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Dynamic Games in Economics

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Preface

The theory of dynamic games provides important instruments for economic analysis. At the same time, the progress of this theory and the associated analytical and numerical methods is largely driven by problems arising in dynamic economic considerations. With the aim of promoting and facilitating the development of optimal control and dynamic games and their applications in economics, the Vienna University of Technology founded a conference series named "Viennese Workshop on Optimal Control, Dynamic Games and Nonlinear Dynamics" (abbreviated as VW), in which specialists in optimal control, dynamic games and dynamical systems gathered with economists, demographers, and social scientists. The research covered by the 12 VWs organized to date ranges from "strange and chaotic behavior" (in the first few workshops) to games involving stochastic dynamics that dominate the dynamic games component (in the last VW).

The present book originates from the most recent, 12th VW, held in Vienna between May 30th and June 2nd, 2012. However, the aim of the editors was to collect papers that present, together, a broader view of the state of the art of dynamic games in economics. Therefore, along with contributions of selected participants in the 12th VW, the book includes several additional contributions by specially invited distinguished scientists in the area. Each chapter consists of a single contribution (paper) and the chapters are ordered alphabetically according to the name of the first author.

The first chapter ("Robust Markov Perfect Equilibria in a Dynamic Choice Model with Quasi-hyperbolic Discounting") deals with intergenerational game setup with an infinite (countable) number of descendants (copies) of an agent as players against Nature in which each copy represents a generation. The utility of each generation depends on its own choice as well as on the utility of consumption of all descendants. Unlike existing publications, in which the transition probability function is completely known, in the present chapter this function depends on uncertain parameters. The chapter applies the concept of quasi-hyperbolic discounting to an infinite horizon stochastic game and proves two existence theorems for a robust Markov perfect equilibrium.

Chapter "Stochastic Differential Games and Intricacy of Information Structures" deals with the analysis of information structural problems in the context with two-player zero-sum stochastic dynamic games. First, the chapter recapitulates results for constructing saddle point equilibria (SPE) for stochastic games with a noisy measurement channel, especially through the concept of certainty equivalence. The innovative part of the chapter extends existing results for the construction of SPE to a case in which the noisy measurement channel fails intermittently. The general analysis of the information structural problem is illustrated by the complete solution of a two-stage zero-sum game.

The main contribution of chapter "Policy Interactions in a Monetary Union: An Application of the OPTGAME Algorithm" is the analysis of a small nonlinear two-country macroeconomic model of a monetary union in which the governments control the fiscal policy while the central bank controls the monetary policy (the central bank sets monetary instruments). It is assumed that the players have different objective functions and the conflict is analyzed using concepts of dynamic game theory. The chapter follows a numerical approach based on a previous study (forthcoming) in which the authors have described the algorithm OPTGAME. The algorithm proved to be flexible enough to accommodate several scenarios and four game strategies (one cooperative and three non-cooperative).

Chapter "The Dynamics of Lobbying Under Uncertainty: On Political Liberalization in Arab Countries" presents an extension of a topical lobbying differential game between a conservative elite and a reformist group by introducing uncertainties to the model; the conservative elite pushes against political liberalization in opposition to the reformist group. It applies a rarely used approach of differential games that introduces multiple equilibria in different kinds and through a different mechanism.

Chapter "A Feedback Stackelberg Game of Cooperative Advertising in a Durable Goods Oligopoly" analyses a deterministic infinite horizon hierarchical game, in which the manufacturer as the leader decides strategically what fraction of retailers' advertising expenditures will be recompensed/subsidized. The retailers, themselves, determine as followers their individual advertising strategy. Postulating durable goods the authors use the concept of feedback Stackelberg equilibrium to compute optimal advertising policies and subsidy rates for various setups, for example, in case of N identical or in case of two non-identical retailers. In the case of a retail channel with two retailers, the authors explore the impact of cooperative advertising on channel and supply chain coordination.

Chapter "Strategies of Foreign Direct Investment in the Presence of Technological Spillovers" focuses on the effects of technological spill-overs, generated by foreign direct investments, have on the evolution of the technology gap. More specifically, a differential game is employed to model the dynamic strategic interaction between two competing firms located in high and low-tech countries, respectively. Due to the highly non-linear structure, numerical methods are utilized to characterize the Markov perfect equilibria of the game.

Chapter "Differential Games and Environmental Economics" provides a review of several publications (including such by the author) that aims to explain several concepts and techniques in the differential games and their applications to environmental and resource economics. However, it is more than a simple compilation of results. The chapter moves from the basics of differential games to recent scientific outcomes in the resolution of two very well-known examples: the game of international pollution and the lake game. The chapter presents the main questions and results in a unified framework. These examples are simple enough to have some analytical solutions, but rich enough to capture the principal techniques and the informational difficulties when solving differential games. The chapter makes it clear that differential games are not a simple and straightforward extension of optimal control problems to the case of several agents.

Chapter "Capacity Accumulation Games with Technology Constraints" considers a dynamic bilateral monopoly of two firms, one of which is the provider of input to the other, where the firms must work together to obtain surplus. Taking the strategy of the other firm into account, the firms decide on their own investment strategies in order to gain higher individual payoffs. A crucial point of the chapter is that, given overall technology constraints, technology interdependences are allowed. The authors investigate how different types of contracts (based on input quantities and on final revenues, respectively) affect efficiency and market power. In a framework of a linear-quadratic non-cooperative deterministic two-player dynamic game example, the authors numerically derive Markov perfect equilibria and point out the influence of the two types of long-term contracting.

Chapter "Dynamic Analysis of an Electoral Campaign" considers a deterministic differential game, in which political parties as players invest in order to maximize their individual aggregated benefits resulting from their particular patronage of voters over a finite planning horizon. The scientific work of this chapter investigates, which impact political parties' strategies and the number of political parties have on the social optimum. The social optimum is defined as to minimize the number of non-voters in a cooperative solution of the game. The main result is that a political party will have lesser votes in its noncooperative optimum than will have in the cooperative case, as long as its campaign is aggressive enough to destroy political rivals' consensus substantially. Further the chapter shows that in the social optimum (cooperative game) the optimal number of political parties is lower than the number of political parties that gain a positive share of consensus (votes) in the noncooperative game.

Chapter "Multi-agent Optimal Control Problems and Variational Inequality Based Reformulations" deals with multi-agent dynamic games, the novelty of which is that each player's cost functional and strategy set are dependent on her rivals' decisions. In this context, a publication in the journal *Mathematical Programming* studies a reformulation of the game as a system of differential equations constrained by parameterized variational inequalities, along with some boundary conditions. This chapter of the book extends this reformulation to stochastic multi-agent dynamic games in which the state dynamics is noisy.

Chapter "Time-Consistent Equilibria in a Differential Game Model with Time Inconsistent Preferences and Partial Cooperation" studies differential games with time-inconsistent preferences. Non-cooperative Markovian Nash equilibria are derived as a benchmark. Time-consistent solutions under partial cooperation—in which players can cooperate at every instant of time—are also obtained. Cooperation is partial in the sense that, although players cooperate at every moment t

forming a coalition, due to the time inconsistency of the time preferences, coalitions at different times value the future in a different way and are treated as different agents. Finally, Markovian subgame perfect equilibria in the cooperative sequential game are derived.

Chapter "Interactions Between Fiscal and Monetary Authorities in a Three-Country New-Keynesian Model of a Monetary Union" presents important issues concerning the macroeconomic policy coordination of fiscal (governmental) and monetary (central bank) authorities in the European Monetary Union in the presence of different types of economic shocks. The authors have used continuous-time linear-quadratic differential games based on a multi-country New-Keynesian monetary union framework to investigate strategic interactions of n heterogeneous countries that are both cooperative or in conflict with the (single) central bank. The novelty of this chapter is that the authors consider various types of coalitions including non-cooperative regimes, partial fiscal cooperations, full fiscal cooperation of all countries, and the grand coalition (including the central bank). Numerical simulations for different types of shocks reveal some interesting results, including unexpected main results and policy suggestions, and the fact that full cooperation without an appropriate transfer system is not a stable configuration.

The final chapter ("Subgame Consistent Cooperative Provision of Public Goods Under Accumulation and Payoff Uncertainties") deals with discrete-time dynamic games, in which both state dynamics and payoffs are uncertain. In detail, the authors consider noisy stock accumulation dynamics and derive subgame consistent cooperative solutions for n asymmetric players, who try to optimize distributed expected future payoffs gained from public goods. To ensure subgame perfect solutions upon optimality principle, the authors develop a suitable payoff distribution procedure.

We are confident that the material presented in this book will be appreciated by researchers and graduate students in applied mathematics and economics. For the latter group especially, we recommend chapters "Policy Interactions in a Monetary Union: An Application of the OPTGAME Algorithm," "A Feedback Stackelberg Game of Cooperative Advertising in a Durable Goods Oligopoly," "Strategies of Foreign Direct Investment in the Presence of Technological Spillovers," "Differential Games and Environmental Economics," "Dynamic Analysis of an Electoral Campaign," and the first two sections of chapter "Multi-agent Optimal Control Problems and Variational Inequality Based Reformulations."

Finally, we would like to thank all the contributors and referees for the time and the efforts they have devoted to this book.

Vienna, Austria

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Robust Markov Perfect Equilibria in a Dynamic Choice Model with Quasi-hyperbolic Discounting

Łukasz Balbus, Anna Jaśkiewicz, and Andrzej S. Nowak

Abstract A stochastic dynamic choice model with the transition probability depending on an unknown parameter is specified and analysed in this chapter. The main feature in our model is an application of the quasi-hyperbolic discounting concept to describe the situation in which agent's preferences may hinge on time. This requirement, in turn, leads to a non-cooperative infinite horizon stochastic game played by a countably many *selves* representing him during the play. As a result, we provide two existence theorems for a robust Markov perfect equilibrium (*RMPE*) and discuss its properties.

1 Introduction

In a number of real-life problems, the preferences of an economic agent change over time. Rational behaviour of such agents was analysed by Strotz (1956) and Pollak (1968), who considered so-called "consistent plans". In a related paper, Phelps and Pollak (1968) introduced the notion which is nowadays called "quasi-hyperbolic discounting" (Montiel Olea and Strzalecki 2014). This concept is a modification of the classical paradigm (discounted utility), proposed in 1937 by Samuelson (1937), that was extensively used in the analysis of intertemporal choice for a great deal of time (Stokey et al. 1989). Within such a framework an economic agent is represented by a sequence of *selves*, who play a non-cooperative dynamic game with appropriate defined payoff functions. A Markov perfect equilibrium in this game,

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if exists, constitutes a time-consistent, optimal (in a certain sense) plan for the agent. The meaning of the equilibrium is adequately explained in Maskin and Tirole (2001). The concept of capturing time-inconsistency, suggested by Phelps and Pollak (1968), was applied in various intergenerational games (Fudenberg and Tirole 1991), with altruism between generations. In such games it is assumed that each generation lives, saves and consumes over just one period. Moreover, each generation cares about the consumption of the following generations, in the sense that it wants to leave a bequest to the successors. Therefore, such generation derives utility from its own consumption and those of its descendants. The next generation's inheritance or capital is described by a certain production function. Dynamic games with quasi-hyperbolic discounting (or intergenerational games) have numerous applications in economics or management, for instance, see Di Corato (2012), Nowak (2006b, 2010) as well as Haurie (2005), Karp (2005), Krusell and Smith (2003), and Krusell et al. (2003).

The existence of a Markov perfect equilibrium in an intergenerational game with an uncountable state space is equivalent to the existence of a fixed point in an appropriately defined function space. This problem was successfully solved by Bernheim and Ray (1987), Leininger (1986), and Harris (1985) for certain classes of deterministic games. Some extensions to continuous-time models are given in Marín-Solano and Shevkoplyas (2011). The models with finite time horizon and production uncertainty, on the other hand, were first examined in Bernheim and Ray (1986). Furthermore, Amir (1996a), Nowak (2006a), and Balbus et al. (2012a) are among those who dealt with stochastic bequest games that possess specific transition structure. The results in these papers are formulated for intergenerational games, where each generation has finitely many descendants. The intergenerational games involving infinitely many descendants were studied by Alj and Haurie (1983) (with finite state space), Harris and Laibson (2001), Balbus and Nowak (2008), Jaśkiewicz and Nowak (2014), and Balbus et al. (2012b). In all the aforementioned works it is assumed that the transition probability function is completely known. A novel feature in this chapter is a dependence of transition probabilities on an unknown parameter. Then, the natural solution for such a model is a concept stemming from robust control theory, called a robust Markov perfect equilibrium. Roughly speaking, this solution is based upon the assumption that the players involved in the game are risk-sensitive and accept some kind of a maxmin utility. For the application of the concept of a robust Markov perfect equilibrium in various economic models the reader is referred to Gilboa and Schmeidler (1989), Hansen and Sargent (2001, 2003), Jaśkiewicz and Nowak (2011), Maccheroni et al. (2006), Strzalecki (2011) and references cited therein. In this chapter, we provide sufficient conditions for the existence of a *RMPE* in models with non-atomic transitions and in models where some atoms are admissible. The question of monotonicity of the functions determined by a *RMPE* is also addressed. A detailed discussion of our assumptions and their relationships with the conditions used in the literature is given in Remarks 1 and 2. Finally, we would like to emphasise that this paper, to the best of our knowledge, is a first study of a *RMPE* in stochastic games with quasi-hyperbolic discounting.

2 The Model and Main Results

2.1 The Dynamic Game with Quasi-hyperbolic Discounting

We start with some preliminaries and notation. Let R be the set of all real numbers and $\underline{R} = R \cup \{-\infty\}$. Let N denote the set of all positive integers. By a Borel space Y we mean a non-empty Borel subset of a complete separable metric space endowed with the Borel σ-algebra $\mathcal{B}(Y)$. We use $P(Y)$ to denote the space of all probability measures on Y endowed with the weak topology and the Borel σ-algebra, see Billingsley (1968) or Chap. 7 in Bertsekas and Shreve (1978). If Y is a Borel space, then $P(Y)$ is a Borel space too, see Corollary 7.25.1 in Bertsekas and Shreve (1978). Further, let us assume that X and Y are Borel spaces. A transition probability or a stochastic kernel from X to Y is a function $\varphi : \mathcal{B}(Y) \times X \mapsto [0, 1]$ such that $\varphi(B|\cdot)$ is a Borel measurable function on X for every $B \in \mathcal{B}(Y)$ and $\varphi(\cdot|x) \in P(Y)$ for each $x \in X$. It is well-known that every Borel measurable mapping $g : X \mapsto P(Y)$ induces a transition probability φ from X to Y. Namely, $\varphi(D|x) = g(x)(D)$, $D \in \mathcal{B}(Y)$, $x \in X$, see Proposition 7.26 in Bertsekas and Shreve (1978). We shall write $g(dy|x)$ instead of $g(x)(dy)$.

The set $B \subset X$ is universally measurable, if it is measurable with respect to every complete probability measure on X that measures all Borel subsets of X, i.e., it is measurable with respect to the σ-algebra $\mathcal{U} := \bigcap_{p\in P(X)} \mathcal{B}_p(X)$, where $\mathcal{B}_p(X)$ is the completion of $\mathcal{B}(X)$ with respect to p. We say that the function $f : X \mapsto \underline{R}$ is universally measurable, if $f^{-1}(B)$ is universally measurable in X for every $B \in \mathcal{B}(\underline{R})$.

Definition 1 The function $f : X \mapsto \underline{R}$ is lower semianalytic, if the set $\{x \in X : f(x) < c\}$ is analytic for each $c \in R$.

Since every analytic set is universally measurable, we conclude that any lower semianalytic function is universally measurable. For further properties of universally measurable and lower semianalytic functions and their applications, the reader is referred to Bertsekas and Shreve (1978) and Shreve and Bertsekas (1979).

Put $S := [0, \overline{s}]$ and $S_+ := (0, \overline{s}]$ for some fixed $\overline{s} > 0$. Let $\underline{a}(\cdot)$ and $\overline{a}(\cdot)$ be non-decreasing and continuous functions on S such that $\underline{a}(0) = \overline{a}(0) = 0$ and $0 \le \underline{a}(s) < \overline{a}(s) \le s$ for each $s \in S_+$. We set

$$A(s) := \left[\underline{a}(s), \overline{a}(s)\right] \quad \text{and} \quad \hat{A}(s) = \left[s - \overline{a}(s), s - \underline{a}(s)\right]$$

for each $s \in S$ and

$$D := \left\{(s, a) : s \in S, a \in A(s)\right\} \quad \text{and} \quad \hat{D} := \left\{(s, y) : s \in S, y \in \hat{A}(s)\right\}.$$

In a *dynamic choice model* with *quasi-hyperbolic preferences* and unknown transition probabilities, we envision an individual consumer as a sequence of autonomous temporal *selves*. These selves are indexed by the respective periods

$t \in T := N$ in which they make their consumption choices. More precisely, for a given state $s_t \in S$ at the beginning of t-th period, self t chooses a consumption level $a_t \in A(s_t)$ and the remaining part $y_t := s_t - a_t$ is invested for future *selves*. Self t's satisfaction is reflected in some way in an *instantaneous utility function* $u : S \mapsto \underline{R}$ that remains unchanged over all periods. Let Θ be a non-empty Borel subset of the Euclidean space R^m ($m \geq 1$). The next state s_{t+1} is determined by a transition probability q from $S \times \Theta$ to S and depends on $y_t \in \hat{A}(s_t)$ and a parameter $\theta_t \in \Theta$. This parameter is chosen according to a certain probability measure $\gamma_t \in \mathcal{P}$, where $\mathcal{P}$ denotes the action set of *Nature* and it is assumed to be a Borel subset of $P(\Theta)$.

Let Φ be the set of Borel measurable functions $\phi : S \mapsto S$ such that $\phi(s) \in A(s)$ for each $s \in S$. A *strategy* for self t is a function $c_t \in \Phi$. If $c_t = c$ for all $t \in T$ and some $c \in \Phi$, then we say that the selves employ a *stationary strategy*. The transition probability $q(\cdot|i(s), \xi)$ induced by q, any $c \in \Phi$ and $\xi \in \mathcal{P}$ is defined as follows

$$q\big(B|i(s), \xi\big) = \int_{\Theta} q\big(B|i(s), \theta\big)\xi(d\theta)$$

where $i(s) = s - c(s)$ and $B \in \mathcal{B}(S)$.

Let Γ be the set of all sequences $(\gamma_t)_{t \in T}$ of Borel measurable mappings $\gamma_t : D \mapsto \mathcal{P}$. For any $t \in T$ and $\gamma = (\gamma_t)_{t \in T} \in \Gamma$, we set $\gamma^t := (\gamma_\tau)_{\tau \geq t}$. Clearly, $\gamma^t \in \Gamma$. A *Markov strategy* for *Nature*[1] is a sequence $\gamma = (\gamma_t)_{t \in T} \in \Gamma$. Note that γ^t can be called a Markov strategy used by Nature from period t onwards.

For any $t \in T$, define H^t as the set of all sequences

$$h^t = (a_t, \theta_t, s_{t+1}, a_{t+1}, \theta_{t+1}, \ldots), \quad \text{where } (s_k, a_k) \in D \text{ and } k \geq t.$$

H^t is the set of feasible future histories of the process from period t onwards. Endow H^t with the product σ-algebra. Assume that $u \leq 0$ and is Borel measurable. Assume that the selves employ a stationary strategy $c \in \Phi$ and Nature chooses some $\gamma \in \Gamma$. Then the choice of Nature is a probability measure depending on $(s_t, c(s_t))$. The Ionescu–Tulcea theorem (see Proposition V.1.1 in Neveu 1965 or Chap. 7 in Bertsekas and Shreve 1978) guarantees the existence of a unique probability measure $P_{s_t}^{c,\gamma^t}$ on H^t induced by a stationary strategy $c \in \Phi$ used by each self τ ($\tau \geq t$), a Markov strategy of Nature $\gamma^t \in \Gamma$ and the transition probability q. Let $E_{s_t}^{c,\gamma^t}$ denote the expectation operator corresponding to the measure $P_{s_t}^{c,\gamma^t}$.

If self t knew γ, his expected utility would be

$$\hat{W}\big(c, \gamma^t\big)(s_t) := E_{s_t}^{c,\gamma^t}\left(u\big(c(s_t)\big) + \alpha\beta \sum_{\tau=t+1}^{\infty} \beta^{\tau-t-1} u\big(c(s_\tau)\big)\right),$$

where $\beta \in (0, 1)$ is a *long-run discount factor* and α ($\alpha > 0$) is a *short-run discount factor* (see Harris and Laibson 2001; Montiel Olea and Strzalecki 2014). Assuming

[1] One can allow for a general class of strategies for Nature, i.e., strategies that may depend on the history of the game. However, this extension does not change our main results, see Remark 3.

that in each period k ($k \geq t$) Nature chooses a probability γ_k from the set $\mathcal{P}$ with the objective of minimising self t's utility and that the choice of Nature may depend on a current state and consumption level performed by self k, we may accept an idea coming from robust dynamic programming programming (see for example Jaśkiewicz and Nowak 2011),[2] and say that the preferences of self t are represented by the following utility function

$$W(c)(s_t) := \inf_{\gamma^t \in \Gamma} \hat{W}\big(c, \gamma^t\big)(s_t).$$

This interpretation of a dynamic choice model with quasi-hyperbolic preferences provides an intuitive notion of ambiguity aversion, which can be regarded as the selves' diffidence for any lack of precise definition of uncertainty, something that provides room for the malevolent influence of Nature.

For any $c \in \Phi$, $\gamma \in \Gamma$, $j \geq 2$ and $s_j \in S$, put

$$\hat{J}\big(c, \gamma^j\big)(s_j) = E_{s_j}^{c,\gamma^j}\left(\sum_{\tau=j}^{\infty} \beta^{\tau-j} u\big(c(s_\tau)\big)\right); \tag{1}$$

then we have

$$\hat{W}\big(c, \gamma^t\big)(s_t) = u\big(c(s_t)\big) + \alpha\beta \int_{\Theta} \int_S \hat{J}\big(c, \gamma^{t+1}\big)(s_{t+1}) q\big(ds_{t+1}|s_t - c(s_t), \theta\big)\gamma_t\big(d\theta|s_t, c(s_t)\big).$$

By Theorem 1 in Shreve and Bertsekas (1979), the function $s_j \mapsto J(c)(s_j)$, defined by

$$J(c)(s_j) := \inf_{\gamma^j \in \Gamma} \hat{J}\big(c, \gamma^j\big)(s_j),$$

is universally measurable (lower semianalytic) on S. Using standard dynamic programming arguments (see Bertsekas and Shreve 1978 or Shreve and Bertsekas 1979), one can show that

$$W(c)(s_t) = u\big(c(s)\big) + \inf_{\xi \in \mathcal{P}} \alpha\beta \int_S J(c)(s_{t+1}) q\big(ds_{t+1}|s_t - c(s_t), \xi\big).$$

For any $s \in S$, $a \in A(s)$ and $c \in \Phi$, let us define

$$P(a, c)(s) = u(a) + \inf_{\xi \in \mathcal{P}} \alpha\beta \int_S J(c)\big(s'\big) q\big(ds'|s - a, \xi\big).$$

If $s = s_t$, then $P(a, c)(s)$ is the utility for self t choosing $a \in A(s_t)$ in this state when all future selves employ a stationary strategy $c \in \Phi$.

[2] Similar concepts have been also examined in Hansen and Sargent (2001, 2003), Maccheroni et al. (2006), and Strzalecki (2011).

Definition 2 A Robust Markov Perfect Equilibrium (*RMPE*) is a function $c^* \in \Phi$ such that for every $s \in S$ we have

$$\sup_{a \in A(s)} P\big(a, c^*\big)(s) = P\big(c^*(s), c^*\big)(s) = W\big(c^*\big)(s). \tag{2}$$

Note that (2) says that if the followers of any self t are going to employ c^*, then the best choice for him in state $s = s_t \in S$ is to choose $c^*(s)$.

The model considered in this chapter can be described in terms of intergenerational stochastic games as studied for example in Balbus and Nowak (2008) or Nowak (2010). Then every self t represents a short-lived generation. The utility of each generation t depends on its own choice and consumptions of all (infinitely many) descendants. The number $\alpha > 0$ is called an altruism factor towards following generations (descendants). The equilibrium concept is the same.

2.2 The Existence of Markov Perfect Equilibria

We now formulate our basic assumptions:

(A0) The functions $s \mapsto \underline{a}(s)$, $s \mapsto \overline{a}(s)$ and $s \mapsto s - \underline{a}(s)$, $s \mapsto s - \overline{a}(s)$ are non-decreasing and continuous.

(A1) $u \le 0$ is unbounded from below and $u(0) = -\infty$. Moreover u is strictly concave, increasing and continuous on S_+ and $\underline{a}(s) > 0$ for each $s \in S_+$.

(A2) $u \le 0$ is strictly concave, increasing and continuous on S and $\underline{a}(s) \ge 0$ for each $s \in S_+$.

(A3) There exist probability measures $\mu_1^\theta, \dots, \mu_l^\theta$ on S and functions $g_1, \dots, g_l : S \mapsto [0,1]$ such that

$$q(\cdot|s-a,\theta) = \sum_{k=1}^{l} g_k(s-a)\mu_k^\theta(\cdot),$$

where g_k are continuous and

$$\sum_{k=1}^{l} g_k(s-a) = 1 \quad \text{for all } (s,a) \in D.$$

(A4) There exist probability measures $\nu_1, \dots, \nu_l$ such that $\mu_k^\theta \ll \nu_k$ for each $k = 1, \dots, l$. In other words, each μ_k^θ has a density function $\widetilde{f}_k(\cdot,\theta)$. It is assumed that $\widetilde{f}_k(\cdot,\cdot)$ is Borel measurable on $S \times \Theta$. In addition, there exist functions $f_k : S \mapsto [0,\infty)$ such that

$$\widetilde{f}_k(\cdot,\theta) \le f_k(\cdot) \quad \text{and} \quad \int_S f_k(s)\nu_k(ds) < +\infty$$

for all $\theta \in \Theta$ and $k = 1, \dots, l$.

(A5) $\max_{1\le k\le l}\int_S |u(\underline{a}(s))| f_k(s)\nu_k(ds) < +\infty$.

Remark 1

(a) Let $S = [0, 1]$ and $u(a) = \log a$ for $a \in A(s)$, $s \in S$. Assume that ν_k is a uniform distribution on S and $0 < \varepsilon_1 \le \widetilde{f}_k(s', \theta) \le \varepsilon_2$ for all (s', θ). Let $c(s') = e^{-1/s'}$ for $s' \in S_+$. Then $\int_S u(c(s'))q(ds'|s - a, \theta) = -\infty$, if such consumption function is employed. This example explains the role of our assumptions (A1) and (A5). In the case of unbounded from below function u, one can observe that very small consumption may lead to minus infinite expectations calculated with respect to the transition probability. A typical example of $\underline{a}$ is a linear function, i.e., $\underline{a}(s) = \lambda s$, where $\lambda > 0$.

(b) Under condition (A0), D and $\hat{D}$ are complete lattices with the usual componentwise order.

(c) The density functions in (A4) are assumed to depend on $\theta \in R^m$. However, a specific function $\widetilde{f}_k(\cdot, \theta)$ may only hinge on some coordinates of θ. We avoid describing it for convenience of our notation.

(d) By adding a positive constant to u one can extend the results of this chapter to instantaneous utilities bounded from above.

Remark 2 The additivity assumption (A3) was extensively used in the study of Nash equilibria in standard stochastic games Nowak (2003) and stochastic bequest games Balbus et al. (2012a), and Nowak (2006a). Recently, Jaśkiewicz and Nowak in Jaśkiewicz and Nowak (2014) proposed a pretty general model of intergenerational stochastic game with additive transitions and risk-sensitive players. Related conditions together with additional stronger requirements were suggested in Balbus and Nowak (2008), and Balbus et al. (2012b). In particular, Balbus et al. (2012b) assume that μ_1 is a Dirac measure concentrated at zero, u is bounded, $u(0) = 0$ and the functions g_k are strictly concave and increasing. These assumptions, however, allow them to examine extra aspects of equilibria such as uniqueness or computational methods. Finally, it is worth mentioning that the transition probability function used by Harris and Laibson (2001) is of different type. Namely, they study a model whose dynamics evolves according to the following equation $s_{t+1} = \rho(s_t - a_t) + \theta_t$, where $\rho > 0$ and θ_t is a random shock that occurred at time t. Moreover, it is assumed that the sequence (θ_t) is i.i.d. and satisfies some boundedness condition. We would like to point out that all aforementioned papers deal with completely known transition functions, which correspond to the case with Θ being a singleton.

Remark 3 Let Σ^j be the set of general (history dependent) strategies used by Nature from period j onwards. Such a strategy $\sigma^j \in \Sigma^j$ is defined in a usual manner as in the discrete-time Markov control processes (see Bertsekas and Shreve 1978). The expected utility (1), for any $\sigma^j \in \Sigma^j$, is then also well-defined with the aid of the Ionescu–Tulcea theorem. By Theorem 3 in Shreve and Bertsekas (1979),

$$J(c)(s_j) = \inf_{\gamma^j \in \Gamma} \hat{J}\big(c, \gamma^j\big)(s_j) = \inf_{\sigma^j \in \Sigma^j} \hat{J}\big(c, \sigma^j\big)(s_j).$$

Hence, our results remain true, if we assume that Nature uses general strategies.

Let I denote the set of non-decreasing lower semicontinuous functions $\phi : S \mapsto R$ such that $\phi(s) \in \hat{A}(s)$ for each $s \in S$. Note that every $\phi \in I$ is continuous from the left and has a countable set of discontinuity points. Define

$$F := \big\{c \in \Phi : c(s) = s - i(s), i \in I\big\}. \tag{3}$$

Every $c \in F$ is upper semicontinuous and continuous from the left.

Our first main results concerns the model with non-atomic transitions.

Theorem 1 *Assume that either* (A0)–(A1), (A3)–(A5) *or* (A0), (A2)–(A4) *hold. If, in addition, the probability measures* $\nu_1, \dots, \nu_l$ *are non-atomic, then there exists a RMPE* $c^* \in F$.

Our second results allows for transitions having some atoms, but then we make some additional assumptions:

(C) The functions $g_2, \dots, g_l$ in (A3) are continuous, non-decreasing and concave. Moreover, $\mu_k^\theta \succ \mu_1^\theta$ for all $\theta \in \Theta$ and $k = 2, \dots, l$, that is, μ_k^θ (first order) stochastically dominates μ_1^θ.

Note that

$$g_1(s - a) = 1 - \sum_{k=2}^{l} g_k(s - a), \quad (s, a) \in D.$$

Recall that $\mu_k^\theta \succ \mu_1^\theta$ if and only if for any non-decreasing function $v : S \mapsto R$, having finite integrals with respect to every μ_j^θ, we have

$$\int_S v(s)\mu_k^\theta(ds) \geq \int_S v(s)\mu_1^\theta(ds).$$

If $c \in \Phi$, then $i \in \Phi$ is defined as $i(s) := s - c(s)$ for each $s \in S$. Define

$$FL := \{c \in \Phi : c \text{ and } i \text{ are non-decreasing}\}. \tag{4}$$

It is easy to see that FL consists of Lipschitz functions with constant one.

Here is our second main result.

Theorem 2 *Assume that either* (A0)–(A1), (A3)–(A5) *or* (A0), (A2)–(A4) *hold. If in addition* (C) *holds, then there exists a RMPE* $c^* \in FL$ *and the corresponding equilibrium functions are non-decreasing and continuous on* S_+ (S) *in the unbounded* (*bounded*) *case.*

Clearly, Theorem 2 can also be applied to the non-atomic case, but at the cost of an additional condition (C). However, in that case we obtain a stronger assertion concerning the monotonicity and continuity of both *RMPE* and the equilibrium utility functions.

Remark 4 Bernheim and Ray (1987) were the first who proposed the set F of strategies in the search of equilibria in deterministic bequest games. Their idea was successfully applied to the study of certain classes of dynamic games, for instance, see Dutta and Sundaram (1992). Lipschitz equilibria, on the other hand, were extensively examined in the following papers Amir (1996a, 1996b), Balbus et al. (2012a, 2012b), Curtat (1996), Harris and Laibson (2001), Jaśkiewicz and Nowak (2014), Leininger (1986), Nowak (2006a), and references cited therein.

3 Proofs

We recall that the function u is strictly concave and continuous on S_+. Let $w : S \mapsto R$ be a *continuous* function. Define

$$\hat{U}(s, y) := u(s - y) + w(y), \quad (s, y) \in \hat{D}.$$

Put

$$\hat{A}_0(s) := \arg\max_{y \in \hat{A}(s)} \hat{U}(s, y) \quad \text{and} \quad i_0(s) := \min \hat{A}_0(s), \quad s \in S.$$

Clearly, $\hat{A}_0(s) \neq \emptyset$ and compact for each $s \in S$ and $\hat{A}_0(0) = \{0\}$. Therefore the function i_0 is well-defined.

The following result is related to Theorem 6.3 in Topkis (1978).

Lemma 1 *Assume that* (A0) *holds. Then the correspondence* $s \mapsto \hat{A}_0(s)$ *has a closed graph and is strongly ascending, i.e., if* $s_1 < s_2$ *and* $y_1 \in \hat{A}_0(s_1)$, $y_2 \in \hat{A}_0(s_2)$, *then* $y_1 \leq y_2$. *Moreover, the function* i_0 *is lower semicontinuous and non-decreasing.*

Proof Suppose that the correspondence $s \mapsto \hat{A}_0(s)$ is not strongly ascending. Then there exist $s_1 < s_2$ and $y_1 \in \hat{A}_0(s_1)$, $y_2 \in \hat{A}_0(s_2)$ such that $y_1 > y_2$. Under assumption (A0), $\hat{D}$ is a lattice. Thus (s_2, y_1) and (s_1, y_2) belong to $\hat{D}$. Since u is strictly concave, from the arguments given in Lemma 2 in Nowak (2006a) or Lemma 0.2 in Amir (1996b), we obtain

$$u(s_2, y_1) - u(s_2, y_2) > u(s_1, y_1) - u(s_1, y_2). \tag{5}$$

Adding $w(y_1) - w(y_2)$ to both sides of (5) and knowing that $y_1 \in \hat{A}_0(s_1)$ and $y_2 \in \hat{A}_0(s_2)$, we obtain

$$0 \geq \hat{U}(s_2, y_1) - \hat{U}(s_2, y_2) > \hat{U}(s_1, y_1) - \hat{U}(s_1, y_2) \geq 0.$$

This contradiction implies that the correspondence $s \mapsto \hat{A}_0(s)$ is strongly ascending. Obviously, it has a closed graph. Thus, the function i_0 is non-decreasing and continuous from the left. Hence i_0 is lower semicontinuous. □

Remark 5 Lemma 1 does not directly follow from Theorem 6.3 in Topkis (1978). The reason is that we should know that $\hat{U}$ can be extended from $\hat{D}$ to a supermodular function on the product space $S \times S$. That is not true, for example, if $S = [0, 1]$ and $u(s, y) = \sqrt{s - y}$.

Lemma 2 *Assume that* (A0) *holds. Let* $\phi : S \mapsto S$ *be a non-decreasing function such that* $\phi(s) \in \hat{A}_0(s)$ *for each* $s \in S$*. If* $s_0 \in S_+$ *is a continuity point of* ϕ*, then* $\hat{A}_0(s_0)$ *is a singleton.*

Proof Suppose that y_1 and y_2 belong to $\hat{A}_0(s_0)$ and $y_1 < y_2$. Since $s \mapsto \hat{A}_0(s)$ is strongly ascending, we get that $\lim_{s \to s_0 -} \phi(s) \le y_1 < y_2 \le \lim_{s \to s_0 +} \phi(s)$. This contradicts our assumption that ϕ is continuous at the point $s_0 \in S_+$. □

Example 1 Let $S = [0, 1]$, $\underline{a}(s) = 0$ and $\overline{a}(s) = s$ for each $s \in S$. Let $u(s - y) = 2(s - y) - (s - y)^2$ and $w(y) = 2y^2$ for $(s, y) \in \hat{D}$. It is easy to verify that $\hat{A}_0(s) = \{0\}$ for $s \in [0, 2/3)$, $\hat{A}_0(s) = \{s\}$ for $s \in (2/3, 1]$ and $\hat{A}_0(2/3) = \{0, 2/3\}$. Hence $i_0(s) = 0$ for $s \in [0, 2/3]$ and $i_0(s) = s$ for $s \in (2/3, 1]$.

Assume now that w is *concave* and continuous on S. Then

$$\hat{A}_0(s) = \arg \max_{i \in \hat{A}(s)} \hat{U}(s, i) = \big\{ i_0(s) \big\} \quad \text{for every } s \in S.$$

Let $U(s, a) := u(a) + w(s - a)$ for all $(s, a) \in D$ and

$$A_0(s) := \arg \max_{a \in A(s)} U(s, a) \quad \text{for } s \in S.$$

Clearly, $A_0(s)$ is a singleton, so there exists a function $c_0 : S \mapsto S$ such that $A_0(s) = \{c_0(s)\}$ for each $s \in S$. Moreover from our concavity assumptions on u and w, it follows that $c_0(s) = s - i_0(s)$ for every $s \in S$. Additional useful information on the function c_0 is given below.

Lemma 3 *Assume that* (A0) *holds and* w *is continuous concave on* S*. Then the function* c_0 *defined above is Lipschitz with constant one.*

Proof Note that $A_0(s)$ is a singleton for each $s \in S$. Write $A_0(s) = \{c_0(s)\}$. Under assumption (A0), D is a lattice. Assume first that w is strictly concave. A simple modification of the arguments used in the proof of Lemma 1 yields that the correspondence $s \mapsto A_0(s)$ is strongly ascending. Thus, c_0 is non-decreasing. Since $s \mapsto i_0(s) = s - c_0(s)$ is also non-decreasing (Lemma 1) we conclude that c_0 (and also i_0) is Lipschitz with constant one; for this assume that $s_1 < s_2$, then we have

$$\begin{aligned} 0 \le c_0(s_2) - c_0(s_1) &= \big| c_0(s_2) - c_0(s_1) \big| = \big| s_2 - i_0(s_2) - \big(s_1 - i_0(s_1) \big) \big| \\ &= s_2 - i_0(s_2) - \big(s_1 - i_0(s_1) \big) \le s_2 - s_1 = |s_2 - s_1|. \end{aligned}$$

Assume now that w is concave on S. Then there exists a sequence (w_n) of strictly concave functions converging uniformly to w on the set S. Define $U_n(s, a) :=$

$u(a) + w_n(s-a)$ for all $(s,a) \in D$ and $A_0^n(s) := \arg\max_{a \in A(s)} U_n(s,a)$ for $s \in S$. Then, for each $n \in N$, there exists a Lipschitz function $c_0^n : S \mapsto S$ such that $A_0^n(s) = \{c_0^n(s)\}$ for all $s \in S$. Without loss of generality (by the Arzèla–Ascoli theorem), we can assume that the sequence (c_0^n) converges uniformly to some function $\tilde{c}_0$ on S. Clearly, $A_0(s) = \{\tilde{c}_0(s)\}$ for each $s \in S$ and $\tilde{c}_0$ is Lipschitz with constant one. □

3.1 Non-atomic Transition Probability Functions

Let X be the vector space of continuous from the left real-valued functions with bounded variations on S. Let (h_n) be a sequence of functions in X. It is said that (h_n) *converges weakly* to $h \in X$, if $\lim_{n\to\infty} h_n(s) = h(s)$ for any continuity point of h. The weak convergence of (h_n) to h is denoted by $h_n \overset{\omega}{\to} h$.

We endow $I \subset X$ with the *topology of weak convergence* (see (3)). Let $\mathcal{M}$ be the space of all regular signed measures on S with bounded variation and $C(S)$ the Banach space of all continuous real-valued functions on S endowed with the supremum norm. It is well-known that $\mathcal{M}$ is the dual of $C(S)$ (see Theorem 14.14 in Aliprantis and Border 2006) and is a linear metrizable topological space when equipped with the weak-star topology. Moreover, there is a homeomorphism $\mathcal{H}$ between I and a set of measures μ such that $\overline{s} - \overline{a}(\overline{s}) \le \mu(S) \le \overline{s} - \underline{a}(\overline{s})$. Denote this set of measures by $\mathcal{M}_S$. An example of such a homeomorphism is the mapping $\mathcal{H}(\mu) = \phi_\mu(\cdot)$, where $\phi_\mu(x) = \mu([0,x))$, $x \in S_+$, $\phi_\mu(0) = 0$. Using the Banach–Alaoglu theorem, we infer that $\mathcal{M}_S$ is compact in the weak-star topology, see also Helly's theorem in Billingsley (1968). Since $I = \mathcal{H}(\mathcal{M}_S)$, I is compact in the space X endowed with the topology of weak convergence. It is obvious that $F \subset X$ is convex and is obtained by a continuous transformation of I, namely: $c(s) = s - i(s)$, $s \in S$, $i \in I$. Thus, we have the following auxiliary result.

Lemma 4 *F is a convex sequentially compact subset of the space X endowed with the topology of weak convergence.*

For $c \in F$ and any lower semianalytic function $v : S \mapsto \underline{R}$ integrable with respect to each measure μ_k^θ, where $k = 1, \dots, l$ and $\theta \in \Theta$, define the operator T_c as follows

$$T_c v(s) := u\big(c(s)\big) + \inf_{\xi \in \mathcal{P}} \beta \int_S v(y) q\big(dy|s - c(s), \xi\big), \quad s \in S. \tag{6}$$

By Propositions 7.47 and 7.48 in Bertsekas and Shreve (1978), it follows that $T_c v$ is also lower semianalytic. Let us now consider $v_0 \equiv 0$, where v_0 is a function that assigns 0 to each $s \in S$. Taking n-th composition of the operator T_c with itself on v_0 and using dynamic programming argument (see Proposition 8.2 in Bertsekas and

Shreve 1978) we obtain that

$$T_c^n v_0(s_j) = \inf_{\gamma^j \in \Gamma} E_{s_j}^{c,\gamma^j} \left(\sum_{\tau=j}^{j+n-1} \beta^{\tau-j} u\big(c(s_\tau)\big) \right). \tag{7}$$

Moreover, the function $s_j \mapsto T_c^n v_0(s_j)$ is lower semianalytic. Since u is non-positive we have that

$$T_c^n v_0(s_j) \geq J(c)(s_j) \quad \text{for every } n \in N. \tag{8}$$

On the other hand, for any $\hat{\gamma}^j \in \Gamma$

$$E_{s_j}^{c,\hat{\gamma}^j} \left(\sum_{\tau=j}^{j+n-1} \beta^{\tau-j} u\big(c(s_\tau)\big) \right) \geq T_c^n v_0(s_j).$$

From the monotone convergence theorem we have that

$$E_{s_j}^{c,\hat{\gamma}^j} \left(\sum_{\tau=j}^{j+n-1} \beta^{\tau-k} u\big(c(s_\tau)\big) \right) \to \hat{J}\big(c, \hat{\gamma}^j\big)(s_j) \quad \text{as } n \to \infty.$$

Since $\hat{\gamma}^j$ was arbitrary it follows that

$$J(c)(s_j) \geq \lim_{n\to\infty} T_c^n v_0(s_j). \tag{9}$$

Letting n tend to infinity in (7) and combining (8) and (9) we infer that

$$J(c)(s_j) = \lim_{n\to\infty} T_c^n v_0(s_j). \tag{10}$$

Assume for the moment that u is unbounded. Then, by assumption (A5) and the monotone convergence theorem for any $c \in F$ and $\gamma^j \in \Gamma$ we get that

$$\begin{aligned} \big|J(c)(s_j)\big| &\leq \big|\hat{J}\big(c, \gamma^j\big)(s_j)\big| \leq E_{s_j}^{c,\gamma^j} \left(\sum_{\tau=j}^{\infty} \beta^{\tau-j} \big|u\big(c(s_\tau)\big)\big| \right) \\ &= \left(\sum_{\tau=j}^{\infty} \beta^{\tau-j} E_{s_j}^{c,\gamma^j} \big|u\big(c(s_\tau)\big)\big| \right) \leq \left(\sum_{\tau=j}^{\infty} \beta^{\tau-j} E_{s_j}^{c,\gamma^j} \big|u\big(\underline{a}(s_\tau)\big)\big| \right). \end{aligned}$$

Next observe that using assumptions (A3)–(A5) for $m \geq 1$ we obtain that

$$\begin{aligned} E_{s_j}^{c,\gamma^j} \big|u\big(\underline{a}(s_{j+m})\big)\big| &= E_{s_j}^{c,\gamma^j} \left[\int_S \big|u\big(\underline{a}(s')\big)\big| q\big(ds' | i(s_{j+m-1}), \gamma_{j+m-1}\big) \right] \\ &\leq \max_{k=1,\ldots,l} \int_S \big|u\big(\underline{a}(s')\big)\big| f_k(s') \nu_k(ds') =: C < +\infty, \end{aligned} \tag{11}$$

where $i(s_{j+m-1}) = s_{j+m-1} - c(s_{j+m-1})$. Therefore, by assumption (A1)

$$\begin{aligned} \left|J(c)(s_j)\right| &\leq \left|u\big(\underline{a}(s_j)\big)\right| + C\sum_{\tau=j+1}^{\infty}\beta^{\tau-j} \\ &= \left|u\big(\underline{a}(s_j)\big)\right| + \frac{C\beta}{1-\beta} < +\infty \end{aligned}$$

for all $s_j \in S_+$. Clearly, if u is bounded, i.e., assumption (A2) holds instead of (A1), then $\sup_{c\in F}|J(c)(s_j)|$ is finite for all $s_j \in S$.

In the proofs of Lemmas 5–7 we heavily exploit the assumption that the measures $\nu_1, \ldots, \nu_l$ are non-atomic. This means that the weak convergence denotes the convergence almost everywhere.

Lemma 5 *Let* (A0)–(A1), (A3)–(A5) *or* (A0), (A2)–(A4) *hold. Assume that* $c^m \stackrel{\omega}{\to} c$ *in* F. *Then, for each* $n \in N$, *and* $k = 1, \ldots, l$

$$\sup_{\theta\in\Theta}\int_S\left|T^n_{c^m}v_0(s) - T^n_c v_0(s)\right|\mu^\theta_k(ds) \to 0 \quad \text{as } m\to\infty. \tag{12}$$

Proof Let us first consider the case with the unbounded function u and apply the induction argument. The proof for bounded function u is analogous. Let $n = 1$. Then, by (A1) $u(c^m) \stackrel{\omega}{\to} u(c)$. Obviously, the function $u(c)$ has at most countable number of discontinuity points. Therefore, making use of (A3)–(A4) and the dominated convergence theorem it follows that

$$\begin{aligned} &\int_S\left|u\big(c^m(s)\big) - u\big(c(s)\big)\right|\mu^\theta_k(ds) \\ &\quad\leq \int_S\left|u\big(c^m(s)\big) - u\big(c(s)\big)\right| f_k(s)\nu_k(ds) \to 0 \end{aligned} \tag{13}$$

for each $k = 1, \ldots, l$.

Assume now that (12) holds true for some $n \in N$. We show that it is satisfied for $n + 1$. Indeed, recall first that

$$T^{n+1}_c v_0(s) = u\big(c(s)\big) + \inf_{\xi\in\mathcal{P}}\beta\int_S T^n_c v_0\big(s'\big)q\big(ds'|i(s),\xi\big)$$

with $i(s) = s - c(s)$. Similarly, we put $i^m(s) = s - c^m(s)$. Hence,

$$\begin{aligned} &\int_S\left|T^{n+1}_{c^m}v_0(s) - T^{n+1}_c v_0(s)\right|\mu^\theta_k(ds) \\ &\quad\leq \int_S\left|u\big(c^m(s)\big) - u\big(c(s)\big)\right|\mu^\theta_k(ds) + \int_S Z_m(s)\mu^\theta_k(ds), \end{aligned} \tag{14}$$

where

$$Z_m(s) := \sup_{\theta\in\Theta}\left|\sum_{k=1}^{l} g_k\big(i^m(s)\big)\int_S T^n_{c^m} v_0\big(s'\big)\mu_k^\theta\big(ds'\big) - \sum_{k=1}^{l} g_k\big(i(s)\big)\int_S T^n_c v_0\big(s'\big)\mu_k^\theta\big(ds'\big)\right|.$$

Obviously,

$$Z_m(s) \le Y_m(s) + \sup_{\theta\in\Theta}\left|\sum_{k=1}^{l} g_k\big(i^m(s)\big)\int_S T^n_c v_0\big(s'\big)\mu_k^\theta\big(ds'\big) - \sum_{k=1}^{l} g_k\big(i(s)\big)\int_S T^n_c v_0\big(s'\big)\mu_k^\theta\big(ds'\big)\right|, \tag{15}$$

where

$$Y_m(s) = \sup_{\theta\in\Theta}\left|\sum_{k=1}^{l} g_k\big(i^m(s)\big)\int_S T^n_{c^m} v_0\big(s'\big)\mu_k^\theta\big(ds'\big) - \sum_{k=1}^{l} g_k\big(i^m(s)\big)\int_S T^n_c v_0\big(s'\big)\mu_k^\theta\big(ds'\big)\right|.$$

The induction hypothesis and (A3) imply that

$$\sup_{s\in S}\big|Y_m(s)\big| \to 0 \quad \text{as } m \to \infty. \tag{16}$$

Observe that for each $k = 1, \dots, l$

$$g_k\big(i^m\big) \xrightarrow{\omega} g_k(i), \tag{17}$$

which implies by the dominated convergence theorem that

$$\int_S\left|\sum_{k'=1}^{l} g_{k'}\big(i^m(s)\big) - \sum_{k'=1}^{l} g_{k'}\big(i(s)\big)\right|\mu_k^\theta(ds) \le \int_S\left|\sum_{k'=1}^{l} g_{k'}\big(i^m(s)\big) - \sum_{k'=1}^{l} g_{k'}\big(i(s)\big)\right| f_k(s)\nu_k(ds) \to 0. \tag{18}$$

Making use of (7) and (11), the second term in (15) can be estimated as follows

$$\left|\sum_{k=1}^{l} g_k\big(i^m(s)\big) - \sum_{k=1}^{l} g_k\big(i(s)\big)\right| \int_S \left|T_c^n v_0\big(s'\big)\right| \mu_k^\theta\big(ds'\big)$$
$$\leq \left|\sum_{k=1}^{l} g_k\big(i^m(s)\big) - \sum_{k=1}^{l} g_k\big(i(s)\big)\right| \frac{C}{1-\beta}.$$

Therefore, the above display, (18), (16) and (15) yield that

$$\int_S \left|Z_m(s)\right| \mu_k^\theta(ds) \leq \int_S \left|Z_m(s)\right| f_k(s)\nu_k(ds) \to 0 \quad \text{as } m \to \infty.$$

This fact, (13) and (14) finish the proof. □

Lemma 6 *Let* (A0)–(A1), (A3)–(A5) *or* (A0), (A2)–(A4) *hold. Assume that* $c^m \overset{\omega}{\to} c$ *in* F. *Then, for each* $k = 1, \ldots, l$

$$\sup_{\theta \in \Theta} \int_S \left|J\big(c^m\big)(s) - J(c)(s)\right| \mu_k^\theta(ds) \to 0 \quad \textit{as } m \to \infty.$$

Proof Assume first that u is unbounded. In view of Lemma 5, it is sufficient to show that the convergence in (10) is uniform with respect to $c \in F$ and $s \in S_+$. Observe that

$$\sup_{c \in F} \left|J(c)(s_j) - T_c^n v_0(s_j)\right| \leq \sup_{c \in F} \sup_{\gamma^j \in \Gamma} \beta^n \left| E_{s_j}^{c,\gamma^j} \left(\sum_{\tau=0}^{\infty} \beta^\tau u\big(c(s_{\tau+j+n})\big) \right) \right|$$
$$\leq \sup_{c \in F} \sup_{\gamma^j \in \Gamma} \beta^n E_{s_j}^{c,\gamma^j} \left(\sum_{\tau=0}^{\infty} \beta^\tau \left|u\big(c(s_{\tau+j+n})\big)\right| \right).$$

Assumption (A1) and the monotone convergence theorem yield that

$$E_{s_j}^{c,\gamma^j} \left(\sum_{\tau=0}^{\infty} \beta^\tau \left|u\big(c(s_{\tau+j+n})\big)\right| \right) = \sum_{\tau=0}^{\infty} \beta^\tau E_{s_j}^{c,\gamma^j} \left|u\big(c(s_{\tau+j+n})\big)\right|.$$

Making use of (11), for any $c \in F$, $\gamma^j \in \Gamma$ and $n \in N$ we obtain that

$$E_{s_j}^{c,\gamma^j} \left|u\big(c(s_{j+n+\tau})\big)\right| \leq C,$$

and consequently,

$$\sup_{s \in S_+} \sup_{c \in F} \left|J(c)(s) - T_c^n v_0(s)\right| \leq \beta^n \sum_{\tau=0}^{\infty} \beta^\tau C = \beta^n \frac{C}{1-\beta} \to 0$$

as $n \to \infty$. This fact yields the proof. We note again that the case with bounded u can be treated analogously. □

Lemma 7 *Let* (A0)–(A1), (A3)–(A5) *or* (A0), (A2)–(A4) *hold. Assume that* $c^m \xrightarrow{\omega} c$ *in* F. *Then,*

$$\sup_{a\in A(s)} P\big(a, c^m\big)(s) \to \sup_{a\in A(s)} P(a,c)(s) \quad \textit{for each } s \in S.$$

Proof Note that

$$\Big|\sup_{a\in A(s)} P\big(a, c^m\big)(s) - \sup_{a\in A(s)} P(a,c)(s)\Big| \\ \le \sup_{\theta\in\Theta} \alpha\beta \sum_{k=1}^{l} \int_S \big|J\big(c^m\big)\big(s'\big) - J(c)\big(s'\big)\big|\mu_k^\theta\big(ds'\big).$$

Thus, the result follows from Lemma 6. □

Proof of Theorem 1 Let us set

$$\hat{A}_0(c)(s) := \arg\max_{y\in\hat{A}(s)} \hat{P}(y,c)(s),$$

where

$$\hat{P}(y,c)(s) = u(s-y) + \alpha\beta \inf_{\xi\in\mathcal{P}} \int_S J(c)\big(s'\big)q\big(ds'|y,\xi\big).$$

Define

$$i_0(s) = \min \hat{A}_0(c)(s). \tag{19}$$

From Lemma 1, it follows that $i_0 \in I$. Let $c_0(s) = s - i_0(s)$. Then, $c_0 \in F$. Moreover, we define

$$Lc(s) := c_0(s).$$

We show that L is weakly continuous. Assume that $c^m \xrightarrow{\omega} c$ in F. Let us consider $c_0(s) = Lc(s)$ and $c_0^m(s) = Lc^m(s)$. Clearly, (c_0^m) is relatively compact in F. Let $\widetilde{c}_0$ be any accumulation point of (c_0^m) in the sequentially compact space F (see Lemma 4). By S_d we denote the set of discontinuity points of $\widetilde{c}_0$. If $s \in S_+ \setminus S_d$, then from Lemma 7, we conclude that

$$P\big(\widetilde{c}_0(s), c\big)(s) = \max_{a\in A(s)} \Big[u(a) + \alpha\beta \inf_{\xi\in\mathcal{P}} \int_S J(c)\big(s'\big)q\big(ds'|s-a,\xi\big)\Big].$$

Hence, $\widetilde{c}_0(s) \in \arg\max_{a\in A(s)} P(a,c)(s)$. Consequently, $\widetilde{i}_0(s) = s - \widetilde{c}_0(s) \in \hat{A}_0(c)(s)$ and s is a continuity point of $\widetilde{i}_0$. From Lemma 2, we deduce that $\hat{A}_0(c)(s)$ is a singleton. Therefore, $\widetilde{c}_0(s) = c_0(s)$. If, on the other hand, $s \in S_d$ and $s \ne 0$, then we may take a sequence (s_k) such that $s_k < s$, $s_k \in S_+ \setminus S_d$. Applying the above reasoning to every s_k, we get that $\widetilde{c}_0(s_k) = c_0(s_k)$. Since both $\widetilde{c}_0$ and c_0 are continuous from the left at s, we obtain that

$$\widetilde{c}_0(s) = \lim_{k\to\infty} \widetilde{c}_0(s_k) = \lim_{k\to\infty} c_0(s_k) = c_0(s).$$

Hence, $\widetilde{c}_0(s) = Lc(s)$ for all $s \in S$. Thus L is weakly continuous on F.

Now from the Schauder–Tychonoff fixed point theorem there exists $c^* \in F$ such that $c^* = Lc^*$ and this fact completes the proof. □

3.2 Transition Probability Functions with Atoms

We endow the space FL (see (4)) with the topology of uniform convergence. By the Arzèla–Ascoli theorem FL is a compact metric space. Obviously, FL is a convex subset of $C(S)$.

Lemma 8 *Let* (A0)–(A1), (A3)–(A5) *or* (A0), (A2)–(A4) *hold. Assume that* $c^m \to c$ *in* FL. *Then, for each* $n \in N$, *and* $k = 1, \ldots, l$

$$\sup_{\theta\in\Theta} \int_S \left| T^n_{c^m} v_0(s) - T^n_c v_0(s) \right| \mu_k^\theta(ds) \to 0 \tag{20}$$

as $m \to \infty$.

Proof Observe first that assumption (A5) implies that in the case of unbounded u, $\mu_k^\theta(\{0\}) = 0$ for each $k = 1, \ldots, l$ and all $\theta \in \Theta$. This fact yields that $u(c^m(s)) \to u(c(s))$ for all $s \in S$. Therefore, by the dominated convergence theorem it follows that (13) holds.

In the case of bounded u, the convergence of $u(c^m)$ to $u(c)$ is uniform on S. Hence, (13) is also satisfied. The remaining part of the proof goes by induction and is analogous to the proof of Lemma 5 except that the weak convergence in (17) is replaced by the uniform convergence of $g_k(i^m) \to g_k(i)$ for each $k = 1, \ldots, l$. □

The next result is a repetition of Lemma 6.

Lemma 9 *Let* (A0)–(A1), (A3)–(A5) *or* (A0), (A2)–(A4) *hold. Assume that* $c^m \to c$ *in* FL. *Then, for each* $k = 1, \ldots, l$

$$\sup_{\theta\in\Theta} \int_S \left| J\left(c^m\right)(s) - J(c)(s) \right| \mu_k^\theta(ds) \to 0 \quad \textit{as } m \to \infty.$$

Lemma 9 implies the following fact.

Lemma 10 *Let* (A0)–(A1), (A3)–(A5) *or* (A0), (A2)–(A4) *hold. Assume that* $c^m \to c$ *in FL. Then,*

$$\sup_{a \in A(s)} P\big(a, c^m\big)(s) \to \sup_{a \in A(s)} P(a, c)(s) \quad \textit{for each } s \in S.$$

Lemma 11 *Let* (A0)–(A1), (A3)–(A5) *or* (A0), (A2)–(A4) *hold. If, in addition, Assumption* (C) *is satisfied, then for any* $c \in FL$ *it follows that the function* $s \mapsto J(c)(s)$ *is non-decreasing.*

Proof We consider the operator T_c defined in (6). We show that the function $s \mapsto T_c^n v_0(s)$ described in (7) is non-decreasing for every $n \in N$. We proceed by induction. Clearly, for $n = 1$, the function

$$T_c v_0(\cdot) = u\big(c(\cdot)\big) \text{ is non-decreasing.} \tag{21}$$

Assume now that $T_c^n v_0(\cdot)$ is non-decreasing and consider $T_c^{n+1} v_0$. Observe that

$$\begin{aligned}
T_c^{n+1} v_0(s) &= u\big(c(s)\big) + \inf_{\xi \in \mathcal{P}} \beta \int_S T_c^n v_0\big(s'\big) q\big(ds' | s - c(s), \xi\big) \\
&= u\big(c(s)\big) + \inf_{\xi \in \mathcal{P}} \beta \int_\Theta \Bigg[\Bigg(1 - \sum_{k=2}^{l} g_k\big(s - c(s)\big) \Bigg) \int_S T_c^n v_0\big(s'\big) \mu_1^\theta\big(ds'\big) \\
&\quad + \sum_{k=2}^{l} \int_S T_c^n v_0\big(s'\big) \mu_k^\theta\big(ds'\big) g_k\big(s - c(s)\big) \Bigg] \xi(d\theta) \\
&= u\big(c(s)\big) + \inf_{\xi \in \mathcal{P}} \beta \int_\Theta \Bigg[\int_S T_c^n v_0\big(s'\big) \mu_1^\theta\big(ds'\big) \\
&\quad + \sum_{k=2}^{l} \bigg(\int_S T_c^n v_0\big(s'\big) \mu_k^\theta\big(ds'\big) - \int_S T_c^n v_0\big(s'\big) \mu_1^\theta\big(ds'\big) \bigg) \\
&\quad \times g_k\big(s - c(s)\big) \Bigg] \xi(d\theta).
\end{aligned}$$

By (C) and induction assumption, we have that

$$\int_S T_c^n v_0\big(s'\big) \mu_k^\theta\big(ds'\big) - \int_S T_c^n v_0\big(s'\big) \mu_1^\theta\big(ds'\big) \geq 0$$

for all $\theta \in \Theta$ and $k = 2, \ldots, l$. This inequality, and the fact that $s \mapsto s - c(s)$ is non-decreasing, allow to deduce that the function

$$s \mapsto \bigg(\int_S T_c^n v_0\big(s'\big) \mu_k^\theta\big(ds'\big) - \int_S T_c^n v_0\big(s'\big) \mu_1^\theta\big(ds'\big) \bigg) g_k\big(s - c(s)\big)$$

is non-decreasing for every $\theta \in \Theta$ and $k = 2, \ldots, l$. This fact and (21), in turn, imply that $T_c^{n+1} v_0(\cdot)$ is non-decreasing. From (10) it follows that the function $J(c)(\cdot)$ is non-decreasing. □

Proof of Theorem 2 Observe first that for any $c \in FL$ the function

$$y \mapsto \inf_{\xi \in \mathcal{P}} \int_S J(c)(s') q(ds'|y, \xi) \text{ is concave.} \tag{22}$$

Indeed, by (A3)

$$\begin{aligned}
&\int_S J(c)(s') q(ds'|y, \xi) \\
&\quad = \int_\Theta \Bigg[\int_S J(c)(s') \mu_1^\theta(ds') \\
&\qquad + \sum_{k=2}^{l} \left(\int_S J(c)(s') \mu_k^\theta(ds') - \int_S J(c)(s') \mu_1^\theta(ds') \right) g_k(y) \Bigg] \xi(d\theta)
\end{aligned}$$

for any $\xi \in \mathcal{P}$. By Assumption (C) and Lemma 11, we obtain that

$$\int_S J(c)(s') \mu_k^\theta(ds') - \int_S J(c)(s') \mu_1^\theta(ds') \geq 0 \tag{23}$$

for all $\theta \in \Theta$ and $k = 2, \ldots, l$. Since $g_2, \ldots, g_k$ are concave, we infer that

$$y \mapsto \sum_{k=2}^{l} \left(\int_S J(c)(s') \mu_k^\theta(ds') - \int_S J(c)(s') \mu_1^\theta(ds') \right) g_k(y)$$

is concave for all $\theta \in \Theta$, and therefore (22) is follows.

For $c \in FL$ let us set

$$A_0(c)(s) := \arg \max_{a \in A(s)} P(a, c)(s).$$

From (22) and the strict concavity of u, we deduce that $A_0(c)(s)$ is a singleton, and therefore there exists $c_0 : S \mapsto S$ such that $A_0(c)(s) = \{c_0(s)\}$. Moreover, our concavity assumptions imply that $c_0(s) = s - i_0(s)$, where i_0 is defined in (19). From Lemma 3, we have that $c_0(\cdot)$ is Lipschitz with constant one. As in the proof of Theorem 1 we define $Lc(s) := c_0(s)$.

Assume now that $c^m \to c$ in FL and that u is unbounded. We show that L is continuous on FL. Set $c_0^m = Lc^m$ and $c_0 = Lc$. Clearly, (c_0^m) is relatively compact. Let $\widetilde{c}_0$ be an accumulation point of (c_0^m) in FL. Since $\widetilde{c}_0$ is continuous on S_+, from Lemma 10 it follows that

$$P\big(\widetilde{c}_0(s), c\big)(s) = \max_{a \in A(s)} \left[u(a) + \alpha\beta \inf_{\xi \in \mathcal{P}} \int_S J(c)(s') q(ds'|s - a, \xi) \right].$$

But $A_0(c)(s)$ is a singleton and therefore $\widetilde{c}_0(s) = c_0(s)$ for all $s \in S$.

By the Schauder–Tychonoff fixed point theorem there exists $c^* \in FL$ such that

$$P\big(c^*(s), c^*\big)(s) = \max_{a \in A(s)}\left[u(a) + \alpha\beta \inf_{\xi \in \mathcal{P}} \int_S J\big(c^*\big)\big(s'\big) q\big(ds'|s-a, \xi\big)\right].$$

Finally, we have to show that $s \mapsto P(c^*(s), c^*)(s)$ is continuous and non-decreasing. But it follows from the following two facts that both

$$s \mapsto u\big(c^*(s)\big) \quad \text{and} \quad s \mapsto \int_S J\big(c^*\big)\big(s'\big) q\big(ds'|s - c^*(s), \xi\big)$$

are non-decreasing and continuous on S_+. Indeed, by (A3) we have that

$$\begin{aligned}
&\int_S J\big(c^*\big)\big(s'\big) q\big(ds'|s - c^*(s), \xi\big) \\
&\quad = \int_\Theta \left[\int_S J\big(c^*\big)\big(s'\big) \mu_1^\theta\big(ds'\big)\right. \\
&\qquad \left. + \sum_{k=2}^{l} \left(\int_S J\big(c^*\big)\big(s'\big)\mu_k^\theta\big(ds'\big) - \int_S J\big(c^*\big)\big(s'\big)\mu_1^\theta\big(ds'\big)\right) g_k\big(s - c^*(s)\big)\right] \xi(d\theta).
\end{aligned}$$

Now the conclusion easily follows from Assumption (C) and (23) with c replaced by c^*. □

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Stochastic Differential Games and Intricacy of Information Structures

Tamer Başar

Abstract This chapter discusses, in both continuous time and discrete time, the issue of certainty equivalence in two-player zero-sum stochastic differential/dynamic games when the players have access to state information through a common noisy measurement channel. For the discrete-time case, the channel is also allowed to fail sporadically according to an independent Bernoulli process, leading to intermittent loss of measurements, where the players are allowed to observe past realizations of this process. A complete analysis of a parametrized two-stage stochastic dynamic game is conducted in terms of existence, uniqueness and characterization of saddle-point equilibria (SPE), which is shown to admit SPE of both certainty-equivalent (CE) and non-CE types, in different regions of the parameter space; for the latter, the SPE involves mixed strategies by the maximizer. The insight provided by the analysis of this game is used to obtain through an indirect approach SPE for three classes of differential/dynamic games: (i) linear-quadratic-Gaussian (LQG) zero-sum differential games with common noisy measurements, (ii) discrete-time LQG zero-sum dynamic games with common noisy measurements, and (iii) discrete-time LQG zero-sum dynamic games with intermittently missing perfect state measurements. In all cases CE is a generalized notion, requiring two separate filters for the players, even though they have a common communication channel. Discussions on extensions to other classes of stochastic games, including nonzero-sum stochastic games, and on the challenges that lie ahead conclude the chapter.

1 Introduction

In spite of decades long research activity on stochastic differential games, there still remain some outstanding fundamental questions on existence, uniqueness, and characterization of non-cooperative equilibria when players have access to noisy state information. Even in zero-sum games and with common measurement channel that feeds noisy state information to both players, derivation of saddle-point poli-

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cies is quite an intricate task, as first identified in Başar (1981). That paper also addressed the issue of whether saddle-point equilibria (SPE) in such games is of the certainty-equivalent (CE) type (Witsenhausen 1971a), that is whether the solution of a similarly structured game but with perfect state measurements for both players can be used in the construction of SPE for the stochastic game with noisy measurement, by simply replacing the state with an appropriately constructed conditional estimate. The answer was a "cautious conditional yes," in the sense that not all SPE are of the CE type, and when they are in both the construction of the conditional estimate and the derivation of conditions for existence many perils exist. This chapter picks up where Başar (1981) had left, and develops further insights into the intricacies and pitfalls in the derivation of SPE of the CE as well as non-CE types. It also provides a complete solution to a two-stage stochastic game of the linear-quadratic-Gaussian (LQG) type where the common measurement channel is not only noisy but also fails intermittently.

Research on stochastic differential games with noisy state measurements goes back to the 1960's, where two-person zero-sum games with linear dynamics and measurement equations, Gaussian statistics, and quadratic cost functions (that is, LQG games) were addressed when players have access to different measurements, within however some specific information structures (Behn and Ho 1968; Rhodes and Luenberger 1969; Willman 1969). A zero sum differential game where one player's information is nested in the other player's was considered in Ho (1974), and a class of zero-sum dynamic games where one player has noisy state information while the other one plays open loop was considered in Başar and Mintz (1973) which showed that the open-loop player's saddle-point strategy is *mixed*. A class of zero-sum stochastic games where the information structure is of the nonclassical type was considered in Başar and Mintz (1972), which showed that some zero-sum games could be tractable even though their team counterparts, as in Witsenhausen (1968), Bansal and Başar (1987), Ho (1980) are not; see also Başar (2008).

When a game is not of the zero-sum type, derivation of equilibria (which in this case would be Nash equilibria) is even more challenging, even when players have access to common noisy measurements, with or without delay, as discussed in Başar (1978a) where an indirect approach of the backward-forward type was developed and employed; see also Başar (1978b) for a different formulation and approach for derivation. Recently, a new class of discrete-time nonzero-sum games with *asymmetric information* was introduced in Nayyar and Başar (2012), where the evolution of the local state processes depends only on the global state and control actions and not on the current or past values of local states. For this class of games, it was possible to obtain a characterization of some Nash equilibria by *lifting* the game and converting it to a symmetric one, solving the symmetric one in terms of Markov equilibria, and then converting it back. Among many others, two other papers of relevance to stochastic nonzero-sum dynamic games are Altman et al. (2009) and Hespanha and Prandini (2001), and one of relevance to teams with delayed sharing patterns is Nayyar et al. (2011).

The paper is organized as follows. In the next section, we introduce LQG zero-sum stochastic differential/dynamic games (ZSDGs) with common noisy measurements, first in continuous time and then in discrete time, and for the latter we also include the possibility of intermittent failure of the measurement channel (modeled through a Bernoulli process), leading to occasionally missing measurements. In the section we also introduce the concept of certainty equivalence, first in the context of the classical LQG optimal control problem and then generalized (in various ways) to the two classes of games formulated. In Sect. 3, we introduce a two-stage stochastic dynamic game, as a special case of the general discrete time LQG game of Sect. 2, which is solved completely for its SPE in both pure and mixed strategies, some of the CE type and others non-CE (see Theorem 1 for the complete solution). Analysis of the two-stage game allows us to develop insight into the intricate role information structures play in the characterization and existence of SPE for the more general ZS-DGs of Sect. 2, and what CE means in a game context. This insight is used in Sect. 4 in the derivation of generalized CE SPE for the continuous-time LQG ZSDG with noisy state measurements (see Theorem 2 for the penultimate result) as well as for the continuous-time LQG ZSDG with noisy state measurements and perfect state measurements with intermittent losses. The paper ends with a recap of the results of the paper and a discussion on extensions and open problems, in Sect. 5.

2 Zero-Sum Stochastic Differential and Discrete-Time Games with a Common Measurement Channel and Issue of Certainty Equivalence

2.1 Formulation of the Zero-Sum Stochastic Differential Game

We first consider the class of so-called Linear-Quadratic-Gaussian zero-sum differential games (LQG ZSDGs), where the two players' actions are inputs to a linear system driven also by a Wiener process, and the players have access to the system state through a common noisy measurement channel which is also linear in the state and the driving Wiener noise process. The objective function, to be minimized by one player and maximized by the other, is quadratic in the state and the actions of the two players.

For a precise mathematical formulation, let $\{x_t, y_t, t \geq 0\}$, be respectively the n-dimensional state and m-dimensional measurement processes, generated by

$$dx_t = (Ax_t + Bu_t + Dv_t)dt + Fdw_t, \quad t \geq 0, \tag{1}$$

$$dy_t = Hx_t dt + Gdw_t, \quad t \geq 0, \tag{2}$$

where $\{u_t, t \geq 0\}$ and $\{v_t, t \geq 0\}$ are respectively Player 1's and Player 2's controls (say of dimensions r_1 and r_2, respectively), nonanticipative with respect to the measurement process, and generated by measurable control policies $\{\gamma_t\}$ and $\{\mu_t\}$, respectively, that is

$$u_t = \gamma_t(y_{[0,t)}), \qquad v_t = \mu_t(y_{[0,t)}), \quad t \geq 0. \tag{3}$$

In (1) and (2), x_0 is a zero-mean Gaussian random vector with covariance Λ_0 (that is $x_0 \sim N(0, \Lambda_0)$), $\{w_t, t \geq 0\}$ is a vector-valued standard Wiener process independent of x_0, and A, B, D, F, H, G are constant[1] matrices of appropriate dimensions, with (to avoid singularity) $FF^T > 0$, $GG^T > 0$, and $FG^T = 0$, where the last condition assures that system and channel noises are independent. We let Γ and $\mathcal{M}$ denote the classes of admissible control policies for Player 1 and Player 2, respectively, with elements $\gamma := \{\gamma_t\}$ and $\mu := \{\mu_t\}$, as introduced earlier. The only restriction on these policies is that when (3) is used in (1), we have unique second-order stochastic process solutions to (1) and (2), with almost sure continuously differentiable sample paths. Measurability and uniform Lipschitz continuity will be sufficient for this purpose.

To complete the formulation of the differential game, we now introduce a quadratic performance index over a finite interval $[0, t_f]$:

$$J(\gamma, \mu) = E\left\{|x_{t_f}|^2_{Q_f} + \int_0^{t_f} \left[|x_t|^2_Q + \lambda |u_t|^2 - |v_t|^2\right] dt \,\Big|\, u = \gamma(\cdot), v = \mu(\cdot)\right\}, \quad (4)$$

where expectation $E\{\cdot\}$ is over the statistics of x_0 and $\{w_t\}$; further, $|x|^2_Q := x^T Q x$, Q and Q_f are non-negative definite matrices, and $\lambda > 0$ is a scalar parameter. Note that any objective function with nonuniform positive weights on u and v can be brought into the form above by a simple rescaling and re-orientation of u and v and a corresponding transformation applied to B and D, and hence the structure in (4) does not entail any loss of generality as a quadratic performance index.

The problem of interest in the context of LQG ZSDGs is to find conditions for existence and characterization of saddle-point strategies, that is $(\gamma^* \in \Gamma, \mu^* \in \mathcal{M})$ such that

$$J(\gamma^*, \mu) \leq J(\gamma^*, \mu^*) \leq J(\gamma, \mu^*), \quad \forall \gamma \in \Gamma, \mu \in \mathcal{M}. \quad (5)$$

A question of particular interest in this case is whether the saddle-point equilibrium (SDE) has the *certainty equivalence* property, that is whether it can be obtained directly from the perfect state-feedback SPE of the corresponding deterministic differential game, by simply replacing the state by its "best estimate," as in the one-player version, the so-called LQG optimal control problem. This will be discussed later in the section.

If a saddle-point equilibrium (SDE) does not exist, then the next question is whether the upper value of the game is bounded, and whether there exists a control strategy for the minimizer that achieves it, that is existence of a $\bar{\gamma} \in \Gamma$ such that

$$\inf_\gamma \sup_\mu J(\gamma, \mu) = \sup_\mu J(\bar{\gamma}, \mu). \quad (6)$$

Note that the lower value of the game, $\sup_\mu \inf_\gamma J(\gamma, \mu)$, is always bounded away from zero, and hence its finiteness is not an issue.

[1] They are taken to be constant for simplicity in exposition; the main message of the paper and many of the expressions stand for the time-varying case as well, with some obvious modifications.

2.2 Formulation of the Discrete-Time Zero-Sum Stochastic Dynamic Game with Failing Channels

A variation on the LQG ZSDG is its discrete-time version, which will allow us also to introduce intermittent failure of the common measurement channel. The system equation (1) is now replaced by

$$x_{t+1} = Ax_t + Bu_t + Dv_t + Fw_t, \quad t = 0, 1, \ldots, \tag{7}$$

and the measurement equation (2) by

$$y_t = \beta_t(Hx_t + Gw_t), \quad t = 0, 1, \ldots, \tag{8}$$

where $x_0 \sim N(0, \Lambda_0)$; $\{w_t\}$ is a zero-mean Gaussian process, independent across time and of x_0, and with $E\{w_t w_t^T\} = I$, $\forall t \in [0, T-1] := \{0, 1, \ldots, T-1\}$; and $\{\beta_t\}$ is a Bernoulli process, independent across time and of x_0 and $\{w_t\}$, with Probability$(\beta_t = 0) = p$, $\forall t$. This essentially means that the channel that carries information on the state to the players fails with equal probability p at each stage, and these failures are statistically independent. A different expression for (8) which essentially captures the same would be

$$y_t = \beta_t Hx_t + Gw_t, \quad t = 0, 1, \ldots, \tag{9}$$

where what fails is the sensor that carries the state information to the channel and not the channel itself. In this case, when $\beta_t = 0$, then this means that the channel only carries pure noise, which of course is of no use to the controllers.

Now, if the players are aware of the failure of the channel or the sensor when it happens (which we assume to be the case), then what replaces (3) is

$$u_t = \gamma_t(y_{[0,t]}, \beta_{[0,t]}), \qquad v_t = \mu_t(y_{[0,t]}, \beta_{[0,t]}), \quad t = 0, 1, \ldots, \tag{10}$$

where $\{\gamma_t\}$ and $\{\mu_t\}$ are measurable control policies; let us again denote the spaces where they belong respectively by Γ and M.

The performance index replacing (4) for the discrete-time game, over the interval $\{0, 1, \ldots, T-1\}$ is[2]

$$J(\gamma, \mu) = E\left\{ \sum_{t=0}^{T-1} \left[|x_{t+1}|_Q^2 + \lambda |u_t|^2 - |v_t|^2 \right] dt \,\middle|\, u = \gamma(\cdot), v = \mu(\cdot) \right\}, \tag{11}$$

where the expectation is over the statistics of x_0, $\{w_t\}$ and $\{\beta_t\}$.

The goal is again the one specified in the case of the LQG ZSDG—to study existence and characterization of SPE (defined again by (5), appropriately interpreted),

[2]We are using "T" to denote the number of stages in the game; the same notation was used to denote *transpose*. These are such distinct usages that no confusion or ambiguity should arise.

boundedness of upper value if a SPE does not exist, and certainty-equivalence property of the SPE. We first recall below the certainty-equivalence property of the standard LQG optimal control problem, which is a special case of the LQG ZSDG obtained by leaving out the maximizer, that is by letting $D = 0$. We discuss only the continuous-time case; an analogous result holds for the discrete-time case (Witsenhausen 1971a; Yüksel and Başar 2013).

2.3 The LQG Optimal Control Problem and Certainty Equivalence

Consider the LQG optimal control problem, described by the linear state and measurement equations

$$dx_t = (Ax_t + Bu_t)dt + F dw_t, \quad t \geq 0, \tag{12}$$

$$dy_t = Hx_t dt + G dw_t, \quad t \geq 0, \tag{13}$$

and the quadratic cost function

$$J(\gamma) = E\left\{ |x_{t_f}|^2_{Q_f} + \int_0^{t_f} \left[|x_t|^2_Q + \lambda |u_t|^2 \right] dt \,\middle|\, u = \gamma(\cdot) \right\}, \tag{14}$$

where F and G satisfy the earlier conditions, and as before $\gamma \in \Gamma$.

It is a standard result in stochastic control (Fleming and Soner 1993) that there exists a unique $\gamma^* \in \Gamma$ that minimizes $J(\gamma)$ defined by (14), and $\gamma_t^*(y_{[0,t)})$ is linear in $y_{[0,t)}$. Specifically,

$$u^*(t) = \gamma_t^*(y_{[0,t)}) = \tilde{\gamma}_t(\hat{x}_t) = -\frac{1}{\lambda} B^T P(t) \hat{x}_t, \quad t \geq 0, \tag{15}$$

where P is the unique non-negative definite solution of the retrograde Riccati differential equation

$$\dot{P} + PA + A^T P - \frac{1}{\lambda} PBB^T P + Q = 0, \quad P(t_f) = Q_f, \tag{16}$$

where $\{\hat{x}_t\}$ is generated by the Kalman Filter:

$$d\hat{x}_t = (A\hat{x}_t + Bu_t)dt + K(t)(dy_t - H\hat{x}_t dt), \quad \hat{x}_0 = 0, t \geq 0, \tag{17}$$

$$K(t) = \Lambda(t) H^T \left[GG^T \right]^{-1} \tag{18}$$

with Λ being the unique non-negative definite solution of the forward Riccati differential equation

$$\dot{\Lambda} - A\Lambda - \Lambda A^T + \Lambda H^T \left[GG^T \right]^{-1} H \Lambda - FF^T = 0, \quad \Lambda(0) = \Lambda_0. \tag{19}$$

Note that this is a certainty-equivalent (CE) controller, because it has the structure of the optimal controller for the deterministic problem, that is $-\frac{1}{\lambda}B^T P(t)x_t$, with the state x_t replaced by its conditional mean, $E[x_t|y_{[0,t)}, u_{[0,t)}]$, which is given by (17). The controller gain $(-\frac{1}{\lambda}B^T P(t))$ is constructed independently of what the estimator does, while the estimation or filtering is also essentially an independent process with however the past values of the control taken as input to the Kalman filter. Hence, in a sense we have a *separation* of estimation and control, but not complete decoupling. In that sense, we can say that the controller has to cooperate with the estimator as the latter needs to have access to the output of the *control box* for the construction of the conditional mean. Of course, an alternative representation for (17) would be the one where the optimal controller is substituted in place of u:

$$d\hat{x}_t = \left(\left(A - \frac{1}{\lambda}BB^T P(t)\right)\hat{x}_t\right)dt + K(t)(dy_t - H\hat{x}_t dt), \quad \hat{x}_0 = 0, t \geq 0, \quad (20)$$

but in this representation also there is a need for collaboration or sharing of information, since the estimator has to have access to $P(\cdot)$ or the cost parameters that generate it. Hence, the solution to the LQG problem has an implicit cooperation built into it, but this does not create any problem or difficulty in this case, since the estimator and the controller are essentially a single unit.

2.4 The LQG ZSDG and Certainty Equivalence

Now we move on to the continuous-time (CT) LQG ZSDG, to obtain a CE SPE, along the lines of the LQG control problem discussed above. The corresponding deterministic LQ ZSDG, where both players have access to perfect state measurements, admits the state-feedback SPE (Başar and Olsder 1999):

$$u^*(t) = \gamma_t^*(x_{[0,t]}) = \tilde{\gamma}_t(x_t) = -\frac{1}{\lambda}B^T Z(t)x_t, \quad t \geq 0, \quad (21)$$

$$v^*(t) = \mu_t^*(x_{[0,t]}) = \tilde{\mu}_t(x_t) = D^T Z(t)x_t, \quad t \geq 0, \quad (22)$$

where Z is the unique non-negative definite continuously differentiable solution of the following Riccati differential equation (RDE) over the interval $[0, t_f]$:

$$\dot{Z} + A^T Z + ZA - Z\left(\frac{1}{\lambda}BB^T - DD^T\right)Z + Q = 0, \quad Z(t_f) = Q_f. \quad (23)$$

Existence of such a solution (equivalently nonexistence of a conjugate point in the interval $(0, t_f)$, or *no finite escape*) to the RDE (23) is also a necessary condition for existence of any SPE (Başar and Bernhard 1995), in the sense that even if any (or both) of the players use memory on the state, the condition above cannot be further relaxed. This conjugate-point condition translates, in this case, on a condition on λ, in the sense that there exists a critical value of λ, say λ^* (which will depend on the parameters of the game and the length of the time horizon, and could actually be any value in $(0, \infty)$), so that for each $\lambda \in (0, \lambda^*)$, the pair (21)–(22) provides a SPE to the corresponding deterministic ZSDG.

Now, if a natural counterpart of the CE property of the solution to the LQG optimal control problem would hold for the LQG ZSDG, we would have as SPE:

$$u^*(t) = \gamma_t^*(y_{[0,t)}) = \tilde{\gamma}_t(\hat{x}_t) = -\frac{1}{\lambda} B^T Z(t)\hat{x}_t, \quad t \geq 0, \tag{24}$$

$$v^*(t) = \mu_t^*(y_{[0,t)}) = \tilde{\mu}_t(\hat{x}_t) = D^T Z(t)\hat{x}_t, \quad t \geq 0, \tag{25}$$

where

$$\hat{x}_t := E\big[x_t | y_{[0,t)}, \{u_s = \gamma_s^*(y_{[0,s)}), v_s = \mu_s^*(y_{[0,s)}), 0 \leq s < t\}\big]$$

is generated by (as counterpart of (20)):

$$d\hat{x}_t = \hat{A}\hat{x}_t dt + K(t)(dy_t - H\hat{x}_t dt), \quad \hat{x}_0 = 0, t \geq 0, \tag{26}$$

where

$$\hat{A} := A - \left(\frac{1}{\lambda} BB^T - DD^T\right) Z(t) \tag{27}$$

and K is the Kalman gain, given by (18), with Λ now solving

$$\dot{\Lambda} - \hat{A}\Lambda - \Lambda\hat{A}^T + \Lambda H^T \big[GG^T\big]^{-1} H\Lambda - FF^T = 0, \quad \Lambda(0) = \Lambda_0. \tag{28}$$

The question now is whether the strategy pair (γ^*, μ^*) above constitutes a SPE for the LQG ZSDG, that is whether it satisfies the pair of inequalities (5). We will address this question in Sect. 4, after discussing in the next section some of the intricacies certainty equivalence entails, within the context of a two-stage discrete-time stochastic dynamic game. But first, we provide in the subsection below the counterpart of the main result of this subsection for the discrete-time dynamic game.

2.5 The LQG Discrete-Time ZS Dynamic Game and Certainty Equivalence

Consider the discrete-time (DT) LQG ZS dynamic game (DG) formulated in Sect. 2.2, but with non-failing channels (that is, with $p = 0$). We provide here a candidate CE SPE for this game, by following the lines of the previous subsection, but in discrete time. First, the corresponding deterministic LQ ZSDG, where both players have access to perfect state measurements admits the state-feedback SPE (Başar and Olsder 1999) (as counterpart of (21)–(22))

$$u_t^* = \gamma_t^*(x_{[0,t]}) = \tilde{\gamma}_t(x_t) = -\frac{1}{\lambda} B^T Z_{t+1} \left(N_t^{-1}\right)^{-1} A x_t, \quad t = 0, 1, \ldots, \tag{29}$$

$$v_t^* = \mu_t^*(x_{[0,t]}) = \tilde{\mu}_t(x_t) = D^T Z_{t+1} \left(N_t^{-1}\right)^{-1} A x_t, \quad t = 0, 1, \ldots, \tag{30}$$

where

$$N_t = I + \left(\frac{1}{\lambda} B B^T - D D^T\right) Z_{t+1}, \quad t = 0, 1, \ldots, \tag{31}$$

and Z_t is a non-negative definite matrix, generated by the following discrete-time game Riccati equation (DTGRE):

$$Z_t = Q + A^T Z_{t+1} \left(N_t^{-1}\right)^T A, \quad Z(T) = Q. \tag{32}$$

Under the additional condition

$$I - D^T Z_{t+1} D > 0, \quad t = 0, 1, \ldots, T-1, \tag{33}$$

which also guarantees invertibility of N, the pair (29)–(30) constitutes a SPE. If, on the other hand, the matrix (33) has a negative eigenvalue for some t, then the upper value of the game is unbounded (Başar and Bernhard 1995). As in the CT conjugate point condition, the condition (33) translates into a condition on λ, in the sense that there exists a critical value of λ, say λ_c (which will depend on the parameters of the game and the number of stages in the game), so that for each $\lambda \in (0, \lambda_c)$, the pair (29)–(30) provides a SPE to the corresponding deterministic ZSDG.

Now, the counterpart of (24)–(25), as a candidate CE SPE, would be

$$u_t^* = \gamma_t^*(y_{[0,t]}) = \tilde{\gamma}_t(\hat{x}_{t|t}) = -\frac{1}{\lambda} B^T Z_{t+1} \left(N_t^{-1}\right)^T A \hat{x}_{t|t}, \quad t = 0, 1, \ldots, \tag{34}$$

$$v_t^* = \mu_t^*(y_{[0,t]}) = \tilde{\mu}_t(\hat{x}_{t|t}) = D^T Z_{t+1} \left(N_t^{-1}\right)^T A \hat{x}_{t|t}, \quad t = 0, 1, \ldots, \tag{35}$$

where

$$\hat{x}_{t|t} := E\left[x_t | y_{[0,t]}, \left\{u_s = \gamma_s^*(y_{[0,s]}), v_s = \mu_s^*(y_{[0,s]}), s = 0, \ldots, t-1\right\}\right]$$

is generated by, with $\hat{x}_{0|-1} = 0$:

$$\begin{aligned} \hat{x}_{t|t} &= \hat{x}_{t|t-1} + \Lambda_t H^T \left(H \Lambda_t H^T + G G^T\right)^{-1} (y_t - H \hat{x}_{t|t-1}) \\ \hat{x}_{t+1|t} &= (N_t)^{-1} A \hat{x}_{t|t-1} \\ &\quad + (N_t)^{-1} A \Lambda_t H^T \left(H \Lambda_t H^T + G G^T\right)^{-1} (y_t - H \hat{x}_{t|t-1}), \end{aligned} \tag{36}$$

where the sequence $\{\Lambda_t, t = 1, 2, \ldots, T\}$ is generated by

$$\begin{aligned} \Lambda_{t+1} = {} & (N_t)^{-1} A \Lambda_t \left[I - H^T \left(H \Lambda_t H^T + G G^T\right)^{-1} H \Lambda_t\right] A^T \left((N_t)^{-1}\right)^T \\ & + F F^T, \end{aligned} \tag{37}$$

with the initial condition Λ_0 being the covariance of x_0.

Now, if instead of the noisy channel, we have intermittent failure of a channel which otherwise carries perfect state information, modeled as in Sect. 2.2 but with *clean* transmission when the channel operates (failure being according to an independent Bernoulli process as before), then as a candidate CE SPE, the pair (34)–(35) is replaced by

$$u_t^* = \gamma_t^*(y_{[0,t]}) = \tilde{\gamma}_t(\zeta_t) = -\frac{1}{\lambda} B^T Z_{t+1} \big(N_t^{-1}\big)^T A \zeta_t, \quad t = 0, 1, \ldots, \tag{38}$$

$$v_t^* = \mu_t^*(y_{[0,t]}) = \tilde{\mu}_t(\zeta_t) = D^T Z_{t+1} \big(N_t^{-1}\big)^T A \zeta_t, \quad t = 0, 1, \ldots, \tag{39}$$

where the stochastic sequence $\{\zeta_t, t = 0, 1, \ldots\}$ is generated by

$$\zeta_t = \beta_t y_t + (1 - \beta_t) \bigg(I - \Big(\frac{1}{\lambda} B B^T + D D^T \Big) Z_{t+1} \big(N_t^{-1}\big)^T \bigg) A \zeta_{t-1}, \quad \zeta_0 = y_0. \tag{40}$$

We will explore in Sect. 4 whether these CE policies are in fact SPE policies, and under what conditions.

3 A Two-Stage Discrete-Time Game with Common Measurement

To explicitly demonstrate the fact that certainty equivalence generally fails in games (but holds in a restricted sense), we consider here a specific 2-stage version of the formulation (7), (8), (11):

$$x_2 = x_1 - u + w_1; \qquad x_1 = x_0 + 2v + w_0, \tag{41}$$

$$y_1 = \beta_1(x_1 + r_1); \qquad y_0 = \beta_0(x_0 + r_0), \tag{42}$$

$$J(\gamma, \mu) = E\big\{(x_2)^2 + \lambda u^2 - v^2 | u = \gamma(\cdot), v = \mu(\cdot)\big\}, \tag{43}$$

with $u = \gamma(y_1, y_0; \beta_1, \beta_0)$, $v = \mu(y_0; \beta_0)$, where the random variables x_0, w_0, w_1, r_0, r_1 are independent, Gaussian, with zero mean and unit variance, and β_1, β_0 are independent Bernoulli random variables with Probability$(\beta_t = 0) = p$, for $t = 0, 1$.

3.1 Certainty-Equivalent SPE

The deterministic version of the game above, with $u = \gamma(x_1, x_0)$, $v = \mu(x_0)$, admits a unique saddle-point solution (Başar and Olsder 1999), given by

$$\gamma^*(x_1, x_0) = \frac{1}{1 + \lambda} x_1, \qquad \mu^*(x_0) = \frac{2\lambda}{1 - 3\lambda} x_0, \tag{44}$$

whenever

$$0 < \lambda < \frac{1}{3}, \tag{45}$$

and for $\lambda > 1/3$, the upper value is unbounded.

Now, a certainty-equivalent (CE) SPE for the original stochastic game, if exists, would be one obtained from the SPE of the related deterministic dynamic game, as above, by replacing x_0 and x_1 by their conditional means, which in the case of x_1 would require the SP policy at the earlier stage (that is, stage 0). Carrying this out, we have

$$\mu^*(y_0; \beta_0) = \frac{2\lambda}{1-3\lambda} E[x_0 | y_0; \beta_0] = \frac{\lambda}{1-3\lambda} y_0, \tag{46}$$

and

$$\begin{aligned} \gamma^*(y_{[0,1]}; \beta_{[0,1]}) &= \frac{1}{1+\lambda} E\big[x_1 | y_1, y_0; \beta_1, \beta_0; v = \mu^*(y_0; \beta_0)\big] \\ &= \frac{1}{1+\lambda}\left[\beta_1\left(\frac{2}{3}y_1 - \frac{3}{10}y_0 - \frac{6}{5}\mu^*(y_0; \beta_0)\right)\right. \\ &\quad \left. + \beta_0\left(-\frac{1}{15}y_1 + \frac{1}{2}y_0 + 2\mu^*(y_0; \beta_0)\right)\right]. \end{aligned} \tag{47}$$

Note that if the channel does not fail at all (that is, $\beta_0 = \beta_1 = 1$), then one can have a simpler expression for (47), given by:

$$\gamma^*(y_1, y_0) = \frac{3}{5(1+\lambda)} y_1 + \frac{1}{5(1-3\lambda)} y_0. \tag{48}$$

3.2 Analysis for $p = 0$ for CE SPE and Beyond

We assume in this subsection that $p = 0$, in which case the CE SPE (whose SPE property is yet to be verified) is given by (46)–(48). It is easy to see that $J(\gamma^*, v)$, with γ^* as in (48) is unbounded in v unless $\lambda < 3/25$, which means that the CE pair (46)–(48) cannot be a SPE for $\lambda \in [3/25, 1/3)$, even though the pair (44) was for the deterministic game. For the interval $\lambda \in (0, 3/25)$, however, the CE pair (46)–(48) is a SPE for the stochastic game, as it can easily be shown to satisfy the pair of inequalities (5). Further, for this case, since μ^* is the unique maximizer to $J(\gamma^*, \mu)$, and γ^* is the unique minimizer to $J(\gamma, \mu^*)$, it follows from the *ordered interchangeability* property of multiple SP equilibria (Başar and Olsder 1999) that the SPE is *unique*. Hence, for the parametrized stochastic dynamic game, a "restricted" CE property holds—*restricted* to only some values of the parameter.

Now the question is whether the parametrized stochastic game admits a SPE for $\lambda \in [3/25, 1/3)$. Clearly, it cannot be a CE SPE, that is the SPE of the deterministic version cannot be used to obtain it. Note that, for $\lambda \in [3/25, 1/3)$, if one picks

$\gamma(y_1, y_0) = [1/(1+\lambda)]y_1$ in $J(\gamma, \mu)$, then the maximum of this function with respect to μ exists and is bounded, which means that the upper value of the stochastic game is bounded. Its lower value is also clearly bounded (simply pick $v = 0$).

Again working with the special case $p = 0$ (that is no failure of the noisy channels), we now claim that there indeed exists a SPE for $\lambda \in [3/25, 1/3)$, but it entails a mixed strategy for the maximizer (Player 2) and still a pure strategy for the minimizer (Player 1). These are:

$$\begin{aligned} v = \mu^*(y_0) &= \frac{2\lambda}{1-3\lambda} E[x_0|y_0] + \xi = \frac{\lambda}{1-3\lambda} y_0 + \xi, \\ \xi \sim N\big(0, \sigma^2\big), \quad \sigma^2 &= \frac{4 - 5\sqrt{1-3\lambda}}{8\sqrt{1-3\lambda}}, \end{aligned} \tag{49}$$

and

$$\begin{aligned} u = \gamma^*(y_1, y_0) &= \frac{1}{1+\lambda} E\big[x_1|y_1, y_0, v = \mu^*(y_0)\big] \\ &= \frac{2 - \sqrt{1-3\lambda}}{2(1+\lambda)} y_1 + \frac{1}{4\sqrt{1-3\lambda}} y_0. \end{aligned} \tag{50}$$

First note that $\sigma^2 > 0$ for $\lambda \in (3/25, 1/3)$, and $\sigma^2 = 0$ at $\lambda = 3/25$, and further that the policies (49)–(50) agree with (46)–(47) at $\lambda = 3/25$, and hence transition from CE SPE to the non-CE one is continuous at $\lambda = 3/25$. Now, derivation of (49)–(50) as a SPE uses the *conditional equalizer* property of the minimizer's policy (that is (50)). One constructs a policy γ for the minimizer, under which (that is, with $u = \gamma(y_0, y_1)$) the conditional cost

$$E\big\{(x_2)^2 + \lambda u^2 - v^2 | y_0\big\}$$

becomes independent of v, and (50) is such a policy; it is in fact the unique such policy in the linear class. Hence, any choice of μ, broadened to include also mixed policies, would be a maximizing policy for Player 2, and (49) is one such policy. This establishes the left-hand-side inequality in (5). For the right-hand-side inequality, it suffices to show that (50) minimizes $J(\gamma, \mu^*)$; this is in fact a strictly convex LQG optimization problem, whose unique solution is (50). Because of this uniqueness, and *ordered interchangeability* of multiple SPE (Başar and Olsder 1999), the SPE (49)–(50) is *unique*.

3.3 The Case $p > 0$

We now turn to the case where the channels fail with positive probability, for which a candidate pair of SPE policies, based on CE, was given by (46)–(47). Their SPE property is yet to be shown, as well as the range of values of λ for which it is valid as a SPE is yet to be determined, which we address in this subsection. Toward that end, let us first consider, as a benchmark, the special case with noise-free (but still failure prone) measurement channel. This would therefore correspond to the formulation where (42) is replaced by

$$y_1 = \beta x_1; \qquad y_0 = \beta_0 x_0.$$

The counterpart of (46)–(47) in this case would be (this can also be obtained directly from the perfect-state SPE):

$$v = \mu^*(y_0; \beta_0) = \frac{2\lambda}{1-3\lambda} y_0, \tag{51}$$

$$u = \gamma^*(y_{[0,1]}; \beta_{[0,1]}) = \beta_1 \frac{1}{1+\lambda} y_1 + (1-\beta_1)\frac{1}{1+\lambda}\big[y_0 + 2\mu^*(y_0)\big]. \tag{52}$$

Now, if Player 2 employs (51), then the unique response of Player 1 will be $(1/(1+\lambda))x_1$ for $\beta_1 = 1$, and $(1/(1+\lambda))E[x_1|x_0] = (1/(1+\lambda))(x_0 + 2\mu^*(x_0))$ if $\beta_1 = 0$ and $\beta_0 = 1$, which agrees with (52); if both β's are *zero*, then clearly Player 1's response will also be *zero*. Note that the responses by Player 1 in all these cases are unique.

If Player 1 employs (52), then the conditional cost (conditioned on y_0, β_0) seen and to be maximized by Player 2 is:

For $\beta_0 = 1$ (after some simplifications):

$$1 + p + (1-p)\frac{\lambda}{1+\lambda}\big[1 + (x_0 + 2v)^2\big] + p\left[\frac{\lambda}{1+\lambda}x_0 + 2v - \frac{2}{1+\lambda}\mu^*(y_0)\right]^2 p$$
$$+ \frac{\lambda}{(1+\lambda)^2}\big[x_0 + 2\mu^*(y_0)\big]^2 p - v^2 \tag{53}$$

and for $\beta_0 = 0$ (after some simplifications):

$$1 + p + \frac{\lambda}{1+\lambda}(2-p) + \frac{3\lambda + 4p - 1}{1+\lambda}v^2. \tag{54}$$

Both (53) and (54) are strictly concave in v if and only if

$$p < \frac{1}{4} \quad \text{and} \quad \lambda < \frac{1-4p}{3}, \tag{55}$$

in which case the unique maximizing solution to (54) is $v^* = 0$, and likewise to (53) is $v^* = \mu^*(x_0) = (2\lambda/1-3\lambda)x_0$, both of which agree with (51). Hence, the CE pair (51)–(52) indeed provides a SPE if the condition (55) holds, that is the failure probability should be less than $1/4$, and the parameter λ should be less than a specific threshold, which decreases with increasing p. Note that if $p = 0$, we recover the earlier condition (45) for the deterministic game, where we know that if $\lambda > 1/3$, then the upper value is unbounded. The question is whether the same applies here, for $\lambda > (1-4p)/3$. This indeed is the case, as one can easily argue that the concavity condition of (54) cannot be improved further as universally optimal choice for γ when Player 1 has access to x_1 but not x_0 has led to that conditional cost. Hence, the pair (51)–(52) is the complete set of SPE for the game of this subsection (with channel failure but no noise in the channel), and the condition (55) is tight.

We are now in a position to discuss the SPE of the original stochastic game of this section, to find the conditions (if any) under which the CE policies (46)–(47) are in SPE, and whether those conditions can be relaxed by employing structurally different policies (as in Sect. 3.2).

