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HOW CHINESE TEACH MATHEMATICS

Perspectives from Insiders

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HOW CHINESE TEACH MATHEMATICS

Perspectives from Insiders

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Focusing on Chinese Mathematics Teaching, Teachers and Teacher Education: An Introduction

Since our first book, *How Chinese Learn Mathematics: Perspectives from Insiders* (hereafter called "*Insiders 1*"), was published in 2004, we as editors have received much encouraging feedback from educational researchers and practitioners, as well as general readers, from different countries. We have also gladly noticed that *Insiders 1* has been reviewed in leading research journals and other publication avenues in a number of countries including the US, UK, Germany, Singapore, and China. [1] In particular, soon after the publication of the book, we began to receive encouragement from our publisher and our advisors, as well as from some readers to publish a new book, or *Insiders 2*, with a particular focus on Chinese mathematics teaching.

Meanwhile, the last decade has continuously witnessed a growing interest in Chinese education and, in particular, Chinese mathematics education. This growth of interest is arguably related to the fact that, in

[1] For example, see Bishop, A. J. (2005). Review of "*How Chinese Learn Mathematics: Perspectives from Insiders*" [in Chinese]. *Journal of Mathematics Education, 14*(2), 100-102; Xie, L., & Zhao, X. (2006). How do we teach mathematics: Perspectives from *How Chinese Learn Mathematics*? [in Chinese]. *Hunan Education*, Issue No. 6, 44-45; Bishop, A. J. (2006). How Chinese learn mathematics perspectives from insiders (book review). *Asia Pacific Journal of Education*, *26*(1), 127-129; Star, J. R., & Chang, K. -L. (2008). Looking inside Chinese mathematics education: A review of "*How Chinese learn mathematics: Perspectives from insiders*". *Journal for Research in Mathematics Education*, *39*(2), 213-216; and Jones, K. (2008). Book review—Windows on mathematics education research in mainland China: A thematic review. *Research in Mathematics Education*, *10*(1), 107-113. Also see S. Lancaster's (2005) online review on the Mathematical Association of American website at http://www.maa.org/.

the Programme for International Student Assessment (PISA) 2009 and 2012 assessments, the Chinese participating students from Shanghai schools, were the very best performers.[2] As a matter of fact, the Shanghai students' average scores in mathematics were 600 and 613 in PISA 2009 and 2012 respectively, while the average scores of the second best performers, Singapore students, were 562 and 573, with the international average for each assessment being 500. Although there exist different interpretations and views about the top performance of Shanghai students, the gaps measured using the average scores in these two large-scale assessments between the Chinese students and their counterparts from other countries are too large to be ignored. It is also notable that, after PISA 2009, Tucker and his colleagues published a book entitled *Surpassing Shanghai: An Agenda for American Education Built on the World's Leading Systems* (Tucker, 2011),[3] and after PISA 2012, the UK government announced, in March 2014, its intention to recruit about 60 Shanghai mathematics teachers to work in English state schools to raise the standards of mathematics and close the gap between the two countries (Parton, 2014).[4] It seems apparent that the interest in Chinese mathematics education will continue for many years, and in terms of academic research, many issues about Chinese mathematics education and Chinese students' performance need to be thoroughly examined.

The present book, or *Insiders 2*, is not only a continuation of our first book, *Insiders 1*, which focused on the learning side of Chinese mathematics education, but also in a sense a response, from a research perspective, to the on-going interest and scholarly discourse about Chinese mathematics teaching and learning.

[2] PISA is organized by the Organisation for Economic Co-operation and Development (OECD) starting from 2000 on a 3-year cycle. PISA 2009 is the first time that students from the Chinese mainland took part in the assessment (see: www.oecd.org/pisa). So far Chinese students from the mainland have not participated in another international comparative study, the Trends in Mathematics and Science Study (TIMSS).
[3] Tucker, M. S. (Ed.). (2011). *Surpassing Shanghai: An agenda for American education built on the world's leading systems.* Cambridge, MA: Harvard Education Publishing Group.
[4] Paton, G. (2014, March 14). Chinese teachers sent into English schools to boost results. *The Daily Telegraph*. Retrieved from http://www.telegraph.co.uk

The focus of this book is on the teaching side of Chinese mathematics education. More specifically, it is about teaching, teachers, and teacher education and professional development relating to Chinese mathematics education.

The book is organized into three main sections. In Section 1, **Historical and Contemporary Perspectives**, we start with Chapter 1 by DAI and CHEUNG, which aims to provide readers with a broad and historical perspective on traditional mathematical teaching in ancient China by examining the key values, thoughts, and approaches as documented in ancient Chinese mathematics texts. In contrast, Chapter 2 by FAN, MIAO and MOK examines contemporary international research and presents an up-to-date review on how modern Chinese mathematics teachers teach and pursue their pre-service training and in-service professional development, in which the crucial role of the Teaching Research Group in schools and Teaching Research Office at different government levels is worth particular attention.

Section 2, **Understanding the Chinese Ways of Teaching Mathematics**, contains 12 chapters, investigating a wide range of issues at both the macro- and micro- levels on how Chinese mathematics teachers teach mathematics. These investigations were undertaken by different researchers in different regions.

Both Chapters 3 and 4 involve some new theoretical models and frameworks to analyze and understand the Chinese ways of teaching mathematics. Chapter 3 by HUANG Rongjin, MILLER and TZUR presents a hybrid model consisting of a tripartite theoretical lens and hence provides a fine-grained examination of learning opportunities created via Chinese classroom instruction, particularly the general features of teaching with variation. Chapter 4 by WANG, CAI and HWANG presents another careful analysis of a model Chinese mathematics lesson and explores the discourse strategies the teacher used to achieve instructional coherence and more generally the features of classroom instruction in China (mainland), for which the authors also put forward a framework for examining instructional coherence.

Chapters 5 through 10 provide readers with an in-depth look into how Chinese mathematics teachers teach a variety of specific mathematics topics. In Chapter 5, HUANG Hsin-Mei examines 12 instructional cases collected from elementary schools in Taiwan using videotaping and interviews and provides a portrait of how teachers conducted lessons and what they were concerned about when teaching length, area, and volume measurements, which the author intends to help in understanding such classroom practices. In Chapter 6, FANG looks at how an experienced secondary teacher in Shanghai explained student homework on geometric proofs and uncovers the hidden dimensions of mathematics teaching in Shanghai classrooms mediated through homework practice. The chapter sheds new light on the role and potential of homework in mathematics instruction.

Chapter 7 by LI Titus reports on primary students' ability in solving time interval problems in Macau, Hong Kong and the Netherlands, and reveals, through interviews and lesson observation, how time interval calculations are taught in each place. Chapter 8 by YANG Der-Ching, CHEN Pei-Chieh, TSAI Yi Fang and HSIEH reports how number sense was taught using interactive multimedia in a primary classroom in Taiwan and shows the differences in students' use of number sense strategies before and after the instruction. The results suggest that interactive multimedia can be an effective tool both in helping children develop number sense and in promoting children's motivation for learning. In Chapter 9, DING, JONES and ZHANG Dianzhou present a case study and analyze how an expert teacher in Shanghai used the "Shen Tou" ("permeation") method to teach theorems in geometry to an eighth-grade mathematics class, in which two key features of the instruction were identified: one is the complex learning support structure and the other the repetition and accumulation of practices of hierarchically-ordered skills and gradual understanding of the systematical connections of knowledge within the multiple-layered teaching procedures. In Chapter 10, HUANG Xingfeng, YANG and LI Shiqi examine three experienced Chinese teachers' teaching of the use of letters to represent numbers based on the new curriculum standards in four dimensions (strands), i.e., knowledge and skills, mathematical thinking, solving problem, and affect and attitude.

In contrast to the previous chapters, Chapters 11 to 14 focus more on general teaching approaches in Chinese mathematics classrooms. In

Chapter 11, XU and ZHU Guangtian look into two cases of project-based instruction, a relatively new development in Chinese mathematics teaching practices, and examine multiple aspects of students' engagement in the project-based classroom environment. In Chapter 12, LEE Yuan-Shun and LIN Fou-Lai investigate the teaching behaviors of Taiwanese mathematics teachers using a large-scale video survey of the fourth-grade classrooms across Taiwan, and reveal the general features of their mathematics teaching. Similarly, in Chapter 13, MA and ZHAO also examine the features of exemplary lessons in the Chinese mainland under the curriculum reform by studying 13 such lessons in elementary mathematics. The results show that these exemplary lessons not only practiced the advocated ideas of the new current reform, but also embodied some elements that might reflect the stable characteristics of Chinese mathematics classrooms. Finally, in Chapter 14, GU, YANG Yudong and HE present the Qingpu experiment, a very well-known mathematics teaching reform in the Chinese mainland, and examine its impact on the eighth-grade students' learning in mathematics.

Section 3, **Chinese Mathematics Teachers, Teacher Education and Teacher Professional Development**, comprises seven chapters, focusing on issues about Chinese mathematics teachers' knowledge, beliefs, and their professional development.

In Chapter 15, ZHANG Qiaoping and WONG first provide a review of the literature on how beliefs and knowledge influence mathematics teachers' teaching with a particular focus on studies conducted in the Chinese regions and then introduce readers to a series of five studies with different research methods to address how beliefs and knowledge affect the teaching of mathematics in the Chinese context. In Chapter 16, FAN, ZHU Yan and TANG investigate more than 30 mathematics master teachers in seven different regions of China through questionnaires and interviews, and examine the reasons behind the success of those master teachers in their acclaimed teaching careers. The results reveal that the master teachers valued internal (personal) factors more than external ones in their professional growth. In Chapter 17, CHEN Qian and LEUNG investigate three Chinese teachers' mathematics beliefs that were espoused and enacted in the context of a constructivism-oriented curriculum reform, and reveal the different beliefs of these teachers.

Focusing on teacher professional development, Chapter 18 by YUAN and LI Xuhui use a case study method to explore how "Same Content Different Designs" (SCDD, or *Tong Ke Yi Gou* in Chinese), which has become popular as a new form of teacher professional activity in the Chinese mainland, has an impact on the professional development of prospective mathematics teachers. They propose a model that characterizes the key components and stages of SCDD activities. In Chapter 19, JIN, LU and ZHONG examine Chinese mathematics teachers' perceptions of concept map, and their incorporation, after brief training, of concept map in mathematics teaching through lesson plans and practical teaching. Their study reveals the importance of operational training and professional development for teachers to adopt new tools and ideas in their teaching. In Chapter 20, LIN Pi-Jen and TSAI Wen-Huan present a study on how the use of research-based cases in a teacher training programme in Taiwan enhanced mathematics teachers' awareness of and abilities in maintaining high-level cognitive demands of mathematical tasks in classroom practice. The final chapter of this book, by LEU, CHAN and WONG, takes an in-depth look at the relationships between religious beliefs and teaching among mathematics teachers in the Chinese mainland, Taiwan and Hong Kong through a comprehensive review of the related literature and two empirical studies they conducted. It is a must read for any readers who are interested in this very specialized and under-researched topic.

It has taken quite a long time to reach the completion of this book since 2010 when we started calling for contributions to the book after having reached official agreement with the publisher in early 2010. For a project of this magnitude, it would be impossible without much support and help along the way from many people, and for this we wish to offer here our deep appreciation.

First, we would like to thank all the contributors who submitted their proposals and later initial manuscripts, and all the authors of accepted chapters, who we think have made great efforts in writing and revising their chapters.

Second, all the contributed manuscripts have gone through blind peer review by at least two and, in some cases, three or four colleagues, who are all university-based academics and/or hold doctoral degrees. We are

much indebted to all the reviewers for offering their generous help and time. The list of the reviewers is as follows:

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Third, we wish to express our sincere appreciation to our four advisors, ZHANG Dianzhou, LEE Peng Yee, LIN Fou-Lai, and GU Lingyuan for their advice and continuous support for this book.

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the completion of this book. We also wish to record our thanks to Manahel ALAFALEQ, CHEN Yangting, LU Jitan, ZHANG Ji, Angeline FONG, KWONG Lai Fun, Elizabeth LIE, Lionel PEREIRA-MENZODA, Ida MOK, HUA Qiong, WANY Yi and XIANG Kun for their help at different stages of work on the book.

This book intends to present another concerted effort from an international group of researchers from different parts of the world on the study of Chinese mathematics education. Nevertheless, it remains clear, or in a sense has even become clearer, to us that given the complex nature of teaching and learning, particularly in connection to the large variety of social, economic, cultural and even religious backgrounds of Chinese schools, students and teachers, many issues remain to be further studied in this area. Like our first book, *Insiders 1*, we hope that this present book, *Insiders 2*, will also make a meaningful contribution to the advancement of research in Chinese mathematics education, and hence more generally to that in the international mathematics education. We continue to welcome exchanges and feedback from colleagues and general readers as well.

FAN Lianghuo
WONG Ngai-Ying
CAI Jinfa
LI Shiqi

July 2014

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Section 1

HISTORICAL AND CONTEMPORARY PERSPECTIVES

Chapter 1

The Wisdom of Traditional Mathematical Teaching in China

DAI Qin CHEUNG Ka Luen

For the past 3,000 years mathematics education in China has developed its own traditions in which abundant and profound teaching thoughts have been accumulated and summarized. Throughout the history of mathematical education in China, there are many typical and inspiring teaching examples which fully display the wisdom of heuristic pedagogy, concise explanation of mathematical concepts with abundant practice, the emphasis of "practice makes perfect", and the idea of alternative solutions to the same mathematical problem. Based on the examples in the mathematics masterpieces in ancient China such as *The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*, *The Nine Chapters on the Mathematical Art*, *Mathematical Treatise in Nine Sections*, and *Yang Hui's Algorithms*, this chapter examines and discusses the traditional values, thoughts, and approaches used in the mathematical education in ancient China.

Keywords: traditional Chinese mathematics education, Chinese ancient mathematics, mathematics education thoughts

1. Understanding Traditional Chinese Mathematics Education

Mathematics education in traditional China has a long history of over 3,000 years during which a lot of outstanding mathematicians were cultivated. Due to geopolitical reasons, for most part of her history, China developed her own education system and philosophy during which influence from the West virtually played no direct role in the development. Scientific knowledge and technology from the Western world began to germinate in China only briefly in the 16th and 17th centuries, when Jesuits began their mission to the Far East (Katz, 2009). Such interaction was soon paused abruptly due to the isolation policy adopted by the newly-established Qing Dynasty (1644-1911). Nevertheless this is also the time when Chinese mathematics started to decline. As a result, we will use in this chapter the term "Traditional China" and "Ancient China" interchangeably[1] to refer to the country until around 17th century. During this long period Chinese mathematicians and educators have broken a number of records in world history. For instance, China has the record of establishing the first institute of higher education for mathematics in the world, called *Suanxue Guan* (the Computation Institute) in the Tang dynasty (618-907). Chinese mathematicians succeeded in approximating π correctly to 6 decimal places, a record which was not broken until almost a millennium later. They also succeeded in discovering numerous theorems and properties long before Western literature made the same discoveries (Wu, 1998). Hence, from a historical perspective it is worthwhile to study mathematics education in traditional China in order to glimpse at the success of Chinese mathematicians and educators. Though mathematics education in China nowadays is strongly affected by the West, the spirit and methodology created in traditional China still shed light on modern mathematics education. Thus, from an education perspective it is also important to investigate how traditional Chinese teachers taught mathematics, so as to learn from their teaching experience and even to apply some of their methodologies to contemporary classroom teaching.

In recent years, there are numerous papers and books focusing on various aspects and examples in mathematics education in ancient China. Mathematics classics such as *Zhou Bi Suan Jing* (*the Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*) and *Jiu Zhang Suan Shu* (*the Nine Chapters on the Mathematical Art*) have been translated into English so that more people can access this literature. Also, there are many papers and books discussing about the learning and

[1] This period of time is usually referred to as "Ancient and Medieval China" (see Katz, 2009, Ch. 7).

teaching of mathematics in ancient China (cf. Wei, 1987; Lam & Ang, 1992; Li & Qian, 1998; Tong et al., 2007; Dai & Matsumiya, 2011). In short, the discussion on mathematics education in ancient China is no longer limited to some typical cases, such as some stories about Chinese mathematicians, or sporadic discussion on mathematical problems like *Wu Bu Zhi Qi Shu* (the Chinese Remainder Theorem) and *Gou Gu Ding Li* (the Chinese version of Pythagorean theorem) (Wu, 1998; Guo & Li, 2010).

Croce (1866-1952), a well-known Italian historian, asserted that "however remote in time events there recounted may seem to be, the history in reality refers to present needs and present situations wherein those events vibrate" (Croce, 1949, p. 19). He also pointed out how historical works can bring inspirations in the present time: "[i]t often happens that the historical sense of a book is lifeless to us…until suddenly it springs to life through new experience gained out of the course of events and through new requirements born in us which have their counterpart in, and bear a more or less intimate resemblance to those of former times" (Croce, 1949, p. 18). In a similar fashion, the thought of "gaining new insights through reviewing old materials" has also been advocated and practiced since ancient times in China: Importance has been attached to reviewing known facts and past experience so as to acquire new knowledge and insights; On a broader sense, Chinese people even emphasize history as a means of reflection during the development of their nation.

In this chapter, we will basically ask two fundamental questions with regard to ancient Chinese mathematics education:

A. What is the philosophy and principles of mathematics education in ancient China?
B. How does the education philosophy affect the methodologies and pedagogies adopted by Chinese mathematics teachers at that time? And what are the features of mathematics education in ancient Chinese that result from this education philosophy?

We will try to give the answer to the first question in Section 2 while for the second question, we will go over a number of examples presented in various mathematics classics in ancient China so as to get a general picture of the mathematics education at that time.

2. Traditional Mathematics Education Thought in China

As a subject in the "*liu yi*" (six arts)[2], mathematics is an indispensable part of traditional Chinese education. Traditional mathematics education in China shares the same principles as all of the traditional education in China. In other words, the philosophy of traditional Chinese education is the soul of the traditional mathematics education in China. The traditional Chinese education is influenced heavily by Confucianism, Daoism, Mohism and Legalism. Since the official teaching of Confucianism started from the Han dynasty (206-220 BC), we can state that Confucianism is the most important underlying philosophy guiding the mathematics education in ancient China (see also: Wong et al., 2012). This is reflected by the emphasis on eternal themes such as "respecting teachers and esteeming the truth" and "teaching benefits teachers as well as students" in traditional mathematics education in China. In this regard, the first author of this chapter has expounded in the first chapter of his book (Dai, 2009) on mathematics education values of ancient China in great detail. Generally speaking, mathematics education in ancient China has the following philosophy and principle:

(a) The emphasis on the dominant role of students and teachers. This is in turn reflected by the following principles:

- Teaching in a way according to the ability and interest of the students;
- Emphasis of stimulation and inspiration on students;

(b) Emphasis of the equal importance of the thinking and learning processes.

(c) Inductive reasoning: Students are encouraged to understand the underlying mathematics principles through studying many examples, so that afterwards they can solve new yet similar questions themselves (Dai, 2009).

Firstly, the traditional mathematics education of China advocates both the dominant role of students and teachers in the teaching process. Since ancient China, teachers have been highly respected by the general public,

---

[2] During the Zhou Dynasty (ca 1122–256 BCE), students were required to master the *Liu yi* (six arts) in the ancient China. They are: Rites ( ), Music (乐), Archery ( ), Charioteering (御), Calligraphy ( ), and Mathematics ( ). These six arts have their roots in the Confucian philosophy (Tong et al., 2007).

enjoying a lofty social status in Chinese ethics. Here are some of the brilliant expositions of ancient Chinese philosophers in this aspect.

Record on Subject of Learning in *Book of Rites* (记・学记 ; see Legge, 1885):

> "When the proper reverence for the master is secured, the course which the master inculcates is regarded with honour. When the course is regarded with honour, the people know how to respect learning."

Da Lue in *Xun Zi* (子・大略 ; see Knoblock, 1994):

> "When a country is on the verge of a great florescence, it is certain to prize its teachers and give great importance to breadth of learning…When a country is on the verge of decay, then it is sure to show contempt for teachers and slight masters."

Taigong Family Education in Lost Ancient Books found in Mingsha Mountain (鸣 室 , 太 教 ; see Luo):

> "He who teaches me for one day is my father for life."

Revised Constitution of Liu Yang Mathematics School, written by TAN Sitong[3] (嗣 : 算学 增 程 ; see Tan):

> "As to the academic students, nothing is more important than respecting their teachers."

"Esteeming the truth" means that knowledge is presented as it is to the students, and such presentation cannot be realized unless the dominant role of students is guaranteed. However, some people misunderstand it as denying the principal status of students during teaching process. Being strict with students cannot be confused with denying their dominant roles in education.

On the aspect of teaching, Confucius emphasized teachers' devotion with an attitude of "the silent treasuring up of knowledge; learning without satiety" (*Shu Er* in *The Analects of Confucius*; see Legge, 1861) and argued that it's a teacher's fault to be lax in teaching, while the

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[3] Names of historical figures in China are translated using pinyin, starting with family name followed by first name, unless in some cases where the family names of some ancient people are uncertain. Family names are capitalized at their first appearance in this chapter.

importance of learning is analogous to that of gem-cutting: gems unwrought can do nothing useful. If a student cannot learn well, a teacher must be responsible for it to some extent. Hence, teachers view the leading function in teaching as their responsibility.

The dominant role of students in the education system in ancient China is realized by the requirement of teaching according to students' ability, characteristics and interest. Education is delivered from the perspective of the needs of the students. To achieve this, teachers should have a thorough understanding of the students themselves. Ji Kang Zi once asked Confucius whether some of his students could be suitable candidates for government officials, the latter could clearly replied that his students Zhong You was determined, while Zi Gong was smart and Ran Qiu was talented, and hence all of them would become good officials.

The dominant role of students is also realized by the emphasis of bringing stimulation and inspiration to students. Education is not merely rote-memorization of facts and rules; rather, it promotes understanding of these facts and rules in a way that students can apply them appropriately to new situations. To achieve this, students must actively think about the materials they learn, and teachers are here to help stimulate the process of active thinking. In particular, when students raise a question, the teacher should be careful of whether they should reply, and how far the teacher should tell to the students. In this regard, Confucius gave three general guidelines. He would advise not to answer students' queries unless (1) they are eager to learn; (2) they are anxious to explain the problems themselves and (3) they could understand the greater picture of the problem provided that they are given some hints about a corner of the problem itself. We will see in the later part of this section that some Chinese mathematicians agreed with these guidelines that they simply repeated these when they gave advice to readers in the commentaries.

The idea of "teaching benefits teachers as well as students" was introduced by Confucius over 2,000 years ago. It is the education philosophy which concisely highlighted the integration of both the dominant role of students and the leading function of teachers. In simple words, Confucius advocated that teaching and learning can enhance each other, that interaction can be achieved between teachers and students, and that improvement can be made on both the pedagogy and the self-learning of teachers. The idea of "teaching benefits both teachers and students" disapproves the dichotomy of "student-centered" or "teacher-centered" viewpoints, and encourages the balance and harmony between teaching and learning. It is reiterated in different manners in various teaching scenarios. For instance, "When I walk along with two others, they may serve me as my teachers" (*Shu Er* in *The Analects of Confucius*; see Legge, 1861) and "So pupils are not necessarily inferior to their teachers, nor are teachers better than their pupils. Some learn the truth earlier than the others, while different people have different specialties." (*On the Teacher,* written by *HAN Yu* (768-824)). The concept of "teaching benefits teachers as well as students" is always the backbone of the practice and development of traditional mathematics education in China.

On the aspect of learning, Confucius valued the combination of learning and thinking: "Learning without thought is labor lost; thought without learning is perilous" (*Wei Zheng* in *The Analects of Confucius*; see Legge, 1861). As an old Chinese saying goes, "it is the master who leads you to the door of a profession, but it is yourself who can train your own skills". All of these encourage the initiative of students. Above all, students should place equal emphasis on learning and thinking.

We should point out that other tradition philosophies like Daoism, Mohism and Legalism have also played interesting roles in the forming of above philosophy and principles as stated in (a)-(c) too. One is referred to Wong et al. (2012) and the references therein for more details.

The thought of "teaching benefits teachers as well as students" can be realized by the discussion and communication between teachers and students, which is similar to the Socratic method of elicitation teaching. Many typical teaching cases of mathematics in ancient China were done in the form of conversation and discussion. The following case from *Zhou Bi Suan Jing* (Cullen, 1966) will illustrate this point vividly.

> Long time ago, Rong Fang asked Chen Zi "Master, I have recently heard something about your Way ( ). Is it really true that your Way is able to comprehend the height and size of the sun, the [area] illuminated by its radiance, the amount of its daily motion, the figures for its greatest and least distances, the extent of human vision, the limits of the four poles, the lodges into which

the stars are ordered, and the length and breadth of heaven and earth?"

"It is true" said Chen Zi.

Rong Fang asked "Although I am not intelligent, Master, I would like you to favour me with an explanation. Can someone like me be taught this Way?"

Chen Zi replied "Yes. All these things can be attained to by mathematics. Your ability in mathematics is sufficient to understand such matters if you sincerely give reiterated thought to them."

At this Rong Fang returned home to think, but after several days he had been unable to understand, and going back to see Chen Zi he asked "I have thought about it without being able to understand. May I venture to enquire further?"

Chen Zi replied "You thought about it, but not to [the point of] maturity. This means you have not been able to grasp the method of surveying distances and rising to the heights, and so in mathematics you are unable to extend categories [i.e. unable to think analogically about similar problems]. This is a case of limited knowledge and insufficient spirit. Now amongst the methods [which are included in] the Way, it is those which are concisely worded but of broad application which are the most illuminating of the categories of understanding. If one asks about one category, and applies [this knowledge] to a myriad affairs, one is said to know the Way. Now what you are studying is mathematical methods, and this requires the use of your understanding. Nevertheless you are in difficulty, which shows that your understanding of the categories is [no more than] elementary. What makes it difficult to understand the methods of the Way is that when one has studied them, one [has to] worry about lack of breadth. Having attained breadth, one [has to] worry about lack of practice. Having attained practice, one [has to] worry about lack of ability to understand. Therefore one studies similar methods in comparison with each other, and one examines similar affairs in comparison with each other. This is what makes the difference between stupid and intelligent scholars, between the worthy and the unworthy. Therefore, it is the ability to distinguish categories in order to unite categories which is the substance of how the worthy one's scholarly patrimony is pure, and of how he applies himself to the practice of understanding. When one studies the same patrimony but cannot enter into the spirit of it, this indicates that the unworthy one lacks wisdom and is unable to apply himself to practice of the patrimony. So if you cannot apply yourself to the practice of mathematics, why should I confuse you with the Way? You must just think the matter out again."

Rong Fang went home again and considered the matter, but after several days he had been unable to understand, and going back to Chen Zi he asked "I have exerted my powers to the utmost, but my understanding does not go far enough, and my spirit is not adequate. I cannot reach understanding, I implore you to explain to me."

Chen Zi said "Sit down again and I will tell you." At this Rong Fang returned to his seat and repeated his request. Chen Zi explained to him […].

Just as ZHAO Shuang (around the 3rd century CE) remarked in his commentaries on *Zhou Bi Suan Jing* (Cullen, 1966), "[a teacher] should not open up the truth to one who is not eager to get knowledge, not help out anyone who is not anxious to explain himself…[When a teacher] has presented one corner of a subject to anyone, he should be able to learn the other three from it." This viewpoint is exactly what has been pointed out at *Shu Er* in *The Analects of Confucius* (see Legge, 1861) some centuries before Zhao, and is quite similar to LIU Hui's (225-295) ideas put forward in his commentaries in the chapters of *Su Mi* (Maize) and *Fang Cheng* (Equations) in *The Nine Chapters on the Mathematical Art* (Shen, Crossley, & Lun, 1999), which emphasize the importance of inductive reasoning in the mathematical application. As described in the above quoted dialogue between Chen Zi and Rong Fang, traditional mathematics education in China implicitly groups different mathematical problems into different "categories", with problems in the same category share some similar features and can be solved by the same set of mathematical techniques. A teacher not just instructs the students to rote-memorize the skills used in solving a particular example ("a corner of a subject"); instead, he should also inspire the students to internalize the mathematical tools introduced in that example so that when given another similar example in the same category, the students can modify and apply the learned tools to solve the new problem themselves.

Though the examples mentioned above are special ones, they all reflect the philosophy of Chinese mathematics pedagogies. The following three methodologies and pedagogies can be inferred: First, in the mathematics teaching of ancient China, heuristic teaching method was applied through conversation between teachers and students. This kind of interaction is similar to Socrates' "elicitation teaching theory" in his teaching of geometry as described in *Menon* by Plato of ancient Greece. Second, while Chen Zi advocated independent study, Rong Fang clarified the importance of reflection in mathematics learning. Third, Chen Zi didn't pass on his knowledge to Rong Fang directly until the latter pondered on it over and over again for a long time.

In brief, it is emphasized in *Zhou Bi Suan Jing* that in Chinese mathematics learning, inductive reasoning on similar problems, the drill and cultivation of thinking ability as well as some basic mathematical knowledge are all essential. The heuristic education in *Zhou Bi Suan Jing* is comparable to Socrates' "elicitation teaching theory" (Ferrari, 2000) and both of them play a positive role in mathematics education history in the East and West, then and now.

3. Common Features of Traditional Chinese Mathematics Education and Some Examples

Throughout the history of ancient China, education is provided by various schools and academies to teach young people the relevant skills and knowledge. Some of these institutions are state-run, while the others are privately-run. With the setup of "keju" (a state examination or recommendation system to select state officials) and "Guozijian" (national central institution for higher education) in the 7th century, education in China focuses more on preparing young candidates for serving the civil service administration (Siu, 2004; Tong et al., 2007). Amongst the various skills and knowledge delivered in these institutions, mathematical knowledge and techniques are also covered in the education system in ancient China as the sixth arts. Though there are abundant sources of mathematical texts throughout the long history of ancient China, the source for the general mathematics education is best summarized by *Suan Jing Shi Shu* (Ten Mathematical Manuals), a list of ten mathematics classics (refer to Table 2 below for the names of the ten texts). Amongst these ten textbooks, *Zhou Bi Suan Jing* and *The Nine Chapters on the Mathematical Art* are the first to appear and stay on the top of the list. Hence, we will firstly examine the writing of these two texts and the explanatory notes provided by famous ancient Chinese mathematicians to highlight some common features of the mathematics teaching in ancient China.

3.1 *Instructional Style in Traditional Chinese Mathematical Texts — Generality and Conciseness over Abstraction*

The presentation of logical deductions in Chinese mathematical texts is somewhat different from that in classical Western mathematical literature. Following the tradition of the *Elements* of Euclid, Western mathematicians almost always start with a list of definitions and axioms to verify some clearly stated theorems and propositions, providing the details of those necessary steps and lemmas explicitly (Heath, 1956). This is a process of abstraction. However, this practice is not followed in traditional Chinese mathematical texts and the axiomatic approach is not adopted at all. The treatment of mathematical deductions in ancient China invites a lot of misunderstanding, all of which are best summarized by Matteo Ricci as "[the Chinese] propose all kinds of propositions but without demonstration" (Ricci, 1953).

Nevertheless, traditional Chinese mathematicians do write their "general techniques or procedures" when presenting their solutions to problems (Chemla, 2003). Despite the absence of clearly defined axioms and definitions, the authors of various ancient mathematics texts in China still give sufficient details so that readers can follow and work them out. A typical situation is as follows: Very often the presentation starts with a *particular* problem (*wen*, in Chinese) being stated in words. Then an answer (*da*, in Chinese) is given immediately after that, and from time to time, when it is necessary, the technique or algorithm (*shu*, in Chinese) for solving the problem will be outlined. Very rare, some lines of very concise calculations (*cao*, in Chinese) are also given hinting how the answer is worked out, but usually there are still some gaps and details where readers should fill in themselves. One may say that traditional Chinese mathematicians and mathematics teachers usually provide just enough ingredients for the "backbone" and the students and readers should provide the "flesh" in between. Compared to the abstract axiomatic approach, the Chinese way of demonstration is more in line with the actual logical activities when one is confronted by an unsolved mathematical problem: very often one outlines the general "*shu*" of attacking the problem before going into the formality.

It should be noted that a traditional mathematical text in China is usually a collective work of many different mathematicians who lived in different centuries. Both *Zhou Bi Suan Jing* and *the Nine Chapters on the Mathematical Art* are examples. They are believed to be complied during the Han dynasty (Katz, 2009). The book *Zhou Bi Suan Jing* already existed around 200 BC but its original author is not certain. It is suspected that the book had been written much earlier and there is more than one author. The original version of *Zhou Bi Suan Jing* is very concise: It is simply a collection of mathematical problems together with the final answers (Cullen, 1966). *The Nine Chapters on the Mathematical Art* is the first Chinese treatise especially on mathematics, and in influence on the development of mathematics it is comparable only the *Elements* (Needham, 1959, p. 25). While Euclid's *Elements* is often considered to be the basis of the Western branch of mathematics, *Nine Chapters* is considered the cornerstone of its Chinese counterpart. It provides arithmetical rules focused on practical applications, compiled in *wen* (question) – *da* (answer) – *shu* (technique or algorithm) format (Shen et al., 1999). Later, mathematicians added comments and remarks on the two books and published their commented versions. These comments and remarks have several functions as a “teacher”: (1) to give explanatory notes to the original text; (2) to state, explain and demonstrate the mathematical ideas, techniques and algorithms used in the derivation and (3) to inspire the readers to “think outside the box” by integrating techniques and algorithms learnt from different problems. *Zhou Bi Suan Jing* has been commented by mathematicians/educators including Zhao Shuang, ZHEN Luan (535-566) and LI Chunfeng (602-670). *The Nine Chapters on the Mathematical Art* has been commented by Liu Hui and Li Chunfeng. Zhao and Liu lived in around the same era while Zhen and Li were three centuries later. This is indeed a good example of the accumulation of knowledge, especially in teaching which we are going to address later.

3.2 *Traditional Chinese Mathematical Teaching — Highly Real Life and Application Oriented*

Mathematical teaching in ancient China always echoes the various aspects and applications in the daily life, and traditional mathematics educators in China emphasize the applications of mathematics besides the logical deductions behind. This is reflected by two aspects: (A) the use of problem-oriented teaching approach, and (B) the use of *shu* (techniques or algorithms) in unifying mathematical problems.

(A) The use of problem-oriented teaching approaches

Traditional Chinese mathematics educators put more emphasis on the applications of mathematical techniques to real-life problems. For instance, though both ancient Western and Chinese made various discoveries in geometry, the former is more interested in the logical foundations behind all those geometrical facts and leads to the development of Euclid's *Elements* which starts from a handful of explicitly stated axioms and definitions, while the latter continues delving into various mathematical problems arising from daily life and tries to conclude some laws and formulas that are universally true or at least true in most of the cases that are carefully framed. An example is *The Nine Chapters on the Mathematical Art*. Like many other typical traditional Chinese mathematical texts, it extracted and investigated 246 mathematical problems that are frequently encountered in ancient China. These problems are grouped into the following nine chapters (and hence the name of the book):

(1) Survey of land (*fang tian*): mainly on the calculation of the area of farmlands of various shapes.
(2) Millet and rice (*su mi*): mainly on the exchange and pricing of commodities at different rates.
(3) Distribution by progressions (*shuai fen*): mainly on distribution of commodities in proportion.
(4) Diminishing breadth (*shao guang*): mainly on extractions of square and cubic roots, and on volume of spheres.
(5) Consultation on engineering works (*shang gong*): mainly on volumes of various shapes.
(6) Imperial taxation (*jun shu*)
(7) Excess and deficiency (*ying bu zu*)
(8) Calculating by tabulation (*fang cheng*): mainly on solving linear equations
(9) Right triangles (*gou gu*)

Indeed, from the title of each chapter one can infer that the original author of *The Nine Chapters on the Mathematical Art* tries to encompass every aspect in the real life, from the administrative to the economic side of the society, by looking into the 246 problems (Shen et al., 1999). The problem-oriented approach presented in traditional Chinese mathematics texts is actually the starting point of the inductive reasoning in ancient Chinese mathematics education as what we shall discuss below.

(B) The emphasis on shu in unifying mathematical problem—distinguish categories in order to unite categories

As mentioned above, mathematics education in ancient China is usually problem-oriented, and starts from problems in real applications. Knowledge from Chinese mathematicians were gradually accumulated by solving numerous typical questions, leading to some summaries and generalizations of mathematical skills, algorithms or techniques (*shu*) that are frequently used in tackling those problems. This kind of education philosophy is very typical in Liu Hui's commentaries on *The Nine Chapters on the Mathematical Art*. Indeed, Liu remarked on the preface by saying that "[t]hings are related to each other through logical reasoning so that like branches of a tree, diversified as they are, they nevertheless come out of a single trunk. If we elucidated by prose and illustrated by pictures, then we may be able to attain conciseness as well as understand the rest".

From the eyes of Liu Hui (and many other ancient Chinese mathematicians), albeit the fact that many different mathematics problems may have different "appearances", very often they can be solved by the same set of mathematical techniques, and hence can be "unified" or categorized into the same group of problems. As a result, these *shu* or techniques are stated, one by one after each of the 246 mathematical problems in *The Nine Chapters on the Mathematical Art* when Liu wrote his commentaries, and they are altogether 202 different *shu*. Indeed, Liu Hui places so much emphasis on these *shu* that he named most of them one by one, for instance, *ge yuan shu* for the technique of approximating π using inscribed *N*-gons and *chong cha shu* for the trigonometric method of finding heights and lengths (Shen et al., 1999). Besides, as pointed out in Chemla (2009) that mathematical problems in *The Nine Chapters on the Mathematical Art* were more than questions to be solved, they also played a key part in conducting proofs of correctness of algorithms.

It should be noted that these *shu* are important not only due to their essential role in solving those mathematical problems, but also due to the fact that *The Nine Chapters on the Mathematical Art* are categorized by the *shu*. For instance, problems in Chapters 2 and 3 in the book are mainly solved by techniques that deal with rates and ratios, while methods of finding volumes of various geometrical objects are discussed in Chapters 4 and 5. Furthermore, Liu also gives the insight that one should not be confined by any particular *shu*; instead, he also proposes "inter-disciplinary" techniques which are applicable to problems in different chapters. Amongst the various techniques he introduces, *jin you shu* (the rule of three) is the mathematical tool that Liu valued most (Tong et al., 2007). In modern terms, *jin you shu* is the technique for solving problems involving ratios. Liu Hui states about this *shu* as a universal techniques or algorithm (Shen et al., 1999). He proposes that every mathematical problem can be solved by *jin you shu* provided that one can find the relationship among various quantities in terms of ratios. Indeed, he succeeded in transforming a number of problems originally solved by other *shu* (including techniques for solving equations, excess and deficiency, taxation problems, etc.) into equivalent problems that are solvable by *jin you shu*. To a certain extent, *jin you shu* is one of the "trunks" which branch out to, and have applications in various mathematical problems in the daily life.

In the case of *Zhou Bi Suan Jing*, as we have seen from Chen Zi that he wanted his student to be able to apply the techniques learnt from one problem to a new yet similar problem of the "same category". The ultimate aim was to "distinguish categories in order to unite categories". In other words, ancient Chinese mathematicians and educators try to distinguish different problem types (according to the *shu* used), and attempt to find out some common structures underlying different problem categories. A comparison with the Western axiomatic approach can help clarify this. Western mathematics starts with a small number of definitions and axioms to derive numerous theorems and facts through deductive logic. Chinese mathematics, on the other hand, goes in the reverse direction by first considering (possibly) infinitely many problems and tries to reduce them to a handful of "categories" (Cullen, 2002). This is exactly the inductive reasoning reflected in ancient Chinese mathematics education!

3.3 *Teaching Style of Ancient Chinese Mathematicians and Mathematics Educators —Explaining Mathematical Ideas by Combining Logical Deduction and Intuitive Analysis*

(A) "Multiple approaches to the same problem" and "multiple proofs to the same theorem" —the accumulation of explanations of mathematics ideas and principles from commentary notes

Mathematics educators in ancient China emphasize the importance of "learning the other three corners of a subject when one is presented one corner of it". To achieve this goal, they tried to provide multiple proofs to theorems and multiple approaches to the same problem so as to help the students when they need to apply the same principle to other similar problems. In this part we are going to use the demonstrations of the Pythagoras' theorem shown in *Zhou Bi Suan Jing* and *The Nine Chapters on the Mathematical Art*, and their commentaries to illustrate this point.

The Pythagoras' theorem is stated clearly in *Zhou Bi Suan Jing* as follows: "If we require the oblique distance [from our position] to the sun, take [the distance to] the subsolar point as the base, and take the height of the sun as the altitude. Square both base and altitude, add them and take the square root, which gives the oblique distance to the sun." The ninth chapter *gou gu* (right triangles) in *The Nine Chapters on the Mathematical Art* is on the application of the Pythagoras' theorem. Although the original text of both books does not give the proof for this important theorem, later commentaries on both books have supplemented a number of different proofs to it. The different versions of proofs also serve as a purpose to illustrate that there are usually multiple approaches to the same problem. The role of a proof is not only for verification of theorems; it also serves the important function of enlightenment. Theorems with multiple proofs are everywhere in Chinese literature. Siu (1993) also gives another example of *Jiu Zhang Suan Shu* in which Liu Hui gives three different proofs for finding the diameter of an inscribed circle of a right-angled triangle whose three sides are known. Here are two different demonstrations to the Pythagoras' theorem given by Zhao Shuang for the *Zhou Bi Suan Jing* and Liu Hui for *The Nine Chapters on the Mathematical Art*.

