 > t_0 + 2$ and hence, $t_1 > t_3 - 1 > t_0 + 1$. Therefore, $|y(t)| \leq \rho$ holds for $t \in [t_0 - 1, t_1]$. Thus by Lemma 3.2.1 we have $|y(t)| \leq \rho$ for all $t \geq t_1$, which contradicts the assumption that $|y(t_3)| > \rho$. Hence if $t_0 > 0$ and $\|\phi\| = \sup_{t\in[t_0-1,t_0]} |\phi(t)| < \delta$, then

$$|y(t; t_0, \phi)| \leq \rho < \varepsilon, \text{ for all } t \geq t_0.$$

The proof is complete. ∎

3.2.2 $\frac{3}{2}$-Global Stability

In this subsection we examine the $\frac{3}{2}$-global stability of (3.69). We consider the solution of (3.69) which corresponds to the initial condition (3.70). The results in this section are adapted from [68].

Before we state and prove the main results we prove the following lemma which will be used in the proof of the main results.

Lemma 3.2.2. *Let* $0 < \beta < 1/2$. *The system of inequalities*

$$\begin{cases} u \le e^{v-\beta v^2} - 1, \\ v \le 1 - e^{-u-\beta u^2}, \end{cases} \tag{3.73}$$

has a unique solution $(u, v) = (0, 0)$ *in the nonnegative quadrant* $\{u, v) : v \ge 0,$ $u \ge 0\}$.

Proof. Assume that (3.73) has another solution in the first quadrant of the $v - u$ plane besides $(0, 0)$, say (v_0, u_0). Then $u_0 > 0$ and $0 < v < 1$. Define Γ_1 to be the curve:

$$u = e^{v-\beta v^2} - 1,$$

and Γ_2 to be the curve:

$$v = 1 - e^{-u-\beta u^2}.$$

Clearly

$$\begin{aligned} \left.\frac{du}{dv}\right|_{(0,0)} &= 1, \left.\frac{d^2u}{dv^2}\right|_{(0,0)} = 1 - 2\beta, \\ \left.\frac{d^3u}{dv^3}\right|_{(0,0)} &= 1 - 6\beta, \quad \text{for } \Gamma_1, \\ \left.\frac{du}{dv}\right|_{(0,0)} &= 1, \left.\frac{d^2u}{dv^2}\right|_{(0,0)} = 1 - 2\beta, \\ \left.\frac{d^3u}{dv^3}\right|_{(0,0)} &= 12\beta^2 - 6\beta + 2, \quad \text{for } \Gamma_2. \end{aligned}$$

Hence Γ_2 lies above Γ_1 near $(0, 0)$. The existence of (v_0, u_0) implies that the curves Γ_1 and Γ_2 must intersect at a point in the first quadrant besides $(0, 0)$. Let (v_1, u_1) be the first such point, i.e. v_1 is smallest. Then the slope of Γ_1 at (u_1, v_1) is no less than the slope of Γ_2 at (u_1, v_1), i.e.

$$(1-2\beta v_1)e^{v_1-\beta v_1^2} \geq \frac{1}{1+2\beta u_1}e^{u_1+\beta u_1^2},$$

or

$$(1-2\beta v_1)(1+2\beta u_1) \geq e^{u_1-v_1+\beta(v_1^2+u_1^2)}.$$

Let

$$\phi(x) = 1 - e^{-x-\beta x^2} - x.$$

Then $\phi(0) = 0$ and $\phi'(x) < 0$, for $x > 0$, since $2\beta < 1$. Thus

$$\phi(x) < 0 \ \text{ for } x > 0 \text{ and } \ v_1 = \phi(u_1) + u_1 < u_1.$$

Then $u_1 > v_1$. Using the inequality $e^x > 1 + x$ $(x > 0)$, we have

$$1 + 2\beta(u_1 - v_1) - 4\beta^2 u_1 v_1 > 1 + u_1 - v_1 + \beta(v_1^2 + u_1^2),$$

or

$$(-1+2\beta)(u_1 - v_1) - 4\beta^2 u_1 v_1 > \beta(v_1^2 + u_1^2),$$

which is a contradiction since $0 < \beta < 1/2$. This completes the proof. ■

Lemma 3.2.3. *Assume that*

$$\int_{t-1}^{t} \alpha(s)ds \leq \frac{3}{2}, \quad \textit{for all large } t. \tag{3.74}$$

Let $y(t)$ be an oscillatory solution of (3.69), (3.70). Then $y(t)$ is bounded above and below from -1 for $t \geq 0$.

Proof. Let $t_0 > 0$ be large enough so that (3.74) holds for all $t \geq t_0$. Let t^* be a local maximum point of $y(t)$ $(t \geq t_0 + 1)$. Then $y'(t^*) = 0$ and by (3.69) $y(t^* - 1) = 0$. Integrating (3.69) from $t^* - 1$ to t^*, we have

$$1 + y(t^*) = e^{-\int_{t^*-1}^{t^*} \alpha(s)y(s-1)ds}.$$

Since $y(s-1) > -1$, by (3.74)

$$1 + y(t^*) \leq e^{\int_{t^*-1}^{t^*} \alpha(s))ds} \leq e^{3/2},$$

and $y(t^*) \le e^{3/2} - 1$. Consequently,

$$\limsup_{t\to\infty} y(t) \le e^{3/2} - 1.$$

Next, let t_* be a local minimum point of $y(t)$ $(t \ge t_0 + 3)$. Then

$$y'(t_*) = 0 \text{ and } y(t_* - 1) = 0.$$

Integrating (3.69) from $t_* - 1$ to t_* and using the fact that

$$y(s-1) \le e^{3/2} - 1,$$

we have

$$1 + y(t_*) \ge e^{\int_{t_*-1}^{t_*} \alpha(s)(1-e^{3/2})ds} = e^{-(e^{3/2}-1)\int_{t_*-1}^{t_*} \alpha(s)ds} \ge e^{-(e^{3/2}-1)\frac{3}{2}}.$$

Hence,

$$y(t_*) \ge e^{-(e^{3/2}-1)\frac{3}{2}} - 1,$$

and

$$\liminf_{t\to\infty} y(t) \ge e^{-(e^{3/2}-1)\frac{3}{2}} - 1 > -1.$$

The proof is complete. ■

The following inequalities were used previously and will be used in the following theorem. Note

$$\frac{3}{2}v + \frac{(1-v)}{v}\ln\left(\frac{1}{1-v}\right) - 1 \le v - \frac{1}{6}v^2, \ 0 \le v < 1, \tag{3.75}$$

$$\frac{3}{2} - \frac{1}{v}\left[1 - e^{-\frac{3}{2}v}\right] \le v - \frac{1}{6}v^2, \ \ 0 \le v < 1. \tag{3.76}$$

Note that if $u_1 \ge 3$, then we have

$$\frac{3}{2}u_1 - \ln(1+u_1) \le u_1 + \frac{1}{6}u_1^2.$$

For $2 < u_1 < 3$, we have

$$\frac{1}{2}u_1 < \frac{3}{2} < \frac{2}{3} + \ln 3 < \frac{1}{6}u_1^2 + \ln(1+u_1),$$

and this implies that

$$\frac{3}{2}u_1 - \ln(1+u_1) \le u_1 + \frac{1}{6}u_1^2,$$

and then from $u_1 > 2$, we have

$$\frac{3}{2}u_1 - \ln(1+u_1) \le u_1 + \frac{1}{6}u_1^2, \tag{3.77}$$

$$\frac{3}{2}u - \frac{(1+u)\ln(1+u)}{u} + 1 \le u + \frac{1}{6}u^2, \quad 0 \le u. \tag{3.78}$$

Theorem 3.2.2. *If (3.74) holds, and*

$$\int_0^\infty \alpha(t)dt = \infty, \tag{3.79}$$

then every solution $y(t)$ of (3.69), (3.70) satisfies

$$\lim_{t\to\infty} y(t) = 0. \tag{3.80}$$

Proof. First, let $y(t)$ be an nonoscillatory solution of (3.69), (3.70). Then there exists a t_0 such that $y(t)$ is of one sign for $t \ge t_0$. Without loss of generality we consider the case when $y(t) \le 0$ for $t \ge t_0$ (since the case when $y(t)$ is nonnegative is similar). Since $1+y(t) \ge 0$, by (3.69) we have $y'(t) \ge 0$. Thus $y(t)$ is increasing and

$$\lim_{t\to\infty} y(t) = -c \le 0, \quad \text{exists.}$$

Integrating (3.69) from t_0+1 to t, we get

$$\begin{aligned} &-\ln(1+y(t)) + \ln(1+y(t_0+1)) \\ &= \int_{t_0+1}^{t} \alpha(s)y(s-1)ds \le -c\int_{t_0+1}^{t} \alpha(s)ds. \end{aligned}$$

From (3.79), the right hand side tends to $-\infty$ as $t \to \infty$ unless $c = 0$. On the other hand, the left-hand side has a finite limit. Therefore $c = 0$. Hence every nonoscillatory solution $y(t)$ of (3.69) satisfies (3.80).

To complete the proof we need to prove also that every oscillatory solution satisfies (3.80). Let $y(t)$ be an oscillatory solution of (3.69). By Lemma 3.2.3 $y(t)$ is bounded above and below away from -1 for $t \ge 0$.

Let

$$u = \limsup_{t\to\infty} y(t), \quad -v = \liminf_{t\to\infty} y(t). \tag{3.81}$$

Then

$$0 \le v < 1 \text{ and } 0 \le u < \infty.$$

To complete the proof it suffices to prove that $u = v = 0$. For any ε, choose $t_0 = t_0(\varepsilon)$ such that

$$-v_1 \equiv -v - \varepsilon < y(t-1) < u + \varepsilon \equiv u_1, \quad \text{for } t \ge t_0.$$

We assume that ε is small enough so that $0 < v_1 < 1$ and t_0 is large enough so that (3.74) holds for $t \ge t_0 - 2$. Using (3.69), we have

$$y'(t) \le \alpha(t)[1 + y(t)]v_1, \quad t \ge t_0, \tag{3.82}$$

and

$$y'(t) \ge -\alpha(t)[1 + y(t)]u_1, \quad t \ge t_0. \tag{3.83}$$

Let $\{t_n^*\}$ be an increasing sequence such that $t_n^* \ge t_0$, $y'(t_n^*) = 0$,

$$\lim_{n\to\infty} t_n^* = \infty \text{ and } \lim_{n\to\infty} y(t_n^*) = u.$$

By (3.69), $y(t_n^* - 1) = 0$. For $t \in [t_n^* - 1, t_n^*]$, we can integrate (3.82) from $t-1$ to $t_n^* - 1$ and get

$$-\ln[1 + y(t-1)] \le v_1 \int_{t-1}^{t_n^*-1} \alpha(s)ds,$$

or

$$y(t-1) \ge -1 + e^{-v_1 \int_{t-1}^{t_n^*-1} \alpha(s)ds}, \quad \text{for } t \in [t_n^* - 1, t_n^*].$$

By (3.69), it follows that

$$y'(t) \le \alpha(t)[1 + y(t)]\left[1 - e^{-v_1 \int_{t-1}^{t_n^*-1} \alpha(s)ds}\right], \quad t \in [t_n^* - 1, t_n^*].$$

Combining this with (3.82), we have

$$(\ln[1 + y(t)])' \le \min\left\{\alpha(t)v_1,\ \alpha(t)\left[1 - e^{-v_1 \int_{t-1}^{t_n^*-1} \alpha(s)ds}\right]\right\}, \tag{3.84}$$

for $t \in [t_n^* - 1, t_n^*]$. We prove that

$$\ln[1 + y(t_n^*)] \le v_1 - \frac{1}{6} v_1^2.$$

There are two possibilities.

Case 1. $\int_{t_n^*-1}^{t_n^*} \alpha(s)ds \le -\frac{\ln(1-v_1)}{v_1}$.

Then by (3.84) we have

$$\begin{aligned}
\ln[1 + y(t_n^*)] &\le \int_{t_n^*-1}^{t_n^*} \alpha(t) \left[1 - e^{-v_1 \int_{t-1}^{t_n^*-1} \alpha(s)ds}\right] dt \\
&= \int_{t_n^*-1}^{t_n^*} \alpha(t) \left[1 - e^{-v_1 \int_{t-1}^{t} \alpha(s)ds + v_1 \int_{t_n^*-1}^{t} \alpha(s)ds}\right] dt \\
&\le \int_{t_n^*-1}^{t_n^*} \alpha(t) \left[1 - e^{-\frac{3}{2} v_1} e^{v_1 \int_{t_n^*-1}^{t} \alpha(s)ds}\right] dt \\
&= \int_{t_n^*-1}^{t_n^*} \alpha(t)dt - e^{-\frac{3}{2} v_1} \int_{t_n^*-1}^{t_n^*} \alpha(t) e^{v_1 \int_{t_n^*-1}^{t} \alpha(s)ds} dt \\
&= \int_{t_n^*-1}^{t_n^*} \alpha(t)dt - e^{-\frac{3}{2} v_1} \frac{1}{v_1} \left[e^{v_1 \int_{t_n^*-1}^{t_n^*} \alpha(s)ds} - 1\right] \\
&= \int_{t_n^*-1}^{t_n^*} \alpha(t)dt - \frac{1}{v_1} e^{-v_1 \left(\frac{3}{2} - \int_{t_n^*-1}^{t_n^*} \alpha(s)ds\right)} \left[1 - e^{-v_1 \int_{t_n^*-1}^{t_n^*} \alpha(s)ds}\right].
\end{aligned}$$

The function

$$\phi(x) = x - \frac{1}{v_1} e^{-v_1(\frac{3}{2} - x)} (1 - e^{-v_1 x}),$$

is increasing for $0 \le x \le 3/2$. Thus for

$$\int_{t_n^*-1}^{t_n^*} \alpha(t)dt \le -\frac{\ln(1 - v_1)}{v_1} \le \frac{3}{2},$$

we have

$$\begin{aligned}
\ln[1 + y(t_n^*)] &\le -\frac{\ln(1 - v_1)}{v_1} - \frac{1}{v_1} e^{-v_1 \left(\frac{3}{2} + \frac{\ln(1-v_1)}{v_1}\right)} \left[1 - e^{\ln(1-v_1)}\right] \\
&= -\frac{\ln(1 - v_1)}{v_1} - e^{-v_1 \left(\frac{3}{2} + \frac{\ln(1-v_1)}{v_1}\right)}.
\end{aligned}$$

Using the fact that $e^{-x} > 1 - x$ for $x > 0$ and (3.75) we have

$$\ln[1 + y(t_n^*)] \le \frac{3}{2}v_1 - \frac{(1 - v_1)\ln(1 - v_1)}{v_1} - 1 \le v_1 - \frac{1}{6}v_1^2.$$

For

$$\int_{t_n^*-1}^{t_n^*} \alpha(t)dt \le \frac{3}{2} < -\frac{\ln(1 - v_1)}{v_1},$$

we have

$$\ln[1 + y(t_n^*)] \le \int_{t_n^*-1}^{t_n^*} \alpha(t)dt - \frac{1}{v_1}\left[e^{-\frac{3}{2}v_1} e^{v_1 \int_{t_n^*-1}^{t_n^*} \alpha(s)ds} - e^{-\frac{3}{2}v_1}\right].$$

The function

$$f(x) = x - \frac{1}{v_1} e^{-\frac{3}{2}v_1 e^{v_1 x}},$$

is increasing for $0 \le x \le \frac{3}{2}$. Thus by (3.76), we have

$$\ln[1 + y(t_n^*)] \le \frac{3}{2} - \frac{1}{v_1}\left[1 - e^{-\frac{3}{2}v_1}\right] \le v_1 - \frac{1}{6}v_1^2.$$

Case 2.

$$-\frac{\ln(1 - v_1)}{v_1} \le \int_{t_n^*-1}^{t_n^*} \alpha(s)ds \le \frac{3}{2}.$$

Choose $\tau \in (0, 1)$ such that

$$\int_{t_n^*-\tau}^{t_n^*} \alpha(s)ds = -\frac{\ln(1 - v_1)}{v_1}.$$

Then by (3.84) and (3.74),

$$\begin{aligned}
&\ln[1 + y(t_n^*)] \\
&\le \int_{t_n^*-1}^{t_n^*-\tau} \alpha(s)v_1 ds + \int_{t_n^*-\tau}^{t_n^*} \alpha(t)\left[1 - e^{-v_1 \int_{t-1}^{t_n^*-1} \alpha(s)ds}\right]dt \\
&\le v_1 \int_{t_n^*-1}^{t_n^*-\tau} \alpha(s)ds + \int_{t_n^*-\tau}^{t_n^*} \alpha(s)ds
\end{aligned}$$

$$
\begin{aligned}
&-e^{-\frac{3}{2}v_1}\int_{t_n^*-\tau}^{t_n^*}\alpha(t)e^{v_1\int_{t_n^*-1}^{t}\alpha(s)ds}dt \\
&= v_1\int_{t_n^*-1}^{t_n^*-\tau}\alpha(s)ds + \int_{t_n^*-\tau}^{t_n^*}\alpha(s)ds \\
&\quad -\frac{1}{v_1}e^{-v_1(\frac{3}{2})}\left[e^{v_1\int_{t_n^*-1}^{t_n^*}\alpha(s)ds} - e^{v_1\int_{t_n^*-1}^{t_n^*-\tau}\alpha(s)ds}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= v_1\int_{t_n^*-1}^{t_n^*-\tau}\alpha(s)ds + \int_{t_n^*-\tau}^{t_n^*}\alpha(s)ds \\
&\quad -\frac{1}{v_1}e^{-v_1\left(\frac{3}{2}-\int_{t_n^*-1}^{t_n^*}\alpha(s)ds\right)}\left[1 - e^{-v_1\int_{t_n^*-\tau}^{t_n^*}\alpha(s)ds}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= v_1\int_{t_n^*-1}^{t_n^*-\tau}\alpha(s)ds + \int_{t_n^*-\tau}^{t_n^*}\alpha(s)ds - e^{-v_1\left(\frac{3}{2}-\int_{t_n^*-1}^{t_n^*}\alpha(s)ds\right)} \\
&= v_1\int_{t_n^*-1}^{t_n^*}\alpha(s)ds + (1-v_1)\int_{t_n^*-\tau}^{t_n^*}\alpha(s)ds - e^{-v_1\left(\frac{3}{2}-\int_{t_n^*-1}^{t_n^*}\alpha(s)ds\right)} \\
&\leq \frac{3}{2}v_1 - \frac{(1-v_1)\ln(1-v_1)}{v_1} - 1.
\end{aligned}
$$

Since,

$$
g(s) = v_1 x - e^{-v_1(\frac{3}{2}-x)},
$$

is increasing for $0 \leq x \leq 3/2$ we have

$$
\ln[1 + y(t_n^*)] \leq v_1 - \frac{1}{6}v_1^2.
$$

Letting $n \to \infty$ and $\varepsilon \to 0$, we have

$$
\ln[1 + u] \leq v - \frac{1}{6}v^2.
$$

or

$$
u \leq e^{v-\frac{1}{6}v^2} - 1. \tag{3.85}
$$

Next, let $\{s_n^*\}$ be an increasing sequence such that $s_n^* \geq t_0 + 1$, $y'(s_n^*) = 0$, $\lim_{n\to\infty} y(s_n^*) = -v$ and $\lim_{n\to\infty} s_n^* = \infty$. We show that

$$-\ln[1 + y(s_n^*)] \leq u_1 + \frac{1}{6}u_1^2.$$

For $t \in [s_n^* - 1, s_n^*]$, integrating (3.83) from $t - 1$ to $s_n^* - 1$, we have

$$\ln[1 + y(t-1)] \leq u_1 \int_{t-1}^{s_n^*-1} \alpha(s)ds,$$

or

$$y(t-1) \leq -1 + \exp\left(u_1 \int_{t-1}^{s_n^*-1} \alpha(s)ds\right).$$

By (3.69)

$$[\ln(1 + y(t))]' \geq -\alpha(t)\left[e^{u_1 \int_{t-1}^{s_n^*-1} \alpha(s)ds} - 1\right], \text{ for } t \in [s_n^* - 1, s_n^*]. \tag{3.86}$$

There are three subcases to consider.

Case (I). $\int_{s_n^*-1}^{s_n^*} \alpha(s)ds \leq 1.$

Integrating (3.83) from $s_n^* - 1$ to s_n^*, we have

$$-\ln(1 + y(s_n^*)) \leq u_1 \int_{s_n^*-1}^{s_n^*} \alpha(s)ds \leq u_1 \leq u_1 + \frac{1}{6}u_1^2.$$

Case (II).

$$1 < \int_{s_n^*-1}^{s_n^*} \alpha(s)ds \leq 3/2 - \frac{\ln(1 + u_1)}{u_1}.$$

Clearly $u_1 > 2$ in this case. We have by (3.77) that

$$-\ln(1 + y(s_n^*)) \leq u_1 \int_{s_n^*-1}^{s_n^*} \alpha(s)ds \leq \frac{3}{2}u_1 - \ln(1 + u_1) \leq u_1 + \frac{1}{6}u_1^2.$$

Case (III).

$$3/2 - \frac{\ln(1 + u_1)}{u_1} < \int_{s_n^*-1}^{s_n^*} \alpha(s)ds \leq 3/2.$$

Choose $\tau \in (0,1)$, such that

$$\int_{s_n^*-1}^{s_n^*-\tau} \alpha(s)ds = 3/2 - \frac{\ln(1+u_1)}{u_1}.$$

Then by (3.83) and (3.86), we have

$$-[\ln(1+y(t))]' \le \min\left\{\alpha(t)u_1, \alpha(t)\left[e^{u_1\int_{t-1}^{s_n^*-1}\alpha(s)ds} - 1\right]\right\}.$$

Consequently

$$\begin{aligned}
&-\ln[1+y(s_n^*)] \\
&\le \int_{s_n^*-1}^{s_n^*-\tau} \alpha(s)u_1 ds + \int_{s_n^*-\tau}^{s_n^*} \alpha(t)\left[e^{u_1\int_{t-1}^{s_n^*-1}\alpha(s)ds} - 1\right]dt \\
&\le u_1\left(\frac{3}{2} - \frac{\ln(u_1+1)}{u_1}\right) + e^{\frac{3}{2}u_1}\int_{s_n^*-\tau}^{s_n^*} \alpha(t)e^{-u_1\int_{s_n^*-1}^{t}\alpha(s)ds}dt \\
&- \int_{s_n^*-\tau}^{s_n^*} \alpha(t)dt = u_1\left(\frac{3}{2} - \frac{\ln(u_1+1)}{u_1}\right) \\
&+\frac{1}{u_1}\left[e^{u_1\left(\frac{3}{2}-\int_{s_n^*-1}^{s_n^*-\tau}\alpha(s)d\right)} - e^{u_1\left(\frac{3}{2}-\int_{s_n^*-1}^{s_n^*}\alpha(s)d\right)}\right] - \int_{s_n^*-\tau}^{s_n^*}\alpha(t)dt. \\
&\le u_1\left(\frac{3}{2} - \frac{\ln(u_1+1)}{u_1}\right) \\
&+\frac{1}{u_1}\left[1 + u_1 - 1 - u_1\left(\frac{3}{2} - \int_{s_n^*-1}^{s_n^*}\alpha(s)d\right)\right] - \int_{s_n^*-\tau}^{s_n^*}\alpha(t)dt
\end{aligned}$$

due to the choice of τ and since $e^x \ge 1 + x$ for $x \ge 0$. Thus

$$\begin{aligned}
&-\ln[1+y(s_n^*)] \\
&\le u_1\left(\frac{3}{2} - \frac{\ln(u_1+1)}{u_1}\right) + 1 - \frac{3}{2} + \int_{s_n^*-1}^{s_n^*}\alpha(t)dt - \int_{s_n^*-\tau}^{s_n^*}\alpha(t)dt \\
&= u_1\left(\frac{3}{2} - \frac{\ln(u_1+1)}{u_1}\right) - \frac{1}{2} + \int_{s_n^*-1}^{s_n^*-\tau}\alpha(t)dt \\
&\le 1 - \ln(1+u_1) + \frac{3}{2}u_1 - \frac{\ln(u_1+1)}{u_1} \\
&= 1 - \frac{(1+u_1)\ln(1+u_1)}{u_1} + \frac{3}{2}u_1 \le u_1 + \frac{1}{6}u_1^2
\end{aligned}$$

by (3.78). Thus we have shown that

$$-\ln[1 + y(s_n^*)] \le u_1 + \frac{1}{6}u_1^2.$$

Letting $n \to \infty$ and $\varepsilon \to 0$, we have

$$-\ln(1 - v) \le u + \frac{1}{6}u^2.$$

or

$$1 - v \ge e^{-u - \frac{1}{6}u^2}.$$

Since u, v satisfy the inequalities in (3.73) with $\beta = 1/6$, by Lemma 3.2.2 $u = v = 0$. This completes the proof. ∎

In the following we discuss the periodic delay logistic equation

$$N'(t) = r(t)N(t)\left[1 - \frac{N(t - n\tau)}{K(t)}\right], \tag{3.87}$$

with the assumption that n is a positive integer, τ is a positive constant, r, K are positive continuous periodic functions of period τ. With (3.87) assume

$$\begin{cases} N(t) = \varphi(t), \text{ for } n\tau < t < 0, \\ \varphi \in C[[-n\tau, 0], \mathbf{R}^+], \ \varphi(0) > 0. \end{cases} \tag{3.88}$$

We note that if (3.87) has a periodic solution of period τ, then such a solution is also a periodic solution of the periodic logistic equation

$$N'(t) = r(t)N(t)\left[1 - \frac{N(t)}{K(t)}\right]. \tag{3.89}$$

Conversely if (3.89) has a periodic solution of period τ, then such a solution is also a periodic solution of the periodic logistic equation (3.87). The unique periodic solution N^* of (3.89) is given by

$$\begin{aligned} N^* &= \left[\int_0^\infty \frac{r(t-s)}{K(t-s)} \exp\left(-\int_0^s r(t-u)du\right) ds\right]^{-1} \\ &= \frac{1 - \exp(-\int_0^\tau r(s)ds)}{\int_0^\tau \frac{r(t-s)}{K(t-s)} \exp\left(-\int_0^s r(t-u)du\right) ds}. \end{aligned}$$

In the following, we establish some sufficient conditions for the global stability of (3.87). The result is adapted from [86].

Theorem 3.2.3. *Assume that r and K are positive continuous periodic functions of period $\tau > 0$. If*

$$\int_0^{n\tau} r(t)dt \leq \frac{3}{2}, \tag{3.90}$$

then the periodic delay equation (3.87) has a unique periodic solution $N^*(t)$ and all other solutions $N(t)$ of (3.87), (3.88) satisfies

$$\lim_{t\to\infty} [N(t) - N^*(t)] = 0, \tag{3.91}$$

Proof. Let $N(t)$ be any positive solution of (3.87), (3.88) and define v such that

$$\ln[1 + v(t)] = \ln N(t) - \ln N^*(t). \tag{3.92}$$

and note that v is given by

$$\frac{dv}{dt} = -a(t)[1 + v(t)]v(t - n\tau), \tag{3.93}$$

where

$$a(t) = \frac{r(t)N^*(t)}{K(t)}. \tag{3.94}$$

It is sufficient to prove that the solution of (3.93) with the initial condition

$$1 + v(s) \geq 0,\ 1 + v(0) > 0,\ \text{for } s \in [-n\tau, 0],$$

satisfies

$$\lim_{t\to\infty} v(t) = 0. \tag{3.95}$$

We let

$$w = \sigma(t) = \int_{t_0}^{t} a(s)ds, \tag{3.96}$$

where t_0 is any nonnegative number and note that $w \to \infty$, as $t \to \infty$ and $\sigma^{-1}(t)$ exists. Also

$$\sigma(t - n\tau) = \int_{t_0}^{t-n\tau} a(s)ds = w - \int_{\sigma^{-1}(w)-n\tau}^{\sigma^{-1}(w)} a(s)ds,$$

and hence

$$(t - n\tau) = \sigma^{-1}(w - \int_{\sigma^{-1}(w)-n\tau}^{\sigma^{-1}(w)} a(s)ds). \tag{3.97}$$

If we define

$$v(t) = v(\sigma^{-1}(w)) = z(w), \tag{3.98}$$

then we have from (3.93), (3.97), and (3.98) that

$$\frac{dz(w)}{dw} = -[1 + z(w)]z(w - \eta(w)), \tag{3.99}$$

where

$$\eta(w) = \int_{\sigma^{-1}(w)-n\tau}^{\sigma^{-1}(w)} a(s)ds = \int_{t-n\tau}^{t} a(s)ds = \int_0^{n\tau} a(s)ds. \tag{3.100}$$

From the fact that N^* is a positive periodic solution of (3.89) of period τ we have

$$0 = \int_0^{n\tau} \frac{(N^*)'(s)}{N^*(s)} ds = \int_0^{n\tau} r(s)ds - \int_0^{n\tau} a(s)ds,$$

and hence

$$\int_0^{n\tau} r(s)ds = \int_0^{n\tau} a(s)ds.$$

Thus (3.99) simplifies to

$$\frac{dz(w)}{dw} = -[1 + z(w)]z(w - \int_0^{n\tau} r(s)ds), \tag{3.101}$$

which is the familiar autonomous delay logistic equation. Now, if (3.90) holds, then by Theorem 3.1.2 (here in Sect. 3.1.2, $\alpha = 1 \,.\, \int_0^{n\tau} r(s)ds$) we have $\lim_{w\to\infty} z(w) = 0$ and then by (3.96) it follows that $\lim_{t\to\infty} v(t) = 0$. The proof is complete. ■

3.2.3 Global Exponential Stability

Motivated by (3.68) in this section we consider

$$y'(t) = -r(t)(1 + y(t))y(t - \tau) \tag{3.102}$$

where $\tau > 0$. We are interested in solutions of (3.102) corresponding to the usual initial condition ϕ of the form

$$1 + \phi(s) \geq 0,\ 1 + \phi(0) > 0,\ \ s \in [-\tau, 0].$$

Theorem 3.2.4. *Let r be a non-negative continuous function defined on* $[0, \infty)$ *such that*

$$\int_{t_0}^{\infty} r(s)ds = \infty \ \text{ for any } \ t_0 \geq 0, \tag{3.103}$$

and

$$\lim_{t\to\infty} \int_{t-\tau}^{t} r(s)ds = r^*.$$

If

$$r^* \exp(r^*(1 - \exp(-r^*(e^{-r^*} - 1)))) < 1,$$

then the trivial solution of (3.102) is exponentially globally asymptotically stable.

Proof. Let $x(t)$ be a solution of (3.102). As in the proof of Theorem 3.1.3 we introduce the variables u and δ where (here t_0 is a nonnegative number)

$$u = \delta(t) = \int_{t_0}^{t} r(s)ds,$$

and

$$x(t) = x(\delta^{-1}(u)) = z(u),$$

so that

$$\frac{dz}{du} = -(1 + z(u))z(u - \delta_*(u)),$$

where

$$\delta_*(u) = \int_{t-\tau}^{t} r(s)ds.$$

Since

$$\lim_{u\to\infty} \delta_*(u) = \lim_{t\to\infty} \int_{t-\tau}^{t} r(s)ds = r^*,$$

the conclusion follows from Theorem 3.1.4. The proof is complete. ∎

Motivated from (3.56) we consider

$$x'(t) = -r(t)(1 + x(t))x(t - \tau(t)). \tag{3.104}$$

We consider solutions of (3.104) corresponding to the initial condition ϕ of the form

$$1 + \phi(s) \geq 0,\ 1 + \phi(0) > 0, \quad s \in [-\sup_{u>0} \tau(u), 0].$$

Theorem 3.2.5. *Assume the following.*

(i) *r is a nonnegative continuous function defined for* $t \geq 0$ *such that*

$$\int^{\infty} r(s)ds = \infty, \text{ and } \int^{\infty} r(s)e^{-as}ds < \infty, \text{ for any } a > 0; \tag{3.105}$$

(ii) τ *is a continuous real value for* $t \geq 0$ *such that there exists a positive constant* τ_0 *satisfying*

$$\int^{\infty} |\tau(s) - \tau_0|\, r(s)ds < \infty, \text{ and } \lim_{t\to\infty} \int_{t-\tau_0}^{t} r(s)ds = r^*.$$

If

$$r^* \exp(r^*(1 - \exp(-r^*(e^{-r^*} - 1)))) < 1, \tag{3.106}$$

then the trivial solution of (3.104) is globally asymptotically stable.

Proof. We rewrite (3.104) in the form

$$x'(t) = -r(t)(1+x(t))x(t-\tau_0)+r(t)(1+x(t))(x(t-\tau_0)-x(t-\tau(t)), \tag{3.107}$$

and compare it with

$$y'(t) = -r(t)(1 + y(t))y(t - \tau_0), \tag{3.108}$$

since we know that from Theorem 3.2.4 that the trivial solution of (3.108) is exponentially globally asymptotically stable. The variational system associated with (3.108) is

$$\frac{d\psi(t)}{dt} = -r(t)y(t - \tau_0)\psi(t) - r(t)(1 + y(t))\psi(t - \tau_0), \tag{3.109}$$

where y denotes any solution of (3.108). Since $y(t) \to 0$, as $t \to \infty$ exponentially, we have immediately from (3.105) (second condition)

$$\int^{\infty} r(s)y(s)ds < \infty,$$

implying

$$\lim_{t\to\infty}\int_{t-\tau_0}^{t} r(s)y(s)ds = 0. \tag{3.110}$$

We now let

$$A(t) = r(t)y(t-\tau_0),$$
$$z(t) = \psi(t)\exp\left(\int_T^t A(s)ds\right), \quad \text{for } t \geq T > t_0.$$

Note that (3.109) simplifies to

$$z'(t) = -Q(t)z(t-\tau_0), \tag{3.111}$$

where

$$Q(t) = r(t)(1+y(t))\exp\left(\int_{t-\tau_0}^{t} A(s)ds\right); \tag{3.112}$$

since $y(t)\to 0$, and

$$\exp\left(\int_{t-\tau_0}^{t} A(s)ds\right)\to 0, \quad \text{as } t\to\infty,$$

we have

$$\lim_{t\to\infty}\int_{t-\tau_0}^{t} Q(s)ds = r^*. \tag{3.113}$$

It follows from (3.113) and Theorem 3.1.3 (see the proof in Theorem 3.1.3) that the trivial solution of (3.111) is exponentially globally asymptotically stable. We have from the nonlinear variation of constant formula

$$y(t) = x(t) + \int_t^{\infty}(T(t,s,x_s)X_0)r(s)$$
$$\times(1+x(s))(x(s-\tau_0)-x(s-\tau(s)))ds, \tag{3.114}$$

where x is any solution of (3.104). By the boundedness of all solutions of (3.104) we have for some constant K_1,

$$\left|\int_t^{\infty}(T(t,s,x_s)X_0)r(s)(1+x(s))(x(s-\tau_0)-x(s-\tau(s)))ds\right|$$
$$\leq K_1\int_t^{\infty}\|(T(t,s,x_s)X_0)\|\,|r(s)|\left|x'(\zeta(s))\right|\,|\tau_0-\tau(s)|\,ds$$
$$\leq K_2\int_t^{\infty} r(s)\,|\tau_0-\tau(s)|\,ds\to 0, \quad \text{as } t\to\infty,$$

where $\zeta(s)$ lies between $s - \tau_0$ and $s - \tau(s),\ s \geq t_0$ and K_2 is a positive number such that

$$\|(T(t,s,x_s)X_0)\|\,|r(s)|\left|x^{'}(\zeta(s))\right| \leq K_2, \text{ for some } t \geq s \geq 0.$$

The conclusion follows from (3.114) and (3.105). This completes the proof. ■

3.3 A Generalized Logistic Model

In this section we consider the generalized model

$$N^{'}(t) = r(t)N(t)f\left(1 - \frac{N(t-\tau(t))}{K}\right). \tag{3.115}$$

Motivated by (3.115) in this section we consider

$$y^{'}(t) = r(t)(1+y(t))f(-y(t-\tau(t))) \tag{3.116}$$

where r and τ are continuous functions defined on $[0,\infty)$ such that $r(t) > 0$, $0 \leq \tau(t) < \tau$ (let $\tau_0 \equiv \sup_{t>0}\tau(t)$), f is a continuous function on $(-\infty,\infty)$ such that $yf(y) > 0$ for $y \neq 0$. The results in this section are adapted from [85].

Under the standard type of initial condition

$$1+\phi(s) \geq 0, 1+\phi(0) > 0 \ \text{ for } \ s \in [-\sup_{t>0}\tau(t), 0]),$$

we see that the solutions of (3.116) satisfy $1 + y(t) > 0$ for $t \geq 0$.

Lemma 3.3.1. *Assume that*

$$\int_0^\infty r(s)ds = \infty.$$

Then every solution of (3.116) is either oscillatory or tends to zero as $t \to \infty$ monotonically.

Proof. Assume y is an nonoscillatory solution of (3.116) and suppose that

$$y(t) > 0,\ y(t-\tau(t)) > 0,\ \text{for } t \geq T > 0.$$

From (3.116), since $yf(y) > 0$ for $y \neq 0$, we have $y^{'}(t) < 0$ for $t \geq T$, so

$$\lim_{t\to\infty} y(t) = \alpha \geq 0. \tag{3.117}$$

Suppose $\alpha > 0$. Then $y(t) \geq \alpha$ and $-y(t) \leq -\alpha$ for $t \geq T$. Let

$$-m = \sup_{t \geq T} f(-y(t - \tau(t)).$$

It follows from (3.116) that

$$y^{'}(t) \leq -mr(t)(1 + y(t)) \leq -mr(t)(1 + \alpha),$$

so

$$y(t) - y(T) \leq -m(1 + \alpha) \int_T^t r(s)ds \to -\infty, \text{ as } t \to \infty,$$

showing that it becomes negative for t sufficiently large, and this contradiction implies that $\alpha = 0$. The convergence to zero of an eventually negative solution of (3.116) can be treated similarly and is omitted. The proof is complete. ■

Lemma 3.3.2. *Assume that*

$$\int_{t-\tau(t)}^{t} r(s)ds, \quad \text{is bounded for } t > 0. \tag{3.118}$$

Then every oscillatory solution of (3.116) is bounded for $t \geq 0$, *and if* $y(t_k)$ *is a local maximum then*

$$y(t_k) \leq \exp\left(M \int_{t_k-\tau(t_k)}^{t_k} r(s)ds\right) - 1, \tag{3.119}$$

where

$$M := \sup_{y > -1} f(-y) = \sup_{-y < 1} f(-y).$$

Proof. Let $\{J_k\}$ denote a sequence of nonoverlapping intervals on $[0,\infty)$ such that y is a zero at the end-points of any J_k and y is of the same sign in the interior of J_k. Let t_k denote a typical local maximum for y. This means that $y^{'}(t_k) = 0$, and this implies that

$$f(-y(t_k - \tau(t_k)) = 0, \quad \text{which leads to} \quad y(t_k - \tau(t_k)) = 0.$$

Assuming that $y > 0$ on $(t_k - \tau(t_k), t_k)$ an integration of (3.116) on $[t_k - \tau(t_k), t_k]$ leads to

$$\ln(1 + y(t_k)) = \int_{t_k-\tau(t_k)}^{t_k} r(s) f(-y(s - \tau(s))ds \leq M \int_{t_k-\tau(t_k)}^{t_k} r(s)ds,$$

from which (3.119) follows. This completes the proof. ■

Theorem 3.3.1. *The trivial solution of (3.116) is (locally) uniformly stable if*

$$\int_{t-\tau(t)}^{t} r(s)ds \to 0, \text{ as } t \to \infty. \tag{3.120}$$

Proof. Let y denote any solution of (3.116). Let $\varepsilon > 0$ be given and

$$M(\varepsilon) = \sup\{|f(-y)|\,;\,|y| \le \varepsilon\}.$$

There exists a $T(\varepsilon) > 0$ satisfying

$$\int_{t-\tau(t)}^{t} r(s)ds < \frac{\varepsilon}{2(1+\varepsilon)M(\varepsilon)}, \quad \text{for } t \ge T(\varepsilon). \tag{3.121}$$

We show that for any

$$\phi : [t_0 - \tau_0, t_0] \to \mathbf{R}, t_0 \ge T(\varepsilon), \|\phi\| = \sup_{t\in[t_0-\tau_0,t_0]} |\phi(t)| < \frac{1}{2}\varepsilon,$$

$$|y(t;t_0,\phi)| < \varepsilon, \qquad \text{for all } t \ge t_0. \tag{3.122}$$

Suppose that (3.122) does not hold. Then there exists a solution $y(t) = y(t;t_0,\phi)$ with $t_0 > T(\varepsilon)$ and $\|\phi\| = \sup_{t\in[t_0-\tau_0,t_0]} |\phi(t)| < \frac{1}{2}\varepsilon$ satisfying $|y(t_3)| \ge \varepsilon$ for $t_3 > t_0$. Let $v(y) = y^2$,

$$t_2 = \inf\{t > t_0 : |y(t)| > \varepsilon\}, \quad t_1 = \inf\{t < t_2 : |y(t)| = \frac{\varepsilon}{2}\}.$$

Note $v(y(t_1)) = \frac{1}{4}\varepsilon^2$, $v(y(t_2)) = \varepsilon^2$, $\frac{1}{4}\varepsilon^2 < v(t) < \varepsilon^2$ for $t \in (t_1, t_2)$ and $\frac{dv}{dt}(y(t))|_{t=t_2} > 0$.

We claim $t_1 \ge t_2 - \tau(t_2)$. Suppose $t_1 < t_2 - \tau(t_2)$. Now

$$0 < \frac{dv}{dt}(y(t))_{t=t_2} = 2\,y(t_2)\,r(t_2)\,[1 + y(t_2)]\,f(-y(t_2 - \tau(t_2))). \tag{3.123}$$

However $y(t_2)\,y(t_2-\tau(t_2)) > 0$ since $t_1 < t_2-\tau(t_2) < t_2$, so we have a contradiction with (3.123). Thus $t_1 \ge t_2 - \tau(t_2)$.

On integrating (3.116), we have

$$\frac{1}{2}\varepsilon = |y(t_2)| - |y(t_1)| \le \int_{t_1}^{t_2} r(s)(1 + |y(s)|)\,|f(-y(s - \tau(s)))|\,ds$$

$$\le (1+\varepsilon)M(\varepsilon) \int_{t_2-\tau(t_2)}^{t_2} r(s)ds < \frac{1}{2}\varepsilon, \tag{3.124}$$

which is a contradiction and hence (3.122) holds. It is well known that solutions of (3.116) depend continuously on the initial date, from which it follows that for any $t_0 \in [0, T(\varepsilon)]$, there exists a $\delta(\varepsilon)$ such that if $\phi : [t_0 - \tau_0, t_0] \to \mathbf{R}$ and $\|\phi\| < \delta(\varepsilon)$, then

$$|y(t; t_0, \phi)| < \frac{1}{2}\varepsilon. \tag{3.125}$$

Combining (3.122) together with (3.125), we derive the result and this completes the proof. ∎

Theorem 3.3.2. *Assume that (3.120) holds and*

$$\int_0^\infty r(s)ds = \infty. \tag{3.126}$$

Then the trivial solution of (3.116) is globally attractive.

Proof. It follows from Theorem 3.3.1 that the trivial solution of (3.116 is uniformly (locally) stable. We now show that every solution of (3.116) approaches the trivial solution as $t \to \infty$. By Lemma 3.3.1 and condition (3.126), every nonoscillatory solution of (3.116) tends to zero as $t \to \infty$. By Lemma 3.3.2 all oscillatory solutions are bounded on $(0, \infty)$.

Let $y(t)$ be an oscillatory solution of (3.116) which does not tend to zero as $t \to \infty$. There exist $\varepsilon > 0$ and sequences $\{t_n\}$, $\{t_n^{'}\}$ as $n \to \infty$ such that for each n, either

$$y(t_n) = 0,\; y(t_n^{'}) = \varepsilon, y(t_n^{'}) \geq 0, \text{ and } \; 0 < y(t) < \varepsilon,$$

for $t_n < t < t_n^{'} < t_{n+1}$ or

$$y(t_n) = 0,\; y(t_n^{'}) = -\varepsilon, y^{'}(t_n^{'}) < 0, \text{ and } 0 > y(t) > -\varepsilon,$$

for $t_n < t < t_n^{'} < t_{n+1}^{'}$.

We consider the former case since the later case can be treated similarly. Integrating (3.116) over $(t_n, t_n^{'})$, and using Lemma 3.3.2 for large n, and letting L be the bound for $(1 + y(s))f(-y(s - \tau(s)))$ on $[-\tau_0, \infty)$, yields

$$\varepsilon = y(t_n^{'}) - y(t_n) \leq \int_{t_n^{'}}^{t_n} r(s)(1 + y(s))f(-y(s - \tau(s)))ds$$

$$\leq L \int_{t_n^{'}}^{t_n} r(s)ds \leq L \int_{t_n^{'} - \tau(t_n^{'})}^{t_n} r(s)ds < \varepsilon,$$

which is impossible. The proof is complete. ∎

3.4 Models with Impulses

Consider the impulsive delay logistic model

$$N^{'}(t) = r(t)N(t)\left(1 - \frac{N(t-\tau)}{K}\right), \quad t \geq 0, \quad t \neq t_k \tag{3.127}$$

where $r \in C([0,\infty), \mathbf{R}^+)$, $\tau > 0$, with the following impulsive condition

$$N(t_k^+) - K = b_k \left(N(t_k^- - 0) - K\right), \quad t = t_k, \quad k = 1, 2, \ldots, \tag{3.128}$$

such that $0 \leq t_1 < t_2 < \ldots < t_k < t_{k+1} < \ldots$ with $\lim_{k\to\infty} t_k = \infty$, $\{b_k\}$ is a sequence of positive numbers with $b_k \leq 1$ and $x^{'}(t)$ denotes the left-hand side derivative of $x(t)$. The results in this section are adapted from [82].

We consider solutions of (3.127), (3.128) corresponding to the initial condition

$$N(t) = \phi(t) \geq 0, \text{ where } t \in [-\tau, 0], \ \phi \in C[-\tau, 0], \text{ and } \phi(0) > 0. \tag{3.129}$$

From the method of steps, we see that (3.127), (3.128), and (3.129) have a unique solution $N(t)$ defined on $[-\tau, \infty)$, with $N(t) > 0$ for all $t \geq 0$.

Motivated by (3.127), (3.128) in this section we consider

$$x^{'}(t) + r(t)x(t-\tau)(1 + x(t)) = 0, \quad t \geq 0, \ t \neq t_k, \tag{3.130}$$

$$x(t_k^+) = b_k x(t_k^-), \quad t = t_k, \quad k = 1, 2, \ldots . \tag{3.131}$$

We consider solutions x corresponding to the initial condition

$$\begin{cases} x(t) = \phi(t) \geq -1, \text{ where } t \in [-\tau, 0], \\ \phi \in C[-\tau, 0] \text{ and } \phi(0) > -1, \end{cases} \tag{3.132}$$

and note $x(t) > -1$ for all $t \geq 0$.

Since $0 < b_k \leq 1$, we have only the following two possibilities to consider:

$$\prod_{k=1}^{\infty} b_k = 0, \tag{3.133}$$

and

$$\prod_{k=1}^{\infty} b_k = b \in (0, 1]. \tag{3.134}$$

First, we consider the case when (3.133) holds.

Theorem 3.4.1. *Assume that (3.133) holds, and that*

$$\int_{t-\tau}^{t} r(s) \prod_{s-\tau \le t_k < s} b_k^{-1} ds \le 1 \textit{ for all large } t. \tag{3.135}$$

Then every solution of (3.130), (3.131), (3.132) tends to zero.

Proof. Let $x(t)$ be a solution of (3.130), (3.131), and (3.132). Then $x(t) > -1$ for $t \ge 0$. Set

$$y(t) = x(t) \prod_{0 \le t_k < t} b_k^{-1}, \quad t \ge 0. \tag{3.136}$$

By (3.130), (3.131), we have

$$y'(t) + a(t)y(t-\tau)\left[1 + \prod_{0 \le t_k < t} b_k(t)y(t)\right] = 0, \quad t \ge 0, \tag{3.137}$$

where

$$a(t) = r(t) \prod_{t-\tau \le t_k < t} b_k^{-1}. \tag{3.138}$$

Then $a(t)$ is piecewise continuous on $[0,\infty)$ and by (3.135) we have

$$\int_{t-\tau}^{t} a(s)ds \le 1, \quad t \ge T, \text{ for some large } T \ge 2\tau. \tag{3.139}$$

We need only to prove that $y(t)$ is bounded. If $y(t)$ is nonoscillatory, then by (3.137) $|y(t)|$ is eventually nonincreasing, and so $y(t)$ is bounded. Now we assume that $y(t)$ is oscillatory. If $y(t)$ is unbounded, we prove that

$$\lim_{t\to\infty} \sup y(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} \inf y(t) = -\infty. \tag{3.140}$$

Indeed, if $\lim_{t\to\infty} \inf y(t) > -\infty$, then there is $\alpha \in (0,\infty)$ such that $y(t) \ge -\alpha$ for $t \ge 0$. Thus, by (3.137) we have

$$y'(t) \le \alpha a(t)\left[1 + y(t) \prod_{0 \le t_k < t} b_k\right], \quad t \ge \tau. \tag{3.141}$$

Let $t^* > T + \tau$ be a local left-sided maximum point of $y(t)$. We prove that $y(t^*) \le e^{\alpha} - 1$, which implies that $\lim_{t\to\infty} \sup y(t) \le e^{\alpha} - 1 < \infty$, and so $y(t)$ is bounded. We need only to suppose that $y(t^*) > 0$. Then $y'(t^*) \ge 0$. By (3.137), we have $y(t^* - \tau) \le 0$. Thus, there is a $\eta^* \in [t^* - \tau, t^*)$ such that $y(\eta^*) = 0$ and $y(t) > 0$ for $t \in (\eta^*, t^*]$. Noting that $0 < b_k \le 1$ and using (3.141), we get

$$y^{'}(t) \leq \alpha a(t)\left[1 + y(t)\right], \quad t \in [\eta^*, t^*]. \tag{3.142}$$

Integrating (3.142) from η^* to t^* we have

$$\ln(1 + y(t^*)) \leq \alpha \int_{\eta^*}^{t^*} a(s)ds \leq \alpha,$$

which yields $y(t^*) \leq e^{\alpha} - 1$. Similarly, we may prove that if $\lim_{t\to\infty} \sup y(t) < \infty$ then $\lim_{t\to\infty} \inf y(t) > -\infty$. Therefore, we have shown that if $y(t)$ is unbounded, then (3.140) holds.

Now, let $\{S_n\}$ be an increasing infinite sequence such that $T + 4\tau < S_n$ with $S_n \to \infty$ as $n \to \infty$, and $y(S_n) = \max_{T\leq t\leq S_n}\{y(t)\} > 0$. Clearly, $\{y(S_n)\}$ is increasing, $y(S_n) \to \infty$ as $n \to \infty$, $y^{'}(S_n) \geq 0$. Also, choose $s_n \in (T + 2\tau, S_n)$ such that $y(s_n) = \min_{T\leq t\leq s_n}\{y(t)\} < 0$. Then $s_n \to \infty$, $y(s_n) \to -\infty$ as $n \to \infty$, and $y^{'}(s_n) \leq 0$. By (3.137), we have $y(S_n - \tau) \leq 0$ and $y(s_n - \tau) \geq 0$. Thus there are $\zeta_n \in [S_n - \tau, S_n)$ and $\eta_n \in [s_n - \tau, s_n)$ such that $y(\zeta_n) = y(\eta_n) = 0$, $y(t) > 0$ for $t \in (\zeta_n, S_n]$ and $y(t) < 0$ for $t \in (\eta_n, s_n]$. We easily see that $s_n < \zeta_n$. Set

$$M_n = y(S_n) \prod_{0\leq t_k<\zeta_n} b_k, \qquad m_n = -y(s_n) \prod_{0\leq t_k<\zeta_n} b_k.$$

It is clear that $0 < m_n < 1$. From (3.137), we have

$$y^{'}(t) \leq -a(t)y(s_n)\left[1 + y(t) \prod_{0\leq t_k<\zeta_n} b_k(t)\right], \quad \text{for } \zeta_n \leq t \leq S_n, \tag{3.143}$$

and

$$-y^{'}(t) \leq a(t)y(S_n)\left[1 + y(t) \prod_{0\leq t_k<\zeta_n} b_k(t)\right], \quad \text{for } \eta_n \leq t \leq s_n. \tag{3.144}$$

By integration, we have,

$$\ln\left[1 + y(t) \prod_{0\leq t_k<\zeta_n} b_k(t)\right] \leq m_n \int_{\zeta_n}^{S_n} a(s)ds \leq m_n,$$

and

$$-\ln\left[1 + y(t) \prod_{0\leq t_k<\zeta_n} b_k(t)\right] \leq M_n \int_{\eta_n}^{s_n} a(s)ds \leq M_n.$$

That is

$$\begin{cases} \ln(1+M_n) \le m_n, \\ -\ln(1-m_n) \le M_n, \end{cases}$$

which yields by Lemma 3.2.2 (with $u = m_n$, $v = M_n$ and $\beta = 0$) that $m_n = M_n = 0$. This contradiction implies that $y(t)$ is bounded and so the proof is complete. ∎

Next, we consider that case when (3.134) holds.

Theorem 3.4.2. *Assume that (3.134) holds, and that*

$$\int_0^\infty r(t)dt = \infty, \tag{3.145}$$

and

$$\int_{t-\tau}^t r(s) \prod_{s-\tau \le t_k < s} b_k^{-1} ds \le \frac{3}{2}, \text{ for all large } t. \tag{3.146}$$

Then every solution of (3.130), (3.131), and (3.132) tends to zero.

Proof. Let $x(t)$ be a solution of (3.130), (3.131), and (3.132). Define $y(t)$ as in (3.136). Then $y(t)$ satisfies (3.137). It suffices to show that

$$\lim_{t\to\infty} y(t) = 0. \tag{3.147}$$

If $y(t)$ is nonoscillatory, then by (3.137) $|y(t)|$ is eventually decreasing. In this case, we easily prove (3.147) by using (3.145) and the fact that $0 < b_k \le 1$. Now we assume that $y(t)$ is oscillatory. We shall prove that $y(t)$ is bounded above, and is bounded below away from -1. Set $a(t) = r(t)\prod_{t-\tau\le t_k<t} b_k^{-1}$. By (3.146), choose $T > 2\tau$ such that

$$\int_{t-\tau}^t a(s)\,ds \le \frac{3}{2}, \quad \text{for } t \ge T. \tag{3.148}$$

Let $t^*(> T + 2\tau)$ be a local maximum point of $y(t)$ with $y(t^*) > 0$. Then the left derivative $y'(t^*) \ge 0$, and by (3.137), $y(t^* - \tau) \le 0$. Thus, there exists $\zeta \in [t^* - \tau, t^*)$ such that $y(\zeta) = 0$ and $y(t) > 0$ for $t \in (\zeta, t^*]$. Clearly $y(t) > -b^{-1}$ for $t \ge 0$. Thus from (3.137), we have

$$y'(t) \le -b^{-1}a(t)(1 + y(t)), \quad \text{for } \zeta \le t \le t^*,$$

which yields

$$\ln(1 + y(t^*)) \le b^{-1}\int_\zeta^{t^*} a(s)ds \le 3/(2b),$$

or

$$y(t^*) \le e^{3/(2b)} - 1,$$

which proves that

$$y(t) \le e^{3/(2b)} - 1, \quad \text{for all } t \ge T + 2\tau.$$

Next, let $t_* > T + 3\tau$ be a local minimum point of $y(t)$ with $y(t_*) < 0$. Then $y^{'}(t_*) \le 0$ and $y(t_* - \tau) \ge 0$, by (3.137). Thus, there exists a $\eta \in [t_* - \tau, t_*)$ such that $y(\eta) = 0$ and $y(t) < 0$ for $t \in (\eta, t_*]$. For $\eta \le t \le t_*$, from (3.137) we have

$$-y^{'}(t) \le (e^{3/(2b)} - 1)a(t)(1 + y(t)).$$

Integrating this from η to t_*, we get

$$-\ln(1 + b\, y(t_*)) \le (e^{3/(2b)} - 1)\int_{\eta}^{t_*} a(s)ds \le \frac{3}{2}b(e^{3/(2b)} - 1).$$

That is

$$y(t_*) \ge \frac{\left[-1 + e^{-(3/2)b(e^{3/(2b)}-1)}\right]}{b},$$

which proves that

$$y(t) > \frac{\left[-1 + e^{-(3/2)b(e^{3/(2b)}-1)}\right]}{b}, \text{ for all } \; t \ge T + 3\tau.$$

Now, set

$$\lambda_1 = \limsup_{t\to\infty} y(t) \;\; \text{and} \quad \lambda_2 = \liminf_{t\to\infty} y(t).$$

Then

$$\frac{\left[-1 + e^{-(3/2)b(e^{3/(2b)}-1)}\right]}{b} \le \lambda_2 \le 0 \le \lambda_1 \le e^{3/(2b)} - 1.$$

To complete the proof it suffices to prove that $\lambda_1 = \lambda_2 = 0$. For any $0 < \varepsilon < 1 + b\lambda_2$, there exists $T_1 > T + 2\tau$ such that

$$\frac{-1}{b} < -\mu \equiv \lambda_2 - \varepsilon < y(t - \tau) < \lambda_1 + \epsilon \equiv \lambda, \quad \text{for } t \ge T_1. \tag{3.149}$$

Substituting (3.149) into (3.137), we have

$$y'(t) \le \mu a(t)[1 + y(t) \prod_{0 \le t_k < t} b_k], \quad t \ge T_1, \tag{3.150}$$

$$y'(t) \ge -\lambda a(t)[1 + y(t) \prod_{0 \le t_k < t} b_k], \quad t \ge T_1. \tag{3.151}$$

Let $\{S_n\}$ be an increasing sequence such that $S_n \ge T_1 + 2\tau$, $\lim_{n\to\infty} S_n = \infty$, $y(S_n) > 0$, $\lim_{n\to\infty} y(S_n) = \lambda_1$, $y'(S_n) \ge 0$. By (3.137), we have $y(S_n - \tau) \le 0$. Thus, there is $\zeta_n \in [S_n - \tau, S_n)$ such that $y(\zeta_n) = 0$ and $y(t) > 0$ for $\zeta_n < t \le S_n$. Set

$$\alpha_n = \prod_{0 \le t_k < \zeta_n} b_k.$$

Then $0 < \alpha_n < 1$ and $\lim_{n\to\infty} \alpha_n = b$. Now we show that

$$-\alpha_n y(t) \le 1 - \exp\left[-\mu\alpha_n \int_t^{\zeta_n} a(s)ds\right], \quad \text{for } \zeta_n - \tau \le t \le \zeta_n. \tag{3.152}$$

If $y(t) \ge 0$, then (3.152) is clearly true. Now suppose that $y(t) < 0$. Choose $\bar{\zeta}_n \in (t, \zeta_n]$ such that $y(\bar{\zeta}_n) = 0$ and $y(s) < 0$ for $s \in [t, \bar{\zeta}_n]$, and we have

$$y'(s) \le \mu a(s)(1 + \alpha_n y(s)), \quad \text{for } s \in [t, \bar{\zeta}_n].$$

Integrating this from t to $\bar{\zeta}_n$ we obtain

$$-\ln[1 + \alpha_n y(t)] \le \mu\alpha_n \int_t^{\bar{\zeta}_n} \alpha(s)ds \le \mu\alpha_n \int_t^{\zeta_n} \alpha(s)ds,$$

or

$$\alpha_n y(t) \le -1 + \exp\left[-\mu\alpha_n \int_t^{\zeta_n} a(s)ds\right],$$

which shows (3.152). Thus for $t \in [\zeta_n, S_n]$, we have

$$-\alpha_n\, y(t - \tau) \le -1 + \exp\left[-\mu\alpha_n \int_{t-\tau}^{\zeta_n} a(s)ds\right].$$

Substituting in (3.137) and noting that $y(t) > 0$ for $t \in (\zeta_n, S_n]$, we have

$$\alpha_n y^{'}(t) \leq a(t)[1 + \alpha_n y(t)] \left[1 - \exp\left[-\mu\alpha_n \int_{t-\tau}^{\zeta_n} a(s)ds \right] \right], \quad t \in [\zeta_n, S_n]. \tag{3.153}$$

From (3.150) we have

$$\alpha_n y^{'}(t) \leq \mu\, \alpha_n\, a(t)\, [1 + \mu\alpha_n y(t)]\,, \qquad t \in [\zeta_n, S_n]. \tag{3.154}$$

The rest of the proof is very similar (see [82]) to that of Theorem 3.2.2 and hence is omitted. ∎

Chapter 4
Logistic Models with Piecewise Arguments

There is no philosophy, which is not founded upon knowledge of the phenomena, but to get any profit from this knowledge it is absolutely necessary to be a mathematician.

Daniel Bernoulli (1700–1782).

When a mathematician has no more ideas he pursues axiomatics.

Felix Klein (1849–1925).

Differential equation with piecewise continuous argument (or DEPCA) will be discussed in this chapter. A typical logistic model with a piecewise constant argument is of the form

$$\frac{dN(t)}{dt} = rN(t)\,[1 - N([t])]\,, \quad t \geq 0, \tag{4.1}$$

where $[\cdot]$ denotes the greatest-integer function. On any interval of the form $[n, n+1)$ for $n = 0, 1, 2, \ldots$, by integrating (4.1), we obtain for $n \leq t < n+1$ and $n = 0, 1, 2, \ldots$ that

$$N(t) = N(n)\exp\{1 - N(n)](t - n)\}. \tag{4.2}$$

Taking the limit as $t \to n + 1$ in (4.2), we find

$$N(n + 1) = N(n)\exp\{1 - N(n)]\},\ n = 0, 1, 2, \ldots\ . \tag{4.3}$$

In this chapter we discuss autonomous and nonautonomous logistic equations with piecewise arguments.

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4.1 Oscillation of Autonomous Models

In this section, we establish some sufficient conditions for the oscillation of the logistic model with piecewise constant argument

$$\frac{dN(t)}{dt} = rN(t)\left[1 - \sum_{j=0}^{m} p_j N[t-j]\right], \quad t \geq 0. \tag{4.4}$$

The results in this section are adapted from [27].

By a solution of (4.4), we mean a function $N(t)$ which is defined on the set

$$\{-m, -m+1, \ldots, -1, 0\} \cup (0, \infty),$$

and which possesses the following properties:

(i) $N(t)$ is continuous on $[0,\infty)$.
(ii) The derivative $\frac{dN(t)}{dt}$ exists at each point $t \in [0, \infty)$ with possible exception of the points $t \in \{0, 1, 2, \ldots\}$ where one-sided derivatives exist.
(iii) Equation (4.4) is satisfied on each interval $[n, n+1)$ for $n = 0, 1, 2, \ldots$.

We assume that (4.4) is supplemented with the initial condition

$$N(0) = N_0 > 0 \text{ and } N(-j) = N_{-j} \geq 0, \quad j = 1, 2, 3, \ldots, m. \tag{4.5}$$

Lemma 4.1.1. *Let $N_0 > 0$ and $N(-j) = N_{-j} \geq 0$ for $j = 1, 2, 3, \ldots, m$ be given. The initial value problem (4.4) and (4.5) has a unique positive solution $N(t)$ given by*

$$N(t) = N_n \exp\left\{r\left[1 - \sum_{j=0}^{m} p_j N_{n-j}\right](t-n)\right\}, \; n \leq t < n+1, \tag{4.6}$$

and $n = 0, 1, 2, \ldots$, where the sequence $\{N_n\}$ satisfies the difference equation

$$N_{n+1} = N_n \exp\left\{r\left[1 - \sum_{j=0}^{m} p_j N_{n-j}\right]\right\}, \; n = 0, 1, 2, \ldots. \tag{4.7}$$

Proof. For every $n = 0, 1, 2, \ldots$ and $n \leq t < n+1$, (4.4) becomes

$$\frac{dN(t)}{dt} = rN(t)\left[1 - \sum_{j=0}^{m} p_j N_{n-j}\right], \quad n \leq t < n+1, \tag{4.8}$$

where we use the notations $N_n = N(n)$ for $n \in \{-m, -m+1, \ldots, -1, 0, 1, 2, \ldots\}$. By integrating (4.8) from n to t we obtain (4.6) and by continuity, as $t \to n+1$ (4.6) implies (4.7). Let $\{N_n\}$ be a solution of the difference equation (4.7) defined on

$$\{-m, -m+1, \ldots, -1, 0\} \cup (0, \infty)$$

by (4.5) and (4.6). Then one can show by direct substitution into (4.4) that $N(t)$ satisfies (4.4) and (4.5). It is also clear that $N_0 > 0$ implies that $N(t) > 0$ for $t > 0$. The proof is complete. ■

We note that $N_0 > 0$ implies that $N(t) > 0$ for $t > 0$ for any $N_{-j} \in \mathbf{R}$, $j = 1, 2, \ldots, m$. However, we assume in (4.5) that $N_{-j} \geq 0$. The following lemma is extracted from [30] and will be used in the proof of the main oscillation results.

Lemma 4.1.2. *Consider the equation*

$$\frac{dx(t)}{dt} + \sum_{j=0}^{m} p_j f(x([t-j])) = 0, \tag{4.9}$$

where

$$p_0,\ p_1, \ldots, p_m \geq 0,\ \sum_{j=0}^{m} p_j > 0,\ m + p_0 \neq 1,$$

and the function f satisfies

$$\begin{cases} f \in C(\mathbf{R}, \mathbf{R}), u f(u) > 0 \text{ for } u \neq 0, \\ f(u) \geq u \text{ for } u \leq 0 \text{ (or } f(u) \leq u \text{ for } u \geq 0), \\ \lim_{u \to 0} \frac{f(u)}{u} = 1. \end{cases} \tag{4.10}$$

Then every solution of (4.9) oscillates if and only if the equation

$$\lambda - 1 + \sum_{j=0}^{m} p_j \lambda^{-j} = 0$$

has no positive roots.

Now, we are ready to state and prove the main oscillation theorem of (4.4) which provides necessary and sufficient condition for the oscillation of all positive solution about the positive steady state

$$N^* := \left(\sum_{j=0}^{m} p_j \right)^{-1}. \tag{4.11}$$

Theorem 4.1.1. *Let $N_0 > 0$ and $N_{-j} \geq 0$ for $j = 1, 2, 3, \ldots, m$ be given and*

$$r \in (0, \infty),\ p_0,\ p_1, \ldots, p_m \geq 0,\ \sum_{j=0}^{m} p_i > 0,\ m + r \neq 1.$$

Then every solution of (4.4) and (4.5) oscillates about N^* if and only if the equation

$$\lambda - 1 + \frac{r}{\sum_{j=0}^{m} p_j} \sum_{j=0}^{m} p_j \lambda^{-j} = 0 \tag{4.12}$$

has no positive roots.