3.4 CE SPE and Beyond for the 2-Stage Game

To obtain the complete set of SPE for the original stochastic 2-stage game, our starting point will be the pair of CE policies (γ^*, μ^*) given by (47) and (46), and to find the region in the $\lambda - p$ space for which these policies constitute a SPE. Clearly, we would expect that region (if exists) to be no larger than the one described by (55) since that one corresponded to the noise-free channel.

Let us first consider the right-hand inequality of (5) for this game, with $\mu^*(y_0; \beta_0)$ given by (46). In terms of γ this is a strictly convex quadratic optimization problem, which one minimizes with respect to u after conditioning the cost on $y_{[0,1]}$ and $\beta_{[0,1]}$; the result is the unique solution given by (47). This part of the inequality does not bring in any additional restriction on λ and p, other than the condition $\lambda < 1/3$ needed in the expression for μ^*.

The left-hand inequality of (5) for this game is a bit more involved. We now pick γ^* as given by (47), and maximize the resulting cost over μ, which is equivalent to maximizing the conditional cost with respect to v where conditioning is with respect to y_0 and β_0. Even though this is also a quadratic optimization problem, existence and uniqueness of maximum are not guaranteed for all values of λ and p, and we have to find (necessary and sufficient) conditions for strict concavity (in v). Now, the conditional cost (conditioned on (y_0, β_0), and with $v = \mu(y_0, \beta_0)$) is:

For $\beta_0 = 1$ (after some simplifications):

$$
\begin{aligned}
& p\left[x_0 + 2v - \frac{1}{2(1-3\lambda)} y_0\right]^2 + (1-p)\left[\frac{2+5\lambda}{5(1+\lambda)}(x_0+2v) - \frac{1}{5(1-3\lambda)} y_0\right]^2 \\
& \quad + \lambda(1-p)\left[\frac{3}{5(1+\lambda)}(x_0+2v) + \frac{1}{5(1-3\lambda)} y_0\right]^2 - v^2 \\
& \quad + 2p + (1-p)\frac{50\lambda^2 + 88\lambda + 38}{25(1+\lambda)^2} + \frac{\lambda p}{4(1-3\lambda)^2}(y_0)^2, \qquad (56)
\end{aligned}
$$

and for $\beta_0 = 0$ (after some simplifications):

$$(1-p)\left[\frac{2+5\lambda}{5(1+\lambda)}(x_0+2v)\right]^2+\lambda(1-p)\left[\frac{3}{5(1+\lambda)}(x_0+2v)\right]^2-v^2$$
$$+\,p[x_0+2v]^2+2p+(1-p)\frac{50\lambda^2+88\lambda+38}{25(1+\lambda)^2}. \quad (57)$$

Both (56) and (57) are strictly concave in v if and only if the coefficients of the quadratic terms in v (identical in the two cases) are negative, that is

$$(1-p)\left[\frac{4(2+5\lambda)^2}{25(1+\lambda)^2}+\frac{36\lambda}{25(1+\lambda)^2}\right]^2+4p-1<0,$$

which is equivalent to

$$p<\frac{3}{28} \quad \text{and} \quad \lambda<\frac{3-28p}{25}. \quad (58)$$

We note that the upper bound on the failure probability p is precisely the condition that makes the upper bound on λ in (58) positive. Another point worth making is that we naturally would expect the conditions on p and λ as given above in (58) to be more stringent than the ones in (55), for the noise-free case. Clearly the condition on p is more restrictive, as $3/28 < 1/4$. For the bound on λ, it again immediately follows that

$$\frac{3-28p}{25}<\frac{1-4p}{3}, \quad (59)$$

whenever $p < 1$.

Now, to complete the verification of the SPE property of the pair (47) and (46), we still have to show that the unique maximizers of the strictly concave (under (58)) quadratic conditional costs (57) and (56) are given by (46). For the former, the result follows readily since its maximizer is $v = 0$. For the latter, a simple differentiation with respect to v, and using $E[x_0|y_0, \beta_0 = 1] = (1/2)y_0$, leads after some extensive calculations and simplifications to $v = [\lambda/(1-3\lambda)]y_0$, which is the same as (46).

Hence, the SPE for the 2-stage game of this section (with noisy channels and nonzero failure probabilities) is a CE SPE, but for a more restrictive set of values for p and λ (compare (58) with (55), as we have noted earlier). The question now is whether the gap can be closed by using non-CE policies, as was done in the failure-free case ($p = 0$). Clearly, the upper bound of the game is finite for the entire set of values of p and λ in (55) (simply substitute (52) into the cost for u, with additive noise in y_1 and y_0) and note that the presence of additive noise in the channels does not alter the required concavity condition, and hence we have a well-defined strictly concave quadratic maximization problem for v under the same condition (55).

As already mentioned, the region in the parameter space $\lambda - p$ that corresponds to the CE SPE (47) and (46) is smaller than the region corresponding to the SPE of the noise-free channel case, and the question now is whether region of existence of a SPE can be enlarged by transitioning to a pair of non-CE policies, as it was done in Sect. 3.4 for the case when $p = 0$. We will see that this is indeed the case, and an equalizer policy for Player 1 does the job. Its derivation, however, is a bit more complicated than the one of Sect. 3.4 because the possibility of channel failures brings in an additional element of complexity (even though the basic idea is still the same). Let us first assume that $\beta_0 = 1$, and start with a general linear policy for Player 1:

$$u = \hat{\gamma}(y_1, y_0, \beta_1) = \alpha_1 y_1 + \alpha_0(\beta_1) y_0, \tag{60}$$

where α_1, $\alpha_0(\beta_1 = 1) =: \alpha_0^1$, $\alpha_0(\beta_1 = 0) =: \alpha_0^0$ are constant parameters yet to be determined.[3] They will be determined in such a way that with (60) used in $J(\gamma, v)$, the latter expression becomes independent of v (when conditioned on y_0). Skipping the details, the expression for

$$J(\hat{\gamma}, v) = E\big\{\big(x_1 - \hat{\gamma}(y_1, y_0, \beta_1) + w_1\big)^2 + \lambda\big(\hat{\gamma}(y_1, y_0, \beta_1)\big)^2 - v^2 | y_0, \beta_0 = 1\big\} \tag{61}$$

is

$$\begin{aligned} & p\big[x_0 + 2v - \alpha_0^0 y_0\big]^2 + (1-p)\big[(1-\alpha_1)(x_0 + 2v) - \alpha_0^1 y_0\big]^2 \\ & \quad + \lambda(1-p)\big[\alpha_1(x_0 + 2v) + \alpha_0^1 y_0\big]^2 - v^2 + (1-p)\lambda\big(2(\alpha_1)^2 + \big(\alpha_0^1\big)^2 + 1\big) \\ & \quad + 2p + (1-p)\big((\alpha_1)^2 + (1-\alpha_1)^2 + 1\big) + \lambda p\big(\alpha_0^0\big)^2 (y_0)^2 + 1. \end{aligned} \tag{62}$$

This is a quadratic function of v; the coefficient of v^2 can be annihilated by choosing

$$\alpha_1 = \frac{1}{1+\lambda}\left[1 - \frac{\sqrt{(1-4p-3\lambda)(1-p)}}{2(1-p)}\right], \tag{63}$$

which is a well-defined expression provided that $4p + 3\lambda < 1$, and naturally (since $\lambda > 0$) also $p < 1/4$ which are identical to (55). For annihilation of the coefficient of v, on the other hand, we need the following relationship between α_0^1 and α_0^0:

$$2p\alpha_0^0 + \alpha_0^1 \sqrt{(1-4p-3\lambda)(1-p)} = \frac{1}{4}. \tag{64}$$

Now, we have to show that these are best responses to some policy by Player 2, which will necessarily be a mixed strategy, as in Sect. 3.4. The process now is to assume that v has the form[4]

$$v = \hat{\mu}(y_0) = k_0 y_0 + \xi, \quad \xi \sim N\big(0, \sigma^2\big), \tag{65}$$

for some k_0 and σ^2; find the best response of Player 1 to this by minimizing $J(\gamma, \hat{\mu})$ with respect to γ (which, by strict convexity, will clearly be unique, and be in

[3] We have taken α_1 not to be dependent on β_1, because when $\beta_1 = 0$, $y_1 \equiv 0$, making α_1 superfluous.

[4] One can take any form here, since $\hat{\gamma}$ had annihilated v, but we take linear-plus-Gaussian in anticipation of $\hat{\gamma}$ to be in equilibrium with v.

the structural form (60)); require consistency with (63)–(64); and solve for k_0, σ^2, α_0^1, α_0^0. The outcome is the following set of unique expressions:

$$k_0 = \frac{\lambda}{1-3\lambda}, \qquad \sigma^2 = \frac{\sqrt{(1-p)}}{2\sqrt{(1-4p-3\lambda)}} - \frac{5}{8} \tag{66}$$

$$\alpha_0^0 = \frac{1}{2(1-3\lambda)}, \qquad \alpha_0^1 = \frac{\sqrt{(1-4p-3\lambda)(1-p)}}{4(1-p)(1-3\lambda)}. \tag{67}$$

To complete the construction of the SPE, we still have to find the conditions under which σ^2 as given above is well defined (that is, it is positive). The required condition (both necessary and sufficient, provided that (55) holds, which is a natural condition) is

$$4(1-p) > 5\sqrt{(1-4p-3\lambda)(1-p)} \quad \Leftrightarrow \quad \lambda > \max\left(0, \frac{3-28p}{25}\right). \tag{68}$$

Note that the (lower) bound on λ matches exactly the upper bound in (58), and hence non-CE SPE policies make up for the restriction brought in by the CE SPE.

To gain further insight (for purposes of establishing continuity) we can look at two limiting cases: (i) For $p = 0$, the non-CE solution matches exactly the one given in Sect. 3.2 for the failure-free case. (ii) At $\lambda = (3-28p)/25$, with $p > 3/28$, which is the boundary between the two regions corresponding CE and non-CE SPE, α_1 in (63) is $3/[5(1+\lambda)]$, which is exactly the coefficient of y_1 in (47) with $\beta_1 = \beta_0 = 1$; α_0^1 in (67) is $1/[5(1-3\lambda)]$, which is exactly the coefficient of y_0 in (47) with $\beta_1 = \beta_0 = 1$; and finally, α_0^0 in (67) (which does not depend on p) is exactly the coefficient of y_0 in (47) with $\beta_1 = 0$, $\beta_0 = 1$, and this one is for all λ satisfying all other conditions, and not only at the boundary.

The remaining case to analyze is $\beta_0 = 0$. The CE SPE in this case would be (from (46)–(47)):

$$\gamma^*(y_1; \beta_1) = \frac{2}{3(1+\lambda)} y_1, \qquad v^* = 0, \tag{69}$$

which is a valid one (that is the cost under γ^* is strictly concave in v) if and only if the multiplying term for v^2 is negative, that is

$$(1-p)\left[\frac{4(3\lambda+1)^2}{9(1+\lambda)^2} + \frac{16\lambda}{9(1+\lambda)^2}\right] - 1 + 4p < 0,$$

which simplifies to $\lambda < (5-32p)/27$, for which we need $p < 5/32$ (for positivity). To extend the solution to a larger region, we again have to look for an equalizer policy that annihilates v, and is also best response to $v = \xi \sim N(0, \sigma^2)$ for some σ^2. Following the same process as earlier, we start with $u = \alpha_1 y_1$, and compute the cost faced by Player 2, where the multiplying term for v^2 is:

$$4(1-p)\big[(1-\alpha_1)^2 + \lambda(\alpha_1)^2\big] - 1 + 4p.$$

Setting this equal to zero, and solving for α_1 we obtain the expression given by (63). Now, the best response by Player 1 to $v = \xi$ is

$$u = \frac{1}{1+\lambda} E[x_0 + 2\xi + w_0 | y_1] = \frac{1}{\lambda}\left(1 - \frac{1}{2}\sqrt{1-3\lambda}\right) y_1,$$

where we then invoke the multiplying term above to equal α_1 given by (63), which leads to the following unique expression for σ^2:

$$\sigma^2 = \frac{\sqrt{1-p}}{2\sqrt{1-4p-3\lambda}} - \frac{3}{4}, \tag{70}$$

which is well-defined and positive provided that

$$\max\left(0, \frac{5-32p}{27}\right) < \lambda < \frac{1-4p}{3}. \tag{71}$$

Note that the lower bound on λ matches the upper bound in the case of the CE SPE, and that the SPE policy of Player 1 is continuous across the boundary $\lambda = (5-32p)/27$.

We now collect all this in the following theorem, which is the main result of this section.

Theorem 1 *The two stage discrete-time stochastic game formulated in this section admits a saddle-point equilibrium (SPE) provided that*

$$0 \le p < \frac{1}{4} \quad \text{and} \quad 0 < \lambda < \frac{1-4p}{3};$$

otherwise, the upper value of the game is unbounded. The SPE policies of the players, (γ^*, μ^*), admit different characterizations in two different regions of the parameter space, and also depending on whether $\beta_0 = 1$ or 0:

- *For $\lambda \le (5-32p)/27$ and $p < 5/32$ when $\beta_0 = 0$, and $\lambda \le (3-28p)/25$ and $p < 3/28$ when $\beta_0 = 1$, γ^* and μ^* are given by (47) and (46), respectively; this constitutes a certainty-equivalent (CE) SPE.*
- *For $\max(0, (5-32p)/27) < \lambda < 1-(4p/3)$ and $p < 1/4$ when $\beta_0 = 0$, and $\max(0, (3-28p)/25) < \lambda < 1-(4p/3)$ and $p < 1/4$ when $\beta_0 = 1$, the SPE policies are of the non-CE type, given by*

$$\gamma^*(y_1, y_0; \beta_1, \beta_0) = \alpha_1 y_1 + \left(\beta_1 \alpha_0^1 + (1-\beta_1)\alpha_0^0\right) y_0 \tag{72}$$

$$\mu^*(y_0; \beta_0) = k_0 y_0 + \xi,$$
$$\xi \sim N\left(0, \sigma^2\right), \quad \sigma^2 = \frac{\sqrt{1-p}}{2\sqrt{1-45-3\lambda}} - \frac{3}{4} + \frac{1}{8}\beta_0, \tag{73}$$

where α_1 is given by (63), and the pair (α_0^0, α_0^1) is given by (67).

4 CE SPE of the LQG ZSDG in Continuous and Discrete Time

4.1 Various Approaches Toward Construction of SPE

For a two-person ZSDG (in normal or strategic form), with strategy spaces Γ (for Player 1, the minimizer) and $\mathcal{M}$ (for Player 2, the maximizer), with (expected) cost function J, defined on $\Gamma \times \mathcal{M}$, let us recall from (5) that a pair $(\gamma^* \in \Gamma, \mu^* \in M)$ is in SPE if

$$J(\gamma^*, \mu) \leq J(\gamma^*, \mu^*) \leq J(\gamma, \mu^*), \quad \forall \gamma \in \Gamma, \mu \in \mathcal{M}.$$

The general direct approach toward derivation of a SPE would be:

- Fix $\mu \in \mathcal{M}$ as an *arbitrary* policy for Player 2, and minimize $J(\gamma, \mu)$ with respect to γ on Γ.
- Fix $\gamma \in \Gamma$ as an *arbitrary* policy for Player 1, and maximize $J(\gamma, \mu)$ with respect to μ on $\mathcal{M}$.
- Look for a fixed point, which would then be a SPE.

Even though direct, this approach would entail a very complex process for *dynamic* games (in continuous or discrete time), even if they are of the linear-quadratic type. Unless the information structure is static, the optimization problems involved structurally depend on the selection of arbitrarily fixed policies, rendering the underlying optimization problems unwieldy.

An alternative (still direct) approach would be a recursive (*backward-forward*) one, applicable to discrete-time dynamic games with particular information structures, and possibly extendable to some classes of continuous-time ZSDGs:

- Proceed recursively at $t = T - 1, T - 2, \ldots$.
- At t, solve for SPE (if exists) of the 1-stage game by fixing in J policies for $t + 1, \ldots, T - 1$ at their optimum choices and for $0, \ldots, t - 1$ arbitrarily, with the former possibly depending on the optimum policies (yet to be determined) at $0, 1, \ldots, t$.

Such a construction is doable, but it is quite tedious (and depends on the specific information structure, and applies primarily to discrete-time games); for such a derivation, in a broader Nash equilibrium context, see Başar (1978a).

A third, *indirect* approach, entails expansion of information structures of the players, obtaining a SPE in the induced expanded (richer) policy spaces, and then projecting the solution (*contracting* it) back to the original policy spaces. Such an approach works when the SPE values of the two games (one on original policy spaces and the other one on the expanded ones) are the same, and this is generally the case if the *expansion* involves only past actions of the players. Hence, we have the following process:

- Endow both players with past actions, assuming that they already have access to the past measurements in terms of which the actions were generated.

- Any SPE to the original stochastic dynamic game (SDG) is also a SPE to the new one (but naturally not vice versa).
- Any two SPE of the new SDG are *ordered interchangeable*.
- Solve for some (conveniently constructed) SP policies for the new SDG, and find *representations* (Başar and Olsder 1999) in the original policy spaces.
- *Verify* for the original SDG that the policies arrived at are indeed in SPE (this step is a verification of existence, which is much simpler than verifying characterization).

A further justification of this indirect approach can be found in Başar (1981). In the next two subsections, we illustrate the approach on the two LQ ZSDGs introduced and discussed in Sects. 2.1, 2.2, 2.4, and 2.5. While doing this, we have to keep in mind the features we have observed within the context of the 2-stage ZS SDG of Sect. 3.

4.2 SPE Property of CE Policies of the LQG ZSDG

We turn here to the continuous-time LQG ZSDG of Sect. 2.1, for which the CE policies (24)–(25) were offered as a candidate SPE for the original SDG with noise in the common channel. We now investigate whether these policies are indeed in SPE for at least some region of the parameter space (as was the case for the 2-stage game of Sect. 3). Toward this end, we first enlarge the policy spaces of the players to include also past actions, that is, the players now have access to $(y_{[0,t)}, u_{[0,t)}, v_{[0,t)})$ at time t. Denote the corresponding expanded policy spaces for Players 1 and 2 by $\tilde{\Gamma}$ and $\tilde{\mathcal{M}}$, respectively. If $y_{[0,t)}$ was replaced by x_t (that is, the perfect state measurement case) and still allowing players to have access to past actions, the pair of policies (21)–(22) would still be in SPE (Başar and Olsder 1999), whose CE counterpart in $\tilde{\Gamma} \times \tilde{\mathcal{M}}$ would still be of the form (24)–(25), with however $\{\hat{x}_t, t \geq 0\}$ replaced by $\{\zeta_t, t \geq 0\}$, generated by

$$d\zeta_t = (A\zeta_t + Bu_t + Dv_t)dt + K(t)(dy_t - H\zeta_t dt), \quad \zeta_0 = 0, t \geq 0. \tag{74}$$

Note that the above is still the Kalman filter equation, but driven not only by the measurement but also by the past actions. Now, one can show using the ordered interchangeability property of multiple SPE policies that any pair of SP policies in $\Gamma \times \mathcal{M}$ also constitute a SPE in the expanded policy spaces $\tilde{\Gamma} \times \tilde{\mathcal{M}}$ (but not vice versa) (Başar and Olsder 1999; Başar 1981), and further that by some standard properties of the LQG control problem discussed in Sect. 2.3, the pair (24)–(25) indeed constitutes a SPE for the new SDG with expanded policy spaces, provided that the RDE (23) does not have a conjugate point in the interval $[0, t_f]$, which is exactly the condition of existence of SPE to the LQG ZSDG with perfect-state measurements.

Clearly, however, the SP policies above for the SDG with expanded policy spaces are not implementable even for that game, because they require cooperation on the generation of the conditional mean of x, or that estimate ζ (as in (74)) to be generated by a third party, and supplied to the two antagonistic players, which is not realistic. To make it real-time implementable, and in line with the adversarial aspect of the game, we have to replace these policies with ones that allow players to run their own filters, driven by the common measurement (but not with actions of the players), as given below:

$$u^*(t) = \gamma_t^*(y_{[0,t)}) = \gamma_t^{\text{CE}}(z_t) = -\frac{1}{\lambda} B^T Z(t) z_t, \quad t \geq 0, \tag{75}$$

$$v^*(t) = \mu_t^*(y_{[0,t)}) = \mu_t^{\text{CE}}(\eta_t) = D^T Z(t) \eta_t, \quad t \geq 0, \tag{76}$$

where z and η are generated by (as counterpart of (74)):

$$dz_t = \hat{A} z_t dt + K(t)(dy_t - H z_t dt), \quad z_0 = 0, t \geq 0, \tag{77}$$

$$d\eta_t = \hat{A} \eta_t dt + K(t)(dy_t - H \eta_t dt), \quad \eta_0 = 0, t \geq 0, \tag{78}$$

where

$$\hat{A} := A - \left(\frac{1}{\lambda} B B^T - D D^T\right) Z(t),$$

K is the Kalman gain, given again by (18), with Λ solving (28).

The policies $(\gamma^{\text{CE}}, \mu^{\text{CE}})$ given by (75)–(76) constitute representations of the SP policies in the expanded policy spaces and now belong to $\Gamma \times \mathcal{M}$, and as such also constitute SPE for the original SDG, as argued earlier, provided that the response of Player 1 to (76) and that of Player 2 to (75) are well defined, leading to bounded costs. For the former, it can be shown easily (and in fact argued without any explicit computation) that

$$\min_{\gamma \in \Gamma} J\big(\gamma, \mu^{\text{CE}}\big) = J\big(\gamma^{\text{CE}}, \mu^{\text{CE}}\big),$$

and particularly that the quadratic function $J(u, \mu^{\text{CE}})$ is strictly convex in u. This establishes the right-hand-side of the SP inequality (5). For the left-hand-side inequality, on the other hand, we have the LQG optimal control problem

$$\max_{\mu \in \mathcal{M}} J\big(\gamma^{\text{CE}}, \mu\big),$$

with $2n$-dimensional differential constraints:

$$dx_t = (Ax + Dv_t)dt - \frac{1}{\lambda} B^T Z(t) z_t dt + F dwt, \quad t \geq 0,$$

$$dz_t = \tilde{A} z_t dt + K(t)(dy_t - H z_t dt), \quad z_0 = 0, t \geq 0.$$

The conjugate-point condition on (23) is not sufficient for this LQG optimal control problem to be well defined, as the cost $J(\gamma^{\text{CE}}, v)$ could be non-concave in v. Strict concavity here is in fact the only condition that would be needed for the pair

$(\gamma^{\text{CE}}, \mu^{\text{CE}})$ in (75)–(76) to constitute a SPE. Now note that $J(\gamma^{\text{CE}}, v)$ can be written as

$$J(\gamma^{\text{CE}}, v) = E\left\{|x_{t_f}|^2_{Q_f} + \int_0^{t_f}\left[|x_t|^2_Q + \frac{1}{\lambda}\left|B^T Z(t) z_t\right|^2 - |v_t|^2\right]dt\right\}$$
$$=: E\left\{|m_{t_f}|^2_{\tilde{Q}_f} + \int_0^{t_f}\left[|m_t|^2_{\tilde{Q}} - |v_t|^2\right]dt\right\},$$

where $m := (x^T z^T)^T$, $\tilde{Q}_f :=$ block diag$(Q_f, 0)$, and $\tilde{Q} :=$ block diag$(Q, (1/\lambda) \times ZBB^T Z)$. Further, m evolves according to

$$dm_t = \tilde{A} m_t dt + \tilde{D} v_t dt + \tilde{F} dw_t,$$

where $\tilde{D} := [D^T, 0]^T$, $\tilde{F} := [F^T, G^T K^T]^T$, and $\tilde{A}$ is a $2n \times 2n$ matrix, whose ij-th block is, for $i, j = 1, 2$: $[\tilde{A}]_{11} := A$, $[\tilde{A}]_{21} := KH$, $[\tilde{A}]_{12} := -(1/\lambda)BB^T Z$, and $[\tilde{A}]_{22} := \hat{A} - KH$.

The condition for strict concavity for this optimization problem, regardless of the nature of the information available to Player 2, is (Başar and Bernhard 1995) nonexistence of a conjugate point to the RDE below on the interval $[0, t_f]$:

$$\dot{S} + S\tilde{A} + \tilde{A}^T S + S\tilde{D}\tilde{D}^T S + \tilde{Q} = 0, \quad S(t_f) = \tilde{Q}_f. \tag{79}$$

We now collect all this in the theorem below.

Theorem 2 *The continuous-time LQG ZSDG of Sect.* 2.1 *admits a pure-strategy SPE provided that the RDEs* (23) *and* (79) *have well-defined nonnegative-definite solutions on the interval* $[0, t_f]$, *in which case the corresponding policies for the players, in SPE, are given by* (75)–(76). *These feature a* restricted certainty equivalence *property.*

Remark 1 A number of observations are in order here. First, the policies in SPE for the SDG are not simple CE versions of the SPE of the deterministic game, that is they are not the pair (24)–(25), even though they can be derived from the SPE of the deterministic game by endowing the players with two separate filter equations even though the players have access to a common measurement channel. Second, the condition of existence of a pure-strategy SPE for the SDG is more restrictive than its counterpart for the perfect-state version (or essentially equivalently the deterministic game). This would not be surprising in view of the results of Sect. 3 for perhaps the simplest stochastic dynamic game, where the gap between the two conditions (for existence of pure-strategy SPE in the games with perfect state and noisy state information) was completely covered by allowing for mixed strategies (for the maximizing player). It is quite plausible that the same would hold here, but derivation of such a mixed-strategy SPE for the continuous-time LQG ZSDG still remains a challenging task.

4.3 SPE Property of CE Policies of the LQG Discrete-Time ZSDG

We now proceed with an analysis that is the counterpart of the one above (in Sect. 4.2) for the discrete-time game of Sect. 2.2 and Sect. 2.5, not for the most general case, but for two scenarios: (i) when there is no failure of channels (that is $p = 0$, as in Sect. 3.2), and (ii) the channel provides perfect state measurement, but intermittently fails (as in Sect. 3.3). In both cases, we obtain restricted CE SPE. The derivation is a direct counterpart of the one in Sect. 4.2), and hence to avoid duplication we will just provide the basic results without providing details of the reasoning and the pathway.

Let us first discuss case (i). In Sect. 2.5, we had offered (34)–(35) as a candidate SPE pair for this scenario, but as we have argued in the previous subsection, having a single filter to be shared by both players is not a realistic situation, and hence we will have to introduce individualized compensators. In view of this, (34)–(35) will have to be modified as follows:

$$u_t^* = \gamma_t^*(y_{[0,t]}) = \gamma_t^{\mathrm{CE}}(z_{t|t}) = -\frac{1}{\lambda} B^T Z_{t+1}\left(N_t^{-1}\right)^T A z_{t|t}, \quad t = 0, 1, \ldots, \tag{80}$$

$$v_t^* = \mu_t^*(y_{[0,t]}) = \mu_t^{\mathrm{CE}}(\eta_t) = D^T Z_{t+1}\left(N_t^{-1}\right)^T A \eta_{t|t}, \quad t = 0, 1, \ldots, \tag{81}$$

where $z_{t|t}$ and $\eta_{t|t}$ are generated by, with $z_{0|-1} = 0$:

$$\begin{aligned} z_{t|t} &= z_{t|t-1} + \Lambda_t H^T\left(H\Lambda_t H^T + GG^T\right)^{-1}(y_t - Hz_{t|t-1}) \\ z_{t+1|t} &= (N_t)^{-1} A z_{t|t-1} \\ &\quad + (N_t)^{-1} A \Lambda_t H^T\left(H\Lambda_t H^T + GG^T\right)^{-1}(y_t - Hz_{t|t-1}), \end{aligned} \tag{82}$$

and, with $\eta_{0|-1} = 0$,

$$\begin{aligned} \eta_{t|t} &= \eta_{t|t-1} + \Lambda_t H^T\left(H\Lambda_t H^T + GG^T\right)^{-1}(y_t - H\eta_{t|t-1}) \\ \eta_{t+1|t} &= (N_t)^{-1} A \eta_{t|t-1} \\ &\quad + (N_t)^{-1} A \Lambda_t H^T\left(H\Lambda_t H^T + GG^T\right)^{-1}(y_t - H\eta_{t|t-1}), \end{aligned} \tag{83}$$

and the sequence $\{\Lambda_t, t = 1, 2, \ldots, T\}$ is as in (37). By going through similar arguments as in the previous subsection, the pair (80)–(81) provides a SPE, provided that (33) holds and the quadratic function $J(\gamma^{\mathrm{CE}}, v)$ is strictly concave in v. An explicit condition can be obtained for the latter in terms of a $2n \times 2n$ discrete-time Riccati equation, which involves a recursive verification as in (33).

For case (ii), that is when $y_t = \beta_t x_t$, $t = 0, 1, \ldots, T-1$, the starting point is the pair of policies (38)–(39), where as before we endow the players with two separate compensators, with states ζ^1 and ζ^2, generated by, for $i = 1, 2$,

$$\zeta_t^i = \beta_t y_t + (1-\beta_t)\Big(I - \Big(\frac{1}{\lambda}BB^T + DD^T\Big) Z_{t+1}\big(N_t^{-1}\big)^T\Big) A\zeta_{t-1}^i, \quad \zeta_0^i = y_0. \tag{84}$$

Hence, the players' CE policies become

$$u_t^* = \gamma_t^*(y_{[0,t]}) = \tilde{\gamma}_t^{\mathrm{CE}}(\zeta_t) = -\frac{1}{\lambda} B^T Z_{t+1}\big(N_t^{-1}\big)^{-1} A\zeta_t^1, \quad t = 0, 1, \ldots, \tag{85}$$

$$v_t^* = \mu_t^*(y_{[0,t]}) = \tilde{\mu}_t^{\mathrm{CE}}\big(\zeta_t^2\big) = D^T Z_{t+1}\big(N_t^{-1}\big)^{-1} A\zeta_t, \quad t = 0, 1, \ldots, \tag{86}$$

which, by an argument similar to the earlier case, are in SPE provided that (33) holds and the quadratic function $J(\tilde{\gamma}^{\mathrm{CE}}, v)$ is strictly concave in v. As before, an explicit condition can be obtained for the latter in terms of a $2n \times 2n$ discrete-time Riccati equation, which involves a recursive verification as in (33).

In view of the complete set of results of Sect. 3 for a 2-stage version of this game, for case (ii), we would not expect a less stringent condition to be obtained (that is, there would not be any need to expand the policy spaces to include mixed strategies), whereas for case (i) the extra condition introduced in terms of strict concavity of $J(\gamma^{\mathrm{CE}}, v)$ in v can be dispensed with by inclusion of mixed strategies. We do not pursue this any further here.

For the more general case, however, when the channel is noisy and failure probability is $p > 0$, still a restricted CE will hold, with z and η in (82)–(83) now incorporating the possibility of failures, as in the case of derivation of Kalman filters with missing measurements (Shi et al. 2010). Here also a strict concavity condition will be needed for the existence of a pure-strategy SPE, in addition to the one for $p = 0$, which however can be dispensed with by inclusion of mixed strategies.

5 Discussion, Extensions, and Conclusions

One important message that this chapter conveys (which applies to more general differential/dynamic games with similar information structures) is that in zero-sum stochastic differential/dynamic games (ZS SDGs) a restricted certainty equivalence (CE) applies if players have a common measurement channel, but the adversarial nature of the problem creates several caveats not allowing the standard notions of certainty equivalence or separation prominent in stochastic control problems (Witsenhausen 1971a, 1971b; Fleming and Soner 1993; Yüksel and Başar 2013) to find an immediate extension. Expansion of information structures to include also action information compatible with the original information, and without increasing payoff relevant information, appears to be a versatile tool in an indirect derivation of pure-strategy saddle-point equilibria (SPE), which however does not apply to derivation of mixed-strategy SPE, as it relies heavily on the ordered interchangeability property of multiple pure-strategy SPE. For the same reason, the indirect approach does not apply to nonzero-sum dynamic games; in fact, Nash equilibria of genuinely nonzero-sum stochastic games (unless they are strategically equivalent to zero-sum games or team problems) never satisfy CE (Başar 1978b). Now, coming back to ZS SDGs, when a generalized CE SPE exists in some region of the parameter space, this is not the full story because the game may also admit mixed-strategy SPE outside that region, which however has to be obtained using a different approach–using notions of *annihilation* and *conditional equalization*, as it has been demonstrated in Sect. 3. Hence, expansion of strategy (policy) spaces from *pure* to *mixed* helps to recover the missing SPE.

We have deliberately confined our treatment in this paper to ZS SDGs with *symmetric information*, to be able to introduce a *restricted* (and in some sense *generalized*) notion of CE and to show that any attempt of directly extending CE from stochastic optimal control to games is a path full of pitfalls, even though the problem (of derivation of SPE) is still tractable, but using an indirect approach (that makes use of expansion of strategy spaces and ordered interchangeability property of multiple pure-strategy SPE). As indicated earlier, this approach does not extend to nonzero-sum games (NZSGs), because expansion of strategy spaces (through actions) leads to multiplicity of Nash equilibria, and in fact a continuum of them (Başar and Olsder 1999), and multiple Nash equilibria (NE) are not orderly interchangeable. Still, there is another approach to derivation of NE with nonredundant information, as briefly discussed in Sect. 1, provided that we have a discrete-time game, with complete sharing of information (that is, a common measurement channel) or sharing of observations with one step delay (Başar 1978a). The same approach would of course apply to ZSDGs as well (with one-step delayed sharing), but then the SPE will not be of the CE type. If there is no sharing of information (or with delay of two units or more), and players receive noisy state information through separate channels, then the problem remains to be challenging in both ZS and NZS settings, unless there is a specific structure of the system dynamics along with the information structure, as in Nayyar and Başar (2012).

Several fairly direct extensions of the results of this chapter are possible, all in the ZS setting. First, it is possible to introduce intermittent failure of the common measurement channel (2) in the continuous-time case, by mimicking (8):

$$dy_t = \beta_t(Hx_t dt + G dw_t) \quad \text{or} \quad dy_t = \beta_t H x_t dt + G dw_t, \quad t \geq 0,$$

where β_t is an independent two-state Markov jump process (or a piecewise deterministic process) with a given rate (jumps are between $\beta_t = 1$ and $\beta_t = 0$), and both players observe realization of this process. The counterpart of the analysis for the discrete-time case could be carried over to this case also (for a related framework, see Pan and Başar 1995). A variant of this, in both discrete and continuous time, is the more challenging class of problems where the failure of the transmission of the common noisy measurement of the state to the players is governed by two independent Bernoulli processes with possibly different rates. Such ZS SDGs would involve primarily two scenarios: (i) the players are not aware of the failure of links corresponding to each other, and (ii) this information is available (that is players share explicitly or implicitly the failure information) but with one step delay.

Further extensions to (i) multi-player ZS SDGs (with teams playing against teams, where agents in each team do not have identical information), and (ii) nonzero-sum stochastic differential games (with particular type of asymmetric information among the players) constitute yet two other classes of challenging problems. In all these problems, including the ones discussed in Sect. 4, characterization of mixed-strategy SPE (as extension of the analysis of Sect. 3) or NE stand out as challenging but tractable avenues for future research.

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Policy Interactions in a Monetary Union: An Application of the OPTGAME Algorithm

Dmitri Blueschke and Reinhard Neck

Abstract In this chapter we present an application of the dynamic tracking games framework to a monetary union. We use a small stylized nonlinear two-country macroeconomic model (MUMOD1) of a monetary union to analyse the interactions between fiscal (governments) and monetary (common central bank) policy makers, assuming different objective functions of these decision makers. Using the OPTGAME algorithm we calculate equilibrium solutions for four game strategies: one cooperative (Pareto optimal) and three non-cooperative games: the Nash game for the open-loop information pattern, the Nash game for the feedback information pattern, and the Stackelberg game for the feedback information pattern. Applying the OPTGAME algorithm to the MUMOD1 model we show how the policy makers react to demand and supply shocks according to different solution concepts. Some comments are given on possible applications to the recent sovereign debt crisis in Europe.

1 Introduction

The economic situation in the European Monetary Union (EMU) is relatively unstable nowadays due to the economic crisis of 2007–2010 and a wide range of structural problems in the affected countries. At the breakout of the last economic crisis policy makers tried to cooperate and to use coordinated countercyclical fiscal and monetary policies to reduce the negative impact of the crisis, placing great emphasis on the GDP growth rate and unemployment. Unfortunately, the public debt situation worsened dramatically and we have been facing a severe sovereign debt crisis in Europe since 2010. Today, there is no consensus among politicians on what is the best way out of the crisis. The European Monetary Union does not appear to be acting like a union of cooperating partners speaking with one voice but like a pool of independent players seeking gains for their own country only. The core of the problem

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seems to be a lack of agreement about objectives and strategies to pursue. This is a typical problem of dynamic strategic interaction. Hence, it is appropriate to run a study of a monetary union using concepts of dynamic game theory.

The framework of dynamic games is most suitable to describe the dynamics of a monetary union because a monetary union consists of several players with independent and different aims and instruments. Even if there are common, union-wide objectives, each of the players may assign different importance (weights) to these targets. In addition, the willingness to cooperate to achieve the common goal is country-specific as well. For these reasons it is necessary to model the conflicts ('non-cooperation') between the players. Such problems can best be modeled using the concepts and methods of dynamic game theory, which has been developed mostly by engineers and mathematicians but which has proved to be a valuable analytical tool for economists, too (see, e.g., Başar and Olsder 1999; Van Aarle et al. 2002).

In this chapter we present an application of the dynamic tracking game framework to a macroeconomic model of a monetary union. Dynamic games have been used by several authors (e.g., Petit 1990) for modeling conflicts between monetary and fiscal policies. There is also a large body of literature on dynamic conflicts between policy makers from different countries on issues of international stabilization (e.g., Hamada and Kawai 1997). Both types of conflict are present in a monetary union, because a supranational central bank interacts strategically with sovereign governments as national fiscal policy makers in the member states. Such conflicts can be analysed using either large empirical macroeconomic models or small stylized models. We follow the latter line of research and use a small stylized nonlinear two-country macroeconomic model of a monetary union (called MUMOD1) for analysing the interactions between fiscal (governments) and monetary (common central bank) policy makers, assuming different objective functions of these decision makers. Using the OPTGAME algorithm we calculate equilibrium solutions for four game strategies, one cooperative (Pareto optimal) and three non-cooperative game types: the Nash game for the open-loop information pattern, the Nash game for the feedback information pattern, and the Stackelberg game for the feedback information pattern. Applying the OPTGAME algorithm to the MUMOD1 model we show how the policy makers react optimally to demand and supply shocks. Some comments are given about possible applications to the recent sovereign debt crisis in Europe.

2 Nonlinear Dynamic Tracking Games

The nonlinear dynamic game-theoretic problems which we consider in this chapter are given in tracking form. The players are assumed to aim at minimizing quadratic deviations of the equilibrium values (according to the respective solution concept) from given target (desired) values. Thus each player minimizes an objective function

J^i given by:

$$\min_{u_1^i,\dots,u_T^i} J^i = \sum_{t=1}^{T} L_t^i\big(x_t, u_t^1, \dots, u_t^N\big), \quad i = 1, \dots, N, \tag{1}$$

with

$$L_t^i\big(x_t, u_t^1, \dots, u_t^N\big) = \frac{1}{2}\big[X_t - \tilde{X}_t^i\big]' \Omega_t^i \big[X_t - \tilde{X}_t^i\big], \quad i = 1, \dots, N. \tag{2}$$

The parameter N denotes the number of players (decision makers). T is the terminal period of the finite planing horizon, i.e. the duration of the game. X_t is an aggregated vector

$$X_t := \begin{bmatrix} x_t & u_t^1 & u_t^2 & \cdots & u_t^N \end{bmatrix}', \tag{3}$$

which consists of an $(n_x \times 1)$ vector of state variables

$$x_t := \begin{bmatrix} x_t^1 & x_t^2 & \cdots & x_t^{n_x} \end{bmatrix}' \tag{4}$$

and N $(n_i \times 1)$ vectors of control variables determined by the players $i = 1, \dots, N$:

$$\begin{aligned} u_t^1 &:= \begin{bmatrix} u_t^{11} & u_t^{12} & \cdots & u_t^{1n_1} \end{bmatrix}', \\ u_t^2 &:= \begin{bmatrix} u_t^{21} & u_t^{22} & \cdots & u_t^{2n_2} \end{bmatrix}', \\ &\vdots \\ u_t^N &:= \begin{bmatrix} u_t^{N1} & u_t^{N2} & \cdots & u_t^{Nn_N} \end{bmatrix}'. \end{aligned} \tag{5}$$

Thus X_t (for all $t = 1, \dots, T$) is an r-dimensional vector, where

$$r := n_x + n_1 + n_2 + \cdots + n_N. \tag{6}$$

The desired levels of the state variables and the control variables of each player enter the quadratic objective functions (as given by equations (1) and (2)) via the terms

$$\tilde{X}_t^i := \begin{bmatrix} \tilde{x}_t^i & \tilde{u}_t^{i1} & \tilde{u}_t^{i2} & \cdots & \tilde{u}_t^{iN} \end{bmatrix}'. \tag{7}$$

Each player $i = 1, \dots, N$ is assumed to be able to observe and monitor the control variables of the other players, i.e. deviations of other control variables can be punished in one's own objective function. For example, the central bank in a monetary union, which controls monetary policy, can also penalize 'bad' fiscal policies of member countries.

Equation (2) contains an $(r \times r)$ penalty matrix Ω_t^i $(i = 1, \dots, N)$, weighting the deviations of states and controls from their desired levels in any time period t

($t = 1, \ldots, T$). Thus the matrices

$$\Omega_t^i = \begin{bmatrix} Q_t^i & 0 & \cdots & 0 \\ 0 & R_t^{i1} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & R_t^{iN} \end{bmatrix}, \quad i = 1, \ldots, N, t = 1, \ldots, T, \tag{8}$$

are of block-diagonal form, where the blocks Q_t^i and R_t^{ij} ($i, j = 1, \ldots, N$) are symmetric. These blocks Q_t^i and R_t^{ij} correspond to penalty matrices for the states and the controls respectively. The matrices $Q_t^i \geq 0$ are positive semi-definite for all $i = 1, \ldots, N$; the matrices R_t^{ij} are positive semi-definite for $i \neq j$ but positive definite for $i = j$. This guarantees that the matrices $R_t^{ii} > 0$ are non-singular, a necessary requirement for the analytical tractability of the algorithm.

In a frequent special case, a discount factor α is used to calculate the penalty matrix Ω_t^i in time period t:

$$\Omega_t^i = \alpha^{t-1} \Omega_0^i, \tag{9}$$

where the initial penalty matrix Ω_0^i of player i is given.

The dynamic system, which constrains the choices of the decision makers, is given in state-space form by a first-order system of nonlinear difference equations:

$$x_t = f\left(x_{t-1}, x_t, u_t^1, \ldots, u_t^N, z_t\right), \quad x_0 = \bar{x}_0. \tag{10}$$

$\bar{x}_0$ contains the initial values of the state variables. The vector z_t contains non-controlled exogenous variables. f is a vector-valued function where f^k ($k = 1, \ldots, n_x$) denotes the kth component of f. For the algorithm, we require that the first and second derivatives of the system function f with respect to x_t, x_{t-1} and $u_t^1, \ldots, u_t^N$ exist and are continuous.

Equations (1), (2) and (10) define a nonlinear dynamic tracking game problem. The task, for each solution concept, is to find N trajectories of control variables u_t^i, $i = 1, \ldots, N$, which minimize the postulated objective functions subject to the dynamic system. In the next section, the OPTGAME3 algorithm, which is designed to solve such types of problems, is presented.

3 The OPTGAME3 Algorithm

We apply the OPTGAME3 algorithm in order to solve the nonlinear dynamic tracking games as introduced in the previous section. This section briefly describes the OPTGAME3 algorithm; for more details about the solution procedures and the numerical methods used, see Blueschke et al. (2013). OPTGAME3 was programmed in C# and MATLAB. The source code of the algorithm is available from the authors on request. A very simplified structure of the OPTGAME algorithm is presented in Table 1.

Table 1 *Algorithm*: Rough structure of the OPTGAME algorithm

1:	initialize input parameters x_0, $(\mathring{u}^i_t)_{t=1}^T$, $(\tilde{x}^i_t)_{t=1}^T$, $(\tilde{u}^{ij}_t)_{t=1}^T$, $(z_t)_{t=1}^T$ and $f(\cdots)$
2:	calculate tentative paths for states $x_t = f(x_{t-1}, x_t, u^1_t, \ldots, u^N_t, z_t)$, $t = 1, \ldots, T$
3:	**while** the stopping criterion is not met (*nonlinearity loop*) **do**
4:	**for** T to 1 (*backward loop*) **do**
5:	linearize the system of equations: $x_t = A_t x_{t-1} + \sum_{i=1}^N B^i_t u^i_t + c_t$
6:	min J^i, get feedback matrices: G^i_t and g^i_t
7:	**end for**
8:	**for** 1 to T (*forward loop*) **do**
9:	calculate the solution: $u^{i*}_t = G^i_t x^*_{t-1} + g^i_t$ and $x^*_t = f(x^*_{t-1}, x_t, u^{1*}_t, \ldots, u^{N*}_t, z_t)$
10:	**end for**
11:	at the end of the forward loop, the solution for the current iteration of the nonlinearity loop is calculated: $(u^{i*}_t, x^*_t)_{t=1}^T$
12:	**end while**
13:	final solution is calculated: $(u^{i*}_t)_{t=1}^T$, $(x^*_t)_{t=1}^T$, J^{i*}, J^*

The algorithm starts with the input of all required data. As indicated in step (1), tentative paths of the control variables $(\mathring{u}^i_t)_{t=1}^T$ are given as inputs. In order to find a tentative path for the state variables we apply an appropriate system solver like Newton–Raphson, Gauss–Seidel, Levenberg–Marquardt or Trust region in step (2). After that the nonlinearity loop can be started where we approximate the solution of the nonlinear dynamic tracking game. To this end we linearize the nonlinear system f along the tentative path determined in the previous steps. Note that we do not globally linearize the system prior to optimization but repeatedly linearize the system during the iterative optimization process. Accordingly, for each time period t we compute the reduced form of the linearization of equation (10) and approximate the nonlinear system by a linear system with time-dependent parameters in step (5).

The dynamic tracking game can then be solved for the linearized system using known optimization techniques, which results in feedback matrices G^i_t and g^i_t in step (6). These feedback matrices allow us to calculate in a forward loop the solutions (u^{i*}_t and x^*_t) of the current iteration of the nonlinearity loop and, at the end of the nonlinearity loop, the final solutions. The convergence criterion for the nonlinearity loop requires the deviations of solutions of the current from previous iterations to be smaller than a pre-specified number.

The core of the OPTGAME3 algorithm occurs in step (6) where the linearized system has to be optimized. The optimization technique for minimizing the objective functions depends on the type of the game or solution concept. The OPTGAME3 algorithm determines four game strategies: one cooperative (Pareto optimal) and three non-cooperative games: the Nash game for the open-loop information pattern, the Nash game for the feedback information pattern, and the Stackelberg game for the feedback information pattern.

Generally, open-loop Nash and Stackelberg equilibrium solutions of affine linear-quadratic games are determined using Pontryagin's maximum principle. Feedback

Nash and Stackelberg equilibrium solutions are calculated using the dynamic programming (Hamilton–Jacobi–Bellman) technique. A detailed discussion on how to calculate the dynamic game solutions depending on the type of the game is given in Blueschke et al. (2013). Here we apply the algorithm to a model of a monetary union.

4 The MUMOD1 Model

In this chapter we use a simplified model of a monetary union called MUMOD1, which improves on the one introduced in Blueschke and Neck (2011) in order to derive optimal fiscal and monetary policies for the economies in a monetary union. The model is calibrated so as to deal with the problem of public debt targeting (a situation that resembles the one currently prevailing in the European Union), but no attempt is made to describe the EMU in every detail. The model builds on discrete data, which is a popular way in economics but there are similar frameworks in continuous time, see, for example, Van Aarle et al. (2002). One of the most important features of our model is the fact that it allows for different kinds of exogenous shocks acting on the economies in the monetary union in an asymmetric way. Analyzing the impact of these different shocks allows us to gain insights into the dynamics of a monetary union.

In this chapter, we investigate three different shocks on the monetary union: a negative demand side shock and two negative supply side shocks. Before we present these three studies it is appropriate to describe the model in detail.

In the following, capital letters indicate nominal values, while lower case letters correspond to real values. Variables are denoted by Roman letters, model parameters are denoted by Greek letters. Three active policy makers are considered: the governments of the two countries responsible for decisions about fiscal policy and the common central bank of the monetary union controlling monetary policy. The two countries are labeled 1 and 2 or core and periphery respectively. MUMOD1 is a stylized model of a monetary union consisting of two homogeneous blocs of countries, which in the current European context might be identified with the stability-oriented bloc (core) and the PIIGS bloc (countries with problems due to high public debt).

The model is formulated in terms of deviations from a long-run growth path. The goods markets are modeled for each country by a short-run income-expenditure equilibrium relation (IS curve). The two countries under consideration are linked through their goods markets, namely exports and imports of goods and services. The common central bank decides on the prime rate, that is, a nominal rate of interest under its direct control (for instance, the rate at which it lends money to private banks).

Real output (or the deviation of short-run output from a long-run growth path) in country i $(i = 1, 2)$ at time t $(t = 1, \ldots, T)$ is determined by a reduced form demand-side equilibrium equation:

$$y_{it} = \delta_i(\pi_{jt} - \pi_{it}) - \gamma_i(r_{it} - \theta) + \rho_i y_{jt} - \beta_i \pi_{it} + \kappa_i y_{i(t-1)} - \eta_i g_{it} + zd_{it}, \quad (11)$$

for $i \neq j$ $(i, j = 1, 2)$. The variable π_{it} denotes the rate of inflation in country i, r_{it} represents country i's real rate of interest and g_{it} denotes country i's real fiscal surplus (or, if negative, its fiscal deficit), measured in relation to real GDP. g_{it} in (11) is assumed to be country i's fiscal policy instrument or control variable. The natural real rate of output growth, $\theta \in [0, 1]$, is assumed to be equal to the natural real rate of interest. The parameters δ_i, γ_i, ρ_i, β_i, κ_i, η_i, in (11) are assumed to be positive. The variables zd_{1t} and zd_{2t} are non-controlled exogenous variables and represent demand-side shocks in the goods market.

For $t = 1, \dots, T$, the current real rate of interest for country i $(i = 1, 2)$ is given by:

$$r_{it} = I_{it} - \pi^e_{it}, \tag{12}$$

where π^e_{it} denotes the expected rate of inflation in country i and I_{it} denotes the nominal interest rate for country i, which is given by:

$$I_{it} = R_{Et} - \lambda_i g_{it} + \chi_i D_{it} + zhp_{it}, \tag{13}$$

where R_{Et} denotes the prime rate determined by the central bank of the monetary union (its control variable); $-\lambda_i$ and χ_i (λ_i and χ_i are assumed to be positive) are risk premiums for country i's fiscal deficit and public debt level. This allows for different nominal (and hence also real) rates of interest in the union in spite of a common monetary policy due to the possibility of default or similar risk of a country (a bloc of countries) with high government deficit and debt. zhp_{it} allows for exogenous shocks on the nominal rate of interest, e.g. negative after-effects of a haircut or a default (see Blueschke and Neck 2012, for such an analysis).

The inflation rates for each country $i = 1, 2$ and $t = 1, \dots, T$ are determined according to an expectations-augmented Phillips curve, i.e. the actual rate of inflation depends positively on the expected rate of inflation and on the goods market excess demand (a demand-pull relation):

$$\pi_{it} = \pi^e_{it} + \xi_i y_{it} + zs_{it}, \tag{14}$$

where ξ_1 and ξ_2 are positive parameters; zs_{1t} and zs_{2t} denote non-controlled exogenous variables and represent supply-side shocks, such as oil price increases, introducing the possibility of cost-push inflation; π^e_{it} denotes the rate of inflation in country i expected to prevail during time period t, which is formed at (the end of) time period $t - 1$. Inflationary expectations are formed according to the hypothesis of adaptive expectations:

$$\pi^e_{it} = \varepsilon_i \pi_{i(t-1)} + (1 - \varepsilon_i)\pi^e_{i(t-1)}, \tag{15}$$

where $\varepsilon_i \in [0, 1]$ are positive parameters determining the speed of adjustment of expected to actual inflation.

The average values of output and inflation in the monetary union are given by:

Table 2 Parameter values for an asymmetric monetary union, $i = 1, 2$

T	θ	ω	$\delta_i, \beta_i, \eta_i, \varepsilon_i$	$\gamma_i, \rho_i, \kappa_i, \xi_i, \lambda_i$	χ_i
30	3	0.6	0.5	0.25	0.0125

$$y_{Et} = \omega y_{1t} + (1 - \omega) y_{2t}, \quad \omega \in [0, 1], \tag{16}$$

$$\pi_{Et} = \omega \pi_{1t} + (1 - \omega) \pi_{2t}, \quad \omega \in [0, 1]. \tag{17}$$

The parameter ω expresses the weight of country 1 in the economy of the whole monetary union as defined by its output level. The same weight ω is used for calculating union-wide inflation in equation (17).

The government budget constraint is given as an equation for government debt of country i $(i = 1, 2)$:

$$D_{it} = (1 + r_{i(t-1)}) D_{i(t-1)} - g_{it} + zh_{it}, \tag{18}$$

where D_i denotes real public debt of country i measured in relation to (real) GDP. No seigniorage effects on governments' debt are assumed to be present. zh_{it} allows us to model an exogenous shock on public debt; for instance, if negative it may express default or debt relief (a haircut).

Both national fiscal authorities are assumed to care about stabilizing inflation (π), output (y), debt (D) and fiscal deficits of their own countries (g) at each time t. This is a policy setting which seems plausible for the actual EMU as well, with full employment (output at its potential level) and price level stability relating to country (or bloc) i's primary domestic goals, and government debt and deficit relating to its obligations according to the Treaty of the European Union. The common central bank is interested in stabilizing inflation and output in the entire monetary union, also taking into account a goal of low and stable interest rates in the union.

Equations (11)–(18) constitute a dynamic game with three players, each of them having one control variable. The model contains 14 endogenous variables and four exogenous variables and is assumed to be played over a finite time horizon. The objective functions are quadratic in the paths of deviations of state and control variables from their desired values. The game is nonlinear-quadratic and hence cannot be solved analytically but only numerically. To this end, we have to specify the parameters of the model.

The parameters of the model are specified for a slightly asymmetric monetary union; see Table 2. Here an attempt has been made to calibrate the model parameters so as to fit for the EMU. The data used for calibration include average economic indicators for the (then) 16 EMU countries from EUROSTAT up to the year 2007. Mainly based on the public finance situation, the EMU is divided into two blocs: a core (country or bloc 1) and a periphery (country or bloc 2). The first bloc has a weight of 60 % in the entire economy of the monetary union (i.e. the parameter ω is equal to 0.6). The second bloc has a weight of 40 % in the economy of the union; it consists of countries with higher public debt and deficits and higher interest and inflation rates on average. The weights correspond to the respective shares in EMU

Table 3 Initial values of the two-country monetary union

$y_{i,0}$	$\pi_{i,0}$	$\pi^e_{i,0}$	$D_{1,0}$	$D_{2,0}$	$R_{E,0}$	$g_{1,0}$	$g_{2,0}$
0	2	2	60	80	3	0	0

Table 4 Target values for an asymmetric monetary union

$\tilde{y}_{it}$	$\tilde{D}_{1t}$	$\tilde{D}_{2t}$	$\tilde{\pi}_{it}$	$\tilde{\pi}_{Et}$	$\tilde{y}_{Et}$	$\tilde{g}_{it}$	$\tilde{R}_{Et}$
0	60↘50	80↘60	1.8	1.8	0	0	3

Table 5 Negative symmetric shock on the demand side

t	1	2	3	4	5	6	⋯	30
zd_1	−2	−4	−2	0	0	0	⋯	0
zd_2	−2	−4	−2	0	0	0	⋯	0

real GDP. For the other parameters of the model, we use values in accordance with econometric studies and plausibility considerations.

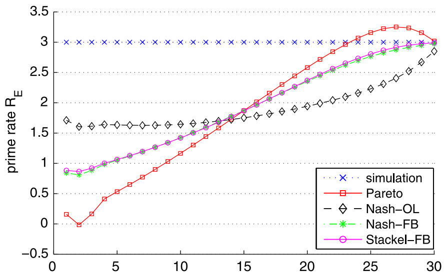
The initial values of the macroeconomic variables, which are the state variables of the dynamic game model, are presented in Table 3. The desired or ideal values assumed for the objective variables of the players are given in Table 4. Country 1 (the core bloc) has an initial debt level of 60 % of GDP and aims to decrease this level in a linear way over time to arrive at a public debt of 50 % at the end of the planning horizon. Country 2 (the periphery bloc) has an initial debt level of 80 % of GDP and aims to decrease its level to 60 % at the end of the planning horizon, which means that it is going to fulfill the Maastricht criterion for this economic indicator. The ideal rate of inflation is calibrated at 1.8 %, which corresponds to the Eurosystem's aim of keeping inflation below, but close to, 2 %. The initial values of the two blocs' government debts correspond to those at the beginning of the Great Recession, the recent financial and economic crisis. Otherwise, the initial situation is assumed to be close to equilibrium, with parameter values calibrated accordingly.

5 Effects of a Negative Demand-Side Shock

The MUMOD1 model can be used to simulate the effects of different shocks acting on the monetary union, which are reflected in the paths of the exogenous non-controlled variables, and the effects of policy reactions towards these shocks. In this section we analyse a symmetric shock which occurs on the demand side (zd_i) as given in Table 5. The numbers can best be interpreted as being measured as percentage points of real GDP.

In the first three periods, both countries experience the same negative demand shock (zd_i) which reflects a financial and economic crisis like the one in 2007–2010. After three periods the economic environment of countries 1 and 2 stabilizes again.

Fig. 1 Prime rate R_{Et} controlled by the central bank



Here, we investigate how the dynamics of the model and the results of the policy game (11)–(18) depend on the strategy choice of the decision makers. For this game, we calculate five different solutions: a baseline solution with the shock but with policy instruments held at pre-shock levels (zero for the fiscal balance, 3 for the central bank's interest rate), three non-cooperative game solutions and one cooperative game solution. The baseline solution does not include any policy intervention and describes a simple simulation of the dynamic system. It can be interpreted as resulting from a policy ideology of market fundamentalism prescribing non-intervention in the case of a recession.

Figures 1–5 show the simulation and optimization results of this experiment. Figures 1–2 show the results for the control variables of the players and Figs. 3, 4, 5 show the results of selected state variables: output, inflation and public debt.

Without policy intervention (baseline scenario, denoted by 'simulation'), both countries suffer dramatically from the economic downturn modeled by the demand-side shock in the first periods. The output of both countries drops by more than 6 %, which for several European countries is a fairly good approximation of what happened in reality. This economic crisis decreases their inflation rates and starting with time period 2 creates a persisting deflation of about -0.5 % to -1 %. Even more dramatic is the development of public debt. Without policy intervention it increases during the whole planning horizon and arrives at levels of 240 % of GDP for country 1 (or core bloc) and 390 % for country 2 (or periphery bloc), which shows a need for policy actions to preserve the solvency of the governments of the monetary union.

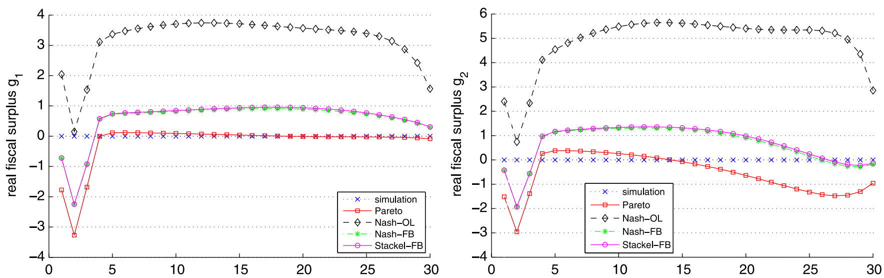


Fig. 2 Country i's fiscal surplus g_{it} (control variable) for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

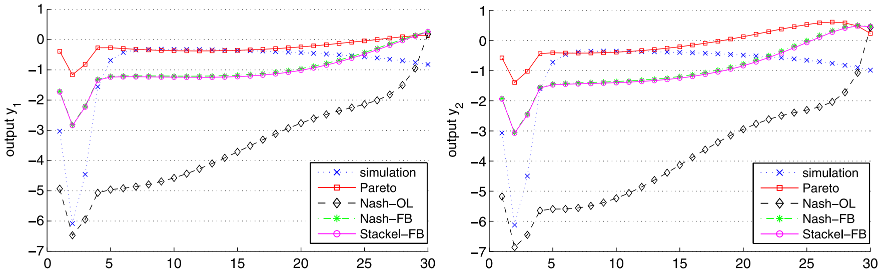


Fig. 3 Country i's output y_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

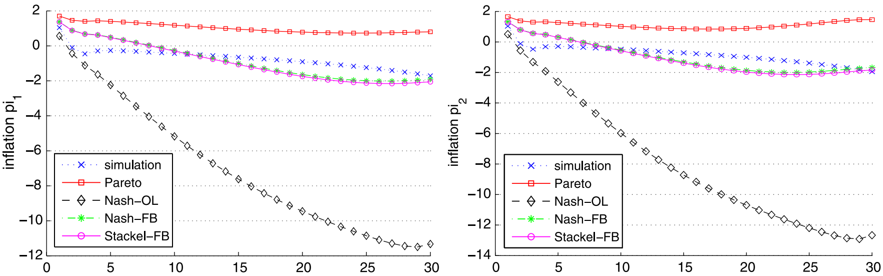


Fig. 4 Country i's inflation rate π_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

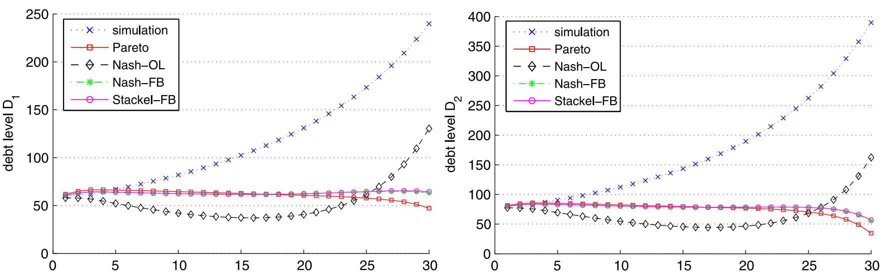


Fig. 5 Country i's debt level D_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

If the players (the central bank and the governments of the countries) want to react optimally to the demand-side shocks, their actions and their intensity depend on the presence or absence of cooperation. For example, optimal monetary policy has to be expansionary (lowering the prime rate) in all solution concepts considered, but in the cooperative Pareto solution it is more active during the first 15 periods. The Nash open-loop solution, in contrast, is more or less constant during the whole optimization period, which causes the central bank to be less active at the beginning and relatively more active at the end of the optimization horizon.

With respect to fiscal policy, both countries are required to set expansionary actions and to create deficits in the first three periods in order to absorb the demand-

Table 6 Negative symmetric persistent shock on the supply side

t	1	2	3	4	5	6	⋯	30
zs_1	10	5	0	0	0	0	⋯	0
zs_2	10	5	0	0	0	0	⋯	0

side shock. After that a trade-off occurs and the governments have to take care of the financial situation and to produce primary surpluses. The only exception is the cooperative Pareto solution: cooperation between the countries and the central bank (which in this strategy runs a more active expansionary monetary policy) and the resulting moderate inflation means that the balance of public finances can be held close to zero. For country 2 it is even optimal to run a slightly expansionary fiscal policy again during the last 15 periods in the Pareto solution. Even so the countries are able to stabilize and to bring down their public debts close to the targeted values under cooperation.

The open-loop Nash solution, which assumes unilateral (not cooperating) commitment for all players, shows a bad performance. The central bank is less active than in all other solutions. The governments are forced to run restrictive fiscal policies which show that the trade-off between output and the public debt target is dominated by the latter one. The lack of cooperation between the players and the open-loop information pattern make the policy makers less flexible and as a result produce huge drops in output and an unsustainable deflation. Here both countries are trapped in a deflationary spiral, the possibility of which is frequently discussed these days for some of the European countries. An economic reason for this result is the lack of (even weak) time consistency of strategies in this solution concept, which implies very restrictive fiscal policies.

The non-cooperative Nash feedback and Stackelberg feedback solutions give very similar results. In comparison to the Pareto optimal solution, the central bank acts less actively and the countries run more active fiscal policies (except during the negative demand shock). As a result, output and inflation are slightly below the values achieved in the cooperative solution, and public debt is slightly higher. Comparing these results with the ones of the Pareto solution the impact of the cooperation can be clearly observed. In the Pareto solution, the central bank cooperates and is willing to be more active in order to support the countries.

6 Effects of a Persistent Negative Supply-Side Shock

In this section we analyze a symmetric shock which occurs on the supply side (zs_i) as given in Table 6.

We call this shock a 'persistent' supply-side shock because after its occurrence there is no exogenous recovery from it and the system has to adjust to the new situation endogenously. This shock could be interpreted as a simplified representation of an oil price shock leading to the worst macroeconomic scenario, stagflation. Here, in

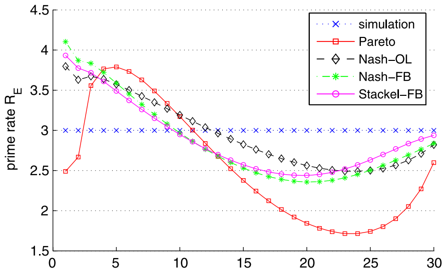


Fig. 6 Prime rate R_{Et} controlled by the central bank

the first two periods both countries experience the same negative shock (zs_i) which directly increases the price levels and the inflation rates in the economies.

Figures 6–10 show the simulation and optimization results of this experiment. Figures 6–7 show the results for the control variables of the players and Figs. 8, 9, 10 show the results of selected state variables: output, inflation and public debt.

Without policy intervention (baseline scenario, denoted by 'simulation'), both countries suffer dramatically from the supply-side shock especially in terms of output drop and high inflation. The output of both countries drops by more than 6 % in the first two periods and improves at very slow rates so that it stays negative (i.e. below the long-run growth path) during the whole planning horizon. The inflation rates start with values of more than 10 % and go back to the 'normal' values of about 2 % very slowly. The one and only positive aspect of these high inflation rates is the resulting development of public debt. Except for the first three periods, where the effect of the negative deviation of output from the steady-state path outweighs the impact of the inflation-led depreciation, the public debt stays even below the targeted values.

If the players want to react optimally to the supply-side shocks, again their actions and their intensity depend on the presence or absence of cooperation. The non-cooperative strategies show very similar optimal solutions. A conflict between the central bank giving high importance to the inflation rate and the local governments caring more about GDP is well observable. The central bank reacts to the shock with a restrictive monetary policy in order to decrease the inflation rate. This restrictive monetary policy becomes less active as time goes by and after 10 to 13

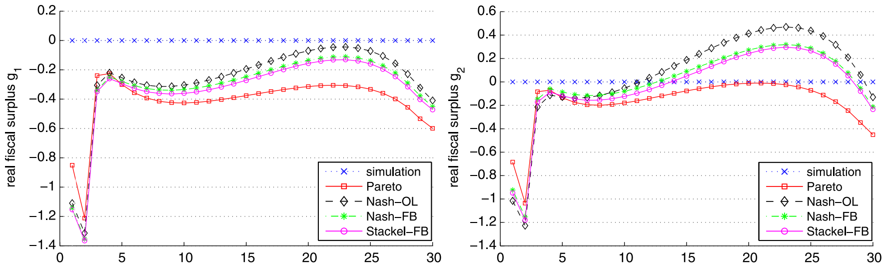


Fig. 7 Country i's fiscal surplus g_{it} (control variable) for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

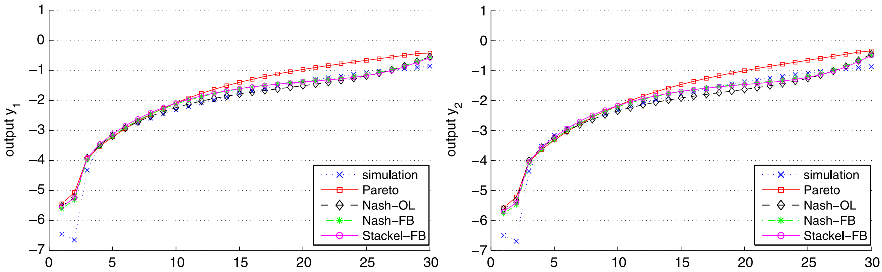


Fig. 8 Country i's output y_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

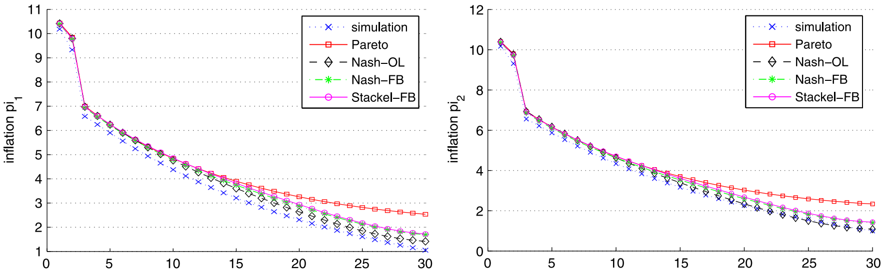


Fig. 9 Country i's inflation rate π_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

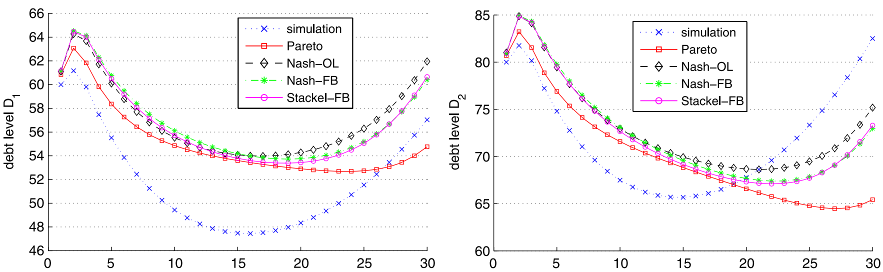


Fig. 10 Country i's debt level D_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

periods (depending on the non-cooperative strategy played) the central bank gradually switches to an active monetary policy. On the other hand, the governments of the countries care about output and run expansionary fiscal policies. While country 1 can concentrate on the output target and therefore runs an expansionary fiscal policy over the whole optimization period, country 2 is forced to take higher public debt into account by running a slightly restrictive fiscal policy for certain periods (between periods 10 and 27). Here the trade-off between the output and public debt target is clearly visible.

From the results of the Pareto optimal solution is clear once again the benefit of the cooperation. In the first two periods, where the impact of the supply-side shock

Table 7 Negative symmetric reverse shock on the supply side

t	1	2	3	4	5	6	...	30
zs_1	10	5	−5	−5	−3	−2	...	0
zs_2	10	5	−5	−5	−3	−2	...	0

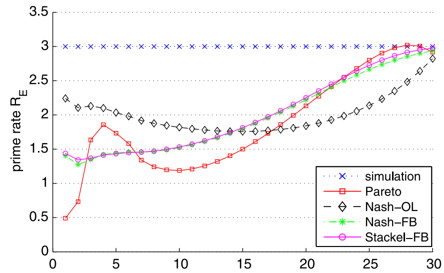


Fig. 11 Prime rate R_{Et} controlled by the central bank

is strongest, the central bank supports the countries in reducing the drop in output by applying an active monetary policy even though the inflation rate stays high. After these two periods the central bank runs a policy similar to the non-cooperative solutions but is slightly more active. As a result the outputs of the countries in the Pareto solution are slightly above and the public debts are slightly below the ones of the non-cooperative solutions.

7 Effects of a Reverse Negative Supply-Side Shock

In this section we analyze another symmetric shock which occurs on the supply side (zs_i) as given in Table 7.

We call this shock a ‘reverse’ supply-side shock because after its occurrence there is a smooth exogenous recovery from it. This shock could be interpreted as a temporary oil price shock with the oil price first going up and then coming back to the initial level. Such a temporary oil price shock occurred in the industrial countries in the 1980s. In the first two periods, both countries experience the same negative shock (zs_i) which directly increases the price levels in the economies and which is similar to the shock described in the previous section. After that the shock changes from the negative to a positive one during four periods.

Figures 11–15 show the simulation and optimization results of this experiment. Figures 11–12 show the results for the control variables of the players and Figs. 13, 14, 15 show the results of selected state variables: output, inflation and public debt.

Without policy intervention, in the first two periods, both economies show the same dynamics as in the case of a persistent supply-side shock with a drop in output by more than 6 % and an increase in the inflation rate to more than 10 %. In contrast to the persistent shock experiment, the reversion of the shock improves the economic situation in both countries very quickly except for the dynamics of their public debts. Now the public debt problem in the uncontrolled scenario grows to dramatic values

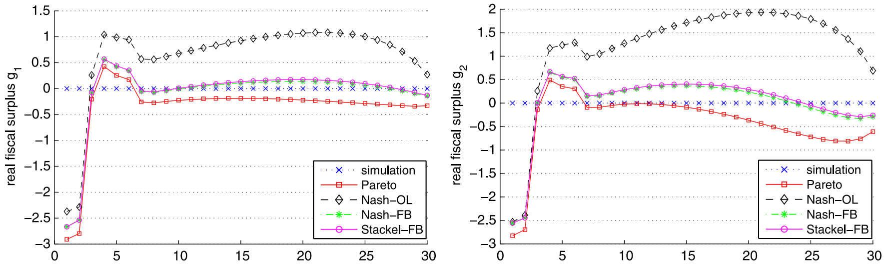


Fig. 12 Country i's fiscal surplus g_{it} (control variable) for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

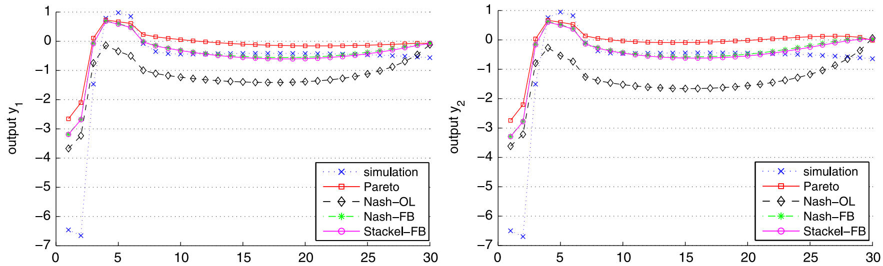


Fig. 13 Country i's output y_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

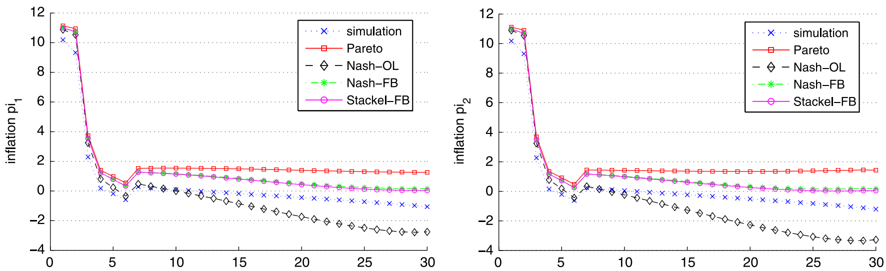


Fig. 14 Country i's inflation rate π_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

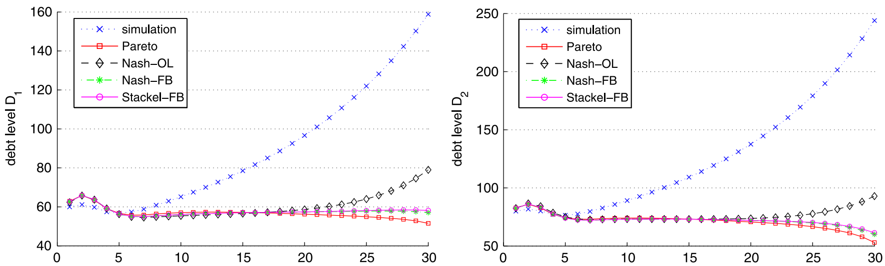


Fig. 15 Country i's debt level D_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

of around 160 % for the core block and 250 % for the periphery block. This means that the policy actions of the players have to deal with the trade-off between the output/inflation problem in the first two periods and the public debt problem for the later periods.

The optimal policies show more or less similar dynamics for all solution concepts. Monetary policy is expansionary during the whole optimization period, with the Pareto solution requiring it to be more active at the beginning and the Nash open-loop solution implying a nearly constant prime rate. The feedback Nash and Stackelberg solutions give results which are in between. Fiscal policy is expansionary during the first part of the supply-side shock for all strategies, requiring the governments to produce deficits in order to improve their outputs. During the second part of the supply-side shock, again all strategies require similar policies, but now restrictive ones. When the crisis runs out after six periods a slight divergence between the proposed solutions can be observed. The Nash open-loop requires restrictive fiscal policies for both countries with slightly higher surpluses for country 2. The feedback solutions (both Nash and Stackelberg) do not require active fiscal policy at all from either country. Only some minor adjustments which are less than 0.2 % for country 1 and 0.5 % for country 2 turn out to be optimal. And due to the cooperation between the players, in the Pareto solution both countries are able to produce some low deficits while still fulfilling the desired targets.

In the case of the output target in all game solutions, the situation is better than in the non-controlled simulation. Instead of the dramatic drop in more than 6 % in the uncontrolled solution, all game solutions allow the impact of the shock to be reduced to a high degree: in the Nash open-loop solution to values between 3 and 4 % and in the feedback Nash and Stackelberg solutions to around 3 %. The Pareto solution gives the best performance and reduces the drop in output to values between 2 and 3 %. Also for the remaining periods the Pareto solution gives the best results with output always being higher than in the other game strategies.

Regarding the inflation target all strategies show similar results during the occurrence of the crisis with the rate of inflation being more than 10 % in the first two periods and decreasing quickly afterwards. After the crisis runs out the Pareto solution is able to stabilize the inflation rate around the target value of 1.8 %. All other solutions produce inflation rates which lie below. In the case of the Nash open-loop solution a deflationary development can be observed.

The public debt situation can be fairly well stabilized as compared to the non-controlled simulation in all game strategies. Only in the last five periods can a slight divergence be observed. In the case of the Nash open-loop solution public debt goes up and for the other solution it goes down. This fact can be partially explained by the well-known effect of the finite horizon on the solution of optimal control problems.

8 Concluding Remarks

In this chapter, we analysed the interactions between fiscal (governments) and monetary (common central bank) policy makers by applying a dynamic game approach to a simple macroeconomic model of a two-country monetary union. Using the OPTGAME3 algorithm, which allows us to find approximate solutions for nonlinear-quadratic dynamic tracking games, we obtained some insights into the design of economic policies facing negative shocks on the demand and the supply side. To this end we introduce three different shocks on the monetary union: a negative demand-side shock and two negative supply-side shocks, a persistent and a reverse one. The monetary union is assumed to be asymmetric in the sense of consisting of a core with less initial public debt and a periphery with higher initial public debt, which is meant to reflect the situation in the EMU.

Our results show strong trade-offs between the targets of output and public debt stabilization. Immediately at the start of the crisis nearly all results propose a countercyclical fiscal policy for the countries, with a quick switch to public debt stabilization afterwards. The 'best' results (in terms of the objective function values or losses) are achieved by the cooperative Pareto solution with a more active role played by the central bank. The trade-off between the targets price stability and output stabilization in the case of the supply shocks is less pronounced and is generally resolved in favor of output stabilization, which is due to the relatively strong reaction of output to the shock. The cooperative solution differs from the noncooperative ones more markedly in the supply-side scenarios than in the demand-side scenario. Altogether, the main policy conclusion consists in recommending coordinated fiscal and monetary policies, which may be interpreted (with caution) as recommending a fiscal pact involving governments and the common central bank.

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The Dynamics of Lobbying Under Uncertainty: On Political Liberalization in Arab Countries

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Abstract We consider a framework *à la* Wirl (Public Choice 80:307–323, 1994) where political liberalization is the outcome of a lobbying differential game between a conservative elite and a reformist group, the former player pushing against political liberalization in opposition to the latter. In contrast to the benchmark model, we introduce uncertainty. We consider the typical case of an Arab resource-exporting country where oil rents are fiercely controlled by the conservative elite. We assume that the higher the oil rents, the more reluctant to political liberalization the elite is. Two states of nature are considered (high vs. low resource rents). We then compute the Markov-perfect equilibria of the corresponding piecewise deterministic differential game. It is shown that introducing uncertainty in this manner increases the set of strategies compared to Wirl's original setting. In particular, the cost of lobbying might be significantly increased under uncertainty with respect to the benchmark. This highlights some specificities of the political liberalization in Arab countries and the associated risks.

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1 Introduction

Rent-seeking activities in countries with developed extraction sectors are abundantly documented. Examples range from timber industries in the Philippines and Malaysia (as detailed in Ross 2001) to fossil energy-related sectors like in OPEC countries (see a recent paper by Gylfason 2001). In general, the rents deriving from the exploitation of natural resources fall under the fierce control of conservative elites. These elites typically manipulate national legislation (pretty much in the sense given by Tullock 1967 to rent-seeking) to perpetuate themselves in power.[1] Empirical evidence show that the so-called "resource curse" can be a consequence of the latter behavior. Bad governance and weak institutions are the main reasons behind the failures of several resource-rich countries to launch a sustainable growth process (see Gylfason 2001; Mehlum et al. 2006; Cabrales and Hauk 2011; Tsui 2011). The "resource curse" is by no way the mere outcome of an automatic mechanism penalizing these otherwise blessed countries.

On the other hand, the impact of rent-seeking behavior on economic efficiency is a quite old idea tracing back to Tullock (1967). Key aspects of the theory are the strategic and non-strategic behaviors of the players involved in rent-seeking and their implication to public policy. As players are roughly the representatives of interest groups in practice, the theory ends up modeling the determinants and outcomes of lobbying in different theoretical contexts (see Becker 1983; Linster 1994; and Kohli and Singh 1999; for more recent examples of the literature stream opened by Tullock). An influential contribution is the one by Becker (1983). He modeled lobbying in a two-player setting, each player with his own lobbying cost and productivity. It was assumed that the larger lobbying expenditures, the stronger the lobby and the more effective a player can be in orienting public (fiscal) policy. However, Becker's model does not entail any strategic behavior of any sort: each player acts as if the lobbying effort exerted by the opponent is independent of his own choice.[2] Researchers after Becker have tried to get rid of this shortcoming. To our knowledge, Wirl (1994) is the first to use differential games in this stream of literature. Wirl uses a linear-quadratic model to investigate the impact of the game structure on the outcomes expressed in terms of players' strategies. Though the government is passive in this framework (in other words, public policy only changes in response to lobbying actions), the paper has two important contributions. First, the game structure matters (the open-loop equilibria are, indeed, carefully compared to the subgame-perfect equilibria derived as linear Markov strategies). Second, in the subgame-perfect equilibria, optimal lobbying expenditures are remarkably lower (than those observed in the open-loop case). This provides a rationale for a conjecture made by Tullock. The cost of rent-seeking activities are rather small compared

[1] The recent Arab Spring uprising shed light on another form of these long lasting rent-seeking activities, not related to natural resources but to the control of financial and trade flows as it was the case in Tunisia under the presidency of Benali.

[2] The main point made by Becker is that increasing competition among interest groups should improve the efficiency of the tax system.

to the rents, therefore implying not too high social costs. The reason behind this striking result is inherent in the feedback nature of the Markovian strategies, which discourages too aggressive lobbying strategies (see Wirl 1994, for more details).

This paper qualifies this important claim by Wirl by introducing uncertainty. If the players do not know with certainty the future politico-economic environment, and provided they are not too averse to risk, they might well depart from the overly cautious behavior described in Wirl (1994). This is especially the case if they anticipate a favorable evolution of the environment. We apply our framework to the process of liberalization in oil exporting countries, and more specifically to Arab countries. The Arab Spring has shown the deep inequalities that characterize the Arab world. On one hand, there are ruling dynasties who usually control all types of economic and political activities. On the other, there is a majority of Arab citizens which are partially or totally excluded from relevant decision-making. A fundamental characteristic of these countries is the essential role played by the oil rents both on the political and economic grounds (see Caselli and Cunningham 2009). The larger these rents are, the bigger the incentives of the elites to stay in power and to block any initiative to open the political game.[3] In many Arab countries, starting with the Gulf emirates and kingdoms, a lot has been already done towards economic liberalization, notably in order to attract more foreign direct investment. However, no significant move has been made in favor of political liberalization (see Dunne and Revkin 2011, on Egypt).[4] We shall consider a framework *à la* Wirl where political liberalization is the outcome of a piecewise deterministic differential game between a conservative elite and a reformist group: oil rents may be high or low (two states of nature). In the former state of nature, the elite is more reluctant to political liberalization. This volatility of the benefits from oil rents is inherent in resource-dependent economies. For instance, van der Ploeg and Poelhekke (2009) show that liquidity constraints are exacerbated when oil rents are volatile. In a subsequent study, van der Ploeg and Poelhekke (2010) has also observed that natural resources worsen macroeconomic volatility and thus impede output growth. Taking into account this context, we thus revisit Wirl's findings. We particularly show how uncertainty alter the optimal strategies in the Markov-perfect equilibria. Incidentally, we highlight some of the specific risks inherent in the current political liberalization process in Arab countries.

This paper is structured as follows. Section 2 introduces the dynamic model of political liberalization. Section 3 considers a setting with uncertainty and derives the MPE of a piecewise deterministic game. Finally, Sect. 4 concludes.

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[3]Gylfason (2001) makes the point that the elites would eventually block human capital education to perpetuate themselves in power. As outlined by Boucekkine and Bouklia-Hassane (2011), this is certainly not the case of Tunisia, the starter in the Arab Spring uprising: more than 20 % of the Tunisian budget has gone to public education in the last decade, much better than many advanced European countries.

[4]Algeria is a case where even economic liberalization efforts have been tightly linked to the level of the oil barrel, as explained in Boucekkine and Bouklia-Hassane (2011).

2 Benchmark Model

In this section, the differential game on lobbying proposed by Wirl (1994) is adapted to the context of the Arab Spring. For the meantime, the case with no uncertainty is discussed. In the next section, we extend Wirl's model by considering a stochastic environment with two states of nature. Throughout the paper, we consider only symmetric games (in the precise sense of Wirl, see Sect. 2.1 just below). This is done for algebraic amenability, as no analytical solution is allowed outside this class of games. Realistically, players engaged in the political liberalization struggle in Arab countries do not have equal power since they do not have equal access to oil rents, etc.[5] Nonetheless, the symmetric set-up adopted includes two important features of political liberalization: the conservative elite is reluctant to liberalization, while the reformist minority pushes for it. This reluctance is an increasing function of oil rents. The former point will be apparent in the stochastic extension of the benchmark.

2.1 The Setup

We consider two competing players (denoted as $i = 1, 2$) who engage in investment efforts x_1 and x_2. Player 1 is a reformist who exerts pressure towards greater political liberalization. On the other hand, Player 2 prefers a conservative system. In the context of the Arab Spring, Player 2 can be considered as the elite government who wants to retain the political status quo. Player 1 represents the groups who prefer regime change. The state of liberalization is measured by $z \in (-\infty, \infty)$. As in Wirl, $z = 0$ is the neutral level of political liberalization. Consequently, the following differential equation captures the evolution of z in response to the efforts of players 1 and 2:

$$\dot{z} = x_1 - x_2, \tag{1}$$

with $z(0) = z_0$ given. As a reformist, Player 1 prefers a higher level of political liberalization. A high value of z, on the other hand, is not beneficial to the conservative stance of Player 2. Thus, the investment x_1 of Player 1 increases z, while Player 2 exerts effort x_2 to lower z.

The benefit from the current level of liberalization is denoted by $\alpha_i(z)$ with: $\alpha_1(z) = a_0 + a_1 z + \frac{a_2}{2} z^2$ and $\alpha_2(z) = a_0 - a_1 z + \frac{a_2}{2} z^2$. We follow Wirl (1994) by qualifying this game as a symmetric one. The opposite signs of the second term in the players' benefit functions represent their antagonistic interests with regard to liberalization. Without loss of generality, we assume that $a_1 > 0$. We also assume

[5]For example, in the Algerian case, the conservative elites benefit from the support of the powerful National Popular Army and the intelligence services (DRS).

that $a_2 \leq 0$ to ensure concavity. Meanwhile, efforts x_1 and x_2 are also associated with cost $\gamma(x_i) = \frac{d}{2}(x_i)^2$.

Players maximize the present value of benefits from liberalization minus the associated costs, $F_i = \alpha_i(z) - \gamma(x_i)$. With an interest rate $r > 0$, players choose effort levels to maximize the following objective function subject to the evolution of z (equation (1)):

$$\max_{x_i(t)} \int_0^\infty e^{-rt}\left\{\alpha_i\big(z(t)\big) - \gamma\big(x_i(t)\big)\right\}dt \tag{2}$$

The solution to this differential game is essentially the same as the symmetric version found in Wirl (1994). In the next subsection, we will summarize the resulting open-loop and feedback strategies. In Sect. 3, we will provide a comprehensive solution to a game under uncertainty and provide analytical comparisons.

2.2 *Open-Loop and Feedback Strategies*

As mentioned above, this subsection provides an overview of the open-loop and Markov-perfect equilibrium (MPE) solutions to the political liberalization game with no uncertainty. Similar to Wirl (1994), the strategy pair $\{x_1^O(t), x_2^O(t), t \in [0, \infty)\}$ comprises an open-loop Nash equilibrium (OLNE) if both strategies, which depend on t, maximize the respective objective functions of the players. In summary, the open-loop case (presented in the feedback form) at a symmetric equilibrium results to:

$$\begin{aligned} x_1^O &= \frac{a_1}{rd} + \frac{1}{4}\left[r - \sqrt{\left(r^2 - \frac{8}{d}a_2\right)}\right]z \\ x_2^O &= \frac{a_1}{rd} - \frac{1}{4}\left[r - \sqrt{\left(r^2 - \frac{8}{d}a_2\right)}\right]z, \end{aligned} \tag{3}$$

which leads the system to a unique steady state characterized by:

$$x_{1\infty}^O = \frac{a_1}{rd} = x_{2\infty}^O; \qquad z_\infty = 0. \tag{4}$$

While the open-loop equilibrium is time-consistent, it is not subgame perfect. That is, using open-loop strategies might not make sense when considering an anticipated change in the evolution of the game. Thus, following literature (Dockner et al. 2000), feedback strategies are deemed suitable. Utilizing the usual Hamilton–Jacobi–Bellman (HJB) equations (refer to Wirl 1994, p. 315, for a detailed discussion), the resulting MPE strategies in the case without regime switching are (the superscript N is used here):

$$
\begin{aligned}
x_1^N &= \frac{6a_1}{d[5r + \sqrt{(r^2 - (12/d)a_2)}]} + \frac{1}{6}\left[r - \sqrt{\left(r^2 - \frac{12}{d}a_2\right)}\right]z \\
x_2^N &= \frac{6a_1}{d[5r + \sqrt{(r^2 - (12/d)a_2)}]} - \frac{1}{6}\left[r - \sqrt{\left(r^2 - \frac{12}{d}a_2\right)}\right]z,
\end{aligned}
\tag{5}
$$

which leads the system to a steady state characterized by:

$$
x_{1\infty}^N = \frac{6a_1}{d[5r + \sqrt{(r^2 - (12/d)a_2)}]} = x_{2\infty}^N; \qquad z_\infty = 0. \tag{6}
$$

The strategies computed have some interesting implications. First, note that in the MPE, the strategy of Player 1 is decreasing in z. This is in strong contrast to Player 2. In terms of our political liberalization framework, it means that the reformist would exert less effort when the level of political freedom is rising. The conservative takes the opposite strategy. Much more interestingly, one can use the previous feedback rules to conclude that $x_{i\infty}^O > x_{i\infty}^N$, for $i = 1, 2$, which is the main result of Wirl's benchmark. Lobbying activities are lower in the MPE compared to the open loop, at least in the steady state. Therefore, the social cost of lobbying activities are less significant than one may expect. This finding is confirmed through some quantitative exercises.[6]

3 Political Liberalization Game Under Uncertainty

We now consider the dynamic game of political liberalization under a setting with uncertainty.

3.1 MPE of the Piecewise Deterministic Game

The symmetric case found in Wirl (1994) is extended by taking into account the possibility of regime switching. A stochastic differential game is analyzed. More specifically, we derive the Markov-perfect Nash equilibria of a piecewise deterministic game.[7]

The pay-offs of players 1 and 2 are altered to:

[6] In the numerical cases studied by Wirl, the comparison is quantitatively striking. The ratio of total lobbying expenditures in the MPE compared with the open loop is only around one third for the symmetric case, and even much less in some asymmetric configurations considered.

[7] We do not consider the piecewise open-loop equilibria as closed-form solutions for this case are rarely derived in literature (Dockner et al. 2000). For analytical tractability, we thus focus on feedback strategies.

$$F_1^j = a_0 + a_1^j z + \frac{a_2}{2} z^2 - \frac{d}{2}(x_1)^2$$
$$F_2^j = a_0 - a_1^j z + \frac{a_2}{2} z^2 - \frac{d}{2}(x_2)^2 \tag{7}$$

Uncertainty is characterized in the coefficient representing the linear benefits incurred from liberalization, a_i^j. There exist two states of the world, denoted by j. In Regime 1, $a_1^1 = \overline{a}_1$. On the other hand, $a_1^2 = \underline{a}_1$ for Regime 2. We assume that $\underline{a}_1 < \overline{a}_1$. In the context of the Arab Spring in predominantly oil-rich economies, Regime 1 can be the state when resource windfalls are high.[8] Meanwhile, Regime 2 can be considered as the scenario during which gains from oil are low. Only the linear term of benefits is dependent on the regime. This assumption is sufficient to characterize resource volatility inherent in many Arab countries (van der Ploeg and Poelhekke 2009). More importantly, it captures the heterogeneity in players' sensitiveness to regime change.[9]

In Regime 1, oil revenues are high. This makes Player 2 even more reluctant to liberalization. This relatively higher reluctance translates into the fact that $\alpha_2(z)$ worsens in Regime 1, compared to Regime 2. This is due to a higher a_1, in absolute terms. This means that, by symmetry, Player 1's gains from liberalization are higher in the first regime. Furthermore, the probability to switch from Regime 1 to 2 is denoted as $q_{12} \in (0, 1)$. Similarly, the probability of switching from Regime 2 to 1 is $q_{21} \in (0, 1)$. Depending on the current regime and taking into account the switching probabilities, players maximize the discounted net payoffs in (7) subject to (1).

As discussed in Dockner et al. (2000), the HJB equations are modified and solved for each regime. The HJB equations for the piecewise deterministic game take the following form:[10]

$$rV_i^j = \max_{x_i}\left\{F_i^j + \frac{\partial V_i^j}{\partial z}\dot{z} + q_{j,-j}\left[V_i^{-j} - V_i^j\right]\right\} \tag{8}$$

Suppose we are in Regime 1, the HJB equation for Player 1 is denoted as:

$$rV_1^1 = \max_{x_1}\Big\{a_0 + a_1^j z + \frac{a_2}{2}z^2 - \frac{d}{2}(x_1)^2 + \left(B_1^1 + C_1^1 z\right)(x_1 - x_2)$$
$$+ q_{12}\Big[\left(A_1^2 - A_1^1\right) + \left(B_1^2 - B_1^1\right)z + \frac{(C_1^2 - C_1^1)}{2}z^2\Big]\Big\} \tag{9}$$

[8] In most oil-dependent Arab countries, natural resource rents are usually received by the governing political elite (Caselli and Cunningham 2009).

[9] In addition, it allows us to get analytical solutions, which would not be possible by, for instance, making the entire payoffs be regime dependent.

[10] Compared to the general form of HJBs utilized in Wirl (1994), there is an additional (last) term which accounts for the possibility of uncertain regime switching from one regime, j, to the other, $-j$.

where we guess that the value function has the following form

$$V_i^j(z) = A_i^j + B_i^j z + \frac{C_i^j}{2} z^2 \quad i, j = 1, 2.$$

The first-order condition yields:

$$x_1^1 = \frac{B_1^1 + C_1^1 z}{d} \tag{10}$$

Similarly, from Player 2's HJB equation, we derive:

$$x_2^1 = -\frac{B_2^1 + C_2^1 z}{d} \tag{11}$$

Substituting x_1^1 and x_2^1 by the expressions given in (10) and (11) in (9), we obtain for Player 1 (disregarding the constant terms):

$$\begin{aligned} r\left(B_1^1 z + \frac{C_1^1}{2} z^2\right) = a_1^1 z + \frac{a_2}{2} z^2 + \frac{1}{2d}\left(B_1^1 + C_1^1 z\right)^2 \\ + \frac{(B_1^1 + C_1^1 z)(B_2^1 + C_2^1 z)}{d} \\ + q_{12}\left(\left(B_1^2 - B_1^1\right) z + \frac{(C_1^2 - C_1^1)}{2} z^2\right). \end{aligned} \tag{12}$$

Let's now proceed with the identification step. From the equation above, we have the following for Player 1:

$$B_1^1\left(r + q_{12} - \frac{C_1^1 + C_2^1}{d}\right) = a_1^1 + \frac{B_2^1 C_1^1}{d} + q_{12} B_1^2 \tag{13}$$

$$\frac{(C_1^1)^2}{2d} - C_1^1\left(\frac{r + q_{12}}{2} - \frac{C_2^1}{d}\right) + q_{12}\frac{C_1^2}{2} + \frac{a_2}{2} = 0. \tag{14}$$

Similarly, for Player 2:

$$B_2^1\left(r + q_{12} - \frac{C_1^1 + C_2^1}{d}\right) = -a_1^1 + \frac{B_1^1 C_2^1}{d} + q_{12} B_2^2 \tag{15}$$

$$\frac{(C_2^1)^2}{2d} - C_2^1\left(\frac{r + q_{12}}{2} - \frac{C_1^1}{d}\right) + q_{12}\frac{C_2^2}{2} + \frac{a_2}{2} = 0. \tag{16}$$

Suppose instead players are in Regime 2. Following the same methodology as before, we get:

$$B_1^2\left(r+q_{21}-\frac{C_1^2+C_2^2}{d}\right)=a_1^2+\frac{B_2^2C_1^2}{d}+q_{21}B_1^1 \tag{17}$$

$$\frac{(C_1^2)^2}{2d}-C_1^2\left(\frac{r+q_{21}}{2}-\frac{C_2^2}{d}\right)+q_{21}\frac{C_1^1}{2}+\frac{a_2}{2}=0 \tag{18}$$

$$B_2^2\left(r+q_{21}-\frac{C_1^2+C_2^2}{d}\right)=-a_1^2+\frac{B_1^2C_2^2}{d}+q_{21}B_2^1 \tag{19}$$

$$\frac{(C_2^2)^2}{2d}-C_2^2\left(\frac{r+q_{21}}{2}-\frac{C_1^2}{d}\right)+q_{21}\frac{C_2^1}{2}+\frac{a_2}{2}=0. \tag{20}$$

To identify the parameters relevant for each player, we first consider the system (14), (16), (18) and (20). Let us assume that C^j parameters are identical for players in any regime j: $C_1^j=C_2^j$ for $j=1,2$. *This sounds reasonable since the game is entirely symmetric in each regime*. Substituting these relationships into our system, we are left with a system of two equations

$$\begin{aligned}&\frac{3}{d}\left(C_1^1\right)^2-(r+q_{12})C_1^1+q_{12}C_1^2+a_2=0\\&\frac{3}{d}\left(C_1^2\right)^2-(r+q_{21})C_1^2+q_{21}C_1^1+a_2=0\end{aligned} \tag{21}$$

in two unknowns (C_1^1,C_1^2). Taking the difference between these two equations, one obtains:

$$\frac{3}{d}\left[\left(C_1^1\right)^2-\left(C_1^2\right)^2\right]-(r+q_{12}+q_{21})\left(C_1^1-C_1^2\right)=0 \tag{22}$$

Observing (22), two cases are possible: 1. $C_1^1\neq C_1^2$ and 2. $C_1^1=C_1^2$.

- First, suppose that $C_1^1\neq C_1^2$. Then, (22) simplifies to:

$$C_1^2=\frac{d}{3}(r+q_{12}+q_{21})-C_1^1. \tag{23}$$

Using (23), the first equation in (21) can be rewritten as

$$\frac{3}{d}\left(C_1^1\right)^2-(r+2q_{12})C_1^1+\frac{dq_{12}}{3}(r+q_{12}+q_{21})+a_2=0. \tag{24}$$

Assuming that $\Delta_1=r^2-\frac{12}{d}a_2-4q_{12}q_{21}>0$, two solutions thus exist

$$\begin{aligned}C_1^{1-}&=C_2^{1-}=C^{1-}=\frac{d}{6}(r+2q_{12}-\sqrt{\Delta_1})\\C_1^{1+}&=C_2^{1+}=C^{1+}=\frac{d}{6}(r+2q_{12}+\sqrt{\Delta_1}),\end{aligned} \tag{25}$$

each corresponding to a particular C_1^2

$$\begin{aligned} C_1^{2+} = C_2^{2+} = C^{2+} = \frac{d}{6}(r + 2q_{21} + \sqrt{\Delta_1}) \\ C_1^{2-} = C_2^{2-} = C^{2-} = \frac{d}{6}(r + 2q_{21} - \sqrt{\Delta_1}). \end{aligned} \tag{26}$$

Specifically, solutions are (C^{1-}, C^{2+}) and (C^{1+}, C^{2-}).