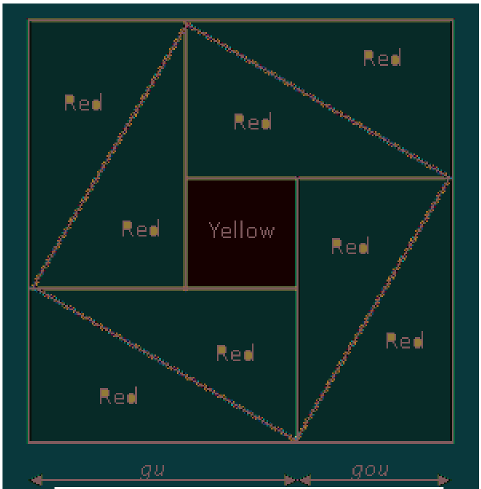

Figure 1. Gou Gu Yuan Fang Tu

(a) The demonstration by Zhao Shuang (Cullen, 1966). While making the commentaries on *Zhou Bi Suan Jing*, Zhao Shuang created **"by words"** the *Gou Gu Yuan Fang Tu*, a figure which makes use of the fact that area is invariant under a series of translations and rotations. Note that although the invariance of area under rigid motions (and many other "facts" and "definitions") is implicitly assumed without stating, the proof itself is still valid and such omission does not hamper students' understanding of the main proof. Knowing that the area of a rectangle is a product of its length and breadth, Zhao proves in the figure by rearranging and re-assembling triangles into rectangles, a principle summarized as *chu ru xiang bu yuanli* (out-in complementary principle: those in deficiency are compensated by those in excess). He then explained the logic as follows: Obtain a rectangle (*xian shi*) whose dimensions are the *gou* (the shorter length of a right-angled triangle) and *gu* (the longer length of a right-angled triangle). Its area is twice that of the right-angled triangle with base and height equal to *gou* and *gu*, and each of these triangles is painted red in the figure. Make four copies of the *xian shi* and arrange them so that there is a small square, painted yellow in the figure, in the middle. This small square has sides of length which is the difference of *gou* and *gu*. The resulting *Gou Gu Yuan Fang Tu* is depicted as in Figure 1.

In modern mathematical language, we can express the argument of Zhao Shuang as follows: Let a, b and c be the *gou*, *gu* and *xian* (hypotenuse) of the triangle, then the red triangle has an area of ½ ab, so that the four rectangles have a total area of $2ab$, and the middle yellow square has an area of $(b-a)^2$. As a result, calculate the area of the largest square in the figure in two different ways, we have $c^2 = 2ab + (b-a)^2 = a^2 + b^2$, i.e. $c^2 = a^2 + b^2$.

(b) The demonstration by Liu Hui (Shen et al., 1999). Liu Hui also applies the *chu ru xiang bu yuanli*, but with a different rearrangement of rectangles. He constructs three different squares whose sides are *gou, gu* and *xian* respectively also "**by words**", as shown in Figure 2[4]. Again, let *a*, *b* and *c* be the *gou, gu* and *xian* of the right-angled triangle. Then, by rearranging the triangles labeled I, II and III, he also concludes that the sum of the areas of two smaller squares is equal to the area of the largest square, i.e. $c^2 = a^2 + b^2$.

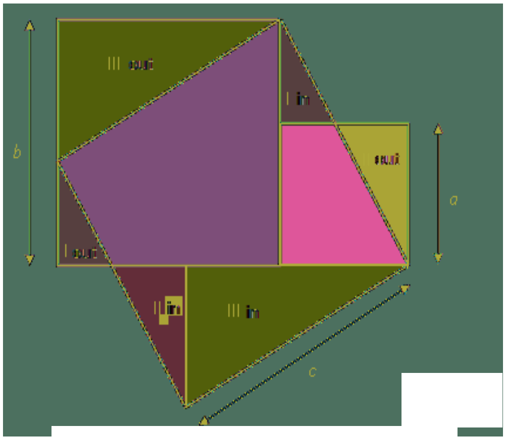

Figure 2. Diagram proof by Liu Hui

The two demonstrations of the Pythagoras' theorem given by Zhao Shuang and Liu Hui highlight the importance of applying known results (in this case the area of rectangles) to deduce some unknown facts (the Pythagoras' theorem). Offering two proofs to the same problem help reinforce the understanding of the problem by looking into it from different perspectives. Students can be inspired by this way to actively look for other alternative approaches when they encounter mathematical problems in the future.

The accumulation of commentaries also changed the writing style of the ancient Chinese mathematics. On top of the *wen– da – shu* or *wen– da – shu – cao* formats as discussed before, one more section *tu* (illustrations or diagrams to illustrate the ideas of the proof or approaches) for explanation purpose was often added later. For example, the commentary book for *The Nine Chapters on the Mathematical Art*, in

[4] Figure 2 was given by Li Huang (Qing dynasty) in his book: *Jiu Zhang Suan Shu Xi Cao Tu Shuo* (Detailed calculations and illustrations for the Nine Chapters on the Mathematical Art). It is really a "proof without words"!

which Figure 2 is taken from: *Jiu Zhang Suan Shu Xi Cao Tu Shuo* (Detailed calculations and illustrations for *The Nine Chapters on the Mathematical Art*) by Li Huang at the Qing dynasty (1644-1912) was written in this *wen – da – shu – cao – tu* format with emphasis on the last two explanatory sections!

Another feature of traditional Chinese mathematical teaching that is shown in the proof of Zhao Shuang is the use of colors for illustration and as a visual proof. Although chromolithography and other extensive color-printing techniques were not available until a millennium after *Zhou Bi Suan Jing* was first written, various portions in Figure 1 are still "colored" by labeling each portion with the name of the color. In this way students can realize which triangles should be matched and combined without actually coloring them. Such treatment is clever and makes any further explanation on the proof, if any, concise and highly understandable.

(B) The emphasis on the intuitive understanding of mathematical ideas and meanings—the use of manipulatives

Ancient Chinese mathematics education uses a number of teaching tools to enhance the intuitive understanding of mathematical concepts. Besides visual aids such as "coloring by label", shown in the discussion of demonstrations of Pythagoras' theorem as part of the teaching strategy suggested by Liu Hui: Elucidating by *ci*[5] and illustrating by *tu*, he also developed the ideas of finding the volumes of some complicated solids such as *yangma* (pyramids with a square base) and *bienao* (a tetrahedron of a particular type); by disintegrating the solids into some assembling blocks (see Straffin, 1998; Siu & Volkov, 1999; Shen, 2006; and Chemla, 2009 for details). Mathematics educators in ancient China also used the idea of *qi*, building blocks or building pieces, when explaining the method of finding the volumes of some solids. The idea is to decompose the solid into some visual building blocks or pieces (labeled by different colors) with their cross sectional areas or volumes are known. The way of decomposition depends on the shape of the solid. In some circumstances,

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[5] *Ci* (辞, prose or words)

one may consider slicing it (perpendicular to the axis of revolution) into thin disc like objects. If one can keep track of these pieces then one can keep track of the volume of the original solid. Roughly speaking, one can try to know more about the volume of a solid from a stack of *qi*'s (thin disc like pieces) generated from that solid.

A very good example of mathematics education in ancient China is the demonstration of finding the volume of a sphere. The whole story begins from a wrong statement in *Jiu Zhang Suan Shu*. There, it was stated that the volume V of a sphere is equal to 9/16 times the cube of its diameter D, i.e., $V = 9D^3/16$. Later, Liu Hui noticed that this is wrong and mentioned it in his commentaries. What is rare in mathematical literature at that time is that Liu also provided a proof by contradiction in order to justify why the original statement should be incorrect. Before correcting it, Liu first explained how the original author came up with the formula $V = 9D^3/16$: The original author used the approximation that $\pi = 3$. With this approximated value, the ratio of the area of a square to the area of its inscribed circle would be 4:3. Thus, it can be concluded that the ratio of the volume of a cube to that of its inscribed circular cylinder would also be 4:3. Liu proposed that it was likely that the original author assumed that the volume of this inscribed cylinder to that of its inscribed sphere would also be 4:3 which would lead to the original formula $V = 9D^3/16$. Liu pointed out that this assumption is wrong, and he explained the reasoning by considering an intermediate" solid which he called it as *mou he fang gai*, literally meaning two square umbrellas/lids piecing together, i.e. Steinmetz solid obtained by intersecting two circular cylinders of same radius at right angle (See Figure 3).

Liu noticed two facts: (1) the ratio of the cross section of *mou he fang gai* to that of its inscribed sphere at every altitude is 4:3. (2) Thus, with the Cavalieri's principle in mind (at that time he did not explicitly state this fact. Cavalieri's principle is discussed below), the ratio 4:3 should be the ratio of volume of *mou he fang gai* to that of its inscribed sphere. Since the *mou he fang gai* is contained in the cylinder, the ratio of volume of cylinder to that of its inscribed sphere should not be 4:3, as the original author may have assumed. As a result, this proof by contradiction disapproves the original statement that $V = 9D^3/16$.

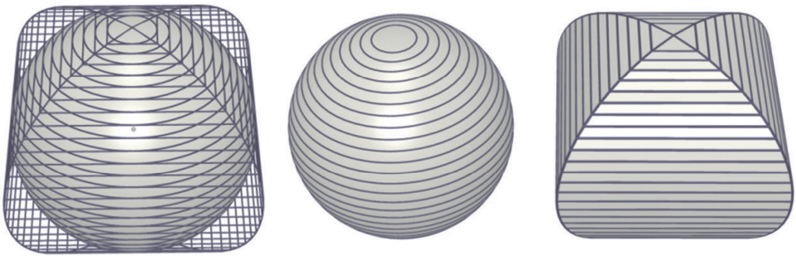

Figure 3. Mou he fang gai and its inscribed sphere. The cross sections for *mou he fang gai* and the sphere are squares and circles respectively, with a ratio of 4 : π.

If Liu could find out the volume of *mou he fang gai*, then he would have succeeded in getting the correct formula for the volume of a sphere. Unfortunately, Liu got stuck at finding the volume of *mou he fang gai* but he honestly stated that this problem was not yet solved, hoping that mathematicians in later generations could help find this missing piece of the puzzle. About 200 years later, ZU Geng (around the 6th century CE) elaborated on the idea of *qi* and stated a fact which is equivalent to Cavalieri's principle. In Zu Geng's word, the principle says that "if blocks (*qi*'s) are piled up to form volumes, and corresponding areas are equal, then the volumes cannot be unequal."[6] (see Figure 4). Zu then used his principle and applied it correctly to obtain the correct formula for the volume of *mou he fang gai*, and hence that of a sphere (see Shen, 2006 or Ke, 2007 for details). In other words, the quest for the formula for the volume of a sphere is a concerted effort of several generations of Chinese mathematicians.

In short, the story of finding the volume of a sphere highlights the following features in mathematics education in ancient China:

(1) Chinese mathematicians and mathematics educators use manipulatives such as *qi* and *mou he fang gai* to explain and visualize some geometrical principles and facts. This is

[6] The English translation is based on Wagner (1978).

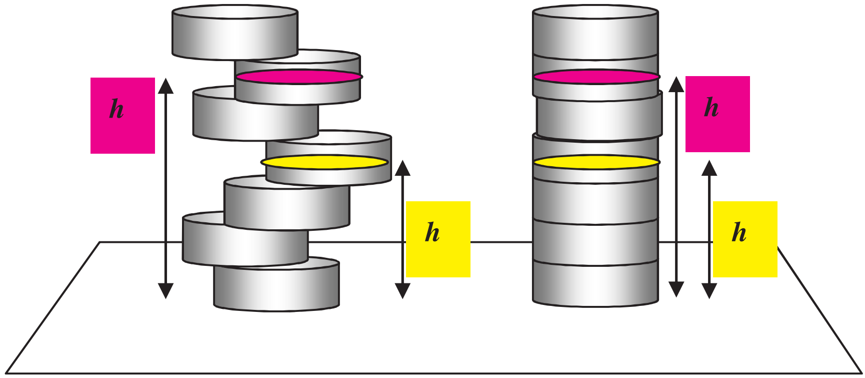

Figure 4. Illustration of Cavalieri's or Zu Geng's principle

well reflected in Lui Hui's commentaries to *The Nine Chapters on the Mathematical Art* about introducing the idea of *qi*: "Speech cannot exhaust the **meaning** (*yi*, in Chinese), hence to dissect/analyze (*jie,* in Chinese) this (volume), we should use *qi*; so that we can get to **understand**"(Shen et al., 1999).

(2) They do not hesitate in stating what they do not understand. Instead, they have strong reservation making statements which they are unsure of, as this would bring confusion and misunderstanding to students and readers.

(3) Though ancient China did not place as much emphasis on the development of a logical system for proof as in ancient Greece, Chinese mathematics educators could still manage to demonstrate their logical argument, say proof by contradiction, to their students, and such demonstration was rare in all known mathematics literature, both East and West, at that time.

(4) The development of Chinese mathematics shows a clear inheritance from generations to generations. It is a common heritage to all mathematics learners in ancient China, with knowledge being accumulated and passed on by the contribution from each generation.

WU Wen-Tsun pointed out that in the instructional strategy proposed by Liu Hui: Elucidating by *ci* and illustrating by *tu*, *ci* should be interpreted as logic, while *tu* should be understood as graphical intuition. The strategy should mean: combining logical deduction and intuitive analysis to deduce the truth of the mathematical conclusions (Wu, 1988, Book 1, p. 101). This is exactly what we have seen from the above examples (in both (A) and (B) of 3.2) on how mathematicians in ancient China managed to explain ingeniously about why mathematical facts and principles were true!

(C) The use of mathematical games in mathematics education to enhance the learning and teaching results

Ancient Chinese mathematics educators do not only focus on the applicability of mathematical skills in daily life, but also place great emphasis on how to make mathematical teaching interesting to the students. From time to time, they incorporated various mathematical games into the mathematics curriculum so that students can learn through games, develop interest in mathematics and stimulate their mathematical thinking effectively.

This reflects that mathematics teaching in ancient China is always student-oriented and is delivered in a way according to the attitude and ability of the students. Amongst the various mathematical games introduced in ancient China, tangram (known as *qi qiao ban*, in Chinese) and magic squares are the most accessible to Chinese children at all social classes.

Tangram was first invented in the Song dynasty (960-1279). As a dissection puzzle, it consists of seven different polygonal shapes. The objective is to find a specific shape given only its outline. For instance, shapes in Figure 5 can be obtained by rearranging the tangram. Through playing tangram, children can experience

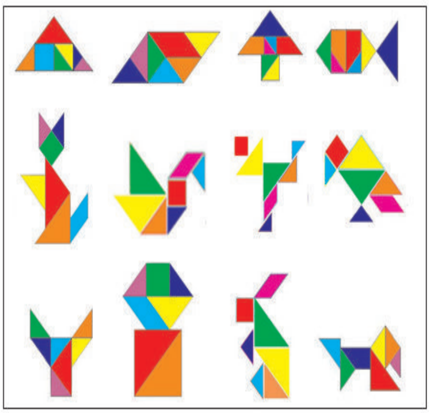

Figure 5. Tangram shapes

assembling and dissembling polygons and grasp the concept of different geometric shapes.

The history of magic squares in ancient China can be dated back to a much earlier time. It is said that the first magic square was obtained by a semi-mythical emperor *Yu* ( ) around the third millennium BC (Joseph, 2011, p. 208). The legend says that he obtained two gifts from a magical dragon-horse, one is called *he tu* which is a crucifix array of numbers 1 to 10 so that except the central 5 and 10 both the odd and even sequences of numbers add up to 20, and the other is called *luo shu*, a regular magic square of order 3. The first general discussion of magic squares, or luo shu, appears in 1275 when YANG Hui (1238-1298) wrote his *Xu Gu Zhai Qi Suan Fa* (Continuation of Ancient Mathematical Methods for Elucidating the Strange Properties of Numbers). In this book he not only demonstrates some magic squares of order up to 10, but also gives explicitly the algorithms for constructing luo shu of orders 3, 4 and 5 so that students can follow and elaborate on it. We have more to say about the role of algorithms in ancient Chinese mathematics education in the next section.

3.4 *Procedural Approach in Ancient Chinese Mathematics Education*

Mathematics education in ancient China is highly procedural. This is reflected by four different aspects: (A) a well-defined mathematics curriculum emphasized on procedures; (B) algorithmic or mechanical approach to mathematical problems; (C) The development and the teaching of *rod calculus* and (D) the extensive use of mnemonic poets and rhymes in teaching.

(A) Mathematics curriculum in ancient China emphasized procedures

Since the education in ancient China is to serve mainly for the recruitment of administrative staff, and the examination system is the main way to select possible candidates, a well-defined curriculum is necessary to help shape the materials to be covered. As mathematics is also examined in some of the selection processes, mathematics educators in ancient China also developed some curricula for mathematics education. During the long history of China before the Western science began to influence the scientific development of China in the 17th century, mathematics curriculum was slowly developed, and modified to suit the needs of the society and to reflect the latest collection of mathematics literature when new texts were written. For details about the change of mathematics curriculum in China, one can refer to Tong et al. (2007).

Table 1. Yang Hui's suggestion of time allocation on learning basic arithmetic (multiplication and division) skills

Stage	Skills to learn	Time required
I	Multiplication:	
	● Examples and placing digits, with exercises	1 day
	● Revision of questions on multiplication of 1- to 6-digit numerals, with exercises	5 days
II	Division:	
	● Examples and placing digits, with exercises	1 day
	● Revision of questions on divison of 1- to 6-digit numerals, with exercise	half a month
III	Further practice: Students should study two books: *Wu Cao*, *Ying Yong Suan Fa* and do two to three exercises in those books each day. Students can also check the 13 questions in Chapter 1 of *Xiang Jie Suan Fa*.	Less than two months
IV	Various shortcuts in multiplication and division:	
	● Study *jia* (addition literally, but here it actually means multiplication[7]) method and placing digits, with exercises	1 day
	● Revision of *jia* method	3 days
	● Study *jian* (subtraction literally, but here it actually means division) method and placing digits, with exercises	1 day
	● Revision of *jian* method	5 days
	● Study *jiu gui* (tables of division). Yang also commented that one needs 5 to 7 days to recite those 44 sentences, but can be shortened to 1 day if one carefully studies the commentaries on *jiu gui* in *Xiang Jie Suan Fa*.	1 to 7 days depending on approaches
	● Study *qiu yi* (this is a shortcut which makes multiplication and division easier by making the first digit of the multiplier/divisor to be 1 through suitable multiplication)	Revision takes 1 day

[7] The *jia* method makes use of the distributive property of multiplication: For instance, when computing 11x29, one can move 29 by one digit forward (hence 290) and add 29, i.e. 11x29=10x29+1x29=290+29. The principle of jian method is similar.

Here we will concentrate on the example called *Xi Suan Gang Mu* (A General Outline of Mathematical Studies), where Yang Hui gave a very detailed study plan of mathematics, and *Suan Jing Shi Shu* (Ten Mathematical Manuals), a curriculum for higher education of mathematics suggested by Li Chunfeng.

Xi Suan Gang Mu is the title of the preface to the first chapter of a book called *Cheng Chu Tong Bian Ben Mo* (Alpha and Omega of Variations on Multiplication and Division). In the preface, Yang gave very detailed procedures (a plan!) of studying basic arithmetic. Besides the topics and skills to learn, he also suggested a study schedule so that students can follow and check their progress and moreover, some other mathematics texts so that students can refer to for further practice. Some of the suggestions by Yang is summarized in Table 1.

It should be noted that though basic arithmetic is indeed very "basic" in modern mathematics, it was not so at the time when Yang Hui wrote his *Xi Suan Gang Mu*; instead, arithmetic used in the daily life and commerce at that time can be very laborious and difficult. Hence, his study plan is very crucial for those beginners in mathematics.

Besides study schedule on a particular theme, there is also a well-defined list of mathematics textbooks for students who would like to sit for the state examination. A typical example is *Suan Jing Shi Shu*

Table 2. Li Chunfeng's suggestion on study scheme for sitting for the state examination of mathematics in Tang dynasty

Text	Translated Meaning of Text	Time Spent on Studying the Text
Zhou Bi Suan Jing	The Arithmetical Classic of the Gnomon and the Circular Paths	1 year
Wu Jing Suan Shu	Arithmetic in the Five Classics	
Jiu Zhang Suan Shu	Nine Chapters on the Mathematical Art	3 years
Hai Dao Suan Jing	Sea Island Mathematical Manual	
Wu Cao Suan Jing	Mathematical Manual of the Five Government Departments	1 year
Sun Zi Suan Jing	Master Sun's Mathematical Manual	
Xia Hou Yang Suan Jing	Xia Hou Yang's Mathematical Manual	1 year
Zhang Qiu Jian Suan Jing	Zhang Qiu Jian's Mathematical Manual	1 year
Qi Gu Suan Jing	Continuation of Ancient Mathematics	3 years
Zhui Shu	Art of Mending	4 years

suggested by Li Chunfeng. According to *Xin Tang Shu* (The New History of the Tang Dynasty) and *Tang Liu Dian* (The Six Codes of the Tang Dynasty), there is even a breakdown of a study schedule (procedures for the studies) telling how one should spend on each text (see Table 2).

(B) Algorithmic/mechanical approach to mathematical problems

As mentioned before, mathematical techniques and algorithms, or *shu*, play an important role in ancient mathematics teaching in China. In addition to that, we can also notice that these *shu* are usually algorithms. Very often Chinese mathematicians study a mathematical problem, devise a mathematical model for this problem, and derive an algorithm for this model. Then next time when they come across another problem which fits the model, they can directly apply the algorithm to find the final answer quickly. Take *qi tong shu* as an example. It is a technique introduced by Liu Hui to handle addition of two fractions with different denominators, say a/b and c/d. It states that the sum of two such fractions is simply a fraction $(ad + bc)/bd$. So this is the algorithm for the mathematical model in which we need to sum two fractions. Liu then discussed the problem: Given that a wild duck takes 7 days to travel from South Sea to North Sea, and a goose takes 9 days to travel in the reverse direction. Suppose also that both the wild duck and goose starts to fly at the same time, when will they meet each other?

To solve this goose and duck problem, Liu suggested that one can rephrase the given information as in one day's time, the wild duck and the goose finish 1/7 and 1/9 of their respective journey. He then applied the *qi tong shu* by rewriting the two fractions as 9/63 and 7/63. So in a period of 63 days, they should fly 9 + 7 = 16 times. In other words, it takes 63/16 days to meet each other. In other words, he noticed that this goose and duck problem fits into the mathematical model where he can apply *qi tong shu*.

By algorithm we mean it is procedural, step-by-step, consistent and mechanical, provided that the right mathematical model (and hence the corresponding right algorithm) is applied to the given problem. Indeed, many of the algorithms in various Chinese mathematics texts can be transformed easily into computer programmes. One such example is the method of finding the highest common factor of two integers, called *geng xiang jian sun shu*, described in *Jiu Zhang Suan Shu* as a method to simplify a fraction. It is similar to the Euclidean algorithm, as stated in Proposition 1 of Book VII in Euclid's *Elements* (Heath, 1956), consists of the following steps.

Step 1: If both numbers are even, divide each of them by two; otherwise, go to Step 2.
Step 2: Subtract the smaller number q from the larger number Q to obtain the difference r. If r is zero, stop and r is the common factor; otherwise, go to Step 3.
Step 3: Repeat Step 2 by applying subtraction once between r and q.

In essence, this method is almost the same as the Euclidean algorithm, with the exception that the latter adopts a division approach rather than subtractions, making it faster than the Chinese version (Guo & Li, 2010).

Another typical example which displays mechanicalization is the *Yang Hui* triangle (Figure 6). This is the Chinese version of the Pascal triangle for binomial coefficients. *Yang Hui* clearly states that the numbers in the triangle correspond to the coefficients of the binomial expansion of $(x + a)^n$, where a is a constant, explains how to write down the terms in the expansion, and explicitly gives the famous algorithm (Figure 7):

$$C^n{}_r + C^n{}_{r-1} = C^{n+1}{}_r$$

This discovery predates the first European documentation of the same result (by Petrus Apianus, 1495-1522) by almost five centuries.

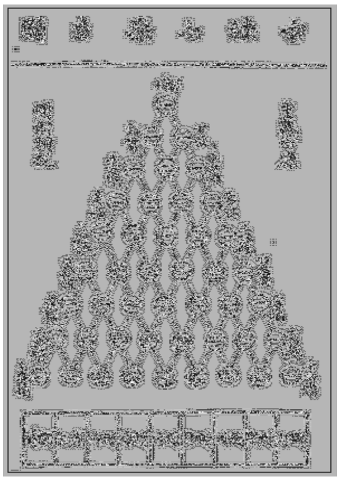

Figure 6. Yang Hui triangle as described by Zhu Shijie ( ) in his book *Si Yuan Yu Jian* (玉).

Aided with the geometrical insight, ancient Chinese mathematicians

$(x+a)^0 =$	1	1
$(x+a)^1 =$	$x+a$	1 1
$(x+a)^2 =$	$x^2+2ax+a^2$	1 2 1
$(x+a)^3 =$	$x^3+3ax^2+3a^2x+a^3$	1 3 3 1
$(x+a)^4 =$	$x^4+4ax^3+6a^2x^2+4a^3x+a^4$	1 4 6 4 1
$(x+a)^5 =$	$x^5+5ax^4+10a^2x^3+10a^3x^2+5a^4x+a^5$	1 5 10 10 5 1
$(x+a)^6 =$	$x^6+6ax^5+15a^2x^4+20a^3x^3+15a^4x^2+6a^5x+a^6$	1 6 15 20 15 6 1

Figure 7. Binomial Expansion as suggested by the Yang Hui triangle

went one step further by developing an iterative approach of finding the n-th root of a number. Here we will discuss how ancient Chinese find out the square root of a number through an iterative algorithm. Suppose we are given a natural number A and would like to find out its positive square root x so that $x^2 = A$. Liu Hui gave the following detailed procedure (rephrased using modern terminology):

Step 1: The number of digits of the root is the smallest integer greater than or equal to the number of digits of A divided by 2. If A is 1- or 2-digit number its root must be single-digit, and if A is 3- or 4-digit number its root must be of a size of ten.

Step 2: Estimate the first digit a_1 of the root x so that a_1 so that $A - 10^{2n}a_1^2 < 10^{2n}a_1^2$ ($< A$). This estimate is obtained when A is divided by $10^{2n}a_1$ the quotient is a_1 and the remainder is smaller than $10^{2n}a_1^2$. Yang also explained that this is equivalent to extracting of a square of side $10^n a_1$ from a square of area A.

Step 3: Estimate the second digit a_2 (if any) of the root x so that $A - (10^n a_1+10^{n-1}a_2)^2 < (2(10^{2n-1})a_1+10^{2n-2}a_2)a_2$. This estimate is obtained when $A - 10^{2n}a_1^2$ is divided by $2(10^{2n-1})a_1+10^{2n-2}a_2$ the quotient is a_2 and the remainder is smaller than $(2(10^{2n-1})a_1+10^{2n-2}a_2)a_2$. Yang also explained that this is equivalent to extracting a square of side $10^{n-1}a_2$ from a region obtained after deleting the square of side $10^n a_1$ and two rectangles each of which has length $10^n a_1$ and breadth $10^{n-1}a_2$.

Step 4: If the remainder in Step 2 or 3 is zero, then the iteration stops. Otherwise, proceed further on with the geometrical picture (Figure 8) in mind.

With the experience of finding square roots, Liu Hui proceeded to consider a similar approach of finding cubic roots iteratively. The generalization is a natural extension: This time, he extracted cubes and cuboids, instead of squares and rectangles from a cube of but volume A. We will not go into details; readers can refer to Guo (1991) or Joseph (2011). Finding n-th root in general, however, requires the knowledge of binomial expansion (i.e. *Yang Hui* triangle) and is usually tedious for higher orders. A more efficient algorithm is proposed by Jia Xian, which avoids finding the binomial terms of higher orders by iterative additions and multiplications. Jia Xian even wrote down how he obtained the 4th root of 1336336 using his method. In modern terms, it means that we want to find x such that $x^4 - 1336336 = 0$. We first make a guess on x, say a. Then all we need to do is to substitute x by $y + a$ to get $(y + a)^4 - 1336336 = 0$.

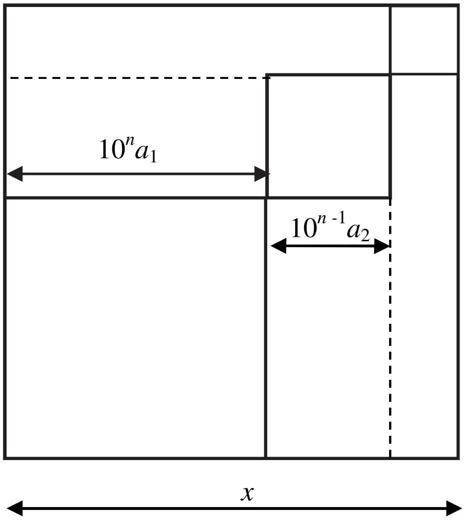

Figure 8. Finding the side x of a square whose area is given.

However, instead of expanding $(y + a)^4$ using binomial terms such as $6y^2a^2$, Jia noticed that

$$(y + a)^4 = a^4 + (4a^3 + (6a^2 + (4a + y)y)y)y,$$

and did the following (the first guess is $a = 30$):

1	+	0	+	0	+	0	–	1336336	30
	+	30	+	900	+	27000	+	810000	
1	+	30	+	900	+	27000	–	526336	
	+	30	+	1800	+	81000			
1	+	60	+	2700	+	108000			
	+	30	+	2700					
1	+	90	+	5400					
	+	30							
1	+	120							
1	+	120	+	5400	+	108000	–	526336	4
	+	4	+	496	+	23584	+	526336	
1	+	124	+	5896	+	131584	+	0	

Hence, the answer is 34. This method is exactly the same as the Horner's method discovered by William George Horner (1786-1837) and Paolo Raffini (1765-1822) 600 years later (Wang & Needham, 1955).

The interested readers can refer to Wu (1988), Guo & Li (2010) and Tong et al. (2007) for more discussions about the algorithmization /mechanicalization of mathematics in ancient China.

(C) The development and the teaching of rod calculus

In modern language, if algorithm is the software part of a computation process, then the ancient Chinese mathematicians also invented *suan chou* (counting rods) as the hardware part to support their computations. *Suan chou* was used to represent numbers and fractions for the purpose of counting and arithmetic at the very beginning (see Joseph, 2011, pp. 198-206; Needham, 1959). Later, with different color, different labeling and arrangements in different relative positions, *suan chou* could also be used to perform complicated calculations that can involve negative numbers (Needham, 1959). However, ancient Chinese did not stop there, they proceed to use *suan chou* further to develop "rod calculus" or rod calculation to perform formula calculations. Ancient Chinese seldom use "pen calculations", and they did not have a system of mathematical symbols for the formula calculations. To perform such task, they rely on the sophisticated operations with *suan chou*. So, rod calculus could be regarded as a symbolic computation system for ancient Chinese. ZHU Shijie, one of the most famous Chinese mathematicians in Yuan dynasty (1271-1368) has developed rod calculus to include polynomial equations of two to four unknowns (Wu, 1988; Li, 2007). "Rod calculus", the highly procedural operational system, was widely taught by mathematics teachers to students in ancient China till the Ming dynasty (1368-1644). To facilitate even more effective performing of arithmetic operations for application and commercial purposes, Chinese abacus was invented and further developed later. A series of poems and mnemonics were also written, summarizing the steps when using the abacus, they were well recorded in many ancient Chinese mathematics texts (Guo & Li, 2010).

(D) The use of mnemonics in Chinese mathematics teaching

It is commonly believed by ancient Chinese that firm rote-memorization is one way that can lead to understanding of concepts. To help facilitate memorization of some hard-facts, procedures and algorithms, and to make the process of memorizing more interesting, ancient Chinese people developed various mnemonics including simple poems and rhymes. The same is true in the mathematical education in ancient China. Chinese poems and rhymes usually have tail-rhymes and equal number of words (and hence syllables, since Chinese language is monosyllabic), which make them highly rhythmic. The introduction of Chinese poets and rhymes in the mathematical education in ancient China is an organic synergy between Chinese art and mathematics. Many mathematical problems and their solutions that appear in ancient Chinese texts are written in the form of poems and rhymes. The first mathematical poem in ancient China that is known to date appears in the preface of *Zhou Bi Suan Jing*:

平
偃
覆

圆

Translated in English, the above verse reads: "The level trysquare is used to set lines true. The supine trysquare is used to sight on heights. The inverted trysquare is used to plumb depths. The recumbent trysquare is used to find distances. The rotated trysquare is used to make circles, and joined trysquares are used to make square" (Cullen, 1966). The first four lines of the verse explain the method of surveying using the idea of similar triangles, while the last two lines give the general construction of circles and squares. In modern terminology,

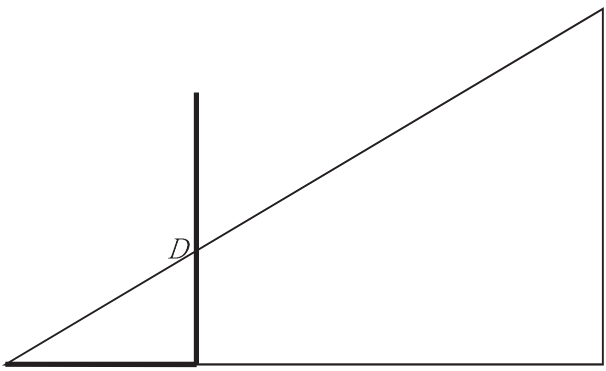

Figure 9. Diagram of trysquare

the surveying is done in the following way as shown in Figure 9: Suppose we put the trysquare on the ground so that its side AC coincides with the horizon AE. The side BC should be perpendicular to AC. Now look at an object F (say, the sun, as the text assumes the distance to the sun is known) from point A and record the point D on BC so that A, D and F are collinear. Then, as $\Delta ACD \backsim \Delta AEF$, $\frac{AC}{CD} = \frac{AE}{EF}$, and hence the height $EF = \frac{CD \times AE}{AC}$.

Another famous application of poets for educational purpose occurs in the case of *da yan qiu yi shu*. Originally proposed by QIN Jiushao 1208-1261) in 1247 in his book *Shu Shu Jiu Zhang*, *da yan qiu yi shu* is a technique for solving systems of linear congruence equations. This serves to answer an old question asked in *Sun Zi Suan Jing* at the 4th century: "Given a number whose reminders are 2, 3 and 2 after division by 3, 5 and 7 respectively, what is that number?" Qin himself gave a complete solution to this kind of questions. Later in 1592, CHENG Dawei (1533-1606) in *Suan Fa Tong Zong* (*Systematic Treatise on Algorithm*) summarized the technique into the following poem (Tong et al., 2007):

十	(It is rare to find a person of 70 years of age amongst a group of three)
花 枝	(There are 21 plum flowers on 5 trees)
	(It takes half a month (i.e. 15 days) to gather 7 masters)
零 知	(The answer can be obtained after division by 105)

In modern terminology, it explains Qin's algorithm as follows: Find out the smallest multiple of 35 (which is the LCM, least common multiple, of 5 and 7) so that its reminder after division by 3 is 1. In this case it is 70. Then find out the smallest multiple of 21 (the LCM of 3 and 7) whose reminder after division by 5 is 1. This time we get 21. Do the same for the smallest multiple of 15 (the LCM of 3 and 5), and we obtain 15. Now,

$$70 \times 2 + 21 \times 3 + 15 \times 2 = 233$$

and the division by 105 will give the final answer, which is 23.

4. Conclusion and Prospects

We have seen from the above that ancient Chinese officials adopted the Confucianism as the major working philosophy and took a practical approach in the development of mathematics and mathematics education. Consequently, as suggested by many Chinese scholars that (i) putting emphasis on calculation, (ii) making extensive use of calculation tool (rod calculus), (iii) building an algorithmic system, and (iv) "hiding the theory [or principle, *li*] in calculation" (*yu li yu suan*, in Chinese) are the main features of the mathematics development along this approach in ancient China (Wu, 1998; Li, 2007; Guo & Li, 2010). Mathematics was used to solve problems encountered by the highly civilized society of ancient China, and the traditional Chinese mathematical teaching is highly life and application oriented. As driven by this working philosophy and the development of mathematics, traditional Chinese mathematical texts (canons) were written in such a way that emphasis was put on effective calculations and general algorithms rather than on building mathematical theories. Also, ancient Chinese officials took a procedural approach for their mathematics education as what we have observed in Section 3.4 so as to increase the teaching effectiveness of the "official system" and at the same time to prepare students to adapt to the mathematical as well as the social development like sitting for the *keju* examination to become a government official at that time.

However, as we all know that there is always a mathematical theory behind an algorithm. Educators cannot teach students mathematics without explaining mathematics principles to them. Actually, teachers of ancient China, no matter from state-run or privately-run institutions, would provide oral explanations on mathematics principles to students in their teaching of mathematics. However, such explanations would seldom be transformed into written records as they would respect the writing of the original mathematics "canons" (Li, 2007). Luckily, we can still find valuable commentaries for those mathematics canons in ancient China, by many eminent Chinese mathematicians including Liu Hui and Yang Hui. There, not only can we learn the accumulated "wisdom of traditional mathematical teaching", but also, on how ancient Chinese mathematicians can provide ingenious "proofs" to mathematics principles by combining logical deduction and intuitive analysis as we have discussed in Section 3.3. We would like to use the following remark suggested by Siu (2004) that is worth our deep reflections:

> *Although the official system did produce tens of thousands of capable "mathocrats" who were employed as officials or imperial astronomers, almost all the eminent mathematicians who left their footprints in the history of mathematics seem to have been nurtured through other channels. An historian of mathematics once listed 50 Chinese mathematicians of fame who flourished between the 4th century B.C. and the end of the 19th century, with only two who can be labeled as educated in the official system* (p. 163).

In 1676, Newton said "If I have seen farther than others, it is because that I was standing on the shoulders of giants". "The shoulders of giants" is the scientific traditions and achievements in Europe. Traditions are the sources and the basis for new ideas and inspiration. Based on its own traditions, the development of mathematics education in China should gain the experience from other countries and bring into suitable positive elements of new ideas and thoughts from other cultures. Chinese mathematics education has its own strength that might serve as a solution to educational problems faced by other countries on one hand, and its own weaknesses and problems could be remedied by ideas from other education systems on the other hand. There is no point of denying traditional Chinese mathematics education. As Zhang had questioned (Zhang & Zhao, 2012), "in recent years, the so called 'traditional' educational methods are almost the synonyms of 'backward' and 'obsolete'." In our opinion, the tradition is inseparable from, and indeed intertwined with the present and the future. "Innovation" does not mean the inevitable separation from tradition; instead, while we surpass and violate traditions as a prerequisite of innovation, there should be some circumstances in which modern and traditional values are compatible. On the contrary, any "rootless" social or cultural development may have the trouble of repeating past failure and the result can be catastrophic. Second, there is no point of insisting on whether Chinese education is "better" than that in the West, or vice versa. Instead, one should realize that each education system has its own set of strengths and weaknesses, and because of this fact it is important to interact with other cultures and learn from their experience!

In recent decades, there has been a trend of introducing materials on mathematics history in mathematics teaching. Many researchers suggested that incorporating history of mathematics in classroom can have positive effects on mathematics learning (e.g., Fauvel, 1991; Gulikers & Blom, 2001).

First, the use of anecdotes and biographies of mathematicians make lessons more interesting and dynamic (Perkins, 1991; Siu, 1997). Students become more actively involved in classroom activities when mathematics history is sprinkled in. Students are more motivated to learn about mathematics if they are able to identify the important role that mathematics play in human culture through history (Tymoczko, 1994). Also, students are able to appreciate the usefulness of mathematics in real life through history, as mathematical concepts are often developed to solve real-life problems in the past (Burton, 1998). Consequently, the use of history of mathematics can arouse students' interest on mathematics, improve their perception about the value of mathematics to mankind, and enhance the learning atmosphere in general.

Second, through explaining the stories behind the mathematical theories, students can obtain a better understanding of the theories. This helps clarify the historical development of mathematical theories, which somehow reflects the logical development of the theories as well (Katz, 1993). Learning takes place more effectively when a learner retraces the key steps in the historical development of the subject (Gulikers & Blom, 2001). As mathematical concepts are often oversimplified in textbooks and by teachers (Freudenthal, 1991; Siegel & Borasi, 1994), students may not be able to understand these concepts which are often broken up into smaller parts and presented to them from an expert's viewpoint (Tall & Vinner, 1981). Showing students the historical development of mathematical concepts can help them to see the links between the broken parts and improve their understanding of these concepts (Furinghetti, 2000).

Third, incorporating mathematics history into lesson planning can bring out the construction of mathematical concepts in students' mind. Such mental reflection on mathematical concepts is helpful to students.

Also, teachers can better understand the common difficulties faced by current students by examining the errors and misconceptions of past mathematicians. They can then take preemptive measures to ensure more effective learning. At the same time, when students realize that it is common for mathematicians to commit errors and learn from their own or others' mistakes, they appreciate that collaboration and perseverance are necessary to derive mathematical concepts which they often feel are beyond their ability to derive or understand initially (Lim, 2010). This gives them the confidence to explore and participate in mathematical activities (Siu & Siu, 1979).