Proof. Let $N(t)$ be the positive solution of (4.4) and (4.5), and set

$$N(t) = N^* e^{x(t)}, t \geq 0.$$

Then $x(t)$ satisfies the equation

$$\frac{dx(t)}{dt} + \sum_{j=0}^{m} rN^* p_j f(x([t - j])) = 0,\ t \geq n, \tag{4.13}$$

where

$$f(u) = e^u - 1, \tag{4.14}$$

together with the initial condition

$$x(j) = \log\left[\frac{N(j)}{N^*}\right],\ \text{ for } j = 0, 1, 2, \ldots, m.$$

Clearly $N(t)$ oscillates about N^* if and only if $x(t)$ oscillates about zero. Also we observe that the function f defined in (4.14) satisfies the conditions in (4.10). To complete the proof apply Lemma 4.1.2. ■

4.2 Stability of Autonomous Models

In this section, we are concerned with the global attractivity of the logistic models with piecewise constant argument

$$\frac{dN(t)}{dt} = rN(t)\left[1 - \sum_{j=0}^{m} p_j N([t - j])\right],\quad t \geq 0, \tag{4.15}$$

where $dN(t)/dt$ means the right-hand side derivative at t of the function $N(t)$. As usual, we assume that (4.15) is supplemented with an initial condition

$$N(0) = N_0 > 0 \text{ and } N(-j) = N_{-j} \geq 0, \; j = 1, 2, 3, \dots, m. \tag{4.16}$$

In Sect. 4.1, we proved that the initial value problem (4.15) and (4.16) has a unique positive solution $N(t)$.

The results in this section are adapted from [27].

Theorem 4.2.1. *Assume the following:*

(i) $r \in (0, \infty)$, $p_0, p_1, \dots, p_m \geq 0$, $\sum_{j=0}^{m} p_j > 0$, $m + r \neq 1$;
(ii) $e^{r(m+1)} < 2$.

Then all solutions of (4.15) corresponding to (4.16) satisfy

$$\lim_{t \to \infty} N(t) = N^*. \tag{4.17}$$

Proof. By using the change of variables (see Sect. 4.1) it is sufficient to prove that every solution $x(t)$ of the equation

$$\frac{dx(t)}{dt} + \sum_{j=0}^{m} rN^* p_j f(x([t - j])) = 0, \; t \geq n, \tag{4.18}$$

where

$$f(u) = e^u - 1, \tag{4.19}$$

satisfies

$$\lim_{t \to \infty} x(t) = 0. \tag{4.20}$$

First, we assume that $x(t)$ is eventually nonnegative. From (4.18) we see that

$$\frac{dx(t)}{dt} \leq 0 \text{ for } n \leq t < n + 1, \tag{4.21}$$

where n is sufficiently large, say $n \geq n_0$. It follows that $x(t)$ is nonincreasing for $n \geq n_0$ and so

$$\lim_{t \to \infty} x(t) = l \geq 0 \text{ exists.}$$

We claim $l = 0$. Assume that, for of the sake of contradiction that $l > 0$. Then

$$\alpha \equiv \sum_{j=0}^{m} rN^* p_j (e^l - 1) = r\,(e^l - 1) > 0,$$

and (4.18) yields

$$\frac{dx(t)}{dt} \le -\alpha, \quad n \le t < n+1, \quad \text{for } n \ge n_0.$$

We note that

$$x(t) - x(n) \le -\alpha(t-n),$$

and as $t \to n+1$, we have

$$x(n+1) - x(n) \le -\alpha, \quad \text{for } n \ge n_0. \tag{4.22}$$

As $n \to \infty$, (4.22) implies that $0 = l - l \le -\alpha < 0$ which is impossible and so (4.20) holds for nonnegative solutions.

In a similar way, it follows that (4.20) is true for nonpositive solutions.

To complete the global attractivity it remains to prove that (4.20) is also true for oscillatory solutions. Now, assume that $x(t)$ is neither eventually nonnegative nor eventually nonpositive. Hence, there exists a sequence of points $\{\zeta_n\}$ such that

$$m < \zeta_1 > \zeta_2 < \ldots < \zeta_{\zeta_n} < \zeta_{n+1} \ldots,$$

$$\lim_{n\to\infty} \zeta_n = \infty, \quad x(\zeta_n) = 0, \; n = 0, 1, 2, \ldots,$$

and in each interval (ζ_n, ζ_{n+1}) the function $x(t)$ assumes both positive and negative values. Let t_n and s_n be points in (ζ_n, ζ_{n+1}) such that for $n = 1, 2, \ldots$

$$x(t_n) = \max[x(t)], \quad \zeta_n, < t < \zeta_{n+1}$$

and

$$x(s_n) = \min[x(t)], \quad \zeta_n, < t < \zeta_{n+1}.$$

Then for $n = 1, 2, \ldots$

$$x(t_n) > 0 \text{ and } D^- x(t_n) \ge 0, \tag{4.23}$$

while

$$x(s_n) < 0 \text{ and } D^- x(s_n) \le 0, \tag{4.24}$$

where $D^- x$ is the left derivative of x. Furthermore, if $t_n \notin \mathbf{N}$,

$$0 = \frac{dx(t_n)}{dt} = D^- x(t_n) = -\sum_{j=0}^{m} rN^* p_j \left[e^{x([t_n - j])} - 1 \right], \tag{4.25}$$

and if $t_n \in \mathbf{N}$,

$$0 \le D^- x(t_n) = -\sum_{j=0}^{m} rN^* p_j \left[e^{x(t_n-j-1)} - 1\right]. \tag{4.26}$$

Similarly, if $s_n \notin \mathbf{N}$

$$0 = \frac{dx(s_n)}{dt} = D^- x(t_n) = -\sum_{j=0}^{m} rN^* p_j \left[e^{x([s_n-j])} - 1\right], \tag{4.27}$$

and if $s_n \in \mathbf{N}$

$$0 \ge D^- x(s_n) = -\sum_{j=0}^{m} rN^* p_j \left[e^{x(t_n-j-1)} - 1\right]. \tag{4.28}$$

Next, we claim that for each $n = 1, 2, \ldots$

$$x(t) \text{ has a zero } T_n \in [\zeta_n, t_n) \cap [t_n - m - 1, t_n) \tag{4.29}$$

and

$$x(t) \text{ has a zero } S_n \in [\zeta_n, s_n) \cap [s_n - m - 1, s_n). \tag{4.30}$$

If for example (4.29) were false, then (4.25) and the hypothesis that

$$\sum_{j=0}^{m} p_j > 0$$

together would lead to a contradiction. Also (4.30) is true due to a similar reason. By integrating (4.13) from T_n to t_n and using the fact that $t_n - T_n \le m + 1$, we note immediately that

$$\begin{aligned} 0 &= x(t_n) - x(T_n) + \sum_{j=0}^{m} rN^* p_j \int_{T_n}^{t_n} \left[e^{x([s-j])} - 1\right] ds \\ &\ge x(t_n) - \sum_{j=0}^{m} rN^* p_j (t_n - T_n) \ge x(t_n) - r(m+1). \end{aligned}$$

That is,

$$x(t_n) < r(m+1),\ n = 1, 2, \ldots$$

and

$$x(t) < r(m+1), \quad \text{for } t \geq \zeta_1.$$

By integrating (4.14) from S_n to s_n and using the fact that $s_n - S_n \leq m+1$ and using the hypothesis (ii) of the theorem, we find

$$\begin{aligned} 0 &= x(s_n) - x(S_n) + \sum_{j=0}^{m} rN^* p_j \int_{S_n}^{s_n} \left[e^{x([s-j])} - 1\right] ds \\ &\leq x(s_n) + \sum_{j=0}^{m} rN^* p_j \left[e^{r(m+1)} - 1\right](m+1) \\ &\leq x(s_n) + r(m+1). \end{aligned}$$

That is

$$x(s_n) \geq -r(m+1),\ n = 1, 2, \ldots,$$

and so

$$x(t) \geq -r(m+1), \quad t \geq \zeta_1.$$

Then we have established that

$$-M < x(t) < M, \quad \text{for } t \geq \zeta_1, \tag{4.31}$$

where

$$M = r(m+1).$$

By using (4.31) and an argument similar to that used above we find that

$$-M(-e^{-M} + 1) < x(t) < M(e^M - 1),\ t \geq \zeta_1.$$

Using induction, we can prove that

$$-L_n < x(t) < R_n, \tag{4.32}$$

where

$$L_0 = R_0 = M, \quad -L_{n+1} = M(-e^{-L_n} + 1),\ R_{n+1} = M(e^{R_n} - 1), \tag{4.33}$$

along with

$$-M \leq -L_n \leq -L_{n+1} < 0 < R_{n+1} \leq R_n \leq M. \tag{4.34}$$

Set

$$L := \lim_{n\to\infty} L_n, \ R := \lim_{n\to\infty} R_n.$$

In view of (4.32), we have that $\lim_{t\to\infty} x(t) = 0$ holds if we show that

$$L = R = 0. \tag{4.35}$$

To this end, from (4.33) and (4.34), we have

$$-L = M(e^{-L} - 1), \ R = M(e^R - 1), \ -M \le -L \le 0 \le R \le M. \tag{4.36}$$

Hence, $-L$ and R are the zeros of the function

$$\varphi(\lambda) = M(e^\lambda - 1) - \lambda$$

in the interval $-M \le \lambda \le M$. We have

$$\varphi(-\infty) = \varphi(\infty) = \infty, \ \varphi(0) = 0,$$

and also φ is decreasing in $(-\infty, -\log M)$ and increasing in $(-\log M, \infty)$. Note also that in view of hypothesis (ii), $M \in (0, 1)$ and

$$\varphi(M) = M(e^M - 1) - M < M(2 - 1) - M = 0.$$

Therefore, $\varphi(\lambda)$ has exactly one zero in $(-\infty, M)$ namely $\lambda = 0$. Thus, $-L$ and R the zeros of $\varphi(\lambda)$ in $[-M, M]$ are both zero. This proves (4.35) and completes the proof of theorem. ∎

Now we establish some sufficient conditions for the global attractivity of N^* of (4.15). The results in this section are adapted from [75]. To prove the main results it is sufficient, as in the proof of Theorem 4.2.1, to prove that every solution of (4.13) satisfies condition (4.20).

We consider a sufficient condition for the global attractivity of the solution $x(t) = 0$ for the general differential equation with piecewise constant arguments

$$\frac{dx(t)}{dt} = -\sum_{j=0}^{m} rN^* p_j f(x([t-j])), \ \ t \ge 0, \tag{4.37}$$

where $p_0 > 0$ and

$$\begin{cases} f(x) \in C^1(-\infty, \infty), \ f(0) = 0, \ f'(x) > 0, \ x \in (-\infty, \infty), \\ \lim_{x\to-\infty} f(x) = -1 \ \text{ and } \ \lim_{x\to\infty} f(x) = \infty. \end{cases} \tag{4.38}$$

As a special case when $f(x) = e^x - 1$, we establish the main results for (4.15). As usual by integrating both sides of (4.37) from n to t on the interval $[n, n+1)$, $n = 0, 1, 2, \ldots,$ we find

$$x(t) - x(n) = -\sum_{j=0}^{m} rN^* p_j \int_n^t f(x(n-j))dt$$
$$= -rN^* \sum_{j=0}^{m} p_j f(x(n-j))(t-n).$$

Now, the solution of (4.37) is written for $0 \le n \le t < n+1$ as

$$x(t) = x(n) - rN^* \sum_{j=0}^{m} p_j f(x(n-j))(t-n).$$

As $t \to n+1$, we have

$$x_{n+1} = x_n - rN^* \sum_{j=0}^{m} p_j f(x_{n-j}),\ n = 0, 1, 2, \ldots, \tag{4.39}$$

where

$$x_n = x(n), \quad n = 0, 1, 2, \ldots.$$

To show that

$$\lim_{t\to\infty} x(t) = 0,$$

it is enough to show that

$$\lim_{n\to\infty} x_n = 0.$$

For simplicity, we put

$$r_1 = rN^* p_0 > 0,\ r_2 = rN^* \sum_{j=1}^{m} p_j \ge 0 \ \text{ and } \ \varphi(x) = x - r_1 f(x). \tag{4.40}$$

Then $r = r_1 + r_2$ and (4.39) is written as

$$x_{n+1} = \varphi(x_n) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j}),\ n = 0, 1, 2, \ldots\ . \tag{4.41}$$

Lemma 4.2.1. *In (4.39), if x_n is eventually nonpositive or eventually nonnegative, then* $\lim_{n\to\infty} x_n = 0$.

Proof. In (4.39) assume there exists an integer n_0 such that x_n is eventually nonpositive for $n \geq n_0$. From (4.37) and (4.38) it is easy to see that $x_n \leq x_{n+1} \leq 0$ for $n \geq n_0 + m - 1$. Let $\alpha = \lim_{n\to\infty} x_n$. Then $f(\alpha) = 0$ so $\alpha = 0$. The other case is similar. ■

To establish the main global stability results we need the following useful lemmas.

Lemma 4.2.2. *Assume that $\varphi(x)$ attains a unique local maximum at*

$$L^* < 0, \tag{4.42}$$

and for $L \leq 0$, put

$$\begin{aligned} F(L) \equiv \min\{&\varphi(L), \varphi(\varphi(\max\{L^*, L\}) - r_2 f(L))\} \\ &-r_2 f(\varphi(\max\{L^*, L\}) - r_2 f(L)). \end{aligned} \tag{4.43}$$

If $F(L) > L$ for any $L < 0$, then

$$\lim_{n\to\infty} x_n = 0.$$

Proof. In the case when x_n is eventually nonpositive or eventually nonnegative by Lemma 4.2.1 we have $\lim_{n\to\infty} x_n = 0$. Now, assume that x_n is not eventually nonnegative or eventually nonpositive. Then as in the proof of Theorem 4.2.1, we can take a sequence $\{\xi_k\}_{k=1}^{\infty}$ such that

$$m < \xi_1 < \xi_2 < \ldots < \xi_n < \xi_{n+1} < \ldots, \quad \lim_{n\to\infty} \xi_n = \infty, \; x(\xi_n) = 0$$

and

$$x(t) > 0 \;\text{ on } (\xi_{2n-1}, \xi_{2n}), \; x_n < 0, \text{ on } \; (\xi_{2n}, \xi_{2n+1})$$

and

$$\xi_{n+1} - \xi_n > m + 1, \quad n = 1, 2, \ldots .$$

Let t_n be a point that attains a maximal value of $x(t)$ on (ξ_{2n-1}, ξ_{2n}) and s_n be a point that attains a minimal value of $x(t)$ on $(\zeta_{2n}, \zeta_{2n+1})$, that is for $n = 1, 2, \ldots$

$$x(t_n) = \max[x(t)], \qquad \zeta_{2n-1} < t < \zeta_{2n}$$

and

$$x(s_n) = \min[x(t)], \qquad \zeta_{2n} < t < \zeta_{2n+1}.$$

Then, we have for $n = 1, 2, \ldots,$ t_n and s_n are positive integers,

$$x(t_n) > 0, \; D^- x(t_n) \geq 0, \;\text{ while } x(s_n) < 0, \;\text{ and }\; D^- x(s_n) \leq 0,$$

where $D^{-}x$ is the left-hand derivative of x at t. Then.

$$0 \leq D^{-}x(t_n) = -rN^{*}\sum_{j=0}^{m} p_j f(x(t_n - j - 1)) \tag{4.44}$$

and

$$0 \geq D^{-}x(s_n) = -\sum_{j=0}^{m} rN^{*} p_j f(x(s_n - j - 1)). \tag{4.45}$$

Following the reasoning in the proof of Theorem 4.2.1, we show that for each $n = 1, 2, \ldots$

$$x(t) \text{ has a zero } T_n \in [t_n - m - 1, t_n) \tag{4.46}$$

and

$$x(t) \text{ has a zero } S_n \in [s_n - m - 1, s_n). \tag{4.47}$$

By integrating (4.37) from T_n to t_n and using the fact that $t_n - T_n \leq m + 1$ we note that

$$\begin{aligned} 0 &= x(t_n) - x(T_n) + \sum_{j=0}^{m} rN^{*} p_j \int_{T_n}^{t_n} f(x([s - j]))ds \\ &\geq x(t_n) - \sum_{j=0}^{m} rN^{*} p_j (t_n - T_n) \geq x(t_n) - r(m + 1). \end{aligned}$$

That is,

$$x(t_n) \leq r(m + 1),\ n = 1, 2, \ldots$$

and

$$x(t) \leq r(m + 1),\ t \geq \zeta_1.$$

By integrating (4.37) from S_n to s_n and using the fact that $s_n - S_n \leq m + 1$ we find

$$\begin{aligned} 0 &= x(s_n) - x(S_n) + \sum_{j=0}^{m} rN^{*} p_j \int_{S_n}^{s_n} f(x([s - j]))ds \\ &\leq x(s_n) + r(m + 1) f(r(m + 1)). \end{aligned}$$

That is

$$x(s_n) \geq -r(m + 1) f(r(m + 1)),\ n = 1, 2, \ldots,$$

and so

$$x(t) \geq -r(m+1)f(r(m+1)),\ t \geq \zeta_2.$$

Then

$$\begin{cases} x_n \leq R_1 = r(m+1), & n \geq \zeta_1 \\ x_n \geq L_1 = -r(m+1)f(r(m+1)), & n \geq \zeta_2. \end{cases}$$

Next, let L_k be a lower bound of x_n for $n > \zeta_{2k}$. Then,

$$x_n \geq L_k, \quad \text{for } n > \zeta_{2k}.$$

We know that $\varphi(x)$ has a unique local maximum at $x = L^* < 0$. We consider an upper bound of x_n for $n > \zeta_{2(k+1)-1}$. Then,

$$\begin{aligned} x_n &= \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j-1}) \leq \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(L_k) \\ &\leq \varphi(\max\{L^*, L_k\}) - r_2 f(L_k). \end{aligned}$$

That is,

$$x_n \leq R_{k+1}, \ \text{ for } \ n > \zeta_{2(k+1)-1},$$

where

$$R_{k+1} = \varphi(\max\{L^*, L_k\}) - r_2 f(L_k).$$

Next, we consider a lower bound of x_n for $n > \zeta_{2(k+1)}$. Then

$$\begin{aligned} x_n &= \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j-1}) \\ &\geq \min\{\varphi(L_k), \varphi(R_{k+1})\} - r_2 f(R_{k+1}) \\ &= \min\{\varphi(L_k), \varphi(\varphi(\max\{L^*, L_k\}) - r_2 f(L_k))\} \\ &\quad -r_2 f(\varphi(\max\{L^*, L_k\}) - r_2 f(L_k)). \end{aligned}$$

Put $L_{k+1} = F(L_k)$. Then,

$$x_n \geq L_{k+1} \ \text{ for } \ n > \zeta_{2(k+1)}.$$

By assumption,

$$L_k < F(L_k) = L_{k+1}.$$

Finally, we show that

$$\lim_{k\to\infty} R_k = 0 \text{ and } \lim_{k\to\infty} L_k = 0.$$

Assume that L_k is a lower bound of x_n for $n > \zeta_{2k}$. Since

$$R_{k+1} = \varphi(\max\{L^*, L_k\}) - r_2 f(L_k)$$

is an upper bound of x_n for $n > \zeta_{2(k+1)-1}$, we have that if $\lim_{k\to\infty} L_k = 0$, then

$$\lim_{k\to\infty} R_k = \varphi(0) - r_2 f(0) = 0.$$

Thus it is sufficient to show that $\lim_{k\to\infty} L_k = 0$. By (4.43), $F(0) = 0$ and

$$L_k < L_{k+1} = F(L_k) \le 0, \text{ for any } L_k < 0,$$

and hence we see $\lim_{k\to\infty} L_k = 0$ by successive iterations. Thus, we get $\lim_{n\to\infty} x_n = 0$. The proof is complete. ■

Lemma 4.2.3. *Assume that $\varphi(x)$ attains a unique local maximum at $R^* > 0$, and*

$$R^* \ge \varphi(R^*) + r_2. \tag{4.48}$$

Put

$$H(L) = r_1 f(L) + r_2 f(\varphi(R^*) - r_2 f(L)), \textit{ for } L \le 0. \tag{4.49}$$

If

$$r_1 > r_2 \ge 0 \quad \textit{and} \quad \lim_{L\to-\infty} H(L) < 0, \tag{4.50}$$

then $\lim_{n\to\infty} x_n = 0$.

Proof. In the case when x_n is eventually nonpositive or eventually nonnegative by Lemma 4.2.2, we have $\lim_{n\to\infty} x_n = 0$. Now, assume that x_n is not eventually nonnegative nor eventually nonpositive. Then as in the proof of Theorem 4.2.1, we can take a sequence $\{\xi_k\}_{k=1}^{\infty}$ such that

$$m < \xi_1 < \xi_2 < \ldots < \xi_n < \xi_{n+1} < \ldots, \ \lim_{n\to\infty} \xi_n = \infty, \ x(\xi_n) = 0$$

and

$$x(t) > 0, \text{ on } (\xi_{2n-1}, \xi_{2n}), \ x(t) < 0, \text{ on } (\xi_{2n}, \xi_{2n+1})$$

and

$$\xi_{n+1} - \xi_n > m + 1, \ n = 1, 2, \ldots$$

and for $n > \xi_2$

$$-r(m+1)f(r(m+1)) \le x_n.$$

Now, let the local maximum of $\varphi(x)$ be attained at $x = R^* > 0$. Then, for

$$n > \xi_1,\ x_n < \varphi(R^*) + r_2 \le R^*.$$

Since

$$0 < \varphi(R^*) - r_2 f(L) \le \varphi(R^*) + r_2 \le R^*,$$

we have

$$(1 - r_1 f'(\varphi(R^*) - r_2 f(L)))r_2 f'(L) \ge 0, \quad \text{for} \quad L < 0.$$

Therefore,

$$f'(\varphi(R^*) - r_2 f(L)) \le 1/r_1.$$

Thus

$$\begin{aligned} H'(L) &= f'(L)\{r_1 - r_2^2(f'(\varphi(R^*) - r_2 f(L))\} \\ &\ge f'(L)\left(r_1 - \frac{r_2^2}{r_1}\right) > 0. \end{aligned}$$

Since $H(L)$ is a strictly monotone increasing function of L on $(-\infty, 0]$ and $\lim_{L\to-\infty} H(L) < 0$, there exists an $L_1 < 0$ such that

$$L_1 < -r(m+1)f(r(m+1)) \ \text{ and } \ H(L_1) < 0.$$

Then

$$\varphi(L_1) - r_2 f(\varphi(R^*) - r_2 f(L_1)) = L_1 - H(L_1) > L_1. \tag{4.51}$$

Thus, L_1 is a lower bound of x_n for $n > \xi_2$, that is $x_n > L_1$ for $n > \xi_2$. Next, let us consider an upper bound of x_n for $n > \xi_3$. Since for $n > \xi_3$,

$$x_n = \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j-1}) \le \varphi(R^*) - r_2 f(L_1),$$

and we have that for

$$R_2 = \varphi(R^*) - r_2 f(L_1) > 0,\ x_n \le R_2, \ \text{for} \ n > \xi_3.$$

Moreover, we have that for $L_1 < 0$,

$$\begin{aligned} R^* - R_2 &= R^* - (\varphi(R^*) - r_2 f(L_1)) \\ &> R^* - (\varphi(R^*) + r_2) \geq 0, \end{aligned}$$

from which we get $0 < R_2 < R^*$. Let us consider a lower bound of x_n for $n > \xi_4$. Since $0 < R_2 < R^*$, we see that for $n > \xi_4$,

$$x_n = \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j-1}) \geq \varphi(L_1) - r_2 f(R_2).$$

Then, for

$$L_2 = \varphi(L_1) - r_2 f(R_2) < 0,\ x_n \geq L_2, n > \xi_4,$$

and by (4.51), we have $L_1 < L_2 < 0$. Next assume that for $k \geq 1$,

$$\begin{cases} R_k = \varphi(R_{k-1}) - r_2 f(L_{k-1}), \ 0 < R_k < R_{k-1} \leq R^*, \\ L_k = \varphi(L_{k-1}) - r_2 f(R_k), \quad L_1 \leq L_{k-1} < L_k < 0, \end{cases}$$

and

$$\begin{cases} x_n \leq R_k,\ n > \xi_{2k-1}, \\ x_n \geq L_k,\ n > \xi_{2k}. \end{cases}$$

We consider an upper bound of x_n for $n > \xi_{2(k+1)-1}$. Then

$$x_n = \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j-1}) \leq \varphi(R_k) - r_2 f(L_k).$$

Therefore, for

$$R_{k+1} = \varphi(R_k) - r_2 f(L_k) > 0, x_n \leq R_{k+1}, n > \xi_{2(k+1)-1},$$

and

$$R_{k+1} = \varphi(R_k) - r_2 f(L_k) < \varphi(R_{k-1}) - r_2 f(L_{k-1}) = R_k.$$

Similarly, let us consider a lower bound of x_n for $n > \xi_{2(k+1)}$. Then, for $n > \xi_{2(k+1)}$,

$$x_n = \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j-1}) \geq \varphi(L_k) - r_2 f(R_{k+1}),$$

and for

$$L_{k+1} = \varphi(L_k) - r_2 f(R_{k+1k}) < 0,\ x_n \geq L_{k+1},\ n > \xi_{2(k+1)}.$$

Moreover,

$$L_{k+1} = \varphi(L_k) - r_2 f(R_{k+1}) > \varphi(L_{k-1}) - r_2 f(R_k) = L_k.$$

Finally, we show that

$$R = \lim_{k\to\infty} R_k = 0 \text{ and } \quad L = \lim_{k\to\infty} L_k = 0.$$

Since

$$R = \varphi(R) - r_2 f(L),\ L = \varphi(L) - r_2 f(R),$$

we have that

$$r_1 f(R) + r_2 f(L) = 0,\ \ r_1 f(L) + r_2 f(R) = 0.$$

By assumption

$$0 \leq r_2 < r_1 < 1,\ f(R) = f(L) = 0,$$

and hence $R = L = 0$. Thus, we get $\lim_{n\to\infty} x_n = 0$. ■

Remark 3. In the proof of Lemma 4.2.3, we need the condition $\lim_{L\to-\infty} H(L) < 0$ in (4.50) which becomes

$$r_1 + r_2 - \frac{r_2}{r_1} e^{r_1+r_2-1} > 0.$$

Lemma 4.2.4. *Assume that $\varphi(x)$ attains a unique local maximum at $R^* > 0$, and*

$$R^* < \varphi(R^*) + r_2. \tag{4.52}$$

Then, there exists a unique $L^* < 0$ such that

$$R^* = \varphi(R^*) - r_2 f(L^*) \tag{4.53}$$

and

$$R^* > \varphi(R^*) - r_2 f(L), \quad L^* < L \leq 0. \tag{4.54}$$

Define, for $L \leq L^*$,

$$\begin{aligned} G(L) :&= \min\{\varphi(L), \varphi(\varphi(R^*) - r_2 f(L))\} \\ &\quad - r_2 f(\varphi(R^*) - r_2 f(L)). \end{aligned} \tag{4.55}$$

If

$$r_1 > r_2 \quad \text{and} \quad G(L) > L, \;\; \text{for any } L \le L^*, \tag{4.56}$$

then $\lim_{n\to\infty} x_n = 0$.

Proof. Since

$$0 < R^* < \varphi(R^*) + r_2 \quad \text{and} \quad f^{'}(L) > 0,$$

we have

$$\lim_{L\to-\infty}(r_1 f(R^*) + r_2 f(L)) = r_1 f(R^*) - r_2 < r_1 f(R^*)$$
$$= \lim_{L\to 0}(r_1 f(R^*) + r_2 F(L)).$$

Hence, by the mid-point theorem, there exists a unique $L^* < 0$ such that

$$r_1 f(R^*) + r_2 F(L^*) = 0,$$

that is

$$R^* = \varphi(R^*) - r_2 f(L^*).$$

In the case when x_n is eventually nonpositive or eventually nonnegative by Lemma 4.2.1 we have $\lim_{n\to\infty} x_n = 0$. Therefore we assume that x_n is not eventually nonnegative nor eventually nonpositive. Then there exists a sequence $\{\xi_k\}_{k=1}^{\infty}$ (as in Lemma 4.2.2) such that

$$\begin{aligned} x_n &\le R_1 = r(m+1), && n \ge \zeta_1, \\ x_n &\ge -r(m+1)f(r(m+1)), && n \ge \zeta_2. \end{aligned}$$

Put

$$L_1 = -r(m+1)f(r(m+1)).$$

Now, we consider the following two cases:

Case 1. $L^* < L_1$.

Then we have $L_1 < 0$ and $x_n \ge L_1$ for $n \ge \zeta_2$. Next, for $n \ge \zeta_3$, consider an upper bound of x_n. Since $n - j - 1 > \zeta_2$, $1 \le j \le m$,

$$x_n = \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j-1}) \le \varphi(R^*) - r_2 f(L_1).$$

Put

$$R_2 = \varphi(R^*) - r_2 f(L_1).$$

Then

$$0 \leq R_2 < \varphi(R^*) - r_2 f(L^*) = R^*,$$

and we have $x_n \leq R_2$ for $n > \xi_3$. Next, for $n \geq \zeta_4$, consider a lower bound of x_n. Since

$$\varphi(L_1) < 0 \leq \varphi(R_2), \min\{\varphi(L_1), \varphi(R_2)\} = \varphi(L_1).$$

For $n > \xi_4$, $n - j - 1 > \zeta_3$, $1 \leq j \leq m$ and

$$x_n = \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j-1}) \geq \varphi(L_1) - r_2 f(R_2).$$

Put

$$L_2 = \max\{L_1 , \varphi(L_1) - r_2 f(R_2)\}.$$

Then $L_1 \leq L_2 < 0$ and $x_n \geq L_2$, $n > \xi_4$. Similarly, consider an upper bound of x_n for $n > \zeta_5$. Then,

$$x_n = \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j-1}) \leq \varphi(R_2) - r_2 f(L_2).$$

Put

$$R_3 = \varphi(R_2) - r_2 f(L_2).$$

Then

$$0 < R_3 = \varphi(R_2) - r_2 f(L_2) < \varphi(R^*) - r_2 f(L_1) = R_2 < R^*$$

and $x_n \leq R_3$ for $n > \zeta_5$.

Next let us assume that for some positive integer $k \geq 2$,

$$\begin{cases} R_k = \varphi(R_{k-1}) - r_2 f(L_{k-1}), & 0 < R_k < R_{k-1}, \\ L_k = \max\{L_{k-1} , \varphi(L_{k-1}) - r_2 f(R_k)\}, & L_{k-1} \leq L_k < 0, \end{cases}$$

and

$$\begin{cases} x_n \leq R_k, \ n > \xi_{2k-1}, \\ x_n \geq L_k, \ n > \xi_{2k}. \end{cases}$$

Consider an upper bound of x_n for $n > \xi_{2(k+1)-1}$. Then

$$x_n = \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j-1}) \leq \varphi(R_k) - r_2 f(L_k).$$

Put

$$R_{k+1} = \varphi(R_k) - r_2 f(L_k).$$

Then

$$R_{k+1} = \varphi(R_k) - r_2 f(L_k) < \varphi(R_{k-1}) - r_2 f(L_{k-1}) = R_k,$$

and

$$x_n \leq R_{k+1}, n > \xi_{2(k+1)-1}.$$

Similarly, let us consider a lower bound of x_n for $n > \xi_{2(k+1)}$. Then, for $n > \xi_{2(k+1)}$,

$$x_n = \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j-1}) \geq \varphi(L_k) - r_2 f(R_{k+1}).$$

Put

$$L_{k+1} = \max\{L_k\ ,\ \varphi(L_k) - r_2 f(R_{k+1})\}.$$

Then

$$L_{k+1} \geq L_k \ \text{ and } \ x_n \geq L_{k+1}, \ \ n > \xi_{2(k+1)-1}.$$

Thus by induction, we get a strictly monotone decreasing sequence $\{R_k\}_{k=1}^{\infty}$ and a monotone increasing sequence $\{L_k\}_{k=1}^{\infty}$. Now, put

$$R = \lim_{k\to\infty} R_k \ \text{ and } \quad L = \lim_{k\to\infty} L_k.$$

Then, we have

$$R = \varphi(R) - r_2 f(L),\ L = \max\{L, \varphi(L) - r_2 f(R)\} \geq \varphi(L) - r_2 f(R).$$

Thus

$$r_1 f(R) + r_2 f(L) = 0,\ \ r_1 f(L) + r_2 f(R) \geq 0.$$

Since $f(R) = -(r_2/r_1) f(L)$ and by assumption

$$r_1 > r_2, (r_1 - r_2^2/r_1) f(L) \geq 0,$$

we get that $f(R) = f(L) = 0$, and hence $R = L = 0$. Thus, we get

$$\lim_{n\to\infty} x_n = 0.$$

Case 2. $L_1 \leq L^*$.

Then we have $x_n \geq L_1$ for $n \geq \zeta_2$. Next, for $n \geq \zeta_3$, consider an upper bound of x_n. Then

$$x_n = \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j-1}) \leq \varphi(R^*) - r_2 f(L_1).$$

Put

$$R_2 = \varphi(R^*) - r_2 f(L_1).$$

Then we have $x_n \leq R_2$ for $n > \xi_3$. Next, for $n \geq \zeta_4$, consider a lower bound of x_n. Then

$$\begin{aligned} x_n &= \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j-1}) \\ &\geq \min\{\varphi(L_1), \varphi(R_2)\} - r_2 f(R_2). \end{aligned}$$

Put

$$L_2 = \min\{L_1\,,\ \varphi(L_1)\}, \varphi(R_2)\} - r_2 f(R_2).$$

Then, we have $x_n \geq L_2$, $n > \xi_4$. Now, we restrict our attention to the lower bound of x_n and assume that for some positive integer k, $x_n \geq L_k$ for $n \geq \xi_{2k}$. Suppose that $L_k \leq L^*$. Consider an upper bound of x_n for $n > \xi_{2(k+1)-1}$. Then,

$$x_n = \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j-1}) \leq \varphi(R^*) - r_2 f(L_k).$$

Put

$$R_{k+1} = \varphi(R^*) - r_2 f(L_k).$$

Then $x_n \leq R_{k+1}$, $n > \xi_{2(k+1)-1}$. Now, consider a lower bound of x_n for $n > \xi_{2(k+1)}$. Then, for $n > \xi_{2(k+1)}$,

$$\begin{aligned} x_n &= \varphi(x_{n-1}) - rN^* \sum_{j=1}^{m} p_j f(x_{n-j-1}) \\ &\geq \min\{\varphi(L_k), \varphi(R_{k+1})\} - r_2 f(R_{k+1}). \end{aligned}$$

Put

$$L_{k+1} = \min\{\varphi(L_k), \varphi(R_{k+1})\} - r_2 f(R_{k+1}).$$

Then

$$L_{k+1} = \min\{\varphi(L_k), \varphi(\varphi(R^*) - r_2 f(L_k))\} - r_2 f(\varphi(R^*) - r_2 f(L_k)).$$

Then

$$x_n \geq L_{k+1},\ n > \xi_{2(k+1)+1}.$$

Thus by the assumption, we have

$$L_{k+1} = G(L_k) > L_k.$$

Since $G(L) > L$ for any $L < L^*$, there exists some positive integer k_0 such that

$$L_{k_0-1} \leq L^* < L_{k_0}.$$

Then

$$x_n \geq L_{k_0} > L^*,\ n \geq \zeta_{2k_0+1}.$$

For

$$L > L^*, \varphi(R^*) - r_2 f(L) < \varphi(R^*) - r_2(f(L^*) = R^*.$$

Hence as before we obtain $\lim_{n\to\infty} x_n = 0$. The proof is complete. ∎

Lemma 4.2.5. *Assume that $\varphi(x)$ attains a unique local maximum at $R^* = 0$. Then*

$$R^* < \varphi(R^*) + r_2, \tag{4.57}$$

and there exists a unique $L^* = 0$ such that

$$R^* = \varphi(R^*) - r_2 f(L^*). \tag{4.58}$$

For any $L < 0$ and $G(L)$ in Lemma 4.2.4, if

$$G(L) > L, \tag{4.59}$$

then $\lim_{n\to\infty} x_n = 0$.

Proof. The proof of this lemma is similar to that in Lemma 4.2.4 and hence is omitted. ∎

Next we consider the special case $f(x) = e^x - 1$ and establish the conditions $F(L) > L$ for any $L < 0$ in Lemma 4.2.3, $G(L) > L$ for any $L < L^*$ in Lemma 4.2.4 and $G(L) > L$ for any $L < 0$ in Lemma 4.2.5.

Lemma 4.2.6. *Put*

$$\varphi^*(x) = x - (r_1 + r_2)f(x), \quad -\infty < x < \infty. \tag{4.60}$$

Assume

$$0 < r_1 + r_2 \le 2. \tag{4.61}$$

Then

$$\begin{cases} (\varphi^*)^2(L) > L, & \text{for any } L < 0, \\ (\varphi^*)^2(R) < R, & \text{for any } R > 0, \end{cases} \tag{4.62}$$

and for (4.39)–(4.41) with $r_2 = 0$, $\lim_{n\to\infty} x_n = 0$.

Proof. First, consider the following function:

$$g_1(t) = t + t\,e^{(r_1+r_2)(1-t)}, \quad 0 < t < \infty.$$

Then, for $f(x) = e^x - 1$,

$$\begin{aligned}(\varphi^*)^2(x) &= \varphi^*(\varphi^*(x)) = \varphi^*(x) - (r_1 + r_2)(e^{\varphi^*(x)} - 1) \\ &= x + (r_1 + r_2)\{2 - e^x - e^{x-(r_1+r_2)(e^x-1)}\}.\end{aligned}$$

Thus,

$$\begin{cases} (\varphi^*)^2(x) - x = (r_1 + r_2)\{2 - g_1(e^x)\}, \\ g_1'(t) = 1 + \{1 - (r_1 + r_2)t\}e^{(r_1+r_2)(1-t)}, \\ g_1''(t) = (r_1 + r_2)\{(r_1 + r_2)t - 2\}e^{(r_1+r_2)(1-t)}. \end{cases}$$

Therefore, we have

$$g_1'(t) \ge g_1'\left(\frac{2}{r_1 + r_2}\right) = 1 - e^{(r_1+r_2)-2} \ge 0, \; 0 < t < \infty,$$

and hence, $g_1(t)$ is a strictly monotone increasing function of t on $(0, \infty)$. Thus

$$\begin{cases} g_1(t) < g_1(1) = 2, & t < 1, \\ g_1(t) > g_1(1) = 2, & t > 1, \end{cases}$$

which implies (4.62).

Now, putting $r_2 = 0$ in (4.61), we have that $r \le 2$ which implies that (see for example [47]) the solution of (4.39)–(4.41) satisfies $\lim_{n\to\infty} x_n = 0$. The proof is complete. ∎

Remark 4. Lemma 4.2.6 implies that if

$$0 < r = r_1 \le 2 \;\text{ and }\; r_2 = 0, \tag{4.63}$$

then $\lim_{n\to\infty} x_n = 0$.

Lemma 4.2.7. *Assume in (4.40),*

$$r_1 > r_2 > 0,\ \ r_1 > 1 \quad \text{and}\ r_1 + r_2 - \frac{r_2}{r_1} e^{(r_1+r_2-1)} \geq 0. \tag{4.64}$$

Then, $\varphi(x)$ attains a unique local maximum at $L^* = -\log r_1 < 0$.

(a) For $L \leq 0$, put

$$G_1(L) = \varphi(L) - r_2 f(R_L^*) - L,\ \ G_1^*(L) = r_1 f(L) + r_2 f(R_L^*),$$
$$R_L^* = \varphi(L^*) - r_2 f(L).$$

Then, each of the following holds:

(i) $\lim_{L\to-\infty} G_1^*(L) \leq 0$,
(ii) $G_1^*(L^*) < 0$,
(iii) $\left(G_1^*\right)'(L) = 0$ *for some $L < L^*$, then $G_1^*(L) < 0$.*

Hence, $G_1^*(L) < 0$ and $G_1(L) > 0$, for any $L \leq L^*$.

(b) For $L \leq 0$, put

$$G_2(L) = \varphi(R_L^*) - r_2 f(R_L^*) - L,\ \ R_L^* = \varphi(L^*) - r_2 f(L). \tag{4.65}$$

Then, each of the following holds:

(i) $\lim_{L\to-\infty} G_2(L) = +\infty$,
(ii) $G_2(L^*) = (\varphi^*)^2(L^*) - L^* > 0$,
(iii) $(G_2)'(L) < 0$ *for any $L < L^*$.*

Hence, $G_2(L) > 0$ for any $L \leq L^*$.

(c) For $L \leq 0$, put

$$G_3(L) = \varphi(R_L) - r_2 f(R_L) - L,\ \ R_L = \varphi(L) - r_2 f(L). \tag{4.66}$$

Then, $G_3(L) = (\varphi^*)^2(L) - L > 0$ for any $L^* \leq L < 0$.

Proof. (a) (i) By assumptions, we have

$$\lim_{L\to-\infty} G_1^*(L) = \frac{r_2}{r_1} e^{(r_1+r_2)-1} - (r_1 + r_2) \leq 0.$$

(ii) Since $\varphi'(x) = 1 - r_1 e^x$, we have from (4.64), $L^* = -\log r_1 < 0$,

$$R_L^* = -\log r_1 + (r_1 + r_2) - 1 - r_2 e^L,$$

and

$$G_1^*(L^*) = 1 - (r_1 + r_2) + \frac{r_2}{r_1} e^{(r_1+r_2)(1-1/r_1)}.$$

Now, consider

$$g_2(x) = 1 - (x + r_2) + \frac{r_2}{x} e^{(x+r_2)-1-r_2/x},\ 1 < x \le 2 - r_2.$$

Then,

$$g_2'(x) = -1 + \left(-\frac{x - r_2}{x^2} + 1\right) \frac{r_2}{x} e^{(x+r_2)-1-r_2/x} \le -1 + \frac{r_2}{x} e^{1-r_2/x},$$
$$1 < x \le 2 - r_2.$$

For

$$g_3(t) = te^{1-t},\ \frac{r_2}{2 - r_2} \le t < 1,$$

we have

$$g_3'(t) = (1 - t)e^{1-t} > 0,\ \frac{r_2}{2 - r_2} \le t < 1,$$

and hence,

$$g_3(t) < g_3(1),\ \frac{r_2}{2 - r_2} \le t < 1.$$

Therefore, we get

$$g_2'(x) < -1 + g_3(1) = 0,\ 1 < x \le 2 - r_2.$$

Thus, $g_2(x)$ is a strictly decreasing function on $[1, 2 - r_2]$ and $G_1^*(L) < g_2(1) = 0$.

(iii) Since

$$\left(G_1^*\right)'(L) = r_1 e^L + \frac{r_2}{r_1} e^{r_1+r_2-1-r_2 e^L} (-r_2 e^L),$$

$\left(G_1^*\right)'(L) = 0$ implies that

$$r_1 e^L = \frac{r_2^2}{r_1} e^L e^{r_1+r_2-1-r_2 e^L}.$$

Therefore, if $\left(G_1^*\right)'(L) = 0$, for some $L^* \le L$, then, since

$$e^x > 1 + x \text{ for } x > 0 \text{ and } (r_1 + r_2) > \frac{r_2}{r_1} e^{r_1+r_2-1}$$

we have

$$\begin{aligned} G_1^*(L) &= r_1 e^L + \frac{r_2}{r_1} e^{r_1+r_2-1-r_2 e^L} - (r_1+r_2) \\ &\le (r_1+r_2)\left(\frac{r_2 e^L + 1}{e^{r_2 e^L}} - 1\right) < 0. \end{aligned}$$

Hence, from (a) (i)–(iii), we get $G_1(L) > 0$, for $L \le L^*$.

(b) (i) We have that

$$R_L^* = -\log r_1 - (r_1+r_2) - 1 - r_2 e^L,$$

and hence

$$\begin{cases} G_2(L) = -\log r_1 + 2(r_1+r_2) - 1 - r_2 e^L - \frac{r_1+r_2}{r_1} e^{r_1+r_2-1-r_2 e^L} - L, \\ \lim_{L\to-\infty} G_2(L) = \infty. \end{cases}$$

(ii) Since $L^* < 0$ by Lemma 4.2.6, we see that

$$G_2(L^*) = (\varphi^*)^2(L^*) - L^* > 0.$$

(iii) We have that

$$\begin{aligned} G_2'(L) &= -r_2 e^L \left(1 - \frac{r_1+r_2}{r_1} e^{r_1+r_2-1-r_2 e^L}\right) - 1, \\ \lim_{L\to-\infty} G_2'(L) &= -1 < 0. \end{aligned}$$

Thus

$$\begin{aligned} G_2'(L^*) &= -\frac{r_2}{r_1}\left(1 - \frac{r_1+r_2}{r_1} e^{(r_1+r_2)(1-1/r_1)}\right) - 1 \\ &= \frac{r_1+r_2}{r_1}\left(\frac{r_2}{r_1} e^{(r_1+r_2)(1-1/r_1)} - 1\right). \end{aligned}$$

Put

$$g_4(t) = (2t-1)e^{2(1-t)} - 1,\ \frac{1}{2} \le t < 1.$$

Then, we can easily see that

$$g_4(t) < g_4(1) = 0,\ \frac{1}{2} \le t < 1.$$

Therefore, we have that

$$\frac{r_2}{r_1} e^{(r_1+r_2)(1-1/r_1)} - 1 \le \frac{2-r_1}{r_1} e^{2(1-1/r_1)} - 1 = g_4(1/r_1) < g_4(1) = 0,$$

from which we get

$$G_2'(L^*) < 0, \text{ and } L^* < 0.$$

Now, we have for $L \le L^*$,

$$G_2''(L) = -r_2 e^L \left\{ 1 - (1 - r_2 e^L) \frac{r_1 + r_2}{r_1} e^{r_1 + r_2 - 1 - r_2 e^L} \right\}.$$

Thus, for $L_* < 0$, $G_2''(L_*) = 0$ implies that by $r_2 > 0$,

$$\frac{r_1 + r_2}{r_1} e^{r_1 + r_2 - 1} = \frac{e^{r_2 e^{L*}}}{1 - r_2 e^{L*}}.$$

Now, the equations

$$r_1 + r_2 = 2, \quad r_1 + r_2 - \frac{r_2}{r_1} e^{r_1 + r_2 - 1} = 0$$

have the unique solution (r_1^*, r_2^*) such that

$$r_1^* = \frac{2e}{e+2} < 2, \quad r_2^* = \frac{4}{e+2} < 1.$$

Then, for any r_1 and r_2 such that (4.64) holds, we easily see that

$$0 \le r_2 \le r_2^*, \quad 1 \le r_1 \le 2 - r_2.$$

For a fixed number r_2 such that $0 < r_2 \le r_2^* = 4/(e+2) < 1$, the function $p(r_1, r_2) = ((r_1 + r_2)/r_1) e^{r_1 + r_2 - 1}$ is a strictly monotone increasing function of r_1 on $[1, 2 - r_2]$. Thus, for $0 < r_2 \le r_2^* < 1$,

$$p(r_1, r_2) \le p(2 - r_2, r_2) = \frac{2e}{2 - r_2} \le \frac{2e}{2 - r_2^*} = e + 2.$$

The equation $e^x/(1-x) = e + 2$ has a unique positive solution $x^* = 0.60995\ldots < 1$, and the function $h_1(x) = e^x/(1-x)$ is a strictly monotone increasing function on $[0, 1)$. Thus, if $G_2''(\bar{L}) = 0$ for some $\bar{L} < 0$, then

$$\frac{r_1 + r_2}{r_1} e^{r_1 + r_2 - 1 - r_2 e^{\bar{L}}} = \frac{1}{1 - r_2 e^{\bar{L}}}$$

and $r_2 e^{\bar{L}} \le x^* < 1$ for any r_1 and r_2 which satisfy (4.64). Then,

$$G_2'(\bar{L}) = -r_2 e^{\bar{L}} \left(1 - \frac{1}{1 - r_2 e^{\bar{L}}} \right) - 1 = \frac{(r_2 e^{\bar{L}})^2 + r_2 e^{\bar{L}} - 1}{1 - r_2 e^{\bar{L}}}$$

and

$$\left(r_2 e^{\bar{L}}\right)^2 + r_2 e^{\bar{L}} - 1 < (x^*)^2 + x^* - 1 = -0.01800\ldots < 0.$$

Thus $G_2'(\bar{L}) < 0$ so $G_2'(L) < 0$ for $L \leq L^*$. Hence from (b) (ii), we get $G_2(L) \geq G_2(L^*) > 0$ for any $L \leq L^*$.

(c) By Lemma 4.2.6, we see that $G_3(L) = (\varphi^*)^2(L) - L > 0$ for any $L^* \leq L < 0$. ∎

Remark 5. Lemma 4.2.7 implies that if (4.64) holds, then the conditions in Lemma 4.2.2 are satisfied and then $\lim_{n\to\infty} x_n = 0$.

Similar reasoning as in the proof of Lemma 4.2.7 (see [75]) yields the following results.

Lemma 4.2.8. *Assume that*

$$1 > r_1 > r_2 > 0 \ \ \text{and}\ r_1 + r_2 - \frac{r_2}{r_1} e^{(r_1+r_2-1)} \geq 0. \tag{4.67}$$

Then, $\varphi(x)$ attains a unique local maximum at $R^* = -\log r_1 > 0$.

(a) For $L \leq 0$, put

$$G_4(L) = \varphi(L) - r_2 f(R_L^*) - L, \ \ G_4^*(L) = r_1 f(L) + r_2 f(R_L^*),$$
$$R_L^* = \varphi(R^*) - r_2 f(L).$$

Then there exists a unique $\bar{L} < 0$ such that

$$R^* = \varphi(R^*) - r_2 f(\bar{L}), \tag{4.68}$$

and each of the following holds:

(i) $\lim_{L\to-\infty} G_4^*(L) \leq 0,$

(ii) $G_4^*(\bar{L}) < 0,$

(iii) $\left(G_4^*\right)'(\bar{L}) > 0.$

Hence, $G_4(L) > 0$ for any $L \leq \bar{L} < 0$.

(b) For $L \leq 0$, put

$$G_5(L) = \varphi(\bar{R}_L^*) - r_2 f(\bar{R}_L^*) - L, \ \bar{R}_L^* = \varphi(R^*) - r_2 f(L). \tag{4.69}$$

Then, $G_5(\bar{R}_L^*) > \varphi(L)$ and $G_5(L) = \varphi^*(\bar{R}_L^*) - L > G_4(L) > 0$ for any $L \leq \bar{L}$.

Remark 6. Lemma 4.2.8 implies that if (4.67) holds, then the conditions in Lemma 4.2.3 are satisfied and then $\lim_{n\to\infty} x_n = 0$.

Lemma 4.2.9. *Assume in (4.40),*

$$r_1 = 1,\ r_2 > 0,\ \textit{and}\ r_2 e^{(r_2-1)} \le 1. \tag{4.70}$$

Then, $\varphi(x)$ attains a unique local maximum at $R^* = 0$, (4.57)–(4.59) hold, and we have the following:

(a) In Lemma 4.2.8 (a), for $G_4(L)$ with $R^* = 0$, $G_4(L) > 0$ for any $L < \bar{L} = 0$.
(b) For (4.69) with $R^* = 0$, $G_5(L) = \varphi^*(\bar{R}^*_L) - L > 0$ for any $L < \bar{L} = 0$.

Remark 7. Lemma 4.2.9 implies that if (4.70) holds, then (4.57)–(4.59) in Lemma 4.2.5 are satisfied and then $\lim_{n\to\infty} x_n = 0$.

Theorem 4.2.2. *Assume that*

$$r \in (0,\infty),\ p_0,\ p_1,\ldots,p_m \ge 0,\ \textit{and}\ \sum_{j=0}^{m} p_i > 0 \tag{4.71}$$

holds. Also suppose either (4.63) or (4.64) or (4.67) or (4.70) hold. Then for any solution $N(t)$ of (4.15) we have $\lim_{t\to\infty} N(t) = N^*$.

Proof. The result follows from the previous four Remarks. ■

4.3 Stability of Nonautonomous Models

In this section we examine nonautonomous logistic models with piecewise constant arguments. We begin with the logistic model

$$N'(t) = r(t)N(t)\left(1 - \frac{N([t])}{K}\right),\ t \ge 0, \tag{4.72}$$

with $N(0) = y_0 > 0$, $r : [0,\infty) \to [0,\infty)$ is a continuous function and K is a positive constant. First of all, by using the method of steps, we see that every solution $y(t)$ of (4.72) is positive for all $t \ge 0$. Thus, the change of variables $x(t) = \log(N(t)/K)$ reduces (4.72) to

$$x'(t) = -r(t)\left(e^{x([t])} - 1\right),\ t \ge 0. \tag{4.73}$$

We begin by presenting some global attractivity results from [46]. Let k be a nonnegative integer. On any interval $[k, k+1)$, (4.73) is expressed as

$$x'(t) = -r(t)\left(e^{x([k])} - 1\right). \tag{4.74}$$

It is clear that $x(t)$ is monotone on $[k, k+1]$ and that $x(k) = 0$ implies $x(t) = 0$ on $[k, \infty)$. Also integrating (4.74) from k to t, we have

$$x(t) - x(k) = -\left(e^{x(k)} - 1\right) \int_k^t r(s)ds. \tag{4.75}$$

We therefore obtain as $t \to k+1$, in (4.75),

$$x(k+1) = x(k) - \left(e^{x(k)} - 1\right) \int_k^{k+1} r(s)ds. \tag{4.76}$$

Lemma 4.3.1. *Let $x(t)$ be a solution of (4.73). Assume that*

$$\int_k^{k+1} r(s)ds \le 2, \text{ for } k = 0, 1, 2, \ldots, \tag{4.77}$$

and that there exists an increasing sequence of positive integers $\{t_n\}$ such that $(-1)^n x(t_n) > 0$ and $x(t_n)x(t_n+1) < 0$, for $n = 1, 2, \ldots$. Then the following is valid

$$-\frac{1}{2} < x(t_{2m+3}) < 0 < x(t_{2m+2}) < \frac{1}{3}, \quad \text{for } m = 0, 1, 2 \ldots .$$

Proof. In view of the definition of $\{t_n\}$ and the monotonicity of $x(t)$ on $[k, k+1]$, we notice that if $t_n + 1 < t_{n+1}$ then $|x(t)|$ is nonincreasing on $[t_n + 1, t_{n+1}]$, which implies

$$\begin{cases} x(t_n+1) \le x(t_{n+1}) < 0, \text{ if } \; n \text{ is even}, \\ 0 < x(t_{n+1}) \le x(t_n+1), \; \text{ if } n \text{ is odd}. \end{cases} \tag{4.78}$$

Note that,

$$x(t_{2m+1}) < 0 < x(t_{2m+1}+1), \text{ for } m = 0, 1, \ldots .$$

Let $A_m = -x(t_{2m+1})$. Then $A_m > 0$. It follows immediately from (4.77) and (4.76) that

$$\begin{aligned} x(t_{2m+1}+1) &= x(t_{2m+1}) - \left(e^{x(t_{2m+1})} - 1\right) \int_{t_{2m+1}}^{t_{2m+1}+1} r(s)ds \\ &\le -A_m + 2(1 - e^{-A_m}). \end{aligned} \tag{4.79}$$

Let

$$\phi(x) = -x + 2(1 - e^{-x}),$$

and observe that $\phi'(x) = -1 + 2e^{-x}$ (also $\phi'(x) = 0$ when $x = \log 2$). Note

$$\phi(A_m) \leq \phi(\log 2) = 1 - \log 2 < \frac{1}{3}.$$

Hence, by (4.78) and (4.79) we obtain that

$$0 < x(t_{2m+2}) < x(t_{2m+1} + 1) < \frac{1}{3}, \quad \text{for } m = 0, 1, \dots .$$

Next, let $B_m = x(t_{2m+2})$. Then $0 < B_m < \frac{1}{3}$ and

$$\begin{aligned} x(t_{2m+1} + 1) &= x(t_{2m+2}) - \left(e^{x(t_{2m+2})} - 1\right) \int_{t_{2m+2}}^{t_{2m+2}+1} r(s)ds \\ &\geq B_m - 2(e^{B_m} - 1). \end{aligned} \tag{4.80}$$

Let

$$\psi(x) = x - 2(e^x - 1).$$

Since $\psi'(t) = 1 - 2e^x < 0$ for $x > 0$, it follows that

$$\psi(B_m) \geq \psi(\frac{1}{3}) = \frac{7}{3} - 2e^{\frac{1}{3}} > -\frac{1}{2}.$$

Thus, by (4.78) and (4.80) we conclude that

$$0 > x(t_{2m+3}) \geq x(t_{2m+2} + 1) > -\frac{1}{2}, \text{ for } m = 0, 1, \dots .$$

The proof is complete. ■

Theorem 4.3.1. *Assume that (4.77) holds and*

$$\int_0^\infty r(s)ds = \infty. \tag{4.81}$$

Then the positive steady state $N(t) = K$ of (4.72) is globally attractive.

Proof. It suffices to show that if (4.77) and (4.81) are satisfied, then every solution $x(t)$ of (4.73) tends to 0 as $t \to \infty$. Suppose the solution $x(t)$ of (4.73) is nonoscillatory. Suppose $x(t)$ is eventually positive (the case when $x(t)$ is eventually negative is similar). Then from (4.73) we have eventually that $x(t)$ is decreasing, so $\lim_{t\to\infty} x(t) = \alpha \geq 0$. If $\alpha > 0$ we get as usual (i.e., there exists $t_0 \geq 0$ with $x'(t) \leq -r(t)\,(e^{\alpha} - 1)$ for $t \geq t_0$) a contradiction if we use (4.81). Thus $\lim_{t\to\infty} x(t) = 0$.

It remains to consider the case when $x(t)$ is oscillatory. Let $\{\xi_n : \xi_n < \xi_{n+1} : n = 1, 2, \ldots\}$ be the zeros of $x(t)$. We only have to consider the case when ξ_n is not integer for $n = 1, 2, \ldots$. Then there exists a sequence of positive integers $\{t_n\}$ such that

$$t_n < \xi_n < t_{n+1} \text{ and } \; x(t_n)x(t_n + 1) < 0, \; \text{for } n = 1, 2, \ldots .$$

Without loss of generality, we may assume that

$$(-1)^n x(t_n) > 0, \; \text{for } \; n = 1, 2, \ldots .$$

Put $A_m = -x(t_{2m+1})$ and $B_m = x(t_{2m+2})$ and then A_m and B_m are positive for $m = 0, 1, \ldots$. From Lemma 4.3.1, we may assume that

$$0 < A_m < \frac{1}{2}, \text{ for } m = 0, 1, \ldots .$$

We claim that A_m tends to 0 as $m \to \infty$. We first show that $0 < A_{m+1} < A_m$ for $m = 0, 1, \ldots$. Using

$$e^{-x} > 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3, \;\; \text{for } \; x > 0,$$

we have from (4.79) that

$$\begin{aligned} B_m &\leq x(t_{2m+1} + 1) \leq -A_m + 2(1 - e^{-A_m}) \\ &< -A_m + 2(A_m - \frac{A_m^2}{2} + \frac{A_m^3}{6}) \\ &= A_m - A_m^2 + \frac{A_m^3}{3} \equiv g^*(A_m). \end{aligned} \tag{4.82}$$

It is easy to see that $g^*(x)$ is an increasing function on $[0,\infty)$ and so

$$B_m < g^*(A_m) < g^*(\frac{1}{2}) < \frac{1}{3}. \tag{4.83}$$

Next, for $0 < x < \frac{1}{3}$, we have for $\theta \in (0, 1)$ that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{e^{\theta x}}{6}x^3 < 1 + x + \frac{x^2}{2} + \frac{x^3}{3},$$

because $e^x < 2$. Using the above fact, we have from (4.80) that

$$\begin{aligned} A_{m+1} &\le -x(t_{2m+2}+1) \le -B_m + 2(e^{B_m}-1) \\ &< -B_m + 2(B_m + \frac{B_m^2}{2} + \frac{B_m^3}{6}) \\ &= B_m + B_m^2 + \frac{2B_m^3}{3} \equiv h^*(B_m). \end{aligned} \tag{4.84}$$

Thus, (4.82) and (4.84) become

$$A_{m+1} < h^*(g^*(A_m)), \quad \text{for } m = 0, 1, \dots . \tag{4.85}$$

Furthermore, notice that

$$g^*(x) < x, \quad \text{if } 0 < x < \frac{1}{2}, \tag{4.86}$$

and we obtain

$$\begin{aligned} x - h^*(g^*(x)) &= x - \{(x - x^2 + \frac{x^3}{3}) + (x - x^2 + \frac{x^3}{3})^2 \\ &\quad + \frac{2}{3}(x - x^2 + \frac{x^3}{3})^3\} \\ &> x - (x - x^2 + \frac{x^3}{3}) - (x - x^2 + \frac{x^3}{3})^2 - \frac{2}{3}x^3 \\ &= x^2(x - \frac{5}{3}x^2 + \frac{2}{3}x^3 - \frac{1}{9}x^4) \\ &= \frac{x^3}{3}(3-5x) + \frac{x^5}{9}(6-x) \ > \ 0, \text{ for } 0 < x < \frac{1}{2}. \end{aligned} \tag{4.87}$$

This implies, together with (4.85), that

$$0 < A_{m+1} < h^*(g^*(A_m)) < A_m, \text{ for } m = 0, 1, \dots . \tag{4.88}$$

Consequently, there exists $\alpha \in [0, \frac{1}{2})$ such that $A_m \to \alpha$, as $m \to \infty$. If $\alpha > 0$, then we have as $m \to \infty$ in (4.85) $\alpha \le h^*(g^*(\alpha))$, which contradicts (4.87). Hence $A_m \to 0$ as $m \to \infty$.

Now we look at the behavior of $x(t)$. Recall that $|x(t)|$ is nonincreasing on $[t_n + 1, t_{n+1}]$ if $t_n + 1 < t_{n+1}$ and that

$$|x(t_n)| = \max_{t_n \le t \le \xi_n} |x(t)| \ \text{ and } \ |x(t_n+1)| = \max_{\xi_n \le t \le t_n+1} |x(t)| \, .$$

If follows from (4.82) and (4.86) that $0 < B_m < g^*(A_m) < A_m$, and hence $|x(t)| \leq A_m$ for all $t \in [t_{2m+1}, t_{2m+3}]$. Finally, by (4.88), we conclude that $x(t)$ tends to 0 as $t \to \infty$. The proof is complete. ■

In the following, we consider the logistic model with piecewise constant arguments

$$\frac{dN(t)}{dt} = r(t)N(t)\left\{1 - \sum_{j=0}^{m} a_j N[t-j]\right\}, \quad t \geq 0 \tag{4.89}$$

and establish some sufficient conditions for an arbitrary solution $N(t)$ satisfying the initial conditions of the form

$$N(0) = N_0 > 0 \text{ and } N(-j) = N_{-j} \geq 0, \; j = 1, 2, \ldots, m, \tag{4.90}$$

to converge to the positive equilibrium $N^* = 1/(\sum_{j=0}^{m} a_j)$ as $t \to \infty$; here $a_j \geq 0$, $j = 0, 1, \ldots, m-1$, $a_m > 0$, $\sum_{j=0}^{m} a_j > 0$ and $r : [0, \infty) \to (0, \infty)$ is a continuous function. The results are adapted from [69]. Using a method similar to Lemma 4.1.1, one can easily see that (4.89) together with (4.90) has a unique solution $N(t)$ which is positive for all $t \geq 0$. On any interval of the form $[n, n+1)$ for $n = 0, 1, 2, \ldots$, we can integrate (4.89) and obtain for $n \leq t < n+1$ and $n = 0, 1, 2, \ldots$

$$N(t) = N(n) \exp\left\{\left[1 - \sum_{j=0}^{m} a_j N(n-j)\right] \int_n^t r(s)ds\right\}. \tag{4.91}$$

Letting $t \to n+1$, we get that

$$N(n+1) = N(n) \exp\left\{r_n \left[1 - \sum_{j=0}^{m} a_j N(n-j)\right]\right\}, \tag{4.92}$$

where $r_n = \int_n^{n+1} r(s)ds$.

Lemma 4.3.2. *Let $N(t)$ be a solution of (4.89), (4.90). If $N(t)$ is eventually greater (respectively less) than N^* (i.e., N is a nonoscillatory solution about N^*) then* $\lim_{t\to\infty} N(t)$ *exists and is positive. Furthermore if*

$$\int_0^{\infty} r(t)dt = \infty, \tag{4.93}$$

then $\lim_{t\to\infty} N(t) = N^*$.

Proof. From (4.91), we know that $N(t)$ is positive for $t \geq 0$. Assume that $N(t)$ is eventually greater than N^* (the case when $N(t)$ is eventually less than N^* is similar and the proof is omitted). By (4.89) we have eventually

$$\frac{dN(t)}{dt} \leq r(t)N(t)\left\{1 - \sum_{j=0}^{m} a_j N^*\right\},$$

which implies that $N(t)$ is eventually decreasing and so $\lim_{t\to\infty} N(t)$ exists. Set $\alpha = \lim_{t\to\infty} N(t)$. We will show that (4.93) implies that $\alpha = N^*$. Indeed suppose $\alpha > N^*$. Then there exists $t_0 > m$ such that $N(t-m) \geq \alpha$ for $t \geq t_0$, since $N(t)$ eventually decreases to α. Using this in (4.89), we have

$$\begin{aligned}\frac{dN(t)}{dt} &\leq r(t)N(t)\left\{1 - \alpha\sum_{j=0}^{m} a_j\right\}\\ &= -\left(\frac{\alpha}{N^*} - 1\right) r(t)N(t), \quad \text{for } t \geq t_0.\end{aligned}$$

Integrating from t_0 to t, we have

$$\ln\frac{N(t)}{N(t_0)} \leq -\left(\frac{\alpha}{N^*} - 1\right)\int_{t_0}^{t} r(s)ds,$$

which implies that $\lim_{t\to\infty} \ln\frac{N(t)}{N(t_0)} = -\infty$. Then $\lim_{t\to\infty} N(t) = 0$, contradicting $\alpha > 0$. The proof is complete. ■

In the following, we first prove that the oscillatory solutions of (4.89) are bounded.

Lemma 4.3.3. *Assume that a solution $N(t)$ (4.89), (4.90) is oscillatory about N^*. If for some constant $M > 0$, we have*

$$\int_{n-m}^{n+1} r(s)ds \leq M, \text{ for all } n = m, m+1, \ldots, \tag{4.94}$$

then $N(t)$ is bounded above and is bounded below away from zero.