- Second, consider that $C_1^1 = C_1^2$. Then, the C parameter is the same for both regimes and for both players. It is equal to

$$\begin{aligned} C^+ = \frac{d}{6}(r + \sqrt{\Delta_2}) \\ C^- = \frac{d}{6}(r - \sqrt{\Delta_2}) \end{aligned} \tag{27}$$

with $\Delta_2 = r^2 - \frac{12}{d}a_2 > 0$ if $\Delta_1 > 0$. In this case, players' response to a change in z is similar to one of Wirl, obtained in the problem with no uncertainty (Sect. 2).

We now turn to the identification of B-parameters by solving the system (13), (15), (17), and (19). Guessing that $B_2^j = -B_j^i$ for $j = 1, 2$, this system simplifies to:

$$\begin{aligned} B_1^1\left(r + q_{12} - \frac{C^1}{d}\right) = a_1^1 + q_{12}B_1^2 \\ B_1^2\left(r + q_{21} - \frac{C^2}{d}\right) = a_1^2 + q_{21}B_1^1 \end{aligned}$$

Combining these equations, we obtain the general solution for B coefficients:

$$B_1^1 = \frac{(r + q_{21} - C^2/d)a_1^1 + q_{12}a_1^2}{(r - C^2/d)(r - C^1/d) + q_{12}(r - C^2/d) + q_{21}(r - C^1/d)} \tag{28}$$

$$B_1^2 = \frac{(r + q_{12} - C^1/d)a_1^2 + q_{21}a_1^1}{(r - C^2/d)(r - C^1/d) + q_{12}(r - C^2/d) + q_{21}(r - C^1/d)} \tag{29}$$

Depending on the particular C considered, there are four potential solutions to our uncertain problem. The first type of solution exhibits identical C-parameters in both regimes. Each player adapts her strategy to changes in the liberalization level in the same way, whatever the regime. In this sense, this solution looks like Wirl's outcome. There also exist solutions for which C-parameters change from one regime to the other, which gives rise to more considerable differences in players' behavior. The next section investigates the properties of these two types of solutions. Particular attention will be paid to the impact of uncertainty on players' strategies through the comparison between solutions for the cases with and without uncertainty.

3.2 *Markov Perfect Equilibria with Regime-Independent Responses to Political Liberalization*

Wirl (1994) has a unique MPE in his deterministic problem. Indeed, he uses a stability argument to select, among the two possible values of C given in (27), the negative one. For the sake of comparison, we report players' strategies at our MPE with identical Cs, given that $C = C^-$ (and $\Delta_2 = r^2 - \frac{12}{d}a_2$):[11]

Proposition 1 *Players' efforts, at MPE, are*

$$\begin{aligned}
x_1^1 &= \frac{6[(5r + 6q_{21} + \sqrt{\Delta_2})a_1^1 + 6q_{12}a_1^2]}{d[5r + \sqrt{\Delta_2})][5r + \sqrt{\Delta_2} + 6(q_{12} + q_{21})]} + \frac{1}{6}[r - \sqrt{\Delta_2}]z,\\
x_2^1 &= \frac{6[(5r + 6q_{21} + \sqrt{\Delta_2})a_1^1 + 6q_{12}a_1^2]}{d[5r + \sqrt{\Delta_2}][5r + \sqrt{\Delta_2} + 6(q_{12} + q_{21})]} - \frac{1}{6}[r - \sqrt{\Delta_2}]z,\\
x_1^2 &= \frac{6[(5r + 6q_{12} + \sqrt{\Delta_2})a_1^2 + 6q_{21}a_1^1]}{d[5r + \sqrt{\Delta_2}][5r + \sqrt{\Delta_2} + 6(q_{12} + q_{21})]} + \frac{1}{6}[r - \sqrt{\Delta_2}]z,\\
x_2^2 &= \frac{6[(5r + 6q_{12} + \sqrt{\Delta_2})a_1^2 + 6q_{21}a_1^1]}{d[5r + \sqrt{\Delta_2}][5r + \sqrt{\Delta_2} + 6(q_{12} + q_{21})]} - \frac{1}{6}[r - \sqrt{\Delta_2}]z.
\end{aligned} \tag{30}$$

For each regime separately, the dynamics drive the system toward a steady state with:

$$z_\infty^1 = z_\infty^2 = 0, \qquad x_{i\infty}^1 = \frac{6[(5r + 6q_{21} + \sqrt{\Delta_2})a_1^1 + 6q_{12}a_1^2]}{d[5r + \sqrt{\Delta_2})][5r + \sqrt{\Delta_2} + 6(q_{12} + q_{21})]} \quad \textit{and}$$
$$x_{i\infty}^2 = \frac{6[(5r + 6q_{12} + \sqrt{\Delta_2})a_1^2 + 6q_{21}a_1^1]}{d[5r + \sqrt{\Delta_2}][5r + \sqrt{\Delta_2} + 6(q_{12} + q_{21})]} \quad \textit{for } i = 1, 2. \tag{31}$$

By assuming $q_{12} = q_{21} = 0$, $a_1^2 = a_1^1 = a_1$, one can check that strategies in (30) reduce to Wirl-type MPE, (x_1^N, x_2^N) defined in (5). These strategies share similarities with the ones of the deterministic situation. In particular, for the solution with identical Cs, the effort of Player 1 is always decreasing in z. Regardless of the regime, the opposite holds for Player 2. When the level of liberalization is higher, Player 1 would have less incentive to call for reforms as the system is already more favorable to his interests. On the other hand, a higher z hurts the conservative stance of Player 2. Hence, in order to counteract this, he exerts more effort.

[11] In our stochastic framework, we also have a solution corresponding to $C = C^+$, which can't be eliminated using the stability argument. However, straightforward calculations reveal that this solution has undesirable features: the level of liberalization goes to infinity, which implies that the liberalization effort of Player 2 goes to $-\infty$ (in the absence of nonnegativity constraint on x). That is why we choose to focus on the other solution, that is also more consistent with Wirl's outcome.

However, there are notable differences between equilibrium strategies found above and those derived for the Wirl-type, symmetric case in Sect. 2.2. The existence of uncertainty plays an integral role in determining the effort levels of players. In what follows, results found in Sects. 2.2 and 3.1 are compared analytically. For ease of notation, we again denote "MPE" as the ones found for the uncertain case (with identical and different Cs) and "Wirl-type MPE" for the certain case. With $a_1^1 > a_1^2$, the following proposition can be established.

Proposition 2

- MPE with identical Cs vs. Wirl-type MPE: $x_i^N > x_i^j$ *for* $i = 1, 2$ *and* $j = 1, 2$ *iff the deterministic economy is associated with* $a_1 = a_1^1$. *The opposite holds, that is* $x_i^N < x_i^j$ *iff* $a_1 = a_1^2$.
- MPE with identical Cs vs. OLNE at the steady state: *When* $a_1 = a_1^1$, *it is straightforward that* $x_{i\infty}^O > x_{i\infty}^j$ *for all* i *and all* j *since* $x_{i\infty}^O > x_{i\infty}^N$ *and* $x_i^N > x_i^j$ *for all* z. *When* $a_1 = a_1^2$, $x_{i\infty}^j > x_{i\infty}^O$, *for all* i, *for all* j, *if* $a_1^2 < \hat{a}_1^2$ *with*

$$\hat{a}_1^2 = a_1^1 \frac{36rq_{21}}{(5r + \sqrt{\Delta_2})(5r + \sqrt{\Delta_2} + 6(q_{12} + q_{21}) - 36rq_{12})}. \tag{32}$$

The proof is relegated to the appendix (see the Appendix). Proposition 2 has several implications. First, recall that from (30) it can be shown that $x_i^1 > x_i^2$. In the MPE with identical Cs, the efforts of players are greater when they are in a state with high windfalls than when they are in the low regime. This finding is analogous to taking $\frac{\partial x_i^N}{\partial a_1}$ for the deterministic, Wirl-type case. An incremental increase in the coefficient representing the linear benefits from the liberalization level z implies an increase in the effort levels. All other things constant, the reformist's investment will rise when a_1 goes up. Knowing that this increase in a_1 may hurt him, the conservative will invest more to counteract Player 1's action.

Second, the impact of uncertainty on the comparative relationship between the MPE with identical Cs and the Wirl-type MPE is not clear-cut. Uncertainty lowers the equilibrium investment levels in comparison to the case when a_1 is surely in a high state. Assume that players are in Regime 1 at the present. Knowing that there is a probability that the regime will shift to a setting with low windfalls, players have less incentive to impact liberalization (i.e. relative to the scenario when they are certain that they will always be in Regime 1). Consequently, we find the following relationship: $x_i^N > x_i^j$ when $a_1 = a_1^1$. Contrast this to the case when $a_1 = a_1^2$. The opposite is observed when comparing our MPE to the Wirl-type MPE for the low state. Suppose players are in Regime 2. Since there is a possibility that the regime will alter to a system with higher windfalls, they invest more. Due to an anticipation of a potential shift to the high state, the MPE with identical Cs is higher relative to the Wirl-type MPE for the low state: $x_i^N < x_i^j$.

Third, the steady state levels of the MPE with identical Cs and the OLNE can be compared. When $a_1 = a_1^1$ (high state regime), the open-loop equilibrium investments are greater than the MPE with identical Cs when $z_\infty = 0$. Similar to the deterministic case, players exert relatively less effort into affecting the level of political liberalization. This is because feedback strategies among players are characterized by a dynamic retaliation mechanism. Whenever Player 1 succeeds in shifting the liberalization level towards his favor, she knows that Player 2 will retaliate more. As Wirl (1994) argued, this common knowledge deters aggressive strategies. However, this is not the case when $a_1 = a_1^2$. In particular, the above-mentioned observation does not apply when a_1^2 is low enough. At the steady state, the OLNE for the symmetric case in the low state is below the MPE with identical Cs. Even in the potential presence of retaliation, the existence of uncertainty induces players to exert more effort compared to the OLNE in the low state. Remember that for the Wirl-type solution, players know that they will always be in the low state. Compare this when they are facing uncertainty. Suppose they are initially in Regime 2. The possibility of shifting to Regime 1 may imply more aggressive investment. As a result, the cost of lobbying along the MPE equilibria under uncertainty might be significantly increased with respect to Wirl's deterministic benchmark. In the context of the political liberalization process at stake in Arab countries, this highlights the property that oil volatility is likely to generate significant social costs inherent in the game. This is contrary to what is predicted by standard deterministic theory. The higher the uncertainty, the larger the social costs as strategies will become more aggressive. Furthermore, independent of the economic costs associated with resource prices in exporting countries, this volatility will make the political liberalization process itself more costly. Another complication of uncertainty is the emergence of alternative strategies which do not show up in deterministic frameworks.

3.3 Markov Perfect Equilibria with Regime-Driven Responses to Changes in Liberalization

The solution discussed in the preceding section can be contrasted with a (C^1, C^2)-type of solution, with C^1, C^2 given in (25)–(26). Players' reaction to a change in the liberalization level is dependent on the current regime of the economy. Given that a certain regime is more favorable to a player than the other, it will be useful to investigate how this regime-driven reaction affects the properties of the solution. In the next proposition, we present equilibrium strategies for the case where $C^1 = C^{1-}$, $C^2 = C^{2+}$, and $\Delta_1 = r^2 - \frac{12}{d}a_2 - 4q_{12}q_{21}$. A discussion about the features of this solution and how it compares to the Wirl-type MPE is later conducted.[12]

[12]The conclusions drawn from analysis of the other MPE candidate, corresponding to (C^{1+}, C^{2-}), are qualitatively similar to the ones obtained for (C^{1-}, C^{2+}). For this reason, this case is not dealt with by the subsequent study.

Proposition 3 *Suppose there exists a MPE with regime-driven response to changes in liberalization, then the strategies are given by*

$$\begin{aligned}
x_1^1 &= \frac{6[(5r+4q_{21}-\sqrt{\Delta_1})a_1^1+6q_{12}a_1^2]}{d[(5r-\sqrt{\Delta_1})(5r+\sqrt{\Delta_1})+4\sqrt{\Delta_1}(q_{21}-q_{12})+20(r(q_{12}+q_{21})-q_{12}q_{21})]} \\
&\quad + \frac{1}{6}(r+2q_{12}-\sqrt{\Delta_1})z, \\
x_2^1 &= \frac{6[(5r+4q_{21}-\sqrt{\Delta_1})a_1^1+6q_{12}a_1^2]}{d[(5r-\sqrt{\Delta_1})(5r+\sqrt{\Delta_1})+4\sqrt{\Delta_1}(q_{21}-q_{12})+20(r(q_{12}+q_{21})-q_{12}q_{21})]} \\
&\quad - \frac{1}{6}(r+2q_{12}-\sqrt{\Delta_1})z, \\
x_1^2 &= \frac{6[(5r+4q_{12}+\sqrt{\Delta_1})a_1^2+6q_{21}a_1^1]}{d[(5r-\sqrt{\Delta_1})(5r+\sqrt{\Delta_1})+4\sqrt{\Delta_1}(q_{21}-q_{12})+20(r(q_{12}+q_{21})-q_{12}q_{21})]} \\
&\quad + \frac{1}{6}(r+2q_{21}+\sqrt{\Delta_1})z, \\
x_2^2 &= \frac{6[(5r+4q_{12}+\sqrt{\Delta_1})a_1^2+6q_{21}a_1^1]}{d[(5r-\sqrt{\Delta_1})(5r+\sqrt{\Delta_1})+4\sqrt{\Delta_1}(q_{21}-q_{12})+20(r(q_{12}+q_{21})-q_{12}q_{21})]} \\
&\quad - \frac{1}{6}(r+2q_{21}+\sqrt{\Delta_1})z.
\end{aligned} \tag{33}$$

It is out of the scope of this paper to provide a full analysis of the ergodicity properties of the solutions. However, we can mention some distinctive features of the alternative MPEs through a separate analysis of our two regimes involved. Paying attention to Regime 2, from (33), we observe that the limit value of z is infinite, positive or negative depending on the sign of the initial level of liberalization z_0. As mentioned in footnote 11, this means that if the economy were to stay in Regime 2 for a sufficiently long interval of time, then Player 2's effort would become negative. It is also worth checking how the system behaves in Regime 1. Indeed, it turns out that Regime 1's dynamics are qualitatively similar to the ones of Regime 2 when one assumes

$$-3\frac{a_2}{d} < q_{12}(r+q_{12}+q_{21}), \tag{34}$$

because under this condition, $C^{1-}, C^{2+} > 0$.

Given that the economy randomly switches from Regime 1 to Regime 2, and vice-versa, one may prefer imposing the opposite of (34). The resulting dynamics of Regime 1 are similar to the ones holding at the Wirl-type MPE or at our MPE with identical response to changes in liberalization. It implies that the limit value of z would be zero whereas x_1^1 and x_2^1 would reach finite values.[13]

Several remarks can be discussed from the comparison of the solution in Proposition 3 and the Wirl-type MPE. First, the impact of an increase in z on effort levels is

[13]Thus, in some sense, the dynamical system valid in Regime 1 offsets the explosive trend of Regime 2.

different from those observed from the Wirl-type MPE (and the MPE with identical Cs). From (33), notice the obvious effect of a higher z on the efforts of players in the second regime. In Regime 2, Player 1's (Player 2) investment increases (decreases) with z. Regardless of the switching probabilities, Regime 2 is always characterized by the above-mentioned results. The low state of a_1^2 gives greater incentive to Player 1 to exert more effort when z increases. This is because he wants to take more advantage from political liberalization. There exists a form of intensified reinforcement. In contrast, when z goes up, Player 2 knows it becomes more favorable to Player 1. Knowing that exerting effort is costly, it is actually strategic for Player 2 to lessen his investment. When z already acquired a much higher level, it might be more difficult for him to shift the system to his favor. There is deterrence in his incentive to change the system.

From the discussion above, the reasoning is less obvious for Regime 1. The findings are similar to those in Regime 2 only when Condition (34) is satisfied. This is more likely, given that switching probabilities are high enough. In this case, the strategy of Player 1 increases with respect to the state z while the opposite is relevant for Player 2. When the C-parameters are different, the impact of uncertainty becomes more prominent. Interestingly, the results become the inverse of those observed for the deterministic, Wirl-type MPE. Suppose players are currently in Regime 1. Given a relatively high probability of switching to Regime 2, an incremental increase in z induces Player 1 to exert more effort. This happens because Player 1 knows that he obtains less linear benefits from liberalization in Regime 2 (due to lower a_1). With the anticipation that he might be in Regime 2 the next period, Player 1 tries to compensate and invests more aggressively in the favorable Regime 1. In contrast, Player 2's effort in Regime 1 decreases with z when the likelihood of switching to Regime 2 is high enough. Given that Regime 2 is more favorable to Player 2, i.e. a_1 is reasonably lower, then he has less incentive to invest in Regime 1.

If Condition (34) does not hold, then the results in Regime 1 are similar to those found in the MPE with identical Cs and the Wirl-type MPE. Indeed, when players are in Regime 1 and the probability of switching to Regime 2 is rather low, their incentives are different from those observed when they are Regime 2. Knowing that there is a higher likelihood that he will stay in the favorable Regime 1, Player 1 invests less when political liberalization is more prevalent. Meanwhile, a higher z combined with being in Regime 1 harms the other player more. Player 2 mitigates this by trying to shift the system to his favor, i.e. exert more effort against liberalization.

Finally, it is worth noting that when $a_2 = 0$, the MPE strategies are constant for the Wirl-type MPE and the solution with identical Cs. However, because switching probabilities appear in the solution for different Cs, this is not the case for the MPE with dissimilar C-parameters. The strategies of players in the MPE with different Cs still vary with z. Taking into account the role of uncertainty (i.e. C varies for each regime), the effort levels do not remain constant. Player 1's (Player 2) effort is always increasing (decreasing) in z. The explanation for this result utilizes a similar logic as above.

4 Conclusion

In this paper, we have developed a dynamic game of political liberalization under uncertainty. This is done by using the context of the Arab Spring in resource-rich countries. It has been observed that effort levels of reformists (those who benefit from greater liberalization) and conservatives (those who are against liberalization) tend to differ depending on the setup of the game. In the case with no uncertainty, the strategy of the reformist decreases with respect to the liberalization level while the opposite is true for the conservative. In striking difference, opposite results were observed in the case with uncertainty. When the regime switching probabilities are high enough, the reformist's effort increases with respect to the state z. On the other hand, the conservative's investment decreases with intensified political liberalization. In the presence of uncertainty and greater likelihood of regime shift, an increase in z reinforces the reformist's incentive to induce change. In contrast, when z goes up, the conservative is in a less favorable position and is surprisingly deterred from altering the system. Finally, it was observed that in certain circumstances, the cost of lobbying might be significantly increased under uncertainty with respect to Wirl's benchmark. In the context of the political liberalization in Arab countries, this implies oil volatility is likely to generate significant social costs. Increased uncertainty in rents will make the political liberalization process itself more costly. This aggravates the economic costs associated with volatility of resource prices in exporting countries.

Subject to analytical tractability, the present model may be extended in the following directions. First, one may introduce uncertainty in the cost functions, e.g. it is less costly to invest in Regime 1 than in 2. Second, one may explore a different stochastic environment by incorporating a Wiener-type process that may affect the evolution of political liberalization.

Appendix

A.1 Proof of Proposition 1

A.1.1 MPE with Identical Cs vs. Wirl-Type MPE

Here we compare the MPE in the deterministic case (Wirl-type results)

$$x_i^N = 6\frac{a_1}{d[5r + \sqrt{(r^2 - (12/d)a_2)}]} \pm \frac{1}{6}\left[r - \sqrt{\left(r^2 - \frac{12}{d}a_2\right)}\right]z,$$

with the MPE obtained with uncertain regime switching and identical Cs. In case of Regime 1,

$$x_i^1 = 6\frac{[5r + 6q_{21} + \sqrt{(r^2 - (12/d)a_2)}]a_1^1 + 6q_{12}a_1^2}{d[5r + \sqrt{(r^2 - (12/d)a_2)}][5r + \sqrt{(r^2 - (12/d)a_2)} + 6(q_{12} + q_{21})]}$$
$$\pm \frac{1}{6}\left[r - \sqrt{\left(r^2 - \frac{12}{d}a_2\right)}\right]z,$$

and in case of Regime 2,

$$x_i^2 = 6\frac{[5r + 6q_{12} + \sqrt{(r^2 - (12/d)a_2)}]a_1^2 + 6q_{21}a_1^1}{d[5r + \sqrt{(r^2 - (12/d)a_2)}][5r + \sqrt{(r^2 - (12/d)a_2)} + 6(q_{12} + q_{21})]}$$
$$\pm \frac{1}{6}\left[r - \sqrt{\left(r^2 - \frac{12}{d}a_2\right)}\right]z.$$

- Let us first consider that the deterministic a_1 is the high one: $a_1 = a_1^1$. Then, it is trivial to show that $x_i^j < x_j^N \Leftrightarrow a_1^2 < a_1^1$ for $i, j = 1, 2$ and for all z. This is satisfied by definition.
- Next, suppose that the deterministic a_1 is the one corresponding to Regime 2: $a_1 = a_1^2$. Then, one can check easily that $x_i^j > x_i^N \Leftrightarrow a_1^2 < a_1^1$ for $i, j = 1, 2$ and for all z, which is true by definition.

A.1.2 MPE with Identical *C*s vs. OLNE at the Steady State

Again, we make a distinction between two cases, depending on whether the deterministic a_1 is the high one or not. Following Wirl (1994), attention is paid only to the steady state.

When $a_1 = a_1^1$, the comparison is straightforward: from what we learnt in the preceding appendix, we know that $x_i^j < x_i^N$ for all z. In particular, it holds that $x_{i\infty}^j < x_{i\infty}^N$ (recall that in both cases, $z_\infty = 0$). In addition, Wirl (1994) has shown that $x_{i\infty}^N < x_{i\infty}^O$. So, we have $x_{i\infty}^j < x_{i\infty}^O$ for all $i, j = 1, 2$.

When $a_1 = a_1^2$, the comparison is less obvious because, at the same time, $x_{i\infty}^j > x_{i\infty}^N$ and $x_{i\infty}^N < x_{i\infty}^O$. In Regime 1, from the definition of the open-loop solution (see equation (3)),

$$x_{i\infty}^1 < x_{i\infty}^O \quad \Leftrightarrow \quad \left[(5r + \sqrt{\Delta_2})\big(5r + 6(q_{12}q_{21}) + \sqrt{\Delta_2}\big) - 36rq_{12}\right]a_1^2$$
$$< 6r(5 + 6q_{21} + \sqrt{\Delta_2})a_1^1.$$

Note that the coefficient in the LHS is larger than the one in the RHS. So, given that $a_1^1 > a_1^2$, $x_{i\infty}^1 < x_{i\infty}^O$ is equivalent to

$$a_1^2 < \frac{6r(5 + 6q_{21} + \sqrt{\Delta_2})}{[(5r + \sqrt{\Delta_2})(5r + 6(q_{12}q_{21}) + \sqrt{\Delta_2}) - 36rq_{12}]}a_1^1,$$

this defines an upper bound $\tilde{a}_1^2$ on the coefficient a_1 valid in the low regime.

In Regime 2, following the same approach, we obtain that

$$x_{i\infty}^2 < x_{i\infty}^O$$
$$\Leftrightarrow \quad \big[(5r+\sqrt{\Delta_2})\big(5r+6(q_{12}q_{21})+\sqrt{\Delta_2}\big)-6r(5r+6q_{12}+\sqrt{\Delta_2})\big]a_1^2$$
$$< 36rq_{21}a_1^1,$$

the coefficient in the LHS being again larger than the one in the RHS. Hence, $x_{i\infty}^2 < x_{i\infty}^O$ is equivalent to

$$a_1^2 < \frac{36rq_{21}}{[(5r+\sqrt{\Delta_2})(5r+6(q_{12}q_{21})+\sqrt{\Delta_2})-36rq_{12}]}a_1^1,$$

this defines a second boundary $\hat{a}_1^2$ on the coefficient a_1 valid in the low regime.

Now, given that $\tilde{a}_1^2 > \hat{a}_1^2$, $a_1^2 < \hat{a}_1^2$ implies that $x_{i\infty}^j < x_{i\infty}^O$ for $i, j = 1, 2$ when $a_1 = a_1^2$.

This completes the proof.

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A Feedback Stackelberg Game of Cooperative Advertising in a Durable Goods Oligopoly

Anshuman Chutani and Suresh P. Sethi

Abstract Cooperative advertising is an important mechanism used by manufacturers to influence retailers' promotional decisions. In a typical arrangement, the manufacturer agrees to reimburse a fraction of a retailer's advertising cost, known as the subsidy rate. We consider a case of new product adoption of a durable good with retail oligopoly, in which a manufacturer sells through a number of independent and competing retailers. We model the problem as a Stackelberg differential game with the manufacturer as the leader and the retailers as followers. The manufacturer announces his subsidy rates for the retailers, and the retailers in response play a Nash differential game to increase their cumulative sales and choose their optimal advertising efforts. We obtain feedback Stackelberg strategies consisting of manufacturer's subsidy rates and retailers' optimal advertising efforts. We obtain the conditions under which it is optimal for the manufacturer to not offer any advertising subsidy and study the role of retail competition on the manufacturer's subsidy rates decisions. For a special case of two retailers, using a linear demand formulation, we present managerial insights on issues such as: dependence of subsidy rates on key model parameters, impact on channel profits and channel coordination, and finally, a case of an anti-discrimination legislation which restricts the manufacturer to offer equal subsidy rates to the two retailers.

1 Introduction

Firms spend huge sums of money in advertising, particularly in competitive markets. For some product categories, a firm's market performance and competitiveness over its competitor relies heavily upon its advertising and promotional strategy. Quite

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often, the responsibility of local advertising lies with the retailers as they usually have much better knowledge about customers and local advertising channels such as TV stations, local newspapers, radio stations, etc. Since advertising can be quite expensive, a retailer might not advertise to the extent desired by the manufacturer, whose product the retailer is selling. In such a case, the manufacturer may consider providing some incentive to the retailer to advertise more. An important incentive comes in the form of cooperative advertising, an important and commonly used arrangement in which the manufacturer agrees to reimburse a fraction of the retailer's advertising expenditures in selling his product (Bergen and John 1997). This fraction is commonly known as the 'subsidy rate.'

Cooperative advertising is a fast growing activity amounting to billions of dollars a year, and it can be a significant portion of the advertising budgets of a manufacturer. Nagler (2006) found that the total expenditure on cooperative advertising in 2000 was estimated at $15 billion, compared with $900 million in 1970. Recent estimates put a figure of more than $25 billion for 2007. According to Dant and Berger (1996), as many as 25–40 % of local advertisements and promotions are cooperatively funded. In addition, Dutta et al. (1995) reported that the subsidy rates differ from industry to industry: it was 88.38 % for consumer convenience products, 69.85 % for other consumer products, and 69.29 % for industrial products.

Many researchers in the past have used static models to study cooperative advertising. Berger (1972) modeled cooperative advertising in the form of a wholesale price discount offered by a manufacturer to his retailer as an advertising allowance. He concluded that both the manufacturer and the retailer can do better with cooperative advertising. Dant and Berger (1996) extended the Berger model to incorporate demand uncertainty. Kali (1998) studied cooperative advertising from the perspective of coordinating a manufacturer-retailer channel. Huang et al. (2002) allowed for advertising by a manufacturer in addition to cooperative advertising. They also justified their static model by making a case for short-term effects of promotion.

Jørgensen et al. (2000) formulated a dynamic model with cooperative advertising as a Stackelberg differential game between a manufacturer and his retailer with the manufacturer as the leader. They considered short term as well as long term forms of advertising efforts made by the retailer as well as the manufacturer. They showed that the manufacturer's support of both types of retail advertising benefits both channel members more than the support of only one type, and support of one type is better than no support at all. Jørgensen et al. (2001) modified the above model by introducing decreasing marginal returns to goodwill and studied two scenarios: a Nash game without advertising support and a Stackelberg game with support from the manufacturer as the leader. They characterized stationary feedback policies in both cases. Jørgensen et al. (2003) explored the possibility of advertising cooperation even when the retailer's promotional efforts may erode the brand image. Karray and Zaccour (2005) extended the above model to consider both the manufacturer's national advertising and the retailer's local promotional effort. All of these papers use the Nerlove–Arrow (1962) model, in which goodwill increases linearly in advertising and decreases linearly in goodwill, and there is no interaction term between the sales and the advertising effort in the dynamics of sales. He et al. (2009) solved a manufacturer-retailer Stackelberg differential game with cooperative advertising using the Sethi (1983) model. He et al. (2011) considered a cooperative advertising channel consisting of a manufacturer selling its product through two retailers. In their study, they used a Lanchester-style extension of the Sethi model, in which the two competitors split the total market.

In this chapter, we study cooperative advertising in the case of durable goods. A durable good can be defined as a commodity which, once purchased by the consumer, does not need to be repurchased for a lengthy period of time. Examples of durable goods include cars, TV's, microwave ovens, washing machines, etc. The market potential of such items depletes with time as cumulative sales increase and, eventually, saturation is reached. The advertising decisions for such products can be crucial, particularly in the early stages of their diffusion in the market. The modeling of durable goods sales dynamics is important in economics and management science. Many researchers in the past have studied the sales-advertising dynamics to study new product adoption for durable goods. Mahajan et al. (1990) review some of these models. A well known example of such a model is the Bass (1969) model of innovation diffusion, given by

$$\dot{X}(t) = a\big(1 - X(t)\big) + bX(t)\big(1 - X(t)\big), \tag{1}$$

where $X(t)$ is the cumulative sales by time t, and a and b are positive constants. Many researchers have extended this model by highlighting the dependence of these constants on pricing and advertising policies. Feichtinger et al. (1994) reviewed such models. From the point of view of our research, we use the following model developed recently by Sethi et al. (2008):

$$\dot{X}(t) = \rho u(t) D\big(p(t)\big)\sqrt{1 - X(t)}, \quad X(0) = X_0 \in [0, 1], \tag{2}$$

where $X(t)$ is the cumulative sales by time t with the total market potential normalized to one, $D(p(t))$ is the demand as a function of price $p(t)$ with $\partial D(p(t))/\partial p(t) < 0$, $u(t)$ is the advertising effort rate at time t, and ρ is the effectiveness of advertising. Krishnamoorthy et al. (2010) presented its duopolistic extension in which the sales-dynamics is given by

$$\dot{X}_i(t) = \rho_i u_i(t) D_i\big(p_i(t)\big)\sqrt{1 - X_1(t) - X_2(t)}, \quad X_i(0) = X_{i0} \in [0, 1], i = 1, 2, \tag{3}$$

where the subscript i refers to firm i, $i = 1, 2$.

We study a dynamic cooperative advertising model for a retail market oligopoly of a durable product. We use an oligopolistic extension of (3), specified in the next section as our sales dynamics. The manufacturer sells his product through n independent and competing retailers and may choose to share their advertising costs. We model the problem as a Stackelberg differential game in which the manufacturer, as the leader announces his subsidy rates for the n retailers, and the retailers, acting as followers, respond by choosing their respective advertising efforts. The retailers, thus compete among themselves to increase their cumulative sales and play a Nash differential game to find their optimal advertising efforts.

To the best of our knowledge, with the exception of Chutani and Sethi (2012a), there has not been much work addressing the issue of manufacturer's promotional support decisions for a dynamic market of durable goods. Chutani and Sethi (2012a) studied optimal pricing and advertising decisions for a retailer duopoly of durable goods. They considered the wholesale and retail prices, the retailers' advertising efforts, and the manufacturer's subsidy rates to the retailers' advertising efforts as decision variables. They found that for a linear demand formulation, the manufacturer's optimal subsidy rates are constant and independent of the model parameters. In this chapter, we study an oligopoly of n retailers with only the retailers' advertising efforts and the manufacturer's subsidy rates to those efforts as decision variables. By keeping the wholesale and retail prices as exogenously given, we can focus only on the advertising decisions. This allows us to obtain important managerial insights on such key issues as dependence of subsidy rates on various model parameters, threshold conditions for non-zero subsidy rates, channel coordination with optimal subsidy rates, and the impact of an anti-discrimination legislation when applied to subsidy rates.

In comparison to Chutani and Sethi (2012b) who also study cooperative advertising in a retailer oligopoly setting for a perishable goods market, we focus on durable goods such as refrigerators and vacuum cleaners. In our paper, the state variable is cumulative sales to account for the fact that those who have already purchased the good are no longer in the market, and its derivative, namely the sales rate, enters into the objective function. On the other hand, with frequently purchased goods such as soft drinks and soaps, the customers do not exit the market after their purchases, although they may switch to other brands for their future purchases. Thus in the perishable goods setting, the state is the rate of sales, expressed often times as a fraction of the market potential. It is this sales rate that enters directly into the objective function and makes the model of Chutani and Sethi (2012b) quite different from the model discussed in this chapter. There is another difference between the two models, i.e., the absence of the decay term in (3). This is due to the fact that the cumulative sales, which have already taken place, do not decay. In the perishable goods case, on the other hand, the decay term is ascribed to the effect of factors such as competition, product obsolescence, forgetting, etc., on the change in the rate of sales.

Thus, we make contributions to the existing literature in areas of cooperative advertising, durable goods sales-advertising dynamics, and supply chain coordination by answering the following key research questions. First, what are the optimal subsidy rates of the manufacturer and the optimal advertising responses by the retailers in feedback form for a durable goods retail oligopoly? Second, is cooperative advertising always optimal for the manufacturer, or are there cases under which it is optimal for the manufacturer not to offer any subsidy to the retailers? Moreover, whenever possible, we find a threshold condition, based on model parameters, which delineates the cases under which no subsidy is optimal for the manufacturer. Third, what role does retail level competition play on the subsidy rate policies? Fourth, how do subsidy rates depend on various model parameters? Fifth, what is the impact of a coop advertising program on the profits of all the members in the channel?

How does the channel profit with coop program compare to that without coop advertising, and to the integrated channel profit? Can coop advertising lead to better channel coordination? Finally, sixth, what are the effects of an anti-discrimination legislation that restricts the manufacturer to offer equal subsidy rates to his retailers. How does it impact the optimal subsidy rates, profits of all the members in the channel, and the total channel profit?

The rest of the chapter is organized as follows. In Sect. 2 we describe the model in detail, followed by some preliminary results in Sect. 3. In Sect. 4, we consider a special case of identical retailers and obtain some explicit analytical results along with some useful insights. In Sect. 5, we study a special case of two competing retailers. We perform numerical analysis in a general case and examine the effect of various model parameters on the optimal subsidy rates in the special case of linear demand. In the same section, we discuss the issue of channel coordination with cooperative advertising. In this special case of two retailers, we discuss an extension in which the manufacturer is required to offer equal subsidy rates, if any, to both the retailers. We also study the impact of such an anti-discriminatory act on the profits of all the channel members and on the performance of the channel as a whole. Finally, we conclude the chapter and summarize our findings in Sect. 6.

2 The Model

We consider a model in which a manufacturer sells his product through n independent and competing retailers, labeled $1, 2, \ldots, n$. The manufacturer may subsidize the advertising expenditures of the retailers. The subsidy, expressed as a fraction of a retailer's total advertising expenditure, is referred to as the manufacturer's subsidy rate to that retailer. We now introduce key notation used in the chapter:

t Time $t \in [0, \infty)$,
i Indicates retailer i, $i = 1, 2, \ldots, n$, when used as a subscript,
$X_i(t) \in [0, 1]$ Cumulative normalized sales of retailer i,
$\bar{X}(t) = \sum_{j=1}^{N} X_i(t)$ Total cumulative sales combined over n retailers,
$X(t) \equiv (X_1(t), X_2(t), \ldots, X_n(t))$ Cumulative sales vector of n retailers at time t,
$u_i(t)$ Retailer i's advertising effort rate at time t,
w_i Wholesale price for retailer i,
p_i Retail price of retailer i,
$p_i - w_i = m_i$ Margin of retailer i,
$\theta_i(t) \geq 0$ Manufacturer's subsidy rate for retailer i at time t,
$\Theta(X(t)) \equiv (\theta_1(X(t)), \ldots, \theta_n(X(t)))$ Subsidy rate vector in feedback form at time t,
D_i Demand of goods sold by retailer i, $D_i \geq 0$,
$\rho_i > 0$ Advertising effectiveness parameter of retailer i,
$r > 0$ Discount rate of the manufacturer and the retailers,
V_i, V_m Value functions of retailer i and of the manufacturer, respectively,
V^I Value function of the integrated channel.

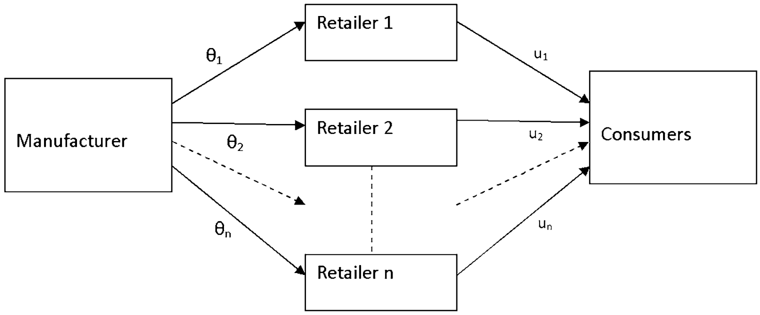


Fig. 1 Sequence of events

Without any loss of generality, we assume that the manufacturing cost of the product is zero. Thus, the margin for the manufacturer from retailer i is equal to the wholesale price w_i. Furthermore, we use the standard notations, i.e., $V_{iX_j} = \partial V_i/\partial X_j$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, and $V_{mX_i} = \partial V_m/\partial X_i$ and $V_{X_i} = \partial V/\partial X_i$, $i = 1, 2, \ldots, n$. For simplicity, $X(t)$ and $\Theta(X(t))$ are also referred to as X, and $\Theta(X)$, respectively

We normalize the total market potential to be one and the cumulative normalized sales of the retailer i, at time t to be denoted by $X_i(t)$, $i = 1, 2, \ldots, n$. The rate of change of cumulative units sold, which is the instantaneous rate of sales, is denoted by $\dot{X}_i(t)$, and is given by

$$\dot{X}_i(t) = \frac{dX_i(t)}{dt} = \rho_i u_i(t) D_i \sqrt{1 - \bar{X}(t)}, \quad X_i(0) = X_i \in [0, 1], i = 1, 2, \ldots, n, \tag{4}$$

where $\bar{X}(t) = \sum_{j=1}^{n} X_j(t)$ is the cumulative sales of the manufacturer at time t, $u_i(t)$ is the retailer i's advertising effort at time t, ρ_i is the effectiveness of firm i's advertising, and D_i is the demand of retailer i. The state of the system is denoted by the cumulative sales vector, i.e., $X(t) \equiv \{X_i(t)\} = (X_1(t), X_2(t), \ldots, X_n(t))$. The sequence of events is shown in Fig. 1. The manufacturer as the Stackelberg leader announces the subsidy rate $\theta_i(t)$ for retailer i, $i = 1, 2, \ldots, n$, at time t. The retailers, acting as followers respond by choosing their respective advertising efforts $u_i(t)$, $i = 1, 2, \ldots, n$, by playing a Nash differential game to increase their sales.

For a solution of our game, we adopt the concept of a feedback Stackelberg equilibrium (see, e.g., Başar and Olsder 1999, and Bensoussan et al. 2014). This type of equilibrium is subgame perfect as well as strongly time-consistent (see Başar and Olsder 1999). Accordingly, the manufacturer's subsidy rates policy, denoted by its subsidy rate vector $\Theta(X) \equiv (\theta_1(X), \theta_2(X), \ldots, \theta_n(X))$, is expressed as a function of the cumulative sales vector $X \equiv (X_1, X_2, \ldots, X_n)$. Thus, the subsidy rates at time $t \geq 0$ are $\theta_i(X(t))$, $i = 1, 2, \ldots, n$. The retailers in response choose their optimal advertising efforts by solving their respective optimization problems. The cost of advertising is quadratic in the advertising effort, representing a marginal diminishing effect of advertising. The use of such quadratic cost function is common in the literature. Retailer i's optimal control problem is to maximize the present value of his profit stream over the infinite horizon, i.e.,

$$V_i(X) = \max_{u_i(t)\geq 0, t\geq 0} \int_0^\infty e^{-rt}\Big((p_i - w_i)\dot{X}_i(t) \\ - \big(1-\theta_i\big(X(t)\big)\big)u_i^2(t)\Big)dt, \quad i = 1, 2, \ldots, n, \tag{5}$$

subject to (4), where $p_i - w_i = m_i$ equals the margin of retailer i and $V_i(X)$ is referred to as the value function of retailer i. The vector $X = (X_1, X_2, \ldots, X_n)$ is the vector of initial conditions such that $X_i \geq 0$, $\forall i = 1, 2, \ldots, n$ and $\sum_{i=1}^n X_i \leq 1$. The solution to the Nash differential game defined by (4)–(5) gives retailer i's feedback advertising effort $u_i(X(t))$, $i = 1, 2, \ldots, n$, which, with a slight abuse of notation, can be written as $u_i(X_1, X_2, \ldots, X_n \mid \theta_1(X_1, X_2, \ldots, X_n), \ldots, \theta_n(X_1, X_2, \ldots, X_n)) \equiv u_i(X \mid \theta(X))$, $i = 1, 2, \ldots, n$.

The manufacturer anticipates the retailers' optimal responses and incorporates them into his optimization problem, which is a stationary infinite horizon optimal control problem:

$$V_m(X) = \max_{\substack{0\leq\theta_i(t)\leq 1\\ i=1,2,\ldots,n, t\geq 0}} \int_0^\infty e^{-rt} \sum_{j=1}^n \Big[w_j \dot{X}_j(t) \\ - \theta_j(t)\big[u_j\big(X(t) \mid \theta_1(t), \ldots, \theta_n(t)\big)\big]^2\Big]dt, \tag{6}$$

subject to for $i = 1, 2, \ldots, n$

$$\dot{X}_i(t) = \rho_i u_i\big(X(t) \mid \theta_1(t), \ldots, \theta_N(t)\big) D_i \sqrt{1-\bar{X}}, \quad X_i(0) = X_i \in [0, 1]. \tag{7}$$

The solution to the optimal control problem (6)–(7) gives the optimal subsidy policy in feedback form, which is expressed as $\theta_i^*(X_1, X_2, \ldots, X_n) \equiv \theta_i^*(X)$. We can also write retailer i's feedback advertising policy, with a slight abuse of notation, as $u_i^*(X) \equiv u_i^*(X \mid \theta_1^*(X), \ldots, \theta_n^*(X)) \equiv u_i^*(X \mid \Theta^*(X))$, $i = 1, 2, \ldots, n$.

The subsidy rate and advertising policies, $\theta_i^*(X)$ and $u_i^*(X)$, $i = 1, 2, \ldots, n$, respectively, constitute a feedback Stackelberg equilibrium of the problem (4)–(7). Substituting these policies into the state equations (4) gives the cumulative sales vector $X^*(t) = (X_1^*(t), X_2^*(t), \ldots, X_n^*(t))$, $t \geq 0$, and the decisions of the manufacturer and the retailers, as $\theta_i^* = \theta_i^*(t) = \theta_i^*(X(t))$ and $u_i^* = u_i^*(t) = u_i^*(X^*(t))$, $i = 1, 2, \ldots, n$, $t \geq 0$, respectively.

3 Preliminary Results

We first solve retailer i's problem to find the optimal advertising policy $u_i^*(X \mid \Theta(X))$, given the subsidy rates $\theta_i(X)$, $i = 1, 2, \ldots, n$, announced by the manufacturer. The Hamilton–Jacobi–Bellman (HJB) equation for the value function of retailer i, $i = 1, 2, \ldots, n$, is

$$rV_i(X) = \max_{u_i \geq 0}\Bigg[(p_i - w_i)\rho_i u_i D_i\sqrt{1-\bar{X}} - \big(1-\theta_i(X)\big)u_i^2 + \sum_{j=1}^{n} V_{iX_j}\rho_j u_j D_j\sqrt{1-\bar{X}}\Bigg], \tag{8}$$

where V_{iX_j} represents the marginal increase in the total discounted profit of retailer i, $i = 1, 2, \ldots, n$, with respect to an increase in the cumulative sales of retailer j, $j = 1, 2, \ldots, n$.

Remark 1 Although we have restricted $\theta_i(X)$, $i = 1, 2, \ldots, n$, to be nonnegative, it is obvious that $0 \leq \theta_i(X) < 1$, $i = 1, 2, \ldots, n$. This is because, if the optimal subsidy rate for a retailer were greater than or equal to one, then that retailer would choose to have an infinite level of advertising, resulting in the manufacturer's value function to be $-\infty$. This would mean that the manufacturer would have even less profit than he would have by not subsidizing any retailer at all. Since the manufacturer is the leader, it also follows that optimal subsidy rates are less than one.

We now obtain the optimal advertising policy of a retailer i, given the subsidy rate policy of the manufacturer.

Proposition 1 *For a given subsidy rate policy $\theta_i(X)$, $i = 1, 2, \ldots, n$, the optimal feedback advertising decision of retailer i is*

$$u_i^* = u_i^*(X \mid \Theta) = \frac{(p_i - w_i + V_{iX_i})\rho_i D_i\sqrt{1-\bar{X}}}{2(1-\theta_i(X))}, \quad i = 1, 2, \ldots, n, \tag{9}$$

and the value function $V_i(X)$ satisfies the partial differential equation

$$rV_i(X) = (1-\bar{X})\left[\frac{(p_i - w_i + V_{iX_i})^2\rho_i^2 D_i{}^2}{4(1-\theta_i(X))} + \sum_{j \neq i}\frac{V_{jX_j}(p_j - w_j + V_{jX_j})\rho_j^2}{2(1-\theta_j(X))}\right]. \tag{10}$$

Proof Using the first-order conditions w.r.t. u_i in (8), $i = 1, 2, \ldots, n$, we obtain (9), and then use (9) in (8) to obtain (10). The second order conditions are also satisfied as it can be seen that $V_i(X)$ is concave in u_i. □

We can see that the advertising effort by retailer i increases with his demand D_i and with the marginal benefit of his own market share. Moreover, the advertising effort is greater for a higher un-captured market $(1-\bar{X})$. Taking into account retailers' optimal responses to his subsidy rates policy, the manufacturer solves his problem to obtain his optimal subsidy rates. The HJB equation for the manufacturer's value function $V_m(X)$ is

$$rV_m(X) = \max_{\theta_i \geq 0, i=1,2,\ldots,n} \sum_{j=1}^{n} \big[(w_j + V_{mX_j})\rho_j u_j^* D_j \sqrt{1-\bar{X}} - \theta_j u_j^{*2}\big].$$

Using (9), we can rewrite the above HJB equation as

$$\frac{rV_m(X)}{(-1+\bar{X})} = \max_{\substack{\theta_i \geq 0 \\ i=1,2,\ldots,n}} \sum_{j=1}^{n} \Big[(p_j - w_j + V_{jX_j}) \times \big(2(w_j + V_{mX_j})(-1+\theta_j) + (p_j - w_j + V_{jX_j})\theta_j\big)\rho_j^2 D_j^2 \Big/ \big(4(-1+\theta_j)^2\big)\Big]. \tag{11}$$

We can now obtain the manufacturer's optimal subsidy rates policy as shown below.

Proposition 2 *The manufacturer's optimal subsidy rate for retailer i is*

$$\theta_i^*(X) = \max\{\hat{\theta}_i(X), 0\}, \quad i = 1, 2, \ldots, n, \tag{12}$$

where

$$\hat{\theta}_i(X) = \frac{2(w_i + V_{mX_i}) - (p_i - w_i + V_{iX_i})}{2(w_i + V_{mX_i}) + (p_i - w_i + V_{iX_i})}, \quad i = 1, 2, \ldots, n, \tag{13}$$

and the manufacturer's value function $V_m(X)$ satisfies

$$\frac{rV_m(X)}{(-1+\bar{X})} = \sum_{j=1}^{n} \Big[(p_j - w_j + V_{jX_j}) \times \big(2(w_j + V_{mX_j})\big(-1+\theta_j^*(X)\big) + (p_j - w_j + V_{jX_j})\theta_j^*(X)\big)\rho_j^2 D_j^2 \Big/ \big(4\big(-1+\theta_j^*(X)\big)^2\big)\Big]. \tag{14}$$

Proof The first-order conditions w.r.t. θ_i, $i = 1, 2, \ldots, n$, in (11) give a unique solution, i.e., $\hat{\theta}_i$, $i = 1, 2, \ldots, n$, as shown in (13). This along with Remark 1 yields the optimal subsidy rates policy as in (12). Finally, we obtain (14) by using (12) in (11). In order to verify the second-order conditions for the optimality of the subsidy rates, we compute the Hessian matrix $\frac{\partial^2 V_m(X)}{\partial\theta_i \partial\theta_j}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$. We find that $\frac{\partial^2 V_m(X)}{\partial\theta_i \partial\theta_j} = 0$ for $i \neq j$, and $\frac{\partial^2 V_m(X)}{\partial\theta_i^2} < 0$ when we use $\theta_i = \hat{\theta}_i$, $i = 1, 2, \ldots, n$. Therefore, the Hessian matrix is negative definite for $\theta_i = \hat{\theta}_i$, $\forall i = 1, 2, \ldots, n$, and the second-order conditions are satisfied. □

Equation (13) shows that the optimal subsidy rate offered by the manufacturer to retailer i increases as the manufacturer's marginal profit with respect to the cumulative sales of retailer i increases. Thus, the manufacturer provides more support to the retailer who offers a higher marginal profit from his sales to the manufacturer. However, as a retailer's own marginal profit with respect to his cumulative sales increases, then the subsidy rate offered by the manufacturer to that retailer decreases. This is because the manufacturer is aware that the retailer has his own incentive to increase his sales by advertising more, and so the manufacturer would lower his subsidy rate to that retailer. Moreover, by using (13) in (9), we see that $u_i^*(X) = \frac{1}{4}(\rho_i D_i(p_i)(2(w_i + V_{mX_i}) + (p_i - w_i + V_{iX_i}))\sqrt{1-\bar{X}})$, which shows that the advertising effort by retailer i increases with the marginal profit of the retailer as well as that of the manufacturer with respect to his cumulative sales.

To obtain the optimal advertising and subsidy rate strategies which constitute a feedback Stackelberg equilibrium, we must find continuously differentiable functions $V_i(X), i = 1, 2, \ldots, n$, and $V_m(X)$ that satisfy equations (10) and (14), respectively. As in Sethi et al. (2008), we look for affine value functions

$$V_i(X) = \beta_i(1-\bar{X}), \qquad i = 1, 2, \ldots, n, \tag{15}$$

$$V_m(X) = \alpha(1-\bar{X}), \tag{16}$$

where α and β_i, $i = 1, 2, \ldots, n$, are constants, and later show that these solve (10) and (14). In order to obtain the coefficients α and β_i, $i = 1, 2, \ldots, n$, we see from (15) and (16) that

$$V_{iX_i} = V_{iX_j} = \beta_i, \quad \text{and} \quad V_{mX_i} = \alpha, \quad i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, n, \; j \neq i. \tag{17}$$

We can also see that with (15)–(16), $\hat{\theta}_i(X)$ and $\theta_i^*(X)$, $i = 1, 2, \ldots, n$, given in (12) and (13) will be constants, and thus, can be simply denoted as $\hat{\theta}_i$ and θ_i^*, respectively, $i = 1, 2, \ldots, n$.

We compare the coefficients of X_i, $i = 1, 2, \ldots, n$, and the constant term of the value functions $V_i(X)$, and $V_m(X)$ in equations (10), (14), and (15)–(16), and obtain the following nonlinear system of equations to be solved for the coefficients α and β_i, $i = 1, 2, \ldots, n$:

$$\begin{aligned} 4r\beta_i = &-\frac{(p_i - w_i - \beta_i)^2 \rho_i^2 D_i^2}{(-1+\theta_i^*)} \\ &+ \sum_{j \neq i} \frac{2\beta_i (p_j - w_j - \beta_j)\rho_j^2 D_j^2}{(-1+\theta_j^*)}, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{18}$$

$$\begin{aligned} 4r\alpha = &\sum_{j=1}^{n} \Bigg[(p_j - w_j - \beta_j)\rho_j^2 D_j^2 \\ &\times \frac{(2(w_j - \alpha)(1-\theta_j^*) - (p_j - w_j - \beta_j)\theta_j^*)}{(1-\theta_j^*)^2} \Bigg], \end{aligned} \tag{19}$$

$$\theta_i^* = \max\left\{ \frac{2(w_i - \alpha) - (p_i - w_i - \beta_i)}{2(w_i - \alpha) + (p_i - w_i - \beta_i)}, 0 \right\}, \quad i = 1, 2, \ldots, n. \tag{20}$$

Using (17) and (20), we can obtain a condition under which the manufacturer will support retailer i. We define

$$\begin{aligned} P_i &= 2(w_i + V_{mX_i}) - (p_i - w_i + V_{iX_i}) \\ &= 2(w_i - \alpha) - (p_i - w_i - \beta_i), \quad i = 1, 2, \ldots, n. \end{aligned} \tag{21}$$

The optimal subsidy rate for retailer i will clearly depend on the sign of P_i, $i = 1, 2, \ldots, n$. Thus, when $P_i > 0$, the manufacturer supports retailer i, otherwise he does not. When $P_i \leq 0$, $\forall i$, no retailer gets advertising support from the manufacturer. In this case, $\theta_i^* = 0$, $i = 1, 2, \ldots, n$, and the set of equations given by (18) can be solved independently of (19) for the coefficients β_i, $i = 1, 2, \ldots, n$. By computing the coefficients α and β_i, $i = 1, 2, \ldots, n$, when $\theta_i^* = 0$, $i = 1, 2, \ldots, n$, we can write the conditions for a zero subsidy rate for each retailer, i.e., $P_i \leq 0$, $i = 1, 2, \ldots, n$, in terms of the model parameters p_i, w_i, ρ_i and D_i, $i = 1, 2, \ldots, n$.

In general, it is difficult to obtain an explicit solution of the system of equations (18)–(20). However, in the special case of identical retailers, defined by $m_1 = m_2 = \cdots = m_n$ (i.e., $p_1 - w_1 = p_2 - w_2 = \cdots = p_n - w_n$), $D_1 = D_2 = \cdots = D_n$, and $\rho_1 = \rho_2 = \cdots = \rho_n$, we can obtain some explicit results, including the values of P_i, $i = 1, 2, \ldots, n$. In addition to this, when $M_1 = M_2 = \cdots = M_n$ i.e., $w_1 = w_2 = \cdots = w_n$), more explicit results can be obtained. In the general case, nevertheless, it is easy to solve the system numerically and study the dependence of the subsidy rates on the various model parameters. We now consider some special cases to get some insights into the problem.

4 Special Case: *n* Identical Retailers

Let $m_i = p_i - w_i = m$, $D_i = D$, and $\rho_i = \rho$, $i = 1, 2, \ldots, n$. Without loss of generality, we can assume that $w_1 > w_2 > w_3 > \cdots > w_{n-1} > w_n$, which is equivalent to $p_1 < p_2 < \cdots < p_n$. In order to obtain the condition under which none of the retailers will be supported (i.e., $P_i \leq 0$, $i = 1, 2, \ldots, n$), we set $\theta_i^* = 0$, $i = 1, 2, \ldots, n$, in equations (18)–(19) and then solve for β_i, $i = 1, 2, \ldots, n$, and α in an explicit form to obtain P_i, $i = 1, 2, \ldots, n$, as follows:

$$\begin{aligned} P_i = 2w_i &\left[1 - \frac{2D^2m^2\rho^2}{2r + (n+1)D^2m\rho^2 + \sqrt{4r^2 + 4nrD^2m\rho^2 + (n-1)^2D^4m^2\rho^4}}\right] \\ &- 2W_{-i}\left[\frac{2D^2m^2\rho^2}{2r + (n+1)D^2m\rho^2 + \sqrt{4r^2 + 4nrD^2m\rho^2 + (n-1)^2D^4m^2\rho^4}}\right] \\ &- \frac{(n-1)D^2m\rho^2 - 2r + \sqrt{4r^2 + 4nrD^2m\rho^2 + (n-1)^2D^4m^2\rho^4}}{(2n-1)D^2\rho^2}, \end{aligned} \tag{22}$$

where $W_{-i} = \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j$. The derivation of (22) is shown in the Appendix. We can now conclude the following result.

Proposition 3 *When $P_i \leq 0$, $i = 1, 2, \ldots, n$, we have a non-cooperative equilibrium in which it is optimal for the manufacturer not to support any retailer. Furthermore, if $P_i > 0$ and $P_j \leq 0$, $j = 1, 2, \ldots, n$, $j \neq i$, we have $\theta_i^* > 0$ and $\theta_j^* = 0$, $j = 1, 2, \ldots, n$, $j \neq i$, that is, the manufacturer supports retailer i only.*

We can observe that P_i is linear in w_j, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$. In P_i, the coefficient of w_i is positive and that of w_j, $j \neq i$ is negative. Thus, P_i increases as the margin of the manufacturer from retailer i (which is the same as the wholesale price charged from retailer i) increases, and it decreases as the margin from any other retailers decreases. As retailer i pays a higher wholesale price to the manufacturer, his likelihood of receiving advertising support from the manufacturer increases. Moreover, this increase in w_i further hampers the case of retailer j, $j \neq i$, in getting support from the manufacturer. This is intuitive because it is beneficial for the manufacturer to support the retailer who is more profitable to him and increase his sales, and since the n retailers compete for the same market, it comes at a cost for the other $n - 1$ retailers. Indeed, it can be seen that

$$P_i - P_j = 2(w_i - w_j), \tag{23}$$

which means that $w_i > w_j$ implies $P_i > P_j$. Thus, when $P_i \leq 0$ and $P_j \leq 0$ for $i \neq j$, retailer i will be the first to start receiving a positive subsidy rate, whenever changes in the parameters (m, D, ρ) cause the sign of P_i to change from negative to positive, and retailer j will never receive any support as long as $w_i > w_j$. In other words, a retailer who pays a higher wholesale price is more likely to get a positive subsidy rate from the manufacturer when compared to a retailer who offers a lower wholesale price.

To further enhance the understanding of our results, we assume that the discount rate is very small, i.e., $r \approx 0$. Under this condition, the expressions for P_i can be simplified to

$$P_i = \frac{2w_i(n-1)}{n} - \frac{2W_{-i}}{n} - \frac{2m(n-1)}{(2n-1)}, \tag{24}$$

where $W_{-i} = \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j$. Equation (24) yields some useful insights from our analysis of the case of identical retailers for small values of the discount rate. We can see from (24) that if w_i is less than the average wholesale price of other $n - 1$ retailers, i.e., $\frac{W_{-i}}{(n-1)}$, then retailer i will not be supported. In addition, if the retailers are also symmetric (i.e., $w_1 = w_2 = \cdots = w_n$), then $P_i < 0$, $i = 1, 2, \ldots, n$, and no retailer will be supported. If we assume that $w_j = 0$, $j = 2, 3, \ldots, n$, $j \neq 1$, so that only retailer 1 sells the manufacturer's product and all other retailers compete with retailer 1, then, under competition, the condition for support for retailer 1 is $w_1 \frac{(n-1)}{n} \geq m \frac{(n-1)}{(2n-1)} = (p_1 - w_1) \frac{(n-1)}{(2n-1)}$. Furthermore, if the number of retailers n is very large, then retailer i receives advertising support when $w_i > m/2 = (p_i - w_i)/2$, i.e. when the manufacturer's margin from retailer i is at-least half of retailer i's margin.

5 Special Case: Two Non-identical Retailers

In this section, we further explore our model in the case of two non-identical retailers, to get some useful managerial insights. We look into issues such as dependence of subsidy rates on different model parameters, issue of channel coordination and profits of the channel members with cooperative advertising, and a case of anti-discriminatory legislation.

5.1 Numerical Analysis

We perform numerical analysis to study the dependence of the manufacturer's subsidy rates on wholesale prices (w_1, w_2), retailers' margins $(p_1 - w_1, p_2 - w_2)$, and advertising effectiveness coefficients (ρ_1, ρ_2). We consider a linear demand form and study the impact of the price sensitivity of demand on the subsidy rates. In this analysis, we first take a base case with a value for each parameter and then vary different parameters one by one to study their impacts on θ_1^* and θ_2^*. To study the effect of retailer 1's margin, we vary p_1 and keep all other parameters unchanged. Similarly, by changing w_1 and keeping all other parameters constant, we study the impact of manufacturer's margin. We consider the following demand specification for given retail prices:

$$D_i = 1 - \eta_i p_i, \quad i = 1, 2, \tag{25}$$

where η_i represents the price sensitivity of the demand. The linear demand function is popular in the literature (e.g., Petruzzi and Dada 1999; Sethi et al. 2008; Krishnamoorthy et al. 2010). We perform numerical analysis for a wide range of parameters and present some representative results for a base case of $w_1 = w_2 = 0.3$, $p_1 = p_2 = 0.6$, $\eta_1 = \eta_2 = 1$, and $\rho_1 = \rho_2 = 1$. Thus in the base case $m_1 = m_2 = 0.3$.

(a) Effect of the manufacturer's margin Fig. 2: We vary w_1 to change the manufacturer's margin from retailer 1, but also change p_1 accordingly to keep retailer 1's margin $(p_1 - w_1)$ constant. Note that as p_1 increases retailer 1's demand D_1 decreases. We find that as w_1 increases, the manufacturer starts offering a higher subsidy rate to retailer 1, rewarding him for providing a higher margin. Retailer 2's subsidy rate decreases initially and then increases. The retailer who offers a higher margin (wholesale price) to the manufacturer gets a higher subsidy rate.

(b) Effect of the manufacturer's margin and changing retailer 1's margin Fig. 3: We vary w_1 to change the manufacturer's margin and keep p_1 constant, thereby changing retailer 1's margin as well. As w_1 increases, the manufacturer offers a higher subsidy rate to retailer 1 and reduces the subsidy rate to retailer 2. Since p_1 remains constant, retailer 1's margin decreases as w_1 increases, so he has less incentive to advertise. The manufacturer rewards retailer 1, now that he gives him a higher margin, by increasing his subsidy rate, and simultaneously

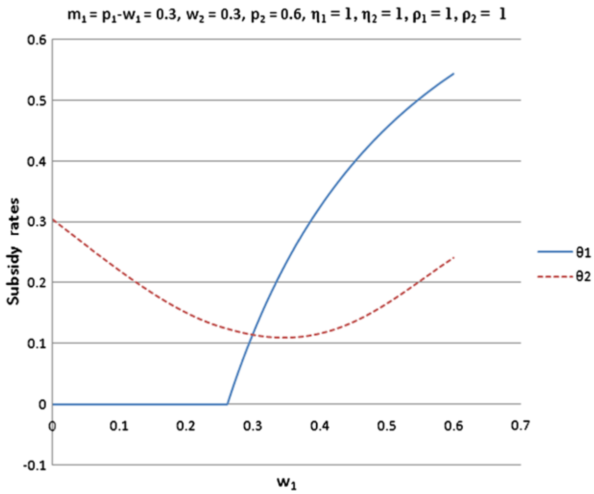


Fig. 2 Subsidy rates vs. w_1, fixed $p_1 - w_1$

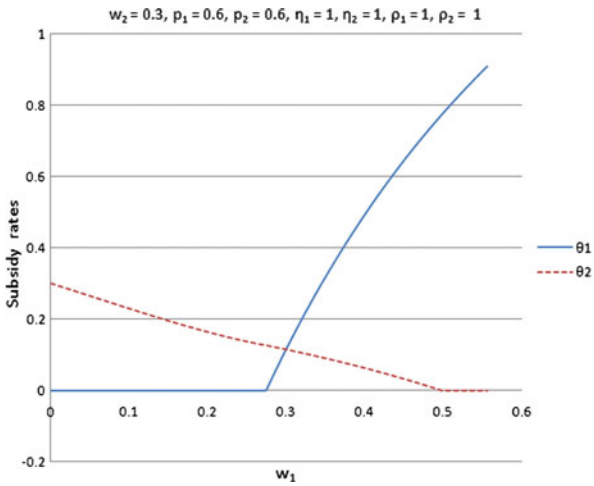


Fig. 3 Subsidy rates vs. w_1

reduces the subsidy rate of retailer 2. The manufacturer gives a higher subsidy rate to the retailer who gives a higher margin.

(c) Effect of retail price (and hence retailer's margin) Fig. 4: As p_1 increases, retailer 1's margin increases and demand D_1 decreases. With a higher p_1, the manufacturer knows that retailer 1 has a higher incentive of his own to advertise more. Moreover as retailer 1's demand decreases, the manufacturer sees a greater possibility of increase in its sales through retailer 2. The combined effect of these factors makes the subsidy rate for retailer 1 to decrease and that of retailer 2 to increase gradually. The retailer with the lower retail price gets a higher subsidy rate.

(d) Effect of the advertising effectiveness parameter Fig. 5: As the advertising effectiveness of retailer 1 increases, the subsidy rates for both retailers decrease.

Fig. 4 Subsidy rates vs. p_1

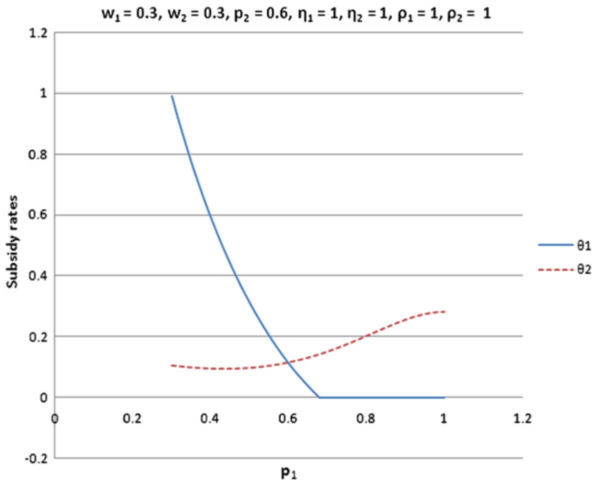
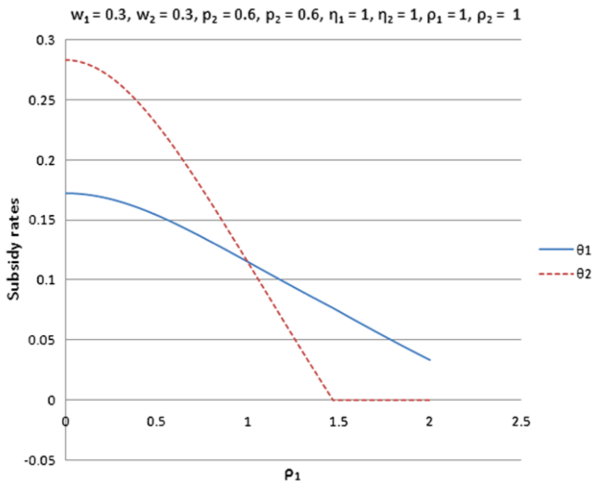


Fig. 5 Subsidy rates vs. ρ_1



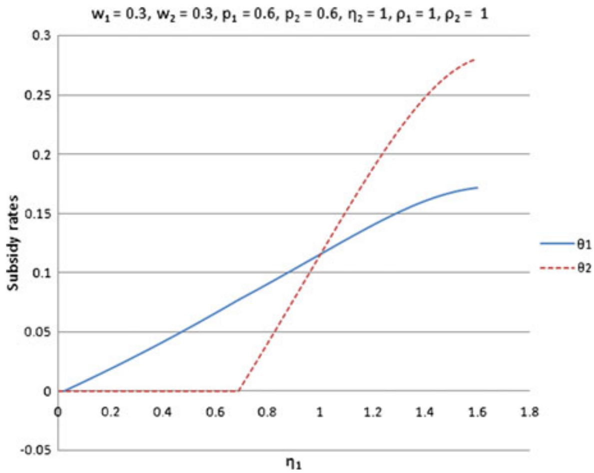
The rate of decrease is higher for retailer 2 than for retailer 1. All other parameters being the same, the retailer with the higher advertising effectiveness gets a higher subsidy rate.

(e) Effect of the price sensitivity of demand Fig. 6: As η_1 increases, $D_1 = 1 - \eta_1 p_1$ decreases. The manufacturer increases the subsidy rate for both retailers. The retailer with the higher price sensitivity gets a lower subsidy rate.

5.2 Channel Coordination

In this section, we analyze the impact of cooperative advertising arrangement on the profits of the manufacturer and the retailers, and thereby investigate the role

Fig. 6 Subsidy rates vs. η_1



of cooperative advertising in coordinating the channel and improving the overall channel profit. We compare the value functions of the channel members and that of the channel as a whole in three cases: (i) an integrated channel in which the advertising decisions are taken on the basis of maximization of the total combined profit of the manufacturer and the retailers, (ii) a decentralized channel with optimal subsidy rates, where the manufacturer chooses the optimal subsidy rates and the retailers decide their optimal levels of advertising, and (iii) a decentralized channel without any cooperative advertising.

In the integrated channel case, the optimization problem to decide the optimal level of advertising can be written as follows:

$$V^I(X_1, X_2) = \max_{u_1(t)\geq 0, u_1(t)\geq 0, t\geq 0} \int_0^{\infty} e^{-rt}\big(p_1\dot{X}_1(t) + p_2\dot{X}_2(t) - u_1^2(t) - u_2^2(t)\big)dt \tag{26}$$

subject to

$$\dot{X}_i(t) = \frac{dX_i(t)}{dt} = \rho_i u_i(t) D_i \sqrt{1 - X_1(t) - X_2(t)}, \qquad X_i(0) = X_i \in [0, 1], i = 1, 2. \tag{27}$$

The HJB equation for the integrated channel value function V^I is

$$rV^I(X_1, X_2) = \max_{u_1(t)\geq 0, u_1(t)\geq 0, t\geq 0} \big[p_1\dot{X}_1 + p_2\dot{X}_2 - u_1^2 - u_2^2 + V_{X_1}^I\dot{X}_1 + V_{X_2}^I\dot{X}_2\big], \tag{28}$$

where $\dot{X}_1$ and $\dot{X}_2$ are given by (27). Using (27) in the HJB equation (28) and applying the first-order conditions for maximization with respect to u_1 and u_2, we obtain the following

Proposition 4 *The optimal feedback advertising policies for the integrated channel are*

$$u_1^* = \frac{\rho_1}{2} D_1(p_1 + V_{X_1})\sqrt{1 - X_1 - X_2},$$
$$u_2^* = \frac{\rho_2}{2} D_2(p_2 + V_{X_2})\sqrt{1 - X_1 - X_2}, \tag{29}$$

and the value function of the integrated channel satisfies the partial differential equation

$$4rV^I(X_1, X_2) = (1 - X_1 - X_2)\big[(p_1 + V_{X_1})^2\rho_1^2 D_1^2 + (p_2 + V_{X_2})^2\rho_2^2 D_2^2\big]. \tag{30}$$

Proof We obtain (29) by applying the first-order conditions with respect to u_1 and u_2 in the HJB equation (28), and (30) can be obtained by using (29) and (27) in (28). □

Once again, we show that

$$V^I(X_1, X_2) = \alpha^I(1 - X_1 - X_2), \tag{31}$$

solves (30), for some α^I to be determined. Since $\alpha^I = -V^I_{X_1} = -V^I_{X_2}$ is a constant, it must satisfy the equation

$$\alpha^I = \big(p_1 - \alpha^I\big)^2\rho_1^2 D_1^2 + \big(p_2 - \alpha^I\big)^2\rho_2^2 D_2^2. \tag{32}$$

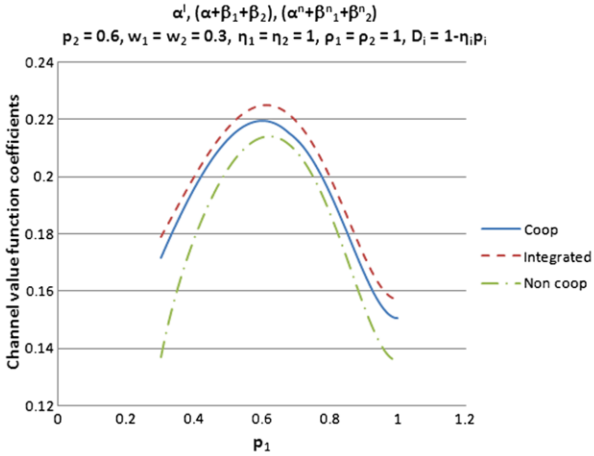
This is a quadratic equation in α^I which gives two real roots. We choose the one which gives $p_1 - \alpha^I \geq 0$ and $p_2 - \alpha^I \geq 0$, as it ensures $u_1^* \geq 0$ and $u_2^* \geq 0$. We therefore have

$$\alpha^I = \frac{2r + p_1\rho_1^2 D_1^2 + p_2\rho_2^2 D_2^2}{\rho_1^2 D_1^2 + \rho_2^2 D_2^2} - \frac{\sqrt{4r^2 + 4r(p_1\rho_1^2 D_1^2 + p_2\rho_2^2 D_2^2) - (p_1 - p_2)\rho_1^2 D_1^2\rho_2^2 D_2^2}}{\rho_1^2 D_1^2 + \rho_2^2 D_2^2}. \tag{33}$$

In the second case, we consider a *decentralized channel with cooperative advertising*, where the manufacturer chooses the optimal subsidy rates. We define the value function in this case as $V^c(X_1, X_2) = V_m^c(X_1, X_2) + V_r^c(X_1, X_2)$, where V_m^c is the manufacturer's value function (given by (16)) and V_r^c is the sum of the value functions of the two retailers obtained by (15).

The third case is of a *decentralized channel with no cooperation*, with the channel value function defined as $V^n(X_1, X_2) = V_m^n(X_1, X_2) + V_r^n(X_1, X_2)$, where V_m^n and V_r^n are the manufacturer's value function and the sum of the two retailers' value functions, respectively, in the non-cooperative setting. These value functions are computed by setting $\theta_1^* = \theta_2^* = 0$ in (18)–(19), and then using the resulting values of α, β_1, and β_2 in (15)–(16). We term the values of these coefficients in the non-cooperative case as α^n, β_1^n, and β_2^n, respectively.

Fig. 7 Channel value functions in integrated, coop, and non-cooperative cases



Since the manufacturer is the leader and decides the subsidy rates by maximizing his total discounted profit, it is obvious that

$$V_m^c(X_1, X_2) \geq V_m^n(X_1, X_2). \tag{34}$$

In general, it is difficult to establish explicit analytical relationships between the value functions of the retailers and the channel as a whole. We therefore use numerical analysis to study the effect of cooperative advertising on the profits of all the parties in the channel. Recall that $V^I = \alpha^I(1 - X_1 - X_2)$, $V^c = (\alpha + \beta_1 + \beta_2)(1 - X_1 - X_2)$, and $V^n = (\alpha^n + \beta_1^n + \beta_2^n)(1 - X_1 - X_2)$. Thus, in order to do a comparison of any two value functions, it is sufficient to compare their respective coefficients of $(1 - X_1 - X_2)$. We study V^I, V^c and V^n with respect to the changes in the optimal subsidy rates brought about by changes in the model parameters. In the results shown, the changes in the value functions correspond to the changes in the margin of retailer 1 (caused by changes in p_1). As p_1 increases, we know from Fig. 4 that θ_1 decreases, and θ_2 increases gradually. Figure 7 depicts the values of α^I, $(\alpha + \beta_1 + \beta_2)$, and $(\alpha^n + \beta_1^n + \beta_2^n)$, and thus compares V^I, V^c, and V^n, respectively, for $D_1 = 1 - \eta_1 p_1$, $D_2 = 1 - \eta_2 p_2$, $p_2 = 0.6$, $w_1 = 0.3$, $w_2 = 0.3$, $\eta_1 = 1$, $\eta_2 = 1$, $\rho_1 = 1$, and $\rho_2 = 1$. Thus, for any point in Fig. 7, the values of the optimal subsidy rates are the same as the corresponding values in Fig. 4. We find that under all instances, $\alpha^I > (\alpha + \beta_1 + \beta_2) > (\alpha^n + \beta_1^n + \beta_2^n)$, and thus $V^I > V^c > V^n$. The result that V^I is the highest of all is obvious, as we expect the integrated channel value function to be greater than that in the decentralized channel, with or without cooperative advertising. This result also indicates that through a cooperative advertising mechanism, as proposed in our model, total channel profit can be increased and better channel coordination can be achieved. The level of partial coordination measured by the ratio $(V^c - V^n)/(V^I - V^n)$ is found to be as high as 82.5 %.

Figure 8 shows the difference in the value function coefficients between cooperative and non-cooperative settings for the manufacturer, the two retailers, and the

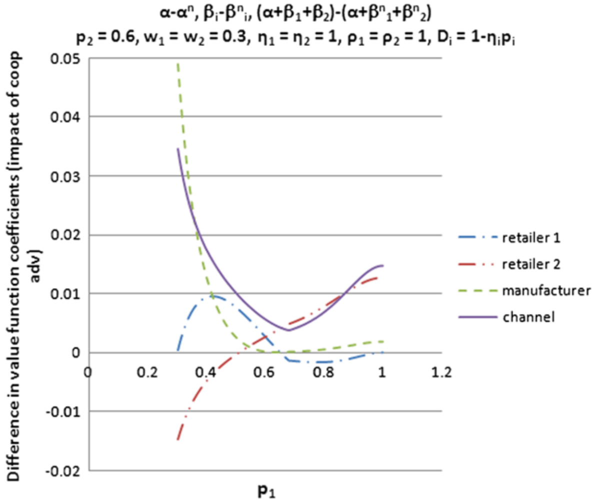


Fig. 8 Difference between the value functions in the coop case and the non-coop case

total channel, i.e., $\alpha - \alpha^n$, $\beta_1 - \beta_1^n$, $\beta_2 - \beta_2^n$, and $(\alpha + \beta_1 + \beta_2) - (\alpha^n + \beta_1^n + \beta_2^n)$, respectively. Once again, we use $D_1 = 1 - \eta_1 p_1$, $D_2 = 1 - \eta_2 p_2$, $p_2 = 0.6$, $w_1 = 0.3$, $w_2 = 0.3$, $\eta_1 = 1$, $\eta_2 = 1$, $\rho_1 = 1$, and $\rho_2 = 1$. As expected, the manufacturer always benefits from cooperative advertising. In view of results in Fig. 4, we can see that the manufacturer's benefit is higher when, roughly speaking, the difference between the subsidy rates of the two retailers is higher. The retailers, however, do not seem to benefit always from cooperative advertising. It is found that when a retailer receives a much lower subsidy rate in comparison to his competitor, he does not seem to benefit from this arrangement. In other words, when $\theta_1^* - \theta_2^*$ is high, retailer 2 does not benefit from cooperative advertising, and vice versa. Figure 8 also shows that the region in which both retailers benefit from cooperative advertising is a small range of values of p_1, around the point when $p_1 = p_2$, which is when both retailers receive almost equal subsidy rates. Thus, the retailer which has a higher margin relative to his competitor and thus gets a significantly lower subsidy rate from the manufacturer, might not prefer a cooperative advertising arrangement.

These observations raise the issue of the manufacturer preferring one retailer over the other, in terms of subsidy rates, particularly when it seems that the retailer receiving a significantly lower subsidy rate might make less profit from cooperative advertising than without it. Next, we study the effect of an anti-discriminatory legislation, such as the Robinson–Patman Act of 1936, which would compel the manufacturer to offer equal subsidy rates to both retailers.

5.3 Equal Subsidy Rate for Both Retailers

We consider the case when the manufacturer is required to offer equal subsidy rates to both retailers. We let $V_m^{RP}(X_1, X_2)$, $V_1^{RP}(X_1, X_2)$, $V_2^{RP}(X_1, X_2)$, and

$V^{RP}(X_1, X_2)$ denote the value functions of the manufacturer, retailer 1, retailer 2, and the total channel, respectively, with the superscript RP standing for Robinson and Patman. These value functions solve the control problems defined by (4)–(6) with $\theta_1 = \theta_2 = \theta$, as the manufacturer's optimization problem now has only one subsidy rate decision. As in the general model, we obtain value functions that are linear in X_1 and X_2 and are a multiple of $(1 - X_1 - X_2)$, and we express them as in (15)–(16). The value function coefficients for the manufacturer and the two retailers are now defined as α^{rp}, β_1^{rp}, and β_2^{rp}, respectively. The coefficients solve the system of equations obtained by setting $\theta_1^* = \theta_2^* = \theta^*$ in (18)–(19). We thus have the following system of equations:

$$4r\beta_i^{rp} = -\frac{(p_1 - w_1 - \beta_i^{rp})^2 \rho_i^2 D_i^2}{(-1+\theta^*)} + \frac{2\beta_i^{rp}(p_{3-i} - w_{3-i} - \beta_{3-i}^{rp})\rho_{3-i}^2 D_{3-i}^2}{(-1+\theta^*)}, \quad i = 1, 2, \tag{35}$$

$$4r\alpha^{rp} = -\sum_{i=1}^{2}\Bigg[\left(p_i - w_i - \beta_i^{rp}\right)\rho_i^2 D_i^2 \times \frac{((p_i - w_i - \beta_i^{rp})\theta^* + 2(w_i - \alpha^{rp})(-1+\theta^*))}{(-1+\theta^*)^2}\Bigg], \tag{36}$$

$$\theta^* = \max\left\{\frac{(2A_1 - B_1)B_1\rho_1^2 D_1^2 + (2A_2 - B_2)B_2\rho_2^2 D_2^2}{(2A_1 + B_1)B_1\rho_1^2 D_1^2 + (2A_2 + B_2)B_2\rho_2^2 D_2^2}, 0\right\}, \tag{37}$$

where

$$A_1 = w_1 - \alpha^{rp}, \qquad A_2 = w_2 - \alpha^{rp}, \qquad B_1 = p_1 - w_1 - \beta_1^{rp} \quad \text{and} \quad B_2 = p_2 - w_2 - \beta_2^{rp}. \tag{38}$$

The threshold for no cooperation with both retailers is that

$$P = (2A_1 - B_1)B_1\rho_1^2 D_1^2 + (2A_2 - B_2)B_2\rho_2^2 D_2^2 \le 0. \tag{39}$$

We perform numerical analysis to study the behavior of θ^* with respect to different model parameters and compare it with the optimal subsidy rates in the general model with no restriction on the subsidy rates. Figures 9, 10, 11, 12, and 13 show the dependence of θ^* on w_1 with fixed $p_1 - w_1$, w_1 with fixed p_1, p_1, ρ_1, and η_1, respectively, and compare θ^* with θ_1^* and θ_2^* (optimal subsidy rates with no legislation in effect) for linear demand, i.e., $D_i = 1 - \eta_i p_i$. We find that as w_1 and p_1 increase while keeping retailer 1's margin $(p_1 - w_1)$ constant, the common subsidy rate for the two retailers increases, but at a decreasing rate Fig. 9. However, when the increase in w_1 is accompanied by a fixed p_1, thereby reducing retailer 1's margin, we find that the common subsidy rate first increases and then decreases. As p_1 increases, θ^* changes in a more complicated decreasing-increasing fashion as shown in Fig. 11. Recall that an increase in p_1 causes D_1 to decrease. The dependence of

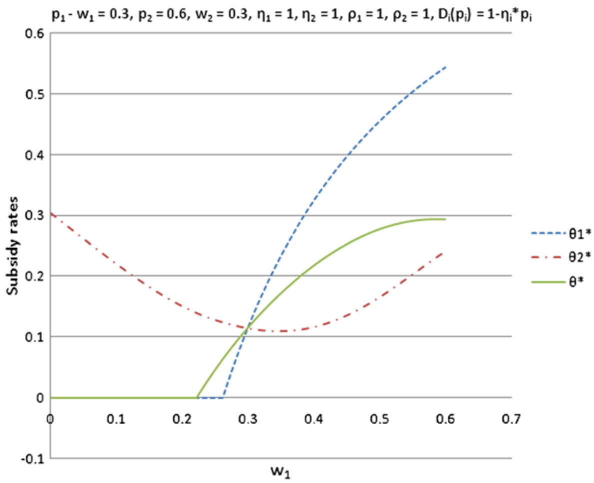


Fig. 9 Subsidy rate vs. w_1, constant $p_1 - w_1$

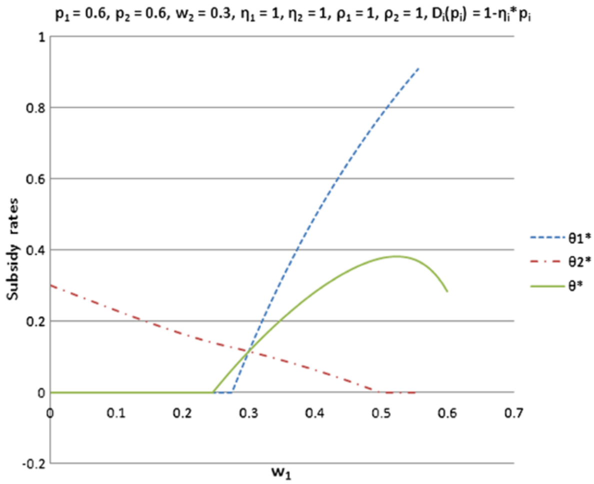


Fig. 10 Subsidy rate vs. w_1, constant p_1

the common subsidy rate on ρ_1 and η_1 is similar to the dependence of the optimal subsidy rates in the unrestricted model, i.e., decreasing with ρ_1 and increasing with η_1, as shown in Fig. 12, and Fig. 13.

We now investigate the impact of an anti-discriminatory legislation on the value functions of all the parties in the supply chain and on the channel value function. We compare the value functions in three cases: a channel without any cooperative advertising, a channel with no anti-discriminatory act and optimal subsidy rates, and a channel with an anti-discriminatory act and optimal common subsidy rate for both retailers. Recall that the value functions in our model take the form of a constant times $(1 - X_1 - X_2)$, and thus we compare the value of these coefficients (α, β_1, β_2, α^{rp}, β_1^{rp}, β_2^{rp}, α^n, β_1^n, β_2^n) via numerical analysis to understand the comparison between the various value functions. Figures 14 and 15 show a comparison of these

Fig. 11 Subsidy rate vs. p_1

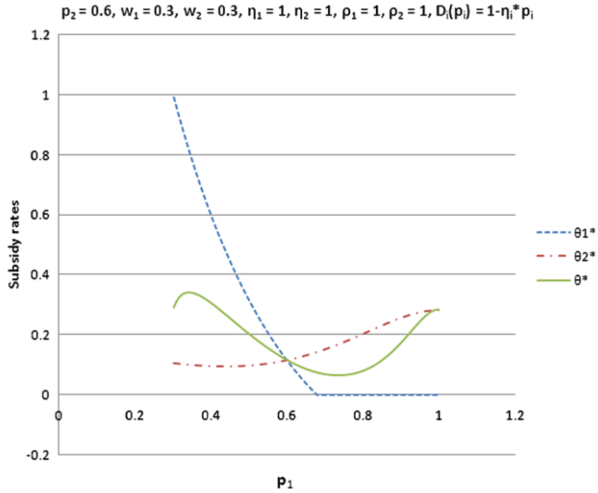
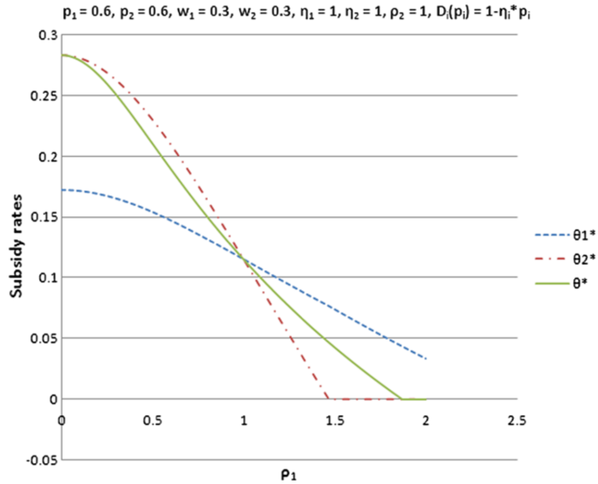


Fig. 12 Subsidy rate vs. ρ_1



coefficients with changes in p_1. The values of the parameters are chosen so that for any point in these curves, the optimal subsidy rates $(\theta_1^*, \theta_2^*, \theta^*)$ are the same as the corresponding values in Fig. 11. Figure 14 shows the impact of a Robinson–Patman like legislation on the profits of all the parties in the channel by plotting the difference between the value function coefficients with and without the legislation. As is obvious, the manufacturer does not benefit from this legislation because of the additional constraint on his optimization problem. The manufacturer's loss is high when p_1 is very low, i.e., when the difference $\theta_1^* - \theta_2^*$ is high. The manufacturer's loss is low when the difference between the two optimal subsidy rates in the unconstrained problem is low. We find that the retailer receiving a higher subsidy rate in the absence of the legislation, does not benefit either. However, a less efficient retailer who would have received a lower subsidy rate without the legislation, benefits

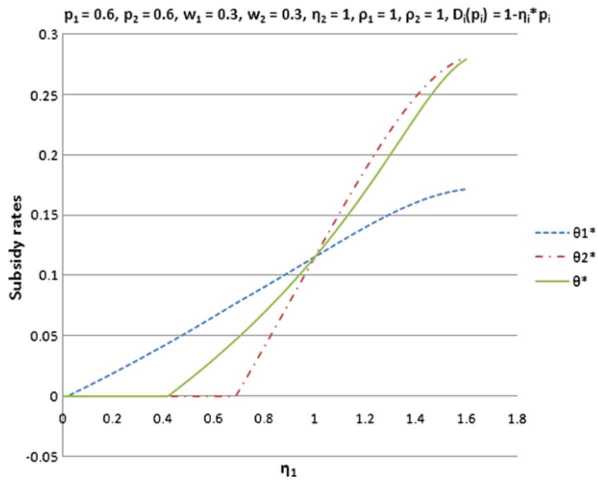


Fig. 13 Subsidy rate vs. η_1

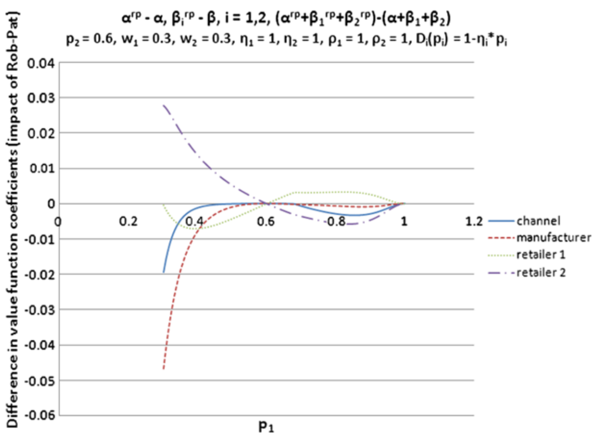


Fig. 14 Impact of anti-discriminatory legislation on the value functions

as his subsidy rate is increased under the act. Thus, when p_1 is low, retailer 1 loses and retailer 2 benefits, and when p_1 is high, retailer 1 loses and retailer 2 benefits from the legislation. Noticeably though, in all the instances studied, the gain of one retailer was not able to offset the losses of the other two parties and the total channel suffered as a whole. These results indicate that an anti-discriminatory legislation in the context of cooperative advertising could be beneficial to only one of the two retailers and not to the other parties, and could also result in a lower channel profit.

Figure 15 compares the total profit of an integrated channel (V^I) with the total channel profit in three cases: no advertising cooperation (V^n), cooperation with no legislation (V^c), and cooperation with equal subsidy rates (V^{rp}), with a view of comparing the level of channel coordination possible in these three scenarios. Here again, we see that while an unrestricted cooperative advertising arrangement can coordinate the channel to a great extent (up to 85 %, as found previously), the enforcement of an anti-discriminatory law on the manufacturer can decrease the

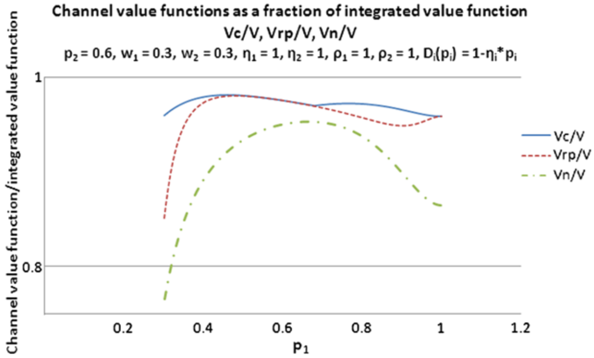


Fig. 15 Value functions in different cases divided by the integrated channel value function

channel profit and thereby reduce the level of coordination achieved. The case of no advertising cooperation seems to perform worst of all with the lowest channel profit and thus, the lowest level of coordination. Once again, these results suggest that for a durable goods duopoly with a sales dynamics as ours, cooperative advertising with no regulation might be the best alternative of the three from the perspective of total channel profit.

6 Concluding Remarks

We study a cooperative advertising model for durable goods in a retail oligopoly of n independent and competing retailers. We obtain the Stackelberg equilibrium and obtain the optimal subsidy rates policy of the manufacturer and the optimal advertising policy of the retailers in feedback form. We explore the conditions under which it is not optimal for manufacturer to support retailers and compute this explicitly as a function of the model parameters in a special case of n identical retailers, and obtain managerial insights on the role of retail competition. For a special case of two non-identical retailers with linear demand, we numerically study the dependence of the optimal subsidy rate on the model parameters. We investigate the impact of cooperative advertising on the profits of the channel members in a channel with two retailers and explore the extent to which cooperative advertising can coordinate the channel. Our numerical analysis shows that a cooperative advertising arrangement can result in higher channel profit and greater supply chain coordination. However, we find that while the manufacturer always benefits with an arrangement with the optimal subsidy rates, the two retailers may not benefit simultaneously. Indeed, we find that both retailers seem to benefit when the retailer are nearly symmetric and thus the subsidy rates they receive are nearly equal. And finally, we consider a case of anti-discrimination legislation in the case of two retailers, in which the manufacturer is required to offer equal subsidy rates to the two retailers. We find that such a legislation may result in a lower channel coordination.

Appendix: Proof of the Derivation of P_i in the Case of Identical Retailers

Proof Using $m_i = p_i - w_i = m$, $D_i = D$, $\rho_i = \rho$, and $\theta_i^* = 0$, $i = 1, 2, \ldots, n$, in (18)–(19), we get the following system of equations:

$$4r\beta_i = (m - \beta_i)^2\rho^2 D^2 - \sum_{j \neq i} 2\beta_i(m - \beta_j)\rho^2 D^2, \tag{40}$$

$$4r\alpha = \sum_{j=1}^{n} \big[(m - \beta_j)\rho^2 D^2 2(w_j - \alpha)\big]. \tag{41}$$

Equations (40) and (41) can be solved to give the following: For $i = 1, 2, \ldots, n$,

$$\beta_i = \beta = \frac{2r + D^2 mn\rho^2 - \sqrt{4r^2 + 4D^2\rho^2 mnr + D^4\rho^4 m^2(n-1)^2}}{D^2\rho^2(2n-1)} \tag{42}$$

or

$$\beta_i = \beta = \frac{2r + D^2 mn\rho^2 + \sqrt{4r^2 + 4D^2\rho^2 mnr + D^4\rho^4 m^2(n-1)^2}}{D^2\rho^2(2n-1)}. \tag{43}$$

We choose the first value, given by (42), which satisfies $m - \beta \geq 0$, which in turn ensures that $u_i \geq 0$, $i = 1, 2, \ldots, n$. Now, using (43) in (41), we get

$$\alpha = \frac{(\sum_{i=1}^{n} w_i) D^2 m\rho^2}{2r + D^2 m(n+1)\rho^2 + \sqrt{4r^2 + 4D^2\rho^2 mnr + D^4\rho^4 m^2(n-1)^2}}, \tag{44}$$

where $W = \sum_{j=1}^{n} w_j$. Finally, we use (44) and (43) in the equation $P_i = 2(w_i - \alpha) - (p_i - w_i - \beta_i)$ to show that the values of P_i, $i = 1, 2, \ldots, n$, are as given in (22). □

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Strategies of Foreign Direct Investment in the Presence of Technological Spillovers

Herbert Dawid and Benteng Zou

Abstract In this paper we present a differential game model of two firms with different technologies producing the same good and selling in the same world market. The firm equipped with advanced technology is deciding whether to outsource parts of its production to the home country of its competitor, where wages and the level of technology are lower. Outsourcing reduces production costs but is associated with spillovers to the foreign competitor. The degree to which the foreign competitor can absorb these spillovers depends on its absorptive effort. Using numerical methods the properties of a Markov perfect equilibrium of this game are characterized and the implications of the variation of different key parameters are examined.

1 Introduction

Though most of the foreign direct investment (FDI) is still undertaken among the developed countries, more and more FDI flows into newly developing ones. Among the newly industrialized countries (NIC), the competition to FDI has increasingly intensified, particularly after China joined the World Trade Organization (WTO). The overall picture of investment is that FDI flows into Asia more than to other developing countries in other regions, with the highest proportion of money flowing into China compared to other developing economies. FDI inflow is highly appreciated by the NIC, where the main reason are twofold: On the one side, the FDI can generate income for local firms and workers; and on the other side, the spillover effect is identified as very important for local firms and their development. Spillover here is not only transfer of production technology, management skill, marketing and ideas, but also the competitive pressure which may spur local firms to operate more

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efficiently or take more advanced technology (see Kokko 1994). As stated by the Global Economic Prospectus from the World Bank (2008, p. 3):

> . . . The lack of advanced technological competencies in these countries means that technological progress in developing countries occurs through the adoption and adaptation of pre-existing but new-to-the-market or new-to-the-firm technologies

and Chap. 3 of this report identifies a number of important and policy-relevant trends and explores some policy implications.

Although the empirical evidence concerning the existence of positive horizontal spill-overs from FDI is mixed, there exists evidence that FDI is associated with positive spill-overs (see e.g. Görg and Greenaway 2004). Indeed, recent empirical studies find positive horizontal spillovers from FDI using firm level data from Hungary (Halpern and Murakozy 2007), Romania (Smarzynska and Spatareanu 2008), from 17 emerging market economies (Gorodnichenko et al. 2007). From Chinese manufacturing firms, Liu (2008) addresses both short-run negative productivity effects and long-run positive effect of domestic firm due to FDI, and from Lithuania, Javorcik (2004) produces evidence consistent with positive productivity spillovers from FDI taking place through contacts between foreign affiliates and their local suppliers in upstream sectors. Several channels of spill-overs have been discussed in the literature, most prominently the demonstration effect, labor turnover (both inducing horizontal spill-overs), and vertical linkages (see e.g. Saggi 2002, among others).

In order to generate positive technological transfers to local firms based on FDI, the receiving country must have appropriate institutions in place and local firms must be ready to adopt new technologies and to adapt their behavior. As has been discussed extensively in the literature on ‘absorptive capacity’ (see e.g. Cohen and Levinthal 1990), local firms have to invest effort in order to build up their capacity to digest information and knowledge that might spill over from more advanced firms investing in their country and to generate productivity increases. The incentives to invest such efforts are driven by the expectations of the firms with respect to future spillovers they might receive and the economic implications of such incoming spillovers. Hence, the determination of absorptive effort should be based on intertemporal considerations of the local firms.

Similarly, the firms with advanced technology are aware that their activities in foreign countries may yield a technological improvement of their foreign competitors jeopardizing in the long-run their technological advantage.

Taking the above two sides story into consideration, a differential game is presented in which a firm with advanced technology makes the decision of whether to outsource parts of its production to a less developed economy where wages and the level of technology are lower. A firm in the less advanced economy decides how much effect to make to absorb the potential spillover. In our setting, the advanced technology taken by the FDI is fixed and hence catching up in technology is possible, which comes from the idea also mentioned in the report of World Bank (2008) that “The level of technological achievement in developing countries has converged with that of high-income countries over the past 15 years.”

Following some FDI literature, such as Das (1987), Wang and Blomstrom (1992), Dawid et al. (2010), among others, we assume that the change in the host country’s productivity is formulated as an increasing function of the presence of foreign capital stock.

The consideration of the dynamic strategic interplay between FDI decisions and the choice of investments in absorptive capacity gives rise to a differential game with a two-dimensional state space and state dynamics characterized by interaction terms of both states and a control. We are interested in characterizing a Markov-perfect equilibrium (MPE) of this game. Closed form solutions for MPE strategies are however available only for a small set class of differential game, most prominently linear quadratic games. The game considered here does not fall in any of these classes and therefore we rely on numerical methods to characterize MPEs, the induced investment paths as well as their dependence on parameters and initial conditions. In particular, we use collocation methods employing a Chebychev polynomial basis to approximate a solution to the set of Hamilton–Jacobi–Bellman equations characterizing the MPE. A similar technique has been used to study MPEs of non-linear quadratic differential games for example in Vedenov and Miranda (2001), Doraszelski (2003), Dockner and Mosburger (2007).[1]

The rest of the paper is organized as follows. In the next section, we present the model. In Sect. 3, we derive conditions characterizing a Markov-perfect equilibrium and describe our numerical approach. The results of our equilibrium analysis are presented and discussed in Sect. 4 and Sect. 5 concludes.

2 The Model

We consider a dynamic two-country model where country H ('home country') is a developed industrialized country whereas country F ('foreign country') is a newly industrializing country. For simplicity we consider only a singe firm in each country and we denote by $Q_i(t)$ the output of firm i at time t, $i = H, F$. The two firms compete on a common market characterized by an inverse demand function $P(Q_H + Q_F)$, where at each point in time both firms simultaneously choose their output quantities.

Firms produce using labor as the only variable production input. Production capacities of a firm in a country are determined by its capital stock there. For simplicity it is assumed that both firms have sufficiently large production capacities in the their home country to be able to produce the desired output. However, the production of firm H in country F is constrained by the size of the foreign capital stock firm H has accumulated in that country. Output per input unit in the two countries is given by $A_H(t)$ and $A_F(t)$ with $A_H(t) > A_F(t)$. If a firm from country H produces in country F, productivity reads $A_{HF}(t)$ where $A_F(0) < A_{HF}(0) < A_H(0)$. Since our focus is on the effects of technological spill-overs generated by FDI on the evolution of the technology gap between the two countries, we abstract from

[1] See Judd (1998) or Miranda and Fackler (2002) for a more general treatment of collocation methods for dynamic optimization and the survey of Jørgensen and Zaccour (2007) for more information as to using numerical methods to solve differential games.

technological change in the developed country and assume that A_H and A_{HF} are constant over time, whereas $A_F(t)$ may change over time due to spill-over effects. In both countries labor is supplied at wage rates w_H and w_F, where $w_H \gg w_F$. It is assumed that wages stay constant over time. In particular with respect to country F, where productivity might increase over time due to spillovers, this assumption is debatable, however the qualitative features of the model would not change if it would be assumed that wages go up due to productivity increases as long as the wages change at a lower rate than productivity such that the spill-overs induce a reduction in unit costs for firms in country F. Assuming constant wages, substantially simplifies the analysis compared to such a setting.