As a result, it is hoped that by having a glimpse on some of the features of mathematics history and education in ancient China, mathematics teachers today can have some ideas about the development of mathematics and mathematics education in China, and about the possibility of introducing history of Chinese mathematics to the current mathematics classrooms (c.f. Siu, 1995; Katz, 2000; Cai & Cifarelli, 2004; Wang, 2009; Ng, 2006).

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Chapter 2

How Chinese Teachers Teach Mathematics and Pursue Professional Development: Perspectives from Contemporary International Research

FAN Lianghuo MIAO Zhenzhen MOK Ah Chee Ida

This chapter aims to provide readers with a comprehensive review of related literature on how Chinese mathematics teachers pursue pre-service training and in-service professional development, and how they teach in classrooms. The results suggest that China (Mainland) has established a highly unique and unified pre-service mathematics teacher education system; pre-service teachers learned more advanced mathematics courses and showed better motivation toward their training as compared with other countries such as the UK and US. China has also established its unique and well-institutionalized teacher professional development system for in-service teachers, with Teaching Research Groups (TRG) at the school level and Teaching Research Office at different government levels playing a crucial role. About teaching, it was found that Chinese mathematics teachers planned their lessons carefully; they adopted more whole-class teaching strategies, emphasized two basics (basic knowledge and basic skills), teacher-student interaction and students' engagement academically and the method of teaching with variation, and assigned homework daily for reinforcement as well as assessment of students' learning. Some issues and suggestions on future research in these areas are raised at the end of the chapter.

Keywords: Chinese mathematics education, mathematics teaching, mathematics teacher education, teacher professional development

1. Introduction

It is well-known that international comparative studies and large-scale assessments over the last two decades or so have consistently shown that

Chinese students are among the best performers in mathematics (e.g., see Fan & Zhu, 2004). In the latest Programme for International Student Assessment (PISA) conducted in 2012 for students of 15 years old from 65 countries and regions, Chinese students from Shanghai were the top performers with an average score of 613, significantly higher than the second top performers, Singapore students whose average score was 573, and the OECD countries' average of 494 (OECD, 2013). The world has shown growing interest in knowing the reasons behind Chinese students' stellar performance, and in particular, why they have steadily outperformed their peers in the West (Hiebert, Gallimore, Garnier, Givvin, Hollingsworth et al., 2003; Stigler, Gonzales, Kawanaka, Knoll & Serrano, 1999; Stigler & Hiebert, 1999; Stigler, Lee & Stevenson, 1987). Many scholars in different countries have investigated various factors that may have contributed to the results, such as cultural and social values, language systems, parental involvement, and curricula and textbooks (Cai, 2003, 2004; Mok, 2006). Some researchers have also looked into issues regarding Chinese teachers and how they teach in classrooms (e.g., Ma, 1999; Mok, 2006), though some are not necessarily from a comparative perspective. Undoubtedly, teachers play one of the most important roles in the formation of students' performance (e.g., Hiebert et al., 2003). As the National Commission on Teaching and America's Future (1996) pointed out: "What teachers know and can do makes the crucial difference in what children learn." (as cited in Fan, 2014, p. 3).

The main purpose of this chapter is to provide readers with a comprehensive review of the related literature on teaching aspects of Chinese mathematics education with a focus on two main areas[a], that is, how Chinese mathematics teachers teach in classrooms, and how their pre-service training, in-service and professional development take place. By doing so, we hope to present an overall picture of what research has told us in these two areas and what particular patterns or features, if any, are revealed about Chinese mathematics teachers' teaching and

[a] We originally included two other important and related areas, Chinese mathematics teachers' belief and knowledge. We later decided to exclude these two areas in this chapter as they have been well reviewed and discussed in other chapters of this book (see Zhang & Wong, this volume; Chen & Leung, this volume).

professional development, and offer suggestions for further research in the concerned areas. The chapter is also, to some extent, intended to provide readers with a helpful background for the main theme of this book, that is, how Chinese teach mathematics.

2. Methods

The literature surveyed in this chapter was primarily obtained through relevant international research journals, online search engines, and our own accumulation and connections.

There are a great number of different journals in education and mathematics education. In order to keep the literature survey reasonably manageable and focused, we selected research journals based on the following criteria: (1) they are peer-reviewed; (2) they are well-established and truly international (not just focused on one country or region) in terms of their aims and scope, editorial boards, contributors, reviewers and target readers; and (3) they are either highly reputable mathematics education journals covering mathematics teaching, teacher and teacher education, or educational journals mainly devoted to or focusing on the areas of teaching, teachers and teacher education. As a result, our survey mainly included the following journals:

Asia-Pacific Teacher Education
Australian Journal of Teacher Education
Educational Studies in Mathematics
European Journal of Teacher Education
International Journal of Science and Mathematics Education
Journal for Research in Mathematics Education
Journal of Mathematics Teacher Education
Research in Mathematics Education
Teaching and Teacher Education
ZDM-International Journal on Mathematics Education

Some other journal articles were also included in our review, but they were obtained mainly through online search, as described below.

The online search was primarily made through the so-called world's largest digital library for education literature, the Education Resource and Information Centre (ERIC). Firstly, we employed the search terms

"China/Chinese, mathematics, teaching", "China/Chinese, mathematics, lesson", and "China/Chinese, mathematics, class" for the issue how Chinese teachers teach mathematics. Concerning Chinese mathematics teachers' professional development including teacher education, we used the search terms "China/Chinese, mathematics, teacher education", "China/Chinese, mathematics, teacher training", and "China/Chinese, mathematics, prospective teacher". In addition, we also used two themes "mathematics teaching in China" and "mathematics teacher education in China" to carry out the literature search through mainly ERIC, although sometimes other search engines, such as the Web of Science, Informaworld, Science Direct, and Google, were also used as supplementary sources. Finally, a small number of research publications were obtained via our own accumulation and connections (e.g., through email communication).

We must point out that our survey is mainly limited to contemporary research literature on Chinese mathematics teaching and teacher education in Mainland China, published in international research journals and other publications and in English. To conduct a survey on research literature in Chinese is beyond the scope of this chapter, and readers who are interested in knowing more about the literature published in Chinese on these two issues may wish to read the relevant chapters in this book. In addition, although we tried to make our survey as comprehensive as possible, it is still possible that some important research work in the surveyed areas was not included into the review.

Two main criteria guided the selection of relevant literature for our review. The first criterion is based on the quality of studies, including clear research questions, methods, and results. Particular attention was paid to empirical studies reported and published in international peer-reviewed journals. In addition, a small number of book chapters presenting high quality studies were also included and reviewed, but general discussion papers were excluded.

The second criterion is the contemporary-China relevance. In terms of timespan, we decided to focus our survey mainly on the research literature from 1980 onwards, though we also paid some attention to the literature before 1980. Our reason for focusing on the past three decades or so is that before this period China basically closed its door to international exchange, especially to the West, because of a variety of internal and external factors. Consequently there was little international research available on Chinese education including mathematics education and teacher education over this period. The situation has significantly changed since China initiated its reform and adopted the open door policy in the late 1970s.

Using the sources and criteria described above, we obtained 30 journal articles and other types of publications on Chinese mathematics teaching, teacher, and teacher education. Table 1 displays the sources of the literature surveyed in this chapter.

Table 1. Sources of literature surveyed about Chinese mathematics teaching, teacher and teacher education

	Pre-1980	1980-89	1990-99	2000-09	2010-12	Total
Journal articles	0	0	2	18	2	22
Other publications	0	0	2	6	0	8
Total	0	0	4	24	2	30

As we can see from Table 1, there were no studies meeting our criteria on Chinese mathematics teaching, teacher and teacher education before 1980, which is not surprising because of the reason mentioned above. However, we were a bit surprised that there were also no such studies published internationally in the 1980s, which however is arguably related to the fact that even after China adopted its open-door policy in late 1970s, it would still take some time to see the concrete outcome and publication of research and exchange concerning Chinese education (including mathematics education) in the international arena. Moreover, Chinese students' outstanding performance was much less well known before the 1990s, largely due to the fact that all well-known modern large-scale international comparative studies, such as TIMSS and PISA, started either in the 1990s or 2000s.

Below we first report our survey findings about Chinese mathematics teachers' pre-service training, in-service and professional development, followed by how they teach in mathematics classroom. We shall also provide relevant general background along with the research findings.

3. Mathematics Teacher Education and Professional Development in China

According to Chen, Hu and Wang (1996), the Fourth National Conference on Teacher Education held by the Chinese Central Government in 1980 marked the start of a new stage for comprehensive development and reform of teacher education in Mainland China. The conference claimed that the advancement of teacher education should be given the highest priority in the country's educational development with a new series of national targets in preparing teachers for the new era after the Cultural Revolution which was widely considered to have ruined the educational system across the whole country.

Since 1980, Chinese teacher education has been progressing steadily under the influence of several nation-level innovations which not only raised the importance of teacher education to a strategic level but also increased the financial support for teacher preparation. In addition, over the last decade or so, the organizational forms of teacher-preparation institutions have experienced a systematic transformation from the traditional form of normal institutions, which have operated for over half a century, to diversified forms that contain not only normal universities, but also comprehensive universities and other types of institutions (Shi & Englert, 2008).

After the subsequent decades of evolution, the providers of teacher education in China have now been organized into four types of institutions: 1) three-year normal schools accepting junior secondary graduates as candidates, who will mainly become primary school teachers after completing training; 2) normal specialized postsecondary college accepting senior secondary graduates as candidates in two- or three-year programs or junior secondary graduates as candidates in five-year programs, whose graduates will primarily become junior secondary school teachers; 3) four-year normal colleges and universities mainly for the training of senior secondary school teachers, and 4) comprehensive universities running four-year bachelor degree programs since 1998, which are new comers due to the expansion of initial teacher training and education and mainly focus on senior secondary school teachers (Li, Zhao, Huang & Ma, 2008).

3.1 *Pre-service Training Programs for Mathematics Teachers*

Before we turn our attention to specific studies, it should be mentioned that, within the context of China's centralized education system, Chinese teacher-education institutions have for a long time implemented basically the same curriculum and textbooks, and structured their programs with many similarities (Sun, 2000).

Among a very few studies that have been conducted in relation to pre-service training programs for mathematics teachers in China, Ferrucci, Li, and Carter's study (1995) is particularly notable. The researchers reviewed three earlier studies (reports) related to China's pre-service teacher training programs from early 1970s and late 1980s, and highlighted that in the program of study provided in the 1980s by Beijing Normal University for pre-service secondary mathematics teachers, 63% of the curriculum time was devoted to mathematics, 12% foreign language, 6% physics, 6% mathematics methods (mathematics pedagogy) and education, and 12% science; in East China Normal University, the required mathematical courses in the pre-service teacher training program included calculus, real analysis, probability and statistics, analytic geometry, advanced algebra, abstract algebra, differential equations, complex variables, topology, computer and informatics, different geometry and higher geometry. From the topics or numbers of hours they either obtained from the three earlier studies or for their own study about the mathematical courses offered in the pre-service teacher training programs in universities in Beijing and Shanghai from 1970 to 1992, we can see a great similarity in either topics of mathematical classes or hours that prospective teachers are expected to devote to those classes in these Chinese universities.

More importantly, collecting data through a questionnaire survey with the focus being on mathematics content, pedagogy, and the practices in teacher training curricula and the respective course arrangements in six teacher-preparation institutions, two from Beijing, China and four from both the eastern and western regions of the US, Ferrucci et al. (1995) concluded that secondary mathematics teacher training programs in the two countries shared a similar way of organizing their training courses which were mostly delivered through lectures and ended with

Table 2. Mathematics courses required for Chinese and American prospective secondary school teachers

	PRC1	PRC2	USA1	USA2	USA3	USA4
Calculus	378	450	120	224	168	210
Real analysis	90	72	0	42	42	0
Elementary functions	*	72	0	0	42	0
Probability/Statistics	90	72	40	84	84	42
Number theory	0	72	40	0	0	42
Analytic geometry	108	0	40	0	0	0
Euclidean geometry	112	72	40	42	42	84
Linear algebra	0	144	40	42	42	42
Advanced algebra	216	0	0	42	0	0
Abstract algebra	85	72	40	42	42	42
Numerical methods	72	72	0	0	42	0
Differential equations	90	72	0	0	42	0
Complex variables	85	72	0	0	0	0
Topology	68	72	0	0	0	0
Mathematics modelling	0	0	0	42	42	0
History of mathematics	0	72	40	42	42	42
Computers/informatics	72	72	60	84	84	42

* The content in this course is contained in other courses.
Source: Ferrucci et al., (1995, p. 222).

discussions on homework for mathematics sessions and in-class activities for pedagogy sessions. However, Chinese prospective teachers spent substantially more hours in mathematical classes, and consequently studied more advanced mathematics topics than their American counterparts, as shown in Table 2. In other words, the difference found is mainly in the content of training, but not in the method of training.

Concerning prospective primary mathematics teachers' training, Li (2002) conducted a two-stage study in China. The first stage was to examine the content and structure of mathematics teaching method courses, which are the same to all prospective primary teachers in China, and the second stage consisted of two case studies in Beijing and Zhejiang, focusing on how the method courses were actually taught, including informal interviews with the tutors. The study found, in the course textbooks used, a great emphasis was placed on these trainee teachers' deep understanding and thorough integration of mathematical subject knowledge, student cognitive development and pedagogical principles in the learning of the courses. Moreover, elementary school textbooks were also used in the method courses for prospective teachers to consider appropriate teaching methods to teach the knowledge effectively in their future practices. The study found that these mathematics student teachers had already achieved a deep understanding of what and how they were going to teach, before they started formal practices in schools.

More recently, You and Jia (2008) conducted a survey with 136 Chinese student teachers and 134 American counterparts who were studying in teacher-preparation programs in their home countries respectively. Data were collected with two questionnaire instruments measuring learning approaches and learning styles. The researchers found that Chinese student teachers were more willing to learn knowledge for its own sake and read books in a particular depth compared to their American counterparts. In addition, the results indicated that Chinese student teachers were deep learners with more intrinsic motivations to understand the knowledge itself than their peers in the US.

It seems clear from the limited number of studies available that there is great similarity among Chinese mathematics teacher education programs; when compared with the US, Chinese pre-service teacher training emphasizes more on developing trainee teachers' advanced mathematical knowledge, and trainee teachers are better motivated toward their training. It appears to us that the fact that Chinese prospective teachers are better motivated and they studied more mathematics topics is evident, when comparing with the US and many other countries in the West, though more studies in this direction are still needed to further investigate how and why pre-service training courses differ in their influences on teachers' classroom teaching and students' learning of mathematics across different countries. In addition, the results appear also consistent with the findings from Ma's well-known interview-based study, in which she found that Chinese mathematics teachers had a profounder understanding of fundamental subject knowledge and pedagogical content knowledge than their American colleagues (Ma, 1999).

3.2 *In-service Teachers' Professional Development*

Researchers have generally agreed that China has developed a coherent and institutionalized in-service teacher education or professional development system, in which peer collaboration and interaction play a crucial role in developing in-service teachers' expertise and competencies. More specifically, teachers are involved into various programs such as in-service training programs, apprenticeship practices, school-based teaching research activities (Huang, Su, & Xu, 2014), and public lesson development (Han & Paine, 2010; Huang & Li, 2009).

The in-service training programs in China are mainly provided by institutes of education at the province and city/district/county levels, and the professional development is mainly organized by Teaching Research Offices, known as "*Jiao yan shi*" in Chinese (hereafter called "TRO") within the governmental education bureaus at county, district, city, and provincial levels. These institutions or offices are responsible for guiding teaching research activities, overseeing teaching quality in schools on behalf of educational bureaus, providing consultation and teacher professional development programs, and promoting high-quality classroom teaching (also see Li & Ni, 2011; Huang et al., 2014).

In each school, as for the case of other subjects, there is usually a mathematics "Teaching Research Group" (TRG), which is responsible for all mathematics teachers' professional development in the school. TRG is a unique organization in Chinese schools, originally introduced from the Soviet Union in the 1950s (Paine & Ma, 1993). In a sense, the mathematics TRG is equivalent to the "Department of Mathematics" in many other countries, but the TRG focuses more on research on classroom teaching and professional development, and it is subject-based and often consists of a sub-unit — Collective Lesson Planning Groups or simply Lesson Planning Groups (LPG) — a teaching organization usually for teachers teaching at the same grade level (Wang, 2002; Wong, 2010). Working as a team, teachers in the TRG or its LPG meet weekly, prepare lessons together, observe each other's lessons, reflect and comment on observations collectively, and conduct open lessons regularly. In a sense school-based TRG and LPG function as in-service teacher education institutions but in actual classroom situations, foster reflective practitioners, and consequently promote continuous development of the whole teaching team. Many schools also have Grade Level Groups, which is more for administrative purposes beyond the subject matter.

It should also be noted that China has established a professional ranking and promotion system since the mid-1980s. The ranking and promotion system provides teachers, like university academic and teaching staff, with professional titles. They include Senior Teacher (equivalent to Associate Professorial Grade at the university), Intermediate Grade Teacher, and Junior Grade Teacher. Earlier than that, the title Master Teacher was awarded (e.g., see Fan & Shen, 2008; Huang & Li, 2009) and more recently Full Senior Grade Teacher (equivalent to the Professorial Grade in the university) have also been used in some regions in China (e.g., Wei, 2013). This system not only specifies components of teacher professional expertise and other related requirements for different grades of teachers, but also provides teachers with professional recognition and incentives and a culturally supported mechanism for teacher to seek professional development at different levels, e.g., school, district, county, city, provincial, or national levels.

Concerning mentoring programs in schools, Wang (2001) investigated 23 teacher mentors in the US, UK and China with two semi-structured interviews, and examined their beliefs about what novice teachers should learn and how they interacted with novice teachers in practice. Among a considerable proportion of similarities that the three countries' mentors shared, the study revealed some notable differences between Chinese mentors and those in other two countries. Unlike their western colleagues, Chinese mentors more commonly believed that novice teachers should establish a solid base of subject matter knowledge and a deep understanding of curricular requirements, be able to build connections among various subject areas of knowledge, and know how to tailor specific methods in teaching particular knowledge and addressing different teaching targets. Their interactions with novice teachers contained 72% pedagogical discussions focusing on novices' pedagogy, whereas the percentages for their American and British colleagues were respectively 68-69% and 34%. It is believed while the centralized curriculum and assessment criteria provide Chinese teachers with less leeway or autonomy, such a school environment also triggers more collaboration and shared visions between mentors and novice teachers than the case of their counterparts in the US and the UK. In particular, at the beginning of new teachers' teaching career, Chinese schools generally arrange a number of peer observations followed by discussions, based on the observed lessons and focusing on pedagogical matters, between experienced and novice teachers. Such subject-oriented professional communities within schools and the corresponding professional interactions between members of the communities offer an essential contribution to the professional growth of new teachers.

Wang and Paine's case study (2003) also shed light on the effectiveness of school-based collaborative teacher development in China. In the study, they followed a beginning middle-school teacher through her preparation for a public lesson in the second year of her teaching career. They noted that, like many other Chinese teachers, the new teacher belonged to two subject-matter teaching groups based within the school. One was a TRG, as mentioned earlier, which was organized at the school level aiming to promote peer observations and discussions in collaboratively coping with examinations, and the other is an LPG, which was set up at each grade level for teachers to plan lessons, share their understanding of curricular resources, and exchange teaching experiences. Data were collected from interviews with the teacher, the leader of the TRG for mathematics and the school principal and subsequently triangulated with the curricular framework and the lesson itself. The teacher's high-quality lesson was found being significantly influenced by the guidance of curricular resources and the teaching groups where teachers met regularly (usually weekly).

It is worth noticing that professional communities within Chinese schools not only foster new teachers' development but also stimulate other in-service teachers' professional development through various research approaches over time. Some professional communities from different schools also involve educational researchers doing action research collectively with teachers. One such approach is through Ke-Li, literally meaning Lesson Example or Exemplary Lesson Development as described in Huang and Bao (2006). According to Huang and Bao (2006), Chinese mathematics teachers' collaboration are similar to Japanese Lesson Study in that both involve teachers' collaboration on lesson planning, peer observation and post-lesson discussion; they are also quite different from each other because the former gives prominence to experts' input, revising lesson plans and carrying out new lessons subsequently. In Chinese schools, professional communities are subject-oriented and school-based, providing rich opportunities for teachers to consistently learn how to teach better while they are teaching in schools. More recently, the Parallel Lesson Study (PLS) has become a very popular Chinese lesson study model, responding to the call for innovative use of textbooks in the classrooms under the new curriculum. While a lesson study group in Chinese usually consists of a mathematics teaching researcher from a district/county education bureau, a master teacher and a demonstrating teacher, the new development extends the lesson study group activity to research collaboration between at least two independent lesson study groups at the cross-district level (Huang et al., 2014). It seems that such school-based professional communities are continuously evolving and steadily widening external connections along with the country's curricular innovation.

Below we turn our attention to Chinese mathematics teachers' teaching in the classroom.

4. Mathematics Teaching in Chinese Classrooms

Compared with research on Chinese mathematics teachers' education and professional development, there are significantly more studies on how Chinese mathematics teachers teach mathematics. These studies addressed a variety of issues related to teachers' teaching in Chinese classrooms from different angles. This section aims to provide an overall picture about Chinese mathematics teachers' teaching practices and their features as revealed in the available empirical studies on mathematics teaching either solely done in China or cross-nationally comparing lessons from China with those from other countries. To highlight the main features of how Chinese mathematics teachers teach mathematics, we shall organize our review into the following nine aspects: planning lessons systematically, emphasizing two basics, whole-class teaching and interaction, teaching with variation, teacher-student interaction and engagement, assigning and marking homework frequently, using textbooks with deep understanding, structured instruction, and making change in light of curriculum innovations.

4.1 *Planning Lessons Systematically*

Researchers have consistently maintained that Chinese mathematics teachers generally devote a great amount of time and energy in advance to scrutinizing the syllabus, textbooks, teacher manuals and other teaching materials, before they formally give lessons, and they also frequently share their ideas and develop understandings of mathematical content they have taught or are about to teach with their colleagues in the subject-based TRG (Ma, 1999, Ch. 6; Paine & Ma, 1993; Wong, 2010).

According to An's study, during the lesson planning stage, Chinese teachers paid more attention as to how appropriate connections could be built between the content of teaching and classroom activities, to suit different levels of student cognition, and between previous knowledge and new knowledge than their American counterparts who concerned more about classroom activities (An, 2008). Moreover, in lesson plans, Chinese mathematics teachers often focus on the essence of mathematical concepts, deal with important and difficult knowledge points, organize teaching steps to develop knowledge progressively, and pay attention to the readiness of students' knowledge and cognition levels, the appropriateness of the textbook use and the selection of teaching methods (Huang & Li, 2009). Likewise, another comparative study also found that Chinese teachers' lesson plans were much longer with more details regarding contents and procedures than were those of American teachers who generally just drew a lesson outline and prepared a number of worksheets (Cai & Lester Jr, 2005; also see Cai, 2005).

4.2 *Emphasizing Two Basics*

Basic knowledge and basic skills, often jointly called two basics in China, are put at the heart of mathematics teaching and learning in China (Zhang, Li, & Tang, 2004). Teachers are expected to demonstrate a high level of proficiency in two basics themselves, and more importantly, they should be able to help their students obtain solid foundation of knowledge and skills. The dual emphases of two basics are on both understanding and application. Through learning and practice, students are expected to reach six basic goals:

- Rapid and accurate mental calculation of four arithmetic operations involving various types of numbers;
- Speedy and accurate manipulation of "polynomial expressions, algebraic factions, exponential and radical expressions and memorization of rules";
- Precise memory of definitions, properties and formulas of various mathematical topics, for instance, those of quadratic equations, curves of the second order, trigonometry, logarithm, etc.
- Logic, clarity and accuracy of mathematical expressions, classification, and mathematics propositions;
- Clear and accurate presentation of rigorous reasoning in the process of problem solving;
- Awareness of basic solution patterns and ability of applying them to similar problems under changing conditions (Zhang et al., 2004, p. 193).

In order to address the learning goals, mathematics teachers in China generally play a leading role in the process of teaching and learning in a class of around 50 students. Teachers aim at delivering mathematical contents effectively and efficiently in a direct way, so as to leave sufficient time for students to practice new knowledge. Although teachers see both understanding and application of knowledge as equally important, they allocate more time to the latter, as they believe appropriate practice can deepen and consolidate understanding. Moreover, teachers tend to give students exercises which contain varying non-fundamental elements but invariant properties of specific point of knowledge. Teachers' emphasis on rigorous mathematics thinking and interconnection of knowledge also transforms the application of two basics to a higher level. Last but not least, students' mastery of two basics is also realized through systematic and rigorous practice of deductive reasoning. The above teaching characteristics collectively contribute to the formation of two basics among students.

4.3 *Whole-class Teaching and Interaction*

If, traditionally, emphasizing "two basics" (basic knowledge and basic skills) is the most typical feature in terms of teaching content in Chinese mathematics teachers' teaching (e.g., see Zhang, Li, & Tang, 2004), then it seems also reasonable to say that, the most typical feature in term of the teaching format in mathematics teaching by Chinese teachers is "whole-class teaching and interaction", which is used in most of the lesson time. This is most evident in the findings of a study on mathematics lessons in three cities by Leung (1995) who observed 36 lessons in Beijing, 36 in Hong Kong and 40 in London. Many differences have been found in classrooms between locations, with the most significant ones being durations of teaching activities and time off task. As shown in Table 3, Beijing and Hong Kong teachers spent the majority of their lesson time on whole-class activities, with little time spent on off-task activities, whereas London teachers spent either no or less than a half of the lesson time on whole-class teaching and more time on individualized teaching, with more time lost in off-task activities.

Table 3. Durations of various teaching activities in the classrooms in Beijing, Hong Kong and London

Types of activities	Beijing	Hong Kong	London (W)	London (I)
Whole class activities	86.30%	72.52%	42.13%	0.00%
Individual activities	13.21%	18.93%	38.35%	77.84%
Off-task activities	0.25%	8.55%	16.58%	20.37%

Note: London (W) stands for London lessons conducted in a whole-class instruction setting; London (I) stands for lessons following individualized programs.
Source: Leung (1995, p. 321).

Teacher-guided active whole-class teaching is a common feature in Chinese mathematics classrooms where students are fully engaged in academic-oriented activities (Huang & Leung, 2005). Such activities usually consist of enormous (on average 50-120 per lesson) questions and answers between the teacher and students, which means that the instruction is highly interactive. Moreover, teachers often give questions that are easy to answer. By doing so, teachers are able to guide students through small and easy steps towards the grasp of learning targets of each lesson (around 40 minutes) (Zhang et al., 2004).

4.4 *Teaching with Variation*

Another important feature of Chinese mathematics lessons is the so-called teaching with variation which has received many researchers' attention over the last two decades (e.g., An, 2008; Gu, 1994; Gu, Huang & Marton, 2004; Huang & Leung, 2005; Lim, 2007; Mok, 2006). Researchers have argued that teaching with variation helps in increasing the level of student engagement, reducing the amount of disruptive behaviors, and drawing students' attention (Borich, 2004; Emmer & Evertson, 2009; Evertson & Emmer, 2009), and effective teachers have the ability of presenting a lesson with variability or flexibility (Borich, 2011; Brophy, 2002; Brophy & Good, 1986).

In a classroom-based study, Lim observed 19 randomly chosen mathematics lessons from five schools in Shanghai, with student socioeconomic status ranging low to medium (Lim, 2007). Lim found that those lessons shared a list of similar features including the use of variation. Around a mathematical concept, teachers generally presented different examples in a logical sequence to help students understand the concept from superficial to a deeper level, which is called *conceptual variation*. Regarding a mathematical problem, teachers tended to encourage students to tackle the same problem with different methods through various procedures, which is known as *procedural variation*. It should be noted that Lim's study is largely consistent with earlier comparable studies in this line about Chinese mathematics lessons (e.g., Gu et al., 2004; Huang & Leung, 2005; Mok, 2006).

4.5 *Teacher-student Interaction and Engagement*

Researchers have widely considered the effective use of teacher-student interaction including questioning as a crucial factor contributing to effective teaching and therefore learning (Borich, 2011; Evertson, Anderson, Anderson & Brophy, 1980).

Employing classroom observation and video recording methods, Li and Ni (2009) compared the forms and contents of teacher-student dialogue in primary classrooms between expert and novice Chinese mathematics teachers and found notable differences. Overall, 55 lessons were observed and recorded in the classrooms of 16 expert teachers and 16 novice teachers who taught either 4th or 6th grade students. The result showed that expert teachers tended to use analytical and comparative questions frequently to foster students' mathematical reasoning. Their interaction model often involves the following steps: (1) students present an answer, (2) the teacher and other students question the answer, and (3) then the students explain the answer. In other words, expert teachers were more likely to judge students' responses collaboratively with students, and more likely to make adequate use of and transfer student answers into teaching resources. On the other hand, the typical teacher-student interaction in a novice teacher's classroom is more likely to involve the following steps: (1) the teacher asks a question, (2) students answer, and (3) the teacher comments. Li and Ni's study reminds us that one should not forget that, within Chinese classrooms, there also exist differences.

Research evidence also shows that Chinese mathematics teachers differ from their American counterparts in the way they deal with students' errors during the teaching and learning process in class. Through lesson observations (n=44) and teacher interviews, Schleppenbach et al. (2007) conducted a comparative study on participating teachers' responses to students' errors during the lesson time between China and the US. The study found that Chinese mathematics teachers asked more follow-up questions, in addition to informing the students about the errors they made, while the American teachers generally just made simple announcement of errors. Again, what is not so clear is how typical or representative these lessons are, given both China and the US are large countries.

Students' time spent on task in mathematics lessons has been found to have a positive correlation with their academic engagement and learning gains (Fisher, Berliner, Filby, Marliave, Cahen et al., 1980; Stallings, Cory, Fairweather & Needles, 1978). Engaging students in tasks of learning is commonly seen in mathematics lessons in China. Cross-national studies revealed Chinese teachers generally taught in a faster pace and applied lesson time more efficiently than their western colleagues (An, 2008). In another study, through structured observations of 8 mathematics lessons in China and 7 lessons in the US, Lan et al. (2009) found Chinese teachers were more proactive and more able to engage students consistently into academic activities than were their American colleagues, which led to better student engagement and more time-on-task (Lan et al., 2009).

4.6 *Assigning and Marking Homework Frequently*

Both empirical data and systematic reviews suggest that regular homework over short intervals with timely feedback from teachers has positive effects on student achievement and attainment, particularly in mathematics (Cooper, Robinson & Patel, 2006; Good & Grouws, 1979; Muijs & Reynolds, 2003).

Studies on mathematics lessons in China have found homework assigned by teachers a common everyday task for Chinese students (Lim, 2007). Through all-day observations of a Shanghai mathematics teacher's homework-related work over two months, interviews with the teacher, her colleagues, and her students, and video-recorded classroom observations, Fang (2010) investigated in depth the nature and functions of homework errors that the teacher defined, coded and made use of in her daily work either individually or collaboratively with colleagues. According to Fang, teachers considered marking students' homework as an important way to assessing students' daily learning results and accordingly examining their teaching effectiveness from day to day. They developed a shared set of discourse symbols in checking homework items and indicating to students what was right and what was wrong, with hints and clues left on students' homework. Thus, their homework marking was process-oriented, which made their students understand not only *what* but also *why* in dealing with homework errors. Teachers communicated with each other while marking homework, exchanged their understanding of various homework errors, and consequently developed their pedagogical knowledge on specific subject content, as they shared a unified teaching pace across the curriculum of mathematics. Moreover, such collective investigation on information embedded in everyday homework among colleagues promoted pedagogical reasoning and contributed to continuous development of teachers' knowledge in both subject matter and pedagogy.

On the other hand, we should point out that the recent PISA 2012 results revealed that in Shanghai, students of 15 years old spend almost 14 hours per week on the homework of all school subjects set by teachers, while across OECD countries the average is about 4.9 hours (OECD, 2013). It is clear that assigning homework regularly is a common feature of Chinese mathematics classrooms, however, how much homework is adequate is another issue and it is worth further investigation.

4.7 *Using Textbooks with Deep Understanding*

Chinese teachers rely considerably on textbooks in planning and carrying out lessons. Thorough understanding of textbooks is widely accepted as an essential way for mathematics teachers in China to improve their teaching both mathematically and pedagogically (Li, 2004). Focusing on Chinese mathematics teachers' use of textbooks, Fan, Chen, Zhu, Qiu and Hu (2004) collected data with 36 teachers and 272 students from two major cities in South China through questionnaires, classroom observations, and interviews. The study found that all teachers used textbooks as the main source of teaching contents and methods, that the way teachers used textbooks varied slightly as their teaching experience accumulated over time, and that overall teachers used textbooks in a similar rather than different way across the sample regardless of their genders and the geographical locations and the performance levels of their schools.

4.8 *Structured Instruction*

Mathematics lessons in China are in general well structured, with specific types of activities more emphasized. For example, drawn on a sequence of five consecutive lessons by a teacher in Shanghai, Lopez-Real, Mok, Leung and Marton (2004, p. 407) found that about 96% of the total time was covered by three elements: Foundation/Consolidation (32%), Exploration (19%), and Guided Practice (45%). Guided Practice obviously occupied a significant part of the lessons. The contrast between lessons illustrated that the teacher, though with different time spans, gave an equal importance in his teaching to both foundations and explorations, where the students had the opportunity to discuss and express their own ideas. Nevertheless, it was also clear that the overall direction of learning was tightly controlled by the teacher and the explorations were limited to 'mini-explorations'. In addition, "Summarizing", another element recognized as important in the interview with the teacher, although took up only a relatively small amount of time, appeared regularly in the end of the lesson and sometimes even in the middle of the lesson.

4.9 *Making Change in Light of Curriculum Innovations*

It should be noted that new and reformed mathematics teaching methods in Chinese classrooms have developed over the time, which is particularly evident under the new curriculum reform, as the national curriculum usually includes clear explanations and guidance concerning classroom teaching practices.

The latest national mathematics curriculum for China issued in 2001 (Ministry of Education, 2001a, 2001b) emphasizes the necessity of (1) diversifying knowledge components, facilitating active learning, cultivating independent and critical learners, (2) reinforcing inter-subject connections rather than isolating them, (3) transforming abstract and complicated knowledge in the previous version of national curriculum into knowledge largely connected to the real world, and (4) decentralizing the power of choosing curricular materials from the central government to the local authorities and schools. The change in the national expectation on the educational results thus encourages changes at all levels, particularly in the instructional practice at the classroom level.

By comparing two groups of teachers who had either been implementing the new curriculum for over five years (n=32) or been constantly sticking to the previous curriculum (n=26), Li and Ni (2011) investigated the effect of this reform on primary mathematics teachers' classroom practices, with particular focus on two domains: instructional tasks and teacher-student interactions. With lessons video-taped and videos coded systematically afterwards, the study found in the teaching process more high-cognitive-level tasks delivered by the teachers applying the new curriculum than those who had kept implementing the conventional curriculum since the reform. In addition, innovative teachers tended to represent knowledge in a more visualized way and asked more process questions and less product questions than their conventional colleagues. This to a certain extent indicates the impact of the curricular reform on the curricular implementers' practice, i.e., the teachers' teaching. In fact, cooperative learning, constructivism learning, emphasizing realistic mathematics and using of ICT in mathematics learning have also been promoted in the new curriculum reform (Ministry of Education, 2001a, 2001b), and it seems reasonable to say that they have gradually had impact on Chinese teachers' teaching of mathematics in recent years, though further studies are needed to examine to what extent the influence is and will last.

5. Concluding Remarks

This chapter aims to examine research-based evidence to illustrate an as-authentic-as-possible picture of how Chinese mathematics teachers receive pre-service training and pursue professional development, and how they teach mathematics in classrooms.

The review of the available limited number of studies suggests that China has established highly unique and unified pre-service teacher education systems with great similarity among mathematics teacher education programs across different parts of the country, that Chinese pre-service mathematics teachers take more advanced mathematics courses during their pre-service study at teacher-training institutions, and that they also showed better motivation toward their training.

In addition, China has also established its unique and well-institutionalized teacher promotion and professional development system for in-service teachers, with Teaching Research Group (TRG) at the school level and Teaching Research Office at different government levels playing a crucial role in the system. The Chinese mathematics teachers' professional communities and development activities are more subject-oriented and school-based, and provide rich opportunities for peer collaboration, peer observation and collective lesson planning for mathematics teachers to development professionally.

Regarding classroom teaching, available research shows that Chinese mathematics teachers pay much attention to lesson planning before their teaching, and in classrooms they adopt whole-class teaching strategies, emphasize two basics (basic knowledge and basic skills), teacher-student interaction, student engagement and teaching with variation, and finally assign homework daily for reinforcement as well as assessment of students' learning.

On the other hand, it should be also noted that, as aforementioned, there have been only a very limited number of studies available on how Chinese teachers teach mathematics in classrooms and how they develop professionally. Moreover, most of the studies reviewed above are of small scale, often conducted in certain regions of China (particularly Shanghai and Beijing), and focused on certain school grade levels. Most international comparisons were limited to between China and the US or UK. The research issues covered in the available international literature are overall fragmented and unbalanced, leaving many areas and issues untouched. For example, the use of ICT by Chinese mathematics teachers in mathematics teaching is understudied. In particular, while the weakness of the Chinese way of teaching mathematics is widely recognized domestically in China, it has been only occasionally mentioned but not really researched internationally. There is no doubt that international research in these two areas concerning Chinese mathematics education is still at an early stage, and much more remains to be done. It is our hope that this book can make a meaningful contribution to this end.

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Section 2

UNDERSTANDING THE CHINESE WAYS OF TEACHING MATHEMATICS

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Chapter 3

Mathematics Teaching in a Chinese Classroom: A Hybrid-Model Analysis of Opportunities for Students' Learning

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This chapter analyzes mathematics teaching in Chinese classrooms by articulating opportunities for learning (cognitive change) created for students. A hybrid model consisting of a tripartite theoretical lens is presented and used: Reflection on Activity-Effect Relationship (Ref*AER), Hypothetical Learning Trajectory (HLT), and Teaching with Bridging and Variation. The analysis examines how teachers use the latter two strategies to (a) tie goals for students' learning with their extant knowledge, (b) create a need for exploring the new mathematics, and (c) provide situations for action and reflection that promote achieving the learning objects. This analysis inspires a three-tiered model for examining and guiding mathematics instruction. At a macro tier, HLT guides setting learning goals, designing mental activity sequences, and articulating cognitive reorganization processes. At an intermediate tier, teaching with bridging and variation provides tools for the deliberate design of problem situations and tasks within a specific HLT to create opportunities for the intended reorganization and thus achieving goals for students' learning—interrelated conceptual and procedural understandings. At a micro tier, Ref*AER provides a lens to link situations/tasks with changes in students' conceptions.

Keywords: Chinese mathematics classroom, opportunity for learning, hybrid model analysis

1. Introduction

Mathematics teaching and learning in China have become an area of interest for educational researchers (Fan, Wong, Cai, & Li, 2004; Li &

Huang, 2012). Often, mathematics classrooms in China have been characterized as well disciplined, polished (Stevenson & Lee, 1995), and coherent (Chen & Li, 2010). In particular, *Teaching with Variation* has been studied as an effective instructional strategy in China (Gu, Huang, & Marton, 2004; Wong, 2008). Yet, much research on interpreting and theorizing Chinese learning and teaching phenomena needs to be completed. Recently, some scholars have attempted to characterize features of Chinese mathematics teaching in terms of a constructivist theory of learning (e.g., Jin, 2012; Jin & Tzur, 2011). This latter work was rooted in a stance that mathematics classroom instruction needs to be linked to and articulated in terms of learning opportunities it creates and realizes for students (Tzur, 2011b).

This chapter is to further extend this line of inquiry. To this end, we use a tripartite, hybrid model that combines constructivist accounts of learning and teaching—*Reflection on Activity-Effect Relationship* (Ref*AER) (Simon, Tzur, Heinz, & Kinzel, 2004; Tzur, 2007; Tzur & Simon, 2004) and the *Hypothetical Learning Trajectory* (HLT) construct (Simon, 1995; Simon & Tzur, 2004)—with the Bridging and Variation strategies identified in Chinese teachers' practices. This hybrid model of how teaching can promote conceptual learning provides a lens for examining the design and delivery of mathematics lessons in regular classrooms. That is, we analyze mathematics lessons taught in China by focusing on three key aspects of HLT: (a) *goals* teachers set for student learning in terms of conceptions (activity-effect relationships) they are expected to construct, (b) *sequences of mental activities* (and reflections on them) hypothesized to promote students' transformation of their extant conceptions into the intended ones, and (c) *tasks* designed and implemented to fit with and promote hypothetical reorganization processes from available to intended mathematics.

We use data of lesson segments from Clarke, Keitel and Shimizu's (2006) study to examine how a Chinese teacher's deliberate use of a series of mathematical tasks over a few consecutive lessons promoted opportunities for students' conceptualization of a key topic in algebra. This examination provides evidence in support of a twofold, central thesis: pedagogical approaches used by Chinese teachers hold high potential for promoting students' learning (interrelated conceptual and procedural understandings), and the hybrid model (described next) provides a useful tool for explaining and realizing this potential.