Proof. First, we prove that $N(t)$ is bounded from above. Suppose $\limsup_{t\to\infty} N(t) = \infty$. Since $N(t)$ is both unbounded and oscillatory, there exists a $t^* > m$ such that

$$N(t^*) = \max_{0\leq t\leq t^*} N(t) > N^*.$$

Since $N(t) > 0$ for $t \geq 0$ it follows from (4.89) that

$$\frac{dN(t)}{dt} \leq r(t)N(t), \quad \text{for } t \geq m. \tag{4.95}$$

From now on, let $D^-N(t)$ denote the leftsided derivative of $N(t)$. Then

$$D^-N(t^*) = r(t^*)N(t^*)\left\{1 - \sum_{j=0}^{m} a_j N([t^* - j])\right\} \geq 0,$$

if $t^* \notin \{0, 1, 2, \ldots\}$ and so $\sum_{j=0}^{m} a_j N([t^* - j]) \leq 1$. Thus there exists a $\xi \in [[t^*-m], t^*]$ such that $N(\xi) = N^*$ and $N(t) > N^*$ for $t \in (\xi, t^*]$. Integrating (4.95) from ξ to t^*, we have

$$\frac{N(t^*)}{N^*} \leq \exp\left(\int_{[t^*-m]}^{t^*} r(s)ds\right) \leq \exp\left(\int_{[t^*]-m}^{[t^*]+1} r(s)ds\right) \leq e^M.$$

If $t^* \in \{0, 1, 2, \ldots\}$, then

$$0 \leq D^-N(t^*) = r(t^*)N(t^*)\left\{1 - \sum_{j=0}^{m} a_j N(t^* - j - 1)\right\},$$

and so $\sum_{j=0}^{m} a_j N(t^*-j-1) \leq 1$. This implies that there exists a $\xi \in [t^*-m-1, t^*)$ such that $N(\xi) = N^*$ and $N(t) > N^*$ for $t \in (\xi, t^*]$. By (4.95), we have

$$\frac{N(t^*)}{N^*} \leq \exp\left(\int_{\xi}^{t^*} r(s)ds\right) \leq \exp\left(\int_{t^*-m-1}^{t^*} r(s)ds\right) \leq e^M.$$

Consequently, $\limsup_{t\to\infty} N(t) \leq N^*e^M$. This contradiction shows that $N(t)$ is bounded above and satisfies

$$N(t) \leq N^*e^M, \quad \text{for } t \geq m. \tag{4.96}$$

Substituting this into (4.89), we have for $t > 2m$ that

$$\frac{dN(t)}{dt} \geq r(t)N(t)\left\{1 - \sum_{j=0}^{m} a_j N^*e^M\right\} = r(t)N(t)(1 - e^M). \tag{4.97}$$

Next we show $N(t)$ is bounded below away from zero. Suppose $\liminf_{t\to\infty} N(t)=0$. Since $N(t)$ is oscillatory about N^*, there exists $t_* > 3m$ such that $N(t_*) = \min_{0\leq t\leq t_*} N(t) < N^*$. Clearly $D^-N(t_*) \leq 0$. If $t_* \in \{0, 1, 2, \ldots\}$ then

$$D^-N(t_*) = r(t_*)N(t_*)\left\{1 - \sum_{j=0}^{m} a_j N([t_* - j])\right\} \leq 0,$$

which shows that there exists $\eta \in [[t_* - m], t_*)$ such that $N(\eta) = N^*$ and $N(t) < N^*$ for $t \in (\eta, t_*]$. By (4.97), we have

$$\frac{N(t^*)}{N^*} \geq \exp\left((1 - e^M) \int_{\eta}^{t_*} r(s)ds\right) \geq \exp\left((1 - e^M) \int_{[t_* - m]}^{[t_*]+1} r(s)ds\right)$$
$$\geq Me^{(1-e^M)}.$$

If $t_* \in \{0, 1, 2, \ldots\}$, then

$$D^- N(t_*) = r(t_*)N(t_*)\left\{1 - \sum_{j=0}^{m} a_j N(t_* - j - 1)\right\},$$

which shows that there exists $\eta \in [t_* - m - 1, t_*)$ such that $N(\eta) = N^*$ and $N(t) < N^*$ for $t \in (\eta, t_*]$. By (4.97), we have

$$\frac{N(t^*)}{N^*} \geq \exp\left((1 - e^M) \int_{t_*-m-1}^{t_*} r(s)ds\right) \geq Me^{(1-e^M)}.$$

Consequently $\liminf_{t\to\infty} N(t) \geq N^* e^{-M\,(e^M - 1)}$, which is a contradiction. The proof is complete. ∎

Combining Lemma 4.3.2 with Lemma 4.3.3, we immediately see that if (4.94) holds, then the solution $N(t)$ of (4.89) is bounded above and bounded below from zero. Now, we are ready to provide sufficient conditions for the global stability of the positive equilibrium N^* of (4.89).

Theorem 4.3.2. *Let $N(t)$ be a solution of (4.89), (4.90). Assume that*

$$\int_{n-m}^{n+1} r(s)ds \leq \frac{3}{2},\ \textit{for } n = m, m+1, \ldots \tag{4.98}$$

and

$$\int_{0}^{\infty} r(s)ds = \infty. \tag{4.99}$$

Then

$$\lim_{t\to\infty} N(t) = N^*. \tag{4.100}$$

Proof. In view of Lemma 4.3.2, it suffices to prove that (4.100) holds if N is an oscillatory solution about N^*. By Lemma 4.3.3, $N(t)$ is bounded from above and bounded from below away from zero. Let

$$u = \limsup_{t\to\infty} N(t), \quad v = \liminf_{t\to\infty} N(t). \tag{4.101}$$

Then $0 < v \le N^* \le u < \infty$. It suffices to prove that $u = v = N^*$. For any $\varepsilon \in (0, v)$, choose an integer $T = T(\varepsilon) > 0$, such that

$$v_1 \equiv v - \varepsilon < N(t-m) < u + \varepsilon \equiv u_1, \;\text{ for } t \ge T. \tag{4.102}$$

Using (4.89), we have

$$\frac{dN(t)}{dt} \le r(t)N(t)\left\{1 - \frac{v_1}{N^*}\right\}, \;\text{ for } t \ge T, \tag{4.103}$$

and

$$\frac{dN(t)}{dt} \ge -r(t)N(t)\left\{\frac{u_1}{N^*} - 1\right\}, \;\text{ for } t \ge T. \tag{4.104}$$

Let $\{T_n\}$ be an increasing sequence such that $T_n \ge T + 2m$, $D^- N(T_n) \ge 0$, $N(T_n) > N^*$, $\lim_{n\to\infty} N(T_n) = u$ and $\lim_{n\to\infty} T_n = \infty$. If $T_n \notin \{0, 1, 2, \ldots\}$, then by (4.89), we have

$$\sum_{j=0}^{m} a_j N([t^* - j]) \le 1,$$

which implies that there exists $\xi_n \in [[T_n - m], T_n]$ such that $N(\xi_n) = N^*$ and $N(t) > N^*$ for $t \in (\xi_n, T_n]$. If $T_n \in \{0, 1, 2, \ldots\}$ then

$$\sum_{j=0}^{m} a_j N(t^* - j - 1) \le 1,$$

and so there exists $\xi_n \in [T_n - m - 1, T_n)$ such that $N(\xi_n) = N^*$ and $N(t) > N^*$ for $t \in (\xi_n, T_n]$. Thus by (4.98),

$$\int_{\xi_n}^{T_n} r(s)ds \le \frac{3}{2}.$$

For $T \le t \le \xi_n$, by integrating (4.103) from t to ξ_n, we get

$$\ln\left(\frac{N(\xi_n)}{N(t)}\right) \le \left(1 - \frac{v_1}{N^*}\right)\int_t^{\xi_n} r(s)ds$$

or

$$N(t) \ge N^* \exp\left(-\left(1 - \frac{v_1}{N^*}\right)\int_t^{\xi_n} r(s)ds\right), \;\text{ for } T \le t \le \xi_n. \tag{4.105}$$

For $j = 0, 1, 2, \ldots, m$, we define the sets

$$E_{1j} = \{t \in [\xi_n, T_n] : [t - j] \geq \xi_n\},$$
$$E_{2j} = \{t \in [\xi_n, T_n] : [t - j] \leq \xi_n\}.$$

Then $E_{1j} \cup E_{2j} = [\xi_n, T_n]$, $j = 0, 1, 2, \ldots, m$. Note that $t \in [\xi_n, T_n]$ implies $[t - m] \leq \xi_n$. For $t \in E_{1j}$, we have

$$N([t-j]) \geq N^* \geq N^* \exp\left(-\left(1 - \frac{v_1}{N^*}\right)\int_{[t-m]}^{\xi_n} r(s)ds\right),$$

and for $t \in E_{2j}$, by (4.105) we have

$$N([t-j]) \geq N^* \exp\left(-\left(1 - \frac{v_1}{N^*}\right)\int_{[t-j]}^{\xi_n} r(s)ds\right)$$
$$\geq N^* \exp\left(-\left(1 - \frac{v_1}{N^*}\right)\int_{[t-m]}^{\xi_n} r(s)ds\right),$$

since $[t - j] \geq [t - m] \geq [\xi_n - m] \geq [[T_n - m] - m] \geq [[T + m] - m] = T$. Hence

$$\frac{dN(t)}{dt} \leq r(t)N(t)\left(1 - \exp\left(-\left(1 - \frac{v_1}{N^*}\right)\int_{[t-j]}^{\xi_n} r(s)ds\right)\right).$$

Denote $1 - \frac{v_1}{N^*}$ by v^*. Then $0 < v^* < 1$. Thus for $t \in [\xi_n, T_n]$, we have

$$\frac{d \ln N(t)}{dt} \leq \min\left\{r(t)v^*, r(t)\left(1 - \exp\left(-v^*\int_{[t-m]}^{\xi_n} r(s)ds\right)\right)\right\}. \tag{4.106}$$

We now prove that

$$\ln\left(\frac{N(T_n)}{N^*}\right) \leq v^* - \frac{(v^*)^2}{6}. \tag{4.107}$$

There are two possibilities:

Case 1. $\int_{\xi_n}^{T_n} r(s)ds \leq -\frac{\ln\left(\frac{v_1}{N^*}\right)}{v^*} = -\frac{\ln(1-v^*)}{v^*}$.

By (4.106),

$$\ln\left(\frac{N(T_n)}{N^*}\right) \leq \int_{\xi_n}^{T_n} r(t)\left(1 - \exp\left(-v^*\int_{[t-m]}^{\xi_n} r(s)ds\right)\right)dt$$
$$= \int_{\xi_n}^{T_n} r(t)\left(1 - \exp\left(-v^*\left(\int_{[t-m]}^{t} r(s)ds - \int_{\xi_n}^{t} r(s)ds\right)\right)\right)dt$$

$$\begin{aligned}
&\leq \int_{\xi_n}^{T_n} r(t)\left(1-\exp\left(-v^*\left(\frac{3}{2}-\int_{\xi_n}^{t} r(s)ds\right)\right)\right)dt \\
&= \int_{\xi_n}^{T_n} r(t)dt - e^{-\frac{3}{2}v^*}\int_{\xi_n}^{T_n} r(t)\exp\left(v^*\int_{\xi_n}^{t} r(s)ds\right)dt \\
&= \int_{\xi_n}^{T_n} r(t)dt - \frac{e^{-\frac{3}{2}v^*}}{v^*}\int_{\xi_n}^{T_n} r(t)\left(\exp\left(v^*\int_{\xi_n}^{t} r(s)ds\right)-1\right)dt \\
&= \int_{\xi_n}^{T_n} r(t)dt - \frac{e^{-v^*\left(\frac{3}{2}-\int_{\xi_n}^{T_n} r(s)ds\right)}}{v^*}\left(1-\exp\left(-v^*\int_{\xi_n}^{t} r(s)ds\right)\right).
\end{aligned}$$

Note that $g(x) = x - \frac{e^{-v^*\left(\frac{3}{2}-x\right)}}{v^*}(1-e^{-v^*x})$ is increasing for $0 \leq x \leq \frac{3}{2}$.

For $\int_{\xi_n}^{T_n} r(s)ds \leq -\frac{\ln(1-v^*)}{v^*} \leq \frac{3}{2}$, we have

$$\begin{aligned}
&\ln\left(\frac{N(T_n)}{N^*}\right) \\
&\leq -\frac{\ln(1-v^*)}{v^*} - \frac{1}{v^*}\exp\left(-v^*\left(\frac{3}{2}+\frac{\ln(1-v^*)}{v^*}\right)\right)\left(1-e^{\ln(1-v^*)}\right) \\
&= -\frac{\ln(1-v^*)}{v^*} - \exp\left(-v^*\left(\frac{3}{2}+\frac{\ln(1-v^*)}{v^*}\right)\right) \\
&\leq -\frac{\ln(1-v^*)}{v^*} - \left[1-v^*\left(\frac{3}{2}+\frac{\ln(1-v^*)}{v^*}\right)\right] \\
&\leq -1+\frac{3}{2}v^* - \frac{(1-v^*)\ln(1-v^*)}{v^*} \\
&\leq \frac{3}{2}v^* - \frac{1}{v^*}\int_0^{v^*}\left(\int_0^{y}\frac{dx}{1-x}\right)dy \\
&\leq \frac{3}{2}v^* - \frac{1}{v^*}\int_0^{v^*}\int_0^{y}(1+x)dxdy = v^* - \frac{(v^*)^2}{6}.
\end{aligned} \tag{4.108}$$

For $\int_{\xi_n}^{T_n} r(s)ds \leq \frac{3}{2} < -\frac{\ln(1-v^*)}{v^*}$, we have

$$\begin{aligned}
&\ln\left(\frac{N(T_n)}{N^*}\right) \\
&\leq \int_{\xi_n}^{T_n} r(t)dt - \frac{1}{v^*}\left(e^{-\frac{3}{2}v^*}\exp\left(v^*\int_{\xi_n}^{t} r(s)ds\right)-e^{-\frac{3}{2}v^*}\right) \\
&\leq \frac{3}{2v^*} - \frac{1}{v^*}(1-e^{-\frac{3}{2}v^*}) \leq v^* - \frac{(v^*)^2}{6}.
\end{aligned}$$

Case 2. $-\frac{\ln(1-v^*)}{v^*} < \int_{\xi_n}^{T_n} r(s)ds \le \frac{3}{2}$.

Choose $h_n \in (\xi_n, T_n]$ such that

$$\int_{h_n}^{T_n} r(s)ds = -\frac{\ln(1-v^*)}{v^*}.$$

Then by (4.106) and (4.98), we have

$$\begin{aligned}
\ln\left(\frac{N(T_n)}{N^*}\right) &\le \int_{\xi_n}^{h_n} v^* r(s)ds \\
&\quad + \int_{h_n}^{T_n} r(t)\left(1 - \exp\left(-v^* \int_{[t-m]}^{\xi_n} r(s)ds\right)\right)dt \\
&= v^* \int_{\xi_n}^{h_n} r(s)ds + \int_{h_n}^{T_n} r(s)ds \\
&\quad - \int_{h_n}^{T_n} r(t)\exp\left(-v^* \int_{[t-m]}^{t} r(s)ds + v^* \int_{\xi_n}^{t} r(s)ds\right)dt \\
&\le v^* \int_{\xi_n}^{h_n} r(s)ds + \int_{h_n}^{T_n} r(s)ds \\
&\quad - e^{-\frac{3}{2}v^*} \int_{h_n}^{T_n} r(t)\exp\left(v^* \int_{\xi_n}^{t} r(s)ds\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\ln\left(\frac{N(T_n)}{N^*}\right) &\le v^* \int_{\xi_n}^{h_n} r(t)dt + (1-v^*)\int_{h_n}^{T_n} r(t)dt \\
&\quad - \exp\left(-v^*\left(\frac{3}{2} - \int_{\xi_n}^{T_n} r(s)ds\right)\right) \\
&= -\frac{(1-v^*)\ln(1-v^*)}{v^*} + v^* \int_{\xi_n}^{T_n} r(t)dt \\
&\quad - \exp\left(-v^*\left(\frac{3}{2} - \int_{\xi_n}^{T_n} r(s)ds\right)\right) \\
&\le -\frac{(1-v^*)\ln(1-v^*)}{v^*} + \frac{3}{2}v^* - 1,
\end{aligned}$$

since $h(x) = xv^* - e^{-v^*\left(\frac{3}{2}-x\right)}$ is increasing for $0 \le x \le \frac{3}{2}$. Thus according to (4.108),

$$\ln\left(\frac{N(T_n)}{N^*}\right) \le v^* - \frac{(v^*)^2}{6}.$$

This completes the proof of (4.107).

Let $n \to \infty$ and $\varepsilon \to 0$ in (4.107), and we have

$$\ln\left(\frac{u}{N^*}\right) \le \left(1 - \frac{v}{N^*}\right) - \frac{1}{6}\left(1 - \frac{v}{N^*}\right)^2. \tag{4.109}$$

Next, let $\{S_n\}$ be an increasing sequence such that $S_n \ge T + m$, $D^- N(S_n) \le 0$, $N(S_n) < N^*$, $\lim_{n\to\infty} N(S_n) = v$ and $\lim_{n\to\infty} S_n = \infty$. If $S_n \notin \{0, 1, 2, \ldots, m\}$, then by (4.89), we have

$$\sum_{j=0}^{m} a_j N([S_n - j]) \ge 1,$$

which implies that there exists $\eta_n \in [[S_n - m], S_n]$ such that $N(\eta_n) = N^*$ and $N(t) < N^*$ for $t \in (\eta_n, S_n]$. If $S_n \in \{0, 1, 2, \ldots\}$ then

$$\sum_{j=0}^{m} a_j N(S_n - j - 1) \ge 1,$$

and so there exists $\eta_n \in [S_n - m - 1, S_n)$ such that $N(\eta_n) = N^*$ and $N(t) < N^*$ for $t \in (\eta_n, S_n]$. Thus by (4.98),

$$\int_{\eta_n}^{S_n} r(s)ds \le \frac{3}{2}.$$

For $T \le t \le \eta_n$, by integrating (4.104) from t to η_n, we get

$$\ln\left(\frac{N(\eta_n)}{N(t)}\right) \ge -\left(\frac{u_1}{N^*} - 1\right)\int_t^{\eta_n} r(s)ds$$

or

$$N(t) \le N^* \exp\left(\left(\frac{u_1}{N^*} - 1\right)\int_t^{\eta_n} r(s)ds\right), \text{ for } T \le t \le \eta_n. \tag{4.110}$$

For $j = 0, 1, 2, \ldots, m$, we define the sets

$$F_{1j} = \{t \in [\eta_n, T_n] : [t - j] \ge \eta_n\},$$
$$F_{2j} = \{t \in [\eta_n, T_n] : [t - j] \le \eta_n\}.$$

Then $F_{1j} \cup F_{2j} = [\eta_n, S_n]$, $j = 0, 1, 2, \ldots, m$. Note that $t \in [\eta_n, S_n]$ implies $[t-m] \leq \eta_n$. For $t \in F_{1j}$, we have

$$N([t-j]) \leq N^* \leq N^* \exp\left(\left(\frac{u_1}{N^*} - 1\right) \int_{[t-m]}^{\eta_n} r(s)ds\right),$$

and for $t \in F_{2j}$, by (4.110), we have

$$\begin{aligned} N([t-j]) \leq N^* &\leq N^* \exp\left(\left(\frac{u_1}{N^*} - 1\right) \int_{[t-j]}^{\eta_n} r(s)ds\right) \\ &\leq N^* \exp\left(\left(\frac{u_1}{N^*} - 1\right) \int_{[t-m]}^{\eta_n} r(s)ds\right). \end{aligned}$$

Hence

$$\frac{dN(t)}{dt} \geq -r(t)N(t)\left(\exp\left(\left(\frac{u_1}{N^*} - 1\right) \int_{[t-j]}^{\eta_n} r(s)ds\right) - 1\right).$$

Let $u^* = \frac{u_1}{N^*} - 1$. Thus for $t \in [\eta_n, S_n]$, we have

$$\frac{d \ln N(t)}{dt} \geq \max\left\{-r(t)u^*, -r(t)\left(\exp\left(u^* \int_{[t-m]}^{\eta_n} r(s)ds\right) - 1\right)\right\}. \tag{4.111}$$

We now prove that

$$-\ln\left(\frac{N(S_n)}{N^*}\right) \leq u^* + \frac{(u^*)^2}{6}. \tag{4.112}$$

There are three cases to consider:

Case (i). $\int_{\eta_n}^{S_n} r(t)dt \leq 1$.

Then by (4.111),

$$-\ln\left(\frac{N(S_n)}{N^*}\right) \leq u^* \int_{\eta_n}^{S_n} r(t)dt \leq u^* < u^* + \frac{(u^*)^2}{6}.$$

Case (ii). $1 < \int_{\eta_n}^{S_n} r(t)dt \leq \frac{3}{2} - \ln \frac{1+u^*}{u^*}$.

Clearly $u^* > 2$ in this case. We have

$$-\ln\left(\frac{N(S_n)}{N^*}\right) \leq u^* \int_{\eta_n}^{S_n} r(t)dt \leq \frac{3}{2}u^* - \ln\left(1 + u^*\right) < u^* + \frac{(u^*)^2}{6}.$$

Case (iii). $\frac{3}{2} - \ln \frac{1+u^*}{u^*} < \int_{\eta_n}^{S_n} r(t)dt \le \frac{3}{2}$.

Choose $g_n \in (\eta_n, S_n)$ such that

$$\int_{g_n}^{S_n} r(t)dt = \frac{3}{2} - \ln \frac{1+u^*}{u^*}.$$

Then by (4.111),

$$\begin{aligned}
&-\ln\left(\frac{N(S_n)}{N^*}\right)\\
&\le u^* \int_{\eta_n}^{g_n} r(t)dt + \int_{g_n}^{S_n} r(t)\left(\exp\left(u^* \int_{[t-m]}^{\eta_n} r(s)ds\right) - 1\right)dt\\
&\le u^* \int_{\eta_n}^{g_n} r(t)dt - \int_{g_n}^{S_n} r(t) + e^{\frac{3}{2}u^*} \int_{g_n}^{S_n} r(t)\left(-u^* \int_{\eta_n}^{t} r(s)ds\right)dt\\
&= u^* \int_{\eta_n}^{g_n} r(t)dt - \int_{g_n}^{S_n} r(t)\\
&\quad + \frac{1}{u^*} e^{\frac{3}{2}u^*}\left(\exp\left(-u^* \int_{\eta_n}^{g_n} r(s)ds\right) - \exp\left(-u^* \int_{\eta_n}^{S_n} r(s)ds\right)\right)\\
&= u^* \int_{\eta_n}^{g_n} r(t)dt - \int_{g_n}^{S_n} r(t)\\
&\quad + \frac{1}{u^*} \exp\left(u^*\left(\frac{3}{2} - \int_{\eta_n}^{g_n} r(s)ds\right)\right) - \exp\left(u^*\left(\frac{3}{2} - \int_{\eta_n}^{S_n} r(s)ds\right)\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
&-\ln\left(\frac{N(S_n)}{N^*}\right)\\
&\le u^* \int_{\eta_n}^{g_n} r(t)dt - \int_{g_n}^{S_n} r(t)\\
&\quad + \frac{1}{u^*}\left(1 + u^* - 1 - u^*\left(\frac{3}{2} - \int_{\eta_n}^{S_n} r(s)ds\right)\right)\\
&= u^* \int_{\eta_n}^{g_n} r(t)dt - \frac{1}{2} + \int_{\eta_n}^{g_n} r(t)dt = (u^* + 1)\int_{\eta_n}^{g_n} r(t)dt - \frac{1}{2}\\
&= 1 + \frac{3}{2}u^* - \frac{(1+u^*)\ln(1+u^*)}{u^*} = \frac{3}{2}u^* - \frac{1}{u^*}\int_0^{u^*}\left(\int_0^x \frac{dy}{1+y}\right)dx\\
&\le \frac{3}{2}u^* - \int_0^{u^*}\int_0^x (1-y)dydx = u^* + \frac{(u^*)^2}{6}.
\end{aligned}$$

This proves that (4.112) holds.

Let $n \to \infty$ and $\varepsilon \to 0$ in (4.112), and we have

$$-\ln\left(\frac{v}{N^*}\right) \leq \left(\frac{u}{N^*} - 1\right) + \frac{1}{6}\left(\frac{u}{N^*} - 1\right)^2. \quad (4.113)$$

Set $x = \frac{u}{N^*} - 1$ and $y = 1 - \frac{v}{N^*}$. Then $x \geq 0, 0 \leq y < 1$. By (4.109) and (4.113), we see that

$$\begin{cases} -\ln(1-y) \leq x + \frac{1}{6}x^2, \\ \ln(1+x) \leq y - \frac{1}{6}y^2. \end{cases} \quad (4.114)$$

In view of Lemma 3.2.2 we see that the system (4.114) has only solution $x = y = 0$. This shows that $v = u = N^*$ and completes the proof. ■

We will now present another result (see [49]) which is different from Theorem 4.3.2. Consider (4.89), (4.90) with $r(t)$ is continuous on $[0, \infty)$, $r(t) \geq 0,\ r(t) \not\equiv 0$, and

$$a_0 > 0 \text{ and } a_k \geq 0,\ 1 \leq k \leq m. \quad (4.115)$$

Let

$$r_l^t = \int_l^t r(s)ds,\ l \leq t < l+1 \text{ and } r_l = \int_l^{l+1} r(s)ds,\ l = 0, 1, 2, \ldots. \quad (4.116)$$

Let

$$N^* = \frac{1}{\sum_{k=0}^m a_k} \text{ and } t_l = 1 - \sum_{k=0}^m a_k N(l-k),\ l = 0, 1, 2, \ldots. \quad (4.117)$$

From (4.92) we obtain

$$N(l+1) = N(l)\exp(r_l t_l),\ l = 0, 1, 2, \ldots, \quad (4.118)$$

and

$$N(l+1) - N^* = \begin{cases} \{1 - a_0 N(l) R_l\}(N(l) - N^*) - N(l) R_l \sum_{k=1}^m a_k (N(l-k) - N^*), \\ \qquad \text{if } t_l \neq 0, \\ \{1 - a_0 N(l) r_l\}(N(l) - N^*) - N(l) r_l \sum_{k=1}^m a_k (N(l-k) - N^*), \\ \qquad \text{if } t_l = 0, \end{cases} \quad (4.119)$$

where

$$R_l := \frac{\exp(r_l t_l) - 1}{t_l}.$$

Let $k_l = N(l)\tilde{f}(t_l; r_l),\ l = 0, 1, 2, \dots,$

$$\underline{r} = \inf_{l \geq 0}(r_l) \quad \text{and} \quad \tilde{f}(t; r) = \begin{cases} \frac{e^{rt}-1}{t}, & t \neq 0, \\ r, & t = 0. \end{cases} \tag{4.120}$$

Lemma 4.3.4. *Assume that*

$$0 < r_l \leq 1,\ l = 0, 1, 2, \dots . \tag{4.121}$$

Then, for any positive integer l we have

$$N(l+1) \leq \frac{1}{r_l a_0} \text{ and } \ 1 - a_0 k_l \geq 0. \tag{4.122}$$

Moreover, if $\underline{r} > 0$, then there exists a positive constant $\underline{k}$ such that for any sufficiently large positive integer l,

$$\underline{k} \leq k_l \leq \frac{1}{a_0}, \tag{4.123}$$

and

$$\limsup_{l \to \infty} N(l) \leq \frac{1}{a_0} \text{ and } \liminf_{l \to \infty} N(l) \geq \left(\frac{1}{a_0}\right)^2 \left(a_0 - \sum_{k=1}^{m} a_k\right). \tag{4.124}$$

Proof. We easily see that (4.121) implies (4.94) and hence by Lemma 4.3.3 (and Lemma 4.3.2) $N(l)$ is bounded above and is bounded below from 0. For $f(x) = xe^{r-ax}$, where r and a are positive constants, we have

$$\max_{0 \leq x < \infty} f(x) = \frac{e^{r-1}}{a}.$$

Thus, by (4.118) and (4.121), we see for $l \geq 0$,

$$N(l+1) \leq N(l)\exp\{r_l(1 - a_0 N(l))\} \leq \frac{\exp(r_l - 1)}{r_l a_0} \leq \frac{1}{r_l a_0}.$$

Put

$$\bar{t}_l = 1 - r_l a_0 N(l), \text{ for any } l \geq 1, \tag{4.125}$$

and note $\bar{t}_l \geq 0$ for any $l \geq 1$. Consider the function

$$g(x) = \begin{cases} \frac{e^x - 1}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Then,

$$g'(x) = \begin{cases} \frac{1}{x^2}\{(x-1)e^x + 1\}, & x \neq 0, \\ \frac{1}{2}, & x = 0, \end{cases}$$

and for $h(x) = (x-1)e^x + 1$, $h'(x) = xe^x$. Then, $h(x) \geq h(0) = 0$, $g'(x) \geq 0$ and hence, $g(x)$ is a strictly monotone increasing function of x on $(-\infty, +\infty)$. Hence, by (4.117), (4.121) and (4.125), we have that $r_l t_l \leq \bar{t}_l$ and

$$a_0 k_l \leq r_l a_0 N(l) g(r_l t_l) \leq (1 - \bar{t}_l) g(\bar{t}_l), \ l = 1, 2, \ldots .$$

We easily see that

$$(1-x)\frac{e^x - 1}{x} \leq 1, \text{ for any } 0 < x \leq 1.$$

Hence, we have (4.122). In (4.117), we see

$$t_l \geq \underline{t}_l \equiv 1 - \sum_{k=0}^{m} a_k \left(\limsup_{l \to \infty} N(l)\right) > -\infty.$$

Put

$$\underline{k} \equiv \begin{cases} \frac{1}{2}(\liminf_{l\to\infty} N(l)) \frac{\exp(\underline{r}\,\underline{t}_l)-1}{\underline{t}_l} > 0, & \underline{t}_l \neq 0, \\ \frac{1}{2}(\liminf_{l\to\infty} N(l))\underline{r} > 0, & \underline{t}_l = 0. \end{cases}$$

Then, for any sufficiently large positive integer l,

$$0 < \underline{k} \leq k_l \leq \frac{1}{a_0}.$$

Next, let a sequence $\{l_p\}_{p=1}^{\infty}$ satisfy

$$0 < \lim_{l \to \infty} N(l_p) = \limsup_{l \to \infty} N(l) < +\infty.$$

Then, by (4.119) and (4.123), we have that for any sufficiently large positive integer p,

$$\begin{aligned} &N(l_{p+1}) - N^* \\ \leq\ &(1 - a_0 k_{l_{p+1}-1})(N(l_{p+1}-1) - N^*) \\ &-k_{l_{p+1}-1}\left(\sum_{k=1}^{m} a_k\right)\left(\min_{1 \leq k \leq m+1} N(l_{p+1}-k) - N^*\right). \end{aligned}$$

Let $p \to +\infty$, in the above equation and so

$$\lim_{p\to\infty} \max_{1\le k\le m+1} N(l_{p+1}-k) \le \limsup_{l\to\infty} N(l)\ ,$$

$$\lim_{p\to\infty} \min_{1\le k\le m+1} N(l_{p+1}-k) \ge \liminf_{l\to\infty} N(l),$$

and by (4.123), we have

$$a_0(\limsup_{l\to\infty} N(l) - N^*) + (\sum_{k=1}^{m} a_k)(\liminf_{l\to\infty} N(l) - N^*) \le 0.$$

Then, we get

$$\begin{aligned}\limsup_{l\to\infty} N(l) - N^* &\le -\left\{(\sum_{k=1}^{m} a_k)/a_0\right\} (\liminf_{l\to\infty} N(l) - N^*) \\ &\le \left\{(\sum_{k=1}^{m} a_k)/a_0\right\} N^*,\end{aligned}$$

and hence, we have

$$\limsup_{l\to\infty} N(l) \le \left\{1 + (\sum_{k=1}^{m} a_k)/a_0\right\} N^* = \frac{1}{a_0}.$$

Similarly, let a sequence $\{l_p\}_{p=1}^{\infty}$ satisfy

$$\lim_{p\to\infty} N(l_p) = \liminf_{l\to\infty} N(l) > 0.$$

By (4.119) and (4.123), we have that for any sufficiently large positive integer l,

$$\begin{aligned}N(l_{p+1}) - N^* \ge\ & (1 - a_0 k_{l_{p+1}-1})(N(l_{p+1}-1) - N^*) \\ & -k_{l_{p+1}-1}(\sum_{k=1}^{m} a_k)(\max_{1\le k\le m+1} N(l_{p+1}-k) - N^*).\end{aligned}$$

Therefore from a similar argument to the above we get

$$a_0(\liminf_{l\to\infty} N(l) - N^*) + (\sum_{k=1}^{m} a_k)(\limsup_{l\to\infty} N(l) - N^*) \ge 0.$$

Then,

$$\liminf_{l\to\infty} N(l) - N^* \ge -\left\{(\sum_{k=1}^{m} a_k)/a_0\right\} (\limsup_{l\to\infty} N(l) - N^*)$$

and

$$\liminf_{l\to\infty} N(l) \geq \{1 + (\sum_{k=1}^{m} a_k)/a_0)\}N^* - \{(\sum_{k=1}^{m} a_k)/a_0\} \limsup_{l\to\infty} N(l)$$
$$= \frac{1}{a_0} - \{(\sum_{k=1}^{m} a_k)/a_0)\} \limsup_{l\to\infty} N(l).$$

Thus, by the first part of (4.124),

$$\liminf_{l\to\infty} N(l) \geq \frac{1}{a_0} - \left\{(\sum_{k=1}^{m} a_k)/a_0\right\} \frac{1}{a_0} = \left(\frac{1}{a_0}\right)^2 \left\{a_0 - \sum_{k=1}^{m} p_k\right\}.$$

The proof is complete. ■

Now we have our main result.

Theorem 4.3.3. *Assume that*

$$a_0 > \sum_{k=1}^{m} a_k \text{ and } 0 < r_l \leq 1,\ l = 0, 1, 2, \ldots . \tag{4.126}$$

Then

$$|N(l+1) - N^*| \leq \max_{0\leq k\leq m} |N(l-k) - N^*|,\ l = 0, 1, 2, \ldots, \tag{4.127}$$

which implies that solutions of (4.89), (4.90) have the contractivity property. Moreover, if $\underline{r} = \inf_{l\geq 0} r_l > 0$*, then the positive equilibrium* N^* *of (4.89), (4.90) is globally asymptotically stable.*

Proof. Note

$$|N(l+1) - N^*|$$
$$\leq (1 - a_0 k_l)|N(l) - N^*| + k_l(\sum_{k=1}^{m} a_k)(\max_{1\leq k\leq m} |N(l-k) - N^*|)$$
$$\leq \{1 - k_l(a_0 - \sum_{k=1}^{m} a_k)\}(\max_{0\leq k\leq m} |N(l-k) - N^*|),\ l = 0, 1, 2, \ldots,$$

from which by (4.126), we get (4.127). Moreover, if $\underline{r} > 0$, then by Lemma 4.3.4,

$$1 - k_l(a_0 - \sum_{k=1}^{m} a_k) < 1 - \underline{k}(a_0 - \sum_{k=1}^{m} a_k) < 1,$$

for any large positive integer l, and hence, $\lim_{l\to\infty} N(l) = N^*$. The proof is complete. ■

Next we discuss the equation

$$\begin{cases} \frac{dN(t)}{dt} = N(t)r(t)\left\{1 - aN(t) - \sum_{i=0}^{m} b_i N(n-i)\right\}, \\ n \le t < n+1,\ n = 0, 1, 2, \dots, \\ N(0) = N_0 > 0,\ \text{and } N(-j) = N_{-j} \ge 0,\ j = 1, 2, \dots, m, \end{cases} \tag{4.128}$$

where $r(t)$ is a nonnegative continuous function on $[0, +\infty)$,

$$r(t) \not\equiv 0, \sum_{i=0}^{m} b_i > 0,\ b_i \ge 0,\ i = 0, 1, 2, \dots, m \text{ and } a + \sum_{i=0}^{m} b_i > 0.$$

The results below are adapted from [48]. Note the positive equilibrium of (4.128) is

$$N^* = \frac{1}{a + \sum_{j=0}^{m} b_j}.$$

For $r > 0$ and $-1 < \alpha < 1$, put

$$f(t; r) = \begin{cases} (1-t)\frac{e^{rt}-1}{t}, & t \ne 0, \\ r, & t = 0, \end{cases} \tag{4.129}$$

and consider the conditions of $r > 0$ such that

$$f(t; r) \le \frac{2}{1-\alpha}, \text{ for any } t < 1. \tag{4.130}$$

For $-1 < \alpha < 1$, consider the function $g(Y; \alpha)$ of Y on $(-1, 1)$:

$$g(Y; \alpha) = \begin{cases} \frac{1}{2(\alpha+Y)} \ln \frac{(1+\alpha)(1+Y)}{(1-\alpha)(1-Y)}, & Y \ne -\alpha \\ \frac{1}{1-\alpha^2}, & Y = -\alpha. \end{cases} \tag{4.131}$$

Lemma 4.3.5. *Under the conditions that $0 < \hat{Y} < \alpha$ for $0 < \alpha < 1$, and $\alpha < \hat{Y} < 0$ for $-1 < \alpha < 0$, there exists a unique solution $\hat{Y} = \hat{Y}(\alpha)$ of the equation:*

$$\frac{1}{1-\hat{Y}^2} = g(\hat{Y}; \alpha),\ -1 < \alpha < 1. \tag{4.132}$$

In particular, $\hat{Y}(0) = 0$ and

$$\lim_{\alpha\to 0} \hat{Y}(\alpha) = \hat{Y}(0). \tag{4.133}$$

Proof. For $Y \neq -\alpha$,

$$g'(Y;\alpha) = \frac{1}{\alpha+Y}\left\{\frac{1}{1-Y^2} - \frac{1}{2(\alpha+Y)}\ln\frac{(1+\alpha)(1+Y)}{(1-\alpha)(1-Y)}\right\}$$

$$= \frac{1}{\alpha+Y}\{\frac{1}{1-Y^2} - g(Y;\alpha)\}.$$

Moreover, for $\alpha \neq 0$,

$$g(0;\alpha) = g(\alpha;\alpha) = \frac{1}{2\alpha}\ln\frac{1+\alpha}{1-\alpha} = 1 + \frac{\alpha^2}{3} + \frac{\alpha^4}{5} + \ldots,$$

and

$$\begin{cases} g'(0;\alpha) = \frac{1}{\alpha}\{1 - g(0;\alpha)\} = -\alpha(\frac{1}{3} + \frac{\alpha^2}{5} + \cdots), \\ g'(\alpha;\alpha) = \frac{1}{2\alpha}\{\frac{1}{1-\alpha^2} - g(\alpha;\alpha)\} = \alpha(\frac{1}{3} + \frac{2\alpha^2}{5} + \ldots). \end{cases}$$

Thus, for $-1 < \alpha < 1$ and $\alpha \neq 0$, there is a solution $\hat{Y} = \hat{Y}(\alpha)$ such that $0 < \hat{Y} < \alpha$ for $0 < \alpha < 1$ and $\alpha < \hat{Y} < 0$ for $-1 < \alpha < 0$, and it satisfies the equation

$$g'(\hat{Y};\alpha) = 0,$$

and hence

$$\frac{1}{1-\hat{Y}^2} = g(\hat{Y};\alpha),$$

and, since $0 < \hat{Y} < \alpha$ for $0 < \alpha < 1$ or $\alpha < \hat{Y} < 0$ for $-1 < \alpha < 0$, we have

$$g''(\hat{Y};\alpha) = \frac{1}{\alpha+\hat{Y}}\{-\frac{(-2\hat{Y})}{(1-\hat{Y}^2)^2} - g'(\hat{Y};\alpha)\}$$

$$= \frac{1}{\alpha+\hat{Y}}\frac{2\hat{Y}}{(1-\hat{Y}^2)^2} > 0.$$

Hence, under the conditions that $0 < \hat{Y} < \alpha$ for $0 < \alpha < 1$ and $\alpha < \hat{Y} < 0$ for $-1 < \alpha < 0$, this solution $\hat{Y} = \hat{Y}(\alpha)$ of (4.132), is unique for $0 < |\alpha| < 1$. We see that $\hat{Y} = \hat{Y}(0) = 0$ is a unique solution of (4.132). Further, we have

$$\lim_{\alpha\to+0}\hat{Y}(\alpha) \geq 0, \text{ and } \lim_{\alpha\to-0}\hat{Y}(\alpha) \leq 0,$$

and hence, $\lim_{\alpha\to 0}\hat{Y}(\alpha) = 0$, from which we have (4.133). The proof is complete. ■

Remark 8. Note that for $0 < |\alpha| < 1$, the equation

$$\frac{1}{1-Y^2} = g(Y;\alpha),$$

has another solution $\hat{Y} = -\alpha$, but this solution does not satisfy the conditions $0 < \hat{Y} < \alpha$ for $0 < \alpha < 1$ and $\alpha < \hat{Y} < 0$ for $-1 < \alpha < 0$.

Lemma 4.3.6. *For $-1 < \alpha < 1$, let $\hat{Y}(\alpha)$ be defined as in Lemma 4.3.5 and put*

$$\hat{r}(\alpha) = \frac{2(1+\alpha)}{1-\hat{Y}^2(\alpha)} \text{ and } \hat{t}(\alpha) = \frac{\alpha+\hat{Y}(\alpha)}{1+\alpha}. \tag{4.134}$$

Then, $\hat{r}(\alpha)$ is a strictly monotone increasing function of α on the interval $(-1, 1)$, and

$$\lim_{\alpha\to -1+0} \hat{r}(\alpha) = 0 \text{ and } \lim_{\alpha\to 1-0} \hat{r}(\alpha) = +\infty, \tag{4.135}$$

and hence,

$$\lim_{\alpha\to -1+0} \hat{Y}(\alpha) = -1 \text{ and } \lim_{\alpha\to 1-0} \hat{Y}(\alpha) = 1. \tag{4.136}$$

Moreover,

$$\begin{cases} \hat{t}(\alpha) < 1, \ f'(\hat{t}(\alpha);\hat{r}(\alpha)) = 0, \\ f'(t;\hat{r}(\alpha)) > 0, \ -\infty < t < \hat{t}(\alpha), \\ f'(t;\hat{r}(\alpha)) < 0, \ \hat{t}(\alpha) < t < 1. \end{cases}$$

Hence, for any $0 < r \le \hat{r}(\alpha)$, we have

$$\begin{cases} f(t;r) \le f(t;\hat{r}(\alpha)) \le f(\hat{t}(\alpha);\hat{r}(\alpha)) = \frac{2}{1-\alpha}, \text{for } t < 1, \\ f(t;\hat{r}(\alpha)) < \frac{2}{1-\alpha}, \qquad \text{for } t < 1, \ t \ne \hat{t}(\alpha). \end{cases} \tag{4.137}$$

Further, for $-1 < \alpha < 0$, we have that $\hat{r}(\alpha) < \hat{r}(1+2\alpha)$, and for any $r < \hat{r}(1+2\alpha)$,

$$1 + \alpha f(t;r) > 0, \text{ for any } t < 1. \tag{4.138}$$

Proof. From (4.129), we have for $t \ne 0$,

$$f'(t;r) = \frac{e^{rt}}{t^2}[e^{-rt} - \{1 - rt(1-t)\}] \tag{4.139}$$

and

$$\lim_{t\to 0} f'(t;r) = r(\frac{r}{2} - 1).$$

We see that

$$f'(0;2) = 0,$$

and for $r \le 2$,

$$f(t;r) \le f(t;2) \le f(0;2) = 2 = \frac{2}{1-0}, \text{ for any } t < 1,$$

from which we get (4.137) for $\alpha = 0$. Now consider the case $0 < |\alpha| < 1$. Since $\hat{Y}(\alpha) \ne -\alpha$, we have

$$\frac{2(\alpha + \hat{Y}(\alpha))}{1 - \hat{Y}^2(\alpha)} = \ln \frac{(1+\alpha)(1+\hat{Y}(\alpha))}{(1-\alpha)(1-\hat{Y}(\alpha))},$$

and hence

$$e^{-\frac{2(\alpha+\hat{Y}(\alpha))}{1-\hat{Y}^2(\alpha)}} - \frac{(1-\alpha)(1-\hat{Y}(\alpha))}{(1+\alpha)(1+\hat{Y}(\alpha))} = 0.$$

Note

$$\hat{r}(\alpha)\hat{t}(\alpha) = \frac{2(\alpha + \hat{Y}(\alpha))}{1 - \hat{Y}^2(\alpha)},$$

and

$$\begin{aligned} 1 - \hat{r}(\alpha)\hat{t}(\alpha)(1-\hat{t}(\alpha)) &= 1 - \frac{2(\alpha + \hat{Y}(\alpha))}{(1+\alpha)(1+\hat{Y}(\alpha))} \\ &= \frac{(1-\alpha)(1-\hat{Y}(\alpha))}{(1+\alpha)(1+\hat{Y}(\alpha))}. \end{aligned}$$

Hence, from (4.139), we have $f'(\hat{t}(\alpha);\hat{r}(\alpha)) = 0$. Further,

$$e^{\hat{r}(\alpha)\hat{t}(\alpha)} - 1 = \frac{(1+\alpha)(1+\hat{Y}(\alpha))}{(1-\alpha)(1-\hat{Y}(\alpha))} - 1 = \frac{2(\alpha + \hat{Y}(\alpha))}{(1-\alpha)(1-\hat{Y}(\alpha))},$$

and hence,

$$f(\hat{t}(\alpha);\hat{r}(\alpha)) = (1-\hat{t}(\alpha))\frac{e^{\hat{r}(\alpha)\hat{t}(\alpha)} - 1}{\hat{t}(\alpha)} = \frac{2}{1-\alpha}.$$

Since

$$\begin{cases} \lim_{t\to-\infty} f(t;\hat{r}(\alpha)) = 1, \ f'(t;\hat{r}(\alpha)) > 0, \\ \text{for } t < 0 \ \text{which has a sufficiently large } |t|, \\ f(1;r) = 0 \ \text{ and } \ f'(1;r) = 1 - e^{-r} < 0, \ \text{ for } r > 0, \end{cases}$$

we see that

$$f'(t;\hat{r}(\alpha)) > 0, \quad \text{for} \quad -\infty < t < \hat{t}(\alpha),$$
$$\text{and } f'(t;\hat{r}(\alpha)) < 0, \quad \text{for} \quad \hat{t}(\alpha) < t < 1.$$

Hence, from (4.134), we get (4.137) for $0 < |\alpha| < 1$. From (4.134) and (4.137), we can see that $\hat{r}(\alpha)$ is a strictly monotone increasing function of α on the interval $(-1, 1)$, and hence (4.135) and (4.136) hold.

If $-1 < \alpha < 0$, then $\frac{2}{1-\alpha} < \frac{1}{-\alpha}$. Hence, we obtain (4.138). The proof is complete. ∎

From Lemma 4.3.6, we have the following corollary for a fixed $r > 0$.

Corollary 4.3.1. *For any $r > 0$, we have*

$$\begin{cases} -1 < \hat{r}^{-1}(r) < 1,\ \hat{t}(\hat{r}^{-1}(r)) < 1,\ \textit{and}\ f'(\hat{t}(\hat{r}^{-1}(r));r) = 0 \\ \quad \textit{and} \\ \qquad \begin{cases} f(t;r) \le f(\hat{t}(\hat{r}^{-1}(r));r) = \frac{2}{1-\hat{r}^{-1}(r)},\ \textit{for}\ t < 1, \\ f(t;r) < \frac{2}{1-\hat{r}^{-1}(r)},\ \textit{for}\ t < 1\ \textit{and}\ t \ne \hat{t}(\hat{r}^{-1}(r)), \end{cases} \end{cases} \tag{4.140}$$

where for $r > 0$, $\alpha = \hat{r}^{-1}(r)$ means $\hat{r}(\alpha) = r$ and $\hat{r}(\alpha)$ is defined as in Lemma 4.3.6.

Proof. The proof is derived directly from Lemma 4.3.6. ∎

Corollary 4.3.2. *For a fixed $r > 0$, let $1 > t = \hat{t} \ne 0$ be the solution of the equation*

$$G(t;r) \equiv e^{-rt} - \{1 - rt(1-t)\} = 0, \tag{4.141}$$

and put

$$\hat{r} = 1 - \frac{2}{f(\hat{t};r)}. \tag{4.142}$$

Then,

$$\hat{r}^{-1}(r) = \hat{r}\ \textit{and}\ \hat{t}(\hat{r}^{-1}(r)) = \hat{t}. \tag{4.143}$$

Proof. By Lemma 4.3.6 and Corollary 4.3.1, we see that there exists a unique solution $1 > \hat{t} \ne 0$ of (4.141). Since

$$f'(t;r) = \frac{e^{rt}}{t^2} G(t;r),$$

we have $f'(\hat{t};r) = 0$, and by (4.142), $f(\hat{t};r) = \frac{2}{1-\hat{r}}$, from which by (4.140), we get the conclusion. ∎

Corollary 4.3.3. *For a fixed* $-1 < \alpha < 1$ *and* $\alpha \neq 0$, *let* $q = \hat{q} \neq 0$ *be a solution of the equation*

$$H(q;\alpha) \equiv (1-\alpha)e^{q} - 2(q-\alpha) - (1+\alpha)e^{-q} = 0, \tag{4.144}$$

and put

$$\hat{t} = \frac{e^{-\hat{q}} - (1-\hat{q})}{\hat{q}} \text{ and } \hat{r} = \frac{\hat{q}}{\hat{t}}. \tag{4.145}$$

Then,

$$\hat{r}(\alpha) = \hat{r} \text{ and } \hat{t}(\alpha) = \hat{t}. \tag{4.146}$$

Proof. By Lemma 4.3.6, we see that there exists a unique solution $\hat{q} \neq 0$ of (4.144). For $\hat{p} = \frac{1-e^{-\hat{q}}}{\hat{q}} \neq 1$, put

$$\hat{t} = 1 - \hat{p} \neq 0 \text{ and } \hat{r} = \frac{\hat{q}}{1-\hat{p}}. \tag{4.147}$$

Then, $\hat{p} = 1 - \hat{t}$, $\hat{q} = \hat{r}\hat{t}$, and by (4.144),

$$(1-\alpha)(e^{\hat{q}} - 2 + e^{-\hat{q}}) = 2\{e^{-\hat{q}} - (\hat{q} - 1)\}$$

and

$$\frac{\hat{p}}{1-\hat{p}} = \frac{1-e^{-\hat{q}}}{e^{-\hat{q}} - (1-\hat{q})}.$$

Hence,

$$\frac{\hat{p}}{1-\hat{p}}(e^{\hat{q}} - 1) = \frac{2}{1-\alpha} \text{ and } e^{-\hat{q}} = 1 - \hat{p}\hat{q}. \tag{4.148}$$

Therefore, we have $\hat{t} \neq 0$ and

$$f(\hat{t};\hat{r}) = \frac{2}{1-\alpha} \text{ and } f'(\hat{t};\hat{r}) = \frac{e^{\hat{r}\hat{t}}}{\hat{t}^2} G(\hat{t};\hat{r}) = 0, \tag{4.149}$$

from which by Lemma 4.3.6, we get the conclusion. ∎

Lemma 4.3.7. *Let* $\beta\gamma > 0$, *and*

$$\tilde{\tilde{f}}(x;r,\beta,\gamma) = x\frac{e^{r(\beta-\gamma x)} - 1}{\beta - \gamma x}. \tag{4.150}$$

Then, for $t = 1 - \frac{\gamma}{\beta}x$ and $\tilde{r} = \beta r$, we have

$$\tilde{\tilde{f}}(x; r, \beta, \gamma) = \frac{1}{\gamma} f(t; \tilde{r}). \tag{4.151}$$

Proof. Since $r(\beta - \gamma x) = \tilde{r}t$ and $\beta - \gamma x = \beta t$, we get (4.150). ■

Note from (4.128) after integrating from n to t we have that

$$N(t) = N(n) \exp\left\{\int_n^t r(s)(1 - aN(s) - \sum_{i=0}^{m} b_i N(n-i))ds\right\},$$
$$n \le t < n+1, \; n = 0, 1, 2, \dots, \tag{4.152}$$

and so $N(t) > 0$ for all $t > 0$. An easy computation yields that for $t \in [n, n+1)$,

$$\frac{d}{dt}\left[\frac{1}{N(t)} \exp\left\{\int_n^t r(s)ds(1 - \sum_{i=0}^{m} b_i N(n-i))\right\}\right]$$
$$= ar(t) \exp\left\{\int_n^t r(s)ds(1 - \sum_{i=0}^{m} b_i N(n-i))\right\}. \tag{4.153}$$

Put

$$\begin{cases} r_n = \int_n^{n+1} r(\tau)d\tau, \; t_n = 1 - \sum_{i=0}^m b_i N(n-i), \\ N^* = \frac{1}{(a + \sum_{i=0}^m b_i)}. \end{cases} \tag{4.154}$$

Lemma 4.3.8. *If*

$$1 + aN(n)\frac{\exp\left\{\int_n^t r(s)ds \; t_n\right\} - 1}{t_n} > 0, \; \textit{for} \; t_n \ne 0,$$

and

$$1 + aN(n)\int_n^t r(s)ds > 0, \; \textit{for} \; t_n = 0,$$

then we have for $n \le t < n+1$,

$$N(t) = \begin{cases} \dfrac{N(n)\exp\left\{\int_n^t r(s)ds \; t_n\right\}}{(1 + aN(n))\left[\frac{\left(\exp\left\{\int_n^t r(s)ds \; t_n\right\} - 1\right)}{t_n}\right]}, & t_n \ne 0, \\ \dfrac{N(n)}{1 + aN(n)\int_n^t r(s)ds}, & t_n = 0, \end{cases} \tag{4.155}$$

and

$$N(t) - N^* = \begin{cases} \frac{1-b_0 N(n)\frac{\exp(r_n^t t_n)-1}{t_n}}{1+aN(n)\frac{\exp(r_n^t t_n)-1}{t_n}}(N(n)-N^*) - \sum_{i=1}^m A(n)\, b_i\, (N(n-i) - N^*), \\ if\ t_n \neq 0, \\ \frac{1-b_0 N(n) r_n^t}{1+aN(n)r_n^t}(N(n)-N^*) - \sum_{i=1}^m B(n)\, b_i\, (N(n-i)-N^*),\ if\ t_n = 0, \end{cases} \tag{4.156}$$

where

$$A(n) = \frac{N(n)\frac{\exp(r_n^t t_n)-1}{t_n}}{1+aN(n)\frac{\exp(r_n^t t_n)-1}{t_n}},\ B(n) = \frac{N(n)r_n^t}{1+aN(n)r_n^t},\ r_n^t = \int_n^t r(s)ds. \tag{4.157}$$

In particular,

$$N(n+1) = \begin{cases} \frac{N(n)\exp\{r_n t_n\}}{1+aN(n)\frac{\exp\{r_n t_n\}-1}{t_n}}, & for\ t_n \neq 0, \\ \frac{N(n)}{1+aN(n)r_n}, & for\ t_n = 0, \end{cases} \tag{4.158}$$

and

$$N(n+1) - N^* = \begin{cases} \frac{1-b_0 N(n)\frac{\exp(r_n t_n)-1}{t_n}}{1+aN(n)\frac{\exp(r_n t_n)-1}{t_n}}(N(n) - N^*) - \sum_{i=1}^m C(n)\, b_i (N(n-i) - N^*), \\ if\ t_n \neq 0, \\ \frac{1-b_0 N(n) r_n}{1+aN(n)r_n}(N(n) - N^*) - \sum_{i=1}^m D(n)\, b_i\, (N(n-i) - N^*),\ if\ t_n = 0, \end{cases} \tag{4.159}$$

where

$$C(n) = \frac{N(n)\frac{\exp(r_n t_n)-1}{t_n}}{1+aN(n)\frac{\exp(r_n t_n)-1}{t_n}}\ and\ D(n) = \frac{N(n)r_n}{1+aN(n)r_n}.$$

Proof. If $\int_n^t r(s)ds = 0$, then $N(t) = N(n),\ t \in [n, n+1)$.
Assume

$$\int_n^t r(s)ds > 0,\ t \in [n, n+1).$$

From (4.153), we have (4.155). Then

$$N(t) = \begin{cases} \frac{N(n)+\{(a+\sum_{i=0}^m b_i)N^* - \sum_{i=0}^m b_i N(n-i)\}N(n)(e^{\int_n^t r(s)ds t_n}-1)/t_n}{1+aN(n)\{(e^{\int_n^t r(s)ds t_n}-1)/t_n\}}, & t_n \neq 0, \\ \frac{N(n)+\{(a+\sum_{i=0}^m b_i)N^* - \sum_{i=0}^m b_i N(n-i)\}N(n)\int_n^t r(s)ds}{1+aN(n)\int_n^t r(s)ds}, & t_n = 0, \end{cases}$$

from which we have (4.156). ■

If $a \geq \sum_{i=0}^{m} b_i > 0$ then for $0 < r_n < +\infty$,

$$\begin{cases} \dfrac{|1 - b_0 N(n)\frac{\exp(r_n t_n)-1}{t_n}| + \sum_{i=1}^{m} b_i N(n)\frac{\exp(r_n t_n)-1}{t_n}}{1 + aN(n)\frac{\exp(r_n t_n)-1}{t_n}} \\ \qquad < \dfrac{1 + \sum_{i=0}^{m} b_i N(n)\frac{\exp(r_n t_n)-1}{t_n}}{1 + aN(n)\frac{\exp(r_n t_n)-1}{t_n}} \leq 1, \ t_n \neq 0, \\ \dfrac{|1 - b_0 N(n) r_n| + \sum_{i=1}^{m} b_i N(n) r_n}{1 + aN(n) r_n} < \dfrac{1 + \sum_{i=0}^{m} b_i N(n) r_n}{1 + aN(n) r_n} \leq 1, \ t_n = 0. \end{cases}$$

We easily get the next result when $a \geq \sum_{i=0}^{m} b_i > 0$.

Theorem 4.3.4. *If*

$$a \geq \sum_{i=0}^{m} b_i > 0, \tag{4.160}$$

then the solutions of (4.128) have the contractivity property, that is,

$$|N(n+1) - N^*| \leq \max_{0 \leq i \leq m} |N(n-i) - N^*|. \tag{4.161}$$

Moreover, if

$$\limsup_{n \to \infty} r_n > 0, \tag{4.162}$$

then

$$\lim_{n \to \infty} N(n) = N^*, \tag{4.163}$$

and hence, the positive equilibrium $N^* = 1/(a + b)$ of (4.128) is globally asymptotically stable.

Hereafter with (4.128) we consider the case $-\sum_{i=0}^{m} b_i < a < \sum_{i=0}^{m} b_i$. Note that if $r_n = 0$, then $N(n+1) = N(n)$.

For simplicity, we assume $r_n > 0$ and put

$$\begin{cases} x(n) = (\sum_{i=0}^{m} b_i) N(n), \ x^* = (\sum_{i=0}^{m} b_i) N^* = \frac{1}{(1+\alpha)}, \\ \alpha = \frac{a}{(\sum_{i=0}^{m} b_i)} > -1, \ a_i = \frac{b_i}{(\sum_{i=0}^{m} b_i)} \geq 0, \ 0 \leq i \leq m, \\ \tilde{f}(t; r) = \begin{cases} \frac{e^{rt}-1}{t}, & t \neq 0, \\ r, & t = 0, \end{cases} \\ \bar{r} = \sup_{n \geq 0} r_n \text{ and } \underline{r} = \inf_{n \geq 0} r_n. \end{cases} \tag{4.164}$$

Then,

$$t_n = 1 - \sum_{i=0}^{m} a_i x(n-i),$$

and (4.158) and (4.159) become respectively

$$x(n+1) = \frac{x(n)\exp\{r_n t_n\}}{1 + \alpha x(n)\tilde{f}(t_n; r_n)} \tag{4.165}$$

and

$$x(n+1) - x^* = \frac{1 - a_0 x(n)\tilde{f}(t_n; r_n)}{1 + \alpha x(n)\tilde{f}(t_n; r_n)}(x(n) - x^*)$$
$$- \sum_{i=1}^{m} \frac{a_i x(n)\tilde{f}(t_n; r_n)}{1 + \alpha x(n)\tilde{f}(t_n; r_n)}(x(n-i) - x^*). \tag{4.166}$$

Theorem 4.3.5. *Let $N(t)$ denote any solution of (4.128) and $\hat{r}(\alpha)$ be defined as in Lemma 4.3.6. If $a \geq 0$ or $-b_0 < a < 0$ and*

$$\begin{cases} \bar{r} < +\infty, \text{ for } a \geq 0, \\ \underline{r} > 0 \text{ and } \bar{r} < \hat{r}(1 + \frac{2a}{b_0}), \text{ for } -b_0 < a < 0, \end{cases} \tag{4.167}$$

then

$$\liminf_{n\to\infty} N(n) > 0. \tag{4.168}$$

Proof. By our assumptions, $\bar{r} > 0$.

For the case $a \geq 0$, the proof is similar to those in Lemmas 4.3.2 and 4.3.3. Now assume $-b_0 < a < 0$, $\underline{r} > 0$, and $\bar{r} < \hat{r}(1 + \frac{2a}{b_0})$. In (4.165), first we see $t_n \leq 1 - a_0 x(n)$, and for

$$g(x) = xe^{\bar{r}(1-a_0 x)},$$

we see

$$g'(x) = (1 - \bar{r}a_0 x)e^{\bar{r}(1-a_0 x)},$$

and hence,

$$g(x) \leq \frac{1}{\bar{r}a_0}e^{\bar{r}(1-\frac{1}{\bar{r}})} = \frac{1}{\bar{r}a_0}e^{\bar{r}-1}.$$

Hence, for $\underline{r} \leq r_n \leq \bar{r}$,

$$x(n)e^{r_n t_n} \leq x(n)e^{r_n(1-a_0 x(n))} \leq \max(\frac{1}{\underline{r}a_0}e^{\underline{r}-1}, \frac{1}{\bar{r}a_0}e^{\bar{r}-1}).$$

By assumption, we see that there is a constant δ such that

$$0 < \delta < \min\{\frac{1-\hat{r}^{-1}(\bar{r})}{2/a_0} + \alpha, \alpha + a_0\}.$$

Then,

$$-1 < 1 + \frac{2}{a_0}(\alpha - \delta) < 1,\ \bar{r} < \hat{r}(1 + \frac{2}{a_0}(\alpha - \delta)),$$

and by (4.134) and (4.137) in Lemma 4.3.6,

$$\begin{aligned} 1 + \alpha x(n)\tilde{f}(t_n; r_n) &\geq 1 + \frac{\alpha}{a_0} f(1 - a_0 x(n); r_n) \\ &> 1 + \frac{\alpha}{a_0}\frac{2}{1 - \{1 + \frac{2}{a_0}(\alpha - \delta)\}} \\ &= \frac{\delta}{\delta - \alpha} > 0. \end{aligned}$$

Thus, from (4.165), we have that

$$x(n+1) \leq \max(\frac{1}{\underline{r}a_0}e^{\underline{r}-1}, \frac{1}{\bar{r}a_0}e^{\bar{r}-1})\frac{\delta - \alpha}{\delta}.$$

Hence, in any case considered, we have $x(t) \leq \bar{M} < +\infty$.

Next we prove

$$\liminf_{n\to\infty} x(n) > 0 \text{ for } 0 < \underline{r} \leq r_i \leq \bar{r} < \hat{r}(1 + \frac{2a}{b_0}).$$

Let us consider the solution $c > 0$ of the following equation:

$$a_0 c \leq 1 \ \text{ and } \ a_0 c\tilde{f}(1 - a_0 c; \bar{r}) = 1. \tag{4.169}$$

We easily see that there exists a unique solution $c > 0$ of this equation. Put $\hat{c} = \min(c, x^*) > 0$. If

$$x(n-j) < \hat{c},\ \ 0 \leq j \leq m,\ n \geq 0,$$

then by (4.169),

$$0 < \frac{1 - a_0 x(n)\tilde{f}(t_n; r_n)}{1 + \alpha x(n)\tilde{f}(t_n; r_n)} = 1 - \frac{(\alpha + a_0)x(n)\tilde{f}(t_n; r_n)}{1 + \alpha x(n)\tilde{f}(t_n; r_n)} < 1,$$

and by (4.165) and (4.166),

$$x(n+1) - x^* \\ \geq \left(1 - \frac{(\alpha + a_0)x(n)\tilde{f}(t_n; r_n)}{1 + \alpha x(n)\tilde{f}(t_n; r_n)}\right)(x(n) - x^*) > x(n) - x^*,$$

and hence,

$$x(n+1) > x(n),\ n \geq 0.$$

If $x(n) \geq \hat{c},\ n \geq 0$, then by (4.165),

$$x(n+1) \geq M_{\hat{c}},\ \ n \geq 0,$$

where

$$M_{\hat{c}} = \hat{c}\exp\{\bar{r}\min(1 - a_0\hat{c} - (\sum_{i=1}^{m} a_i)\bar{M}, 0)\}\frac{\delta - \alpha}{\delta}.$$

Similarly, if for some $1 \leq j \leq m$, $x(n-j) \geq \hat{c},\ n \geq j$, then by (4.165), we see $x(n+1) \geq M_{\hat{c}}^{j},\ n \geq j$. Hence, if for some $0 \leq j \leq m$, $x(n-j) \geq \hat{c},\ n \geq m$, then

$$x(n+1) \geq \underline{M},\ n \geq m,$$

where

$$\underline{M} = \min_{0 \leq j \leq m} M_{\hat{c}}^{j} > 0.$$

Suppose that there is a subsequence $\{n_l\}_{l=1}^{\infty}$ such that

$$\lim_{l\to\infty} x(n_l + 1) = 0.$$

Then, there is an $l \geq 1$ such that $n_l \geq m$ and

$$x(n_l + 1) < x(n_l)\ \text{ and }\ x(n_l + 1) < \underline{M}.$$

Therefore, by the above discussions, we see that

$$x(n_l - j) < \hat{c},\ \ 0 \leq j \leq m,$$

and hence, we have

$$x(n_l + 1) > x(n_l),$$

which is a contradiction. Thus, we get $\liminf_{n\to\infty} x(n) > 0$ and (4.168). ■

Remark 9. If $-b_0 < a < 0$ and $\bar{r} \geq \hat{r}(1 + \frac{2a}{b_0})$, then it may occur that $1 + \alpha x(n)\tilde{f}(t_n; r_n) \leq 0$ for some r_n and $x(n-i),\ 0 \leq i \leq m,\ n \geq 0$. In particular, if $m = 0$ and

$$r_n \geq \hat{r}\left(1 + \frac{2a}{b_0}\right), \text{ for } -b_0 < a < 0,$$

then for $-1 < \alpha = \frac{a}{b_0} < 0$,

$$\begin{aligned} 1 + \alpha f(\hat{t}(1+2\alpha); r_n) &\leq 1 + \alpha f(\hat{t}(1+2\alpha); \hat{r}(1+2\alpha)) \\ &= 1 + \frac{2\alpha}{1-(1+2\alpha)} = 0, \end{aligned}$$

and hence there exists an

$$0 < N(n) < (1 - \hat{t}(1+2\alpha))/b_0$$

such that

$$1 + \alpha f(1 - b_0 N(n); r_n) = 0. \tag{4.170}$$

In this case, we cannot define $N(n+1)$ by (4.158) [see also (4.172)].