We assume that the firm in country H can reduce its unit production costs if it produces in the foreign country, i.e.

$$\frac{w_H}{A_H} > \frac{w_F}{A_{HF}}. \tag{1}$$

In order to produce abroad, firm H has to invest to build up production capacities in country F. We denote by $I(t) \in \mathcal{R}$ foreign investment of firm H and by $K_F(t)$ the capital stock of firm H in country F at time t. It should be noted that we also allow for negative investment, and due to the spillover-effects described below disinvestment might in principle be optimal for firm H. The capital accumulation equation is given by

$$\dot{K}_F(t) = I(t) - \delta K_F(t) \tag{2}$$

where δ is the depreciation rate of capital and $I \in \mathbb{R}$.

Foreign direct investments of country H firm in country F generates technological spill-overs. Following Findlay (1978), Wang and Blomstrom (1992), and Dawid et al. (2010), we posit that the change in technology level in the foreign country is given by

$$\dot{A}_F(t) = \lambda(t) K_F(t)\big(A_{HF} - A_F(t)\big). \tag{3}$$

The assumption that the speed of the change of the technology level in country F depends linearly both on the size of the foreign capital stock and the size of the technological gap is made for convenience. Clearly, one could also imagine non-linear specifications of these relationships, however since such non-linearities should not affect qualitatively the findings we will discuss, we stick to the most simple linear formulation here. The speed of absorption is determined by the absorption rate λ which is assumed to be

$$\lambda(t) = a + b\alpha(t),$$

with $a \geq 0$, $b \geq 0$ and $\alpha(t) \in \mathbb{R}_+$ denotes the effort of Firm F in order to absorb knowledge brought into country F by the FDI of firm H. Notice that we allow also for the case, if $a > 0$, where spillovers are positive even if the absorptive effort of firm F is $\alpha = 0$. However, with effort of firm F, it will speed up the catching up process. On the other hand, for $a = 0$, which we will consider as the default case, there is no absorption of knowledge of firm F unless it invests positive effort.

In addition to investment (for firm H) and absorptive effort (for firm F), both firms at each point in time also choose their output quantities. However, since the output choices do not have any intertemporal implications, it is obvious that the quantities are chosen according to Cournot equilibrium with marginal costs $c_H = \frac{w_H}{A_H}$, $c_F(t) = \frac{w_F}{A_F(t)}$. It should be noted that using these marginal costs we assume that firm H is not able to produce its entire output with its foreign capital stock in country F. We have verified that this assumption holds in the equilibrium we calculate in the next section.

In the following, we assume a linear inverse demand function given by

$$P(t) = \bar{P} - \big(Q_H(t) + Q_F(t)\big) \tag{4}$$

where $\bar{P} > 0$ is the reservation price. Then, the equilibrium outputs and profits in the oligopoly market are given as follows,[2] where we write these expressions as functions of the state $A_F(t)$:

$$Q_H^*(A_F) = \frac{\overline{P} - 2(w_H/A_H) + w_F/A_F}{3},$$

$$Q_F^*(A_F) = \frac{\overline{P} - 2(w_F/A_F) + w_H/A_H}{3},$$

$$\Pi_H^*(K_F, A_F) = \big(Q_H^*(A_F)\big)^2 + K_F A_{HF}\left(\frac{w_H}{A_H} - \frac{w_F}{A_{HF}}\right) \tag{5}$$

$$\Pi_F^*(A_F) = \big(Q_F^*(A_F)\big)^2. \tag{6}$$

Inserting these expressions into the objective functions of both firms allows us to consider only the intertemporally relevant controls I and α when formulating the maximization problems of the firms.

Firm H's objective then is to choose its investment strategy in order to maximize its discounted payoff stream given by

$$\max_{I(\cdot)} J_H = \int_0^\infty e^{-rt}\big[\Pi_H^*(K_F, A_F) - \big(\beta_H I + \gamma_H I^2\big)\big]dt, \tag{7}$$

where firm H's market profit Π_H^* is given by (5). Firm F's problem is to choose its effort strategy, α, to maximize its present value

$$\max_{\alpha(\cdot)} J_F = \int_0^\infty e^{-rt}\big[\Pi_F^*(A_F) - \big(\beta_F \alpha + \gamma_F \alpha^2\big)\big]dt, \tag{8}$$

with market profit $\Pi_F^*(A_F)$ given by (6). The optimization problems are subject to the state dynamics (2), (3) and the initial conditions

$$K_F(0) = 0, \qquad A_F(0) = A_F^{ini} \in (0, A_{HF}).$$

3 Markov-Perfect Equilibria

Our analysis is based on the consideration of Markov-perfect equilibria of the game described in the previous section. Given that both firms have infinite planing hori-

[2] See Dawid et al. (2010) for detailed calculations.

zons and time-autonomous instantaneous objective functions, we assume that firms use stationary Markovian feedback strategies of the form $I(K_F, A_F) : X \mapsto \mathbb{R}$ (for firm H) and $\alpha(K_F, A_F) : X \mapsto \mathbb{R}_+$ (for firm F), where $X = [0, \bar{K}] \times [\underline{A}, A_{HF}]$ is the considered state space with $\bar{K}$ sufficiently large and $\underline{A}$ smaller than A_F^{ini}. A pair of strategies (I^*, α^*) is a Markov-perfect equilibrium if for each firm its feedback strategy induces a control path which solves the dynamic optimization problem (7) respectively $V^F(K_F, A_F)$ given that the opponent sticks to its equilibrium feedback strategy. It is well known that Markov perfect equilibria are strongly time consistent (see Dockner et al. 2000) and hence this is the standard concept for the characterization of the dynamic strategic interaction of firms which are not able to commit ex-ante to certain control paths.

Due to the time autonomous nature of the objective and the infinite time horizon also the value functions of both firms in a (stationary) MPE do not explicitly depend on t and can be written as $V^H(K_F, A_F)$ respectively $V^F(K_F, A_F)$. The value functions of firm H has to solve the following Hamilton–Jacob–Bellman (HJB) equation

$$rV^H = \max_{I\in\mathbb{R}} \Big\{ \big[\Pi_H(K_F, A_F) - \big(\beta_H I + \gamma_H I^2\big) \big] + \big[V^H_{K_F}(I - \delta K_F) + V^H_{A_F}\big(a + b\alpha^*(K_F, A_{HF})\big) K_F (A_{HF} - A_F) \big] \Big\} \quad (9)$$

where $V^H_{K_F}$ ($V^H_{A_F}$) represents the partial derivative of V^H with respect to K_F (A_F).

Since the right hand side of the above HJB equation is strictly concave with respect to I, the first order condition is necessary and sufficient for maximization problem, which gives

$$I^* = \frac{V^H_{K_F} - \beta_H}{2\gamma_H}. \quad (10)$$

Similarly, the value function of firm F solves the HJB equation

$$rV^F = \max_{\alpha\in\mathbb{R}_+} \Big\{ \big[\Pi_F(A_F) - \big(\beta_F \alpha + \gamma_F \alpha^2\big) \big] + \big[V^F_{K_F}\big(I^*(K_F, A_F) - \delta K_F\big) + V^F_{A_F}(a + b\alpha) K_F (A_{HF} - A_F) \big] \Big\}, \quad (11)$$

and the optimal effort is

$$\alpha^* = \frac{\max[bV^F_{A_F} K_F (A_{HF} - A_F) - \beta_F, 0]}{2\gamma_F}. \quad (12)$$

Substituting the optimal choice (10) and (12) into the HJB equations (9) and (11), we obtain the two Bellman equation system

$$\begin{aligned} &rV^H - \Pi_H(A_F, K_F) + \frac{\beta_H (V^H_{K_F} - \beta_H)}{2\gamma_H} + \frac{(V^H_{K_F} - \beta_H)^2}{4\gamma_H} \\ &\quad - V^H_{K_F}\left(\frac{V^H_{K_F} - \beta_H}{2\gamma_H} - \delta K_F \right) \\ &\quad - V^H_{A_F}\left[a + b\left(\frac{\max[bV^F_{A_F} K_F (A_{HF} - A_F) - \beta_F, 0]}{2\gamma_F} \right) \right] K_F (A_{HF} - A_F) \\ &= 0, \end{aligned} \quad (13)$$

and

$$
\begin{aligned}
&rV^F - \Pi^F(A_F, K_F) + \frac{\beta_F \max[bV^F_{A_F} K_F(A_{HF} - A_F) - \beta_F, 0]}{2\gamma_F} \\
&\quad + \frac{\max[bV^F_{A_F} K_F(A_{HF} - A_F) - \beta_F, 0]^2}{4\gamma_F} - V^F_{K_F}\left(\frac{V^H_{K_F} - \beta_H}{2\gamma_H} - \delta K_F\right) \\
&\quad - V^F_{A_F}\left[a + b\left(\frac{\max[bV^F_{A_F} K_F(A_{HF} - A_F) - \beta_F, 0]}{2\gamma_F}\right)\right] K_F(A_{HF} - A_F) \\
&= 0.
\end{aligned}
\tag{14}
$$

Any pair of value functions satisfying (13), (14) and the transversality conditions

$$\lim_{t\to\infty} e^{-rt} V^H\big(K_F^*(t), A_F^*(t)\big) = \lim_{t\to\infty} e^{-rt} V^F\big(K_F^*(t), A_F^*(t)\big) = 0,$$

where $(K_F^*(t), A_F^*(t))$ denotes the state trajectory induced by the pair of value functions corresponds to a Markov-perfect equilibrium of the game. Due to the non-linear structure of the system of partial differential equations (13) and (14) no closed form solutions can be obtained. Therefore, we numerically determine a pair of value functions that approximately solves (13)–(14) on the state space X and calculate approximations of the equilibrium feedback strategies using these value function approximations. We briefly describe the procedure in the following subsection.

3.1 Numerical Approach

We employ a collocation method using Chebychev polynomials to obtain the approximation of the value functions and the equilibrium feedback strategies. To this end we generate a set of n_K Chebychev nodes $\mathcal{N}_{K_F}$ in $[0, \bar{K}]$ and a set of n_A Chebychev nodes $\mathcal{N}_{A_F}$ in the interval $[\underline{A}, A_{HF}]$ (see e.g. Judd 1998, for the definition of Chebychev nodes and Chebychev polynomials) and define the set of interpolation node in the state space X as

$$\mathcal{N} = \big\{(k_f, a_f) | k_f \in \mathcal{N}_{K_F}, a_f \in \mathcal{N}_{A_F}\big\}.$$

Note that the cardinality of $\mathcal{N}$ is $n_K n_A$. In what follows we calculate polynomial approximations of V^H and V^F which satisfy (13) and (14) on the set of interpolation nodes $\mathcal{N}$. It is well known that the choice of Chebychev interpolation nodes avoids large oscillations of the interpolating polynomial between the interpolation node (as could occur e.g. for equi-distant nodes) and implies that the interpolating polynomials approximately solve the HJB equations on the entire state space.

The set of basis functions for the polynomial approximation is determined as $\mathcal{B} = \{B_{j,k}, j = 1, \dots, n_K, k = 1, \dots, n_A\}$ with

$$B_{j,k}(K_F, A_F) = T_{j-1}\left(-1 + \frac{2K_F}{\bar{K}_F}\right) T_{k-1}\left(-1 + \frac{2(A_F - \underline{A})}{(A_{HF} - \underline{A})}\right),$$

where $T_j(x)$ denotes the j-th Chebychev polynomial (see e.g. Judd 1998, for the definition of the Chebychev polynomial basis). Since Chebychev polynomials are defined on $[-1, 1]$ the state variables have to be transformed in the way shown above.

The value function is approximated by

$$\begin{aligned} V^i(K_F, A_F) &\approx \hat{V}^i(K_F, A_F) \\ &= \sum_{j=1}^{n_K} \sum_{k=1}^{n_A} C^i_{j,k} B_{j,k}(K_F, A_F), \quad (K_F, A_F) \in X, i = H, F, \end{aligned} \tag{15}$$

where $C = \{C^i_{j,k}\}$ with $j = 1, \dots, n_K$, $k = 1, \dots, n_A$, $i = H, F$ is the set of $2n_K n_A$ coefficients to be determined.

To determine these coefficients we set up a system of non-linear equations derived from the condition that $(\hat{V}^H, \hat{V}^F)$ satisfies the HJB equations (13) and (14) on the set of interpolation nodes $\mathcal{N}$. This system consists of $2n_K n_A$ equations with $2n_K n_A$ unknowns (i.e. the coefficients $C^i_{j,k}$) and is solved by a recursive algorithm, where based on an initial guess $\tilde{C}^0 = \{C^{i,0}_{j,k}, j = 1, \dots, n_K, k = 1, \dots, n_A, i = H, F\}$ of the coefficients in iteration $l \geq 1$ the coefficients $\tilde{C}^{l-1}$ are used to calculate approximations of the value functions and their partial derivatives at each node in $\mathcal{N}$. These approximations are inserted for all terms that occur in (13) and (14) where the value functions or their derivatives appear in a non-linear form. Inserting the approximation (15) with C replaced by $\tilde{C}^l$ for all terms in (13) and (14) where the value functions and their derivatives occur in a linear way, yields a linear system of equations for the coefficients $\tilde{C}^l$, which even for large values of $2n_K n_A$ can be solved efficiently using standard methods as long as the coefficient matrix is well conditioned. The solution of this linear system gives the new set of coefficient values $\tilde{C}^l$. To complete the iteration the new approximations of the value functions and their derivatives are inserted into all (including the non-linear) corresponding terms in (13) and (14) and the resulting absolute value of the left hand side of these equations relative to the corresponding value function is determined for all nodes in $\mathcal{N}$. If the maximum of this relative error is below a given threshold ε the algorithm is stopped, we set $C = \tilde{C}^l$ and the current approximation of the value functions is used to calculate the feedback strategies of the players and the equilibrium dynamics.

Unfortunately, no general conditions can be given that guarantee the existence of a stable fixed point of the described algorithm, which corresponds to an economically meaningful Markov-perfect equilibrium of the game. Also, starting with an appropriate initial guess for the coefficients is often crucial for convergence to a meaningful fixed point, even if there exists such a stable fixed point. To obtain the numerical results discussed in the next section a continuation method was applied by starting with a simplified problem without strategic interaction ($a = b = 0$) and then increasing the variable b in small steps to arrive at the default parameter setting with $b = 0.2$ (see below), where in each step the value function approximations from the previous steps are used as the initial guess for the current step. Similar methods were used to obtain results for the different parameter variations reported below.

Table 1 Standard parameter setting

$A_H = 4$	$A_{HF} = 2$	$w_H = 4$	$w_F = 1$
$\beta_H = 0$	$\gamma_H = 250$	$\beta_F = 0$	$\gamma_F = 0.03$
$\delta = 0.06$	$r = 0.03$	$\bar{P} = 5$	
$a = 0$	$b = 0.2$		
$\bar{K} = 0.6$	$\underline{A}_F = 1.5$	$K^{ini} = 0$	$A_F^{ini} = 1.55$
$n_K = 8$	$n_A = 8$	$\varepsilon = 0.003$	

For all numerical solutions reported it was checked that the state dynamics under the equilibrium strategies does not leave X for any initial conditions in X, which implies that the considered state space is sufficiently large to allow the correct calculation of the value functions under the considered equilibria and also implies that the transversality conditions are satisfied.

4 Results

The results presented below are based on the default parameter setting given in Table 1 which to a large extend follows the values used in Dawid et al. (2010). These values are not based on a serious empirical calibration of the model, but, by sticking to the parametrization in Dawid et al. (2010) allows us to highlight the implication of the consideration of dynamic strategic interaction between the firms in the two countries, which was not considered in that paper. The ratio of wages in the two countries is four to one, and the monetary unit is normalized in a way that the wage in country F is $w_F = 1$. Unit costs of production for firm H in country F ($w_F/A_{HF} = 0.5$) are well below the unit costs at home ($w_H/A_H = 1$).

The upper bound for the foreign capital stock of firm H is set to $\bar{K} = 0.6$, which under the considered cost parameters is sufficient to ensure that firm H never has incentives to build a stock larger than $\bar{K}$. Also, by setting $A_{HF} = 2$ and $\underline{A} = 1.5$, we restrict attention to the case where the initial productivity in country F is relatively close to A_{HF}, which in the absence of the consideration of absorptive effort choice always leads to a convergence of productivity in country F to A_{HF}. The robustness of the qualitative findings reported below with respect to variations in this parameter setting has been tested.

Figure 1 shows the feedback strategies of the two firms in the MPE under the default parameter setting. The qualitative features of these feedback strategies are quite intuitive. The foreign investment of firm H becomes smaller the larger the technological gap of firm F is because the lager the gap the larger the marginal effect of an additional unit of foreign capital on the future increase in A_F. This increase reduces marginal costs of firm F, which negatively affects the future profits of firm H. Furthermore, investments are an increasing function of the stock of foreign capital firm H has. The reason for this observation is similar to that just given with respect to changes in A_F. The larger foreign capital stock of firm H the faster the productivity of firm F will catch-up and the smaller will be the technology gaps

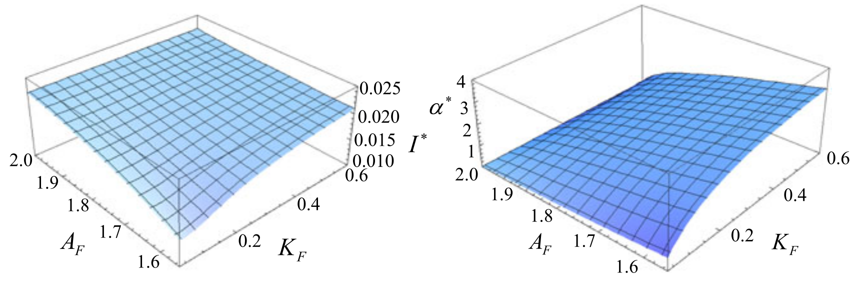

Fig. 1 Equilibrium strategies of firm H (**a**) and firm F (**b**) for the default parameter setting

in the future. Smaller future technology gaps imply that the negative future effects of an additional unit of foreign capital for firm H are smaller and hence investments of that firm increase. Absorptive effort of firm F is larger the larger the capital stock K_F is because the marginal effect of absorptive effort grows with K_F. On the other hand, the marginal effect of absorptive effort declines as A_F comes closer to A_{HF} and therefore effort is a decreasing function of A_F.

Figure 2 shows the dynamics of the state and control variables in the MPE for the default parameter setting and an initial productivity of firm F of $A_F(0) = 1.55$. It can be clearly seen that under the default setting firm H builds up a positive capital stock in country F and keeps a positive stock in the long run. Firm F chooses positive absorption effort throughout the run and is therefore able to completely close the gap between its own productivity A_F and the productivity A_{HF} of the high-tech firm H when producing in country F. The amount of effort invested in absorption is non-monotone with a steep initial increase in effort and a long phase of decreasing effort after an early peak at approximately $t = 15$. The intuition for the non-monotone dynamics of the absorptive effort is that the marginal effect of an increase of K_F on the incentives to invest in absorptive effort is large in the initial periods when the technological gap is still large. The fast catch-up of A_F towards A_{HF} reduces the marginal future value of an investment in absorptive effort and therefore reduces the incentives for firm F to choose a high α. This effect starts to dominate after $t = 15$ inducing a steady decline in α. However, the absorptive effort always stays positive because under the default parameter setting marginal costs of effort converge to zero as α becomes small. As we will see below the property that marginal costs of effort are zero for $\alpha = 0$ is the crucial property that induces a full catch-up steady state.

In order to understand the implications of a relaxation of the assumption that marginal costs of effort are zero at $\alpha = 0$ we now consider the effects of an increase of the parameter β_F. For positive β_F marginal costs of effort are strictly positive on the entire control space $\mathbb{R}_+$. In Fig. 3 we show the feedback functions in the MPE for $\beta_F = 0.2$. Setting β_F to that positive value induces that $\alpha = 0$ on a substantial part of the state space. In particular, for small foreign capital stocks of firm H and for a small technological gap, the marginal (present and future) returns of absorptive

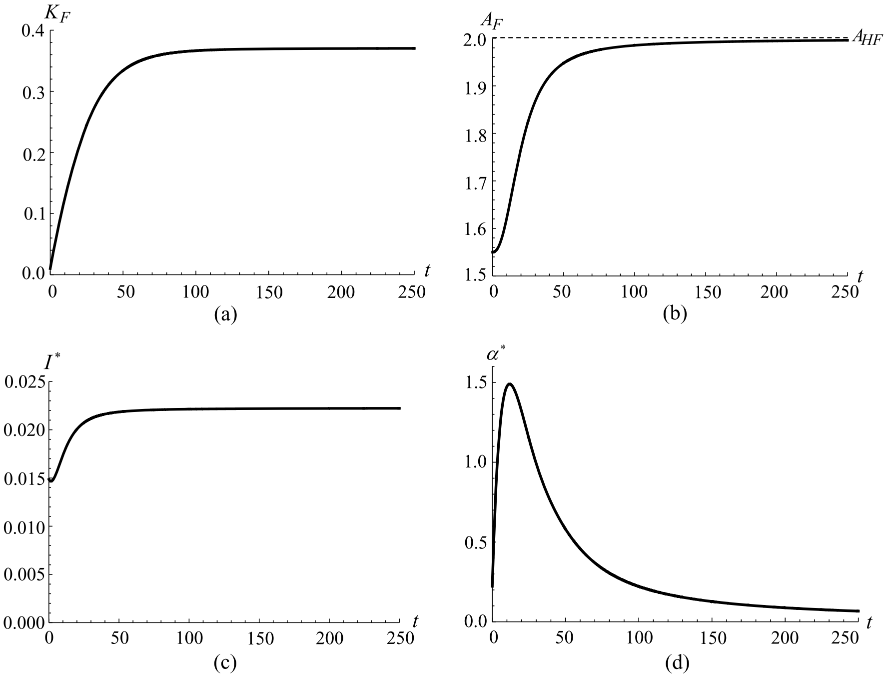

Fig. 2 Equilibrium dynamics under the default parameter setting: **a** foreign capital stock of firm H, **b** productivity of firm F, **c** investment of firm H, **d** absorptive effort of firm F

effort are not sufficient to outweigh the marginal costs and therefore firm F does not invest any effort to absorb the potential spillovers from firm H. In equilibrium firm H takes this behavior of its opponent into account and therefore reduces investment in the areas of the state space close to the line where firm F starts to invest positive effort. In the interior of the region where firm F invests positive effort the same logic as in the case of $\beta_F = 0$ implies that investment of firm H is an increasing function of K_F and this gives rise to the rather complex and non-monotone shape of the feedback function of firm H.

The implications of these changes in the feedback strategies on the equilibrium dynamics of states and controls can be seen in Fig. 4. First, considering the dynamics of K_F and A_F we observe that no full catch-up of the productivity of firm F emerges although firm H keeps a positive foreign capital stock in the long-run. Hence, contrary to the case with exogenous absorptive capacity studied in Dawid et al. (2010) with endogenous absorptive capacity a steady state exists where the high tech firm H can keep some productivity advantage relative to its local competitor in country F although it keeps exploiting the wage advantage in country F with a positive long run stock of foreign capital. Whereas the steady state level of the productivity of firm F is a strictly decreasing function of β_F the dependence of the size of the long run foreign capital stock of firm H from β_F is non-monotonous, where this level is approximately identical for $\beta_F = 0$ and $\beta_F = 0.2$, but substan-

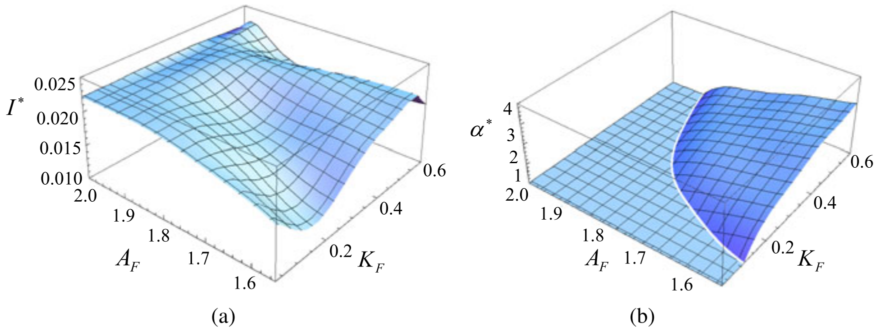

Fig. 3 Equilibrium feedback strategies of firm H (**a**) and firm F (**b**) for $\beta_F = 0.2$

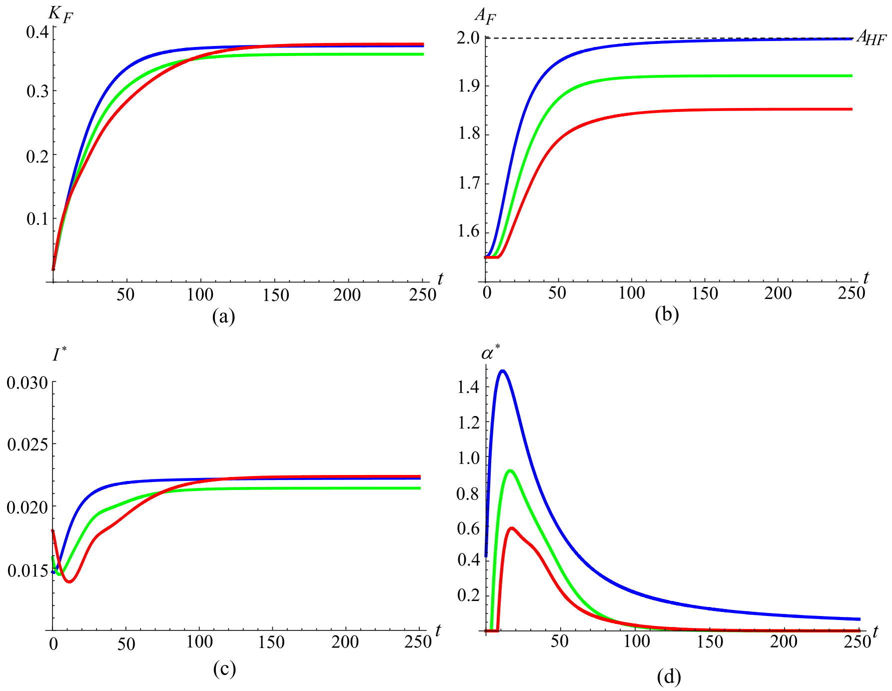

Fig. 4 Equilibrium dynamics for $\beta_F = 0$ (*blue line*), $\beta_F = 0.1$ (*green line*) and $\beta_F = 0.2$ (*red line*): **a** foreign capital stock of firm H, **b** productivity of firm F, **c** investment of firm H, **d** absorptive effort of firm F

tially smaller for $\beta_F = 0.1$. In the initial part of the dynamics a larger value of the absorptive cost parameter β_F induces a *smaller* foreign capital stock of firm H. This might seem counter-intuitive, but can be well explained by considering the dynamics of the controls of the two firms.

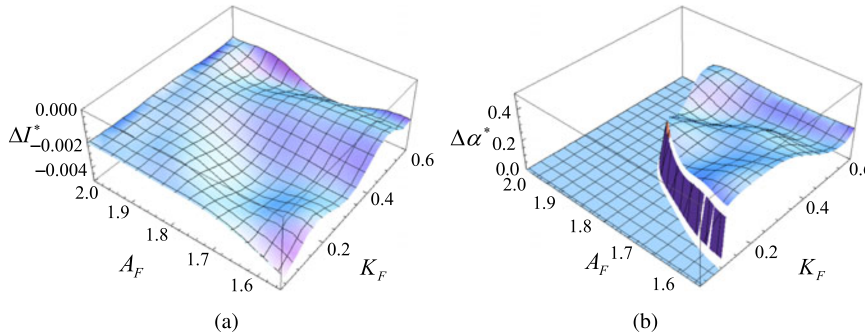

Fig. 5 Difference in the equilibrium strategies of firm H (**a**) and firm F (**b**) between the cases $w_F = 1.1$ and $w_F = 1$

For positive β_F in the initial periods the stock of foreign capital is so small that firm F does not invest any absorptive effort and hence the technological gap remains unchanged. Firm H however foresees that due to the increase in K_F eventually the point in time will be reached where firm F starts investing absorptive effort and the productivity of firm F starts to increase. As this point in time comes closer the implicit (discounted future) costs associated with an additional unit of capital in country F becomes larger and therefore investment of firm H decreases in the initial periods. The time interval on which investment is decreasing is longer the larger absorption costs β_F are and this explains why an increase of absorption costs has a negative impact on the size of the foreign capital stock in the initial periods. Once firm F starts choosing positive α the technological gap shrinks and this reduces the costs for firm H of an additional unit of foreign capital due to future technological catch-up. Once this effect becomes dominant firm H starts to increase its investment and firm F reduces its absorptive effort until a steady state is reached where absorptive effort is again zero. It should be noted that this steady state is located exactly at the line in the state-space where the equilibrium feedback function of firm F switches from zero to positive α.

We now consider the effect of an increase of the wage in country F on the dynamic of FDI and of the technological catch-up. Clearly, the lower labor costs in country F are the main motivation for firm H to move parts of the production of country F and it is therefore important to understand how the economic dynamics is influenced by the amount of the wage disparity. As discussed above we assume that wages stay constant over time, for example due to institutional inertia.

In order to examine the effects of an increase in w_F, we first consider the direct effect of such a parameter change on the equilibrium feedback functions of both firms. We carry out this analysis for a positive value of $\beta_F = 0.2$ where in general no full technological catch-up of firm F occurs. In Fig. 5 we depict the change in the feedback functions as the wage in country F increases by 10 %, i.e. we depict $\Delta I^*(K_F, A_F) = I^*(K_f, A_F; w_f = 1.1) - I^*(K_f, A_F; w_f = 1)$ and $\Delta\alpha^*(K_F, A_F) = \alpha^*(K_f, A_F; w_f = 1.1) - \alpha^*(K_f, A_F; w_f = 1)$. The figure clearly shows that a wage increase in country F induces a downward shift of the foreign

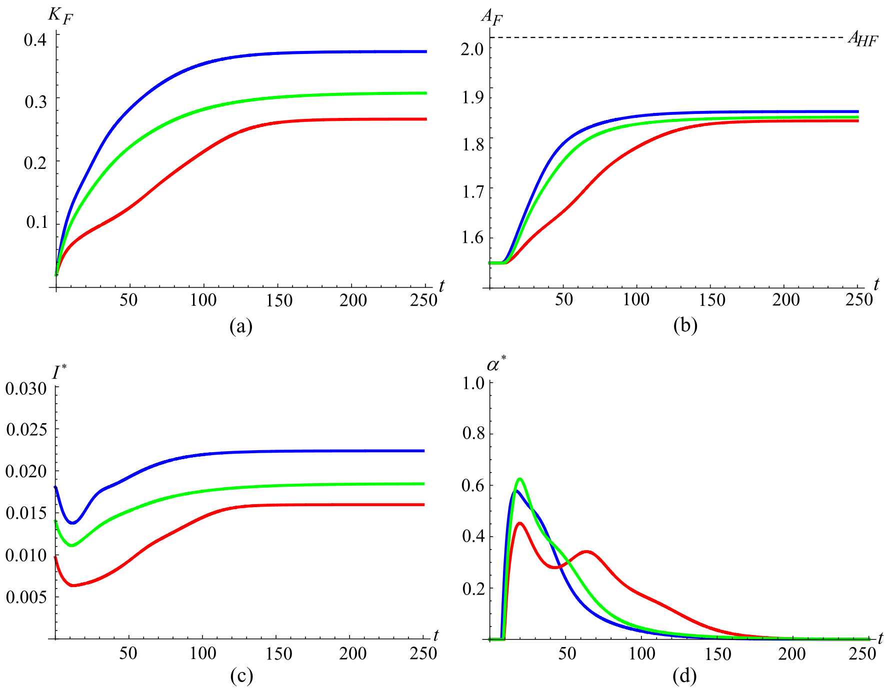

Fig. 6 Equilibrium dynamics for $w_F = 1$ (*blue line*), $w_F = 1.1$ (*green line*) and $w_F = 1.2$ (*red line*): **a** foreign capital stock of firm H, **b** productivity of firm F, **c** investment of firm H, **d** absorptive effort of firm F

investment function of firm H and an upward shift of the feedback function determining the absorptive effort of firm F, thereby enlarging the area in the state space where firm F chooses positive effort. There is a clear intuition for these shifts. An increase of w_F induces higher unit costs of labor for firm F, which increases its incentives to increase labor productivity and this leads to higher absorptive effort of that firm. This increase in absorptive effort reduces the incentives for firm H to invest in country F. Furthermore, the increased wage costs in country F reduces the cost savings of firm H from production in country F, which also negatively affects its incentives for foreign investment.

The effects of these changes in the feedback functions on the equilibrium dynamics of states and controls can be seen in Fig. 6. As expected, foreign investment the size of the foreign capital stock of firm H becomes smaller throughout the run and in the steady state if the wage w_F goes up. More surprisingly, the increase in wages in country F induces a decrease in the productivity of firm F throughout the run and in the steady state. This effect is particularly strong in the initial phase of the dynamics, where first the delay till the catch-up of firm F starts becomes larger for increasing w_F and then the speed of the catch-up is substantially smaller under a larger value w_F. Hence, although the incentives for firm F to improve its labor productivity become stronger if the local wage goes up, in equilibrium the induced

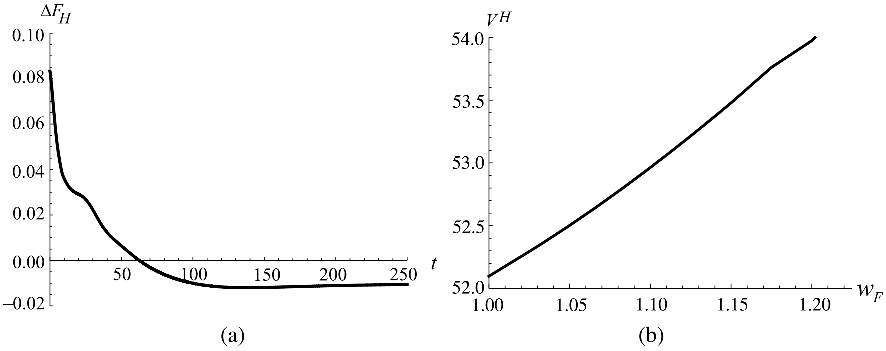

Fig. 7 Effects of changes in w_F on the profit of firm H. **a** Difference in the instantaneous profits of firm H between the cases $w_F = 1.1$ and $w_F = 1$; **b** value function of firm H for the default initial conditions and an increasing value of w_F

slowdown in the foreign investment is so strong that the actual productivity trajectory is shifted downwards. In the initial periods the reduction in foreign capital stock due to a wage increase implies even a reduction of absorptive effort of firm F.

Based on these considerations, it is straight forward to see that the profits of firm F are negatively affected by an increase of the local wage w_F. Numerical results not shown here confirm that an increase in w_F induces a downward shift of the value function of firm F and also a downward shift of the trajectory of instantaneous profits of that firm, which means that an increase of w_F (for given initial conditions) implies not only lower productivity but also a reduction of profits of the firm at each point in time. Furthermore, numerical evidence shows that the total wage income of workers in country F is positively affected by an increase of w_F, but that this effect is smaller than the induced loss of firm profits, such that total income in country F, consisting of the sum of firm profits and wage income, is negatively affected by the increase in w_F.

The effect of an increase in the wage in country F on the profit of firm H is less clear. On the one hand, such a wage increase makes the production of firm H in country F more expensive. Also, as discussed above, the wage increase leads to an upward shift of the absorptive effort of firm F. Both of these effects have negative implications for the profits of firm H. On the other hand, the wage increase induces higher marginal costs for the opponent firm F, which makes it a weaker competitor and induces an upward shift of the price. This increases the profit of firm H. As can be seen in panel (a) of Fig. 7 the interplay of these effects with the adjustment of the optimal strategy of firm H is such that an increase in w_F initially leads to a higher profit of firm H but induces a decrease in the long run profit. The main mechanism for these dynamic implications seems to be the induced reduction of investments of firm H we discussed above. This reduction of investments which, does not only reduce investment costs but also leads to a substantial slowdown of initial technological catch-up of firm F, has particularly strong positive profit implication for firm H in the initial periods, whereas the negative implications of an increase of

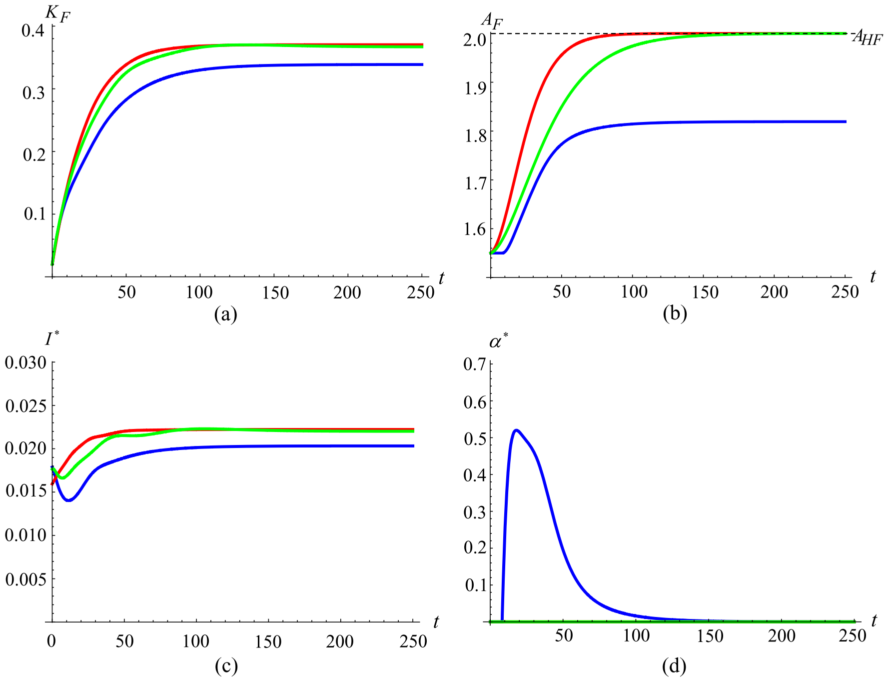

Fig. 8 Equilibrium dynamics for $a = 0$ (*blue line*), $a = 0.1$ (*green line*) and $a = 0.2$ (*red line*): **a** foreign capital stock of firm H, **b** productivity of firm F, **c** investment of firm H, **d** absorptive effort of firm F

w_F grow over time as firm H accumulates more foreign capital. Also, in the long run the increase in the technological gap $A_{HF} - A_F$ induced by the wage increase is much smaller than in the initial periods. However, as can be seen from panel (b) where the value function of firm H is depicted for increasing values of w_H, the initial positive effect outweighs the negative long-run implications of the wage increase, such that the discounted payoff stream of firm H in equilibrium increases with w_H.

Having examined the implications of a wage increase in country F, we conclude our analysis by briefly considering a scenario where even without any absorptive effort of firm F this firm receives technological spillovers as long as firm H has a positive capital stock in country F. With respect to our parametrization this means that we consider the implications of an increase of the parameter a to some positive value. Figure 8 shows the equilibrium dynamics of states and controls for $a = 0$, $a = 0.1$, and $a = 0.2$. Whereas the difference in the dynamics between the cases with positive values of a is very minor, there is a substantial difference between these two cases and our default setting where $a = 0$. For positive values of a, firm F always achieves a complete technological catch-up to $A_F = A_{HF}$. However absorptive effort of the firm is reduced to zero such that the catch-up is due to the 'automatic spillovers' generated by the positive capital stock of firm H. Firm H

could prevent such spillovers only by reducing its foreign capital stock to zero, but, as can be seen in panels (a) and (c) of the figure, the increase of the spillover intensity induces *higher* foreign investment of firm H and also a larger foreign capital stock for almost the entire time interval. Intuitively, there are two reasons for this. First, the faster catch-up of A_F towards A_{HF} reduces the marginal effect of an additional unit of foreign capital on the future dynamics of A_F, and, second, since absorptive effort is constant zero for positive values of a, the strategic effect that an increase in the foreign capital stock of firm H induces higher absorptive effort by firm F is not present for positive values of a and this increases the incentives of the firm for foreign investment.

5 Conclusions

In this paper, we have considered the dynamic strategic interaction between two competing firms located in a high- respectively low-tech country. The firm in the high-tech country can reduce production costs by moving parts of its production to the low-tech country, but by doing so risks to generate technological spillovers which allow its competitor to reduce the technological gap between the two firms in terms of productivity. Taking into account the literature on absorptive capacity we assume that the intensity of the spillovers does not only depend on the size of the foreign capital stock of the high-tech firm and the size of the technological gap, but also on the amount of effort invested by the firm in the low-tech country to absorb the spillovers. The differential game, which captures the strategic interaction between the two firms, is of highly non-linear structure and does not fall into any of the classes of games where analytical treatments of Markov perfect equilibria are feasible. Therefore numerical methods of collocation type were used to examine the characteristics of the Markov perfect equilibria of the game.

Several noteworthy findings result from this numerical analysis. First, it is demonstrated that under weak assumptions on the cost function of effort equilibrium behavior leads to a steady state where the high-tech firm keeps a positive capital stock in the low tech country, but the productivity of the firm located in that country still does not catch-up to the productivity of the high-tech firm. Such a steady state is ruled out in the treatment of Dawid et al. (2010), where the absorptive capacity of the local firm was assumed to be exogenously given. Second, the analysis highlights several interesting implications of changes in the key parameters on the equilibrium dynamics. In particular, it is shown an increase of the costs of absorptive effort of the low-tech firm leads to a *reduction* of the foreign investment of the high-tech firm in the initial part of the dynamics. An increase of the wage rate in the low-tech country induces a slower catch-up of the productivity of the firm in that country and less foreign investment of the high-tech firm. Finally, an increase of the base value (with zero effort) of the absorptive capacity of the low-tech firm to a positive value induces more foreign investment. Overall, this analysis demonstrates that even in games with more than one state and highly non-linear state dynamics the application of numerical collocation methods allows to obtain a clear and comprehensive picture of the qualitative properties of Markov perfect equilibria of the game.

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Differential Games and Environmental Economics

Aart de Zeeuw

Abstract Differential games are the natural framework of analysis for many problems in environmental and resource economics. This chapter will discuss the concepts and techniques of differential games and it will analyze two famous models in environmental and resource economics, the game of international pollution control and the lake game. It will be shown that existing solution techniques can cover some of the gap between non-cooperative Nash equilibria and the cooperative outcome. It will also be shown that the regulation by means of realistic tax rates can cover some of the remaining gap but not all the way.

1 Introduction

Differential game theory extends optimal control theory to situations with more than one decision maker. The controls of each decision maker affect the development of the state of the system, given by a set of differential equations. The objectives of the decision makers are integrals of functions that depend on the state of the system, so that the controls of each decision maker affect the objectives of the other decision makers and the problem turns into a game. A differential game is a natural framework of analysis for many problems in environmental and resource economics. Usually these problems extend over time and have externalities in the sense that the actions of one agent affect welfare of the other agents. For example, emissions of all agents accumulate into a pollution stock and this pollution stock is damaging to all agents. Or, resource extractions by all agents decrease a resource stock and the availability of the resource affects welfare of all agents.

It is logical to identify the pollution stock and the resource stock with the state of the system but one needs to be careful here. In systems theory, the state of the system is a memory concept, comparable to a sufficient statistic. It contains sufficient information from the history of the system, at some point in time, to be able to predict the future of the system, under given inputs. This indeed applies to the

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pollution stock and the resource stock. However, the controls in the Nash equilibrium of the differential game usually differ when they are conditioned on time only, on the current state of the system or on the history of the state of the system. These possible interactions between the controls and the state complicate the conceptual framework. This chapter will first clarify this issue that was already presented in the early stages of the development of differential game theory (Starr and Ho 1969; Başar and Olsder 1982).

Most theory and applications are restricted to linear-quadratic differential games where the state transition is linear and the functions under the objective integral are quadratic. In this case the controls are linear and the value functions are quadratic but again one needs to be careful here. It can happen that non-linear equilibria of the game exist as well (Dockner and Van Long 1993). Moreover, the assumption that the state transition is linear is fine for most standard economic problems with capital accumulation, for example, or for the simple environmental and resource problems mentioned above. However, ecological systems are usually non-linear and if the resource is an ecological system, a differential game with a non-linear constraint has to be solved. For example, a lake reacts in a non-linear way to the release of phosphorus on the lake, so that in case of many users a stock externality occurs that develops in a non-linear way.

An important reason for analyzing differential games in environmental and resource economics is to assess the "tragedy of the commons" (Hardin 1968). The tragedy of the commons means that if a common resource is accessible to all agents and is not jointly managed, it will be overused and joint welfare will not be optimal. It would be collectively rational to use less but in that situation there are individual incentives to use more. One can say that the non-cooperative Nash equilibrium of the differential game characterizes the situation without optimal management. By comparing this to the cooperative outcome of the game, the welfare losses can be assessed. Note, however, that the non-cooperative Nash equilibrium of the differential game is not unique. It depends on the information structure or, more specifically, on whether the controls can be conditioned on the state of the system or not. It also depends on the level of commitment the agents can make with respect to their future strategies. This positions the analysis in what is usually called the "Nash program": a fundamental question in game theory is whether equilibria exist that mimic the cooperative outcome because this combines the highest joint welfare with the stability properties of the equilibrium in the sense that no agent has an incentive to deviate. In that case the players would only have to coordinate on the proper Nash equilibrium.

If equilibrium behavior is not jointly optimal, it can be corrected by taxes or other policies. In principle, a fully corrective mechanism usually exists but this mechanism may be too complicated or too costly to implement. The general question becomes how close one can get to the cooperative outcome with a non-cooperative Nash equilibrium that is based on a realistic set of strategies for the individual agents and a realistic policy, mandated to a policy maker. The main purpose of this chapter is to show how far we can go in case of two typical examples, with the available techniques in differential games.

This chapter will start with an overview of the most important concepts and techniques of differential games. Then two examples in environmental and resource economics will be elaborated. The first one is the game of international pollution control where countries emit, for example, greenhouse gases into a concentration level of pollutants. With some linear natural degradation, the state transition is linear. There are benefits of emissions in a one-to-one relationship with production and there is damage of the stock of pollutants. If these costs and benefits are approximated by quadratic functions, the differential game is a linear-quadratic one. The second example is the lake game with many users. The state is again the stock of pollutants which is now the total amount of phosphorus sequestered in algae. This pollution stock, however, develops in a non-linear way which will be explained below. Such a non-linear differential game with a one-dimensional state can still be solved but requires numerical techniques.

Section 2 will give an overview of the most important concepts and techniques of differential games. Section 3 will analyse the game of international pollution control as an example of a linear-quadratic differential game. Section 4 will analyze the lake game as an example of a non-linear differential game. Section 5 concludes.

2 Concepts and Techniques of Differential Games

It may be fair to say that Bellman's principle of optimality is one of the most important concepts in dynamic optimization theory. It simply states that at each point in time the remainder of an optimal control path is still optimal when starting at the state of the system that is reached by implementing the optimal control path up to that point in time. The proof is easy. If another future control path is better, replacing the remainder of the original optimal control path with this one leads to a control path that is better than the original one so that the original one cannot be optimal. The principle of optimality is simple but powerful because it allows to solve a dynamic optimization problem backwards in time and it yields the technique of dynamic programming and, more specifically, the Hamilton–Jacobi–Bellman equation for the value function of the optimal control problem. Therefore, it was remarkable to find that the equivalent does not hold in differential games.

2.1 Example from 1969

This issue was first put forward in a paper by Starr and Ho (1969) in an instructive example. This example is not a differential game. It has only two periods and two actions for each player but it provides insights that are relevant for the sequel. There are two players. Each player has controls 0 and 1. The game starts in the state x_0 and the state transition is such that the control pair $(1, 0)$ leads to the state x_1, with costs -1 for player 1 and costs 1 for player 2. The control pairs $(0, 0)$ and $(1, 1)$ both lead to x_2, with respective costs $(2, 2)$ and $(5, 0)$, and the control pair $(0, 1)$ leads to x_3, with costs $(2, 2)$. In the second period one of the following three bi-matrix games is

Table 1 The bi-matrix games in the second period

x_1	0	1
0	(5, 2)	(2, 3)
1	(4, 1)	(1, 4)

x_2	0	1
0	(2, 2)	(3, 1)
1	(2, 4)	(0, 3)

x_3	0	1
0	(2, 2)	(1, 3)
1	(4, 1)	(0, 2)

Table 2 One bi-matrix game in the four possible strategies

x_0	00	01	10	11
00	(4, 4)	(5, 3)	(4, 4)	(3, 5)
01	(4, 6)	(2, 5)	(6, 3)	(2, 4)
10	(4, 3)	(1, 4)	(7, 2)	(8, 1)
11	(3, 2)	(0, 5)	(7, 4)	(5, 3)

Table 3 The bi-matrix game in the first period

x_0	0	1
0	(2, 5)	(4, 4)
1	(3, 2)	(5, 3)

played, depending on whether the state x_1, x_2 or x_3 has been reached. The actions of player 1 are represented by the rows of these matrices and the actions of player 2 by the columns. The entries of these matrices represent the respective costs of player 1 and player 2. The objective of the players is to minimize costs. The bi-matrix games in the second period are given by Table 1. The game over both periods can be solved in two ways. First, the game can be written as one bi-matrix game (confer Table 2) in the four possible strategies of the players over two periods (00, 01, 10 and 11), by adding up the costs in the first period (given in the text above) and the costs in the second period (given by the three bi-matrices above). The Nash equilibrium of this game is $(11, 00)$ with total costs $(3, 2)$. Second, the game can be solved backwards in time. The Nash equilibria in the second period are $(1, 0)$ in x_1 with costs (4, 1), $(1, 1)$ in x_2 with costs $(0, 3)$, and $(0, 0)$ in x_3 with costs $(2, 2)$. By adding up the costs in the first period and the Nash equilibrium costs in the second period, we get a bi-matrix game in the first period that is shown in Table 3. The Nash equilibrium of this game is $(0, 1)$ with total costs $(4, 4)$.

The two Nash equilibria are different. The Nash equilibrium of the game in strategies over the two periods is $(11, 00)$. The continuation of these strategies $(1, 0)$, starting in the state x_1 that is reached by implementing the controls $(1, 0)$ in the first period, is also a Nash equilibrium of the remainder of the game (time-consistency). However, this does not imply that this Nash equilibrium can be found by dynamic programming, backwards in time! The reason is that if the players realize that the controls are reconsidered in the second period, conditioned on the state that has been reached, they have incentives to deviate. Player 1 wants to play 0 in the first period, leading to state x_2 and Nash equilibrium $(1, 1)$ with total costs 2 for player 1 that are lower than 3. However, then the total costs of player 2 are 5 and therefore this player wants to play 1 in the first period, leading to state x_3 and Nash equilibrium $(0, 0)$ with total costs 4 for player 2 that are lower than 5. This results in the other Nash equilibrium with total costs $(4, 4)$. There are two mechanisms at work here: one is that the players are not committed to a control in the second period and the other one is that the players can condition their controls on the state of the system. Note that these additional options lead to higher total costs in the Nash equilibrium!

2.2 *Open-Loop and Feedback*

The theory of differential games was developed in the engineering literature and therefore the concepts of open-loop and closed-loop were used to describe the findings in the example above. A controlled system has closed loops if the controller uses observations on the state of the system. One can also say that the observations are fed back into the controller and therefore the concept of feedback is also used, instead of closed-loop. The conclusion of the example above is that the open-loop Nash equilibrium differs from the feedback or closed-loop Nash equilibrium, where the controls are conditioned on the state of the system. Later it was found that more Nash equilibria exist where the controls are conditioned on the history of the state of the system. These are called closed-loop memory Nash equilibria. If these equilibria are considered, the concept of state has to change. If the current and future controls are known, the state contains sufficient information to be able to predict the future of the system but if the current controls depend on the history of the state, this information is not sufficient. The state has to be augmented with the relevant history of the system. For example, the current pollution stock has to be augmented with previous pollution stocks. Başar and Olsder (1982) label open-loop, closed-loop and closed-loop memory as possible information structures and focus on the informational non-uniqueness of the Nash equilibrium of differential games.

Most of the economic applications of differential games, however, are restricted to the open-loop and the feedback Nash equilibrium. This is mainly driven by the available solution techniques. A differential game is basically a set of optimal control problems and therefore the well-known solution techniques of optimal control problems are also at the heart of differential games. When Pontryagin's maximum principle is used, the control is a function of time and the open-loop Nash equilibrium is found. This can be compared to the first Nash equilibrium in the example above. When dynamic programming is used or, more specifically, the Hamilton–Jacobi–Bellman equations in the value functions, the feedback Nash equilibrium is found. In the example above, the value functions give the costs-to-go of the two players in the Nash equilibrium of the second period as a function of the state. These are added to the possible costs in the first period and in this way the feedback Nash equilibrium is found.

2.3 Commitment

It is quite common in economic applications of differential games to label the open-loop Nash equilibrium as the pre-commitment solution, in contrast with the feedback Nash equilibrium. It is clear that the players in the feedback Nash equilibrium are not committed to their future controls and wait with choosing their actions until they observe the state of the system. It is also clear that the players in the open-loop Nash equilibrium are committed to their future controls, simply because they do not get new information. However, it is a bit confusing to distinguish these two equilibria on the basis of commitment because the comparison is not made ceteris paribus. When moving from open-loop to feedback, the information structure is changed and commitment is lost because the dynamic programming framework is used. A comparison on the basis of commitment only would require to keep the information structure fixed but then one cannot rely on a standard solution technique from optimal control theory.

	Open-loop	Feedback
Commitment	Maximum Principle	?
No commitment		Dynamic Programming

This combination of commitment and feedback or closed-loop information structures is simply not very well developed in economic applications of differential games. This is in contrast with another area in dynamic games, namely repeated games. Repeated games do not have a state but strategies can be conditioned on the strategies of the other players. In this way, with commitment, the cooperative outcome can be realized because players can announce to punish other players sufficiently much if they deviate from the cooperative outcome. A similar approach can be chosen for differential games (Tolwinski 1982). Another approach is to solve for closed-loop memory Nash equilibria. Tolwinski et al. (1986) show that for a class of differential games, closed-loop memory Nash equilibria can sustain efficient outcomes. Gaitsgory and Nitzan (1994) consider difference games which are differential games with discrete time. They develop a folk theorem in the sense that closed-loop memory Nash equilibria sustain individually rational outcomes with respect to the open-loop Nash equilibrium. In terms of applications to economics, these approaches are simply not very well developed and will therefore not be further pursued in this chapter. However, the idea of a folk theorem will pop up again below when the possible multiplicity of feedback Nash equilibria is discussed. It will be shown that the steady state of the cooperative outcome can be approached with feedback Nash equilibria if the discount rate goes to zero (see the next section).

If commitment is attached to the open-loop Nash equilibrium, the commitment device is to refrain from observing the state of the system. Commitment in this way may pay as we have seen in the example above where the open-loop Nash equilibrium has lower costs than the feedback Nash equilibrium. However, if we do not think that commitment is treated properly in this way and if we do not think that it is realistic to deliberately refrain from observing the state of the system, this is not an interesting way to go. Moreover, it is not generally true that the feedback Nash equilibrium has lower costs than the open-loop Nash equilibrium, as we will see in the applications below. One may say in general that the feedback Nash equilibrium is the most realistic solution concept. It usually confirms some tragedy of the commons but more optimistic results are also feasible, as we will see in the applications below. Most economic applications of differential games focus on comparing the open-loop Nash equilibrium and the feedback Nash equilibrium. Since the feedback Nash equilibrium is derived with dynamic programming and since the controls are conditioned on the state, it is also called the Markov–Perfect Nash equilibrium.

2.4 Formal Model

It is time to introduce some formalities. An important class of differential games is given by

$$\max_{u_i(\cdot)} W_i = \int_0^{\infty} F_i\big[x(t), u_i(t)\big] \exp(-rt)dt, \quad i = 1, 2, \ldots, n \tag{1}$$

subject to

$$\dot{x}(t) = f\big[x(t), u_1(t), u_2(t), \ldots, u_n(t)\big], \quad x(0) = x_0, \tag{2}$$

where i indexes the n players, x denotes the state of the system, u the controls, r the discount rate, W total welfare, F welfare at time t, and f the state transition. Note that the players only interact through the state dynamics. The problem has an infinite horizon and the welfare function and the state transition do not explicitly depend on time, except for the discount rate. This implies that the problem is stationary. Note also that the objective of the players is to maximize total welfare, whereas in the example in Sect. 2 players were minimizing costs.

In the open-loop Nash equilibrium the controls only depend on time: $u_i(t)$. This implies that player i solves an optimal control problem with Pontryagin's maximum principle and the strategies of the other players as exogenous inputs. This results in an optimal control strategy for player i that is a function of the strategies of the other players. This is, in fact, the rational reaction or the best response of player i. The open-loop Nash equilibrium simply requires consistency of these best responses. Pontryagin's maximum principle gives a necessary condition in terms of a differential equation for the co-state λ_i. If the optimal solution for player i can be characterized by the set of differential equations in x and λ_i, then the open-loop Nash equilibrium can be characterized by the set of differential equations in x and $\lambda_1, \lambda_2, \ldots, \lambda_n$. This is usually the best way to find the open-loop Nash equilibrium. The necessary conditions for player i in terms of the current-value Hamiltonian function

$$H_i(x, u_i, t, \lambda_i) = F_i(x, u_i) + \lambda_i f\big[x, u_1(t), \ldots, u_i, \ldots, u_n(t)\big] \tag{3}$$

are that the optimal $u_i^*(t)$ maximizes H_i and that the state x and the co-state λ_i satisfy the set of differential equations

$$\dot{x}(t) = f\left[x(t), u_1(t), \dots, u_i^*(t), \dots, u_n(t)\right], \quad x(0) = x_0, \tag{4}$$

$$\dot{\lambda}_i(t) = r\lambda_i(t) - H_{ix}\left[x(t), u_i^*(t), t, \lambda_i(t)\right], \tag{5}$$

with a transversality condition on λ_i. Note that the actions of the other players $u_j(t)$, $j \neq i$, are exogenous to player i. If sufficiency conditions are satisfied and if $u_i^*(t)$ can be explicitly solved from the first-order conditions of optimization, the open-loop Nash equilibrium can be found by solving the set of differential equations for x and $\lambda_1, \lambda_2, \dots, \lambda_n$ given by

$$\dot{x}(t) = f\left[x(t), u_1^*(t), u_2^*(t), \dots, u_n^*(t)\right], \quad x(0) = x_0, \tag{6}$$

$$\dot{\lambda}_i(t) = r\lambda_i(t) - H_{ix}\left[x(t), u_i^*(t), t, \lambda_i(t)\right], \quad i = 1, 2, \dots, n, \tag{7}$$

with transversality conditions on $\lambda_1, \lambda_2, \dots, \lambda_n$.

In the feedback Nash equilibrium the controls depend on the current state of the system and since the problem is stationary, they do not depend explicitly on time: $u_i(x)$. The Hamilton–Jacobi–Bellman equations for the current value functions V_i are given by

$$rV_i(x) = \max_{u_i}\left\{F_i(x, u_i) + V_i'(x) f\left[x, u_1(x), \dots, u_i, \dots, u_n(x)\right]\right\}, \quad i = 1, 2, \dots, n. \tag{8}$$

If sufficiency conditions are satisfied and if $u_i^*(x)$ can be explicitly solved from the first-order conditions of optimization, the feedback Nash equilibrium can be found by solving the set of equations in the current value functions V_i given by

$$rV_i(x) = F_i\left(x, u_i^*(x)\right) + V_i'(x) f\left[x, u_1^*(x), u_2^*(x), \dots, u_n^*(x)\right], \quad i = 1, 2, \dots, n. \tag{9}$$

How this works in specific problems will follow below.

This is only a small part of the theory of differential games. The first textbook is Başar and Olsder (1982) which was written from an engineering perspective. A more recent textbook with many economic applications is Dockner et al. (2000). A nice and concise introduction is Van Long (2013).

3 International Pollution Control

This section is strongly based on van der Ploeg and de Zeeuw (1992). The game of international pollution control, as it is formulated in this paper and in other papers such as Dockner and Van Long (1993), is an example of a linear-quadratic differential game where the state transition f is linear in the state and in the controls and where the objective function F is quadratic in the state and in the control. It is interesting to compare the open-loop Nash equilibrium and the linear feedback Nash equilibrium and to interpret the difference. It is shown that the players are worse off in the linear feedback Nash equilibrium. It follows that the additional information does not pay or, to put it differently, that it pays to stick to the open-loop controls. The reason is that, at the margin, players emit more knowing that the other players will partly offset this when they observe the resulting higher stock of pollution. Therefore, in equilibrium, emissions are higher and the difference in terms of welfare with the cooperative outcome is higher. However, it will be shown that also non-linear feedback Nash equilibria exist and that the steady state of the best non-linear feedback Nash equilibrium converges to the steady state of the cooperative outcome when the discount rate converges to zero. Apparently, with non-linear strategies some sort of folk theorem can be achieved.

3.1 The Model

Pollution P is an inevitable by-product of production Y and the stock of pollution damages the environment. In the case of global environmental problems, pollution P crosses national borders but in the absence of a supra-national government that is mandated to implement policies worldwide, these transboundary externalities cannot be internalized in a standard way. For example, climate change affects many countries and is caused by worldwide emissions of greenhouse gases but emissions can only be controlled by policies at the national level. At the international level a game is played between the countries. In the non-cooperative Nash equilibrium of this game the countries do not take the transboundary externalities into account but only focus on the damage by their own emissions in their own country. Of course, they could coordinate their policies and correct the transboundary externalities as well but then incentives to deviate arise and this is the heart of the problem. An important question is how much the countries would gain from cooperation, but this depends on which Nash equilibrium is to be expected.

The relationship between pollution P and production Y is simply modeled by a fixed emission-output ratio α. Pollution P accumulates into a stock of pollution which is partly degrading by natural processes. Damage is caused by the concentration level of pollution S and its development over time or the state transition, in the case of n countries, is simply modeled as

$$\dot{S}(t) = \frac{\alpha}{n}\sum_{i=1}^{n} Y_i(t) - \delta S(t), \quad S(0) = S_0. \tag{10}$$

This is a linear equation in the state S and in the controls Y_i. Note, however, that the linear natural degradation δS of the concentration level may be too simple. Usually processes in the natural system are more complicated but if the state transition is modeled in a non-linear way, the analysis of the differential game becomes much more complicated. We leave this to the next section on lakes.

The objectives are simply modeled as

$$\max_{Y_i(\cdot)} W_i = \int_0^\infty \left[\beta Y_i(t) - \frac{1}{2}Y_i^2(t) - \frac{1}{2}\gamma S^2(t)\right] \times \exp(-rt)dt, \quad i = 1, 2, \ldots, n. \tag{11}$$

The objectives are quadratic in the state S and in the control Y_i. Again, quadratic damage costs $\frac{1}{2}\gamma S^2$ may be too simple. Climate change, for example, is better modeled by some tipping point where damage is sharply increasing but again, the analysis of the differential game would become much more complicated. Note also that the countries are assumed to be the same. We will only consider symmetric Nash equilibria so that the index i will at some point disappear.

3.2 Open-Loop Nash Equilibrium

Using Pontryagin's maximum principle, the current-value Hamiltonian functions become

$$H_i(S, Y_i, t, \lambda_i) = \beta Y_i - \frac{1}{2}Y_i^2 - \frac{1}{2}\gamma S^2 + \lambda_i\left(\frac{\alpha}{n}Y_i + \frac{\alpha}{n}\sum_{j\neq i}^{n} Y_j(t) - \delta S\right) \tag{12}$$

and since sufficiency conditions are satisfied, the open-loop Nash equilibrium conditions become

$$Y_i(t) = \beta + \frac{\alpha}{n}\lambda_i(t), \quad i = 1, 2, \ldots, n, \tag{13}$$

$$\dot{S}(t) = \frac{\alpha}{n}\sum_{i=1}^{n} Y_i(t) - \delta S(t), \quad S(0) = S_0, \tag{14}$$

$$\dot{\lambda}_i(t) = (r + \delta)\lambda_i(t) + \gamma S(t), \quad i = 1, 2, \ldots, n, \tag{15}$$

with transversality conditions on $\lambda_1, \lambda_2, \ldots, \lambda_n$. The symmetric open-loop Nash equilibrium can therefore be characterized by the set of differential equations

$$\dot{S}_{OL}(t) = \alpha\left(\beta + \frac{\alpha}{n}\lambda_{OL}(t)\right) - \delta S_{OL}(t), \quad S_{OL}(0) = S_0, \tag{16}$$

$$\dot{\lambda}_{OL}(t) = (r + \delta)\lambda_{OL}(t) + \gamma S_{OL}(t), \tag{17}$$

with a transversality condition on λ_{OL}, where OL denotes open-loop. This yields a standard phase diagram in the state/co-state plane for an optimal control problem with a stable manifold and a saddle-point-stable steady state, given by

$$S_{OL}^{*} = \frac{\alpha\beta(r+\delta)n}{\delta(r+\delta)n+\alpha^{2}\gamma}. \tag{18}$$

The negative of the shadow value $-\lambda_{OL}$ can be interpreted as the tax on emissions that is required in each country to implement the open-loop Nash equilibrium. Note that this tax only internalizes the externalities within the countries but not the transboundary externalities. This would require a higher tax that can be found from the cooperative outcome of the game.

In the cooperative outcome the countries maximize their joint welfare. This is a standard optimal control problem with objective

$$\max_{Y_1(\cdot),\ldots,Y_n(\cdot)} \sum_{i=1}^{n} W_i. \tag{19}$$

Using Pontryagin's maximum principle, the current-value Hamiltonian function becomes

$$H(S,Y_1,Y_2,\ldots,Y_n,\lambda) = \sum_{i=1}^{n}\left(\beta Y_i - \frac{1}{2}Y_i^2\right) - \frac{1}{2}\gamma n S^2 + \lambda\left(\frac{\alpha}{n}\sum_{i=1}^{n} Y_i - \delta S\right) \tag{20}$$

and since sufficiency conditions are satisfied, the optimality conditions become

$$Y_i(t) = \beta + \frac{\alpha}{n}\lambda(t), \quad i = 1,2,\ldots,n, \tag{21}$$

$$\dot{S}(t) = \frac{\alpha}{n}\sum_{i=1}^{n} Y_i(t) - \delta S(t), \quad S(0) = S_0, \tag{22}$$

$$\dot{\lambda}(t) = (r+\delta)\lambda(t) + \gamma n S(t), \tag{23}$$

with a transversality condition on λ. The cooperative outcome can therefore be characterized by the set of differential equations

$$\dot{S}_C(t) = \alpha\left(\beta + \frac{\alpha}{n}\lambda_C(t)\right) - \delta S_C(t), \quad S_C(0) = S_0, \tag{24}$$

$$\dot{\lambda}_C(t) = (r+\delta)\lambda_C(t) + \gamma n S_C(t), \tag{25}$$

with a transversality condition on λ_C, where C denotes cooperative. This yields a standard phase diagram in the state/co-state plane for an optimal control problem with a stable manifold and a saddle-point-stable steady state, given by

$$S_C^{*} = \frac{\alpha\beta(r+\delta)}{\delta(r+\delta)+\alpha^{2}\gamma} < S_{OL}^{*}. \tag{26}$$

The negative of the shadow value $-\lambda_C$ can be interpreted as the tax on emissions that is required in each country to implement the cooperative outcome and this tax is higher than the tax in the open-loop Nash equilibrium because now the trans-boundary externalities are internalized as well. The steady state of the cooperative outcome is lower than the steady state of the open-loop Nash equilibrium, as is to be expected. This implies that welfare is lower in the open-loop Nash equilibrium than in the cooperative outcome. These results are straightforward. In the next section we will consider the feedback Nash equilibrium.

3.3 Feedback Nash Equilibrium

The Hamilton–Jacobi–Bellman equations in the current value functions V_i become

$$rV_i(S) = \max_{Y_i}\left\{\beta Y_i - \frac{1}{2}Y_i^2 - \frac{1}{2}\gamma S^2 + V_i'(S)\left(\frac{\alpha}{n}Y_i + \frac{\alpha}{n}\sum_{j\neq i}^{n} Y_j(S) - \delta S\right)\right\}, \quad i = 1, 2, \ldots, n, \tag{27}$$

with first-order conditions

$$Y_i^*(S) = \beta + \frac{\alpha}{n}V_i'(S), \quad i = 1, 2, \ldots, n. \tag{28}$$

Since sufficiency conditions are satisfied, the symmetric feedback Nash equilibrium can be found by solving the differential equation in $V = V_i$, $i = 1, 2, \ldots, n$,

$$rV(S) = \beta\left(\beta + \frac{\alpha}{n}V'(S)\right) - \frac{1}{2}\left(\beta + \frac{\alpha}{n}V'(S)\right)^2 - \frac{1}{2}\gamma S^2 + V'(S)\left[\alpha\left(\beta + \frac{\alpha}{n}V'(S)\right) - \delta S\right]. \tag{29}$$

The usual way to solve this equation is to assume that the current value function V is quadratic with the general form

$$V(S) = \sigma_0 - \sigma_1 S - \frac{1}{2}\sigma_2 S^2, \quad \sigma_2 > 0, \tag{30}$$

so that a quadratic equation in the state S results. Since this equation has to hold for all S, the coefficients of S^2 and S on the left-hand side and the right-hand side have to be equal. It follows that

$$\sigma_2 = \frac{-(r+2\delta)n^2 + n\sqrt{(r+2\delta)^2 n^2 + 4\alpha^2\gamma(2n-1)}}{2\alpha^2(2n-1)}, \tag{31}$$

$$\sigma_1 = \frac{\alpha\beta n^2\sigma_2}{(r+\delta)n^2 + \alpha^2(2n-1)\sigma_2}. \tag{32}$$

The feedback Nash equilibrium becomes

$$Y_i^*(S) = \beta + \frac{\alpha}{n}(-\sigma_1 - \sigma_2 S), \quad i = 1, 2, \ldots, n, \tag{33}$$

and the controlled state transition becomes

$$\dot{S}(t) = \alpha(\beta + \frac{\alpha}{n}\big(-\sigma_1 - \sigma_2 S(t)\big) - \delta S(t), \tag{34}$$

which is stable and yields the steady state

$$S_{FB}^* = \frac{\alpha\beta n - \alpha^2\sigma_1}{\delta n + \alpha^2\sigma_2}, \tag{35}$$

where *FB* denotes feedback.

It is tedious but straightforward to show that

$$S_C^* < S_{OL}^* < S_{FB}^*. \tag{36}$$

This implies that in the feedback Nash equilibrium the countries are worse off than in the open-loop Nash equilibrium or, to put it differently, that the gains of cooperation are higher when the non-cooperative model is the feedback model. Since the feedback model, where countries observe the state of the system and are not committed to future actions, is the more realistic model, the tragedy of the commons is more severe than one would think when the open-loop model is used to assess the gains of cooperation. The intuition for this pessimistic result is as follows. A country argues that when it increases emissions, this will increase the concentration level of pollution and this will induce the other countries to lower their emissions, so that part of the increase in emissions will be offset by the other countries. Each country argues the same way so that in equilibrium emissions will be higher than in the case where the concentration level is not observed. With the open-loop model, the gains of cooperation are in fact underestimated.

The bulk of the literature on economic applications of differential games has this type of result. However, a different approach to the issue is possible. Dockner and Van Long (1993) show that non-linear feedback Nash equilibria (with non-quadratic current value functions) for this problem exist which may be better than the open-loop Nash equilibrium. This will be shown in the next section.

3.4 Non-linear Feedback Nash Equilibria

The symmetric feedback Nash equilibrium is given by

$$Y_i^*(S) = \beta + \frac{\alpha}{n}V'(S) := h(S), \quad i = 1, 2, \ldots, n, \tag{37}$$

and using these equations for substituting $V'(S)$, the Hamilton–Jacobi–Bellman equation can be written as

$$rV(S) = \beta h(S) - \frac{1}{2}\big(h(S)\big)^2 - \frac{1}{2}\gamma S^2 + \left[\frac{n}{\alpha}\big(h(S) - \beta\big)\right]\big[\alpha h(S) - \delta S\big]. \tag{38}$$

Assuming that h is differentiable, differentiating this equation with respect to S and substituting $V'(S)$ again yields an ordinary differential equation in the feedback equilibrium control h given by

$$\big[(2n-1)\alpha h(S) + (1-n)\alpha\beta - n\delta S\big]h'(S) = n(r+\delta)h(S) + \alpha\gamma S - n(r+\delta)\beta. \tag{39}$$

This is in fact the Euler–Lagrange equation for this problem. This differential equation may have multiple solutions because the boundary condition is not specified. The steady-state condition yields a boundary condition but the steady state is not determined in a differential game. One can also say that the multiplicity of nonlinear feedback Nash equilibria results from the indeterminacy of the steady state in differential games. Dockner and Van Long (1993) have the same model but with $n = \alpha = 2$ which yields

$$\big[3h(S) - \beta - \delta S\big]h'(S) = (r+\delta)h(S) + \gamma S - (r+\delta)\beta, \tag{40}$$

with the boundary condition in the steady state S^* given by

$$h\big(S^*\big) = \frac{1}{2}\delta S^*. \tag{41}$$

The solutions of this differential equation in the feedback control h must lead to a stable system where the state S converges to the steady state S^*. The stable solutions form a set of hyperbolas in the (S, h)-plane that cut the steady state line $\frac{1}{2}\delta S$ in the interval

$$\frac{2\beta(2r+\delta)}{\delta(2r+\delta)+4\gamma} \le S^* < \frac{2\beta}{\delta}. \tag{42}$$

Rubio and Casino (2002) correct this result by showing that it only holds for a certain set of initial states S_0. The right-hand side of the interval represents the situation where the countries do not have a concern for the environment and choose $Y = \beta$. The left-hand side is the lowest steady-state that can be achieved with a feedback Nash equilibrium. In this equilibrium the hyperbola $h(S)$ is tangent to the steady-state line, so that $h(S^*) = \frac{1}{2}\delta S^*$ and $h'(S^*) = \frac{1}{2}\delta$.

The steady state in the cooperative outcome (for $\alpha = 2$) is still lower,

$$S_C^* = \frac{2\beta(r+\delta)}{\delta(r+\delta)+4\gamma} < \frac{2\beta(2r+\delta)}{\delta(2r+\delta)+4\gamma}, \tag{43}$$

but it is interesting to note that the best steady state in a feedback Nash equilibrium converges to the steady state in the cooperative outcome when the discount rate r converges to zero. This does not imply, however, that welfare in this feedback Nash equilibrium also converges to welfare in the cooperative outcome when the discount rate r converges to zero. We will come back to this in the next section on lakes.

This result in Dockner and Van Long (1993) is important. It shows that when we allow for non-linear equilibria in this linear-quadratic framework, the feedback Nash equilibrium can be better than the open-loop Nash equilibrium, i.e. in terms of the steady states. Apparently, the feedback information structure can also be beneficial for the countries, by keeping each other targeted on a better steady state with a different set of feedback equilibrium controls. These feedback controls are not offsetting part of the earlier extra emissions but are threatening to emit even more. This implies that even if a linear feedback Nash equilibrium exists, the countries can decide to coordinate on another, non-linear, feedback Nash equilibrium because this one leads to a steady state with higher welfare. Note that this result is achieved in a dynamic programming framework and that the equilibrium is Markov perfect. If the discount rate r approaches zero, the steady state approaches the steady state of the cooperative outcome which can be interpreted as some sort of folk theorem in a differential game (see also Rowat 2007). This analysis has only been developed for one-dimensional systems and we have to wait and see how it works out in higher-dimensional systems. In the next section on lakes, we will apply the same technique but now in a model with a non-linear state transition.

4 The Lake Game

This section is strongly based on Mäler et al. (2003) and Kossioris et al. (2008, 2011). The lake game, as it is formulated in these papers and in other papers such as Brock and Starrett (2003), is an example of a differential game where the state transition f is non-linear in the state. Since the linear-quadratic structure is lost anyway, the objective function F is chosen to be logarithmic in the control because this implies that the cooperative outcome is independent of the number of players, which is convenient in the analysis. We will compare the cooperative outcome with the best feedback Nash equilibrium that is derived with the technique in the last sub-section. Furthermore, we will introduce a tax rate on pollution in order to see if that can internalize the externality in this case. Because of the complexity of the problem, at some point we need to resort to numerical solutions.

4.1 The Model

It can be shown that the essential dynamics of eutrophication of lakes can be described by the differential equation

$$\dot{x}(t) = \sum_{i=1}^{n} a_i(t) - bx(t) + \frac{x^2(t)}{x^2(t)+1}, \quad x(0) = x_0, \tag{44}$$

where x denotes the amount of phosphorus sequestered in algae, a_i the loading of phosphorus on the lake by agent i, $i = 1, 2, \ldots, n$, and b the parameter for the rate of loss (which differs across lakes). The last non-linear term reflects an internal positive release of phosphorus, that has been sequestered in sediments and submerged vegetation, due to changes in the condition of the lake (Carpenter and Cottingham 1997). Note that this equation has one or more steady states, depending on the value of b and on the level of total loading a. If $b < 3\sqrt{0.375}$, for a certain range of a, the equation has three steady states: two stable ones and an unstable one in between. A low x is usually referred to as an oligotrophic state and a high x is usually referred to as a eutrophic state. With these multiple steady states, a hysteresis effect can occur. Increasing total loading a from a low level, with a low steady state, will at some point lead to a sudden flip to a high steady state (tipping point). Trying to return to a low steady state, by decreasing total loading a again, requires a substantial decrease in a before the lake flips back. If $b \leq 0.5$, it is even impossible to flip back since total loading a cannot become negative. In this case, the flip is irreversible and the lake is trapped in a eutrophic state. We will assume that $0.5 < b < 3\sqrt{0.375}$, so that hysteresis can occur but a flip to a eutrophic state is reversible. This type of model is also relevant for other natural systems such as coral reefs, rangelands and climate, so that it can be seen as a metaphor for many environmental problems facing us today.

Damage to the lake is caused by the amount of phosphorus sequestered in algae x. We take a simple increasing quadratic form. However, the release of phosphorus on the lake is a by-product of agriculture and in that sense also beneficial (value as a waste sink). The agents can be seen as communities around the lake, which is common property to them. We take a logarithmic form for the benefits of loadings a_i of phosphorus on the lake, because this form has the property that the cooperative outcome is independent of the number of agents, as we will see below. The objectives are simply modeled as

$$\max_{a_i(\cdot)} W_i = \int_0^\infty \left[\ln a_i(t) - cx^2(t)\right] \exp(-rt)dt, \quad i = 1, 2, \ldots, n, \tag{45}$$

where c denotes the relative weight in the objective between the value of the lake as a waste sink and the damage to the lake. For a high c it is to be expected that the resulting state will be oligotrophic. Note that the communities are assumed to be the same. We will only consider symmetric Nash equilibria again so that the index i will at some point disappear.

4.2 Open-Loop Nash Equilibrium

Using Pontryagin's maximum principle, the current-value Hamiltonian functions become

$$H_i(x, a_i, t, \lambda_i) = \ln a_i - cx^2 + \lambda_i \left(a_i + \sum_{j \neq i}^{n} a_j(t) - bx + \frac{x^2}{x^2 + 1} \right) \tag{46}$$

and since sufficiency conditions are satisfied, the open-loop Nash equilibrium conditions become

$$a_i(t) = -\frac{1}{\lambda_i(t)}, \quad i = 1, 2, \ldots, n, \tag{47}$$

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{x^2(t)+1}, \quad a := \sum_{i=1}^{n} a_i, x(0) = x_0, \tag{48}$$

$$\dot{\lambda}_i(t) = \left(r + b - \frac{2x(t)}{(x^2(t)+1)^2}\right)\lambda_i(t) + 2cx(t), \quad i = 1, 2, \ldots, n, \tag{49}$$

with transversality conditions on $\lambda_1, \lambda_2, \ldots, \lambda_n$. The symmetric open-loop Nash equilibrium can therefore be characterized by the set of differential equations in the state x and in the total loadings a, given by

$$\dot{x}_{OL}(t) = a_{OL}(t) - bx_{OL}(t) + \frac{x_{OL}^2(t)}{x_{OL}^2(t)+1}, \quad x_{OL}(0) = x_0, \tag{50}$$

$$\dot{a}_{OL}(t) = -\left(r + b - \frac{2x_{OL}(t)}{(x_{OL}^2(t)+1)^2}\right)a_{OL}(t) + 2\frac{c}{n}x_{OL}(t)a_{OL}^2(t), \tag{51}$$

with a transversality condition on a_{OL}, where OL denotes open-loop. This system may have multiple steady states, depending on the value of the parameters. We return to this issue below.

In the cooperative outcome the communities maximize their joint welfare. This is a standard optimal control problem with objective

$$\max_{a_1(\cdot),\ldots,a_n(\cdot)} \sum_{i=1}^{n} W_i. \tag{52}$$

Using Pontryagin's maximum principle, the current-value Hamiltonian function becomes

$$H(x, a_1, a_2, \ldots, a_n, \lambda) = \sum_{i=1}^{n} \ln a_i - ncx^2 + \lambda\left(\sum_{i=1}^{n} a_i - bx + \frac{x^2}{x^2+1}\right) \tag{53}$$

and since sufficiency conditions are satisfied, the optimality conditions become

$$a_i(t) = -\frac{1}{\lambda(t)}, \quad i = 1, 2, \ldots, n, \tag{54}$$

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{x^2(t)+1}, \quad a := \sum_{i=1}^{n} a_i, x(0) = x_0, \tag{55}$$

$$\dot{\lambda}(t) = \left(r + b - \frac{2x(t)}{(x^2(t)+1)^2}\right)\lambda(t) + 2ncx(t), \tag{56}$$

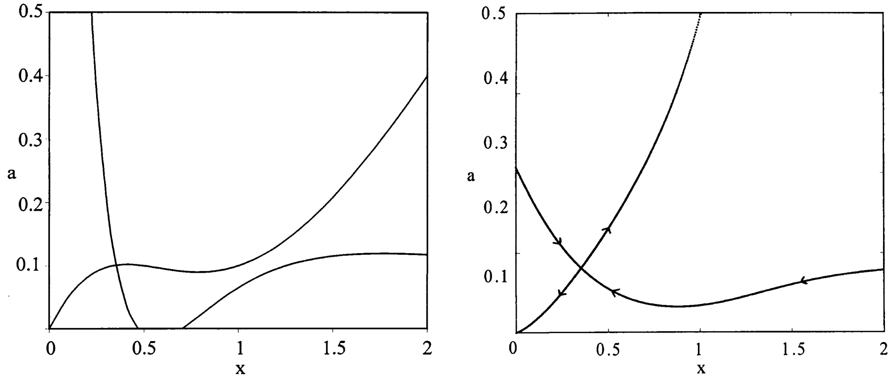

Fig. 1 Phase diagram in the (x, a)-plane for the cooperative outcome

with a transversality condition on λ. The cooperative outcome can therefore be characterized by the set of differential equations in the state x and in the total loadings a, given by

$$\dot{x}_C(t) = a_C(t) - bx_C(t) + \frac{x_C^2(t)}{x_C^2(t)+1}, \quad x_C(0) = x_0, \tag{57}$$

$$\dot{a}_C(t) = -\left(r + b - \frac{2x_C(t)}{(x_C^2(t)+1)^2}\right)a_C(t) + 2cx_C(t)a_C^2(t), \tag{58}$$

with a transversality condition on a_C, where C denotes cooperative.

For $b = 0.6$, $c = 1$ and $r = 0.03$, the phase diagram in the (x, a)-plane for the cooperative outcome is given in Fig. 1.

For these parameter values, the controlled system has one oligothrophic steady state that is saddle-point stable. The phase diagram has a stable and an unstable manifold. The situation is essentially not different from the linear-quadratic case in the previous section. It would be different if we increase r, for example, but it is more interesting to increase n and thus move to the open-loop Nash equilibrium. Note that the open-loop Nash equilibrium can be found by solving the optimal control problem with parameter c/n (a game with this property is called a potential game). For $n = 2$, the phase diagram in the (x, a)-plane for the open-loop Nash equilibrium is given in Fig. 2.

Now we get three steady states: saddle-point stable ones to the left and to the right and an unstable one in between. Stable manifolds curl out from the unstable steady state and go either to the left or to the right steady state. The outcome depends on the initial state. A point x_S exists with the property that if $x_0 < x_S$, the open-loop Nash equilibrium moves towards the oligotrophic steady state at the left and if $x_0 > x_S$, the open-loop Nash equilibrium moves towards the eutrophic steady state at the right. Such a point is called a Skiba point since it was first presented by Skiba (1978) in an optimal growth model with a convex-concave production function. The

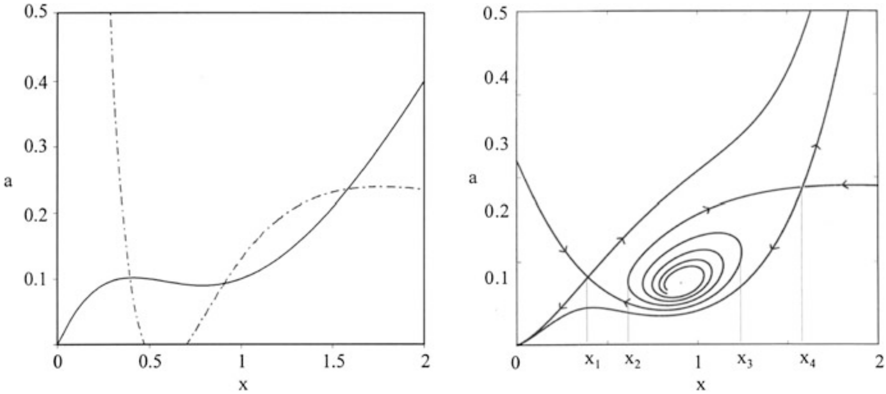

Fig. 2 Phase diagram in the (x, a)-plane for the open-loop Nash equilibrium

intuition is clear: if the lake is already heavily polluted, it does not pay anymore to move to an oligotrophic state.

The policy question is whether a properly chosen tax τ on phosphorus loadings a_i, with an extra cost $\tau(t)a_i(t)$ under the integral in the objective, can regulate the system on the optimal path towards the optimal (oligotrophic) steady state. The answer is yes, because the tax τ should simply bridge the gap between the negatives of the shadow values $-\lambda_i$ and $-\lambda$:

$$\tau(t) - \lambda_i(t) = -\lambda(t). \tag{59}$$

However, such a tax τ is time-dependent, since these shadow values are constantly changing on the optimal path. Such a tax would be very difficult to implement because it would require a regulating institution to continuously change the tax rate. Therefore, the more realistic policy question is what a fixed tax rate can do. By comparing steady-state equations, it is easy to see that in the optimal steady state (a_C, x_C), this (fixed) tax must be equal to

$$\tau^* = \frac{n-1}{a_C}. \tag{60}$$

This implies that the steady state of the open-loop Nash equilibrium under the fixed tax rate τ^* is the same as the steady state of the cooperative outcome. However, the locus of steady states in the resulting phase diagram for the total loading a differs, and the trajectory may differ as well, of course. In Mäler et al. (2003) it is shown that for a small number of communities n $(n \leq 7)$, the phase diagram for the open-loop Nash equilibrium under the fixed tax rate τ^* is qualitatively the same as the phase diagram for the cooperative outcome. It follows that for small n, the optimal steady state can be achieved, although welfare may be lower because of changes in the trajectory. However, for a large number of communities n $(n > 7)$, the phase diagram for the open-loop Nash equilibrium under the fixed tax rate τ^* is complicated and irregular, and multiple steady states may occur. It follows that for large n, it may even not be possible to achieve the optimal steady state, depending on the initial state x_0. We may conclude that this regulation works fine for a small number of communities n but in general does not work for a large number of communities. This situation can be improved in the feedback Nash equilibrium, which we will consider in the next section.

4.3 Feedback Nash Equilibria

The Hamilton–Jacobi–Bellman equations in the current value functions V_i become

$$rV_i(x) = \max_{a_i}\left\{\ln a_i - cx^2 + V_i'(x)\left(a_i + \sum_{j\neq i}^{n} a_j(x) - bx + \frac{x^2}{x^2+1}\right)\right\}, \quad i = 1, 2, \ldots, n. \tag{61}$$

Since sufficiency conditions are satisfied, the symmetric feedback Nash equilibrium with $V = V_i$, $i = 1, 2, \ldots, n$, is given by

$$a_i^*(x) = -\frac{1}{V'(x)} := h(x), \quad i = 1, 2, \ldots, n, \tag{62}$$

and using these equations for substituting $V'(x)$, the Hamilton–Jacobi–Bellman equation in the current value function V can be written as

$$rV(x) = \ln h(x) - cx^2 - \frac{1}{h(x)}\left(nh(x) - bx + \frac{x^2}{x^2+1}\right). \tag{63}$$

Assuming that h is differentiable, differentiating this equation with respect to x and substituting $V'(x)$ again yields an ordinary differential equation in the feedback equilibrium control h given by

$$\left[-h(x) + bx - \frac{x^2}{x^2+1}\right]h'(x) = \left(r + b - 2cxh(x) - \frac{2x}{(x^2+1)^2}\right)h(x). \tag{64}$$

This is, in fact, the Euler–Lagrange equation for this problem. It is a so-called Abel differential equation of the second kind (Murphy 1960) which cannot be solved analytically, but we can solve it numerically with the ode15s solver in Matlab. As in the previous section, this differential equation may have multiple solutions because the boundary condition is not specified. The steady-state condition gives the boundary condition

$$h(x_{FB}) = \frac{1}{n}\left(bx_{FB} - \frac{x_{FB}^2}{x_{FB}^2+1}\right), \tag{65}$$

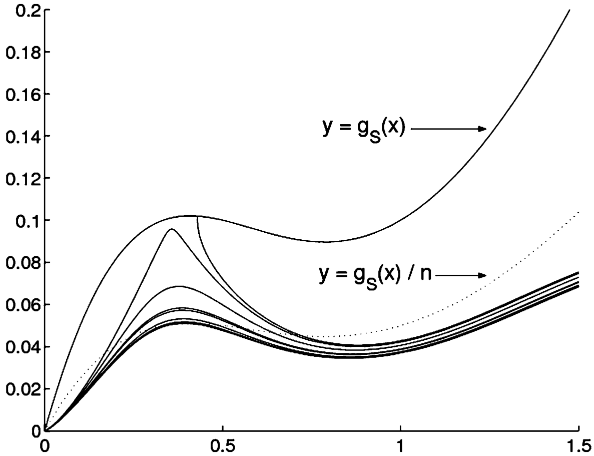

Fig. 3 Phase diagram in the (x, a)-plane for the feedback equilibrium

where *FB* denotes feedback, but the steady state x_{FB} is not determined in a differential game which yields the multiplicity of feedback Nash equilibria. We use the same values for the parameters as in the previous section: $b = 0.6$, $c = 1$, $r = 0.03$ and $n = 2$. The solutions of this differential equation in the feedback equilibrium control h must lead to a stable system where the state x converges to the steady state x_{FB}. The results are shown in Fig. 3.

Figure 3 depicts both the locus of steady states for the state x with total loading a on the y-axis ($y = g_S(x)$) and with individual loading a_i on the y-axis ($y = g_S(x)/n$) where

$$g_S(x) = bx - \frac{x^2}{x^2+1}. \tag{66}$$

Furthermore, solutions $h(x)$ of the differential equation are depicted. These solutions must have an intersection point x_{FB} with $y = g_S(x)/n$ because of the boundary condition. If the derivative $h'(x)$ is negative in this intersection point x_{FB}, the solution $h(x)$ yields a stable system with steady state x_{FB}, at least in a neighborhood of x_{FB}.

A number of conclusions can be drawn from Fig. 3 (see Kossioris et al. 2008). The benchmark is the curve $h(x)$ that is tangent to the steady-state curve $y = g_S(x)/n$ which occurs at the steady state $x^*_{FB} = 0.38$. For this feedback Nash equilibrium, the steady state can be reached from any initial state $x_0 > 0.38$ but not from an initial state $x_0 < 0.38$. It is also possible to reach any steady state $x_{FB} > 0.38$ but welfare is lower in the corresponding feedback Nash equilibria. Low steady states ($x_{FB} < 0.17$) cannot be reached because for the corresponding feedback Nash equilibria, the resulting system is not stable. It is not possible to get a stable solution for initial states $x_0 < 0.17$. For the initial states $0.17 < x_0 < 0.38$, the situation is more complicated. At such an initial state x_0, the feedback equilibrium control $h(x)$ that starts just above the steady-state curve $y = g_S(x)/n$ will steer the system to a stable steady state to the right, in the eutrophic area of the lake (a similar observation was made in Rubio and Casino (2002) for the non-linear feedback Nash equilibria in the game of international pollution control in the previous section). This leads to a type of time-inconsistency: as soon as the system has moved a bit, the incentive occurs to adjust the equilibrium and to jump down to a lower feedback equilibrium control $h(x)$, just above the steady-state curve $y = g_S(x)/n$ at the higher level of x. Moreover, as soon as the system has moved beyond the state $x = 0.38$, the incentive occurs to jump down all the way to the tangent feedback equilibrium control $h(x)$ that steers the system to the steady state $x^*_{FB} = 0.38$. A full picture of possible equilibria for the lake and for similar models can be found in Dockner and Wagener (2008).

For any initial state $x_0 > 0.38$, the best the communities can do, in terms of welfare, is to coordinate on the tangent feedback equilibrium control $h(x)$ that steers the system to the steady state $x^*_{FB} = 0.38$. The steady state of this best feedback Nash equilibrium (0.38) is closer to the steady state of the cooperative outcome (0.353) than the steady states of the two open-loop Nash equilibria (0.393 and 1.58). This is not generally true. If the number of communities is increased to $n = 3$, the steady state of the best feedback Nash equilibrium becomes 0.417 and the steady state of the best open-loop Nash equilibrium becomes 0.412 whereas the steady state of the cooperative outcome remains 0.353. More details can be found in Kossioris et al. (2008). More importantly, however, the feedback Nash equilibrium allows the communities to move to an oligotrophic steady state and they are not trapped in a eutrophic steady state, like in the open-loop Nash equilibrium, when the initial state is above the Skiba point. Furthermore, the result of Dockner and Van Long (1993) for the game of international pollution control also holds for the lake: the best steady state of a feedback Nash equilibrium converges to the steady state of the cooperative outcome when the discount rate r converges to zero. However, this does not imply that welfare is the same. Kossioris et al. (2008) show that the best feedback Nash equilibrium generally performs worse, in terms of welfare, than the open-loop Nash equilibrium and therefore a fortiori worse that the cooperative outcome but differences are small, of course, when the initial state is close to the steady states.

Kossioris et al. (2011) study what a tax can achieve in the feedback information structure. More specifically, they focus on a stationary tax rate $\tau(x)$ that depends on the state of the system. The Hamilton–Jacobi–Bellman equations in the current value functions V_i become

$$r V_i(x) = \max_{a_i} \Biggl\{ \ln a_i - cx^2 - \tau(x) a_i + V_i'(x) \Biggl(a_i + \sum_{j \neq i}^{n} a_j(x) - bx + \frac{x^2}{x^2+1} \Biggr) \Biggr\}, \quad i = 1, 2, \ldots, n. \quad (67)$$

A similar derivation as above yields the differential equation

$$\left[-h(x)-(n-1)\tau(x)h^2(x)+bx-\frac{x^2}{x^2+1}\right]h'(x)$$
$$=\left[\left(r+b-\frac{2x}{(x^2+1)^2}\right)\big(1-\tau(x)h(x)\big)-2cxh(x)\right]h(x)$$
$$+\left((n-1)h(x)-bx+\frac{x^2}{x^2+1}\right)\tau'(x)h^2(x) \tag{68}$$

with the two unknown functions $h(x)$ and $\tau(x)$. Kossioris et al. (2011) take different polynomial functional forms for $\tau(x)$ and fix the parameters of the functional form such that the (tangent) best feedback Nash equilibrium $h(x)$ steers the system to the steady state of the cooperative outcome $x_C = 0.353$, starting at higher initial states. More specifically they focus on a fixed tax rate, a linear tax rate and a quadratic tax rate and they compare the resulting welfare with the welfare in the cooperative outcome. The welfare differences get smaller for higher order polynomials, because the trajectories move closer to the trajectory of the cooperative outcome, but it is not possible to mimic the cooperative outcome with these relatively simple tax rates. The "Nash program" cannot be solved with simple tax rates in this context.

5 Conclusion

The purpose of this chapter is twofold. First, it provides an introduction into some concepts and techniques of differential games that are widely used for economic applications. Second, it analyzes two famous models in environmental and resource economics, the game of international pollution control and the lake game. The analysis fits in what is called the "Nash program." Since Nash equilibria in differential games are usually not unique, the question is which one comes closest to the cooperative outcome or can even mimick it. If the last option is not available, the question is whether some realistic tax rate can regulate the Nash equilibrium in order to cover the remaining welfare gap. The chapter is mainly based on a number of articles that have already appeared in journals but it puts the main conclusions in this general framework.

The basics of differential games goes back more than 40 years and results from the observation that the equivalent of Bellman's principle of optimality does not hold in games. This implies that Nash equilibria in strategies that only depend on time and are derived with Pontryagin's maximum principle are different from Nash equilibria that also depend on the state and are derived with dynamic programming. This leads to the general question whether it is possible to characterize the full set of Nash equilibria that depend on all possible information structures regarding the state and that have different levels of commitment but this issue is far from solved. However, the restriction to Nash equilibria that either result from Pontryagin's maximum principle or from the Hamilton–Jacobi–Bellman equations of dynamic programming already gives interesting results and insights. The first set of equilibria is referred to as open-loop Nash equilibria and the second set is referred to as feedback Nash equilibria.

The game of international pollution control is an example of a differential game where the objective is quadratic and the state transition is linear in the state and in the controls. The open-loop Nash equilibrium is linear. A linear feedback Nash equilibrium exists with a steady state stock of pollution that is higher than in the open-loop Nash equilibrium, and with lower welfare. However, also non-linear feedback Nash equilibria exist. The most favorable one has a steady state stock of pollution that converges to the steady state stock of pollution in the cooperative outcome. The conclusion whether the open-loop or the feedback Nash equilibrium is better is therefore mixed. On the one hand, feedback equilibria can push up the stock of pollution because countries know that extra emissions will be partly offset by the other countries. On the other hand, feedback Nash equilibria can push down the stock of pollution if countries are threatening to emit even more as a reaction to extra emissions.

The lake game is an example of a differential game where the state transition is non-linear in the state. Assuming that the steady state in the cooperative outcome is oligotrophic (good), increasing the number of communities using the lake will at some point lead to the situation that the open-loop Nash equilibrium has both an oligotrophic and a eutrophic (bad) steady state. It depends on the initial condition (below or above the Skiba point) where the equilibrium trajectory will end up. Regulation by means of a fixed tax rate works for a low number of communities but it does not work for a large number of communities, in which case the system gets trapped in the eutrophic area of the lake. Feedback Nash equilibria (that are non-linear, of course) have the same properties as in the game of international pollution control. In addition, with feedback Nash equilibria the system cannot get trapped in the eutrophic area of the lake. Regulation by means of a polynomial state-dependent tax rate can steer the system to the steady state of the cooperative outcome and can move the system closer to the trajectory of the cooperative outcome but it cannot mimic the cooperative outcome and has therefore lower welfare.

Differential games are the natural framework of analysis for many problems in environmental and resource economics. The existing solution techniques can cover some of the gap between non-cooperative Nash equilibria and the cooperative outcome but still not all the way. Regulation by means of realistic tax rates can cover some of the remaining gap but also still not all the way. Further research has to show what is feasible here.

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Capacity Accumulation Games with Technology Constraints

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Abstract This chapter examines the conduct and performance of large mutually dependent firms. Its objective is to study contractual relationships in a dynamic bilateral monopoly, where producers' investment choices must obey a technology constraint. This is in contrast to previous studies of accumulation games, in which technological interdependence was not explicitly allowed for. The analysis focuses on investment incentives and payoff allocation under two regimes: (1) contracting based on input quantities, and (2) contracting based on final revenues. The technologically feasible equilibrium strategies and the terms of trade that support them are characterized with intuitive necessary conditions which reflect the players' intertemporal trade-offs. To assess the factors that influence efficiency and market power, the chapter presents a linear-quadratic example. Our simulations indicate that contracts based on input quantities generate higher joint payoffs and tend to benefit the input producer.

1 Introduction

Closeness of geographic locations and development of relationship-specific assets may give rise to bilateral monopolies. That is, sometimes two firms become "locked in" to one another and can only operate in a tandem. Examples of exclusive production arrangements are numerous and include: coal mine—thermal power generator, power generator—aluminium smelter, timber mill—furniture factory. Familiarity and the desire to avoid renegotiation costs often drive bilateral monopolists into long-term contractual relationships. While the fundamentals of these contracts are rather durable, the terms of trade may undergo periodic adjustments to account for changes in the operating environment.

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The primary objective of the present chapter is to examine long-term contracting in such dynamic bilateral monopolies.[1] We study an infinite-horizon game between an input supplier and a final good producer, where investment in capacity gives rise to intertemporal spillovers. Firms are technologically interdependent: their choices are bound by a "production function." The players' strategies, as well as the terms of trade that govern surplus allocation, are assumed to have a Markovian structure. Thus, each player takes current prices as given, but also accounts for the consequences of his actions for future surplus shares.