2. Conceptual Framework

Teaching involves promoting students' understanding of particular mathematics through designed activities (Marton & Tsui, 2004; Simon, Saldanha, McClintock, Akar, Watanabe, & Zembat, 2009), that is, teaching can and should purposely create opportunities for student learning. To tie mathematics teaching and learning, we use a hybrid model that explicitly articulates the learning process and how teaching can promote it (Tzur, 2007, 2011b). This hybrid model uses a constructivist stance to articulate how teaching with bridging and variation can promote learning opportunities.

Marton and Tsui (2004) explained that variation helps learning by allowing students to develop new ways of experiencing and thinking about the objects of learning. Gu et al. (2004) and Jin (2012) further articulated how Chinese teachers' systematic variation of problems and/or solutions (answers + reasoning) effectively helps students to discern key features of the object of learning. Specifically, Gu et al. (2004) identified two types of variations: conceptual and procedural, which are crucial for promoting students' learning. *Conceptual variation* provides students with a variety of instances of the intended concept—including counterexamples—that orient their attention to critical features of that concept from different perspectives. *Procedural variation* provides a series of scaffoldings aiming at formulating concepts logically and coherently, and/or finding solutions to a problem progressively. For example, to help students form the concept of equation—discerning the two critical features of *balance (equivalence)* and *known*, a teacher could use a prototype word problem: "Peter pays two dollars to buy three same-price erasers, and gets 2 dimes in return. What is the price of one eraser?" To guide students' development of the concept of equation, Chinese teachers may enact learning through three stages: iconic representation (e.g., 2 dollars -3□=2 dimes); letter label representation (e.g., $20\text{-}3E=2$), and abstract unknown representation (e.g., $20\text{-}3x=2$) (Huang & Leung, 2004).

Bridging refers to tasks Chinese teachers use for helping every student in the class *reactivate her or his available knowledge* in a way that supports learning of the intended mathematics (Jin & Tzur, 2011). For example, to teach simplifying algebraic fractions at grade 7 (e.g., $8ab^2c/\text{-}12a^2b$), a teacher engaged the students in simplifying a few numerical fractions (e.g., -16/42). Students learned simplifying such numerical fractions at grade 4 and could successfully complete these tasks. The whole class discussion of their solutions focused on the invariant mental process reactivated for the goal of simplification, namely, finding and dividing both the numerator and the denominator by the greatest common factor. Students were then oriented by the teacher to use this method for solving variations of tasks with algebraic fractions.

To explain how teaching with bridging and variation may impact learning, we anchor it in the recent elaboration of a social-constructivist perspective proposed by Simon and Tzur and their collaborators (Simon et al., 2004; Simon & Tzur, 2004; Tzur, 2007, 2011b; Tzur & Simon, 2004). Building on the seminal work of Piaget (1985), Dewey (1933), and von Glasersfeld (1995), they explained the design and implementation of mathematical tasks/lessons by coordinating a postulated mechanism of cognitive change—reflection on activity-effect relationship (Ref*AER) (Tzur, 2007, 2011a; Tzur & Lambert, 2011) with a Hypothetical Learning Trajectory (HLT) (Simon, 1995). Below, we briefly present aspects of these two key notions relevant to this study.

A foundational constructivist principle, consistent with sociocultural perspectives (Vygotsky, 1986), asserts that learning entails progressive reorganization of knowledge already available in the learner's mind through reflection on goal-directed activities. Building on Piaget's (1985) work, von Glasersfeld (1995) postulated a tripartite mental structure—a scheme: (i) a situation that assimilates and interprets an external stimulus (e.g., mathematical problems) and triggers one's goal, (ii) a mental activity triggered to accomplish that goal, and (iii) a result anticipated to be brought forth via that activity. The reorganization of available schemes into ones that are new to the learner has been explained via the key construct of reflective abstraction (Piaget, 1985), which has been further articulated via the Ref*AER mechanism (Simon & Tzur, 2004). This mechanism commences with a learner's assimilation of (mathematical) situations into her available schemes and setting the learner's goal for which an activity sequence is called up and executed. Regulated by the goal, the learner's mind performs two types of comparison that constitute the Ref*AER mechanism. The first (Reflection Type-1) continually and automatically compares between the effects of the activity and the learner's goal (Tzur, 2011a). When the learner notices effects of the activity that differ from the anticipated goal, results of this comparison are sorted and stored as records of novel activity-effect dyads. The second (Reflection Type-2) is not necessarily automatic, but it can be prompted from within or outside the learner's mental system. It consists of comparison among a variety of situations in which the recorded activity-effect dyads are called upon, which can bring about abstraction of the activity-effect relationship as a reasoned, invariant anticipation. This invariant anticipation involves a reorganization of the situation that brought forth the activity in the first place, that is, of the learners' previous assimilatory conceptions. It is in this sense that the Ref*AER mechanism highlights the key role of teaching with variation; a teacher can promote opportunities for abstracting a new invariant AER by selecting problems and solutions methods that orient either or both types of reflection in students.

In postulating the *Ref*AER* mechanism, Tzur and Simon (2004) also distinguished two stages in the construction of a new conception. The *participatory (first) stage* is characterized by dependence on being prompted for the activity at issue. At this stage the learner forms a provisional anticipation of *AER*, which includes the capability to reason why the effects follow the activity. This stage is marked by the well-known "oops" experience, in which a learner realizes in retrospect that the effect of her activity could have been anticipated without running it. It is postulated that the first type of reflection, between one's goal/anticipation and the actual effects, is necessary to bring forth construction of a new scheme at the participatory stage. The *anticipatory (second) stage* is characterized by a learner's independent calling up and utilizing of an anticipated *AER* proper for solving a given problem situation. At this stage, which necessitates reflection of the second type, the learner has formed an explicit link between the newly formed *AER* and an array of situations that was not abstracted in the participatory stage. That *AER* becomes an object that can be taken as 'input' for other operations (Sfard, 1991; von Glasersfeld, 1995). It should be noted that in both stages the essence of the anticipated relationship is the same; what differs is its availability to the learner.

Based on Simon's (1995) notion of HLT, Simon and Tzur (2004) and Tzur (2011b) have further linked the *Ref*AER* mechanism to a framework for designing and implementing lessons. In their framework, HLTs are made up of three major components: the learning goal that defines the direction for students' learning (the intended mathematics), the mental activities inferred to be involved in learning that mathematics, and the hypothetical learning (reorganization) process — a prediction of how the students' understanding will evolve in the context of the teaching-learning endeavor. Simon and Tzur (2004) articulated a way to design task-based lessons that therefore proceeds from specifying *both* students' current knowledge and the pedagogical goal (intended mathematics), identifying the mental process of change (activity sequence), and selecting a task sequence that can promote students' use of the activity sequence and the two types of reflections (proper to the stage at which students' conceptions seem to be). Building on empirical studies with using such designs in classroom instruction, Tzur (2011b) further theorized the following, seven-step framework for generating Ref*AER-rooted HLTs: (1) specifying student assimilatory conceptions, (2) specifying a goal for students' learning, (3) identifying a mental activity sequence, (4) selecting and sequencing tasks, (5) engaging students in the task sequence, (6) examining students' progress, and (7) introducing follow-up prompts to orient students reflection (1 and 2).

Tzur's (2011b) seven-step framework can be linked to the widely used, four-component structure of mathematics lessons in China: Reviewing recent lessons, bridging ('*xian jie*') to reactivate long-existing knowledge, teaching with variation to introduce and practice new knowledge, and summarizing the lesson (Huang & Leung, 2004; Jin, 2012; Li, Huang, & Yang, 2011; Tzur, 2011a, 2011b). In particular, we propose the following six components for examining classroom instruction in China:

Component 1: *Tailoring old-to-new*—how relevant previous *AERs* (concepts, skills, and thinking/solution methods) available to learners are explicitly linked and likely to promote the learning of the intended mathematics;

Component 2: *Specifying intended mathematics*—how are goals for student learning identified in terms of important content points, critical/focal content points, and difficult concept points (Jin, 2012; Yang & Ricks, 2012);

Component 3: *Articulating mental activity sequences*—how are mental AER considered and progressively interconnected, for promoting students' advances from available to intended mathematics;

Component 4: *Designing variation tasks*—how are initial and follow-up (prompting) tasks selected and sequenced to create opportunities for students' abstraction of the intended understandings (i.e., for orienting both types of reflection);

Component 5: *Engaging students in tasks*—how do teacher-student interactions proceed through the task sequences to individually and collectively compare and contrast solutions (and problems) and thus discern what remains the same across those variations (i.e., the invariant *AER*);

Component 6: *Examining students' progress*—how do teachers continually assess students' actual work and progress through variation problems and shared solutions (Huang, Mok, & Leung, 2006) as a means to tailor the intended mathematics of today's lesson to Component 1 of the next lesson.

Among these six components, Component 1 is fundamentally important for ensuring that students commence the *Ref*AER* mechanism via assimilating tasks into available schemes. Component 2 sets the concrete goals to guide the teacher's design and delivery of the lesson. Component 3 addresses the trajectory by which students may reorganize the new knowledge (*AER*) while Components 4 & 5 constitute enactment of Component 3 in a classroom. Component 6 provides the ongoing, formative, conceptually based assessment of students' learning and thus a basis for designing the next lesson. We note that summarizing different content points during the work on the task sequence is done not just at the end of a lesson but rather embedded throughout Components 4-6. Such summarizing can promote each student's reflection on how (a) her goals were (or were not) accomplished via activities she carried out and (b) how *AER* dyads were used across solutions (one's own, shared by others publicly). The process of executing, reflecting, and summarizing is aimed to cement the learned concept structure and the anticipatory stage and build connections between different concepts.

Figure 1 depicts how the hybrid model links the theoretical perspectives outlined above and the six components of Chinese classroom teaching. This schematic depiction is organized in a three-tiered way by which we propose to establish the links among the parts of this hybrid model: HLT at the macro tier, teaching with bridging and variation at the intermediate tier, and reflection on activity-effect relationship (*Ref*AER*, with its two stages and corresponding types of reflection) at the micro tier.

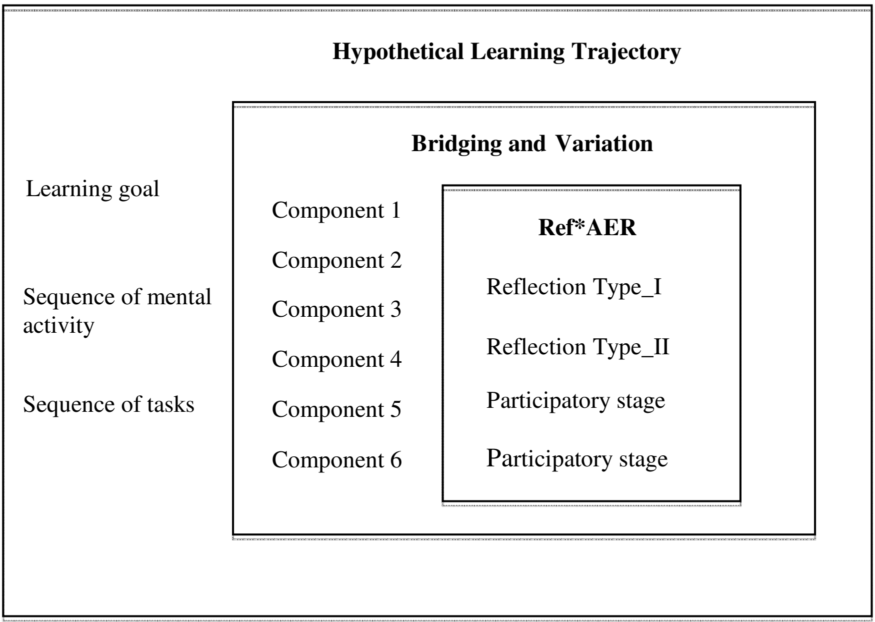

Figure 1. A hybrid model for examining Chinese mathematics teaching in terms of a systematic creation of opportunities for student learning

This hybrid model served to guide our study of the following research questions:

1) What features of enacted HLTs can be identified in the lessons?

2) To what extent do the lessons unfold in line with certain HLTs?

3) How does the unfolding of lessons contribute to the creation of opportunities for students' learning?

3. Methods

3.1 *Data Sources*

The larger data sets we considered for addressing the above research questions were collected in three schools via the Shanghai *Learner's Perspective Study* (LPS) (Clarke et al., 2006). Each of the data sets consists of more than 10 consecutive lessons taught by a competent teacher at his/her respective seventh grade classrooms. In these lessons, the mathematics intended as a goal for students' learning included the concepts of systems of linear equations with two unknowns (SLE2), methods of solving SLE2, and applications of SLE2 methods to solve realistic (or symbolic) problems. For the purpose of our study, we selected one data set from School 3, because it was (a) the most complete set and (b) generated in classrooms in which teachers used the Chinese official curriculum.

All videotaped lessons taken from the Chinese LPS data set focused on that unit of systems of linear equations. According to the textbooks (Curriculum Research Institute, 2005; Shanghai Education Press, 2006), the main goals of learning SLE2 include: (a) understanding the concept of SLE2 and its solution, (b) understanding and mastering various methods of solving SLE2 (substitution method and addition-subtraction elimination method), and (c) applications of solutions to SLE2. This chapter focuses on achieving the first two purposes. In particular, we focused on the middle five lessons in which the concepts of SLE2 and methods of solving SLE2 were taught.

The teacher, Mr. Kang (SH3 in Clarke et al.'s study), taught the concepts of SLE2 in lesson 5, substitution methods in lesson 6, and addition-subtraction methods in lessons 7, 8 and 9. The data for our study consist of lesson plans and videotaped lessons. Mr. Kang earned his bachelor's degree in mathematics from a Teacher Education Institute and had 24 years of teaching experience. In the lessons on which we focused, there were 55 seventh-grade students in the classroom. Each lesson lasted approximately 45 minutes. Within the lesson sequence, some findings have already been reported on the use of mathematical tasks (Huang & Cai, 2010), pedagogical representations (Huang & Cai, 2011), and instructional coherence (Huang & Li, in press). Our analysis in this chapter focuses on how teaching created opportunities for student learning as explained through the hybrid model lens.

3.2 *Data Analysis*

We analyzed data in three iterations. At the macro-tier, we examined lesson plans to identify teachers' intended goals for student learning and classroom situations for promoting it, that is, HLT. Then, we combined this analysis with line-by-line scrutiny of the videotaped (enacted) lessons to identify instructional tasks used in bridging and variation (intermediate tier), while linking those tasks to the HLT. Then, we used the micro-tier lens of *Ref*AER* to explain how the unfolding of Mr. Kang's teaching could have contributed to creating learning opportunities of the intended concepts and skills for his students. Finally, we further articulated a HLT based on the analysis of the enacted lessons to show that the Chinese teacher (implicitly) conducted his lessons in line with the HLT we articulated.

4. Results

In this section we present results of our analysis of algebra lessons intended for 7th grade students' learning to understand and master solutions to systems of linear equations with two unknowns (SLE2).

First, we analyze goals for students' learning during the entire 5-lesson sequence. We gleaned these goals from objectives stated in both the curriculum and the teacher's lesson plans, as well as from the lesson component of *Summarizing* (Jin & Tzur, 2011), in which a teacher typically recaps and emphasizes the focal and important points taught. Next, we present a task sequence used by the teacher during the first lesson and suggest the particular role each task could hypothetically play in student learning. Then, we present data excerpts that enable "zooming-in" on opportunities that the instructional tasks seemed to create for students' learning of the intended ideas. In that part of our analysis we use the *Ref*AER* framework to postulate how student extant conceptions, and teacher-student exchanges when working on particular tasks, could enable the intended learning. Due to space limitation, we selected a few excerpts that provide compelling data for supporting the type of analysis we seek to provide. Finally, we present a schematic HLT that could have served as a basis for the instruction analyzed (noting that neither the curriculum nor the teacher plan/implementation was rooted in this constructivist-based HLT).

4.1 *Intended Mathematics: Goals for Student Learning of SLE2*

The mathematical domain in which students' learning to solve SLE2 is situated includes (a) understandings and mastery of each representation—algebraic (symbolic formula), graphical, and tabular—and translations among these representations, and (b) using those interrelated representations as a means for solving realistic (word) problems. When characterized by the *number of solutions* that SLE2 may have, this domain includes the following three distinct options:

A *single* point (value-pair) solves the SLE2; graphically this means the two lines intersect at the point of solution (e.g., $y=3x$ and $x-y=10$)

Infinitely many points solve the SLE2; graphically this means the two lines fully overlap, as one equation is equivalent to the other but transformed in its written, symbolic form (e.g., $y=3x$ and $2(y-2x+4)=8+2x$);

No point solves the SLE2; graphically this means that the two lines are parallel (e.g., $y=3x$ and $6x-2y-9=0$).

Among these three options, our focus is on teaching that promoted students' learning of algebraic representations of Option 1, because starting with this option makes developmental sense. Within the first option, the Chinese curriculum emphasizes two major algebraic (symbolic) methods. The first method involves *solving SLE2 by the substitution method* (SM; e.g., substitute 3x for y in x-y=*10* to obtain one equation with one unknown: x-$3x$=*10*).The second method involves *elimination* of an unknown via addition or subtraction of equations (ASM; e.g., add y=$3x$ with x-y=*10* to eliminate y and obtain x=$3x$+*10*). Regardless of the method one may use to solve a given SLE2, goals for student learning include (a) recognition of the existence of realistic problems that, when presented algebraically, involve SLE2, (b) understanding that (and why), *as long as equivalence is maintained*, substitution can be applied not only to numbers but also to an entire expression (a critical shift in reasoning, see Kieran, 1992), and (c) knowing when, how, and why to strategically and effectively use steps for finding a solution. Table 1 displays intended and actually enacted goals in the selected lessons, with numbers of lessons corresponding to their indices in Clarke et al.'s (2006) study.

Seen through the lens of *Ref*AER*, an established scheme of Option 1 (single solution) of SLE2 can be described as follows (lessons in which aspects of this scheme were taught are given in parentheses). The situation into which a word problem is assimilated consists of two distinct quantities (symbolized by letters) and given relationship among them (L5). This situation triggers the mental, global goal of finding the values of the asked for, unknown quantities (L5). To accomplish this goal, a first part of the activity sequence is called upon: symbolizing the unknown quantities and the given relationship by a set of two equations, that is, representing the realistic problem in corresponding algebraic formulas (L5). Once two equations are set, they are *re-assimilated* into a threefold anticipation included in the scheme's situation: (a) (infinitely) many value-pairs can satisfy *each equation separately,* but only one such value-pair will satisfy both equations simultaneously (L5), (b) a systematic process, not just guess-and-check, can yield the desired effect of finding this value-pair (L5-L6), and (c) verification will be needed whether or not the found value-pair (effect) is a simultaneous solution

Table 1. Goals for student learning in the selected lessons

Lesson	Intended goals	Enacted goals
L5	- Establish link between realistic problems and SLE2 - Create need and awareness of reason for using SLE2 when solving problems - Grasp what constitutes a SLE2 and its solution - Judge if a system of equations is, or is not, a SLE2 - Judge if an ordered pair is a solution to a given SLE2 (or not)	- Introduced SLE2 and the 'service' they provide via student attempts to solve word problems (initially by guess-and-check) - Restated the definition of SLE2 and its solutions - Emphasized how to check if a solution satisfies both equations
L6	- Know how/when to use the SM for solving a given SLE2	- Experienced and summarized steps of using the SM - Emphasized fundamental reasons for using SM: isolating each unknown (while expressing one in terms of the other)
L7	- Grasp, preliminarily, the Addition-Subtraction Method (ASM) of solving SLE2	- Experienced and summarized steps of using ASM - Summarized conditions of using ASM - Compared and contrasted SM vs. ASM
L8	- Develop facility and flexibility in using ASM to solve SLE2	- Experienced and summarized five steps of using ASM in general
L9	- Develop reasoning for flexibly and strategically selecting a method to solve a SLE2 based on characteristics of coefficients of the unknowns	- Emphasized and summarized key points: simplifying SLE2, standardizing SLE2, and proper selection of method, SM or ASM, based on coefficient characteristics

(L5). This re-assimilation triggers the second part of the activity sequence, namely, choosing and using one of the two methods (SM or ASM) while always maintaining equation equivalence (L6-L9), followed by actually verifying that the found value-pair satisfies both equations. This verification, in turn, triggers returning to the quantities given in the realistic problem and checking the soundness of the value-pair (L5-L9).

Taken together, Table 1 and the articulation of the intended SLE2 scheme showed that in each lesson the teacher identified and enacted very specific learning goals, which were developmentally and mathematically coherent. Thus, these goals could provide direction to the creation of a HLT, which in turn could guide implementing an enacted lesson consistent with the intended goals. Because the last two lessons (L8-L9) focused on promoting students' *flexibility* in using SM and ASM, we shall now turn to data from the first three lessons to analyze tasks and learning opportunities.

4.2 *Activity and Task Sequences for Promoting Student Learning of SLE2*

To articulate an activity sequence and a corresponding task sequence that may promote students' learning of the intended mathematics, one must begin with specifying presumed extant (assimilatory) schemes students are expected to reactivate at the outset of learning (Jin & Tzur, 2011; Simon & Tzur, 2004; Tzur, 2011b). Indeed, we do not have data or knowledge about these extant schemes. Thus, we provide a brief description of what a teacher could have supposed Chinese students have constructed in previous lessons about solving single linear equations. All of these suppositions are rooted in what we know about how solving SLE1 is taught and learned in these classrooms. In particular, we suppose the students have understood the equal sign ("=") as equivalence between two expressions (Kieran, 1992). Thus, they could solve linear equations by operating equivalently on both sides of given equations toward the goal of isolating the unknown. We also suppose that students could anticipate and explain why certain operations maintain (or not) an equivalence, that is, why are such operations *mathematically justified* for creating equivalent forms of the same equation (e.g., "combine like-terms," "add/subtract the same expressions to both sides," "substitute a number for an unknown," etc.). In addition, we suppose that the students could anticipate and explain why using certain operations in a particular order would be *useful* in solving different kinds of linear equations (e.g., "combine like-terms before dividing by the coefficient of an unknown," "multiply all addends above a fraction bar," etc.). Finally, we suppose that students could anticipate the possibility, and equivalence-maintaining effect, of an activity of substituting a number for an unknown (or for an expression), but not necessarily substituting one unknown/expression for another.

Mental activities students could use and reflect upon while capitalizing on the aforementioned extant schemes could include the following sequence. It would be beneficial to begin by reactivating their available processes of recognizing a realistic problem, setting algebraic expressions for the given quantities and relationships, and using guess-and-check to find value-pairs that satisfy *at least one equation* (maybe both). The guess-and-check activity can lead to identifying and discussing why numerous (infinite) solutions are possible. This recognition serves in *creating students' need (perturbation)* for devising a solid method (*Fang Fa*, or in Chinese) for finding the single value-pair that would *necessarily* satisfy both equations simultaneously. Here, an activity of checking a few value pairs—some true and some not—could help to further substantiate this need and how properly substituting values can confirm or disconfirm whether a pair is a solution. This twofold need (find and confirm a value-pair) can lead to recognizing the source of the issue at hand, namely, the infinite number of value-pairs due to the number of unknowns, which can yield a key sub-goal: to reduce the number of unknowns from two to one. This goal can then give good reason for and trigger the use of a substitution activity for SLE2 in which one equation initially expresses one unknown in terms of the other (e.g., $m=k$) and then in gradually more complex variations (e.g., $m=2+k$, $3m+k=5$). The different ways one equation can be operated on to allow substitution can then become the source, invariant anticipation underlying both methods of solution (SM and ASM). That is, students will learn to anticipate that equivalence-maintaining transformations of one or both equations can yield the effect of eliminating one unknown, solving for the other, and then use the found value in the substitution equation to find the necessary, single pairing.

The activity sequence specified above is echoed in the sequence of tasks used by the Chinese teacher in the first lesson on SLE2 (L5). Table 2 presents the tasks (left column) under each of three lesson-components (bold-face) and a possible instructional rationale (right column) for using these activities when seen through the *Ref*AER* lens.

Table 2. Task sequence (L5) for initiating student learning of SLE2

Tasks	Rationale
Task 1: Bridging	
Task 1: Wang goes to the post office to buy some \$1 and \$2 stamps. At least one of each type of stamps will be bought. [Create expressions for] How many stamps of each type can Wang get from the postman?	Reactivate extant knowledge relevant to intended learning (symbolize a word problem by an equation)
Tasks 2-4: Teaching intended ideas with variation	
Task 2: Wang goes to the post office to buy a total of seven (\$1 and \$2) stamps. At least one stamp of each type will be bought. The total amount he spends on these stamps is \$10. How many stamps of each type does Wang get from the postman?	Introduce SLE2 concept as a means for solving realistic problems; trigger guess-and-check, create need for method (*Fang Fa*)
Task 3: For each set of the equations judge if it is a SLE2 or not: (1) x-$2y$=1 and $3x$+y=5; (2) x=2+y and $1/2x$-$3y$=8; (3) x+8=4 and $5x$-$7y$=-2; (4) xy=1 and $2x$+$3y$=1; (5) x+$2y$=4 and x-$2z$=3.	Discriminating "what is a SLE2" from "what is not" (based on definition the teacher provided after class discussion of Task 2)
Task 4: Find possible solutions to each equation in the system of linear equations we created for the word problem in Task 1 (i.e., $2x+y=10$ and $x+y=7$).	Creating awareness of numerous possible solutions and need/concept of a single (value-pair) solution
Tasks 5a-5c: Elaborating with variation	
Task 5a: Given the system of linear equations, (1) $x+y=6$ and (2) $2x+y=8$, judge which of the following value-pairs is a solution to that system: (i) $x=-2$ and $y=8$; (ii) $x=2$ and $y=4$.	Anticipate need to confirm (or disconfirm) if a found value-pair is a solution
Task 5b: Given the value-pair of $x=-2/3$ and $y=2$, identify to which of the following systems of linear equations it is a solution: (i) $3x+y=0$ and $x-2y=-14/3$; (ii) $3x-y=-4$ and $3x+10y=14$.	Anticipate need to confirm (or disconfirm) if a found value-pair is a solution (more 'messy' numbers)
Task 5c: Given the value-pair of $x=1$ and $y=2$, design a system of linear equations so that it has this pair as the solution.	Anticipate overarching structure of *AER*, including multitude of SLE2 for which a given value-pair is a solution

* *Note:* The additional words in brackets indicate the way in which the Chinese teacher seemed to expect his students to interpret the question.

Table 2 shows that the task sequence in the first lesson (L5) corresponded to the presumed extant conceptions and the hypothesized sequence of mental activities we have articulated through the *Ref*AER* lens. Yet, whereas task sequences may provide a glimpse into how a lesson might have been enacted, we believe that finer analysis of learning opportunities requires attention to teacher-student exchanges in the classroom. Analysis of such exchanges is presented next.

4.3 *Promoting Opportunities for Students' Construction of SLE2*

As shown in Table 2, the teacher's use of Task 1 could serve in reactivating extant conceptions of students that would be relevant for the intended learning. Particularly, this "bridging" task provided every student with an opportunity to bring forth her or his extant knowledge about how to symbolize linear relationships among quantities (e.g., setting x and y for the number of \$1 and \$2 stamps bought, respectively, and writing the expression $x+2y$ for the total expenditure). We note that this task was a new-and-easy one for students, not just a review of previously solved homework problems or "warm-up" problems from a different, not relevant topic. Thus, like Mr. Kang, we would have expected students with the extant knowledge as described above to generate such expressions. Moreover, the short whole-class discussion that ensued could reveal key differences and similarities among student solutions to the same problem. Whether capturing the relationship correctly or incorrectly, every student would hear others' (peers, teacher) expressions of the effect of their symbolizing activity. Thus, every student was provided with an opportunity for Type-1 reflection (between one's actual and supposed effect—the expression she or he wrote). Furthermore, students could symbolize correct solutions by different letters (e.g., $a+2b$), which would provide an opportunity for Type-2 reflection (across activity-effect instances). Such a reflection could lead to cementing a previously constructed invariant—that the defining issue for a mathematically justified solution is not the letter chosen but rather the way the given relationship was symbolized (e.g., comparing $x+2y$ with $x+(y+2)$ where letters are the same but relationships differ). Through these two types of reflections on a variety of solutions to Task 1, students who were not yet at the anticipatory stage of the supposedly known symbolizing activity for realistic problem situations received further opportunity to make sense of and master this concept. Once the extant, relevant conceptions were reactivated via Task 1 (bridging), the teacher moved to teaching with variations, using three tasks designed to focus on the lesson's goal.

4.4 *Promoting Learning of What Constitutes SLE2 and Its Solution*

We turn to discussing Tasks 2-4 and how they could promote opportunities for students' learning of the new, intended mathematics. In that 3-task sequence, Task 2 could promote students' understanding of which type of realistic problems is symbolized by a system of linear equations with two unknowns (SLE2) and the issues involved in solving such a system (hence, the realistic problem). These issues would be revealed through students' attempts to create proper equations (SLE2 or single equations), listing several solutions of each linear equation (e.g., making a table), and detecting a solution common to both equations. Following a whole class introduction of the meaning and definition of SLE2 based on students' work on Task 2, Task 3 could then help them clarify this new idea by learning to distinguish what is SLE2 from what it is not (see Xie & Carspecken, 2008), that is, to deepen their understanding of what constitutes SLE2. Task 4 would then promote opportunities to learn via recognizing that there are numerous solutions to each equation separately and hence to the need (perturbation) for a solution method. Below, we further analyze learning opportunities created by each of these three tasks.

Students' work on Task 1 and the class discussion of possible solutions produced an expression with chosen unknowns: $2x+y$. Mr. Kang then moved on and asked them to work individually on Task 2, with the clear norm established in his classroom that different solutions would then be shared. Key in sequencing the tasks is that Task 1 and Task 2 consisted of identical relationships between the given quantities (two types of stamps). Reactivating the *AER* of representing such relationships algebraically could support students' extension of it to a linear equation for the same relationship and a constraint, namely,

$2x+y=10$. Said differently, a learning opportunity was promoted because students have just used their extant schemes to produce one expression $(2x+y)$ needed for the equation, which could enable each student to, at least, start the work independently. At this point, to find out how many stamps of each type would Mr. Wang get, they could begin generating solutions to this single equation (e.g., $x=1$ and $y=8$, or one \$2 and eight \$1 stamps). Some students may have also noticed the other condition and tried producing an equation and solutions for it, too (e.g., $x+y=7$, and value-pairs such as -2 and 9, or 6 and 1, etc.). All in all, the teacher (and observers) could suppose that each student had at least a few value-pairs to solve $2x+y=10$ and possibly value pairs to solve $x+y=7$ (but not necessarily the pair that solves both simultaneously). Thus, Mr. Kang proceeded to the sharing of solutions, as presented in Excerpt 1 below.

Excerpt 1: Whole class discussion of multi-solutions (teaching with variation)

T: .... in the problem we just solved [individually], according to the condition that the total amount of money is \$10, we set this equation as $2x+y=10$, and we found four solutions of the equation by making a table. (Mr. Kang shows a table on the screen, including the solutions: $x=1$ and $y=8$; $x=2$ and $y=6$; $x=3$ and $y=4$; $x=4$ and $y=2$). According to the second condition, that the total number of stamps is 7, we set the equation $x+y=7$; and then we obtained 6 solutions for that equation. (Mr. Kang shows another table on the screen, including the solutions: $x=1$ and $y=6$; $x=2$ and $y=5$; $x=3$ and $y=4$; $x=4$ and $y=3$; $x=5$ and $y=2$; $x=6$ and $y=1$). According to the word problem, the values of x and y must satisfy the first equation $2x+y=10$ and the second equation $x+y=7$, too. That is, the value of the pair of x and y is a solution of the first equation and the second equation as well. Can you find such a solution?

Ss: There is.

T: Which one?

Ss (in unison): $x=3$, $y=4$.

Excerpt 1 indicates how the whole class discussion, following individual students' work on Task 2, could promote attention to and awareness of an essential need for solving SLE2, namely, finding a value-pair that solves *both* equations simultaneously. We note the teacher's start with the $2x+y=10$ equations, which he seemed to suppose all students could use. Different students provided the value-pairs in the table during the sharing of individual solutions. The teacher then proceeded to the second equation ($x+y=7$). Some students may have not produced this equation on their own. However, they have seen it during the sharing by those who did produce the second equation. This provided all students with a learning opportunity as they could *compare and notice an effect* of the activity of producing an equation and value-pairs for both conditions in the realistic word problem that differed from what they might have produced on their own. It is precisely this effect that turned out to be the focus of the teacher's next probing. First, he emphasized the need to find a value-pair for x and y that satisfies both conditions (and hence, both equations), and recast this pair as a solution for *both* equations ("That is, the value of the pair of x and y is a solution of the first equation and the second equation <u>as well</u>"). Then, he asked students if they could find such a value-pair. The students' unison responses ("There is") indicated they could apply the stated, general anticipation to the value-pairs in both tables and identify the pair ("$x=3$, $y=4$") that solved both equations.

Following this individual and whole-class work on the first two tasks, Mr. Kang introduced (PPT on the screen) the concept of SLE2 and the notation used for it. Particularly, he emphasized that such a system needs to include two equations, each consisting of both unknowns—none at a power higher than 1 and each multiplied only by a constant number (including 1 or -1). He then engaged the students in Task 3 (determining which system is SLE2) and in finding a way to solve the SLE2 created for Task 2. The latter move is of specific interest, as the answer (value-pair) has already been discussed. We thus point out the benefit of orienting students' attention onto possible methods, besides guess-and-check, they could use to solve the system. At this point, Mr. Kang moved on to Tasks 5a and 5b, which we postulate could orient student reflection on and anticipation of a central effect of their activity, namely, the simultaneous nature of a solution (value-pair). By asking students to check a few instances of value-pairs, starting with rather simple ones (Task 5a) and progressing to more complex ones (Task 5b), he used variation to create an opportunity for both reflection Type-1 (one's verification to each instance could be correct or incorrect) and Type-2 (comparing the invariant *AER* used across all value-pairs). A particularly important Reflection Type-2 instance can be seen in Excerpt 2, which presents data from the class discussion about whether (-2, 8) was a solution for the SLE2 of $x+y=6$ and $2x+y=8$.

Excerpt 2: Whole class discussion of value-pair (-2, 8)

T: Please think about how to judge whether a pair of [ordered] numbers is the solution to a given system of [linear] equations. … We ask one classmate to express the solution. You [pointing to Eliza], how do you think about this, please tell us your thoughts.

Eliza: (….)

T: How to compute when substituting the value-pair into the equations?

Eliza: $x+y=6$, $2x+y=8$, (…), $-2+8=6$, and then substituting into the second equation.

T: Substituting into the second equation. This means substituting into each of the two equations, is it not? Good. Let us do it together by following this thinking method. Substituting $x=-2$ and $y=8$ into the first equation. Please do not use your pencils; let's just do it together. Let us speak out together … (here, the entire class talked in unison while the teacher wrote down on the board what students said.)

Ss: $-2+8=6$; the left side is equal to 6; because the left equals the right, $x=-2$ and $y=8$ is a solution of the first equation.

T: Is it finished?

Ss: Substituting $x=-2$ and $y=8$ into Equation 2, the left side equals $2\times(-2)+8$, which is 4; the right side is equal to 8. Because the left side is not equal to the right side, $x=-2$, and $y=8$ is not a solution of Equation 2 [The teacher continued writing down what students said].

T: Is it finished?

Ss: No

T: Why?

Ss: Because … so…

T: So, … So… ?

Ss: So $x=-2$, and $y=8$ is not the solution of system of equations $x+y=6$ and $2x+y=8$ [The teacher wrote down what students said].

Excerpt 2 indicates two key points about the teacher's use of value-pair instances to check the validity of solutions. The first and obvious point is orienting students' attention to the key requirement—to be considered a solution of a given SLE2 a value-pair must satisfy each of the equations simultaneously. We do not have data to judge Mr. Kang's knowledge about the student (Eliza) he asked to provide the answer first. It is quite possible that the teacher knew Eliza was still struggling and thus invited her to share (Jin, 2012, had shown how Chinese teachers know who among their students needs further support and thus frequently call upon them). We do get a sense that, for Eliza, the activity of substituting *x-y* values from the pair into each equation (one learned when solving single equations) was not yet anticipatory. But once prompted for the activity, Eliza quickly resumed and successfully completed it for Equation 1. Seen through the *Ref*AER* lens, one would consider the possibility that Eliza has been at the participatory stage of conceptualizing the link between substitution of values in a pair and the validity of that solution.

The second point indicated in Excerpt 2 is the teacher's purposeful use of a non-example—an instance that violates simultaneity. Such a pedagogical move is prevalent among mathematics teachers in China (Jin, 2012), and seems rooted in the Chinese dialectical perspective about knowing and learning (Xie & Carspecken, 2008). This move creates a learning opportunity in that it orients students' attention to distinguishing 'what is' from 'what is not', which provides further impetus and input for Reflection Type-2. Mr. Kang further promoted such a reflection by changing the class activity from one student sharing to a whole class unison (out loud) recitation of the entire activity sequence: substituting the value-pair into Equation 2, figuring out it does not satisfy the required equivalence (left-right sides), and explicitly linking this effect to the global goal of the entire substituting activity, namely, determining the given value-pair was *not* a solution.

In regard to this last point, we note that at this early stage of learning about SLE2 most of the class might still have been at a participatory stage, as indicated in their answer to the teacher's question ("Is it finished?"). While their initial response ("No") could reflect interpretation of Mr. Kang's intonation that they were not yet done, the hesitation that followed (line 12) suggested they needed more thinking to make the conceptual leap from finding that the value-pair does not satisfy one of the equations to concluding that it is therefore not a solution to the SLE2. All in all, by using a few instances in which the value-pair satisfies both equations and other instances in which it satisfies only one equation (i.e., teaching with variation), the teacher created an opportunity for students' abstraction of the goal set for their learning. Here, they learned a key principle (and method) that remains invariant: checking equivalence of both sides in each of the two equations (activity) is anticipated to yield the effect of determining a solution (or not) to a given SLE.

As a move that could further promote students' construction of the new anticipation of SLE2 and what constitutes a valid solution for it, Mr. Kang proceeded to Task 5c (designing SLE2 for which a given pair, $x=1$ and $y=2$, is a valid solution). Students were first working on this task in pairs, and then shared solutions in a whole class discussion. Conceptually, this task is more challenging not just or mainly because it is totally open—it literally has infinitely many answers. Rather, the conceptual challenge is in inverting one's thinking from 'seeing' the anticipated effect of an activity of substitution to determining a process by which solutions can be traced back to a source SLE2. Such reversal of thinking is a hallmark of what Piaget (1985) referred to as a mental operation. By engaging students in starting the activity from what, to this point for them, has been the effect of a different activity, seemed to create a powerful opportunity for cementing the intended concept. Seen through the *Ref*AER* lens, asking students to pose a problem for a given solution entails (a) assimilation of the givens into the newly formed activity-effect relationship dyad (2nd and 3rd parts of a scheme) and (b) relating this dyad back to an instance of a situation that could trigger such an *AER* in the first place. Excerpt 3 presents data from the whole class discussion following students' work on Task 5c in pairs.

Excerpt 3: Whole class discussion of Task 5c (Kang writes what students say)

T: Do you have your design?

Ss (out loud in unison): Yes.

T: Good. Let us exchange our solutions. Who would like to come up first? (The majority of students raise their hands). Good. Felix?

Felix: $x+5=6$, and $x+2y=5$.

T: Classmates, can you judge whether it is correct or not?

Ss (out loud in unison): Correct!

T (to Felix): Please sit down. Who will go next? Denson.

Denson: $3x+y/2=4$, I am still thinking about the second one [equation].

T: You have not finished. It is good to have one. Donnie, can you complement the second equation?

Donnie: $2x-5y=-8$.

T: Class, is it correct or not?

Ss (out loud in unison): Correct.

T: (Invites two more students to share their SLE2, each time the class confirms in unison. Then, he summarizes two key points: the definition of the system and how a given value-pair, such as $x=1$ and $y=2$, could be a solution of many different equations (that is, it simultaneously satisfies both equations in each system.)

Excerpt 3 indicates that most (if not all) students in Mr. Kang's classroom benefited more learning from their work on Task 5c. Key to their learning seemed to be an opportunity to work with peers. While some students might have been struggling on their own, they all received a chance to discuss the creation of two equations. Indeed, not all students solved the challenging task successfully. On one hand, students like Felix could complete the creation of SLE2 on their own (and justify it during the pair work). Furthermore, students like Donnie could assimilate 'half-a-solution' (provided by Denson) and complete it to have a second equation that the given value-pair also satisfied. On the other hand, students like Denson needed more work to abstract the inversion of the *AER* and solidly link it to the SLE2 situation. Nevertheless, Denson's response suggested that he had abstracted the key principle (anticipation) learned in the previous parts of the lesson. He seemed to anticipate not only the need to check that a value-pair satisfied one equation but also that another one would be needed ("$3x+y/2=4$, I am still thinking about the second one [equation]") to satisfy the same value-pair. Critically, Mr. Kang created a learning opportunity for all students, Denson included, by orienting their Reflection Type-2 across different instances in which the same link between the situation and the newly formed *AER* is established and anticipated. These SLE2 instances, produced (one equation after another) by most students during the pair work and shared during whole class discussion, could promote inversion of the *AER* as a means for cementing it within the SLE2 scheme.