Now, let us consider the contractivity of solutions and the global stability for the positive equilibrium N^* of (4.128). First, we study the case $m = 0$ in (4.128). For simplicity, we assume $r_n > 0$ and put

$$-1 < \alpha = \frac{a}{b_0} < 1,\ x(n) = b_0 N(n) > 0, \text{ and } x^* = \frac{1}{1+\alpha} > 0. \tag{4.171}$$

If

$$1 + \alpha x(n)\frac{\exp\{r_n(1-x(n))\} - 1}{1 - x(n)} > 0, \text{ for } x(n) \neq 1$$

and

$$1 + \alpha r_n > 0, \text{ for } x(n) = 1,$$

then from (4.164), (4.165) and (4.129), we have

$$x(n+1) = \frac{x(n)\exp\{r_n(1-x(n))\}}{1 + \alpha f(1-x(n); r_n)}, \tag{4.172}$$

and from (4.166), we have

$$x(n+1) - x^* = F(x(n), r_n; \alpha)(x(n) - x^*), \tag{4.173}$$

where

$$F(x, r, \alpha) = \frac{1 - f(1 - x; r)}{1 + \alpha f(1 - x; r)}. \tag{4.174}$$

We easily see that for $-1 < \alpha < 1$ ($-b_0 < a < b_0$),

$$\begin{cases} 1 + \alpha f(1 - x(n); r_n) > 0, \text{ and } -1 \leq F(x(n), r_n; \alpha) < 1, \\ \Leftrightarrow \\ f(1 - x(n); r_n) \leq \frac{2}{1-\alpha}. \end{cases} \tag{4.175}$$

Remark 10. Note that for $x > 0$ and $r > 0$, $f(1 - x; r) > 0$ and in this case, $F(x, r; \alpha) < 1$, for $-1 < \alpha < 1$. In (4.175), if

$$0 < r_n \leq \hat{r}(\alpha), \ x(n) \neq x^*, \text{ and } F(x(n), r_n; \alpha) = -1,$$

then

$$0 < r_n \leq \hat{r}(\alpha), \ x(n) \neq x^*, \text{ and } f(1 - x(n); r_n) = \frac{2}{1 - \alpha},$$

and by (4.137) in Lemma 4.3.6, we see that

$$1 - x(n) = \hat{t}(\alpha), \ r_n = \hat{r}(\alpha)$$

and

$$x(n + 1) - x^* = x^* - x(n),$$

but $x(n) \neq x(n + 1)$. Then,

$$1 - x(n + 1) \neq \hat{t}(\alpha),$$

and for $0 < r_{n+1} \leq \hat{r}(\alpha)$,

$$0 < f(1 - x(n + 1); \ r_{n+1}) < \frac{2}{1 - \alpha},$$

and hence

$$|x(n + 2) - x^*| < |x(n + 1) - x^*|.$$

Theorem 4.3.6. *Assume $m = 0$ and $-b_0 < a < b_0$, and put $-1 < \alpha = \frac{a}{b_0} < 1$.*

(i) If

$$r_n \leq \hat{r}(\alpha), \tag{4.176}$$

then the solutions of (4.128) have the contractivity property, that is,

$$|N(n+1) - N^*| \le |N(n) - N^*|. \tag{4.177}$$

(ii) If

$$\begin{cases} \bar{r} < \hat{r}(1+2\alpha), \text{ if } -1 < \alpha < 0, \text{ and} \\ 0 < \limsup_{n\to\infty} r_n < \hat{r}(\alpha) \text{ or } \limsup_{n\to\infty} r_n = \hat{r}(\alpha), \end{cases} \tag{4.178}$$

then

$$\lim_{n\to\infty} N(n) = N^*, \tag{4.179}$$

and hence, the positive equilibrium $N^* = 1/(a + b_0)$ of (4.128) is globally asymptotically stable.

(iii) If

$$r_n > \hat{r}(\alpha), \tag{4.180}$$

then there exists an $N(n) > 0$ such that

$$|N(n+1) - N^*| > |N(n) - N^*|. \tag{4.181}$$

Proof. By (4.137) and (4.138) in Lemma 4.3.6, we have that for $-1 < \alpha < 1$ and $r_n \le \hat{r}(\alpha)$,

$$\begin{aligned} 0 < f(1 - x(n); r_n) &\le f(1 - x(n); \hat{r}(\alpha)) \\ &\le \frac{2}{1-\alpha} \text{ and } 1 + \alpha f(1 - x(n); r_n) > 0. \end{aligned}$$

Then, we have (4.175) and by (4.173),

$$|x(n+1) - x^*| \le |x(n) - x^*|,$$

from which (4.177) holds.

Now, assume (4.178). Then, by (4.138) in Lemma 4.3.6, $1+\alpha f(1-x(n); r_n) > 0$. If

$$0 < \limsup_{n\to\infty} r_n < \hat{r}(\alpha),$$

then by Theorem 4.3.5 and (4.137) and (4.138) in Lemma 4.3.6, for a sufficiently large n,

$$F(x(n), r_n; \alpha) < 1,$$

and

$$\liminf_{n\to\infty} F(x(n), r_n; \alpha)$$
$$= \liminf_{n\to\infty} \left(-1 + \frac{2-(1-\alpha)f(1-x(n);r_n)}{1+\alpha f(1-x(n);r_n)}\right) > -1.$$

Hence, there is a subsequence $\{n_l\}_{l=1}^{\infty}$ and a constant γ_1 such that

$$\lim_{l\to\infty} |F(x(n_l), r_{n_l}; \alpha)| \leq \gamma_1 < 1,$$

from which we get (4.179). By (4.137), and (4.138) in Lemma 4.3.6, we have for any $t < 1$,

$$1 + \alpha f(t; \hat{r}(\alpha)) > 0, \;\; f(t; \hat{r}(\alpha)) \leq \frac{2}{1-\alpha},$$

and for any $x > 0$,

$$|F(x^* + F(x, \hat{r}(\alpha); \alpha)(x - x^*), \hat{r}(\alpha); \alpha) F(x, \hat{r}(\alpha); \alpha)|$$
$$= \left| \left(-1 + \frac{2-(1-\alpha)f(1-x^* - F(x,\hat{r}(\alpha);\alpha)(x-x^*);\hat{r}(\alpha))}{1+\alpha f(1-x^* - F(x,\hat{r}(\alpha);\alpha)(x-x^*);\hat{r}(\alpha))}\right) \right.$$
$$\left. \times \left(-1 + \frac{2-(1-\alpha)f(1-x;\hat{r}(\alpha))}{1+\alpha f(1-x;\hat{r}(\alpha))}\right) \right| < 1.$$

Hence, if

$$\limsup_{n\to\infty} r_n = \hat{r}(\alpha),$$

then from (4.173),

$$x(n+2) - x^* = F(x(n+1), r_{n+1}; \alpha)(x(n+1) - x^*)$$
$$= F(x^* + F(x(n), r_n; \alpha)(x(n) - x^*), r_{n+1}; \alpha) F(x(n), r_n; \alpha)(x(n) - x^*),$$

and from the proof of Theorem 4.3.5, there is a constant γ_2 such that

$$\limsup_{n\to\infty} |F(x^* + F(x(n), r_n; \alpha)(x(n) - x^*), \; r_{n+1}; \alpha) F(x(n), r_n; \alpha)|$$
$$\leq \gamma_2 < 1,$$

from which we get (4.179). Now (4.179) implies the positive equilibrium N^* of (4.128) is globally asymptotically stable.

If (4.180) holds, then from (4.128), (4.175), and (4.137), we can easily see that there exists an $N(n) > 0$ such that (4.181) holds. The proof is complete. ■

Theorem 4.3.7. *Assume that $m = 0$ and $r(t) \equiv r > 0$ in (4.128).*

(i) For any solution $N(n)$ of (4.128), $\liminf_{n\to\infty} N(n) > 0$, if and only if,

$$\begin{cases} r < +\infty, \ for\ 0 \le b_0 \le a, \ -a < b_0 < 0 \ or\ 0 \le a < b_0, \\ r < \hat{r}(1+2\alpha), \ for\ -b_0 < a < 0 \ and\ -1 < \alpha = \frac{a}{b_0} < 0. \end{cases} \tag{4.182}$$

(ii) For any solution $N(n)$ of (4.128), we have that $|N(n+1)-N^*| \le |N(n)-N^*|$, if and only if,

$$\begin{cases} r < +\infty, \ for\ a \ge b_0 \ge 0, \ or\ -a < b_0 < 0, \\ r \le \hat{r}(\alpha), \ for\ -b_0 < a < b_0\ and\ \alpha = \frac{a}{b_0}. \end{cases} \tag{4.183}$$

Now, for $m \ge 1$, assume that $b_0 > 0,\ b_i \ge 0,\ 1 \le i \le m$ and $\sum_{i=1}^m b_i > 0$, and we consider sufficient conditions for the contractivity of solutions and the positive equilibrium N^* of (4.128) to be globally asymptotically stable. We have for (4.166),

$$\begin{cases} \left|\dfrac{1-a_0x(n)\tilde{f}(t_n;r_n)}{1+\alpha x(n)\tilde{f}(t_n;r_n)}\right| + \sum_{i=1}^m \dfrac{a_i x(n)\tilde{f}(t_n;r_n)}{1+\alpha x(n)\tilde{f}(t_n;r_n)} \le 1, \\ \Leftrightarrow \\ 0 < \dfrac{x(n)\tilde{f}(t_n;r_n)-1}{1+\alpha x(n)\tilde{f}(t_n;r_n)} \le 1, \text{ for } \dfrac{1-a_0x(n)\tilde{f}(t_n;r_n)}{1+\alpha x(n)\tilde{f}(t_n;r_n)} < 0, \\ \alpha + a_0 \ge \sum_{i=1}^m a_i, \qquad \text{for } \dfrac{1-a_0x(n)\tilde{f}(t_n;r_n)}{1+\alpha x(n)\tilde{f}(t_n;r_n)} \ge 0. \end{cases} \tag{4.184}$$

If (4.184) holds, then from (4.166),

$$\begin{aligned} |x(n+1)-x^*| &\le \max\left(|x(n)-x^*|, \left|\frac{\sum_{i=1}^m a_i x(n-i)}{\sum_{i=1}^m a_i} - x^*\right|\right) \\ &\le \max_{0\le i\le m} |x(n-i)-x^*|. \end{aligned}$$

Note that by (4.164), $a_0 + \sum_{i=1}^m a_i = 1$ and $\alpha > a_0 > -1$. Since $x(n-i) > 0,\ 1 \le i \le m$ and

$$t_n < 1 - a_0x(n) < 1,$$

using (4.129) and (4.164), we get the following sufficient conditions for (4.184):

$$\begin{cases} r_n < +\infty, \ \text{if } \alpha \ge 1, \\ f(1-a_0x(n);r_n) \le \frac{2a_0}{1-\alpha}, \ \text{if } -a_0 + \sum_{i=1}^m a_i < \alpha < 1. \end{cases} \tag{4.185}$$

Thus, for

$$-a_0 + \sum_{i=1}^{m} a_i < \alpha < 1,$$

$$f(1 - a_0 x(n); r_n) \leq \frac{2a_0}{1-\alpha} = \frac{2}{1 - (1 - \frac{1-\alpha}{a_0})}, \tag{4.186}$$

and we see that the condition of r_n to α for $m \geq 1$, corresponds to that of $\tilde{\alpha} = 1 - \frac{1-\alpha}{a_0} > -1$ in place of α in Theorem 4.3.6 for $m = 0$. Note that

$$-1 < \tilde{\alpha} < 1 + \frac{2\alpha}{a_0} < 1, \text{ for } -a_0 < \alpha < 0, \tag{4.187}$$

and if

$$\alpha + a_0 = \sum_{i=1}^{m} a_i,$$

then

$$\tilde{\alpha} = 1 - \frac{1-\alpha}{a_0} = -1.$$

Theorem 4.3.8. *Assume $m \geq 1,\ b_0 > 0$ and $\sum_{i=1}^{m} b_i - b_0 < a < \sum_{i=0}^{m} b_i$, and put*

$$-1 < \tilde{\alpha} = \frac{(a - \sum_{i=1}^{m} b_i)}{b_0} < \alpha = \frac{a}{(\sum_{i=0}^{m} b_i)} < 1. \tag{4.188}$$

(i) If

$$r_n \leq \hat{r}(\tilde{\alpha}), \tag{4.189}$$

then solutions of (4.128) have the contractivity property, that is,

$$|N(n+1) - N^*| \leq \max_{0 \leq i \leq m} |N(n-i) - N^*|. \tag{4.190}$$

(ii) If

$$\begin{cases} \bar{r} \leq \hat{r}(1 + 2\alpha),\ if\ -1 < \alpha < 0, \\ 0 < \limsup_{n\to\infty} r_n < \hat{r}(\tilde{\alpha})\ or\ \limsup_{n\to\infty} r_n = \hat{r}(\tilde{\alpha}), \end{cases} \tag{4.191}$$

then

$$\lim_{n\to\infty} N(n) = N^*, \tag{4.192}$$

and hence, the positive equilibrium $N^* = 1/(a + \sum_{i=0}^{m} b_i)$ of (4.128) is globally asymptotically stable.

Proof. We see $t_n < 1 - a_0 x(n)$. For (4.188) and $r_n \le \hat{r}(\tilde{\alpha})$ we have from (4.137) in Lemma 4.3.6 and (4.186),

$$f(1 - a_0 x(n); r_n) \le f(1 - a_0 x(n); \hat{r}(\tilde{\alpha})) \le \frac{2a_0}{1-\alpha}.$$

Then, by (4.185) and (4.184), we have

$$-1 \le \frac{|1 - a_0 x(n) \tilde{f}(t_n; r_n)| + (\sum_{i=1}^{m} a_i) x(n) \tilde{f}(t_n; r_n)}{1 + \alpha x(n) \tilde{f}(t_n; r_n)} \le 1, \tag{4.193}$$

and by (4.166),

$$\begin{aligned} |x(n+1) - x^*| &\le \max\left(|x(n) - x^*|, \left|\frac{\sum_{i=1}^{m} a_i x(n-i)}{\sum_{i=1}^{m} a_i} - x^*\right|\right) \\ &\le \max_{0 \le i \le m} |x(n-i) - x^*|, \end{aligned} \tag{4.194}$$

which implies (4.190).

Suppose first, $r_n \le \hat{r}(\tilde{\alpha}),\ n \ge 0$. Then, there exists a constant β such that

$$\lim_{n\to\infty} \max_{0 \le i \le m} |x(n-i) - x^*| = \beta \ge 0,$$

and hence,

$$\limsup_{n\to\infty} |x(n) - x^*| = \beta \ge 0.$$

Then, there is a subsequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} |x(n_k + 1) - x^*| = \beta,$$

and by (4.193),

$$\begin{aligned} \beta &= \lim_{k\to\infty} |x(n_k + 1) - x^*| \\ &\le \limsup_{k\to\infty} \frac{|1 - a_0 x(n_k) \tilde{f}(t_{n_k}; r_{n_k})| + (\sum_{i=1}^{m} a_i) x(n_k) \tilde{f}(t_{n_k}; r_{n_k})}{1 + \alpha x(n_k) \tilde{f}(t_{n_k}; r_{n_k})} \\ &\quad \times \limsup_{k\to\infty} (\max_{0 \le i \le m} |x(n_k - i) - x^*|) \\ &\le \limsup_{k\to\infty} \frac{|1 - a_0 x(n_k) \tilde{f}(t_{n_k}; r_{n_k})| + (\sum_{i=1}^{m} a_i) x(n_k) \tilde{f}(t_{n_k}; r_{n_k})}{1 + \alpha x(n_k) \tilde{f}(t_{n_k}; r_{n_k})} \beta \le \beta. \end{aligned}$$

Suppose that $\beta > 0$. Then, by the above inequalities, we have

$$\limsup_{k\to\infty} \frac{|1 - a_0 x(n_k)\tilde{f}(t_{n_k}; r_{n_k})| + (\sum_{i=1}^{m} a_i) x(n_k)\tilde{f}(t_{n_k}; r_{n_k})}{1 + \alpha x(n_k)\tilde{f}(t_{n_k}; r_{n_k})} = 1$$

and

$$\limsup_{k\to\infty} |x(n_k - i) - x^*| = \beta, \quad 0 \le i \le m.$$

Thus, there is a subsequence $\{n_l^1\}_{l=0}^{\infty}$ of $\{n_k\}_{k=0}^{\infty}$ such that

$$\lim_{l\to\infty} |x(n_l^1 + 1 - i) - x^*| = \beta, \; 0 \le i \le 1 \text{ and}$$
$$\limsup_{l\to\infty} |x(n_l^1 - i) - x^*| = \beta, \; 1 \le i \le m.$$

Hence using similar reasoning we see that there are subsequences $\{n_l^j\}_{l=0}^{\infty}$ of $\{n_l^{j-1}\}_{l=0}^{\infty}$, $j = 2, 3, \cdots, m$, such that

$$\lim_{l\to\infty} |x(n_l^j + 1 - i) - x^*| = \beta, \; 0 \le i \le j$$
$$\text{and } \limsup_{l\to\infty} |x(n_l^j - i) - x^*| = \beta, \; j \le i \le m.$$

Finally, we get a subsequence $\{n_l\}_{l=0}^{\infty}$ of $\{n_l^m\}_{l=0}^{\infty}$, such that

$$\lim_{l\to\infty} |x(n_l - i) - x^*| = \beta, \; -1 \le i \le m,$$
$$\lim_{l\to\infty} \frac{|1-a_0 x(n_l)\tilde{f}(t_{n_l}; r_{n_l})| + (\sum_{i=1}^{m} a_i) x(n_l)\tilde{f}(t_{n_l}; r_{n_l})}{1+\alpha x(n_l)\tilde{f}(t_{n_l}; r_{n_l})} = 1,$$

because, if there is a subsequence $\{n_j\}_{j=1}^{\infty}$ of $\{n_l\}_{l=0}^{\infty}$ such that

$$\lim_{j\to\infty} \frac{|1 - a_0 x(n_j)\tilde{f}(t_{n_j}; r_{n_j})| + (\sum_{i=1}^{m} a_i) x(n_j)\tilde{f}(t_{n_j}; r_{n_j})}{1 + \alpha x(n_j)\tilde{f}(t_{n_j}; r_{n_j})} < 1,$$

then

$$\begin{aligned}\beta &= \lim_{j\to\infty} |x(n_j + 1) - x^*| \\ &\le \lim_{j\to\infty} \frac{|1 - a_0 x(n_j)\tilde{f}(t_{n_j}; r_{n_j})| + (\sum_{i=1}^{m} a_i) x(n_j)\tilde{f}(t_{n_j}; r_{n_j})}{1 + \alpha x(n_j)\tilde{f}(t_{n_j}; r_{n_j})} \\ &\quad \times \lim_{j\to\infty} |x(n_j) - x^*| < \beta,\end{aligned}$$

which is a contradiction. Then, by (4.194), we can see that

$$\lim_{n\to\infty} \frac{|1 - a_0 x(n)\tilde{f}(t_n; r_n)| + (\sum_{i=1}^{m} a_i) x(n_j)\tilde{f}(t_{n_j}; r_{n_j})}{1 + \alpha x(n)\tilde{f}(t_n; r_n)} = 1$$

and

$$\lim_{n\to\infty} |x(n) - x^*| = \beta. \tag{4.195}$$

Suppose $-1 < \alpha < 0$, $\sup_{n\geq 0} r_n \leq \hat{r}(1 + 2\alpha)$ and (4.191). Assume that there is a subsequence $\{n_p\}_{p=1}^{\infty}$ such that

$$1 - a_0 x(n_p)\tilde{f}(t_p; r_{n_p}) \leq 0.$$

Then, from (4.194), we have

$$\lim_{p\to\infty} \frac{x(n_p)\tilde{f}(t_{n_p}; r_{n_p}) - 1}{1 + \alpha x(n_p)\tilde{f}(t_{n_p}; r_{n_p})} = 1,$$

that is,

$$\lim_{p\to\infty} x(n_p)\tilde{f}(t_{n_p}; r_{n_p}) = \frac{2}{1-\alpha}.$$

Thus, by Theorem 4.3.5,

$$\lim_{p\to\infty} x(n_p) = x^* - \tilde{\beta} > 0,\ \lim_{p\to\infty} x(n_p + 1) = x^* + \tilde{\beta} > 0, \text{ and } |\tilde{\beta}| = \beta.$$

Since $-1 < \tilde{\alpha} \leq \alpha < 1 + 2\alpha < 1$, and

$$\begin{aligned}\frac{2}{1-\alpha} &= \lim_{p\to\infty} x(n_p)\tilde{f}(t_{n_p}; r_{n_p}) \leq \frac{1}{a_0} f(\hat{t}(\tilde{\alpha}); \hat{r}(\tilde{\alpha})) \\ &= \frac{1}{a_0}\frac{2}{1-\tilde{\alpha}} = \frac{2}{1-\alpha},\end{aligned}$$

by (4.137) in Lemma 4.3.6, we have

$$\lim_{p\to\infty} t_{n_p} = \lim_{p\to\infty}\{1 - a_0 x(n_p)\} = \hat{t}(\tilde{\alpha}) \text{ and } \lim_{p\to\infty} r_{n_p} = \hat{r}(\tilde{\alpha}),$$

and hence,

$$\lim_{p\to\infty} x(n_p) = \frac{1}{a_0}(1 - \hat{t}(\tilde{\alpha})) \text{ and } \lim_{p\to\infty} x(n_p - i) = 0,\ 1 \leq i \leq m,$$

which contradicts Theorem 4.3.5, because

$$-1 < \tilde{\alpha} < 1 + \frac{2\alpha}{a_0} < 1 \text{ for} -1 < \alpha < 0.$$

Hence, we have

$$1 - a_0 x(n)\tilde{f}(t_n; r_n) \geq 0,$$

for a sufficiently large n, and from (4.195),

$$\lim_{n\to\infty} \frac{1 - (a_0 - \sum_{i=1}^{m} a_i)x(n)\tilde{f}(t_n; r_n)}{1 + \alpha x(n)\tilde{f}(t_n; r_n)} = 1.$$

Then, by

$$\alpha + a_0 > \sum_{i=1}^{m} a_i,$$

and Theorem 4.3.5, we get $\lim_{n\to\infty} r_n = 0$, which is a contradiction. Therefore, we have for the case $r_n \leq \hat{r}(\tilde{\alpha})$, $\lim_{n\to\infty} |x(n) - x^*| = 0$ and (4.192) holds. Also by Theorem 4.3.5 there is a positive constant $\underline{M}$ such that

$$t_n \leq 1 - a_0 x(n) - \left(\sum_{i=1}^{m} a_i\right)\underline{M} < 1 - a_0 x(n).$$

Then, the above discussion for the case $r_n \leq \hat{r}(\tilde{\alpha})$, is also applicable to the case

$$\limsup_{n\to\infty} r_n = \hat{r}(\tilde{\alpha}).$$

Hence, we get (4.192), and the proof is complete. ■

Remark 11. Note that $\sum_{i=1}^{m} b_i > 0$ and $r_n \leq \hat{r}(1 + 2\alpha)$ for $-1 < \alpha < 0$, implies that $\sum_{i=1}^{m} a_i x(n-i) > 0$ and $1 + \alpha x(n)\tilde{f}(t_n; r_n) > 0$.

Finally in this section we consider

$$\begin{cases} \frac{dx(t)}{dt} = x(t)r(t)\{1 - ax(t) - b_0 x(t_l) - \sum_{j=1}^{m} b_j x(\tau_j(t))\}, \\ t_l \leq t < t_{l+1},\ l = 0, 1, 2, \ldots, \\ x(t) = \phi(t) \geq 0,\ -\underline{\tau} \leq t \leq t_0 \text{ and } \phi(t_0) > 0, \end{cases} \tag{4.196}$$

where $\frac{dx(t)}{dt}$ means that the right-hand side derivative at t of the function $x(t)$, and $r(t)$ is a nonnegative continuous function on $[t_0, \infty)$, $r(t) \not\equiv 0$, $\phi(t)$ is continuous on the interval $[-\underline{\tau}, t_0]$, $b_0 > 0$, $\tau_0(t)$ is the following piecewise constant delay:

$$\tau_0(t) = t_l,\ t_l \leq t < t_{l+1},\ l = 0, 1, 2, \cdots, \tag{4.197}$$

$\tau_j(t)$ is piecewise continuous on (t_0, ∞), $-\underline{\tau} \leq \tau_j(t) \leq \tau_0(t) \leq t,\ 1 \leq j \leq m$, and

$$\underline{\tau}(t) \equiv \inf_{0\leq j\leq m} \tau_j(t) \to +\infty, \text{ as } t \to +\infty.$$

Assume

$$a + \sum_{j=0}^{m} b_j > 0, \tag{4.198}$$

and put $x^* = 1/(a + \sum_{j=0}^{m} b_j)$ and

$$\begin{cases} b_j = b_j^+ + b_j^-, \ b_j^+ \geq 0, \ b_j^- \leq 0, \ 0 \leq j \leq m, \ b = \sum_{j=0}^{m} b_j, \\ \hat{b} = \sum_{j=0}^{m} |b_j| < +\infty, \ b^+ = \sum_{j=0}^{m} b_j^+ \geq 0, \ \text{and } b^- = \sum_{j=0}^{m} b_j^- \leq 0. \end{cases} \tag{4.199}$$

For simplicity, we assume $r(t) > 0, \ t \geq t_0$, and $\hat{b} > 0$.

Let $D^- x(t)$ be the left-hand side derivative at t of the function $x(t)$.

Lemma 4.3.9. *(a) Assume*

$$a \geq \hat{b}. \tag{4.200}$$

Then,

$$a + b^- > 0, \ \frac{b^+}{a + b^-} \leq \frac{b^+ + |b^-|}{a} \leq 1. \tag{4.201}$$

If for $\bar{t} \geq t_0$,

$$x(\bar{t}) > x^* \text{ and } D^- x(\bar{t}) \geq 0, \tag{4.202}$$

then

$$x(\bar{t}) - x^* \leq \frac{b^+}{a + b^-} \max_{0 \leq j \leq m} |x(\tau_j(\bar{t})) - x^*|, \tag{4.203}$$

and if for $\underline{t} \geq t_0$,

$$x(\underline{t}) < x^* \quad \text{and} \quad D^- x(\underline{t}) \leq 0, \tag{4.204}$$

then

$$x(\underline{t}) - x^* \geq -\frac{b^+}{a + b^-} \max_{0 \leq j \leq m} |x(\tau_j(\underline{t})) - x^*|. \tag{4.205}$$

(b) Suppose

$$a + b^- \geq 0. \tag{4.206}$$

(i) If for $\bar{t} \geq t_0$,

$$x(\bar{t}) > x^* \quad \text{and} \quad D^- x(\bar{t}) \geq 0, \tag{4.207}$$

then

$$|b^-|x(\tau_j(\bar{t})) \leq |b^-|x(\bar{t})\ 0 \leq j \leq m, \text{ imply } b^+ > 0, \min_{0\leq j\leq m} x(\tau_j(\bar{t})) \leq x^*. \tag{4.208}$$

In particular, if $a + b^- > 0$, *then*

$$x(\bar{t}) < \frac{1}{a + b^-}. \tag{4.209}$$

(ii) If for $\underline{t} \geq t_0$,

$$x(\underline{t}) < x^*, \text{ and } D^- x(\underline{t}) \leq 0, \tag{4.210}$$

then

$$|b^-|x(\tau_j(\underline{t})) \geq |b^-|x(\underline{t}),\ 0 \leq j \leq m, \text{ imply } b^+ > 0, \max_{0\leq j\leq m} x(\tau_j(\underline{t})) \geq x^*. \tag{4.211}$$

Proof. By assumption (4.198) and (4.200), we see (4.201). Suppose that

$$\max_{0\leq j\leq m} |b^-|x(\tau_j(\bar{t})) \leq |b^-|x(\bar{t}).$$

Then, from (4.196) and (4.202), we have

$$\begin{aligned} 0 \leq D^- x(\bar{t}) &= x(\bar{t})r(\bar{t})\{a(x^* - x(\bar{t})) + \textstyle\sum_{j=0}^m b_j(x^* - x(\tau_j(\bar{t})))\} \\ &= x(\bar{t})r(\bar{t})\{a(x^* - x(\bar{t})) + \textstyle\sum_{j=0}^m b_j^-(x^* - x(\tau_j(\bar{t}))) \\ &\quad + \textstyle\sum_{j=0}^m b_j^+(x^* - x(\tau_j(\bar{t})))\} \\ &\leq x(\bar{t})r(\bar{t})\{(a + b^-)(x^* - x(\bar{t})) + \textstyle\sum_{j=0}^m b_j^+(x^* - x(\tau_j(\bar{t})))\}. \end{aligned} \tag{4.212}$$

Then, $a \geq \hat{b}$ and

$$\frac{b^+}{a + b^-} \max_{0\leq j\leq m} |x(\tau_j(\bar{t})) - x^*| < x(\bar{t}) - x^*,$$

implies

$$0 \leq D^- x(\bar{t}) < 0,$$

which is a contradiction. Hence, we get (4.203).

Similarly, from (4.204), we can prove (4.205).

Now, suppose (4.206) and (4.207). Then, by (4.212) and (4.198), we can easily obtain (4.208) and (4.209).

Similarly, from (4.206), (4.210) and (4.198), we get (4.211). The proof is complete. ■

Corollary 4.3.4. *Assume $b^- = 0 \le a$, and let $x(t)$ be the solution of (4.196). If $x(t)$ is eventually greater (respectively, less) than x^*, then $x(t)$ is monotone decreasing and greater (respectively, monotone increasing and less) than x^*.*

Lemma 4.3.10. *Assume $a + b^- > 0$ or $a = b^- = 0$, and let $x(t)$ be the solution of (4.196). If $x(t)$is eventually greater (respectively, less) than x^*, then $\lim_{t\to\infty} x(t)$ exists and is positive. Furthermore, if*

$$\int_{t_0}^{\infty} r(t)dt = \infty,$$

then we have $\lim_{t\to\infty} x(t) = x^*$.

Proof. On any interval of the form $[t_l, t_{l+1})$ for $n = 0, 1, 2, \cdots$, we can integrate the differential equation in (4.196) together with the initial conditions, and we obtain for $t_l \le t < t_{l+1}$ and $n = 0, 1, 2, \cdots$

$$x(t) = x(t_l)\exp\{\int_{t_l}^{t} r(s)(1 - ax(s) - \sum_{j=0}^{m} b_j x(\tau_j(s)))ds\}.$$

Thus we see that (4.196) has a unique solution $x(t)$ which is positive for $t \ge t_0$.

Assume that $x(t)$ is eventually greater than x^*. Let

$$\limsup_{t\to\infty} x(t) = x^* + \beta, \text{ where } \beta \ge 0.$$

If $x(t)$ is not eventually decreasing, then by Lemma 4.3.9, it is the case that $a + b^- > 0$ and $b^- < 0$, and hence, $a > 0$. Suppose $\beta > 0$. Then, for

$$0 < \epsilon < (a + b^-)/(a + |b^-|) < 1,$$

there exist a sequence $\{\bar{t}_k\}_{k=1}^{\infty}$ such that

$$\begin{aligned} x(\bar{t}_k) > x^*,\ D^-(\bar{t}_k) \ge 0,\ x(\bar{t}_k) > x^* + \beta(1-\epsilon), \text{ and} \\ x^* \le x(t) \le x^* + \beta(1+\epsilon), \qquad \text{for } t \ge \bar{t}_1 \ge t_0. \end{aligned}$$

Then,

$$\begin{aligned} 0 \le D^-x(\bar{t}_k) &= x(\bar{t}_k)r(\bar{t}_k)\{a(x^* - x(\bar{t}_k)) + \textstyle\sum_{j=0}^{m} b_j(x^* - x(\tau_j(\bar{t}_k)))\} \\ &= x(\bar{t}_k)r(\bar{t}_k)\{a(x^* - x(\bar{t}_k)) + \textstyle\sum_{j=0}^{m} b_j^-(x^* - x(\tau_j(\bar{t}_k))) \\ &\quad + \textstyle\sum_{j=0}^{m} b_j^+(x^* - x(\tau_j(\bar{t}_k)))\} \\ &< x(\bar{t}_k)r(\bar{t}_k)\{-a\beta(1-\epsilon) - b^-\beta(1+\epsilon)\} \\ &< x(\bar{t}_k)r(\bar{t}_k)\{-(a+b^-) + (a+|b^-|)\epsilon\}\beta \\ &< 0, \end{aligned}$$

which is a contradiction. Hence, $\beta = 0$, and

$$\lim_{t\to\infty} x(t) = x^*.$$

Next, let us consider the case that $x(t)$ is eventually decreasing and bounded below by x^*. Then, $\lim_{t\to\infty} x(t)$ exists. Set

$$\beta := \lim_{t\to\infty} x(t) - x^* \geq 0.$$

We will show that

$$\int_{t_0}^{\infty} r(t)dt = +\infty \text{ implies } \beta = 0.$$

Indeed, suppose $\beta > 0$. Take ϵ such that

$$0 < \epsilon < /(-b^- x^*), \ \text{ if } \ b^- < 0, \ \text{and} \ \epsilon > 0, \ \text{if} \ b^- = 0.$$

Then, there exists $\bar{t}_0 \geq t_0$ such that

$$\beta \leq x(\underline{\tau}(t)) - x^* \leq \beta + \epsilon, \ \text{for } t \geq \bar{t}_0,$$

since $x(t) - x^*$ eventually decreases to β. By (4.212), we have

$$\begin{aligned} D^- x(t) &\leq x(t)r(t)\left\{-(a + \sum_{j=0}^{m} b_j)\beta - (\sum_{j=0}^{m} b_j^-)\epsilon\right\} \\ &= -\left\{\frac{\beta}{x^*} + b^-\epsilon\right\} x(t)r(t), \ \text{for } t \geq \bar{t}_0. \end{aligned}$$

Integrating from $\bar{t}_0$ to t, we have

$$\ln \frac{x(t)}{x(\bar{t}_0)} \leq -\left\{\frac{\beta}{x^*} + b^-\epsilon\right\} \int_{\bar{t}_0}^{t} r(s)ds,$$

which in turn implies, due to $\frac{\beta}{x^*} + b^-\epsilon > 0$ and $\int_{t_0}^{\infty} r(t)dt = +\infty$,

$$\lim_{t\to\infty} \ln(\frac{x(t)}{x(\bar{t}_0)}) = -\infty.$$

Hence, $\lim_{t\to\infty} x(t) = 0$, contradicting $x(t) \geq x^* + \beta > x^* > 0$. The case that $x(t)$ is eventually less than x^* is similarly proved. Thus, the proof is complete. ■

From Lemmas 4.3.9 and 4.3.10, we see that under the conditions

$$a + b^- > 0 \text{ or } a = b^- = 0, \text{ and } \int_{t_0}^{\infty} r(t)dt = +\infty,$$

in the analysis of global stability we need to investigate only the case that a solution $x(t)$ is oscillatory about x^*. If there is a point $\bar{t} \geq t_0$ such that for

$$t_0 \leq t \leq \bar{t}, \quad |x(t) - x^*| \leq |x(\bar{t}) - x^*|,$$

then the conditions (4.207) and (4.208), or (4.210) and (4.211) in Lemma 4.3.9, really occur.

The following lemma is elementary.

Lemma 4.3.11. *For $t_l \leq t < t_{l+1}$,*

$$x(t) = \frac{x(t_l)\exp(\int_{t_l}^{t} r(s)\{1 - b_0 x(t_l) - \sum_{j=1}^{m} b_j x(\tau_j(s))\}ds)}{1 + ax(t_l)\int_{t_l}^{t} r(s)\exp(\int_{t_l}^{s} r(\sigma)\{1 - b_0 x(t_l) - \sum_{j=1}^{m} b_j x(\tau_j(\sigma))\}d\sigma)ds}, \tag{4.213}$$

and

$$\begin{cases} x(t) - x^* = \{1 - (a + b_0)k_l\}(x(t_l) - x^*) \\ -\frac{x(t_l)\int_{t_l}^{t} r(s)\{\sum_{j=1}^{m} b_j(x(\tau_j(s)) - x^*)\}\exp(\int_{t_l}^{s} r(\sigma)\{1 - b_0 x(t_l) - \sum_{j=1}^{m} b_j x(\tau_j(\sigma))\}d\sigma)ds}{1 + ax(t_l)\int_{t_l}^{t} r(s)\exp(\int_{t_l}^{s} r(\sigma)\{1 - b_0 x(t_l) - \sum_{j=1}^{m} b_j x(\tau_j(\sigma))\}d\sigma)ds}, \end{cases} \tag{4.214}$$

where

$$k_l = \frac{x(t_l)\int_{t_l}^{t} r(s)\exp(\int_{t_l}^{s} r(\sigma)\{1 - b_0 x(t_l) - \sum_{j=1}^{m} b_j x(\tau_j(\sigma))\}d\sigma)ds}{1 + ax(t_l)\int_{t_l}^{t} r(s)\exp\{\int_{t_l}^{s} r(\sigma)\{1 - b_0 x(t_l) - \sum_{j=1}^{m} b_j x(\tau_j(\sigma))\}d\sigma)ds}. \tag{4.215}$$

Theorem 4.3.9. *Assume $a + b^- > 0$ or $a = b^- = 0$, and*

$$\int_{t_0}^{\infty} r(t)dt = +\infty.$$

Then, for any

$$\bar{r} = \sup_{t \geq t_0} \int_{\underline{\tau}(t)}^{t} r(s)ds < +\infty,$$

there exists $\bar{t}_1 \geq t_0$ such that for any $t \geq \bar{t}_1$,

$$\underline{M} \leq x(t) \leq \bar{M}, \tag{4.216}$$

where

$$\bar{M} = \begin{cases} \frac{1}{a+b^-}, & \text{if } a+b^- > 0, \text{ and } \bar{r} \geq \ln \frac{a+b}{a+b^-}, \\ \frac{1}{a+b} e^{\bar{r}}, & \text{if } a+b^- > 0, \text{ and } \bar{r} < \ln \frac{a+b}{a+b^-}, \\ \frac{1}{b} e^{\bar{r}}, & \text{if } a = b^- = 0, \end{cases} \tag{4.217}$$

and

$$\underline{M} = \frac{\min\{1, e^{\bar{r}(1-b^+\bar{M})}\}}{a+b+a(e^{\bar{r}(1-b^-\bar{M})}-1)/(1-b^-\bar{M})} > 0. \tag{4.218}$$

Hence, for $\bar{r} < +\infty$, any solution $x(t)$ of (4.196) is persistent.

Proof. By Lemmas 4.3.9 and 4.3.10 and (4.213), there exists a $\bar{t}_1 \geq t_0$ such that for any $t \geq \bar{t}_1$,

$$\begin{cases} x(t) < \frac{1}{a+b^-}, \; a+b^- > 0, \\ x(t) \leq x^* e^{\bar{r}}, \; a+b^- \geq 0, \end{cases}$$

and in the case $a + b^- > 0$, we see

$$\frac{1}{a+b^-} \leq \frac{1}{a+b} e^{\bar{r}}, \text{ if and only if } \bar{r} \geq \ln \frac{a+b}{a+b^-}.$$

Then, by Lemmas 4.3.9 and 4.3.10 and (4.213), we have that for $t \geq \bar{t}_1$,

$$x(t) \geq \frac{x^* \min\{1, e^{\bar{r}(1-b^+\bar{M})}\}}{1 + ax^*(\exp\{\bar{r}(1-b^-\bar{M})\}-1)/(1-b^-\bar{M})} = \underline{M} > 0,$$

from which we obtain (4.218). The proof is complete. ■

Theorem 4.3.10. *Assume (4.196)–(4.199), and for (4.213),*

$$\begin{cases} b^- = 0, \text{ and } b - 2b_0 < a, \\ or \\ \max\{-b^-, \hat{b} - 2b_0\} < a. \end{cases} \tag{4.219}$$

Put

$$-1 < \tilde{\alpha} = (a - \sum_{i=1}^{m} b_i)/b_0 < \alpha = a/(\sum_{i=0}^{m} b_i) < 1 \text{ and } \tilde{r}_l = \int_{t_l}^{t_{l+1}} r(t)dt. \tag{4.220}$$

If

$$\begin{cases} \tilde{r}_l < +\infty, & \text{for } a \geq \hat{b}, \\ \tilde{r}_l \leq \hat{r}(\frac{(a+b_0)-b}{b_0}), & \text{for } b^- = 0, \; b - 2b_0 < a < b, \\ \tilde{r}_l \leq \frac{a+b^-}{a}\hat{r}(1 - \frac{(a+b^-)(\hat{b}-a)}{ab_0}), & \text{for } \max\{-b^-, \hat{b} - 2b_0\} < a < \hat{b}, \end{cases} \tag{4.221}$$

then, solutions of (4.196) have the contractivity property, that is,

$$\max_{t_l \le t \le t_{l+1}} |x(t) - x^*| \le \max_{\underline{\tau}(t_l) \le t \le t_l} |x(t) - x^*|. \tag{4.222}$$

If

$$\begin{cases} \sup_{l \ge 0} \tilde{r}_l \le \hat{r}(1+2\alpha), \ \textit{if} -1 < \alpha < 0, \quad \textit{and} \\ 0 < \limsup_{l\to\infty} \tilde{r}_l < \hat{r}(\tilde{\alpha}), \ \textit{or} \ \limsup_{l\to\infty} \tilde{r}_l = \hat{r}(\tilde{\alpha}), \end{cases} \tag{4.223}$$

then

$$\lim_{l\to\infty} x(t_l) = x^*, \tag{4.224}$$

and hence, the positive equilibrium x^* of (4.196) is globally asymptotically stable.

Proof. Assume $t_l < \bar{t}_{l+1} \le t_{l+1}$ and

$$|b^-|x(t) \le |b^-|x(\bar{t}_{l+1}), \ \text{for } \underline{\tau}(t_l) \le t \le t_l \text{ and } D^-x(\bar{t}_{l+1}) \ge 0.$$

Then, from (4.196),

$$0 \le D^-x(\bar{t}_{l+1}) \le x(\bar{t}_{l+1})r(\bar{t}_{l+1})\{1 - (a+b^-)x(\bar{t}_{l+1}) - b_0x(t_l)\}$$

and hence,

$$(a+b^-)x(\bar{t}_{l+1}) \le 1 - b_0x(t_l).$$

Then, in (4.214) and (4.215),

$$\begin{aligned} 1 - b_0x(t_l) - \sum_{j=1}^{m} b_j x(\tau_j(\sigma)) &\le 1 - b_0x(t_l) - \frac{b^-(1-b_0x(t_l))}{a+b^-} \\ &= \frac{a}{a+b^-} - \frac{ab_0}{a+b^-}x(t_l). \end{aligned}$$

Consider $\tilde{\tilde{f}}(x; r, \beta, \gamma)$ in (4.217) in Lemma 4.3.7 with

$$\beta = \frac{a}{a+b^-} \text{ and } \gamma = \frac{ab_0}{a+b^-}.$$

Then,

$$\begin{aligned} &x(t_l)\int_{t_l}^{t_{l+1}} r(s)\exp\Big(\int_{t_l}^{s} r(\sigma)\{1 - b_0x(t_l) - \sum_{j=1}^{m} b_j x(\tau_j(\sigma))\}d\sigma\Big)ds \\ &\le \tilde{\tilde{f}}(x(t_l); \tilde{r}_l, \beta, \gamma). \end{aligned}$$

Hence by Lemma 4.3.7 and the similar proofs in Theorem 4.3.4 for $a \ge \hat{b}$, and Theorem 4.3.8, we can easily obtain the result. ∎

4.4 Global Stability of Models of Volterra Type

In this section we discuss a model of Volterra type, namely

$$\frac{dN(t)}{dt} = N(t)\left[r - cN(t) - \sum_{j=0}^{\infty} d_j N([t-j])\right], \quad t \geq 0, \tag{4.225}$$

where $r > 0$, $c > 0$, d_i $(j = 0, 1, 2, \ldots)$ are nonnegative and $\sum_{j=0}^{\infty} d_j < \infty$. The results in this section are adapted from [42]. Using the following substitution in (4.225)

$$a = \frac{c}{r},\ b = \frac{1}{r}\sum_{j=0}^{\infty} d_j,\ c_j = d_j\left(\sum_{j=0}^{\infty} d_j\right)^{-1}, \tag{4.226}$$

we have

$$\frac{dN(t)}{dt} = rN(t)\left[1 - aN(t) - b\sum_{j=0}^{\infty} c_j N([t-j])\right], \quad t \geq 0, \tag{4.227}$$

where

$$\sum_{j=0}^{\infty} c_j = 1. \tag{4.228}$$

The initial conditions associated with (4.227) are assumed to be of the form

$$N(-j) = \beta_j \geq 0,\ \beta_0 > 0,\ \{\beta_j\} \in \ell^{\infty}. \tag{4.229}$$

It is easily seen that the initial conditions provided for (4.227) guarantee that

$$N(0) > 0,\ \sup_{j\geq 0}\{N(-j)\} < \infty,$$

and integration of (4.227) on an interval of the form $[n, t)$, $n \leq t < n+1$ leads to

$$N(t) = N(n)\exp\left\{\int_n^t r\left(1 - aN(s) - b\sum_{j=0}^{\infty} c_j N(n-j)\right) ds\right\},$$

for $n \leq t < n+1$. For $N(0) = \beta_0 > 0$, it follows that $N(t) > 0$ on $[0, 1)$, now letting $n = 0$ and $t \to 1$ we find that $N(1) > 0$ since the sum of the terms from the initial values β_j remain bounded. Thus $N(1) > 0$ and $N(1)$ is finite. Repetition of this procedure shows that $N(t)$ is defined for $t \geq 0$ and remains continuous for $t \in [0, \infty)$ and satisfies $N(t) > 0$ for $t > 0$.

Consider (4.227) on an interval of the form $[n, n+1)$ for $n = 0, 1, \ldots$, and (4.227) becomes

$$\frac{dN(t)}{dt} = P(n)N(t) - raN^2(t), \quad t \in [n, n+1), \tag{4.230}$$

where

$$P(n) = r\left[1 - b\sum_{j=0}^{\infty} c_j N(n-j)\right], \quad n = 0, 1, 2, \ldots . \tag{4.231}$$

We can rewrite (4.230) in the form

$$\frac{d}{dt}\left(\frac{1}{N(t)}e^{P(n)t}\right) = rae^{P(n)t}, \ t \in [n, n+1). \tag{4.232}$$

An integration on both sides of (4.232) from n to t leads to

$$N(n+1) = \frac{N(n)e^{P(n)(t-n)}}{1 + raN(n)\{e^{P(n)(t-n)} - 1\}/P(n)}, \ t \in [n, n+1). \tag{4.233}$$

Letting $t \to n+1$, we obtain

$$N(n+1) = \frac{N(n)e^{P(n)}}{1 + raN(n)\{e^{P(n)} - 1\}/P(n)}, \ n = 0, 1, 2 \ldots . \tag{4.234}$$

The right-hand side of (4.234) has a removable singularity when $P(n) = 0$, and we will assume that the right-hand side of (4.234) is suitably defined by

$$N(n+1) = \frac{N(n)e^{P(n)}}{1 + raN(n)}, \quad \text{for } P(n) = 0, \tag{4.235}$$

so as to make the right-hand side of (4.234) continuous. It is now easy to see that for the given initial values (4.229) one can calculate successively the values $N(1)$, $N(2)$, $N(3), \ldots$ and with this, one can compute $N(t)$ in (4.233). Thus an iterative solution of (4.227) is possible. The properties of (4.227) are now determined by (4.234) and vice versa.

Lemma 4.4.1. *Let $N(n)$ denote the solution of (4.234). Then*

$$N(n) \leq M = \frac{1}{a}\left(\frac{e^r}{e^r - 1}\right), \ n = 1, 2, \ldots . \tag{4.236}$$

Proof. We note that the function

$$f(p) = \frac{pe^p}{e^p - 1}, \ p \in \mathbf{R},$$

is increasing on $(-\infty, \infty)$ since

$$f'(p) = \frac{p(e^p - 1 - p)}{(e^p - 1)^2}, \; p \in \mathbf{R}.$$

Note

$$P(n) = r\left[1 - b\sum_{j=0}^{\infty} c_j N(n-j)\right] < r.$$

Hence from (4.234) we have

$$\begin{aligned} N(n+1) &\le \frac{1}{ra} f(P(n)) = \frac{P(n)e^{P(n)}}{ra\{e^{P(n)} - 1\}} \\ &\le \frac{1}{ra} f(r) = \frac{1}{ra}\frac{re^r}{\{e^r - 1\}} = \frac{1}{a}\left(\frac{e^r}{e^r - 1}\right) = M, \end{aligned}$$

from which the boundedness of $N(n)$ follows. The proof is complete. ■

Lemma 4.4.2. *Let $N(t)$ be a solution of (4.227). Then*

$$\lim_{t\to\infty} \sup N(t) < \infty. \tag{4.237}$$

Proof. From Lemma 4.4.1, we have $N(t) \le M$, $n = 1, 2, 3, \ldots$. By the continuity of the solution of (4.227), $N(t)$ is bounded in each interval $[n, n+1)$, $n = 1, 2, 3, \ldots$. Suppose now the assertion of (4.237) is not true. Then there exists a sequence $\{t_k\}$, $t_k \to \infty$, as $k \to \infty$ and $t_k \ne n_k$ such that

$$N(t_k) \ge M, \; t_k \in (n_k, n_{k+1}) \; \text{ and } \; \lim_{k\to\infty} N(t_k) = \infty, \; N'(t_k) \ge 0.$$

Hence from (4.227), we have

$$\begin{aligned} 0 \le N'(t_k) &= rN(t_k)\left(1 - aN(t_k) - b\sum_{j=0}^{\infty} c_j N([t_k - j])\right) \\ &\le rN(t_k)\{1 - aN(t_k)\} \le rN(t_k)\left[1 - \frac{e^r}{e^r - 1}\right] < 0, \end{aligned}$$

which is impossible. Hence the result follows. ■

In the following, we study the linear stability of the positive steady state $N^* = 1/(a+b)$ of (4.227).

Theorem 4.4.1. *Assume that $b < a$. Then the positive steady state N^* of (4.227) is uniformly asymptotically stable.*

Proof. Set $y(t) = N(t) - N^*$, and then (4.227) becomes

$$\frac{dy(t)}{dt} = -r(y(t) + N^*)\left[ay(t) + b\sum_{j=0}^{\infty} c_j y([t-j])\right], \quad t \geq 0. \tag{4.238}$$

Thus the stability of N^* of (4.227) is equivalent to the stability of the trivial solution of (4.238). If we ignore the nonlinear terms in (4.238) (and write y as x for convenience), then the linearization of (4.238) is

$$\frac{dx(t)}{dt} = -raN^*x(t) - rbN^*\sum_{j=0}^{\infty} c_j x([t-j]), \quad t \geq 0. \tag{4.239}$$

On the interval $n \leq t < n+1$, (4.239) can be written as

$$\frac{dx(t)}{dt} = -raN^*x(t) - rbN^*\sum_{j=0}^{\infty} c_j x(n-j), \ n \leq t < n+1. \tag{4.240}$$

The solution of (4.240) on the interval $n \leq t < n+1$ is

$$x(t) = e^{-raN^*(t-n)}x(n) - \frac{b}{a}(1 - e^{-raN^*(t-n)})\sum_{j=0}^{\infty} c_j x(n-j). \tag{4.241}$$

We let $t \to n+1$ and obtain for $n = 0, 1, 2, \ldots,$

$$x(n+1) = e^{-raN^*}x(n) - \frac{b}{a}(1 - e^{-raN^*})\sum_{j=0}^{\infty} c_j x(n-j), \tag{4.242}$$

which is a linearization of the difference equation (4.234) at the positive equilibrium $N^* = 1/(a+b)$. In order to determine the stability of the linear difference equation (4.242), we will ignore the nonhomogeneous terms involving the value $N(-j) = \beta_j - N^*$ $(j = 1, 2, \ldots)$, that is, we drop the expression

$$g(n) = -\frac{b}{a}(1 - e^{-raN^*})\sum_{j=n+1}^{\infty} c_j x(n-j). \tag{4.243}$$

From (4.242), we have for $n = 0, 1, 2, \ldots$ the following Volterra difference equation

$$x(n+1) = e^{-raN^*}x(n) - \frac{b}{a}(1 - e^{-raN^*})\sum_{j=0}^{n} c_j x(n-j). \tag{4.244}$$

From the boundedness of solutions of $\{\beta_j\}$ and the convergence of $\sum_{j=0}^{\infty} c_j$, $|g(n)| \to 0$, as $n \to \infty$. Now, the asymptotic stability of zero solution of (4.242) is guaranteed by the asymptotic stability of the zero solution of (4.244), which can be decided by its characteristic equation. As is known the solution $x = 0$ is uniformly asymptotically stable if and only if all the roots of the characteristic equation

$$D(\lambda) = \lambda - e^{-raN^*} + \frac{b}{a}(1 - e^{-raN^*}) \sum_{j=0}^{\infty} c_j \lambda^{-j} = 0 \tag{4.245}$$

lie in the open unit disk of the complex plane. To complete the proof it suffices to show that $D(\lambda)$ has no zeros with $|\lambda| \geq 1$. Note that for $|\lambda| \geq 1$,

$$\left| \sum_{j=0}^{\infty} c_j \lambda^{-j} \right| \leq \sum_{j=0}^{\infty} c_j \left| \lambda^{-j} \right| \leq \sum_{j=0}^{\infty} c_j = 1, \tag{4.246}$$

and if there exists a zero λ_0 of $D(\lambda)$ with $|\lambda_0| \geq 1$, then by (4.246) we obtain

$$\left| \lambda - e^{-raN^*} \right| = \frac{b}{a}(1 - e^{-raN^*}) \left| \sum_{j=0}^{\infty} c_j \lambda^{-j} \right| \leq \frac{b}{a}(1 - e^{-raN^*}). \tag{4.247}$$

Now, by using $b < a$,

$$\begin{aligned} |\lambda_0| &\leq e^{-raN^*} + \frac{b}{a}(1 - e^{-raN^*}) = \frac{b}{a} + (1 - \frac{b}{a})e^{-raN^*} \\ &\leq \frac{b}{a} + (1 - \frac{b}{a}) = 1, \end{aligned}$$

which is a contradiction. The proof is complete. ■

The following lemma is well known.

Lemma 4.4.3. *Let $N(t)$ denote an arbitrary positive bounded solution of (4.227). Suppose that*

$$\lim_{t\to\infty} \sup N(t) = \tilde{N} \quad \textit{and} \quad \lim_{t\to\infty} \inf N(t) = \check{N}.$$

Then there exist sequences $\{t_n\}$ and $\{s_n\}$ such that $t_n \to \infty$, $s_n \to \infty$ as $n \to \infty$ for which

$$\left| N(t_n) - \tilde{N} \right| \leq \frac{1}{n}, \ \frac{dN(t)}{dt} \geq -\frac{2}{n}, \ n = 1, 2, \ldots, \tag{4.248}$$

$$\left| N(s_n) - \check{N} \right| \leq \frac{1}{n}, \ \frac{dN(t)}{dt} \leq \frac{2}{n}, \ n = 1, 2, \ldots. \tag{4.249}$$

Lemma 4.4.4. *Let r, a be positive numbers, let b, c_j $(j = 0, 1, 2, \ldots)$ be nonnegative numbers and c_j satisfy (4.228). If $a > b$, then all positive solutions $N(t)$ of (4.227) satisfy*

$$\lim_{t\to\infty} \inf N(t) \geq \delta > 0. \tag{4.250}$$

Proof. We first prove that

$$\lim_{t\to\infty} \sup N(t) = \tilde{N} < \frac{2}{a+b}. \tag{4.251}$$

From Lemma 4.4.2, $\tilde{N}$ must exist. Suppose on the contrary that

$$\lim_{t\to\infty} \sup N(t) = \tilde{N} \geq \frac{2}{a+b}. \tag{4.252}$$

From Lemma 4.4.3, there exists $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$ for which

$$\left|N(t_k) - \tilde{N}\right| \leq \frac{1}{k}, \;\; N'(t_k) \geq -\frac{2}{k}.$$

From (4.228) and the boundedness of $N([t-j])$, there exists n_0 such that

$$\varepsilon < (1/b)(\tilde{N} - N^*)(a-b),$$

$$\sum_{j=n_0+1}^{\infty} c_j \left|N([t_k - j]) - N^*\right| < \varepsilon.$$

From (4.252), we have for t_k large enough, $0 < N([t_k - j]) < \tilde{N}$. Hence for sufficiently large k we have

$$-\frac{2}{k} \leq N'(t_k) = rN(t_k)\left(1 - aN(t_k) - b\sum_{j=0}^{\infty} c_j N([t_k - j])\right)$$

$$= rN(t_k)\left(-a(N(t_k) - N^*) - b\sum_{j=0}^{\infty} c_j (N([t_k - j]) - N^*)\right)$$

$$\leq rN(t_k)\left(-a(N(t_k) - N^*) + b\sum_{j=0}^{\infty} c_j \left|N([t_k - j]) - N^*\right|\right)$$

$$= rN(t_k)\left(-a(N(t_k) - N^*) + b\sum_{j=0}^{n_0} c_j \left|N([t_k - j]) - N^*\right|\right)$$

$$+brN(t_k)\sum_{j=n_0+1}^{n_0} c_j \left|N([t_k - j]) - N^*\right|.$$

Letting $k \to \infty$, we have

$$\begin{aligned}
0 &\le r\tilde{N}\left(-a(\tilde{N}-N^*) + b\sum_{j=0}^{n_0} c_j(\tilde{N}-N^*) + b\varepsilon\right) \\
&\le r\tilde{N}\left(-a(\tilde{N}-N^*) + b\sum_{j=0}^{\infty} c_j(\tilde{N}-N^*) + b\varepsilon\right) \\
&= r\tilde{N}\left(-a(\tilde{N}-N^*) + b(\tilde{N}-N^*) + b\varepsilon\right) \\
&= r\tilde{N}\left(-(a-b)(\tilde{N}-N^*) + b\varepsilon\right) < 0.
\end{aligned}$$

This is a contradiction and hence (4.251) holds. From (4.251), there is a $T > 0$ such that

$$N(t) < \frac{2}{a+b}, \ t > T. \tag{4.253}$$

Hence for $t > n_0 + T$, we have

$$\begin{aligned}
\frac{dN(t)}{dt} &= rN(t)\left[1 - aN(t) - b\sum_{j=0}^{\infty} c_j N([t-j])\right] \\
&= rN(t)\left[1 - bN^* - aN(t) - b\sum_{j=0}^{\infty} c_j (N([t-j]) - N^*)\right] \\
&\ge rN(t)\left[\frac{a}{a+b} - aN(t) - b\sum_{j=0}^{n_0} c_j (N([t-j]) - N^*) - b\varepsilon\right]
\end{aligned}$$

$$\begin{aligned}
&\ge rN(t)\left[\frac{a}{a+b} - aN(t) - b\sum_{j=0}^{n_0} c_j N^* - b\varepsilon\right] \\
&\ge rN(t)\left[\frac{a}{a+b} - aN(t) - b\varepsilon\right] \\
&= rN(t)\left[\frac{a-b}{a+b} - b\varepsilon - aN(t)\right].
\end{aligned}$$

Since

$$L = \frac{a-b}{a+b} - b\varepsilon > 0$$

and the solution of the equation

$$u'(t) = ru(t)[L - au(t)], \quad u(0) > 0$$

is bounded above from zero, by the comparison principal, we conclude that the result of the lemma holds. The proof is complete. ■

Theorem 4.4.2. *If the conditions of Lemma 4.4.4 hold, then all positive solution $N(t)$ of (4.227) satisfy*

$$\lim_{t\to\infty} N(t) = N^* = \frac{1}{a+b}. \tag{4.254}$$

Proof. In order to prove (4.254), it is sufficient to prove that

$$\lim_{t\to\infty} \sup |N(t) - N^*| = 0. \tag{4.255}$$

Suppose (4.255) is not valid, then at least one of the following holds for some $\rho^* > 0$ and $\rho_* > 0$:

$$\lim_{t\to\infty} \sup N(t) = \tilde{N} = N^* + \rho^*, \tag{4.256}$$

$$\lim_{t\to\infty} \inf N(t) = \check{N} = N^* - \rho_*. \tag{4.257}$$

Suppose that (4.256) holds. Then there exists a sequence $\{t_n\}$ such that

$$t_n \to \infty, \; N'(t_n) \geq -\frac{2}{n}, \; N(t_n) \to \tilde{N} \;\; \text{as } n \to \infty. \tag{4.258}$$

It follows that

$$\begin{aligned}
-\frac{2}{n} \leq \frac{dN(t_n)}{dt} &= rN(t_n)\left[1 - aN(t_n) - b\sum_{j=0}^{\infty} c_j N([t_n - j])\right] \\
&= rN(t_n)\left[-a(N(t_n) - N^*) - b\sum_{j=0}^{\infty} c_j (N([t_n - j]) - N^*)\right] \\
&\leq rN(t_n)\left[-a(N(t_n) - N^*) + b\sum_{j=0}^{\infty} c_j \left|N([t_n - j]) - N^*\right|\right] \\
&\leq rN(t)\left[-a(N(t_n) - N^*) + b\sum_{j=0}^{n_0} c_j \left|N([t_n - j]) - N^*\right| + b\varepsilon\right].
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$0 \le -r\tilde{N}(\left(a\rho^* - b\sum_{j=0}^{\infty} c_j \rho^* - b\varepsilon\right) < 0,$$

and this is not possible. For the case (4.257), we choose a sequence

$$t_n \to \infty,\ N'(t_n) \le \frac{2}{n},\ N(t_n) \to \check{N}, \quad \text{as } n \to \infty.$$

Then, we have

$$\begin{aligned}
\frac{2}{n} \ge \frac{dN(t_n)}{dt} &= rN(t_n)\left[1 - aN(t_n) - b\sum_{j=0}^{\infty} c_j N([t_n - j])\right] \\
&= rN(t_n)\left[-a(N(t_n) - N^*) - b\sum_{j=0}^{\infty} c_j (N([t_n - j]) - N^*)\right] \\
&\ge rN(t_n)\left[a(N^* - N(t_n)) - b\sum_{j=0}^{\infty} c_j\, |N([t_n - j]) - N^*|\right] \\
&= rN(t)\left[a(N^* - N(t_n)) - b\sum_{j=0}^{n_0} c_j\, |N([t_n - j]) - N^*| - b\varepsilon\right].
\end{aligned}$$

We let $n \to \infty$, to obtain

$$0 \ge r\check{N}\left(a\rho_* - b\sum_{j=0}^{n_0} c_j \rho_* - b\varepsilon\right) \ge r\check{N}(a-b)\rho_* - r\check{N}b\varepsilon > 0,$$

which is not possible. Thus (4.254) holds and the proof is complete. ∎

Combining Theorems 4.4.1 and 4.4.2, we have the following global attractivity result.

Theorem 4.4.3. *If $b < a$, then the positive equilibrium $N^* = 1/(a+b)$ of (4.227) is globally asymptotically stable.*

Chapter 5
Food-Limited Population Models

If a nonnegative quantity was so small that is smaller than any given one, then it certainly could not be anything but zero. To those who ask what the infinity small quantity in mathematics is, we answer that it is actually zero. Hence there are not so many mysteries hidden in this concept as they are usually believed to be.

Leonhard Euler (1707–1783)

The real end of science is the honor of the human mind.

Gustav J. Jacobi (1804–1851)

Smith [66] reasoned that a food-limited population in its growing stage requires food for both maintenance and growth, whereas, when the population has reached saturation level, food is needed for maintenance only. On the basis of these assumptions, Smith derived a model of the form

$$\frac{dN(t)}{dt} = rN(t)\frac{K - N(t)}{K + crN(t)} \tag{5.1}$$

which is called the "food limited" population. Here N, r, and K are the mass of the population, the rate of increase with unlimited food, and the value of N at saturation, respectively. The constant $1/c$ is the rate of replacement of mass in the population at saturation. Since a realistic model must include some of the past history of the population, Gopalsamy, Kulenovic and Ladas introduced the delay in (5.1) and considered the equation

$$\frac{dN(t)}{dt} = rN(t)\frac{K - N(t-\tau)}{K + crN(t-\tau)},$$

as the delay "food-limited" population model, where r, K, c, and τ are positive constants.

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In this chapter we discuss autonomous and nonautonomous “food-limited” population models with delay times.

5.1 Oscillation of Delay Models

Motivated by the model

$$N'(t) = r(t)N(t)\frac{K - N(h(t))}{1 + s(t)N(g(t))}, \quad t \geq 0, \tag{5.2}$$

in this section we consider

$$x'(t) = -r(t)x(h(t))\frac{1 + x(t)}{1 + s(t)[1 + x(g(t))]}, \quad t \geq 0, \tag{5.3}$$

with the following assumptions:

$(A1)$ $r(t)$ and $s(t)$ are Lebesgue measurable locally essentially bounded functions such that $r(t) \geq 0$ and $s(t) \geq 0$.

$(A2)$ $h, g : [0, \infty) \to \mathbf{R}$ are Lebesgue measurable functions such that $h(t) \leq t$, $g(t) \leq t$, $\lim_{t\to\infty} h(t) = \infty$, and $\lim_{t\to\infty} g(t) = \infty$.
Note the oscillation (or nonoscillation) of N about K is equivalent to oscillation (nonoscillation) of (5.3) about zero (let $x = N/K - 1$).
One could also consider for each $t_0 \geq 0$ the problem

$$x'(t) = -r(t)x(h(t))\frac{1 + x(t)}{1 + s(t)[1 + x(g(t))]}, \quad t \geq t_0, \tag{5.4}$$

with the initial condition

$$x(t) = \varphi(t), \quad t < t_0, \; x(t_0) = x_0. \tag{5.5}$$

We also assume that the following hypothesis holds:

$(A3)$ $\varphi : (-\infty, t_0) \to \mathbf{R}$ is a Borel measurable bounded function.

An absolutely continuous function $x(: \mathbf{R} \to \mathbf{R})$ on each interval $[t_0, b]$ is called a solution of problems (5.4) and (5.5), if it satisfies (5.4) for almost all $t \in [t_0, \infty)$ and the equality (5.5) for $t \leq t_0$. Equation (5.3) has a nonoscillatory solution if it has an eventually positive or an eventually negative solution. Otherwise, all solutions of (5.3) are oscillatory. The results in this section can be found in [10]. In the following, we assume that $(A1)$–$(A3)$ hold and we consider only such solutions of (5.3) for which the following condition holds:

$$1 + x(t) > 0. \tag{5.6}$$

The proof of the following lemma follows a standard argument (see the proof in Theorem 2.4.1 and see Lemma 2.6.1).

Lemma 5.1.1. *Let* $(A1)$ *and* $(A2)$ *hold for the equation*

$$x'(t) + r(t)x(h(t)) = 0, \quad t \geq 0. \tag{5.7}$$

Then the following hypotheses are equivalent:

(1) *The differential inequality*

$$x'(t) + r(t)x(h(t)) \leq 0, \quad t \geq 0 \tag{5.8}$$

has an eventually positive solution.

(2) *There exists* $t_0 \geq 0$ *such that the inequality*

$$u(t) \geq r(t) \exp\left\{\int_{h(t)}^{t} u(s)ds\right\}, \ t \geq t_0, \ u(t) = 0, t < t_0 \tag{5.9}$$

has a nonnegative locally integrable solution.

(3) *Equation (5.7) has a nonoscillatory solution.*
If

$$\lim_{t\to\infty} \sup \int_{h(t)}^{t} r(s)ds < \frac{1}{e}, \tag{5.10}$$

then (5.7) has a nonoscillatory solution. If

$$\lim_{t\to\infty} \inf \int_{h(t)}^{t} r(s)ds > \frac{1}{e}, \tag{5.11}$$

then all the solutions of (5.7) are oscillatory.