We analyze and compare contract designs that support bilateral trade as a technologically feasible non-cooperative equilibrium. Our focus is on two payoff allocation regimes: (i) contracting based on input quantities, and (ii) contracting based on final revenue. Both arrangements are quite common in the real word. For example, Gazprom (the dominant Russian producer and exporter of natural gas) makes deliveries to European countries on the basis of contracts over input quantities. On the other hand, various software developers have an arrangement with Apple Inc. (a provider of hardware platforms) which specifies shares of final revenues.

We investigate how these regimes affect the division of surplus within the bilateral monopoly, and explore their consequences for firms' behavior. Furthermore, our model sheds light on the efficiency of such contractual arrangements. We show that strategic considerations drive investment away from the plan that maximizes the present value of the stream of joint profits. Last but not least, this study can be used by a central planner to design an allocation mechanism which ensures that neither of the bilateral monopolists will be stuck with unused production capacity in the long run.

Our methodology exhibits two desirable features. First, the Markovian assumption for contracts and investments allows us to endogenously determine the technologically feasible terms of trade.[2] Second, firms' choices are obtained as subgame-perfect equilibrium strategies in a non-cooperative game. This implies that contracts will be *self-enforcing*. Since these strategies specify optimal actions in all periods and for any history, no player will have an incentive to unilaterally deviate at any point in the game.

To quantify and compare the properties of the contractual arrangements, we analyze a numerical example based on a linear-quadratic formulation. It yields terms of trade and investment strategies that are linear in the state variables. The simulations underscore the importance of trading procedures for payoff allocation within the bilateral monopoly: for a given set of parameters, the player who has control over the value of the contractible variable is able to attain a higher surplus share. Moreover, the example shows that the choice of a contractible variable can have important consequences for economic efficiency. Specifically, joint payoffs are higher when contracts are based on input quantities rather than on final revenue.

[1] This chapter draws from and extends Petkov and Krawczyk (2004).

[2] The assumption of Markovian terms of trade relates our chapter to the literature on asset pricing originating from Lucas (1978).

We model bilateral trade using a classical capital accumulation framework that gives rise to intertemporal trade-offs. In this setting, firms are willing to incur instantaneous costs in order to gain a future strategic advantage. Thus, our analysis delivers equilibrium conditions similar to those in the literature on investment games (e.g. Hanig 1986; Reynolds 1987). More generally, our chapter contributes to the theory of dynamic oligopoly (e.g. Maskin and Tirole 1987, 1988), which studies the importance of strategic commitment in market interactions.

Previous work in this field has assumed a particular functional form for payoffs, effectively determining the strategic properties of the game (i.e., whether players' choices would be strategic complements or substitutes). In contrast, our surplus allocating procedure emerges as an equilibrating mechanism that accounts for technological interdependence. It permits the analysis and comparison of various bilateral trading relationships without making ad-hoc assumptions regarding the nature of competition.

There also exists abundant literature on bargaining and bilateral exchange with incomplete contracts which aims to explore the boundaries of the firm and asset ownership. It originates from the seminal work of Coase (1937) and is further developed by Williamson (1975, 1979, 1985), Klein et al. (1978), Grossman and Hart (1986), Hart and Moor (1990), and Whinston (2003). These papers model the allocation of final payoffs as a cooperative bargaining game, usually employing the Nash solution with exogenously fixed bargaining weights. In a more complicated dynamic setup where firms receive *streams of payoffs*, this approach may lack plausibility. It could be argued that intertemporal spillovers will cause bargaining weights to change over time. In particular, their dynamics will reflect forward-looking attempts to strategically influence the future terms of trade. While non-cooperative bargaining games can account for forward-looking behavior, they usually require the specification of restrictive bargaining procedures e.g., alternating offers as in Rubinstein (1982). Our methodology, on the other hand, is amenable to various modifications and environments.

The remainder of the chapter is organized as follows. Section 2 describes the bilateral monopoly setting and our solution concept. Section 3 provides formal analysis of the allocation arrangements and derives general equilibrium conditions. A linear-quadratic algebraic formulation of the model is motivated and solved numerically in Sect. 4. The concluding remarks are presented in Sect. 5.

2 The Setup

2.1 Key Features

The setup below is a mathematical abstraction of the following key aspects of dynamic bilateral monopolies.

- *Technological interdependence*: firms need to operate in a tandem in order to generate surplus.

- *Existence of market power*: each party has the ability to influence the future terms of trade.
- *Strategic conduct*: firms take into account the effect of their investment choices on the opponent's behavior.
- *Noncooperative decision making*: private payoff maximization ensures that contracts will be self-enforcing.

2.2 Technology and Industry Structure

Suppose that, in each period $t = 0, 1, \ldots,$ the market for a final good x is served by a single downstream producer (referred to as firm A). The production process involves the use of an intermediate good y. Input y is supplied by a single upstream firm (referred to as firm B).[3] To manufacture one unit of their goods, the two firms must use one unit of their capacities. The laws of motion of these capacities are given by the state equations

$$x_{t+1} = \mu^A x_t + u_t \tag{1}$$

$$y_{t+1} = \mu^B y_t + v_t, \tag{2}$$

where $u_t, v_t \in R$ are the (non-contractible) investment levels of players A and B, respectively, and $(1 - \mu^A)$ and $(1 - \mu^B)$ are the corresponding depreciation rates. The firms choose u_t, v_t simultaneously and non-cooperatively. They also incur convex investment costs, $C^A(u_t)$ and $C^B(v_t)$. For simplicity we assume that there are no other costs involved in the manufacturing of x and y.

The available technology implies a relationship between input and output quantities that is represented by a production function:

$$x_{t+1} = F(y_{t+1}, x_t, y_t). \tag{3}$$

This dynamic specification accounts for inherently intertemporal phenomena such as congestion, learning-by-doing, etc. The technology constraint imposed by the production function is only required to hold *in equilibrium*. Short-run violations of (3) would not cause discontinuities in the players' payoffs. In particular, we assume that:

- firm A is contractually obligated to purchase the entire production of firm B at the current terms of trade (i.e., firm B can sell y_t units at the current terms of trade even if $x_t < F(y_t, x_{t-1}, y_{t-1})$);

[3] For example, consider a thermal power station (the final good producer) which purchases coal from a nearby coal mine (the intermediate good producer). This power station produces output (i.e. electricity) that is technologically constrained by the available supply of coal, and may as well be the single most important customer of the coal mine. Other examples of such relationships were alluded to in the Introduction.

- firm A has limited reserves of the intermediate good, and thus it can operate in the event of a temporary shortage of y_t (i.e., firm A can produce and sell x_t units even if $x_t > F(y_t, x_{t-1}, y_{t-1})$).

Consequently, both producers can fully exploit their capacities although (3) may not be satisfied in the current period. It should be pointed out that it is in the players' private interest to adhere to this constraint, as any unilateral deviation from the equilibrium will be suboptimal. Even if a violation occurs, it will not persist through time: when firms implement their equilibrium strategies in the following period, (3) will hold again regardless of previous investment decisions.

2.3 Revenue Sharing

For reasons explained in the introduction we study long-term contractual relationships with a time-invariant structure. In our model, this structure is characterized by two elements: (i) an observable and contractible variable z which is agreed on by the firms in a pre-play period, and (ii) a differentiable allocation function g which represents endogenously determined terms of trade.

Our analysis focuses on arrangements where the contractible variable z_t is a function of current input and output levels (capacities) i.e., $z_t \equiv z(x_t, y_t)$. Thus, we assume away trading in futures. Furthermore, we restrict attention to allocation functions that depend only on the current industry state i.e., $g \equiv g(x_t, y_t)$. This assumption implies that firms take the current terms of trade as given, but their investment choices will reflect the desire to affect future surplus allocations. Note that we do not impose restrictions on the functional form of g (other than differentiability): the terms of trade are pinned down by our solution concept.

Let $R(x_t)$ denote the period-t bilateral monopoly revenue generated from the sale of the final good, and let $S^A(z(x_t, y_t), g(x_t, y_t))$ and $S^B(z(x_t, y_t), g(x_t, y_t))$ be the revenue shares of firm A and firm B. The instantaneous period-t payoffs of the two players are defined as

$$\pi_t^A \equiv S^A\big(z(x_t, y_t), g(x_t, y_t)\big) - C^A(u_t) \tag{4}$$

and

$$\pi_t^B \equiv S^B\big(z(x_t, y_t), g(x_t, y_t)\big) - C^B(v_t). \tag{5}$$

Contracts are *allocatively feasible* if

$$S^A\big(z(x_t, y_t), g(x_t, y_t)\big) + S^B\big(z(x_t, y_t), g(x_t, y_t)\big) = R(x_t), \quad \forall x_t, y_t. \tag{6}$$

In order to account for technological interdependence, we also require that surplus allocation induces forward-looking firms to behave in a manner consistent with the production function. Thus, contracts are *technologically feasible* if the terms of trade function $g(x_t, y_t)$ gives rise to equilibrium investment strategies that satisfy (3).

As discussed earlier, we examine two types of revenue-sharing arrangements, distinguished by the specification of the contractible variable.

- Contracting over input quantities: $z_t = y_t$. Under this regime, g can be interpreted as the input price. The instantaneous revenue shares are thus defined as

$$S^A = R(x) - g(x, y)y, \qquad S^B(x, y) = g(x, y)y, \tag{7}$$

where $g(x_t, y_t) : \mathcal{R}_+^2 \to (0, R(x)/y)$.
- Contracting over the realized revenue: $z_t = R(x_t)$. In this case, g can be interpreted as firm B's share of the final revenue. The players' revenue shares are now given by

$$S^A = \big(1 - g(x, y)\big)R(x), \qquad S^B = g(x, y)R(x), \tag{8}$$

where $g(x_t, y_t) : \mathcal{R}_+^2 \to (0, 1)$.

2.4 A Solution Concept

A plausible solution concept for the bilateral monopoly problem at hand needs to allow for strategic behavior of forward-looking players, while also accounting for technological interdependence that requires coordination of investment in order to generate surplus. Given an arbitrary allocation function $g(x, y)$, we have a well-defined two-player dynamic game. We will refer to this game as Γ^g. By assumption, (i) firm A is obligated to purchase the entire production of firm B; and (ii) the final good producer has sufficient reserves to cover temporary input shortages. Therefore, both players can choose any positive or negative investment levels while maintaining full capacity utilization in the short run. Even when the technological constraint is violated in the current period, payoffs would still be given by (4) and (5).

We focus on the Markov perfect equilibrium (MPE) of Γ^g, where the players' strategies are time-invariant functions of the current industry state.[4] Let

$$u_t = f^A(x_t, y_t), \qquad v_t = f^B(x_t, y_t)$$

be the MPE strategies of firm A and firm B when the allocation function is $g(x, y)$. Payoff maximization requires that the players' choices satisfy the Bellman equations for the final good producer,

$$\begin{aligned} V^A(x_t, y_t) = \max_{u_t} \big\{ & R(x_t) - S^B\big(z(x_t, y_t), g(x_t, y_t)\big) - C^A(u_t) \\ & + \delta V^A\big(\mu^A x_t + u_t, \mu^B y_t + f^B(x_t, y_t)\big)\big\}, \end{aligned} \tag{9}$$

[4]This solution concept is also known as feedback-Nash equilibrium.

and for the intermediate good producer,

$$V^B(x_t, y_t) = \max_{v_t}\big\{S^B\big(z(x_t, y_t), g(x_t, y_t)\big) - C^B(u_t) \\ + \delta V^B\big(\mu^A x_t + f^A(x_t, y_t), \mu^B y_t + v_t\big)\big\}. \tag{10}$$

Stationarity of Markovian strategies implies that

$$f^A(x_t, y_t) = \arg\max_{u_t}\big\{S^B\big(z(x_t, y_t), g(x_t, y_t)\big) - C^B(u_t) \\ + \delta V^A\big(\mu^A x_t + u_t, \mu^B y_t + f^B(x_t, y_t)\big)\big\}, \tag{11}$$

and

$$f^B(x_t, y_t) = \arg\max_{v_t}\big\{S^B\big(z(x_t, y_t), g(x_t, y_t)\big) - C^B(u_t) \\ + \delta V^B\big(\mu^A x_t + f^A(x_t, y_t), \mu^B y_t + v_t\big)\big\}. \tag{12}$$

As usual, the MPE strategy functions $f^A(x, y)$, $f^B(x, y)$ of the game Γ^g are a fixed point of the mapping defined by (11), (12).

Remark 1 The Bellman equations (9), (10) will have a well-defined interior solution only if the instantaneous payoffs π_t^A, π_t^B can ensure the concavity of their right-hand sides.[5]

The MPE of the game Γ^g yields investment choices that maximize the players' private payoffs for an arbitrary allocation function g. However, the equilibrium strategies are also bound by the constraint of the existing production technology. Thus, we need to focus on contracts that are technologically feasible. In other words, the terms of trade should give rise to MPE investment levels which are consistent with the production function (3) for *all* possible states (x_t, y_t):

$$\mu^A x_t + f^A(x_t, y_t) = F\big(\mu^B y_t + f^B(x_t, y_t), x_t, y_t\big), \quad \forall x_t, y_t. \tag{13}$$

Definition 1 For a pre-agreed contractible variable $z(x_t, y_t)$, a Markovian allocation equilibrium of the bilateral exchange game described above is characterized by an investment strategy $f^A(x, y)$ of the final good producer, an investment strategy $f^B(x, y)$ of the input producer, and an allocation function $g(x, y)$ with the following properties:

1. contingent on $g(x, y)$, the functions $f^A(x, y)$, $f^B(x, y)$ are the MPE strategies of the game Γ^g, i.e. they are obtained as a fixed point of (11), (12);
2. the allocation function $g(x, y)$ is such that $f^A(x, y)$, $f^B(x, y)$ satisfy the technology constraint (13).

[5]For some results on solutions to concave dynamic games see Krawczyk and Tidball (2006).

The solution concept described above has two desirable features.

- The specification of the terms of trade is quite general. The only requirements for the allocation function are differentiability and Markovian structure. Yet, these mild restrictions enable us to pin down the functional form of g. As a result, the number of possible equilibria is reduced, which boosts the predictive power of our model.
- The investment choices $f^A(x,y)$, $f^B(x,y)$ are the Markov perfect (and therefore subgame-perfect) equilibrium strategies of a non-cooperative dynamic game. Firms will choose to follow these strategies in all periods and for all states. This implies that the agreement between the input supplier and the final good producer is self-enforcing: no player will have an incentive to unilaterally breach the contract at any point in the game regardless of past history.

The above features of the proposed solution concept may lead to inefficient outcomes. In particular, the equilibrium plans may fail to maximize the joint surplus available for allocation between the firms. We will discuss how strategic considerations will distort the players' incentives in Sect. 3.2.

3 The Analysis

In this section, we analyze the bilateral monopoly game defined above. To simplify the notation, we will suppress the arguments of payoffs and strategies. In addition, we will use subscripts to denote partial derivatives. For example, φ_i would signify the derivative of a function $\varphi(r_1,\ldots,r_n)$ with respect to its i-th argument: $\varphi_i = \partial\varphi(r_1,\ldots,r_n)/\partial r_i$.[6] Finally, let φ' and φ'' be the values of φ one and two periods ahead, respectively.

3.1 Characterization of the Equilibrium

When firms choose investment levels, they take into account the direct and strategic effects of their decisions for current and future costs and revenues. The considerations that influence these choices are spelled out by Proposition 1. In this proposition, we use Bellman equations (9) and (10) to derive conditions for the players' equilibrium strategies. The derivations are provided in Appendix A.

Proposition 1 *The Markovian equilibrium strategies $f^A(x,y)$, $f^B(x,y)$ and the allocation function $g(x,y)$ of the bilateral monopoly game satisfy the private Euler*

[6]Since our objective is to derive necessary conditions for the equilibrium strategies and allocation function, we do not need to compute second order derivatives. For a brief discussion on concavity see Remark 1 and footnote 5 on page 167.

equations,

$$-C_1^A + \mu^A \delta C_1^{A\prime} + \delta R_1^{A\prime} + \delta^2 f_1^{B\prime} R_2^{A\prime\prime}$$
$$- \frac{\delta f_1^{B\prime}(\mu^B + f_2^{B\prime\prime})}{f_1^{B\prime\prime}} \big(-C_1^{A\prime} + \mu^A \delta C_1^{A\prime\prime} + \delta R_1^{A\prime\prime}\big) = 0, \tag{14}$$

$$-C_1^B + \mu^B \delta C_1^{B\prime} + \delta R_2^{B\prime} + \delta^2 f_2^{A\prime} R_1^{B\prime\prime}$$
$$- \frac{\delta f_2^{A\prime}(\mu^A + f_1^{A\prime\prime})}{f_2^{A\prime\prime}} \big(-C_1^{B\prime} + \mu^B \delta C_1^{B\prime\prime} + \delta R_2^{B\prime\prime}\big) = 0, \tag{15}$$

and the technological feasibility condition,

$$\mu^A x + f^A(x, y) = F\big(\mu^B y + f^B(x, y), x, y\big), \quad \forall x, y, \tag{16}$$

where $R^A(x, y) \equiv S^A(z(x_t, y_t), g(x_t, y_t))$ *and* $R^B(x, y) \equiv S^B(z(x_t, y_t), g(x_t, y_t))$ *are the revenues of firm A and firm B*, *respectively.*

When players contract over input quantities, the firms' marginal revenues R_1^A, R_1^B, R_2^A, R_2^B take the form

$$R_1^A = R_1 - g_1 y, \qquad R_2^A = -g_2 y - g$$
$$R_1^B = g_1 y, \qquad R_2^B = g_2 y + g.$$

If, instead, the arrangement specifies shares of final revenues, R_1^A, R_1^B, R_2^A, R_2^B become

$$R_1^A = R_1 - g_1 x - g, \qquad R_2^A = -g_2 x$$
$$R_1^B = g_1 x + g, \qquad R_2^B = g_2 x.$$

The Euler equations reflect the economic factors underlying the players' decision making process. In the following description of these factors, we focus on firm A's condition (14). The intuition behind Euler equation (15) is analogous.

The left-hand side of (14) incorporates the effects of a marginal change in firm A's current investment on the costs and revenues over its planning horizon. In equilibrium, the change in costs must be equal to the change in revenues. Thus, these effects will sum up to 0. Their interpretation is provided below.

- An increase in current investment generates an additional adjustment cost of C_1^A in the current period. However, capacity carry-over will create cost savings $\mu^A \delta C_1^{A\prime}$ by reducing the need for future investment.
- The additional capacity has a delayed direct effect of $\delta R_1^{A\prime}$ on future revenues through two channels: the contractible variable, $S_1^{A\prime} z_1'$, and the terms of trade, $S_2^{A\prime} g_1'$.

- Furthermore, a marginal change in firm A's investment will invoke a reaction from its opponent in the subsequent period, which will have repercussions for firm B's future capacity, y'', and induce a strategic revenue effect, $\delta^2 f_1^{B\prime} R_2^{A\prime\prime}$.
- Finally, firm A anticipates B's reaction, and will concurrently re-adjust its capacity. This gives rise to additional strategic cost effects $\delta f_1^{B\prime}(\mu^B + f_2^{B\prime\prime})(C_1^{A\prime} - \mu^A \delta C_1^{A\prime\prime})/f_2^{B\prime\prime}$ and delayed revenue effects $-\delta f_1^{B\prime}(\mu^B + f_2^{B\prime\prime}) R_1^{A\prime\prime}/f_2^{B\prime\prime}$.

Differentiating the technology constraint with respect to x and y yields

$$\mu^A + f_1^A = F_1 f_1^B + F_2,$$
$$f_2^A = F_1\big(f_2^B + \mu^B\big) + F_3.$$

Substitution in (15) shows that technologically feasible input choices must also satisfy

$$-C_1^B + \mu^B \delta C_1^{B\prime} + \delta R_2^{B\prime} + \delta^2\big(F_1'\big(f_2^{B'} + \mu^B\big) + F_3'\big) \times \left(R_1^{B\prime\prime} - \frac{\delta(F_1'' f_1^{B\prime\prime} + F_2')}{F_1''(f_2^{B\prime\prime} + \mu^B) + F_3''}\big(-C_1^{B\prime} + \mu^B \delta C_1^{B\prime\prime} + \delta R_2^{B\prime\prime}\big)\right) = 0. \quad (17)$$

3.2 Efficiency Results

In this subsection we compare the Markovian allocation equilibrium characterized in Sect. 3.1 to an efficient outcome. For presentation purposes, we define efficiency as maximization of the net present value of the stream of joint surplus. The question of interest is whether the strategies defined by (14), (15) and (16) are consistent with this type of efficient behavior.

Let $w_t \equiv R(x_t) - C^A(u_t) - C^B(v_t)$ denote the period-t joint surplus generated by the bilateral monopoly.

Definition 2 A bilateral trade contract (z, g) is efficient if it maximizes the net present value of the stream of joint surplus $W = \sum_{t=1}^{\infty} \delta^{t-1} w_t$.

Now we show that the equilibrium input and output paths generated by (14), (15) and (16) are usually inefficient. Although firms fully utilize the available capacities in each period, their decisions are distorted by strategic considerations regarding the allocation of future revenues. In the game studied here, the Markovian structure of contracts implies that current investments have repercussions for the subsequent values of the terms of trade and the contractible variable. Forward-looking players take these repercussions into account. They choose their actions to bias the division of future surplus in their favor, thus creating inefficiencies. This phenomenon is analogous to the problem of "hold-up," which often arises in the literature on incomplete contracts (see Grossman and Hart 1986; Klein et al. 1978).

To illuminate the underlying reasons, we now derive necessary conditions for the efficient investment policies of firm A and firm B, $h^A(x,y)$ and $h^B(x,y)$. We will argue that the private Euler equations (14), (15) are generically inconsistent with these efficiency conditions.

Technological feasibility requires that

$$u_t = F\big(\mu^B y_t + v_t, x_t, y_t\big) - \mu^A x_t. \tag{18}$$

Substitution of (18) allows us to write the period-t joint surplus as

$$w(x_t, y_t) = R(x_t) - C^A\big(F\big(\mu^B y_t + v_t, x_t, y_t\big) - \mu^A x_t\big) - C^B(v_t). \tag{19}$$

Therefore, efficient investment in input capacity will solve the Bellman equation

$$\begin{aligned} W(x_t, y_t) = \max_{v_t}\big\{&R(x_t) - C^A\big(F\big(\mu^B y_t + v_t, x_t, y_t\big) - \mu^A x_t\big) \\ &- C^B(v_t) + \delta W\big(F\big(\mu^B y_t + v_t, x_t, y_t\big), \mu^B y_t + v_t\big)\big\}. \end{aligned} \tag{20}$$

In Appendix B, we use (20) to derive the following Euler equation:

$$\begin{aligned} &F_1\big(\delta R_1' - C_1^A + \delta\mu^A C_1^{A\prime}\big) - C_1^B + \delta\mu^B C_1^{B\prime} \\ &\quad + \delta\big(F_1 F_2' + F_3'\big)\big(\delta R_1'' - C_1^{A\prime} + \delta\mu^A C_1^{A\prime\prime}\big) \\ &\quad - \frac{\delta F_2''(F_1F_2' + F_3')}{(F_1'F_2'' + F_3'')}\big(F_1'\big(\delta R_1'' - C_1^{A\prime} + \delta\mu^A C_1^{A\prime\prime}\big) - C_1^{B\prime} + \delta\mu^B C_1^{B\prime\prime}\big) \\ &= 0. \end{aligned} \tag{21}$$

Condition (18) in conjunction with (21) defines the efficient investment policies $h^A(x,y)$ and $h^B(x,y)$.

In general, the equilibrium strategy functions that solve Euler equations (14) and (15) will fail to satisfy (21). Hence, the contracts considered here are typically inefficient. As already discussed, this failure is driven by the strategic nature of interactions. More precisely, both players will attempt to influence the future contractible variable and terms of trade in order to increase their payoffs. Such behavior precludes the implementation of efficiency: firms will deviate from the investment policies that maximize joint surplus.

To see this, suppose that the efficient investment policies $h^A(x,y)$ and $h^B(x,y)$ are in fact solutions to (14) and (15). Then we can rewrite the private Euler equations as

$$\delta R_1' - C_1^A + \delta\mu^A C_1^{A\prime} = D^A, \qquad -C_1^B + \delta\mu^B C_1^{B\prime} = D^B,$$

where D^A and D^B embody the strategic payoff effects:

$$\begin{aligned} D^A = {}&\delta\big(S_1^{B\prime} z_1' + S_2^{B\prime} g_1'\big) - \delta^2 h_1^{B\prime} R_2^{A\prime\prime} \\ &+ \frac{\delta h_1^{B\prime}(\mu^B + h_2^{B\prime\prime})}{h_1^{B\prime\prime}}\big(-C_1^{A\prime} + \mu^A\delta C_1^{A\prime\prime} + \delta R_1^{A\prime\prime}\big) \end{aligned} \tag{22}$$

$$D^B = -\delta\big(S_1^{B\prime} z_2' + S_2^{B\prime} g_2'\big) - \delta^2 h_2^{A\prime} R_1^{B\prime\prime} + \frac{\delta h_2^{A\prime}(\mu^A + h_1^{A\prime\prime})}{h_2^{A\prime\prime}}\big(-C_1^{B\prime} + \mu^B \delta C_1^{B\prime\prime} + \delta R_2^{B\prime\prime}\big). \tag{23}$$

Note that (22) and (23) are generically non-zero so long as the players follow state-contingent investment rules. Substitution in (21) shows that if the equilibrium was efficient, it would necessarily imply the following condition:

$$F_1 D^A + D^B + \delta\big(F_1 F_2' + F_3'\big) D^{A\prime} - \frac{\delta F_2''(F_1 F_2' + F_3')}{(F_1' F_2'' + F_3'')}\big(F_1' D^{A\prime} + D^{B\prime}\big) = 0. \tag{24}$$

However, if $D^A \neq 0$ and $D^B \neq 0$, (24) will typically fail to hold.

The features of our model suggest that these inefficiencies would arise for most allocation mechanisms. Nevertheless, some contracts might generate a smaller dead-weight loss than others. We address this issue in Sect. 4, where we compute the Markovian allocation equilibrium in a linear-quadratic example. It allows us to compare the welfare properties of the different contractual arrangements.

4 A Linear-Quadratic Formulation

In this section, we consider a linear-quadratic formulation of the bilateral monopoly game defined above. As expected, it yields a computationally tractable equilibrium with linear strategies and a linear allocation function. We use numerical simulations to explore how the contractual arrangements affect the size and the allocation of joint surplus.

4.1 Payoffs

We believe that quadratic investment costs can adequately capture the observation that marginal costs are often positively related to the magnitude of capacity adjustments. In conjunction with the assumptions of linear final revenue and production function, this specification delivers a computable equilibrium characterized by a linear allocation function and investment strategies.

Specifically, suppose that the investment costs incurred by firm A and firm B are given by

$$C^A(u) = \frac{\psi^A}{2} u^2, \qquad C^B(v) = \frac{\psi^B}{2} v^2. \tag{25}$$

Furthermore, assume a perfectly elastic demand for the final good, which translates into a linear revenue function

$$R(x) = px. \tag{26}$$

The final assumption concerns the technology constraint. Suppose that it has the form

$$x_{t+1} = a + by_{t+1} + dx_t + ey_t. \tag{27}$$

That is, output is produced according to a linear production function.

The above structure motivates the conjecture that the equilibrium strategies are linear in the state variables:

$$u_t = \alpha^A + \beta_1^A x_t + \beta_2^A y_t \tag{28}$$

$$v_t = \alpha^B + \beta_1^B x_t + \beta_2^B y_t. \tag{29}$$

Moreover, we guess a linear allocation function:

$$g(x, y) = \eta + \theta_1 x + \theta_2 y. \tag{30}$$

These conjectures, together with (25) and (26), suggest that contracting over input quantities would generate instantaneous payoffs

$$\begin{aligned} \pi_t^A &= px_t - (\eta + \theta_1 x_t + \theta_2 y_t) y_t - \frac{\psi^A}{2} u_t^2, \\ \pi_t^B &= (\eta + \theta_1 x_t + \theta_2 y_t) y_t - \frac{\psi^B}{2} v_t^2, \end{aligned} \tag{31}$$

while contracting over final revenues would yield

$$\begin{aligned} \pi_t^A &= px_t - (\eta + \theta_1 x_t + \theta_2 y_t) px_t - \frac{\psi^A}{2} u_t^2, \\ \pi_t^B &= (\eta + \theta_1 x_t + \theta_2 y_t) px_t - \frac{\psi^B}{2} v_t^2. \end{aligned} \tag{32}$$

4.2 Existence

Non-negative input and output paths and an allocation profile $\{x_t, y_t, g_t\}_{t=0}^{\infty}$ constitute an equilibrium if the investment strategies and the supporting allocation function satisfy the players' necessary conditions (14), (15), as well as the technology constraint (16). Furthermore, firms will engage in bilateral trade only if it is mutually beneficial. Hence, the equilibrium path must be such that

$$V^A(x_t, y_t) \geq 0, \qquad V^B(x_t, y_t) \geq 0, \quad t = 0, 1, \ldots$$

Finally, we would like our solutions to be dynamically stable. This requirement suggests that the eigenvalues of the capacity transition matrix

$$\begin{pmatrix} \mu^A + \beta_1^A & \beta_2^A \\ \beta_1^B & \mu^B + \beta_2^B \end{pmatrix}$$

should be inside the unit circle.

An equilibrium with these properties may not always exist in the above linear-quadratic setting. The issue of non-existence is particularly serious when contracts are based on final revenues. For some parameter values, the players' problems may not have interior solutions. For example, suppose that firm B's investment is very productive (i.e., b is high) or that it is rather durable (i.e., μ^B is high). Then the linear technology constraint (27) might never be satisfied so long as the allocation function features a positive θ_2. On the other hand, a negative θ_2 would imply that the surplus share of the intermediate good producer, $S_t^B = (\eta + \theta_1 x_t + \theta_2 y_t) p x_t$, is decreasing in his capacity y_t. But then firm B's marginal return on investment is negative, and so Euler equation (36) will not have a solution. Moreover, existence of equilibrium requires concavity of the right hand sides of the private Bellman equations, which may be lost for some parameter values (e.g., when the output price p is low).

4.3 Simulations

4.3.1 Numerical Results

To compute the Markovian allocation equilibrium, we substitute the expressions for payoffs (31), (32) and conjectures (28), (29), (30) in equations (14), (15) and (16).

- When firms contract over input quantities, the private Euler equation (14) and (15) become

$$-\psi^A u_t + \mu^A \delta u_{t+1} + \delta(p - \theta_1 y_{t+1}) - \delta^2 \beta_1^B (\eta + \theta_1 x_{t+2} + 2\theta_2 y_{t+2}) \\ - \delta\big(\mu^B + \beta_2^B\big)\big(-u_{t+1} + \mu^A \delta u_{t+2} + \delta(p - \theta_1 y_{t+2})\big) = 0 \tag{33}$$

  and

$$-\psi^B v_t + \mu^B \delta \psi^B v_{t+1} + \delta(\eta + \theta_1 x_{t+1} + 2\theta_2 y_{t+1}) + \delta^2 \beta_2^A \theta_1 y_{t+2} \\ - \delta\big(\mu^A + \beta_1^A\big)\big(-\psi^B v_{t+1} + \mu^B \delta \psi^B v_{t+2} + \delta(\eta + \theta_1 x_{t+2} + 2\theta_2 y_{t+2})\big) = 0. \tag{34}$$

- If, instead, contracts are based on the final revenues, (14) and (15) are given by

$$-\psi^A u_t + \mu^A \delta \psi^A u_{t+1} + \delta p(1 - \eta - 2\theta_1 x_{t+1} - \theta_2 y_{t+1}) - \delta^2 \beta_1^B \theta_2 p x_{t+2} \\ - \delta\big(\mu^B + \beta_2^B\big) \\ \times \big(-\psi^A u_{t+1} + \mu^A \delta \psi^A u_{t+2} + \delta p(1 - \eta - 2\theta_1 x_{t+2} - \theta_2 y_{t+2})\big) \\ = 0 \tag{35}$$

Table 1 The base case parameter set

Firm A	Firm B	Technology	Other
$\mu^A = 0.8$	$\mu^B = 0.5$	$a = 30, b = 0.35$	$p = 30$
$\psi^A = 0.4$	$\psi^B = 0.4$	$d = 0.2, e = -0.25$	$\delta = 0.9$

and

$$-\psi^B v_t + \mu^B \delta \psi^B v_{t+1} + \delta\theta_2 p x_{t+1} + \delta^2 \beta_2^A p(\eta + 2\theta_1 x_{t+2} + \theta_2 y_{t+2})$$
$$- \delta\big(\mu^A + \beta_1^A\big)\big(-\psi^B v_{t+1} + \mu^B \delta \psi^B v_{t+2} + \delta\theta_2 p x_{t+2}\big) = 0. \quad (36)$$

- Under both arrangements, technological feasibility requires that the equilibrium strategies satisfy the constraint (16) for all possible states:

$$\mu^A x_t + \alpha^A + \beta_1^A x_t + \beta_2^A y_t = a + b\big(\mu^B y_t + \alpha^B + \beta_1^B x_t + \beta_2^B y_t\big) + dx_t + ey_t. \quad (37)$$

Applying the method of undetermined coefficients to (33)–(37) yields nine equations that pin down the nine unknown variables needed to fully describe the equilibrium in each of the two regimes. These variables are the MPE strategy parameters α^A, β_1^A, β_2^A and α^B, β_1^B, β_2^B, as well as the parameters of the allocation function, η, θ_1, θ_2.

We can now compute a baseline scenario with parameters as listed in Table 1. Specifically, we assume a negative value of the parameter e in (27) to reflect input congestion and capacity deterioration caused by previous heavy workloads; the positive sign of d can be attributed to learning-by-doing effects.

The corresponding Markovian allocation equilibria are presented in Table 2 for contracts based on input quantities, and in Table 3—for contracts based on final revenues. The first rows in these tables show the results for our baseline scenario. Then we vary one parameter at a time (denoted by bold font in the first column of each table) and obtain the rest of the table entries.

Figure 1 and Fig. 2 depict the equilibrium transition paths for contracting over input quantities and contracting over final revenues, respectively, as well as the efficient input and output paths. Figure 3 illustrates the evolution of the terms of trade under the two regimes. The initial conditions are set at $y_0 = 10$, $x_0 = 10$. Then Fig. 4 shows a number of transition paths for various values of y_0 and x_0. The input and output paths each converge to an identical (technologically feasible) steady state, as would be expected in a Markov perfect equilibrium.

4.3.2 Equilibrium Investment Strategies

Each player's investment choice will reflect his direct and strategic considerations regarding current and future profits. These considerations are influenced by the terms of trade function, which is constructed so that the technology constraint holds

Table 2 *Contracting over input quantities*. The table shows the equilibrium strategies and allocation function, the steady-state output and input quantities, total surplus, value of the allocation function and the surplus share of firm B

	Strategy of firm A	Strategy of firm B	Allocation function	Steady state
$\mu^A = 0.8$, $\mu^B = 0.5$ $\psi^A = 0.4$, $\psi^B = 0.4$ $p = 30$, $\delta = 0.95$ $a = 30$, $b = 0.35$ $d = 0.2$, $e = -0.25$	$\alpha^A = 45.86$ $\beta_1^A = -0.541$ $\beta_2^A = -0.110$	$\alpha^B = 45.30$ $\beta_1^B = 0.167$ $\beta_2^B = -0.010$	$\eta = -0.775$ $\theta_1 = 0.008$ $\theta_2 = 0.009$	$\hat{x} = 48.63$ $\hat{y} = 89.06$ $\hat{w} = 1043$ $\hat{g} = 0.448$ $\hat{s}^B = 24.6$ %
$\mu^A = 0.8$, $\boldsymbol{\mu^B = 0.55}$ $\psi^A = 0.4$, $\psi^B = 0.4$ $p = 30$, $\delta = 0.95$ $a = 30$, $b = 0.35$ $d = 0.2$, $e = -0.25$	$\alpha^A = 39.04$ $\beta_1^A = -0.565$ $\beta_2^A = -0.074$	$\alpha^B = 25.83$ $\beta_1^B = 0.101$ $\beta_2^B = -0.048$	$\eta = -554$ $\theta_1 = 0.012$ $\theta_2 = 0.006$	$\hat{x} = 45.12$ $\hat{y} = 60.93$ $\hat{w} = 1187$ $\hat{g} = 0.364$ $\hat{s}^B = 28.8$ %
$\mu^A = 0.8$, $\mu^B = 0.5$ $\psi^A = 0.4$, $\boldsymbol{\psi^B = 0.45}$ $p = 30$, $\delta = 0.95$ $a = 30$, $b = 0.35$ $d = 0.2$, $e = -0.25$	$\alpha^A = 42.43$ $\beta_1^A = -0.551$ $\beta_2^A = -0.103$	$\alpha^B = 35.40$ $\beta_1^B = 0.139$ $\beta_2^B = -0.081$	$\eta = -0.666$ $\theta_1 = 0.010$ $\theta_2 = 0.009$	$\hat{x} = 46.52$ $\hat{y} = 72.20$ $\hat{w} = 1085$ $\hat{g} = 0.426$ $\hat{s}^B = 27.8$ %
$\boldsymbol{\mu^A = 0.85}$, $\mu^B = 0.5$ $\psi^A = 0.4$, $\psi^B = 0.4$ $p = 30$, $\delta = 0.95$ $a = 30$, $b = 0.35$ $d = 0.2$, $e = -0.25$	$\alpha^A = 48.85$ $\beta_1^A = -0.585$ $\beta_2^A = -0.114$	$\alpha^B = 53.85$ $\beta_1^B = 0.185$ $\beta_2^B = -0.112$	$\eta = -1.007$ $\theta_1 = 0.008$ $\theta_2 = 0.010$	$\hat{x} = 50.40$ $\hat{y} = 103.23$ $\hat{w} = 968$ $\hat{g} = 0.431$ $\hat{s}^B = 12.2$ %
$\mu^A = 0.8$, $\mu^B = 0.5$ $\boldsymbol{\psi^A = 0.42}$, $\psi^B = 0.4$ $p = 30$, $\delta = 0.95$ $a = 30$, $b = 0.35$ $d = 0.2$, $e = -0.25$	$\alpha^A = 48.42$ $\beta_1^A = -0.536$ $\beta_2^A = -0.113$	$\alpha^B = 52.62$ $\beta_1^B = 0.181$ $\beta_2^B = -0.110$	$\eta = -0.969$ $\theta_1 = 0.008$ $\theta_2 = 0.010$	$\hat{x} = 50.15$ $\hat{y} = 101.22$ $\hat{w} = 971$ $\hat{g} = 0.427$ $\hat{s}^B = 13.4$ %
$\mu^A = 0.8$, $\mu^B = 0.5$ $\psi^A = 0.4$, $\psi^B = 0.4$ $\boldsymbol{p = 35}$, $\delta = 0.95$ $a = 30$, $b = 0.35$ $d = 0.2$, $e = -0.25$	$\alpha^A = 43.97$ $\beta_1^A = -0.542$ $\beta_2^A = -0.110$	$\alpha^B = 39.92$ $\beta_1^B = 0.167$ $\beta_2^B = -0.010$	$\eta = -0.429$ $\theta_1 = 0.007$ $\theta_2 = 0.008$	$\hat{x} = 47.47$ $\hat{y} = 79.77$ $\hat{w} = 1325$ $\hat{g} = 0.538$ $\hat{s}^B = 43.4$ %
$\mu^A = 0.8$, $\mu^B = 0.5$ $\psi^A = 0.4$, $\psi^B = 0.4$ $p = 30$, $\delta = 0.95$ $a = 30$, $\boldsymbol{b = 0.32}$ $d = 0.2$, $e = -0.25$	$\alpha^A = 50.53$ $\beta_1^A = -0.530$ $\beta_2^A = -0.141$	$\alpha^B = 64.17$ $\beta_1^B = 0.218$ $\beta_2^B = -0.159$	$\eta = -1.053$ $\theta_1 = 0.005$ $\theta_2 = 0.012$	$\hat{x} = 47.39$ $\hat{y} = 113.04$ $\hat{w} = 764$ $\hat{g} = 0.540$ $\hat{s}^B = 16.8$ %
$\mu^A = 0.8$, $\mu^B = 0.5$ $\psi^A = 0.4$, $\psi^B = 0.4$ $p = 30$, $\delta = 0.95$ $\boldsymbol{a = 25}$, $b = 0.35$ $d = 0.2$, $e = -0.25$	$\alpha^A = 36.33$ $\beta_1^A = -0.542$ $\beta_2^A = -0.110$	$\alpha^B = 32.37$ $\beta_1^B = 0.167$ $\beta_2^B = -0.010$	$\eta = -0.371$ $\theta_1 = 0.008$ $\theta_2 = 0.009$	$\hat{x} = 39.37$ $\hat{y} = 64.92$ $\hat{w} = 957$ $\hat{g} = 0.553$ $\hat{s}^B = 46.2$ %

Table 3 *Contracting over final revenue shares*. The table shows the equilibrium strategies, allocation function and the steady-state output and input quantities, total surplus, value of the allocation function and the surplus share of firm B

	Strategy of firm A	Strategy of firm B	Allocation function	Steady state
$\mu^A = 0.8$, $\mu^B = 0.5$ $\psi^A = 0.4$, $\psi^B = 0.4$ $p = 30$, $\delta = 0.95$ $a = 30$, $b = 0.35$ $d = 0.2$, $e = -0.25$	$\alpha^A = 39.52$ $\beta_1^A = -0.500$ $\beta_2^A = -0.178$	$\alpha^B = 27.21$ $\beta_1^B = 0.286$ $\beta_2^B = -0.293$	$\eta = -7.41$ $\theta_1 = 0.504$ $\theta_2 = -0.032$	$\hat{x} = 43.76$ $\hat{y} = 50.04$ $\hat{w} = 1172$ $\hat{g} = 13.07$ $\hat{s}^B = 45.1\ \%$
$\mu^A = 0.8$, $\boldsymbol{\mu^B = 0.55}$ $\psi^A = 0.4$, $\psi^B = 0.4$ $p = 30$, $\delta = 0.95$ $a = 30$, $b = 0.35$ $d = 0.2$, $e = -0.25$	$\alpha^A = 39.04$ $\beta_1^A = -0.506$ $\beta_2^A = -0.182$	$\alpha^B = 25.82$ $\beta_1^B = 0.268$ $\beta_2^B = -0.357$	$\eta = -7.88$ $\theta_1 = 0.558$ $\theta_2 = -0.063$	$\hat{x} = 43.30$ $\hat{y} = 46.36$ $\hat{w} = 1197$ $\hat{g} = 13.35$ $\hat{s}^B = 44.4\ \%$
$\mu^A = 0.8$, $\mu^B = 0.5$ $\psi^A = 0.4$, $\boldsymbol{\psi^B = 0.45}$ $p = 30$, $\delta = 0.95$ $a = 30$, $b = 0.35$ $d = 0.2$, $e = -0.25$	$\alpha^A = 38.74$ $\beta_1^A = -0.508$ $\beta_2^A = -0.183$	$\alpha^B = 24.97$ $\beta_1^B = 0.262$ $\beta_2^B = -0.210$	$\eta = -8.14$ $\theta_1 = 0.570$ $\theta_2 = -0.052$	$\hat{x} = 43.10$ $\hat{y} = 44.82$ $\hat{w} = 1165$ $\hat{g} = 14.08$ $\hat{s}^B = 44.4\ \%$
$\boldsymbol{\mu^A = 0.85}$, $\mu^B = 0.5$ $\psi^A = 0.4$, $\psi^B = 0.4$ $p = 30$, $\delta = 0.95$ $a = 30$, $b = 0.35$ $d = 0.2$, $e = -0.25$	$\alpha^A = 39.20$ $\beta_1^A = -0.545$ $\beta_2^A = -0.176$	$\alpha^B = 26.28$ $\beta_1^B = 0.300$ $\beta_2^B = -0.288$	$\eta = -8.17$ $\theta_1 = 0.507$ $\theta_2 = -0.025$	$\hat{x} = 43.75$ $\hat{y} = 50.00$ $\hat{w} = 1179$ $\hat{g} = 12.76$ $\hat{s}^B = 43.5\ \%$
$\mu^A = 0.8$, $\mu^B = 0.5$ $\boldsymbol{\psi^A = 0.42}$, $\psi^B = 0.4$ $p = 30$, $\delta = 0.95$ $a = 30$, $b = 0.35$ $d = 0.2$, $e = -0.25$	$\alpha^A = 39.19$ $\beta_1^A = -0.497$ $\beta_2^A = -0.175$	$\alpha^B = 26.25$ $\beta_1^B = 0.296$ $\beta_2^B = -0.286$	$\eta = -8.04$ $\theta_1 = 0.503$ $\theta_2 = -0.026$	$\hat{x} = 43.73$ $\hat{y} = 49.82$ $\hat{w} = 1172$ $\hat{g} = 12.66$ $\hat{s}^B = 43.3\ \%$
$\mu^A = 0.8$, $\mu^B = 0.5$ $\psi^A = 0.4$, $\psi^B = 0.4$ $\boldsymbol{p = 35}$, $\delta = 0.95$ $a = 30$, $b = 0.35$ $d = 0.2$, $e = -0.25$	$\alpha^A = 41.79$ $\beta_1^A = -0.500$ $\beta_2^A = -0.178$	$\alpha^B = 33.69$ $\beta_1^B = 0.286$ $\beta_2^B = -0.293$	$\eta = -5.45$ $\theta_1 = 0.504$ $\theta_2 = -0.032$	$\hat{x} = 44.83$ $\hat{y} = 58.61$ $\hat{w} = 1381$ $\hat{g} = 15.30$ $\hat{s}^B = 52.5\ \%$
$\mu^A = 0.8$, $\mu^B = 0.5$ $\psi^A = 0.4$, $\psi^B = 0.4$ $p = 30$, $\delta = 0.95$ $a = 30$, $\boldsymbol{b = 0.32}$ $d = 0.2$, $e = -0.25$	$\alpha^A = 38.70$ $\beta_1^A = -0.507$ $\beta_2^A = -0.182$	$\alpha^B = 27.18$ $\beta_1^B = 0.292$ $\beta_2^B = -0.287$	$\eta = -7.58$ $\theta_1 = 0.502$ $\theta_2 = -0.019$	$\hat{x} = 41.88$ $\hat{y} = 50.06$ $\hat{w} = 1117$ $\hat{g} = 12.50$ $\hat{s}^B = 44.8\ \%$
$\mu^A = 0.8$, $\mu^B = 0.5$ $\psi^A = 0.4$, $\psi^B = 0.4$ $p = 30$, $\delta = 0.95$ $\boldsymbol{a = 25}$, $b = 0.35$ $d = 0.2$, $e = -0.25$	$\alpha^A = 35.20$ $\beta_1^A = -0.500$ $\beta_2^A = -0.178$	$\alpha^B = 29.15$ $\beta_1^B = 0.286$ $\beta_2^B = -0.293$	$\eta = -4.21$ $\theta_1 = 0.504$ $\theta_2 = -0.032$	$\hat{x} = 37.53$ $\hat{y} = 50.26$ $\hat{w} = 988$ $\hat{g} = 13.13$ $\hat{s}^B = 54.0\ \%$

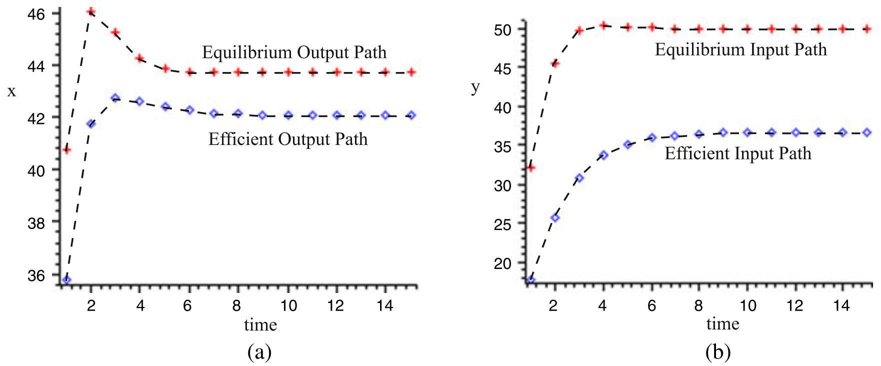

Fig. 1 *Contracting over input quantities* (**a** output paths, **b** input paths). Panel **a** illustrates the equilibrium and efficient output paths. Panel **b** illustrates the equilibrium and efficient input paths. The calculations are based on parameters $p = 30$, $\delta = 0.95$, $\psi^A = 0.4$, $\psi^B = 0.4$, $\mu^A = 0.8$, $\mu^B = 0.5$, $a = 20$, $b = 0.35$, $d = 0.2$, $e = -0.25$ and initial conditions $x_0 = y_0 = 10$

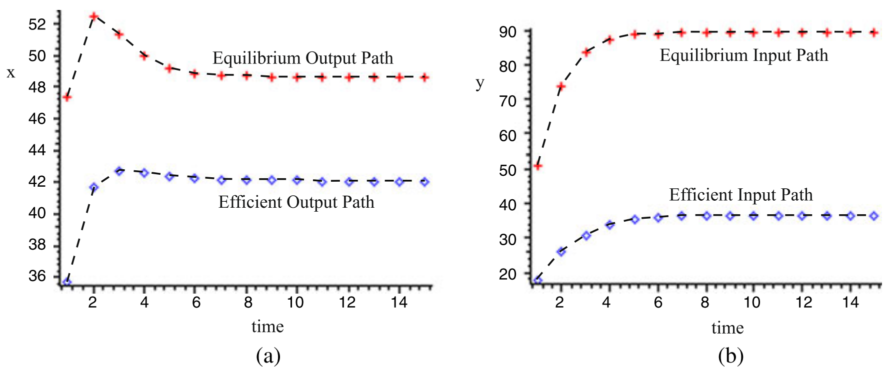

Fig. 2 *Contracting over final revenue shares* (**a** output paths, **b** input paths). Panel **a** illustrates the equilibrium and efficient output paths. Panel **b** illustrates the equilibrium and efficient input paths. The calculations are based on parameters $p = 30$, $\delta = 0.95$, $\psi^A = 0.4$, $\psi^B = 0.4$, $\mu^A = 0.8$, $\mu^B = 0.5$, $a = 20$, $b = 0.35$, $d = 0.2$, $e = -0.25$ and initial conditions $x_0 = y_0 = 10$

for any (x_t, y_t). We focus on strategies that are linear in the state variables.[7] Specifically, they are defined by (28) and (29). The strategy parameters β_1^A, β_2^B, β_2^A, β_1^B capture the effect of the observed output and input capacities x_t, y_t on the players' investment decisions. As Table 2 and Table 3 show, all numerical examples studied here yield $\beta_1^A < 0$, $\beta_2^B < 0$ and $\beta_2^A < 0$, $\beta_1^B > 0$. We contend that the signs of these parameters are in line with economic intuition. Our reasoning is explained below.

[7] As noted earlier, such equilibria may not always exist. Also, there could be equilibria involving non-linear strategies, see e.g. Haurie et al. (2012).

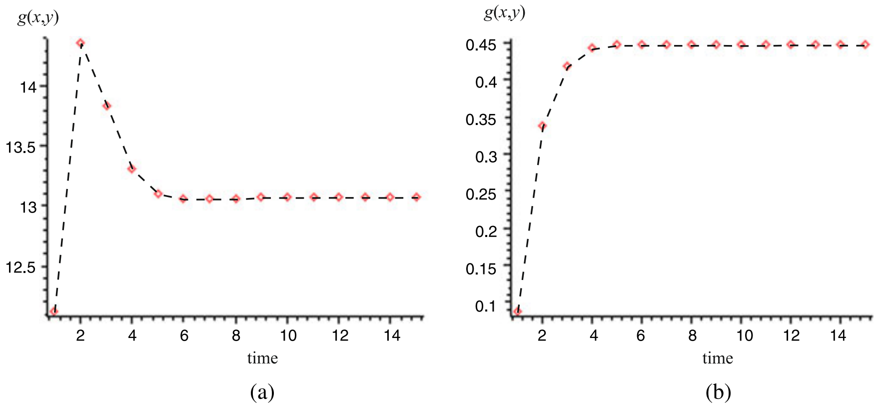

Fig. 3 *Equilibrium terms of trade* (**a** contracting over input quantities, **b** contracting over final revenue). Panel **a** shows the evolution of the equilibrium allocation function with contracting over input quantities. Panel **b** shows the evolution of the equilibrium allocation function with contracting over final revenues. The calculations are based on parameters $p = 30$, $\delta = 0.95$, $\psi^A = 0.4$, $\psi^B = 0.4$, $\mu^A = 0.8$, $\mu^B = 0.5$, $a = 30$, $b = 0.35$, $d = 0.2$, $e = -0.25$ and initial conditions $x_0 = y_0 = 10$

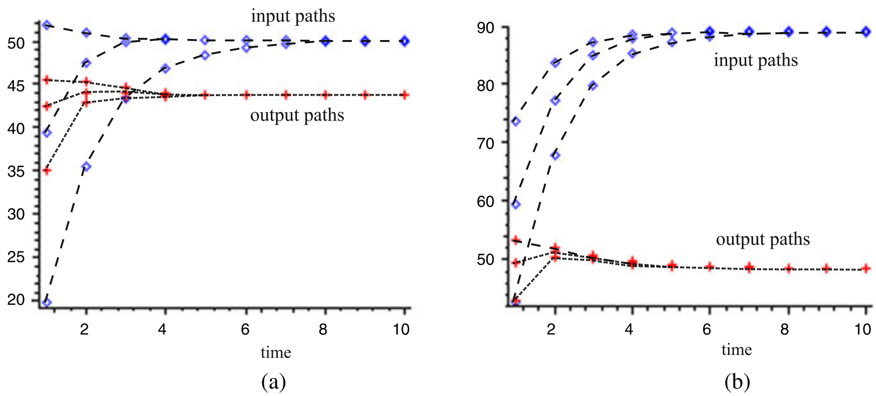

Fig. 4 *Equilibrium paths under alternative initial conditions* (**a** contracting over input quantities, **b** contracting over final revenue). Panel **a** shows the equilibrium input and output paths with contracting over input quantities. Panel **b** shows the equilibrium input and output paths with contracting over final revenues. The calculations are based on parameters $p = 30$, $\delta = 0.95$, $\psi^A = 0.4$, $\psi^B = 0.4$, $\mu^A = 0.8$, $\mu^B = 0.5$, $a = 30$, $b = 0.35$, $d = 0.2$, $e = -0.25$. The respective initial conditions are $x_0 = y_0 = 0$, $x_0 = y_0 = 25$ and $x_0 = y_0 = 50$

- The simulations yield allocation function parameters θ_1 and θ_2 whose signs suggest that marginal payoffs are constant or decreasing in the players' own capacities. However, (25) gives rise to increasing marginal costs of investment. Thus, cost considerations will motivate each firm to cut down on investment if its capacity has gone up, implying that $\beta_1^A < 0$, $\beta_2^B < 0$.

- To satisfy the technology constraint (27) for the assumed values of a, b, d and e, we need to have $\beta_2^A < 0$ and $\beta_1^B > 0$. That is, the investment of firm A should be decreasing in the capacity of the intermediate good producer, while the investment of firm B should be increasing in the capacity of the final good producer. Note that the allocation function (30) must ensure that it is in the players' self interest to behave accordingly. The resulting implications for the parameters θ_1 and θ_2 are discussed in the next subsection.

To assess the efficiency of the contractual arrangements, it may be useful to compare the equilibrium input and output paths to the plans that maximize joint surplus. As shown in Fig. 1 and Fig. 2, both types of contracting will lead to overinvestment relative to the efficient levels in our baseline scenario. However, for other parameter values (e.g., a high b) the equilibrium might involve underinvestment. Whether MPE capacities will exceed or fall short of their efficient levels will depend on the incentives provided by the revenue sharing arrangements. In general, if an incremental change in investment increases a player's private continuation payoff by more than the future joint surplus, then the equilibrium behavior of this player will involve overinvestment.

4.3.3 Equilibrium Allocation Function

Given a linear allocation function (30), the firms' ability to affect the future terms of trade will depend on θ_1 and θ_2. The signs of these parameters determine whether the capacities of the intermediate good producer and the final good producer are strategic complements or substitutes. As already established, the technology constraints in our examples are consistent with linear feedback strategies (28) and (29) whose parameters satisfy $\beta_2^A < 0$ and $\beta_1^B > 0$. Therefore, we would expect to obtain an allocation function such that: (i) firm A's marginal return on investment is decreasing in firm B's output (i.e., $\partial^2\pi_t^A/\partial x_t \partial y_t < 0$), and (ii) firm B's marginal return on investment is increasing in firm A's output (i.e., $\partial^2\pi_t^B/\partial x_t \partial y_t > 0$).

- If the parties contract over input quantities, these complementarity requirements would imply that $\theta_1 > 0$ and $\theta_2 < 0$. That is, the input price should be decreasing in the capacity of the input producer and increasing in the capacity of the final good producer. This appears to be consistent with the standard laws of supply and demand. The numerical results shown in Table 2 confirm this intuition.
- If, on the other hand, contracts are based on final revenues, these complementarity requirements would amount to $\theta_1 > 0$ and $\theta_2 > 0$. In other words, firm B's revenue share should be increasing in both production capacities. This intuition is supported by the results in Table 3.

The transitional dynamics of the terms of trade are illustrated in Fig. 3.

4.3.4 Equilibrium Surplus Allocation

The numerical examples also shed light on the factors that influence the distribution of surplus in bilateral trade. Tables 2 and 3 illustrate the allocative properties of the Markov perfect equilibrium by providing information about the steady-state surplus share of the input producer, $s_t^B = \pi_t^B/(\pi_t^A + \pi_t^B)$.

The simulations suggest that the choice of a contractible variable z plays an important role in payoff allocation. A comparison between the two arrangements shows that the firm which chooses the value of the contractible variable usually attains a higher surplus share. In particular, contracting over input quantities tends to benefit the input producer (high steady-state surplus share $\hat{s}^B$), while contracting over final revenue is more favorable for the final good producer (low steady-state surplus share $\hat{s}^B$). This result is consistent with the observation that in real-world interactions control is often advantageous.

4.3.5 Contract Efficiency

The linear-quadratic formulation also enables us to compare the efficiency of the two regimes as measured by the joint surplus w_t generated by the bilateral exchange. The numerical examples underscore the importance of design and procedure for economic efficiency.

Interestingly, Table 2 and Table 3 show that, in all of the cases studied here, contracting over input quantities yields a higher steady-state joint surplus $\hat{w}$ relative to contracting over final revenue. For some parameter values those welfare differences are rather significant. A brief inspection of Fig. 1 and Fig. 2 shows that contracting over input quantities generates transition paths that are much closer to their efficiency counterparts. On the other hand, contracts based on final revenues seem to cause substantial distortions in the input supply decisions of firm B.

We can use condition (21) to compute the welfare generated by the efficient input and output paths, and compare it to the joint surplus arising in the Markovian allocation equilibrium. Our numerical example shows that contracting over input quantities gives rise to a small deadweight loss: in our baseline scenario, steady-state welfare is only 0.8 % lower than the efficient level. The corresponding number for contracting over final revenues is 11.7 %.

A comparison between Figs. 1 and 2 suggests a possible explanation for this result. While both arrangements cause the intermediate good producer to overinvest in our numerical examples, this problem is exacerbated when contracts are based on final revenues. The smaller deviations from surplus maximization observed in Fig. 1 are likely due to the effect of firm B's investment on the future value of the allocation function (30). As we already established, contracts based on input quantities yield $\theta_2 < 0$. Thus, an increase in investment of the intermediate good producer will worsen his future terms of trade. As a result, his incentives to boost input capacity will be mitigated. On the other hand, when contracts are based on final revenues, we have $\theta_2 > 0$. Therefore, an increase in investment will now improve firm B's future terms of trade. This suggests that the intermediate good producer will overinvest more relative to the other regime, adversely affecting overall efficiency.

5 Concluding Remarks

This chapter offers a novel perspective on contract design and firm conduct in dynamic environments where the production process necessitates bilateral exchange. The analysis illuminates the factors that govern surplus allocation within bilateral monopolies and explores the efficiency of different contractual arrangements.

Our model incorporates dynamic capacity constraints, where technological interdependence causes firms to engage in trade. Two types of surplus allocation procedures are considered: (i) contracting based on input quantities; and (ii) contracting based on final revenues. Furthermore, we impose a Markovian restriction on contracts and strategies. The benefit of this approach is twofold:

- it explicitly accounts for the strategic motives driving the firms' investment decisions;
- it enables us to determine the prevailing terms of trade implied by profit maximization, technological constraints and the surplus allocation mechanism.

Using dynamic programming, we derive necessary conditions for the equilibrium investment strategies that are consistent with the production technology. We argue that strategic concerns will typically prevent the firms from attaining joint surplus maximization. The adoption of a linear-quadratic payoff formulation allows us to characterize numerically the equilibrium investment decisions and the terms of trade. We find that surplus allocation arrangements based on input quantities are more efficient, but tend to benefit the input producer. This helps explain why "dominant" suppliers like Gazprom may want to entice their partners to sign contracts that are tied to input quantities.

Acknowledgements We are grateful to an anonymous referee for an insightful report and improvement requests that have assisted us in clarifying and, hopefully, sharpening our message.

Appendix A: Markov Allocation Equilibrium Conditions of the Bilateral Monopoly Game

This appendix derives the necessary conditions that characterize the Markov equilibrium of the bilateral monopoly game.

A.1 Euler Equation of the Final Good Producer

First consider the problem of firm A. Differentiating Bellman equation (9) yields the first-order condition:

$$V_1^{A\prime} = \frac{C_1^A}{\delta}. \tag{38}$$

By assumption the equilibrium strategies of firm A and firm B are respectively $f^A(x, y)$ and $f^B(x, y)$. Therefore, these strategy functions satisfy the recursive equation

$$\begin{aligned} V^A(x_t, y_t) = R(x_t) - S^B\big(z(x_t, y_t), g(x_t, y_t)\big) - C^A\big(f^A(x_t, y_t)\big) \\ + \delta V^A\big(\mu^A x_t + f^A(x_t, y_t), \mu^B y_t + f^B(x_t, y_t)\big). \end{aligned} \tag{39}$$

Differentiating with respect to x_t gives us

$$\begin{aligned} V_1^A = R_1 - S_1^B z_1 - S_2^B g_1 - C_1^A f_1^A \\ + \delta\big(\mu^A + f_1^A\big)V_1^{A\prime} + \delta f_1^B V_2^{A\prime}. \end{aligned} \tag{40}$$

Substituting $V_1^A(x, y)$ from the first-order condition into (40) forwarded one period yields an equation for $V_2^A(x, y)$:

$$V_2^{A\prime\prime} = -\frac{1}{\delta f_1^{B\prime}}\left\{R_1' - S_1^{B\prime} z_1' - S_2^{B\prime} g_1' - \frac{C_1^A}{\delta} + \mu^A C_1^{A\prime}\right\}. \tag{41}$$

Furthermore, differentiating (39) with respect y_{t-1} delivers

$$\begin{aligned} V_2^A = R_1 - S_1^B z_2 - S_2^B g_2 - C_1^A f_2^A \\ + \delta f_2^A V_1^{A\prime} + \delta\big(\mu^B + f_2^B\big)V_2^{A\prime}. \end{aligned} \tag{42}$$

Substituting $V_1^A(x, y)$ from (38) and $V_2^A(x, y)$ from (41) into (42) yields (14).

A.2 Euler Equation of the Intermediate Good Producer

Now consider the decision problem of firm B. Bellman equation (10) implies that the optimal strategy solves the first-order condition

$$V_2^{B\prime} = \frac{C_1^B}{\delta}. \tag{43}$$

Furthermore, by assumption the optimal strategies of firm A and firm B are respectively $f^A(x, y)$ and $f^B(x, y)$. Therefore, these strategy functions satisfy the recursive equation

$$\begin{aligned} V^B(x_t, y_t) = S^B\big(z(x_t, y_t), g(x_t, y_t)\big) - C^B\big(f^B(x_t, y_t)\big) \\ + \delta V^B\big(\mu^A x_t + f^A(x_t, y_t), \mu^B y_t + f^B(x_t, y_t)\big). \end{aligned} \tag{44}$$

Differentiating (44) with respect to y_t yields

$$V_2^B = S_1^B z_2 + S_2^B g_2 - f_2^B C_1^B + \delta f_2^A V_1^{B\prime} + \delta\big(\mu^B + f_2^B\big) V_2^{B\prime}. \tag{45}$$

Substituting V_2^B from the first-order condition and solving for V_1^B we get

$$V_1^{B\prime\prime} = -\frac{1}{\delta f_2^{A\prime}}\left\{S_1^{B\prime} z_2' + S_2^{B\prime} g_2' - \frac{C_1^B}{\delta} + \mu^B C_1^{B\prime}\right\}. \tag{46}$$

Similarly, differentiating (44) with respect to x_t gives us

$$V_1^B = S_1^B z_1 + S_2^B g_1 - f_1^B C_1^B + \delta\big(\mu^A + f_1^A\big) V_1^{B\prime} + \delta f_1^B V_2^{B\prime}. \tag{47}$$

After substitution of (43) and (46) into (47) we obtain (15).

Appendix B: Dynamically Efficient Investment

This appendix derives the necessary condition for joint surplus maximization.

Bellman equation (20) yields the first-order condition

$$-F_1 C_1^A - C_1^B + \delta F_1 W_1' + W_2' = 0. \tag{48}$$

Differentiation with respect to x gives us the envelope condition

$$W_1 = R_1 - \big(F_2 - \mu^A\big) C_1^A + \delta F_2 W_1'. \tag{49}$$

Furthermore, differentiation with respect to y gives us the envelope condition

$$W_2 = -\big(\mu^B F_1 + F_3\big) C_1^A + \delta\big(\mu^B F_1 + F_3\big) W_1' + \delta \mu^B W_2'. \tag{50}$$

Multiplying (49) by F_1 and adding it to (50) yields

$$\begin{aligned} F_1 W_1' + W_2' &= F_1 C_1^A - C_1^B \\ &= \delta F_1 R_1' - \delta F_1\big(F_2 - \mu^A\big) C_1^{A\prime} - \big(\mu^B F_1' + F_3'\big) C_1^{A\prime} \\ &\quad + \delta^2 \mu^B \big(F_1' W_1'' + W_2''\big) + \delta^2\big(F_1 F_2' + F_3'\big) W_1''. \end{aligned} \tag{51}$$

Substituting $F_1' W_1'' + W_2''$ from the first-order condition into (51) gives us an equation for W_1:

$$\begin{aligned} W_1'' = \frac{1}{\delta^2 (F_1 F_2' + F_3')} \big\{ & F_1 C_1^A - C_1^B - \delta F_1 R_1' \\ & + \delta F_1\big(F_2' - \mu^A\big) C_1'^A + F_3' C_1^{A\prime} - \delta \mu^B C_1^{B\prime}\big\}. \end{aligned} \tag{52}$$

Finally, substituting (52) into (49) delivers Euler equation (21).

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Dynamic Analysis of an Electoral Campaign

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Abstract This chapter proposes a differential game describing electoral campaigns where two candidates invest so as to increase the number of their respective voters. It is shown that parties overinvest as compared to the social optimum, while the total number of votes may be larger or lower than the first best one, depending on the level of negative externalities affecting the campaign. The model is also extended to allow for *n* candidates, so as to derive the socially optimal number of candidates. It appears that the number of candidates maximising the total number of votes on the election day is lower than the number of candidates entering the political game attracted by any non-negative share of consensus.

1 Introduction

The aim of this chapter consists in formalising electoral competition as a differential game where parties invest resources in the electoral campaign so as to increase the consensus for their respective candidates. To begin with, I set out by looking at a two-party system, where parties non-cooperatively invest on a finite horizon, with elections taking place at the terminal date, when the candidate receiving the highest share of votes obtains the office. The campaigning activity is characterised by external effects, in the sense that part of each candidate's investment spills over to rivals. To model parties' investments during the campaign, I examine a technology involving a quadratic cost of investment at each point in time, while the amount of consensus evolves linearly over time.

The ensuing setup is explicitly not designed to outline a connection between campaign investments and the final outcome of elections. Rather, it allows one to address the following issues. First, whether parties invest too much in the electoral

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campaign as compared to what would be socially efficient. Relatedly, the second issue is whether maximising the number of voters (i.e., collective participation to the polls) is a sensible measure of welfare in such a setting, or not. This problem can be reformulated in equivalent but perhaps clearer terms by asking whether it is necessarily true that "the more people express their political opinions, the merrier we are." This sounds like a sensible view, if we attach a positive value to political participation. However, a formal analysis of this aspect may in fact deliver some counterintuitive insight. Indeed, the model presented in the remainder of the chapter points out that each party's incentives involves excess investments at any point in time (except the election date) as compared to the social optimum. However, the planner indeed ends up with a lower political participation if the negative externality characterising the campaign is below a critical threshold (and conversely). This property emerges again in an extension of the model accounting for the presence of n parties, assumed a priori to be fully symmetric.

A third issue is whether it is possible to establish what the optimal number of parties should be, having in mind the objective to induce the highest number of voters to express their political preferences. To address this topic, I use the model with n parties, envisaging two alternative situations. One is the private equilibrium driven by individual incentives, where parties have an incentive to participate in the electoral game as long as their resulting individual number of votes is non-negative. The other scenario is that in which, given the parties' noncooperative behaviour, the planner can intervene to regulate the maximum number of participants or candidates) so as to maximise the overall amount of votes on the election day. It turns out that the privately optimal fragmentation driven by the selfish incentives of parties always exceeds the regulated number a benevolent dictator would impose. Accordingly, there emerges that a reduction in the number of parties/candidates would be welfare-improving, avoiding the perspective of having a political system populated by a very large number of parties, all of them too small in size.

The ensuing analysis is carried out by using a model where the open-loop solution is subgame perfect. This is appealing since it entails that agents may rely upon simple decision rules adopted at the very outset, and then strictly abided by along the game, as these rules are strongly time consistent.

The remainder of the chapter is organised as follows. Section 2 briefly discusses the related streams of literature in the fields of politics and industrial economics, with the aim of clarifying the background to the analysis carried out in the present chapter. The basic setup is laid out in Sect. 3. Private and social optima are characterised and comparatively assessed in Sect. 4. Section 5 extends the analysis to the case of n parties. Section 6 contains concluding remarks.

2 Related Literature

Issues like the cost of campaigning, the search for funds, the value of incumbency and their relation with the outcome of elections have received a large amount of attention by economists and political scientists alike (Jacobsen 1978, 1980, 1987;

Baron 1989; Abramovitz 1991; Austen-Smith 1995; Anderson and Prusa 2001; Prat 2002a, 2002b; Sahuguet and Persico 2006; Stratmann and Aparicio-Castillo 2006, to mention only a few). Indeed, Federal Election Commission data reveal the striking relevance of money in politics in general and electoral campaigns in particular. Candidates raising little or no money have negligible, if any, chances of winning.

To the best of my knowledge, these topics have never been explicitly put in relation with two connected streams of literature which are very familiar to industrial economists, namely those dealing with advertising and R&D activities. Both kinds of investment relate to rent-seeking behaviour, and they have been investigated extensively with and without uncertainty.[1] Firms carry out advertising and R&D to acquire a dominant position in the market, either by increasing market share or by improving production technology. In doing so, some of each firm's effort spills over to rivals. Profit incentives may give rise to overinvestment or underinvestment as compared to the social optimum, depending upon the shape of downstream market competition.[2]

Electoral campaigns share many fundamental features with advertising campaigns and R&D races (see, e.g., Cellini and Lambertini 2003, 2005, 2009; Nair and Narasimhan 2006; Viscolani and Zaccour 2009). Increasing consensus through costly investments is formally equivalent to acquiring a dominant market position through advertising or R&D either in process or in product innovation. In doing so, each party may waste some amount of resources to the benefit of rivals, and the optimal investment effort as well as the outcome of elections will depend upon the size of such external effects. The only relevant difference is that the electoral campaign has a terminal date which is known a priori, while R&D races end at some uncertain date in the future, as soon as the first innovator obtains the new technology or product, and advertising campaigns may never end at all.

These issues also have some relevant connections with the well known discussion concerning of the optimal number of firms in the commons (see Cornes and Sandler 1983; Cornes et al. 1986; Mason and Polasky 1997, inter alia), which, in turn, is connected with the recurrent theme of excess entry in the industrial organisation literature (see Novshek 1980; Jones 1987; Mankiw and Whinston 1990, inter alia).

3 Setup and Definitions

Consider a two-party system where each party has a candidate racing for the presidential office (or premiership), and elections are expected with certainty at date T.

[1] Differential games describing R&D races under uncertainty are in Reinganum (1982, 1989). Deterministic differential games of R&D and advertising are in Cellini and Lambertini (2002, 2003, 2004). Comprehensive surveys on dynamic models of advertising can be found in Jørgensen (1982), Feichtinger et al. (1994), and Dockner et al. (2000). A comprehensive overview of differential games in economics is in Van Long (2010).

[2] For exhaustive accounts, see Tirole (1988), Reinganum (1989), and Martin (2001).

Over $t \in [0, T]$, the fixed number of accessible votes is $\overline{X}$, and each party invests in an advertising campaign so as to increase the number of votes to its candidate. Instantaneous investment is $k_i(t)$ while the share of votes is $x_i(t) \geq 0$, with $x_i(t) + x_j(t) \leq \overline{X}$;[3] $k_i(t)$ is the control variable while $x_i(t)$ is the state variable of player i.[4]

Candidate i's consensus (or vote share) evolves according to:

$$\dot{x}_i(t) = k_i(t) - sk_j(t) - \delta x_i, \tag{1}$$

where $s > 0$ is a constant negative spillover from candidate j's investment onto candidate i's share of votes,[5] and $\delta > 0$ is a constant depreciation rate. This is a slightly simplified version of an advertising model described in Leitmann and Schmitendorf (1978) and Feichtinger (1983).

The gross instantaneous satisfaction associated with the share $x_i(t)$ is $\beta_i x_i(t)$, $\beta_i > 0$. This entails that there exists an ad interim value, measured by parameter β_i, attached to the current stock of voters patronising party (or candidate) i at any instant $t \in [0, T]$. Of course the outcome of the election is directly determined by the relative size of x_i's, but the weights β_i's influence the willingness to invest and, ultimately, the equilibrium outcome.

The instantaneous cost of investment is[6]

$$C_i(t) = \frac{c}{2}\big[k_i(t)\big]^2, \quad c > 0, \tag{2}$$

which amounts to assuming decreasing returns to the advertising activity, at any instant.[7]

Candidate i aims at maximising

$$\Pi_i = \int_0^T e^{\rho t}\left\{\beta_i x_i(t) - \frac{c}{2}\big[k_i(t)\big]^2\right\}dt + e^{\rho T} S\big[x_i(T)\big] \tag{3}$$

[3] The condition (i) for the non-negativity of each $x_i(t)$ and (ii) for $x_i(t)+x_j(t) \leq \overline{X}$ at any $t \in [0, T]$ are illustrated in the appendix. In the remainder of the analysis, I will assume such conditions to hold. For a similar model (dealing with advertising strategies in a dynamic oligopoly) where these constraints are explicitly accounted for, see Viscolani and Zaccour (2009).

[4] In the present setting, the political platforms of parties are left unspecified. This issue is detailedly addressed in the literature using the spatial approach to multiparty competition (see Bartholdi et al. 1991; Page et al. 1993; Anderson et al. 1994; Weber 1998; Adams 1999, 2000; Ansolabehere and Snyder 2000, inter alia).

[5] Unlike standard oligopoly models describing R&D competition (from d'Aspremont and Jacquemin 1988, onwards), here I allow the spillover parameter to exceed unit, as it is possible for a party to more than offset the rivals' efforts during particularly tough electoral campaigns. Several appropriate examples spring to mind, in particular—but not exclusively—with reference to Italy.

[6] The possibility that voting be costly for voters is investigated (in a static model) by Börgers (2004).

[7] In line of principle, parameters c and s could be asymmetric across candidates. Symmetry may be a sensible assumption whenever all candidates has comparable access to media and funds to finance their respective electoral campaigns. This can be the case of two-party political systems where institutional rules ensure competition will be carried out on equal basis, as, e.g., in the US.

under the constraint (1). $S[x_i(T)] \geq 0$ is the scrap (or salvage) value of the state at the terminal date T while $e^{\rho t}$ is the factor at which consensus is being capitalised during the game, in view of the election date, at which is becomes most relevant (relying on Nordhaus 1975, where a similar problem is investigated).[8] The instantaneous discount rate ρ is constant, common to both parties and positive. The initial condition is $x_{i0} = x_i(0) \geq 0$ possibly inherited from the previous (unmodelled) history of this political system.

It is worth stressing that, in the present model, both single party's and overall turnouts are changing over time till the terminal date T (when elections are actually carried out), so that, if at any $t \in [0, T]$ one candidate gains additional votes, this does not necessarily reduce the rival candidates' votes by an equal amount, since some of this additional support comes from citizens who would otherwise abstain. This is a major difference between the present approach and that characterising spatial voting models where the total turnout is fixed (see the references in footnote 4).

4 Equilibrium Analysis

The Hamiltonian of party (or candidate) i is:

$$\mathcal{H}_i(t) = e^{\rho t}\left\{\beta_i x_i(t) - \frac{c}{2}\left[k_i(t)\right]^2\right\} + \lambda_{ii}(t)\left[k_i(t) - sk_j(t) - \delta x_i\right] + \lambda_{ij}(t)\left[k_j(t) - sk_i(t) - \delta x_j\right], \tag{4}$$

where $\lambda_{ij}(t)$ is the co-state variable associated by party i to state $x_j(t)$. I will set out by investigating the privately optimal strategies of parties.

4.1 The Private Optimum

Here I investigate the outcome of the noncooperative simultaneous-move game where each party maximises (4) w.r.t. $k_i(t)$. To begin with, it is worth noting that the Hamiltonian function (4) is linear in the state variables, so that the game is indeed a linear state one, and therefore *perfect* or *state redundant* (see Mehlmann and Willing 1983, inter alia).[9] Accordingly, it can be solved through open-loop techniques to obtain a strongly time consistent solution. This procedure leads to the following result:

[8] Alternatively, one could assume that discounting is nil, as Gavious and Mizrahi (2002) do in investigating the interplay between an elected politician and an interest group. Using such an optimal control model without discounting, Gavious and Mizrahi (2002) show that, as long as elections are sufficiently far away, the politician in office should invest a constant level of resources, while getting closer to the election date, his/her effort increases or decreases depending on the electoral significance of that interest group.

[9] An exhaustive exposition of several classes of games where open-loop equilibria are subgame perfect can be found in Fershtman (1987), Mehlmann (1988, Chap. 4) and Dockner et al. (2000, Chap. 7).

Proposition 1 *The optimal control at any time $t \in [0, T)$ is*

$$k_i^*(t) = \frac{\beta_i(e^{(\rho-\delta)(T-t)} - 1)}{c(\rho - \delta)}.$$

As a result, party i's electoral consensus at the election date T is

$$x_i^*(T) = \max\left\{0, \frac{(\beta_i - s\beta_j)[\delta(e^{(\rho-\delta)T} + e^{\delta T} - 2) - \rho(e^{\delta T} - 1)]}{e^{\delta T}(\rho - \delta)(\rho - 2\delta)} + \frac{x_{i0}}{e^{\delta T}}\right\}.$$

Proof To derive the open-loop solution, I proceed as in Nordhaus (1975) (see also Chiang 1992, pp. 193–199). From the Hamiltonian of party (or candidate) i, one derives the following first order conditions:

$$\frac{\partial \mathcal{H}_i(t)}{\partial k_i(t)} = -ce^{\rho t}k_i(t) + \lambda_{ii}(t) - s\lambda_{ij}(t) = 0 \tag{5}$$

$$-\frac{\partial \mathcal{H}_i(t)}{\partial x_i(t)} = \dot{\lambda}_{ii}(t) \quad \Rightarrow \quad \dot{\lambda}_{ii}(t) = \delta\lambda_{ii}(t) - e^{\rho t}\beta_i \tag{6}$$

$$-\frac{\partial \mathcal{H}_i(t)}{\partial x_j(t)} = \dot{\lambda}_{ij}(t) \quad \Rightarrow \quad \dot{\lambda}_{ij}(t) = \delta\lambda_{ij}(t) \tag{7}$$

plus the initial conditions $x_i(0) = x_{i0}$ and the transversality condition $\lambda_{ij}(T) = 0$ for all i. Observe that indeed (7) admits the solution $\lambda_{ij}(t) = 0$ at all times, so that (5) reduces to $-ce^{\rho t}k_i(t) + \lambda_{ii}(t) = 0$. Then, note that (7) can be rewritten as

$$\dot{\lambda}_{ii}(t) - \delta\lambda_{ii}(t) = -e^{\rho t}\beta_i, \tag{8}$$

i.e., it is a first-order linear differential equation with a constant coefficient and a variable term, the complementary function and the particular integral being $\widehat{\lambda}(t) = A_i e^{\delta t}$ and $\overline{\lambda}(t) = -\beta_i e^{\rho t}/(\rho - \delta)$ respectively. Constant A_i can be identified using the transversality condition, whereby $A_i = \beta_i e^{(\rho-\delta)t}/(\rho - \delta)$. Therefore, the solution of (8) is

$$\lambda_{ii}(t) = \frac{\beta_i(e^{\delta t + (\rho-\delta)T} - e^{\rho t})}{\rho - \delta} \tag{9}$$

and the optimal control is[10]

$$k_i^*(t) = \frac{\beta_i(e^{(\rho-\delta)(T-t)} - 1)}{c(\rho - \delta)} > 0, \quad \forall \rho > 0, t \in [0, T). \tag{10}$$

The resulting state equation writes as follows:

$$\dot{x}_i(t) = \frac{(\beta_i - s\beta_j)(e^{(\rho-\delta)(T-t)} - 1) - c\delta(\rho - \delta)x_i(t)}{c(\rho - \delta)}, \tag{11}$$

[10]Note that at $\delta = \rho$, the value of (10) must be calculated using

$$\lim_{\delta \to \rho} k_i^*(t) = \frac{\beta_i(T - t)}{c} > 0.$$

whose solution is

$$x_i^*(t) = \frac{(\beta_i - s\beta_j)[\delta(2 - e^{(\rho-\delta)(T-t)}) - \rho]}{(\rho - \delta)(\rho - 2\delta)} + e^{-\delta t} Q, \tag{12}$$

where the integration constant Q can be determined by solving $x_{i0} = x_i^*(0)$, obtaining thus

$$Q = x_{i0} + \frac{(\beta_i - s\beta_j)[\delta(e^{(\rho-\delta)T} - 2) + \rho]}{(\rho - \delta)(\rho - 2\delta)}. \tag{13}$$

Hence, the volume of votes accruing to party (or candidate) i at the election time is

$$\begin{aligned} x_i^*(T) &= x_i^*(t)\big|_{t=T} \\ &= \frac{(\beta_i - s\beta_j)[\delta(e^{(\rho-\delta)T} + e^{\delta T} - 2) - \rho(e^{\delta T} - 1)]}{e^{\delta T} c(\rho - \delta)(\rho - 2\delta)\delta} + \frac{x_{i0}}{e^{\delta T}}. \end{aligned} \tag{14}$$

This concludes the proof. □

A straightforward corollary to Proposition 1 is the following:

Corollary 1 *The optimal investment of party i is monotonically decreasing over time.*

Proof This is straightforward, as from (10) one obtains

$$\frac{\partial k_i^*(t)}{\partial t} = -\frac{\beta_i e^{(\rho-\delta)(T-t)}}{c} < 0 \tag{15}$$

always. □

This replicates an analogous feature of the political business cycle model in Nordhaus (1975) and expresses the idea that last-minute efforts are in fact worthless. Now observe that

$$\text{sign}\{\delta(e^{(\rho-\delta)T} + e^{\delta T} - 2) - \rho(e^{\delta T} - 1)\} = \text{sign}\{(\rho - \delta)(\rho - 2\delta)\} \tag{16}$$

for all $\delta, \rho > 0$.[11] Consequently,

[11] Observe that

$$\lim_{\delta \to \rho} x_i^*(T) = \frac{c\rho^2 x_{i0} + (e^{\rho T} - 1 - \rho T)(\beta_i - s\beta_j)}{c\rho^2 e^{\rho T}};$$

and

$$\lim_{\delta \to \rho/2} x_i^*(T) = \frac{c\rho^2 x_{i0} + 2[2 + e^{\rho T/2}(\rho T - 2)](\beta_i - s\beta_j)}{c\rho^2 e^{\rho T/2}}.$$

Lemma 1 $x_i^*(T) > 0$ *for all*

$$x_{i0} > \max\left\{0, \frac{(\beta_i - s\beta_j)[\delta(e^{(\rho-\delta)T} + e^{\delta T} - 2) - \rho(e^{\delta T} - 1)]}{c(\rho-\delta)(\rho-2\delta)\delta}\right\}$$

with

$$\max\left\{0, \frac{(\beta_i - s\beta_j)[\delta(e^{(\rho-\delta)T} + e^{\delta T} - 2) - \rho(e^{\delta T} - 1)]}{c(\rho-\delta)(\rho-2\delta)\delta}\right\} = 0, \quad \forall \beta_i > s\beta_j,$$

and conversely.

The above Lemma implies that $\beta_i > s\beta_j$ suffices to ensure $x_i^*(T) > 0$ irrespective of the value of x_{i0}. However, given the asymmetry among parties, there remains to check whether $x_i^*(T)$ and $x_j^*(T)$ can indeed be simultaneously positive in some admissible portion of the relevant parameter space (s, β_i, β_j). The sufficient condition for $x_j^*(T) > 0$ is $\beta_i \in (0, \beta_j/s)$ for any x_{j0}, and since $\beta_j/s \gtrless s\beta_j$ for all $s \lesseqgtr 1$, one can formulate the following

Corollary 2 *In the parameter region identified by* $s \in (0,1)$ *and* $\beta_i \in (s\beta_j, \beta_j/s)$, $x_i^*(T), x_j^*(T) > 0$ *for any pair of initial conditions* $x_{i0}, x_{j0} \geq 0$.

As anticipated in the previous section, the relative size of β_i's matters in shaping the equilibrium outcome. At first sight, one could argue that this may appear somewhat unrealistic. However, it is worth noting that the presence of a parameter measuring the ad interim satisfaction (i) reflects the observed behaviour of candidates taking part in real-world elections; and (ii) affects, again realistically, the investment efforts.[12]

There remains to assess the electoral outcome if both parties attain positive levels of electoral consensus. To this aim, define $\Delta x_0 = x_{i0} - x_{j0}$, $\Delta\beta = \beta_i - \beta_j$ and

$$\Lambda \equiv \frac{\delta(e^{(\rho-\delta)T} + e^{\delta T} - 2) - \rho(e^{\delta T} - 1)}{c(\rho-\delta)(\rho-2\delta)\delta} > 0. \tag{17}$$

Then,

$$\Delta x^*(T) \equiv x_i^*(T) - x_j^*(T) = \frac{1}{e^{\delta T}}\left[\Delta x_0 + \Lambda(1+s)\Delta\beta\right], \tag{18}$$

from which we draw

Proposition 2 $\Delta x^*(T) > 0$ *for all* $\Delta x_0 > -\Lambda(1+s)\Delta\beta$, *and conversely.*

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[12]With asymmetric values of parameters c and s, the election outcome would also depend on such parameters. However, one can think that (i) both parties be equally able to access media at the same costs, and (ii) reciprocal negative externalities be comparable in size, which would justify the symmetry assumption adopted here. This may be particularly true for countries in which the par condicio applies.

The above Proposition can be spelled out as follows: the party attaching the lower weight to consensus may still win the elections if its initial condition is more favourable then the rival's when the campaign starts off. Of course, if $\Delta x_0, \Delta\beta > 0$, then surely $\Delta x^*(T) > 0$. However, it is also true that, if $\Delta x_0 < 0$, party i ultimately gets the office at stake provided that $\Delta\beta > -\Delta x_0/[\Lambda(1+s)] > 0$, i.e., if its eagerness to acquire consensus during the campaign is strong enough.

Concerning the stability analysis of the dynamic system, the following holds:

Proposition 3 *The open-loop game follows a saddle path.*

Proof The relevant dynamic equations are:

$$\dot{x}_i = k_i(t) - sk_j(t) - \delta x_i(t) \tag{19}$$

$$\dot{k}_i = \frac{1}{c}\left[c(\rho+\delta)k_i(t) - \beta_i\right] \tag{20}$$

and the stability properties of the above system depend upon the sign of the trace and determinant of the following Jacobian matrix:

$$\mathcal{J} = \begin{bmatrix} \frac{\partial \dot{x}_i}{\partial x_i} = -\delta & \frac{\partial \dot{x}_i}{\partial k_i} = 1 \\ \frac{\partial \dot{k}_i}{\partial x_i} = 0 & \frac{\partial \dot{k}_i}{\partial k_i} = \rho + \delta \end{bmatrix} \tag{21}$$

with trace $\mathrm{Tr}(\mathcal{J}) = \rho > 0$ and determinant $\Delta(\mathcal{J}) = -\delta(\rho+\delta) < 0$ for all $\delta, \rho > 0$. Since the determinant is everywhere negative, the game evolves along a saddle path. □

Before passing on to the social planning case, a last remark is in order. Expression (20), describing the dynamics of party i's investment, reveals that it is independent of the rival's. This entails that strategic interaction between parties takes place only through the dynamics of the state variable, as described by (1). This contrasts with some of the acquired wisdom (see, e.g., Anderson and Prusa 2001), where the efforts of parties are strategic complements. On this basis, whether parties overinvest or underinvest as compared to the social optimum, in the present setting, is not obvious from the outset.

4.2 The Social Optimum

Now consider the situation where a benevolent social planner chooses the vector of investments $\{k_i(t)\}$ so as to maximise collective welfare, which is defined as the sum of both parties' discounted payoffs, under the constraint (1). This amounts to assuming that parties (or their candidates) are indeed representative of their electors, to the extent that those individuals who do not vote are irrelevant.[13] In the present

[13]This is somewhat similar to what is often assumed in innovation race models, where the value of innovation is the same irrespectively of whether the firm racing for it is a private or a public one (see Kamien and Schwartz 1982; Reinganum 1989, inter alia).

setting, the planner's behaviour is aimed at assessing the social convenience to put a ceiling to the parties' investments during the electoral campaign, even if this may well entail trading off some voters against a lower expenditure.

Now the relevant Hamiltonian is:

$$\mathcal{H}^{SP}(t) = e^{\rho t}\left\{\beta_i x_i(t) + \beta_j x_j(t) - \frac{c}{2}\big[k_i(t)\big]^2 - \frac{c}{2}\big[k_j(t)\big]^2\right\} + \lambda_i(t)\big[k_i(t) - sk_j(t) - \delta x_i\big] + \lambda_j(t)\big[k_j(t) - sk_i(t) - \delta x_j\big], \quad (22)$$

where superscript *SP* stands for *social planning*. The technical details of the analysis of the social planner's problem are omitted for brevity, as the procedure is the same as in the above game between parties. For brevity, henceforth I will also omit the explicit indication of the time argument for state and control variables. The outcome is summarised by the following:

Proposition 4 *The optimal control at any time $t \in [0, T)$ is*

$$k_i^{SP} = \max\left\{0, \frac{(\beta_i - s\beta_j)(e^{\delta t+\rho T} - e^{\rho t+\delta T})}{c(\rho - \delta)e^{\rho t+\delta T}}\right\}.$$

As a result, party i's electoral consensus at the election date T is

$$x_i^{SP}(T) = \max\left\{0, \frac{[\beta_i(1+s^2) - 2s\beta_j]\Lambda + x_{i0}}{e^{\delta T}}\right\}.$$

In correspondence of $\delta = \rho$, the value of the optimal instantaneous control must be calculated using the limit:

$$\lim_{\delta\to\rho} = k_i^{SP} = \frac{(\beta_i - s\beta_j)(T - t)}{c}. \quad (23)$$

Since $\text{sign}\{e^{\delta t+\rho T} - e^{\rho t+\delta T}\} = \text{sign}\{\rho - \delta\}$ everywhere, controls k_i^{SP} and k_j^{SP} are both positive in the region $\{s \in (0, 1), \beta_i \in (s\beta_j, \beta_j/s)\}$. Anywhere else, at least one control drops to zero, which is a first major difference between the planner's behaviour as against the parties'.

Moreover, studying the difference $x_i^{SP}(T) - x_j^{SP}(T)$ in the region where they are both positive delivers the following:

$$\Delta x^{SP}(T) \equiv x_i^{SP}(T) - x_j^{SP}(T) = \frac{1}{e^{\delta T}}\big[\Delta x_0 + \Lambda(1+s)^2\Delta\beta\big], \quad (24)$$

which implies

Proposition 5 $\Delta x^{SP}(T) > 0$ *for all* $\Delta x_0 > -\Lambda(1+s)^2\Delta\beta$, *and conversely.*

That is, the critical threshold imposed on the pair of initial conditions, above which party i wins, is higher (resp., lower) under planning that at the noncooperative equilibrium of the election game, for all $\Delta\beta < 0$ (resp., $\Delta\beta > 0$). The consequences of this fact will be outlined below.

The stability properties of the steady state under social planning are stated in the following:

Proposition 6 *The social planning problem follows a saddle path.*

The proof is omitted, as it proceeds along much the same lines as for Proposition 6.

The above analysis illustrates that under social planning the winner is the same candidate as in the private optimum. However, a thorough comparison between the two regimes remains to be carried out. In particular, one may wonder whether social planning may prevent parties from performing a wasteful effort duplication, and how this affects the total number of voters at equilibrium.

4.3 Private vs. Social Optimum

Considering first equilibrium investments in advertising in the two regimes. Provided $k_i^{SP}, k_j^{SP} > 0$,

$$k_i^* - k_i^{SP} = \frac{s\beta_j(e^{(\rho-\delta)(T-t)} - 1)}{c(\rho-\delta)} = \frac{s\beta_j k_i^*}{\beta_i} > 0, \tag{25}$$

which entails that individual incentives leads to a socially wasteful excess investment at any time (this being a fortiori true should $k_i^{SP} = 0$ at some instant t). As to the outcome of elections in the two cases, on the basis of Propositions 2 and 5, we have that for all

$$\begin{aligned}\Delta x_0 \in \big(&\min\{-\Lambda(1+s)^2\Delta\beta, -\Lambda(1+s)\Delta\beta\},\\ &\max\{-\Lambda(1+s)^2\Delta\beta, -\Lambda(1+s)\Delta\beta\}\big)\end{aligned} \tag{26}$$

the electoral outcome differs across the two regimes, private and social preferences being not reciprocally aligned:

Proposition 7 *Take $\Delta\beta < 0$. For all*

$$\Delta x_0 \in \big(-\Lambda(1+s)\Delta\beta, -\Lambda(1+s)^2\Delta\beta\big),$$

party i wins the electoral game while party j would take office under social planning. The opposite applies if $\Delta\beta > 0$ and

$$\Delta x_0 \in \big(-\Lambda(1+s)^2\Delta\beta, -\Lambda(1+s)\Delta\beta\big).$$

This result illustrates the presence of an admissible portion of the space $\{\Delta\beta, \Delta x_0\}$ in which the party enjoying a more favourable initial condition (say, $x_{i0} < x_{j0}$) gets the office 'more easily' under planning than in the standard electoral game, when its taste for consensus, measured by β_i is higher than the rival's (and conversely in the opposite situation). This aspect of the model suggests that the planner behaves as if it wanted to reward a sort of political commitment implicit in the size of β_i, or to punish the lack thereof.

The last step of this comparative assessment consists in looking at the overall equilibrium level of consensus in the two cases. To this aim, use $X^*(T) = x_i^*(T) + x_j^*(T)$ and $X^{SP}(T) = x_i^{SP}(T) + x_j^{SP}(T)$ to verify that

$$\text{sign}\left\{X^*(T) - X^{SP}(T)\right\} = \text{sign}\{1 - s\}. \tag{27}$$

That is,

Proposition 8 *For all* $s \in (0, 1)$, *the planner is happy with a lower number of voters, as long as this is more than compensated by a reduction in the wasteful duplication of efforts during the electoral campaign.*

Of course the opposite applies if $s > 1$, precisely because of the higher aggressiveness which goes along with it. To grasp the essence of the above Proposition, one may observe that the planner's objective replicates that of an *entente cordiale* between the two parties, i.e., a cooperative agreement internalising negative externalities, something which doesn't happen in the fully noncooperative game. I will come back to this aspect below.

5 Extension: Optimal Fragmentation

In the foregoing analysis, I have adopted the assumption that there exist only two parties, and I have evaluated the efficiency of such a system. However, multiparty systems are rather common, and to this regard several interesting questions can be addressed in a generalisation of any of the above settings. One such question is whether there should be a limit to the number of parties, and, if so, how to set this limit. Two related questions are (i) whether the optimal number of parties obtains by maximising the overall number of voters or collective welfare, and (ii) whether optimal fragmentation is higher in the social or in the private equilibrium. In particular, question (ii) involves the comparison of two situations. One is the first best where a benevolent social planner controls both the number of parties and their individual investment in the electoral campaign. The other is a second best where the investment is noncooperatively decided by parties, while the number of parties is controlled by a benevolent planner.

Here I reconsider the same model with n parties (and n candidates), where, for the sake of simplicity, I adopt the symmetry assumptions $\beta_i = \beta$ and $x_{i0} = x_0 \geq 0$ for all i. This amounts to excluding the use of a *quorum* so as to determine the optimal number of parties (or candidates), since the equilibrium size (i.e., the volume of votes in steady state) is the same across parties. Additionally, symmetry rules out the possibility of a winner standing out at time T, the issue being instead the optimal number of parties.

The differential equation of the state variable is:

$$\dot{x}_i = k_i - s\sum_{j \neq i} k_j - \delta x_i. \tag{28}$$

Accordingly, candidate i's Hamiltonian now rewrites as:

$$\mathcal{H}_i = e^{\rho t}\left(\beta x_i - \frac{c}{2}k_i^2\right) + \lambda_{ii}\left(k_i - s\sum_{j\neq i} k_j - \delta x_i\right) + \sum_{j\neq i}\lambda_{ij}\left(k_j - s\sum_{m\neq j} k_m - \delta x_j\right). \tag{29}$$

Once again, it can be shown that the open-loop solution is a degenerate feedback solution, and therefore (29) can be reformulated by setting $\lambda_{ii} = \lambda_i$ and $\lambda_{ij} = 0$ for all j.

Solving the game among party yields exactly the same optimal control as in Proposition 1 and (10), the resulting total volume of votes being

$$X^*(T,n) = \max\left\{0, \frac{n[\beta(1-s(n-1))\Lambda + x_0]}{e^{\delta T}}\right\}. \tag{30}$$

The benevolent social planner' Hamiltonian is instead the following:

$$\mathcal{H}^{SP} = e^{\rho t}\left(\sum_{i=1}^{n}\beta x_i - \frac{c}{2}k_i^2\right) + \sum_{i=1}^{n}\lambda_i\left(k_i - s\sum_{j\neq i}k_j - \delta x_i\right), \tag{31}$$

whereby the socially efficient control at a generic instant t is

$$k^{SP}(n) = \max\left\{0, \frac{\beta(s(n-1)-1)(e^{\rho t+\delta T} - e^{\delta t+\rho T})}{c(\rho-\delta)e^{\rho t+\delta T}}\right\} \tag{32}$$

and the number of votes at the election date amounts to

$$X^{SP}(T,n) = \max\left\{0, \frac{n[\beta(1-s(n-1))^2\Lambda + x_0]}{e^{\delta T}}\right\}. \tag{33}$$

Taking $s \in (0, 1/(n-1))$ suffices to ensure that $X^*(T,n)$ be strictly positive (while $X^{SP}(T,n) > 0$ everywhere). This is an intuitive finding, as low levels of s accounts for a comparatively milder attitude of each party as far as the predatory aspects of its campaign are concerned.

$$X^*(T,n) > X^{SP}(T,n), \quad \forall n > \max\left\{\frac{1+s}{s}, 2\right\}, \tag{34}$$

and conversely. Given that $2 > (1+s)/s$ for all $s > 1$, this proves:

Proposition 9 *If the aggressiveness characterising the electoral campaign is sufficiently high, the total number of votes at the social planning outcome is higher than that autonomously generated by the parties' noncooperative strategies in any political system admitting at least two parties.*

This extends Proposition 8 to a (symmetric) multiplicity of parties. The above result doesn't come unexpected, and lends itself to the quite intuitive interpretation whereby the noncooperative attitude of parties, whose activities are partly aimed at destroying the rivals' consensus, generates an inefficient outcome. In particular, if $s > 1$, this happens through an excess of votes generated through excess investment, because the negative spillover generated by each rival's efforts more than offsets the effectiveness of a party in building up its own one.

Less obvious, a priori, is whether the number of parties populating the political system exceeds the social optimum. That is, shall one expect to observe excess fragmentation or not, if parties may freely enter the system? Should a planner regulate such entry process, to limit the number of parties allowed to compete for the office at stake? To answer this question, I will exclusively look at the individual and collective performance of agents at the noncooperative outcome, to perform two related exercises: the first consists in deriving the limit number of parties that may survive at the electoral equilibrium of the noncooperative firms with non-negative volumes of votes (which is the aspect worrying any party interested in entering a system as long as it expects to obtain some consensus). The second consists instead in calculating the number of parties maximising the total number of votes at the same equilibrium (this being a reasonable objective of a planner acting *super partes* and interested in the overall participation of voters). The comparison of these two numbers will reveal the presence of excess fragmentation (or the lack thereof).