In summary, the sequence of tasks used by Mr. Kang in Lesson 5 seemed to promote all students' learning at least up to the point of inverting the method of verifying if a value-pair is a solution. In particular, students have actively abstracted the concept of (what constitutes) a solution to a system of linear equations, including (a) the need to solve a word problem, (b) building the concept of solution to a system of linear equations as transformation of the concept of a solution to one linear equation (Task 1), (c) deepening the concept through resolving a perturbation (not just guess-and-check) and solving a variety of problems (Tasks 2-5b), and (d) applying the concept through open-ended problem solving (Task 5c). The work on the systematically progressing variation of tasks seemed to engender students' development of a deep understanding of the intended concept: its connection to previous concepts and discernment of the critical feature for the new concept (simultaneous solution to both linear equations). The culminating, challenging experience of *creating* many different SLE2 (Task 5c – pair work and sharing in whole class) seemed to further cement the newly abstracted anticipation in close linkage with solving (and posing) problems. This analysis through the *Ref*AER* lens—of opportunities created by Mr. Kang for his students' learning—provides a basis for articulating a HLT that could underlie his teaching.

4.5 *Articulating HLT for the SLE2 Unit*

As explained in the conceptual framework section, a HLT (Simon, 1995; Simon & Tzur, 2004) consists of the following three components: *goal* for student learning, *mental activities* inferred to be involved in accomplishing it, and *hypotheses about the reorganization of the cognitive process* from current to intended mathematics. It is important to note that HLT, a central notion of a pedagogy rooted in a constructivist stance on learning, does not seem to be how Chinese teachers in general and Mr. Kang in particular devise their lessons. Nevertheless, the meticulous selection of tasks and the way lessons unfolded enable us to use HLT as a tool for further portraying a constructivist-based rationale that could potentially inform their practice (Jin, 2012). This theoretical portrayal focused on two key phases of the Chinese lesson structure, *bridging* (Jin & Tzur, 2011) and *variation* (Gu et al., 2004; Wong, 2008).

The goal (scheme) of a HLT potentially underlying the Chinese curricular unit taught by Mr. Kang has been articulated at the start of the Results section (see Table 1 for details). In a nutshell, it consists of an assimilatory situation into which word problems are assimilated by recognizing and quantitatively relating two different quantities in a word problem. This situation triggers a goal of finding a value-pair that satisfies both equations simultaneously. In turn, this goal triggers an activity sequence consisting of equivalence-maintaining operations on one or both equations to (a) eliminate an unknown (transform a SLE2 to a linear equation with one unknown (extant knowledge), (b) solve for the value of the other unknown (working with the extant knowledge), (c) use that value to find the value of the eliminated unknown, (d) check that the value-pair does, in fact, satisfy both equations (building the connections between the new concept and previous knowledge), and (e) check that the value-pair also solves the word problem soundly. The last step of the activity sequence is relevant when SLE2 was not simply introduced and solved algebraically but was produced to reflect on the relationships among quantities in a word problem. The result part of this scheme is therefore a value-pair that can be found methodically, which also demarcates the intersection point of the two lines that correspond to the equations. When such a scheme is established, it is further extended to recognize at least two different methods for eliminating one unknown (substitution method and addition-subtraction method) and strategically choosing one of them depending on the characteristics of coefficients of the two equations. The multiplicity and flexibility in methods of solving SLE2 help students realize the invariant feature of eliminating unknowns, which is the core mathematics thinking method (*Fangfa*) in this unit. Due to space limitations, using data and analysis about this latter part of the curricular unit will appear in a forthcoming paper.

Mental activities inferred as part of HLT for students' learning of SLE2 (here, for the SM method), when assuming they know how to solve a word problem and the value-pair cannot be trivially guessed, include the following (not necessarily in this order):

1. Identifying the quantities that are relevant to the question asked (e.g., the number of \$1 and \$2 stamps bought);
2. Selecting letters to stand for those quantities, including what, precisely, is the meaning of each letter (e.g., x = number of \$2 stamps Mr. Wang buys);
3. Determining the relationships that link the quantities (e.g., addition, division, exponentiation, etc.) and setting corresponding equations for them, which usually includes a visible action of writing down the equations (e.g., total number of stamps is the *sum* of each type separately, hence $x+y=7$; total amount paid is the sum of products of each type multiplied by its price, hence $\$2x+\$1y=\$10$ or simply $2x+y=10$);
4. Selecting possible ways to eliminate one of the unknowns and selecting one that seems useful (e.g., it's easy to "isolate" y in the first equation, as $y=7-x$, and substitute into the second to avoid an extra step of multiplying by 2 if x was isolated);
5. Executing the substitution and solving for the other unknown, which is done mentally and may also be manifested in a visible action such as writing, though this is *not* necessary (e.g., $2x+(7-x)=10$, hence $x=3$);
6. Recognizing that, unlike in a single equation, the equations are not yet solved and thus setting a sub-goal of finding the substituted unknown accomplished by reverse substitution of the found value (e.g., $y=7-3$, hence $y=4$);
7. Checking that the unknowns, as defined in Step #2, satisfy both equations simultaneously (i.e., no errors were made that created a "misfit" value pair), and possibly also graphing the solution (not used by Mr. Kang) to indicate the intersection point;
8. Linking the value-pair back to the given word problem and judging the reasonableness of this solution (e.g., if the solution to SLE2 included zero as a value it does not fit because at least one stamp of each type was bought).
9. Examining and adjusting any of the previous steps in case a 'misfit' of values for the SLE2 and/or problem was found, until all conditions are satisfied.

As the examples given in parentheses illustrate, these nine, goal-directed mental activities were all promoted within the design of lessons found in the Chinese curriculum and taught by Mr. Kang (not including graphing the SLE2 in Step 7). Furthermore, the lessons seemed to sequence student engagement in these activities in a progressive way. The lessons unfolded from reactivating (bridging) available schemes of solving single linear equations with one unknown and using such a solution to create a need for methods. This led to identifying relationships among quantities and generating algebraic representations of those relationships. Before solving SLE2s, students were then engaged in a few activities that promoted, through variation of solutions, their ability to set a proper goal and know when it is accomplished (i.e., when a value-pair is the solution to the SLE2 and thus to the word problem). This identification, through distinguishing "what is" from "what is not" (distinguishing what is invariant from what varies is the key in learning according to variation theory (Marton & Tsui, 2004)), gave way to more variation of problems/SLE2s solved via two different methods (moving from easy, almost trivial, to quite challenging). Consequently, students could strengthen each mental activity and link all of them into a meaningful sequence.

Indeed, the nine mental activities—some of which may be accompanied by an individual's physical actions—are necessarily carried out in one's mind. Nevertheless, a student carrying out each of these activities may greatly benefit from repeatedly being engaged in working with others, or detecting others' observable actions when solving the problem. Key to such benefit would be that the individuals (a) establish compatible goals, which would guide one's making sense of someone else's visible actions (and inferred mental processes) and (b) can meaningfully link the observable actions of others back to one's own particular mental activity and its role/purpose within the entire activity sequence. This last comment is crucial, as too often teachers may engage their students in working cooperatively while, just like a teacher's actions and language, the work of peers may be meaningless to an individual due to *incompatible goals* and/or having no internal activities by which to assimilate and interpret the peers' observable actions. In this regard, we note that Chinese students are educated learners (Cortazzi & Jin, 2001). They seem to understand teachers' intentions (i.e., shared goals), know the instructional flow (Huang & Barlow, 2013), and are willingly and attentively listening to the teacher or peers' talks. As our analysis showed, progress made by students in Mr. Kang's class differed, but more often than not the problems posed and solutions shared seemed to be within every student's reach.

We culminate the could-be HLT by articulating a *hypothesized reorganization of the cognitive process* by which students may build on their existing algebraic knowledge (specified earlier) and transform it into the intended mathematics (SLE2 as explained in the *HLT's goal* above). To initiate this reorganization, students' available scheme of linear equations with one unknown, and word problems such equations can help solve, could be activated (e.g., Tasks 1 and 2 in Lesson 5). After creating single equations to symbolize each of two single relationships among the quantities, a first reorganization may take place through noticing the conditions required by both equations (e.g., Mr. Kang's repeated questions such as "are we done?"). That is, a first step in the reorganization, which seemed well oriented by the curricular unit and its implementation, concerns the new, intended anticipation that some problems consist of two quantities and double relationships among them, and hence required creation of two equations. Through generating different value-pairs for each equation, this new anticipation can in turn orient students' attention onto the variation of these pairs, and hence the need to identify one pair that solves both equations simultaneously. Consequently, through students' use of their known guess-and-check activity, and Reflection Type-2 on its effects, a need (perturbation) could arise for developing a method to figure out that pair. Bridging lesson portions used by Mr. Kang seemed to be directly geared at and quite effectively accomplishing this first set of transformations from single linear equations to SLE2 in Lesson 5.

To further reorganize previous knowledge, variation of substitutions could promote both types of reflection. Initially, a few substitutions could be quite simple, building on previously known ones such as substituting a number for a letter/unknown (guess-check activity). These simple substitutions could give way to gradually more complex ones, in which a letter is substituted for another letter, and then an expression is substituted for a letter and/or expression (e.g., the SLE2 Mr. Kang asked students to solve in Lesson 6: $2x+3y=4$ and $5x-2y=29$). This variation of problems in which substitution is used creates an activity-effect dyad for each problem—the activity sequence used for the solution and the effect it brought about (e.g., a value-pair that, students notice, does *not* solve both equations simultaneously). Reflection Type-1 is continually applied to and its results are recorded in terms of dyads linking solution activity and the value-pair found for it (e.g., first substituting for y or for x to find the value of one of them). Meanwhile, Reflection Type-2 could be oriented across solutions in which similar activity sequences were used (e.g., isolate and eliminate an unknown appropriately in Lesson 6, such as solving systems of linear equations $3x+2y=13$ and $x-y+3=0$; or $3a+4b=0$ and $2a-2b=7$)

Combined, both types of reflection could lead to noticing and abstracting the intended, threefold, new invariant—how two linear equations constitute SLE2, the simultaneity of a justified solution, and the methods (goal-directed activity sequences) that can yield (a) the needed equations and (b) solutions through elimination, isolation, and finally linking of the unknowns.

Moreover, the following word problem was used to reactivate the arithmetic method (learned in elementary school) and the equation method (just learned in previous lessons):

> Siu Ming's family intends to travel to Beijing during the national holiday by train, so they have booked three adult tickets and one student ticket, totaling \$560. After knowing this news, Siu Ming's classmate Siu Wong would like to go to Beijing with them. As a result they bought three adult tickets and two student tickets for a total of \$640, can you calculate how much it costs for each adult and student ticket?

The arithmetic method (e.g., $640-560=80$ is the student's price and $(560-80)\div 3=160$ is the adult's price) lays the foundation for discovering a method of solving $3x+y=560$ and $3x+2y=640$ by subtraction. This unintentional solution creates a need for exploring a new method: eliminating unknowns by addition and subtraction. Reflection (Type-1) on the *AER* for solving a SLE2: $3x+2y=8$ and $3x-2y=4$, leads to an understanding of the process of eliminating unknowns by using addition and subtraction, and the rationale for using the method (Properties of equation). Then reflections (Type-2) on the *AER* with observation of the characteristics of coefficients of SLE2s such as: $5x-4y=7$ and $5x+y=4$;

3x+7y=9 and *4x-7y=5*, leads to noticing the characteristics of coefficients when addition or subtraction could be used. Reflection (Type-2) on the experience in solving *x+2y=8* and *x-2y=4* in three different ways creates a need to develop flexibility in selecting appropriate methods, which was the focus of the following two lessons. Moreover, in contrast with different methods (substitution, addition and subtraction) the invariant principle of eliminating unknowns was discerned. It is the fundamental thinking method underlying all methods of solving SLE2. Thus, taking two types of reflection on various experiences (or *AER*) together provides the opportunities to (1) reactivate students' extant knowledge of the arithmetic method and equation method, (2) create a need for exploring the addition and subtraction method, (3) learn the process of using the method appropriately, and finally (4) create a need for exploring the flexibility in selecting appropriate methods of solving SLE2s.

It seems that, although not informed by the conceptual analysis offered in this chapter, the Chinese curricular unit and the way Mr. Kang enacted it in class, could effectively foster the above reorganization process.

5. Discussion and Conclusion

In this chapter we analyzed data from algebra lessons in Chinese classrooms (Clarke et al., 2006) to demonstrate and support our twofold, central thesis: (a) pedagogical approaches used by Chinese teachers hold high potential for promoting students' learning (interrelated conceptual and procedural understandings), and (b) the hybrid model (*Ref*AER*, teaching with bridging and variation, HLT) provides a useful tool for explaining and realizing this potential. In this final section we discuss each of these.

5.1 *Promoting Conceptual Understanding and Procedural Fluency in Tandem*

This study demonstrates a way of structuring a mathematics lesson that is consistent with a constructivist stance on learning as transformation in students' available conceptions. Chinese teachers use such structuring consistently and competently, via the two key lesson components of bridging and teaching with variation. In bridging, they strategically reactivate relevant extant knowledge and trigger intended learning objects. In variation, they capitalize on and promote change in this knowledge through variation of problems and solutions that create opportunities for students to reorganize and consolidate new, intended knowledge. The deliberate use of these two pedagogical strategies makes it possible to (a) connect the goals for student learning with extant knowledge, (b) create a need for exploring new mathematics, (c) provide effectual situations for action and reflection, and (d) achieve the learning objects.

We have analyzed how these two teaching strategies unfolded (particularly solving variation problems progressively) while promoting both types of reflection postulated to underlie the reorganization of new conceptions. Reflection Type-1 opened the way for students to notice and link novel activity-effect relationship (*AER*) dyads (e.g., there is only a single value-pair that simultaneously satisfies both equations of SLE2). Reflection Type-2 then deepened understanding of the new concept through contrasting different instances of *AER* dyads and discerning what (necessarily) remains invariant across a variety of situations. Thus, our analysis of learning opportunities indicates the power of teaching with variation to deepen and consolidate conceptual understanding and procedural fluency concurrently. In this sense, our study provides further articulation of the theory of variation (Gu et al., 2004 Huang et al., 2006) and of the potential of Chinese mathematics instruction to effectively promote student learning.

5.2 *A Hybrid Theory for Examining Classroom Instruction*

This study provides a fine-grained examination of learning opportunities created via Chinese classroom instruction—particularly the general features of teaching with variation. This examination was situated in a model that combined a constructivist way of explaining learning as cognitive change (*Ref*AER*) with a corresponding stance on teaching (HLT). Both *Ref*AER* and HLT focus on students' conceptual development while teaching with variation is a broad pedagogical practice and overarching strategy of teaching mathematics in China. Thus, the study provides a lens for deepening our understanding of Chinese mathematics classrooms. The *Ref*AER* mechanism postulates how learning may happen, and HLT projects a goal-oriented, task-driven, progressive journey of cognitive reorganization. The importance of our analysis is then seen when these accounts are integrated with the processes of selecting and implementing tasks used by Chinese teachers. That is, teaching via bridging and variation seems to provide feasible and practical strategies for creating, sequencing, using, and adjusting strategically designed mathematical tasks. This study demonstrates the power of the three-tiered, hybrid model (Fig. 1) for examining and guiding mathematics instruction. At the macro tier, HLT provides an overall guidance for setting learning goals, designing mental activity sequences, and articulating processes of cognitive reorganization. At the intermediate tier, teaching with bridging and variation provides practical tools for deliberate design of problem situation and tasks that are likely to create opportunities for such reorganization and thus lead to achieving intended goals for students' learning. Moreover, strategically using variation problems is employed not only in classrooms (Cai & Nie, 2007; Huang et al., 2006) but also in textbooks (Sun, 2011). At the micro tier, *Ref*AER* provides a lens to examine connections between tasks and students' learning, which constitute the crucial process of promoting students' understanding of the intended mathematics. We note that, in China, there is no explicit notion about HLT or *Ref*AER*; yet, Chinese teaching practices seem highly relevant and linkable to these accounts, as they emphasize foundation, connection, and progression in mathematic teaching and learning (Shao, Fan, Huang, Li, & Ding, 2012). Thus, the tradition and practice in China may support teachers in developing relevant ideas about learning progressions and how learning as cognitive reorganization (*Ref*AER*) can be explicated.

5.3 *Understanding Chinese Mathematics Instruction beyond the Classroom*

We culminate this chapter with an extension of our study, and the proposed three-tiered model, to pedagogical aspects beyond the 'walls' of the classroom. Our extension is rooted in the premise that teaching is a cultural activity (Stigler & Hiebert, 1999), which entails that classrooms should reflect relevant aspects of cultures in which they are embedded. Concerning mathematics teaching in China, studies revealed that Chinese mathematics teachers typically have strong mathematics knowledge for teaching (Huang, 2014; Ma, 1999). Moreover, Chinese teachers develop much of their professional competencies through extensively studying the mathematics in textbooks they use (Ding, Li, Li, & Gu, 2012; Ma, 1999) and participating in various teaching research activities such as lesson group planning (Huang, Peng, Wang, & Li, 2010; Li, Qi, & Wang, 2011). These activities support teacher development of strong mathematics and pedagogical content knowledge necessary for teaching with variation and the consistency needed to implement textbooks in classroom with high fidelity (Huang, Ozel, Li, & Rowntree, 2013). Thus, lesson plans transformed from textbooks can possibly be organized as intended HLTs and enacted in the classrooms. Then, Chinese teachers' extensive observations of peers' classroom teaching, and their development of public or exemplary lessons, can provide teachers with opportunities to continuously anchor their expertise in a theoretical stance on learning.

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Chapter 4

Achieving Coherence in the Mathematics Classroom: Toward a Framework for Examining Instructional Coherence

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Coherence has been identified as an important factor in fostering students' learning of mathematics. In this chapter, by applying classroom discourse theories, we propose a framework for examining instructional coherence through a fine-grained analysis of a video-taped lesson from China. The lesson was chosen because it has been recognized as a model lesson for instructional coherence. Based on a careful analysis of instructional coherence on multiple levels of classroom discourse, we explored discourse strategies the teacher used to achieve instructional coherence in the classroom, as well as the features of classroom instruction in China.

Keywords: instructional coherence, classroom discourse, international study, mathematics learning, student voice, mathematics classroom

1. Introduction

A frequently cited result from the Third International Mathematics and Science Study (TIMSS) is that there is a direct relationship between students' exposure to coherent instruction and their performance. This finding has prompted increased interest and research in instructional coherence in mathematics education. It has generally been acknowledged that instructional coherence can enhance student understanding by providing students with connected mathematical ideas. Recent international studies in mathematics education have identified a high degree of instructional coherence as a distinguishing feature in classrooms in China (Cai, 1995, 2005; Cai & Wang, 2010; Cai, Ding, & Wang, 2014; Chen & Li, 2010; Fan, Wong, Cai, & Li, 2004; Ma, 1999; Wang & Murphy, 2004) and other East Asian countries such as Japan (Fernandez, Yoshida, & Stigler, 1992; Sekiguchi, 2006; Shimizu, 2007; Stevenson & Stigler, 1992; Stigler & Hiebert, 1999). Researchers have tried to determine the important features of instructional coherence in these classrooms. So far, most studies have focused on analyzing connections between the pieces of mathematical content taught in a class.

Although content connections are important for instructional coherence, two salient constraints in the literature have limited our understanding of the complex and dynamic process of achieving instructional coherence. First, the content connections described in the literature are quite linear and are based on researchers' general observations with very limited, fine-grained, and systematical analyses of discourse data. According to discourse theory (cf. Tomlin, Forrest, Pu, & Kim, 1997), content connections in discourse are not linear but rather form a complex system with thematic connections on multiple levels. The over-simplified description of content coherence in the classroom limits our understanding of this complexity. Second, some research findings (Stevenson & Stigler, 1992; Truxaw & DeFranco, 2008; Wood, 1998) have revealed that merely having interrelated content does not guarantee student mathematical understanding. This is because learning is realized through complicated intermental interactions mediated by dynamic class discourse (Cobb, Jaworski, & Presmeg, 1996; Cobb, Yackel, & McClain, 2000; Truxaw & DeFranco, 2008; Voigt, 1996). Therefore, in addition to content analysis, it is critical to understand how the teacher uses classroom discourse to make content coherence clear for students learning.

To complement previous research, this chapter uses a fine-grained analysis of discourse data from a sixth grade classroom in China to discuss the ways the teacher uses discourse strategies to achieve instructional coherence. By applying theories of discourse analysis, and particularly of classroom discourse, this chapter attempts to develop a multi-level model for analyzing instructional coherence.

2. Theoretical Framework

In this section, we review the literature from two perspectives. First, we review the research literature on studies of discourse coherence and classroom discourse structure. The aim of this part of review is to set up an analytical framework to analyze coherence in classroom discourse. Second, we review the literature on coherence in mathematics classrooms and in Chinese mathematics classrooms.

However, it is first necessary to explain the construct of instructional coherence. Given that language (both written and spoken) is the main medium through which instructional content is planned and exchanged in the classroom, our use of the term instructional coherence is largely congruent with discourse coherence. As in other studies of discourse, discourse here is treated as both a linguistic structure and a social interactional process. Unlike informal talk at home and in the street, classroom discourse is a "staged, goal-oriented social practice" (Christie, 1995, p. 222). Recognizing its distinct discourse structure and instructional function, Bernstein (1986, 1996) treated classroom discourse as a special genre of discourse and labeled it as pedagogical discourse. Following this vein, we use the term instructional coherence to highlight pedagogical meaning in analyzing thematic and structural connections in classroom discourse.

2.1 *Discourse Coherence and Classroom Structure*

In this study we adopt a multilevel framework to analyze the coherence of mathematics classroom discourse by combining the discourse coherence theory of Tomlin et al. (1997) and the classroom structure theory of Mehan (1979). Here, we will present a review of discourse theory to explain what discourse coherence is and how it can be studied on multiple levels of classroom structure. Then, we will discuss a theoretical structure for discourse strategies used by speakers to achieve coherence.

2.1.1 *Discourse Coherence and its Different Levels*

Discourse is not a sequence of arbitrary utterances with irrelevant topics. Rather, the topics are related each other, thereby making discourses meaningful (van Dijk, 1997). Discourse coherence reflects the degree of relatedness of topics affecting listeners' understanding (Dore, 1985). A coherent story, for example, with thematically related events is easy for the reader or listener to comprehend. However, discourse coherence is not a flat and linear entity because the organization of discourse meaning is hierarchical and interwoven. For example, a story could consist of multiple chapters, each of which consists of several episodes. Each episode could be divided into paragraphs, and further into finer units (e.g., sentences and phrases). Correspondingly, discourse coherence that reveals the connectedness of different semantic units can be discussed at different levels. Tomlin et al. (1997) distinguish discourse coherence on three levels:

1. *Global coherence*: the participants develop a sense of what the overall narrative or procedure or conversation deals with.
2. *Episodic coherence*: the participants are sensitive to smaller scale units which contribute to global coherence but which display an internal gist of their own.
3. *Local coherence*: the participants make sense of the contribution of individual sentences or utterances (p. 66).

Note, however, that the number of levels of coherence can vary according to different structural features of the target discourse and the purposes of the research.

Unlike casual daily conversation, classroom discourse uses unique structures to realize its pedagogical function (Bernstein, 1986; Cazden, 2001; Christie, 2002; Coulthard, 1974; Mehan, 1979). One popular structural model for classroom activity was proposed by Mehan (1979), who suggested investigating classroom activities along sequential and hierarchical dimensions. The sequential dimension follows the flow of the lesson as it unfolds through time from beginning to end. The hierarchical dimension refers to the disassembly of the lesson into its component parts. Along the sequential dimension, Mehan divided the lesson into three phases: opening, instructional, and closing. Discourse in the three phases serves to fulfill goals specific to each phase: to begin class, to exchange academic information, and to end the class, respectively. Along the hierarchical dimension, the activity in each phase is further disassembled into its component parts. The instructional phase consists of a series of topic-related discourse sets (TRS). The most basic unit of dialogue in the class is a three-part IRE sequence: **Initiation**, often by the teacher; **Response**, often by the student(s); and **Evaluation**, often the teacher's feedback.

2.1.2 *Discourse Strategies for Achieving Coherence*

Although the coherence of the content of discourse is important for listeners, the implicit connections in that content do not themselves guarantee that the listeners will be able to integrate the content coherently (Tomlin, et al., 1997). Instead, the ease with which listeners can quickly establish connections between concepts is largely dependent on the speaker's use of discourse strategies that help listeners perceive the thematic connections. Tomlin et al. (1997) noted three kinds of discourse management strategies that a speaker usually considers to help listeners construct a coherent knowledge base: rhetorical, referential, and focus strategies.[1]

A rhetorical management strategy helps listeners to be clear about the goals and main topics of the discourse so they can easily integrate various kinds of information from speech into a coherent picture. A

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[1] Tomlin et al. (1997) included a fourth strategy, the thematic strategy. However its basic function is similar to the focus strategy; that is, to let the listener "know what information is more central" and should be noticed (p. 93). Therefore, we use the term focus strategy in this study to include both the focus and thematic strategies.

referential management strategy helps listeners establish connections between known and unknown knowledge. Typically, this strategy is in play when the speaker frequently refers back and forth between old and new themes. Finally, a focus management strategy highlights the main and important topics in the speech flow.

The fundamental goal in a mathematics classroom is to facilitate student conceptual understanding by establishing connections between different knowledge pieces. Recently, a few studies (e.g., Clarke, 2013; Sekiguchi, 2006) have started to examine instructional coherence from a discourse perspective. However, it remains largely unclear what discourse strategies teachers use to achieve instructional coherence in mathematics classrooms.

2.2 *Comparative Studies of Coherence in Mathematics Education*

Just as a good story keeps readers interested, a mathematics lesson that is organized with highly interrelated themes is more likely to help students achieve conceptual understanding (Fernandez, Yoshida, & Stigler, 1992; Stevenson & Stigler, 1992). Although there has been relatively little research on discourse strategies that contribute to coherent mathematics lessons, in the past two decades educational researchers have started to examine how teachers organize coherent mathematics activities in classrooms. Most of the existing literature has come from cross-national studies comparing U.S. and Asian mathematics classrooms. To compare classroom teaching across cultures, researchers have investigated three aspects of the coherence of activities: coherence of the mathematical topics in classroom activities, discourse transitions explicating the relationship between activities, and interruptions in proceeding with the activities (Fernandez et al., 1992; Stevenson & Stigler, 1992). With this framework, researchers have found that mathematics instruction in China, Japan, and Korea is more coherent than in the U.S.

After observing mathematics classrooms in Taipei (Taiwan), Sendai (Japan), and Chicago (United States), Stevenson and Stigler (1992) noted that the Asian classrooms were structured in a more coherent fashion than the U.S. classrooms. Stigler and Perry (1990) provided further quantitative analysis to illustrate different coherence levels in these classrooms. They believed that a lesson with fewer mathematical topics might be more coherent than one with more topics. Segmenting the classroom flow into 5-minute segments (the median length of segments with ongoing instruction), they counted the number of topics included in each segment. Of all the segments in Sendai classrooms, 75% focused on only one topic, compared with 55% of the segments in Taipei and only 17% of the segments in Chicago. Although this time-sampling method is useful for tabulating the frequency of numbers of topics, it obscures the sequential flow of classroom activity. In particular, marking tallies at regular time intervals fails to reveal the contingent (and sequential) connections among classroom discourse topics (see the theoretical discussion in the previous section).

Other researchers (Grow-Maienza, Hahn, & Joo, 2001; Ma, 1999) have observed that Korean and Chinese teachers also tend to devote an entire 40-minute class to the solution of only one mathematics problem. In such a lesson, a single mathematical topic is discussed from multiple perspectives. For example, Ma (1999) reported that in Chinese primary classrooms, students are often encouraged to solve one mathematics problem using several ways (*yi ti duo jie*). This one-topic design in Asian classrooms helps teachers organize activities around one clear mathematical topic at a time.

In addition to focusing on the number of topics discussed, some researchers have considered the components of mathematics lessons and how they are connected. For example, Leung (1995) observed 36 lessons from grade 1 to grade 3 in four elementary classes in Beijing. He found that in both selected, high-performing inner Beijing schools and remote, below-average rural area schools, all the lessons strictly followed a clear structure that promoted coherence in organizing mathematics activities. Briefly, lessons were organized into four sequential instructional phases: reviewing, teaching new content, student practice, and assigning homework. Although Leung described several teachers' strategies that promoted instructional coherence (e.g., summarizing content from time to time, solving one mathematical problem with various methods), he did not provide discourse data to show how the structure he observed contributed to instructional coherence. For example, it is not clear whether the review phase was connected to the new content teaching that followed. If it was connected, how did the teachers use transitional discourse to connect the mathematical topics in the two instructional phases?

By analyzing a tape of an open-class lesson in a Chinese 5th grade class, Wang and Murphy (2004) investigated the use of transitional discourse between the review and new content phases. They found that the teacher did indeed use transitional discourse to connect the review phase to the new content. For example, she highlighted the connection between old knowledge and new knowledge by explaining the conceptual continuity between the content just reviewed and the forthcoming content. In our own studies comparing U.S. and Chinese mathematics teachers' beliefs about effective teaching, the Chinese teachers more often explicitly emphasized the value of high coherence than their U.S. counterparts (Cai & Wang, 2010; Wang & Cai, 2007) and took pains to achieve such coherence in preparing their lessons (Cai, 2005; Cai & Wang, 2006; Cai et al., 2014). For example, we found that most Chinese teachers' lesson plans tended to look like scripts of a stage play with careful arrangements connecting different activities in classroom.

The existing literature provides a general picture of the coherence in Chinese mathematics classrooms. The teacher tends to use coherent discourse to unfold well-connected mathematical themes carefully. However the details of discourse coherence in this picture are rather sketchy. Moreover, the focus of the studies has mainly been on the teacher. As we noted above, student learning is a social interaction, and both coherence and student engagement affect student learning. Clearly, the lack of student voices in coherence analysis limits our insight into the impact on coherence on student learning. In this analytical framework, one of the focuses is on student voices in coherence analysis.

3. Method

3.1 *The Teacher and the Class*

Our analysis in this chapter is based on one video-taped sixth grade lesson about circles delivered by an experienced mathematics teacher in China. The teacher was recommended by the local educational bureau as an outstanding mathematics teacher who had been teaching elementary mathematics for more than twenty years. She held a top-level certificate for teaching elementary mathematics and had received several honors for her excellent teaching. The local teacher training department selected this teacher several times to give model lessons (*gongkaike*) (see the relevant information in Ma, 1999; Paine, 1990), so other local teachers could learn how to teach in a coherent way. Therefore, her teaching has been recognized as a model of high coherence.

3.2 *Data Analysis*

We used the CLAN software (MacWhinney, 2000) as a tool to transcribe and code the data. One advantage of using CLAN was it allowed us to link utterances in the transcript precisely to the corresponding moments in the video and display both on the same screen. With this tool, we could easily look back and forth between the transcript and the live classroom situation.

3.2.1 *Lesson Discourse Structure Coding*

Following Mehan's (1979) theory of lesson structure, we analyzed the structure of discourse along two dimensions: sequential and hierarchical. Sequentially, this lesson had three phases: opening, instructional, and closing. The boundaries between the phases of lessons in China are often very clear because the beginning and ending phases are formal greetings and close with prescribed physical rituals (standing up and bowing). Therefore our analysis focused on the instructional phase and along the hierarchical dimension. It consisted of several sequential stages. In each stage, the classroom activity was further disassembled into topic-related sets (TRS) (see Figure 1). The boundaries between stages and between TRSs were identified by discourse marks and content in adjacent utterances (see Table 1 for an example).

3.2.2 *Discourse Strategy Coding*

We followed the definitions of the three kinds of discourse management strategies described above (rhetorical, referential, and focus) to identify the teacher's use of different discourse strategies. We paid special attention to those strategies specific to the instructional context and discussed the discourse and instructional functions of those strategies in detail. It should be noted that a speaker normally adopts more than one strategy at a time. Any piece of discourse can have multiple functions (cf. Redeker, 1990), and thus could be multiply coded. For example, blackboard writing could be seen as both a referential and a focus strategy (see the discussion in Results).

4. Results

We present our results in two parts, corresponding to the two main components of our theoretical framework. First, we analyze thematic coherence and discourse in the lesson along both sequential and hierarchical dimensions (see Figure 1). Then, we explore how the teacher used particular discourse strategies to achieve instructional coherence in the lesson.

4.1 *Thematic Coherence and Classroom Discourse*

According to Mehan (1979), the structure of a lesson can be analyzed along two dimensions: sequential and hierarchical. Sequentially, this lesson had three phases: beginning, instructional, and ending. Hierarchically, the instructional phase can be disassembled into four sequential stages: reviewing, teaching new content, practicing, and assigning homework. In every stage except the assigning homework, the activity can be further disassembled into topic-related sets (TRS). In this section, we first trace the thematic coherence and the discourse in the sequential phases. Then, we discuss thematic connections between the hierarchically nested stages within the instructional phase.

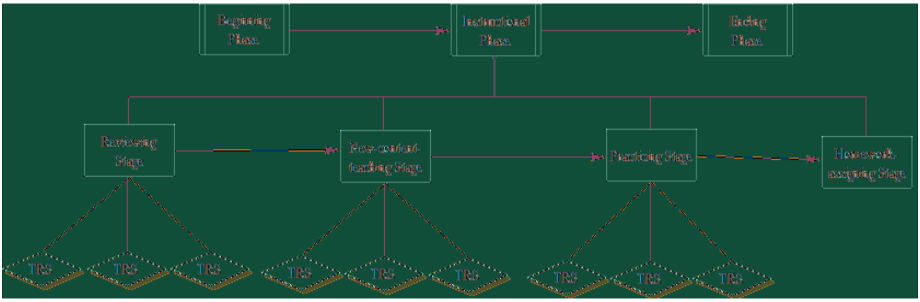


Figure 1. Overall structure of the lesson

4.1.1 *Discourse in the Beginning and Ending Phases*

The discourse in each of the beginning, instructional, and ending phases serves to fulfill goals specific to that phase: to begin class, to exchange academic information, and to end the class, respectively. Like other East Asian classes reported in literature (Stigler & Hiebert, 1999), it began and ended with a short, but formal customary exchange through imperative language use and prescribed physical rituals (standing up and bowing).

Excerpt 1

TEA[2]: Class begins.
ST1: Stand up.
(All students stand up).

---

[2] TEA: Teacher; ST1: an individual student (in this example, the individual student is the student monitor); SSS: student choral response.

TEA: Good morning, students.
SSS: Good morning, teacher.
(All students bow to the teacher).
ST1: Sit down.
(Students sit down).

The discourse in the ending phase mirrors the beginning. At the end of the class, the teachers and students had a customary exchange similar to that at the beginning, saying “Good bye” to each other instead of “Good morning.” The physical behavior was exactly the same (standing up and bowing).

Note that the teacher started the classroom discourse with a short imperative utterance (“Class begins”). In a more casual situation, people often use other linguistic forms to signal an initiation of behavior, for example a request form, “Can we start?” or with a little harsher voice, “Please start.” Sociolinguists Brown and Levinson (1987) explained that in daily life people speak in these ways because people want to show their politeness to one another. The strategy of using these linguistic forms is called a *redressive action*. Redressive actions mitigate the directness of the speech and save face for both the addressor and addressee. For example, with a request form (“Can we start?”) the addressor shows politeness by providing the addressee the chance to refuse. In contrast, direct and imperative speech like the utterance, “Class begins,” used by the teacher in this lesson is speech in a *bald* form (without any redressive actions), and is very face-threatening according to Brown and Levinson.

Why did the teacher use this face-threatening speech form? According to Brown and Levinson’s (1987) theory, this kind of face-threatening speech can be justified in a situation where the addressor enjoys high power. In this classroom setting, by legitimately using this imperative linguistic choice, the teacher enacted a hierarchical social norm for the upcoming event (the formal class), where she had power to direct student attention and behavior. The students’ subsequent physical and verbal responses acknowledged the high authority of the teacher by obeying her order for the initiation of the class. Pragmatically, this exchange initiated the class event. Socio-linguistically, it enacted the social interactional norms acknowledged by the participants.

Classroom discourse, as Bernstein (1986) pointed out, with its pedagogical goal, has two basic functions, a regulative function and instructional function. The regulative function sets up a social context which constrains the way that subject-matter content will be exchanged. Although the opening exchange in this lesson did not include any mathematical content, it set the tone for the rest of the lesson. From a discursive perspective, it functioned as a rhetorical strategy, helping the students be clear about the goals (mathematics instruction in this case) of the subsequent discourse interaction (Tomlin et al., 1997). Therefore this opening exchange contributed to thematic coherence in the instructional stage by tuning student minds to the propositional content of production (formal mathematical knowledge), thereby helping them understand the construal of what was to be heard (see the discussion of rhetorical management below).

4.1.2 *Thematic Coherence and Classroom Discourse in the Instructional Phase*

The instructional phase consisted of four stages: reviewing, teaching new content, practicing, and assigning homework. These four stages can be identified by content and discourse marks used by the teacher. For example, after the beginning ritual exchange, the teacher started the instructional stage with the discourse mark, “Okay,” followed by a clear announcement of the instructional task – recalling specific mathematical knowledge learned in the semester (see Table 1). With linguistic choices like this at various points in the instructional phase, the teacher coherently presented a sequential structure for the lesson by announcing the beginning of the reviewing stage and the content transitions in the next two stages.

The stage of teaching new content was the pedagogical core of the lesson. The other three stages served to support the learning of the new

Table 1. Stage-initiating discourse and discourse marks

Stage	Transitional Discourse	Discourse Marks and clear information of the new stage	Transition of the themes and activity
1. Reviewing	**Okay**, students **recall** what plane figures we have learned in this semester?	Okay, recall	From greeting to mathematical theme
2. Teaching new content	**Today**, we **will learn** about a plane figure shaped with curve. That is a circle.	Today, will learn	From prior knowledge to new knowledge
3. Practicing	**Following**, please look at several circles given by the teacher on the blackboard.	Following	From new content teaching to practice
4. Assigning homework	**That's all** for today's content. **Today's homework** is questions 2 and 3 on page 109.	That's all, Today's homework	From teaching mathematics to assigning homework

content (see Figure 2). Although sequential connections provided a local structural coherence that helped listeners to be aware of the start of a new speech event, the thematic connections between and within stages that were created through the classroom discourse provided students with conceptual connections within the content. The sequence of main themes introduced during the new content stage was as follows: the circle is a two-dimensional curve figure, drawing a circle, the center point of the circle, the radius, the diameter, and the symmetry of the circle. In the remainder of this section, we will use discourse data to examine the thematic connections in this lesson on two levels. First, we will discuss the thematic coherence among the four stages of the instructional phase. Specifically, we will focus on the connections between the stage of teaching new content and each of the three other stages. Then, we will turn our attention to coherence within the stage of teaching new content and explore how the teacher uses discourse to make connections between new knowledge pieces.

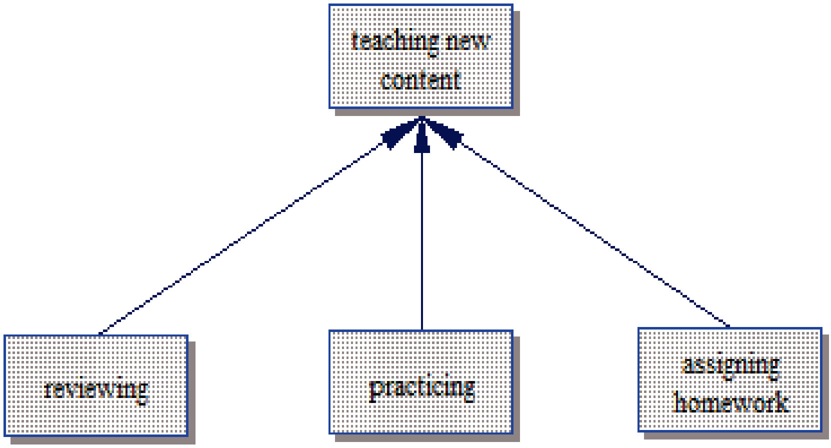


Figure 2. Relationships among the four stages in the instructional phase

4.1.2.1 *Thematic connections between the reviewing stage and the stage of teaching new content*

In the reviewing stage, the teacher asked three questions to review some previously learned knowledge about two-dimensional figures. Three themes emerged from these questions: the names of the polygons, symmetric polygons, and that polygons are composed of line segments (Figure 3). The first theme served as a starting point to initiate the next two themes. The second theme, symmetric polygons, included two sub-themes, the number of axes of symmetry and their locations. During the stage of teaching new content, this symmetry theme was connected to a new theme through the following discourse exchange:

Excerpt 2

TEA: A circle has numerous axes of symmetry. What is the difference between it and other symmetric figures? How is it different from the previously learned

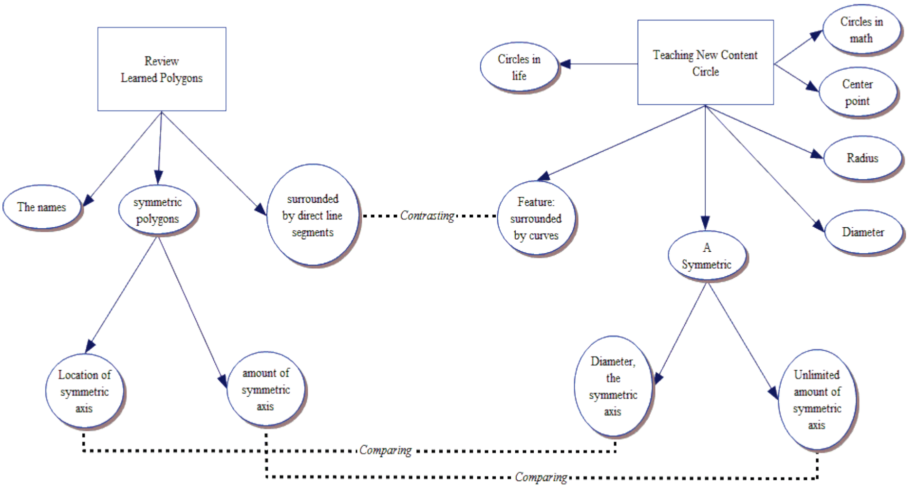


Figure 3. Thematic connections between the reviewing and teaching new content stages.

symmetric figures like a square and isosceles triangle? What is the difference? Jing [a student's name].