Lemma 5.1.2. *Let* $x(t)$ *be a nonoscillatory solution of* (5.3) *and suppose that*

$$\int_0^{\infty} \frac{r(t)}{1+s(t)} dt = \infty. \tag{5.12}$$

Then $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose first $x(t) > 0, t \geq t_1$. Then there exists $t_2 \geq t_1$ such that

$$h(t) \geq t_1, \quad g(t) \geq t_1, \ \text{for } t \geq t_2. \tag{5.13}$$

Let

$$u(t) = -\frac{x'(t)}{x(t)}, \quad t \geq t_2. \tag{5.14}$$

Then $u(t) \geq 0, t \geq t_2$ and

$$x(t) = x(t_2) \exp\left\{-\int_{t_2}^{t} u(s)ds\right\}, \ t \geq t_2. \tag{5.15}$$

Substituting this into (5.3) we obtain

$$u(t) = r(t)e^{\left(\int_{h(t)}^{t} u(s)ds\right)} \frac{\left[1 + c\exp\left\{-\int_{t_2}^{t} u(s)ds\right\}\right]}{\left[1 + s(t)\left(1 + c\exp\left\{-\int_{t_2}^{g(t)} u(s)ds\right\}\right)\right]}, \tag{5.16}$$

where $h(t) \leq t$, $g(t) \leq t$, for $t \geq t_2$, and $c = x(t_2) > 0$. Hence

$$u(t) \geq \frac{r(t)}{(1+c)(1+s(t))}. \tag{5.17}$$

From (5.12) we have $\int_{t_2}^{\infty} u(t)dt = \infty$.

Now suppose $-1 < x(t) < 0, t \geq t_1$. Then there exists $t_2 \geq t_1$ such that (5.13) holds for $t \geq t_2$. With $u(t)$ denoted in (5.14) and $c = x(t_2)$ we have $u(t) \geq 0$ and $-1 < c < 0$. Substituting (5.15) into (5.3) and using (5.16), we have

$$u(t) \geq \frac{(1+c)r(t)}{(1+s(t))}. \tag{5.18}$$

Thus $\int_{t_2}^{\infty} u(t)dt = \infty$. Equation (5.15) implies that $\lim_{t\to\infty} x(t) = 0$. The proof is complete. ∎

Theorem 5.1.1. *Suppose (5.12) holds and for some $\varepsilon > 0$, all solutions of the linear equation*

$$x'(t) + (1-\varepsilon)\frac{r(t)}{1+s(t)}x(h(t)) = 0 \tag{5.19}$$

are oscillatory. Then all solutions of (5.3) are oscillatory.

Proof. First suppose $x(t)$ is an eventually positive solution of (5.3). Lemma 5.1.2 implies that there exists $t_1 \geq 0$ such that $0 < x(t) < \varepsilon$ for $t \geq t_1$. We suppose (5.13) holds for $t \geq t_2 \geq t_1$. For $t \geq t_2$, we have

$$\begin{aligned} \frac{[1+s(t)](1+x(t))}{1+s(t)[1+x(g(t))]} &\geq \frac{(1+s(t))}{1+s(t)(1+\varepsilon)} \\ \geq \frac{(1+s(t))}{(1+s(t))(1+\varepsilon)} &= \frac{1}{(1+\varepsilon)} \geq 1-\varepsilon. \end{aligned} \tag{5.20}$$

Equation (5.3) implies

$$x'(t) + (1-\varepsilon)\frac{r(t)}{1+s(t)}x(h(t)) \leq 0, \quad t \geq t_2. \tag{5.21}$$

Lemma 5.1.1 yields that (5.19) has a nonoscillatory solution. We have a contradiction.

Now suppose $-\varepsilon < x(t) < 0$ for $t \geq t_1$ and (5.13) holds for $t \geq t_2 \geq t_1$. Then for $t \geq t_2$

$$\frac{[1+s(t)](1+x(t))}{1+s(t))[1+x(g(t))]} \geq \frac{(1+s(t))(1-\varepsilon)}{1+s(t)} = 1-\varepsilon. \tag{5.22}$$

Hence, (5.19) has a nonoscillatory solution and we again obtain a contradiction which completes the proof. ■

Corollary 5.1.1. *If*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} \frac{r(\tau)}{1+s(\tau)} d\tau > \frac{1}{e}, \tag{5.23}$$

then all solutions of (5.3) are oscillatory.

Theorem 5.1.2. *Suppose for some $\varepsilon > 0$ there exists a nonoscillatory solution of the linear delay differential equation*

$$x'(t) + (1+\varepsilon)\frac{r(t)}{1+s(t)} x(h(t)) = 0. \tag{5.24}$$

Then there exists a nonoscillatory solution of (5.3).

Proof. Lemma 5.1.1 implies that there exists $t_0 \geq 0$ such that

$$w_0(t) \geq 0, \text{ for } t \geq t_0, \text{ and } w_0(t) = 0, \text{ for } t \leq t_0,$$

and

$$w_0(t) \geq (1+\varepsilon)\frac{r(t)}{1+s(t)} \exp\left\{\int_{h(t)}^{t} w_0(s)ds\right\}. \tag{5.25}$$

Suppose $0 < c < \varepsilon$ and consider two sequences:

$$w_n(t) = r(t)\exp\left\{\int_{h(t)}^{t} w_{n-1}(s)ds\right\}$$
$$\times \frac{1+c\exp\left\{-\int_{t_0}^{t} v_{n-1}(s)ds\right\}}{1+s(t)\left(1+c\exp\left\{-\int_{t_0}^{g(t)} w_{n-1}(s)ds\right\}\right)}$$

and

$$v_n(t) = r(t)\exp\left\{\int_{h(t)}^{t} v_{n-1}(s)ds\right\}$$
$$\times \frac{1+c\exp\left\{-\int_{t_0}^{t} w_{m-1}(s)ds\right\}}{1+s(t)\left(1+c\exp\left\{-\int_{t_0}^{g(t)} v_{n-1}(s)ds\right\}\right)},$$

where w_0 is as defined above and $\upsilon_0(t) \equiv 0$. We have

$$w_1(t) = \frac{r(t)}{1+s(t)} \exp\left\{\int_{h(t)}^{t} w_0(s)ds\right\} \times \frac{(1+s(t))(1+c)}{1+s(t)\left(1+c\exp\left\{-\int_{t_0}^{g(t)} w_0(s)ds\right\}\right)}$$

$$\leq \frac{r(t)}{1+s(t)} \exp\left\{\int_{h(t)}^{t} w_0(s)ds\right\} \times \frac{(1+s(t))(1+\varepsilon)}{1+s(t)\left(1+c\exp\left\{-\int_{t_0}^{g(t)} w_0(s)ds\right\}\right)}$$
$$\leq w_0(t) \tag{5.26}$$

from (5.25). Clearly $\upsilon_1(t) \geq \upsilon_0(t)$ and $w_0(t) \geq \upsilon_0(t)$. Hence by induction

$$\begin{cases} 0 \leq w_n(t) \leq w_{n-1}(t) \leq \ldots \leq w_0(t), \\ \upsilon_n(t) \geq \upsilon_{n-1}(t) \geq \ldots \geq \upsilon_0(t) = 0, \\ \quad w_n(t) \geq \nu_n(t). \end{cases} \tag{5.27}$$

There exist pointwise limits of the nonincreasing nonnegative sequence $w_n(t)$ and of the nondecreasing sequence $\nu_n(t)$. Let

$$w(t) = \lim_{n\to\infty} w_n(t) \text{ and } \upsilon(t) = \lim_{n\to\infty} \upsilon(t).$$

Then by the Lebesgue Convergence Theorem, we conclude that

$$w(t) = r(t)\exp\left\{\int_{h(t)}^{t} w(s)ds\right\} \times \frac{1+c\exp\left\{-\int_{t_0}^{t} \nu(s)ds\right\}}{1+s(t)\left(1+c\exp\left\{-\int_{t_0}^{g(t)} w(s)ds\right\}\right)}$$

and

$$\nu(t) = r(t)\exp\left\{\int_{h(t)}^{t} \nu(s)ds\right\} \times \frac{1+c\exp\left\{-\int_{t_0}^{t} w(s)ds\right\}}{1+s(t)\left(1+c\exp\left\{-\int_{t_0}^{g(t)} \nu(s)ds\right\}\right)}.$$

We fix $b \geq t_0$ and define the operator $T : L_\infty[t_0, b] \to L_\infty[t_0, b]$ by

$$T(u(t)) = e^{\int_{h(t)}^{t} u(s)ds} \frac{r(t)\left(1 + c\exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right)}{1 + s(t)\left(1 + c\exp\left\{-\int_{t_0}^{g(t)} u(s)ds\right\}\right)}. \tag{5.28}$$

For every function u from the interval $\upsilon \leq u \leq w$, we have $\upsilon \leq Tu \leq w$. One can also check that T is a completely continuous operator on the space $L_\infty[t_0, b]$. Then by Schauder's Fixed Point Theorem there exists a nonnegative solution of equation $u = Tu$. Let

$$x(t) = \begin{cases} c\exp\left\{-\int_{t_0}^{t} u(s)ds\right\}, & \text{if } t \geq t_0, \\ 0, & \text{if } t < t_0, \end{cases} \tag{5.29}$$

and then $x(t)$ is a nonoscillatory solution of (5.3) which completes the proof. ■

The results in this section apply to (5.2). For example by applying Theorem 5.1.1 we have the following result.

Theorem 5.1.3. *Suppose (5.12) holds and for some $\varepsilon > 0$, all solutions of the linear equation*

$$N'(t) + (1 - \varepsilon)\frac{r(t)}{1 + s(t)} N(h(t)) = 0 \tag{5.30}$$

are oscillatory. Then all solutions of (5.2) are oscillatory about K.

5.2 Oscillation of Impulsive Delay Models

In this section we consider the impulsive "food-limited" population model

$$\begin{cases} N'(t) = r(t)N(t)\dfrac{K - N(h(t))}{K + \sum_{i=1}^{m} p_i(t)N(g_i(t))}, \quad t \neq t_k, \\ N(t_k^+) - N(t_k) = b_k(N(t_k) - K), \ \text{for } k = 1, 2, \ldots; \end{cases} \tag{5.31}$$

here $N(t_k) = N(t_k^-)$. In this section, we will assume that the following assumptions hold:

$(A1)$ $0 \leq t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ are fixed points with $\lim_{k\to\infty} t_k = \infty$,
$(A2)$ $b_k > -1, k = 1, 2, \ldots$, K is a positive constant,
$(A3)$ $r(t)$ and $p_i, i = 1, 2, \ldots, m$, are Lebesgue measurable locally essentially bounded functions, in each finite interval $[0, b]$, $r(t) \geq 0$ and $p_i(t) \geq 0$, for $i = 1, 2, \ldots, m$,
$(A4)$ $h, g_i : [0, \infty) \to \mathbf{R}$ are Lebesgue measurable functions, $h(t) \leq t, g_i(t) \leq t$, $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} g_i(t) = \infty, i = 1, 2, \ldots, m$.

In this section (motivated by (5.31) with $y(t) = \frac{N(t)}{K} - 1$) we consider the delay model with impulses

$$\begin{cases} y'(t) = -r(t)\dfrac{(1 + y(t))\,y(h(t))}{1 + \sum_{i=1}^{m} p_i(t)\,[1 + y(g_i(t))]}, \ t \neq t_k, \ t \geq T_0 \geq 0 \\ y(t_k^+) - y(t_k) = b_k y(t_k), \ \text{for } k = 1, 2, \ldots, \end{cases} \tag{5.32}$$

where $b_k > -1$ and r, h, p_i for $m = 1, 2, \ldots$ are nonnegative real-valued functions. We consider (5.32) with the initial condition

$$y(t) = \varphi(t) \geq 0, \quad \varphi(T_0) > 0, \ \ t \in [T^-, T_0]. \tag{5.33}$$

Here for any $T_0 \geq 0$, $T^- = \min_{1 \leq i \leq m} \inf_{t \geq T_0}(g_i(t), h(t))$, and $\varphi : [T^-, T_0] \to \mathbf{R}_+$ is a Lebesgue measurable function.

For any $T_0 \geq 0$ and $\varphi(t)$, a function $y : [T^-, \infty] \to \mathbf{R}$ is said to be a solution of (5.32) on $[T, \infty]$ satisfying the initial value condition (5.33), if the following conditions are satisfied:

1. $y(t)$ satisfies (5.33);
2. $y(t)$ is absolutely continuous in each interval $(T_0, t_k), (t_k, t_{k+1}), t_k > T_0, k \geq k_0$, $y(t_k^+), y(t_k^-)$ exist and $y(t_k^-) = y(t_k), k > k_0$;
3. $y(t)$ satisfies the former equation of (5.32) in $[T, \infty) \backslash \{t_k\}$ and satisfies the latter equation for every $t = t_k, k = 1, 2, \ldots$.

For any $t \geq 0$, consider the nonlinear delay differential equation

$$x'(t) = -r(t)\frac{1 + \left(\prod_{T_0 \leq t_k < t} (1 + b_k)\right) x(t)}{1 + \Psi(x(g_i(t))} \times \prod_{h(t) \leq t_k < t} (1 + b_k)^{-1} x(h(t)), \tag{5.34}$$

where

$$\Psi(x(g_i(t)) = \sum_{i=1}^{m} p_i(t) \left[1 + (\prod_{T_0 \leq t_k < g_i(t)} (1 + b_k)) x(g_i(t)) \right].$$

The results in this section are adapted from [77] (in fact as we see below it is easy to extend the theory in the nonimpulsive case in Sect. 5.1 to the impulsive case).

Lemma 5.2.1. *Assume that* $(A1)$–$(A4)$ *hold. Then the solution* $N(t)$ *of (5.31) oscillates about* K *if and only if the solution* $y(t)$ *of (5.32) oscillates about zero.*

The proof (which is elementary and straightforward) of the next lemma can be found in [81].

Lemma 5.2.2. *Assume that* $(A1)$–$(A4)$ *hold. For any* $T_0 \geq 0, y(t)$ *is a solution of (5.32) on* $[T_0, \infty)$ *if and only if*

$$x(t) = \left(\prod_{T_0 \leq t_k < t} (1 + b_k) \right)^{-1} y(t) \tag{5.35}$$

is a solution of the nonimpulsive delay differential equation (5.34).

From Lemmas 5.2.1 and 5.2.2 we see that the solution $N(t)$ of (5.31) is oscillatory about K if and only if the solution $y(t)$ of (5.32) is oscillatory.

We consider only such solutions of (5.32) for which the following condition holds:

$$1 + y(t) > 0, \quad \text{for } t \geq T_0, \tag{5.36}$$

and hence, in view of (5.35),

$$1 + \left(\prod_{T_0 \leq t_k < t} (1 + b_k) \right) x(t) > 0, \quad \text{for } t \geq T_0. \tag{5.37}$$

With $y(t) = \frac{N(t)}{K} - 1$ then from (5.36) and (5.37), we see that

$$N(t) = K \left(1 + \prod_{T_0 \leq t_k < t} (1 + b_k) x(t) \right) > 0, \quad t \geq T_0.$$

Thus for the initial condition $N(t) = \varphi(t) : [T^-, T_0] \to \mathbf{R}_+, \varphi(T_0) > 0$, the solution of (5.31) is positive on $[T_0, \infty)$.

Lemma 5.2.3. *Assume that* $(A1)$–$(A4)$ *hold,*

$$\int_0^\infty r(t) \left(1 + \sum_{i=1}^m p_i(t) \right)^{-1} dt = \infty, \tag{5.38}$$

and

$$\prod_{T_0 \leq t_k < t} (1 + b_k) \textit{ is bounded and } \liminf_{t \to \infty} \prod_{T_0 \leq t_k < t} (1 + b_k) > 0. \tag{5.39}$$

If $y(t)$ *is a nonoscillatory solution of (5.32), then* $\lim_{t \to \infty} y(t) = 0$.

Proof. Suppose first $y(t) > 0$ for $t \geq T_1 \geq 0$. From (5.35) and $(A1)$, $x(t) > 0$ for $t \geq T_1$. Then there exists $T_2 \geq T_1$ such that

$$h(t) \geq T_2, \quad g_i(t) \geq T_2, \quad i = 1, 2, \ldots, m, \text{ for } t \geq T_2. \tag{5.40}$$

Let

$$u(t) = -\frac{x'(t)}{x(t)}, \quad \text{for } t \geq T_2. \tag{5.41}$$

Then $u(t) \geq 0$ for $t \geq T_2$ and

$$x(t) = x(T_2) \exp\left\{-\int_{T_2}^{t} u(s)ds\right\}, \quad \text{for } t \geq T_2. \tag{5.42}$$

Setting $c = x(T_2)$, we have

$$\begin{aligned}
u(t) &= \frac{r(t)}{x(t)} \left(\prod_{h(t)\leq t_k<t} (1+b_k)^{-1} \right) x(h(t)) \\
&\quad \times \frac{1 + (\prod_{T_0\leq t_k<t} (1+b_k))x(t)}{1 + \sum_{i=1}^{m} p_i(t)[1 + (\prod_{T_0\leq t_k<g_i(t)} (1+b_k))x(g_i(t))]} \\
&\geq \frac{r(t)}{x(t)} \left(\prod_{h(t)\leq t_k<t} (1+b_k)^{-1} \right) x(t) \\
&\quad \times \frac{1}{1 + \sum_{i=1}^{m} p_i(t)[1 + \prod_{T_0\leq t_k<g_i(t)} (1+b_k)c]} \\
&= \frac{r(t)}{1 + \sum_{i=1}^{m} p_i(t)} \left(\prod_{h(t)\leq t_k<t} (1+b_k)^{-1} \right) \\
&\quad \times \frac{1 + \sum_{i=1}^{m} p_i(t)}{1 + \sum_{i=1}^{m} p_i(t)[1 + \prod_{T_0\leq t_k<g_i(t)} (1+b_k)c]} \\
&\geq \frac{r(t)}{1 + \sum_{i=1}^{m} p_i(t)} \frac{(\prod_{h(t)\leq t_k<t} (1+b_k))^{-1}}{(1 + \sum_{i=1}^{m} p_i(t)(1 + (\prod_{T_0\leq t_k<g_i(t)} (1+b_k)c))}.
\end{aligned}$$

Then from (5.38) and (5.39), $\int_{T_2}^{\infty} u(t)dt = \infty$.

Now suppose $-1 < y(t) < 0$. Hence in view of (5.36),

$$-1 < \prod_{T_0 \le t_k < g_i(t)} (1 + b_k)x(t) < 0, \quad t \ge T_1.$$

Then there exists $T_2 > T_1$ such that (5.40) holds for $t > T_2$. With $u(t)$ denoted in (5.41) and $c = x(T_2)$, then from (5.37) $u(t) \ge 0$, $-1 < c < 0$, and we obtain

$$u(t) = \frac{r(t)}{x(t)} \left(\prod_{h(t) \le t_k < t} (1 + b_k)^{-1} \right) x(h(t))$$

$$\times \frac{1 + (\prod_{T_0 \le t_k < t} (1 + b_k))x(t)}{1 + \sum_{i=1}^{m} p_i(t)[1 + (\prod_{T_0 \le t_k < g_i(t)} (1 + b_k))x(g_i(t))]}$$

$$\ge \frac{r(t)}{x(h(t))} \left(\prod_{h(t) \le t_k < t} (1 + b_k)^{-1} \right) x(h(t)) \frac{1 + (\prod_{T_0 \le t_k < t} (1 + b_k))c}{1 + \sum_{i=1}^{m} p_i(t)}$$

$$= \left(\prod_{h(t) \le t_k < t} (1 + b_k)^{-1} \right) \left(1 + (\prod_{T_0 \le t_k < t} (1 + b_k))c \right)$$

$$\times \frac{r(t)}{1 + \sum_{i=1}^{m} p_i(t)}.$$

Then by (5.37)–(5.39), we have $\int_{T_2}^{\infty} u(t)dt = \infty$. Equation (5.42) implies $\lim_{t \to \infty} x(t) = 0$. Use (5.35), and then we have $\lim_{t \to \infty} y(t) = 0$. The proof is complete. ■

Theorem 5.2.1. *Assume that* $(A1)$ *and* $(A2)$*, (5.38) hold and for some* $\epsilon > 0$*, all solutions of the linear equation*

$$x'(t) + (1-\epsilon) \prod_{h(t)\le t_k<t} (1+b_k)^{-1} \frac{r(t)x(h(t))}{1+\sum_{i=1}^{m} p_i(t)} = 0 \tag{5.43}$$

are oscillatory. Then all solutions of (5.32) are oscillatory.

Proof. Suppose $y(t)$ is an eventually positive solution of (5.32). Then $x(t)$ is an eventually positive solution of (5.34). Lemma 5.2.3 implies that there exists $T_1 \ge 0$, such that

$$0 < (\prod_{T_0\le t_k<t} (1+b_k))x(t) < \epsilon, \quad \text{for } t \ge T_1.$$

We suppose (5.40) holds for $t \ge T_2$, and we have

$$\begin{aligned}
&\frac{(1+\sum_{i=1}^{m} p_i(t))(1+(\prod_{T_0\le t_k<t}(1+b_k))x(t))}{1+\sum_{i=1}^{m} p_i(t)[1+(\prod_{T_0\le t_k<g_i(t)}(1+b_k))x(g_i(t))]}\\
&\ge \frac{1+\sum_{i=1}^{m} p_i(t)}{1+\sum_{i=1}^{m} p_i(t)(1+\epsilon)} \ge \frac{1+\sum_{i=1}^{m} p_i(t)}{(1+\epsilon)(1+\sum_{i=1}^{m} p_i(t))}\\
&= \frac{1}{1+\epsilon} \ge 1-\epsilon.
\end{aligned} \tag{5.44}$$

Equation (5.34) implies

$$x'(t) + (1-\epsilon) \prod_{h(t)\le t_k<t} (1+b_k)^{-1} \frac{r(t)x(h(t))}{1+\sum_{i=1}^{m} p_i(t)} \le 0, \quad t \ge T_2. \tag{5.45}$$

This implies that the (5.43) has a positive solution, which is a contradiction.

Now, we suppose

$$-\epsilon < (\prod_{T_0\le t_k<t} (1+b_k))x(t) < 0, \quad \text{for } t \ge T_1,$$

and (5.38) holds for $t \ge T_2 \ge T_1$. Then for $t \ge T_2$, we also get

$$\frac{(1+\sum_{i=1}^{m} p_i(t))(1+(\prod_{T_0\le t_k<t}(1+b_k))x(t))}{1+\sum_{i=1}^{m} p_i(t)[1+(\prod_{T_0\le t_k<g_i(t)}(1+b_k))x(g_i(t))]}$$

$$\ge \frac{(1+\sum_{i=1}^{m} p_i(t))(1-\epsilon)}{1+\sum_{i=1}^{m} p_i(t)} = 1-\epsilon. \tag{5.46}$$

Thus (5.43) has a nonoscillatory solution and we again obtain a contradiction. The proof is complete. ■

Theorem 5.2.2. *Assume that* $(A1)$ *and* $(A2)$ *hold and*

$$\prod_{h(t)\le t_k<t}(1+b_k) \quad \textit{is convergent.} \tag{5.47}$$

Moreover, for some $\epsilon > 0$ *if there exists a nonoscillatory solution of the linear delay differential equation*

$$x'(t) + (1+\epsilon)\prod_{h(t)\le t_k<t}(1+b_k)^{-1}\frac{r(t)x(h(t))}{1+\sum_{i=1}^{m} p_i(t)} = 0, \tag{5.48}$$

then there exists a nonoscillatory solution of (5.32).

Proof. Suppose that $x(t) > 0$ for $t > T_0$ is a solution of (5.48). Then by (5.34) there exist $T_0 \ge 0$ and $\omega_0(t) \ge 0, t \ge T_0, \omega_0(t) = 0, T_0^- \le t \le T_0$ such that

$$\omega_0(t) \ge \frac{(1+\epsilon)r(t)}{1+\sum_{i=1}^{m} p_i(t)}(\prod_{h(t)\le t_k<t}(1+b_k)^{-1})\exp\left\{\int_{h(t)}^{t}\omega_0(s)ds\right\}. \tag{5.49}$$

Since $\prod_{T_0\le t_k<g_i(t)}(1+b_k)$ is convergent, there exists a positive constant c such that

$$0 < c(\prod_{T_0\le t_k<g_i(t)}(1+b_k)) < \epsilon.$$

Consider the two sequences:

$$\omega_n(t) = r(t)\left(\prod_{h(t)\le t_k<t}(1+b_k)^{-1}\right)\exp\left\{\int_{h(t)}^{t}\omega_{n-1}(s)ds\right\}$$

$$\cdot\frac{1+c(\prod_{T_0\le t_k<g_i(t)}(1+b_k))\exp\left\{-\int_{T_0}^{t}\upsilon_{n-1}(s)ds\right\}}{1+\sum_{i=1}^{m}p_i(t)(1+c(\prod_{T_0\le t_k<g_i(t)}(1+b_k))\exp-\{\int_{T_0}^{g_i(t)}\omega_{n-1}(s)ds\})},$$

$$n = 1,2,\ldots,$$

$$\upsilon_n(t) = r(t)\left(\prod_{h(t)\le t_k<t}(1+b_k)^{-1}\right)\exp\left\{\int_{h(t)}^{t}\upsilon_{n-1}(s)ds\right\} \tag{5.50}$$

$$\cdot\frac{1+c(\prod_{T_0\le t_k<g_i(t)}(1+b_k))\exp\left\{-\int_{T_0}^{t}\omega_{n-1}(s)ds\right\}}{1+\sum_{i=1}^{m}p_i(t)(1+c\left(\prod_{T_0\le t_k<g_i(t)}(1+b_k)\right)\exp-\{\int_{T_0}^{g_i(t)}\upsilon_{n-1}(s)ds\})},$$

$$n = 1,2,\ldots,$$

where ω_0 is defined above and $\upsilon_0 \equiv 0$. Thus we have

$$\omega_1(t) = \frac{r(t)}{1+\sum_{i=1}^{m}p_i(t)}\left(\prod_{h(t)\le t_k<t}(1+b_k)^{-1}\right)\exp\left\{\int_{h(t)}^{t}\omega_0(s)ds\right\}$$

$$\times\frac{(1+\sum_{i=1}^{m}p_i(t))(1+c(\prod_{T_0\le t_k<g_i(t)}(1+b_k))}{1+\sum_{i=1}^{m}p_i(t)(1+c(\prod_{T_0\le t_k<g_i(t)}(1+b_k))\exp-\{\int_{T_0}^{g_i(t)}\omega_0(s)ds\})}$$

$$\leq \frac{r(t)(\prod_{h(t)\leq t_k<t}(1+b_k)^{-1})}{1+\sum_{i=1}^{m} p_i(t)} e^{\int_{h(t)}^{t}\omega_0(s)ds} \frac{(1+\sum_{i=1}^{m} p_i(t))(1+\epsilon)}{1+\sum_{i=1}^{m} p_i(t)}$$
$$\leq \omega_0(t). \tag{5.51}$$

Clearly $\upsilon_1(t) \geq \upsilon_0(t), \omega_0(t) \geq \upsilon_0(t)$. Hence by induction

$$\begin{cases} 0 \leq \omega_n(t) \leq \omega_{n-1}(t) \leq \ldots \leq \omega_0(t), \\ \upsilon_n(t) \geq \upsilon_{n-1}(t) \geq \ldots \geq \upsilon_0(t) = 0, \quad n = 1, 2, \ldots, \\ \omega_n(t) \geq \upsilon_n(t). \end{cases}$$

There exist pointwise limits of the nonincreasing nonnegative sequence $\omega_n(t)$ and of the nondecreasing sequence $\upsilon_n(t)$. Let $\omega(t) = \lim_{n\to\infty}\omega_n(t)$ and $\upsilon(t) = \lim_{n\to\infty}\upsilon_n(t)$. Then by the Lebesgue Convergence Theorem, we deduce that

$$\omega(t) = r(t)\left(\prod_{h(t)\leq t_k<t}(1+b_k)\right)\exp\left\{\int_{h(t)}^{t}\omega(s)ds\right\}$$
$$\cdot\frac{1+c(\prod_{T_0\leq t_k<g_i(t)}(1+b_k))\exp\left\{-\int_{T_0}^{t}\upsilon(s)ds\right\}}{1+\sum_{i=1}^{m} p_i(t)(1+c(\prod_{T_0\leq t_k<g_i(t)}(1+b_k))\exp-\{\int_{T_0}^{g_i(t)}\omega(s)ds\})},$$

$$\upsilon(t) = r(t)\left(\prod_{h(t)\leq t_k<t}(1+b_k)\right)\exp\left\{\int_{h(t)}^{t}\upsilon(s)ds\right\}$$
$$\times\frac{1+c(\prod_{T_0\leq t_k<g_i(t)}(1+b_k))\exp\left\{-\int_{T_0}^{t}\omega(s)ds\right\}}{1+\sum_{i=1}^{m} p_i(t)(1+c(\prod_{T_0\leq t_k<g_i(t)}(1+b_k))e^{-\int_{T_0}^{g_i(t)}\upsilon(s)ds})}. \tag{5.52}$$

We fix $b \geq T_0$ and define the operator $T : L_\infty[T_0, b] \to L_\infty[T_0, b]$ by the following

$$(Tu)(t) = r(t)\left(\prod_{h(t)\leq t_k<t}(1+b_k)\right)\exp\left\{\int_{h(t)}^{t} u(s)ds\right\}$$
$$\times \frac{1 + c\displaystyle\prod_{T_0\leq t_k<g_i(t)}(1+b_k)\exp\left\{-\int_{T_0}^{t} u(s)ds\right\}}{1+\displaystyle\sum_{i=1}^{m} p_i(t)(1+c\prod_{T_0\leq t_k<g_i(t)}(1+b_k)e^{-\int_{T_0}^{g_i(t)} u(s)ds})}. \quad (5.53)$$

For every function u from the interval $\upsilon \leq u \leq \omega$, we have $\upsilon \leq Tu \leq \omega$. Also T is a completely continuous operator on the space $L_\infty[T_0, b]$, and then by the Schauder Fixed Point Theorem there exists a nonnegative solution of the equation $u = Tu$. Let

$$x(t) = \begin{cases} c\exp\{-\int_{T_0}^{t} u(s)ds\}, & t \geq T_0, \\ c, & T^- \leq t \leq T_0. \end{cases} \quad (5.54)$$

Then $x(t)$ is a nonoscillatory solution of (5.34). Thus by Lemma 5.2.1

$$y(t) = \left(\prod_{h(t)\leq t_k<t}(1+b_k)^{-1}\right)x(t)$$

is a nonoscillatory solution of (5.32) which completes the proof of Theorem 5.2.2. ∎

The results in this section apply to (5.31).

5.3 $\frac{3}{2}$-Global Stability

In this section we examine the global attractivity of the "food-limited" population model

$$N'(t) = r(t)N(t)\frac{1 - N(t-\tau)}{1 + c(t)N(t-\tau)},\ t \geq 0, \quad (5.55)$$

where

$$r(t) \in C([0,\infty),(0,\infty)),\ c(t) \in C([0,\infty),(0,\infty)),\ \tau > 0.$$

We consider solutions of (5.55) which correspond to the initial condition

$$\begin{cases} N(t) = \phi(t), \ t \in [\tau, 0], \\ \phi \in C([\tau, 0], [0, \infty)), \ \phi(0) > 0. \end{cases} \tag{5.56}$$

Motivated by (5.55) in this section, we will study the global stability of the general equation

$$x'(t) + [1 + x(t)][1 - cx(t)]F(t, x(g(t))] = 0, \tag{5.57}$$

where $F(t, \varphi)$ is a continuous functional on $[0, \infty) \times C_t$, such that $F(t, 0) = 0$ for $t \geq 0$ and satisfies a York-type condition

$$-\frac{r(t)}{1+c} M_t(-\varphi) \leq F(t, \varphi) \leq \frac{r(t)}{1+c} M_t(-\varphi), \tag{5.58}$$

where $g : [0, \infty) \to (-\infty, \infty)$ is a nondecreasing continuous function with $g(t) < t$ for $t \geq 0$ and $\lim_{t\to\infty} g(t) = \infty$, $M_t(\varphi) = \max\{0, \sup_{s\in[g(t),t]} \varphi(s)\}$, $c \in (0, \infty)$ and $r \in C([0, \infty), (0, \infty))$. The class C_t is the set of all continuous functions $\varphi : [g(t), t] \to [-1, \infty)$ with the sup-norm $\|\varphi\|_t = \sup_{s\in[g(t),t]} |\varphi(s)|$.

Let $\tau = -g(0)$. We consider solutions of (5.57) which correspond to the initial condition

$$\begin{cases} x(t) = \phi(t), \ t \in [-\tau, 0], \\ \phi \in C([-\tau, 0], [-1, \frac{1}{c})), \ \phi(0) > -1. \end{cases} \tag{5.59}$$

In the following, we will establish a $3/2$-global attractivity condition for (5.57), and then apply this condition on equation (5.55) to establish a $3/2$-global attractivity condition. The results in this section are adapted from [73]. To prove the results, we need the following results (whose proofs are standard; for Lemma 5.3.7 see Lemma 5.7.3 with $c = 1$).

Lemma 5.3.1. *Assume that* $c \in (0, 1]$. *Then for any* $v \in [0, 1)$

$$(1 - v) \ln \frac{(1 + c)e^{-cv(1-cv/2)} - 1}{c} \geq -(1 + c)v \left(1 - \frac{1 + c}{2} v - \frac{1 - c}{6} v^2\right).$$

Lemma 5.3.2. *Assume that* $c \in (0, 1]$. *Then for any* $u \in [0, \infty)$

$$(1 + u) \ln \frac{(1 + c)e^{cu(1+cu/2)} - 1}{c} \geq (1 + c)u \left(1 + \frac{1 + c}{2} u - \frac{1 - c}{6} u^2\right).$$

Lemma 5.3.3. *Assume that* $c \in (0, 1]$ *and* $v \in (0, 1)$. *Then for any* $x \in [0, \infty)$

$$\ln \frac{1 + [(1 + c)e^{-cv(1-cv/2)} - 1]e^{-vx}}{1 + ce^{-vx}} \leq -cv(1 - \frac{cv}{2}) + \frac{cv^2}{1 + c} x.$$

Lemma 5.3.4. *Assume that* $c \in (0, 1]$. *Then for* $0 < v < \left[1 - \frac{c}{2} + \sqrt{\frac{2(1-c)}{3} + \frac{c^2}{4}}\right]^{-1}$

$$-\frac{1}{v}\ln\frac{(1+c)e^{-cv(1-cv/2)}-1}{c} \le \frac{3}{2}(1+c).$$

Lemma 5.3.5. *Assume that* $c \in (0, 1]$. *Then for any* $x \in [0, \infty)$

$$\ln\frac{c+e^x}{1+c} \le \frac{x}{1+c} + \frac{cx^2}{2(1+c)^2} - \frac{c(1-c)x^3}{6(1+c)^3} + \frac{c(1-4c+c^2)x^4}{24(1+c)^4}$$
$$-\frac{c(1-11c+11c^2-c^3)}{120(1+c)^5}x^5 + \frac{c(1+14c^2+c^4)}{720(1+c)^6}x^6.$$

Lemma 5.3.6. *Assume that* $c \in (0, 1]$ *and*

$$1 \ge v \ge \left[1 - \frac{c}{2} + \sqrt{\frac{2(1-c)}{3} + \frac{c^2}{4}}\right]^{-1}.$$

Then

$$\frac{81(1-11c+11c^2-c^3)}{160}v^3 \ge 1 - \frac{19(1-c)v}{6} + \frac{27(1-4c+c^2)v^2}{16}$$
$$+\frac{81(1+14c^2+c^4)}{640}v^4.$$

Lemma 5.3.7. *The system of inequalities*

$$\begin{cases} \ln\frac{1+y}{1-cy} \le (1+c)\left(x - \frac{1-c}{6}x^2\right), \\ -\ln\frac{1-x}{1+cx} \le (1+c)\left(y + \frac{1-c}{6}y^2\right) \end{cases}$$

has only a unique solution $x=y=0$ *in the region* $\{(x, y) : 0 \le x \le 1, 0 \le y < 1/c\}$.

Theorem 5.3.1. *Assume that (5.58) holds. Then the solution* $x(t, 0, \varphi)$ *of (5.57), (5.59) exists on* $[0, \infty)$ *and satisfies* $-1 < x(t, 0, \varphi) < 1/c$.

Theorem 5.3.2. *Assume that (5.58) holds and there exists a function* $r^* \in C([0, \infty), (0, \infty))$ *such that for each* $\varepsilon > 0$ *there is a* $\eta = \eta(\varepsilon) > 0$ *satisfying*

$$\inf_{s\in[g(t),t]} \varphi(s) \ge \varepsilon \Rightarrow F(t, \varphi) \ge \eta r^*(t),\ F(t, -\varphi) \le -\eta r^*(t) \text{ for } t \ge 0 \quad (5.60)$$

and

$$\int_0^\infty r^*(s)ds = \infty. \quad (5.61)$$

Then every nonoscillatory solution of IVP (5.57) and (5.59) tends to zero.

Theorem 5.3.3. *Assume that (5.58), (5.60), and (5.61) hold. If there exists a constant M such that*

$$\int_{g(t)}^{t} r(s)ds \leq M, \tag{5.62}$$

then the solutions of (5.57), (5.59) satisfy

$$\frac{-1+\exp\left(\frac{M(1-e^M)}{1+ce^M}\right)}{1+c\exp\left(\frac{M(1-e^M)}{1+ce^M}\right)} \leq x(t) \leq \frac{e^M-1}{1+ce^M}. \tag{5.63}$$

We now prove our main result in this section.

Theorem 5.3.4. *Assume that (5.58)–(5.61) hold, and*

$$\int_{g(t)}^{t} r(s)ds \leq \frac{3}{2}(1+c) \text{ for large } t. \tag{5.64}$$

Then every solution of (5.57), (5.59) tends to zero.

Proof. Let $x(t)$ be a solution of (5.57) and (5.59) (note also Theorem 5.3.1 so $-1 < x(t) << 1/c$, $t \geq 0$). By Theorem 5.3.2, we only consider the case when $x(t)$ is oscillatory. First assume that $0 < c \leq 1$. Set

$$u = \limsup_{t\to\infty} x(t) \text{ and } v = \liminf_{t\to\infty} x(t). \tag{5.65}$$

By Theorem 5.3.3, $0 \leq u < \infty$ and $0 \leq v < 1$. It suffices to prove that $u = v = 0$. For any $0 < \varepsilon < 1 - v$, by (5.64) and (5.65) there exists a $t_0 = t_0(\varepsilon) > g^{-2}(0)$ such that

$$\int_{g(t)}^{t} r(s)ds \leq \delta_0 \equiv \frac{3}{2}(1+c), \quad t \geq g(t_0), \tag{5.66}$$

$$-v_1 \equiv -(v+\varepsilon) < x(t) < u+\varepsilon \equiv u_1, \quad t \geq g(t_0). \tag{5.67}$$

From (5.57), (5.58), and (5.67), we have

$$\frac{x'(t)}{(1+x(t))(1-cx(t))} \leq \frac{r(t)v_1}{1+c}, \quad t \geq t_0, \tag{5.68}$$

and

$$\frac{x'(t)}{(1+x(t))(1-cx(t))} \geq \frac{-r(t)u_1}{1+c}, \quad t \geq t_0. \tag{5.69}$$

Let $\{l_n\}$ be an increasing infinite sequence of real numbers such that $g(l_n) > t_0$, $x(l_n) > 0$, $x'(l_n) = 0$, and $\lim_{n\to\infty} x(l_n) = u$. We may assume that l_n is a left local maximum point of $x(t)$. It is easy to show that there exists $\zeta_n \in [g(l_n), l_n)$ such that $x(\zeta_n) = 0$ and $x(t) > 0$ for $t \in (\zeta_n, l_n]$. By (5.68), we have

$$x(t) \geq \frac{-1 + \exp\left(-v_1 \int_t^{\zeta_n} r(s)ds\right)}{1 + c\exp\left(-v_1 \int_t^{\zeta_n} r(s)ds\right)}, \;\; t_0 \leq t \leq \zeta_n,$$

and [see also (5.57) and (5.58)] for $\zeta_n \leq t \leq l_n$ we have

$$\frac{x'(t)}{(1 + x(t))\,(1 - cx(t))} \leq \frac{r(t)}{1 + c}\,\frac{1 - \exp\left(-v_1 \int_{g(t)}^{\zeta_n} r(s)ds\right)}{1 + c\exp\left(-v_1 \int_{g(t)}^{\zeta_n} r(s)ds\right)},$$

which together with (5.68) yields for $\zeta_n \leq t \leq l_n$

$$\frac{x'(t)}{(1 + x(t))\,(1 - cx(t))} \leq \min\left\{\frac{r(t)v_1}{1 + c}, \frac{r(t)}{1 + c}\,\frac{1 - \exp\left(-v_1 \int_{g(t)}^{\zeta_n} r(s)ds\right)}{1 + c\exp\left(-v_1 \int_{g(t)}^{\zeta_n} r(s)ds\right)}\right\}. \tag{5.70}$$

There are two cases to consider.

Case 1. $\int_{\zeta_n}^{l_n} r(s)ds \leq -\frac{1}{v_1}\ln\frac{(1+c)e^{-cv_1(1-cv_1/2)} - 1}{c} \equiv A$

Then by (5.66) and (5.70), we have

$$\ln\frac{1 + x(l_n)}{1 - cx(l_n)} \leq \int_{\zeta_n}^{l_n} r(s)ds - \frac{1 + c}{cv_1}\ln\frac{1 + c\exp\left[-v_1\left(\delta_0 - \int_{\zeta_n}^{l_n} r(s)ds\right)\right]}{1 + ce^{-\delta_0 v_1}}. \tag{5.71}$$

If $\int_{\zeta_n}^{l_n} r(s)ds \leq A \leq \delta_0 = \frac{3}{2}(1 + c)$, then by Lemmas 5.3.1 and 5.3.3

$$\ln\frac{1 + x(l_n)}{1 - cx(l_n)} \leq A - \frac{1 + c}{cv_1}\ln\frac{1 + ce^{-v_1(\delta_0 - A)}}{1 + ce^{-\delta_0 v_1}} \leq (1 + c)\left(v_1 - \frac{1 - c}{6}v_1^2\right).$$

If $\int_{\zeta_n}^{l_n} r(s)ds \leq \delta_0 = \frac{3}{2}(1 + c) < A$, then

$$-\frac{1}{v_1}\ln\frac{(1 + c)e^{-cv_1(1-cv_1/2)}}{c} - 1 > \frac{3}{2}(1 + c).$$

From Lemma 5.3.4 we have that

$$v_1 > \left[1 - \frac{c}{2} + \sqrt{\frac{2(1 - c)}{3} + \frac{c^2}{4}}\right]^{-1}.$$

Hence from (5.71), Lemmas 5.3.5 and 5.3.6, we have

$$\ln \frac{1+x(l_n)}{1-cx(l_n)} \le \delta_0 - \frac{1+c}{cv_1} \ln \frac{1+c}{1+ce^{-\delta_0 v_1}} \le (1+c)\left(v_1 - \frac{1-c}{6}v_1^2\right).$$

Case 2. $A < \int_{\zeta_n}^{l_n} r(s)ds \le \delta_0$

Choose $\eta_n \in (\zeta_n, l_n)$ such that $\int_{\eta_n}^{l_n} r(s)ds = A$. Then by (5.66), (5.70), and Lemma 5.3.1 we have

$$\begin{aligned}
&\ln \frac{1+x(l_n)}{1-cx(l_n)} \\
&\le v_1 \int_{\zeta_n}^{\eta_n} r(s)ds + \int_{\eta_n}^{l_n} \frac{r(t)\left[1-\exp\left(-v_1 \int_{g(t)}^{\zeta_n} r(s)ds\right)\right]}{1+c\exp\left(-v_1 \int_{g(t)}^{\zeta_n} r(s)ds\right)}dt \\
&\le -(1+c)\left(1-\frac{3+c}{2}\right) - \frac{1-v_1}{v_1} \ln \frac{(1+c)\,e^{-cv_1(1-cv_1/2)}-1}{c} \\
&\le (1+c)\left(v_1 - \frac{1-c}{6}v_1^2\right).
\end{aligned}$$

Combining the above cases we see that

$$\ln \frac{1+x(l_n)}{1-cx(l_n)} \le (1+c)\left(v_1 - \frac{1-c}{6}v_1^2\right).$$

Letting $n \to \infty$ and $\varepsilon \to 0$, we have

$$\ln \frac{1+u}{1-cu} \le (1+c)\left(v - \frac{1-c}{6}v^2\right). \tag{5.72}$$

Now, we show that

$$-\ln \frac{1-v}{1+cv} \le (1+c)\left(u + \frac{1-c}{6}u^2\right). \tag{5.73}$$

Let $\{s_n\}$ be an increasing infinite sequence of real numbers such that $g(s) > t_0$, $x(s_n) < 0, x'(s_n) = 0$ and $\lim_{n\to\infty} x(s_n) = -v$. We may assume that s_n is a left local minimum point of $x(t)$. It is easy to show that there exists $\eta_n \in [g(s_n), s_n)$ such that $x(\eta_n) = 0$ and $x(t) < 0$ for $t \in (\eta_n, s_n]$. By (5.69), we get

$$x(t) \le \frac{\exp\left(u_1 \int_t^{\eta_n} r(s)ds\right) - 1}{1 + c\exp\left(u_1 \int_t^{\eta_n} r(s)ds\right)}, \quad t_0 \le t \le \eta_n,$$

which together with (5.58) yields

$$\frac{-x'(t)}{(1+x(t))\,(1-cx(t))} \le \frac{r(t)}{1+c}\,\frac{\exp\left(u_1\int_{g(t)}^{\eta_n} r(s)ds\right)-1}{1+c\exp\left(u_1\int_{g(t)}^{\eta_n} r(s)ds\right)},\ \ \eta_n < t < s_n. \tag{5.74}$$

Note that u_1 is bounded and note

$$\frac{1}{u_1}\ln\frac{(1+c)e^{cu_1(1+cu_1/2)}-1}{c} \le \frac{3(1+c)}{2}.$$

We consider two cases.

Case I. $\int_{\eta_n}^{s_n} r(s)ds < \frac{3(1+c)}{2} - \frac{1}{u_1}\ln\frac{(1+c)e^{cu_1(1+cu_1/2)}-1}{c} \equiv B.$

From (5.69) and Lemma 5.3.2, we have

$$\begin{aligned}-\ln\frac{1+x(s_n)}{(1-cx(s_n))} &\le u_1\int_{\eta_n}^{s_n} r(s)ds \\ &\le u_1\frac{3(1+c)}{2} - \ln\frac{(1+c)e^{cu_1(1+cu_1/2)}-1}{c} \\ &\le (1+c)\left(u_1 + \frac{1-c}{6}u_1^2\right).\end{aligned}$$

Case II. $B < \int_{\eta_n}^{s_n} r(s)ds < \frac{3(1+c)}{2}$

Choose $h_n \in (\eta_n, s_n)$ such that $\int_{\eta_n}^{h_n} r(s)ds = B$. Then by (5.69) and (5.74) we have

$$\begin{aligned}-\ln\frac{1+x(s_n)}{(1-cx(s_n))} &\le u_1\int_{\eta_n}^{h_n} r(s)ds + \int_{h_n}^{s_n}\frac{r(t)\left[\exp\left(u_1\int_{g(t)}^{\eta_n} r(s)ds\right)-1\right]}{1+c\exp\left(u_1\int_{g(t)}^{\eta_n} r(s)ds\right)} \\ &\le (1+c) + \frac{(1+c)(3+c)}{2}u_1 \\ &\quad -\frac{1+u_1}{u_1}\ln\frac{(1+c)e^{cu_1(1+cu_1/2)}-1}{c} \\ &\le (1+c)\left(u_1 + \frac{1-c}{6}u_1^2\right).\end{aligned}$$

Combining these two cases we have

$$-\ln\frac{1+x(s_n)}{(1-cx(s_n))} \le (1+c)\left(u_1 + \frac{1-c}{6}u_1^2\right).$$

Letting $n \to \infty$ and $\varepsilon \to 0$ we see that (5.73) holds. In view of Lemma 5.3.7, we see from (5.72) to (5.73) that $u = v = 0$.

Next assume that $c > 1$. Set $y(t) = -cx(t)$. Then (5.57) reduces to

$$y'(t) + [1 + y(t)][1 - c^* y(t)]F^*(t, y(g(t)) = 0, \ t \geq 0, \tag{5.75}$$

where $c^* = 1/c \in (0, 1)$ and $F^*(t, \varphi) = -cF(t, -\frac{1}{c}\varphi)$ satisfies the York-type condition

$$-\frac{r^*(t)}{1 + c^*} M_t(-\varphi) \leq F^*(t, \varphi) \leq \frac{r^*(t)}{1 + c^*} M_t(-\varphi). \tag{5.76}$$

Note for large t that

$$\int_{g(t)}^{t} r^*(s)ds \leq \frac{3}{2}(1 + c^*), \tag{5.77}$$

so we have $\lim_{t\to\infty} y(t) = 0$, and this implies that $\lim_{t\to\infty} x(t) = 0$. The proof is complete. ■

Applying Theorem 5.3.4 on (5.55) we have the following result.

Theorem 5.3.5. *Assume that*

$$\int_0^\infty \frac{r(t)}{1 + c(t)} dt = \infty$$

and

$$\int_{t-\tau}^{t} r(s)ds \leq \frac{3}{2}(1 + c_0) \textit{ for large } t, \tag{5.78}$$

where $c_0 = \inf\{c(t) : t \geq 0\}$. *Then every solution of (5.55), (5.56) tends to 1.*

5.4 $\frac{3}{2}$-Uniform Stability

In this section we discuss the uniform stability of the "food-limited" population model

$$N'(t) = r(t)N(t)\frac{k - N^l(t - \tau)}{k + s(t)N^l(t - \tau)}, \quad t \geq 0, \tag{5.79}$$

where $r(t)$ and $s(t)$ are positive functions, $l, \tau > 0$ are positive constants, and $k^{1/l}$ is the unique positive equilibrium point of (5.79). The results in this section are adapted from [67].

Motivated by (5.79) (let $x(t) = (N(t)/k^{1/l}) - 1$) in this section we examine

$$x'(t) = r(t)[1 + x(t)]\frac{1 - (1 + x(t-\tau))^l}{1 + s(t)(1 + x(t-\tau))^l}, \quad t \geq 0. \tag{5.80}$$

We consider solutions of (5.80), which correspond to the initial condition for any $t_0 \geq 0$

$$\begin{cases} x(t) = \varphi(t), \text{ for } t_0 - \tau \leq t \leq t_0, \ \varphi \in C[t_0 - \tau, t_0] \\ 1 + \varphi(t) \geq 0 \text{ for } t_0 - \tau \leq t \leq t_0 \ \text{ and } \ 1 + \varphi(t_0) > 0. \end{cases} \tag{5.81}$$

The zero solution of (5.80) is said to be uniformly stable if, for $\varepsilon > 0$, there exists a $\delta(\varepsilon)$ such that $t_0 > 0$ and $\|\phi\| = \sup_{s \in [t_0 - \tau, t_0]} |\varphi(s)| < \delta$ imply $|y(t; t_0, \varphi)| < \varepsilon$ for all $t \geq t_0$ where $y(t; t_0, \varphi)$ is a solution of (5.80) with the initial value φ at t_0.

Theorem 5.4.1. *If*

$$l\int_{t-\tau}^{t} \frac{r(u)}{1 + s(u)} du \leq \alpha < \frac{3}{2}, \quad t \geq \tau, \tag{5.82}$$

then the zero solution of (5.80) is uniformly stable.

Proof. Since $\alpha < \frac{3}{2}$, there exist $\alpha_1 > 1$ and $0 < p < 1$, such that

$$\alpha_1 \frac{(1 + p)\alpha}{(1 - p)^l} < \frac{3}{2} \tag{5.83}$$

and

$$\left|(1 + x)^l - 1\right| \leq l\alpha_1 |x|, \qquad \text{for } |x| \leq p.$$

For $0 < \varepsilon < p$, we choose a $\delta = \delta(\varepsilon) > 0$ sufficiently small so that $\delta < p$,

$$p_1 \equiv (1 + \delta)e^{h_1 \alpha} - 1 < \varepsilon, \text{ and } \ p_2 \equiv (1 + p_1)e^{h_2 \alpha} - 1 < \varepsilon,$$

where

$$h_1 \equiv \alpha_1 \delta/(1 - \delta)^l > 0, \text{ and } h_2 \equiv \alpha_1 p_1/(1 - p_1)^l > 0.$$

Clearly, $\delta < p_1 < p_2 < \varepsilon$. Consider a solution $x(t) = x(t; t_0, \varphi)$ of (5.80) with initial condition φ at t_0, where $t_0 \geq 0$ and $\|\varphi\| = \sup_{s \in [t_0 - \tau, t_0]} |\varphi(s)| < \delta$. We need to prove that

$$|x(t)| < \varepsilon, \quad \text{for all } t \geq t_0. \tag{5.84}$$

For $t \in [t_0, t_0 + \tau]$, we have

$$\left|[\ln(1 + x(t))]'\right| \leq h_1 \frac{l r(t)}{1 + s(t)},$$

since

$$\left|1-(1+\varphi)^l\right| \le l\alpha_1\delta$$

and

$$1+s(t)(1+\varphi)^l \ge 1+s(t)(1-\delta)^l \ge (1+s(t))(1-\delta)^l.$$

Hence

$$\left|\ln\frac{1+x(t)}{1+x(t_0)}\right| \le h_1 l\int_{t_0}^{t}\frac{r(u)}{1+s(u)}du \le h_1\alpha,\ \text{for } t\in[t_0,t_0+\tau].$$

It follows that

$$\begin{aligned}1-(1+\delta)e^{h_1\alpha} &< (1-\delta)e^{h_1\alpha}-1\\ &< x(t) < (1+\delta)e^{h_1\alpha}-1,\ \text{for } t\in[t_0,t_0+\tau]\end{aligned}$$

and so

$$|x(t)| < p_1 < \varepsilon,\quad \text{for } t\in[t_0,t_0+\tau].$$

Repeating the previous argument, we have $|x(t)| < p_2 < \varepsilon$ for all $t \in [t_0+\tau, t_0+2\tau]$ and thus

$$|x(t)| < p_2 < \varepsilon,\quad \text{for } t\in[t_0,t_0+2\tau].$$

There are two cases to consider.

Case 1. $x(t)$ has no zeros on $[t_0+\tau, t_0+2\tau]$.

Without loss of generality, we assume that $x(t) > 0$ for $t \in [t_0+\tau, t_0+2\tau]$ (the case when $x(t) < 0$ is similar). Then by (5.80)

$$x'(t) < 0\quad \text{for } t\in[t_0+2\tau,t_0+3\tau].$$

If $x(t) > 0$ for all $t \ge t_0+\tau$, then $x'(t) < 0$ for all $t \ge t_0+2\tau$ and

$$0 < x(t) \le x(t_0+2\tau) < p_2 < \varepsilon, \text{for } \ t\ge t_0+2\tau.$$

Now let t_1 be the smallest zero of $x(t)$ on $(t_0+2\tau, \infty)$. Clearly, $0 < x(t) < p_2$ for $t \in [t_0+2\tau, t_1)$ since $x(t)$ is decreasing on $[t_0+2\tau, t_1)$. Thus $|x(t)| < p_2$ for $t \in [t_0, t_1]$. Assume that (5.84) does not hold. Then there must exist $t_2 > t_1$ such that $|x(t_2)| = p_2$ and $x(t_2)x'(t_2) \ge 0$ and $|x(t)| < p_2$, for $t_0 \le t < t_2$. By (5.80), we have that $x(t)$ has a zero in $[t_2-\tau, t_2]$, which we call ξ. Since

$$\begin{aligned}\left|x'(t_2)\right| &\le (1+p_2)r(t)\frac{l\alpha_1 p_2}{1+s(t)(1-p_2)^l}\\&\le \frac{(1+p_2)\alpha_1 l p_2}{(1-p_2)^l}\frac{r(t)}{1+s(t)},\ \text{for } t_0 \le t < t_2,\end{aligned}$$

we have for $t \in [\xi, t_2]$ that

$$|-x(t-\tau)| \le \frac{(1+p_2)\alpha_1 l p_2}{(1-p_2)^l}\int_{t-\tau}^{\xi}\frac{r(u)}{1+s(u)}du,$$

and so

$$\begin{aligned}\left|x'(t)\right| &\le (1+p_2)\frac{\alpha_1 l}{(1-p_2)^l}\frac{r(t)}{1+s(t)}\left|x(t-\tau)\right|\\&\le \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2 p_2\frac{r(t)}{1+s(t)}\int_{t-\tau}^{\xi}\frac{r(u)}{1+s(u)}du.\end{aligned}$$

Thus, we get for $t \in [\xi, t_2]$ that

$$\left|x'(t)\right| \le \min\left\{\frac{(1+p_2)\alpha_1 l p_2}{(1-p_2)^l}\frac{r(t)}{1+s(t)}, \mu(t,s)\int_{t-\tau}^{\xi}\frac{r(u)}{1+s(u)}du\right\},$$

and therefore

$$|x(t_2)| \le \int_{\xi}^{t_2}\min\left\{\frac{(1+p_2)\alpha_1 l p_2}{(1-p_2)^l}\frac{r(t)}{1+s(t)}, \mu(t,s)\int_{t-\tau}^{\xi}\frac{r(u)}{1+s(u)}du\right\}dt,$$

where

$$\mu(t,s) := \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2 p_2\frac{r(t)}{1+s(t)}.$$

There are two possibilities.

Case I.

$$\int_{\xi}^{t_2}\frac{r(t)}{1+s(t)}dt\frac{(1+p_2)\alpha_1 l}{(1-p_2)^l} \le 1.$$

Then

$$\begin{aligned}|x(t_2)| \le &\left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2\\&\times p_2\int_{\xi}^{t_2}\frac{r(t)}{1+s(t)}\int_{t-\tau}^{\xi}\frac{r(u)}{1+s(u)}du dt\end{aligned}$$

$$= \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2$$
$$\times p_2 \left[\int_\xi^{t_2} \frac{r(t)}{1+s(t)} \left(\int_{t-\tau}^t \frac{r(u)}{1+s(u)}du - \int_\xi^t \frac{r(u)}{1+s(u)}du\right) dt\right]$$

$$< \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2$$
$$\times p_2 \left[\frac{3}{2}\frac{(1-p_2)^l}{\alpha_1 l(1+p_2)} \int_\xi^{t_2} \frac{r(t)}{1+s(t)}dt - \frac{1}{2}\left(\int_\xi^{t_2} \frac{r(t)}{1+s(t)}dt\right)^2\right],$$

since

$$\int_{t-\tau}^t \frac{r(u)}{1+s(u)}du < \frac{3}{2}\frac{(1-p_2)^l}{\alpha_1 l(1+p_2)}$$

and

$$\int_\xi^{t_2} \frac{r(t)}{1+s(t)} \int_\xi^t \frac{r(u)}{1+s(u)}dudt$$
$$= \int_\xi^{t_2} d\left(\frac{1}{2}\left(\int_\xi^{t_2} \frac{r(u)}{1+s(u)}du\right)^2\right) = \frac{1}{2}\left(\int_\xi^{t_2} \frac{r(t)}{1+s(t)}dt\right)^2.$$

Using the fact that $\frac{3}{2}az - \frac{1}{2}z^2$ (here $a > 0$) is an increasing function for $0 < z < \frac{3}{2}a$, we have

$$|x(t_2)|$$
$$< \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2 p_2 \left[\frac{3}{2}\left(\frac{(1-p_2)^l}{(1+p_2)\alpha_1 l}\right)^2 - \frac{1}{2}\left(\frac{(1-p_2)^l}{(1+p_2)\alpha_1 l}\right)^2\right] = p_2,$$

which is a contradiction.

Case II.

$$\int_\xi^{t_2} \frac{r(t)}{1+s(t)}dt \frac{(1+p_2)\alpha_1 l}{(1-p_2)^l} > 1.$$

Choose $\eta \in (\xi, t_2)$ such that

$$\int_\eta^{t_2} \frac{r(t)}{1+s(t)}dt \frac{(1+p_2)\alpha_1 l}{(1-p_2)^l} = 1.$$

Then

$$
\begin{aligned}
&|x(t_2)| \\
&\leq \int_{\xi}^{\eta} \frac{(1+p_2)\alpha_1 l p_2}{(1-p_2)^l} \frac{r(t)}{1+s(t)} \\
&\quad + \int_{\eta}^{t_2} \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2 p_2 \frac{r(t)}{1+s(t)} \int_{t-\tau}^{\xi} \frac{r(u)}{1+s(u)} du dt \\
&= \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2 p_2 \int_{\eta}^{t_2} \frac{r(t)}{1+s(t)} dt \int_{\xi}^{\eta} \frac{r(u)}{1+s(u)} du dt \\
&\quad + \left[\frac{\alpha_1 l(1+p_2)}{(1-p_2)^l}\right]^2 p_2 \int_{\eta}^{t_2} \frac{r(t)}{1+s(t)} \int_{t-\tau}^{\xi} \frac{r(u)}{1+s(u)} du dt
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\frac{(1+p_2)\alpha_1 l}{(1-p_2)^l}\right]^2 p_2 \int_{\eta}^{t_2} \frac{r(t)}{1+s(t)} dt \int_{t-\tau}^{\eta} \frac{r(u)}{1+s(u)} du dt \\
&< \left[\frac{(1+p_2)\alpha_1 l}{(1-p_2)^l}\right]^2 p_2 \left[\int_{\eta}^{t_2} \frac{r(t)}{1+s(t)} \left(\frac{3}{2} \frac{(1-p_2)^l}{\alpha_1 l(1+p_2)} - \int_{\eta}^{t} \frac{r(u)}{1+s(u)} du\right) dt\right] \\
&= \left[\frac{(1+p_2)\alpha_1 l}{(1-p_2)^l}\right]^2 p_2 \left[\frac{3}{2}\left(\frac{(1-p_2)^l}{\alpha_1 l(1+p_2)}\right)^2 - \frac{1}{2}\left(\frac{(1-p_2)^l}{\alpha_1 l(1+p_2)}\right)^2\right] = p_2,
\end{aligned}
$$

which is a contradiction.

This shows that if $x(t)$ has no zero in $[t_0+\tau, t_0+2\tau]$, then $|x(t)| < p_2 < \varepsilon$ for all $t \geq t_0$.

Case 2. $x(t)$ has a zero $\bar{t} \in [t_0+\tau, t_0+2\tau]$.

We prove that

$$|x(t)| < p_2, \text{ for all } t \geq \bar{t}. \tag{5.85}$$

In fact, if (5.85) does not hold, then there must be a point $t^* > \bar{t}$ such that $|x(t^*)| = p_2$, $x(t^*)\ x'(t^*) \geq 0$ and $|x(t)| < p_2$ for $t \in [t_0, t^*)$. Following the reasoning in Case 1 we derive a similar contradiction. The proof of Theorem 5.4.1 is now complete. ■

Theorem 5.4.2. *Assume that*

$$\int_0^{\infty} \frac{r(t)}{1+s(t)} dt = \infty. \tag{5.86}$$

If (5.82) holds, then the zero solution of (5.80) is uniformly and asymptotically stable.

Proof. In view of Theorem 5.4.1, it suffices to prove that there exists a $\delta_0 > 0$ such that the solution of (5.80) with the initial condition $\|\varphi\| = \sup_{t\in[t_0-\tau,t_0]} |\varphi(t)| < \delta_0$ satisfies

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x(t; t_0, \varphi) = 0, \quad t_0 \geq 0.$$

Let $\alpha_1 > 1$ and $0 < p < 1$ be such that

$$\alpha^* \equiv \max\left\{1, \frac{\alpha\alpha_1}{(1-p)^l}\right\} < \frac{3}{2}$$

and

$$\left|(1+x)^l - 1\right| \leq l\alpha_1 |x|, \text{ for } |x| \leq p.$$

Since the zero solution of (5.80) is uniformly stable, it follows that for $0 < \varepsilon < p$, there exists $\delta_0 > 0$ such that

$$|x(t)| = |x(t; t_0, \varphi)| < \frac{\varepsilon}{2}, \quad \text{for } t \geq t_0$$

provided $\|\varphi\| = \sup_{t\in[t_0-\tau,t_0]} |\varphi(t)| < \delta_0$. Set

$$\Delta := \limsup_{t\to\infty} |x(t)|. \tag{5.87}$$

Clearly $0 \leq \Delta < \varepsilon$. We prove that $\Delta = 0$.

If $x(t)$ is eventually nonnegative, then by (5.80), $x(t)$ is eventually decreasing and hence $\lim_{t\to\infty} x(t) = \Delta_1$ exists. Suppose $\Delta_1 > 0$. Then there exists $t_1 > t_0$ such that

$$\frac{1}{2}\Delta_1 < x(t) < 2\Delta_1, \quad \text{for } t \geq t_1.$$

By (5.80), we have for $t \geq t_1 + \tau$ that

$$\begin{aligned}(\ln[1 + x(t)])' &= r(t)\frac{1 - (1 + x(t-\tau))^l}{1 + s(t)(1 + x(t-\tau))^l} \\ &\leq \frac{-[(1 + \frac{1}{2}\Delta_1)^l - 1]}{(1 + 2\Delta_1)^l}\frac{r(t)}{1 + s(t)}.\end{aligned}$$

Using (5.86), we have

$$\ln[1 + x(t)] \to -\infty, \quad \text{as } t \to \infty,$$

which contradicts $\Delta_1 > 0$. Hence $\lim_{t\to\infty} x(t) = \Delta_1 = 0$. Similarly, one can show that if $x(t)$ is eventually nonpositive then $\lim_{t\to\infty} x(t) = 0$.