First, it is easily checked that the maximum number of parties that are able to survive on the election day solves $X^*(T,n) = 0$, since each party, under full symmetry, gets $X^*(T,n)/n$. The unique solution is

$$\overline{n} = \frac{x_0 + \Lambda(1+s)\beta}{\Lambda s\beta}, \tag{35}$$

while

$$\arg\max_n X^*(T,n) = n^* = \frac{x_0 + \Lambda(1+s)\beta}{2\Lambda s\beta} = \frac{\overline{n}}{2}. \tag{36}$$

Now, without dwelling upon the elementary condition ensuring $\overline{n} \geq 2$, the evident implication of (36) can be summarised in the following:

Proposition 10 *The noncooperative election game is affected by excess fragmentation.*

This closely replicates a traditional result known in the theory of industrial organization (at least since Novshek 1980) as *excess entry*, whereby profit incentives drive too many firms into a market, as compared to what would be the socially optimal industry structure, all else equal (that is, taking for granted their profit-maximising behaviour via prices, quantities and other strategic variables). The same problem replicates in the present setting, in which too many parties are lured into the political system even by the smallest slice of electoral consensus. What is also analogous to the acquired wisdom from the IO literature is that the entry process causes the overall and individual consensus to collapse to zero, exactly as the individual and industry output do in a Cournot model replicating perfect competition in the limit of the entry process (see Novshek 1980; Jones 1987; Mankiw and Whinston 1990).

6 Concluding Remarks

I have analysed a differential game describing an electoral campaign where two candidates invest so as to increase the number of their respective voters. The outcomes of the non-cooperative game has been evaluated against the social optimum, where a benevolent social planner chooses the investment levels so as to maximise collective welfare. The private optimum is characterised by overinvestment and, if the externality between parties is low enough, a larger number of voters as compared to the social optimum. Therefore, it appears that if competition is extremely tough, the electoral campaign conducted by aggressive parties ends up increasing abstention.

Then, I have extended the model to account for n candidates, in order to evaluate what is the optimal number of candidates (or parties). There has clearly emerged that the private incentives of parties involve too much fragmentation as against the optimal structure of the political system that would be chosen by a benevolent planner to maximise the total number of votes expressed by citizens on the election day, taking as given the parties' noncooperative behaviour.

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Appendix

The conditions for the non-negativity of consensus levels $x_i^*(T)$ and $x_j^*(T)$ are stated in Lemma 1 and Corollary 2. Here I will identify the conditions that must be satisfied by the state variables in order for the consensus of each party to be non-negative at any instant during the game, and for total consensus not to exceed $\overline{X}$ over the same time interval. For brevity, I will confine to the two-party game and its social planning version.

On the basis of expressions (12)–(13), one can easily ascertain that $x_i^*(t) \geq 0$ for all $x_{i0} \geq \max\{0, (\beta_i - s\beta_j)\Lambda(t)\}$, in which

$$\Lambda(t) \equiv \frac{\delta[e^{(\rho-\delta)T} + e^{\delta t+(\rho-\delta)(T-t)} + 2(e^{\rho t} - 1)] - \rho(e^{\delta t} - 1)}{c(\rho-\delta)(\rho-2\delta)\delta} > 0, \tag{A.1}$$

which coincides with (A.1) at $t = T$. Moreover, $\overline{X} \geq x_i^*(t) + x_j^*(t)$ requires

$$\overline{X} \geq \frac{x_{i0} + x_{j0} + \Lambda(t)(1-s)(\beta_i - \beta_j)}{e^{\delta t}} \equiv \overline{X}^*, \tag{A.2}$$

the r.h.s. of (A.2) being positive if indeed $x_{i0} \geq \max\{0, \Lambda(t)(\beta_i - s\beta_j)\}$.

Under planning, $x_i^{SP}(t) \geq 0$ for all $x_{i0} \geq \max\{0, [2s\beta_j - \beta_i(1+s^2)]\Lambda(t)\}$, while $\overline{X} \geq x_i^{SP}(t) + x_j^{SP}(t)$ writes as follows:

$$\overline{X} \geq \frac{x_{i0} + x_{j0} + \Lambda(t)(1-s)^2(\beta_i + \beta_j)}{e^{\delta t}} \equiv \overline{X}^{SP}. \tag{A.3}$$

Comparing the thresholds appearing on the r.h.s. of (A.2)–(A.3), one gets

$$\overline{X}^{*}-\overline{X}^{SP}=\frac{\Lambda(t)(1-s)s(\beta_i+\beta_j)}{e^{\delta t}} \tag{A.4}$$

whose sign changes at $s=1$, fully reflecting the message conveyed by Propositions 8–9.

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Multi-agent Optimal Control Problems and Variational Inequality Based Reformulations

George Leitmann, Stefan Pickl, and Zhengyu Wang

Abstract The multi-agent optimal control problem involves a decision process with multiple agents, where each agent solves an optimal control problem with the individual cost functional and strategy set, and the cost functional is dependent on all the other agents' state and/or control variables. Here the "agent" can be understood as a true decision maker, or as an abstract optimization criterion. The strategy sets, along with admissible control set, are often described by a system of parameterized ordinary differential/difference equations (the state dynamic) or partial differential equations, and in realistic settings they may be dependent on the rivals's variables due to, for example, certain constraints from the common resources. This chapter describes the multi-agent optimal control problem, and studies the reformulation of a system of differential equations constrained by parameterized variational inequalities, along with some initial and/or boundary conditions. This reformulation presents differential equations, variational inequalities, and equilibrium conditions in a systematic way, and is advantageous since it can be treated as a system of differential algebraic equations, for which abundant theory and algorithms are available.

1 Optimal Control Problems

In a multi-agent optimal control problem each agent solves an optimal control problem that is dependent on the rivals' states and decisions. Let us begin the study with the single-agent case: the standard optimal control problem.

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For more details on optimal control problems and on the multi-agent extension we refer to the basic books by Leitmann (1976, 1981).

1.1 Problem Description

Let the terminal time $T > 0$ and the initial point $x^0 \in \mathbb{R}^n$ be given, let $U \subset \mathbb{R}^n$ be an open bounded set, $\Xi \in \mathbb{R}^m$ be convex and closed, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{W(s)\}_{0\leq t\leq T}$ be a d-dimensional Brownian motion, let $\mathcal{A}$ be a subset of all progressive measurable stochastic processes $u(\cdot) : [0, T] \times \Omega \to \Xi$. Given the following four functions, of which the first two constitute the dynamic and the last two give the cost functional:

- $f : [0, T] \times \mathbb{R}^n \times \Xi \to \mathbb{R}^n$ (drift term),
- $\sigma : [0, T] \times \mathbb{R}^n \times \Xi \to \mathbb{R}^{n\times d}$ (diffusion term),
- $\varphi : [0, T] \times \bar{U} \times \Xi \to \mathbb{R}$ (running cost),
- $\psi : [0, T] \times \bar{U} \times \Xi \to \mathbb{R}$ (terminal cost).

For a $t \geq 0$ and for every $u(\cdot) \in \mathcal{A}$ and $(s, x) \in [t, T] \times \bar{U}$, the *state dynamic* is a stochastic differential equation (SDE for short) which is to find an Itô process $x(s)$ satisfying:

$$\begin{cases} dx(s) = f(s, x(s), u(s))ds + \sigma(s, x(s), u(s))dW(s) \\ x(t) = x \end{cases} \tag{1}$$

for $s \in (t, T]$, where a $u(\cdot) \in \mathcal{A}$ is called the *control*, and $x(\cdot)$ is called the *state*. We define the *cost functional* $J : [0, T] \times \mathbb{R}^n \times \mathcal{A} :\to \mathbb{R}$ by:

$$J\big(t, x, x(\cdot), u(\cdot)\big) := \mathbb{E}\left\{\int_t^T \varphi\big(s, x(s), u(s)\big)ds + \psi\big(T, x(T)\big)\right\}, \tag{2}$$

where $\mathbb{E}$ means the expectation over the statistics of $\{W(s)\}$. Denote $J(x(\cdot), u(\cdot)) = J(0, x^0, x(\cdot), u(\cdot))$ for simplicity if the state $x(\cdot)$ starts from x^0 at $t = 0$. Then the optimal control problem is just to find a pair $(x(\cdot), u(\cdot))$ minimizing $J(x(\cdot), u(\cdot))$ under the constraint given by the SDE (1):

$$\begin{aligned} &\min J\big(x(\cdot), u(\cdot)\big) \\ &\text{s.t. } dx(s) = f\big(s, x(s), u(s)\big)ds + \sigma\big(s, x(s), u(s)\big)dW(s) \\ &\qquad x(0) = x^0. \end{aligned} \tag{3}$$

For the SDE (1), one of the problems of the most interest is its solvability. For the details on this issues we refer to Øksendal (2003). Here we just mention the conditions required in part for guaranteeing the existence and the uniqueness of the strong solution with continuous paths of (1) for any choice of $u(\cdot)$:

$$\big\|f(t,x,u)-f(s,y,u)\big\|_2+\big\|\sigma(t,x,u)-\sigma(s,y,u)\big\|_F\le C\big(\|x-y\|_2+|t-s|\big)$$
$$\big\|f(t,x,u)\big\|_2+\big\|\sigma(t,x,u)\big\|_F\le C\big(1+\|x\|_2\big),$$

where $\|\cdot\|_F$ denotes the Frobenius norm, $f(s,x,u)$ and $\sigma(s,x,u)$ are assumed in $C^0([0,T]\times\mathbb{R}^n\times\Xi)$ and $f(\cdot,\cdot,u)$ and $\sigma(\cdot,\cdot,u)$ are in $C^1([0,T]\times\mathbb{R}^n)$ for every $u\in\Xi$, and $C\ge 0$ is a constant, $u\in\Xi$, $x,y\in\mathbb{R}^n$ and $t,s\in[0,T]$ are arbitrary.

1.2 Hamilton–Jacobi–Bellman Equation

Define the *value function*:

$$\begin{aligned} v(t,x) := \min\ & J\big(t,x,x(\cdot),u(\cdot)\big)\\ \text{s.t. } & dx(s)=f\big(s,x(s),u(s)\big)ds+\sigma\big(s,x(s),u(s)\big)dW(s)\\ & x(t)=x. \end{aligned}$$

Denote $\chi(t,x,u)=\frac{1}{2}\|\sigma(t,x,u)\|_F^2$, and denote

$$H(t,x,u,\nabla v,\Delta v)=\chi(t,x,u)\Delta v(t,x)+\big\langle f(t,x,u),\nabla v(t,x)\big\rangle+\varphi(t,x,u). \quad (4)$$

Suppose that $H(t,x,u,\nabla v,\Delta v)$ is continuously differentiable in u. Then the optimal control problem (3) can be reformulated as the Hamilton–Jacobi–Bellman equation (HJB equation for short):

$$\frac{\partial v(t,x)}{\partial t}+\min_{u\in\Xi}H\big(t,x,u,\nabla v(x,t),\Delta v(t,x)\big)=0, \quad (5)$$

along with the terminal condition $v(T,x)=\psi(T,x)$, where $\Delta v(t,x)$ and $\nabla v(t,x)$ denote the Laplacian and the gradient of v in x, respectively. Normally, the HJB equation does not have a classic solution, for this one has to use another notion of solution: viscosity solution (refer to Fleming and Rishel 1975, for example).

1.3 Constrained Hamilton System

For the deterministic case: $\sigma(t,x,u)\equiv 0$, we introduce the costate variable $p(t)=\nabla v(t,x(t))$. Then the Hamiltonian defined in (4) reads:

$$H(t,x,u,p)=f(t,x,u)^Tp+\varphi(t,x,u), \quad (6)$$

and by simple calculus we obtain the following Hamilton system from the HJB equation:

$$\begin{cases} \dot{p}(t) = -\nabla_x H(t, x(t), u(t), p(t)) \\ \dot{x}(t) = \nabla_p H(t, x(t), u(t), p(t)) \\ u(t) \in \arg\min\{H(t, x(t), z, p(t)), \text{ s.t. } z \in \Xi\} \\ x(0) = x^0 \text{ and } p(T) = \nabla_x \psi(T, x(T)), \end{cases} \tag{7}$$

where $\nabla_x \psi(T, x)$ denotes the gradient of $\psi(t, x)$ with respect to x.

1.4 VI Based Reformulations

Variational inequality (VI for short) is a powerful model for characterizing the optimal condition of optimization problems in a general setting (Facchinei and Pang 2003). Given a closed and convex subset $\Omega \subseteq \mathbb{R}^m$ and a mapping $G : \Omega \to \mathbb{R}$. Then by the minimum principle, we know that a local minimizer x^* of $G(\cdot)$ over the feasible domain Ω must satisfy the variational inequality (VI for short) of the following form

$$\left(x - x^*\right)^T \nabla G\left(x^*\right) \geq 0, \quad \forall x \in \Omega. \tag{8}$$

We remind us that Ξ is assumed convex and closed set. Here we allow us an abuse of the notation u: it means the control variable in the general case and means also a local minimizer of $H(t, x, \cdot, \nabla v, \Delta v)$ in some specific cases, which can be readily distinguished in the context. Then by the minimum principle, we know that u satisfies the VI

$$(z - u)^T \nabla_u H(t, x, u, \nabla v, \Delta v) \geq 0 \quad \forall z \in \Xi, \tag{9}$$

where $\nabla_u H(t, x, u, \nabla v, \Delta v)$ denote the gradient of $H(t, x, u, \nabla v, \Delta v)$ in u. We denote by $\mathrm{SOL}(\Xi, \nabla_u H(t, x, \cdot, \nabla v, \Delta v))$ the solution set of the above VI. Further known is that if moreover $\nabla_u H$ is convex in u, then a solution of the VI is just a global minimizer of H.

Now we arrive at the position for reformulating the HJB equation as the following PDE, which is constrained by a VI

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} + H(t, x, u, \nabla v(t, x), \Delta v(t, x)) = 0 \\ u \in \mathrm{SOL}(\Xi, \nabla_u H(t, x, \cdot, \nabla v(t, x), \Delta v(t, x)) \\ v(T, x) = \psi(T, x). \end{cases}$$

It is well known that $u \in \mathrm{SOL}(\Xi, \nabla_u H(t, x, \cdot, \nabla v, \Delta v))$ if and only if

$$u = \mathrm{Pr}_{\Xi}\left(u - \nabla_u H(t, x, u, \nabla v, \Delta v)\right),$$

where $\mathrm{Pr}_{\Xi}(\cdot)$ denotes the projection onto Ξ. Then the HJB equation can further be reformulated as the PDE constrained by a system of algebraic equations

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} + H(t, x, u, \nabla v(t, x), \Delta v(t, x)) = 0 \\ u = \mathrm{Pr}_{\Xi}(u - \nabla_u H(t, x, u, \nabla v(t, x), \Delta v(t, x))) \\ v(T, x) = \psi(T, x). \end{cases} \tag{10}$$

Note that the projection operation often leads to the nonsmoothness of the algebraic system in the above hybrid system.

For the constrained Hamilton system (7), the VI formulation gives the following system:

$$\begin{cases} \dot{p}(t) = -\nabla_x H(t, x(t), u(t), p(t)) \\ \dot{x}(t) = \nabla_p H(t, x(t), u(t), p(t)) \\ u(t) = \mathrm{Pr}_{\Xi}(u(t) - \nabla_u H(t, x(t), u(t), p(t))) \\ x(0) = x^0 \text{ and } p(T) = \nabla_x \psi(T, x(T)), \end{cases} \tag{11}$$

where $H(t, x, u, p)$ is defined by (6). This is a system of ordinary differential equations constrained by a parameterized VI, called *differential variational inequality* (DVI for short). For a comprehensive treatment of the DVI, we refer to Pang and Stewart (2008).

The system (11) usually has no classic solution, and we have to seek the weak solution $(x(t), p(t), u(t))$, where x and p are absolutely continuous and u is integrable on $[0, T]$ such that $\forall 0 \le s \le t \le T$:

$$x(t) - x(s) = \int_s^t \nabla_p H\big(\tau, x(\tau), u(\tau), p(\tau)\big) d\tau,$$

and

$$p(t) - p(s) = -\int_s^t \nabla_x H\big(\tau, x(\tau), u(\tau), p(\tau)\big) d\tau,$$

and $u(t) = \mathrm{Pr}_{\Xi}(u(t) - \nabla_u H(t, x(t), u(t), p(t)))$ holds for almost all $t \in [0, T]$.

2 Multi-agent Optimal Control Problems

2.1 Problem Description

The multi-agent optimal control problem involves a decision process with multiple agents, where each agent solves an optimal control problem with his own cost functional and admissible control set. Each agent's cost functional is, and its admissible control set may be, dependent on all the other agents' state and control variables. Such a problem is also referred as the Nash equilibrium problem, where the agent is usually called as *player*.

Denote by $x_\nu \in \mathbb{R}^{n_\nu}$ and $u_\nu \in \mathbb{R}^{m_\nu}$ the ν-th player's state and control variables, respectively. The control is also called as strategy, action or decision. Collectively write $x = (x_\nu)_{\nu=1}^N \in \mathbb{R}^n$, $u = (u_\nu)_{\nu=1}^N \in \mathbb{R}^m$, $x_{-\nu} = (x_{\nu'})_{\nu' \neq \nu} \in \mathbb{R}^{n-n_\nu}$ and $u_{-\nu} = (u_{\nu'})_{\nu' \neq \nu} \in \mathbb{R}^{m-m_\nu}$, where $n = \sum_{\nu=1}^N n_\nu$ and $m = \sum_{\nu=1}^N m_\nu$. When we emphasize the ν-th player's state and strategy variables, we use the block form $x = (x_\nu, x_{-\nu})$ and $u = (u_\nu, u_{-\nu})$ to represent x and u, respectively. For the ν-th player, we denote

- admissible control set (the strategy set) by

$$\Xi_\nu(u_{-\nu}) = \big\{u_\nu | h_\nu(u_\nu) \le 0, g(u_\nu, u_{-\nu}) \le 0\big\},$$

  where $h_\nu(\cdot) : \mathbb{R}^{n_\nu} \to \mathbb{R}^{l_\nu}$ and $g(\cdot, u_{-\nu}) : \mathbb{R}^{n_\nu} \to \mathbb{R}^\ell$;
- the state dynamic by

$$\begin{cases} dx_\nu(t) = f_\nu(t, x_\nu(t), u_\nu(t))dt + \sigma_\nu(t, x_\nu(t), u_\nu(t))dW(t) \\ x_\nu(0) = x_\nu^0, \end{cases} \tag{12}$$

  where $x_\nu^0 \in \mathbb{R}^{n_\nu}$ is the initial state, $f_\nu : [0, T] \times \mathbb{R}^{n_\nu} \times \Xi \to \mathbb{R}^n$ is the drift term, $\sigma_\nu : [0, T] \times \mathbb{R}^{n_\nu} \times \Xi \to \mathbb{R}^{n_\nu \times d_\nu}$ is the diffusion term;
- the cost functional by

$$J_\nu\big(x(\cdot), u(\cdot)\big) := \mathbb{E}\left\{\int_0^T \varphi_\nu\big(t, x(t), u(t)\big)dt + \psi_\nu\big(T, x_\nu(T)\big)\right\}, \tag{13}$$

  where $\psi_\nu : [0, T] \times \mathbb{R}^{n_\nu} \to \mathbb{R}$ and $\varphi_\nu : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, and $T > 0$ is the terminal time.

Writing

$$J_\nu\big(x(\cdot), u(\cdot)\big) = J_\nu\big(x_\nu(\cdot), x_{-\nu}(\cdot), u_\nu(\cdot), u_{-\nu}(\cdot)\big),$$

the solution (or called the equilibrium point) of the multi-agent optimal control problem is a state-control pair $(x^*(\cdot), u^*(\cdot))$ satisfying: for fixed $x^*_{-\nu}(\cdot)$ and $u^*_{-\nu}(\cdot)$, $(x^*_\nu(\cdot), u^*_\nu(\cdot))$ is a solution of the following optimal control problem

$$\begin{aligned} &\min\ J_\nu\big(x_\nu(\cdot), x^*_{-\nu}(\cdot), u_\nu(\cdot), u^*_{-\nu}(\cdot)\big) \\ &\text{s.t. } dx_\nu(t) = f_\nu\big(t, x_\nu(t), u_\nu(t)\big)dt + \sigma_\nu\big(t, x_\nu(t), u_\nu(t)\big)dW(t) \\ &\qquad x_\nu(0) = x_\nu^0 \\ &\qquad u_\nu(t) \in \Xi_\nu\big(u^*_{-\nu}(t)\big) \quad \text{for almost all } t \in [0, T]. \end{aligned} \tag{14}$$

Note that $\Xi_\nu(\cdot)$ is a set-valued mapping given by the shared constraint $g(u_\nu, u_{-\nu}) \le 0$, namely, the ν-th player's strategy set is dependent on its rivals' states and controls. Without the shared constraint, the strategy set Ξ_ν is constant, and then the problem (14) reduces to the standard dynamic Nash equilibrium problem.

Here we make the following blanket assumptions on the convexity of the strategy set.

Assumption 1 Suppose that all the components of h_ν and g are convex for any ν.

2.2 Reformulation of System of HJB Equations

Define the value function for the ν-th player:

$$
\begin{aligned}
v_\nu(t,x) := \min\ & J_\nu\big(x_\nu(\cdot), x^*_{-\nu}(\cdot), u_\nu(\cdot), u^*_{-\nu}(\cdot)\big)\\
\text{s.t. }\ & dx_\nu(s) = f_\nu\big(s, x_\nu(s), u_\nu(s)\big)ds + \sigma_\nu\big(s, x_\nu(s), u_\nu(s)\big)dW(s)\\
& x_\nu(t) = x_\nu\\
& u_\nu(s) \in \Xi_\nu(u^*_{-\nu}(s)) \quad \text{for almost all } s \in [t,T].
\end{aligned}
$$

Denote $\chi_\nu(t,x,u) = \frac{1}{2}\|\sigma_\nu(t,x,u)\|_F^2$, and denote

$$
\begin{aligned}
&H_\nu(t,x,u,\nabla_{x_\nu}v_\nu, \Delta_{x_\nu}v_\nu)\\
&\quad = \chi_\nu(t,x_\nu,u_\nu)\Delta_{x_\nu}v_\nu(t,x) + \big\langle f_\nu(t,x_\nu,u_\nu), \nabla_{x_\nu}v_\nu(t,x)\big\rangle + \varphi_\nu(t,x,u), \quad (15)
\end{aligned}
$$

$\Delta_{x_\nu}v_\nu(t,x)$ and $\nabla_{x_\nu}v_\nu(t,x)$ denote the Laplacian and the gradient of v_ν in x_ν, respectively. We suppose in our setting that $H_\nu(t,x,u,\nabla_{x_\nu}v_\nu, \Delta_{x_\nu}v_\nu)$ is continuously differentiable in u_ν. Write $v_\nu(T,x) = v_\nu(T,x_\nu,x_{-\nu})$. Then the HJB equation (5) for the problem (14) has the following form:

$$
\begin{cases}
\frac{\partial v_\nu(t,x)}{\partial t} + \min_{u_\nu \in \Xi_\nu(u_{-\nu})} H_\nu(t,x,u,\nabla_{x_\nu}v_\nu, \Delta_{x_\nu}v_\nu) = 0,\\
v_\nu(T,x_\nu,x_{-\nu}) = \psi_\nu(T,x_\nu).
\end{cases}
\quad (16)
$$

Applying the VI formulation (9) to characterize the optimality of the minimization in (16), it gives

$$
\begin{cases}
\frac{\partial v_\nu(t,x)}{\partial t} + H_\nu(t,x,u,\nabla_{x_\nu}v_\nu, \Delta_{x_\nu}v_\nu) = 0,\\
u_\nu \in \mathrm{SOL}(\Xi_\nu(u_{-\nu}), \nabla_{u_\nu}H_\nu(t,x,u,\nabla_{x_\nu}v_\nu, \Delta_{x_\nu}v_\nu))\\
v_\nu(T,x_\nu,x_{-\nu}) = \psi_\nu(T,x_\nu).
\end{cases}
\quad (17)
$$

Now we have for each player one partial differential equation, which is parameterized by the rivals' states and controls, and is subject to the parameterized VI. We are going to collect all such equations into one system, whose solution may give an equilibrium state of the multi-agent optimal control. Denote

$$
\Xi(u) = \prod_{\nu=1}^{N} \Xi_\nu(u_{-\nu}), \quad \big(\mathrm{R}^m \rightrightarrows \mathbb{R}^m\big)
$$

here we mention that Ξ is a set-valued mapping. Collecting all the value functions, we have the value function profile:

$$
V(t,x) = \big(v_\nu(t,x)\big)_{\nu=1}^N, \quad \big([0,T]\times\mathbb{R}^n \to \mathbb{R}^N\big)
$$

which is to be computed. Collecting all the terminal payoff, we have the profile:

$$\Psi(t,x) = \big(\psi_\nu(t,x_\nu)\big)_{\nu=1}^N. \quad \big([0,T]\times\mathbb{R}^n \to \mathbb{R}^N\big)$$

Collecting the functions defining the HJB equations, we have

$$F(t,x,u,V) = \big(H_\nu(t,x,u,\nabla_{x_\nu} v_\nu, \Delta_{x_\nu} v_\nu)\big)_{\nu=1}^N,$$

and collecting the functions defining the parameterized VIs, we have

$$G(t,x,u,V) = \big(\nabla_{u_\nu} H_\nu(t,x,u,\nabla_{x_\nu} v_\nu, \Delta_{x_\nu} v_\nu)\big)_{\nu=1}^N.$$

Then by concentrating the HJB equations of the form (17) and the parameterized VIs for all the players, we have the following system

$$\begin{cases} \frac{\partial V(t,x)}{\partial t} + F(t,x,u,V) = 0, \\ u \in \mathrm{SOL}(\Xi(u), G(t,x,u,V)) \\ V(T,x) = \Psi(T,x). \end{cases} \tag{18}$$

Here $u \in \mathrm{SOL}(\Xi(u), G(t,x,u,V))$ is meant given t, x, V fixed, it holds

$$(z-u)^T G(t,x,u,V) \geq 0, \quad \forall z \in \Xi(u).$$

This is just a *quasi* variational inequality (QVI for short). Then (18) is a system of partial differential equations constrained by a QVI.

Because of the complex structure of $\Xi(u)$, it is hard to analyze the solvability and the convergence of numerical algorithms for solving (18). Here we try to propose a VI-based formulation, instead of the quasi one. Denote

$$\Xi = \big\{u \in \mathbb{R}^m | h_\nu(u_\nu) \leq 0, g(u_\nu, u_{-\nu}) \leq 0\big\}. \tag{19}$$

Assumption 1 ensures that Ξ is closed and convex. The following lemma states that the solvability of the VI implies the solvability of the quasi VI.

Lemma 1 (Facchinei et al. 2007) *For any fixed t, x and V, we have*

$$\mathrm{SOL}\big(\Xi, G(t,x,\cdot,V)\big) \subseteq \mathrm{SOL}\big(\Xi(u), G(t,x,\cdot,V)\big).$$

This lemma justifies the VI-based reformulation of the multi-agent optimal control problem:

$$\begin{cases} \frac{\partial V(t,x)}{\partial t} + F(t,x,u,V) = 0, \\ u \in \mathrm{SOL}(\Xi, G(t,x,u,V)) \\ V(T,x) = \Psi(T,x). \end{cases}$$

Moreover, using the projection formulation of the VI, we equivalently rewrite the above system of HJB equations as the following form, which is a system of partial differential equations, along with the boundary value conditions and the algebraic constraints:

$$\begin{cases} \frac{\partial V(t,x)}{\partial t} + F(t,x,u,V) = 0, \\ u = \mathrm{Pr}_{\Xi}(u - G(t,x,u,V)) \\ V(T,x) = \Psi(T,x). \end{cases} \tag{20}$$

Note that the algebraic constraints is defined by a system of equations that is nonsmooth, as the projection operator is nonsmooth.

2.3 Reformulation of Hamilton System

For the deterministic case: $\sigma_\nu(t, x_\nu, u_\nu) \equiv 0$ for $\nu = 1, \ldots, N$, we introduce the costate variable $p_\nu = \nabla_{x_\nu} v_\nu(t, x)$. Then for the ν-th player the Hamiltonian reads

$$H_\nu(t,x,u,p_\nu) = \langle f_\nu(t,x_\nu,u_\nu), p_\nu \rangle + \varphi_\nu(t,x,u),$$

and we have the following constrained Hamilton system

$$\begin{cases} \dot{p}_\nu(t) = -\nabla_{x_\nu} H_\nu(t,x,u,p_\nu) \\ \dot{x}_\nu(t) = \nabla_{p_\nu} H_\nu(t,x,u,p_\nu) \\ u_\nu(t) \in \mathrm{SOL}(\Xi_\nu(u_{-\nu}), \nabla_{u_\nu} H_\nu(t,x,\cdot,u_{-\nu},p_\nu)), \\ x_\nu(0) = x_\nu^0 \text{ and } p_\nu(T) = \nabla_{x_\nu} \psi_\nu(T, x(T)), \end{cases} \tag{21}$$

where we write $H_\nu(t,x,u,p_\nu) = H_\nu(t,x,u_\nu,u_{-\nu},p_\nu)$ for emphasizing the dependence of the mapping $H_\nu(t,x,u,p_\nu)$ on the rivals' control variables $u_{-\nu}$, and where $\nabla_{x_\nu} \psi_\nu(t,x_\nu)$ denotes the gradient of $\psi_\nu(t,x_\nu)$ with respect to x_ν.

Collectively write

$$G(t,x,u,p) = \left(\nabla_{u_\nu} H_\nu(t,x,u,p_\nu)\right)_{\nu=1}^N$$

and

$$\Gamma\big(x(0), p(0), x(T), p(T)\big) = \begin{pmatrix} x_\nu(0) - x_\nu^0 \\ p_\nu(T) - \nabla_{x_\nu} \psi_\nu(T, x(T)) \end{pmatrix}_{\nu=1}^N .$$

Concatenating (21) for $\nu = 1, \ldots, N$, we can formulate the multi-agent optimal control problem as the following differential quasi VI:

$$\begin{aligned} \dot{p}(t) &= \left(-\nabla_{x_\nu} H_\nu(t,x,u,p_\nu)\right)_{\nu=1}^N, \\ \dot{x}(t) &= \left(\nabla_{p_\nu} H_\nu(t,x,u,p_\nu)\right)_{\nu=1}^N, \\ u(t) &\in \mathrm{SOL}\big(\Xi(u), G(t,x,\cdot,p)\big) \\ 0 &= \Gamma\big(x(0), p(0), x(T), p(T)\big). \end{aligned} \tag{22}$$

Again, Lemma 1 implies a VI-based reformulation of the multi-agent optimal control problem:

$$\begin{aligned} \dot{p}(t) &= \big(-\nabla_{x_\nu} H_\nu(t,x,u,p_\nu)\big)_{\nu=1}^N, \\ \dot{x}(t) &= \big(\nabla_{p_\nu} H_\nu(t,x,u,p_\nu)\big)_{\nu=1}^N, \\ u(t) &\in \mathrm{SOL}\big(\Xi, G(t,x,\cdot,p)\big) \\ 0 &= \Gamma\big(x(0), p(0), x(T), p(T)\big), \end{aligned} \tag{23}$$

where Ξ is defined by (19). Moreover, we use the projection operator to reformulate the system (23) into the following system of differential algebraic equations:

$$\begin{aligned} \dot{p}(t) &= \big(-\nabla_{x_\nu} H_\nu(t,x,u,p_\nu)\big)_{\nu=1}^N, \\ \dot{x}(t) &= \big(\nabla_{p_\nu} H_\nu(t,x,u,p_\nu)\big)_{\nu=1}^N, \\ u(t) &= \mathrm{Pr}_\Xi\big(u - G(t,x,u,p)\big) \\ 0 &= \Gamma\big(x(0), p(0), x(T), p(T)\big). \end{aligned} \tag{24}$$

Write $\varphi_\nu(t,x,u) = \varphi_\nu(t,x_\nu,x_{-\nu},u_\nu,u_{-\nu})$. Suppose for any $\nu = 1,\dots,N$ that $\psi_\nu(T,\cdot)$ and each components of h_ν and $g(\cdot,u_{-\nu})$ are convex, and suppose that $\varphi_\nu(t,\cdot,x_{-\nu},\cdot,u_{-\nu})$ and each component of $\nabla_{p_\nu} H_\nu(t,x,u,p_\nu)$ are convex and continuously differentiable for any fixed $x_{-\nu}$ and $u_{-\nu}$, suppose that $\nabla_{p_\nu} H_\nu(t,x,u,p_\nu)$ is linear with respect to (x_ν,u_ν). Here we call (x,u) as a *feasible pair* of the multi-agent optimal control problem if $u \in \Xi$ and $\dot{x}_\nu(t) = \nabla_{p_\nu} H_\nu(t,x,u,p_\nu)$ for $\nu = 1,\dots,N$. Then we can show that

Theorem 1 *Let* (x^*,u^*) *be a weak solution of* (23). *Then* (x^*,u^*) *is a solution of the multi-agent optimal control problem in the following sense*: *for any feasible pair* (x,u), *we have for* $\nu = 1,\dots,N$:

$$J_\nu\big(x_\nu(\cdot), x^*_{-\nu}(\cdot), u_\nu(\cdot), u^*_{-\nu}(\cdot)\big) \ge J_\nu\big(x^*_\nu(\cdot), x^*_{-\nu}(\cdot), u^*_\nu(\cdot), u^*_{-\nu}(\cdot)\big).$$

Proof The details of the proof can be found in Chen and Wang (2013b). □

3 Approximation of Variational Inequality

The systems (20) and (24) concern the projection equation $u = \mathrm{Pr}_\Xi(u - G(t,x,u,V))$ and $u = \mathrm{Pr}_\Xi(u - G(t,x,u,p))$, respectively, which may have no solution, or have multiple (possibly infinitely many) solutions. Finding a solution of the systems involves solving optimization problems without standard constraint qualifications at each grid. Let $G : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$ be given for defining the concerned parameterized VI, where the parameter vector is taken in the space of $\mathbb{R}^k$. Denote

$$\Phi(u,\alpha) = u - \mathrm{Pr}_\Xi\big(u - G(u,\alpha)\big),$$

where α is a parameter, and we are interested in finding for a given parameter α a vector u satisfying

$$\Phi(u,\alpha) = 0. \tag{25}$$

Here we propose a regularized smoothing method to find a solution of (25). Our main idea is to replace $\Phi(u,\alpha)$ by the following regularized and smoothing function

$$\Phi_{\lambda,\mu}(u,\alpha) = \int_R \big[u - \mathrm{Pr}_{\Xi}\big(u - G(u,\alpha) - \lambda u - \mu s e\big)\big]\rho(s)ds, \tag{26}$$

where $\lambda > 0$ and $\mu > 0$ are the regularization and smoothing parameters. The integration is performed componentwise with $e = (1, 1, \ldots, 1)^T$ and $\rho(\cdot)$ is a density function with

$$\kappa = \int_R |s|\rho(s)ds < \infty.$$

Suppose that $G(\cdot,\alpha)$ is monotone for any fixed α. Then when $\mu = 0$, the regularized system

$$\Phi_{\lambda,0}(u,\alpha) := u - \mathrm{Pr}_{\Xi}\big(u - G(u,\alpha) - \lambda u\big) = 0$$

has a unique solution u for any fixed α, but $\Phi_{\lambda,0}$ and u may not be differentiable with respect to (t,x). To overcome the non-smoothness of the projection operator, we adopt the smoothing approximation. The regularized smoothing function $\Phi_{\lambda,\mu}(u,\alpha)$ has the following properties

$$\big\|\Phi_{\lambda,0}(u,\alpha) - \Phi(u,\alpha)\big\|_2 \le \lambda\|u\|_2$$

and

$$\big\|\Phi_{\lambda,\mu}(u,\alpha) - \Phi_{\lambda,0}(u,\alpha)\big\|_2 \le \kappa\sqrt{m}\mu.$$

For fixed $\alpha \in \mathbb{R}^k$, $\lambda > 0$ and $\mu > 0$ the mapping $\Phi_{\lambda,\mu}(\cdot,\alpha)$ is continuously differentiable and the system

$$\Phi_{\lambda,\mu}(u,\alpha) = 0 \tag{27}$$

has a unique solution $u_{\lambda,\mu}(\alpha)$, which is continuously dependent on α. For $\lambda, \mu \downarrow 0$ (λ, μ chosen in an appropriate way—see also the second point in the summary) we approximate the solution of (25).

Namely, we approximate (20) by the following differential algebraic system

$$\begin{cases} \frac{\partial V(t,x)}{\partial t} + F(t,x,u,V) = 0, \\ \Phi_{\lambda,\mu}(t,x,u,V) = 0 \\ V(T,x) = \Psi(T,x), \end{cases} \tag{28}$$

and approximate (24) by the following differential algebraic system

$$\begin{aligned}\dot{p}(t) &= \big(-\nabla_{x_\nu} H_\nu(t,x,u,p)\big)_{\nu=1}^{N},\\ \dot{x}(t) &= \big(\nabla_{p_\nu} H_\nu(t,x,u,p)\big)_{\nu=1}^{N},\\ 0 &= \Phi_{\lambda,\mu}(t,x,u,p)\\ 0 &= \Gamma\big(x(0),p(0),x(T),p(T)\big).\end{aligned} \tag{29}$$

We mention four points on this methodology.

- Finding an equilibrium point of the multi-agent optimal control problem is of the great practical importance, which is quite hard because the problem is coupled by the optimization problems, dynamical systems and the side constraints. Existing methods normally can not treat the generalized case: the problem with coupled strategy sets (see for example Krabs et al. 2000; Krabs 2005; Krabs and Pickl 2010). The methodology proposed here is promising since it reformulates the multi-agent optimal control problem as a differential algebraic equation, for which abundant theory and algorithms can be utilized. This new approach will be extended in the future.
- The convergence of the solution of the approximating system to the original one is of the most interest. Suitable notions of convergence, for example the Γ-convergence, have to be carefully selected. The convergence may be considerably dependent on the dependence between λ and μ, different dependence defines different regularized smoothing system, and therefore the different system of differential algebraic equations. Now we are in the position to touch the next point.
- Smoothing approximation and regularization have been studied extensively in solving the static VI (Facchinei and Pang 2003). However, to the best of our knowledge, the impact of the dependence between the smoothing and regularization parameters on the convergence behavior has not been studied. An example can be found in Chen and Wang (2013a, 2013b), which shows that for different relations of the two parameters λ, μ, the solution $u_{\lambda,\mu}(\alpha)$ of (27) can be divergent, or convergent to different solutions of the original projection equation (25). For the system (27), if $\mu = o(\lambda)$ is taken, then $u_{\lambda,\mu}(\alpha)$ is convergent to the least norm element of the solution set of (25). Note that finding the least norm solution is significant since it can provide a stable solution (refer to Chen and Wang 2013b, for more details).
- Our methodology is variational, which employs the comparison of solutions in a neighborhood of the optimal one to derive the necessary conditions, and to obtain candidate of the optimal solutions. Of course we need to impose additional conditions for ensuring the optimality. A different approach, namely the direct method, is also available, which offers global optima by using coordinate transformations instead of comparison techniques (Leitmann 1962). This method can also be applied to a class of differential games (Leitmann 1976). It is our aim to combine these destinguished approaches in the future.

- As mentioned before the multi-agent optimal control problem involves a certain decision process with multiple agents, where each agent solves an optimal control problem with the individual cost functional and strategy set. As a specialty the cost functional itself is dependent on all the other agents' state and/or control variables.

  In a forthcoming contribution we would like to apply this specific model to decision problems in the context of complex aviation management processes. It is our aim to apply the gained algorithms to the solution of concrete decision problems which occur in this innovative context of Operations Research.

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Time-Consistent Equilibria in a Differential Game Model with Time Inconsistent Preferences and Partial Cooperation

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Abstract Differential games with time-inconsistent preferences are studied. Noncooperative Markovian Nash equilibria are described. If players can cooperate at every instant of time, time-consistent equilibria are analyzed for the problem with partial cooperation. Cooperation is partial in the sense that, although players cooperate at every moment t forming a coalition, due to the time inconsistency of the time preferences, coalitions at different times value the future in a different way, and they are treated as different agents. Time-consistent equilibria are obtained by looking for the Markovian subgame perfect equilibria in the corresponding noncooperative sequential game. The issue of dynamic consistency is then considered. In order to guarantee the sustainability of cooperation, players should bargain at every instant of time their weight in the whole coalition, and nonconstant weights are introduced. The results are illustrated with two examples: a common property resource game and a linear state pollution differential game.

1 Introduction

In the study of intertemporal choices it is customary in economics to consider the so-called Discounted Utility (DU) Model, introduced in Samuelson (1937). In the DU model, agent's time preferences are time-consistent and they are characterized by a single parameter, the constant discount rate of time preference. However, empirical observations seem to show that predictions of the DU model sometimes disagree with the actual behavior of decision makers (we refer to Frederick et al. 2002, for a review on the topic). In addition, if there are several players, although it is typically assumed that all economic agents have the same rate of time preference, there is no reason to believe that consumers, firms or countries have identical time preferences for utility streams. For instance, in a non-cooperative setting, Van Long et al. (1999) studied feedback Nash equilibria for the problem of extraction of exhaustible resources under common access in the case of different but constant discount rates.

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There are also problems—for instance, in the analysis of international trade agreements, climate change policies, or the exploitation of common property natural resources; we refer to Jørgensen et al. (2010) and Van Long (2011) for two recent surveys on dynamic games in these topics—in which it is natural to assume that players can communicate and coordinate their strategies in order to optimize their collective payoff. In this cooperative framework, if time preferences of players in the problem are time-inconsistent, or they apply different discount rates (constant or not), the notion of Pareto efficiency is lost. For the case of constant but different discount rates in a discrete time setting, Sorger (2006) proposed a recursive Nash bargaining solution. Also in a discrete time setting, Breton and Keoula (2014) studied the stability of coalitions in a resource economics model. In a continuous time setting, De Paz et al. (2013) (see also Marín-Solano and Shevkoplyas 2011) studied the problem of asymmetric players under two fundamental assumptions: all players commit themselves to cooperate at every instant of time t, and the different t-coalitions (with different time-preferences) lack precommitment power. Equilibria were computed by finding subgame perfect equilibria in a noncooperative sequential game where agents are the different t-coalitions (representing, for instance, different generations). Hence, this solution to the problem is time-consistent provided that all players commit to cooperate at every instant of time t.

The objectives of this chapter are the following. First, results derived, in a continuous time setting, in Karp (2007) and Ekeland and Lazrak (2010) for the case of time-distance and nonconstant discount functions are extended to a noncooperative differential game with more general discount functions. This is in fact a straightforward generalization of the results in Marín-Solano and Shevkoplyas (2011). Then, attention is addressed to extend the setting of partial cooperation among players studied in De Paz et al. (2013). In order to guarantee the stability of the grand coalition, nonconstant weights are introduced, so that players can bargain their weight in the grand coalition at every instant of time. Strictly speaking, although the proposed solution assumes cooperation among players at every instant of time t, it is a noncooperative Markovian Nash equilibrium for the non-cooperative sequential game defined by these infinitely many t-coalitions. In this sense we call this solution a time-consistent equilibrium with partial cooperation. It is important to realize that, in the standard case with a common and constant discount rate for all players, if weights are constant, standard dynamic optimization techniques (the Pontryagin's maximum principle, or the Hamilton–Jacobi–Bellman equation) provide time-consistent solutions. However, there are cases in which no constant weights exist guaranteeing the sustainability of cooperation along time (see e.g. Yeung and Petrosyan 2006, and references therein). For this standard problem, the introduction of nonconstant weight provides a way to construct dynamically consistent solutions guaranteeing the stability of the grand coalition. The price to be paid is that the proposed solution with nonconstant weights is found for a problem with time-inconsistent preferences, and this makes the problem less computationally tractable. Maybe, what is more relevant is to check which are the effects of introducing time-inconsistent preferences in economic models. First, a simple common access resource game solved in Clemhout and Wan (1985) is studied by introducing heterogeneity and nonconstancy in the discount rates. Finally, a linear state pollution differential game with the same kind of time preferences model is also studied. Along the paper we will assume that players are rational, in the sense that they are aware of the changing preferences and they look for time-consistent solutions.

The chapter is organized as follows. In Sect. 2, we describe the noncooperative problem. The problem with partial cooperation and nonconstant weights is studied in Sect. 3. Finally, Sect. 4 analyzes the two above mentioned models coming from the field of environmental and resource economics.

2 Markovian Nash Equilibria in Noncooperative Differential Games with Time-Inconsistent Preferences

Within the framework of the (β, δ)-preferences introduced in Phelps and Pollak (1968), differential games with time-inconsistent preferences were already studied in Alj and Haurie (1983). In that paper the authors analyzed intergenerational equilibria, extending previous definitions and results to stochastic games and intragenerational conflicts. More recent references on the topic are Haurie (2005), Nowak (2006) and Balbus and Nowak (2008). In this section we study Markovian Nash equilibria in differential games for a rather general model with time-inconsistent preferences.

First, let us review the problem in case of just one decision maker. Let $x = (x^1, \ldots, x^n) \in X \subset \mathbf{R}^n$ be the vector of state variables, $u = (u^1, \ldots, u^m) \in U \subset \mathbf{R}^m$ the vector of control (or decision) variables, $L(x(s), u(s), s)$ the instantaneous utility function at time s, T the planning horizon (terminal time) and $S(x(T), T)$ the final (scrap or bequest) function. Let $d(s,t)$ be an arbitrary discount function representing how the agent a time t (the so-called *t-agent* in the hyperbolic discounting literature) discounts utilities enjoyed at time s. For instance, if $d(s,t) = e^{-r(s-t)}$ we recover the standard problem with a constant instantaneous discount rate of time preference. In the case of time-distance discounting with a nonconstant discount rate, $d(s,t) = \theta(s-t) = \exp(-\int_0^{s-t} r(\tau)d\tau)$. For our general problem, an agent taking decisions at time t (the t-agent) aims to maximize

$$J(x,u,t) = \int_t^T d(s,t)L\big(x(s), u(s), s\big)ds + d(T,t)S\big(x(T), T\big), \tag{1}$$

with

$$\dot{x}^i(s) = g^i\big(x(s), u(s), s\big), \quad x^i(t) = x^i_t, \text{for } i = 1, \ldots, n.$$

In Problem (1) we assume that functions $L(x,u,s)$, $S(x,T)$ and $g^i(x,u,s)$, $i = 1, \ldots, n$, are continuously differentiable in all their arguments. In the following we will also assume that $d(s,t)$ is continuously differentiable in both arguments. In general, unless the discount function is multiplicatively separable in time s and the planning date t, i.e. $d(s,t) = d_1(s)d_2(t)$, for all $t \in [0, T]$, $s \in [t, T]$, the optimal solution from the viewpoint of the agent at time t will be no longer optimal for future s-agents. Hence, the solution provided by the use of standard optimal control techniques (such as the Pontryagin's maximum principle, or the Hamilton–Jacobi–Bellman equation) is time inconsistent. In this paper we center our interest in the search of time-consistent solutions (agents are sophisticated, according to the literature of hyperbolic preferences).

In order to solve Problem (1), an intuitive way to derive a dynamic programming equation is to discretize it, find later on the Markov perfect equilibrium in the corresponding sequential game and define finally the equilibrium rule of the original problem by passing to the continuous time limit (provided that it exists). This is the approach followed in Karp (2007) in the derivation of a dynamic programming equation extending the classical Hamilton–Jacobi–Bellman equation for the problem of time-distance discounting with a nonconstant discount rate of time preference. Alternatively, we can follow the approach introduced in Ekeland and Lazrak (2010) (later on extended in Ekeland and Pirvu 2008, to a stochastic setting) for the same problem. Next we briefly describe the latter procedure and the corresponding results derived in Marín-Solano and Shevkoplyas (2011).

If $u^*(s) = \phi(s, x(s))$ is the equilibrium rule, then the value function is given by

$$V(x,t) = \int_t^T d(s,t)L\big(x(s), \phi\big(x(s), s\big), s\big)ds + d(T,t)S\big(x(T), T\big) \tag{2}$$

where $\dot{x}(s) = g(x(s), \phi(x(s), s), s)$, $x(t) = x_t$. Next, for $\varepsilon > 0$, let us consider the variations

$$u_\varepsilon(s) = \begin{cases} v(s) & \text{if } s \in [t, t+\varepsilon], \\ \phi(x, s) & \text{if } s > t+\varepsilon. \end{cases}$$

If the t-agent has the ability to precommit her behavior during the period $[t, t+\varepsilon]$, the value function for the perturbed control path u_ε is given by

$$\begin{aligned} V_\varepsilon(x,t) = \max_{\{v(s), s\in[t,t+\varepsilon]\}} \bigg\{ & \int_t^{t+\varepsilon} d(s,t)L\big(x(s), v(s), s\big)ds \\ & + \int_{t+\varepsilon}^T d(s,t)L\big(x(s), \phi\big(x(s), s\big), s\big)ds + d(T,t)S\big(x(T), T\big)\bigg\}. \end{aligned} \tag{3}$$

Definition 1 A decision rule $u^*(s) = \phi(s, x(s))$ is called an equilibrium rule if

$$\lim_{\varepsilon\to 0^+} \frac{V(x,t) - V_\varepsilon(x,t)}{\varepsilon} \geq 0.$$

This definition of equilibrium rule is rather weak, as explained, e.g., in Ekeland et al. (2012), and in particular is satisfied by the optimal solutions in a classical optimal control problem. Concerning regularity conditions, in Karp (2007) and Ekeland and Lazrak (2010) it was assumed that decision rules were differentiable. This condition was not assumed in Ekeland and Pirvu (2008). In fact, the differentiability of the decision rule is not needed in the derivation of the following result (see Marín-Solano and Shevkoplyas 2011, for a proof): if the value function is of class C^1, then the solution $u = \phi(x,t)$ to the integral equation (2) with

$$u^* = \phi(x,t) = \arg\max_u \big[L(x,u,t) + \nabla_x V(x,t) g(x,u,t)\big]$$

is an equilibrium rule, in the sense that it satisfies Definition 1.

If there is no final function and $T = \infty$, in Marín-Solano and Shevkoplyas (2011) it was proved that, if there exists a bounded value function of class C^1 solving the integral equation

$$V(x,t) = \int_t^\infty d(s,t) L\big(x(s), \phi\big(x(s),s\big),s\big) ds \tag{4}$$

where

$$u^* = \phi(x,t) = \arg\max_u \big[L(x,u,t) + \nabla_x V(x,t) g(x,u,t)\big], \tag{5}$$

then $u^* = \phi(x,t)$ is an equilibrium rule, in the sense that it satisfies Definition 1.

In order to guarantee the finiteness of the integral in (4), Ekeland and Lazrak (2010) restrict their attention to convergent policies (i.e. equilibrium rules such that the corresponding state variables converge to an stationary state).

We can easily generalize the previous results to multi-agent problems. Let us consider a differential game defined on $[0,T]$. The state of the game at time t is described by a vector $x \in X \subseteq \mathbf{R}^n$. The initial state is fixed, $x(0) = x_0$. There are N players. Let $u_i(t) \in U_i \subseteq \mathbf{R}^{m_i}$ be the control variables of player i. Each agent i at time t seeks to maximize in u_i her objective functional

$$J_i\big(x,t,u_1(s),\ldots,u_N(s)\big) = \int_t^T d_i(s,t) L_i\big(x(s),u_1(s),\ldots,u_N(s),s\big) ds \\ + d_i(T,t) S_i\big(x(T),T\big)$$

subject to

$$\dot{x}(s) = g\big(x(s),u_1(s),\ldots,u_N(s),s\big), \quad x(t) = x_t. \tag{6}$$

In a noncooperative setting with simultaneous play, we restrict our attention to the case when players apply Markovian strategies, $u_i(t) = \phi_i(x,t)$, for $i = 1,\ldots,N$. Note that open-loop strategies are not appropriate for our problem, since time-consistent players with time-inconsistent preferences decide at each time t according to their new time preferences, and taking into account the value of the state variable at time t, x_t. Time-consistent Markovian Nash equilibria in a noncooperative differential game can be obtained as a generalization of the results for one decision maker. Let $(\phi_1^{nc},\ldots,\phi_N^{nc})$ be a N-tuple of functions $\phi_i^{nc} : X \times [0,T] \to \mathbf{R}^{m_i}$, $i = 1,\ldots,N$, such that the following assumptions are satisfied:

1. There exists a unique absolutely continuous curve $x : [0,T] \to X$ solution to

$$\dot{x}(t) = g\big(x(t), \phi_1\big(x(t),t\big),\ldots,\phi_N\big(x(t),t\big)\big), \quad x(0) = x_0.$$

2. For all $i = 1, 2, \ldots, N$, there exists a continuously differentiable function $V_i^{nc} : X \times [0, T] \to \mathbf{R}$ verifying the integral equation

$$V_i^{nc}(x,t) = \int_t^T d_i(s,t) L_i\big(x(s), \phi_1 nc\big(x(s),s\big), \ldots, \phi_N^{nc}\big(x(s),s\big), s\big) ds$$
$$+ d_i(T,t) S_i\big(x(T),T\big), \quad V_i(x,T) = S_i(x,T),$$

where

$$\begin{aligned} u_i^{nc} &= \phi_i^{nc}(x,t) \\ &= \arg\max_{\{u_i\}} \big\{ L_i\big(x, \phi_1^{nc}(x,t), \ldots, \phi_{i-1}^{nc}(x,t), u_i, \phi_{i+1}^{nc}(x,t), \ldots, \phi_N^{nc}(x,t), t\big) \\ &\quad + \nabla_x V_i^{nc}(x,t) \\ &\quad \times g\big(x, \phi_1^{nc}(x,t), \ldots, \phi_{i-1}^{nc}(x,t), u_i, \phi_{i+1}^{nc}(x,t), \ldots, \phi_N^{nc}(x,t), t\big) \big\}. \end{aligned} \quad (7)$$

Then the strategy $(\phi_1^{nc}(x,t), \ldots, \phi_N^{nc}(x,t))$ is a time-consistent Markov Nash equilibrium, and $V_i^{nc}(x,t)$, $i = 1, \ldots, N$, are the corresponding value functions for all players in the noncooperative differential game.

In an infinite horizon setting ($T = \infty$ and there is no final function), equations (4) and (5) generalize as follows. Let $(\phi_1^{nc}, \ldots, \phi_N^{nc})$ be a N-tuple of functions $\phi_i^{nc} : X \times [0, \infty) \to \mathbf{R}^{m_i}$ such that the following assumptions are satisfied:

1. There exists a unique absolutely continuous curve $x : [0, \infty) \to X$ solution to

$$\dot{x}(t) = g\big(x(t), \phi_1^{nc}\big(x(t),t\big), \ldots, \phi_N^{nc}\big(x(t),t\big)\big), \quad x(0) = x_0,$$

2. For all $i = 1, 2, \ldots, N$, there exists a bounded continuously differentiable function $V_i : X \times [0, \infty) \to \mathbf{R}$ verifying the integral equation

$$V_i^{nc}(x,t) = \int_t^\infty d_i(s,t) L_i\big(x(s), \phi_1^{nc}\big(x(s),s\big), \ldots, \phi_N^{nc}\big(x(s),s\big), s\big) ds,$$

where $u_i^{nc} = \phi_i^{nc}(x,t)$ solves (7).

Then the strategy $(\phi_1^{nc}(x,t), \ldots, \phi_N^{nc}(x,t))$ is a time-consistent Markov Nash equilibrium, and $V_i^{nc}(x,t)$, $i = 1, \ldots, N$, are the corresponding value functions.

3 Time-Consistent Solutions in a Differential Game with Asymmetric Players Under Partial Cooperation

In the analysis of intertemporal decision problems with several agents, when players can communicate and coordinate their strategies in order to optimize their collective payoff, cooperative solutions are introduced. If there is a unique and constant discount rate of time preference for all agents, the Pareto efficient solution is easily obtained by solving a standard optimal control problem. However, in the case of different discount rates or time inconsistent preferences, when looking for time-consistent cooperative solutions, standard dynamic optimization techniques fail. In these cases, when agents lack commitment power but they decide to cooperate at every instant of time, they act at different times t as sequences of independent coalitions (the t-coalitions). The solution we propose in this chapter, which is an extension of that in De Paz et al. (2013) (see also Marín-Solano and Shevkoplyas 2011) assumes cooperation among players at every time t, but is a non-cooperative (Markovian Nash) equilibrium for the non-cooperative sequential game defined by these infinitely many t-coalitions.

In this section, we tackle the problem of maximizing

$$J^c = \sum_{i=1}^{N} \lambda_i(x_t, t) \int_t^T d_i(s,t) L_i\big(x(s), u_1(s), \ldots, u_N(s), s\big) ds \tag{8}$$

subject to (6), where $\lambda_i(x_t, t) \geq 0$, for every $i = 1, \ldots, N$, and $\sum_{i=1}^{N} \lambda_i(x_t, t) = N$. Coefficients $\lambda_i(x_t, t)$ represent the bargaining power of agent i at time t.

Note that, in general, there are two sources of time-inconsistency in Problem (8). First, there is the time-consistency problem related to the changing time preferences of the different t-coalitions, as we have discussed in the previous paragraphs. In addition, if players are not committed themselves to cooperate at every instant of time t, a problem of dynamic inconsistency or time-inconsistency can arise, independently of the form of the discount function: it is possible that players initially agree on a cooperative solution that generates incentives for them, but it is profitable for some of them to deviate from the cooperative behavior at later periods. Haurie (1976) proved that the extension of the Nash bargaining solution to differential games is typically not dynamically consistent. We refer to Zaccour (2008) for a recent review on the topic. For the case of transferable utilities, if the agents can redistribute the joint payoffs of players in any period, Petrosyan proposed in a series of papers a payoff distribution procedure in order to solve this problem of dynamic inconsistency (see e.g. Yeung and Petrosyan 2006; Petrosyan and Zaccour 2003, and references therein).

In De Paz et al. (2013) this issue of dynamic consistency (related to the stability of the whole coalition) was not considered. In that paper it was assumed that weights are given and constant. Agents commit themselves to cooperate at every instant of time t. There are several problems in which this seems to be a rather reasonable assumption, since players necessarily cooperate. Consider for instance, the intra-personal problem of a decision maker who faces how to allocate her money in order to buy different goods that she values in a different way (different utility functions and different impatience degree or discount rate). In a similar way, there is the problem of a family whose members take consumption decisions according to different preferences. There are also problems in which it is always profitable to cooperate, because if they do not cooperate they obtain nothing. For this kind of problems in which cooperation is guaranteed, equilibria were computed by finding subgame perfect equilibria in a noncooperative sequential game where players are the different t-coalitions (representing, for instance, different generations). However, in general the sustainability of cooperation can not be assured. For instance, in a discrete time setting, Breton and Keoula (2014) illustrated how, for a simple model of management of a renewable natural resource, if players apply different discount rates and have equal weights, the sustainability of cooperation is lost. If utilities are transferable, payoff (imputation) distribution procedures can be introduced in order to guarantee the stability of the whole coalition, extending in an easy way this method to the problem with asymmetric players and time inconsistent preferences, as in the case of differential games with time-distance non-constant discounting (see Marín-Solano and Shevkoplyas 2011).

If utilities are not transferable, we refer to Yeung and Petrosyan (2006) for a study in some models of constant weights guaranteeing the dynamic consistency of the whole coalition. Non surprisingly, they found that there are problems in which such constant weights guaranteeing the sustainability of cooperation do not exist. Sorger (2006) proposed, in a multiperiod (discrete time) setting with two asymmetric players, a recursive Nash bargaining solution which gives rise to a dynamically consistent equilibrium, by assuming that weights are bargained at each period of time and are therefore state-dependent. In this paper we depart from the model in De Paz et al. (2013) and consider the possibility that weights depend in general on the moment t at which the decision is taken, and also on the current state x_t. Hence, at time t, given the initial state x_t, and knowing which will be the equilibrium decision rule of future s-agents, $s > t$, the members of the coalition decide their decision rule and bargain their current weight in the coalition. Since the equilibrium rule of future s-agents depends on the changing preferences and, also, on the changing weights, the members of the coalition decide at time t their decision rule and also their current weights by taking into account this information. Non surprisingly, in our model, in order to guarantee the sustainability of the cooperation, weights λ_m of players in whole coalition should be non-constant, in general, but a result of a bargaining procedure at every time t. This applies also to the problem with constant and equal discount rates of time preference. As we present in the Introduction, the price to pay if weights are assumed to be of the form $\lambda_i(x, t)$ is that the solution obtained by applying the standard optimal control techniques is time-inconsistent also in the case of equal and constant discount rates, hence the problem should be solved always as a problem with time-inconsistent preferences.

Let us briefly analyze first the problem in which all players have equal (and constant) weights in the whole coalition. The objective of the whole coalition is then to find a time-consistent solution to the problem of maximizing

$$J^c = \sum_{i=1}^{N} \lambda_m \int_t^T d_i(s,t) L_i\big(x(s), u_1(s), \ldots, u_N(s), s\big) ds \tag{9}$$

subject to (6). As we prove later, for this problem, if $V_i^c(x, t)$, $i = 1, \ldots, N$, is a set of continuously differentiable functions in (x, t) characterizing the value function of all agents in the problem, then the decision rule $(u_1^c, \ldots, u_N^c) =$

$(\phi_1^c(x,t), \dots, \phi_N^c(x,t))$ solving

$$\max_{\{u_1,\dots,u_N\}} \left\{ \sum_{i=1}^{N} \lambda_i L_i(x, u_1, \dots, u_N, t) + \sum_{i=1}^{N} \lambda_i \nabla_x V_i^c(x,t) g(x, u_1, \dots, u_N, t) \right\}$$

with

$$V_i^c(x,t) = \int_t^T d_i(s,t) L_i\big(x(s), \phi_1^c\big(x(s), s\big), \dots, \phi_N^c(x,s), s\big) ds, \tag{10}$$

for every $i = 1, \dots, N$, is a (time-consistent) Markov Perfect Equilibrium for the problem with partial cooperation (9). The extension to the infinite horizon problem is straightforward.

Next, let us consider Problem (8). If $u_i^c(s) = \phi_i^c(s, x(s))$, $i = 1, \dots, N$, is the equilibrium rule, then the joint value function is

$$V^c(x,t) = \sum_{i=1}^{N} \lambda_i(x,t) \int_t^\tau d_i(s,t) L_i\big(x(s), \phi_1^c\big(x(s), s\big), \dots, \phi_N^c(x,s), s\big) ds. \tag{11}$$

The planning horizon τ can be finite or infinite. We assume that, if $\tau = \infty$, along the equilibrium rule, the value function (11) is finite (i.e. the integral converges). This is guaranteed if we restrict our attention to convergent policies (along the equilibrium rule the state variables converge to a stationary state). Hence we have:

Theorem 1 *If there exists a value function of class C^1 solving the set of N integral equations* (10) *where*

$$\begin{aligned}
\big(u_1^c, \dots, u_N^c\big) &= \big(\phi_1^c(x,t), \dots, \phi_N^c(x,t)\big) \\
&= \arg \max_{\{u_1,\dots,u_N\}} \left\{ \sum_{i=1}^{N} \lambda_i(x,t) \big(L_i(x, u_1, \dots, u_N, t) \right. \\
&\qquad \left. + \nabla_x V_i^c(x,t) g(x, u_1, \dots, u_N, t)\big) \right\},
\end{aligned} \tag{12}$$

and there exists a unique absolutely continuous curve $x : [0, \tau] \to X$ solution to $\dot{x}(t) = g(x(t), \phi_1^c(x(t), t), \dots, \phi_N^c(x(t), t))$, $x(0) = x_0$, then $(u_1^c, \dots, u_N^c) = (\phi_1^c(x,t), \dots, \phi_N^c(x,t))$ is an equilibrium rule for Problem (8), *in the sense that it satisfies Definition* 1.

Proof According to Definition 1, for $\varepsilon > 0$, let us consider the variations

$$u_i^\varepsilon(s) = \begin{cases} v_i(s) & \text{if } s \in [t, t+\varepsilon], \\ \phi_i^c(x,s) & \text{if } s > t+\varepsilon, \end{cases}$$

for $i=1,\dots,N$. In the following, we denote $u=(u_1,\dots,u_N)$, $u^\varepsilon=(u_1^\varepsilon,\dots,u_N^\varepsilon)$, $v=(v_1,\dots,v_N)$ and $\phi^c(x,t)=(\phi_1^c(x,t),\dots,\phi_N^c(x,t))$. Let

$$V_i^\varepsilon(x,t)=\int_t^\tau d_i(s,t)L_i\big(x^\varepsilon(s),u^\varepsilon(s),s\big)ds,$$

where $x^\varepsilon(s)$ denotes the state trajectory obtained from equation (6) when the decision rule $u^\varepsilon(s)$ is applied. By definition,

$$\begin{aligned}
&V^c(x,t)-V^\varepsilon(x,t)\\
&\quad=\sum_{i=1}^N\lambda_i(x,t)\big[V_i^c(x,t)-V_i^\varepsilon(x,t)\big]\\
&\quad=\sum_{i=1}^N\lambda_i(x,t)\bigg[\int_t^{t+\varepsilon}d_i(s,t)\big[L_i\big(x(s),\phi^c\big(x(s),s\big),s\big)-L_i\big(x^\varepsilon(s),v(s),s\big)\big]ds\\
&\qquad+\int_{t+\varepsilon}^\tau d_i(s,t)\big[L_i\big(x(s),\phi^c\big(x(s),s\big),s\big)-L_i\big(x^\varepsilon(s),\phi^c\big(x^\varepsilon(s),s\big),s\big)\big]ds\bigg].
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{t+\varepsilon}^\tau d_i(s,t)L_i\big(x(s),\phi^c\big(x(s),s\big),s\big)ds\\
&\quad=V_i^c\big(x(t+\varepsilon),t+\varepsilon\big)\\
&\qquad-\int_{t+\varepsilon}^\tau\big[d_i(s,t+\varepsilon)-d_i(s,t)\big]L_i\big(x(s),\phi^c\big(x(s),s\big),s\big)ds.
\end{aligned}$$

In a similar way,

$$\begin{aligned}
&\int_{t+\varepsilon}^\tau d_i(s,t)L_i\big(x^\varepsilon(s),\phi^c\big(x^\varepsilon(s),s\big),s\big)ds\\
&\quad=V_i^c\big(x^\varepsilon(t+\varepsilon),t+\varepsilon\big)\\
&\qquad-\int_{t+\varepsilon}^\tau\big[d_i(s,t+\varepsilon)-d_i(s,t)\big]L_i\big(x^\varepsilon(s),\phi^c\big(x^\varepsilon(s),s\big),s\big)ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&V^c(x,t)-V^\varepsilon(x,t)\\
&\quad=\sum_{i=1}^N\lambda_i(x,t)\bigg[\int_t^{t+\varepsilon}d_i(s,t)\big[L_i\big(x(s),\phi^c\big(x(s),s\big),s\big)-L_i\big(x^\varepsilon(s),v(s),s\big)\big]ds\\
&\qquad+V_i^c\big(x(t+\varepsilon),t+\varepsilon\big)-V_i^c\big(x^\varepsilon(t+\varepsilon),t+\varepsilon\big)
\end{aligned}$$

$$+\int_{t+\varepsilon}^{\tau}\big[d_i(s,t)-d_i(s,t+\varepsilon)\big]$$
$$\times\Big[L_i\big(x(s),\phi^c\big(x(s),s\big),s\big)-L_i\big(x^\varepsilon(s),\phi^c\big(x^\varepsilon(s),s\big),s\big)\Big]ds\Bigg].$$

Hence,

$$\lim_{\varepsilon\to 0^+}\frac{V^c(x,t)-V^\varepsilon(x,t)}{\varepsilon}=(A)+(B)+(C),$$

where

$$(A)=\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\Bigg(\sum_{i=1}^{N}\lambda_i(x,t)\int_t^{t+\varepsilon}d_i(x,t)$$
$$\times\Big[L_i\big(x(s),\phi^c\big(x(s),s\big),s\big)-L_i\big(x^\varepsilon(s),v(s),s\big)\Big]ds\Bigg)$$
$$=\sum_{i=1}^{N}\lambda_i(x,t)\big[L_i\big(x,\phi^c(x,t),t\big)-L_i(x,v,t)\big],$$

$$(B)=\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\Bigg(\sum_{i=1}^{N}\big[V_i^c\big(x(t+\varepsilon),t+\varepsilon\big)-V_i^c\big(x^\varepsilon(t+\varepsilon),t+\varepsilon\big)\big]\Bigg)$$
$$=\sum_{i=1}^{N}\lambda_i(x,t)\big[\nabla_x V_i^c(x,t)\big(g\big(x,\phi^c(x,t),t\big)-g(x,v,t)\big)\big],$$

and

$$(C)=\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\Bigg(\sum_{i=1}^{N}\Bigg[\int_{t+\varepsilon}^{\tau}\big[d_i(s,t)-d_i(s,t+\varepsilon)\big]$$
$$\times\Big[L_i\big(x(s),\phi^c\big(x(s),s\big),s\big)-L_i\big(x^\varepsilon(s),\phi^c\big(x^\varepsilon(s),s\big),s\big)\Big]ds\Bigg]\Bigg)$$
$$=0.$$

Summarizing

$$\lim_{\varepsilon\to 0^+}\frac{V^c(x,t)-V^\varepsilon(x,t)}{\varepsilon}$$
$$=\sum_{i=1}^{N}\lambda_i(x,t)\big[\big(L_i\big(x,\phi^c(x,t),t\big)+\nabla_x V_i^c(x,t)g\big(x,\phi^c(x,t),t\big)\big)$$

$$- \left(L_i(x, v, t) + \nabla_x V_i^c(x, t) g(x, v, t)\right)\Big]$$
$$\geq 0$$

and the result follows. □

Remark 1 It is important to realize that, unless $d_i(s,t) = \alpha(t)\beta_i(s)$ (or, in particular, $d_i(s,t) = e^{-\rho(s-t)}$, i.e. all players discount the future by using the same constant discount rate of time preference) and weights λ_i are constant, for $i = 1, \ldots, N$, the time-consistent solution provided by condition (12) in Theorem 1 is different to that obtained by applying the classical Pontryagin maximum principle (or the Hamilton–Jacobi–Bellman equation) to the problem of maximizing (8) subject to (6) from the viewpoint of the time preferences of all players at time $t = 0$.

4 Examples

In this section we illustrate our results with two simple models coming from the field of environmental and resource economics. In the first example we solve a common property resource model studied in Clemhout and Wan (1985) with time-distance nonconstant discounting. For this model we compute both the Markovian Nash equilibria and the time-consistent equilibria with partial cooperation. We restrict our attention to the particular case of constant weights. The second example is a pollution linear state differential game whose equilibria are state independent. Although this is not a nice property from an economic viewpoint, it has the advantage that, in the computation of time-consistent equilibria within the framework of partial cooperation, in the problem with nonconstant weights for the different players, it is rather natural to restrict our attention to time dependent but state independent weights. In this case we are able to derive explicit formula for time consistent equilibria with arbitrary weights.

4.1 A Common Property Resource Game

Let us consider the problem of exploitation of a renewable natural resource in which, if $x(t)$ represents the stock of natural resource at time t, and $h_i(t)$ is the harvest rate at time t of player i, for $i = 1, \ldots, N$, the state dynamics is described by the equation

$$\dot{x}(s) = x(s)\big(a - b \ln x(s)\big) - \sum_{i=1}^{N} h_i(s), \quad x(t) = x_t. \tag{13}$$

Players have logarithmic instantaneous utility functions depending just on their harvest rates, and they discount the future according to different distance-based nonconstant discount rates of time preference. Hence, the intertemporal utility function for player i is given by

$$J_i = \int_t^\infty \theta_i(s-t)\ln h_i(s)ds.$$

4.1.1 Noncooperative Markovian Nash equilibrium

In players do not cooperate, let us look for stationary strategies. According to the results in Sect. 2, player i aims to look for the solution to

$$\max_{\{h_i\}}\left\{\ln h_i + \left(V_i^{nc}(x)\right)'\left[x(a-b\ln x) - h_i - \sum_{j\neq i}\phi_j^{nc}(x)\right]\right\},$$

where $\phi_j^{nc}(x),\ j=1,\ldots,N$, denotes the equilibrium strategy of player j in feedback form. Hence $h_i = ((V_i^{nc}(x))')^{-1}$. We look for a value function of the form $V_i^{nc}(x) = \alpha_i^{nc}\ln x + \beta_i^{nc}$, for $i=1,\ldots,N$. Then $h_i^{nc} = \phi_i^{nc}(x) = (\alpha_i^{nc})^{-1}x$. By substituting in equation (13) and solving we obtain

$$x(s) = \exp\left[\left(\ln x_t + \frac{\sum_{j=1}^N 1/\alpha_j^{nc} - a}{b}\right)e^{-b(s-t)} + \frac{a - \sum_{j=1}^N 1/\alpha_j^{nc}}{b}\right].$$

Hence,

$$\ln\phi_i\big(x(s)\big) = e^{-b(s-t)}x_t + \frac{a - \sum_{j=1}^N 1/\alpha_j^{nc}}{b}\left(1-e^{-b(s-t)}\right) - \ln\alpha_i^{nc},$$

for every $i=1,\ldots,N$. Therefore, since

$$V_i(x) = \int_t^\infty \theta_i(s-t)\ln\phi_i\big(x(s)\big)ds,$$

then

$$\begin{aligned}\alpha_i^{nc}\ln x + \beta_i^{nc} = {} & \left[\int_t^\infty \theta_i(s-t)e^{-b(s-t)}ds\right]\ln x \\ & + \frac{a - \sum_{j=1}^N 1/\alpha_j^{nc}}{b}\int_t^\infty \theta_i(s-t)\left[1-e^{-b(s-t)}\right]ds \\ & - \ln\alpha_i^{nc}\int_t^\infty \theta_i(s-t)ds.\end{aligned}$$

By simplifying we obtain

$$\alpha_i^{nc} = \int_0^\infty \theta_i(s)e^{-bs}ds,$$

$$\beta_i^{nc} = \frac{1}{b}\left(a - \sum_{j=1}^{N} \frac{1}{\int_0^\infty \theta_j(s)e^{-bs}ds}\right)\int_0^\infty \theta_i(s)\left[1 - e^{-bs}\right]ds$$
$$- \ln\left(\int_0^\infty \theta_i(s)e^{-bs}ds\right)\int_0^\infty \theta_i(s)ds$$

and

$$h_i^{nc}(x) = \frac{x}{\int_0^\infty \theta_i(s)e^{-bs}ds},$$

for $i = 1, \ldots, N$.

4.1.2 Time-Consistent Equilibrium with Partial Cooperation

Next, let us compute the time consistent equilibria in case players at every time t decide to cooperate among them, but coalitions taking decisions at different times do not cooperate. We restrict our attention to stationary strategies, and weights are assumed to be constant. According to Theorem 1, we look for the solution to

$$\max_{\{h_1,\ldots,h_N\}}\left\{\sum_{j=1}^{N}\lambda_j \ln h_j\right.$$
$$\left. + \left(\sum_{i=1}^{N}\lambda_i\left(V_i^c(x)\right)'\right)\left[x(a - b\ln x) - h_i - \sum_{j\neq i}\phi_j^c(x)\right]\right\}.$$

Therefore, $h_j^c = \lambda_j(\sum_{i=1}^N \lambda_i(V_i^c(x))')^{-1}$. We look for a set of value functions of the form $V_i^c(x) = \alpha_i^c \ln x + \beta_i^c$, for $i = 1, \ldots, N$. Then

$$h_j^c = \phi^c(x) = \frac{\lambda_j x}{\sum_{i=1}^N \lambda_i \alpha_i^c}.$$

By substituting in equation (13) and solving we obtain

$$x(s) = \exp\left[\left(\ln x_t + \frac{\sum_{j=1}^N \lambda_j - a\sum_{j=1}^N \lambda_j\alpha_j^c}{b\sum_{j=1}^N \lambda_j\alpha_j^c}\right)e^{-b(s-t)}\right.$$
$$\left. - \frac{\sum_{j=1}^N \lambda_j - a\sum_{j=1}^N \lambda_j\alpha_j^c}{b\sum_{j=1}^N \lambda_j\alpha_j^c}\right].$$

Hence, proceeding as in the noncooperative case, we easily obtain

$$\alpha_i^c \ln x + \beta_i^c = \left[\int_t^\infty \theta_i(s-t)e^{-b(s-t)}ds\right]\ln x$$
$$+ \frac{a\sum_{j=1}^N \lambda_j \alpha_j^c - \sum_{j=1}^N \lambda_j}{b\sum_{j=1}^N \lambda_j \alpha_j^c}\int_t^\infty \theta_i(s-t)\big[1-e^{-b(s-t)}\big]ds$$
$$- \ln\left(\frac{\sum_{j=1}^N \lambda_j \alpha_j^c}{\lambda_i}\right)\int_0^\infty \theta_i(s)ds.$$

By simplifying we obtain

$$\alpha_i^c = \int_0^\infty \theta_i(s)e^{-bs}ds,$$
$$\beta_i^c = \frac{a\sum_{j=1}^N \lambda_j \int_0^\infty \theta_j(s)ds - \sum_{j=1}^N \lambda_j}{b\sum_{j=1}^N \lambda_j \int_0^\infty \theta_j(s)ds}\int_0^\infty \theta_i(s)\big[1-e^{-bs}\big]ds \qquad (14)$$
$$- \ln\left(\frac{\sum_{j=1}^N \lambda_j \int_0^\infty \theta_j(s)e^{-bs}ds}{\lambda_i}\right)\int_0^\infty \theta_i(s)ds$$

and

$$h_i^c(x) = \frac{\lambda_i x}{\sum_{j=1}^N \lambda_j \int_0^\infty \theta_j(s)e^{-bs}ds},$$

for $i = 1, \ldots, N$. Note that $\alpha_i^{nc} = \alpha_i^c$, as in the case of constant and equal discount rates.

4.2 A Linear State Differential Game of Pollution Control

As a second example, we consider the environmental problem studied in Jørgensen et al. (2003) where N countries (the players of the game) coordinate their pollution strategies to optimize their joint payoff. Let us denote by $E_i(t)$, for $i = 1, \ldots, N$, the emissions of country i at time t. The evolution of the stock of pollution $S(t)$ is described by the differential equation

$$\dot{S}(\tau) = \sum_{i=1}^N E_i(\tau) - \delta S(\tau), \quad S(0) = S_0, \qquad (15)$$

where $\delta > 0$ represents the natural absorption rate of pollution. The emissions are assumed to be proportional to the production and hence the revenue from production can be expressed as a function of the emissions. In particular, the revenue function of country i is assumed to be logarithmic. The damage cost is a linear function on the stock of pollution. The intertemporal utility function for player i is given by

$$J_i = \int_t^\infty \theta_i(\tau - t)\big(\ln E_i(\tau) - \varphi_i S(\tau)\big)d\tau$$

Next we compute both the time-consistent Markovian noncooperative and with partial cooperation equilibria.

4.2.1 Noncooperative Markovian Nash Equilibrium

In this case, player i aims to maximize

$$\max_{\{E_i\}}\left\{\ln E_i - \varphi_i S + \big(V_i^{nc}(S)\big)'\left(E_i + \sum_{j\neq i}\phi_j^{nc}(S) - \delta S\right)\right\},$$

where $E_j^{nc} = \phi_j^{nc}(S)$ is the equilibrium rule. Then $E_i^{nc} = (-(V_i^{nc})'(S))^{-1}$. We look for a value function of the form $V_i(S) = \alpha_i^{nc} S + \beta_i^{nc}$. Then $E_i^{nc} = \phi_i^{nc} = (-\alpha_i^{nc})^{-1}$. By substituting in (15) we obtain $\dot{S}(\tau) = \sum_{j=1}^N (-\alpha_j^{nc})^{-1} - \delta S(\tau)$, whose solution with the initial condition $S(t) = S_t$ is

$$S(\tau) = e^{-\delta(\tau-t)}S_t - \sum_{j=1}^N \frac{1}{\delta\alpha_j^{nc}}\big(1 - e^{-\delta(\tau-t)}\big).$$

By identifying the value functions we obtain

$$\begin{aligned}
\alpha_i^{nc} S + \beta_i^{nc} &= \int_t^\infty \theta_i(\tau - t)\big[\ln\phi_i^{nc}\big(S(\tau)\big) - \varphi_i S\big]d\tau \\
&= \int_t^\infty \theta_i(\tau - t)\Bigg[-\ln\big(-\alpha_i^{nc}\big) \\
&\quad - \varphi_i\left(e^{-\delta(\tau-t)}S - \sum_{j=1}^N \frac{1}{\delta\alpha_j^{nc}}\big(1 - e^{-\delta(\tau-t)}\big)\right)\Bigg]d\tau.
\end{aligned}$$

By simplifying we obtain

$$\begin{aligned}
\alpha_i^{nc} &= -\varphi_i \int_0^\infty \theta_i(\tau)e^{-\delta\tau}d\tau, \\
\beta_i^{nc} &= -\ln\left(\varphi_i\int_0^\infty \theta_i(\tau)e^{-\delta\tau}d\tau\right)\int_0^\infty \theta_i(\tau)d\tau \\
&\quad - \frac{\varphi_i}{\delta}\sum_{j=1}^N \frac{1}{\varphi_j\int_0^\infty \theta(\tau)e^{-\delta\tau}}\left(\int_0^\infty \theta_i(\tau)\big(1 - e^{-\delta\tau}\big)d\tau\right)
\end{aligned}$$

and the emission rule becomes

$$E_i^{nc} = \frac{1}{\varphi_i \int_0^\infty \theta_i(\tau) e^{-\delta\tau} d\tau}.$$

4.2.2 Time-Consistent Equilibrium with Partial Cooperation

Finally, let us compute the time-consistent equilibrium for the problem with partial cooperation. In comparison with the previous example on the management of a common property access resource, we consider the case of nonconstant weights for this problem. Since the decision rule for linear state games is typically independent on the state variable (the pollution stock), it seems natural to restrict our attention to weights $\lambda_i(t)$, i.e. independent on the state variable. This simplification allows to solve the model. The payoff for the grand coalition is given by

$$J^c = \sum_{j=1}^{N} \lambda_j^c(t) \int_t^\infty \theta_j(\tau - t)\big(\ln E_j(\tau) - \varphi_j S(\tau)\big) d\tau.$$

According to Theorem 1, we must solve

$$\max_{\{E_1,\ldots,E_N\}} \left\{ \sum_{j=1}^{N} \lambda_j^c(t) \left[\ln E_j - \varphi_j S + \big(V_j^c(S)\big)' \left(\sum_{i=1}^{N} E_i - \delta S \right) \right] \right\}.$$

The equilibrium rule is given by

$$E_i = -\frac{\lambda_i(t)}{\sum_{j=1}^{N} \lambda_j(t) (V_j^c(S))'}.$$

We look for a family of value functions of the form $V_j(S) = \alpha_j^c(t) S + \beta_j^c(t)$, for $j = 1, \ldots, N$. Then the emission rules become

$$E_i = -\frac{\lambda_i(t)}{\sum_{j=1}^{N} \lambda_j(t) \alpha_j^c(t)}. \tag{16}$$

By substituting (16) in (15) we obtain the linear differential equation

$$\dot{S}(\tau) = -\frac{\sum_{j=1}^{N} \lambda_j(t)}{\sum_{i=1}^{N} \lambda_i(t) \alpha_i^c(t)} - \delta S(\tau) = -\frac{N}{\sum_{i=1}^{N} \lambda_i(t) \alpha_i(t)} - \delta S(\tau),$$

whose solution with the initial condition $S(t) = S_t$ is given by

$$S(\tau) = e^{-\delta(\tau - t)} S_t - \int_t^\tau \frac{e^{-\delta(\tau - z)}}{\sum_{i=1}^{N} \lambda_i(z) \alpha_i^c(z)} dz.$$

In order to compute the values of the (nonconstant) coefficients $\alpha_i^c(t)$, $\beta_i^c(t)$, proceeding as in the previous example, note that

$$\begin{aligned}\alpha_i(t)S + \beta_i(t) &= \int_t^\infty \theta_i(\tau - t)\big(\ln E_i(\tau) - \varphi_i S(\tau)\big)d\tau \\ &= \int_t^\infty \theta_i(\tau - t)\Bigg[\ln\Bigg(-\frac{\lambda_i(\tau)}{\sum_{j=1}^N \lambda_j(\tau)\alpha_j^c(\tau)}\Bigg) \\ &\quad - \varphi_i\Bigg(e^{-\delta(\tau-t)}S_t - \int_t^\tau \frac{e^{-\delta(\tau-z)}}{\sum_{j=1}^N \lambda_j(z)\alpha_j^c(z)}dz\Bigg)\Bigg]d\tau.\end{aligned}$$

By identifying terms we obtain

$$\alpha_i^c(t) = -\varphi_i \int_0^\infty \theta_i(\tau)e^{-\delta\tau}d\tau$$

and

$$\begin{aligned}\beta_i^c(t) = \int_t^\infty \theta_i(\tau - t)\Bigg[&\ln \frac{\lambda_i(\tau)}{\sum_{j=1}^N \varphi_j \int_0^\infty \theta_j(z)e^{-\delta z}dz} \\ &- \varphi_i \int_t^\tau \frac{e^{-\delta(\tau-z)}}{\sum_{j=1}^N \varphi_j \int_0^\infty \theta_j(s)e^{-\delta s}ds}dz\Bigg]d\tau.\end{aligned}$$

From (16), the emission rule of country i is given by

$$E_i^c(t) = \frac{\lambda_i(t)}{\sum_{j=1}^N \lambda_j(t)\varphi_j \int_0^\infty \theta_j(\tau)e^{-\delta\tau}d\tau}.$$

For instance, in the case of a constant and common discount rate for all players but nonconstant weights, $\theta_i(\tau) = e^{-\rho\tau}$, emissions of country i become

$$E_i^c(t) = \frac{(\rho + \delta)\lambda_i(t)}{\sum_{j=1}^N \varphi_j \lambda_j(t)}.$$

In Jørgensen et al. (2003) parameter conditions ensuring the time consistency of the coalition (so that payoffs obtained when they cooperate are higher than payoffs in the case of non cooperation) were established when players are not symmetric. By introducing nonconstant weights obtained from a bargaining procedure at every t (by using e.g. the Nash bargaining solution), a time-consistent solution guaranteeing the stability of the coalition can be found.

5 Conclusions

In this chapter, differential games with time-inconsistent preferences generated by the introduction of general (not necessarily time-distance) discount functions are studied. Both the noncooperative setting and a framework with partial cooperation are analyzed. The corresponding dynamic equations for the derivation of time-consistent equilibria are obtained. In order to guarantee the stability of the grand coalition, nonconstant weights are introduced, so that players can bargain their weight in the grand coalition at every instant of time. In particular, the introduction of nonconstant weights provides a way to construct dynamically consistent solutions guaranteeing the stability of the grand coalition in problems in which the players discount the future by using constant (and not necessarily different) discount rates. The price to be paid is that the use of nonconstant (time and/or state dependent) weights induces time-inconsistent preferences. The results in the chapter are illustrated with two simple examples coming from the field of environmental and resource economics. In a first example, a simple common access resource game is studied by introducing heterogeneous and time-distance nonconstant discount rates. Weights of players in the problem with partial cooperation are assumed to be constant. The second example analyzes a linear state pollution differential game with the same kind of time preferences. For this problem, nonconstant weights in the problem with partial cooperation are introduced.

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Interactions Between Fiscal and Monetary Authorities in a Three-Country New-Keynesian Model of a Monetary Union

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Abstract In this paper we consider the effectiveness of various coordination arrangements between monetary and fiscal authorities within a monetary union if an economic shock has occurred. We address this problem using a multi-country New-Keynesian model of a monetary union cast in the framework of linear quadratic differential games. Using this model we study various coordination arrangements between fiscal and monetary players, including partial fiscal cooperation between only a subgroup of countries, which, to the best of our knowledge, has not been considered yet in the New-Keynesian literature. Using a simulation study we show that, in many cases and from the global point of view, partial fiscal cooperation between a subgroup of fiscal players is more efficient than non-coordination and that, in general, full cooperation without an appropriate transfer system is not a stable configuration. Furthermore, in case there is no full cooperation we show that the optimal configuration of the coordination structure depends on the type of shock that has occurred. We present a detailed analysis of the relationship between coordination structures and type of shock.

1 Introduction

The creation of the (multi-country) European Monetary Union (EMU), with a common central bank, yet independent national fiscal policies, urged the ongoing discussion about the need and the feasibility of macroeconomic policy coordination within a monetary union. Since the 'one-size-fits-all' policy of the European Central Bank

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(ECB) cannot address country-specific shocks, and the other stabilisation mechanisms in the euro-zone (such as labour force mobility and financial assets mobility) are limited, the general consensus is that the main burden of stabilisation should be born by fiscal policies. However, the abuse of fiscal policies can be detrimental to both financial and economic stability and may result in undesirable suboptimal outcomes. Consequently, budgetary positions in the EMU Member States are constrained (mainly by the provisions of the Stability and Growth Pact, SGP) and are monitored by the European Commission (EC). Due to the recent experience with the current economic crises it is to be expected that this monitoring will be much more strict in the future. This situation gives rise to several relevant questions, the most important of them being, whether the coordination of monetary and fiscal policies is desirable in the aftermath of a shock.

Many studies analysing the desirability of policy coordination in a monetary union have been performed and they provide mixed results. On the one hand, many authors support the classic result of the Optimal Currency Area (OCA) that policy coordination is desirable in case of symmetric shocks. For instance, Buti and Sapir (1998) argue that coordination of fiscal policies should be implemented to tackle large symmetric shocks. However, on the other hand, there are a number of studies that demonstrate that policy coordination can provide inferior levels of welfare. Notably, in a two-country model of a monetary union, Beetsma et al. (2001) find fiscal cooperation to be counter-productive as a result of the elevated conflict with the central bank.

Against this background, in this paper we present and analyse a multi-country New-Keynesian (NK) model of a monetary union which is cast in the framework of open-loop linear quadratic differential games (LQDGs) including multi-player strategic elements. To the best of our knowledge, it is the first model in the New(-Keynesian) Open Economy Macroeconomics (NOEM) spirit to feature strategic elements between more than three players. Essentially, the starting point of the NK approach is the explicit derivation of macroeconomic relationships from underlying microeconomic foundations. This principle is largely shared with New Classical macroeconomics, although the former includes a great deal of imperfections in the goods and labour markets. Recently, NK macroeconomics has constituted the core of the macroeconomic paradigm world-wide, with a great deal of research effort directed towards the issue of optimal monetary policy. However, until now, relatively little attention has been paid to the interactions between fiscal and monetary policies when stabilising an economy after a shock; something that is especially important in the EMU context.[1] The strength of our NK model is its multi-player (monetary union countries and the central bank) strategic dimension, which allows for extensive analysis of desirability of policy coordination.

In particular, within this framework of a dynamic multi-country NK model we try to identify coordination configurations that outperform others under different types of shocks. We show that, in many cases, partial fiscal coordination of a sub-

[1] See the next section for a selection of papers that analyse this issue in the spirit of NOEMs.

group of fiscal players is more efficient, from the social point of view, than non-coordination. In other words, coordination of fiscal policies is likely to be counter-productive when the coordinating group of countries is large enough, thus increasing the conflict with the central bank. The intuition for this result is that, in case countries cooperate in smaller groups, their policies are less likely to be completely symmetric. Therefore, there are less likely to be in conflict with the central bank that targets only union averages.

Our approach adds new dimensions to the current literature. First of all, in our approach, strategic interactions between various parties become dynamic; and this within a model with rich NK specification. Secondly, a multi-country model enables us to study various intermediate coordination regimes between fiscal players (partial fiscal cooperation). Finally, we consider cooperation between the central bank and multiple fiscal players (i.e. the grand coalition) in the same manner in which other recent, but only two-country NK studies, have done.[2]

The remainder of the paper is organised as follows. The next section gives a brief overview of the literature on cyclical stabilisation in a monetary union and related issues. Section 3 outlines the model of a monetary union whereas Sect. 4 introduces an alternative concept for social loss. Section 5 presents results of numerical simulations and discussion. The last section concludes and proposes directions of future research.

2 Policy Coordination in a Monetary Union

The inability of the common monetary policy to tackle country-specific shocks is generally considered to be the single and most important macroeconomic cost of monetary unification. However, the size of this cost depends on how good other mechanisms, which may be helpful in adjusting to idiosyncratic shocks, are functioning too.[3] Unfortunately, factor markets (especially the labour market) are relatively immobile in Europe and capital flows are limited (Buti 2001). Consequently, these two basic mobility mechanisms, which play an important role in the US economy, are rather dysfunctional in the EMU case.[4] This means that, the stabilisation burden in the case of country specific shocks should be placed on other policies. One such policy is the federal tax-transfer system; another mechanism that is vital

[2]See next section for a short literature review on policy coordination in a monetary union.

[3]It should also be noted that from the macroeconomic perspective the real cost of accessing a monetary union is reflected by the shadow cost of the abandonment of an own monetary policy so that it cannot be used as an adjustment tool in the case of an idiosyncratic shock. In other words, the cost of entering the union depends, to a large extent, on the effectiveness of the national central bank to tackle idiosyncratic shocks. It is especially important in the environment of closely integrated economies, like the EU, and in the case of small countries, such as Belgium.

[4]See, for instance, Pierdzioch (2004) who presents a dynamic general equilibrium two-country optimising sticky-price model to analyse the consequences of international financial market integration for the propagation of asymmetric productivity shocks in the EMU.

to the US state-specific adjustments (see Mélitz and Zumer 1998; von Hagen 2000). Also, in the EU framework, tax-transfers already take place, however, in spite of public perception, the amounts of funds transferred are limited and are related to the long-term economic development, rather than to short and medium term conditions. A deeper tax-system integration, which may be able to address idiosyncratic shocks, seems to be politically implausible in the EMU, at least in the near future.[5] In the light of this, the national fiscal policy is considered to be the most important instrument left to policymakers.[6] In theory, this policy might be able to circumvent the problem of country specific shocks in the euro-zone. However, whether the incurred cost of stabilisation via national fiscal policies in a (relatively) integrated economic environment like the EMU is lower than the benefits, remains to be answered, especially since spillovers from national fiscal policies may be counteractive w.r.t. each other and lead to suboptimal outcomes. Another closely related problem is the response of a common monetary authority which will react to national fiscal policies. Such a reaction can be counteractive as the objectives of fiscal and monetary policies tend to be dissimilar. All of these issues warrant a discussion on the profitability and feasibility of policy coordination in a monetary union.

Aside from certain institutional issues, such as the effectiveness of enforcing the cooperation agreement, profitability of any coordination arrangement depends on the nature of the shock, i.e. whether it is symmetric or asymmetric, inflation or output gap, etc. We will focus on symmetric and asymmetric inflation shocks. In the case of symmetric shocks, policy coordination was traditionally considered to be beneficial because it internalised externalities emerging from individual policies (as argued by Uhlig 2003; Plasmans et al. 2006a). In particular, the usual argument in favour of international policy coordination is based on direct positive demand spillovers. In contrast to this, more recent micro-founded models of the EMU tend to conclude in favour of negative fiscal spillovers by emphasising the adverse terms-of-trade effects of balanced-budget foreign fiscal expansion on the domestic economy. For example, should governments perceive negative spillovers from other countries, they would reconsider a non-cooperative ("beggar-thy-neighbour") policy in response to bad economic shocks and would agree on a more restrictive stance in all countries.[7] Conversely, should governments perceive positive spillovers, coordination should eliminate free-riding behaviour of individual countries and promote

[5] The issue of fiscal transfers within the EMU has been studied by a number of authors such as Kletzer and von Hagen (2000), van Aarle et al. (2004), Evers (2006). The latter considers direct transfers among private sectors and indirect transfers among national fiscal authorities showing relative efficiency of such solutions.

[6] Already Kenen (1969) emphasised the possible role which fiscal policies might play in a monetary union as potential chock-adjustment mechanisms.

[7] One country attempts to improve its output-inflation trade-off by running a "beggar-thy-neighbour" policy. This is followed by the reaction of (an)other country(-ies) and the resulting non-cooperative outcome is a deflationary bias with all countries worse off with regard to a cooperative situation in which each country takes care of domestic inflation without attempting to affect the exchange rate (Cooper 1985).

a more expansionary policy in response to some economic shocks. Evidently, incentives for fiscal policy coordination in a monetary union are directly linked to the sizes and signs of the spillovers resulting from national fiscal policies. Fiscal spillovers are crucial, since they ultimately determine whether coordination should lead to a more expansionary or a more restrictive fiscal stance in the member states.

Other arguments in favour of coordination are that (see e.g. Hughes-Hallett and Ma 1996): (i) it restores policy effectiveness; (ii) speeds up an economy's response to policy actions; and (iii) enables to exploit comparative policy advantages.

In contrast to the above traditional arguments, more recent works question the profitability of policy coordination. Beetsma et al. (2001) show that policy coordination in a monetary union can be counter-productive in case of a symmetric shock.[8] Their model emphasises the conflict between governments and the central bank which share the stabilisation burden.[9] They argue that the coordination of budgetary stance between countries in the union makes fiscal policy more effective, thus governments are more willing to accept changes in their deficits. In other words, cooperation increases the use of fiscal policies. This, in turn, has important consequences on the behaviour of the central bank, which, generally has objectives different from those of fiscal policymakers. For instance, in case of a negative demand shock, both fiscal and monetary authorities are interested in stabilising output, which, from the central bank's point of view, is a means of stabilising inflation. Since cooperating fiscal authorities are eager to bear more stabilisation burden, the (non-cooperating) central bank free-rides on their efforts and does not loosen monetary policy as much as when governments do not collude. So, under this type of shock this cooperation structure is probably not optimal. But, at least, both authorities do not enter into a conflict situation. However, in case a supply shock occurs it is clear that this cooperation structure is far from optimal. Supply shocks make output and inflation move in opposite directions. The stronger fiscal response encouraged by cooperation exacerbates the conflict between price and output stability and, therefore, between monetary and fiscal authorities. As a result of this, the central bank is more restrictive when governments cooperate, compared to a case where governments would not act like that.

In other words, for both types of shocks, fiscal coordination may turn out to be counter-productive, albeit for different reasons.

In contrast, fiscal cooperation may be beneficial in the case of a country-specific shock. The free-riding of (or the conflict with) the central bank can in such a case largely be avoided as the monetary authority is interested in aggregate inflation (and

[8]It should be noted that prior to Beetsma et al. (2001), it has been observed in empirical studies (see e.g. Neck et al. 1999) that coordination does not necessarily lead to superior results as a result of either time-inconsistency and/or coalition formation of the fiscal policymakers against the monetary authority (see also next footnote). Furthermore, Beetsma and Bovenberg (1998) show that fiscal coordination may have a negative influence on a tax and public spending discipline, i.e. they may reduce the positive effects of monetary unification.

[9]Rogoff (1985) already stated that there is a potential for a negative impact of coordination among a subset of actors (in this case the two fiscal authorities, leaving out the common central bank).

possibly, to some extent, aggregate output). Therefore, the reaction of the monetary authority to idiosyncratic shock is limited.

Lombardo and Sutherland (2004) confirm the above point of view by showing, in a micro-founded model, that fiscal coordination is advantageous when country-specific shocks are negatively correlated. This study also suggests that the best results are delivered by an appropriate mix of both fiscal and monetary instruments.

Other works defend the view of desirability of policy coordination. von Hagen and Mundschenk (2003) argue that, in the long run, there is little need for coordination, however, in the short term, there are substantial gains from fiscal cooperation. Furthermore, if the central bank also pursues a goal of output stabilisation, the grand coalition of all the authorities together is advisable. Buti et al. (2001), Engwerda et al. (2002), Beetsma and Jensen (2004), Kirsanova et al. (2007) also support the active role of fiscal policy in stabilisation.

Cavallari and Di Gioacchino (2005), in the framework of a two-country static model, show that coordination of fiscal policies can only reduce output and inflation volatility w.r.t. the non-cooperative regime in the case of a demand shock, and that it can be potentially counter-productive otherwise. This adverse effect of union-wide coordinated fiscal measures can be circumvented by "global coordination," i.e. grand coalition. In more complex micro-founded general equilibrium models, Galí and Monacelli (2005b), Beetsma and Jensen (2004, 2005) also consider the case for fiscal and monetary policies' coordination. Specifically, Beetsma and Jensen (2005) extend the framework developed by Benigno and Lopez-Salido (2002) and develop an NK two-country monetary union model whereby national fiscal authorities pursue active stabilisation policies using public spending. Their model reveals that the relative advantage of using fiscal stabilisation policy is unchanged when the correlation of the supply shocks decreases. However, from a welfare point of view, the use of fiscal policy for the purpose of stabilisation appears to be relevant. Beetsma and Jensen (2005) argue that the governments should be active in situations in which a restriction on fiscal policy in order to equalise this policy with its natural level leads to welfare losses being equivalent to a permanent reduction in consumption of the order from 0.5 to 1 percentage point. A similar view is shared by Galí and Monacelli (2005b) who argue in favour of active fiscal policies.

In addition to cooperative scenarios, Forlati (2007) focuses on a non-cooperative regime showing that, in such a situation, the central bank does not stabilise the average monetary union inflation as it has to accommodate the distortions caused by non-cooperative national governments. At the same time, the non-existence of an agreement between countries calls for an active fiscal stance, even in case of symmetric shocks.

In conclusion, a lot of work already has been done in this area of policy coordination. Further, though some results seem to be contradictory at first sight, in most cases these differences can be attributed to (not formalised) assumptions being made. As already indicated in the previous section we will drop some of these assumptions here and study the consequences w.r.t. optimal cooperation configurations.

Most of the initial works (like the static model of Beetsma et al. 2001) were tractable enough to deliver analytical solutions. However, the much more complex dynamic general equilibrium modelling with a higher number of cooperation arrangements requires resorting to numerical methods. These were used in van Aarle et al. (2002a, 2002b), Beetsma and Jensen (2004, 2005), Plasmans et al. (2006a), among others, and will also be applied in this work. More specific, we will use the numerical toolbox developed by Michalak et al. (2011) to perform simulations for infinite-planning horizon affine linear quadratic open-loop differential games. For an introduction on dynamic games we refer to Başar and Olsder (1999) and, more specific, on LQDG, to Engwerda (2005).

3 A Multi-country NKOEM Model of a Monetary Union

During recent years the theoretical and empirical research in NK macroeconomics has been extended steadily and produced a whole new series of results and insights about the workings of the macroeconomy. Essentially, the starting point of the NK approach is the explicit derivation of macroeconomic relationships from underlying microeconomic foundations. This principle is shared with New Classical macroeconomics, although the former includes a great deal of imperfections in the goods and labour markets. The NK approach now constitutes the core of the macroeconomic paradigm world-wide.