ST1: Squares and isosceles triangles have a limited number of axes of symmetry. Circles have an unlimited number of axes of symmetry.

TEA: Was Jing's answer correct?

SSS: Yes.

TEA: All right. Let's repeat Jing's statement together. Squares and isosceles triangles' axes of symmetry… Start.

SSS: Squares and isosceles triangles have a limited number of axes of symmetry. Circles have an unlimited number of axes of symmetry.

It seems that knowledge about symmetry in polygons from the reviewing stage was intentionally selected to serve as an introduction to the new but related content about symmetry in circles. The connection was established by comparing the number and locations of axes of symmetry between the circle and the two previously-learned polygons. In this manner, a wider conceptual coherence was realized through the device of local discourse coherence.

The third theme that arose in the reviewing stage was that the previously-learned polygons were composed of line segments. After summarizing this feature of polygons, the teacher used it as a springboard to launch the new content stage by contrasting this feature to the shape of circle.

Excerpt 3

TEA: The five figures we learned are squares, rectangles, parallelograms, trapezoids, and triangles. They are all shaped with straight lines. Right?

SSS: Right.

TEA: Today, we will learn about a plane figure shaped with curve. That is a circle.

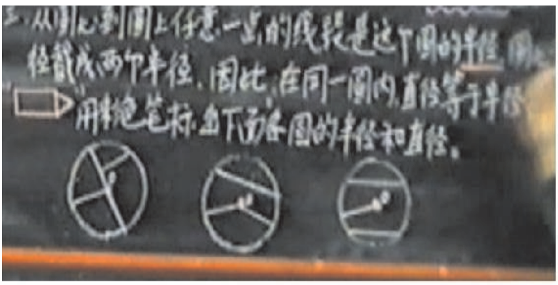
It is clear that the reviewing stage served two pedagogical functions: reviewing previously learned material and setting up a conceptual context for new content. In our previous research (Wang & Cai, 2007), some Chinese teachers tended to highlight the second function by naming this stage as 导 (*daoru*). The literal meaning of this word is to gradually put something into an existing container. Thus, this word describes a process of introducing a new concept into the students' existing knowledge structure. We found that the Chinese teachers explicitly pointed out that this reviewing stage is extremely important for an effective lesson because it provides a process to connect old and new knowledge. For example, one Chinese teacher explained the function of the reviewing stage in this way:

> It is crucial to clarify the connection between today's content and previously learned content. You should study extremely carefully when you design the review because it paves the road for teaching the new content (p. 296).

In the same spirit, the teacher in the present study purposefully selected some previously learned content to discuss during the reviewing stage that would set up a conceptual context for the stage of teaching new content.

4.1.2.2 *Thematic connections between the practicing stage and the stage of teaching new content*

After the new content was taught, the teacher asked the students to use their just-learned knowledge to solve some problems. Different questions in the exercises provided opportunities to practice the themes just learned in a new context (see Figures 4 and 5). In Exercise 1, the students were asked whether certain line segments are radiuses. In two of the given figures, the segments were not radiuses because they violate the definition of a radius (one segment does not start from the center and the other does not stop at a point on the circle). Students applied the newly learned definition in a concrete context to explain their judgments. The second exercise follows a similar pattern, highlighting that diameters pass through the center and have both endpoints on the circle. Exercise 3 provides three more complex situations with various examples that violate the definitions of radius and diameter. Within the practicing stage these three exercises are connected to each other and are carefully arranged from easy to difficult. At the same time, the concepts introduced during the stage of teaching new content are applied in different contexts. Figure 6 shows a map of the thematic connections between the stage of teaching new content and the practicing stage.



Use a color pen to identify and mark radiuses and diameters in the following circles.

Exercise 3

Figure 4. Exercises 1 and 2 on the blackboard.

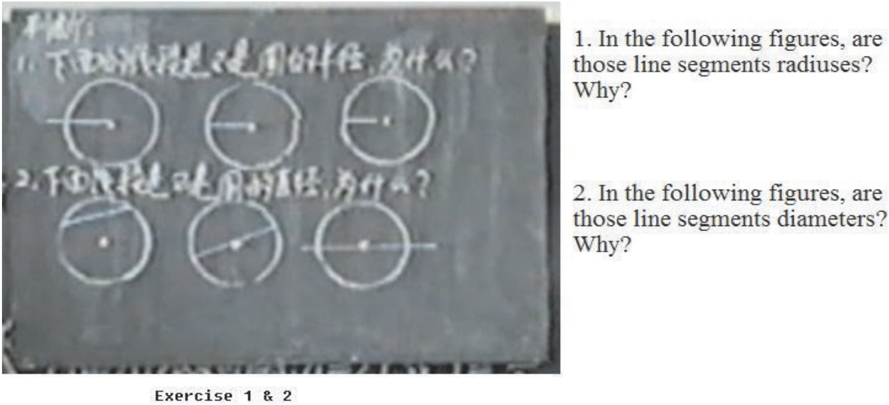


Figure 5. Exercise 3 on the blackboard.

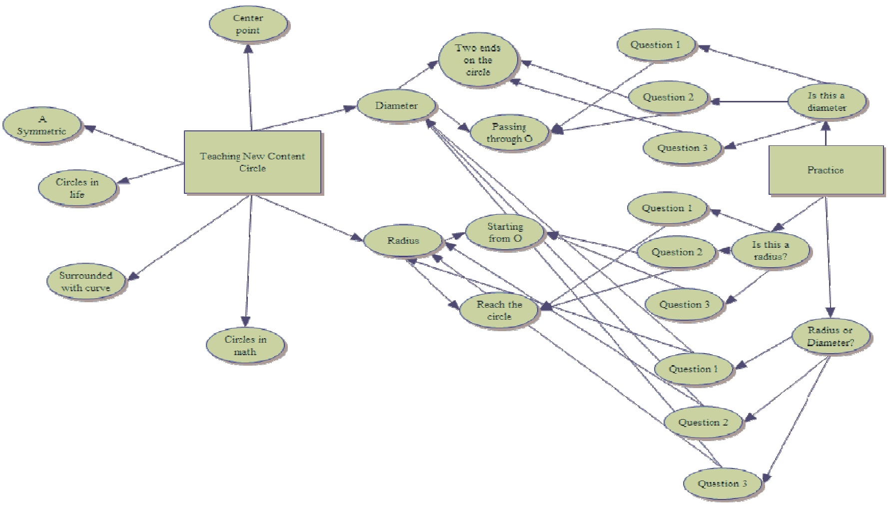


Figure 6. Thematic connections between the practicing and teaching new content stages

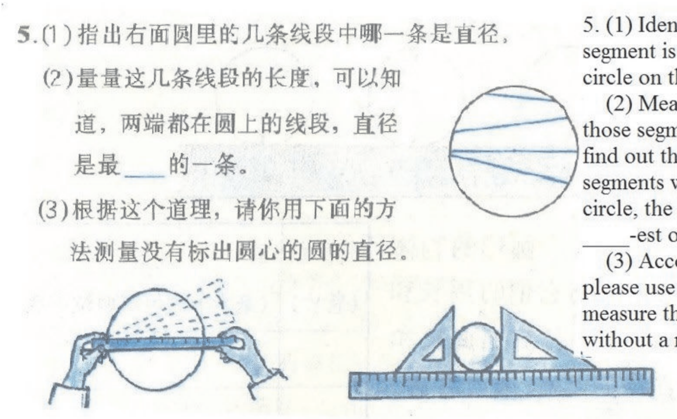
4.1.2.3 *Thematic connections between the assigning homework stage and the stage of teaching new content*

Although doing homework is not part of the classroom activities, it is an indispensible extension of learning that involves both the school and the home. Thus the thematic coherence between homework and learned content also plays an important role in facilitating learning. Usually the instructional purpose of homework assignments is to provide the student with an opportunity to practice or review relevant material that has already been presented in class (Cooper, 1989). Indeed most exercises in the homework that were assigned in this lesson reviewed some important concepts taught in class. For example, Exercise 4 required the students to apply the formula $d = 2r$ to fill in the blank cells shown in Table 2.

Table 2. Exercise 4 from homework assignment

r (meters)	0.24		1.42		2.6
d (meters)		0.86		1.04	

In addition to the function of reviewing, some questions in the homework were intended to extend the students' conceptual understanding. For example, in Exercise 5, the first question required the students to identify which line segment was a diameter. The second question then asked the students to measure the four segments and to conclude that the diameter is the longest among all the segments with two ends on the circle. The third question required the students to apply this new knowledge to measuring the length of the diameter in a circle without a marked center point (see Figure 7). These assigned homework problems strongly connected to the main themes taught in the class by practicing and even extending the relevant concepts.



5. (1) Identify which line segment is a diameter in the circle on the right side.

(2) Measure the length of those segments. Then you can find out that among all the segments with two ends on the circle, the diameter is the ____-est one.

(3) According to this finding, please use following method to measure the length of a circle without a marked center.

Figure 7. Exercise 5 from the homework assignment

In summary, the mathematical themes in the reviewing, practicing, and homework-assigning stages within the instructional phase were all closely connected to the new content taught in the stage of teaching new content. However each of those three stages served different pedagogical functions. The reviewing stage set up a conceptual context for introducing the new content. The practicing stage provided a concrete context with different variations for applying the newly-learned content. Finally, the homework reviewed the concepts and extended the students' understanding of those concepts.

4.1.2.4 *Connections within and among themes*

In addition to the thematic connections across the four stages of the instructional phase, the teacher also built connections within the stage of teaching new content. One set of these connections involved a series of activities that linked the students' concrete and abstract understandings

of circles. Once she had introduced the new central topic of circles, the teacher proceeded with the following sequence of activities: talking about round figures in real life, drawing a circle, folding a circle, teaching concepts of the components of circles, and teaching about the symmetry of circles. The core instructional activity involved the concepts of the circle components, including the center, radius, and diameter. The other four activities in the sequence served to connect with and reinforce the teaching of these concepts
(see Figure 8).

After pointing out that the new figure for today's lesson was a curved figure, the teacher led the class to talk about circular objects in daily life. Through this activity, she connected the new content of circles to the students' life experiences while at the same time highlighting the curved feature of the shape. The teacher then summarized the connection and initiated the next activity, drawing a circle.

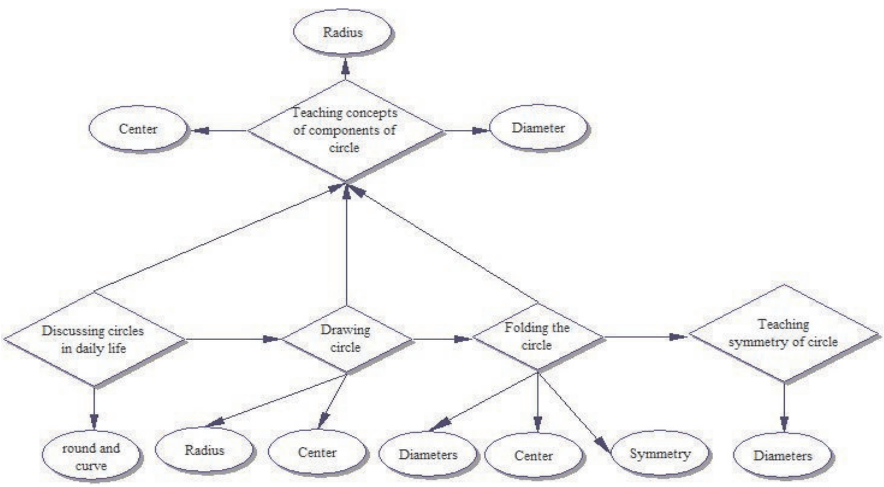


Figure 8. Classroom activities and mathematical themes within the stage of teaching new content

Excerpt 4

TEA: This is a circle [Placing a paper circle on the blackboard].

TEA: A circle is surrounded by a curve. [Writing on the blackboard: A figure surrounded by a curve].

TEA: All right. It is surrounded by a curve. Then, how to draw a circle? How to draw a circle? Look at what the teacher is going to do.

The teacher demonstrated how to draw a circle on the blackboard and then asked the students to follow her instructions to draw another circle.

Excerpt 5

TEA: Now you have seen how the teacher drew the circle. Follow the teacher's approach and draw a circle by yourself. The teacher's approach is, as the teacher just said, first open the two legs of the compass, fix the leg with the pin onto the paper; move the other leg with the pencil around. (Pointing to the blackboard) Okay, now everyone follow this method of drawing a circle to draw a circle on your own paper.

The steps of this drawing process embodied two important mathematical concepts about circles to the class: the center (fixing the leg with the pin onto the paper) and the radius (opening the two legs a certain distance). Although the teacher did not start the conceptual discussion at this moment, she did connect back to this activity when teaching the concept of radius in various ways. For example, after teaching the concept of radius, she asked the students to draw a circle with a radius of 2 cm. Then she asked the students to discuss the difference between the two drawing processes (drawing without a specific radius and drawing with a specific radius of 2 cm).

After the students finished drawing their circles, the teacher asked them to use scissors to cut them out. She then began the folding activity.

Excerpt 6

TEA: Okay. Now students, please fold your circle symmetrically, then open it. Then, change a direction to fold again. All right, after several folds, students, what do you find?

TEA: What do you find? Cheng.

ST1: I found that all the creases pass through a center point of the circle.

TEA: Is Cheng right?

SSS: Right.

At the end of this folding process, the teacher finally started to teach the concept of the first component of circle, the center. Once the teacher described the center, she soon led the class to define and discuss the concepts of two other circle components, radius and diameter. During the defining and discussing process, the connections among these three main thematic concepts were established (see Table 3).

Table 3. Thematic relationships between center point, radius, and diameter

Concepts	Connection
Radius and Center	Radius starts from the center
Diameter and Center	Diameter passes through the center
Radius and Diameter	$d = 2r$
Radius, Diameter, and Center	Center point is in the middle of the diameter and divides it into two radiuses

In these activities, the teacher gradually introduced the mathematical concepts of the circle by connecting mathematical circles to students' life experiences (round objects in daily life), and abstracting specific mathematical concepts of the circle (center and radius) from hands-on manipulative processes (drawing and folding). In addition, the teacher developed connections among the three main thematic concepts of center, radius, and diameter.

4.2 *The Teacher's Use of Discourse Strategies to Achieve Thematic Coherence*

As we noted above, the implicit connections in discourse content do not necessarily guarantee that the listener can integrate the content into a coherent version (Tomlin et al., 1997). However, the speaker's use of discourse strategies can help the listener perceive those thematic connections. In this section, we consider how the teacher in this lesson made use of three types of discourse management strategies (rhetorical, referential, and focus) for this purpose. Recall that a rhetorical management strategy helps the listeners to be clear about the goals and main topics of the discourse. This allows the listeners to integrate different pieces of information from speech into a coherent picture. A referential management strategy helps the listeners establish connections between new knowledge and already-held knowledge. Typically the speaker will refer back and forth between old and new themes. Finally, a focus management strategy highlights the main topics in the speech flow. In this lesson on circles, the teacher clearly used these discourse strategies, intentionally or unintentionally, to make connections explicit for her students. We will explain the three strategies and their manifestations below.

4.2.1 *Rhetorical Management Strategy*

In a lesson, the teacher has several goals or intentions and wants to produce discourse interactions that promote them. This dimension of coherence between goals and discourse production is called rhetorical management. The teacher in this study used this strategy from time to time at various levels of the lesson structure. As we described above, the lesson had three phases: beginning, instructional, and ending. The formal ritual exchange at the beginning announced that the subsequent speech event would be formal mathematics teaching. Once the instructional phase started, the main topic, concepts about circles, unfolded in a four-stage script: reviewing, teaching new content, practicing, and assigning homework. At the beginning of each stage, the teacher explicitly marked a shift in the pedagogical goal (see Table 1). In addition, within each stage the teacher often announced the main goal of the next activity. For example, in the stage of teaching new content, the teacher first asked students to give examples of round shapes from daily life. Then she explicitly shifted the discourse to the theme of drawing circles:

Excerpt 7

TEA: Then, how do we draw a circle? How to do it? In the following, let's see what the teacher is going to do. Usually we use a compass to draw a circle. How to draw it?

The teacher used a question to capture the students' attention at this point. It is clear that the topic in this discourse move (drawing a circle) differed from the previous topic (finding round objects in daily life). Then, the teacher followed up with the discourse mark "next" to further remind the students that their main responsibility was to observe the teacher. Finally, before the teacher started to draw a circle, she refined the topic to drawing a circle with a specific tool, a compass.

In addition to announcing the pedagogical goals at the beginning of each event, the teacher used a rhetorical strategy when she periodically summarized just-learned content. This also served to keep the discourse coherent with her pedagogical goals. For example after discussing some features of the radius, the teacher summarized what had just been said about the radius before moving on to new content.

Excerpt 8

TEA: Okay, just now, we have learned about the radius of a circle. A circle can have numerous radiuses. All the radiuses have same length. In the following, we will continue learning about diameter.

In the first utterance, the teacher used two discourse marks, "Okay" and "just now." While the "Okay" served to mark the start of a TRS, the "just now" signaled that the new event would be summarizing the contents of previous speech events. This summarizing strategy further highlighted the teacher's pedagogical goals by recapitulating them. Then, after she had finished the summary, the teacher used another discourse mark, "Next," to announce that the new topic in the next event would be diameter. From time to time throughout the lesson, the teacher used this summarizing and announcing rhetorical management strategy to help the students recognize the boundaries of different TRSs and the main topic in each TRS.

4.2.2 *Referential Management Strategy*

In addition to using rhetorical management strategies to point out the main topics in each TRS, the teacher also used referential strategies, moving back and forth between already-learned knowledge and new knowledge to help make her instruction coherent. Indeed, one of the characteristics of connected and coherent discourse is that entities, once introduced at a given point in text, are often referred to again at a later point (Tomlin et al. 1997). In the stage of teaching new content, the teacher moved through the new topics in the following sequence: center, radius, and diameter. Once the center was introduced, it soon became given information from which the other two new concepts (radius and diameter) were introduced.

Excerpt 9

TEA:	Okay, some students have already marked it [the center].
TEA:	Okay, let us have another look. From the center, from the center to any point on the circle, this line segment, from the center to any point on the circle is called a radius.

In this discourse segment, after seeing that the students could find the center of the circle, the teacher said, "From the center, from the center to any point on the circle…" Instead of immediately finishing her statement, the teacher then repeated her words, "this line segment, from the center point to any point on the circle, is called a radius." It is clear that the teacher introduced the new concept by repeatedly referring to the previous concept, the center.

Tomlin et al. (1997) have observed that "one important problem in reference management has been understanding how speaker and listener keep track of referents during discourse production and comprehension" (p. 80). Unlike standard samples of discourse in linguistics, there are many strategies and tools available to help the participants keep track of referents in classroom discourse. One tool that the teacher made use of in this lesson was the blackboard. The blackboard is one of the most effective tools for coping with the limitation of human short-term memory in classroom lessons. For example, Figure 9 shows how the teacher wrote down important mathematical content on the blackboard and then referred the students to the blackboard whenever needed.

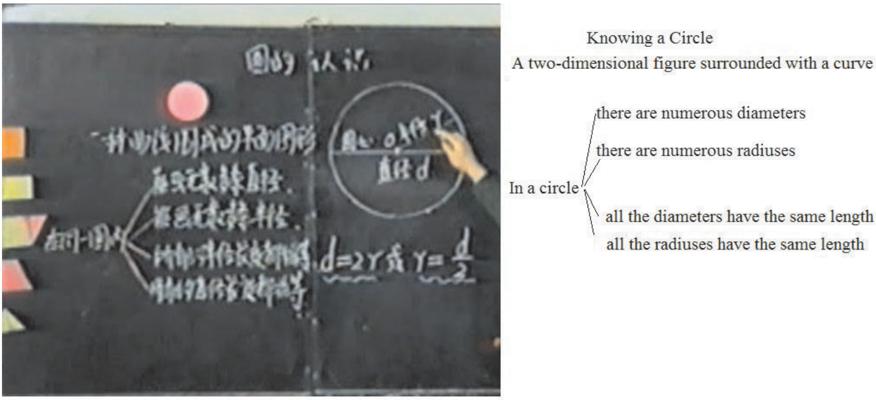


Figure 9. Important mathematical information written on the blackboard.

In our previous study (Cai, 2005; Cai & Lester, 2005), we found that experienced Chinese teachers carefully planned their blackboard writing ("*banshu*") in their lesson plans by recording exactly what should be written on the blackboard and when it should be written. These teachers used large blackboards in such a carefully planned manner that the students could easily understand and remember what had been discussed and take well-organized notes for review. Chinese teachers often write important mathematical concepts on the blackboard as this teacher did in Figure 9. Therefore, in addition to the function of coping with the limitations of human short-term memory and allowing the teacher to easily make references to previously-discussed content, blackboard writing also serves to highlight and foreground important mathematical concepts. This function helps the students to focus appropriately on the important concepts.

4.2.3 *Focus Management Strategy*

A single utterance can often embody multiple themes. Some of these themes are more important and central than others. In a long speech flow, a key strategy to make discourse coherent is to direct the listener's focus to the themes which are most central to the discourse. In this lesson, the teacher used various tactics to manage students' focus. For example, in the previous section we noted that the teacher highlighted the important concepts by writing them on the blackboard. In addition to blackboard writing, the teacher also used the following strategies: comparing and contrasting themes, recurring themes, and using choral responding.

4.2.3.1 *Comparing and contrasting*

One approach to make certain information more prominent or salient is to compare or contrast it with other information in the discourse. This can be an important approach to establishing thematic connections, for example using questions to help students focus on similarities and differences between mathematical concepts and features. Sometimes, this strategy is used at the beginning of teaching a new concept so that the new concept can become the central focus. For example, in Excerpt 3 the teacher directed the students' attention to the concept of the circle by contrasting its curved nature to the straight line segments that made up the five previously-studied polygons. This strategy made the circle the central focus at the beginning of the teaching.

In addition to using the comparing and contrasting strategy at the beginning of teaching a new concept, sometimes the teacher can use it after a new concept has been taught. For example, in Excerpt 2 after the students had learned that a diameter is an axis of symmetry of the circle and that a circle has numerous diameters, the teacher contrasted this just-learned knowledge with the knowledge learned in previous lessons about the limited number of axes of symmetry of squares and isosceles triangles.

Another typical situation where the teacher can use the contrasting strategy is during the practicing stage. In this lesson, the teacher assigned three exercises (Figures 4 and 5) that were designed to contrast the radius and diameter with other line segments on the circle that did not satisfy their definitions. Again, this comparing and contrasting task served to focus the students on the conceptual features of radius and diameter.

4.2.3.2 *Recurring theme(s)*

An important feature of a long but coherent speech flow is its great topic persistency (Givon, 1983). Thus, a flow of coherent classroom discourse should maintain a persistent focus on one topic or a limited number of main topics. One approach to measuring the level of topic persistence in a discourse is simply to count the frequency of main terms. We have already pointed out that the whole lesson we are analyzing in this chapter was designed with one clear central topic, the circle. Under this central topic are three related subtopics, the center, radius, and diameter. In the 40 minutes of instruction, the term "circle" was used 220 times, more than five times in every minute on average. The three-subtopic terms were also used frequently through the discourse. "Center" was used 58 times, radius 78 times, and diameter 72 times. On average, these terms were mentioned more than once every minute. The frequent occurrences of these themes may reflect a great deal of thematic centrality and topic persistency in the classroom discourse of this lesson. Note however, that the teacher did not simply repeat these main concept terms through the discourse. Instead the terms were introduced in various learning contexts including students' life experience, hands-on activities, mathematical definitions, and practice with applications.

4.2.3.3 *Choral response*

In addition to the strategies of comparing and contrasting and recurring themes, the teacher also used a special discourse format, student choral response, to direct the students' attention to important mathematical information. Choral response is defined as a pattern of student responses when the teacher opens the speaking floor to every student in classroom (Wang, 2010). In this lesson, we can identify two variations of choral response: choral reading and answering simple questions. Both variations play an important role in directing student attention to important mathematical themes.

Although choral reading was used only four times throughout the lesson, each instance focused on core mathematical content. In the first and second choral readings, the definitions of radius and diameter were read by the whole class loudly. The third choral reading concerned some of the features of radius and diameter that had been written on the blackboard (see Figure 9). The last choral reading was to repeat a student's response about the difference between circles and other previously-learned symmetric figures (see Excerpt 2). Here the teacher initiated the choral reading with a short clear cue in an imperative tone, "Start." This explicit cue set a serious and formal tone for the exchange and thus implied a strong obligation for student participation. As a consequence, the whole class responded to the teacher by reading the content in unison in a rhetorically exaggerated voice with a slow pace and high pitch. These discourse features indicate the high level of formality of choral reading, which orients students' attention fully on the content being read at the moment.

Compared to choral reading, answering simple questions is less formal in terms of discourse structure and student participation level. However, it happened much more frequently in the lesson than choral reading. Throughout the lesson it happened 49 times, or on average more than once per minute. Often the student response was only a word or a short sentence. Nevertheless, the students' choral responses in answer to simple questions often included important mathematical information that the teacher wanted to highlight. For example, the following exchange happened as the teacher reminded the students of the correct approach for drawing a diameter.

Excerpt 10

TEA:	Pay attention to the approach for drawing a diameter.
TEA:	First, it should pass through what?
SSS:	The center point.
TEA:	And what else?
SSS:	The two endpoints are on the circle.
TEA:	Correct.

These student choral responses highlighted the exact two features of the diameter that the teacher had emphasized earlier in the lesson. In general, the high frequency and selectively targeted content of the answering simple questions variation of choral response served to increase student participation in the lesson and to foreground important mathematical information.

5. Discussion

Instructional coherence is a complex phenomenon achieved both through careful planning and through a spontaneous and dynamic process in the classroom. In order to study instructional coherence, at least two interrelated dimensions must be attended to: the complex connections of content and thematic coherence across various levels of discourse, and the teacher's use of discourse management strategies to achieve coherent instruction that helps the students to perceive those complex connections. In this chapter, we have closely analyzed the discourse in one model sixth-grade lesson about circles in order to identify the thematic connections that are made across stages of instruction. We have also used a theoretical framework of discourse management strategies (Tomlin et al., 1997) to help us understand how the teacher used discourse to help her students perceive the desired thematic connections.

Our analysis of coherence and discourse in this lesson on circles reveals that the lesson is very well-structured in several ways. As others have observed, coherent lessons often focus on very few themes or even a single topic (Ma, 1999; Stevenson & Stigler, 1992). We found a similar sharpness of focus in this lesson. There was one clear main theme, circles, with three well-defined sub-topics, center, radius, and diameter. In addition, these concepts were presented and discussed within a coherent lesson structure. As in Leung's (1995) observations of Chinese lessons, the instructional phase of this lesson comprised four sequential stages: reviewing, teaching new content, practicing, and assigning homework. Moreover, our analysis of the discourse data from this lesson shows that the teacher made a number of specific thematic connections across these stages. The new content about circles was introduced in a conceptual context that the teacher had set up in the reviewing stage. And, the exercises given to the students in the practicing and assigning homework stages were connected both to the themes introduced in the stage of teaching new content and to one another, forming a well-designed progression designed to extend the students' understanding of the themes of the lesson. This arrangement of carefully sequenced exercises that highlight different important features of the content reflects a commonly-used teaching technique in China known as teaching with variation (Gu, Huang, & Marton, 2004).

With respect to the teachers' use of discourse strategies, we found examples of rhetorical, referential and focus management strategies. Each type of strategy served a different purpose within the discourse, but all supported coherence in instruction. For example, the teacher used several focus management strategies including directing the participation of her students in the discourse through choral responses to highlight important mathematical information. In the process of teaching the new content, she drew attention to connections between new and old knowledge and among the knowledge pieces within the new content. This finding accords with the practice of making connections between old and new knowledge that Wang and Murphy (2004) observed. Indeed, the teacher made effective use of the blackboard as a tool to help the students manage referents within the lesson, thus helping to make the thematic connections of the content salient for the students.

It is interesting to ask whether the nature of the mathematical content in this lesson may have been particularly amenable to building connections through discourse management strategies such as these. For example, the definition of a radius depends naturally on having established what the center of a circle means. Thus, one might expect that the teacher would necessarily organize her instruction so that her discussion of the radius would refer back to her previous discussion of the center. The mathematical structure of the lesson content may thus prompt the teacher's structuring of the discourse to some degree. However as we have noted above, listeners are not guaranteed to be able to integrate content coherently simply because there are implicit connections in the content (Tomlin, et al., 1997). Thus, it is critical for the teacher to recognize and deeply understand the implicit connections of the mathematical content and then make them explicit for students through strategic management of the discourse.

Finally, given that it was the teacher who planned the content and controlled the classroom discourse in this model lesson, this chapter focused mainly on the teacher's contributions to the discourse. However, it remains important to ask how the students contributed to achieving this high instructional coherence. In this lesson, the teacher largely stated information rather than developing ideas with her students. She appeared to treat students as passive receivers and employed a highly controlled discourse structure (e.g., choral response). This discursive positioning of the teacher and students reflects socially-established student and teacher roles in China that are "developed through reinforcement, social contracts, conformities, and social negotiations" common in Confucian heritage culture classrooms (Wong, 2008, p. 976). Nevertheless, to further understand the relationship between instructional coherence and student learning, whenever possible and relevant we should include analyses of the contributions of student voices to achieving instructional coherence.

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Chapter 5

Elementary School Teachers' Instruction in Measurement: Cases of Classroom Teaching of Spatial Measurement in Taiwan

HUANG Hsin-Mei Edith

This study demonstrates how elementary school teachers in Taiwan enacted lessons involving spatial measurement of length, area, and volume in different grade levels based on 12 instructional cases collected using videotaping and interviews. The analyses of the videotaped lessons reveal that these teachers have formed a consensus about the importance of actual measuring manipulations and workbook reviewing about students' construction of measurement concepts. The processes of teaching measurement were reviewed sequentially, from visual perception, to direct and indirect comparisons, and to the use of nonstandard measures, and finally, to the application of a standard measure. Two groups were mixed together in the instructional cases—collaborative-learning groups working on measuring activities and teacher-led instruction. Teacher-guided explorations were specifically evident in the case of teaching area formulas and displaced volume for upper-grade students. Although the teachers strived to create lesson enactments and offer opportunities for students to participate in actual measuring manipulations, more teacher effort is needed for making measurement a thought-provoking activity that stimulates students to communicate their ideas about measuring and estimating, and reflecting thinking. This is an essential intervention for developing students' conceptual understanding and sense of measurement.

Keywords: Taiwan mathematics classroom, teaching of spatial measurement, primary mathematics education

1. Introduction

Mathematically, spatial measurement of length, area, and volume involves spatial structures of units of one to three dimensions. Mathematics educators have suggested that once teachers understand how to develop students' knowledge of the spatial attribute, they may later facilitate generalizing the general measurement concepts to other measurement domains (e.g., time and angle) because these three measurements consist of similar learning stages—the identification of the attribute, informal measurement, and unit structure (e.g., Outhred, Mitchelmore, McPhail, & Gould, 2003; Owens & Outhred, 2006). Thus, many studies have pinpointed what teachers need to teach in the classroom for measurement lessons (Dickson, 1989). Yet, few studies have focused on the nature of teachers' teaching practices for developing students' understanding of length, area, and volume measurement. Seeing this inadequacy, this chapter aims at looking into the features of teaching practices about spatial measurement of linear, area, and volume in the primary grades, as exhibited by elementary school teachers in Taiwan.

Recent research and documents regarding curriculum and instruction call for teachers' attentions for taking advantage of measurement activities which engage students in doing actual measurements, measuring estimations, and reflecting on their ideas of measurement (National Council of Teachers of Mathematics [NCTM], 2006; Owens & Outhred, 2006). Within the domain of measurement, the subject matter of measurement that should be included in elementary school textbook curricula mandated in the four sets of curriculum standards (or guideline), as initiated by the Ministry of Education in Taiwan (referred to Taiwan Ministry of Education, TME hereafter) (TME, 1975, 1993, 2000, 2003) remained stable (Chu, 2000; Chung, 2003; Huang, 2012a). In contrast, in the aspect of teaching measurement, the vision set forth by the recent curriculum documents emphasizes equally developing students' conceptual understanding of measurement and procedural skills by the use of instructional strategies and technology rather than memorization of factual knowledge of measurement and calculations (TME, 1993, 2010). Therefore, the extent to which teachers provide activities that involve actual measurement manipulations, measurement estimations, and questions that demand high-level thinking processes for developing students' competence in conducting measurement, were examined separately in the study. Through analyses of teachers' classroom practices, this study may provide a window with regard to trends in teachers' teaching of measurement during the past two decades in Taiwan.

The purposes of the study are to look into the instructional processes and approaches in which teachers have adapted to teaching length, area, and volume measurement lessons in classroom practices on the basis of a theoretical framework. Thus, three questions were included in the study:

1) Through what sequential process do teachers fashion their instructional activities for teaching spatial measurement of length, area, and volume?
2) What approaches associated with measurement estimation do teachers frequently use to help students develop a sense of measurement?
3) To what extent do teachers provide meaningful inquiries into students' understanding of measurement concepts and skills for post-measurement activity discussion?

2. Theoretical Framework

2.1 *Research-Based Perspectives for Teaching and Learning Length, Area, and Volume Measurements*

Spatial measurement combines activities with cognitive thinking, geometry, and arithmetic (Clements, 1999). Moreover, length, area, and volume measurements include the repetition of a particular unit of measurement throughout the extent of whatever it is that is being measured. Additionally, the process of measurement consists of selecting attributes for an object that may be compared and measured by units which can be counted and reported (Owens & Outhred, 2006). Thus, there are three mathematical components that support much of measurement concepts and measuring skills. These three components include: a. partitioning, b. unit iteration, c. tiling and accumulating units that pertain to space and then are related to number (Lehrer, 2003; Stephan & Clements, 2003). In addition, an understanding of transitivity and conservation are commonly regarded as important reasoning abilities for solving problems involving measurement (Stephan & Clements, 2003; Wilson & Rowland, 1993). Although some researchers (e.g., Clements, 1999) have debated the necessity of the development of conservation and transitivity as prerequisites for learning length measurement, which was suggested by Piaget's research (Piaget, Inhelder, & Szeminska, 1960)—conservation and transitivity and the components mentioned above should be developed through measurement activities in which students are required to think about, "What is being measured?" (e.g., the attributes of length, area, and volume measurement), "What is a good unit for measuring and making comparisons?" (e.g., the use of units for making comparisons), and "What is a suitable procedure for measurement?" (e.g., iterating, covering, counting, and the use of tools and formulas).

In the domain of length measurement, students need to know that length means the distance between two points and that the distance of a line segment can be quantified by a number. Conceptualizations of unitization, iterations, and scale are the core content of length measurement (Lehrer, 2003; Stephan & Clements, 2003). To learn concepts of area measurement requires acquisition of concepts about shapes, as well as the ability to incorporate units about two-dimensional regions—based on knowledge of length measurement (Fendel, 1987).

Research-based instructional sequences for teaching area measurement commonly suggest that one should begin with a tiling activity in which students use a square as a unit for covering a region. Next, after sufficient experience has been obtained in covering various squares with squared units, students can be led to find the area formula for rectangles, the basis for understanding the formulas for other basic figures (parallelograms, triangles, trapezoids) and circles (Lehrer, 2003).

Volume measurement includes measuring the space occupied by three-dimensional objects and measuring the capacity of containers that refer to interior volume (Cathcart, Pothier, Vance, & Bezuk, 2003; Wilson & Rowland, 1993). For teaching volume measurement, the concepts of internal volume, occupied space, and the formula for volume measurement of a rectangular solid (container) are essential subject matter components with respect to volume measurement. Furthermore, direct-measurement activities, which include stacking the Cuisenaire cubes and counting the number of cubes structured in a rectangular solid (container), are suggested for helping fifth-grade students learn volume measurement. Understanding a concept of volume as a certain number of unit cubes, and the idea about occupied space, lays the foundation for further understanding of displacement volume. It is a topic in school mathematics frequently included in fifth-grade or six-grade textbook units (TME, 2010).

Therefore, many conventional activities designed in the textbook curricula for introducing volume measurement seem to suggest the following instructional sequence: counting the number of cubes built into rectangular solids and finding the spaces of containers at the beginning, and then proceeding to search for formulas and numerical volume calculations for determining the volume of solids (or finding capacity of containers) (e.g., Nan-I Publishing Group, 2008; Kang-Hsuan Educational Publishing Group, 2012). Nevertheless, more activities involving finding the volume of solids and measuring occupied space, few opportunities were provided for students to experiment with displaced volume. To understand the idea that the volume of the object which is immersed in a container of water leads to the volume of liquid displaced is difficult for students (Dickson, Brown, & Gibson, 1984). However, research on developing students the concept of displaced volume is limited.

2.2 *Sense of Measurement*

In addition to the measuring concepts and skills mentioned above, measurement sense, which fosters the sense of getting a "feeling" for units of measurement and processing a set of meaningful reference points or benchmarks for these units, should be taken into account for developing students' competence in measurement (Joram, 2003). Estimation instruction for developing students' ideas about measurement and making reasonable approximations of the measurement is highly advocated (NCTM, 2006; TME, 1993, 2003, 2010).

According to Joram, Gabriele, Bertheau, Gelman, and Subrahmanyam's (2005) and Joram's (2003) suggestions, guess-and-check and finding benchmarks are two approaches that have been recommended as effective ways to train students for measurement estimation, which in turn helps deepen their sense of measurement. a. Guess-and-check. Students were asked to make (write down) an estimate and then measure the to-be-estimated item. b. Building references (or finding benchmarks) for estimating. Students were asked to identify a personal reference point (or an item that is available) and use it for representing measurement units for generating estimates. Students may gradually learn how to produce estimates and make improvements in estimation accuracy with an incremental approach to measuring experiences.

Although measurement estimation needs a repertoire of a wide variety of everyday measurement referents, children are able to utilize perception to make judgments about relative size without using tools (Joram, 2003; Joram, Subrahmanyam, & Gelman, 1998). Moreover, measuring estimation should be a part of measurement instruction from the beginning of school (TME, 1993).

2.3 *Teacher-and-Student Interaction for Sharing and Reflection on Measurement Ideas*

To develop students' conceptual understanding of measurement and sense of measuring, the use of an appropriate language to represent and communicate mathematical ideas is considered as important as the use of actual measurement manipulations while teaching measurement (NCTM, 2006; Outhred et al., 2003; Van den Heuvel-Panhuizen & Buys, 2008). For learning measurement, students' sense of measurement will not necessarily arise from their manipulations carried out with a lack of reflection about measurement. This can be accomplished through mathematical discussions about instructional purposes, done within a shared learning context (Ball, 1992). In other words, doing measurement activities that require students to do manipulations do not necessarily ensure that they understand the concepts underlying measurement tasks. Consequently, there must be ample classroom discourse in which teachers shift students' attention toward making sense of their measuring experiences. This can be accomplished through posing questions that demand high-level thinking processes (Grant & Kline, 2003). For example, questions such as "How did you arrive at that solution?" "Which units (tools) would be appropriate for measuring the area of the window?" "Why do you think they are suitable?" These types of questions demand students' explanations, clarifications, and reflections on the problems with which they are engaged.

The nature of problems (or tasks) provided by teachers can potentially influence and structure the way students think about or view subject matter with which they are engaged (Henningsen & Stein, 1997). The questions that require explanations, verifications, and reflections elicited for instructional discussion by teachers in accordance with instructional goals may serve as an effective means of communicating a deeper conceptual grasp of mathematical knowledge and developing more higher-order thinking in students (Hicks, 1996). Thus, teachers' engagement in meaningful inquiry into their students' measurement experiences is an essential intervention which may enhance students' powers of thinking and reasoning (Owen & Outhred, 2006).