Now assume that $x(t)$ is oscillatory. For any $0 < \eta < \varepsilon - \Delta$, by (5.87) there exists $t_2 > t_0$ such that $|x(t)| < \Delta + \eta$ for $t \geq t_2$. Let $\{t_n^*\}$ be an increasing sequence such that $t_n^* \geq t_2 + 2\tau$, $x'(t_n^*) = 0$, $\lim_{n\to\infty} |x(t_n^*)| = \Delta$ and $t_n^* \to \infty$ as $n \to \infty$. By (5.80), $x(t_n^* - \tau) = 0$. Thus, we have

$$\begin{aligned} &\left|(\ln[1 + x(t)])'\right| \\ &\leq \frac{l\alpha_1}{(1-\Delta-\eta)^l}\frac{r(t)}{1+s(t)}\left|x(t-\tau)\right|, \quad \text{for } t \geq t_2 + \tau. \end{aligned} \tag{5.88}$$

This yields

$$\begin{aligned} &|-\ln(1 + x(t-\tau))| \\ &\leq \frac{l(\Delta+\eta)\alpha_1}{(1-\Delta-\eta)^l}\int_{t-\tau}^{t_n^*-\tau}\frac{r(u)}{1+s(u)}du, \text{ for } t \in [t_n^* - \tau, t_n^*]. \end{aligned}$$

Consequently,

$$|x(t-\tau)| \leq \exp\left(\frac{l(\Delta+\eta)\alpha_1}{(1-\Delta-\eta)^l}\int_{t-\tau}^{t_n^*-\tau}\frac{r(u)}{1+s(u)}du\right) - 1,$$

since $|\ln(1 + z)| \leq a$ implies $|z| \leq e^a - 1$. Thus for $t \in [t_n^* - \tau, t_n^*]$

$$\begin{aligned} &\left|(\ln[1 + x(t)])'\right| \\ &\leq \frac{l\alpha_1}{(1-\Delta-\eta)^l}\frac{r(t)}{1+s(t)} \\ &\quad\times\left[\exp\left(\frac{l(\Delta+\eta)\alpha_1}{(1-\Delta-\eta)^l}\int_{t-\tau}^{t_n^*-\tau}\frac{r(u)}{1+s(u)}du\right) - 1\right], \end{aligned}$$

which implies for $t \in [t_n^* - \tau, t_n^*]$ that

$$\left|(\ln[1 + x(t)])'\right| \leq \min\{C_1, C_2\}, \tag{5.89}$$

where

$$\begin{aligned} C_1 &:= \frac{l(\Delta+\eta)\alpha_1}{(1-\Delta-\eta)^l}\frac{r(t)}{1+s(t)}, \\ C_2 &:= \frac{l\alpha_1}{(1-\Delta-\eta)^l}\frac{r(t)}{1+s(t)}\left[\exp\left(\frac{l(\Delta+\eta)\alpha_1}{(1-\Delta-\eta)^l}\int_{t-\tau}^{t_n^*-\tau}\frac{r(u)}{1+s(u)}du\right) - 1\right]. \end{aligned}$$

There are three cases to consider:

Case I.

$$\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(t)}{1+s(t)}dt \leq 1.$$

Then

$$
\begin{aligned}
&|\ln(1+x(t_n^*))| \\
&\le \frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(t)}{1+s(t)} \\
&\quad\times\left[\exp\left(\frac{l(\Delta+\eta)\alpha_1}{(1-\Delta-\eta)^l}\int_{t-\tau}^{t_n^*-\tau}\frac{r(u)}{1+s(u)}du\right)-1\right]dt \\
&\le \frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(t)}{1+s(t)} \\
&\quad\times\left[\exp\left((\Delta+\eta)\left(\alpha^*-\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t}\frac{r(u)}{1+s(u)}du\right)\right)-1\right]dt \\
&= \frac{-1}{\Delta+\eta}\int_{t_n^*-\tau}^{t_n^*} d\left[\exp\left(-\frac{l\alpha_1(\Delta+\eta)}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t}\frac{r(u)}{1+s(u)}du\right)-1\right]e^{(\Delta+\eta)\alpha^*} \\
&\qquad -\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(t)}{1+s(t)}dt \\
&\quad= \frac{1}{\Delta+\eta}e^{(\Delta+\eta)\alpha^*}\left[1-\exp\left(-\frac{l\alpha_1(\Delta+\eta)}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t}\frac{r(u)}{1+s(u)}du\right)\right] \\
&\qquad -\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(t)}{1+s(t)}dt \\
&\quad\le \frac{1}{\Delta+\eta}e^{(\Delta+\eta)\alpha^*}(1-e^{(\Delta+\eta)})-1,
\end{aligned}
$$

since the function

$$z \to \frac{1}{\Delta+\eta}e^{(\Delta+\eta)\alpha^*}[1-e^{(\Delta+\eta)z}]-z$$

is increasing for $0 \le z \le \alpha^*$ and

$$\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(u)}{1+s(u)}du \le 1 \le \alpha^*.$$

Thus,

$$|x(t_n^*)| \le \exp\left(\frac{1}{\Delta+\eta}e^{(\Delta+\eta)\alpha^*}(1-e^{(\Delta+\eta)})-1\right)-1.$$

Case II.

$$1 < \frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(t)}{1+s(t)}dt \le \alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}.$$

Then

$$\begin{aligned}|\ln(1+x(t_n^*))| &\le \frac{l(\Delta+\eta)\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(t)}{1+s(t)}dt\\ &\le \alpha^*(\Delta+\eta) - \ln(1+\Delta+\eta)\end{aligned}$$

or

$$|x(t_n^*)| \le \frac{1}{1+\Delta+\eta}e^{(\Delta+\eta)\alpha^*} - 1.$$

Case III.

$$\alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta} < \frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(t)}{1+s(t)}dt \le \alpha^*.$$

Choose $h \in (0,\tau)$ such that

$$\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*-h}\frac{r(t)}{1+s(t)}dt = \alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}.$$

Then by (5.89)

$$\begin{aligned}&|\ln(1+x(t_n^*))|\\ &\le \int_{t_n^*-\tau}^{t_n^*-h}\frac{r(t)}{1+s(t)}dt\,\frac{l(\Delta+\eta)\alpha_1}{(1-\Delta-\eta)^l}\\ &\quad+\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-h}^{t_n^*}\frac{r(t)}{1+s(t)}\\ &\quad\times\left[\exp\left(\frac{l(\Delta+\eta)\alpha_1}{(1-\Delta-\eta)^l}\int_{t-\tau}^{t_n^*-\tau}\frac{r(u)}{1+s(u)}du\right)-1\right]dt\\ &\quad\le(\Delta+\eta)\left(\alpha^*-\frac{\ln(1+\Delta+\eta)}{\Delta+\eta}\right)\\ &\qquad+e^{(\Delta+\eta)\alpha^*}\int_{t_n^*-h}^{t_n^*}\frac{r(t)}{1+s(t)}\end{aligned}$$

$$\times \exp\left(-\frac{l(\Delta+\eta)\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t}\frac{r(u)}{1+s(u)}du\right)dt$$
$$-\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-h}^{t_n^*}\frac{r(t)}{1+s(t)}dt$$

$$= (\Delta+\eta)\left(\alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}\right)$$
$$+\frac{e^{(\Delta+\eta)\alpha^*}}{(\Delta+\eta)}\exp\left(-\frac{l\alpha_1(\Delta+\eta)}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*-h}\frac{r(u)}{1+s(u)}du\right)$$
$$-\frac{e^{(\Delta+\eta)\alpha^*}}{(\Delta+\eta)}\exp\left(-\frac{l\alpha_1(\Delta+\eta)}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(u)}{1+s(u)}du\right)$$
$$-\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-h}^{t_n^*}\frac{r(t)}{1+s(t)}dt$$

$$= (\Delta+\eta)\left(\alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}\right)$$
$$+\frac{1}{(\Delta+\eta)}\exp\left((\Delta+\eta)\left(\alpha^* - \frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*-h}\frac{r(u)}{1+s(u)}du\right)\right)$$
$$-\frac{1}{(\Delta+\eta)}\exp\left((\Delta+\eta)\left(\alpha^* - \frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(u)}{1+s(u)}du\right)\right)$$
$$-\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-h}^{t_n^*}\frac{r(t)}{1+s(t)}dt, \quad \text{since } e^x \geq 1+x \text{ for all } x,$$

$$\leq (\Delta+\eta)\left(\alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}\right) + \frac{1+\Delta+\eta-1}{(\Delta+\eta)}$$
$$-(\Delta+\eta)\left(\alpha^* - \frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*}\frac{r(u)}{1+s(u)}du\right)$$
$$-\frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-h}^{t_n^*}\frac{r(t)}{1+s(t)}dt$$

$$= (\Delta+\eta)\left(\alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}\right)$$
$$+1-\alpha^* + \frac{l\alpha_1}{(1-\Delta-\eta)^l}\int_{t_n^*-\tau}^{t_n^*-h}\frac{r(t)}{1+s(t)}dt$$
$$= (\Delta+\eta)\left(\alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}\right)$$
$$+1-\alpha^* + \alpha^* - \frac{\ln(1+\Delta+\eta)}{\Delta+\eta}$$
$$= 1+\alpha^*(\Delta+\eta) - \frac{(1+\Delta+\eta)\ln(1+\Delta+\eta)}{\Delta+\eta}$$

and so

$$|x(t_n^*))| \le \exp\left(1+\alpha^*(\Delta+\eta) - \frac{(1+\Delta+\eta)\ln(1+\Delta+\eta)}{\Delta+\eta}\right) - 1.$$

Combining all the three cases, we have

$$|x(t_n^*))| \le \max\{A,\ B, C\}, \tag{5.90}$$

where

$$A = \exp\left(\frac{1}{\Delta+\eta}e^{(\Delta+\eta)\alpha^*}(1-e^{(\Delta+\eta)}) - 1\right) - 1,$$
$$B = \frac{1}{1+\Delta+\eta}e^{(\Delta+\eta)\alpha^*} - 1,$$
$$C = \exp\left(1+\alpha^*(\Delta+\eta) - \frac{(1+\Delta+\eta)\ln(1+\Delta+\eta)}{\Delta+\eta}\right) - 1.$$

Since

$$\lim_{z\to 0}\frac{1}{z}\left\{\exp\left(\frac{1}{z}e^{\alpha^* z}(1-e^z) - 1\right) - 1\right\} = \alpha^* - \frac{1}{2} < 1,$$

$$\lim_{z\to 0}\frac{1}{z}\left\{\frac{1}{z+1}e^{\alpha^* z} - 1\right\} = \alpha^* - 1 < 1,$$

and

$$\lim_{z\to 0}\frac{1}{z}\left\{\exp\left(1+\alpha^* z - \frac{(1+z)\ln(1+z)}{z}\right) - 1\right\} = \alpha^* - \frac{1}{2} < 1$$

it follows that there exists $\alpha_0 < 1$ such that, for sufficiently small $\varepsilon > 0$, we have

$$\exp\left(\frac{1}{z}e^{\alpha^* z}(1-e^{-z})-1\right)-1 < \alpha_0 z,\ \ \frac{1}{z+1}e^{\alpha^* z}-1 < \alpha_0 z,$$

and

$$\exp\left(1+\alpha^* z-\frac{(1+z)\ln(1+z)}{z}\right)-1 < \alpha_0 z,\ \text{ for all } 0 < z < \varepsilon.$$

Thus by (5.90), we get

$$|x(t_n^*))| < \alpha_0(\Delta+\eta).$$

Letting $n \to \infty$ and $\eta \to 0$, we have

$$\Delta \le \alpha_0 \Delta,$$

which, together with $\alpha_0 < 1$, implies $\Delta = 0$. The proof is now complete. ■

5.5 Models with Periodic Coefficients

The variation of the environment plays an important role in many biological and ecological dynamical systems. The assumption of periodicity of the parameters in the system (in a way) incorporates the periodicity of the environment. It is realistic to assume that the parameters in the models are periodic functions of period ω. We consider the nonautonomous "food-limited" population model

$$\frac{dN(t)}{dt} = r(t)N(t)\frac{K(t)-N(t-m\omega)}{K(t)+c(t)r(t)N(t-m\omega)}. \tag{5.91}$$

In this section we discuss (5.91) when K is a periodic function. The results in this section are adapted from [28]. We first consider the nondelay case.

Theorem 5.5.1. *Suppose r, c, and K are continuous and positive periodic function of period ω. Then there exists a unique ω-periodic solution $N^*(t)$ of the periodic differential equation*

$$\frac{dN(t)}{dt} = r(t)N(t)\frac{K(t)-N(t)}{K(t)+c(t)r(t)N(t)}, \tag{5.92}$$

such that all other positive solutions of (5.92) satisfy

$$\lim_{n\to\infty}[N(t)-N^*(t)] = 0. \tag{5.93}$$

Proof. Let $N(t, 0, N_0)$ denote the unique solution of (5.92) through the initial point $(0, N_0)$. Let

$$K_* = \min_{0\leq t\leq\omega} K(t) \quad \text{and} \quad K^* = \max_{0\leq t\leq\omega} K(t).$$

Then it follows from (5.92) that

$$N_0 \in [K_*, K^*] \Rightarrow N(t, 0, N_0) \in [K_*, K^*], \text{ for } t \geq 0$$

and in particular

$$N_\omega \equiv N(\omega, 0, N_0) \in [K_*, K^*].$$

Define the function

$$f : [K_*, K^*] \to [K_*, K^*]$$

by

$$f(N_0) = N_\omega.$$

As $N(t; 0, N_0)$ depends continuously on N_0, it follows that f is a continuous function mapping $[K_*, K^*]$ into itself. Therefore f has a fixed point N_0^*. In view of the ω-periodic of r, c, and K, it follows that the unique solution $N^*(t) \equiv N(t, 0, N_0^*)$ of (5.92) through the initial point $(0, N_0^*)$ is positive and ω-periodic. This completes the proof of the existence of a positive and ω-periodic solution $N^*(t)$ of (5.92).

Let $N(t)$ be an arbitrary positive solution of (5.92). We let

$$N(t) = N^*(t)e^{x(t)} \tag{5.94}$$

and note

$$\frac{dx(t)}{dt} = F(N^*(t)e^{x(t)}) - F(N^*(t)), \tag{5.95}$$

where

$$F(u) = r(t)\frac{K(t) - u}{K(t) + c(t)r(t)u}.$$

By the mean-value theorem of differential calculus, we can rewrite (5.95) in the form

$$\frac{dx(t)}{dt} = -A(t)[e^{x(t)} - 1], \tag{5.96}$$

where

$$A(t) = \frac{1 + r(t)c(t)}{[K(t) + r(t)c(t)\xi(t)]^2} r(t)N^*(t)K(t), \tag{5.97}$$

and $\xi(t)$ lies between $N^*(t)$ and $N^*(t)e^{x(t)}$. Define a Lyapunov function V for (5.96) in the form

$$V(t) = V(x(t)) = [e^{x(t)} - 1]^2.$$

Calculating the rate of change of V along the solutions of (5.96) we obtain for $x(t) \neq 0$ that

$$\frac{dV(t)}{dt} = -2A(t)[e^{x(t)} - 1]^2 e^{x(t)} < 0. \tag{5.98}$$

One can easily see that every positive solution of this equation is bounded. Therefore $x(t)$ is also bounded. As r, K, and N^* are positive functions and $\xi(t)$ lies between $N^*(t)$ and $N^*(t)e^{x(t)}$, it follows from (5.97) that there exists a positive number μ such that

$$A(t) \geq \mu, \quad \text{for } \ t \geq 0.$$

Thus from (5.98) we have

$$\frac{dV(t)}{dt} \leq -2\mu e^{x(t)}[e^{x(t)} - 1]^2,$$

so

$$V(t) + 2\mu \int_0^t e^{x(s)}[e^{x(s)} - 1]^2 ds \leq V(0) < \infty.$$

Hence

$$e^{x(t)}[e^{x(t)} - 1]^2 \in L_1(0, \infty).$$

Since $x(t)$ and $\dot{x}(t)$ are bounded in $[0, \infty)$, it follows from Barbalats' Theorem (see Sect. 1.4) that

$$e^{x(t)}[e^{x(t)} - 1]^2 \to 0 \quad as \ \ t \to \infty.$$

Thus $x(t) \to 0$ as $t \to \infty$ and the result follows from (5.94). This completes the proof. ■

Now we consider the periodic delay differential equation (5.91), namely

$$N'(t) = r(t)N(t)\frac{K(t) - N(t - m\omega)}{K(t) + c(t)r(t)N(t - m\omega)}, \tag{5.99}$$

together with the initial condition

$$\begin{cases} N(t) = \varphi(t), & \text{for } \ -m\omega \leq t \leq 0, \\ \varphi \in C[[-m\omega, 0], \mathbf{R}^+], & \text{and } \ \varphi(0) > 0. \end{cases} \tag{5.100}$$

Note the unique positive periodic solution $N^*(t)$ of (5.92) is also a periodic solution of (5.99).

For convenience, we introduce the notations

$$r^* = \max\{r(t) : t \in [0, \omega]\}, \quad r_* = \min\{r(t) : t \in [0, \omega]\},$$

$$K^* = \max\{K(t) : t \in [0, \omega]\}, \quad K_* = \min\{K(t) : t \in [0, \omega]\},$$

$$N^u = K^* \exp[K^* (\frac{r}{K})_{av} m\omega], \text{ where } (\frac{r}{K})_{av} = \frac{1}{m\omega}\int_0^{m\omega} \frac{r(s)}{K(s)} ds, \tag{5.101}$$

$$N_l = K_* \exp[\frac{K_* - N^u}{K_*} r_{av} m\omega], \text{ where } r_{av} = \frac{1}{m\omega}\int_0^{m\omega} r(s) ds. \tag{5.102}$$

Theorem 5.5.2. *If $N(t)$ is a solution of the initial value problems (5.99) and (5.100) then there exists a number $T = T(\varphi)$ such that*

$$N_l \le N(t) \le N^u, \quad \text{for } t \ge T. \tag{5.103}$$

Proof. We note that any solution of (5.99) satisfies the differential inequality

$$N'(t) \le \frac{r(t)N(t)[K^* - N(t - m\omega)]}{K(t) + c(t)r(t)N(t - m\omega)}. \tag{5.104}$$

Solutions of (5.104) can be either oscillatory or nonoscillatory about K^*.

First, suppose that $N(t)$ is oscillatory about K^*. Then there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ of zeros of $N(t) - K^*$ such that $N(t) - K^*$ takes both positive and negative values on (t_n, t_{n+1}) for $n = 1, 2, \ldots$. Let $N(t_n^*)$ denote a local maximum of $N(t)$ on (t_n, t_{n+1}). Then from (5.104), we obtain

$$0 = N'(t_n^*) \le \frac{r(t_n^*)N(t_n^*)[K^* - N(t_n^* - m\omega)]}{K(t_n^*) + c(t_n^*)r(t_n^*)N(t_n^* - m\omega)},$$

which implies that

$$N(t_n^* - m\omega) \le K^*.$$

This shows that there exists a point $\xi \in [t_n^* - m\omega, t_n^*]$ such that $N(\xi) = K^*$. Integrating (5.104) over $[\xi, t_n^*]$ we obtain

$$\ln \frac{N(t_n^*)}{N(\xi)} \le \int_{\xi}^{t_n^*} K^* \frac{r(s)}{K(s)} ds \le K^* \int_{t_n^* - m\omega}^{t_n^*} \frac{r(s)}{K(s)} ds$$

and

$$N(t_n^*) \le K^* \exp[K^* (r/K)_{av} m\omega]. \tag{5.105}$$

Since the right side of (5.105) is independent of t_n, we conclude that

$$N(t) \le K^* \exp[K^*(r/K)_{av} m\omega] = N^u, \quad \text{for } t > t_1 + 2m\omega. \tag{5.106}$$

Next assume that $N(t)$ is non oscillatory about K^*. Then it is easily seen that for every $\varepsilon > 0$ there exists a $T_1 = T_1(\varepsilon)$ such that

$$N(t) < K^* + \varepsilon, \quad \text{for } t > T_1.$$

This and (5.106) imply that there exists a $T = T(\varphi)$ such that

$$N(t) \le N^u \quad for \ t > T.$$

In a similar way we can derive a lower bound for positive solutions of (5.99). In fact from (5.99) we find

$$N'(t) \ge r(t)N(t)\frac{K_* - N(t - m\omega)}{K(t) + c(t)r(t)N(t - m\omega)}. \tag{5.107}$$

Let $N(t)$ be an oscillatory solution about K_* and let $\{s_n\} \to \infty$ as $n \to \infty$ be such that

$$N(s_n) - K_* = 0, \ \text{for } n = 1, 2, \dots ,$$

and $N(t) - K_*$ takes both positive and negative values on (t_n, t_{n+1}). Let s_n^* be such that $N(s_n^*)$ is a local minimum of $N(t)$. Then from (5.107), we obtain

$$0 = N'(s_n^*) \ge r(s_n^*)N(s_n^*)\frac{K_* - N(s_n^* - m\omega)}{K(s_n^*) + c(s_n^*)r(s_n^*)N(s_n^* - m\omega)},$$

which implies that

$$N(s_n^* - m\omega) \ge K_*.$$

This show that there exists a point $\eta \in [s_n^* - m\omega, s_n^*]$ such that $N(\eta) = K_*$. Integrating (5.107) over $[\eta, s_n^*]$ we find

$$\begin{aligned} \ln \frac{N(s_n^*)}{K_*} &\ge \int_\eta^{s_n^*} \frac{r(s)(K_* - N^u)}{K_*} ds \\ &= \frac{K_* - N^u}{K_*}\int_\eta^{s_n^*} r(s) \ge \frac{K_* - N^u}{K_*}\int_{s_n^* - m\omega}^{s_n^*} r(s)ds \end{aligned}$$

and

$$N(s_n^*) \ge K_* \exp\left(\frac{K_* - N^u}{K_*}\int_{s_n^* - m\omega}^{s_n^*} r(s)ds\right) = N_l.$$

Hence

$$N(s) \geq N_l, \quad \text{for} \;\; t \geq t_1 + 2m\omega. \tag{5.108}$$

Next, assume that $N(t)$ is nonoscillatory about K_*. One can easily show in this case that for every positive ε there exists a $T_2 = T_2(\varepsilon)$ such that

$$N(t) > K_* - \varepsilon, \quad \text{for} \;\; t > T_2.$$

This and (5.108) imply that there exists a $T_2 = T_2(\varphi)$ such that

$$N(t) \geq N_l - \varepsilon, \quad \text{for} \;\; t \geq T_2.$$

The proof is complete. ■

We will derive sufficient conditions for the global attractivity of $N^*(t)$ with respect to all other positive solutions of (5.99) and (5.100). As before we set

$$N(t) \equiv N^*(t)e^{x(t)}, \tag{5.109}$$

in (5.99) and note that

$$x'(t) = G(x(t - m\omega)) - G(0), \tag{5.110}$$

where

$$G(u) = r(t)\frac{K(t) - N^*(t)e^u}{K(t) + c(t)r(t)N^*(t)e^u}. \tag{5.111}$$

We can rewrite (5.110) in the form

$$x'(t) = -B(t)\, x(t - m\omega), \tag{5.112}$$

where

$$B(t) = \frac{K(t)r(t)[1 + r(t)c(t)]\zeta(t)}{[K(t) + c(t)r(t)\zeta(t)]^2} \tag{5.113}$$

and $\zeta(t)$ lies between $N^*(t)$ and $N(t - m\omega)$. Clearly

$$B_l = \frac{K_* r_*(1 + r_* c_*)N_l}{(K^* + c^* r^* N^u)^2} \leq B(t) \leq \frac{K^* r^*(1 + r^* c^*)N^u}{(K_* + c_* r_* N_l)^2} = B^u. \tag{5.114}$$

Theorem 5.5.3. *Assume that the positive periodic functions $r(t)$, $K(t)$, and $c(t)$ satisfy the condition*

$$\mu \equiv K^* \exp\left[K^* \left(\frac{r}{K}\right)_{av} m\omega\right] \int_0^{m\omega} [1 + r(s)c(s)]\frac{r(s)}{K(s)} ds < 1. \tag{5.115}$$

Then every solution of (5.99) and (5.100) satisfies

$$\lim_{t\to\infty}[N(t) - N^*(t)] = 0. \tag{5.116}$$

Proof. It suffices to prove that every solution x of (5.112) and (5.113) satisfies

$$\lim_{t\to\infty} x(t) = 0. \tag{5.117}$$

Consider $V(t) = V(x(t))$ given by

$$V(t) = \left[x(t) - \int_{t-m\omega}^{t} B(s+m\omega)x(s)ds\right]^2 + \int_{t-m\omega}^{t} B(s+2m\omega)\left(\int_{s}^{t} B(u+m\omega)x^2(t)du\right)ds, \tag{5.118}$$

which in view of (5.112) yields

$$\begin{aligned}\frac{dV(t)}{dt} &= 2\left[x(t) - \int_{t-m\omega}^{t} B(s+m\omega)x(s)\,ds\right][-B(t+m\omega)x(t)] \\ &\quad + B(t+m\omega)x^2(t)\int_{t-m\omega}^{t} B(s+2m\omega)ds \\ &\quad - B(t+m\omega)\int_{t-m\omega}^{t} B(u+m\omega)x^2(u)du. \end{aligned} \tag{5.119}$$

Using the inequality

$$2x(t)x(s) \le x^2(t) + x^2(s),$$

and simplifying (5.119) we obtain

$$\begin{aligned}\frac{dV(t)}{dt} &\le -B(t+m\omega)x^2(t) \\ &\quad \times\left[2 - \int_{t-m\omega}^{t} B(s+m\omega)ds - \int_{t-m\omega}^{t} B(s+m\omega)ds\right] \\ &\le -B(t+m\omega)x^2(t)(1-\mu). \end{aligned} \tag{5.120}$$

It follows from (5.115) that V is eventually nonincreasing say for $t \ge T$. Clearly all solutions of (5.99) are bounded and so by (5.109) and (5.110), x is uniformly continuous on $[0,\infty)$. Integrating (5.120) over $[T,t]$ and taking into account the inequality (5.115), we get

$$V(t) + 2B_l(1-\mu)\int_{T}^{t} x^2(s)ds \le V(T) < \infty.$$

Hence $x^2 \in L_1(T,\infty)$ and by Barbalat's Theorem (see Sect. 1.4)

$$\lim_{t\to\infty} x^2(t) = 0.$$

The proof is complete. ■

5.6 Global Stability of Models with Impulses

In this section, we are concerned with the global stability of "food-limited" population models with impulsive effects. We consider the model

$$\begin{cases} N'(t) = p(t)N(t)\dfrac{1-N(t-\tau)}{1+\lambda N(t-\tau)}, & t \geq 0, \ \ t \neq t_k, \\ N(t_k^+) = N(t_k)^{1+b_k}, \ \ k \in \mathbf{N}, \end{cases} \tag{5.121}$$

where $p \in C[0,\infty)$ with $p > 0$, $\lambda \in (0,\infty)$, $\tau > 0$, $b_k > -1$ for all $k \in \mathbf{N}$. The aim in this section is to establish some sufficient conditions which ensure that every solution of (5.121) tends to 1 as $t \to \infty$. The results in this section are adapted from [41]. Let the sequence $t_k (k \in \mathbf{N})$ be fixed and satisfy the condition,

$$0 < t_1 < t_2 < \ldots < t_{k+1} \to \infty, \ \ \text{as } k \to \infty.$$

We only consider solutions of (5.121) with initial conditions of the form

$$\begin{cases} N(t) = \phi(t), & \text{for} \quad -\tau \leq t \leq 0, \\ \phi \in C([-\tau,0],[0,\infty)), & \text{and } \phi(0) > 0. \end{cases} \tag{5.122}$$

Lemma 5.6.1. *Suppose that any* $\epsilon > 0$ *there exists an integer* N *such that*

$$\prod_{k=n}^{n+m}(1+b_k) < 1+\epsilon, \ \ \textit{for } n > N \textit{ and } m \geq 0. \tag{5.123}$$

If in addition

$$\int_0^{+\infty} p(s) \prod_{0\leq t_k<s} (1+b_k)^{-1} ds = \infty, \tag{5.124}$$

then every non-oscillatory solution of

$$\begin{cases} x'(t) = p(t)\dfrac{1-e^{x(t-\tau)}}{1+\lambda e^{x(t-\tau)}}, & t \neq t_k, \\ x(t_k^+) = (1+b_k)x(t_k), & k \in \mathbf{N} \end{cases} \tag{5.125}$$

tends to zero as t *tends to infinity.*

Proof. Without loss of generality, suppose that $x(t)$ is an eventually positive solution of (5.125). Then there is a $T_1 \geq 0$ such that $x(t-\tau) > 0$ for $t \geq T_1,\ t \neq t_k$. Thus (5.125) implies that $x(t)$ is decreasing in $(t_k, t_{k+1}]$ with $t_k \geq T_1$. Let

$$\lim_{t\to+\infty} \inf x(t) = \alpha.$$

Then $\alpha \geq 0$. First we prove $\alpha = 0$. Since $x(t_k)$ is a left locally minimum value of $x(t)$, there is a subsequence $\{x(t_{k_j})\}$ such that

$$\lim_{j\to+\infty} x(t_{k_j}) = \alpha.$$

If $\alpha \neq 0$, then $\alpha > 0$. Choose $\epsilon > 0$ such that $\alpha - \epsilon > 0$. Again there is a $T > T_1,\ T \neq t_k$ such that $x(t-\tau) > \alpha - \epsilon$, for $t \geq T$. Hence (5.125) implies

$$x'(t) \leq p(t)\frac{1-e^{\alpha-\epsilon}}{1+\lambda e^{\alpha-\epsilon}},\quad t \geq T,\quad t \neq t_k.$$

Integrating the above inequality from T to t_{k_j}, we get

$$\prod_{T\leq t_k<t_{k_j}} (1+b_k)^{-1} x(t_{k_j}) - x(T)$$

$$\leq \frac{1-e^{\alpha-\epsilon}}{1+\lambda e^{\alpha-\epsilon}} \int_T^{t_{k_j}} p(s) \prod_{T\leq t_k<s} (1+b_k)^{-1} ds.$$

Let either

$$\lim_{j\to+\infty} \sup \prod_{T\leq t_k<t_{k_j}} (1+b_k) = 0 \ \ or \ \ \lim_{j\to+\infty} \sup \prod_{T\leq t_k<t_{k_j}} (1+b_k) \neq 0,$$

and it follows that $\infty \leq -\infty$ or $-x(T) \leq -\infty$, a contradiction. Then $\alpha = 0$.

Now for any $t \geq T$, there is a t_{k_j} such that $t_{k_j} \leq t < t_{k_{j+1}}$. Suppose that $t_{k_j} < t_{k_j+1} < \ldots < t_{k_j+l} \leq t$. Then

$$\begin{aligned}
0 < x(t) < x(t^+_{k_j+l}) &= (1+b_{k_j+l})x(t_{k_j+l}) \\
&\leq (1+b_{k_j+l})x(t^+_{k_j+l-1}) \\
&= (1+b_{k_j+l})(1+b_{k_j+l-1})x(t_{k_j+l-1}) \\
&\leq \ldots \leq \prod_{s=0}^{l}(1+b_{k_j+s})x(t_{k_j}).
\end{aligned}$$

From (5.123), there is a constant $A > 0$ such that $\prod_{s=0}^{l}(1 + b_{k_j+s}) \leq A$ for any l and any k_j. Thus $0 < x(t) \leq Ax(t_{k_j})$. Then $\lim_{t\to+\infty} x(t) = 0$. The proof is complete. ∎

Lemma 5.6.2. *Suppose that (5.123), (5.124) hold and there is a constant $M > 0$ such that*

$$\int_{t-\tau}^{t} p(s) \prod_{s\leq t_k<t} (1 + b_k)ds \leq M, \quad t \geq 0. \tag{5.126}$$

Then every oscillatory solution of (5.125) is bounded.

Proof. Let $x(t)$ be oscillatory solution of (5.125). Equation (5.125) implies

$$x'(t) \leq p(t), \ t \geq 0, \quad t \neq t_k. \tag{5.127}$$

Choose a sequence $\{c_n\}$ such that

$$x(c_n) = 0, \ \text{where } 0 < c_1 < c_2 < \ldots, \ \text{with } \lim_{n\to+\infty} c_n = +\infty,$$

$$x(t) \geq 0, \ \text{for } t \in [c_{2i-1}, c_{2i}], \ \text{and } x(t) \leq 0, \ \text{for } t \in [c_{2i}, c_{2i+1}].$$

Let

$$\hat{x}_i = \sup_{t\in[c_{2i-1},c_{2i}]} x(t) \ \text{and} \ \tilde{x}_i = \inf_{t\in[c_{2i},c_{2i+1}]} x(t).$$

It suffices to prove that $\{\hat{x}_i\}$ and $\{\tilde{x}_i\}$ are bounded. First, we prove that $\{\hat{x}_i\}$ is bounded above. In this step, there are two cases to consider.

Case 1. $\hat{x}_i$ is the maximum value of $x(t)$ in $[c_{2i-1}, c_{2i}]$.

In this case, there is a $c \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(c) > 0$, $x'(c) \geq 0$. Equation (5.125) implies $x(t - \tau) \leq 0$. Then there is a $\xi \in (c - \tau, c)$ such that $x(\xi) = 0$. Integrating (5.127) from ξ to c, we get

$$\hat{x}_i = x(c) \leq \int_{\xi}^{c} p(t) \prod_{t\leq t_k<c} (1 + b_k)dt \leq M.$$

Case 2. $\hat{x}_i$ is not the maximum value of $x(t)$ in $[c_{2i-1}, c_{2i}]$.

In this case, there is a $t_{k+l} \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(t^+_{k+l})$. We suppose that

$$c_{2i-1} < t_{k+1} < \ldots < t_{k+l}.$$

There are two cases to consider.

Subcase 2.1: $x(t_{k+j-1}^{+}) \geq x(t_{k+j}),\ j = 2, \ldots, l$

Then $x(t)$ has maximum $x(c)$ in $[c_{2i-1}, t_{k+1}]$. By Case 1 we have $x(c) \leq M$. Hence

$$\hat{x}_i = x(t_{k+l}^{+}) = (1 + b_{k+l})x(t_{k+l}) \ldots \leq \prod_{s=1}^{l}(1 + b_{k+s})x(t_{k+1})$$
$$\leq M \prod_{s=1}^{l}(1 + b_{k+s}).$$

Subcase 2.2: There is an integer $j^{\star} \in \{2, \ldots, l\}$ with $x(t_{k+j^{\star}-1}^{+}) < x(t_{t+j^{\star}})$ and $x(t_{k+j-1}^{+}) \geq x(t_{k+j}),\ j = j^{\star} + 1, \ldots, l.$

Then $x(t)$ has maximum $x(c)$ in $[t_{k+j^{\star}-1}, t_{k+j^{\star}}]$. By Case 1 we have $x(c) \leq M$. Hence

$$\hat{x}_i = x(t_{k+l}^{+}) = (1 + b_{k+l})x(t_{k+l}) \leq \ldots \leq \prod_{s=j^{\star}}^{l}(1 + b_{k+s})x(t_{k+j^{\star}})$$
$$\leq M \prod_{s=j^{\star}}^{l}(1 + b_{k+s}).$$

From condition (5.123), from Cases 1 and 2, one gets that there is a constant $A > 0$ such that

$$\hat{x}_i = x(t_{k+l}) \leq M \quad \text{or} \quad \hat{x}_i = x(t_{k+l}) \leq AM. \tag{5.128}$$

Next, we prove that $\{\tilde{x}_i\}$ is bounded below. From (5.128), there is a constant $B > 0$ such that $x(t) \leq B$, for all $t \geq 0$. Equation (5.125) implies

$$x'(t) \geq \frac{1 - e^B}{1 + \lambda e^B} p(t), \quad t \geq 0, \quad t \neq t_k. \tag{5.129}$$

Using a method similar to that in Cases 1 and 2, we get

$$\tilde{x}_i \geq \frac{1 - e^B}{1 + \lambda e^B} M$$

or

$$\tilde{x}_i \geq \frac{1 - e^B}{1 + \lambda e^B} AM.$$

This shows that $\{\tilde{x}_i\}$ is bounded below. The proof is complete. ■

The following result is well known.

Lemma 5.6.3. *The system of inequalities*

$$v \le (1+\lambda)\frac{1-e^{u}}{1+\lambda e^{u}} \text{ and } u \ge (1+\lambda)\frac{1-e^{v}}{1+\lambda e^{v}}$$

has only a unique solution $u = v = 0$ in the region $-\infty < u \le 0 \le v < +\infty$.

Lemma 5.6.4. *Suppose that $\lambda \in (0, 1]$ and (5.123), (5.124) hold. If*

$$\limsup_{t\to+\infty} \int_{t-\tau}^{t} p(s) \prod_{s \le t_k < t} (1+b_k)ds \le 1+\lambda, \tag{5.130}$$

then every oscillatory solution of (5.125) tends to zero as t tends to infinity.

Proof. Let $x(t)$ be an oscillatory solution of (5.125). By Lemma 5.6.2, $x(t)$ is bounded. Let

$$\lim \inf_{t\to+\infty} x(t) = u \quad \text{and} \quad \lim \sup_{t\to+\infty} x(t) = v.$$

Then

$$-\infty < u \le 0 \le v < +\infty.$$

For any $\epsilon > 0$, (5.123) implies that there is a $N > 0$ such that

$$\prod_{k=n}^{n+m} (1+b_k) < 1+\epsilon, \quad \text{for } n \ge N \text{ and } m \ge 0.$$

In addition, for this ϵ there is a $T > t_N$ such that

$$\int_{t-\tau}^{t} p(s) \prod_{s \le t_k < t} (1+b_k)ds < (1+\lambda)(1+\epsilon), \quad \text{for all } t \ge T,$$

and

$$u_1 \equiv u - \epsilon < u(t-\tau) < v + \epsilon \equiv v_1.$$

Then (5.125) implies

$$x'(t) \le p(t)\frac{1-e^{u_1}}{1+\lambda e^{u_1}}, \quad t \ge T, \quad t \ne t_k, \tag{5.131}$$

and

$$x'(t) \ge p(t)\frac{1-e^{v_1}}{1+\lambda e^{v_1}}, \quad t \ge T, \quad t \ne t_k.$$

Choose a sequence $\{c_n\}$ such that $x(c_n) = 0$, $T < c_1 < c_2 < \ldots$, $c_n \to +\infty$, $x(t) \geq 0$, for $t \in (c_{2i-1}, c_{2i})$ and $x(t) \leq 0$ for $t \in (c_{2i}, c_{2i+1})$. Let

$$\hat{x}_i = \sup_{t \in (c_{2i-1}, c_{2i})} x(t), \quad \tilde{x}_i = \inf_{t \in (c_{2i}, c_{2i+1})} x(t).$$

Then

$$\lim_{i \to \infty} \sup \hat{x}_i = v, \quad \lim_{i \to \infty} \inf \tilde{x}_i = u.$$

We divide the proof into two steps.

Case 1. $\hat{x}_i$ is the maximum value of $x(t)$ in (c_{2i-1}, c_{2i}).

In this case, there is a $c \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(c) > 0$, $x'(c) \geq 0$, and $x(t - \tau) \leq 0$. Then there is a $\xi \in (c - \tau, c)$ such that $x(\xi) = 0$. Integrating (5.131) from ξ to c, we get

$$\begin{aligned} \hat{x}_i = x(c) &\leq \frac{1 - e^{u_1}}{1 + \lambda e^{u_1}} \int_{\xi}^{c} p(s) \prod_{s \leq t_k < c} (1 + b_k) ds \\ &\leq (1 + \lambda)(1 + \epsilon) \frac{1 - e^{u_1}}{1 + \lambda e^{u_1}}. \end{aligned}$$

Case 2. $\hat{x}_i$ is not the maximum value of $x(t)$ in (c_{2i-1}, c_{2i}).

In this case, there is a $t_{k+l} \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(t_{k+l}^+)$. Suppose $c_{2i-1} < t_{k+1} < \ldots < t_{k+l}$. As in Case 2 in Lemma 5.6.2, there is a $c \in (c_{2i-1}, t_{k+l})$ such that $x(c)$ is a left locally maximum value of $x(t)$, and we have that there is a $j \in \{1, 2, \ldots, l\}$ such that

$$\hat{x}_i \leq \prod_{s=j}^{l} (1 + b_{k+s}) x(c) \leq \prod_{s=j}^{l} (1 + b_{k+s})(1 + \epsilon)(1 + \lambda) \frac{1 - e^{u_1}}{1 + \lambda e^{u_1}}.$$

Then by (5.123), we get

$$\hat{x}_i \leq (1 + \epsilon)^2 (1 + \lambda) \frac{1 - e^{u_1}}{1 + \lambda e^{u_1}}.$$

Let $i \to +\infty$, $\epsilon \to 0$, and we get

$$v \leq (1 + \lambda) \frac{1 - e^{u}}{1 + \lambda e^{u}}. \tag{5.132}$$

Similarly, we have

$$u \geq (1 + \lambda) \frac{1 - e^{v}}{1 + \lambda e^{v}}. \tag{5.133}$$

From Lemma 5.6.3, we get from (5.132) and (5.133) that $u = v = 0$. Then $\lim_{t \to +\infty} x(t) = 0$. This completes the proof. ■

Lemma 5.6.5. *Suppose that $\lambda > 1$ and (5.123), (5.124), and (5.130) hold. Then every oscillatory solution of (5.125) tends to zero as t tends to infinity.*

Proof. Since $\lambda \in (1, +\infty)$, let $M(t) = \dfrac{1}{N(t)}$, and (5.121) becomes

$$M'(t) = \frac{1}{\lambda} p(t) M(t) \frac{1 - M(t-\tau)}{1 + \frac{1}{\lambda} M(t-\tau)}. \tag{5.134}$$

We note $\dfrac{1}{\lambda} \in (0,1)$. Then by Lemma 5.6.4, we get Lemma 5.6.5. The proof is complete. ■

Lemma 5.6.6. *Suppose that $\lambda \in (0,1]$, and (5.123), (5.124) holds. If*

$$\limsup_{t\to+\infty} \int_{t-\tau}^{t} p(s) \prod_{t-\tau \le t_k < t} (1+b_k)^{-1} ds \le \frac{3}{2}(1+\lambda), \tag{5.135}$$

then every oscillatory solution of (5.125) tends to zero as $t \to +\infty$.

Proof. Let $x(t)$ be an oscillatory solution of (5.125). By Lemma 5.6.2, $x(t)$ is bounded. Let

$$\lim\sup_{t\to+\infty} x(t) = v \text{ and } \lim\inf_{t\to+\infty} x(t) = u.$$

Then

$$-\infty < u \le 0 \le v < +\infty.$$

From (5.123), for any $\epsilon > 0$, there is a N such that

$$\prod_{k=n}^{n+m} (1+b_k) < 1 + \epsilon, \quad n \ge N, \quad m \ge 0.$$

Again for this $\epsilon > 0$, there is a $T \ge t_N$ such that

$$\begin{cases} \displaystyle\int_{t-\tau}^{t} \frac{p(s)}{\prod\limits_{t-\tau \le t_k < s} (1+b_k)} ds \le \frac{3}{2}(1+\lambda)(1+\epsilon) := \delta(1+\epsilon), \ t \ge T, \\ u_1 \equiv u - \epsilon < x(t-\tau) < v + \epsilon \equiv v_1, \qquad t \ge T. \end{cases} \tag{5.136}$$

Then (5.125) implies

$$x'(t) \le \frac{1 - e^{u_1}}{1 + \lambda e^{u_1}} p(t), \quad t \ge T, \quad t \ne t_k. \tag{5.137}$$

Choose a sequence $\{c_n\}$ such that $x(c_n) = 0,\ T < c_1 < c_2 < \ldots,\ c_n \to +\infty, n \to +\infty$, $x(t) \geq 0$ for $t \in (c_{2i-1}, c_{2i})$ and $x(t) \leq 0$ for $t \in (c_{2i}, c_{2i+1})$. Let

$$\hat{x}_i = \sup_{t \in (c_{2i-1}, c_{2i})} x(t), \quad \tilde{x}_i = \inf_{t \in (c_{2i}, c_{2i+1})} x(t).$$

Then

$$\lim_{i \to \infty} \sup \hat{x}_i = v, \ \lim_{i \to \infty} \inf \tilde{x}_i = u.$$

We first prove

$$\hat{x}_i \leq (1+\lambda)\left(A - \frac{1-\lambda}{6}A^2\right)(1+\epsilon) \tag{5.138}$$

or

$$\hat{x}_i \leq (1+\lambda)(1+\epsilon)^2\left(A - \frac{1-\lambda}{6}A^2\right), \text{ where } A = \frac{1-e^{u_1}}{1+\lambda e^{u_1}}. \tag{5.139}$$

There are two cases to be considered.

Case 1. $\hat{x}_i$ is the maximum value of $x(t)$ in (c_{2i-1}, c_{2i}).

In this case, there is a $c \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(c) > 0,\ x'(c) \geq 0$. By (5.125) we have $x(t-\tau) \leq 0$. Then there is a $\xi \in (c-\tau, c)$ such that $x(\xi) = 0$. If $t \in [\xi, c]$, then $t - \tau \leq \xi$. Integrating (5.137) from $t-\tau$ to ξ, one gets

$$-\prod_{t-\tau \leq t_k < \xi}(1+b_k)x(t-\tau) \leq A\int_{t-\tau}^{\xi} p(s)\prod_{s \leq t_k < \xi}(1+b_k)ds. \tag{5.140}$$

Equation (5.125) implies for $t \geq 0$ that

$$x'(t) \leq p(t)\frac{1-\exp(-A\int_{t-\tau}^{\xi} p(s)\prod_{t-\tau \leq t_k < s}(1+b_k)^{-1}ds)}{1+\lambda\exp(-A\int_{t-\tau}^{\xi} p(s)\prod_{t-\tau \leq t_k < s}(1+b_k)^{-1}ds)}. \tag{5.141}$$

Integrating (5.141) from ξ to c and noting that $\dfrac{1-e^x}{1+\lambda e^x}$ is decreasing, we get

$$x(c) \le \int_{\xi}^{c} p(t) \frac{1 - \exp(-A\int_{t-\tau}^{\xi} p(s) \prod_{t-\tau \le t_k < s} (1+b_k)^{-1} ds)}{1 + \lambda \exp(-A\int_{t-\tau}^{\xi} p(s) \prod_{t-\tau \le t_k < s} (1+b_k)^{-1} ds)} \prod_{t \le t_k < c} (1+b_k) dt$$

$$\le \int_{\xi}^{c} p(t) \prod_{t \le t_k < c} (1+b_k)$$

$$\times \frac{1 - e^{-A\delta} \exp(A\int_{\xi}^{t} p(s) \prod_{t-\tau \le t_k < s} (1+b_k)^{-1} ds)}{1 + \lambda e^{-A\delta} \exp(A\int_{\xi}^{t} p(s) \prod_{t-\tau \le t_k < s} (1+b_k)^{-1} ds)} dt$$

$$\le \int_{\xi}^{c} p(t) \frac{1 - e^{-A\delta} \exp(A\int_{\xi}^{t} p(s) \prod_{s \le t_k < c} (1+b_k) ds \prod_{t-\tau \le t_k < c} (1+b_k)^{-1})}{1 + \lambda e^{-A\delta} \exp(A\int_{\xi}^{t} p(s) \prod_{s \le t_k < c} (1+b_k) ds \prod_{t-\tau \le t_k < c} (1+b_k)^{-1})}$$

$$\times \prod_{t \le t_k < c} (1+b_k) dt$$

$$\le \int_{\xi}^{c} p(t) \prod_{t \le t_k < c} (1+b_k) \frac{1 - e^{-A\delta} \exp(A(1+\epsilon)^{-1} \int_{\xi}^{t} p(s) \prod_{s \le t_k < c} (1+b_k) ds)}{1 + \lambda e^{-A\delta} \exp(A(1+\epsilon)^{-1} \int_{\xi}^{t} p(s) \prod_{s \le t_k < c} (1+b_k) ds)} dt$$

$$= \int_{\xi}^{c} p(t) \prod_{t \le t_k < c} (1+b_k) dt - \frac{1+\lambda}{\lambda A (1+\epsilon)^{-1}}$$

$$\times \ln \frac{1 + \lambda e^{-A\delta} \exp(A(1+\epsilon)^{-1} \int_{\xi}^{c} p(s) \prod_{s \le t_k < c} (1+b_k) ds)}{1 + \lambda e^{-A\delta}} dt.$$

Subcase 1.1:

$$\int_{\xi}^{c} p(t) \prod_{t \le t_k < c} (1+b_k)dt \le -\frac{1}{A} \ln \frac{(1+\lambda)e^{-\lambda A(1-\frac{\lambda}{A})}}{\lambda}(1+\epsilon)$$
$$\equiv \alpha(1+\epsilon) \le \delta(1+\epsilon).$$

By the monotone property of the function

$$x - \frac{(1+\lambda)}{\lambda A(1+\epsilon)^{-1}} \ln\left(1+\lambda e^{-A\delta + Ax(1+\epsilon)^{-1}}\right),$$

and using $\lambda e^{-A\alpha} = (1+\lambda)e^{-\lambda A(1-\frac{\lambda A}{2})} - 1$, we get that

$$x(c) \le (1+\epsilon)\left(\alpha - \frac{1+\lambda}{\lambda A} \ln \frac{1+\lambda e^{-A\delta + A\alpha}}{1+\lambda e^{-A\delta}}\right)$$
$$= (1+\epsilon)(\alpha + \frac{1+\lambda}{\lambda A} \ln \frac{1+((1+\lambda)e^{-\lambda A(1-\frac{\lambda A}{2})} - 1)e^{-A\delta + A\alpha}}{1+\lambda e^{-A\delta + A\alpha}}).$$

Then Lemma 5.3.3 gives us that

$$\hat{x}_i = x(c) \le (1+\epsilon)\left[\alpha + \frac{1+\lambda}{\lambda A}(-\lambda A(1-\frac{\lambda A}{2}) + \frac{\lambda A^2}{1+\lambda}(\delta - \alpha))\right]$$
$$= (1+\epsilon)\left[\alpha - (1+\lambda)(1-\frac{\lambda A}{2}) + A\delta - A\alpha\right]$$
$$= -(1+\epsilon)(1+\lambda)\left[1 - \frac{\lambda A}{2} - \frac{3}{2}A\right]$$
$$-(1+\epsilon)\frac{1-A}{A} \ln \frac{(1+\lambda)e^{-\lambda A(1-\frac{\lambda A}{2})} - 1}{\lambda}$$
$$= (1+\epsilon)\left[-(1+\lambda)\left(1 - \frac{3+\lambda}{2}A\right) - \frac{1-A}{A} \ln \frac{(1+\lambda)e^{-\lambda A(1-\frac{\lambda A}{2})}-1}{\lambda}\right].$$

Then from Lemma 5.3.1

$$x(c) \le -(1+\lambda)(1+\epsilon)\left(1 - \frac{3+\lambda}{2}A\right)$$
$$+(1+\epsilon)\frac{1+\lambda}{A}A\left(1 - \frac{1+\lambda}{2}A - \frac{1-\lambda}{6}A^2\right)$$
$$= (1+\epsilon)(1+\lambda)\left(A - \frac{1-\lambda}{6}A^2\right),$$

i.e.,

$$x(c) \le (1+\epsilon)(1+\lambda)\left(A - \frac{1-\lambda}{6}A^2\right). \tag{5.142}$$

Subcase 1.2:

$$\int_{\xi}^{c} p(t) \prod_{t \le t_k < c} (1+b_k)dt \le \delta(1+\epsilon) < \alpha(1+\epsilon).$$

In this case $\alpha > \frac{3}{2}(1+\lambda)$, i.e.,

$$-\frac{1}{A}\ln\frac{(1+\lambda)e^{-\lambda A(1-\frac{\lambda}{A})} - 1}{\lambda} > \frac{3}{2}(1+\lambda).$$

From Lemma 5.3.4 we have that

$$A > \left(1 - \frac{\lambda}{2} + \sqrt{\frac{2(1-\lambda)}{3} + \frac{\lambda^2}{4}}\right)^{-1}.$$

Integrating (5.141) from ξ to c, we get

$$\begin{aligned}
\hat{x}_i &= x(c) \le \delta(1+\epsilon) - \frac{1+\lambda}{\lambda A(1+\epsilon)^{-1}} \\
&\quad \times \ln\frac{1+\lambda e^{-A\delta}\exp(A(1+\epsilon)^{-1}\delta(1+\epsilon))}{1+\lambda e^{-A\delta}} \\
&= (1+\epsilon)\left(\delta - \frac{1+\lambda}{\lambda A}\ln\frac{1+\lambda}{1+\lambda e^{-A\delta}}\right) \\
&= (1+\epsilon)\left(\delta + \frac{1+\lambda}{\lambda A}\left(\ln\frac{\lambda+e^{A\delta}}{1+\lambda} - A\delta\right)\right).
\end{aligned}$$

By a method similar to that in Lemmas 5.3.5 and 5.3.6, we get

$$\begin{aligned}
\hat{x}_i &= x(c) \le (1+\epsilon)(1+\lambda) \\
&\quad \times A\,[1 - \frac{1-\lambda}{6}A + \frac{1}{8}(1 - \frac{19(1-\lambda)}{6}A + \frac{27(1-4\lambda+\lambda^2)}{16}A^2 \\
&\quad - \frac{81(1-11\lambda+11\lambda^2-\lambda^3)}{160}A^3 + \frac{81(1+14\lambda^2+\lambda^4)}{640}A^4)],
\end{aligned}$$

i.e.,

$$x(c) \le (1+\epsilon)(1+\lambda)\left(A - \frac{1-\lambda}{6}A^2\right). \tag{5.143}$$

Subcase 1.3:

$$\delta(1+\epsilon) \geq \int_{\xi}^{c} p(t) \prod_{t \leq t_k < c} (1+b_k)dt > \alpha(1+\epsilon).$$

Choose $\eta \in (\xi, c)$ such that

$$\int_{\eta}^{c} p(t) \prod_{t \leq t_k < c} (1+b_k)dt = \alpha(1+\epsilon).$$

Integrating (5.137) from ξ to η, one gets

$$x \leq A \int_{\xi}^{\eta} p(t) \prod_{t \leq t_k < \eta} (1+b_k)dt.$$

Integrating (5.137) from η to c, we get

$$x(c) - x(\eta) \prod_{\eta \leq t_k < c} (1+b_k)$$

$$\leq \int_{\eta}^{c} p(t) \prod_{t \leq t_k < c} (1+b_k) \frac{1-\exp\left(-A \int_{t-\tau}^{\xi} p(s) \prod_{t-\tau \leq t_k < s} (1+b_k)^{-1} ds\right)}{1+\lambda \exp\left(-A \int_{t-\tau}^{\xi} p(s) \prod_{t-\tau \leq t_k < s} (1+b_k)^{-1} ds\right)} dt.$$

By deleting $x(\eta)$ and noting

$$e^{-A\alpha} = \frac{(1+\lambda)e^{-\lambda A(1-\frac{\lambda A}{2})} - 1}{\lambda},$$

we have

$$\begin{aligned}\hat{x}_i &= x(c) \\ &\leq A \int_{\xi}^{\eta} p(t) \prod_{t \leq t_k < \eta} (1+b_k)dt + \int_{\eta}^{c} p(t) \prod_{t \leq t_k < c} (1+b_k) \\ &\quad \times \frac{1+\lambda e^{-A\delta} \exp(A(1+\epsilon)^{-1} \int_{\xi}^{c} p(s) \prod_{s \leq t_k < c} (1+b_k)ds)}{1+\lambda e^{-A\delta} \exp(A(1+\epsilon)^{-1} \int_{\xi}^{\eta} p(s) \prod_{s \leq t_k < c} (1+b_k)ds)} dt\end{aligned}$$

$$= A\int_{\xi}^{\eta} p(t)\prod_{t\le t_k<c}(1+b_k)dt + \int_{\eta}^{c} p(t)\prod_{t\le t_k<c}(1+b_k)dt$$

$$-\frac{1+\lambda}{\lambda A(1+\epsilon)^{-1}}\ln\frac{1+\lambda e^{-A\delta}\exp(A(1+\epsilon)^{-1}\int_{\xi}^{c} p(s)\prod_{s\le t_k<c}(1+b_k)ds)}{1+\lambda e^{-A\delta}\exp(A(1+\epsilon)^{-1}\int_{\xi}^{\eta} p(s)\prod_{s\le t_k<c}(1+b_k)ds)}.$$

Using the monotone property of the function

$$Ax - \frac{(1+\lambda)}{\lambda A(1+\epsilon)^{-1}}\ln\frac{1+\lambda e^{-A\delta+Ax(1+\epsilon)^{-1}}}{1+\lambda e^{-A\delta-A\alpha+Ax(1+\epsilon)^{-1}}},\ \text{on } [0,\delta(1+\epsilon)]$$

and by Lemma 5.3.1, it follows that

$$\begin{aligned}\hat{x}_i &= x(c)\\ &\le (1+\epsilon)\left(A\delta+(1-A)\alpha-\frac{1+\lambda}{\lambda A}\ln\frac{1+\lambda}{1+\lambda e^{-A\alpha}}\right)\\ &= (1+\epsilon)\left(A\delta+(1-A)\alpha-(1+\lambda)(1-\frac{\lambda A}{2})\right)\\ &= (1+\epsilon)\left(-(1+\lambda)(1-\frac{3+\lambda}{2}A)-\frac{1-A}{A}\varpi\right)\\ &\le (1+\epsilon)(1+\lambda)(A-\frac{1-\lambda}{6}A^2),\end{aligned}$$

where

$$\varpi = \ln\frac{(1+\lambda)e^{-\lambda A(1-\frac{\lambda A}{2})}-1}{\lambda},$$

i.e.,

$$x(c) \le (1+\epsilon)(1+\lambda)(A-\frac{1-\lambda}{6}A^2). \tag{5.144}$$

Case 2. $\hat{x}_i$ is not the maximum value of $x(t)$ in (c_{2i-1}, c_{2i}).

In this case, there is a $t_{k+l} \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(t_{k+l}^+)$. Suppose $c_{2i-1} < t_{k+1} < \ldots < t_{k+l}$. As in Case 2 in Lemma 5.6.2, there is a $c \in (c_{2i-1}, t_{k+l})$ such that $x(c)$ is a locally maximum value of $x(t)$, and there is a $j \in \{1, 2, \ldots, l\}$ such that

$$\hat{x}_i \le \prod_{s=j}^{l}(1+b_{k+s})x(c),$$

where $x(c)$ satisfies (5.138). Then by (5.123), we get

$$\hat{x}_i \le (1+\epsilon)x(c) \le (1+\epsilon)^2(1+\lambda)(A - \frac{1-\lambda}{6}A^2).$$

Let $i \to +\infty,\ \epsilon \to 0$ in (5.138) and (5.139) to obtain

$$v \le (1+\lambda)\left(\frac{1-e^u}{1+\lambda e^u} - \frac{1-\lambda}{6}\left(\frac{1-e^u}{1+\lambda e^u}\right)^2\right). \tag{5.145}$$

Next we prove

$$u \ge (1+\lambda)\left(\frac{1-e^u}{1+\lambda e^u} - \frac{1-\lambda}{6}\left(\frac{1-e^u}{1+\lambda e^u}\right)^2\right). \tag{5.146}$$

Let $B = \dfrac{1-e^v}{1+\lambda e^v}$. Then by (5.125), we have

$$x'(t) \ge Bp(t),\quad t \ge T,\quad t \ne t_k. \tag{5.147}$$

There are two cases to consider.

Case 1. $\tilde{x}_i$ is the minimum value of $x(t)$ in (c_{2i}, c_{2i+1}).

In this case, there is a $c \in (c_{2i}, c_{2i+1})$ such that $x(c) = \tilde{x}_i < 0,\ x'(c) \le 0$, and then there is a $\xi \in (c-\tau, c)$ such that $x(\xi) = 0$. If $t \in [\xi, c]$, then $t - \tau \le \xi$. Integrating (5.137) from $t-\tau$ to c, we get

$$-\prod_{t-\tau\le t_k<\xi}(1+b_k)x(t-\tau) \ge B\int_{t-\tau}^{\xi} p(s)\prod_{s\le t_k<\xi}(1+b_k)ds.$$

Then, we get for $t \in [\xi, c],\ t \ne t_k$, that

$$x'(t) \ge p(t)\frac{1-\exp(-B\int_{t-\tau}^{\xi} p(s)\prod_{t-\tau\le t_k<s}(1+b_k)^{-1}ds)}{1+\lambda\exp(-B\int_{t-\tau}^{\xi} p(s)\prod_{t-\tau\le t_k<s}(1+b_k)^{-1}ds)}. \tag{5.148}$$

We consider two subcases.

Subcase 1.1:

$$\int_{\xi}^{c} p(t) \prod_{t \le t_k < c} (1+b_k)dt \le (1+\epsilon)\left(\delta + \frac{1}{B}\ln \frac{(1+\lambda)e^{-\lambda B(1-\frac{\lambda B}{2})}-1}{\lambda}\right).$$

In this case, it is easy to see that

$$\begin{aligned} &-\frac{(1+\lambda)B}{1-B}\left(1-\frac{1+\lambda}{2}B-\frac{1-\lambda}{6}B^2\right) \\ &> -\frac{(1+\lambda)B}{2}\left(1+\frac{1-\lambda}{3}B\right). \end{aligned}$$

Then by Lemma 5.3.2, we get

$$\ln \frac{(1+\lambda)e^{-\lambda B(1-\frac{\lambda B}{2})}-1}{\lambda} > \frac{1+\lambda}{2}(B-\frac{1-\lambda}{3}B^2).$$

Integrating (5.147) from ξ to c, one gets

$$\begin{aligned} \tilde{x}_i &= x(c) \ge B\int_{\xi}^{c} p(t) \prod_{t \le t_k < c}(1+b_k)dt \\ &\ge \left[\delta B + \ln \frac{(1+\lambda)e^{-\lambda B(1-\frac{\lambda B}{2})}-1}{\lambda}\right](1+\epsilon) \\ &\ge (1+\lambda)(1+\epsilon)(B-\frac{1-\lambda}{6}B^2). \end{aligned}$$

Then

$$x(c) = \tilde{x}_i \ge (1+\lambda)(1+\epsilon)(B-\frac{1-\lambda}{6}B^2). \tag{5.149}$$

Subcase 1.2:

$$\begin{aligned} \delta(1+\epsilon) &\ge \int_{\xi}^{c} p(t) \prod_{t \le t_k < c}(1+b_k)dt \\ &> (\delta + \frac{1}{B}\ln \frac{(1+\lambda)e^{-\lambda B(1-\frac{\lambda B}{2})}-1}{\lambda})(1+\epsilon). \end{aligned}$$

Choose $\eta \in (\xi, c)$ such that

$$\int_{\eta}^{c} p(t) \prod_{t \le t_k < c} (1 + b_k) dt = \left[\delta + \frac{1}{B} \ln \frac{(1+\lambda) e^{-\lambda B (1 - \frac{\lambda B}{2})} - 1}{\lambda} \right] (1 + \epsilon).$$

Integrating (5.147) from ξ to η, integrating (5.148) from η to c, and deleting $x(\eta)$, we get

$$\begin{aligned}
\tilde{x}_i &= x(c) \\
&\ge B \int_{\xi}^{\eta} p(t) \prod_{t \le t_k < \eta} (1 + b_k) dt + \int_{\eta}^{c} p(t) \prod_{t \le t_k < c} (1 + b_k) \\
&\quad \times \frac{1 - \exp(-B \int_{t-\tau}^{\eta} p(s) \prod_{t-\tau \le t_k < s} (1 + b_k)^{-1} ds)}{1 + \lambda \exp(-B \int_{t-\tau}^{\eta} p(s) \prod_{t-\tau \le t_k < s} (1 + b_k)^{-1} ds)} dt \\
&\ge B \int_{\xi}^{\eta} p(t) \prod_{t \le t_k < \eta} (1 + b_k) dt \\
&\quad + \int_{\eta}^{c} p(t) \prod_{t \le t_k < c} (1 + b_k) dt \\
&\quad - \frac{1+\lambda}{\lambda B (1+\epsilon)^{-1}} \ln \frac{1 + \lambda e^{-B\delta} \exp\left(B(1+\epsilon)^{-1} \int_{\xi}^{c} p(s) \prod_{s \le t_k < c} (1 + b_k) ds \right)}{1 + \lambda e^{-B\delta} \exp\left(B(1+\epsilon)^{-1} \int_{\xi}^{\eta} p(s) \prod_{s \le t_k < c} (1 + b_k) ds \right)} \\
&= B \int_{\xi}^{\eta} p(t) \prod_{t \le t_k < \eta} (1 + b_k) dt + \int_{\eta}^{c} p(t) \prod_{t \le t_k < c} (1 + b_k) dt \\
&\quad - \frac{1+\lambda}{\lambda B (1+\epsilon)^{-1}} \ln \frac{1 + \lambda e^{-B(1+\epsilon)^{-1}\delta} \exp\left(B \int_{\xi}^{c} p(s) \prod_{s \le t_k < c} (1 + b_k) ds \right)}{(1+\lambda) e^{-B\lambda (1 - \frac{\lambda B}{2})}}
\end{aligned}$$

$$= -(1-B)\int_{\xi}^{\eta} p(t)\prod_{t\le t_k<\eta}(1+b_k)dt$$
$$+\int_{\eta}^{c} p(t)\prod_{t\le t_k<c}(1+b_k)dt-(1+\lambda)(1+\epsilon)(1-\frac{\lambda B}{2})$$
$$-\frac{(1+\lambda)(1+\epsilon)}{\lambda B}$$
$$\times\ln\frac{1+\lambda e^{-B\delta}\exp(B(1+\epsilon)^{-1}\int_{\xi}^{c} p(s)\prod_{s\le t_k<c}(1+b_k)ds)}{1+\lambda}.$$

Using the monotone property of the function

$$x-\frac{(1+\lambda)(1+\epsilon)}{\lambda B}\ln\frac{1+\lambda e^{-B\delta}e^{B(1+\epsilon)^{-1}x}}{1+\lambda},\ x\in[0,\delta(1+\epsilon)],$$

we get

$$x(c)$$
$$\ge -(1-B)\int_{\xi}^{\eta} p(t)\prod_{t\le t_k<c}(1+b_k)dt$$
$$+\delta(1+\epsilon)-(1+\lambda)(1+\epsilon)(1-\frac{\lambda B}{2})$$
$$=(1+\epsilon)\left[-(1+\lambda)+\tfrac{(1+\lambda)(3+\lambda)}{2}B-\frac{1-B}{B}\ln\tfrac{(1+\lambda)e^{-\lambda B(1-\frac{\lambda B}{2})}-1}{\lambda}\right].$$

By Lemma 5.3.2, we get

$$\tilde{x}_i=x(c)\ge(1+\epsilon)(1+\lambda)(B-\frac{1-\lambda}{6}B^2). \tag{5.150}$$

Case 2. $\tilde{x}_i$ is not the minimum value of $x(t)$ in (c_{2i},c_{2i+1}).

In this case, there is a $t_{k+l}\in(c_{2i-1},c_{2i})$ such that $\tilde{x}_i=x(t_{k+l}^{+})$. Suppose $c_{2i}<t_{k+1}<\ldots<t_{k+l}$. As in Case 2 in Lemma 5.6.2, there is a $c\in(c_{2i-1},t_{k+l})$ such that $x(c)$ is a locally minimum value of $x(t)$, and $x(c)$ satisfies (5.149) [(5.150)]. Then there is a $j\in\{1,2,\ldots,l\}$ such that

$$\tilde{x}_i\ge\prod_{s=j}^{l}(1+b_{k+s})(1+\epsilon)x(c).$$

By (5.123), we have

$$\tilde{x}_i \geq (1+\epsilon)x(c) \geq (1+\epsilon)^2(1+\lambda)(B - \frac{1-\lambda}{6}B^2). \tag{5.151}$$

Let $i \to +\infty$, $\epsilon \to 0$ in (5.149) and (5.151) and we get (5.146). Let

$$\frac{1-e^u}{1+\lambda e^u} = x, \quad \frac{1-e^v}{1+\lambda e^v} = -y.$$

Then (5.145) and (5.146) become

$$\begin{cases} \ln \dfrac{1+y}{1-\lambda y} \leq (1+\lambda)(x - \dfrac{1-\lambda}{6}x^2), \\ \ln \dfrac{1-x}{1+\lambda x} \geq (1+\lambda)(-y - \dfrac{1-\lambda}{6}y^2). \end{cases} \tag{5.152}$$

By Lemma 5.3.7, then $x = y = 0$. Thus $u = v = 0$. Then $x(t)$ tends to zero as t tends to infinity. The proof is complete. ∎

Lemma 5.6.7. *Suppose that $\lambda \in (1, \infty)$ and (5.123), (5.130) holds. Then every oscillatory solution of (5.125) tends to zero as t tends to infinity.*

Theorem 5.6.1. *Assume $-1 < b_k \leq 0$ for every $k \in \mathbf{N}$ and $\sum_{k=1}^{\infty} b_k = -\infty$. In addition if*

$$\int_{t-\tau}^{t} p(s) \prod_{s \leq t_k < t} (1+b_k)ds$$

is bounded, then every positive solution of (5.121) tends to 1 as t tends to infinity.