Our modelling objective is to cast the NK model of a monetary union in the LQDG framework in order to analyse strategic interactions between a comparatively large number of players. By definition, LQDGs concern continuous-time models but, unfortunately, the vast majority of NK/NOEM models were constructed in the discrete-time framework. Notable exceptions are Benhabib et al. (2001), Linnemann and Schabert (2002), Buiter (2004), Kirsanova et al. (2006). However, with the exception of the last one, all these are single economy models, thus would obviously require extensions to allow them to be applied to a monetary union setting.[10] In line with this, our strategy will be to transform a discrete-time NK model of a monetary union into its continuous-time counterpart. This methodology is also convenient from the point of view of model parametrisation as most of the empirical studies, useful for calibration purposes, concern discrete-time models. The second important modelling issue is the computational complexity of an LQDG, which grows with the number of dynamic equations of the model and/or the number of players. Having this in mind, we aim to describe every country in a monetary union in a manner as concise as possible. In fact, as explained in Svensson (1997) and Ball (1999), short-term macrodynamics can be analysed using a relatively simple system consisting of an aggregate demand (AD) curve showing the evolution of the output gap driven by the real interest rate (see e.g. Woodford 2003), and a Phillips curve describing the dynamics of inflation. Despite its relative simplicity, such models have been widely used to understand the basic mechanisms of macroeconomic policies. Consequently,

[10]There are two countries in the model developed by Kirsanova et al. (2006) but this particular framework becomes computationally difficult when we add another, third country.

our multi-country monetary union model will consist of as many AD equations as countries, as many Phillips curves as countries, and a number of real exchange rate relationships.

Due to space constraints, we will refrain from deriving the NK model of a monetary union from micro-foundations. Instead, we will refer to results from various studies in the literature. Let fiscal and monetary players from a set N be divided in two groups: n countries i ($i \in F$) and one central bank b ($b = B$, with $N = F \cup B$). Following, among others, Lindé et al. (2004), AD equations are:[11]

$$\begin{aligned} y_{i,t} = {} & \kappa_{i,y} E_t y_{i,t+1} + (1-\kappa_{i,y}) y_{i,t-1} - \gamma_i (i_{U,t} - E\pi_{i,t+1}) + \eta_i f_{i,t} \\ & + \sum_{j \in F/i} \rho_{ij} \left[-\kappa_{i,y} E_t y_{j,t+1} + y_{j,t} - (1-\kappa_{i,y}) y_{j,t-1} \right] \\ & + \sum_{j \in F/i} \delta_{ij} \left[-\kappa_{i,y} E_t s_{ij,t+1} + s_{ij,t} - (1-\kappa_{i,y}) s_{ij,t-1} \right] + v^y_{i,t}, \end{aligned} \tag{1}$$

where E_t is an expectation operator at time t; $y_{i,t}$, $p_{j,t}$, $\pi_{i,t}$, $f_{i,t}$, $s_{ij,t} := p_{j,t} - p_{i,t}$ denote the (logarithmic) output gap, price level, inflation, fiscal policy in country i and competitiveness between countries i and j, respectively, whereas $i_{U,t}$ denotes the union-wide common nominal interest rate. All parameters are non-negative. The current output gap in country i depends positively on the expected output gap, the past output gap, the real interest rate, the government's fiscal deficit, the dynamics of other countries' output gaps and competitiveness, defined as the difference between respective price levels. Finally, $v^y_{i,t}$ is an output gap shock. This functional form of the AD equation may be obtained from a linearised model of optimisation behaviour on the part of consumers, in particular, from the resulting Euler equation, in which consumption is replaced with output gap, as shown, for instance, in Lindé et al. (2004). Inertia term $(1-\kappa_{i,y}) y_{i,t-1}$ reflects so-called "habit formation" in consumption (see for example Smets and Wouters 2002; Plasmans et al. 2006b), which measures the sluggishness of households in changing their choices over time. Foreign output gap and competitiveness elements in (1) reflect the economic linkages between countries. In particular, the first one is a trade channel, where, intuitively, higher foreign output gaps contribute to higher domestic output gaps as a result of increased import. Similarly, domestic export increases when a foreign price level becomes higher than the domestic one. The forward-looking and backward-looking dynamics of foreign output gap and competitiveness spillovers result from habit formation in consumption and have a similar form as the dynamics of domestic output gaps in the AD equations.

The second set of equations in our model are NK Phillips curves, which relate inflation to cyclical activity. In the New-Keynesian model, these are derived from optimising firms' price-setting decisions subject to constraints on the frequency of

[11] By F/i we denote the set of all countries except for country i.

price adjustment. We assume Phillips curves of the form:

$$\pi_{i,t} = \beta_i\left[\kappa_{i,\pi} E_t \pi_{i,t+1} + (1-\kappa_{i,\pi})\pi_{i,t-1}\right] + \xi_i\left(y_{i,t} + \sum_{j\in F/i} \varsigma_{ij} s_{j,t}\right) + v^{\pi}_{i,t}, \tag{2}$$

where we follow various studies in the literature allowing for some degree of price inertia for $0 < \kappa_{i,\pi} < 1$.[12] Inflation shock $v^{\pi}_{i,t}$ in (2) independent, exogenous, stationary, zero mean AR(1) shock with damping parameter $0 < \psi_{i,\pi} < 1$, i.e. $v^{\pi}_{i,t} = \psi_{i,\pi} v^{\pi}_{i,t-1} + \varepsilon^{\pi}_{i,t}$, where $\varepsilon^{\pi}_{i,t}$ is an independently and identically distributed (*i.i.d.*) error term.

For n countries there are as many as $n(n-1)$ competitiveness relationships s_{ij}, however, as shown in the Appendix, it is possible to rewrite all of them only in terms of $s_{1j,t} := p_{j,t} - p_{1,t}$ where $j \in F/1$:

$$s_{1j,t+1} = s_{1j,t} + \pi_{j,t+1} - \pi_{1,t+1}, \tag{3}$$

which are only $n-1$ dynamic equations.

In his seminal work, Taylor (1995) demonstrated that actual US monetary policy could be described by a simple rule that relates the real interest rate to inflation and to output gap. This relationship became known as the (monetary) Taylor rule. In the monetary union case, the Taylor rule of the central bank might be written in the form:

$$i_{U,t} = \theta^U_\pi \pi_{U,t} + \theta^U_y y_{U,t}, \tag{4}$$

where $\pi_{U,t} := \sum_{i=1}^n \omega_i \pi_{i,t}$ is the average union inflation and $y_{U,t} := \sum_{i=1}^n \omega_i y_{i,t}$ is the average union output gap with parameter ω_i indicating the relative weight of country i in a monetary union ($\sum_{i=1}^n \omega_i = 1$).[13] The first term in the Taylor rule shows that the central bank responds to the rise in average inflation with a more restrictive monetary policy in order to weaken demand across the union. This, in turn, should hinder the growth in inflation. The second term shows that the real interest rate is also raised if output rises as this indicates a future inflation acceleration.

Taylor (2000) also points out that fiscal policy can be approximated by a policy rule (for further discussion see van Aarle et al. 2004). The fiscal Taylor rule can be written as:

$$f_{i,t} = \theta^i_\pi \pi_{i,t} + \theta^i_y y_t. \tag{5}$$

We extend the above definition of both rules so that:

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[12] See, among others, Fuhrer and Moore (1995), Galí and Gertler (1999), Woodford (2003), Lindé et al. (2004), Evans and McGough (2005) or Plasmans et al. (2006b).

[13] For a similar formulation of the monetary policy rule in a model of a monetary union see van Aarle et al. (2004).

$$\tilde{\imath}_{U,t} = i_{U,t} + \hat{\imath}_{U,t} = \theta_\pi^U \pi_{U,t} + \theta_y^U y_{U,t} + \hat{\imath}_{U,t}, \quad \text{and} \tag{6}$$

$$\tilde{f}_{i,t} = f_{i,t} + \hat{f}_{i,t} = \theta_\pi^i \pi_{i,t} + \theta_y^i y_t + \hat{f}_{i,t}, \tag{7}$$

where $\hat{f}_{i,t}$ and $\hat{\imath}_{U,t}$ are the players' control variables in the LQDG and denote deviations of the fiscal deficit and nominal common interest rate from (4) and (5), respectively. In particular, as Taylor (2000) argues, a simple fiscal rule can be used to explain most fluctuations in fiscal deficits. The starting point of his analysis is the division of the fiscal deficit into a cyclical component and a structural component. The first part can be interpreted as the systematic response of fiscal policy to output fluctuations (the so-called automatic stabilisers); the second part contains structural and discretionary components of fiscal policy. In our case, the standard Taylor fiscal rule $\theta_\pi^i \pi_{i,t} + \theta_y^i y_t$ is to be interpreted as an automatic stabiliser, whereas $\hat{f}_{i,t}$ is a discretionary component. For the monetary Taylor rule, $\hat{\imath}_{U,t}$ is the discretional component of the central bank's policy.

In order to reduce the number of equations, it is convenient to substitute $i_{U,t}$ and $f_{i,t}$ in (1) with (4) and (5). The resulting system consists of n AD curves (1), n Philips curves (2), and $n-1$ competitiveness equations, which, together with the inflation shock AR processes, constitute a hybrid (forward- and backward-looking) stochastic NK Model (SNKM henceforth) of a closed monetary union.

This completes the description of the discrete-time NK model. Next, we used some standard transformation techniques to recast the reduced form of this model into its equivalent continuous-time counterpart. In the Appendix, Sect. 5, we explained the details of this transformation procedure.

In order to complete the construction of the LQDG, we propose the following fiscal players' objectives:

$$\min_{\hat{f}_i(t)} J_i(t_0) = \min_{\hat{f}(t)} \frac{1}{2} \int_{t_0}^{\infty} \left\{ \alpha_i \pi_i^2(t) + \beta_i y_i^2(t) + \chi_i \hat{f}_i^2(t) \right\} e^{-\theta(t-t_0)} dt, \tag{8}$$

for $i = 1, 2, \ldots, n$, where α_i, β_i, χ_i indicate fiscal players' relative preferences concerning deviations of national inflation rates, output gaps and fiscal deficits.[14] The common central bank's objective function is defined in a similar way as:

$$\min_{\hat{\imath}} J_{CB}(t_0) = \min_{\hat{\imath}} \frac{1}{2} \int_{t_0}^{\infty} \left\{ \alpha_{CB} \pi_U^2(t) + \beta_{CB} y_U^2(t) + \chi_{CB} \hat{\imath}^2(t) \right\} e^{-\theta(t-t_0)} dt, \tag{9}$$

where α_U and β_U indicate the central bank's relative preferences concerning deviations of inflation, output gap and interest rate in the MU as a whole.

[14] Since the seminal works of Kydland and Prescott (1977) and Barro and Gordon (1983), the quadratic loss functions are commonly used in the literature on strategic behaviour of fiscal and monetary authorities. See also Schellekens and Chadha (1999) for a more recent analysis supporting the quadratic form of the loss function.

4 Social Loss

Usually, it is assumed that the entire union's loss, also (often) called the social loss, is represented by the total sum of monetary and fiscal authorities' losses:

$$J_U(t_0, \Pi) := \sum_{i \in F} J_i(t_0) + J_{CB}(t_0), \tag{10}$$

where $J_i(t_0)$ and $J_{CB}(t_0)$ are defined by loss functions (8) and (9), respectively, and Π is a cooperation regime in which the combined loss is computed. Whereas the above definition seems to be plausible for a two-country model, it is not appealing in more complex settings. Since, in a general formulation of our model, there are n countries in the union and only one central bank, the relative importance of the monetary policymaker in $J_U(t_0, \Pi)$ and, hence, also of the monetary instruments gets smaller with increasing n. It is rather difficult to see the rationale behind it as the relative cost related to the interest rate volatility should be irrelevant of the size of the union.[15] To circumvent the above concerns, we propose the next MU loss function:

$$J_U^*(t_0, \Pi) := \frac{1}{2} \int_{t_0}^{\infty} \left[\sum_{i=1}^{n} \omega_i \left(\alpha_U \hat{\pi}_i^2(t) + \beta_U \hat{y}_i^2(t) + \chi_{f,U} \widehat{\hat{f}}_i^2(t) \right) + \chi_{r,U} \widehat{\imath_U}^2(t) \right] \times e^{-\theta(t-t_0)} dt. \tag{11}$$

Here α_U, β_U, $\chi_{f,U}$ and $\chi_{r,U}$ are preference parameters in the monetary union concerning deviations of inflation, output gap and both types of control instruments. Averages of variables' squares instead of squares of variables' averages guarantee that negative deviations of inflations and output gaps do not cancel out with positive ones. Furthermore, taking into account the average value of fiscal control instruments across a monetary union guarantees that volatility of interest rate is well represented in the loss (as it corresponds to its actual relative importance in a single economy).

Whether $J_U^*(t_0, \Pi)$ is smaller, equal or greater than $J_U(t_0, \Pi)$ depends largely on the preference parameters in loss functions (8)–(9) and (11). In a basic case when all these preferences coincide, i.e. $\alpha_i = \alpha_{CB} = \alpha_U$, $\beta_i = \beta_{CB} = \beta_U$ and $\chi_i = \chi_{CB} = \chi_{f,U} = \chi_{r,U}$, it is trivial to show that the (more) conventional social loss $J_U(t_0, \Pi)$ will always be higher than $J_U^*(t_0, \Pi)$ irrelevant of $\hat{\pi}_i$, $\hat{y}_i$, $\widehat{\hat{f}}_i$ and $\hat{\imath_U}$ adjustment paths. Otherwise, the result of this comparison is case dependent. In particular, it may vary with the type of shock considered.

The formula in (11) is similar to the one proposed by Beetsma et al. (2001); however, we extended the definition of cross monetary union loss with the devia-

[15]Formula (10) applied to the 50-State US and to the 13-member state EMU would show that (ceteris paribus) the relative importance of interest rate volatility for the American economy is much lower than for the Euro-zone.

tion of the interest rate, as this is an important factor influencing the welfare of the representative citizen in each country.

Furthermore, formula (11) can be interpreted in terms of NK microfoundations. Specifically, as shown by, for instance, Amato and Laubach (2003) and Woodford (2003), it is possible to derive a quadratic loss function of a form similar to (8) and (9) from a second-order Taylor series approximation to the representative household's welfare. Thus, function (11) can be interpreted as an average welfare function of all households in a monetary union. Taking this social point of view, the output gap volatility is related to the number of hours worked (i.e. employment) in the representative household's utility function whereas inflation volatility to purchasing power. As shown by Woodford (2003), a nominal interest rate in the social welfare function is related to the presence of real money balances in a representative household's utility function. In this respect, Friedman (1968) argued that high nominal interest rates result in transaction costs. Furthermore, the fiscal debt element in (11) can be attributed to the cost of excessive fiscal deficit for a society, that results in an increased price of servicing accumulated debt.

There remains an open question about the values of α_U, β_U, $\chi_{f,U}$ and $\chi_{r,U}$ that define preferences of the society. In the numerical simulations presented in the next section, it is assumed that values of these parameters correspond to preferences of fiscal and monetary authorities. The underlying assumption here is that both the governments' and the central bank's objectives are, to a large extent, representative for the public's preferences. Such an interpretation of formula (11) as a true social loss function is subject to various assumptions. In particular, we assume that both types of authorities aggregate heterogeneous preferences of a society in such a way that the aggregation is equal to the outcome of a voting mechanism or a utilitarian social welfare function determining society's weights on inflation and employment.[16]

Coalition structures for which $J_U^*(t_0, \Pi)$ is the lowest will be called social optima and will be denoted by Π^{*SOP}. Similarly, those regimes that are characterised by the conventional lowest loss $J_U(t_0, \Pi)$ will be denoted by Π^{SOP}. It is straightforward to show that, in the LQDG framework considered here, a coalition composed of all the players in the game (i.e. the grand coalition) always belongs to a set of social optima Π^{SOP} (for more details on this issue see Plasmans et al. 2006a, Chap. 2). However, as the definition of $J_U^*(t_0, \Pi)$ is not necessarily based on the same players' preferences as those used in the optimisation process, the grand coalition does not necessarily belong to Π^{*SOP}. This will be evident from various numerical simulations presented in the subsequent section.

5 Numerical Simulations

For clarity and space concerns we will focus our simulations on the three-country application of the model from Sect. 3. This number is sufficient to consider partial

[16] For further discussions see, for instance, Rogoff (1985), Persson and Tabellini (1993) and McCallum (1997).

Table 1 List of considered coalition structures

CS	Long notation	Acronym	Description
Π_1	$[C1\|C2\|C3\|CB]$ or $[1\|2\|3\|4]$	N	Non-cooperative regime
Π_2	$[C1, C2, C3, CB]$ or $[1234]$	C	The grand coalition
Π_3	$[C1, C2, C3\|CB]$ or $[123\|4]$	F	Full fiscal cooperation
Π_4	$[C1, C2\|C3\|CB]$ or $[12\|3\|4]$	P (or 4)	Partial fiscal cooperation
Π_5	$[C1, C3\|C2\|CB]$ or $[13\|2\|4]$	P (or 5)	Partial fiscal cooperation
Π_6	$[C1\|C2, C3\|CB]$ or $[1\|23\|4]$	P (or 6)	Partial fiscal cooperation

Table 2 Baseline parameters ($i, j \in \{1, 2, 3\}, i \neq j$)

Structural parameters:				
$\kappa_{i,\pi/y} = 2/3$	$\gamma_i = 0.5$	$\eta_i = 0.75$	$\rho_{ij} = 0.5$	$\delta_{ij} = 0.25$
$\beta = 0.99$	$\zeta_i = 0.06$	$\varsigma_{ij} = 0.5$		
Policy rules parameters:				
$\theta_y^i = -0.5$	$\theta_\pi^i = 0$	$\theta_y^U = 0.5$	$\theta_\pi^U = 1.5$	
Preference parameters:				
$\alpha_i = 0.02$	$\beta_i = \alpha_i/5$	$\chi_i = 0.1$	$\alpha_U = 0.02$	$\beta_U = 0.02$
$\alpha_{CB,i} = \beta_{CB,i}/5$	$\beta_{CB,i} = 0.02$	$\chi_{CB,i} = 0.1$	$\chi_{f,U} = \chi_{r,U} = 0.1$	

cooperation between fiscal authorities but still small enough to be computationally tractable. Furthermore, throughout this chapter we assume that cooperation between the central bank and only a subgroup of countries is not allowed, which yields 6 feasible coalition structures listed in Table 1. $C1$, $C2$ and $C3$ and CB denote governments within the union and the central bank, respectively.

5.1 Parametrisation

Table 2 lists all the parameters of the benchmark model. In the baseline scenario (denoted sc_1), countries are assumed to be symmetric with respect to all 7 structural parameters.

The parameters listed in Table 2 are comparable to other simulation studies, in particular, Batini and Haldane (1999), Leith and Wren-Lewis (2001) and van Aarle et al. (2004). When calibrating the IS equation with both backward- and forward-looking behaviour for the UK, they assumed $\kappa_{i,y}$ equal to 0.8 and 0.9, respectively, which are plausible values for quarterly data. For an average EMU economy we set this value to be $2/3$ in the benchmark model; however, we will pay special attention to this parameter in our sensitivity analysis. McCallum (2001), for the US case, suggests that for the interest rate elasticity of output in the IS curve (γ_i in our model), a value of 0.4 is more appropriate than Rotemberg and Woodford's (1999) value of

0.6 or McCallum and Nelson's (1999) value of 0.2. However, Cecchetti et al. (2002) estimate its average value to be 0.7 in the EU. In our case it is assumed that $\gamma_i = 0.5$ which is the value in between the above studies and, for example, corresponds to the parametrisation of Batini and Haldane (1999). In the sensitivity analysis both lower and higher values of this parameter will be considered.

The fiscal multiplier (η_i) measures the impact of changes in fiscal deficit on output gap and is estimated by the European Commission (2001, 2002) in the framework of the Commission's QUEST model and the OECD's Interlink model. The first simulations suggest an average value of 0.6 (± 0.1) in the EU countries while second ones yield values of 0.6 in France, 0.9 in Italy, 1.0 in Germany and the UK, and 1.3 in the US. The difference is to be attributed to the forward-looking nature of the first model. Having these values in mind, we assume η_i to be equal to 0.75. Parameters ρ_{ij} measure the elasticity of domestic import w.r.t. the foreign output gap and is estimated to be equal on average to 0.4 for the EU countries (Equipe MIMOSA 1996) and about 0.35 for Sweden by Lindé et al. (2004). We follow van Aarle et al. (2004) and assume the value of 0.5, which implies relative high trade integration of the economies in a monetary union. This regards also the competitiveness parameter δ_{ij} that is set to 0.25.

There is no consensus in the literature regarding inflation persistence in the Phillips-curve. It is generally recognised that a backward-looking element plays an important role in this equation, but various empirical studies deliver different estimates of $\kappa_{i,\pi}$. Whereas Galí and Gertler (1999) and Benigno and Lopez-Salido (2002) find a predominantly forward-looking specification of the Phillips curve ($\kappa_{i,\pi}$ around 0.7 for Germany, 0.64 for France, 0.4 for Italy, etc.), Mehra (2004) finds a predominantly backward-looking specification ($\kappa_{i,\pi}$ around 0.1). Furthermore, Mankiw (2001) argues that stylised empirical facts are inconsistent with the predominantly forward-looking Phillips curve. In the benchmark we assume the same value of $\kappa_{i,\pi}$ as $\kappa_{i,y}$ i.e. 0.66 but we will consider different specifications later. The elasticity of inflation w.r.t. the output gap is an important parameter of the Philips curve as it ultimately determines short-run adjustment between inflation and output gap. McCallum and Nelson (1999) and Galí and Monacelli (2005a) assume this value to be 0.3, Batini and Haldane (1999), 0.2, Leith and Wren-Lewis (2001) for the UK and Rotemberg and Woodford (1999) for the US set this value to be 0.1, whereas Beetsma and Jensen (2005) choose the value of 0.04. Furthermore, Lindé et al. (2004) estimate it to be at most 0.0158 for the Swedish economy. Again, we choose the value in between the above values setting $\zeta_i = 0.06$, however, it will be one of the main parameters on which we are going to focus our sensitivity check. Gagnon and Ihrig (2002) estimate the import price pass-through parameter ($\zeta_i \times \varsigma_{ij}$ in our model) to be between 0.05 and 0.23 for most OECD countries. On the other hand, Lindé et al. (2004) estimates this value for the Swedish economy to be smaller than 0.003. We calibrate this value to be 0.03, i.e. $\varsigma_{ij} = 0.5$.

Structural model parameters are assumed to be symmetric, however, policymakers' preferences are not. The central bank's preferences differ from those of the (assumed identical) national governments. As it is common in the literature (see, for instance, Beetsma and Bovenberg 1998; Dixit and Lambertini 2001; Engwerda et al. 2002; Uhlig 2003), we assume that the central bank puts a larger weight on inflation stabilisation than on output-gap stabilisation. In contrast, fiscal players are more concerned with output-gap stabilisation than with inflation stabilisation. Moreover, as laid down in the Maastricht Treaty, the central bank's objectives concern aggregate output and inflation in the monetary union while the fiscal players are only concerned about own output and inflation. Parameter β_U is often regarded as a (counter proportional) measure of the central bank independence and it is argued, that a fully independent central bank should be concerned only about inflation, i.e. $\beta_{CB} = 0$.[17] In the benchmark we do not take such a restrictive position and we assume that the central bank is 5 times more concerned about inflation than about an output gap, however, β_{CB} is still positive. Fiscal authorities, in turn, are 5 times more concerned about output gap than about inflation. Thus, calibrated preferences appear to be the most appropriate in our model as they guarantee that no variable is overrepresented in the total loss of any player. For the social loss function $J_U^*(t_0)$, it is assumed that society should be concerned about the output gap as much as the government is, whereas it should be concerned about inflation as much as the central bank is. Hence, $\alpha_U = \alpha_{CB} = 5\alpha_i$ and $\beta_U = \beta_i = 5\beta_{CB}$. The preference parameters of control instruments are set the same as in loss functions (8)–(9), i.e. $\chi_i = \chi_{CB} = \chi_{f,U} = \chi_{r,U}$.

As far as both policy rules are concerned, for the monetary rule, we assume the parametrisation originally proposed by Taylor (1993a, 1993b) for the US, i.e.: $\theta_\pi^U = 1.5$ and $\theta_y^U = 0.5$. For the fiscal policy rule, we assume that $\theta_y^i = -0.5$, which is the value found for the sensitivity of the fiscal deficit in relation to the cyclical variation by the European Commission (2001) for the Euro-area. It is used, for instance, in the simulations of van Aarle et al. (2004). Furthermore, $\theta_\pi^i = 0$. The value of a discount factor $\theta = 0.01$ in the loss functions (8)–(9) is coherent with the assumed structural discount parameter $\beta = 0.99$, which implies a 1 % (steady-state) real rate of return on a quarterly basis. Finally, in the benchmark, we assume symmetric bargaining power in every coalition, i.e. $\tau_{C1/C2/C3} = \tau_{CB} = \frac{1}{4}$ in the grand coalition, $\tau_{C1/C2/C3} = \frac{1}{3}$ in the fiscal coalition under F, and $\tau_{C1} = \tau_{C2} = \frac{1}{2}$, $\tau_{C1} = \tau_{C3} = \frac{1}{2}$, and $\tau_{C2} = \tau_{C3} = \frac{1}{2}$ in regimes 4, 5, and 6, respectively.

5.2 Symmetric Inflation Shock

The first four rows of Table 3 contain the (optimal) welfare losses in the various coalitional arrangements for the symmetric benchmark scenario and the common inflation shock, $v_{0S}^\pi := [v_{1,0}^\pi, v_{2,0}^\pi, v_{3,0}^\pi]^T = [1, 1, 1]^T$.[18] The next two rows show social losses J_U and J_U^* whereas, in the rest of the table, a decomposition of players'

[17] This opinion was also expressed by Lars Svensson at the conference "Inflation Targeting, Central Bank Independence and Transparency," 15–16 June 2007, Trinity College, Cambridge.

[18] All (optimal) losses are multiplied by the factor 10^3.

Table 3 Optimal losses for a symmetric inflation shock, baseline parametrisation (see Table 7 in the Appendix for the number of equilibria)

(sc_1, v_{0S}^{π})	N	C	F	[12|3|4]	[13|2|4]	[1|23|4]
$C1$	2.1948	2.1211	9.3016	2.5328	2.5328	1.7296
$C2$	2.1948	2.1211	9.3016	2.5328	1.7296	2.5328
$C3$	2.1948	2.1211	9.3016	1.7296	2.5328	2.5328
CB	6.2456	5.3308	21.843	5.6049	5.6049	5.6049
$J_U(t_0)$	12.8300	11.6940	49.7480	12.4000	12.4000	12.4000
$J_U^*(t_0)$	7.1445	6.3675	29.9340	6.6853	6.6853	6.6853
$\alpha_{F,C1}\hat{\pi}_{C1}^2$	1.0782	0.8725	0.9979	0.9605	0.9605	0.9792
$\beta_{F,C1}\hat{y}_{C1}^2$	1.0880	1.0596	1.0612	1.3182	1.3182	0.7167
$\chi_{F,C1}\widehat{\hat{f}}_{C1}^2$	0.0285	0.1890	7.2424	0.2541	0.2541	0.0336
$\alpha_{F,C2}\hat{\pi}_{C2}^2$	1.0782	0.8725	0.9979	0.9605	0.9792	0.9605
$\beta_{F,C2}\hat{y}_{C2}^2$	1.0880	1.0596	1.0612	1.3182	0.7167	1.3182
$\chi_{F,C2}\widehat{\hat{f}}_{C2}^2$	0.0285	0.1890	7.2424	0.2541	0.0336	0.2541
$\alpha_{F,C3}\hat{\pi}_{C3}^2$	1.0782	0.8725	0.9979	0.9792	0.9605	0.9605
$\beta_{F,C3}\hat{y}_{C3}^2$	1.0880	1.0596	1.0612	0.7167	1.3182	1.3182
$\chi_{F,C3}\widehat{\hat{f}}_{C3}^2$	0.0285	0.1890	7.2424	0.0336	0.2541	0.2541
$\alpha_{CB}\hat{\pi}_{CB}^2$	5.3912	4.3627	4.9896	4.8337	4.8337	4.8337
$\beta_{CB}\hat{y}_{CB}^2$	0.2176	0.2119	0.2122	0.2181	0.2181	0.2181
$\chi_{CB}\widehat{\hat{i}}_U^2$	0.6367	0.7561	16.641	0.5530	0.5530	0.5530

losses into constituting elements is presented.[19] Reporting inflation, output gap and instrument shares in the total loss aims to provide additional intuition for our results.

As mentioned before, the grand coalition C is always a standard social optimum Π^{SOP} in the LQDG framework. In this particular case this regime constitutes also the social optimum as in Π^{*SOP}. For every coalition structure, $J_U^*(t_0)$ is approximately two times smaller than $J_U(t_0)$ which is caused by the following two reasons: (i) $J_U^*(t_0)$ contains only averages of inflation, output gap and fiscal debt deviations whereas $J_U(t_0)$ is composed of nominal values; (ii) $J_U(t_0)$ includes additionally the loss of the central bank from output and inflation. For some combinations of preference parameters it could be theoretically possible to obtain $J_U^*(t_0) > J_U(t_0)$ but this condition would hold only in an extreme case.

[19]For instance, $C1$'s loss $\frac{1}{2}\int_{t_0}^{\infty}\{\alpha_i\hat{\pi}_i^2(t) + \beta_i\hat{y}_i^2(t) + \chi_i\widehat{\hat{f}}_i^2(t)\}e^{-\theta(t-t_0)}dt$ reported in the top of Table 3 is decomposed into $\frac{1}{2}\int_{t_0}^{\infty}\{\alpha_i\hat{\pi}_i^2(t)\}e^{-\theta(t-t_0)}dt$, $\frac{1}{2}\int_{t_0}^{\infty}\{\beta_i\hat{y}_i^2(t)\}e^{-\theta(t-t_0)}dt$ and $\frac{1}{2}\int_{t_0}^{\infty}\{\chi_i\widehat{\hat{f}}_i^2(t)\}e^{-\theta(t-t_0)}dt$ in the lower part of the table.

In general, the structural symmetry of the model, the symmetry of fiscal players' preferences and the shocks make all the fiscal players' losses to be the same in the N, C and F regimes. Naturally, this symmetry is broken up under partial fiscal cooperation. The decomposition of losses shows that in nominal terms squared inflation deviation over time is about 5 times higher than squared output gap deviation over time. That is why inflation deviation contributes to the total fiscal loss as much as output gap deviation, even though fiscal players care 5 times less about inflation than about output gap. This observation validates our choice of benchmark weights in the loss functions.[20]

Regarding the form of the AD curve and the comment of Lambertini and Rovelli (2003), quoted in Sect. 2, that both types of authorities target the same variable, it is interesting to note that, in our dynamic setting, there is no straightforward relationship between changes in total volatility of output and changes in total volatility of inflation. Intuitively, we would expect that, since the volatility of inflation is directly linked with volatility of output gap via the Phillips Curve just as the volatility of output gap is linked to the volatility of inflation in the AD curve, the relationship between changes in the total loss of both variables should be one-directional. In other words, diminished inflation volatility would be related to either diminished or increased output gap volatility only. However, in our relatively rich dynamic setting, diminished (total) inflation volatility can be associated both with decreased (total) volatility of output gap (e.g. $\beta_{F,C1}\hat{y}_{C1}^2$ and $\alpha_{F,C1}\hat{\pi}_{C1}^2$ in F vs. N) and with increased (total) volatility of output gap (e.g. $\beta_{F,C1}\hat{y}_{C1}^2$ and $\alpha_{F,C1}\hat{\pi}_{C1}^2$ in [12|3|4] vs. N). Thus, it is clear that complex patterns of economic conditions can emerge in our model, which emphasises the need for an accurate policy regime.

The dynamics of all relevant variables in regimes N, C and F is compared in Fig. 1. Symmetric supply side (positive) inflation shocks cause output gap to decline which urges expansionary fiscal policies in all countries. In contrast, the central bank reacts to positive inflation by increasing interest rates. Thus, there is an obvious policy conflict between fiscal and monetary authorities. From decomposed losses in Table 3 we see that in the non-cooperative regime N there is a strong free-riding effect compared to the grand coalition regime C, which results from the positive fiscal spillovers characterising our setting.[21] When authorities do not cooperate, each of them tries to free-ride on the others' stabilisation efforts. The same phenomenon occurs in Beetsma et al. (2001) as their model also features positive fiscal spillovers. Consequently, all the authorities do not stabilise the economy strongly enough and the output gap and inflation deviations under regime N are comparatively high. This results, for instance, in the following total loss for fiscal players from output gap and inflation deviations: 1.0880 and 1.0782, respectively. In contrast, under regime C,

[20]More conventional preferences in which fiscal authorities care only two times as much about output gap as about inflation will be studied later on in this chapter.

[21]The term "free-riding" refers here to taking advantage of others' stabilisation policies during the stabilisation game (i.e. from time t_0, when the shock occurs, onwards). However, this term will be also used in the context of individual players breaking-up coalitional arrangements (with the same objective to take advantage of others' cooperative stabilisation policies but themselves playing non-cooperatively and constraining own costly policies).

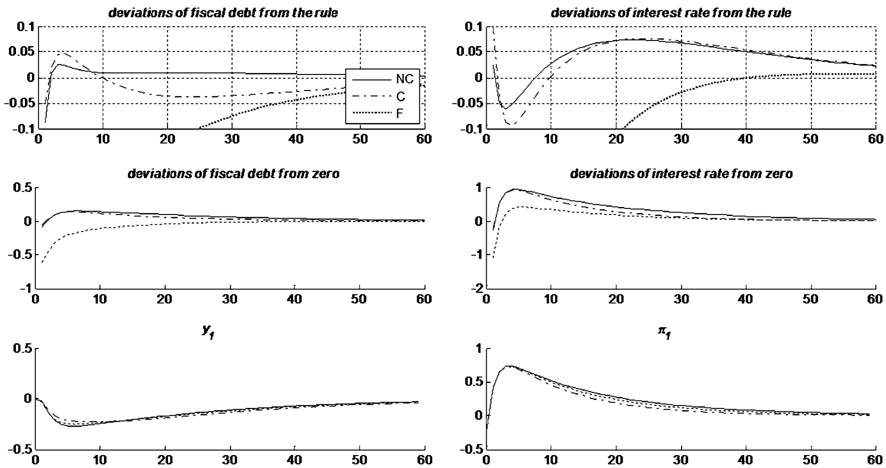

Fig. 1 Benchmark model, v_{0S}^{π}, comparison of regimes N, C and F

fiscal players: (i) do not try to free-ride on each other and (ii) take into account positive spillovers from other fiscal policies. Accordingly, they pursue a more active fiscal policy which is associated with the increased fiscal stabilisation cost from 0.0285 to 0.1890 and diminished losses from output gap and inflation deviations (1.0596 and 0.8725 respectively). The attitude of the central bank is crucial at this point. As under a symmetric inflation shock, there is clearly a policy conflict between fiscal and monetary authorities, increased fiscal activity should urge the central bank to more restrictive monetary policy than under the non-cooperative regime N. However, in the grand coalition all players, including the central bank, cooperate, and, secondly, the objective of every player is different than under non-cooperation as all of them aim to minimise the joint loss function which is a weighted sum of individual losses. Thus, in our benchmark, where all players have an equal weight in every coalition, the central bank's objective function (9) counts only for a quarter of the common loss function in regime C. Consequently, this function is "biased" towards fiscal preferences. Interestingly, under regime C these are fiscal players which find it profitable to change their policies so that there is no conflict between authorities about the direction of the stabilisation policies. Deviations of fiscal debts and the interest rate from the rules is shown in the upper part of Fig. 1. Under regime N both authorities deviate positively from the rules, which is counter-active as fiscal debt influences output gap in the opposite way to (nominal and real) interest rates. In contrast, when fiscal and monetary authorities cooperate, fiscal authorities deviate positively from the rule until period 10 and negatively since then. The sign of the central bank's deviations is exactly opposite; hence, both control instruments influence the output gap always in the same direction. In addition to the consent on policy direction, the lack of free-riding makes all the policies more active. All in all, cooperation makes the grand coalition the most attractive regime from the social point of view.

As far as now we have analysed our results mainly from the social perspective. However, even the regime which is the most desirable from the social point of view, can be very difficult to attain in the self-oriented environment. This is, for instance, the case for the grand coalition in Table 3. In a self-oriented myopic environment it could be very difficult to enforce this form of coordination, because every fiscal player has an incentive to unilaterally deviate from C.[22] For instance, $C1$ prefers [1|23|4] to C, thus, would break up the grand coalition, with hope that $C2$ and $C3$ maintain cooperation. In other words, there are strong free-riding incentives in the case of symmetric inflation shock to deviate from full cooperation. On the other hand, if the assumption of myopic behaviour is waived regime C is more likely to be stable, as it is clear that no partial fiscal cooperation is sustainable as any fiscal player involved in a coalition would prefer to break it and play in the non-cooperative regime N.

The situation is completely different under regime F, where the central bank does not cooperate with the coalition of all fiscal authorities. As before, coordination of policies among governments eliminates the free-riding and alleviates the use of fiscal instruments. However, it is clear from the decomposed losses in Table 3 that it exacerbates the conflict with the central bank as the loss from control effort is much higher in this regime both for governments and for the central bank. The policies are counter-active (see the upper part of Fig. 1) and the payoff for an increased control action is limited to only a little lower inflation and output gap deviations, which, by far, cannot make up for the increased loss. As a consequence, full fiscal coordination is worse than both cooperative and non-cooperative regimes. This latter result is exactly in line with the conclusions of Beetsma et al. (2001).[23] The difference between the results of both analyses is in the direction of the policies. In our model, when all fiscal players decide to cooperate, the central bank pursues more expansionary monetary policies than the assumed Taylor rule, whereas, the fiscal authorities pursue a more restrictive fiscal policy than the assumed fiscal rule. The direction of policies is, therefore, opposite to Beetsma et al. (2001), where supply shocks lead to more restrictive monetary policy vs. further reaching fiscal expansion.[24] However, in both studies a regime in which fiscal players cooperate against the central bank is counter-productive.

The ordering of social preferences over the regimes in Table 3 is $C \overset{J_U/J_U^*}{\succ} P \overset{J_U/J_U^*}{\succ} N \overset{J_U/J_U^*}{\succ} F$, i.e. cooperation in the grand coalition or in a partial fiscal coalition is better than playing alone. The analysis of P regimes follows similar lines as the dis-

[22] Deviations from a coalition are related to the coalition formation theory concept of internal stability (see Plasmans et al. 2006a, for further details).

[23] Note that Beetsma et al. (2001) do not consider coordination in a grand coalition.

[24] It might be argued that the above high cost of stabilisation is caused by the specific choice of policy rules which is so far away from optimum that players are forced to deviate much. In other words, it might be argued that θ_π^M should be closer to 1 and θ_π^i closer to 0. However, in other regimes, even in the fully non-cooperative regime, players are able to choose paths of stabilisation instruments close to assumed policy rules. This clearly overlures such an objection. Furthermore, the results reported in Table 3 were checked also for other parameterisations of policy rules such as: $(\theta_\pi^M = 1.5; \theta_\pi^i = 0)$; $(\theta_\pi^M = 1; \theta_\pi^i = -0.5)$; $(\theta_\pi^M = 1.25; \theta_\pi^i = 0)$; or $(\theta_\pi^M = 1; \theta_\pi^i = 0)$ and produce similar results (under all the above assumptions).

cussion of the grand coalition case. Starting from the non-cooperation, the creation of partial fiscal coalition eliminates free-riding incentives between two governments involved in cooperation which increases their activity. For example, when [12|3|4] is created, $\chi_{F,C1}\widehat{\hat{f}}^2_{C1}$ and $\chi_{F,C1}\widehat{\hat{f}}^2_{C2}$ increase symmetrically from 0.0285 to 0.2541. This increase is higher than the increase for the grand coalition, thus, it cannot be justified only by the diminished free-riding incentives. At the same time $\chi_{F,C1}\widehat{\hat{f}}^2_{C3}$ also increases (from 0.0285 to 0.0336), instead of decreasing.[25] Both the above results are to be explained by the more constrained activity of the central bank, which is caused by the asymmetry of output gap and inflation in this regime that makes union-wide averages less volatile and, therefore, not affecting the loss of a monetary authority to such an extent as before. In other words, the central bank is able to free-ride even more than in regime N and this, in turn, increases the use of control instrument in both fiscal players involved in a coalition and playing non-cooperatively.

5.3 Asymmetric Inflation Shock

Table 4 is constructed in a similar manner as Table 3 and presents (optimal) welfare losses for the asymmetric (country-specific) inflation shock, $v^{\pi}_{0S} := [v^{\pi}_{1,0}, v^{\pi}_{2,0}, v^{\pi}_{3,0}]^T = [1, 0, 0]^T$. Clearly, not all fiscal losses are now symmetric, as the shock directly hits the first country only and other member-states of a monetary union have to deal with its indirect consequences. However, in general, it can be said that the pattern of losses in Table 4 is quite similar to Table 3: again the ordering of social preference over regimes is $C \overset{J_U/J^*_U}{\succ} P \overset{J_U/J^*_U}{\succ} N \overset{J_U/J^*_U}{\succ} F$; full fiscal cooperation is the worst regime for everybody; the grand coalition performs reasonably well but myopic fiscal players would have an incentive to break up this regime; partial fiscal coordination is not sustainable as players involved in cooperation would prefer the non-cooperative regime. Finally, it should be noted that cooperation of a country which is not affected directly by the shock with a country which is affected (regimes [12|3|4] or [13|2|4]) is very unprofitable for both of them but especially for the former one. Furthermore, also cooperation of both countries which are not affected directly by the shock (regime [1|23|4]) is also not profitable when compared to non-cooperation. Consequently, any form of partial fiscal cooperation when the correlation of shocks gets smaller seems to be unsustainable in case of a symmetric inflation shock.

Losses in this case are smaller than for the symmetric inflation shock because there is a different reaction of the common monetary policy and national fiscal policies in countries not affected by the shock. This happens because the central bank

[25] The decrease in $C3$ control effort would be expected as the cooperation between $C1$ and $C2$, by increasing their activism, gives even more incentives to free-ride.

Table 4 Optimal losses for an asymmetric inflation shock, baseline parametrisation

(sc_1, v_{0S}^{π})	N	C	F	[12|3|4]	[13|2|4]	[1|23|4]
$C1$	0.3409	0.3245	1.0848	0.3921	0.3921	0.2807
$C2$	0.2369	0.2329	1.0495	0.3006	0.1763	0.2896
$C3$	0.2369	0.2329	1.0495	0.1763	0.3006	0.2896
CB	0.6939	0.5923	2.4270	0.6024	0.6024	0.6084
$J_U(t_0)$	1.5092	1.3827	5.6109	1.4715	1.4715	1.4684
$J_U^*(t_0)$	0.8483	0.7618	3.3804	0.7898	0.7898	0.7932
$\alpha_{F,C1}\hat{\pi}_{C1}^2$	0.1659	0.1461	0.1584	0.1509	0.1509	0.1541
$\beta_{F,C1}\hat{y}_{C1}^2$	0.1681	0.1679	0.1607	0.2183	0.2183	0.1100
$\chi_{F,C1}\widehat{\hat{f}}_{C1}^2$	0.0069	0.0104	0.7656	0.0228	0.0228	0.0165
$\alpha_{F,C2}\hat{\pi}_{C2}^2$	0.1067	0.0823	0.0970	0.0896	0.0924	0.0899
$\beta_{F,C2}\hat{y}_{C2}^2$	0.1266	0.1215	0.1253	0.1573	0.0760	0.1512
$\chi_{F,C2}\widehat{\hat{f}}_{C2}^2$	0.0035	0.0291	0.8270	0.0536	0.0078	0.0483
$\alpha_{F,C3}\hat{\pi}_{C3}^2$	0.1067	0.0823	0.0970	0.0924	0.0896	0.0899
$\beta_{F,C3}\hat{y}_{C3}^2$	0.1266	0.1215	0.1253	0.0760	0.1573	0.1512
$\chi_{F,C3}\widehat{\hat{f}}_{C3}^2$	0.0035	0.0291	0.8270	0.0078	0.0536	0.0483
$\alpha_{CB}\hat{\pi}_{CB}^2$	0.5990	0.4847	0.5544	0.5220	0.5220	0.5230
$\beta_{CB}\hat{y}_{CB}^2$	0.0241	0.0235	0.0235	0.0242	0.0242	0.0242
$\chi_{CB}\widehat{\hat{i}}_U^2$	0.0707	0.0840	1.8490	0.0561	0.0561	0.0612

targets average inflation which in the case of an asymmetric shock is clearly much smaller than in case of a symmetric one (see Fig. 2). Also, the national fiscal policies of $C2$ and $C3$ react more moderately since these countries are only affected by cross border-spillovers and the restrained reaction of the common monetary policy.

The above findings, in general, correspond to the main arguments of Beetsma et al. (2001). Indeed, the clash between authorities is diminished under an asymmetric inflation shock because many effects cancel each other and policymakers have less incentives to excaberate the dispute. However, in our model, the extent of the conflict is still so substantial, that the excessive use of control instruments make the regime of fiscal cooperation, by far, unprofitable.

5.4 Sensitivity Analysis

Thus far, we have studied a number of cases characterised by different forms of coordination. It is interesting to perform sensitivity analyses of the model in order to identify which results (possibly) prevail and which parameters contribute mostly

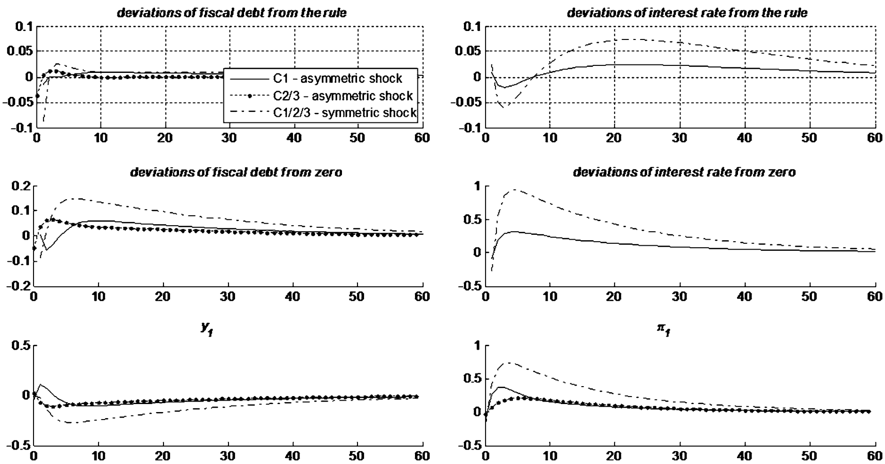

Fig. 2 Benchmark model, v_{0S}^{π} vs. v_{0A}^{π}, regime N

to particular outcomes. Some elements of a sensitivity analysis have been already present in the discussion above. In particular, we have inquired what happens to the social loss when alternative weights of the central bank in the grand coalitions are assumed. Additionally, we performed sensitivity analysis in three other dimensions:

1. Similarly to Beetsma et al. (2001) or Beetsma and Jensen (2005) we vary one structural parameter of the model at a time, assuming that it has either a *high* or a *low* value. Additionally, we perform not only one but a number (usually 4 or 5) of simulations in the neighbourhood of each high and low value;
2. Furthermore, we study various combinations of preference parameters in the loss functions of players. In particular, different ratios of preferences towards output gap and inflation are considered;
3. In the next step, the sensitivity analysis of governments' preferences is performed. A particular attention will be paid to the government preference parameter χ_i which different values can be interpreted as levels of the SGP stringency.

A detailed description of the obtained results is available in the Appendix (Sections A.1–A.4). The structural parameter sensitivity check reveals that the degree of output gap backward-lookingness is the key parameter, which can either magnify or diminish the conflict between fiscal and monetary authorities. Counter-intuitively, it turns out that cooperation can be more effective when the economies are relatively more backward-looking. Backward-lookingness makes economies more rigid and therefore more difficult to control which, in turn, should lead to increased control effort. This is the main factor inducing conflict with the central bank in the benchmark and resulting in higher losses in the full fiscal cooperation regime. However, when a certain threshold is triggered, the players realise that there is no point in setting off each other's policies as their influence on the economy becomes too limited. Consequently, they all decide to refrain from excessive actions and instead of colliding their policies, governments try to free-ride on the central bank's control effort and vice versa. Interestingly, this is not the case under the non-cooperative regime. Instead, in this type of regime, conflict between authorities increases together with the backward-lookingness of the output gap. Consequently, when the rigidity of the output gap becomes high enough, non-cooperation starts to become inferior w.r.t. fiscal cooperation. In other words, the situation is the exact reversal of that of the baseline analysis. In the benchmark case, characterised by high forward-lookingness, free-riding diminishes the conflict between fiscal and monetary authorities under non-coordination. However, under fiscal coordination, the struggle between governments and the central bank increases everybody's loss. When high backward-lookingness is assumed, the conflict is exaggerated under non-cooperation, whereas under fiscal cooperation, authorities try to free-ride on each others' policies.

The second factor which heavily influences the results of our benchmark analysis is the relative conflict between preferences of fiscal and monetary authorities. The more governments exclusively focus on output gap stabilisation and the more central banks focus on inflation stabilisation, the more pronounced the conflict between them becomes. Consequently, the counter-profitable effects of full fiscal cooperation are even greater. On the other hand, when the preferences of both types of players are more alike, there is less reason for conflict and fiscal cooperation becomes, from the social point of view, beneficial. More specifically, full fiscal cooperation initially becomes beneficial w.r.t. the non-cooperative regime, before subsequently becoming more beneficial w.r.t. partial fiscal cooperation regime and finally, more beneficial w.r.t. the grand coalition. This last effect is counter-intuitive as, by definition, the grand coalition always minimises the sum of the losses of all the players in the LQDGs. However, in our analysis we mainly refer to our own definition of the social loss in which, as has been mentioned previously, the cost of interest rate deviation from the rule is properly weighted to correspond to the one-country case. Consequently, our social optimum does not necessarily agree with the grand coalition. In fact, when the loss functions of governments and central banks become very alike, social loss obtained under fiscal cooperation tends to be smaller than under the grand coalition. This is caused by the fact that, in the simple sum of players' losses, the weight of the control instrument of a monetary authority is relatively small w.r.t. fiscal debts of individual governments. This creates an incentive to use it more extensively. If the importance of the interest rate is appropriately rescaled in the social loss, the grand coalition is no longer the most profitable regime.

Using numerical simulations, we study the way in which various combinations of policy rules' parameters influence output gap and inflation volatility. We also determine which of such combinations are likely to result in a(n) (near) optimal outcome from the social point of view. In the non-cooperative regime, and under a symmetric inflation shock, the proximity of the social optimum is reached for the combination of rules in which the central bank follows the standard Taylor rule, yet there is no automatic fiscal stabiliser to output. However, we show that if players in a monetary union were able to unilaterally choose their rules, the social optimum combination would not be sustainable. This is because the monetary authority has incentives to increase its automatic reaction to inflation and, at the same time, the government has incentives to increase its reaction to output gap. When these things occur simultaneously, the economies end up in a position that is suboptimal, not only from the social point of view, but also from the perspective of the individual.

Finally, we study various scenarios characterised by different levels of the SGP stringency and show that it is the third factor that is pivotal to the benchmark results. The increased SGP stringency reduces the incentives for fiscal players to use control instruments. Therefore, in situations where high social losses were driven by the conflict between authorities (notably full fiscal cooperation regime), this firmer stance is beneficial to the union-wide economic interest. However, in situations where free-riding is present (notably non-cooperative regime under benchmark parametrisation), increased SGP stringency may lead to more extensive free-riding of governments, since controlling the economy becomes much more costly. This, in turn, forces the central bank to intervene and increases social loss of the union. In other words, the stringent SGP has both positive and negative effects in our context and is able to render unprofitable full fiscal cooperation regime profitable w.r.t. non-cooperation.

6 Conclusions

In this work we considered a number of important issues concerning the policy coordination in the monetary union which have been discussed in the literature. In relation to this, we proposed a (stylised) Multi-Country New-Keynesian Monetary Union Model cast in the framework of linear quadratic differential games which can be used to simulate strategic interactions between an arbitrary number of fiscal authorities who interact in coalitions either in cooperation with or against the common central bank. In the above setting, we studied various coordination arrangements, including partial fiscal cooperation between only a subgroup of countries, which, to the best of our knowledge, had not previously been considered in the literature.

Our results are comparable to those of Beetsma et al. (2001) but in a much richer dynamic setting. Whereas free-riding in economics and social sciences is usually associated with inefficiency and losses, in our model, for many parameter combinations, fiscal cooperation between all countries in a monetary union turns out to be counter-productive as a result of exacerbated conflict with the central bank. Thus, the non-cooperative regime, in which policy-makers free-ride on each others' stabilisation efforts, is more profitable. The relative performance of fiscal cooperation is worst in the case of a symmetric inflation shock. When the shock is asymmetric, the response of the central bank to an average inflation in the union is more moderate, as are the losses of all the authorities.

In addition to the above results, in our multi-country framework, we were able to study intermediate regimes, in which only a subgroup of the fiscal players cooperate. Such a solution turns out to be interesting, especially from the common perspective. When a unity of governments is broken and they no longer optimise the common objective function, then the free-riding element is back into play and the ultimate choice of optimal policies is much less intense. Nevertheless, such regimes can be difficult to sustain, as they lead to a deteriorated position of some individual countries.

Furthermore, we discussed the effectiveness of the grand coalition, i.e. cooperation between all fiscal authorities and the monetary authority. Although this regime is profitable from the social point of view, it seems unlikely that it could be sustained without the creation of a central control or an effective transfer system. Since the situation is caused by the fact that individual countries are often worse off in the grand coalition than if they were in the non-cooperative regime, these countries are not willing to accept such a regime without compensation.

Finally, we performed an extensive sensitivity check of our results and determined three variables that are pivotal to the results we obtained for the benchmark of our model: (i) the degree of forward-lookingness in the union's economies; (ii) the preference conflict between fiscal and monetary authorities; and (iii) the SGP stringency. The last analysis reveals the most interesting finding as long as the issue of fiscal cooperation is concerned. Since the main source of the relative inefficiency of full fiscal cooperation is an increased conflict with the central bank due to a more intensive use of the relevant control instrument. Higher levels of SGP stringency mean that the use of fiscal debt becomes less and less attractive. This, in turn, removes the reason for conflict between two types of authorities.

The following policy conclusion can be derived from our analysis. The grand coalition is, in general, the most effective regime; however, only if the design of a cooperative arrangement takes into account the specific nature of a central bank and its policy instruments. However, this regime is difficult to implement due to various problems. In particular, the very nature of such a coalition jeopardises the idea of the independent central bank. On the other hand, under special circumstances, like present financial crises that spreads to other sectors of the global economy, we already witness various forms of cooperation between the ECB and the EMU member states as well as EU member states. However, in general, both de jure and de facto state of the affairs in the EMU can be described best by the non-cooperative regime. Since the grand coalition is rather out of question in the long term, intergovernmental coordination appears to be another interesting alternative. With respect to this, our results show that obligatory fiscal coordination between all the countries within the union can be counter-productive and that smaller coalitions of countries should be considered. This corresponds to the findings of the literature on voluntary environmental agreements that suggest that local solutions are more stable than a centralised approach as for instance the Kyoto Protocol. Similarly, partial fiscal cooperation can be an interesting option as far as macroeconomic policy coordination is concerned. However, if (possibly multiple) partial agreements are not feasible for political reasons, then full fiscal coordination can be considered again in conjunction with the increased stringency of the SGP. The SGP should prevent excessive increases in fiscal activity that might be induced by cooperation.

Our analysis can be extended in several important directions. Although we considered both symmetric and asymmetric shocks in our analysis, we assumed full symmetry of monetary union member states. Following on from this, it would be beneficial to carry out an analysis of an asymmetric model, in terms of both the economies structures and the players preferences. Furthermore, since our results suggest that more attention should be devoted to partial forms of cooperation between the governments of a monetary union, we would like to extend our analysis to (at least) a four-country case, where two non-trivial fiscal coalitions may coexist in one single coalition structure. Finally, it could be interesting to consider how the results obtained from our model are altered by the introduction of a federal fiscal transfer system in a monetary union.[26]

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Appendix

A.1 Sensitivity Analysis with Respect to Structural Parameters

The assumed changes of one structural parameter at a time are presented in the first column of Table 5. The next columns show the social preference ordering over different regimes for four shocks.[27] The following notation is used: P—stands for regimes 4, 5 and 6; regime in bold means that players in a coalition are better off than in N; $\hat{C}/\hat{F}$—means that all fiscal players in grand/full fiscal coalition are better off than in N but the CB is not in this regime; $\check{C}/\check{F}$—vice versa; **F**—all players are better off than in N. In most of the cases the ordering based on J_U^* was the same as based on J_U, with only few minor exception, therefore, only the preference ordering based on the former social loss is reported in Table 5.

For all combinations of parameters except for $\kappa_{i,y/\pi} = 0.5$, from the social point of view, regime N is preferred over F; in other words full fiscal coordination in counter-productive. Thus, it can be said that for the large set of parameters our model confirms results of Beetsma et al. (2001).

The ordering *CPNF* prevails for symmetric inflation shock v_{0S}^{π} and is robust to changes in parametrisation (except for lower value of $\kappa_{i,y/\pi}$).[28] The same ordering is valid for country-specific inflation shock v_{0A}^{π}, however, due to asymmetry partial

[26] An example of such an analysis (albeit only in a two-country setting) can be found in Plasmans et al. (2006a, Chap. 3).

[27] For the time being we focus on structural parameters of the model excluding policy rules, which together with preference parameters will be discussed in the next section.

[28] Note that in the case of symmetric shock regimes 4, 5 and 6 denoted jointly by P are symmetric; thus, are characterised by the same social loss. However, for asymmetric shocks which always hit $C1$ only regimes 4 and 5 are symmetric to each other where as they are, in general, asymmetric to regime 6. Consequently, for asymmetric shocks we do not the joint P-notation for partial fiscal cooperation regimes.

Table 5 Sensitivity analysis of the benchmark model

	$J_U^*(t_0)$			
	v_{0S}^{π}	v_{0A}^{π}	v_{0S}^{y}	v_{0A}^{y}
Benchmark	$\mathbf{C}PNF$	$\mathbf{C}456NF$	$P\hat{C}NF$	$CN456F$
$\kappa_{i,y/\pi} \approx 0.5$	$\mathbf{C}\check{F}\mathbf{P}N$	$\mathbf{C6}\check{F}\mathbf{45}N$	$\mathbf{C}P\mathbf{F}N$	$\mathbf{C6}45\mathbf{F}N$
$\kappa_{i,y/\pi} \approx 0.8$	$\mathbf{CP}NF$	$\mathbf{C645}NF$	$\mathbf{C}PNF$	$\mathbf{C645}NF$
$\gamma_i \approx 0.45$	$\mathbf{C}PNF$	$C45\mathbf{6}NF$	$N\hat{C}PF$	$CN456F$
$\gamma_i \approx 0.55$	$\mathbf{C}PNF$	$\mathbf{C}45\mathbf{6}NF$	$N\hat{C}PF$	$CN456F$
$\eta_i \approx 0.65$	$\mathbf{CP}NF$	$\mathbf{C}456NF$	$\hat{C}PNF$	$CN45\mathbf{6}F$
$\eta_i \approx 1$	$\mathbf{C}PNF$	$\mathbf{C}456NF$	$P\hat{C}NF$	$CN456F$
$\rho_{ij} \approx 0.4$	$\mathbf{C}PNF$	$\mathbf{C}456NF$	$N\hat{C}PF$	$CN456F$
$\rho_{ij} \approx 0.5$	$CPNF$	$C456NF$	$CPNF$	$CN456F$
$\delta_{ij} \approx 0.075$	$\mathbf{C}PNF$	$\mathbf{C}45\mathbf{6}NF$	$P\hat{C}NF$	$C45N6F$
$\delta_{ij} \approx 2/3$	$\mathbf{C}PNF$	$\mathbf{C}456NF$	$N\hat{C}PF$	$CN456F$
$\xi_i \approx 0.03$	$\mathbf{CP}NF$	$\mathbf{C}645NF$	$\mathbf{C}PNF$	$\check{C}\mathbf{645}NF$
$\xi_i \approx 0.125$	$\mathbf{CP}NF$	$\mathbf{C}45\mathbf{6}NF$	$\mathbf{C}PNF$	$\check{C}\mathbf{P}N\mathbf{F}$
$\varsigma_{ij} \approx 0.025$	$\mathbf{C}PNF$	$\mathbf{C}456NF$	$N\hat{C}PF$	$\check{\mathbf{C}}\mathbf{6}N45F$
$\varsigma_{ij} \approx 0.1$	$\mathbf{C}PNF$	$\mathbf{C}456NF$	$\mathbf{C}PNF$	$\check{\mathbf{C}}\mathbf{6}N45F$

fiscal coalition 6 is (in general) characterised by different social loss than 4 and 5; thus, preference ordering over partial fiscal coalitions might vary. This leads to the conclusion that the grand coalition or partial fiscal coalitions should be sought as the socially efficient regimes in the case of inflation shock. The question whether such forms of coordination would be sustainable remains open. In Table 5 under v_{0S}^{π} and v_{0A}^{π} the grand coalition is preferred over non-cooperation by every player from an individual point of view. However, it does not tell us whether any player would like to deviate from this arrangement with hope that remaining fiscal players maintain cooperation and a partial fiscal regime emerges. In contrast, the fact that P-regimes are usually inferior to N for (fiscal) player(s) being in coalitions allows us to draw a conclusion that, certainly, these regimes are not sustainable in self-oriented (and myopic) environment.

The picture is less clear for output gap shocks as parameter changes have more influence on regimes' social ordering here. Full fiscal cooperation is always least preferred (except for lower value of $\kappa_{i,y/\pi}$) so the results of Beetsma hold also in this case. However, in contrast to inflation shocks, the grand coalition scores often worse than non-cooperation or P-regimes in the case of symmetric output gap shock.

To summarise, sensitivity analysis of the benchmark model confirms the result of Beetsma et al. (2001) as in all the cases but one full fiscal coordination is worse than non-cooperation. Furthermore, this result can be extended further, as it comes out that F is the worst of all regimes, including partial fiscal cooperation. For inflation shocks, the grand coalition is the socially optimal outcome, and this regime is better than non-cooperation from the individual point of view. However, whether

Table 6 Optimal losses for $(v_{0S}^{\pi}, \kappa_{i,y} = 0.5)$

(sc_1, v_{0S}^{π})	N	C	F	[12|3|4]	[13|2|4]	[1|23|4]	N^F
$C1$	4.2650	3.5500	3.5033	3.7917	3.7917	3.4779	4.0750
$C2$	4.2650	3.5500	3.5033	3.7917	3.4779	3.7917	4.0750
$C3$	4.2650	3.5500	3.5033	3.4779	3.7917	3.7917	4.0750
CB	11.7566	6.9454	8.5717	10.1221	10.1221	10.1221	12.3601
$J_U(t_0)$	24.5516	17.5957	19.0817	21.1836	21.1836	21.1836	24.5859
$J_U^*(t_0)$	13.9057	8.9598	10.3530	11.8533	11.8533	11.8533	13.7293
$\beta_{F,C1}\hat{y}_{C1}^2$	1.7332	2.1976	2.0190	2.0272	2.0272	1.4935	1.4647
$\alpha_{F,C1}\hat{\pi}_{C1}^2$	1.7692	1.0961	1.3181	1.5834	1.5834	1.5980	2.4136
$\chi_{F,C1}\widehat{\hat{f}}_{C1}^2$	0.7625	0.2562	0.1661	0.1810	0.1810	0.3863	0.1966
$\beta_{F,C2}\hat{y}_{C2}^2$	1.7332	2.1976	2.0190	2.0272	1.4935	2.0272	1.4647
$\alpha_{F,C2}\hat{\pi}_{C2}^2$	1.7692	1.0961	1.3181	1.5834	1.5980	1.5834	2.4136
$\chi_{F,C2}\widehat{\hat{f}}_{C2}^2$	0.7625	0.2562	0.1661	0.1810	0.3863	0.1810	0.1966
$\beta_{F,C3}\hat{y}_{C3}^2$	1.7332	2.1976	2.0190	1.4935	2.0272	2.0272	1.4647
$\alpha_{F,C3}\hat{\pi}_{C3}^2$	1.7692	1.0961	1.3181	1.5980	1.5834	1.5834	2.4136
$\chi_{F,C3}\widehat{\hat{f}}_{C3}^2$	0.7625	0.2562	0.1661	0.3863	0.1810	0.1810	0.1966
$\beta_{CB}\hat{y}_{CB}^2$	0.3466	0.4395	0.4038	0.3677	0.3677	0.3677	0.2929
$\alpha_{CB}\hat{\pi}_{CB}^2$	8.8462	5.4808	6.5908	7.9414	7.9414	7.9414	12.0680
$\chi_{CB}\widehat{\hat{i}}_U^2$	2.5637	1.0251	1.5770	1.8128	1.8128	1.8128	0

C is sustainable remains still an open question. Partial fiscal cooperation is suboptimal w.r.t. the grand coalition but gives better results than non-cooperation from the social perspective. Unfortunately, in many cases these regimes are suboptimal from individual point of view, thus, possibly unsustainable in the non-cooperative environment, especially if players are myopic. The ineffectiveness results about full fiscal coordination hold also for output gap shocks, but it more difficult to draw some definite conclusions about the preference ordering over the other regimes as the differences in social loss between them are small and, therefore, sensitive, to changes in parametrisation.

It is apparent from Table 5 that the influence of forward-lookingness on our model calls for more attention, as lower values of this parameter may have an important impact on the result obtained above. Table 6 shows the optimal losses for symmetric inflation shock v_{0S}^{π} and benchmark parametrisation but with $\kappa_{i,y/\pi} = 0.5$. What is the reason for the improved effectiveness of F regime w.r.t. N when the economies in a monetary union are characterised by lower forward-lookingness?

Change of an important model parameter certainly influenced the reduced form matrices $\tilde{B}_4$ and $\tilde{B}_5$ which show the influence of control instruments on output gaps

Table 7 Symmetric inflation shock, the number of equilibria in LQDGs

(sc_1, v_{0S}^{π})	N	C	F	[12|3|4]	[13|2|4]	[1|23|4]
LQDGE	158	1	1	140	140	140
PUNE	1	1	1	13	13	13

and inflations, respectively.

$$\tilde{B}_4 = \begin{bmatrix} 0.7007 & 0.1468 & 0.1468 & -0.6631 \\ 0.1468 & 0.7007 & 0.1468 & -0.6631 \\ 0.1468 & 0.1468 & 0.7007 & -0.6631 \end{bmatrix}, \quad \text{and}$$

$$\tilde{B}_5 = \begin{bmatrix} 0.0134 & 0.0026 & 0.0026 & -0.0125 \\ 0.0026 & 0.0134 & 0.0026 & -0.0125 \\ 0.0026 & 0.0026 & 0.0134 & -0.0125 \end{bmatrix}.$$

The pattern is closely comparable to the benchmark (all the values are now more or less 15 % lower than before and all the signs are preserved). With a bigger backward-looking component, economies are more persistent and converge slower to the equilibrium. This is reflected by nearly two times as more losses in regimes N, C and P for $\kappa_{i,y/\pi} = 0.5$ compared to benchmark with $\kappa_{i,y/\pi} = \frac{2}{3}$ in Table 3. As far as the cost of the control instruments is concerned, we observe the following pattern. For regimes where only fiscal players fully or partially cooperate (i.e. F and P), (total) cost of control of those players involved in the coalition gets lower when backward-looking component becomes more eminent. In the case of full fiscal cooperation. This reduction is drastic, from 7.2424 to 0.1974. In contrast, the loss of the fiscal players from the control effort (in P regimes) increases more than 10 times, from 0.0336 to 0.3682. The same holds for the non-cooperative regime, where the (non-cooperative) fiscal players have their control effort increased from 0.0285 to 0.6618. In spite of the somewhat increased volatility of inflation and output gap, the above decrease in control cost in the F regime accompanied by the increase in control cost in N regime (see Fig. 3) makes full fiscal cooperation more attractive than previously.

The reason for these more moderate actions should be sought in the fact that more backward-lookingness in the model means that economies are more persistent and more slowly converging to equilibrium. This means that it is more costly to control them as the use of instruments is less efficient, which successfully diminishes the conflict between authorities, which choose more reasonable policies.

To improve our understanding of the meaning of forward-lookingness in the model similar sensitivity check to the previous one was conducted but now assuming that $\kappa_{i,y/\pi}$ is set to 0.5 as the benchmark. The results are presented in Table 8 which was constructed in the same way as Table 5.[29]

[29] In Table 8 the orderings based on $J_U(t_0)$ are in all cases but 8 exactly the same as those based on $J_U^*(t_0)$, where the differences occurred only in the ordering of partial fiscal cooperation regimes 4, 5 and 6 under asymmetric shocks. Consequently, we report only orderings based on $J_U^*(t_0)$.

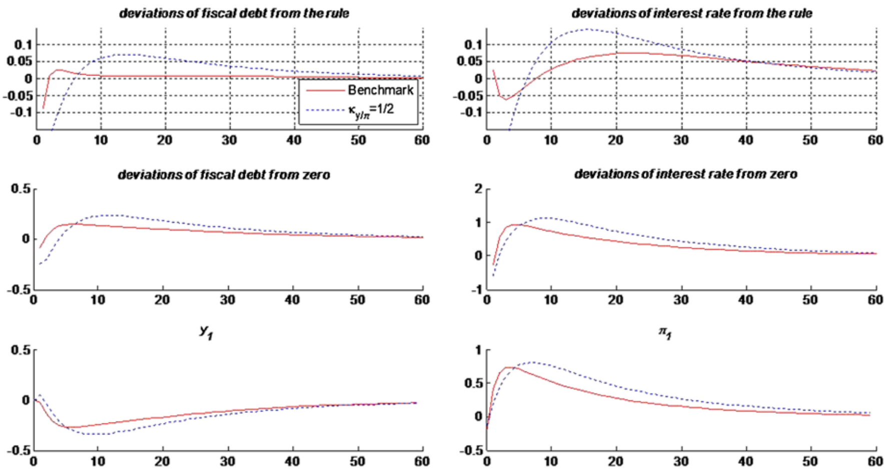

Fig. 3 Benchmark vs. model with lower forward-lookingness, v_{0S}^{π}, regime N

Table 8 Sensitivity analysis, benchmark model but with lower forward-lookingness

	$J_U^*(t_0)/J_U(t_0)$			
	v_{0S}^{π}	v_{0A}^{π}	v_{0S}^{y}	v_{0A}^{y}
Benchmark	$\mathbf{C}\check{F}\mathbf{P}N$	$\mathbf{C6}\check{F}\mathbf{45}N$	$\mathbf{CPF}N$	$\mathbf{C}6\mathbf{45F}N$
$\gamma_i \approx 0.45$	$\mathbf{CF456}N$	$\mathbf{CF645}N$	$\mathbf{C456F}N$	$\check{C}\mathbf{645F}N$
$\gamma_i \approx 0.66$	$\mathbf{CF456}N$	$\mathbf{CF645}N$	$\mathbf{C}NF456$	$\check{C}NF456$
$\eta_i \approx 0.65$	$\mathbf{CF456}N$	$\mathbf{CF456}N$	$\mathbf{C}N\hat{F}456$	$\check{C}\hat{F}N456$
$\eta_i \approx 1$	$\mathbf{CF456}N$	$\mathbf{CF645}N$	$\mathbf{C456F}N$	$\check{C}\mathbf{45F}N\mathbf{6}$
$\rho_{ij} \approx 0.35$	$\mathbf{CF456}N$	$\mathbf{CF645}N$	$\mathbf{C456F}N$	$\mathbf{C645F}N$
$\rho_{ij} \approx 0.55$	$\mathbf{CF456}N$	$\mathbf{CF645}N$	$\mathbf{C}FN456$	$\check{C}\mathbf{645}\hat{F}N$
$\delta_{ij} \approx 0.075$	$\mathbf{CF456}N$	$\mathbf{CF645}N$	$\mathbf{C}456FN$	$\check{C}45\hat{F}N\mathbf{6}$
$\delta_{ij} \approx 0.5$	$\mathbf{CF456}N$	$\mathbf{CF645}N$	$\mathbf{C645F}N$	$\check{C}\mathbf{456F}N$
$\xi_i \approx 0.03$	$\mathbf{CF456}N$	$\mathbf{CF456}N$	$\mathbf{C456}\check{F}N$	$\check{C}\mathbf{6}\check{F}N\mathbf{45}$
$\xi_i \approx 0.125$	$\mathbf{CF456}N$	$\mathbf{CF645}N$	$\mathbf{CF456}N$	$\mathbf{C645F}N$
$\varsigma_{ij} \approx 0.02$	$\mathbf{CF456}N$	$\mathbf{CF645}N$	$\mathbf{C456F}N$	$\mathbf{C645F}N$
$\varsigma_{ij} \approx 0.1$	$\mathbf{CF456}N$	$\mathbf{CF645}N$	$\mathbf{C645F}N$	$\check{C}\mathbf{456F}N$

In most of the cases, $J_U^*(t_0, F)$ is preferred over $J_U^*(t_0, N)$ which confirms that forward-lookingness is decisive for the social profitability of full fiscal cooperation in our model. Similarly to Table 6 the social ordering is especially robust for both symmetric and asymmetric inflation shocks. In all these cases full fiscal coordination is the second best regime just after the grand coalition. Regimes P with smaller fiscal coalitions score worse, where as the worst result is obtained for non-cooperation. Consequently, in contrast to previous findings and Beetsma et al. (2001), the above results strongly advocate the need for coordination in the case of inflation shocks.

Furthermore, any form of coordination is better than non-cooperation, not only from the social point of view, but also from the perspective of individual authorities.

The results are a little less clear for output gap shocks as, for a few parameter combinations, non-cooperation comes back to the position it took in Table 5, i.e. just after C but before N. It happens when the CB gets more powerful w.r.t. fiscal players in its influence on the output gap (either higher values of γ_i or lower value of η_i).

A.2 Sensitivity Analysis w.r.t. Preference Parameters

Thus far, our sensitivity analysis was performed only w.r.t. structural parameters of the economies (excluding policy rules). It is interesting to take a closer look into preference parameters in players' loss functions. In the benchmark, fiscal players cared five times more about development of output gaps than inflation whereas the CB cared five times more about inflation than output gap. We argued that such a preferences were in line with other parameters in the model as they guarantee that no variables are overrepresented in the total loss of players (as decompositions in previous sections confirm). In this section we vary relative preferences of fiscal and monetary authorities regarding output gap and inflation. In particular, let $r_F^{\pi/y}$ and $r_{CB}^{\pi/y}$ denote the ratio between α_i and β_i and between $\beta_{CB,i}$ and $\alpha_{CB,i}$, respectively, i.e. $r_F^{\pi/y} := \frac{\alpha_i}{\beta_i}$ and $r_{CB}^{\pi/y} := \frac{\beta_{CB,i}}{\alpha_{CB,i}}$. The sensitivity analysis will be performed for $r_F^{\pi/y}$ and $r_{CB}^{\pi/y}$ simultaneously changing from 0 to 1 with step 0.1. For $r_F^{\pi/y} = r_{CB}^{\pi/y} = 0$ we have a situation where governments are concerned only about output gaps (i.e. are very liberal in stabilisation sense) while the CB is concerned only about inflation (i.e. is very conservative). In other words, this is the situation when preferences are totally opposite. In contrast, when $r_F^{\pi/y} = r_{CB}^{\pi/y} = 1$ fiscal authorities as well as the monetary ones are equally interested in deviations of both variables, which, taking into account that the weight of the control instrument does not change and is equal between all the players, means that under symmetric shocks their objectives are the same in this extreme case. The results of the sensitivity check of the benchmark model with preferences amended in the above way are presented in Table 9. The ordering in this table, similarly to previous tables, is based on $J_U^*(t_0)$.

It is evident from Table 9 that the grand coalition is the socially optimal regime for lower values of $r_{F/CB}^{\pi/y}$, when preferences of various authorities are opposite. The second best choice are partial fiscal cooperation regimes, where as fiscal cooperation scores worst, even worse than non-cooperative regime N. This pattern is observed in the neighbourhood of the benchmark for all shocks except for v_{0S}^y. In contrast, when $r_{F/CB}^{\pi/y}$ becomes larger, first partial fiscal cooperation becomes more socially profitable than C, then F more profitable than N and, finally, when preferences of governments and the CB coincide, F becomes the most profitable outcome of all. This last result is interesting as previously, in the majority of situations, C was the

Table 9 Sensitivity analysis, benchmark model with altered preference

$r^{\pi/y}_{F/CB}$	$J^*_U(t_0)$							
	v^{π}_{0S}	$\overline{J^*_U}$	v^{π}_{0A}	$\overline{J^*_U}$	v^{y}_{0S}	$10^2\overline{J^*_U}$	v^{y}_{0A}	$10^2\overline{J^*_U}$
0	$\mathbf{C}PNF$	17.16	$\check{C}645NF$	1.93	$CNPF$	4.23	$CN456F$	3.17
0.1	$\mathbf{C}PNF$	13.14	$\check{C}645NF$	1.54	$\hat{C}NPF$	7.78	$CN456F$	3.38
0.2[a]	$\mathbf{C}PNF$	10.58	$\mathbf{C}456NF$	1.23	$P\hat{C}NF$	7.37	$CN456F$	3.32
0.3	$\mathbf{C}PNF$	8.97	$\mathbf{C}456NF$	1.06	$\mathbf{P}\hat{C}NF$	7.13	$C45N6F$	3.31
0.4	$\mathbf{C}PNF$	8.05	$\mathbf{C}45\mathbf{6}NF$	0.96	$\mathbf{P}\hat{C}NF$	6.98	$CN456F$	3.28
0.5	$P\mathbf{C}NF$	7.58	$45\mathbf{C6}NF$	0.91	$\mathbf{P}\hat{C}NF$	6.90	$\hat{C}45NF\mathbf{6}$	3.27
0.6	$P\mathbf{C}NF$	7.11	$45\mathbf{C6}NF$	0.84	$\mathbf{P}\hat{C}N\hat{F}$	6.85	$\hat{C}\hat{F}45N\mathbf{6}$	3.28
0.7	$P\mathbf{C}NF$	6.80	$45\mathbf{6C}NF$	0.81	$\mathbf{P}\hat{F}N\hat{C}$	6.84	$\hat{C}\hat{F}N45\mathbf{6}$	3.24
0.8	$\mathbf{PCF}N$	6.66	$45\mathbf{6CF}N$	0.79	$\mathbf{FP}N\hat{C}$	6.82	$\mathbf{F}\hat{C}\mathbf{45}N6$	3.28
0.9	$\mathbf{FPC}N$	6.58	$\mathbf{F6}45\mathbf{C}N$	0.79	$\mathbf{FP}N\hat{C}$	6.82	$\mathbf{F}45\hat{C}N\mathbf{6}$	3.29
1.0	$\mathbf{FPC}N$	6.61	$\mathbf{F6}45\mathbf{C}N$	0.79	$\mathbf{FP}N\hat{C}$	6.85	$\mathbf{F}45CN\mathbf{6}$	3.26

[a]Benchmark

most socially desirable outcome. However, when $r^{\pi/y}_{F/CB} \approx r^{\pi/y}_{F/CB}$ this regime is not so efficient any more because under equal bargaining power assumption the loss of the CB is under-represented in the joint loss of the grand coalition. This leads to the situation in which the interest rate is less important in the joint loss than fiscal debts of individual countries and, therefore, is used more extensively than under F, where free-riding between fiscal coalition and the CB prevents both groups from an overuse of their control instruments. It is easily visible in Table 10 which shows the decomposed players losses for symmetric price shock and symmetric preferences, i.e. $r^{\pi/y}_{F} = r^{\pi/y}_{CB} = 1$. Loss from $\chi_{F,C3}\widehat{\hat{f}^2_{C3}}$ under F is bigger than under C and, at the same time, $\chi_{CB}\widehat{\hat{i}^2_U}$ is lower under F than under C as fiscal players in F cannot rely so much as in C on interest rate to stabilise economies due to free-riding of the CB. Since, in social loss based on J^*_U interest rate has relatively bigger share than in the joint loss of the grand coalition, full fiscal coordination scores better than the grand coalition where interest rate is relatively overused. Another important observation from Table 9 is that partial fiscal cooperation, as in most of the cases analysed before, is very often the second best choice.

Next to every column with social preference ordering we show average social loss obtained for different levels of $r^{\pi/y}_{F/CB}$. Obviously, the less conflicting preferences are the lower average common loss suffered by the union is. Thus, the percentage difference between the average losses for $r^{\pi/y}_{F/CB} = 0$ and $r^{\pi/y}_{F/CB} = 1$ is the highest for symmetric and asymmetric inflation shocks (61.5 % and 59 %, respectively), and much more moderate for both output shock (13.5 % and around 0), which confirms are previous results that the biggest gains from choosing an appropriate regime is to be expected in the former case.

Table 10 Optimal losses for v^{π}_{0S}, $\alpha_{F,i} = \alpha_{CB} = \frac{1}{2}\alpha_{CB} = \frac{1}{2}\alpha_{F,i}$

(sc_1, v^{π}_{0S})	NC	C	F	[12|3|4]	[13|2|4]	[1|23|4]
$C1$	6.3331	5.0871	5.8439	6.3071	6.3071	5.4947
$C2$	6.3331	5.0871	5.8439	6.3071	5.4947	6.3071
$C3$	6.3331	5.0871	5.8439	5.4947	6.3071	6.3071
CB	6.9370	6.5147	5.5070	6.2236	6.2236	6.2236
$J_U(t_0)$	25.9363	21.7763	23.0388	24.3328	24.3328	24.3328
$J^*_U(t_0)$	7.0018	6.9906	6.1134	6.5194	6.5194	6.5194
$\beta_{F,C1}\hat{y}^2_{C1}$	1.0667	1.1640	1.0812	1.3336	1.3336	0.6526
$\alpha_{F,C1}\hat{\pi}^2_{C1}$	5.2015	3.4472	4.1563	4.6365	4.6365	4.7429
$\chi_{F,C1}\widehat{\hat{f}}^2_{C1}$	0.0648	0.4758	0.6063	0.3369	0.3369	0.0992
$\beta_{F,C2}\hat{y}^2_{C2}$	1.0667	1.1640	1.0812	1.3336	0.6526	1.3336
$\alpha_{F,C2}\hat{\pi}^2_{C2}$	5.2015	3.4472	4.1563	4.6365	4.7429	4.6365
$\chi_{F,C2}\widehat{\hat{f}}^2_{C2}$	0.0648	0.4758	0.6063	0.3369	0.0992	0.3369
$\beta_{F,C3}\hat{y}^2_{C3}$	1.0667	1.1640	1.0812	0.6526	1.3336	1.3336
$\alpha_{F,C3}\hat{\pi}^2_{C3}$	5.2015	3.4472	4.1563	4.7429	4.6365	4.6365
$\chi_{F,C3}\widehat{\hat{f}}^2_{C3}$	0.0648	0.4758	0.6063	0.0992	0.3369	0.3369
$\beta_{CB}\hat{y}^2_{CB}$	1.0667	1.1640	1.0812	1.0688	1.0688	1.0688
$\alpha_{CB}\hat{\pi}^2_{CB}$	5.2015	3.4472	4.1563	4.6717	4.6717	4.6717
$\chi_{CB}\widehat{\hat{i}}^2_U$	0.6686	1.9034	0.2695	0.4831	0.4831	0.4831

A number of interesting conclusions can be drawn the above analysis. First of all, the relative antagonism between the CB and governments in the monetary union is an important factor which strongly determines the profitability of full fiscal coordination. In contrast to other various findings from the literature, in our model, strongly independent bank of a monetary union is not so profitable from the common perspective and more intermediate arrangements are advisable. Secondly, if bargaining power in the grand coalition do not coincide with socially optimal preferences, this regime might be counter-productive w.r.t. full-fiscal coalition, which in turn, can turn out to be optimal due to free-riding.

The analyses in Table 9 is made under the assumption that both types of authorities simultaneously change their preferences from the most conflicting to the same ones. This was rather theoretical simulation as having little chances to be realised in (the European) practice as the ECB independence is strongly safeguarded by relevant treaties. It is also interesting to study the more realistic situation in which the strong CB's focus on inflation remains unchanged while governments, at the beginning fixed only at inflation, become gradually interested in inflation.

Table 11 Sensitivity analysis, benchmark model with altered preference

$r_{CB}^{\pi/y}=0$	$J_U^*(t_0)$							
	v_{0S}^{π}	$\overline{J_U^*}$	v_{0A}^{π}	$\overline{J_U^*}$	v_{0S}^{y}	$10^2\overline{J_U^*}$	v_{0A}^{y}	$10^2\overline{J_U^*}$
$r_F^{\pi/y}=0$	**C***PNF*	17.16	*Č*645*NF*	1.93	*CNPF*	4.23	*CN*456*F*	3.17
$r_F^{\pi/y}=0.1$	**C***PNF*	13.29	*Č*645*NF*	1.53	*ČPNF*	7.83	*C*45*N*6*F*	3.31
$r_F^{\pi/y}=0.2$	**C***PNF*	10.72	**C**645*NF*	1.25	*ČPNF*	7.77	*C*45*N*6*F*	3.33
$r_F^{\pi/y}=0.3$	**C***PNF*	9.35	**C**645*NF*	1.10	**C***PNF*	7.65	*Č*45*N*6*F*	3.32
$r_F^{\pi/y}=0.4$	**C***PNF*	8.63	**C**645*NF*	1.02	**C***PNF*	7.50	**C6**45*NF*	3.31
$r_F^{\pi/y}=0.5$	**C***PNF*	8.17	**C**645*NF*	1.00	**CP***NF*	7.21	**C**45**6***NF*	3.30
$r_F^{\pi/y}=0.6$	*P***C***NF*	7.93	**645C***NF*	0.99	**CP***NF*	7.24	**C6**45*NF*	3.29
$r_F^{\pi/y}=0.7$	*P***C***NF*	7.66	**645C***NF*	0.96	**PC***NF*	7.13	**C**45**6***FN*	3.27
$r_F^{\pi/y}=0.8$	*P***C***NF*	7.62	**645C***NF*	0.94	**PC***NF*	7.14	**C**45**6***FN*	3.27
$r_F^{\pi/y}=0.9$	*P***C***NF*	7.41	**645C***NF*	0.88	**PCF***N*	7.14	**645CF***N*	3.27
$r_F^{\pi/y}=1$	*P***C***NF*	7.29	**645C***NF*	0.86	**PČF***N*	7.15	**645CF***N*	3.27

More formally, we consider the case where $r_{CB}^{\pi/y}$ is kept constant at 0 where as $r_F^{\pi/y}$ changes from 0 to 1. One possible interpretation of such simulations in Table 11, which one can think of, are more and more stringent provisions of the SGP, which additionally to fiscal debt issues regulates also inflation in the EMU Member States.

In general, the outcomes in Table 11 are reasonably similar to those from Table 9 as far as main trends are considered, i.e. the less intensive conflict between authorities makes partial fiscal cooperation regimes as well as full fiscal cooperation more interesting from the social point of view. Of course, always restrictive CB makes it impossible to reach the same outcome as in the previous case. For $r_{CB}^{\pi/y}=0$ and $r_F^{\pi/y}=1$ (i.e. the last row of Table 11) the social orderings are similar $r_{F/CB}^{\pi/y}=0.5$ previously (the middle of the Table 9). Accordingly, minimal average loss obtained for the last case is now higher than when authorities' preferences were more alike. However, what is important, social losses at the end of both tables are not much different which shows that similar low social welfare can be obtained either by making preferences of fiscal and monetary authorities more parallel, or by safeguarding the CB independence and making government to be more equally oriented about inflations and output gaps. The first proposition seems to be rather unacceptable by the modern economic school, but the second one seems not only to be acceptable from this point of view, but actually implemented in the current European practice (in the form of the strongly independent ECB and the SGP, which makes governments more "inflation-aware").

The issue related to the SGP will be discussed further in this paper but first we will consider (nearly) optimal policy rules.

Table 12 Optimal losses for $\theta_\pi^M = 1.25$ and $\theta_y^F = 0, 0.1, 0.3, 0.5, 0.8, 1.0$

$\theta_\pi^U = 1.25$	−0	−0.1	−0.3	−0.5	−0.8	−1.0
$C1, C2, C3$	2.9554	5.8427	9.0471	3.3611	2.8931	2.8148
CB	6.3700	4.2528	19.2972	12.7490	12.5669	13.0460
$J_U(t_0)$	15.2364	21.7811	46.4388	22.8324	21.2463	21.4904
$J_U^*(t_0)$	**8.3235**	**9.1524**	**25.6551**	**14.1157**	**13.3184**	**13.5231**
$\beta_{F,C1}\hat{y}_{C1}^2$	1.1145	1.6788	0.8843	0.4886	0.3842	0.3234
$\alpha_{F,C1}\hat{\pi}_{C1}^2$	0.7790	0.6073	2.5124	1.8967	2.0648	2.2729
$\chi_{F,C1}\widehat{f}_{C1}^2$	1.0618	3.5565	5.6504	0.9757	0.4440	0.2184
$\beta_{CB}\hat{y}_{CB}^2$	0.2229	0.3357	0.1768	0.0977	0.0768	0.0646
$\alpha_{CB}\hat{\pi}_{CB}^2$	3.8953	3.0368	12.5621	9.4835	10.3240	11.3648
$\chi_{CB}\widehat{i}_U^2$	2.2517	0.8801	6.5582	3.1677	2.1659	1.6165

A.3 Nearly Optimal Policy Rules

In the LQDG framework it is not possible to analytically optimise certain parameters of the model, however, an approximate analyses can be performed numerically. We will use this method to study how various combinations of policy rules' parameters influence output gap and inflation volatility and which of them are likely to bring (nearly) optimal outcome from the social point of view. Due to the space constraints we will focus mainly on the symmetric inflation shock. Tables 12, 13, 14 show (optimal) losses together with their decomposition in the non-cooperative regime for different values of θ_y^i and θ_π^U. More specifically, Table 12 shows cases in which $\theta_\pi^U = 1.25$ and θ_y^i changes from 0 to -1; Table 13 cases in which $\theta_\pi^U = 1.5$ and θ_y^i as before; and, finally, Table 14 shows cases in which $\theta_\pi^U = 1.75$ and as before. Such an analysis, albeit approximate, may give us an important insight into efficiency of different policy rules' combinations.

Figure 4 compares J_U^* losses for different combinations of θ_π^U and θ_y^i. In general, from the monetary authority perspective, a rule less focused on inflation (i.e. $\theta_\pi^U = 1.25$, Table 12) results in higher losses than for the benchmark value (i.e. $\theta_\pi^U = 1.50$, Table 13), whereas a rule more focused on inflation (i.e. $\theta_\pi^U = 1.75$, Table 14) generates lower losses. The only exception from this pattern is a combination of coefficients $\theta_\pi^U = 1.50$ and $\theta_y^i = 0$ which produces the lowest social loss, i.e. is an optimal Taylor rules parameters' combination (ceteris paribus) for the non-cooperative regime. From the fiscal authority perspective the stronger reaction to output, the higher loss and vice versa. Finally, it should be mentioned that for combinations ($\theta_\pi^U = 1.25, \theta_y^i = -0.3$) and ($\theta_\pi^U = 1.25, \theta_y^i = -0.5$) strong irregularities

Table 13 Optimal losses for $\theta_\pi^M = 1.50$ and $\theta_y^F = 0, 0.1, 0.3, 0.5, 0.8, 1.0$

$\theta_\pi^U = 1.50$	−0	−0.1	−0.3	−0.5	−0.8	−1.0
$C1, C2, C3$	2.4891	2.3665	2.2435	2.1948	2.4202	2.9156
CB	4.2403	4.5948	5.4647	6.2456	8.0400	9.8299
$J_U(t_0)$	11.7078	11.6945	12.1955	12.8300	15.3007	18.5770
$J_U^*(t_0)$	**5.7327**	**5.9191**	**6.5302**	**7.1445**	**8.9168**	**11.0600**
$\beta_{F,C1}\hat{y}_{C1}^2$	1.7336	1.5703	1.2982	1.0880	0.8648	0.7523
$\alpha_{F,C1}\hat{\pi}_{C1}^2$	0.6500	0.7282	0.9183	1.0782	1.3704	1.5350
$\chi_{F,C1}\widehat{\hat{f}_{C1}^2}$	0.1054	0.0679	0.0269	0.0285	0.1848	0.6282
$\beta_{CB}\hat{y}_{CB}^2$	0.3467	0.3140	0.2596	0.2176	0.1729	0.1504
$\alpha_{CB}\hat{\pi}_{CB}^2$	3.2501	3.6410	4.5919	5.3912	6.8523	7.6753
$\chi_{CB}\widehat{\hat{i}_U^2}$	0.6434	0.6397	0.6131	0.6367	1.0147	2.0041

Table 14 Optimal losses for $\theta_\pi^M = 1.75$ and $\theta_y^F = 0, 0.1, 0.3, 0.5, 0.8, 1.0$

$\theta_\pi^U = 1.75$	−0	−0.1	−0.3	−0.5	−0.8	−1.0
$C1, C2, C3$	3.4359	3.1001	2.8075	2.6060	2.4825	2.4264
CB	4.1017	3.9972	4.4681	5.0434	6.0023	6.5807
$J_U(t_0)$	14.4096	13.2978	12.8907	12.8614	13.4500	13.8602
$J_U^*(t_0)$	**6.5822**	**6.1072**	**6.1979**	**6.4953**	**7.1816**	**7.5970**
$\beta_{F,C1}\hat{y}_{C1}^2$	2.3617	2.1732	1.8597	1.6176	1.3284	1.1784
$\alpha_{F,C1}\hat{\pi}_{C1}^2$	0.4830	0.5555	0.7057	0.8305	1.0375	1.1745
$\chi_{F,C1}\widehat{\hat{f}_{C1}^2}$	0.5911	0.3714	0.2419	0.1578	0.1165	0.0735
$\beta_{CB}\hat{y}_{CB}^2$	0.4723	0.4346	0.3719	0.3235	0.2656	0.2356
$\alpha_{CB}\hat{\pi}_{CB}^2$	2.4154	2.7777	3.5289	4.1525	5.1879	5.8725
$\chi_{CB}\widehat{\hat{i}_U^2}$	1.2139	0.7848	0.5672	0.5673	0.5486	0.4725

emerge, explanation of which should be sought in mathematical properties of the model.[30]

Let us now focus on the individual players' perspective. Due to the irregularities mentioned above we will exclude from our analysis Table 12. For $\theta_\pi^U = 1.50$, the loss of the fiscal players is not monotonic and reaches its minimum for the

[30] Such an analysis goes far beyond the scope of this paper. It has been numerically checked that in the neighbourhood of $\theta_y^F = 0.3$ for $\theta_\pi^M = 1.25$ social loss goes to the (nearly) infinite limiting value.

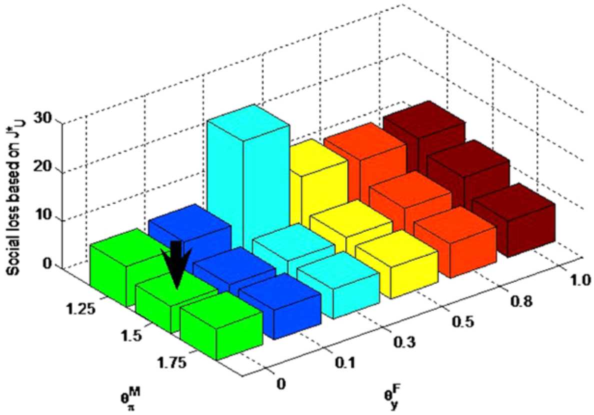

Fig. 4 Social loss for various combinations of policy rules parameters, regime *N*

benchmark parametrisation (i.e. $\theta_y^i = 0.50$). However, for $\theta_\pi^U = 1.75$ the results are more clear-cut as the stronger reaction of governments to output always leads to the lower loss. This is totally at odds with the CB losses which behave in the exactly opposite way. The reasons of this difference can be found in the decomposition of losses. Stronger reaction of fiscal debt to output gap leads to its lower volatility (in Table 14 $\beta_{F,C1}\hat{y}_{C1}^2$, and, consequently, $\beta_{CB}\hat{y}_{CB}^2$ decrease with θ_y^i), however, this positive effect is reached at the very expense of inflation volatility which grows accordingly. This is detrimental for the CB as this authority is mainly concerned about inflation under benchmark parametrisation. Overall, the conflict of interest between both types of authorities is clearly visible here. Highest value of θ_π^U without a counter-response in fiscal rule is damaging to loss of governments as the CB's strong reaction to inflation makes output gap very volatile. Thus, governments use the most of their control effort to improve the situation, however, only higher (absolute) values of θ_y^i make them increasingly better off. This pattern is robust even for θ_y^i reaching minus one. In contrast, as mentioned above, for the more moderate CB's policy rule (i.e. $\theta_\pi^U = 1.5$) and for θ_y^i higher than half, fiscal loss start to increase. This means that, if fiscal rule responses to tempered monetary rule too strongly, there is an effect of overshooting the fiscal policy rule. As a result, governments are pushed to deviate from such a rule much stronger than before because the (overshot) rule must be discretionary corrected. Consequently, $\chi_{F,C1}\widehat{\hat{f}}_{C1}^{2}$ grows from 0.0269 in benchmark to 0.6282 for $\theta_y^i = -1$. Comparing the CB's losses between Tables 12, 13, 14 it is evident that more reactionary stance is in the interest of the CB as its loss decreases with increasing θ_π^U due to the lower inflation deviation in the union.

To sum up, from the individual point of view, we have a situation where CB has incentives to increase θ_π^U and, at the same time, for high values of θ_π^U, governments have incentives to increase θ_y^i. When both types of authorities do it at the same time, the economies end up in a position which is not only suboptimal from the social point of view (right-down corner in Fig. 4), but also from individual ones. For $\theta_\pi^U = 1.75$ and $\theta_y^i = -1.0$ governments obtain loss of 2.4264 and the CB of 6.5807, which is the outcome *Pareto*-dominated by other combinations, e.g. the benchmark. Unfortunately, the socially optimal combination ($\theta_\pi^U = 1.5, \theta_y^i = 0$) does not *Pareto*-dominate combination ($\theta_\pi^U = 1.75, \theta_y^i = -1.0$) so the mutual agreement to move toward the optimum seems unlikely to be obtained.

A.4 SGP Analysis

Within our model we can also investigate the effects of the major policy-surveillance institution of the EMU, namely the SGP. The SGP imposes a framework of fiscal stringency and coordination measures that aim at securing the implementation of the BEPGs. In our model the effects of various levels of the SGP stringency can be studied by considering (i) different levels of the countercyclical parameter θ_y^i in the fiscal rule; and (ii) different weights associated with the domestic fiscal deficit, χ_i, in the objective functions of the fiscal players. We compare the following three cases, each characterised by stricter SGP provisions than the other:[31]

I. In the fiscal rule the coefficient measuring a countercyclical reaction of fiscal debt to deviation of output gap is two times smaller than in the benchmark, i.e.: $\theta_y^{i,new} = -0.25$;
II. As above, but, additionally, deviations from the rule are more costly (i.e. are more severely punished by the SGP provisions), $\chi_i^{new,II} = 1.5\chi_i$;
III. As in point I, but, additionally, $\chi_i^{new,III} = 3\chi_i$.

It is expected that smaller countercyclical reaction of the fiscal rule is going to force fiscal authorities to deviate stronger from the rule than in the benchmark. On the other hand, more costly deviations from the rule in cases II and III are likely to diminish the use of fiscal instrument w.r.t. case I. As far as individual losses of players are concerned, it is possible to directly compare new cases to the benchmark, however, it is not exactly obvious whether we can do so with the social loss. Whereas governments, as public authorities, might be bound by tougher SGP provisions, it does not have to lead to an automatic increase of the social loss. In the benchmark we assumed that $\chi_U = \chi_i$. Now, we are going to compute "adjusted" social loss of the entire union $J_U^*(t_0)$, denoted $J_U^{*A}(t_0)$, by assuming that cost of the deviation of the fiscal instrument from the rule is unchanged, i.e. equals to $\chi_U = \chi_i$ as in the benchmark, instead of $\chi_U = \chi_i^{new,II}$ or $\chi_U = \chi_i^{new,III}$. By doing so we will see

[31]Naturally, there is problem with interpretation of the SGP as well as other issues related to the control variables caused by the (linear-) quadratic form of the loss functions. In reality, a negative deviation of fiscal debt from the rule, i.e. more restrictive budgetary policy, is not likely to be considered so "bad" or "undesirable" as the same positive deviation which, eventually, is going to increase public debt. It could be possible to partially take into account such issues also in the quadratic loss function but in the much complex model, which is far our of the scope of this paper.