2.4 *A Brief Description of the Curriculum Standards and Mathematics Curricula in Taiwan*

In Taiwan, the elementary school mathematics curriculum and instruction was modified because of the efforts of an educational reform movement that took place during the past two decades (Chung, 2003; Huang, 2012a). During the time period of educational reform, between the 1990s and 2000s, the mathematics curriculum, as enacted in the classroom, can be classified into four types of textbooks, according to the disparate sets of mathematics curriculum standards (or guidelines) initiated by Ministry of Education in Taiwan in 1975, 1993, 2000, and 2003 that were undertaken for textbook design. The four sets of curriculum standards (or guidelines) embodied in the textbooks included are as follows. First, the curriculum standards for elementary school mathematics, initiated in 1975 (referred to hereafter as 1975-Standards, TME, 1975). Second, the curriculum standards for elementary school mathematics, initiated in 1993 (referred to hereafter as 1993-Standards, TME, 1993), which was a revision of 1975-Standards (TME, 1993, pp. 346-347). It is noteworthy that the guidelines regarding instructional approaches mandated in the 1993-Standards, which highlighted conceptual understanding, problem solving, and student-centered learning, were different from those in suggested in the 1975-Standards (Chung, 2005). Third, the Temporary Grade 1-9 Curriculum Guidelines for Junior High School and Elementary School- Learning Stage I, which maintained the characteristics of instructional approaches that were highlighted in the 1993-Standards, initiated in 2000 (referred to hereafter as 2000-Temporary Guidelines, TME, 2000). Fourth, the Grade 1-9 Curriculum Guidelines for Junior High School and Elementary School, which was a revision of the 2000-Temporary Guidelines, initiated in 2003 (referred to hereafter as 2003-Guidelines, TME, 2003, 2010).

As mentioned in the previous section, the subject matter components of spatial measurement and the specific instructional sequence for introducing the concepts included in the strand of quantity and measurement that were highlighted in the four sets of curriculum standards (or Guidelines) mentioned above were relatively similar (Chu, 2000; Chung, 2003). In contrast, the instructional approach was gradually shifted from memorization of factual knowledge of measurement and practices of procedures--which were emphasized in the 1975-Standards--toward instruction that stressed incorporating students' mathematical intuition, and levels of awareness and mathematical inquiry by using mathematical language to developing their construction of measurement knowledge from 1993 to 2000 (Chung, 2003, 2005; Huang, 2001a, 2012a). However, compared to the instructional features that emphasized conceptual understanding and mathematics competence in the 1993-Standards and 2000-Temporary Guidelines, a swing back toward the mastery of arithmetical skills of instruction, which was stressed in the 1975-Standards, can be seen in the 2003-Guidelines (Chung, 2005; Hsu & Chang, 2008). Thus, the 2003-Guidelines are comparatively different from both the 1993-Standards and 2000-Temporary Guidelines. Fostering students' basic competency in mathematics, both conceptual understanding and arithmetic ability, along with problem-solving ability—are given equal stress with regard to current mathematics curriculum and instruction (TME, 2003, 2010).

3. Methodology

3.1 *Data Source*

Researchers suggest that making comparisons and syntheses across studies can provide new insights, which in turn facilitate deeper understanding and further elaboration of theory (e.g., Griffin & Reddick, 2011; Noblit & Hare, 1988). Based on the full data sets regarding teaching practices of spatial measurement which were readily accessible, a new theoretical lens on instructional actions that incorporate types of questions asking for developing students' understanding of measurement can be taken, for the purpose of understanding the nature of elementary school teachers' classroom practices.

Therefore, the research questions are addressed through a study that synthesize three parts of the existing data, which involves instruction on spatial measurement, as collected by the author, and spanning the 1997-1998 through 2009-2010 school years. This includes the interview data about teachers' perspectives on teaching measurement, as collected from the teachers who enacted the videotaped lessons.

First, there are two sets of data containing eight teachers' teaching practices and perspectives regarding teaching and learning length measurement for grades 1 through 4 (i.e., G1A's, G1B's, G2A's, G2B's, G3A's, G3B's, G4A's, and G4B's instructional videotapes), as collected by Huang (1999, 2001b, 2004). To obtain the eight teachers' viewpoints about teaching and learning length measurement (or measurement estimation), each teacher took part in face-to-face interviews, which were twice administrated by the author–during or after teaching lessons.

Second, there are sets of data involving three teachers' instruction regarding area measurement and perspectives on teaching and learning area measurement (i.e., G2C's, G5A's, and G6A's instructional videotapes). These cases were respectively adopted from websites and instructional sources provided by researchers and Compulsory Education Advisory Groups, which serve to supporting elementary school teachers' professional development under permissions. The sources of the videotapes collected for the study included: a. The videotaped lesson on area measurement for grade-2 students, which was videotaped in 2001, was collected by Huang (2012b). b. The videotaped lesson on area formula of trapezoids for grade-5 students, which was videotaped in 2010, was adopted from instructional sources provided by Taipei City Compulsory Education Advisory Groups. c. The videotaped lesson on the formula for area measurement of circles for grade-6 students, which was videotaped in 2008, was adopted from the website of Tao-Yuan County Compulsory Education Advisory Groups (http://mod.tyc.edu.tw/MediaInfo.aspx?MID=2307&ChID=90&CateID=743,745,751).

Finally, an additional videotaped lesson related to displaced volume (measuring the volume of irregular-shaped objects) for grade-5 students, which was videotaped in 2010, was included.

In the study, the set of interview data for collecting the teachers' perspectives on teaching area measurement and displaced volume (i.e., G2C's, G5A's, G6A's, and G5B's viewpoints) were collected after their lessons was completed.

Overall, the data in the study consisted of 12 instructional cases with regard to the teaching practices of spatial measurement, including eight cases of length measurement, three cases of area measurement, and one case of volume measurement. A brief description of the demographic characteristics of the 12 teachers investigated and the main mathematical content involved in the videotaped lessons are summarized as Table 1.

3.2 *Demographic Characteristics of the Teachers Investigated*

Table 1 shows the lessons related to length measurement, across four grade levels. The grade-1 teachers (G1A and G1B), grade-3 teachers (G3A and G3B), and grade-4 teachers (G4A and G4B) were from the same public school in Taipei, whereas the grade-2 teachers (G2A and G2B) were from another public school in Taipei.

For the instructional cases of area measurement, the grade-2 teacher (G2C) and the grade-5 teacher (G5A) were listed separately from two public schools in Taipei, while the grade-6 teacher (G6A) was from a public school in Tao-Yuan County. For the instructional case of volume measurement, the grade-5 teacher (G5B) was from a public school in Taipei, which is different from the schools mentioned above.

The number of students in a class who received the measurement instruction was between 19 and 32 (M = 26.00, SD = 4.49). The range of number of years of teaching experience of the 12 teachers was between 8 and 35 (M = 19.75, SD = 7.84).

3.3 *The Number of Lessons Enacted and Mathematical Content and Length of Instruction*

As can be seen in Table 1, for the cases of length measurement, the length of instruction for each unit took place from between three to eight 40-minute class sessions. All of the instructional cases of length measurement for grades 2 through 4 contained one textbook lesson enacted, whereas the two grade-1 cases included two lessons enacted. The main content of length measurement involved in the lessons for grades 1 through 4 included: (a) the attribute of length measurement and direct and indirect comparisons of lengths for grade 1, (b) standard units of metric system for grade 2, (c) unit conversions within the metric system for grade 3, and (d) estimation of length for grade 4.

As to the three instructional cases of area measurement, teacher G2C's instruction focused on acquisition of the attribute of area measurement and a direct comparison of the area of figures for grade-2 students, whereas teachers G5A's and G6A's instruction focused on introducing the respective formulae for the area measurement of trapezoids and circles. The length of instruction time on area measurement took place between one and three sessions. Finally, the instructional case of teacher G5B, focusing on volume measurement of irregular-shaped objects, took place in about one session.

3.4 *The Curriculum Standards Embodied in the Mathematics Textbooks Adopted*

The curriculum standards embodied in the mathematics textbooks that were adopted by the 12 instructional cases included three sets of curriculum standards (or guidelines). As Table 1 shows, the textbooks

Table 1. The demographic characteristics of the 12 teaching cases of spatial measurement of length, area, and volume, and the curriculum standards embodied in the textbooks enacted

Code of teacher	Years of teaching	Grade	No. of students	No. of curriculum unit and core content	Length of instruction	Curriculum standards embodied in textbooks enacted
G1A	10	1	*n*=24	2 units (The attribute of length measurement)	6 sessions	1993-Standards
G1B	12	1	*n*=24	2 units (The attribute of length measurement)	8 sessions	1993-Standards
G2A	22	2	*n*=31	1 unit (Standard units of metric system)	4 sessions	1993-Standards
G2B	24	2	*n*=32	1 unit (Standard units of metric system)	3 sessions	1993-Standards
G3A	20	3	*n*=23	1 unit (Unit conversions within the metric system)	4 sessions	1975-Standards
G3B	30	3	*n*=27	1 unit (Unit conversions within the metric system)	4 sessions	1975-Standards
G4A	35	4	*n*=21	1 unit (Estimation of length)	4 sessions	1975-Standards
G4B	18	4	*n*=24	1 unit (Estimation of length)	3 sessions	1975-Standards
G2C	22	2	*n*=31	1 unit (Basic concepts of area measurement)	3 sessions	1993-Standards
G5A	20	5	*n*=24	Partial unit (The formula for area measurement of trapezoids)	1session	2003-Guidelines
G6A	8	6	*n*=32	Partial unit (The formula for area measurement of circles)	1 session	2003-Guidelines
G5B	16	5	*n*=19	Partial unit (measuring the volume of irregular-shaped objects)	1 session	2003-Guidelines

that were adopted for teaching third- and four-grade length measurement (G3A, G3B, G4A, and G4B) were designed based on the 1975-Standards, whereas the textbooks that were adopted for teaching first-and second-grade length measurement and area measurement (G2A, G2B, and G2C) were designed based on the 1993-Standards. Finally, the textbooks that were adopted for teaching area formulas and volume measurement for upper-grades (G5A, G5B, and G6A) were designed in alignment with the 2003-Guidelines.

3.5 *Coding and Analysis Procedures*

To conduct a more thorough analysis of the data, all of the respective detailed transcriptions of the videotaped instruction and interview data, as well as field notes taken from classroom observations during the times of data collection were reread and identified by the author and a rater. The rater has a qualification of being an elementary school teacher with a master's degree in education.

In addition to describing the modes of sequential process of teaching spatial measurement, as exhibited by the teachers investigated, the coding processes that were administered to analyze the data included two domains. a. Teachers' approaches involving measurement estimation utilized in the lessons; b. Types of teachers' questions asked during post-measurement activity discussions. The coding scheme adopted for each aforementioned analysis is described as follows.

3.6 *The Mode of Sequential Teaching Process*

Seeing that the types of teaching processes provide insights into teachers' patterns of teaching measurements, the sequence of teaching processes were investigated and analyzed, and then presented as flow charts. To describe the nature of instruction in spatial measurement, teaching actions or activities that teachers carried out for fulfilling the lesson's instructional goals and objectives were identified and organized according to the sequential teaching process they employed. The teaching process included the following four steps. a. Beginning of Lesson. This step includes a beginning activity in which a brief review of a previously learned concept, or a brief talk about a new topic to-be-learned, without an elaborated explanation. b. Lesson Development. This step included teachers' activities in which they attempt to engage students in physical manipulations or solving measurement problems for the purpose of developing their mathematical understanding and skills through individual work, or in pairs or small groups. c. Post-Activity Discussion. This step included activities in which the teacher and students engaged in sharing ideas about measurement or solutions to the problems they were solving, and reflection on previously completed measurements, as well as verbal interactions for making summaries and drawing conclusions. d. Workbook (WB) or Worksheet (WS) Reviewing. This review consisted of checking student WB (WS) solutions and correcting mistakes.

Additionally, the types of instructional settings, manipulatives, instruments, and types of measurement activities that the teachers assigned students to work on measuring were identified.

3.7 *Teachers' Approaches Involving Measurement Estimations*

Taiwanese teachers' method of mathematics lesson enactment, in general, is characterized by staying quite close to the textbook content that is provided in the textbooks or curriculum guides (Askew, Hodgen, Hossain, Bretscher, 2010; Tan, 1999). Seeing that engaging students in solving problems given in the textbook and WB was a part of instructional activities involved in each cases, for estimation problems, only the supplementary problems that required measurement estimation provided beyond the textbook curriculum were coded for each instructional case. Moreover, the problems in WB or WS provided for the lesson by the individual teacher and the two instructional cases of G5A and G6A, which centered on the area formulas, were excluded from the coding of estimation problems. In sum, all estimation questions posed extendedly by the 10 teachers were identified by comparing the detailed transcripts of an enacted lesson with written problems suggested in textbooks and curriculum guides.

In all, three types of estimation problems were identified and coded with reference to Joram et al.'s (1998) and Joram's (2003) studies. a. Guess-and-check: Students were asked to write an estimate and then measure the to-be-estimated item. b. Finding benchmarks for estimating: Students were asked to find an item that could serve as a benchmark for a centimeter, a meter, a square centimeter, and a cubic centimeter, or to find a container that could serve a benchmark for a cup. c. Guessing without checking: Students were asked only to make a guess or to show a length/distance/size by gesticulations or body movements, without additional types of measurement, for checking the particular estimate. Once all of the types of estimation problems had been identified, every extensive problem involving measurement estimation posed by the teachers investigated was coded according to these types. Moreover, the frequency of each type of approach was calculated.

3.8 *Types of Teachers' Questions Asking for Post-activity Discussion*

In the study, an analysis was done on the types of teachers' questions which were intended to reconvene the class to share findings and reflections on measuring manipulations for generating post measuring activity discussions. The types of questions were categorized with reference to the synthesis of theoretical literature on types of questions that might initiate different types of student responses (Hicks, 1996; Walshaw & Anthony, 2008). a. High-level thought process (HTP) question. The questions that students were asked to verbally explain their meaning or to demonstrate their understanding of the concepts underlying their measure activities, may be laid as a foundation for a follow-up teacher-and-student verbal discussion. The HTP questions may potentially support teachers in identifying out students' levels of understanding or clarifying students' misconceptions. These questions might also highlight the content or generate an extended analysis of reasoning by listening to student explanations (Henningsen & Stein, 1997; Walshaw & Anthony, 2008). Moreover, the HTP questions that were asked to initiate teacher-student discussion (about the measurement manipulations that students engaged in each session) may, to some extent, facilitate follow-up mathematical discourses for instruction. For example, "Here, we have lots of different objects which have been used as units for measuring the length (or area) of the item," or, "How can we make sure that these results of measurement are correct?" and so on.

In contrast, low-level thought processes questions (LTP) are those in which students are requested to respond by repeating memorized facts without explanations, or providing step-by-step algorithmic procedures without any connection to underlying meaning. An example of this would be recognizing units and measuring with a ruler and reporting the results, or offering a yes-or-no short answer. This type of question is frequently adopted in a classroom in which the direct instruction of skills is conducted or when teachers focus on seeking the "correct answer" (Hicks, 1996).

The types of teachers' questions asked in post-activity discussions were coded according to the coding scheme mentioned above. The frequency of each type of questions was then calculated.

4. Results

For assessing inter-rater reliability for the coding of categories of teachers' approaches involving measurement estimations and types of questions asking for post-activity discussion, the results of Kappa analyses were found to be .74, $p < .001$ and .86, $p < .001$, respectively.

4.1 *The Teaching Process and Characteristics Involved in the Teachers' Measurement Instruction*

After a detailed rereading of all 12 transcripts of instructional cases and field notes, the four steps commonly included in each lesson enacted included: Beginning of Lesson, Lesson Development, Post-Activity Discussion, and WB reviewing. In the Beginning of Lesson phase, teachers frequently began their lessons by either asking questions (e.g., "Which one is longer? Can you tell by looking?" "Do you know how to measure the length of the item held in my hand?" "How tall are you? How can you measure your height?"), or providing a brief reviewing of previously learned concepts (e.g., "What are the characteristics of a parallelogram (or a rectangle or a trapezoid)?" "How do you measure the circumference of a circle?"). The length of Beginning of Lesson time varied from teacher to teacher. The majority of the teachers took about two to four minutes each session for starting their lesson.

Next, the teachers proceed to the Lesson Development phase by assigning measurement activities in which students became engaged in processing actual measurements, recordings, and the reporting of results of their various measurements. For example, the objects provided by the teacher (e.g., strings of a given length, 30-cm rulers, cloth tape, metal tape) and readily available objects (12-cm or 15-cm rulers, erasers, edge of a mathematics textbook) or body parts were commonly used to measure various lengths. The objects frequently used to measure areas included paper, the face of a student notebook, transparent grids, unit squares, a face of a 1-cm^3 cube, sets of figure cards and paper manipulatives attached to student workbooks. For measuring volumes, different sizes of containers, 1-cm^3 cubes and small rocks were used. Basically, the content of textbooks and curriculum guides (teaching manuals) were the primary sources for teachers' lesson enactments.

When students worked on measuring, the majority of the teachers circulated through the classroom to assess students' performance in measuring manipulations, gave feedback, and gave support for individual student's or small groups' needs. The instruction time that the teachers allocated for student's actual measuring manipulations varied from teacher to teacher. About half to two thirds of the total amount of instruction time that was scheduled for unit teaching was allocated by the majority of teachers in order to facilitate students to delve into measuring activities.

Next, teachers reconvened with their students to share measuring results or to argue about the validity of particular measurement ideas. They also, as a group, discussed questions that emerged from measuring activities. The types of questions that demanded different levels of thinking, which were asked by the teachers at the time of post-activity discussions, are presented later in this paper. The length of time for post-activity discussion depended upon types of questions asked by the teachers, as well as students' responses. The teachers commonly used about five to 10 minutes for post-activity discussions.

Finally, there was the matter of reviewing and checking WB (or WS) answers. With regard to the WB reviewing session, all of the teachers investigated reviewed the problems in WB (or WS) at the end of instruction, excluding teachers G5A and G5B who integrated the lessons with the problems of WS, which were edited and provided specifically for the lesson enacted. Generally, aside from the three teachers who taught area measurement, the remainder of the teachers investigated took about one session to check students' WB answers and discuss how to solve their problems in WB. Although there were some minor differences in classroom teaching practices among the 12 instructional cases, two examples of the modes of sequential teaching process implemented in the teachers' measurement instructions are summarized and represented briefly as flow charts, as seen in Figure 1 and Figure 2.

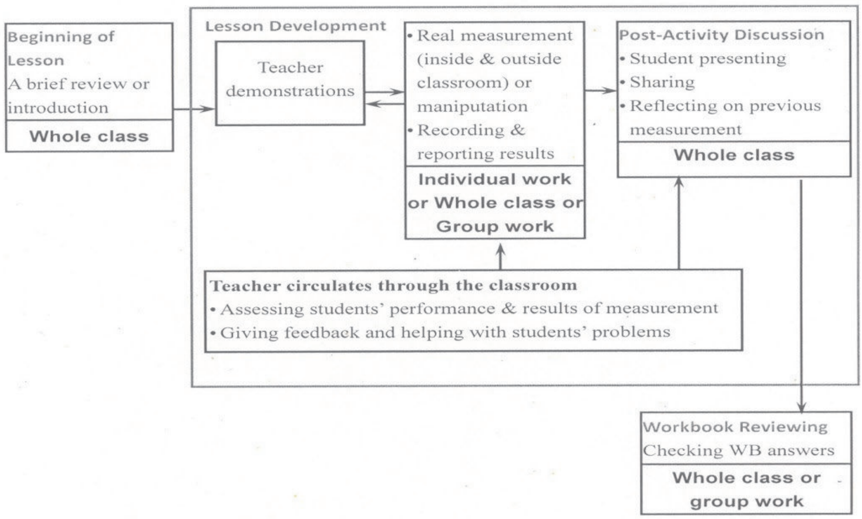

Figure 1. An example of the mode of sequential teaching process for lower-grade students and occasions of teacher-led instruction

The mode of sequential teaching process in Figure 1 was frequently exhibited in teaching measurements for first-grade students and occasions of teacher-led instruction, in which the teachers were more likely to dominate the teaching processes. Furthermore, teacher-student interaction and student-tool interaction was contained in the mode in Figure 1. In contrast, the mode in Figure 2 was frequently exhibited in teaching measurement for middle-grade students and occasions of

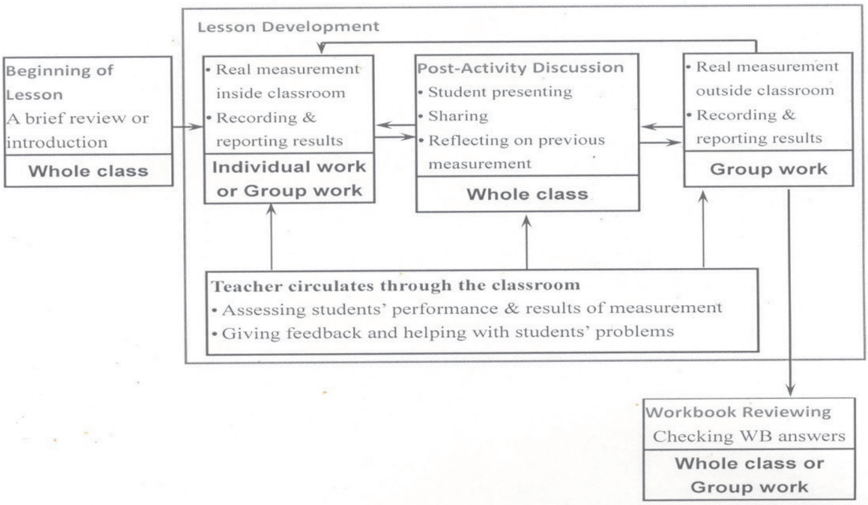

Figure 2. An example of the mode of sequential teaching process for middle-grade students and occasions of collaborative group work

collaborative group work, in which more student-to-student interactions were involved for the measurement lessons.

In the study, the teachers who taught length measurement at the first-grade level and the teachers who taught area formulas, as well as the teacher who taught displaced volume, were observed to frequently deliver concepts and principles of measurement to their students and then engaged students in solving measurement problems through individual work or whole class or group work, with the exception of activities involving outdoor measurement. They believed that teachers' explanations and demonstrations of concepts of measurement and measuring techniques are important for students' learning of measurement. Conversely, the teachers who taught length measurement at the second-grade level, and the teachers who taught middle-grade levels, as well as the teacher who taught the basic concepts of area measurement, tended to engage students in collaborative group work and have discussions referring to students' ideas about measurement. They tended to highlight the importance of collaborative group work, as well as discussions for solving measurement problems that demanded measuring the lengths of large-sized objects or large distances.

The common characteristics of classroom practice enacted in teaching measurement as exhibited by these teachers can be derived based on an analysis of the videotaped lessons. This is true regardless of some distinctions that occurred from differences in the content of the lessons as they were enacted, and the grade levels of the students who participated in the instructional cases. These characteristics include the following three aspects.

First, activities integrating with real measuring manipulations inside (or outside) the classroom, coupled with a review of the student WB, were perceived by the teachers investigated as effective mediums in which to develop students' measurement concepts and skills. For the instructional cases of length measurement, all eight of the teachers investigated stimulated interest in measuring daily life objects or body parts, through employing activities that involves actual measuring manipulations. In addition to indoor activities, seven of the eight teachers were found to have their students work in pairs or small groups, measuring larger distances outside the classroom by means of pacing-and-counting, recording, and reporting the approximate lengths of measured objects. Furthermore, teachers G2A, G2B, and G2C tended to accentuate the importance of collaborative group work in problem-solving explorations and carrying out measurement.

For the instructional cases of area measurement, teacher G2C highlighted the meanings of area and measuring the area of an enclosed region, as well as skills of covering, such as no overlapping. She conducted activities for area comparisons, which students engaged in by tiling and counting 1-cm^3 cubes (or grids) covered by the given figures. Teacher G5A engaged the students in exploring the relationships between triangles and either parallelograms and trapezoids through rearranging the various triangles given, whereas teacher G6A engaged the students in arranging eight sectors of a circle and 16 sectors of a circle into approximate parallelograms.

For the instructional case of volume measurement, Teacher G5B got the students involved in observing, recording, and comparing changes in water levels prior to and after putting an object into a container of water (throughout the session of actual manipulations). The questions involving the equivalence between the volume of an object when it was dropped into a container of water, and the amount of water level raised, were discussed through the entire session.

Evidence that teachers' emphasis on the importance of real measurement can be supported from the interview data. For example:

> "Real measurement manipulations are crucial for developing students' sense of measurement though preparing for the measurement materials needs teachers' efforts and lesson preparatory time. Students need experiences of measuring manipulations even though the fact that measuring manipulations often take much instruction time and that teachers strive for meeting the lesson schedule." (Teacher G2B)

In addition to activities of actual measurement, most of the teachers investigated discussed how to solve the problems on WB (or WS) and carefully checked students' written solutions. Specifically, some of the teachers investigated (including G1A, G2A, G3A, G3B, and G5B) explicitly indicated that WB served as an important tool to ensure that students grasped core of knowledge that is learned from actual manipulation activities. Most of the teachers who taught length measurement added measuring daily objects as part of homework, such as measuring the length/width of a table or requesting that children work cooperatively with their parents.

Second, the subject matters that the teachers concerned about was students' learning of measurement manipulations. In the observations of the instructional cases, all of the teachers who taught length and volume measurement, including teacher G2C, focused their attention on students' conceptions of unit, unit iteration, and addition that are combined, so that numerical values are assigned. Yet, compared with checking students' measuring manipulations and skills of using rulers, the teachers provided fewer activities involving comparing attributes of a to-be-measured object, as well as logical comparisons with third objects. This was especially true in the cases of length measurement and volume measurement. For example, only teachers G4A and G2C were found to explicitly discuss what it means to measure with students and explain meanings (e.g., "What does the length (or area) measure mean?" "What does measurement estimation mean?") at the time they enacted their lessons. Conversely, the remaining seven teachers who taught length measurement—and Teacher G5B, who taught volume measurement—seemed to believe that the meaning of the attribute to be measured could be attained through manipulative activities which involve much perception and direct comparisons.

Moreover, the teachers were prone to having their students measure given objects (or using a given unit) and report both the number of units and the name of the units. Conversely, a lack of the activities that involve making objects of given sizes or representing objects of a given size through drawing or the use of written reports, were provided in the instructional cases. Additionally, the teachers who taught length measurement for the first- and second-grade students did not seem to pay much attention to examining students' ideas about length conservation, which is associated with the ideas of additivity property and partition, and transitive property reasoning, which is a comparison idea.

Third, there was a concern about the use of a variety of manipulatives (e.g., figure cards) and technology integrated with two-dimensional geometry. Equipments such as over-head projectors, electronic whiteboards, and computers were commonly used in classroom teaching. For example, over-head projectors were commonly utilized in the instructional cases of length and area measurement for the grade-2 students. To help the students explore the relationships among the areas of basic shapes and circles, teachers G5A and G6A integrated technologies of dynamic geometry programs with manipulations. For example, Teacher G5A demonstrated the dynamic processes of figure transformations and rearrangements by means of an electronic whiteboard, whereas Teacher G6A applied technology which combined PowerPoint program, Flash, and Activate Mind Attention (AMA) programs to demonstrated how figures become closer and closer to the shape of a rectangle through respective arrangements of 50 sectors or 100 sectors of a circle. Moreover, by experimenting with cut-up circles and changing them into a rectangle, Teacher G6A guided his students to see the correspondences between: a. the circumference of a circle as corresponding to the lengths (horizontal sides) of the rectangle whose top and bottom was formed by the circumference of a circle, so that the length is half the circumference, and b. the radius of the circle as corresponding to the width (vertical sides) of the rectangle.

On the basis of the interview data, teachers G5A and G6A highly recommended the importance of the connections between knowledge of geometry and concepts of area measurement for developing students' understanding of area formulas for trapezoids and circles. In particular, it is crucial that students take the advantage of dynamic geometry programs to explicitly demonstrate the processes of figure transformations and rearrangements—essential content involving relationships between the areas of basic shapes and circles.

In sum, the findings indicate that all of the teachers were devoted to actual measuring activities and WB reviewing. Moreover, in our interview data, they expressed their struggle with the pressure of maintaining the schedule of their lesson plans. Due to time constraints and the extra efforts needed to prepare a measurement activity, these activities could be simplified or displaced by teacher demonstrations or exhibitions that would be performed by calling on a few students. Such simplification may occur in teaching area and volume measurement in the upper-grade levels.

4.2 *Teachers' Approaches Involving Students Engaged in Measurement Estimations*

Table 2 shows the frequency of the various approaches that teachers adopted for teaching measurement estimation. As can be seen in this table, with the exception of the two instructional cases of area formulas (G5A and G6A), seven of the 10 teachers provided students with supplementary estimation problems beyond the textbook unit—whereas three teachers did not provide any extra estimation problems in their lesson enactments. All of the teachers investigated tended to highlight numbers used with the correct unit labeled, and posed questions such as "Which unit can be suitable for expressing the length of a pencil, 'centimeter' or 'meter'?" and the like. However, the two approaches "guess and check" and "finding benchmarks," which contain a complete process of estimation, seemed not to be typically adopted. Only teachers G4B and G2C adopted both of the two approaches mentioned above into their lesson enactments. Moreover, some of the teachers (G1A, G2A, G2B, G3A) showed a preference for using "guessing without checking" which contains an incomplete process of estimation rather than that of "guess and check" or "finding benchmarks."

Table 2. Frequency of measurement estimation conducted by each teacher

Spatial Measurement	Length								Area	Volume	Total
Teachers	G1A	G1B	G2A	G2B	G3A	G3B	G4A	G4B	G2C	G5B	
Number of sessions	6	8	6	3	4	4	4	3	3	1	42
1. Guess and check	-	-	1	4	3	-	2	1	1	-	12
2. Finding a reference point (or benchmarks)	-	-	-	-	-	-	-	2	3	-	5
3. Guessing without checking	3	-	6	1	2	-	-	2	-	-	14
Total	3	-	7	5	5	-	2	5	4	-	31

It is noteworthy that teachers G4A and G4B provided guess-and-check problems and went through the process of estimating problems given in the textbook unit—in which "measurement estimation" comprised the main content. However, they did not provide additional measurement estimation problems beyond the textbook unit. Instead, they were inclined toward providing informational feedback, which allowed estimators to hone their skills; however, questions such as "How can you obtain a reasonable estimation?" seemed to be ignored.

Looking closely at the estimation activities in which teachers and students were engaged, students were frequently asked to estimate measurements of remembered and familiar objects, such as the length of the students' desks that were used in the classroom. Teachers then evaluated students' responses to these questions, which demanded a certain amount of correct answers. Conversely, the students had few opportunities to discuss about questions that required making reasonable measurement estimates, and justifying their reasoning. For example, "How do you make the estimation? Why do you think that your estimation is reasonable?" or "How do you improve estimation accuracy?" or "How do you find an item that could serve as a benchmark for a centimeter and a meter?"

4.3 *Types of Questions That Teachers Asked for Post-Activity Discussion*

Table 3 shows the frequency of the two types of questions asked by the 12 teachers investigated for post-activity discussion. Most of the teachers conducted a whole-class discussion and provided more LTP questions than HTP questions in a session. Only four teachers (G2B, G2C, G5A, and G6A) asked more than two HTP questions for post-activity discussion in each session. Conversely, four of the 12 teachers (G3B, G4A, G4B, and G5A) asked solely LTP questions rather than HTP questions. When comparing the average number of HTP questions asked in a session among the 12 teachers investigated, the frequency of HTP questions provided by teachers G2C, G5A, and G6A seemed to be slightly higher than those provided by the other teachers.

A close look at the content of teacher-and-student interaction in the post-activity discussion phase shows that, with some exceptions (of the teachers who taught area measurement), the teachers in our study were more likely to call on some students to share their measurement results by reporting a numerical answer or representing arithmetical equations, or to ask questions which required short answers without explanations. For example, the questions asked by the teachers who taught grade-1 and grade-3 and -4 length measurement focused on the results of comparisons, the use of standard units, the amount of units used for the measure, the correct method for using rulers, and reading the scales. Teacher G5B tended to asked questions which demanded quantitative reports with regard to how much the water level rose when different amounts of 1-cm^3 blocks were dropped into a container.

Table 3. Frequency of the types of questions asked by the teachers for post-activity discussion

Spatial Measurement	Length								Area			Vol-ume
Teachers	G1A	G1B	G2A	G2B	G3A	G3B	G4A	G4B	G2C	G5A	G6A	G5B
Number of sessions	6	8	6	3	4	4	4	3	3	1	1	1
1. HTP questions	5	2	6	7	3	-	-	-	11	3	5	-
2. LTP questions	8	15	6	7	8	3	3	4	1	9	3	8
3. Average number of HTP questions asked in a session	5/6	2/8	6/6	7/3	3/4	-	-	-	11/3	3/1	5/1	-
4. Average number of LTP questions asked in a session	8/6	15/8	6/6	7/3	8/4	3/4	3/4	4/3	1/3	9/1	3/1	8/1

Teachers who inquired about students' understanding of and reflection about their measurement experiences were inclined to ask HTP questions. For example, frequently asked questions included: "How was the measurement done?" "How did you measure the length of a to-be-measured object when the amount of objects that are used as units for measuring was fewer than what was needed?" "How did the measurer ensure the accuracy of the measurement?" and "Is there another way of finding out the area, and which figure is bigger?" and so on. Moreover, Teacher G2C who taught area measurement brought students' misconceptions to a later discussion. She explicitly discussed what it means to measure an area, and devoted time to clarifying and correcting students' confusion about whether they should count points or squares inside the given rectangles, which were created using geoboards.

It is noteworthy that some of the teachers tended to ask HTP questions, but the response time offered for students' thinking or further teacher-and-student discussion seemed to be inadequate. This was evident in the cases of upper-grade lessons. For example, both teachers G5A and G6A were inclined to guiding their students to find the answers follow a series of questions asked sequentially by the teachers though they did pose some HTP questions.

5. Discussion and Conclusion

5.1 *Discussion*

At the level of classroom practice, this study demonstrates how elementary school teachers enacted lessons involving spatial measurement across different grade levels. Based on the above results, the 12 instructional cases analyzed in the study reveal that all of the teachers were devoted to students' acquisition of measurement and focused much of their attention on students' skills of measuring, using tools, and recording and representing the results of measurement by using units and numerical calculations. For teaching estimation skills, only seven of the 10 teachers included estimation activities in their lessons, with the exceptions of the two teachers who taught area formulas. The approaches of teaching estimations as adopted by the seven teachers included guess-and-check, guessing-without-checking, and finding benchmarks. All of the teachers reconvened the class to share findings of actual measurement and took post-activity discussions seriously. However, they were inclined to ask more LTP questions than HTP questions. The questions which demanded students' explanations, reasoning, and reflections about measuring manipulations regarding properties being measured did not seems to be adequately addressed by the teachers.

Additionally, teacher-led instruction and collaborative-learning group working on measuring activities were conducted in a mixed fashion in the instructional cases. Teacher-guided explorations were frequently undertaken to help students pay attention to some complicated principles that would demand additional contemplation. This is specifically evident in the case of teaching basic concepts of length measurement for first-grade students and in the cases of teaching concepts of area formulas and displaced volume for upper-grade students. As to the use of technology and teaching aids, aside from the use of various concrete manipulatives (e.g., 1-centimeter cubes, figure cards, geoboards, or containers), the activities that integrated manipulations (transformed figures) with dynamic geometry programs on the computer were adopted for developing students' understanding of area formulas.

The findings from the analyses of the instructional cases of spatial measurement showed that the teachers investigated were competent to teach spatial measurement of length, area, and volume. They displayed a consensus about the importance of actual measuring manipulations with respect to students' construction of measurement concepts. The processes of teaching lessons involving length and area measurement were carried out sequentially, from visual perception, to direct (and indirect) comparisons, and then to the use of nonstandard measures—and, finally, to the application of a standard measure. The concepts of volume measurement and formulas were then to be introduced into the upper grades on the basis of knowledge of length and area measurement, and geometry concepts. These teaching practices regarding developing students' measurement knowledge are in accordance with recent instructional recommendations that begin with comparing and ordering, and then move on to repeated use of a unit to find quantity--and, finally, to read off a scaled value by using a tool (Sarama & Clements, 2009; Van den Heuvel-Panhuizen & Buys, 2008).

With respect to developing students' understanding of area formulas, the two upper-grade teachers helped students derive formulas for the area measurement of trapezoids and circles by means of using computer-generated activities which integrated figure transformations with dynamic geometry programs. These approaches, integrated with geometry and manipulations (Huang & Witz, 2011) and dynamic geometry programs, are also recommended by mathematics educators as an effective method for facilitating students' reasoning and understanding when the subject matter is complex and a particular manipulating task is quite demanding (Huang, 2012; Kordaki & Balomenou, 2006). For example, teachers might lead the students in perceiving the relationships among the family of basic shapes and to discover the area formula of circles based on the process of figure transformations and generalization.

As to estimation instruction, the teachers adopted two approaches, guess-and-check and finding benchmarks, as useful strategies for developing skills in measurement estimation (Joram et al., 1998; Fendel, 1987). In addition to the two approaches mentioned above, guessing-without-checking also was utilized by some of the teachers investigated.

Given the fact that teachers showed a preference for using guessing-without-checking, the effectiveness on improving students' refinement of making reasonable approximations of measurement may be limited because of a lack of any checking process (Joram, Gabriele, Bertheau, Gelman & Subrahmanyam, 2005).

As Sovchik (1996) suggested, lower-and middle-grades students need to intuitively develop experience with measurement through explorations and estimations rather than via precision of measurement. The analyses of the present study reveal that the students were exposed to more opportunities for doing actual measuring manipulations, as assigned by the teachers, but fewer activities involving explorations and measurement estimations. Although estimation skills can be fostered through experiences of actual measurement (e.g., establishing standard reference measures based on familiar objects) (Joram et al., 2005), a sense of measurement that develops from ample experience with estimating measures may not occurs naturally for most students within the regular curriculum (Hope, 1989). Indeed, students' development of estimation skills heavily relies upon teachers' instruction, namely, that in which various strategies used for guiding students to obtain reasonable answers are employed. Thus, to some extent, the findings of this study may provide a reason for explaining why Taiwanese students' estimation skills are not well developed, as previously indicated by Tan (1998).

The teachers in this study have demonstrated a strong tendency to engage their students in actual measuring manipulations, such as skills of using rulers and reviewing student's work through workbooks or worksheets. Students' participation, collaboration, and idea-sharing were appreciated by these teachers. Nevertheless, the use of questioning techniques for stimulating students' high-level thinking and reflection about measurement tasks seemed insufficient. Such inadequacy was displayed in most of the instructional cases in the study regardless of the modes of sequential teaching processes the teachers had adopted. Actually, teaching practices that highlighted measuring skills and quantity information may lead to apparent procedural competence; however, these factors are not adequate for enhancing students' conceptual understanding of measurement (Lehrer, 2003). As Owens and Outhred (2006) suggested, making sense of measurement depends heavily upon the use of actual manipulations and language (or symbols) to interpret practical experience in doing manipulations and reflecting upon learning. Examples of the above would include thinking how to answer questions related to "what if" or "finding a relationship among different units," or "drawing a conclusion from measuring data." Seeing that fourth-grade students in Taiwan were less successful in doing problems underlying measurement concepts rather than solving problems underlying number and data display knowledge of the sort addressed in the assessments of Trends in International Mathematics and Science Study (TIMSS) in 2003 (Lin & Tsai, 2003) and 2007 (Mulliss, Martin, & Foy, 2008). That is, more teacher effort is needed for making measurement a thought-provoking activity that stimulates students to communicate their ideas about measuring and estimating, as well as reflecting thinking.

Moreover, curriculum guidelines and instructional suggestions given in teaching guides (or teaching manual) are assumed to impact on teachers' instructional approaches for implementing curriculum (e.g., Shkedi, 2009). Interestingly, the findings of the study reveal two examples of modes of sequential teaching processes adopted by the teachers across different content of lessons, and the grade levels of students who participated in the classes, and sets of curriculum standards (or guidelines). It implies that teachers' adoption of instructional approaches depends on their viewpoints on what and how concepts and skills of measurement should be learned and their individual preference in teaching measurement. However, the findings of the study were obtained from a small sample of in-service teachers' instructional videotapes for teaching spatial measurement of length, area, and volume, the descriptions of teachers' instructional approaches discussed in this study might be viewed as researchable presumptions. These presumptions merit further research with larger samples to investigate instructional mode adopted by teachers for teaching length, area, and volume measurements.

A related issue concerns the use of the geoboards or dot-papers for teaching area measurement: these two aids are frequently suggested for helping students explore areas (e.g., Fendel, 1987; Van Voorst, 2000). Observations of the videotaped lesson involving grade-2 students found that they become confused while counting the pegs and a square enclosed in a rectangle, as formed by rubber bands on a geoboards. The students' confusion probably was generated from insufficient experience in measuring shapes, a process that entails covering square pieces of paper or 1-cm^2 squares, which is an instructional step suggested for beginning learners (Lehrer, 2003; Wilson & Rowland, 1993). For guiding grade-2 students who are beginning to learn area measurement to explore area, the appropriateness of using figures represented on dot-paper or the geoboards needs further investigation.