Proof. It follows from $-1 < b_k \leq 0$ and $\int_{t-\tau}^{t} p(s) \prod_{s \leq t_k < t} (1+b_k)ds$ is bounded that (5.123) holds. Let

$$y(t) = x(t) \prod_{0 \leq t_k < t} (1+b_k)^{-1}.$$

An argument similar to that in the proof of Lemma 5.6.2 yields that $y(t)$ is bounded. If $-1 < b_k \leq 0$, then $\prod_{k=1}^{\infty}(1+b_k) = 0$, if and only if $\sum_{k=1}^{\infty} b_k = -\infty$. Hence

$$x(t) = y(t) \prod_{0 \leq t_k < t} (1+b_k),$$

and the conditions of this theorem imply that $x(t)$ tends to zero as t tends to infinity. This completes the proof. ∎

Theorem 5.6.2. *Suppose (5.123), (5.124), and (5.135) hold. Then every positive solution of (5.121) tends to 1 as t tends to infinity.*

5.7 Global Stability of Generalized Models

In this section we establish some global attractivity conditions of the generalized "food-limited" population model

$$N'(t) = r(t)N(t)\left(\frac{1 - N(t-\tau)}{1 + \lambda(t)N(t-\tau)}\right)^{\alpha}, \quad t \geq 0, \tag{5.153}$$

where

$$r \in C([0,\infty),(0,\infty)), \ \lambda(t) \in C([0,\infty),[0,\infty)), \ \tau > 0,$$

and α is a ratio of two odd positive integers so that $\alpha \geq 1$. The results in this section are adapted from [39]. We consider solutions of (5.153) under the initial condition

$$\begin{cases} N(t) = \phi(t), \ t \in [-\tau, 0], \\ \phi \in C([-\tau,0],[0,\infty)), \ \phi(0) > 0. \end{cases} \tag{5.154}$$

Lemma 5.7.1. *For any $v \in [0,1)$,*

$$\ln(2e^{-v(1-v/2)} - 1) \geq -2v,$$

and for any $u \in [0,\infty)$,

$$\ln(2e^{u(1+u/2)} - 1) \geq 2u.$$

Proof. Let

$$f(v) = 2e^{-v(1-v/2)} - e^{-2v} \text{ and } g(v) = (1-v)e^{v(1+v/2)}.$$

It is easy to see that

$$g(0) = 1, g'(v) = -v^2 e^{v(1+v/2)} \leq 0$$

and

$$f'(v) = 2e^{-2v}[1 - g(v)] = -2e^{-2v}g'(\xi)v \geq 0, \text{ for some } \xi \in (0,v).$$

It follows that $f(v) \geq f(0) = 1$ for $v \in [0,1)$. The other assertion can be similarly proved. The proof is complete. ■

Lemma 5.7.2. *Assume that* $v \in (0, 1)$. *Then for any* $x \in [0, \infty)$,

$$\ln \frac{1 + [2e^{-v(1-v/2)} - 1]e^{-vx}}{1 + e^{-vx}} \leq -v\left(1 - \frac{v}{2}\right) + \frac{v^2}{2}x \tag{5.155}$$

Proof. Set

$$a := 2e^{-v(1-v/2)} - 1$$

and

$$f(x) := \ln((1 + ae^{-vx})/(1 + e^{-vx})).$$

Note

$$f(0) = -v(1 - v/2), \quad f'(0) = \frac{v}{2}[e^{-v(1-v/2)} - 1],$$

and

$$f''(x) = \left[\frac{a}{(a + e^{vx})^2} - \frac{1}{(1 + e^{vx})^2}\right] v^2 e^{vx}.$$

Since $\alpha \leq 1$, it follows that $f''(x) \leq 0$ for $x \geq 0$. By the mean-value theorem and the fact that

$$e^{x(1-x/2)} \leq 1 + x, \quad \text{for } x \geq 0,$$

we have

$$f(x) \leq f(0) + f'(0)x = -v(1 - \frac{v}{2}) + \frac{vx}{2}[e^{v(1-v/2)} - 1]$$
$$\leq -v(1 - \frac{v}{2}) + \frac{v^2 x}{2}.$$

The proof is complete. ■

The following result follows the usual argument in the literature (for completeness we include it here; see also Lemma 5.3.7).

Lemma 5.7.3. *The system of inequalities*

$$\begin{cases} \ln \frac{1+u}{1-u} \leq 2v, \\ -\ln \frac{1-v}{1+v} \leq 2u \end{cases} . \tag{5.156}$$

has a unique solution $(u, v) = (0, 0)$ *in the region* $\{(u, v) : -1 < v \leq 0 \leq u < 1\}$.

Proof. Set

$$g(x) = \exp(2(1-x)/(1+x)),\ \ f(x) = x - g(g(x))$$

and

$$h(x) = (1+x)^2[1+g(x)]^2 - 16g(x)g(g(x)).$$

Observe that $h(1) = 0$,

$$f'(x) = 1 - g'(x)g'(g(x)) = 1 - \frac{16g(x)g(g(x))}{(1+x)^2[1+g(x)]^2},$$

and for $x > 1$

$$\begin{aligned} h'(x) &= 2[1+g(x)][(1+x)(1+g(x)) - 4g(x)] \\ &\quad + \frac{64}{(1+x)^2} g(x)g(g(x)) \frac{[1-g(x)]^2}{[1+g(x)]^2} \\ &> 0. \end{aligned}$$

It follows that $h(x) > h(1) = 0$ for $x > 1$, and so $f'(x) > 0$ for $x > 1$. This shows that $f(x) > f(1) = 0$ for $x > 1$. From (5.156), we have

$$g(\mu) \le \lambda \le 1 \le \mu \le g(\lambda),$$

where

$$\lambda = (1-v)/(1+v) \text{ and } \mu = (1+u)/(1-u).$$

If $u > 0$, then $\mu > 1$, and so

$$\mu \le g(\lambda) \le g(g(\mu)) < \mu.$$

This contradiction implies that $u = v = 0$. The proof is complete. ∎

The following result follows the usual argument.

Lemma 5.7.4. *Suppose that*

$$\int_0^{+\infty} \frac{r(t)}{[1+\lambda(t)]^\alpha} dt = \infty. \tag{5.157}$$

Then every solution of (5.153) and (5.154) that does not oscillate about 1 tends to 1 as $t \to \infty$.

Lemma 5.7.5. *Suppose* $0 < \lambda(t) \leq 1$ *for* $t \geq 0$ *and*

$$\limsup_{t\to+\infty} \int_{t-\tau}^{t} \frac{r(s)}{(\lambda(s))^{\alpha}} ds \leq 3. \tag{5.158}$$

Let $N(t) = N(t;0,\phi)$ *be a solution of (5.153) and (5.154) which is oscillatory about 1. Then* $N(t)$ *is bounded above and is strictly bounded below by 0.*

Proof. Let t_0 be large enough so that

$$\int_{t-\tau}^{t} \frac{r(s)}{(\lambda(t))^{\alpha}} ds \leq 4, \text{ for all } t \geq t_0.$$

Let t^* be a local maximum point of $N(t)$ for $t \geq t_0 + \tau$. Then

$$N'(t^*) = 0 \text{ and } N(t^* - \tau) = 1.$$

Integrating (5.153) from $t^* - \tau$ to t^* yields

$$\begin{aligned} N(t^*) &= \exp\left(\int_{t^*-\tau}^{t^*} r(s)N(s) \left[\frac{1 - N(s-\tau)}{\lambda(s)N(s-\tau)}\right]^{\alpha} ds \right) \\ &\leq \exp\left(\int_{t^*-\tau}^{t^*} r(s) ds \right) \leq e^4. \end{aligned}$$

Consequently,

$$\limsup_{t\to\infty} N(t) \leq e^4.$$

Next, let t_* be a local minimum point of $N(t)$ for $t \geq t_0 + 3\tau$. Then $N'(t_*) = 0$ and $N(t_* - \tau) = 1$. Proceeding as before and using the fact that

$$\frac{1 - N(t-\tau)}{1 + \lambda(t)N(t-\tau)} \geq \frac{1 - e^4}{1 + \lambda(t)e^4} \geq \frac{1 - e^4}{\lambda(t)(1 + e^4)},$$

for $t \geq t_0 + \tau$, we have

$$\begin{aligned} N(t_*) &\geq \exp\left(\int_{t_*-\tau}^{t^*} \frac{r(s)}{\lambda^{\alpha}(s)} \left[\frac{1 - e^4}{\lambda(s)(1 + e^4)}\right]^{\alpha} ds \right) \\ &\geq \exp\left(4 \left[\frac{1 - e^4}{1 + e^4}\right]^{\alpha} \right). \end{aligned}$$

Hence

$$\liminf_{t\to\infty} N(t) \geq \exp\left(4\left[\frac{1-e^4}{1+e^4}\right]^{\alpha}\right) > 0.$$

The proof is complete. ■

The proof of next result is similar to the proof of Lemma 5.7.5 and is thus omitted.

Lemma 5.7.6. *Assume that* $\lambda(t) \geq 1$ *for* $t \geq 1$ *and*

$$\limsup_{t\to+\infty} \int_{t-\tau}^{t} r(s)ds \leq 3. \tag{5.159}$$

Let $N(t) = N(t, 0, \phi)$ *be a solution of (5.153) and (5.154) which is oscillatory about 1. Then* $N(t)$ *is bounded above and strictly bounded below by 0.*

Theorem 5.7.1. *Suppose* $0 < \lambda(t) \leq 1$, *for* $t \geq 0$, *and (5.157) holds. If (5.158) holds, then every solution of (5.153) and (5.154) tends to* 1 *as* t *tends to* $+\infty$.

Proof. Let

$$u = \limsup_{t\to\infty} N(t) \text{ and } v = \liminf_{t\to\infty} N(t).$$

Then by Lemma 5.7.5, $0 < v \leq 1$ and $u \geq 1$. It suffices to show that $u = v = 1$. For any $\varepsilon \in (0, v)$, choose $t_0 = t_0(\varepsilon)$ such that

$$v_1 \equiv v - \varepsilon < N(t-\tau) < u + \varepsilon \equiv u_1, \ t \geq t_0 \tag{5.160}$$

and

$$\int_{t-\tau}^{t} \frac{r(s)}{\lambda^{\alpha}(t)} ds \leq 3 + \varepsilon, \quad t \geq t_0 - \tau. \tag{5.161}$$

Note that

$$\frac{(1-x)}{(1+\lambda x)} \leq \frac{(1-x)}{(\lambda(1+x))} \quad \text{for } x \leq 1$$

and

$$\frac{(1-x)}{(1+\lambda x)} \geq \frac{(1-x)}{\lambda(1+x)} \quad \text{for } x \geq 1.$$

Thus

$$N'(t) \le r(t)N(t)\left(\frac{1-v_1}{1+\lambda(t)v_1}\right)^\alpha \le r(t)N(t)\left(\frac{1-v_1}{\lambda(t)(1+v_1)}\right)^\alpha, \ t \ge t_0, \tag{5.162}$$

and

$$N'(t) \ge r(t)N(t)\left(\frac{1-u_1}{1+\lambda(t)u_1}\right)^\alpha \ge r(t)N(t)\left(\frac{1-u_1}{\lambda(t)(1+u_1)}\right)^\alpha, \ t \ge t_0. \tag{5.163}$$

Consequently,

$$N'(t) \le \frac{r(t)}{\lambda^\alpha(t)}N(t)\frac{1-v_1}{1+v_1}, \quad t \ge t_0, \tag{5.164}$$

and

$$N'(t) \ge \frac{r(t)}{\lambda^\alpha(t)}N(t)\frac{1-u_1}{1+u_1}, \quad t \ge t_0. \tag{5.165}$$

Let $R(t) = r(t)/\lambda^\alpha(t)$. Let $\{p_n\}$ be an increasing sequence such that $p_n \ge t_0 + \tau$

$$\lim_{n\to\infty} p_n = +\infty, \ N'(p_n) = 0 \text{ and } \lim_{n\to\infty} N(p_n) = u.$$

By (5.153), $N(p_n - \tau) = 1$. For $p_n - \tau \le t \le p_n$, by integrating (5.164) from $t - \tau$ to $p_n - \tau$, we get

$$N(t-\tau) \ge \exp\left(-\frac{1-v_1}{1+v_1}\int_{t-\tau}^{p_n-\tau} R(s)ds\right), \ (p_n - \tau) \le t \le p_n.$$

Substituting this into (5.153), if $N(t - \tau) \le 1$, we have

$$N'(t) \le R(t)N(t)\left[\frac{1-N(t-\tau)}{1+N(t-\tau)}\right]^\alpha \le R(t)N(t)\frac{1-N(t-\tau)}{1+N(t-\tau)}$$
$$\le R(t)N(t)\frac{1-\exp\left(-\dfrac{1-v_1}{1+v_1}\displaystyle\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}{1+\exp\left(-\dfrac{1-v_1}{1+v_1}\displaystyle\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}.$$

If $N(t - \tau) > 1$, by (5.153), $N'(t) < 0$, and thus

$$N'(t) \le R(t)N(t)\frac{1-\exp\left(-\dfrac{1-v_1}{1+v_1}\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}{1+\exp\left(-\dfrac{1-v_1}{1+v_1}\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}.$$

If $t \in (p_n - \tau, p_n)$, we have

$$N'(t) \le \min\left\{ R(t)N(t)\frac{1-v_1}{1+v_1}, R(t)N(t)A(t) \right\}, \tag{5.166}$$

where

$$A(t) = \frac{1 - \exp\left(-\frac{1-v_1}{1+v_1}\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}{1 + \exp\left(-\frac{1-v_1}{1+v_1}\int_{t-\tau}^{p_n-\tau} R(s)ds\right)}.$$

Since

$$0 < x = (1-v_1)/(1+v_1) < 1,$$

it follows from Lemma 5.7.1 that

$$\ln 2e^{-x(1-x/2)-1} \ge -2x,$$

and so

$$0 < -\frac{1}{x}\ln(2e^{-x(1-x/2)} - 1) \le 2.$$

There are two possibilities.

Case 1.

$$\int_{p_n-\tau}^{p_n} R(s)ds \le -\frac{1}{v_0}\ln(2e^{-v_0(1-v_0/2)} - 1) \equiv A \le 3 + \varepsilon,$$

where $v_0 = (1-v_1)/(1+v_1)$.

Then

$$\ln N(p_n) \le \int_{p_n-\tau}^{p_n} \frac{R(t)\left[1 - \exp\left(-v_0\int_{t-\tau}^{p_n-\tau} R(s)ds\right)\right]}{1 + \exp\left(-v_0\int_{t-\tau}^{p_n-\tau} R(s)ds\right)} dt$$

$$= \int_{p_n-\tau}^{p_n} \frac{R(t)\left[1 - \exp\left(-v_0\left(\int_{t-\tau}^{t} r(s)ds - \int_{p_n-\tau}^{t} R(s)ds\right)\right)\right]}{1 + \exp\left(-v_0\left(\int_{t-\tau}^{t} R(s)ds - \int_{p_n-\tau}^{t} R(s)ds\right)\right)} dt$$

$$\leq \int_{p_n-\tau}^{p_n} \frac{R(t)\left[1-\exp\left(-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{t} R(s)ds\right)\right)\right]}{1+\exp\left(-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{t} R(s)ds\right)\right)}dt$$

$$= \int_{p_n-\tau}^{p_n} R(s)ds - \frac{2}{v_0}\ln\frac{1+\exp\left(-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{t} R(s)ds\right)\right)}{1+e^{-(3+\varepsilon)v_0}}.$$

Note that the function

$$f(x) = x - \frac{(2\ln[1+e^{-v_1(3+\varepsilon-x)}])}{v_1}$$

is increasing in $[0, 3+\varepsilon]$ and we have by Lemmas 5.7.1 and 5.7.2, that

$$\begin{aligned}
\ln N(p_n) &\leq A - \frac{2}{v_0}\ln\frac{1+e^{-v_0(3+\varepsilon-A)}}{1+e^{-(3+\varepsilon)v_0}}\\
&= A + \frac{2}{v_0}\ln\frac{1+[2e^{-v_0(1-v_0/2)}-1]e^{-v_0(3+\varepsilon-A)}}{1+e^{-v_0(3+\varepsilon-A)}}\\
&\leq A + \frac{2}{v_0}\left[-v_0\left(1-\frac{v_0}{2}\right)+\frac{v_0^2}{2}(3+\varepsilon-A)\right]\\
&= -2+(4+\varepsilon)v_0 - \frac{1-v_0}{v_0}\ln(2e^{-v_0(1-v_0/2)}-1)\\
&\leq (2+\varepsilon)v_1.
\end{aligned}$$

Case 2.

$$A < \int_{p_n-\tau}^{p_n} R(s)ds \leq 3+\varepsilon.$$

Choose $\xi_n \in (p_n-\tau, p_n)$ such that

$$\int_{\xi_n}^{p_n} R(s)ds \equiv A.$$

Then by (5.166) and Lemma 5.7.1,

$$\ln N(p_n) \leq \int_{p_n-\tau}^{\xi_n} R(s)ds + \int_{\xi_n}^{p_n} \frac{R(t)\left[1-\exp\left(-v_0 \int_{t-\tau}^{p_n-\tau} R(s)ds\right)\right]}{1+\exp\left(-v_0 \int_{t-\tau}^{p_n-\tau} R(s)ds\right)} dt$$

$$\leq v_0 \int_{p_n-\tau}^{\xi_n} R(s)ds + \int_{\xi_n}^{p_n} \frac{R(t)\left[1-\exp\left(-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{t} R(s)ds\right)\right)\right]}{1+\exp\left(-v_0\left(3+\varepsilon-\int_{p_n-\tau}^{t} R(s)ds\right)\right)} dt$$

$$= v_0 \int_{p_n-\tau}^{\xi_n} R(s)ds + \int_{\xi_n}^{p_n} R(s)ds - \frac{2}{v_0}\ln B_0$$

$$= v_0 \int_{p_n-\tau}^{p_n} R(s)ds + (1-v_0)A - \frac{2}{v_0} B_0$$

$$\leq (3+\varepsilon)v_0 + (1-v_0)A - \frac{2}{v_0}\ln\frac{2}{1+e^{-Av_0}}$$

$$= -2 + (4+\varepsilon)v_0 - \frac{1-v_0}{v_0}\ln(2e^{-v_0(1-v_0/2)} - 1)$$

$$\leq (2+\varepsilon)v_1,$$

where

$$B_0 = \frac{1+\exp\left(-v_0\left(3+\varepsilon-\int\limits_{p_n-\tau}^{p_n} R(s)ds\right)\right)}{1+\exp\left(-v_0\left(3+\varepsilon-\int\limits_{p_n-\tau}^{\xi_n} R(s)ds\right)\right)}$$

and we have used the fact that the function

$$g(x) = -\frac{2}{v_1}\ln\frac{1+\exp[-v_1\,(3+\varepsilon-x)]}{1+\exp[-v_1\,(3+\varepsilon+A-x)]}+v_1 x$$

is increasing on $[0, 3+\varepsilon]$.

In either cases, we have proved that

$$\ln N(p_n) \le (2+\varepsilon)v_1 \text{ for } n = 1, 2, \ldots .$$

Letting $n \to \infty$ and $\varepsilon \to 0$, we have

$$\ln u \le 2\frac{1-v}{1+v}. \tag{5.167}$$

Next, let $\{q_n\}$ be an increasing sequence such that $q_n \ge t_0 + \tau$, $\lim\limits_{n\to\infty} q_n = +\infty$, $N'(q_n) = 0$, and $\lim_{n\to\infty} N(q_n) = -v$. By (5.153), $N(q_n - \tau) = 1$. For $q_n - \tau \le t \le p_n$, integrating (5.165) from $t-\tau$ to $q_n - \tau$, we have

$$N(t-\tau) \le \exp\left(-\frac{1-u_1}{1+u_1}\int\limits_{t-\tau}^{p_n-\tau} R(s)ds\right),\ q_n-\tau \le t \le q_n.$$

Substituting this into (5.153), if $N(t-\tau) \ge 1$, we have

$$\begin{aligned} N'(t) &= r(t)N(t)\left[\frac{1-N(t-\tau)}{1+\lambda(t)N(t-\tau)}\right]^{\alpha} \\ &\ge R(t)N(t)\frac{1-N(t-\tau)}{1+\lambda(t)N(t-\tau)} \\ &\ge R(t)N(t)\frac{1-\exp(-u_0\int\limits_{t-\tau}^{q_n-\tau} R(s)ds)}{1+\exp\left(-u_0\int\limits_{t-\tau}^{q_n-\tau} R(s)ds\right)} \end{aligned}$$

for $q_n - \tau \le t \le q_n$. If $N(t-\tau) < 1$, then by (5.153), $N'(t) > 0$, and thus

$$N'(t) \geq R(t)N(t)\frac{1-\exp\left(-u_0\int\limits_{t-\tau}^{q_n-\tau}R(s)ds\right)}{1+\exp\left(-u_0\int\limits_{t-\tau}^{q_n-\tau}R(s)ds\right)},$$

where $u_0 = (1-u_1)/(1+u_1)$. Thus

$$-N'(t) \leq \min\left\{-R(t)N(t)u_0, -R(t)N(t)\frac{1-\exp\left(-u_0\int\limits_{t-\tau}^{q_n-\tau}R(s)ds\right)}{1+\exp\left(-u_0\int\limits_{t-\tau}^{q_n-\tau}R(s)ds\right)}\right\} \tag{5.168}$$

for $q_n - \tau \leq t \leq q_n$. Note that $0 < -u_0 < 1$, and one can easily see that

$$0 < -\frac{1}{u_0}\ln(2e^{-u_0(1-u_0/2)}-1) < 3.$$

There are two cases to consider.

Case 1.

$$\int\limits_{q_n-\tau}^{q_n} R(s)ds \leq (3+\varepsilon) + \frac{1}{u_0}\ln(2e^{-u_0(1-u_0/2)}-1) \equiv B.$$

By (5.168) and Lemma 5.7.1,

$$\begin{aligned}-\ln N(q_n) &\leq -u_0\int\limits_{q_n-\tau}^{q_n} R(s)ds \leq -(3+\varepsilon)u_0 - \ln(2e^{-u_0(1-u_0/2)}-1)\\ &\leq -(1+\varepsilon)u_0.\end{aligned}$$

Case 2.

$$B < \int\limits_{q_n-\tau}^{q_n} R(s)ds \leq 3+\varepsilon.$$

We choose $\eta_n \in (q_n-\tau, q_n)$ such that

$$B = \int\limits_{q_n-\tau}^{\eta_n} R(s)ds.$$

Then by (5.155) and Lemma 5.7.1, we have

$$-\ln N(q_n) \le -u_0 \int_{q_n-\tau}^{\eta_n} R(s)ds + \int_{\eta_n}^{q_n} \frac{R(t)[\exp(-u_0 \int_{t-\tau}^{q_n-\tau} R(s)ds) - 1]}{1 + \exp(-u_0 \int_{t-\tau}^{q_n-\tau} R(s)ds)} dt$$

$$\le -u_0 \int_{q_n-\tau}^{\eta_n} R(s)ds$$

$$+ \int_{\eta_n}^{q_n} \frac{R(t)[\exp -u_0(3 + \varepsilon - \int_{q_n-\tau}^{t} R(s)ds)] - 1}{1 + \exp(-u_0(3 + \varepsilon - \int_{q_n-\tau}^{t} R(s)ds))} dt$$

$$= -u_0 \int_{q_n-\tau}^{\eta_n} R(s)ds - \int_{\eta_n}^{q_n} R(s)ds$$

$$-\frac{2}{u_0} \ln \frac{1 + \exp(-u_0(3 + \varepsilon - \int_{q_n-\tau}^{q_n} R(s)ds))}{1 + \exp(-u_0(3 + \varepsilon - \int_{q_n-\tau}^{\eta_n} R(s)ds))} - \ln N(q_n)$$

$$= (1 - u_0)B - \int_{\eta_n}^{q_n} R(s)ds + 2\left(1 - \frac{u_0}{2}\right)$$

$$+\frac{2}{u_0} \ln \frac{1 + \exp(-u_0(3 + \varepsilon - \int_{q_n-\tau}^{q_n} R(s)ds))}{2}$$

$$\le 2 - (4 + \varepsilon)u_0 + \left(\frac{1 - u_0}{u_0}\right) \ln\left(2e^{-u_0(1-u_0/2)-1}\right)$$

$$\le (2 + \varepsilon)u_0,$$

where we have used the fact that

$$h(x) = -x - \frac{2}{u_0} \ln \frac{1 + \exp(-u_0(3 + \varepsilon - x))}{2}$$

is increasing on $[0, 3 + \varepsilon]$.

In either cases, we have proved that $-\ln N(p_n) \leq -(2+\varepsilon)u_0$ for $n = 1, 2, \ldots$. Letting $n \to \infty$ and $\varepsilon \to 0$, we have

$$-\ln v \leq -2\frac{1-u}{1+u}. \tag{5.169}$$

Let

$$y = -(1-u)/(1+u)$$

and

$$x = (1-v)/(1+v),$$

then in view of (5.167), (5.169), and Lemma 5.7.3, we get $x = y = 0$. This shows that $u = v = 1$. The proof is complete. ■

By methods similar to those in the proof of Theorem 5.7.1, and by noting that if $\lambda \geq 1$, then

$$(1-x)/(1+\lambda x) \leq (1-x)/(1+x), \text{ for } x \leq 1,$$

and

$$(1-x)/(1+\lambda x) \geq (1-x)/(1+x), \text{ for } x \geq 1,$$

one can prove the next result. The details are omitted.

Theorem 5.7.2. *Suppose $\lambda(t) \geq 1$ for $t \geq 0$, (5.157), and (5.159) hold. Then every solution of (5.153) and (5.154) tends to 1 as t tends to $+\infty$.*

5.8 Existence of Periodic Solutions

In this section, we consider the equation

$$\frac{dN(t)}{dt} = N(t)\frac{r(t) - a(t)N(t) - b(t)N(t-\tau(t))}{k(t) + c(t)N(t) + d(t)N(t-\tau(t))} \tag{5.170}$$

and establish some sufficient condition which ensures the existence of periodic solutions. Here a, b, c, d, k, r are continuous ω-periodic functions with $r > 0$, $k > 0$, $a > 0$, $b \geq 0$, $c \geq 0$, and $d \geq 0$. The results in this section are adapted from [22]. Considering the biological significance of system (5.170), we always assume that $N(0) > 0$. The main results will be proved by applying Theorem 1.4.11. To prove the main results we present some useful lemmas.

Let f be a ω-periodic function and define

$$f^l = \min_{t\in[0,\omega]} f(t), \quad f^u = \max_{t\in[0,\omega]} f(t).$$

Lemma 5.8.1. *There exists a unique* $u^* > 0$ *such that*

$$\int_0^\omega \frac{r(t) - [a(t) + b(t)]u^*}{k(t) + [c(t) + d(t)]u^*} dt = 0.$$

Proof. Let

$$f(u) = \int_0^\omega \frac{r(t) - [a(t) + b(t)]u}{k(t) + [c(t) + d(t)]u} dt.$$

It is clear that

$$f(0) = \int_0^\omega \frac{r(t)}{k(t)} dt > 0,$$

$$\begin{aligned} f\left(\frac{r^u + 1}{a^l + b^l}\right) &= \int_0^\omega \frac{r(t) - [a(t) + b(t)]\dfrac{r^u + 1}{a^l + b^l}}{k(t) + [c(t) + d(t)]\dfrac{r^u + 1}{a^l + b^l}} dt \\ &\le \int_0^\omega \frac{r(t) - (r^u + 1)}{k(t) + [c(t) + d(t)]\dfrac{r^u + 1}{a^l + b^l}} dt < 0, \end{aligned}$$

and then from the zero point theorem, it follows that there exists a $u^* \in \left(0, \dfrac{r^u + 1}{a^l + b^l}\right)$ such that $f(u^*) = 0$. Moreover,

$$\frac{df}{du} = -\int_0^\omega \frac{k(t)[a(t) + b(t)] + r(t)[c(t) + d(t)]}{\{k(t) + [c(t) + d(t)]u\}^2} dt < 0,$$

that is, $f(u)$ is monotonically decreasing with respect to u, and hence u^* is unique. The proof is complete. ■

Theorem 5.8.1. *Equation (5.170) has at least one positive periodic solution of period* ω

Proof. Let $N(t) = \exp\{x(t)\}$. Then (5.170) may be reformulated as

$$\frac{dx(t)}{dt} = \frac{r(t) - a(t)\exp\{x(t)\} - b(t)\exp\{x(t-\tau(t))\}}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x(t-\tau(t))\}}. \tag{5.171}$$

In order to apply Theorem 1.4.11 to (5.171), we first let

$$\mathbb{X} = \mathbb{Y} = \{x(t) \in C(\mathbb{R}, \mathbb{R}),\ x(t+\omega) = x(t)\}$$

and

$$\|x\| = \max_{t\in[0,\omega]} |x(t)|, \quad x \in \mathbb{X}\ (or\ \mathbb{Y}).$$

Then $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces with the norm $\|.\|$. Let

$$N\,x = \frac{r(t) - a(t)\exp\{x(t)\} - b(t)\exp\{x(t-\tau(t))\}}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x(t-\tau(t))\}}, \quad x \in \mathbb{X},$$

$$L\,x = x' = \frac{dx(t)}{dt}, \qquad P\,x = \frac{1}{\omega}\int_0^{\omega} x(t)dt, \quad x \in \mathbb{X},$$

$$Q\,z = \frac{1}{\omega}\int_0^{\omega} z(t)dt, \quad z \in \mathbb{Y}.$$

Then it follows that

$$Ker\ L = \mathbb{R}, \ \ \text{Im}\ L = \left\{z \in \mathbb{Y} : \int_0^{\omega} z(t)dt = 0\right\} \text{ is closed in } \mathbb{Y},$$

$$\dim\ Ker\ L = 1 = co\dim\ \text{Im}\ L,$$

and $P,\ Q$ are continuous projectors such that

$$\text{Im}\,P = Ker\ L, \quad Ker\ Q = \text{Im}\ L = \text{Im}\,(I - Q).$$

Therefore, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (of L)

$$K_P : \text{Im}\ L \to KerP \cap Dom\ L$$

is

$$K_P(z) = \int_0^t z(s)ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t z(s)\ ds\ dt.$$

Also

$$QN\,x = \frac{1}{\omega}\int_0^{\omega} \frac{r(s) - a(s)\exp\{x(s)\} - b(s)\exp\{x(s-\tau(s))\}}{k(s) + c(s)\exp\{x(s)\} + d(s)\exp\{x(s-\tau(s))\}}ds$$

and

$$K_P(I-Q)N\,x = \int_0^{t} \frac{r(s) - a(s)\exp\{x(s)\} - b(s)\exp\{x(s-\tau(s))\}}{k(s) + c(s)\exp\{x(s)\} + d(s)\exp\{x(s-\tau(s))\}}ds$$

$$-\frac{1}{\omega}\int_0^{\omega}\int_0^{t} \frac{r(s) - a(s)\exp\{x(s)\} - b(s)\exp\{x(s-\tau(s))\}}{k(s) + c(s)\exp\{x(s)\} + d(s)\exp\{x(s-\tau(s))\}}ds\,dt$$

$$-\left(\frac{t}{\omega} - \frac{1}{2}\right)\int_0^{\omega} \frac{r(s) - a(s)\exp\{x(s)\} - b(s)\exp\{x(s-\tau(s))\}}{k(s) + c(s)\exp\{x(s)\} + d(s)\exp\{x(s-\tau(s))\}}ds.$$

By the Arzela–Ascoli Theorem, it is easy to see that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded subset Ω of $\mathbb{X}$ and $QN(\overline{\Omega})$ is bounded. Thus, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \in \mathbb{X}$.

Consider the operator equation $L\,x = \lambda N\,x,\ \ \lambda \in (0,1)$, that is,

$$\frac{dx(t)}{dt} = \lambda\frac{r(t) - a(t)\exp\{x(t)\} - b(t)\exp\{x(t-\tau(t))\}}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x(t-\tau(t))\}}. \tag{5.172}$$

Let $x = x(t) \in \mathbb{X}$ be a solution of (5.172) for a certain $\lambda \in (0,1)$. Integrating (5.172) with respect to t over the interval $[0,\omega]$ yields

$$\int_0^{\omega} \frac{r(t) - a(t)\exp\{x(t)\} - b(t)\exp\{x(t-\tau(t))\}}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x(t-\tau(t))\}}dt = 0, \tag{5.173}$$

and therefore

$$\int_0^{\omega} \frac{a(t)\exp\{x(t)\} + b(t)\exp\{x(t-\tau(t))\}}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x(t-\tau(t))\}}dt$$

$$= \int_0^{\omega} \frac{r(t)}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x(t-\tau(t))\}}dt$$

$$\leq \int_0^{\omega} \frac{r(t)}{k(t)}dt \leq \frac{\omega r^u}{k^l}, \tag{5.174}$$

which together with (5.172) implies

$$\int_0^{\omega} |x'(t)|\,dt = \lambda \int_0^{\omega} \left| \frac{r(t) - a(t)\exp\{x(t)\} - b(t)\exp\{x(t-\tau(t))\}}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x(t-\tau(t))\}} \right| dt < \frac{2\omega r^u}{k^l}.$$

From (5.173) and the mean-value theorem for integral, we see that there exists $\xi \in [0, \omega]$ such that

$$\frac{r(\xi) - a(\xi)\exp\{x(\xi)\} - b(\xi)\exp\{x(\xi-\tau(\xi))\}}{k(\xi) + c(\xi)\exp\{x(\xi)\} + d(\xi)\exp\{x(\xi-\tau(\xi))\}}\omega = 0,$$

and therefore

$$r(\xi) = a(\xi)\exp\{x(\xi)\} + b(\xi)\exp\{x(\xi-\tau(\xi))\}. \tag{5.175}$$

Since $x(t) \in \mathbb{X}$, there exist $t_1, t_2 \in [0, \omega]$ such that $x(t_1) = x^l,\ x(t_2) = x^u$, and then from (5.175) it follows that

$$x(t_1) \le \ln\left\{\frac{r(\xi)}{a(\xi) + b(\xi)}\right\} \le \ln\left\{\frac{r^u}{a^l + b^l}\right\},$$

$$x(t_2) \ge \ln\left\{\frac{r(\xi)}{a(\xi) + b(\xi)}\right\} \ge \ln\left\{\frac{r^l}{a^u + b^u}\right\},$$

from which we derive

$$x(t) \le x(t_1) + \int_0^{\omega} |x'(t)|\,dt \le \ln\left\{\frac{r^u}{a^l + b^l}\right\} + \frac{2\omega r^u}{k^l} := M_1,$$

$$x(t) \ge x(t_2) - \int_0^{\omega} |x'(t)|\,dt \ge \ln\left\{\frac{r^l}{a^u + b^u}\right\} - \frac{2\omega r^u}{k^l} := M_2,$$

and hence

$$\|x\| = \max_{t\in[0,\omega]} |x(t)| \le \max\{|M_1|, |M_2|\} := B_1.$$

Clearly, B_1 is independent of the choice of λ. Take $B = B_1 + B_2$, where $B_2 > 0$ is taken sufficiently large such that $|\ln(u^*)| < B_2$ and define

$$\Omega := \{x(t) \in \mathbb{X} :\ \|x\| < B\}.$$

When $x \in \partial\Omega \cap Ker\ L = \partial\Omega \cap \mathbb{R},\ x = B$ or $x = -B$, and then

$$QN\,x = \frac{1}{\omega}\int_0^{\omega} \frac{r(t) - a(t)\exp\{x(t)\} - b(t)\exp\{x\}}{k(t) + c(t)\exp\{x(t)\} + d(t)\exp\{x\}}dt \neq 0.$$

Furthermore, a direct calculation reveals that

$$\deg\{JQN, \Omega \cap Ker\ L, 0\}$$
$$= sign\left\{-\frac{1}{\omega}\int_0^{\omega} \frac{k(t)[a(t) + b(t)] + r(t)[c(t) + d(t)]}{\{k(t) + [c(t) + d(t)]u^*\}^2}dt\right\} \neq 0;$$

here J is the identity mapping since $\Im P = KerL$. Thus all the requirements in Theorem 1.4.11 are satisfied. Hence (5.171) has at least one solution $x^*(t) \in Dom\ L \cap \overline{\Omega}$. Set $N^*(t) = \exp\{x^*(t)\}$. Then $N^*(t)$ is a positive ω-periodic solution of (5.170). The proof is complete. ■

Chapter 6
Logistic Models with Diffusions

You know that I write slowly. This is chiefly because I am never satisfied until I have said as much as possible in a few words, and writing briefly takes far more time than writing at length.

Carl F. Gauss (1777–1855)

He who does not employ mathematics for himself will some day find it employed against himself.

Johann F. Herbart (1776–1841)

Population dispersal plays an important role in the population dynamics which arises from environmental and ecological gradients in the habitat. We assume that the systems under consideration are allowed to diffuse spatially besides evolving in time. The spatial diffusion arises from the tendency of species to migrate towards regions of lower population density where the life is better. The most familiar model systems incorporating these features are reaction diffusion equations.

This chapter discusses oscillation, global stability, and periodicity of some diffusive logistic models.

6.1 Introduction

A diffusion mechanism models the movement of many individuals in an environment or media. The individuals can be very small such as basic particles in physics, bacteria, molecules, or cells or very large objects such as animals, plants, or certain kind of events like epidemics, or tumors. The particles reside in a region, which we call Ω, and we assume that it is an open subset of $\mathbf{R}^n$ (the n^{th} dimensional space with Cartesian coordinate system) with $n \geq 1$. In particular, we are interested in the cases of $n = 1, 2$, and 3. The main mathematical variable we consider here is

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the density function of the particles $N(t,x)$, where t is the time and $x \in \Omega$ is the location. The dimension of the population density usually is the number of particles or organisms per unit area (if $n = 2$) or unit volume (if $n = 3$).

Technically, we define the population density function $N(t,x)$ as follows. Let x be a point in the habitat Ω and let $\{O_n\}_{n=1}^{\infty}$ be a sequence of spatial regions (which have the same dimension as Ω) surrounding x; here O_n is chosen in a way that the spatial measurement $|O_n|$ of O_n (length, area, volume, or mathematically, the Lebesgue measure) tends to zero as $n \to \infty$, and $O_n \supset O_{n+1}$. Then

$$N(t,x) = \lim_{n\to\infty} \frac{\text{number of organisms in } O_n \text{ at time } t}{|O_n|}, \tag{6.1}$$

if the limit exists. The total population in any subregion O of Ω at time t is

$$\int_O N(t,x)dx. \tag{6.2}$$

The movement of $N(t,x)$ is called the flux of the population density, which is a vector. The "high to low" principle now means that the flux always points to the most rapid decreasing direction of $N(t,x)$, which is the negative gradient of $N(t,x)$. This principle is called Fick's law, and it can be represented as

$$J(t,x) = -d(x)\nabla x N(t,x), \tag{6.3}$$

where J is the flux of N, $d(x)$ is called diffusion coefficient at x, and ∇_x is the gradient operator

$$\nabla_x f(x) = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}).$$

The number of particles at any point may change because of other reasons like birth, death, hunting, or chemical reactions. We assume that the rate of change of the density function due to these reasons is $f(t,x,N)$, which we usually call the reaction rate. Now, we present a differential equation using the balanced law. We choose any region O. Then the total population in O is $\int_O N(t,x)dx$, and the rate of change of the total population is

$$\frac{d}{dt}\int_O N(t,x)dx. \tag{6.4}$$

The net growth of the population inside the region O is

$$\int_O f(t,x,N(t,x))dx, \tag{6.5}$$

and the total out flux is

$$\int_{\partial O} J(t,x).\, n(x) dS, \tag{6.6}$$

where ∂O is the boundary of O and $n(x)$ is the outer normal direction at x. Then the balance law implies

$$\frac{d}{dt}\int_O N(t,x)dx = -\int_{\partial O} J(t,x)\cdot n(x)dS + \int_O f(t,x,N(t,x))dx. \tag{6.7}$$

From the divergence theorem in multivariable calculus, we have

$$\int_{\partial O} J(t,x)\cdot n(x)dS = \int_O div(J(t,x))dx. \tag{6.8}$$

Combining (6.3), (6.7), and (6.8) and interchanging the order of differentiation and integration, we obtain

$$\int_O \frac{\partial}{\partial t} N(t,x)dx = \int_O [div(d(x)\nabla_x N(t,x)) + f(t,x,N(t,x))]dx. \tag{6.9}$$

Since the choice of the region O is arbitrary, then the differential equation

$$\frac{\partial}{\partial t} N(t,x) = div(d(x)\nabla_x N(t,x)) + f(t,x,N(t,x)), \tag{6.10}$$

holds for any (t,x). The (6.10) is called a reaction diffusion equation. Here $div(d(x)\nabla_x N(t,x))$ is the diffusion term which describes the movement of the individuals, and $f(t,x,N(t,x))$ is the reaction term which describes the birth-death or reaction occurring inside the habitat or reactor. The diffusion coefficient $d(x)$ is not a constant in general since the environment is usually heterogeneous. But when the region of the diffusion is approximately homogeneous, we can assume that $d(x) \equiv d$, and then (6.10) can be simplified to

$$\frac{\partial}{\partial t} N(t,x) = d\Delta N + f(t,x,N), \tag{6.11}$$

where $\Delta N = div(\nabla N)$ is the Laplacian operator. Sometimes Eq. (6.11) is called a nonlinear heat equation.

The most popular reaction diffusion equations are

$$\frac{\partial}{\partial t} N(t,x) = D\Delta N + kN, \text{ diffusive Malthus model,} \tag{6.12}$$

and

$$\frac{\partial}{\partial t} N(t,x) = D\Delta N + kN\left(1 - \frac{N}{K}\right), \text{diffusive logistic model.} \tag{6.13}$$

In general there are three commonly used boundary conditions:

$$N(t,x) = \phi(x), \quad t > 0, x \in \partial\Omega, \text{ (Dirichlet)},$$
$$\nabla N(t,x)\cdot n(x) + a(x)N(t,x) = \phi(x), \quad t > 0, x \in \partial\Omega, \text{ (Robin)},$$
$$\nabla N(t,x)\cdot n(x) = \phi(x), \quad t > 0, x \in \partial\Omega \text{ (Neumann)}.$$

6.2 Oscillation of the Malthus Equation

In some applications, some diffusion processes are modeled by the diffusive Malthus equation

$$\frac{\partial u(x,t)}{\partial t} = a(t)\Delta u - pu(x,t) + qu(x,t),\ (x,t) \in \Omega \times [t_0,\infty) \equiv G, \tag{6.14}$$

where a, p, q are nonnegative coefficients representing the phenomena which underlie the diffusion process. For example, in population dynamics the term $a\Delta u$ corresponds to the diffusion due to local concentration, while $-pu$ and qu correspond to death and birth rates, respectively. Since such phenomena may lead to instantaneous changes in population size, it is natural to include delays in the models under consideration. Consider the delay diffusive Malthus equation

$$\frac{\partial u(x,t)}{\partial t} = a(t)\Delta u - p(x,t)u(x,t-\sigma) + q(x,t)u(x,t-\tau), \tag{6.15}$$

where $(x,t) \in \Omega \times [t_0,\infty) \equiv G$ the delays τ, σ are nonnegative constants, Ω is a bounded domain in $\mathbf{R}^n$ with a piecewise smooth boundary $\partial\Omega$, and $\Delta u(x,t) = \sum_{i=1}^{n} \frac{\partial_i^2 u(x,t)}{\partial x_i^2}$.

In this section, we are concerned with the oscillation of the diffusive Malthus model with several coefficients and several delays

$$\frac{\partial u(x,t)}{\partial t} = a(t)\Delta u - \sum_{i=1}^{n} p_i(x,t)u(x,t-\sigma_i) + \sum_{j=1}^{m} q_j(x,t)u(x,t-\tau_j), \tag{6.16}$$

where

(H1) $a, p_i, q_j \in C([t_0,\infty), R^+)$, $\sigma_i, \tau_j \in [0,\infty)$ for $i = 1,\dots,n$ and $j = 1,\dots,m$;

(H2) there exist a positive number $p \leqslant n$ and a partition of the set $\{1,\dots,m\}$ into p disjoint subsets $J_1, J_2, J_3, \dots, J_p$ such that $j \in J_i$; implies that $\tau_j \le \sigma_i$;

(H3) $P_i(t) > \sum_{k\in J_i} Q_k(t+\tau_k-\sigma_i)$ *for* $t \geqslant t_0+\sigma_i-\tau_k$, and $i=1,\ldots,p$;

(H4) $\sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\sigma_i}^{t-\tau_k} Q_k(s)ds \leqslant 1$, for $t \geq t_0+\sigma_i$;

here

$$P_i(t) = \min_{x\in\Omega} p_i(x,t) \text{ and } \quad Q_j(t) = \max_{x\in\Omega} q_j(x,t).$$

Together with (6.16), we consider three kinds of the boundary conditions:

$$\frac{\partial u(x,t)}{\partial N} = 0, \quad \text{on } (x,t) \in \partial\Omega \times [t_0,\infty), \tag{6.17}$$

$$u(x,t) = 0, \quad \text{on } (x,t) \in \partial\Omega \times [t_0,\infty), \tag{6.18}$$

$$\frac{\partial u(x,t)}{\partial N} + \gamma u = 0, \quad \text{on } (x,t) \in \partial\Omega \times [t_0,\infty), \tag{6.19}$$

where N is the unit exterior normal vector to $\partial\Omega$ and $\gamma(x,t)$ is a nonnegative continuous function on $\partial\Omega \times [t_0,\infty)$.

In this Section, we establish some sufficient conditions for the oscillation of all solutions of (6.16) subject to the boundary conditions (6.17), (6.18), (6.19), respectively. A function $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ is said to be a solution of the problem (6.16) and (6.17) (for example) if it satisfies (6.16) in the domain G and satisfies the boundary condition (6.17). The solution $u(x,t)$ of the problem (6.16) is said to be oscillatory in the domain $G \equiv \Omega \times [t_0,\infty)$ if for any positive number μ there exists a point $(x_1,t_1) \in \Omega \times [\mu,\infty)$ such that the equality $u(x_1,t_1) = 0$ holds. A function $U(t)$ is called eventually positive (negative) if there exists a number $t_1 \geq t_0$ such that $U(t) > 0$ (< 0) holds for all $t_1 \geq t_0$.

6.2.1 Oscillation of the Neumann Problem

In this section, we will establish some sufficient conditions for the oscillation of all solutions of (6.16), (6.17).

In our next theorem we will use the following well-known result [30].

Lemma 6.2.1. *Let* $a \in (-\infty,0)$, $\tau \in (0,\infty)$, $t_0 \in (-\infty,\infty)$ *and suppose* $x(t) \in C[t_0,\infty)$ *satisfies*

$$x(t) \leq a + \max_{s\in[t-\tau,t]} x(s).$$

Then $x(t)$ *cannot be a nonnegative function.*

Theorem 6.2.1. *Assume that* $(H1)-(H4)$ *hold, and every solution of*

$$z^{'}(t)+\sum_{i=1}^{p}\left[P_i(t)-\sum_{k\in J_i}Q_k(t+\tau_k-\sigma_i)\right]z(t-\sigma_i)=0 \tag{6.20}$$

oscillates. Then every solution of (6.16), (6.17) is oscillatory in G.

Proof. Assume that (6.16), (6.17) has a nonoscillatory solution. Since the negative solution of (6.16), (6.17) is also a solution, then without loss of generality we assume that (6.16), (6.17) has a solution $u(x,t)>0$, $u(x,t-\sigma_i)>0$, and $u(x,t-\tau_j)>0$ in $\Omega\times[t_1,\infty)$ for some $t_1\geq t_0$. Set

$$U(t)=\int_{\Omega}u(x,t)dx,\quad t\geq t_1,$$

and then $U(t)>0$ for $t\geq t_1$. Integrating (6.16) with respect to x over the domain Ω, we have

$$\frac{d}{dt}\left[\int_{\Omega}u(x,t)dx\right]=a(t)\int_{\Omega}\Delta u(x,t)dx-\int_{\Omega}\sum_{i=1}^{n}p_i(x,t)u(x,t-\sigma_i)dx$$
$$+\int_{\Omega}\sum_{j=1}^{m}q_j(x,t)u(x,t-\tau_j)dx. \tag{6.21}$$

From Green's formula and the boundary condition (6.17), it follows that

$$\int_{\Omega}\Delta u(x,t)dx=\int_{\partial\Omega}\frac{\partial u(x,t)}{\partial N}dS=0,\quad t\geq t_1,$$

where dS is the surface element on $\partial\Omega$. Then (6.21) reduces to

$$\frac{d}{dt}\left[\int_{\Omega}u(x,t)dx\right]=-\int_{\Omega}\sum_{i=1}^{n}p_i(x,t)u(x,t-\sigma_i)dx+\int_{\Omega}\sum_{j=1}^{m}q_j(x,t)u(x,t-\tau_j)dx. \tag{6.22}$$

Using the definition of $U(t)$, $P_i(t)$, and $Q_j(t)$, we have

$$U^{'}(t)+\sum_{i=1}^{n}P_i(t)U(t-\sigma_i)-\sum_{j=1}^{m}Q_j(t)U(t-\tau_j)\leq 0,\quad t\geq t_1. \tag{6.23}$$

Set

$$z(t) = U(t) - \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\sigma_i}^{t-\tau_k} Q_k(s+\tau_k)U(s)ds,\ t \geqslant t_2 = t_0 + \sigma_i - \tau_k. \quad (6.24)$$

Then from (6.23)

$$z^{'}(t) + \sum_{i=1}^{p} [P_i(t) - \sum_{k \in J_i} Q_k(t + \tau_k - \sigma_i)]U(t - \sigma_i) \le 0, \quad t \ge t_2. \quad (6.25)$$

Thus from (H3) we see that $z(t)$ is eventually strictly decreasing. Then an easy contradiction argument using (H4) and Lemma 6.2.1 guarantees that $z(t)$ is eventually positive. This implies that $U(t) \ge z(t)$. This with (6.25) yields that $z(t)$ is a positive solution of the delay differential inequality

$$z^{'}(t) + \sum_{i=1}^{p} \left[P_i(t) - \sum_{k \in J_i} Q_k(t + \tau_k - \sigma_i) \right] z(t - \sigma_i) \le 0, \quad t \ge t_2. \quad (6.26)$$

Now the usual standard result (see [30]) guarantees that (6.20) has an eventually positive solution which contradicts the assumption that every solution of (6.20) oscillates. The proof is complete. ∎

6.2.2 Oscillation of the Dirichlet Problem

In this subsection, we will establish some sufficient conditions for the oscillation of all solutions of (6.16), (6.18). For the following Dirichlet problem in the domain Ω

$$\Delta u + \alpha u = 0, \qquad \text{in } (x,t) \in \Omega \times [t_1, \infty), \quad (6.27)$$

$$u = 0, \qquad \text{on } (x,t) \in \partial\Omega \times [t_1, \infty), \quad (6.28)$$

in which α is a constant, it is well known that the smallest eigenvalue α_1 of problem (6.27) and (6.28) is positive and the corresponding eigenfunction $\Phi(x)$ is also positive on $x \in \Omega$. With each solution $u(x,t)$ of problem (6.16), (6.18) we associate a function $V(t)$ defined by

$$V(t) = \int_{\Omega} u(x,t)\Phi(x)dx, \quad t \ge t_1.$$

Theorem 6.2.2. *Assume that* $(H1) - (H4)$ *hold, and every solution of*

$$z'(t) + \sum_{i=1}^{p} \left[\overline{P_i}(t) - \sum_{k \in J_j} \overline{Q}_k(t + \tau_k - \sigma_i) \right] z(t - \sigma_i) = 0, \tag{6.29}$$

oscillates, where

$$\overline{P_i}(t)(t) = P_i(t) \exp(\alpha_1 \int_{t-\sigma_i}^{t} a(s)ds), \ \overline{Q}_k(t) = Q_j(t) \exp(\alpha_1 \int_{t-\tau_i}^{t} a(s)ds).$$

Then every solution of (6.16), (6.18) is oscillatory in G.

Proof. Assume that (6.16), (6.18) has a nonoscillatory solution. Since the negative solution of (6.16), (6.18) is also a solution, then without loss of generality we assume that (6.16), (6.18) has a solution $u(x,t) > 0$, $u(x, t - \sigma_i) > 0$, and $u(x, t - \tau_j) > 0$ in $\Omega \times [t_1, \infty)$ for some $t_1 \geq t_{0.}$ Multiplying (6.16) by $\Phi(x)$ and integrate with respect to x over the domain Ω, we have

$$\begin{aligned} \frac{d}{dt}\left[\int_\Omega u(x,t)\Phi(x)dx \right] &= a(t) \int_\Omega \Delta u(x,t)\Phi(x)dx \\ &\quad - \int_\Omega \sum_{i=1}^{n} p_i(x,t) u(x, t - \sigma_i)\Phi(x)dx \\ &\quad + \int_\Omega \sum_{j=1}^{m} q_j(x,t) u(x, t - \tau_j)\Phi(x)dx). \end{aligned} \tag{6.30}$$

Using Green's formula and boundary condition (6.18), we obtain

$$\begin{aligned} \int_\Omega \Delta u(x,t)\Phi(x)dx &= \int_{\partial\Omega} \left(\Phi(x)\frac{\partial u}{\partial N} - u\frac{\partial \Phi(x)}{\partial N} \right) dS + \int_\Omega u(x,t)\Delta\Phi(x)dx \\ &= -\alpha_1 \int_\Omega u(x,t)\Phi(x)dx, \quad t \geq t_1, \end{aligned}$$

where dS is the surface element on $\partial\Omega$. From the definitions of $V(t)$, $P_i(t)$, and $Q_j(t)$, we get

$$V'(t) + \alpha_1 a(t) V(t) + \sum_{i=1}^{n} P(t) V(t - \sigma_i) - \sum_{j=1}^{m} Q_j(t) V(t - \tau_j) \leq 0, \quad t \geq t_1. \tag{6.31}$$

Set

$$V(t) = v(t)\exp(-\alpha_1 \int_{t_0}^{t} a(s)ds),$$

which reduces inequality (6.31) to

$$v'(t) + \sum_{i=1}^{n} \overline{P}_i(t)v(t-\sigma) - \sum_{j=1}^{m} \overline{Q}_j(t)v(t-\tau_j) \le 0, \quad t \ge t_1. \tag{6.32}$$

Set

$$z(t) = v(t) - \sum_{i=1}^{p} \sum_{k \in J_j} \int_{t-\sigma_i}^{t-\tau_k} \overline{Q}_k(s+\tau_k)v(s)ds, \; t \geqslant t_2 = t_0 + \sigma_i - \tau_k. \tag{6.33}$$

Then as in Theorem 6.2.1, we have

$$z'(t) + \sum_{i=1}^{p} [\overline{P}_i(t) - \sum_{k \in J_i} \overline{Q}_k(t + \tau_k - \sigma_i)] z(t-\sigma_i) \le 0, \quad t \ge t_2. \tag{6.34}$$

The reminder of the proof is similar to that of Theorem 6.2.1 and will be omitted. ■

6.2.3 Oscillation of the Rodin Problem

In this subsection, we establish some sufficient conditions for the oscillation of all solutions of (6.16) and (6.19)

Theorem 6.2.3. *Assume that* $(H1)-(H4)$ *hold, and every solution of*

$$z'(t) + \sum_{i=1}^{p} [P_i(t) - \sum_{k \in J_i} Q_k(t + \tau_k - \sigma_i)] z(t-\sigma_i) = 0, \tag{6.35}$$

oscillates. Then every solution of (6.16), (6.19) is oscillatory in G.

Proof. Assume that (6.16), (6.19) has a nonoscillatory solution. Since the negative solution of (6.16), (6.19) is also a solution, then without loss of generality we assume that (6.16), (6.19) has a solution $u(x,t) > 0$, $u(x,t-\sigma_i) > 0$, and $u(x,t-\tau_j) > 0$ in $\Omega \times [t_1, \infty)$ for some $t_1 \ge t_0$. Set

$$U(t) = \int_{\Omega} u(x,t)dx, \quad t \ge t_1,$$

then $U(t) > 0$ for $t \geq t_1$. Integrating (6.16) with respect to x over the domain Ω, we have

$$\frac{d}{dt}[\int_\Omega u(x,t)dx] = a(t)\int_\Omega \Delta u(x,t)dx - \int_\Omega \sum_{i=1}^{n} p_i(x,t)u(x,t-\sigma_i)dx$$
$$+ \int_\Omega \sum_{j=1}^{m} q_j(x,t)u(x,t-\tau_j)dx. \tag{6.36}$$

From Green's formula and the boundary condition (6.19), it follows that

$$\int_\Omega \Delta u(x,t)dx = -\int_{\partial\Omega} \nu u dS \leq 0, \quad t \geq t_1,$$

where dS is the surface element on $\partial\Omega$. Then (6.36) reduces to

$$\frac{d}{dt}\left[\int_\Omega u(x,t)dx\right] + \int_\Omega \sum_{i=1}^{n} p_i(x,t)u(x,t-\sigma_i)dx$$
$$-\int_\Omega \sum_{j=1}^{m} q_j(x,t)u(x,t-\tau_j)dx \tag{6.37}$$
$$= -\int_{\partial\Omega} \gamma u dS \leq 0,$$

and by using the definition of $U(t)$, $P_i(t)$, and $Q_j(t)$ as above and substituting in (6.37), we have

$$U'(t) + \sum_{i=1}^{n} P_i U(t-\sigma_i) - \sum_{j=1}^{m} Q_j U(t-\tau_j) \leq 0, \quad t \geq t_1. \tag{6.38}$$

The reminder of the proof is similar to that of Theorem 6.2.1 and will be omitted. ■

6.3 Oscillation of an Autonomous Logistic Model

In this section, we are concerned with the oscillation of the diffusive delay logistic model

$$\frac{\partial N(x,t)}{\partial t} = D\frac{\partial^2 N(x,t)}{\partial x^2} + rN(x,t)\left[1 - \frac{N(x,t-\tau)}{K}\right], \tag{6.39}$$

for $t > 0$ and $x \in (0, l)$, supplemented with the homogeneous Neumann-type boundary conditions

$$\frac{\partial N(0,t)}{\partial x} = 0 = \frac{\partial N(l,t)}{\partial x}, \quad t \geq -\tau, \tag{6.40}$$

and the initial condition

$$N(x,s) = \varphi(x,s), \quad \varphi(x,0) > 0, \quad x \in [0,l], \quad s \in [-\tau, 0], \tag{6.41}$$

where φ is a assumed to be suitably smooth. Our aim is to establish some sufficient conditions for the oscillation of all positive solutions about K. The results in this section are adapted from [29].

A function $N(x,t)$ defined on $[0,l] \times [-\tau, T)$ is said to be a classical solution of the initial boundary value problem (6.39), (6.40), and (6.41) if N is continuously differentiable in t on $(0,T)$, twice continuously differentiable on x on $(0,l)$ and N satisfies Eqs. (6.39), (6.40), and (6.41) in a pointwise sense. If $T = \infty$, then the solution N is called a globally defined classical solution of (6.39), (6.40) and (6.41). In fact one can show that solutions of (6.39), (6.40), and (6.41) remain nonnegative for all $t \geq 0$, $x \in [0,l]$. Suppose this is not the case. There exist $t_0 \geq 0$ and $x_0 \in [0,l]$ such that $N(x_0, t_0) < 0$. Let

$$(x,t) \in \Gamma = \{(x,t);\ (x,t) \in [0,l] \times [0,t_0]\},$$

and define m by

$$m(x,t) := N(x,t)e^{-\lambda t}, \tag{6.42}$$

where $\lambda > 0$ will be suitably selected below. It follows from (6.39) and (6.42) that

$$\frac{\partial m(x,t)}{\partial t} = D\frac{\partial^2 m(x,t)}{\partial x^2} + m(x,t)\left[r - \lambda - r\frac{N(x,t-\tau)}{K}\right]. \tag{6.43}$$

Let

$$L = \frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}, \quad h(x,t) = r - \lambda - r\frac{N(x,t-\tau)}{K}.$$

Then (6.43) can be written as follows:

$$L[m] = h(x,t)m(x,t).$$

If λ is chosen large enough, then the coefficients of m in (6.43) can be made negative for $t \in [0,t_0]$ and $x \in [0,l]$. By the continuity of $m(x,t)$ on Γ, $m(x,t)$ must have a negative minimum in Γ, say at (x^*, t^*). Hence by the parabolic maximum principle, we know that $(x^*, t^*) \notin \Gamma$, since otherwise, $m(x,t)$ is a constant for $(x,t) \in [0,l] \times [0,t^*]$ which is impossible from (6.42) and (6.43). From the initial condition (6.41), we have

$$(x^*, t^*) \notin \{(x,t);\ 0 < x < l,\ \ t = 0\},$$

and hence (x^*, t^*) must belong to G where

$$G = \{(x,t) : 0 \le t \le t_0,\ x = 0,\ x = l\}.$$

Let us suppose that $x^* = 0$ and $m(0,t^*)$ is a negative minimum of m on Γ. We have from the homogeneous Neumann boundary condition for N that $\partial m(0,t^*)/\partial x = 0$. The slope of the curve $m(x,t^*)$ is either concave upward or horizontal when $x \in [0,\delta]$ for $\delta > 0$ with t^* fixed, and as a consequence, for any $\varepsilon > 0$, we can find a $\delta_\varepsilon > 0$ such that

$$\frac{\partial^2 m(x,t^*)}{\partial x^2} \ge 0, \quad \frac{\partial m(x,t^*)}{\partial t} < \varepsilon,\ m(x,t^*) < 0, \quad \text{for } x \in (0,\delta_\varepsilon). \tag{6.44}$$

We note that $x^* = 0$ and h is negative on Γ, and hence for a suitable choice of $\lambda > 0,\ h$ has a negative maximum on Γ. We choose a positive number ε_0 such that

$$4\varepsilon_0 = m(x^*, t^*) \max_{(x,t)\in\Gamma} h(x,t).$$

Then from the continuity of $m(x,t)$, there exists a δ_0 such that

$$m(x,t^*)h(x,t^*) \ge m(x,t^*) \max_{(x,t)\in\Gamma} h(x,t) \ge 2\varepsilon_0, \quad \text{for}\ \ x \in (0,\delta_0).$$

Let $\delta_1 = \min\{\delta_0, \delta_{\varepsilon_0}\}$. From (6.43) and (6.44), we have

$$\begin{aligned}\varepsilon_0 > \frac{\partial m(x,t^*)}{\partial t} &= D\frac{\partial^2 m(x,t^*)}{\partial x^2} + m(x,t^*)\left[r - \lambda - r\frac{N(x,t^*-\tau)}{K}\right]\\ &\ge m(x,t^*)h(x,t^*) \ge 2\varepsilon_0, \quad \text{for } x \in (0,\delta_1).\end{aligned}$$

This is impossible and so m cannot have a negative minimum at $x = 0$. A similar analysis can be used to show that m cannot have a negative minimum at $x = l$. We conclude that $m \ge 0$ for $x \in [0,l]$ and $t \ge -\tau$. It now follows from (6.42) that it is impossible for $N(x,t)$ to become negative for $x \in [0,l]$ and $t \ge 0$. We also remark that if $N(x,0) > 0$ for $x \in [0,l]$, then in fact $N(x,t) > 0$, for $t > 0$ and $x \in [0,l]$.

For convenience we let $u(x,t) = N(x,t)/K - 1$ and note u is governed by

$$\left.\begin{aligned}&\tfrac{\partial u(x,t)}{\partial t} = D\tfrac{\partial^2 u(x,t)}{\partial x^2} - r(1 + u(x,t))u(x,t-\tau),\ t > 0;\ x \in (0,l),\\ &\left.\tfrac{\partial u(x,t)}{\partial x}\right|_{x=0} = \left.\tfrac{\partial u(x,t)}{\partial x}\right|_{x=l}, \qquad\qquad t \ge -\tau.\end{aligned}\right\} \tag{6.45}$$

In order to study the oscillation of (6.39) and (6.40) about K it suffices to investigate similar characteristics of the trivial solution of (6.45). We note that the positivity of N of (6.39) implies that any solution of (6.45) satisfies

$$1 + u(x,t) > 0, \quad \text{for } x \in (0,l), \quad t > 0. \tag{6.46}$$

To prove the main results, we need the following two lemmas.

Lemma 6.3.1. *Let* $f : [t_0,\infty) \to [0,\infty)$ *be continuously differentiable on* (t_0,∞) *such that*

$$f \in L_1[t_0,\infty) \text{ and } \frac{df}{dt} \in L_1[t_0,\infty).$$

Then

$$\lim_{t\to\infty} f(t) = 0.$$

Proof. Since $\frac{df}{dt} \in L_1[t_0,\infty)$, for every $\varepsilon > 0$ there exists a positive number T such that for $t_1,\ t_2$ and $t_2 > t_1 > T$

$$\left|\int_{t_1}^{\infty} \frac{df}{dt} dt\right| < \varepsilon/2, \quad \left|\int_{t_2}^{\infty} \frac{df}{dt} dt\right| < \varepsilon/2.$$

We have

$$\begin{aligned} |f(t_2) - f(t_1)| &= \left|\int_{t_1}^{t_2} \frac{df}{dt} dt\right| = \left|\int_{t_1}^{\infty} \frac{df}{dt} dt - \int_{t_2}^{\infty} \frac{df}{dt} dt\right| \\ &\le \left|\int_{t_1}^{\infty} \frac{df}{dt} dt\right| + \left|\int_{t_2}^{\infty} \frac{df}{dt} dt\right| < \varepsilon. \end{aligned}$$

It follows that $\lim_{t\to\infty} f(t)$ exists, and this together with the facts

$$f(t) \ge 0 \text{ and } f \in L_1[t_0,\infty)$$

implies that $\lim_{t\to\infty} f(t) = 0$. The proof is complete. ∎

Lemma 6.3.2. *If* Q *is differentiable function defined on* $[0,\infty)$ *such that both the limits*

$$\lim_{t\to\infty} Q(t) \text{ and } \lim_{t\to\infty} \frac{dQ(t)}{dt}$$

exist, then

$$\lim_{t\to\infty} \frac{dQ(t)}{dt} = 0.$$

Proof. Suppose that the result is not true and that $\lim_{t\to\infty} \frac{dQ(t)}{dt} = c \neq 0$. If $c > 0$, then there exists a $T > 0$ such that $dQ(t)/dt > (c/2)$ for $t > T$. This implies that

$$Q(t) - Q(T) > c/2(t - T).$$

This yields that $Q(t) \to \infty$ as $t \to \infty$ which is a contradiction. By a similar argument one can deduce that $\lim_{t\to\infty} Q(t) = -\infty$, which again is a contradiction. The proof is complete. ■

In the following, we prove that every nonoscillatory solution of (6.39) and (6.40) converges to K and we establish some sufficient conditions for the oscillation of all positive solutions about K.