Table 15 Regimes N, C and F for different levels of the SGP stringency

(sc_1, v_{0S}^{π})	N	N^I	N^{II}	N^{III}	C	C^I	C^{II}	C^{III}
$C1, C2, C3$	2.19	4.22	6.18	4.73	2.12	2.23	2.22	2.31
CB	6.24	8.65	16.82	12.88	5.33	3.66	3.64	3.65
$J_U(t_0)$	12.80	21.30	35.36	27.07	11.70	10.35	10.31	10.57
$J_U^*(t_0)$	7.14	11.60	21.67	16.05	6.37	5.02	4.99	5.05
$J_U^{A*}(t_0)$	**7.14**	**11.60**	**20.48**	**14.84**	**6.37**	**5.02**	**4.93**	**4.89**
$\beta_{F,C1}\hat{y}_i^2$	1.09	1.57	1.60	1.71	1.06	1.47	1.47	1.47
$\alpha_{F,C1}\hat{\pi}_i^2$	1.08	0.94	1.01	1.22	0.88	0.58	0.57	0.61
$\chi_{F,C1}\widehat{\hat{f}}_i^2$	0.03	1.71	2.38	0.60	0.19	0.18	0.12	0.08
$\chi_{F,C1}^{new,I/II}\widehat{\hat{f}}_i^2$	–	–	3.57	1.80	–	–	0.18	0.23
$\beta_{CB}\hat{y}_{CB}^2$	0.22	0.31	0.32	0.34	0.22	0.29	0.29	0.29
$\alpha_{CB}\hat{\pi}_{CB}^2$	5.40	4.72	5.06	6.10	4.37	2.89	2.87	3.05
$\chi_{CB}\widehat{\hat{i}}_U^2$	0.64	3.62	11.44	6.43	0.76	0.47	0.47	0.30

what is the contribution of a change in use of a fiscal instrument in the total change of the loss from stabilisation effort.[32]

Players' losses in the first three regimes in the case of a symmetric inflation shock and benchmark parametrisation are shown in Tables 15 and 16. In spite of vast differences between cases a few general conclusions can be drawn. First of all, lower counter-cyclical reaction of fiscal debt (case I) always makes the losses of fiscal players higher than in the benchmark, which means that the assumed value $\theta_y^i = -0.5$ was chosen relatively well for the initial simulations and which confirms our findings from the previous section.[33] Secondly, in all the new cases higher SGP stringency leads to increasing losses of fiscal players from output gap volatility. This is natural as fiscal authorities refrain from using the more expensive control instruments. Interestingly, in different regimes we obtain different relationships between SGP stringency and the amount of the control instrument used. Whereas in all regimes with any form of cooperation (i.e. C, F, and P-regimes) the higher χ_i, the less control instrument is used (compare cases II to I and III to II), then under non-cooperation this relationship is highly non-linear. For $\chi_i^{new,II}$ governments decide

[32]A change in total loss from stabilisation effort caused by a change in the value of the relevant preference parameters can be decomposed into two effects. First is the change in the use of the stabilisation instrument as it becomes more/less expensive w.r.t. to other elements of the loss. Second change is directly caused by the increased/decreased cost.

[33]If the absolute value of θ_y^i is too high, the counter-cyclical output gap stabilisation effort can be overshot, i.e. output gap can be (ceteris paribus) more volatile than for lower values of θ_y^i, and would probably require additional pro-cyclical (and costly) control effort from fiscal authorities (see previous section for more details).

Table 16 Regimes N, C and F for different levels of the SGP stringency

(sc_1, v_{0S}^{π})	F	F^I	F^{II}	F^{III}
$C1$	9.30	13.48	9.05	5.61
$C2$	9.30	13.48	9.05	5.61
$C3$	9.30	13.48	9.05	5.61
CB	22.00	29.30	14.20	6.66
$J_U(t_0)$	50.00	69.73	41.30	23.50
$J_U^*(t_0)$	**30.00**	**41.84**	**22.30**	**11.30**
$J_U^{A*}(t_0)$	**30.00**	**41.48**	**20.70**	**20.01**
$\beta_{F,C1}\hat{y}_{C1}^2$	1.06	1.48	1.49	1.50
$\alpha_{F,C1}\hat{\pi}_{C1}^2$	1.00	0.63	0.65	0.67
$\chi_{F,C1}\widehat{\hat{f}}_{C1}^2$	7.00	11.37	4.61	1.15
	–	–	6.91	3.44
$\beta_{CB}\hat{y}_{CB}^2$	0.20	0.30	0.30	0.30
$\alpha_{CB}\hat{\pi}_{CB}^2$	5.00	3.17	3.27	3.36
$\chi_{CB}\widehat{\hat{i}}_U^2$	16.00	25.83	10.60	3.00

to use $\frac{\chi_{C1}\widehat{\hat{f}}_i^2}{\chi_{C1}} = \widehat{\hat{f}}_i^2 = 1190$ which is nearly 40 % more (not less) than for χ_i in case I, however when χ_i increases to $\chi_i^{new,III}$ they contract substantially control action to $\frac{\chi_{C1}\widehat{\hat{f}}_i^2}{\chi_{C1}} = \widehat{\hat{f}}_i^2 = 300$. The SGP regulating the use of control instrument influences also the use of interest rate by the CB of the union. In many cases when control action of fiscal authorities is diminished the response of the CB also fades out, i.e. conflict between both types of authorities is hampered. In relative terms, the biggest reductions in the control effort of the monetary authority is obtained under F^{III} where cost of the control effort is lowered from 25.83 to 3.00. On the other hand, under N^{III} we also witness quite a reduction in the fiscal control effort w.r.t. N^I, but the main driving force in this case is a free-riding of fiscal players, which forces the CB to increase its engagement in the union economy not stabilised enough by national governments. As the loss from the CB's control instrument is an important part of $J_U^*(t_0)$ and $J_U^{A*}(t_0)$, this leads to higher social loss under N^{III} than under F^{III}.

To summarise, we established the third factor (next to the degree of backward-lookingness and loss functions' preferences) which heavily determined the results obtained for the benchmark parametrisation of the model. The increased SGP stringency reduces incentives of fiscal players to use control instruments, therefore, in situations where high social losses where driven by the conflict between authorities (notably regime F), such a firmer stance is beneficial to the union-wide economic interest. However, in situations in which free-riding is present (notably regime N under benchmark) increased SGP stringency may lead to more extensive free-riding of governments as undertaking any actions become more costly. This, in turn, makes the CB to intervene and increases social loss of the union. In other words, the stringent SGP has both positive and negative effects in the context of this paper and is able to make unprofitable regime to become profitable.

Similar analysis has been performed for 3 other shocks. Since the conflict under v^{π}_{0A} is less eminent also the social gains from higher SGP stringency are lower. As before, both output shocks are characterised by the lower variability of losses between different regimes, however, still SGP stringency is able to make non-cooperation inferior to fiscal cooperation, at least, in the case of the symmetric shock.

A.5 Model Derivations

A.5.1 Reduced Form of the Model

Defining

$$K_y := \begin{bmatrix} \kappa_{1,y} & 0 & \cdots & 0 \\ 0 & \kappa_{2,y} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \kappa_{n,y} \end{bmatrix}, \qquad K_\pi := \begin{bmatrix} \kappa_{1,\pi} & 0 & \cdots & 0 \\ 0 & \kappa_{2,\pi} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \kappa_{n,\pi} \end{bmatrix},$$

$$G := \begin{bmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \gamma_n \end{bmatrix}, \qquad E := \begin{bmatrix} \eta_1 & 0 & \cdots & 0 \\ 0 & \eta_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \eta_n \end{bmatrix},$$

$$\Xi := \begin{bmatrix} \xi_1 & 0 & \cdots & 0 \\ 0 & \xi_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \xi_n \end{bmatrix}, \qquad B := \begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \beta_n \end{bmatrix},$$

$$R := \begin{bmatrix} 0 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 0 & \cdots & \rho_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{n1} & \rho_{n2} & \cdots & 0 \end{bmatrix},$$

$$D := \begin{bmatrix} \delta_{12} & \delta_{13} & \cdots & \delta_{1n} \\ -\sum_{j\in F/2} \delta_{2j} & \delta_{23} & \cdots & \delta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \delta_{n2} & \delta_{n3} & \cdots & -\sum_{j\in F/n} \delta_{nj} \end{bmatrix},$$

$$\Psi_y := \begin{bmatrix} \psi_{1,y} & 0 & \cdots & 0 \\ 0 & \psi_{2,y} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \psi_{n,y} \end{bmatrix}$$

$$V := \begin{bmatrix} \varsigma_{12} & \varsigma_{13} & \cdots & \varsigma_{1n} \\ -\sum_{j \in F/2} \varsigma_{2j} & \varsigma_{23} & \cdots & \varsigma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \varsigma_{n2} & \varsigma_{n3} & \cdots & -\sum_{j \in F/n} \varsigma_{nj} \end{bmatrix},$$

$$\Psi_\pi := \begin{bmatrix} \psi_{1,y} & 0 & \cdots & 0 \\ 0 & \psi_{2,y} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \psi_{n,y} \end{bmatrix}$$

$$\Psi := \begin{bmatrix} \Psi_y & 0 \\ 0 & \Psi_\pi \end{bmatrix}, \quad \text{and} \quad \upsilon_t := \begin{bmatrix} \upsilon_t^y \\ \upsilon_t^\pi \end{bmatrix},$$

$$\iota_n := \begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix}_n, \qquad S := \begin{bmatrix} -\iota_{(n-1)} & I_{n-1} \end{bmatrix},$$

$$\Theta_\pi^F := \begin{bmatrix} \theta_\pi^1 & \theta_\pi^2 & \cdots & \theta_\pi^n \end{bmatrix}^T \quad \text{and} \quad \Theta_y^F := \begin{bmatrix} \theta_y^1 & \theta_y^2 & \cdots & \theta_y^n \end{bmatrix}^T$$

the SNKM model can be rewritten as:

$$\begin{aligned} y_t = {} & K_y E_t y_{t+1} + (I_n - K_y) y_{t-1} - G(\iota_n i_t - E\pi_{t+1}) + Ef_t \\ & - K_y R E_t y_{t+1} + R y_t - (I_n - K_y) R y_{t-1} - K_y D E_t s_{t+1} \\ & + D s_t - (I_n - K_y) D s_{t-1} + \upsilon_t^y, \end{aligned} \tag{12}$$

$$\pi_t = K_\pi B E_t \pi_{t+1} + (I_n - K_\pi) B \pi_{t-1} + \Xi y_t + \Xi V s_t + \upsilon_t^\pi, \tag{13}$$

$$s_{t+1} = s_t + S\pi_{t+1}, \tag{14}$$

$$\upsilon_{t+1} = \Psi \upsilon_t + \varepsilon_{t+1}, \tag{15}$$

where $\mathbf{I}_m$ is $m \times m$ identity matrix ($m = n-1, n$), $s_t := [s_{12,t} \cdots s_{1n,t}]$ and y_t, π_t, υ_t^y and υ_t^π are appropriately defined vectors of size n each. In particular, it can be shown that every $s_{ij,t} := p_{j,t} - p_{i,t}$ $(j \neq i)$ can be expressed in terms of $s_{12,t}, \ldots, s_{1n,t}$. For example, in a three-country monetary union we have six bilateral real exchange rates: $s_{12,t} = p_{2,t} - p_{1,t}$, $s_{13,t} = p_{3,t} - p_{1,t}$, $s_{21,t} = p_{1,t} - p_{2,t}$, $s_{23,t} = p_{3,t} - p_{2,t}$, $s_{31,t} = p_{1,t} - p_{3,t}$, and $s_{32,t} = p_{2,t} - p_{3,t}$. Clearly, the last four variables can be expressed as a combination of the first two, i.e. $s_{21,t} = -s_{12,t}$, $s_{23,t} = s_{13,t} - s_{12,t}$, $s_{31,t} = -s_{13,t}$ and $s_{32,t} = s_{12,t} - s_{13,t}$.

Defining fiscal and monetary policy rule vectors as:

$$f_t := \Theta_\pi^F \pi_t + \Theta_y^F y_t, \quad \text{and} \tag{16}$$

$$i_t := \theta_\pi^U \omega^T \pi_t + \theta_y^U \omega^T y_t, \tag{17}$$

substituting them into system (12)–(15) and rearranging we get:

$$\begin{aligned} &-K_y(I_n-R)E_t y_{t+1} - GE_t\pi_{t+1} + K_y DE_t s_{t+1} \\ &\quad = -\big(I_n - E\Theta_y^F + G\iota_n\theta_y^U\omega^T - R\big)y_t - \big(G\iota_n\theta_\pi^U\omega^T - E\Theta_\pi^F\big)\pi_t \\ &\qquad + (I_n-K_y)(I_n-R)y_{t-1} + Ds_t - (I_n-K_y)Ds_{t-1} + I_n\upsilon_t^y, \end{aligned} \tag{18}$$

$$-K_\pi BE_t\pi_{t+1} = \Xi y_t - \pi_t + (I_n - K_\pi)B\pi_{t-1} + \Xi V s_t + I_n\upsilon_t^\pi, \tag{19}$$

$$s_{t+1} - SE_t\pi_{t+1} = s_t, \tag{20}$$

$$\upsilon_{t+1} = \Psi\upsilon_t + \varepsilon_{t+1}. \tag{21}$$

Introducing three additional vectors of variables $a_{t+1} := y_t$, $b_{t+1} := \pi_t$ and $c_{t+1} := s_t$ we may rewrite system (18)–(21) as:

$$\begin{aligned} &-K_y(I_n-R)E_t y_{t+1} - GE\pi_{t+1} + K_y Ds_{t+1} \\ &\quad = -\big(I_n - E\Theta_y^F + G\iota_n\theta_y^U\omega^T - R\big)y_t - \big(G\iota_n\theta_\pi^U\omega^T - E\Theta_\pi^F\big)\pi_t \\ &\qquad + Ds_t + (I_n-K_y)(I_n-R)a_t - (I_n-K_y)Dc_t + I_n\upsilon_t^y \end{aligned} \tag{22}$$

$$-K_\pi BE\pi_{t+1} = \Xi y_t - \pi_t + \Xi V s_t + (I_n - K_\pi)Bb_t + I_n\upsilon_t^\pi, \tag{23}$$

$$-S\pi_{t+1} + s_{t+1} = s_t, \tag{24}$$

$$a_{t+1} = y_t, \tag{25}$$

$$b_{t+1} = \pi_t, \tag{26}$$

$$c_{t+1} = s_t, \tag{27}$$

$$\upsilon_{t+1} = \Psi\upsilon_t + \varepsilon_{t+1}. \tag{28}$$

Defining: $A_{11} = -K_y(I-R)$, $A_{12} = -G$, $A_{13} = K_y D$, $B_{11} = -(I - E\Theta_y^F + G\iota_n\theta_y^U\omega^T - R)$, $B_{12} = -(G\iota_n\theta_\pi^U\omega^T - E\Theta_\pi^F)$, $B_{13} = D$, $B_{14} = (I_n - K_y)(I_n - R)$, $B_{16} = -(I_n - K_y)D$, $B_{17} = [0_{n\times n}\ I_n]$, $A_{22} = -K_\pi B$, $B_{21} = \Xi$, $B_{22} = -I_n$, $B_{23} = \Xi V$, $B_{25} = (I_n - K_\pi)B$, $B_{27} = [I_n\ 0_{n\times n}]$, $A_{32} = -S$, $B_{77} = \Psi$, the system (22)–(28) in state-space form as:

$$E_t z_{t+1} = A^{-1}Bz_t + F\upsilon_t, \tag{29}$$

where $z_t := [y_t^T\ \pi_t^T\ s_t^T\ a_t^T\ b_t^T\ c_t^T\ \upsilon_t^T]^T$ or $z_t := [z_{1,t}^T\ z_{2,t}^T\ \upsilon_t^T]^T$ with $z_{1,t} := [a_t^T\ b_t^T\ c_t^T]^T$, $z_{2,t} := [y_t^T\ \pi_t^T\ s_t^T]^T$,

$$\upsilon_t := \big[0_{1\times n}\quad 0_{1\times n}\quad 0_{1\times(n-1)}\quad 0_{1\times n}\quad 0_{1\times n}\quad 0_{1\times(n-1)}\quad \varepsilon_t^T\big]^T,$$

$$A := \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0_1 & 0_1 & 0_2 & 0_3 \\ 0_1 & A_{22} & 0_2 & 0_1 & 0_1 & 0_2 & 0_3 \\ 0_4 & A_{32} & I_{(n-1)} & 0_4 & 0_4 & 0_5 & 0_6 \\ 0_1 & 0_1 & 0_1 & I_n & 0_1 & 0_2 & 0_3 \\ 0_1 & 0_1 & 0_1 & 0_1 & I_n & 0_2 & 0_3 \\ 0_4 & 0_4 & 0_5 & 0_4 & 0_4 & I_{(n-1)} & 0_6 \\ 0_7 & 0_7 & 0_8 & 0_7 & 0_7 & 0_8 & I_{2n} \end{bmatrix},$$

$$B := \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & 0_1 & B_{16} & B_{17} \\ B_{21} & B_{22} & B_{23} & 0_1 & B_{25} & 0_2 & B_{27} \\ 0_4 & 0_4 & I_{(n-1)} & 0_4 & 0_4 & 0_5 & 0_6 \\ 0_1 & 0_1 & 0_1 & I_n & 0_1 & 0_2 & 0_3 \\ 0_1 & 0_1 & 0_1 & 0_1 & I_n & 0_2 & 0_3 \\ 0_4 & 0_4 & 0_5 & 0_4 & 0_4 & I_{(n-1)} & 0_6 \\ 0_7 & 0_7 & 0_8 & 0_7 & 0_7 & 0_8 & \Psi \end{bmatrix},$$

$$F := \begin{bmatrix} 0_1 & 0_1 \\ 0_1 & 0_1 \\ 0_4 & 0_4 \\ 0_1 & 0_1 \\ 0_1 & 0_1 \\ 0_4 & 0_4 \\ I_n & 0_1 \\ 0_1 & I_n \end{bmatrix},$$

where 0_1, 0_2, 0_3, 0_4, 0_5, 0_6, 0_7, 0_8, 0_9 are zero matrices of dimensions $n \times n$, $n \times (n-1)$, $n \times 2n$, $(n-1) \times n$, $(n-1) \times (n-1)$, $n \times 2n$, $2n \times n$, $2n \times (n-1)$ and $2n \times 1$.

In order to obtain LQDG NKM, we assume that $E_t z_{1,t+1} = z_{1,t+1}$, i.e. that economic agents in the deterministic NKM make neither systematic nor random errors when predicting the future. Furthermore, substituting monetary and fiscal rules (6)–(7) in which deviation is possible into system (12)–(15) in the way presented above and performing similar transformations we obtain the system:

$$\begin{aligned} &-K_y(I_n - R)E_t y_{t+1} - GE\pi_{t+1} + K_y D s_{t+1} \\ &\quad = -\big(I_n - E\Theta_y^F + G\iota_n\theta_y^U\omega^T - R\big)y_t - \big(G\iota_n\theta_\pi^U\omega^T - E\Theta_\pi^F\big)\pi_t \\ &\qquad + Ds_t + (I_n - K_y)(I_n - R)a_t - (I_n - K_y)Dc_t + I_n v_t^y + E\hat{f}_t - G\hat{\imath}_t \end{aligned} \tag{30}$$

$$-K_\pi BE\pi_{t+1} = \Xi y_t - \pi_t + \Xi V s_t + (I_n - K_\pi)Bb_t + I_n v_t^\pi \tag{31}$$

$$-S\pi_{t+1} + s_{t+1} = s_t \tag{32}$$

$$a_{t+1} = y_t \tag{33}$$

$$b_{t+1} = \pi_t \tag{34}$$

$$c_{t+1} = s_t \tag{35}$$

$$v_{t+1} = \Psi v_t, \tag{36}$$

which, compared to the system (22)–(28), has two additional vectors of control variables $\hat{f}_t$ and $\hat{\imath}_t$. System (30)–(36) in state-space form can we written as:

$$z_{t+1} = A^{-1}Bz_t + A^{-1}Cu_t, \tag{37}$$

where $u_t := [\hat{f}_t^T \ \hat{\imath}_t]^T$ and

$$C := \begin{bmatrix} E & -G\iota_n \\ 0_1 & 0_1 \\ 0_1 & 0_1 \\ 0_1 & 0_1 \\ 0_4 & 0_4 \\ 0_7 & 0_9 \end{bmatrix}.$$

A.5.2 Initial Condition Derivation

Initial condition z_0 should position the system on the saddle-path so that the model would converge to the equilibrium. We propose two alternative ways of deriving this initial condition:

1. One way to obtain z_0 which positions the system on the saddle-path is to solve the RE version of the model and then use the initial state obtained. This initial state, by definition (if RE-model is stable), meets the required condition because it positions the system on the saddle path. In particular, at $t = 0$ vector of endogenous non-predetermined variables $z_{1,t}$ will "jump" to a saddle path whereas vector of endogenous state (predetermined) variables $z_{2,t}$ will have a value of 0. The initial value of shock vector v_t should follow the same assumptions made while solving the RE SNKM, i.e. its initial value should equal to standard deviation of ε_t. A number of freeware applications is available to solve RE model with DYNARE by Juillard (1996) being probably the most famous.[34]
2. Another method to position the system on the saddle-path is to calculate the orthogonal projection of the shock v_t onto the stable subspace at time $t = 1$. This method will be described below in more details.

Let

$$z_{t+1} = \bar{A}z_t, \quad z(0) = z_0, \tag{38}$$

be the deterministic NKM, where $\bar{A} := A^{-1}B$.

Now, let $\bar{A} = SJS^{-1}$ be a Jordan decomposition of $\bar{A}$ such that $J = \text{diag}(\Lambda_S, \Lambda_U)$ and $S = [S_S \ S_U]$, where Λ_S contains all stable eigenvalues of $\bar{A}$ and Λ_U all unstable eigenvalues of $\bar{A}$ and S_S (S_U) is the with Λ_S (Λ_U) corresponding stable

[34] For DYNARE website with the most current version of the software see: www.dynare.org.

(unstable) subspace of $\bar{A}$. Then, if z_0 belongs to S_S we have $z(0) = S_S y$ for some y ($y = (S_S^T S_S)^{-1} S_S^T z_0$).[35] In that case we may write:

$$\begin{aligned} z(t) &= e^{\bar{A}t} z(0) = S e^{Jt} S^{-1} S_{\bar{S}} y \\ &= S e^{Jt} \begin{bmatrix} I \\ 0 \end{bmatrix} y = S \begin{bmatrix} e^{\Lambda_S t} \\ 0 \end{bmatrix} y = S \begin{bmatrix} e^{\Lambda_S t} (S_S^T S_S)^{-1} S_S^T z_0 \\ 0 \end{bmatrix}, \end{aligned}$$

which is the solution for $t \geq 0$. In our simulations we always consider such orthogonal projection of $\mathbf{z}$ onto the stable subspace at time $t = 1$ as the initial condition $\tilde{z}_0$.

A.5.3 Change from a Discrete- to a Continuous-Time Model

Following Kwakernaak (1976) let the continuous time system be:

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

Under the assumptions that $u(t) = u(t_i)$, $t_i \leq t \leq t_{i+1}$ and $\Delta = t_{i+1} - t_i$ the equivalent discrete-time system is:

$$x(i+1) = A_{Cl} x(i) + B_d u(i) \tag{39}$$
$$y(i) = C_d x(i) + D_d u(i), \tag{40}$$

where

$$\begin{aligned} A_{Cl} &= e^{A\Delta}, \\ B_d &= \left(\int_0^{\Delta} e^{A\tau} d\tau \right) B, \\ C_d &= C e^{A\Delta} \quad \text{and} \\ D_d &= C \left(\int_0^{\Delta} e^{A\tau} d\tau \right) B + D. \end{aligned}$$

Assuming $\Delta = 1$ we may rewrite the continuous time system in terms of discrete time system matrices as:

$$A = \log A_{Cl} = \log(I + A_{Cl} - I) \approx A_{Cl} - I, \tag{41}$$

$$B = \left[\int_0^1 e^{A\tau} d\tau \right]^{-1} B_d = \left(e^A - I \right)^{-1} A B_d, \tag{42}$$

[35] In case x_0 does not belong to S_A, vector $y = (S_S^T S_S)^{-1} S_S^T x_0$ is such that the distance between S_S and x_0 is minimal (y is the least-squares solution of $x_0 = S_S y$, i.e. $\|x_0 - S_S y\| \leq \|x_0 - S_S \tilde{y}\|$ for all $\tilde{y}$).

$$C = C_d e^{-A}, \tag{43}$$

$$\begin{aligned} D &= D_d - C\left(\int_0^{\Delta} e^{A\tau} d\tau\right)B \\ &= D_d - C\left[A^{-1}\left(e^{A} - I\right)\right]B. \end{aligned} \tag{44}$$

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Subgame Consistent Cooperative Provision of Public Goods Under Accumulation and Payoff Uncertainties

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Abstract The provision of public goods constitutes a classic case of market failure which calls for cooperative optimization. However, cooperation cannot be sustainable unless there is guarantee that the agreed-upon optimality principle can be maintained throughout the planning duration. This paper derives subgame consistent cooperative solutions for public goods provision by asymmetric agents in a discrete-time dynamic game framework with uncertainties in stock accumulation dynamics and future payoff structures. In particular, subgame consistency ensures that as the game proceeds agents are guided by the same optimality principle and hence they do not possess incentives to deviate from the previously adopted optimal behavior. A "payoff distribution procedure" leading to subgame-consistent solutions is derived and an illustration is presented. This is the first time that subgame consistent cooperative provision of public goods with uncertainties in stock dynamics and future payoffs is analyzed.

1 Introduction

The provision of public goods constitutes a classic case of market failure. Examples of public goods include clean environment, national security, scientific knowledge, openly accessible public capital, technical know-how and public information. The non-exclusiveness and positive externalities of public goods constitutes major factors for markets to malfunction in their efficient provision. Problems concerning pri-

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vate provision of public goods are studied in Bergstrom et al. (1986). Static analysis on provision of public goods are found in Chamberlin (1974), McGuire (1974) and Gradstein and Nitzan (1989). Fershtman and Nitzan (1991) and Wirl (1996) studied differential games of voluntary public goods provision by symmetric agents. Wang and Ewald (2010) introduced stochasticity into the dynamics of public goods accumulation elements into these games. Dockner et al. (2000) presented a game model with two asymmetric agents in which knowledge is a public good. These studies on dynamic game analysis focus on the noncooperative equilibria and the collusive solution that maximizes the joint payoffs of all agents.

Cooperation suggests the possibility of socially optimal solutions to the public goods provision problem. However, one may find it hard to be convinced that dynamic cooperation can offer a long-term solution unless there is guarantee that participants will always be better off throughout the entire cooperation duration and the agreed-upon optimality principle be maintained from the beginning to the end. To enable a cooperation scheme to be sustainable throughout the agreement period, a stringent condition is needed—that of subgame consistency. This condition requires that the optimality principle agreed upon at the outset must remain effective in any subgame starting at a later starting time with a state brought about by prior optimal behavior. Hence the players do not have incentives to deviate from the cooperative scheme throughout the cooperative duration. Moreover, a subgame consistent solution must also satisfy individual rationality and group optimality. Individual rationality ensures that the payoff allocated to an agent under cooperation will be no less than his noncooperative payoff. Group optimality ensures that all potential gains from cooperation are exhausted. The notion of subgame consistency in cooperative stochastic differential games was originated by Yeung and Petrosyan (2004).

Yeung and Petrosyan (2013a) analyzed subgame consistent cooperative provision of public goods with transferable payoffs in a stochastic differential game framework in which the accumulation dynamics of the public capital is stochastic. Another, often more common, uncertainty facing decision makers is the uncertain changes in the payoff structures. This kind of uncertainties arises because the changes in preferences, technologies, demographic structures, institutional arrangements and political and legal frameworks are not known with certainty. Yeung (2001 and 2003) introduced the class of randomly furcating stochastic differential games which allows the future payoff structures of the game to furcate (branch-out) randomly in addition to the game's stochastic dynamics. Yeung and Petrosyan (2013b) examined cooperative stochastic dynamic games with randomly furcating payoffs and presented a theorem characterizing their subgame consistent solutions. A continuous-time analog can be found in Petrosyan and Yeung (2007). The presence of random elements in future payoff structures and stock dynamics reflects an important element of reality in cooperative provision of public goods.

This paper considers subgame consistent cooperative solutions for public goods provision by asymmetric agents in a discrete-time stochastic dynamic game framework with randomly furcating future payoff structures. In addition, agents' payoffs are transferable. The noncooperative game outcome is characterized and dynamic cooperation is considered. Group optimal strategies are derived and subgame consistent solutions are characterized. A "payoff distribution procedure" leading to subgame-consistent solutions is derived. An Illustration is presented to demonstrate the explicit derivation of subgame consistent solution for public goods provision game. This is the first time that subgame consistent solution on cooperative provision of public goods with stochastic dynamics and uncertain future payoffs is studied.

The chapter is organized as follows. The analytical framework and the non-cooperative outcome of public goods provision are provided in Sect. 2. Details of a Pareto optimal cooperative scheme are presented in Sect. 3. A payment mechanism ensuring subgame consistency is derived in Sect. 4 and an illustration is given in Sect. 5. Section 6 concludes the chapter.

2 Analytical Framework and Non-cooperative Outcome

Consider the case of the provision of a public good in which a group of n agents carry out a project by making contributions to the building up of the stock of a productive public good. The game involves T stages of operation and a terminal stage in which each agent received a terminal payment. We use K_t to denote the level of the productive stock and I_t^i the public capital investment by agent i at stage $t \in \{1, 2, \dots, T\}$. The stock accumulation dynamics is governed by the stochastic difference equation:

$$K_{t+1} = K_t + \sum_{j=1}^{n} I_t^i - \delta K_t + \vartheta_t, \quad K_1 = K^0, \tag{1}$$

for $t \in \{1, 2, \dots, T\}$, where δ is the depreciation rate and ϑ_t is a sequence of statistically independent random variables.

The payoff of agent i at stage t is affected by a random variable θ_t. In particular, the payoff to agent i at stage t is

$$R^i(K_t, \theta_t) - C^i\left(I_t^i, \theta_t\right), \quad i \in N = \{1, 2, \dots, n\}, \tag{2}$$

where $R^i(K_t, \theta_t)$ is the revenue/payoff to agent i, $C^i(I_t^i, \theta_t)$ is the cost of investing $I_t^i \in X^i$, and θ_t for $\{1, 2, \dots, T\}$ are independent discrete random variables with range $\{\theta_t^1, \theta_t^2, \dots, \theta_t^{n_t}\}$ and corresponding probabilities $\{\lambda_t^1, \lambda_t^2, \dots, \lambda_t^{n_t}\}$, where n_t is a positive integer for $t \in \{1, 2, \dots, T\}$. In stage 1, it is known that θ_1 equals θ_1^1 with probability $\lambda_1^1 = 1$.

Marginal revenue product of the productive stock is positive, that is $\partial R^i(K_t, \theta)/\partial K_t > 0$, before a saturation level $\bar{K}$ has been reached; and marginal cost of investment is positive and non-decreasing, that is $\partial C^i(I_t^i, \theta_t)/\partial I_t^i > 0$ and $\partial^2 C^i(I_t^i, \theta_t)/\partial {I_t^i}^2 > 0$.

The objective of agent $i \in N$ is to maximize its expected net revenue over the planning horizon, that is

$$E_{\theta_1,\theta_2,\ldots,\theta_T;\vartheta_1,\vartheta_2,\ldots,\vartheta_T}\left\{\sum_{s=1}^{T}\left[R^i(K_s,\theta_s)-C^i\left(I_s^i,\theta_s\right)\right](1+r)^{-(s-1)} + q^i(K_{T+1})(1+r)^{-T}\right\} \tag{3}$$

subject to the stock accumulation dynamics (1), where $E_{\theta_1,\theta_2,\ldots,\theta_T;\vartheta_1,\vartheta_2,\ldots,\vartheta_T}$ is the expectation operation with respect to the random variables $\theta_1,\theta_2,\ldots,\theta_T$ and $\vartheta_1,\vartheta_2,\ldots,\vartheta_T$; r is the discount rate, and $q^i(K_T)\geq 0$ is an amount conditional on the productive stock that agent i would received at stage $T+1$. Since there is no uncertainty in stage $T+1$, we use θ_{T+1}^1 to denote the condition in stage $T+1$ with probability $\lambda_{T+1}^1=1$.

Acting for individual interests, the agents are involved in a stochastic dynamic game with randomly furcating payoffs (see Yeung and Petrosyan 2013b). Let $I_t^{(\sigma_t)i}$ denote the strategy of agent i at stage t given that the realized random variable affecting the payoff function is $\theta_t^{\sigma_t}$. In a stochastic dynamic game framework, a strategy space with state-dependent property has to be considered. In particular, a pre-specified class Γ^i of mapping $\phi_t^{(\sigma_t)i}(\cdot):K\to I_t^{(\sigma_t)i}$ with the property $I_t^{(\sigma_t)i}=\phi_t^{(\sigma_t)i}(K)\in\Gamma^i$ is the strategy space of agent i and each of its elements is a permissible strategy.

To solve the game, we follow Yeung and Petrosyan (2013b) and begin with the subgame starting at the last operating stage, that is stage T. If $\theta_T^{\sigma_T}\in\{\theta_T^1,\theta_T^2,\ldots,\theta_T^{\eta_T}\}$ has occurred at stage T and the public capital stock is $K_T=K$, the subgame becomes:

$$\max_{I_T^i} E_{\vartheta_T}\left\{\left[R^i\left(K_T,\theta_T^{\sigma_T}\right)-C^i\left(I_T^i,\theta_T^{\sigma_T}\right)\right](1+r)^{-(T-1)} + q^i(K_{T+1})(1+r)^{-T}\right\} \quad \text{for } i\in N \tag{4}$$

$$\text{subject to} \quad K_{T+1}=K_T+\sum_{\substack{j=1\\ j\neq i}}^{n} I_T^j-\delta K_T+\vartheta_T, \quad K_T=K. \tag{5}$$

The subgame (4)–(5) is a stochastic dynamic game. Invoking the standard techniques for solving stochastic dynamic games, a feedback Nash equilibrium solution can characterized as follows:

Lemma 1 *A set of strategies*

$$\phi_T^{(\sigma_T)^*}(K)=\left\{\phi_T^{(\sigma_T)1^*}(K),\phi_T^{(\sigma_T)2^*}(K),\ldots,\phi_T^{(\sigma_T)n^*}(K)\right\}$$

provides a Nash equilibrium solution to the subgame (4)–(5), *if there exist functions* $V^{(\sigma_T)i}(t,K)$, *for* $i \in N$ *and* $t \in \{1,2\}$, *such that the following conditions are satisfied*:

$$V^{(\sigma_T)i}(T,K) = \max_{I_T^i} E_{\vartheta_T}\Bigg\{\big[R^i\big(K_T,\theta_T^{\sigma_T}\big) - C^i\big(I_T^i,\theta_T^{\sigma_T}\big)\big](1+r)^{-(T-1)} + V^{(\sigma_{T+1})i}\Bigg[T+1, K + \sum_{\substack{j=1\\ j\neq i}}^{n} \phi_T^{(\sigma_T)j^*}(K) + I_T^i - \delta K + \vartheta_T\Bigg]\Bigg\}, \quad (6)$$

$$V^{(\sigma_{T+1})i}(T+1,K) = q^i(K)(1+r)^{-T} \quad \textit{for } i \in N.$$

Proof The system of equations in (6) satisfies the standard stochastic dynamic programming property and the Nash property for each agent $i \in N$. Hence a Nash equilibrium of the subgame (4)–(5) is characterized. Details of the proof of the results can be found in Theorem 6.10 in Başar and Olsder (1995). □

We sidestep the issue of multiple equilibria and focus on games in which there is a unique noncooperative Nash equilibrium in each subgame. Using Lemma 1, one can characterize the value functions $V^{(\sigma_T)i}(T,K)$ for all $\sigma_T \in \{1,2,\ldots,\eta_T\}$ if they exist. In particular, $V^{(\sigma_T)i}(T,K)$ yields agent i's expected game equilibrium payoff in the subgame starting at stage T given that $\theta_T^{\sigma_T}$ occurs and $K_T = K$.

Then we proceed to the subgame starting at stage $T-1$ when $\theta_{T-1}^{\sigma_{T-1}} \in \{\theta_{T-1}^1, \theta_{T-1}^2, \ldots, \theta_{T-1}^{\eta_{T-1}}\}$ occurs and $K_{T-1} = K$. In this subgame, agent $i \in N$ seeks to maximize his expected payoff

$$E_{\theta_T;\vartheta_{T-1},\vartheta_T}\Bigg\{\sum_{s=T-1}^{T}\big[R^i(K_s,\theta_s) - C^i\big(I_s^i,\theta_s\big)\big](1+r)^{-(s-1)} + q^i(K_{T+1})(1+r)^{-T}\Bigg\}$$
$$= E_{\vartheta_{T-1}}\Bigg\{\big[R^i\big(K_{T-1},\theta_{T-1}^{\sigma_{T-1}}\big) - C^i\big(I_{T-1}^i,\theta_{T-1}^{\sigma_{T-1}}\big)\big](1+r)^{-(T-2)} + \sum_{\sigma_T=1}^{\eta_T}\lambda_T^{\sigma_T}\big[R^i\big(K_T,\theta_T^{\sigma_T}\big) - C^i\big(I_T^i,\theta_T^{\sigma_T}\big)\big](1+r)^{-(T-2)} + q^i(K_{T+1})(1+r)^{-T}\Bigg\}, \quad (7)$$

subject to the capital accumulation dynamics

$$K_{t+1} = K_t + \sum_{j=1}^{n} I_t^j - \delta K_t + \vartheta_t, \quad K_{T-1} = K \text{ for } t \in \{T-1, T\}. \tag{8}$$

If the functions $V^{(\sigma_T)i}(T,K)$ for all $\sigma_T \in \{1,2,\dots,\eta_T\}$ characterized in Lemma 1 exist, the subgame (7)–(8) can be expressed as a game in which agent i seeks to maximize the expected payoff

$$E_{\vartheta_{T-1}}\Bigg\{\big[R^i(K_{T-1},\theta_{T-1}) - C^i\big(I_{T-1}^i,\theta_{T-1}\big)\big](1+r)^{-(T-2)} + \sum_{\sigma_T=1}^{\eta_T} \lambda_T^{\sigma_T} V^{(\sigma_T)i}\Bigg[T, K_{T-1} + \sum_{j=1}^{n} I_{T-1}^j - \delta K_{T-1} + \vartheta_{T-1}\Bigg]\Bigg\},$$
$$\text{for } i \in N, \tag{9}$$

using his control I_{T-1}^i.

A Nash equilibrium of the subgame (9) can be characterized by the following lemma.

Lemma 2 *A set of strategies*

$$\phi_{T-1}^{(\sigma_{T-1})*}(K) = \big\{\phi_{T-1}^{(\sigma_{T-1})1*}(K), \phi_{T-1}^{(\sigma_{T-1})2*}(K), \dots, \phi_{T-1}^{(\sigma_{T-1})n*}(K)\big\}$$

provides a Nash equilibrium solution to the subgame (9) *if there exist functions* $V^{(\sigma_T)i}(T,K_T)$ *for* $i \in N$ *and* $\sigma_T = \{1,2,\dots,\eta_T\}$ *characterized in Lemma* 1, *and functions* $V^{(\sigma_{T-1})i}(T-1,K)$, *for* $i \in N$ *such that the following conditions are satisfied*:

$$V^{(\sigma_{T-1})i}(T-1,K) = \max_{I_{T-1}^i} E_{\vartheta_{T-1}}\Bigg\{\big[R^i\big(K_{T-1},\theta_{T-1}^{\sigma_{T-1}}\big) - C^i\big(I_{T-1}^i,\theta_{T-1}^{\sigma_{T-1}}\big)\big](1+r)^{-(T-2)} + \sum_{\sigma_T=1}^{\eta_T} \lambda_T^{\sigma_T} V^{(\sigma_T)i}\Bigg[T, K + \sum_{\substack{j=1\\ j\neq i}} \phi_{T-1}^{(\sigma_{T-1})j*}(K) + I_{T-1}^i - \delta K + \vartheta_{T-1}\Bigg]\Bigg\}$$
$$\textit{for } i \in N. \tag{10}$$

Proof The conditions in Lemma 1 and the system of equations in (10) satisfies the standard discrete-time stochastic dynamic programming property and the Nash property for each agent $i \in N$. Hence a Nash equilibrium of the subgame (9) is characterized. □

Using Lemma 2 one can characterize the functions $V^{(\sigma_T)i}(T-1,K)$ for all $\theta_{T-1}^{\sigma_{T-1}} \in \{\theta_{T-1}^1, \theta_{T-1}^2, \dots, \theta_{T-1}^{\eta_{T-1}}\}$, if they exist. In particular, $V^{(\sigma_{T-1})i}(T-1,K)$

yields agent i's expected game equilibrium payoff in the subgame starting at stage $T-1$ given that $\theta_{T-1}^{\sigma_{T-1}}$ occurs and $K_{T-1} = K$.

Consider the subgame starting at stage $t \in \{T-2, T-3, \ldots, 1\}$ when $\theta_t^{\sigma_t} \in \{\theta_t^1, \theta_t^2, \ldots, \theta_t^{\eta_t}\}$ occurs and $K_t = K$, in which agent $i \in N$ maximizes his expected payoff

$$E_{\vartheta_t}\Bigg\{\big[R^i\big(K, \theta_t^{\sigma_t}\big) - C^i\big(I_t^i, \theta_t^{\sigma_t}\big)\big](1+r)^{-(t-1)} + \sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} V^{(\sigma_{t+1})i}\left[t+1, K + \sum_{j=1}^{n} I_t^j - \delta K + \vartheta_t\right]\Bigg\}, \quad \text{for } i \in N, \tag{11}$$

subject to the public capital accumulation dynamics

$$K_{t+1} = K_t + \sum_{j=1}^{n} I_t^j - \delta K_t + \vartheta_t, \quad K_t = K. \tag{12}$$

A Nash equilibrium solution for the game (1)–(3) can be characterized as follows:

Theorem 1 *A set of strategies*

$$\phi_i^{(\sigma_t)^*}(K) = \big\{\phi_t^{(\sigma_t)1^*}(K), \phi_t^{(\sigma_t)1^*}(K), \ldots, \phi_t^{(\sigma_t)n^*}(K)\big\},$$

for $\sigma_t \in \{1, 2, \ldots, \eta_t\}$ *and* $t \in \{1, 2, \ldots, T\}$, *constitutes a Nash equilibrium solution to the game* (1)–(3), *if there exist functions* $V^{(\sigma_t)i}(t, K)$, *for* $\sigma_t \in \{1, 2, \ldots, \eta_t\}$, $t \in \{1, 2, \ldots, T\}$, *and* $i \in N$, *such that the following recursive relations are satisfied*:

$$V^{(\sigma_T)i}(T+1, K) = q^i(K_{T+1})(1+r)^{-T},$$

$$V^{(\sigma_t)i}(t, K) = \max_{I_t^i} E_{\vartheta_t}\Bigg\{\big[R^i\big(K_t, \theta_t^{\sigma_t}\big) - C^i\big(I_t^i, \theta_t^{\sigma_t}\big)\big](1+r)^{-(t-1)} + \sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} V^{(\sigma_{t+1})i}\left[t+1, K + \sum_{\substack{j=1\\ j\neq i}}^{n} \phi_t^{(\sigma_t)j^*}(K) + I_t^i - \delta K_t + \vartheta_t\right]\Bigg\}, \tag{13}$$

for $\sigma_t \in \{1, 2, \ldots, \eta_t\}$, $t \in \{1, 2, \ldots, T\}$, *and* $i \in N$.

Proof The results in (13) characterizing the game equilibrium in stage T and stage $T-1$ are proved in Lemma 1 and Lemma 2. Invoking the subgame in stage $t \in \{1, 2, \ldots, T-1\}$ as expressed in (11)–(12), the results in (13) satisfy the optimality conditions in stochastic dynamic programming and the Nash equilibrium property for each agent in each of these subgames. Therefore, a feedback Nash equilibrium of the game (1)–(3) is characterized. □

Hence, the noncooperative outcome of the public capital provision game (1)–(3) can be obtained.

3 Pareto Optimal Cooperative Scheme

It is well-known that non-cooperative provision of public goods would, in general lead to inefficiency. Cooperation suggests the possibility of socially optimal and group efficient solutions. Now consider the case when the agents agree to cooperate and enhance their gains from cooperation. In particular, they act cooperatively to maximize their expected joint payoff and distribute the joint payoff among themselves according to an agreed-upon optimality principle. If any agent deviates from the cooperation scheme, all agents will revert to the noncooperative framework to counteract the free-rider problem in public goods provision. Moreover, group optimality, individual rationality and subgame consistency are three crucial properties that sustainable cooperative scheme has to satisfy.

3.1 Pareto Optimal Provision

To fulfill group optimality the agents would seek to maximize their expected joint payoff. In particular, they have to solve the discrete-time stochastic dynamic programming problem of maximizing

$$E_{\theta_1,\theta_2,\ldots,\theta_T;\vartheta_1,\vartheta_2,\ldots,\vartheta_T}\left\{\sum_{j=1}^{n}\sum_{s=1}^{T}\left[R^j(K_s,\theta_s)-C^j\left(I_s^j,\theta_s\right)\right](1+r)^{-(s-1)}\right.$$
$$\left.+\sum_{j=1}^{n}q^j(K_{T+1})(1+r)^{-T}\right\} \tag{14}$$

subject to dynamics (1).

To solve the dynamic programming problem (1) and (14), we first consider the problem starting at stage T. If $\theta_T^{\sigma_T}\in\{\theta_T^1,\theta_T^2,\ldots,\theta_T^{\eta_T}\}$ has occurred at stage T and the state $K_T=K$, the problem becomes:

$$\max_{I_T^1,I_T^2,\ldots,I_T^n} E_{\vartheta_T}\left\{\sum_{j=1}^{n}\left[R^j\left(K,\theta_T^{\sigma_T}\right)-C^j\left(I_T^j,\theta_T^{\sigma_T}\right)\right](1+r)^{-(T-1)}\right.$$
$$\left.+\sum_{j=1}^{n}q^j(K_{T+1})(1+r)^{-T}\right\} \tag{15}$$

$$\text{subject to}\quad K_{T+1}=K_T=\sum_{j=1}^{n}I_T^j-\delta K_T+\vartheta_T,\quad K_T=K. \tag{16}$$

An optimal solution to the stochastic control problem (15)–(16) can be characterized by the following lemma.

Lemma 3 *A set of controls*

$$I_T^{(\sigma_T)^*} = \psi_T^{(\sigma_T)^*}(K) = \left\{\psi_T^{(\sigma_T)1^*}(K), \psi_T^{(\sigma_T)2^*}(K), \ldots, \psi_T^{(\sigma_T)n^*}(K)\right\}$$

provides an optimal solution to the stochastic control problem (15)–(16), *if there exist functions* $W^{(\sigma_{T+1})}(T, K)$ *such that the following conditions are satisfied*:

$$\begin{aligned} &W^{(\sigma_T)}(T, K) \\ &= \max_{I_T^{(\sigma_T)1}, I_T^{(\sigma_T)2}, \ldots, I_T^{(\sigma_T)n}} E_{\vartheta_T}\left\{\sum_{j=1}^{n}\left[R^j\left(K, \theta_T^{\sigma_T}\right) - C^j\left(I_T^j, \theta_T^{\sigma_T}\right)\right](1+r)^{-(T-1)}\right. \\ &\quad \left. + \sum_{j=1}^{n} q^j\left(K + \sum_{h=1}^{n} I_T^h - \delta K + \vartheta_T\right)(1+r)^{-T}\right\}, \\ &W^{(\sigma_{T+1})i}(T+1, K) = \sum_{j=1}^{n} q^j(K)(1+r)^{-T}. \end{aligned} \tag{17}$$

Proof The system of equations in (17) satisfies the standard discrete-time stochastic dynamic programming property. Details of the proof of the results can be found in Başar and Olsder (1995). □

Using Lemma 3, one can characterize the functions $W^{(\sigma_T)}(T, K)$ for all $\theta_T^{\sigma_T} \in \{\theta_T^1, \theta_T^2, \ldots, \theta_T^{\eta_T}\}$, if they exist. In particular, $W^{(\sigma_T)}(T, K)$ yields the expected cooperative payoff starting at stage T given that $\theta_T^{\sigma_T}$ occurs and $K_T = K$.

Following the analysis in Sect. 2, the control problem starting at stage t when $\theta_t^{\sigma_t} \in \{\theta_t^1, \theta_t^2, \ldots, \theta_t^{\eta_t}\}$ occurs and $K_t = K$ can be expressed as:

$$\begin{aligned} &\max_{I_t^{(\sigma_t)1}, I_t^{(\sigma_t)2}, \ldots, I_t^{(\sigma_t)n}} E_{\vartheta_t}\left\{\sum_{j=1}^{n}\left[R^j\left(K, \theta_t^{\sigma_t}\right) - C^j\left(I_t^j, \theta_t^{\sigma_t}\right)\right](1+r)^{-(t-1)}\right. \\ &\quad \left. + \sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} W^{(\sigma_{t+1})}\left[t+1, K + \sum_{h=1}^{n} I_t^h - \delta K + \vartheta_t\right]\right\}, \end{aligned} \tag{18}$$

where $W^{(\sigma_{t+1})}[t+1, K + \sum_{h=1}^{n} I_t^h - \delta K + \vartheta_t]$ is the expected optimal cooperative payoff in the control problem starting at stage $t+1$ when $\theta_{t+1}^{\sigma_{t+1}} \in \{\theta_{t+1}^1, \theta_{t+1}^2, \ldots, \theta_{t+1}^{\eta_{t+1}}\}$ occurs.

An optimal solution for the stochastic control problem (14) can be characterized as follows.

Theorem 2 *A set of controls*

$$\psi_t^{(\sigma_t)*}(K) = \{\psi_t^{(\sigma_t)1^*}(K), \psi_t^{(\sigma_t)2^*}(K), \dots, \psi_t^{(\sigma_t)n^*}(K)\},$$

for $\sigma_t \in \{1, 2, \dots, \eta_t\}$ *and* $t \in \{1, 2, \dots, T\}$, *provides an optimal solution to the stochastic control problem* (1) *and* (14), *if there exist functions* $W^{(\sigma_t)}(t, K)$, *for* $\sigma_t \in \{1, 2, \dots, \eta_t\}$ *and* $t \in \{1, 2, \dots, T\}$, *such that the following recursive relations are satisfied*:

$$W^{(\sigma_T)}(T+1, K) = \sum_{j=1}^{n} q^j(K)(1+r)^{-T},$$

$$\begin{aligned} &W^{(\sigma_T)}(t, K) \\ &= \max_{I_t^{(\sigma_t)1}, I_t^{(\sigma_t)2}, \dots, I_t^{(\sigma_t)n}} E_{\vartheta_t} \Bigg\{ \sum_{j=1}^{n} \big[R^j(K, \theta_t^{\sigma_t}) - C^j(I_t^j, \theta_t^{\sigma_t}) \big](1+r)^{-(t-1)} \\ &\quad + \sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} W^{(\sigma_{t+1})} \Bigg[t+1, K + \sum_{h=1}^{n} I_t^h - \delta K + \vartheta_t \Bigg] \Bigg\}, \end{aligned} \tag{19}$$

for $\sigma_t \in \{1, 2, \dots, \eta_t\}$ *and* $t \in \{1, 2, \dots, T\}$.

Proof Invoking Lemma 3 and the specification of the control problem starting in stage $t \in \{1, 2, \dots, T-1\}$ as expressed in (18), the results in (19) satisfy the optimality conditions in discrete-time stochastic dynamic programming. Therefore, an optimal solution of the stochastic control problem is characterized in Theorem 2. □

Substituting the optimal control $\{\psi_t^{(\sigma_t)i^*}$, for $t \in \{1, 2, \dots, T\}$ and $i \in N\}$ into (1), one can obtain the dynamics of the cooperative trajectory of public capital accumulation as:

$$K_{t+1} = K_t + \sum_{j=1}^{n} \psi_t^{(\sigma_t)j^*}(K_t) - \delta K_t + \vartheta_t, \quad K_t = K, \text{ if } \theta_t^{\sigma_t} \text{ occurs at stage } t, \tag{20}$$

for $t \in \{1, 2, \dots, T\}$, $\sigma_t \in \{1, 2, \dots, \eta_t\}$.

We use X_t^* to denote the set of realizable values of K_t at stage t generated by (20). The term $K_t^* \in X_t^*$ is used to denote an element in X_t^*. The term $W^{(\sigma_t)}(t, K_t^*)$ gives the expected total cooperative payoff over the stages from t to T if $\theta_t^{\sigma_t}$ occurs and $K_t^* \in X_t^*$ is realized at stage t.

3.2 Individually Rational Condition

The agents then have to agree to an optimality principle in distributing the total cooperative payoff among them. For individual rationality to be upheld the expected payoffs an agent receives under cooperation have to be no less than his expected noncooperative payoff along the cooperative state trajectory $\{K_t^*\}_{t=1}^{T+1}$. For instance, the agents may (i) share the total expected cooperative payoff proportional to their expected noncooperative payoffs, or (ii) share the excess of the total expected cooperative payoff over the expected sum of individual noncooperative payoffs equally.

Let $\xi^{(\sigma_t)}(t, K_t^*) = [\xi^{(\sigma_t)1}(t, K_t^*), \xi^{(\sigma_t)2}(t, K_t^*), \ldots, \xi^{(\sigma_t)n}(t, K_t^*)]$ denote the imputation vector guiding the distribution of the total expected cooperative payoff under the agreed-upon optimality principle along the cooperative trajectory given that $\theta_t^{\sigma_t}$ has occurred in stage t, for $\sigma_t \in \{1, 2, \ldots, \eta_t\}$ and $t \in \{1, 2, \ldots, T\}$. In particular, the imputation $\xi^{(\sigma_t)i}(t, K_t^*)$ gives the present value of expected cumulative payments that agent i will receive from stage t to stage $T + 1$ under cooperation.

If for example, the optimality principle specifies that the agents share the expected total cooperative payoff proportional to their non-cooperative payoffs, then the imputation to agent i becomes:

$$\xi^{(\sigma_t)i}\big(t, K_t^*\big) = \frac{V^{(\sigma_t)i}(t, K_t^*)}{\sum_{j=1}^{n} V^{(\sigma_t)j}(t, K_t^*)} W^{(\sigma_t)}\big(t, K_t^*\big),$$

for $i \in N$ and $t \in \{1, 2, \ldots, T\}$.

For individual rationality to be guaranteed in every stage $k \in \{1, 2, \ldots, T\}$, it is required that the imputation satisfies:

$$\xi^{(\sigma_t)i}\big(t, K_t^*\big) \geq V^{(\sigma_t)i}\big(t, K_t^*\big), \tag{21}$$

for $i \in N$, $\sigma_t \in \{1, 2, \ldots, \eta_t\}$ and $t \in \{1, 2, \ldots, T\}$.

To ensure group optimality, the imputation vector has to satisfy

$$W^{(\sigma_t)}\big(t, K_t^*\big) = \sum_{j=1}^{n} \xi^{(\sigma_t)j}\big(t, K_t^*\big), \tag{22}$$

for $\sigma_t \in \{1, 2, \ldots, \eta_t\}$ and $t \in \{1, 2, \ldots, T\}$.

Hence, a valid imputation scheme $\xi^{(\sigma_t)i}(t, K_t^*)$, for $i \in N$, $\sigma_t \in \{1, 2, \ldots, \eta_t\}$ and $t \in \{1, 2, \ldots, T\}$, has to satisfy conditions (21)–(22).

4 Subgame Consistent Payment Mechanism

As demonstrated in Yeung and Petrosyan (2004 and 2013b), to guarantee dynamical stability in a stochastic dynamic cooperation scheme, the solution has to satisfy the property of subgame consistency in addition to group optimality and individual rationality. In particular, an extension of a subgame-consistent cooperative solution policy to a subgame starting at a later time with a feasible state brought about by prior optimal behavior would remain effective. Thus subgame consistency ensures that as the game proceeds agents are guided by the same optimality principle at each stage of the game, and hence they do not possess incentives to deviate from the agree-upon optimal behavior. For subgame consistency to be satisfied, the imputation according to the original optimality principle has to be maintained at all the T stages along the cooperative trajectory $\{K_t^*\}_{t=1}^T$. In other words, the imputation

$$\xi^{(\sigma_t)}\big(t, K_t^*\big) = \big[\xi^{(\sigma_t)1}\big(t, K_t^*\big), \xi^{(\sigma_t)2}\big(t, K_t^*\big), \ldots, \xi^{(\sigma_t)n}\big(t, K_t^*\big)\big] \tag{23}$$

has to be upheld for $\sigma_t \in \{1, 2, \ldots, \eta_t\}$, $t \in \{1, 2, \ldots, T\}$, and $K_t^* \in X_t^*$.

4.1 Payoff Distribution Procedure

Following the analysis of Yeung and Petrosyan (2013b), we formulate a Payoff Distribution Procedure (PDP) so that the agreed-upon imputation (23) can be realized. Let $B_t^{(\sigma_t)i}(K_t^*)$ denote the payment that agent i will received at stage t under the cooperative agreement, if $\theta_t^{\sigma_t} \in \{\theta_t^1, \theta_t^2, \ldots, \theta_t^{\eta_t}\}$ occurs and $K_t^* \in X_t^*$ is realized at stage $t \in \{1, 2, \ldots, T\}$. The payment scheme $\{B_t^{(\sigma_t)i}(K_t^*)$ for $i \in N$ contingent upon the event $\theta_t^{\sigma_t}$ and state K_t^*, for $t \in \{1, 2, \ldots, T\}\}$ constitutes a PDP in the sense that the imputation to agent i over the stages 1 to T can be expressed as:

$$\begin{aligned}
&\xi^{(\sigma_1)i}\big(1, K^0\big) \\
&\quad = B_1^{(\sigma_1)i}\big(K^0\big) \\
&\qquad + E_{\theta_2,\ldots,\theta_T;\vartheta_1,\ldots,\vartheta_T}\left(\sum_{\zeta=2}^{T} B_\zeta^{(\sigma_\zeta)i}\big(K_\zeta^*\big) + q^i\big(K_{T+1}^*\big)(1+r)^{-T}\right) \quad \text{for } i \in N.
\end{aligned}$$

Moreover, according to the agreed-upon optimality principle in (23), if $\theta_t^{\sigma_t}$ occurs and $K_t^* \in X_t^*$ is realized at stage t the imputation to agent i is $\xi^{(\sigma_t)i}(t, K_T^*)$. Therefore the payment scheme $B_t^{(\sigma_t)}(K_t^*)$ has to satisfy the conditions

$$\begin{aligned}
&\xi^{(\sigma_t)i}\big(t, K_t^*\big) \\
&\quad = B_t^{(\sigma_t)i}\big(K_t^*\big) \\
&\qquad + E_{\theta_{t+1},\theta_{t+2},\ldots,\theta_T;\vartheta_t,\vartheta_{t+1},\ldots,\vartheta_T}\left(\sum_{\zeta=t+1}^{T} B_\zeta^{(\sigma_\zeta)i}\big(K_\zeta^*\big) + q^i\big(K_{T+1}^*\big)(1+r)^{-T}\right)
\end{aligned} \tag{24}$$

for $i \in N$ and all $t \in \{1, 2, \ldots, T\}$.

For notational convenience the term $\xi^{(\sigma_{T+1})i}(T+1, K_{T+1}^*)$ is used to denote $q^i(K_{T+1}^*)(1+r)^{-T}$. Crucial to the formulation of a subgame consistent solution is the derivation of a payment scheme $\{B_t^{(\sigma_t)i}(K_t^*)$, for $i \in N, \sigma_t \in \{1, 2, \ldots, \eta_t\}$, $K_t^* \in X_t^*$ and $t \in \{1, 2, \ldots, T\}\}$ so that the imputation in (24) can be realized.

A theorem for the derivation of a subgame consistent payment scheme can be established as follows.

Theorem 3 *A payment equaling*

$$B_t^{(\sigma_t)i}\big(K_t^*\big)$$
$$= (1+r)^{(t-1)}\Bigg(\xi^{(\sigma_t)i}\big(t, K_t^*\big)$$
$$- E_{\vartheta_t}\left\{\sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} \xi^{(\sigma_{t+1})i}\left[t+1, K_t^* + \sum_{h=1}^{n} \psi_t^{(\sigma_t)h^*}\big(K_t^*\big) - \delta K_t^* + \vartheta_t\right]\right\}\Bigg),$$

given to agent $i \in N$ at stage $t \in \{1, 2, \ldots, T\}$, if $\theta_t^{\sigma_t}$ occurs and $K_t^* \in X_t^*$, leads to the realization of the imputation in (24).

Proof To construct the proof of Theorem 3, we first express the term

$$E_{\theta_{t+1},\theta_{t+2},\ldots,\theta_T;\vartheta_t,\vartheta_{t+1},\ldots,\vartheta_T}\left\{\sum_{\zeta=t+1}^{T} B_\zeta^{(\sigma_\zeta)i}\big(K_\zeta^*\big)(1+r)^{-(\zeta-1)}\right.$$
$$\left. + q^i\big(K_{T+1}^*\big)(1+r)^{-T}\right\}$$
$$= E_{\vartheta_{t+1}}\left\{\sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}}\left[B_{t+1}^{(\sigma_{t+1})i}\big(K_{t+1}^*\big)(1+r)^{-(t-1)}\right.\right.$$
$$+ E_{\theta_{t+2},\theta_{t+3},\ldots,\theta_T;\vartheta_{t+2},\vartheta_{t+3},\ldots,\vartheta_T}\left(\sum_{\zeta=t+2}^{T} B_\zeta^{(\sigma_\zeta)i}\big(K_\zeta^*\big)(1+r)^{-(\zeta-1)}\right.$$
$$\left.\left.\left. + q^i\big(K_{T+1}^*\big)(1+r)^{-T}\right)\right]\right\}. \tag{25}$$

Then, using (24) we can express the term $\xi^{(\sigma_{t+1})i}(t+1, K_{t+1}^*)$ as

$$\xi^{(\sigma_{t+1})i}\big(t+1, K_{t+1}^*\big)$$
$$= B_{t+1}^{(\sigma_{t+1})i}\big(K_{t+1}^*\big)(1+r)^{-t}$$
$$+ E_{\theta_{t+2},\theta_{t+3},\ldots,\theta_T;\vartheta_{t+2},\vartheta_{t+3},\ldots,\vartheta_T}\left\{\sum_{\zeta=t+2}^{T} B_\zeta^{(\sigma_\zeta)i}\big(K_\zeta^*\big) + q^i\big(K_{T+1}^*\big)(1+r)^{-T}\right\}. \tag{26}$$

The expression on the right-hand-side of equation (26) is the same as the expression inside the square brackets of (25). Invoking equation (26) we can replace the expression inside the square brackets of (25) by $\xi^{(\sigma_{t+1})i}(t+1, K^*_{t+1})$ and obtain:

$$E_{\theta_{t+1},\theta_{t+2},\ldots,\theta_T;\vartheta_t,\vartheta_{t+1},\ldots,\vartheta_T}\left\{\sum_{\zeta=t+1}^{T} B_\zeta^{(\sigma_\zeta)i}\left(K^*_\zeta\right)(1+r)^{-(\zeta-1)} + q^i\left(K^*_{T+1}\right)(1+r)^{-T}\right\}$$

$$= E_{\vartheta_t}\left\{\sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} \xi^{(\sigma_{t+1})i}\left[t+1, K^*_{t+1}\right]\right\}$$

$$= E_{\vartheta_t}\left\{\sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} \xi^{(\sigma_{t+1})i}\left[t+1, K^*_t + \sum_{h=1}^{n}\psi^{(\sigma_t)h^*}\left(K^*_t\right) - \delta K^*_t + \vartheta_t\right]\right\}.$$

Substituting the term

$$E_{\theta_{t+1},\theta_{t+2},\ldots,\theta_T;\vartheta_t,\vartheta_{t+1},\ldots,\vartheta_T}\left\{\sum_{\zeta=t+1}^{T} B_\zeta^{(\sigma_\zeta)i}\left(K^*_\zeta\right)(1+r)^{-(\zeta-1)} + q^i\left(K^*_{T+1}\right)(1+r)^{-T}\right\}$$

by

$$E_{\vartheta_t}\left\{\sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} \xi^{(\sigma_{t+1})i}\left[t+1, K^*_t + \sum_{h=1}^{n}\psi^{(\sigma_t)h^*}\left(K^*_t\right) - \delta K^*_t + \vartheta_t\right]\right\}$$

in (24) we can express (24) as:

$$\xi^{(\sigma_t)i}\left(t, K^*_t\right) = B_t^{(\sigma_t)i}\left(K^*_t\right)(1+r)^{-(t-1)} + E_{\vartheta_t}\left\{\sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} \xi^{(\sigma_{t+1})i}\left[t+1, K^*_t + \sum_{h=1}^{n}\psi_t^{(\sigma_t)h^*}\left(K^*_t\right) - \delta K^*_t + \vartheta_t\right]\right\}. \tag{27}$$

For condition (27), which is an alternative form of (24), to hold it is required that:

$$B_t^{(\sigma_t)i}\left(K_t^*\right)$$
$$= (1+r)^{t-1}\Bigg(\xi^{(\sigma_t)i}\left(t, K_t^*\right)$$
$$- E_{\vartheta_t}\left\{\sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} \xi^{(\sigma_{t+1})i}\left[t+1, K_t^* + \sum_{h=1}^{n} \psi_t^{(\sigma_t)h^*}\left(K_t^*\right) - \delta K_t^* + \vartheta_t\right]\right\}\Bigg),$$

for $i \in N$ and $t \in \{1, 2, \ldots, T\}$.

Therefore by paying $B_t^{(\sigma_t)i}(K_t^*)$ to agent $i \in N$ at stage $t \in \{1, 2, \ldots, T\}$, if $\theta_t^{\sigma_t}$ occurs and $K_t^* \in X_t^*$ is realized, leads to the realization of the imputation in (24). Hence Theorem 3 follows. □

For a given imputation vector

$$\xi^{(\sigma_t)}\left(t, K_t^*\right) = \left[\xi^{(\sigma_t)1}\left(t, K_t^*\right), \xi^{(\sigma_t)2}\left(t, K_t^*\right), \ldots, \xi^{(\sigma_t)n}\left(t, K_t^*\right)\right],$$

for $\sigma_t \in \{1, 2, \ldots, \eta_t\}$ and $t \in \{1, 2, \ldots, T\}$, Theorem 3 can be used to derive the PDP that leads to the realization this vector.

4.2 Transfer Payments

When all agents are using the cooperative strategies given that $K_t^* \in X_t^*$, and $\theta_t^{\sigma_t}$ occur, the payoff that agent i will directly received at stage t becomes

$$\left[R^i\left(K_t^*, \theta_t^{\sigma_t}\right) - C^i\left(\psi^{(\sigma_t)i^*}\left(K_t^*\right), \theta_t^{\sigma_t}\right)\right](1+r)^{-(t-1)}$$

However, according to the agreed upon imputation, agent i is supposed to received $B_t^{(\sigma_t)i}(K_t^*)$ at stage t as given in Theorem 3. Therefore a transfer payment (which can be positive or negative)

$$\varpi_t^{(\sigma_t)i}\left(K_t^*\right) = B_t^{(\sigma_t)i}\left(K_t^*\right) - \left[R^i\left(K_t^*, \theta_t^{\sigma_t}\right) - C^i\left(\psi_t^{(\sigma_t)i^*}\left(K_t^*\right), \theta_t^{\sigma_t}\right)\right](1+r)^{-(t-1)},$$

for $t \in \{1, 2, \ldots, T\}$ and $i \in N$, will be assigned to agent i to yield the cooperative imputation $\xi^{(\sigma_t)}(t, K_t^*)$.

5 An Illustration

In this section, we provide an illustration of the derivation of a subgame consistent solution of public goods provision under accumulation and payoff uncertainties in a multiple asymmetric agents situation. The basic game structure is a discrete-time analog of an example in Yeung and Petrosyan (2013a) but with the crucial addition of uncertain future payoff structures to reflect probable changes in preferences, technologies, demographic structures and institutional arrangements. This is the first time that an explicit dynamic game model on cooperative public good provision under uncertain future payoffs is presented.

5.1 Multiple Asymmetric Agents Public Capital Build-up

We consider an n asymmetric agents economic region in which the agents receive benefits from an existing public capital stock K_t at each stage $t \in \{1, 2, \ldots, T\}$. The accumulation dynamics of the public capital stock is governed by the stochastic difference equation:

$$K_{t+1} = K_t + \sum_{j=1}^{n} I_t^j - \delta K_t + \vartheta_t, \quad K_1 = K^0, \text{ for } t \in \{1, 2, 3\}, \tag{28}$$

where ϑ_t is a discrete random variable with non-negative range $\{\vartheta_t^1, \vartheta_t^2, \vartheta_t^3\}$ and corresponding probabilities $\{\gamma_t^1, \gamma_t^2, \gamma_t^3\}$, and $\sum_{j=1}^{3} \gamma_t^j \vartheta_t^j = \varpi > 0$.

At stage 1, it is known that $\theta_1^{\sigma_1} = \theta_1^1$ has happened with probability $\lambda_1^1 = 1$, and the payoff of agent i is

$$\alpha_1^{(\sigma_t)i} K_1 - c_1^{(\sigma_1)i} \left(I_1^i\right)^2.$$

At stage $t \in \{2, 3\}$, the payoff of agent i is

$$\alpha_t^{(\sigma_t)i} K_t - c_t^{(\sigma_t)i} \left(I_t^i\right)^2,$$

if $\theta_1^{\sigma_t} \in \{\theta_t^1, \theta_t^2, \theta_t^3, \theta_t^4\}$ occurs.

In particular, $\alpha_t^{(\sigma_t)i} K_t$ gives the gain that agent i derives from the public capital at stage $t \in \{1, 2, 3\}$, and $c_t^{(\sigma_t)i} (I_t^i)^2$ is the cost of investing I_t^i in the public capital.

The probability that $\theta_t^{\sigma_t} \in \{\theta_t^1, \theta_t^2, \theta_t^3, \theta_t^4\}$ will occur at stage $t \in \{2, 3\}$ is $\lambda_t^{\sigma_t} \in \{\lambda_t^1, \lambda_t^2, \lambda_t^3, \lambda_t^4\}$. In stage 4, a terminal payment contingent upon the size of the capital stock equaling $(q^i K_4 + m^i)(1 + r)^{-3}$ will be paid to agent i. Since there is no uncertainty in stage 4, we use θ_4^1 to denote the condition in stage 4 with probability $\lambda_4^1 = 1$.

The objective of agent $i \in N$ is to maximize the expected payoff:

$$E_{\theta_1, \theta_2, \theta_3; \vartheta_1, \vartheta_2, \vartheta_3} \left\{ \sum_{\tau=1}^{3} \left[\alpha_\tau^{(\sigma_\tau)i} K_\tau - c_\tau^{(\sigma_\tau)i} \left(I_\tau^i\right)^2\right](1 + r)^{-(\tau-1)} + \left(q^i K_4 + m^i\right)(1 + r)^{-3} \right\}, \tag{29}$$

subject to the public capital accumulation dynamics (28).

The noncooperative outcome will be examined in the next subsection.

5.2 Noncooperative Outcome

Invoking Lemma 2, one can characterize the noncooperative Nash equilibrium strategies for the game (28)–(29) as follows. In particular, a set of strategies $\{I_t^{(\sigma_t)i*} = \phi_t^{(\sigma_t)i*}(K)$, for $\sigma_1 \in \{1\}, \sigma_2, \sigma_3 \in \{1, 2, 3, 4\}, t \in \{1, 2, 3\}$ and $i \in N\}$ provides a Nash equilibrium solution to the game (28)–(29), if there exist functions $V^{(\sigma_t)i}(t, K)$, for $i \in N$ and $t \in \{1, 2, 3\}$, such that the following recursive relations are satisfied:

$$V^{(\sigma_4)i}(4, K) = \left(q^i K + m_i\right)(1+r)^{-3};$$

$$V^{(\sigma_t)i}(t, K) = \max_{I_t^i} E_{\vartheta_t}\Bigg\{\left[\alpha_t^{(\sigma_t)i} K - c_t^{(\sigma_t)i}\left(I_t^i\right)^2\right](1+r)^{-(t-1)} + \sum_{\sigma_{t+1}=1}^{4} \lambda_{t+1}^{\sigma_{t+1}}$$
$$\times V^{(\sigma_{t+1})i}\Bigg[t+1, K + \sum_{\substack{j=1\\ j\neq i}}^{n} \phi_t^{(\sigma_t)j*}(K) + I_t^i - \delta K + \vartheta_t\Bigg]\Bigg\}$$
$$= \max_{I_t^i}\Bigg\{\left[\alpha_t^{(\sigma_t)i} K - c_t^{(\sigma_t)i}\left(I_t^i\right)^2\right](1+r)^{-(t-1)} \tag{30}$$
$$+ \sum_{y=1}^{3}\gamma_t^y \sum_{\sigma_{t+1}=1}^{4} \lambda_{t+1}^{\sigma_{t+1}}$$
$$\times V^{(\sigma_{t+1})i}\Bigg[t+1, K + \sum_{\substack{j=1\\ j\neq i}}^{n} \phi_t^{(\sigma_t)j*}(K) + I_t^i - \delta K + \vartheta_t^y\Bigg]\Bigg\},$$

for $t \in \{1, 2, 3\}$.

Performing the indicated maximization in (30) yields:

$$I_t^i = \phi_t^{(\sigma_t)i*}(K)$$
$$= \frac{(1+r)^{t-1}}{2c_t^{(\sigma_t)i}} \sum_{y=1}^{3}\gamma_t^y \sum_{\sigma_{t+1}=1}^{4} \lambda_{t+1}^{\sigma_{t+1}}$$
$$\times V_{K_{t+1}}^{(\sigma_{t+1})i}\Bigg[t+1, K + \sum_{j=1}^{n} \phi_t^{(\sigma_t)j*}(K) - \delta K + \vartheta_t^y\Bigg], \tag{31}$$

for $i \in N$, $t \in \{1, 2, 3\}$, $\sigma_1 = 1$, and $\sigma_\tau \in \{1, 2, 3, 4\}$ for $\tau \in \{2, 3\}$.

Proposition 1 *The value function which represents the expected payoff of agent i can be obtained as*:

$$V^{(\sigma_t)i}(t,K)=\big[A_t^{(\sigma_t)i}K+C_t^{(\sigma_t)i}\big](1+r)^{-(t-1)},$$

for $i\in N$, $t\in\{1,2,3\}$, $\sigma_1=1$, *and* $\sigma_\tau\in\{1,2,3,4\}$ *for* $\tau\in\{2,3\}$, *where*

$$A_3^{(\sigma_3)i}=\alpha_3^{(\sigma_3)i}+q^i(1-\delta)(1+r)^{-1},\quad \textit{and}$$

$$C_3^{(\sigma_3)i}=-\frac{(q^i)^2(1+r)^{-2}}{4c_3^{(\sigma_3)i}}+\left[q^i\sum_{j=1}^{n}\frac{q^j(1+r)^{-1}}{2c_3^{(\sigma_3)j}}+q^i\varpi_3+m^i\right](1+r)^{-1};$$

$$A_2^{(\sigma_2)i}=\alpha_2^{(\sigma_2)i}+\sum_{\sigma_3=1}^{4}\lambda_3^{\sigma_3}A_3^{(\sigma_3)i}(1-\delta)(1+r)^{-1},\quad \textit{and}$$

$$\begin{aligned}C_2^{(\sigma_2)i}&=-\frac{1}{4c_2^{(\sigma_2)i}}\left(\sum_{\sigma_3=1}^{4}\lambda_3^{\sigma_3}A_3^{(\sigma_3)i}(1+r)^{-1}\right)^2\\&\quad+\sum_{\sigma_3=1}^{4}\lambda_3^{\sigma_3}\left[A_3^{(\sigma_3)i}\left(\sum_{j=1}^{n}\sum_{\hat\sigma_3=1}^{4}\lambda_3^{\hat\sigma_3}\frac{A_3^{(\hat\sigma_3)j}(1+r)^{-1}}{2c_2^{(\sigma_2)j}}+\varpi_2\right)+C_3^{(\sigma_3)i}\right]\\&\quad\times(1+r)^{-1};\end{aligned}$$

$$A_1^{(\sigma_1)i}=\alpha_1^{(\sigma_1)i}+\sum_{\sigma_2=1}^{4}\lambda_2^{\sigma_2}A_2^{(\sigma_2)i}(1-\delta)(1+r)^{-1},\quad \textit{and}$$

$$\begin{aligned}C_1^{(\sigma_1)i}&=-\frac{1}{4c_1^{(\sigma_1)i}}\left(\sum_{\sigma_2=1}^{4}\lambda_2^{\sigma_2}A_2^{(\sigma_2)i}(1+r)^{-1}\right)^2\\&\quad+\sum_{\sigma_2=1}^{4}\lambda_2^{\sigma_2}\left[A_2^{(\sigma_2)i}\left(\sum_{j=1}^{n}\sum_{\hat\sigma_2=1}^{4}\lambda_2^{\hat\sigma_2}\frac{A_2^{(\hat\sigma_2)j}(1+r)^{-1}}{2c_1^{(\sigma_1)j}}+\varpi_1\right)+C_2^{(\sigma_2)i}\right]\\&\quad\times(1+r)^{-1};\end{aligned}$$

for $i\in N$.

Proof See Appendix. □

Substituting the relevant derivatives of the value functions in Proposition 1 into the game equilibrium strategies (31) yields a noncooperative Nash equilibrium solution of the game (28)–(29).

5.3 Cooperative Provision of Public Capital

Now we consider the case when the agents agree to cooperate and seek to enhance their gains. They agree to maximize their expected joint gain and distribute the cooperative gain proportional to their expected non-cooperative gains. The agents would first maximize their expected joint payoff

$$E_{\theta_1,\theta_2,\theta_3;\vartheta_1,\vartheta_2,\vartheta_3}\left\{\sum_{j=1}^{n}\sum_{\tau=1}^{3}\left[\alpha_\tau^{(\sigma_\tau)j}K_\tau - c_\tau^{(\sigma_\tau)j}\left(I_\tau^j\right)^2\right](1+r)^{-(\tau-1)}\right.$$
$$\left.+\sum_{j=1}^{n}\left(q^jK_4+m^j\right)(1+r)^{-3}\right\}, \tag{32}$$

subject to the stochastic dynamics (28).

Invoking Theorem 2, one can characterize the solution of the stochastic dynamic programming problem (28) and (32) as follows. In particular, a set of control strategies $\{u_t^{(\sigma_t)i*} = \psi_t^{(\sigma_t)i*}(K)$, for $t \in \{1,2,3\}$ and $i \in N, \sigma_1 = 1, \sigma_\tau \in \{1,2,3,4\}\}$ for $\tau \in \{2,3\}$, provides an optimal solution to the problem (28) and (32), if there exist functions $W^{(\sigma_t)}(t,K)$, for $t \in \{1,2,3\}$, such that the following recursive relations are satisfied:

$$W^{(\sigma_4)}(4,K) = \sum_{j=1}^{n}\left(q^jK+m^j\right)(1+r)^{-3};$$

$$W^{(\sigma_t)}(t,K) = \max_{I_t^1,I_t^2,\ldots,I_t^n} E_{\vartheta_t}\left\{\sum_{j=1}^{n}\left[\alpha_t^{(\sigma_t)j}K - c_t^{(\sigma_t)j}\left(I_t^j\right)^2\right](1+r)^{-(t-1)}\right.$$
$$\left.+\sum_{\sigma_{t+1}=1}^{4}\lambda_{t+1}^{\sigma_{t+1}}W^{(\sigma_{t+1})}\left[t+1, K+\sum_{j=1}^{n}I_t^j-\delta K+\vartheta_t\right]\right\} \tag{33}$$
$$=\max_{I_t^i}\left\{\sum_{j=1}^{n}\left[\alpha_t^{(\sigma_t)j}K - c_t^{(\sigma_t)j}\left(I_t^j\right)^2\right](1+r)^{-(t-1)}\right.$$
$$\left.+\sum_{y=1}^{3}\gamma_t^y\sum_{\sigma_{t+1}=1}^{4}\lambda_{t+1}^{\sigma_{t+1}}W^{(\sigma_{t+1})i}\left[t+1, K+\sum_{j=1}^{n}I_t^j-\delta K+\vartheta_t^y\right]\right\}$$
$$\text{for } t \in \{1,2,3\}.$$

Performing the indicated maximization in (33) yields:

$$I_t^i = \psi_t^{(\sigma_t)i*}(K)$$
$$=\frac{(1+r)^{t-1}}{2c_t^{(\sigma_t)i}}\sum_{y=1}^{3}\gamma_t^y\sum_{\sigma_{t+1}=1}^{4}\lambda_{t+1}^{\sigma_{t+1}}$$
$$\times W_{K_{t+1}}^{(\sigma_{t+1})}\left[t+1, K+\sum_{j=1}^{n}\psi_t^{(\sigma_t)j*}(K)-\delta K+\vartheta_t^y\right], \tag{34}$$

for $i \in N$, $t \in \{1,2,3\}$, $\sigma_1 = 1$, and $\sigma_\tau \in \{1,2,3,4\}$ for $\tau \in \{2,3\}$.

Proposition 2 *The value function which represents the expected joint payoff of agents can be obtained as*:

$$W^{(\sigma_t)}(t,K) = \left[A_t^{(\sigma_t)} K + C_t^{(\sigma_t)}\right](1+r)^{-(t-1)},$$

for $t \in \{1,2,3\}$, $\sigma_1 = 1$, *and* $\sigma_\tau \in \{1,2,3,4\}$ *for* $\tau \in \{2,3\}$, *where*

$$A_3^{(\sigma_3)} = \sum_{j=1}^{n} \alpha_3^{(\sigma_3)j} + \sum_{j=1}^{n} q^j (1-\delta)(1+r)^{-1}, \quad \textit{and}$$

$$C_3^{(\sigma_3)} = -\sum_{j=1}^{n} \frac{(\sum_{h=1}^{n} q^h (1+r)^{-1})^2}{4c_3^{(\sigma_3)j}} + \sum_{j=1}^{n} \left[q^j \left(\sum_{\ell=1}^{n} \frac{\sum_{h=1}^{n} q^h (1+r)^{-1}}{2c_3^{(\sigma_3)\ell}} + \varpi_3 \right) + m^j \right] (1+r)^{-1};$$

$$A_2^{(\sigma_2)} = \sum_{j=1}^{n} \alpha_2^{(\sigma_2)j} + \sum_{\sigma_3=1}^{4} \lambda_3^{\sigma_3} A_3^{(\sigma_3)} (1-\delta)(1+r)^{-1}, \quad \textit{and}$$

$$C_2^{(\sigma_2)} = -\sum_{j=1}^{n} \frac{1}{4c_2^{(\sigma_2)j}} \left(\sum_{\sigma_3=1}^{4} \lambda_3^{\sigma_3} A_3^{(\sigma_3)} (1+r)^{-1} \right)^2 + \sum_{\sigma_3=1}^{4} \lambda_3^{\sigma_3} \left[A_3^{(\sigma_3)i} \left(\sum_{j=1}^{n} \sum_{\hat{\sigma}_3=1}^{4} \lambda_3^{\hat{\sigma}_3} \frac{A_3^{(\hat{\sigma}_3)j} (1+r)^{-1}}{2c_2^{(\sigma_2)j}} + \varpi_2 \right) + C_3^{(\sigma_3)i} \right] \times (1+r)^{-1};$$

$$A_1^{(\sigma_1)} = \sum_{j=1}^{n} \alpha_1^{(\sigma_1)j} + \sum_{\sigma_2=1}^{4} \lambda_2^{\sigma_2} A_2^{(\sigma_2)} (1-\delta)(1+r)^{-1}, \quad \textit{and}$$

$$C_1^{(\sigma_1)} = -\sum_{j=1}^{n} \frac{1}{4c_1^{(\sigma_1)j}} \left(\sum_{\sigma_2=1}^{4} \lambda_2^{\sigma_2} A_2^{(\sigma_2)} (1+r)^{-1} \right)^2 + \sum_{\sigma_2=1}^{4} \lambda_2^{\sigma_2} \left[A_2^{(\sigma_2)} \left(\sum_{j=1}^{n} \sum_{\hat{\sigma}_2=1}^{4} \lambda_2^{\hat{\sigma}_2} \frac{A_2^{(\hat{\sigma}_2)} (1+r)^{-1}}{2c_1^{(\sigma_1)j}} + \varpi_1 \right) + C_2^{(\sigma_2)} \right] \times (1+r)^{-1}.$$

Proof Follow the proof of Proposition 1. □

Using (34) and Proposition 2, the optimal cooperative strategies of the agents can be obtained as:

$$\psi_3^{(\sigma_3)i*}(K) = \frac{\sum_{h=1}^{n} q^h (1+r)^{-1}}{2c_3^{(\sigma_3)i}},$$

$$\psi_2^{(\sigma_2)i*}(K) = \sum_{\sigma_3=1}^{4} \lambda_3^{\sigma_3} \frac{A_3^{(\sigma_3)}(1+r)^{-1}}{2c_2^{(\sigma_2)i}}, \tag{35}$$

$$\psi_1^{(\sigma_1)i*}(K) = \sum_{\sigma_2=1}^{4} \lambda_2^{\sigma_2} \frac{A_2^{(\sigma_2)}(1+r)^{-1}}{2c_1^{(\sigma_1)i}}, \quad \text{for } i \in N.$$

Substituting $\psi_t^{(\sigma_t)i*}(K)$ from (35) into (28) yields the optimal cooperative accumulation dynamics:

$$K_{t+1} = K_t + \sum_{j=1}^{n} \sum_{\sigma_{t+1}=1}^{4} \lambda_{t+1}^{\sigma_{t+1}} \frac{A_{t+1}^{(\sigma_{t+1})}(1+r)^{-1}}{2c_t^{(\sigma_t)j}} - \delta K_t + \vartheta_t, \quad K_1 = K^0, \tag{36}$$

if $\theta_t^{\sigma_t}$ occurs at stage t, for $t \in \{1, 2, 3\}$.

5.4 Subgame Consistent Cooperative Solution

Given that the agents agree to share the cooperative gain proportional to their expected non-cooperative payoffs, an imputation

$$\begin{aligned} \xi^{(\sigma_t)i}\left(t, K_t^*\right) &= \frac{V^{(\sigma_t)i}(t, K_t^*)}{\sum_{j=1}^{n} V^{(\sigma_t)j}(t, K_t^*)} W^{(\sigma_t)}\left(t, K_t^*\right) \\ &= \frac{[A_t^{(\sigma_t)i} K_t^* + C_t^{(\sigma_t)i}]}{\sum_{j=1}^{n}[A_t^{(\sigma_t)i} K_t^* + C_t^{(\sigma_t)i}]}\left[A_t^{(\sigma_t)} K_t^* + C_t^{(\sigma_t)}\right](1+r)^{-(t-1)}, \\ &\quad \text{for } i \in N, \end{aligned} \tag{37}$$

if $\theta_t^{\sigma_t}$ occurs at stage t for $t \in \{1, 2, 3\}$ has to be maintained.

Invoking Theorem 3, if $\theta_t^{\sigma_t}$ occurs and $K_t^* \in X_t^*$ is realized at stage t a payment equaling

$$\begin{aligned} B_t^{(\sigma_t)i}\left(K_t^*\right) = (1+r)^{(t-1)} \Bigg\{ & \xi^{(\sigma_t)i}\left(t, K_t^*\right) \\ & - \left[\sum_{y=1}^{3} \gamma_t^y \sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} \right. \\ & \left. \times \left(\xi^{(\sigma_{t+1})i}\left[t+1, K_t^* + \sum_{h=1}^{n} \psi_t^{(\sigma_t)h*}\left(K_t^*\right) - \delta K_t^* + \vartheta_t^y\right]\right)\right]\Bigg\} \end{aligned}$$

$$= \frac{A_t^{(\sigma_t)i} K_t^* + C_t^{(\sigma_t)i}}{\sum_{j=1}^n [A_t^{(\sigma_t)i} K_t^* + C_t^{(\sigma_t)i}]} [A_t^{(\sigma_t)} K_t^* + C_t^{(\sigma_t)}]$$

$$- \sum_{y=1}^3 \gamma_t^y \sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} \frac{A_{t+1}^{(\sigma_{t+1})i} K_{t+1}(\sigma_{t+1}, \vartheta_t^y) + C_{t+1}^{(\sigma_{t+1})i}}{\sum_{j=1}^n [A_{t+1}^{(\sigma_{t+1})i} K_{t+1}(\sigma_{t+1}, \vartheta_t^y) + C_{t+1}^{(\sigma_{t+1})i}]}$$

$$\times \big[A_{t+1}^{(\sigma_{t+1})} K_{t+1}\big(\sigma_{t+1}, \vartheta_t^y\big) + C_{t+1}^{(\sigma_t)}\big](1+r)^{-1}, \quad (38)$$

where

$$K_{t+1}\big(\sigma_{t+1}, \vartheta_t^y\big) = K_t^* + \sum_{j=1}^n \sum_{\sigma_{t+1}=1}^4 \lambda_{t+1}^{\sigma_{t+1}} \frac{A_{t+1}^{(\sigma_{t+1})}(1+r)^{-1}}{2c_t^{(\sigma_t)j}} - \delta K_t^* + \vartheta_t^y,$$

given to agent i at stage $t \in \{1, 2, 3\}$ if $\theta_t^{\sigma_t}$ occurs would lead to the realization of the imputation (37).

A subgame consistent solution and the corresponding payment schemes can be obtained using Propositions 1 and 2 and conditions (35)–(38).

Finally, since all agents are adopting the cooperative strategies, the payoff that agent i will directly received at stage t is

$$\alpha_t^{(\sigma_t)i} K_t^* - \frac{1}{4c_t^{(\sigma_t)i}} \left(\sum_{\sigma_{t+1}=1}^4 \lambda_{t+1}^{\sigma_{t+1}} A_{t+1}^{(\sigma_{t+1})} (1+r)^{-1} \right)^2,$$

if $\theta_t^{\sigma_t}$ occurs at stage t.

However, according to the agreed upon imputation, agent i is to receive $\xi^{(\sigma_t)i}(t, K_t^*)$ in (38), therefore a transfer payment (which can be positive or negative) equaling

$$\pi^{(\sigma_t)i}\big(t, K_t^*\big) = \xi^{(\sigma_t)i}\big(t, K_t^*\big) - \alpha_t^{(\sigma_t)i} K_t^* + \frac{1}{4c_t^{(\sigma_t)i}} \left(\sum_{\sigma_{t+1}=1}^4 \lambda_{t+1}^{\sigma_{t+1}} A_{t+1}^{(\sigma_{t+1})} (1+r)^{-1} \right)^2$$

will be given to agent $i \in N$ at stage t.

6 Concluding Remarks

An essential characteristic of decision making over time is that though the decision-maker gathered all past and present information available, the precise state of the future, in general, could not be foreseen with absolute certainty. An empirically meaningful theory must therefore incorporate relevant uncertainties in an appropriate manner. This paper resolves the classical problem of market failure in the provision of public goods with a subgame consistent cooperative scheme taking into consideration two types of commonly observed uncertainties—stochastic stock accumulation dynamics and uncertain future payoff structures. A scheme that guarantees the agreed-upon optimality principle be maintained in any subgame and provides the basis for sustainable cooperation is derived. A "payoff distribution procedure" leading to subgame-consistent solutions is developed. An illustrative example is presented to demonstrate the derivation of subgame consistent solution for public goods provision game under these uncertainties. The analysis can be readily extended into a multiple public capital goods paradigm. This is the first time that subgame consistent cooperative provision of public goods is analysed under uncertainties in both the accumulation dynamics and future payoff structures. Further research and applications are expected.

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Appendix

Proof of Proposition 1 Consider first the last stage, that is stage 3, when $\theta_3^{\sigma_3}$ occurs. Invoking that

$$V^{(\sigma_3)i}(3, K) = \big[A_3^{(\sigma_3)i} K + C_3^{(\sigma_3)i}\big](1+r)^{-2} \quad \text{and}$$
$$V^{(\sigma_4)i}(4, K_4) = \big(q^i K + m^i\big)(1+r)^{-3}$$

from Proposition 1, the condition governing $t = 3$ in equation (30) becomes

$$\begin{aligned}
&\big[A_3^{(\sigma_3)i} K + C_3^{(\sigma_3)i}\big](1+r)^{-2} \\
&\quad = \max_{I_3^i} \Bigg\{ \big[\alpha_3^{(\sigma_3)i} K - c_3^{(\sigma_3)i} \big(I_3^i\big)^2\big](1+r)^{-2} \\
&\qquad + \sum_{y=1}^{3} \gamma_3^y \sum_{\sigma_4=1}^{4} \lambda_4^{\sigma_4} \Bigg[q^i \Bigg(K + \sum_{\substack{j=1 \\ j\neq i}}^{4} \phi_3^{(\sigma_3)j*}(K) + I_3^i - \delta K + \vartheta_3^y \Bigg) + m^i \Bigg] \\
&\qquad \times (1+r)^{-3} \Bigg\}, \quad \text{for } i \in N.
\end{aligned} \tag{39}$$

Performing the indicated maximization in (39) yields the game equilibrium strategies in stage 3 as:

$$\phi_3^{(\sigma_3)i*}(K) = \frac{q^i (1+r)^{-1}}{2c_3^{(\sigma_3)i}}, \quad \text{for } i \in N. \tag{40}$$

Substituting (40) into (39) yields:

$$\begin{aligned}&\left[A_3^{(\sigma_3)i}K + C_3^{(\sigma_3)i}\right]\\&= \alpha_3^{(\sigma_3)i}K - \frac{(q^i)^2(1+r)^{-2}}{4c_3^{(\sigma_3)i}}\\&\quad + \sum_{y=1}^{3}\gamma_3^y\left[q^i\left(K + \sum_{j=1}^{n}\frac{q^j(1+r)^{-1}}{2c_3^{(\sigma_3)j}} - \delta K + \vartheta_t^y\right) + m^i\right]\\&\quad \times (1+r)^{-1}, \quad \text{for } i \in N. \end{aligned} \tag{41}$$

Note that both sides of equation (41) are linear expressions of K. For (41) to hold it is required that:

$$\begin{aligned}A_3^{(\sigma_3)i} &= \alpha_3^{(\sigma_3)i} + q^i(1-\delta)(1+r)^{-1}, \quad \text{and}\\C_3^{(\sigma_3)i} &= -\frac{(q^i)^2(1+r)^{-2}}{4c_3^{(\sigma_3)i}}\\&\quad + \left[q^i\sum_{j=1}^{n}\frac{q^j(1+r)^{-1}}{2c_3^{(\sigma_3)j}} + q^i\varpi_3 + m^i\right](1+r)^{-1}, \quad \text{for } i \in N.\end{aligned} \tag{42}$$

Now we proceed to stage 2, using $V^{(\sigma_3)i}(3,K) = [A_3^{(\sigma_3)i}K + C_3^{(\sigma_3)i}](1+r)^{-2}$ with $A_3^{(\sigma_3)i}$ and $C_3^{(\sigma_3)i}$ given in (42), the conditions in equation (30) become

$$\begin{aligned}&\left[A_2^{(\sigma_2)i}K + C_2^{(\sigma_2)i}\right](1+r)^{-1}\\&= \max_{I_2^i}\Bigg\{\left[\alpha_2^{(\sigma_2)i}K - c_2^{(\sigma_2)i}\left(I_2^i\right)^2\right](1+r)^{-1}\\&\quad + \sum_{y=1}^{3}\gamma_2^y\sum_{\sigma_3=1}^{4}\lambda_3^{\sigma_3}\Bigg[A_3^{(\sigma_3)i}\Bigg(K + \sum_{\substack{j=1\\j\neq i}}^{n}\phi_2^{(\sigma_2)j*}(K) + I_2^i - \delta K + \vartheta_2^y\Bigg)\\&\quad + C_3^{(\sigma_3)i}\Bigg](1+r)^{-2}\Bigg\}, \quad \text{for } i \in N.\end{aligned} \tag{43}$$

Performing the indicated maximization in (43) yields the game equilibrium strategies in stage 2 as:

$$\phi_2^{(\sigma_2)i*}(K) = \sum_{\sigma_3=1}^{4}\lambda_3^{\sigma_3}\frac{A_3^{(\sigma_3)i}(1+r)^{-1}}{2c_2^{(\sigma_2)i}}, \quad \text{for } i \in N. \tag{44}$$

Substituting (44) into (43) yields:

$$\begin{aligned}
&\left[A_2^{(\sigma_2)i} K + C_2^{(\sigma_2)i}\right] \\
&= \alpha_2^{(\sigma_2)i} K - \frac{1}{4c_2^{(\sigma_2)i}} \left(\sum_{\sigma_3=1}^{4} \lambda_3^{\sigma_3} A_3^{(\sigma_3)i} (1+r)^{-1} \right)^2 \\
&\quad + \sum_{y=1}^{3} \gamma_2^y \sum_{\sigma_3=1}^{4} \lambda_3^{\sigma_3} \Bigg[A_3^{(\sigma_3)i} \left(K + \sum_{j=1}^{n} \sum_{\hat{\sigma}_3=1}^{4} \lambda_3^{\hat{\sigma}_3} \frac{A_3^{(\hat{\sigma}_3)j} (1+r)^{-1}}{2c_2^{(\sigma_2)j}} - \delta K + \vartheta_2^y \right) \\
&\quad + C_3^{(\sigma_3)i} \Bigg] (1+r)^{-1} \quad \text{for } i \in N.
\end{aligned} \tag{45}$$

Both sides of equation (45) are linear expressions of K. For (45) to hold it is required that:

$$\begin{aligned}
A_2^{(\sigma_2)i} &= \alpha_2^{(\sigma_2)i} + \sum_{\sigma_3=1}^{4} \lambda_3^{\sigma_3} A_3^{(\sigma_3)i} (1-\delta)(1+r)^{-1}, \quad \text{and} \\
C_2^{(\sigma_2)i} &= -\frac{1}{4c_2^{(\sigma_2)i}} \left(\sum_{\sigma_3=1}^{4} \lambda_3^{\sigma_3} A_3^{(\sigma_3)i} (1+r)^{-1} \right)^2 \\
&\quad + \sum_{\sigma_3=1}^{4} \lambda_3^{\sigma_3} \Bigg[A_3^{(\sigma_3)i} \left(\sum_{j=1}^{n} \sum_{\hat{\sigma}_3=1}^{4} \lambda_3^{\hat{\sigma}_3} \frac{A_3^{(\hat{\sigma}_3)j} (1+r)^{-1}}{2c_2^{(\sigma_2)j}} + \varpi_2 \right) \\
&\quad + C_3^{(\sigma_3)i} \Bigg] (1+r)^{-1}, \quad \text{for } i \in N.
\end{aligned} \tag{46}$$

Now we proceed to stage 1, using $V^{(\sigma_2)i}(2, K) = [A_2^{(\sigma_2)i} K + C_2^{(\sigma_2)i}](1+r)^{-1}$ with $A_2^{(\sigma_2)i}$ and $C_2^{(\sigma_2)i}$ given in (46), the conditions in equation (30) become

$$\begin{aligned}
&\left[A_1^{(\sigma_1)i} K + C_1^{(\sigma_1)i}\right] \\
&= \max_{I_1^i} \Bigg\{ \left[\alpha_1^{(\sigma_1)i} K - c_1^{(\sigma_1)i} \left(I_1^i\right)^2\right] \\
&\quad + \sum_{y=1}^{3} \gamma_1^y \sum_{\sigma_2=1}^{4} \lambda_2^{\sigma_2} \Bigg[A_2^{(\sigma_2)i} \left(K + \sum_{\substack{j=1 \\ j \neq i}}^{n} \phi_1^{(\sigma_1)j^*}(K) + I_1^i - \delta K + \vartheta_1^y \right) \\
&\quad + C_2^{(\sigma_2)i} \Bigg] (1+r)^{-1} \Bigg\}, \quad \text{for } i \in N.
\end{aligned} \tag{47}$$

Performing the indicated maximization in (47) yields the game equilibrium strategies in stage 1 as:

$$\phi_1^{(\sigma_1)i*}(K) = \sum_{\sigma_2=1}^{4} \lambda_2^{\sigma_2} \frac{A_2^{(\sigma_2)i}(1+r)^{-1}}{2c_1^{(\sigma_1)i}}, \quad \text{for } i \in N. \tag{48}$$

Substituting (48) into (47) yields:

$$\begin{aligned}
&\left[A_1^{(\sigma_1)i} K + C_1^{(\sigma_1)i}\right] \\
&= \alpha_1^{(\sigma_1)i} K - \frac{1}{4c_1^{(\sigma_1)i}} \left(\sum_{\sigma_2=1}^{4} \lambda_2^{\sigma_2} A_2^{(\sigma_2)i} (1+r)^{-1} \right)^2 \\
&\quad + \sum_{y=1}^{3} \gamma_1^y \sum_{\sigma_2=1}^{4} \lambda_2^{\sigma_2} \left[A_2^{(\sigma_2)i} \left(K + \sum_{j=1}^{n} \sum_{\hat{\sigma}_2=1}^{4} \lambda_2^{\hat{\sigma}_2} \frac{A_2^{(\hat{\sigma}_2)j}(1+r)^{-1}}{2c_1^{(\sigma_1)j}} - \delta K + \vartheta_1^y \right) \right. \\
&\quad \left. + C_2^{(\sigma_2)i} \right] (1+r)^{-1}, \quad \text{for } i \in N.
\end{aligned} \tag{49}$$

Both sides of equation (49) are linear expressions of K. For (49) to hold it is required that:

$$A_1^{(\sigma_1)i} = \alpha_1^{(\sigma_1)i} + \sum_{\sigma_2=1}^{4} \lambda_2^{\sigma_2} A_2^{(\sigma_2)i} (1-\delta)(1+r)^{-1}, \quad \text{and}$$

$$\begin{aligned}
C_1^{(\sigma_1)i} &= -\frac{1}{4c_1^{(\sigma_1)i}} \left(\sum_{\sigma_1=1}^{4} \lambda_2^{\sigma_2} A_2^{(\sigma_2)i} (1+r)^{-1} \right)^2 \\
&\quad + \sum_{\sigma_2=1}^{4} \lambda_2^{\sigma_2} \left[A_2^{(\sigma_2)i} \left(\sum_{j=1}^{n} \sum_{\hat{\sigma}_2=1}^{4} \lambda_2^{\hat{\sigma}_2} \frac{A_2^{(\hat{\sigma}_2)j}(1+r)^{-1}}{2c_1^{(\sigma_1)j}} + \varpi_1 \right) \right. \\
&\quad \left. + C_2^{(\sigma_2)i} \right] (1+r)^{-1}, \quad \text{for } i \in N.
\end{aligned}$$

Hence Proposition 1 follows. □

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