Theorem 6.3.1. *Let D, r, τ, l be positive numbers. If the boundary value problem (6.39) and (6.40) has a solution (say N) which is nonoscillatory (eventually positive or negative) about K, then*

$$N(x,t) \to K, \quad \text{as } t \to \infty, \text{ uniformly in } x \in (0,l). \tag{6.47}$$

Proof. To prove that (6.47) holds it is sufficient to prove that every nonoscillatory solution of (6.45) satisfies

$$u(x,t) \to 0, \quad \text{as } t \to \infty, \text{ uniformly in } x \in (0,l).$$

Suppose u is an eventually positive solution of (6.45) (if u is eventually negative the proof is similar). There exists a $T^* > 0$ such that

$$u(x,t) > 0, \quad \text{for } t > T^* \text{ and } x \in (0,l).$$

Define v as follows:

$$v(t) = \int_0^l u(x,t)dx, \quad t > T^* + \tau. \tag{6.48}$$

Then from (6.45), we have

$$\frac{dv(t)}{dt} = D\int_0^l \frac{\partial^2 u(x,t)}{\partial x^2}dx - r\int_0^l (1 + u(x,t))u(x,t-\tau)dx. \tag{6.49}$$

Using the boundary condition in (6.45), we get

$$\frac{dv(t)}{dt} = -r\int_0^l (1 + u(x,t))u(x,t-\tau)dx < 0, \quad \text{for } t > T^* + \tau. \tag{6.50}$$

Since $v(t) > 0$ for $t > T^* + \tau$, it follows from (6.50) that $v(t) \to v^* \geq 0$ as $t \to \infty$. We note from (6.49) that

$$v(t) - v(T^* + \tau) + r \int_{T^*+\tau}^{t} \left(\int_0^l (1 + u(x,s))u(x,s-\tau)dx \right) ds = 0. \quad (6.51)$$

Since $\lim_{t\to\infty} v(t)$ exists, we can conclude from (6.51) that

$$\lim_{t\to\infty} r \int_{T^*+\tau}^{t} \left(\int_0^l (1 + u(x,s))u(x,s-\tau)dx \right) ds \text{ exists,}$$

and therefore

$$\lim_{t\to\infty} r \int_0^t \left(\int_0^l (1 + u(x,s))u(x,s-\tau)dx \right) ds \text{ exists.} \quad (6.52)$$

For convenience, we define m as

$$m := \frac{1}{l} \left[\int_0^l u(x,0)dx + \int_0^t \int_0^l f(x,s)ds \right], \quad (6.53)$$

where

$$f(x,t) = -r(1 + u(x,t))u(x,t-\tau), \ x \in (0,l), \ \ t > 0. \quad (6.54)$$

It can be found from (6.52) to (6.54) that there exists a number m^* such that

$$\lim_{t\to\infty} m(t) = m^*. \quad (6.55)$$

Let $G(x,t,\zeta,s)$ denote the Green's function associated with the Neumann boundary condition for (6.45). Then any solution of (6.45) satisfies

$$u(x,t) = \begin{cases} \int_0^l G(x,t,\zeta,0)\phi(\zeta,0)d\zeta + \int_0^t \int_0^l G(x,t,\zeta,s) f(\zeta,s)d\zeta ds, & t > 0, \\ \phi(\zeta,s), & s \in [-\tau,0], \ \ x \in [0,l]. \end{cases}$$

Using (6.53), we then have for $x \in [0,l]$ and $t > 0$ that

$$\begin{aligned} u(x,t) - m(t) = & \int_0^l G\left[(x,t,\zeta,0) - \frac{1}{l}\right] u(\zeta,0)d\zeta \\ & + \int_0^t \int_0^l \left[G(x,t,\zeta,s) - \frac{1}{l}\right] f(\zeta,s)d\zeta ds. \end{aligned} \quad (6.56)$$

It is known that the Green function $G(x,t,\zeta,s)$ satisfies

$$\begin{cases} \left|G(x,t,\zeta,0)-\frac{1}{l}\right| \le c_1 e^{-c_2(t-s)}, & t-s\ge 1, \\ \left|G(x,t,\zeta,0)-\frac{1}{l}\right| \le c\left(\frac{1}{t-s}\right)^{1/2}, & t-s>0, \end{cases} \quad x\in(0,l),\ \zeta\in(0,l), \tag{6.57}$$

where $c_1,\ c_2$, and c are positive constants. It is easy from (6.57) to see that

$$\int_0^l G\left[(x,t,\zeta,0)-\frac{1}{l}\right]u(\zeta,0)d\zeta \to 0,\ \text{as } t\to\infty, \tag{6.58}$$

and the convergence in (6.58) is uniform in $x \in (0,l)$. Before we consider the limiting behavior as $t\to\infty$ of the second integral on the right-hand side of (6.56), we shall show that

$$\int_0^t |f(x,t)|\,dx \to 0, \quad \text{as } t\to\infty. \tag{6.59}$$

It is seen from the eventual positivity of u that if we define v as

$$v(t)=\int_0^l u(x,t)dx,\ \ t>0,$$

then for all large $t>0$

$$\frac{dv}{dt} = -r\int_0^l (1+u(x,t))u(x,t-\tau)dx \tag{6.60}$$

$$< -r\int_0^l u(x,t-\tau)dx < 0. \tag{6.61}$$

It follows from (6.60) and (6.61) that

$$\begin{cases} \int_0^l (1+u(x,t))u(x,t-\tau)dx \in L_1(0,\infty), \\ \int_0^l u(x,t)dx \in L_1(0,\infty). \end{cases} \tag{6.62}$$

Since $u>0$ eventually, we also have from (6.62) that

$$\int_0^l u(x,t)u(x,t-\tau)dx \in L_1(0,\infty). \tag{6.63}$$

We now let

$$P(t)=\int_0^l u^2(x,t)dx,\ \ t>0, \tag{6.64}$$

and note that

$$
\begin{aligned}
\frac{dP(t)}{dt} &= 2\int_0^l u(x,t)\frac{\partial u(x,t)}{\partial t}dx \\
&= 2\int_0^l u(x,t)\left[D\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)\right]dx \\
&= -2D\int_0^l \left(\frac{\partial u(x,t)}{\partial x}\right)^2 dx + 2\int_0^l u(x,t)f(x,t)dx < 0. \quad (6.65)
\end{aligned}
$$

We have from (6.65) that

$$
\begin{cases} \int_0^l \left(\frac{\partial u(x,t)}{\partial x}\right)^2 dx \in L_1(0,\infty), \\ \int_0^l u(x,t)[1+u(x,t)]u(x,t-\tau)dx \in L_1(0,\infty). \end{cases} \quad (6.66)
$$

If F is defined by

$$
F(t) = r\int_0^l [1+u(x,t)]u(x,t-\tau)dx, \quad t>\tau, \quad (6.67)
$$

then from (6.62), we conclude that $F \in L_1(0,\infty)$. We now establish that

$$
\left|\frac{dF(t)}{dt}\right| \in L_1(0,\infty). \quad (6.68)
$$

By direct calculation, we have from (6.67), (6.45), and (6.54) that

$$
\begin{aligned}
\frac{dF(t)}{dt} &= r\int_0^l u(x,t-\tau)\left\{D\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)\right\}dx \quad (6.69) \\
&\quad + r\int_0^l [1+u(x,t)]\left\{D\frac{\partial^2 u(x,t-\tau)}{\partial x^2} + f(x,t-\tau)\right\}dx. \quad (6.70)
\end{aligned}
$$

After simplifying, we have

$$
\begin{aligned}
\frac{dF(t)}{dt} &= -rD\int_0^l \frac{\partial u(x,t)}{\partial x}\frac{\partial u(x,t-\tau)}{\partial x}dx + r\int_0^l u(x,t-\tau)f(x,t)dx \\
&\quad -rD\int_0^l \frac{\partial u(x,t)}{\partial x}\frac{\partial u(x,t-\tau)}{\partial x}dx \\
&\quad + r\int_0^l [1+u(x,t)]f(x,t-\tau)dx, \quad (6.71)
\end{aligned}
$$

and this leads to

$$\frac{dF(t)}{dt} = -2rD \int_0^l \frac{\partial u(x,t)}{\partial x} \frac{\partial u(x,t-\tau)}{\partial x} dx + r \int_0^l u(x,t-\tau) f(x,t) dx$$
$$+ r \int_0^l [1 + u(x,t)] f(x,t) dx. \tag{6.72}$$

We can now obtain that

$$F(t) + \int_T^t \left[r \int_0^l u(x,s-\tau) |f(x,s)| \, dx + r \int_0^l [1+u(x,s)] \, |f(x,s)| \, dx \right] ds$$
$$\leq F(T) - 2rD \int_T^t \int_0^l \frac{\partial u(x,s)}{\partial x} \frac{\partial u(x,s-\tau)}{\partial x} dx ds$$
$$\leq F(T) + Dr \int_T^t \left[\int_0^l \left(\frac{\partial u(x,s)}{\partial x} \right)^2 + \left(\frac{\partial u(x,s-\tau)}{\partial x} \right)^2 dx \right] ds < \infty. \tag{6.73}$$

It follows from (6.73) that

$$\begin{cases} \int_0^l (1 + u(x,t)) \, |f(x,t)| \, dx \in L_1(0,\infty), \\ \int_0^l u(x,t-\tau) \, |f(x,t)| \, dx \in L_1(0,\infty). \end{cases} \tag{6.74}$$

As a consequence of (6.72), (6.74) we see that (6.68) holds. Thus both F and $\frac{dF}{dt}$ belong to $L_1(0,\infty)$. By Lemma 6.3.1 it follows that

$$F(t) = r \int_0^l [1+u(x,t)] u(x,t-\tau) dx = \int_0^l |f(x,t)| \, dx \to 0, \text{ as } t \to \infty. \tag{6.75}$$

To investigate the asymptotic behavior of the second integral on the right-hand side of (6.56), we let $w(t)$ be defined by

$$w(t) = \int_0^t \int_0^l \left[G(x,t,\zeta,s) - \frac{1}{l} \right] f(\zeta,s) d\zeta ds \tag{6.76}$$

and proceed to estimate $w(t)$ as follows:

$$|w(t)| \leq \int_0^T \int_0^l \left| G(x,t,\zeta,s) - \frac{1}{l} \right| |f(\zeta,s)| \, d\zeta ds$$
$$+ \int_T^{t-1} \int_0^l \left| G(x,t,\zeta,s) - \frac{1}{l} \right| |f(\zeta,s)| \, d\zeta ds$$
$$+ \int_{t-1}^t \int_0^l \left| G(x,t,\zeta,s) - \frac{1}{l} \right| |f(\zeta,s)| \, d\zeta ds. \tag{6.77}$$

Using the properties of G in (6.57), we have

$$\begin{aligned}
&\int_{t-1}^{t}\int_{0}^{l}\left|G(x,t,\zeta,s)-\frac{1}{l}\right||f(\zeta,s)|\,d\zeta ds\\
&\le \int_{t-1}^{t}\int_{0}^{l} c\left(\frac{1}{t-s}\right)^{1/2}|f(\zeta,s)|\,d\zeta ds\\
&\le c\int_{0}^{1}\left(\frac{1}{\alpha}\right)^{1/2}\left(\int_{0}^{l}|f(\zeta,t-\alpha)|\,d\zeta\right)d\alpha \to 0, \qquad (6.78)
\end{aligned}$$

as $t \to \infty$ by using (6.75). For an arbitrary $\varepsilon > 0$,

$$\begin{aligned}
&\int_{T}^{t-1}\int_{0}^{l}\left|G(x,t,\zeta,s)-\frac{1}{l}\right||f(\zeta,s)|\,d\zeta ds\\
&\le c_1\int_{T}^{t-1}\int_{0}^{l} e^{-c_2(t-s)}|f(\zeta,s)|\,d\zeta ds\\
&\le c_1\int_{T}^{t}\int_{0}^{l}|f(\zeta,s)|\,d\zeta ds < \varepsilon, \text{ if } T \text{ is sufficiently large.} \qquad (6.79)
\end{aligned}$$

By using (6.75) and L' Hospital's rule, we have as $t \to \infty$ that

$$\begin{aligned}
&\int_{0}^{T}\int_{0}^{l}\left|G(x,t,\zeta,s)-\frac{1}{l}\right||f(\zeta,s)|\,d\zeta ds\\
&\le c_1\int_{0}^{T}\int_{0}^{l} e^{-c_2(t-s)}|f(\zeta,s)|\,d\zeta ds\\
&\le \frac{c_1\int_0^t e^{c_2 s}\int_0^l|f(\zeta,s)|\,d\zeta ds}{e^{c_2 t}} \to 0. \qquad (6.80)
\end{aligned}$$

Thus each of the three integrals on the right side of (6.77) can be made arbitrarily small for large enough t by a suitable choice of T and this leads to

$$\lim_{t\to\infty} w(t) = \int_0^t\int_0^l\left[G(x,t,\zeta,s)-\frac{1}{l}\right]f(\zeta,s)d\zeta ds = 0. \qquad (6.81)$$

It follows from (6.56), (6.58) and (6.81) that

$$u(x,t) \to m(t), \text{ as } t \to \infty \text{ uniformly in } x \in (0,l).$$

We know that

$$m(t) \to m^*, \quad \text{as } t \to \infty, \qquad (6.82)$$

and therefore

$$u(x,t) \to m^*, \quad \text{as } t \to \infty \text{ uniformly in } x \in (0,l). \tag{6.83}$$

From (6.53) and (6.83), we have

$$\begin{aligned}\frac{dm(t)}{dt} &= \frac{-r}{l}\int_0^l (1+u(x,t))u(x,t-\tau)dx \\ &\to \frac{-r}{l} l(1+m^*)m^*, \quad \text{as } t \to \infty.\end{aligned} \tag{6.84}$$

Thus

$$m(t) \to m^* \text{ and } \frac{dm(t)}{dt} \to \frac{-r}{l} l(1+m^*)m^*, \quad \text{as } t \to \infty, \tag{6.85}$$

and Lemma 6.3.2 implies that

$$\frac{dm(t)}{dt} \to 0, \quad \text{as } t \to \infty, \tag{6.86}$$

which leads to

$$(1+m^*)m^* = 0. \tag{6.87}$$

Since $m^* \geq 0$, it follows from (6.87) that $m^* = 0$ and thus we have

$$u(x,t) \to m(t) \to m^* = 0 \text{ as } t \to \infty \text{ uniformly in } x \in (0,l).$$

This implies that $N(x,t) \to K$ as $t \to \infty$. The proof is complete. ■

The following result is well known [30].

Lemma 6.3.3. *Let p and τ be positive constants, and let $z(t)$ be an eventually positive solution of $z'(t)+pz(t-\tau) \leq 0$. Then for t sufficiently large $z(t-\tau) < \beta z(t)$ where $\beta = 4/(p\tau)^2$.*

Theorem 6.3.2. *Let D, r, τ, l be positive constants and suppose that all positive solutions of*

$$\frac{dy(t)}{dt} = ry(t)\left[1 - \frac{y(t-\tau)}{K}\right] \tag{6.88}$$

are oscillatory about the positive equilibrium K. Then all positive solutions of (6.39) and (6.40) are oscillatory about the positive equilibrium of (6.39).

Proof. It is known that a necessary and sufficient condition for the oscillation of all positive solutions of (6.88) about the positive equilibrium is that the equation

$$\lambda = -re^{-\lambda\tau}$$

has no real roots. This is equivalent to

$$re^{\lambda\tau} > \lambda \ \text{ for all } \lambda \in (0, \infty). \tag{6.89}$$

It follows from (6.89) that there exists a $\mu > 0$ such that

$$re^{\lambda\tau} > \lambda + \mu, \ \text{ for all } \lambda > 0. \tag{6.90}$$

It is enough to show that when (6.89) holds all solutions of (6.45) and (6.46) are oscillatory. Suppose that u is an eventually positive solution of (6.45) and (6.46) (if u is an eventually negative solution of (6.45) and (6.46) the proof is similar). Since u is not oscillatory we have by Theorem 6.3.1 that

$$r[1 + u(t)] \to r, \ \text{as } t \to \infty.$$

Thus, for $0 < \varepsilon < r$, there exists a T such that

$$r + \varepsilon > r[1 + u(t)] > r - \varepsilon = p_0, \ \text{ for } t > T. \tag{6.91}$$

We have from (6.45) to (6.46) that

$$\int_0^l u(x,s)dx = -r \int_0^l (1 + u(x,s))u(x, s - \tau)dx. \tag{6.92}$$

Let

$$v(t) := \int_0^l u(x,s)dx. \tag{6.93}$$

Define Λ_v of real numbers as follows:

$$\Lambda_v = \{\lambda \geq 0 : v'(t) + \lambda v(t) \leq 0, \ \text{eventually for large } t \geq 0\}. \tag{6.94}$$

We note from (6.93) that $\lambda = 0 \in \Lambda_v$ and that Λ_v is a subinterval of $[0,\infty)$. The proof of the theorem will be completed by showing that the set Λ_v has the following contradictory properties P_1 and P_2.

$P_1 :$ Λ_v is bounded;
$P_2 : \lambda \in \Lambda_v \Rightarrow \lambda + \mu \in \Lambda_v$.

Let us first verify P_1. We have from (6.91) and (6.92) that for $t > T + \tau$

$$\frac{dv}{dt} + p_0 v(t-\tau) \le 0, \tag{6.95}$$

which from Lemma 6.3.3 implies that

$$v(t-\tau) \le \frac{4}{(p_0\tau)^2} v(t). \tag{6.96}$$

We have from (6.92)

$$0 = \int_0^l u(x,s)dx + r\int_0^l (1+u(x,s))u(x,s-\tau)dx, \tag{6.97}$$

so

$$0 \le v^{'}(t) + (r+\varepsilon)v(t-\tau) \le v^{'}(t) + (r+\varepsilon)\frac{4}{(p_0\tau)^2}v(t). \tag{6.98}$$

From (6.98) the set Λ_v is bounded above which verifies P_1.

In order to verify (P_2), let $\lambda \in \Lambda_v$ and set

$$\varphi(t) = e^{\lambda t} v(t), \tag{6.99}$$

and note that

$$\varphi'(t) = e^{\lambda t}[v^{'}(t) + \lambda v(t)] \le 0, \text{ eventually for large } t. \tag{6.100}$$

This shows that $\varphi(t)$ is positive and eventually nonincreasing and that

$$\begin{aligned}
& v^{'}(t) + (\lambda+\mu)v(t) \\
&= -r\int_0^l (1+u(x,s))u(x,s-\tau)dx + (\lambda+\mu)v(t) \\
&\le -r\int_0^l u(x,s-\tau)dx + (\lambda+\mu)v(t) \\
&\le -re^{-\lambda(t-\tau)}\phi(t-\tau) + (\lambda+\mu)e^{-\lambda\tau}\phi(t) \\
&\le e^{-\lambda t}\phi(t)\left[-re^{\lambda\tau} + \lambda + \mu)\right] \le 0,
\end{aligned}$$

and hence it follows that $\lambda + \mu \in \Lambda_v$ showing that P_2 holds.

Note P_1 and P_2 are mutually contradictory. Thus (6.45) cannot have eventually positive solutions. The proof is complete. ■

Corollary 6.3.1. *If all solutions of the linear delay differential equation*

$$\frac{dy(t)}{dt} = -ry(t-\tau)$$

are oscillatory, then all solutions of the (6.39) and (6.40) are oscillatory about K.

6.4 Oscillation of a Nonautonomous Logistic Model

In this section we discuss the oscillation of the diffusive logistic model with several delays

$$\frac{\partial N(x,t)}{\partial t} = d(t)\Delta N(x,t) + c(t)N(x,t)\left[a(t) - \sum_{i=1}^{n} b_i(t)N(x,t-\tau_i(t))\right], \tag{6.101}$$

where $(x,t) \in \Omega \times (0,\infty)$, Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, Δ is the Laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2},$$

and $\tau_i(t)$, $1 \le i \le n$, are positive continuous functions defined on $[0,\infty)$, $a(t)$, $c(t)$, $d(t)$, $b_1(t)$, $b_2(t), \ldots, b_n(t)$ are positive, bounded, and continuous functions on $[0,\infty)$ and $0 < d_0 \le d(t)$, $0 < b_0 \le c(t)b_i(t)$ for some $i \in \{1,2,\ldots,n\}$.

We consider boundary conditions of the form

$$\begin{cases} \frac{\partial N(x,t)}{\partial \mu} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ N(x,t) = \phi(x,t), & (x,t) \in \partial\Omega \times [-\tau, 0], \end{cases} \tag{6.102}$$

where μ is a the outward unit normal vector, $\phi(x,t)$ is a nonnegative and nontrivial continuous function, and $\tau = \max_i\{\max_t\{\tau_i(t)\}\}$. We will assume that there is a positive constant N^* such that

$$\sum_{i=1}^{n} b_i(t)N^* = a(t), \quad t \ge 0, \tag{6.103}$$

so that $N(x,t) = N^*$ is a stationary solution of (6.101). We establish some sufficient conditions for the oscillation of all positive solutions of the boundary value problem (6.101) and (6.102) about N^*. The results in this section are adapted from [79].

Existence and uniqueness theorems for solution of (6.101) and (6.102) follow from the existence of a unique "heat kernel" $g(x,t,\zeta,\nu)$ associated with the differential operator $L[N] = N_t = d(t)N_{xx}$ and the boundary condition (6.102). By means of this kernel (6.101) and (6.102) can be transformed into an integral equation which is well posed and can be solved by the method of steps.

By a solution of (6.101) and (6.102) we mean a function $N(x,t)$ which is continuously differentiable on the closure of $\Omega \times [-\tau, \infty)$ and twice continuously differentiable on $\Omega \times [-\tau, \infty)$. Let $N(x,t)$ be a real continuous function defined on $\bar{\Omega} \times [t_0, \infty)$.

Lemma 6.4.1. *Suppose that (6.101) and (6.102) has a positive solution $N(x,t)$ such that $N(x,t) - N^*$ is eventually positive. Then the first-order delay differential inequality*

$$y^{'}(t) \leq -\sum_{i=1}^{n} N^* c(t) b_i(t) y(t - \tau_i(t)) \tag{6.104}$$

has an eventually positive solution.

Proof. Suppose there is a positive number t_1 such that $N(x,t) - N^* > 0$ on $\Omega \times [t_1, \infty)$. For convenience, let

$$w(x,t) = N(x,t) - N^*.$$

Then from (6.101) and (6.103), we have

$$\frac{\partial w(x,t)}{\partial t} = d(t)\Delta w(x,t) - c(t)[w(x,t) + N^*] \sum_{i=1}^{n} b_i(t) w(x, t - \tau_i(t)). \tag{6.105}$$

Integrate both sides of (6.105) with respect to x to obtain

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} w(x,t)dx \\ &= d(t) \int_{\Omega} \Delta w(x,t)dx \\ &\quad -c(t) \int_{\Omega} [w(x,t) + N^*] \sum_{i=1}^{n} b_i(t) w(x, t - \tau_i(t))dx. \end{aligned} \tag{6.106}$$

By the Green formula and the boundary condition in (6.102), we obtain

$$\int_{\Omega} \Delta w(x,t)dx = \int_{\partial\Omega} \frac{w(x,t)}{\partial \mu} ds = 0. \tag{6.107}$$

Pick a number $t_2 > t_1 + \tau$. Then

$$w(x,t) > 0 \text{ and } w(x, t - \tau_i(t)) > 0, \text{ for } (x,t) \in \Omega \times [t_2, \infty).$$

In view of (6.106) and (6.107) we have

$$\frac{d}{dt}\int_{\Omega} w(x,t)dx \le -\sum_{i=1}^{n} c(t)N^* b_i(t)\int_{\Omega} w(x,t-\tau_i(t))dx,\ t \ge t_2.$$

Set

$$y(t) = \int_{\Omega} w(x,t)dx, \quad t \ge t_2.$$

Then $y(t)$ is an eventually positive solution of (6.104). The proof is complete. ■

Lemma 6.4.2. *Suppose that (6.101) and (6.102) has a positive solution $N(x,t)$ such that $N(x,t) - N^*$ is eventually negative. Then for any $\beta \in (0,1)$, the first-order delay differential inequality*

$$y'(t) \le -\sum_{i=1}^{n} \beta N^* c(t) b_i(t) y(t-\tau_i(t)) \tag{6.108}$$

has an eventually positive solution.

Proof. Suppose there is a positive number t_1 such that $N(x,t) - N^* < 0$ on $\Omega \times [t_1,\infty)$. For convenience, let

$$p(x,t) = \ln\left(\frac{N(x,t)}{N^*}\right) < 0,\ (x,t) \in \Omega \times [t_1,\infty).$$

We assert

$$y(t) = \int_{\Omega} -p(x,t)dx$$

is an eventually positive solution of (6.108). To prove this, note first that from (6.101) and (6.103) we have

$$\begin{aligned}\frac{\partial p(x,t)}{\partial t} &= d(t)e^{-p(x,t)}\Delta e^{p(x,t)} + c(t)\left[a(t) - \sum_{i=1}^{n} b_i(t)N^* e^{p(x,t-\tau_i(t))}\right] \\ &= d(t)e^{-p(x,t)}\Delta e^{p(x,t)} - N^* c(t)\sum_{i=1}^{n} b_i(t)\left[e^{p(x,t-\tau_i(t))} - 1\right].\end{aligned}$$

Integrate the last equality with respect to x over Ω and we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\Omega} p(x,t)dx &= d(t)\int_{\Omega} e^{-p(x,t)}\Delta e^{p(x,t)}dx \\ &\quad -N^* c(t)\sum_{i=1}^{n} b_i(t)\int_{\Omega}\left[e^{p(x,t-\tau_i(t))} - 1\right]dx.\end{aligned} \tag{6.109}$$

Since

$$\frac{\partial}{\partial \mu}\{e^{\pm p(x,t)}\} = \pm e^{\pm p(x,t)}\frac{\partial p(x,t)}{\partial \mu},\ (x,t) \in \Omega \times [t_1, \infty), \tag{6.110}$$

$$\left.\frac{\partial p(x,t)}{\partial \mu}\right|_{\partial\Omega} = \frac{1}{N(x,t)}\left.\frac{\partial N(x,t)}{\partial \mu}\right|_{\partial\Omega} = 0 \tag{6.111}$$

and

$$e^{p(x,t)}\Delta e^{-p(x,t)} = |\triangledown p(x,t)|^2 - \Delta p(x,t),\ (x,t) \in \Omega \times [t_1, \infty),$$

we obtain

$$\begin{aligned}\int_\Omega e^{-p(x,t)}\Delta e^{p(x,t)}dx &= \int_\Omega e^{p(x,t)}\Delta e^{-p(x,t)}dx \\ &= \int_\Omega |\triangledown p(x,t)|^2\,dx - \int_\Omega \Delta p(x,t)dx \\ &= \int_\Omega |\triangledown p(x,t)|^2\,dx - \int_{\partial\Omega} \frac{\partial p(x,t)}{\partial \mu} \\ &= \int_\Omega |\triangledown p(x,t)|^2\,dx. \end{aligned} \tag{6.112}$$

From (6.109) we also obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega -p(x,t)dx &= -d(t)\int_\Omega |\triangledown p(x,t)|^2\,dx \\ &\quad + N^*c(t)\sum_{i=1}^n b_i(t)\int_\Omega \left[e^{p(x,t-\tau_i(t))} - 1\right]dx \\ &\le N^*c(t)\sum_{i=1}^n b_i(t)\int_\Omega \left[e^{p(x,t-\tau_i(t))} - 1\right]dx. \end{aligned} \tag{6.113}$$

To complete the proof, it is suffices to show that for any $\beta \in (0, 1)$, there is some T_0 such that

$$\left[e^{p(x,t-\tau_i(t))} - 1\right] \le \beta p(x, t - \tau_i(t)),\ \ t \ge T_0.$$

In order to do this, pick $t_2 > t_1 + \tau$ so that

$$p(x,t) < 0,\ \text{and}\ \ p(x, t - \tau_i(t)) < 0 \text{ for } (x,t) \in \Omega \times [t_2, \infty).$$

For $(x,t) \in \Omega \times [t_2, \infty)$ consider the positive functional V defined by

$$V[p](t) := N^*\int_\Omega \int_0^{p(x,t)} (e^y - 1)dydx.$$

By (6.110)–(6.112), the derivative of V with respect to (6.101) satisfies

$$\begin{aligned}\frac{dV}{dt} &= N^* \int_\Omega (e^{p(x,t)} - 1)\frac{\partial p(x,t)}{\partial t}dx \\ &= N^* d(t)\int_\Omega \Delta e^{p(x,t)}dx - N^* d(t)\int_\Omega e^{-p(x,t)}\Delta e^{p(x,t)}dx \\ &\quad - \left(N^*\right)^2 c(t)\sum_{i=1}^n b_i(t)(e^{p(x,t)} - 1)\left[e^{p(x,t-\tau_i(t))} - 1\right]dx \\ &\le -N^* d(t)\int_\Omega |\nabla p(x,t)|^2 dx \\ &\quad - \left(N^*\right)^2 c(t)\sum_{i=1}^n b_i(t)\int_\Omega (e^{p(x,t)} - 1)\left[e^{p(x,t-\tau_i(t))} - 1\right]dx.\end{aligned}$$

We note from (6.101)

$$\begin{aligned}&\frac{d}{dt}\int_\Omega N(x,t)dx \\ &= d(t)\int_\Omega \Delta N(x,t)dx \\ &\quad + \int_\Omega c(t)N(x,t)\sum_{i=1}^n b_i(t)\left[N^* - N(x,t-\tau_i(t))\right]dx \\ &= \int_\Omega c(t)N(x,t)\sum_{i=1}^n b_i(t)\left[N^* - N(x,t-\tau_i(t))\right]dx \ge 0,\end{aligned}$$

and so

$$\int_\Omega (N(x,t) - N(x,t-\tau_i(t)))dx \ge 0,$$

which implies that

$$\begin{aligned}&\int_\Omega (e^{p(x,t)} - 1)\left[e^{p(x,t-\tau_i(t))} - 1\right]dx \\ &= \int_\Omega (e^{p(x,t)} - 1)^2 dx + \int_\Omega (e^{p(x,t)} - 1)\left[e^{p(x,t-\tau_i(t))} - e^{p(x,t)}\right]dx \\ &= \int_\Omega (e^{p(x,t)} - 1)^2 dx \\ &\quad + \frac{1}{(N^*)^2}\int_\Omega (N(x,t) - N^*)(N(x,t-\tau_i(t)) - N(x,t))dx \\ &\ge \int_\Omega (e^{p(x,t)} - 1)^2 dx,\end{aligned}$$

by the first mean-value theorem of integrals. As a consequence, we see that

$$\begin{aligned}
&\frac{dV}{dt}\\
&\leq -N^*d(t)\int_\Omega |\triangledown p(x,t)|^2\,dx - \left(N^*\right)^2 c(t)\sum_{i=1}^n b_i(t)\int_\Omega (e^{p(x,t)}-1)^2 dx\\
&\leq -N^*d(t)\int_\Omega |\triangledown p(x,t)|^2\,dx\\
&\quad -\left(N^*\right)^2 c(t)\sum_{i=1}^n b_i(t)\int_\Omega (N(x,t)-N^*)^2 dx, \qquad (6.114)
\end{aligned}$$

for $t \geq t_2$. Integrate both sides of (6.114) and recall the assumptions that $0 < d_0 \leq d(t)$ and $0 < b_0 \leq b_i(t)c(t)$, and we obtain

$$\begin{aligned}
V(t_2) \geq V(t) &+ N^*d_0\int_{t_2}^t\int_\Omega |\triangledown p(x,s)|^2\,dxds\\
&+ \left(N^*\right)^2 b_0\int_{t_2}^t\int_\Omega (N(x,s)-N^*)^2 dxds.
\end{aligned}$$

Hence, by writing $\left(\int_\Omega |\,.\,|^2\,dx\right)^{1/2} = \|.\|$ we have

$$\int_{t_2}^\infty\int_\Omega (N(x,s)-N^*)^2 dxds = \int_{t_2}^\infty \|N(x,s)-N^*\|^2\,ds < \infty,$$

and

$$\int_{t_2}^\infty\int_\Omega |\triangledown p(x,s)|^2\,dxds = \int_{t_2}^\infty \|\triangledown p(x,s)\|^2\,ds < \infty,$$

so that

$$\|N(x,s)-N^*\| \in L_1(0,\infty) \text{ and } \left\|\triangledown p(x,s)^2\right\| \in L_1(0,\infty).$$

But from the assumption that $N(x,t) < N^*$ for $(x,t) \in \Omega \times [t_1,\infty)$, we have

$$\frac{1}{(N^*)^2}\,\|\triangledown N(x,t)\|^2 \leq \|\triangledown p(x,t)\|^2,$$

so that $\|\triangledown N(x,t)\|^2 \in L_1(0,\infty)$. Now,

$$\begin{aligned}
&\frac{d}{dt}\,\|\triangledown N(x,t)\|^2\\
&= \left(\frac{\partial N(x,t)}{\partial t}, -\Delta N(x,t)\right.
\end{aligned}$$

$$= -d(t)\,\|\Delta N(x,t)\|^2 + \left(\nabla\left\{c(t)N(x,t)\left[a(t)-\sum_{i=1}^{n} b_i(t)N(x,t-\tau_i(t))\right]\right\}\cdot\nabla N(x,t)\right)$$

$$\le -d_0\,\|\Delta N(x,t)\|^2 + c(t)a(t)\,\|\nabla N(x,t)\|^2 + N^* c(t)\sum_{i=1}^{n} b_i\,\|\nabla N(x,t-\tau_i(t))\|\,\|\nabla N(x,t)\|, \tag{6.115}$$

where

$$|a(t)| \le a, |c(t)| \le a, |b_i(t)| \le b_i$$

are bounded functions. Integrate both sides of (6.115) from t_2 to T, and we obtain

$$\|\nabla N(x,T)\|^2 - \|\nabla N(x,t_2)\|^2 + d_0\int_{t_2}^{T}\|\Delta N(x,t)\|^2\,dt$$
$$\le ca\int_{t_2}^{T}\|\nabla N(x,t)\|^2\,dt + N^* c\sum_{i=1}^{n} b_i\int_{t_2}^{T}\|\nabla N(x,t-\tau_i(t))\|\,\|\nabla N(x,t)\|\,dt$$
$$\le ca\int_{t_2}^{T}\|\nabla N(x,t)\|^2\,dt + N^* c\sum_{i=1}^{n} b_i\left\{\int_{t_2}^{T}\|\nabla N(x,t-\tau_i(t))\|^2\,dt\right\}^{1/2}\left\{\int_{t_2}^{T}\|\nabla N(x,t)\|^2\,dt\right\}^{1/2}. \tag{6.116}$$

We may now infer from $\|\nabla N(x,t)\|^2 \in L_1(0,\infty)$ and the above inequality that $\|\Delta N(x,t)\|^2 \in L_1(0,\infty)$ and $\|\nabla N(x,t)\|^2$ is bounded on $[t_2,\infty)$. In a similar fashion, we obtain

$$\int_{t_2}^{T}\left|\frac{d}{dt}\|\nabla N(x,t)\|^2\right|dt$$
$$\le d_0\int_{t_2}^{T}\|\Delta N(x,t)\|^2\,dt + ca\int_{t_2}^{T}\|\nabla N(x,t)\|^2\,dt + N^* c\sum_{i=1}^{n} b_i\left\{\int_{t_2}^{T}\|\nabla N(x,t-\tau_i(t))\|^2\,dt\right\}^{1/2}\times\left\{\int_{t_2}^{T}\|\nabla N(x,t)\|^2\,dt\right\}^{1/2}.$$

Since $\|\bigtriangledown N(x,t)\|^2 \in L_1(0,\infty)$ and $\|\Delta N(x,t)\|^2 \in L_1(0,\infty)$, we may deduce the fact that $\frac{d}{dt}\|\bigtriangledown N(x,t)\|^2 \in L_1(0,\infty)$. Also we see

$$\lim_{t\to\infty}\int_\Omega |\bigtriangledown N(x,t)|^2\,dt = \lim_{t\to\infty}\|\bigtriangledown N(x,t)\|^2 = 0. \tag{6.117}$$

Furthermore, since

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|N(x,t)-N^*\|^2 \\
&= \int_\Omega (N(x,t)-N^*)\frac{\partial N(x,t)}{\partial t}dx \\
&= d(t)\int_\Omega (N(x,t)-N^*)\Delta N(x,t)dx \\
&\quad -\int_\Omega (N(x,t)-N^*)c(t)N(x,t)\sum_{i=1}^n b_i(t)(N(x,t-\tau_i(t))-N^*)dx,
\end{aligned}$$

we have

$$\begin{aligned}
&\left|\frac{1}{2}\frac{d}{dt}\|N(x,t)-N^*\|^2\right| \\
&\le d\int_\Omega |N(x,t)-N^*|\,|\Delta N(x,t)|\,dx \\
&\quad +c\sum_{i=1}^n b_i N^*\int_\Omega |N(x,t)-N^*|\,|N(x,t-\tau_i(t))-N^*|\,dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_2}^T \left|\frac{1}{2}\frac{d}{dt}\|N(x,t)-N^*\|^2\right| dt \\
&\le d\left\{\int_{t_2}^T \|N(x,t)-N^*\|^2\,dt\right\}^{1/2}\left\{\int_{t_2}^T \|\Delta N(x,t)\|^2\,dt\right\}^{1/2} \\
&\quad +c\sum_{i=1}^n b_i N^*\left\{\int_{t_2}^T \|N(x,t)-N^*\|^2\,dt\right\}^{1/2} \\
&\quad \times\left\{\int_{t_2}^T \|N(x,t-\tau_i(t))-N^*\|^2\,dt\right\}^{1/2}.
\end{aligned}$$

Consequently we have $\frac{d}{dt}\|N(x,t)-N^*\| \in L_1(0,\infty)$ in view of the previous facts that $\|\Delta N(x,t)\|^2 \in L_1(0,\infty)$ and $\|N(x,t)-N^*\|^2 \in L_1(0,\infty)$. Also we see that

$$\lim_{t\to\infty}\|N(x,t)-N^*\| = 0. \tag{6.118}$$

Next, from $N(x,t) < N^*$ for $t > t_2$ and the boundedness of $\|\triangledown N(x,t)\|$ on (t_2, ∞), we see that $\|N(x,t) - N^*\|_\infty$ and $\|\triangledown N(x,t)\|_\infty$ are bounded (where $\|w\|_\infty = ess \sup |w|$), and then from the inequality

$$\|w\|_\sigma \leq \|w\|_\infty^{(\sigma-2)/\sigma} \|w\|_2^{2/\sigma} \quad \text{for all } \sigma \geq 2$$

and (6.117) as well as (6.118), we obtain

$$\lim_{t\to\infty} \|N(x,t) - N^*\|_\sigma = \lim_{t\to\infty} \|\triangledown N(x,t)\|_\sigma = 0, \quad \sigma > m. \tag{6.119}$$

Next, from the Sobolev inequality

$$\|u\|_{L^q(\Omega)} \leq C \, \|\triangledown u\|_{L^p(\Omega)},$$

where C is a constant independent of u, we obtain

$$\|N(x,t) - N^*\|_\infty \leq M(\Omega, m, \sigma)\{\|N(x,t) - N^*\|_\sigma + \|\triangledown N(x,t) - N^*\|_\sigma\}, \tag{6.120}$$

where $M(\Omega, m, \sigma)$ is a positive constant independent of N. Finally, from (6.119) to (6.120), we see that $\|N(x,t) - N^*\|_\infty \to 0$ as $t \to \infty$, so that

$$\lim_{t\to\infty} N(x,t) = N^*, \quad \text{uniformly in } x \in \bar{\Omega},$$

which implies

$$\lim_{t\to\infty} p(x,t) = 0, \quad \text{uniformly in } x \in \bar{\Omega}. \tag{6.121}$$

Now, for any $t > t_2$ and $t_3 > t_2$

$$e^{p(x,t-\tau_i(t))} - e^{p(x,t_3)} = \{p(x, t - \tau_i(t)) - p(x, t_3)\}e^{p(x,\xi_i(t))},$$

where $e^{p(x,\xi_i(t))} \to 1$ as $t \to \infty$. Thus for any $\beta \in (0, 1)$, we can find a t_4 such that $\beta < e^{p(x,\xi_i(t))} < 1,\ t \geq t_4$, which implies

$$e^{p(x,t-\tau_i(t))} - 1 \leq \beta p(x, t - \tau_i(t)),$$

as required. The proof is complete. ■

6.5 Stability of an Autonomous Logistic Model

In this section, we will establish some sufficient conditions for all positive solutions of (6.39) and (6.40) to converge as $t \to \infty$ to the positive equilibrium of (6.39). The results in this section are adapted from [29].

Theorem 6.5.1. *Let p and q denote the solutions of the following:*

$$\begin{cases} \frac{dp(t)}{dt} = rp(t)\left[1 - \frac{p(t-\tau)}{K}\right], \\ p(s) = \max_{x\in[0,l]} N(x,s), \quad s \in [-\tau, 0], \end{cases} \tag{6.122}$$

$$\begin{cases} \frac{dq(t)}{dt} = rq(t)\left[1 - \frac{q(t-\tau)}{K}\right], \\ q(s) = \min_{x\in[0,l]} N(x,s), \quad s \in [-\tau, 0]. \end{cases} \tag{6.123}$$

Then every solution N of (6.39) and (6.40) satisfies

$$q(t) \le N(x,t) \le p(t), \quad t > 0, x \in [0,l]. \tag{6.124}$$

Proof. We shall prove that

$$N(x,t) \le p(t), \quad t > 0, x \in [0,l]. \tag{6.125}$$

Let $p_\varepsilon(t)$ denote the solution of

$$\begin{cases} \frac{dp_\varepsilon(t)}{dt} = rp_\varepsilon(t)\left[1 - \frac{p_\varepsilon(t-\tau)}{K}\right], \\ p_\varepsilon(s) = \max_{x\in[0,l]} N(x,s) + \varepsilon, \quad s \in [-\tau, 0], \end{cases} \tag{6.126}$$

where ε is an arbitrary positive number. It is sufficient to prove that

$$N(x,t) \le p_\varepsilon(t), \quad t > 0. \tag{6.127}$$

The result will then follow from (6.127) by the continuous dependence of solutions of (6.126) on the initial conditions and

$$N(x,t) \le \lim_{\varepsilon\to 0} p_\varepsilon(t) = p(t). \tag{6.128}$$

Suppose (6.127) does not hold. Then there exists $x_0 \in (0,l),\ t_0 > 0$ such that

$$p_\varepsilon(t_0) - N(x_0,t_0) < 0. \tag{6.129}$$

Define a function M as follows

$$M(x,t) = [p_\varepsilon(t) - N(x,t)]\, e^{-\lambda t}, \quad (x,t) \in [0,l] \times [0,t_0], \tag{6.130}$$

where λ is a positive number to be suitably selected. It follows from (6.129) that

$$M(x_0,t_0) < 0. \tag{6.131}$$

As a consequence, $M(x,t)$ will have a negative minimum on $[0,l]\times[0,t_0]$. Suppose that such a minimum of $M(x,t)$ occurs at (x^*,t^*) where $x^* \in (0,l),\ t^* \in (0,t_0]$. Then, we have

$$\left.\frac{\partial M(x,t)}{\partial t}\right|_{(x^*,t^*)} < 0, \quad \left.\frac{\partial^2 M(x,t)}{\partial x^2}\right|_{(x^*,t^*)} \geq 0.$$

From (6.39), (6.126), and (6.130) we have that

$$\begin{aligned}
&\frac{\partial M(x,t)}{\partial t} - D\frac{\partial^2 M(x,t)}{\partial x^2}\\
&= M(x,t)\left[r - \lambda - \frac{r}{K}p_\varepsilon(t-\tau)\right]\\
&\quad -\frac{r}{K}M(x,t)N(x,t)\left[\frac{p_\varepsilon(t-\tau-N(x,t-\tau)}{p_\varepsilon(t)-N(x,t)}\right].
\end{aligned} \tag{6.132}$$

We can choose λ large and positive such that

$$\left|r-\lambda-\frac{r}{K}p_\varepsilon(t^*-\tau)\right| > \left|\frac{r}{K}N(x^*,t^*)\frac{p_\varepsilon(t^*-\tau)-N(x^*,t^*-\tau)}{p_\varepsilon(t^*)-N(x^*,t^*)}\right|, \tag{6.133}$$

which is possible since the left side of (6.133) can be made arbitrarily large by a suitable choice of $\lambda > 0$. By choosing λ appropriately one can thus make the right side of (6.132) positive at (x^*, t^*) while the left side of (6.132) remains nonpositive at (x^*, t^*) and this is a contradiction. Hence it follows that an interior negative minimum of $M(x,t)$ cannot exist for $x \in (0, l)$. Since

$$M(x,0) > 0 \ \text{ and } \ \frac{\partial M(x,0)}{\partial x} = 0 \quad \text{at } x = 0,\ l,$$

and we can prove (as before) that M cannot have a negative minimum at the endpoints of the interval $(0,l)$. Hence it follows that M cannot have a negative minimum on the closed set $[0,l]\times[0,t_0]$ for $t_0 \geq 0$ from which (6.127) follows. The conclusion (6.125) is now a consequence of (6.128). The proof of the other half is similar and the details are omitted. ∎

Theorem 6.5.2. *Suppose that* $D \in [0,\infty)$, $\tau \in [0,\infty)$, $r \in (0,\infty)$. *If*

$$r\tau < 1, \tag{6.134}$$

then all positive solutions of the Neumann problem (6.39) and (6.40) satisfy

$$\lim_{t\to\infty} N(x,t) = K, \tag{6.135}$$

and convergence is uniform in $x \in [0,l]$.

Proof. In view of the result of Theorem 6.5.1, it is sufficient to show that all positive solutions of (6.88) satisfy

$$\lim_{t\to\infty} y(t) = K.$$

For convenience we let

$$y(t) = K[1 + w(t)]$$

to obtain

$$\frac{dw(t)}{dt} = -r[1 + w(t)]w(t - \tau). \tag{6.136}$$

It is sufficient to prove that $w(t) \to 0$ as $t \to \infty$. We consider here two cases.

Case (i). Suppose that w is a nonoscillatory solution and $|w(t)| > 0$ eventually and hence $|dw(t)/dt| > 0$ eventually for large t. It follows from $w(t) > 0$ and $dw(t)/dt < 0$ that $\lim_{t\to\infty} w(t)$ exists and this implies by using (6.136) that $\lim_{t\to\infty} dw(t)/dt$ exists. As a consequence if

$$\lim_{t\to\infty} w(t) = w^*,$$

then by Lemma 6.3.2

$$\lim_{t\to\infty} \frac{dw(t)}{dt} = 0 = r[1 + w^*]w^*,$$

and since $w^* \geq 0$, we conclude that $w^* = 0$. If w is eventually negative, a similar argument will show that

$$\lim_{t\to\infty} w(t) = 0.$$

Case (ii). Suppose that $w(t)$ is oscillatory. Let $\{u_n\}$ and $\{v_n\}$ $(n = 1, 2, \ldots)$ denote the respective magnitudes of the successive minima and maxima of w. One can derive from (6.136) that these sequences satisfy for $n = 2, 3, \ldots$

$$\begin{cases} 1 + v_{n+1} \leq e^{r\tau u_n}, \\ 1 - u_n \geq e^{-r\tau v_n}. \end{cases} \tag{6.137}$$

It follows from (6.137) that

$$v_{n+1} \leq \exp[r\tau(1 - e^{-r\tau v_n})] - 1. \tag{6.138}$$

We shall now show that $v_n \to 0$ as $n \to \infty$. Let us consider a map $V : [0, \infty) \to (-\infty, \infty)$ defined by

$$V(v) = \exp[r\tau(1 - e^{-r\tau v_n})] - 1. \tag{6.139}$$

We note that $V(0) = 0$ and by using (6.134) we have

$$\frac{dV(v)}{dv} = (r\tau)^2 \exp[r\tau(1 - v - e^{-r\tau v})] \leq (r\tau)^2. \tag{6.140}$$

It follows from (6.138) to (6.140) and by the mean-value theorem that there exists $\theta \in [0, v_n]$ such that

$$\begin{aligned} v_{n+1} &\le V(v_n) = V(v_n) - V(0) = v_n V'(\theta) \\ &\le (r\tau)^2 v_n \le (r\tau)^4 v_{n-1} \le \ldots \le (r\tau)^{2n+2} v_0 \to 0 \text{ as } n \to \infty. \end{aligned}$$

Since v_n denotes the magnitude of the sequence of maxima of the oscillatory solution w, it follows from

$$0 \le v_{n+1} \le (r\tau)^{2n+2} v_0$$

that

$$\lim_{n\to\infty} v_n = 0.$$

By (6.137) we have

$$0 \le u_n \le 1 - e^{-r\tau v_n},$$

and this implies that

$$\lim_{n\to\infty} u_n = 0.$$

Since the sequences of successive maxima and minima of the oscillatory solution w converge to zero, we can conclude that

$$\lim_{t\to\infty} w(t) = 0 \Longrightarrow \lim_{t\to\infty} p(t) = K = \lim_{t\to\infty} q(t). \qquad (6.141)$$

By Theorem 6.5.1

$$q(t) \le N(x,t) \le p(t), \quad t > -\tau, x \in [0, l],$$

and hence the result follows from (6.141). The proof is complete. ■

6.6 Global Stability of a Volterra-Type Model

Nonlinear periodic equations with diffusion arise naturally in population models where the birth and death rates, rates of diffusion, rates of interaction, and environmental carrying capacities are periodic on a seasonal scale. In this section, we are concerned with periodicity and global stability of the periodic parabolic logistic model with instantaneous and delay effects of Volterra-type of the form

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - Au(t,x) \\ = u(t,x)\left[a(t,x) - b(t,x)u(t,x) - \sum_{r=1}^{m} c_r(t,x)u(t-rT,x)\right], \\ \qquad (t,x) \in [0,\infty) \times \Omega, \\ B[u](t,x) = 0, \qquad (t,x) \in [0,\infty) \times \partial\Omega, \\ u(s,x) = u_0(s,x), \qquad (s,x) \in [-mT,0]) \times \Omega, \end{cases} \tag{6.142}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, and the differential operator A is defined by

$$Af(x) = \sum_{i,j=1}^{n} \alpha_{i,j}(t,x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{j=1}^{n} \beta_j(t,x)\frac{\partial f(x)}{\partial x_j}. \tag{6.143}$$

The results here are adapted from [24]. The system will be studied under the following assumptions:

(H_1) The coefficients $\alpha_{i,j}$ and β_j are Hölder continuous in x and t and T-periodic functions in t.
(H_2) The functions $a(t,x)$ and $b(t,x)$ are T-periodic in T, positive and Hölder continuous on $[0,\infty) \times \bar{\Omega}$.
(H_3) The functions $c_r(t,x)$ are nonnegative and Hölder continuous on $[0,\infty) \times \bar{\Omega}$, with $c(t,x) = \sum_{r=1}^{m} c_r(t,x)$ positive and T-periodic.

We also assume that

$$\begin{cases} B[u] = u, \\ B[u] = \dfrac{\partial u}{\partial \nu} + \gamma(x)u, \end{cases}, \tag{6.144}$$

with $\gamma \in C^{1+\alpha}(\partial\Omega)$ and $\gamma(x) \geq 0$ on $\partial\Omega$.

The corresponding periodic-parabolic boundary-value problem of (6.142) without delay is

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} - Av(t,x) = v(t,x)\left[a(t,x) - h(t,x)u(t,x)\right], & (t,x) \in [0,\infty) \times \Omega, \\ B[v](t,x) = 0, & (t,x) \in [0,\infty) \times \partial\Omega, \end{cases} \tag{6.145}$$

where $h(t,x) = b(t,x) + c(t,x)$ has been studied by Hess [31] and some sufficient condition for global stability of the periodic solution was established. These results are summarized in the following lemma.

Lemma 6.6.1. *The eigenvalue problem*

$$\frac{\partial \phi(t,x)}{\partial t} - A\phi(t,x) - a(t,x)\phi(t,x) = \sigma\phi(t,x), \quad (t,x) \in [0,\infty) \times \Omega,$$
$$B[\phi](t,x) = 0, \qquad (t,x) \in [0,\infty) \times \partial\Omega,$$
$$\phi \text{ is } T\text{-periodic},$$

has a principle eigenvalue σ_1 *with positive eigenfunction.*

(i). If $\sigma_1 \geq 0$, *then the trivial solution 0 is globally asymptotically stable in (6.144) with respect to every nonnegative initial condition.*
(ii). If $\sigma_1 < 0$, *then the problem (6.144) admits a positive T-periodic solution* $\theta(t,x)$ *which is globally asymptotically stable with respect to every nonnegative nontrivial initial function.*

We begin with the following comparison lemma for the delay system (6.142).

Lemma 6.6.2. *If there exists a pair of smooth functions* $\bar{U}$ *and* $\tilde{U}$ *(called upper and lower solutions of* U*) such that* $\bar{U} \geq \tilde{U}$ *on* $[-mT,\infty) \times \bar{\Omega}$ *and they satisfy the following inequalities:*

$$\begin{aligned}
&\frac{\partial \bar{U}(t,x)}{\partial t} - A\bar{U}(t,x)\\
&\geq \bar{U}(t,x)\left[a(t,x) - b(t,x)\bar{U}(t,x) - \textstyle\sum_{r=1}^{m} c_r(t,x)\bar{U}(t-rT,x)\right],\\
&\qquad (t,x) \in [0,\infty) \times \Omega,\\
&\tfrac{\partial \tilde{U}(t,x)}{\partial t} - A\tilde{U}(t,x)\\
&\leq \tilde{U}(t,x)\left[a(t,x) - b(t,x)\tilde{U}(t,x) - \textstyle\sum_{r=1}^{m} c_r(t,x)\tilde{U}(t-rT,x)\right],\\
&\qquad (t,x) \in [0,\infty) \times \Omega,\\
&B[\bar{U}](t,x) \geq 0 \geq B[\tilde{U}](t,x), \quad (t,x) \in [0,\infty) \times \partial\Omega,\\
&\bar{U}(s,x) \geq u_0(s,x) \geq \tilde{U}(s,x), \quad (s,x) \in [-mT,0] \times \Omega,
\end{aligned}$$

then the delay system (6.142) has a unique solution u *with* $\bar{U} \geq u \geq \tilde{U}$ *on* $[-mT,0] \times \bar{\Omega}$.

Theorem 6.6.1.

(i) If $\sigma_1 \geq 0$, *then the trivial solution 0 is globally asymptotically stable in (6.142) with respect to every nonnegative initial condition* u_0.
(ii) Let $L = max_{[0,T]\times\bar{\Omega}}[c(t,x)/b(t,x)]$. *If* $\sigma_1 < 0$ *and* $L < 1$ *then the positive* T*-periodic solution* $\theta(t,x)$ *is globally asymptotically stable in (6.142) with respect to every nonnegative nontrivial initial function* u_0.

Proof. To prove (i) we will use Lemmas 6.6.1 and 6.6.2. Let U^* be the nonnegative solution of the following parabolic problem:

$$\begin{aligned}
&\frac{\partial U^*(t,x)}{\partial t} - AU^*(t,x)\\
&= U^*(t,x)\left[a(t,x) - b(t,x)U^*(t,x)\right], \ (t,x) \in [0,\infty) \times \Omega,\\
&B[U^*](t,x) = 0, \qquad (t,x) \in [0,\infty) \times \partial\Omega,\\
&U^*(0,x) = u_0(0,x), \qquad x \in \Omega.
\end{aligned}$$

Define the function $\tilde{U}$ as $\tilde{U} = u_0$ on $[-mT,0] \times \bar{\Omega}$ and $\tilde{U} = U^*$ on $(0,\infty) \times \bar{\Omega}$. Then $(\tilde{U},0)$ is a pair of upper and lower solutions of the time delay system (6.142) on $[-mT, 0] \times \bar{\Omega}$. Therefore, by Lemma 6.6.2 there exists a unique solution of u for (6.142) with $0 \le u \le U^*$ on $[-mT, 0] \times \bar{\Omega}$. When $\sigma_1 \ge 1$, it follows from Lemma 6.6.1 that

$$\lim_{t\to\infty} \|u(t,.)\|_{C(\bar{\Omega})} = \lim_{t\to\infty} \|U^*(t,.)\|_{C(\bar{\Omega})} = 0.$$

In order to prove (ii), we assume that $\sigma_1 < 0$ and $L < 1$. Then $c(t,x) \le Lb(t,x)$ on $[0,\infty) \times \bar{\Omega}$. Hence we obtain on $[0,\infty) \times \bar{\Omega}$ that

$$c(t,x) \le \frac{L[b(t,x)+c(t,x)]}{(L+1)} \quad \text{and} \quad b(t,x) \ge \frac{[b(t,x)+c(t,x)]}{(L+1)}. \tag{6.146}$$

Denote by $U(t,x)$ the solution of (6.145) with initial data $u_0(0,x)$. From the nonnegativity of u, we have for $(t,x) \in [0,\infty) \times \Omega$ that

$$\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} - Au(t,x) \\
&\le u(t,x)\,[a(t,x) - b(t,x)u(t,x)] \\
&\le u(t,x)[a(t,x) - \frac{1}{L+1}(b(t,x)+c(t,x))u(t,x)].
\end{aligned}$$

By Lemma 6.6.1 and the comparison principle for parabolic equations, we have

$$u(t,x) \le (L+1)U(t,x), \quad \text{on } [0,\infty) \times \bar{\Omega}.$$

Therefore

$$\begin{aligned}
&\lim_{t\to\infty} \sup \|u(t,.) - (L+1)\theta(t,.)\|_{C(\bar{\Omega})} \\
&\le \lim_{t\to\infty} (L+1)\,\|U(t,.) - \theta(t,.)\|_{C(\bar{\Omega})} = 0.
\end{aligned} \tag{6.147}$$

For each $\epsilon > 0$, there exists a T_ϵ such that when $(t,x) \in (T_\epsilon, \infty) \times \Omega$ and for each $0 < \alpha < L+1$

$$\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} - Au(t,x) \\
&\ge u(t,x)\left[a(t,x) - b(t,x)u(t,x) - (L+1+\epsilon)\sum_{r=1}^{m} c_r(t,x)\theta(t-rT,x)\right] \\
&= u(t,x)\,[a(t,x) - b(t,x)u(t,x) - (L+1+\epsilon)c(t,x)\theta(t,x)] \\
&= u(t,x)\begin{bmatrix} a(t,x) - b(t,x)u(t,x) - \alpha c(t,x)\theta(t,x) \\ -(L+1+\epsilon-\alpha)c(t,x)\theta(t,x) \end{bmatrix}
\end{aligned}$$

$$\geq u(t,x)[a(t,x) - b(t,x)u(t,x) - \alpha c(t,x)\theta(t,x)$$
$$-\frac{L(L+1+\epsilon-\alpha)}{L+1}(b(t,x)+c(t,x))\theta(t,x)].$$

Then by a comparison argument we have $u(t,x) \geq U_1(t,x)$ on $[T_\epsilon, \infty) \times \bar{\Omega}$, where U_1 is the solution of the parabolic problem

$$\begin{cases} \dfrac{\partial U_1(t,x)}{\partial t} - AU_1(t,x) \\ = U_1(t,x)[a(t,x) - b(t,x)U_1(t,x) - \alpha c(t,x)\theta(t,x) \\ \qquad -\frac{L(L+1+\epsilon-\alpha)}{L+1}(b(t,x)+c(t,x))\theta(t,x)], \\ \qquad B[U_1](t,x) = 0 \quad (t,x) \in [T_\epsilon, \infty) \times \partial\Omega, \end{cases} \tag{6.148}$$

with $u(T_\epsilon, x) = U_1(T_\epsilon, x)$ in Ω. If there exists $0 < \alpha < 1$ such that

$$\frac{L(L+1+\epsilon-\alpha)}{L+1} = 1 - \alpha, \tag{6.149}$$

then it is known from Lemma 6.6.1 that $\alpha\theta(t,x)$ is a positive T-periodic solution of (6.148) which is globally asymptotically stable. Relation (6.149) is equivalent to $\alpha = 1 - L^2 - L\epsilon > 0$. The arbitrariness of ϵ implies that for $L < 1$

$$\liminf_{t\to\infty} \left\|u(t,.) - (1-L^2)\theta(t,.)\right\|_{C(\bar{\Omega})} \geq \lim_{t\to\infty} \left\|U(t,.) - (1-L^2)\theta(t,.)\right\|_{C(\bar{\Omega})} = 0. \tag{6.150}$$

Hence from (6.147) and (6.150) we have

$$\limsup_{t\to\infty} \|u(t,.) - (1+L)\theta(t,.)\|_{C(\bar{\Omega})} \leq 0,$$
$$\liminf_{t\to\infty} \left\|u(t,.) - (1-L^2)\theta(t,.)\right\|_{C(\bar{\Omega})} \geq 0. \tag{6.151}$$

Assume by induction that for some integer k

$$\limsup_{t\to\infty} \left\|u(t,.) - (1+L^{k-1})\theta(t,.)\right\|_{C(\bar{\Omega})} \leq 0,$$
$$\liminf_{t\to\infty} \left\|u(t,.) - (1-L^{k+1})\theta(t,.)\right\|_{C(\bar{\Omega})} \geq 0. \tag{6.152}$$

Then for any $\epsilon > 0$, there exists a T_ϵ such that

$$\frac{\partial u(t,x)}{\partial t} - Au(t,x)$$
$$\leq a(t,x)u(t,x) - b(t,x)u^2(t,x)$$
$$-(1-L^k-\epsilon)u(t,x)\sum_{r=1}^{m} c_r(t,x)\theta(t-rT,x)$$
$$= u(t,x)\left[a(t,x) - b(t,x)u(t,x) - (1-L^k-\epsilon)c(t,x)\theta(t,x)\right] \tag{6.153}$$

in $(T_\epsilon, \infty) \times \Omega$. Hence for any $\beta > 1$

$$\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} - Au(t,x) \\
&\le u(t,x)\left[a(t,x) - b(t,x)\left(1 - \frac{1-L^k-\epsilon}{\beta}\right)u(t,x)\right. \\
&\quad \left. -b(t,x)\left(\frac{1-L^k-\epsilon}{\beta}\right)u(t,x) - (1-L^k-\epsilon)c(t,x)\theta(t,x)\right] \\
&\le u(t,x)\left[a(t,x) - \frac{b(t,x)+c(t,x)}{L+1}\left(1 - \frac{1-L^k-\epsilon}{\beta}\right)u(t,x)\right. \\
&\quad \left. -b(t,x)\left(\frac{1-L^k-\epsilon}{\beta}\right)u(t,x) - (1-L^k-\epsilon)c(t,x)\theta(t,x)\right],
\end{aligned}$$

in $(T_\epsilon, \infty) \times \Omega$. Then by a comparison argument we have $u(t,x) \le U_2(t,x)$ on $[T_\epsilon, \infty) \times \overline{\Omega}$ where U_2 is the solution of the parabolic problem

$$\begin{aligned}
\frac{\partial U_2(t,x)}{\partial t} - AU_2(t,x) &= U_2(t,x)a(t,x) - \frac{b(t,x)+c(t,x)}{L+1} \\
&\quad \left(1 - \frac{1-L^k-\epsilon}{\beta}\right)U_2(t,x) \\
&\quad -b(t,x)\left(\frac{1-L^k-\epsilon}{\beta}\right)U_2(t,x) \\
&\quad -(1-L^k-\epsilon)c(t,x)\theta(t,x), \\
&\qquad \text{in } (T_\epsilon, \infty) \times \Omega, \\
B[U_2](t,x) = 0 \quad & (t,x) \in [T_\epsilon, \infty) \times \partial\Omega,
\end{aligned} \tag{6.154}$$

with $u(T_\epsilon, x) = U_2(T_\epsilon, x)$ in Ω. If there exists $\beta > 1$ such that

$$\frac{\beta}{L+1}\left(1 - \frac{1-L^k-\epsilon}{\beta}\right) + (1-L^k-\epsilon) = 1, \tag{6.155}$$

then it is known by Lemma 6.6.1 that $\beta\theta(t,x)$ is the positive solution of (6.154) which is globally asymptotically stable. The relation (6.155) is equivalent to $\beta = 1 + L^{k+1} + L\epsilon > 1$. Therefore, from the arbitrariness of ϵ we have

$$\begin{aligned}
&\lim_{t\to\infty} \sup \left\| u(t,.) - (1+L^{k+1})\theta(t,.)\right\|_{C(\overline{\Omega})} \\
&\le \lim_{t\to\infty} \left\| U_2(t,.) - (1+L^{k+1})\theta(t,.)\right\|_{C(\overline{\Omega})} \le 0.
\end{aligned} \tag{6.156}$$

Again for any $\epsilon > 0$, there exists a T_ϵ such that

$$\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} - Au(t,x) \\
&\geq a(t,x)u(t,x) - b(t,x)u^2(t,x) \\
&\quad -(1 + L^{k+1} + \epsilon)u(t,x)\sum_{r=1}^{m} c_r(t,x)\theta(t - rT, x) \\
&= u(t,x)\left[a(t,x) - b(t,x)u(t,x) - (1 + L^{k+1} + \epsilon)c(t,x)\theta(t,x)\right],
\end{aligned}$$

in $(T_\epsilon, \infty) \times \Omega$. Hence for any $0 < \delta < 1$ and $(t,x) \in (T_\epsilon, \infty) \times \Omega$, we have

$$\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} - Au(t,x) \\
&\geq u(t,x)\left[a(t,x) - b(t,x)u(t,x) - \delta c(t,x)\theta(t,x)\right. \\
&\quad \left. -(1 + L^{k+1} + \epsilon - \delta)c(t,x)\theta(t,x)\right] \\
&\geq u(t,x)\left[a(t,x) - b(t,x)u(t,x) - \delta c(t,x)\theta(t,x)\right. \\
&\quad \left. -\frac{L(1 + L^{k+1} + \epsilon - \delta)}{L+1}(b(t,x) + c(t,x))\theta(t,x)\right].
\end{aligned}$$

Then by a comparison argument we have $u(t,x) \geq U_3(t,x)$ on $[T_\epsilon, \infty) \times \bar{\Omega}$ where U_3 is the solution of the parabolic problem

$$\begin{aligned}
&\frac{\partial U_3(t,x)}{\partial t} - AU_3(t,x) \\
&= u(t,x)\left[a(t,x) - b(t,x)U_3(t,x) - \delta c(t,x)\theta(t,x)\right. \\
&\quad \left. -\frac{L(1 + L^{k+1} + \epsilon - \delta)}{L+1}(b(t,x) + c(t,x))\theta(t,x)\right] \\
&\qquad\qquad \text{in } (T_\epsilon, \infty) \times \Omega, \\
&B[U_3](t,x) = 0, \quad (t,x) \in [T_\epsilon, \infty) \times \partial\Omega,
\end{aligned} \tag{6.157}$$

with $u(T_\epsilon, x) = U_3(T_\epsilon, x)$ in Ω. If there exists $0 < \delta < 1$ such that

$$\delta + \frac{L(1 + L^{k+1} + \epsilon - \delta)}{L+1} = 1, \tag{6.158}$$

then it is known by Lemma 6.6.1 that $\delta\theta(t,x)$ is the positive solution of (6.157) which is globally asymptotically stable. The relation (6.155) is equivalent to $\delta = 1 - L^{k+2} - L\epsilon < 1$. Therefore, from the arbitrariness of ϵ we have

$$\begin{aligned}
&\lim_{t\to\infty} \inf \left\| u(t,.) - (1 - L^{k+2})\theta(t,.) \right\|_{C(\bar{\Omega})} \\
&\geq \lim_{t\to\infty} \left\| U_2(t,.) - (1 - L^{k+1})\theta(t,.) \right\|_{C(\bar{\Omega})} \geq 0.
\end{aligned}$$

The induction argument as above shows that relation (6.152) holds for any positive even integer k. Letting $k \to \infty$ in (6.152) yields

$$\lim_{t\to\infty} \sup \|u(t,.) - \theta(t,.)\| \to 0 \text{ uniformly on } \bar{\Omega}.$$

The proof is complete. ■

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