5.2 *Conclusion*

This study covers sets of data about teachers' teaching practices in length, area, and volume measurement across different grade-levels. The findings show that some ideas about teaching spatial measurement that were advocated by recent research and documents of curriculum and instruction (e.g., Lehrer, 2003; NCTM, 2006; Owens & Outhred, 2006; TME, 2010) were implemented in their instructional practices. For example, the task of measuring manipulations is inclined to be partially skipped in classroom teaching because of teachers' struggle in getting students to become fully involved in real measuring activities (e.g., Van den Heuvel-Panhuizen & Buys, 2008). Nonetheless, it was not found to be the case with the teachers investigated in this study. They tended to offer many activities involving actual measuring manipulations, and took advantage of using manipulatives, dynamic geometry programs, and electronic tools to affect learning. Notwithstanding that the teachers made great efforts to fulfill the instructional objectives preseted in textbook units, offering additional opportunities for students to participate in measurement estimations and extending teacher-and-students discussions were recommendations put forth by recent research and curriculum documents, as mentioned above. None of these approaches were significantly evident in the instructional cases.

On the other hand, the data in this study provide a portrait of what teachers concern about when teaching and learning length, area, and volume measurements—a picture which may facilitate mathematics researchers in understanding such classroom practices. A limitation of the study is its small sample size. Thus, any generalization from the findings is limited. In particular, the three instructional cases of area measurement which were videotaped for supporting teacher professional development should not be taken as necessarily representative of other teachers outside the context of those in the study. Moreover, teaching measurement in a computer-based environment heavily depends upon teachers' competence in using technology and supportive sources, as afforded by school contexts. At most, the modes of the teaching process and lesson enactment shown in the study may represent many of the teaching practices prevalent in urban schools in Taiwan. Further studies are needed about instructional cases for teaching spatial measurement that incorporate an emphasis on students' learning outcomes.

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Chapter 6

Pedagogical and Curriculum Potentials of Homework: A Case Study About Geometric Proofs in Shanghai

FANG Yanping

This study reports how an experienced secondary teacher in Shanghai explained student homework on geometric proofs. A discourse analysis reveals a structured set of her routine instructional actions in involving students in an IRE/F-patterned instructional discourse to get students to recall, reconnect and reconstruct their earlier learning. Marking student homework, 'analyzing' student thinking and tutoring individual students made the explaining detailed, multifaceted, structured and targeted at the mathematical substance. Analysis of curriculum materials and marked student work finds that difficulties in learning were with the fundamental norms to follow and habit of thinking required in writing geometric proofs. Finally, the tradition of design with variation was found to have students discern and master the fundamental deductive reasoning skills under diverse, carefully designed problem settings. These uncover the hidden dimensions of math teaching in Shanghai mediated through homework practice and help us rethink the role and potentials of homework.

Keywords: mathematics homework, mathematics errors, design with variation; geometric proof, pedagogical reasoning and action

1. Introduction

After Tr. Wang and the students bid each other good morning, she summoned the lesson. "I continued marking your homework as soon as I arrived this morning," she shared. "In doing proofs," she continued, "our classmates again missed

> *either an arm or a leg." She read the names of the students who did well and then directed the class attention to the first figure that she drew on the blackboard. She used examples of students' wrong methods in guiding them, step by step, to understand why they were wrong and how they should approach their own correction. She explained three of the proof writing assignments that required students to apply the perpendicular bisector theorem that she taught on the previous day. Explaining homework allowed her to review the topic in a contextualized way. In less than ten minutes, she started the new lesson. "Today, we're going to study another bisector theorem: an angle bisector and its converse theorem," she announced and wrote the topic on the blackboard. By capitalizing on the review, she made a smooth transition to the related topic, angle bisector theorem.*

The above captures a glimpse of a typical activity at the beginning of Teacher (Tr.) Wang's lessons on the days when she taught the unit on *geometric proofs,* in one of her two eighth grade classrooms. As usual, she would explain and comment on student homework to the whole class focusing on the problems or errors she identified from homework assigned the day before that students had submitted and she had carefully marked. In less than 10 minutes she helped students attend to and make sense of the issues in learning the topic and get them ready for the related new topic of the day. Compared to her teaching the subsequent topic on *functions*, the level of engagement with homework was found to be more intensive in teaching geometric proofs because of the large class size and more importantly, the nature of proof writing that demands the rigor of logic and thus meticulous attention to the process. In this chapter, I draw on my observations on one of the sections, *Converse Propositions and Theorems*, which she taught in mid-November 2002, to demonstrate how she used homework in teaching students' deductive thinking. Detailed analysis of this activity across those teaching days reveals a clear structure of her actions and visible discourse patterns in explaining and commenting on homework *(jiangping zuoye)*, a common teaching practice in China, hereafter referred to as explaining homework.

Homework use is common to mathematics teaching in schools in both Western and Eastern countries (Mullis et al., 2000; Stigler & Stevenson, 1991; Stigler & Hiebert, 1999). In the U.S., homework research in the past three decades has been focused on whether homework is useful and effective in promoting student academic achievement (Cooper et al., 2003) given the public's concerns about its disruptive role in children's life (Vatterott, 2009). In East Asian countries, however, homework's supporting role in teaching and learning has been taken for granted. Nevertheless, in both the East and the West, there has been a dearth of literature on how homework is used effectively in teaching (Cooper, 1989; Cooper et al., 2003).

Meanwhile, teaching is culturally embedded and so is use of homework in teaching and learning. Berliner (1986) and Leinhardt (1990) found that mathematics teachers in the U.S. use homework checking as a routine activity at the beginning of a lesson. Leinhardt and Greeno's study (1991) of the knowledge gap between the homework-checking practices of novice and expert teachers in U.S. schools found that "homework correction [checking]" performed by an expert teacher "is an ideal example of how one rather small lesson component (it lasts 2 to 5 minutes and is rarely mentioned by teachers, student teachers, or texts) can help achieve multiple goals" such as taking attendance, knowing who has not completed the day's assignment, finding out what mistakes there are, and deciding how to adapt the lesson to overcome existing problems (pp. 238-241). In both situations, they found that homework checking is largely a disciplinary tool for teachers to monitor homework completion. For mathematics teachers in China, as reflected in Tr. Wang's lessons, homework explaining focused on errors identified by the teacher would go way beyond simply checking homework and when conducted in structured ways to all students, it would provide timely and targeted feedback to support students' learning of fundamental mathematical ideas. As a veteran teacher and Head of the school's Mathematics Teaching Research Group, her excellence in teaching was highly regarded by her colleagues. Her example would speak to what Hattie and Timperley (2007) advocated, the powerful role of timely feedback provided to students in promoting effective learning.

As detailed in this chapter, explaining homework as a typical lesson component for Tr. Wang has a routine activity structure and discourse patterns. Hours of marking, tutoring individual students and talking with her colleagues about problems in homework would have made it possible for her to explain homework in structured and detailed ways (Fang, 2010). It is hoped that Tr. Wang's case would open up a window of a hidden homework-mediated activity system to help reveal more fully mathematics teaching practice in China (Fang, 2010).

To build Tr. Wang's case, classroom discourse analysis and document and artifact analysis were conducted to tease out patterns of her teaching actions and goals, structures of discourse, homework tasks and patterns of their design and use. I use the notion of variation and draw on key research in teaching and learning of geometric proofs to examine the homework tasks and how they are designed in ways to focus teachers' and students' attention to discerning the critical dimensions of geometric proofs (Marton & Tsui, 2004). In doing so I aim to offer a more systematic understanding of the values behind the curriculum and pedagogical traditions and the culturally embedded meaning that support the mediating role of homework in mathematics teaching and learning in China.

2. Method

This was an ethnographic study with focused participant observations of a teacher's teaching practice both in her classrooms and her grade-level office as well as weekly meetings. A case study approach was adopted to provide an in-depth examination of the teacher's homework-related practice because it "offers a means of investigating complex social units consisting of multiple variables of potential importance in understanding the phenomenon" (Meriam, 1998, p. 41). These variables were examined by collecting a full range of evidence, including documents, artifacts, interviews in addition to observations to uncover their relationships to the curriculum and cultural contexts. Classroom observations were able to surface how the explained homework, marked homework artifacts, curriculum documents and open-ended interviews with Tr. Wang were able to uncover why she explained those errors, what mathematics is entailed in the errors that she explained as well as the design features of the tasks. In addition, key literature on teaching of geometric proofs was consulted to shed more light on the teaching actions and goals, structures of discourse, homework tasks and the patterns of their design and use.

The analysis was conducted on two levels to form a general picture of the object of explanation and the multiple dimensions revealed through the teacher's selecting and explaining processes. The first level focused on the discourse and structure of Tr. Wang's pedagogical actions in explaining, that is, how she explained the errors in ways to make the mathematics entailed accessible to students and how she used the information she collected from student errors to assist her explaining process. The IRF/IRE (initiate–response– feedback/evaluation) discourse structure (Cazden, 1988) was examined to understand the dynamic interaction with students in providing her feedback: how the teacher initiated the activity, how her explanation was enriched with the help of the information she collected from marking homework, and how students responded and how she probed for more understanding about student problems of learning. Furthermore, the teacher-student interaction routines (such as students' choral responses) showed how she manipulated the information she gathered from marking homework to orchestrate the explanation, engage the students and move the explanation forward. These reflect some of the cultural dimensions of Tr. Wang's teaching practice given the large class size and teacher's leading role that requires a different kind of student participation in the form of active mental participation (Briggs, 1996).

The second level of analysis is aimed at understanding the nature and role of the errors that Tr. Wang chose to explain. The errors and her explanations were coded for knowledge and skills involved, the curricular location, the mathematics entailed, the purposes for student learning as well as the pedagogical characteristics. Research on teaching and student learning of the topic was drawn on to shed more light on the importance of the assignment, the errors and her explanations to teaching and learning. On both levels, interviews and other observation data were drawn wherever necessary to help cross-reference the coding and better understand the construction of the object as well as its transforming process.

3. Findings

Three major findings are reported. First, a discourse analysis of Tr. Wang's activity of explaining and commenting on student homework reveals a structured set of routine actions under reviewing, introducing the problem context, explaining and commenting on the errors, and generalize to conclude. These were enacted in clear IRE/F patterns involving students to recall, reconnect and reconstruct their earlier learning in ways to allow them to make timely corrections. Hours of marking student homework and 'analyzing' student thinking had, to a great extent, made the explaining detailed, multifaceted, and structured. The patterned IRE/F discourse allowed the teacher and her students to focus on the substance embedded in the errors. Second, analysis of the curriculum materials and marked student work vis-à-vis the errors explained finds that students did not have problem with the new theorems they learned but struggled with the fundamental norms of geometric proofs, which was reflected in the errors they made. Third, the tradition of design with variation is found to have students surface those errors to discern the fundamental mathematical ideas while allowing the teacher to pick them up to explain and help address through marking, explaining, and commenting. Together these findings mark a hidden practice of mathematics teaching in Shanghai mediated through teachers' homework practice and helps us rethink the role of homework in mathematics teaching and learning.

3.1 *The Activity Structure of Explaining Homework*

Coding the classroom discourse with a focus on the brief beginning segments of Tr. Wang's lessons found a fairly consistent structure in the way she explained homework to her students—it consists of three or sometimes four routine sets of actions: reviewing the previously taught content; introducing the context of the problem or errors in the assigned task; explaining the problems or errors; and summarizing and generalizing her feedback to conclude. The following details what is entailed in each set of these actions.

Reviewing previously taught content. Reviewing previously taught content is common to mathematics teaching as a way to bridge between related prior knowledge and the new knowledge to be taught (Compiling Committee of Record of Famous Teachers' Lessons—Secondary Math, 1999). Very often, Tr. Wang's review was either initiated with or followed by explaining homework. For instance, in the opening narrative of this chapter, on November 19, 2002, she started explaining homework after asking students to recall the perpendicular bisector theorem and its converse taught on the previous day: all points of equidistance to the endpoints of a line segment are on the perpendicular bisector of the line segment. She then moved from the "two points" (referred to as the endpoints of a given or constructed segment) to "three points"—"the three residential areas" in the first assignment (see Table 1, Row 2, Ex.1 for this assignment). She then transitioned to explaining homework:

> *Then, the homework we did yesterday. There are three points given here (pointing to the board at the already drawn points), now we need find the point, so that its distances to these three points are all equal. Now here, there are* ***three points*** *(emphatically), how should we solve this problem?*

As discussed later, the use of variation in curriculum and task design has allowed students to use homework to both practice and apply the theorem taught in the lesson in a new problem setting. In turn, marking student homework would enable Tr. Wang to identify the difficulties they tended to encounter in trying to use it accurately in proof writing. She could then provide timely feedback to assist them in mastering the fundamental steps of sound deductive reasoning.

Introducing the problem contexts and error(s). With or without a review proceeding her explaining homework, Tr. Wang would give a complete introduction to each of the major assignments that she had chosen to explain. For instance, a few days earlier, on Nov. 15, 2002, she started explaining and commenting on student homework by sharing her feedback (see the original assignment in Table 1, Row 1):

T: Our classmates in proof, I marked homework of 3 groups as soon as I arrived in the morning, in applying the theorems you just learned, like these two corollaries, you all used very well. But, some, in fact, the majority of our classmates, in proof writing, missed either an arm or a leg. For example, yesterday, you wrote the "Given" and 'To prove' statements on your own. This, AB=AC, $\angle C = 15°$, and the altitude on the side, BD. It requires us to prove that BD is half of the side, right? Some classmates wrote their proof in this way: the first sentence, because angle C=angle ABC=15 degrees…

S: *(A few students)* Wrong. *(More students joined in noisily.)*

In this brief introduction, she did several things: first, she shared with students the status of her marking homework (attaching a sense of importance to homework as she usually would do). To relate to the review, she commented that students used the two corollaries quite well in their homework and then focused directly on the problem of proof writing (missing of conditions) that she found in their homework. She then provided an example to illustrate her point. She went through the proof writing virtually line by line, as shown later, and then zoomed in on the first sentence in which the error was commonly found. She also drew the sketch on the blackboard and referred to it while explaining step by step. This sequence of actions was more apparent in the introduction she gave on homework at the beginning of her lesson on Nov. 20, 2002 (see Table 1 for more details of this assignment) after she taught angle bisector theorem, as shown below:

T: *(Holding a set of marked workbooks in her hand)* These many students have all got wrong. *(She then read the names of a smaller pile one by one)* Jiang XXX… These students all paid attention to when to use perpendicularity. Those classmates who got it wrong, please make corrections during recess after lunch. *(She started drawing the figure on board while the monitor was passing the workbooks back to the students.)* Those who have got your workbooks, please take a look at where you got wrong. Where *(have you got)* wrong? In general, mistakes were found in Problem No. 2. Do you know what you got wrong? Some students after drawing the perpendiculars *(she drew them on the blackboard, pause a while),* and then said, because, AB = CD, $\therefore$OE = OF. This so statement, is it right or wrong?

S: *(A boy first and more students joined him.)* Wrong.

In introducing the problem, the teacher brought students to the same page through her drawing on the blackboard, helped them to recall from memory of their thinking in doing those tasks the night before, engaged them with questions to initiate their responses, and then she evaluated the responses before moving on. Brief as it was, she communicated clearly to students how well they learned and what they missed or ignored.

Developing explanations. This essential step involved interactions between the teacher and her students to come up with the explanations even though the responses given by students are often brief and obvious since they could recall how they completed the homework tasks. Continuing with her introducing the problem on November 15, 2002 as mentioned earlier, she engaged her students in developing explanations for the error involved:

I	T:	Some classmates wrote their proof in this way: the first sentence, because $\angle C=\angle ABC=15\ °$…
R	S:	*(A few students)* Wrong. *(More students joined in noisily.)*
I	T:	This first sentence, is it right or not? *(With emphatic tone)*
R	S:	*(More students)* No, not right.
I	T:	Why not?
R	S:	*(Noisily)* You can't immediately say it *(it's not immediately known)…*
E/I	T:	$\angle C=\angle ABC$, what's its reason?
R	S:	*(A few students)* AB=AC.
Ev/Ex	T:	**AB=AC. Then angle C equals angle ABC. Right? Included in this step there is a logical deductive segment** (syllogism). Right? There is a syllogism (segment) like this.
	S:	*(A student repeating the teacher)* A logical deductive segment.
I/ Ev/ Ex	T:	Then how can you start by stating that the two angles are equal? **This is not a given condition, is it? They are deducted from AB=AC.** So we should say…

19	R	S:	*(Trying to answer at the same time)* Because AB=AC…
20 21 22 23 24 25 26 27 28	Summarize & generalize	T:	*(Picking up)* So we should say: **(because) AB=AC, ∠C=∠ABC; and (because) ∠C =15°, (then) ∴ ∠C=∠ABC=15°.** Is it like this? So our class all tends to err at such small links… With a small problem like this in the first sentence, the entire proof you write is not valid. But in practicing the theorems like these *(pointing at the board),* our classmates did generally well. So we really need to pay attention to the small details in writing our proofs. Please correct your errors after lunch.
	Transition to new lesson		[Then, next, we proceed to our new lesson. Like the two theorems, *(slower and softer to engage students)* what kind of relationship exists between their statements and conclusions?]

This brief segment of explanation (Line 1-22) has a clear pattern of IRE (teacher initiation-student response-teacher evaluation). Even though student responses are very brief, such as Line 3, a one-word response while Line 5, a three-word response to a yes-or-no question, each time, the teacher made use of students' responses for her to continue initiation or evaluation. For instance, in Line 4 of the quick IR exchange from Line 1-11, when she heard only a proportion of them answered (Line 3) she questioned with a more emphatic tone about whether they thought the step was wrong until more students answered "No" (Line 5). In Line 7, when students answered why, "You cannot immediately say it…," the teacher followed with "∠C=∠ABC" echoing the "it" in students' response and initiated with another question (Line 8). When students responded why in Line 10, "AB=AC", the teacher confirmed their answer by repeating it in Line 11.

This brief and quick exchange illustrates a co-construction by the teacher and her students, which was made possible by capitalizing on students' fresh memory and recall from doing their own homework the night before. The teacher held the key but instead of just telling students what went wrong and how to correct it, her goal was obviously to engage students in an interaction by questioning and probing on the errors step by step and sentence by sentence to get them to recall from their solution steps and push them to think over again. Therefore, the explaining would not have been accomplished by the teacher alone—she needed the students' involvement, their being there attentively with refreshed memory of how they performed the homework tasks explained. As discussed later, the explaining was made possible by her thinking during hours of marking student homework.

Summarize and generalize to conclude. In developing the explanations, Tr. Wang would often lead her students to open up different ways of viewing an error or multiple dimensions of the errors, such as in the segment presented above. She would often share different ways an error was made – "some students did this…, other students wrote that …, and still others …." As written earlier (Fang, 2010), she achieved this capacity through long hours of marking homework, a process in which she found she was always analyzing student thinking, tutoring individual students coming to her grade-level office and conversing with her 'desk-mate' math colleague who was often marking student work from his classes (Fang, 2010).

Most of her explanations did not end there; instead, in the way she introduced the problem context and the error, she would close up by summarizing the key ideas (such as she did from Line 20-22 above) and highlighting the important role that the error would play for determining a successful proof writing (Line 23-24 above). She would always end by reminding students to make corrections and ask them to show her their corrections which required their understanding to make and it was this understanding that her error explaining aimed to achieve. This final step of explaining is found at the end of all her explaining to geometric proof assignments, which could give students a sense of structure required in a proof writing and model for them to achieve such structure and completion.

3.2 *Mapping out the Activity Structure*

Figure 1 summarizes the activity structure of the actions and the goals guiding them. This diagram uncovers a clear set of goals that direct Tr. Wang's explicit pedagogical actions she would usually take in explaining homework as a form of feedback. It is quite visible that these goal-directed actions move towards building connections between the old

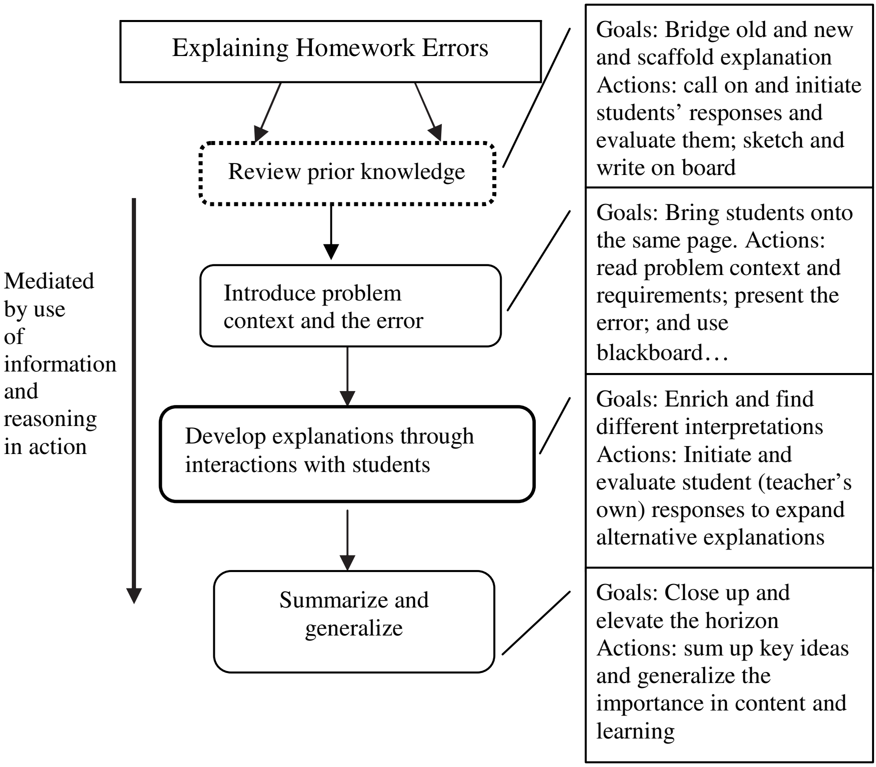


Figure 1. Activity structure and pedagogical actions in Teacher Wang's explaining homework

content and the new, between problem contexts and errors, between different forms and representations of errors and misunderstandings to get students to understand the key mathematical concepts and procedures behind such errors and misunderstandings. These routine actions that the teacher and her students performed together regularly "allow relatively low-level activities to be carried out efficiently" through a discourse of repeated IRF/IRE structure, "without diverting significant mental resources from the more general substantive activities or goals of teaching" (Leinhardt & Greeno, 1991, p. 235), which is to establish, little by little, the habit of deductive thinking. Through these clearly structured actions over time, the teacher modeled to her students how to approach a proof writing in structured ways.

3.3 *Nature and Substance of the Feedback Provided by Explaining Homework*

Through the above discourse analysis, the pedagogical values of explaining homework are clearly shown: it allowed Tr. Wang to communicate her feedback to students interactively and in structured ways within a very limited amount of time. The importance she attached to conveying her feedback timely before moving on with her teaching of the subsequent content points us naturally to the importance of such feedback to student learning. Analysis of the homework tasks by matching them with the errors explained to students has revealed the embedded fundamental mathematical ideas and their roles in student learning to do geometric proofs. These tasks were designed with strategic variations in order for students to commit those errors and for the teacher to attend to them via student homework. Such a design allows Tr. Wang to tap into the curriculum and pedagogical values of the errors systematically to promote students' habit of deductive thinking.

3.3.1 *Locating the Errors in the Curricular Sequence*

The learning of new topics or concepts depends on the mastery of previously learned ones (Ausubel, 1963). As *Converse Propositions and Theorems* is the last section in the Unit on *Geometric Proof*, it builds on teaching of the previous concepts, such as construction of figures, definition of lines and shapes, adding of auxiliaries and procedures of proof writing, which can all be found in the assigned tasks chosen by Tr. Wang to explain in Table 1 below. The curriculum also prepares for this section by transitioning from the converse relation of the two corollaries of the right triangle theorem taught on November 14, 2002. As the errors unfold, one can notice the intricate relationships between the errors and these previous knowledge and skills. As the teacher explained the errors, she assisted students in brushing up and reinforcing the understanding of the previous concepts and building connection to extend the current learning. Table 1 has located the errors identified by Tr. Wang in the homework that she assigned and explained in this section.

Tr. Wang selected the errors to explain to all students either based on their being typical or difficult or both. Her beliefs about what needed to be emphasized in student learning were both determined by availability of teaching time and whether the error(s) represented typical student learning difficulty or important points that the curriculum stipulated. As she shared, "Because of time, I cannot explain all the problematic ones; I choose the most typical ones, the ones not necessarily just typical or difficult; they are the ones I believe that need to be emphasized again" (interview, 11-19-02).

3.3.2 *Trivial at First Sight but Fundamental to Deductive Reasoning*

At a first glance, the selected errors given in Table 1 do not bear directly upon the learning of the newly taught theorems or concepts: for instance, Error 1 is not directly related to the corollaries of the right triangle theorems taught on the day when the homework was assigned or Error 5, not immediately related to angle bisector theorem and its converse. Instead, they appeared to be trivial. For example, Error 1 does not have to do with the context of the assignment, applying the theorem in an isosceles triangle: students wrote in the first sentence of the proof that the base angles are equal instead of the isosceles sides are equal. Error 5 has to do with the auxiliary distances (see the two dotted segments in the figure, OE and OF) that students drew from the center of the circle, O, to the chords (AB and CD) but failed to use the auxiliaries to indicate perpendicularity as distances.

Table 1. Locating the explained errors and the design features of these homework assignments in the section, *Converse Proposition and Theorems*

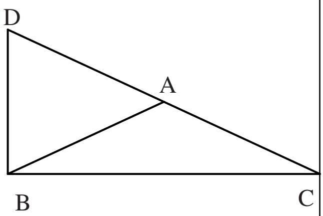
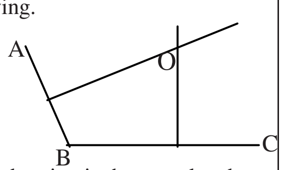
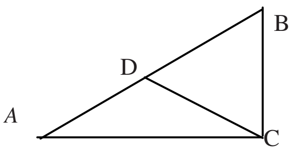
Date/Topic of assignments	**Exercise and problems chosen to explain**	**Error chosen to explain**	**Assignments with variations**
Friday, Nov. 15, 2002 *Two theorems* (Nov. 13) and *two corollaries of right triangles** (Nov. 14) Ex. 1-4, Vol. A/p.35-36:	**Ex. 2**. An isosceles triangle with a base angle equal to 15°. To prove: the altitude on one side is half of the side. (Students are to do the drawing) [Vol. A/p. 35. Ex 22.4 (8) 2.] 	**Error 1** **Ex. 2**. Opening the proof writing by citing the congruent base angles: $\angle$ C= $\angle$ ABC=15°	Apply the theorems* in Rt Δ created by altitudes in other shapes: altitude on the hypotenuse of a Rt Δ ; on the side and the base of an isosceles Δ ; and a Rt trapezoid with aof 60- degree angle – need an auxiliary altitude to make a Rt Δ .
Tuesday, Nov. 19, 2002 *Perpendicular bisector theorem* Both classes: Ex. 1, 2 & 3 in Vol. A (p. 37). Extra for Class 4: Ex. 2, 3 & 4, Textbook (p. 79)	**Ex.1.** Say how to find a point of equidistance to the three residential sites shown in the drawing.  (The drawing is the completed version of the construction) [Vol. A/p. 37, Ex. 22.5 (2) 1.]	**Error 2** **Ex.1.** (1) Two perpendicular bisectors (PB) suffice but many drew a third one for AC; (2) Failure to write complete construction methods and conclusion in standard language.	One construction, one filling blanks (calculate lengths of sides and degrees of angles), and 4 proof writing. Apply PB theorem and its converse in different types of exercises: construction--find a point of equidistance to a given segment or points; calculate angles and proportion of sides in given right Δ ; and proof writing.
	Ex. 2. The given: See drawing, in ΔABC , $\angle ACB = 90°$, $\angle$ 1= $\angle$ B. Prove: D is on the perpendicular bisector of AC.  [Vol. A/p. 37, Ex. 22.5 (2) 3.]	**Error 3** **Ex.2** Drawing an unnecessary auxiliary, the PB of AC and used it as median with D not given as midpoint. Creates the givens and mixing up what is the given with what is to prove.	

Table 1. (*Continued*)

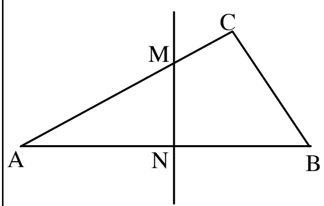
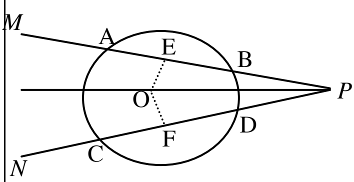
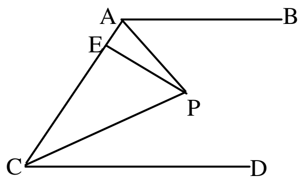
Tuesday, Nov. 19, 2002 (Continued)	**Ex. 3.** The given: See drawing, $\angle C = 90°$, the perpendicular bisector of AC intersects with AC and AB at point M and N, and AM=2CM. To prove: $\angle A = 30°$  [Textbook/p. 79. Ex. 22.5 (2): 4]	**<u>Error 4</u>** **Ex. 3.** A few students failed to use the concept of PB to draw the auxiliary by connecting M and B.	In proof writing, apply the theorem in a quadrilateral with perpendicular diagonals and in triangles: Given relationships between sides or angles to prove that a point is on a PB; or given the PB and sides, to get the degree of an angel.
Wed. Nov. 20, 2002 *Angle bisector* For both classes: Ex. 1, 2, & 3, Vol. B (p. 42). Extra for Class 4: Ex. 1, 2 &3, Textbook (p. 81-82)	**Ex. 2.**[**] The given: See the drawing, Circle O intersects $\angle MPN$ to get AB=CD. To prove: PO is bisector of $\angle MPN$. [Vol. B/p. 42. Ex. 22.5 (3) 2.]  **In this exercise, 3 errors were identified and explained.**	**<u>Error 5</u>** **Ex. 2** 1) Drew distances from O to AB and CD but failed to write construction in accurate language; 2) Failed to indicate them as perpendiculars and use them as sufficient conditions to justify the two equal distances; 3) Failed to use the concept of distance as perpendicularity to justify O on the bisector of angle MPN.	Apply the angle bisector theorem and its converse to find the distances from a point to two intersecting rays and to sides of an angle. Find the distances from the intersecting point of a vertex of a triangle intersected in an angle to the sides of the angle. Prove that center of a circle intersecting an angle is on the angle's bisector: given either the intersected arcs are equal or the intersected chords are equal.

Table 1. (*Continued*)

Wed. Nov. 20, 2002 (Continued)	**<u>Ex. 1.</u>** As in the figure below, AB//CD, AP and CP respectively bisect ∠ BAC and ∠ DCA. If the altitude of ΔPAC, PE=8cm, then the distances from AB and CD are respectively ___________.  (Take advantage of a shared side to allow substitution of equal distances.) Note the day's two errors are arranged in the order that was explained.	**<u>Error 6</u>** The exercise requires filling in the lengths of two distances. A number of students only filled in one. They were misled by being given only one blank. They did not pay attention to the word, "respectively".	Two parallel lines intersected by a third line forming two angles. Find the distance from the intersecting point of the angle's bisector to the two parallel lines.

* Theorem: In a right triangle, the median on the hypotenuse is half of the hypotenuse. Corollary 1: In a right triangle, if an acute angle is 30 degrees, the right side it faces is half of the hypotenuse. Corollary 2: In a right triangle, if a right side is half of the hypotenuse, the angle it faces is 30 degrees.

** *The Teaching Reference Material* gives a rationale for choosing to represent the content in the concept of set versus locus (to be taught in the second semester of 8th grade) and compares their similarities. Note that Schoenfeld (1991)'s chapter in *Informal reasoning and education* edited by Voss, Perkins, and Segal used this similar construction on some college students and Fawcett's (1938) use of it on high school students. The later was both based on the concept of locus. They used it to let students construct by drawing on the theorems while this exercise was used in Shanghai's 8th grade as a situation to practice the theorem of an angle bisector.

Careful reading of the explanations that Tr. Wang gave to her students on these errors, however, reveals important but fundamental mathematics behind them that the students are unfamiliar with. Both Error 1 and Error 5 have important implications for student learning of the nature of the deductive system: the need to understand that the axiomatic proof is established on the basis of definitions or some "rock bottom self-evident facts upon which the whole structure is to rest" (Davis & Hersh, 1981, p. 149). As she questioned her students, "What's the reason for $\angle C=\angle ABC$ (the two base angles)?" She wanted to let her students understand that the two equal base angles are derived from the two equal sides that define an isosceles triangle and "there is a logical syllogism in this step" that cannot be missed. Included in Error 5 are two key requirements for learning to write a good and rigorous proof: to justify a statement with sufficient evidence and to put the conditions and steps in logical sequence. The mathematics entailed in the errors and the explanations given by the teacher and research related to student learning of related topics are summarized in Table 2, which illustrates that these seemingly trivial errors entail important mathematics and considerations about student learning of the mathematics involved.

The rest of the selected errors are also emblematic of other different dimensions of a geometric proof, such as the need for deduction in doing construction (Error 2), the ability to see where an auxiliary is needed and how to draw them (Error 4, 5 & 6), use of geometric language in both construction and proof (Error 2, 4 & 5) and use of language in mathematics (Error 6). These errors also represent issues related to student learning of geometric proofs widely identified and documented in important research on mathematic education. In the study done by Fuys and colleagues (1988), for example , students, like those of Tr. Wang's who are at the transition from van Hiele Level 2 to Level 3, typically do not recognize the need for definitions in a proof. Schoenfeld (1991, p. 149) found that college and high school students do not see that constructions and proofs are connected when they are given a construction before a proof.

3.3.3 *Homework Assignments Designed with Variations*

The last column on the right in Table 1 indicates the design features of the homework assignments selected by Tr. Wang to explain to her students. They are designed to consolidate and extend teaching and learning by changing a problem in multiple forms to increase the

Table 2. Mathematics and student learning entailed in the errors in the proof writing

Types of error/date	**Mathematics entailed**	**Role of the error in student learning**
Nature of axiomatic system: Error 1 (11-15-02); Error 5 (11-20-02)	*"There is a logical reasoning segment (syllogism) in this step (that you cannot miss)."* Deductive and axiomatic system starts with a definition or axiom; justification based on sufficient conditions.	Students transitioning from van Hiele Level 2-3, "…do not grasp the meaning of deduction in an axiomatic sense, e.g., do not see the need for definitions and basic assumptions" (Fuys et al., 1988).
Deductive reasoning is needed in a construction problem: Error 2 (11-19-02); Error 3 & 4 (11-19-02)	**Ex. 1** *"Do we need to draw a third perpendicular bisector?"* Deductive reasoning is needed to justify that it is sufficient to connect two segments (instead of three) and draw their perpendicular bisectors intersecting at one point. Procedures of construction; language of construction; and writing of the procedures in clear construction language.	"…students do not often see the connection between construction and proof problem when a construction problem is given before a proof." (Schoenfeld, 1991:319) "Have we found the point?" "Many students drew 3 PBs. Is it Necessary?" questioned Tr. Wang repeatedly. "… Does this exercise need an auxiliary?" Wang questioned students.
How to decide where and what auxiliary is needed: Error 3 & 4	**Ex.2** **"**When is an auxiliary needed?"—use of counter examples; knowledge and skills to find auxiliary lines to assist finding a proof; writing the construction in geometric language.	**Ex. 2 & 3.** "(F)ind the lines is part of finding a proof, and this may be no easy matter" (Davis & Hersh, 1981, p.150). According to the Teaching Reference Material, knowing when and how to tell if an auxiliary is necessary is both an important and difficult point.
Logical sequence in proof writing: Error 5 (11-20-02); Error 1 (11-15-02)	*"Distance has to be used twice in constructing this proof"* The rigor of deductive proof demands justification of a statement with sufficient conditions and put them in a logical sequence.	The place of a concept in the axiomatic chain or the "chains of deductive proof" (Brumfiel, 1973:102): how it becomes part of the deductive chain and how it is used to extend the chain. (Farrell, 1987:239). Some, after drawing the perpendiculars, wrote, because AB=CD, (so) OE=OF (E on AB and F on CD). "Is this because statement right or wrong?" asked Tr. Wang.
Understanding of language Error 6	"Respectively" Language use in assisting understanding of mathematics	

pedagogical value. This is similar to the design of one problem (or concept) with multiple changes in format (Cai, 1995). For instance, the assignments for Nov. 15, 2002 aim to practice the two theorems of right triangles and their converses by using the concept (approach) of an altitude (perpendicularity) located in different geometric shapes: a right triangle, isosceles triangle and a trapezoid. Assignments for Nov. 19, 2002 apply the perpendicular bisector in different triangles and the relationships of their sides. Those for Nov. 20, 2002 are designed to apply the angle bisector theorem and its converse in the context of angles formed by two rays, intersected by circles or in between two parallel lines. The design also used different types of exercises, such as construction, filling blanks and proof writing. It goes as what Zhang, Li, and Tang (2004) put it, "The focal point of variation is the procedure or form in which problems are proposed. It is carefully designed such that only the non-fundamental elements of knowledge and skills are changed in a variety of ways. By comparing and differentiating, students struggle to identify invariant properties: the essence of mathematics ideas and procedures" (pp. 196-197). In this case, the fundamentals of deductive reasoning and proof writing are discerned and understood.

What is noticeable, however, as mentioned earlier, although the designed changes provide difficulty for students, what they got wrong was not typically with the newly taught concepts or topics that they needed to practice through these changes. In certain ways the design of variations surfaced errors and problems of learning in proof writing (beyond simply applying the theorems), such as use of conditions or evidence to support a claim, ways of expression, so that the teacher take the advantage to hone them again through homework to achieve real understanding. The goal of learning geometric proofs is to be able to write proofs with the required deductive reasoning habit not simply applying the theorems. In this way, the writing mechanism got practiced in different problem settings for students to see that the rules and norms remain the same even though the problems situations vary. Such application can also help avoid a sense of repetition and repeated drilling in practice.

4. Discussion and Conclusion

Tr. Wang's case allows us to rethink about homework's role in teaching and learning, particularly as a vehicle for providing feedback not only for learning but, more importantly, for the purpose of informing teaching. The case makes us reconsider too what constitutes feedback and how it works. A case of teaching geometric proofs, a topic requiring special attention to the process of the working, provides an ideal example to shed light on the curriculum and pedagogical values that the Chinese traditions of teaching assign to homework and how homework, in turn, designed and used with variation, mediates teaching and learning by capitalizing on these values to promote teaching and learning. To understand the process, a broadened view of teaching is needed.

First of all, when homework mediates teaching and learning in substantive ways, it weaves a coherent pedagogy with numerous connection building. One connection is between feedback provision and instruction. According to Hattie and Timperley (2007), "when feedback is combined with effective instruction in classrooms, it can be very powerful in enhancing learning" (p. 104). But they were unable to show what this combination looks like with their review on a large number of quantitative studies. In fact, explaining homework is both a powerful way of providing feedback and a brief but structured instructional segment to review and transition to new content. For feedback provision, it helps students recall, reconceptualize and redo for timely correction and building a habit of deductive thinking; for instruction, it was conducted through dynamic interaction with students addressing directly errors and faulty solutions to reinforce understanding and procedural knowledge. Tr. Wang's "analysis" and reasoning during long hours of marking homework allowed her to grapple with "the routes to understanding that student had experienced" by "getting inside the head of the learner" (Beveridge & Rimmershaw, 1991, p. 289). Her explanations of the problems of learning identified from homework was elaborate, multi-faceted and highly structured, targeted at the central objects of learning. For the teacher herself, homework mediates her pedagogical reasoning and action process to create comprehension of the subject matter content from student learning perspectives, assessing their misconceptions in different representations and engaging her in the evidence-informed decision making and timely corrective measures. Through Tr. Wang, we can visualize a teacher being engaged in Shulman's (1987) model of pedagogical reasoning and action overtime, a process concretized by homework mediation, enabling a connected and thorough understanding of the fundamental mathematics of teaching that Ma (1999) found many mathematics teachers in China may possess.

Second, explaining homework to convey her feedback to students connects classroom learning and learning outside classroom weaving them together into a coherent and consolidated learning experience for students. As part of the routine 45-minute classroom instruction and given the brevity of time, a pattern of IRE/F questioning, choral responses followed by teacher evaluation or summary enabled the teacher to share or 'co-construct' feedback and reach most of the students in a large class of 58 students in ways that drew their participation. This kind of "clear and consistent event structure" "allow(s) participants to attend to content rather than procedures" (Cazden, 1988, p. 47). With students' being able to recollect their solution or proof writing process they did the night before in doing their homework, such engagement was made possible to a great extent by the boundary role of the homework activity. Leading a review and also a formal lesson by numerous questioning and student choral responses was a cultural approach of mathematics teaching well documented by Gu and his colleagues (1999) in their observation of middle school mathematics classroom practices in Shanghai. While this approach appeared to highly engage students to follow the teacher's orchestrating, the researchers are also concerned that, with the teacher acting as the leader of the scene most of the time, it hinders students from more active participation.

Third, as mentioned earlier, the tradition of design with variations has created the space for students to struggle and discern the essential properties of a concept by applying it in diverse and carefully designed contexts in homework tasks and for teachers to identify the issues of learning in marking homework and explain them to students in a next lesson. Design and teach with variation has a lot to do with a curriculum and pedagogical discourse found in the curriculum materials and teachers' daily work in Shanghai structured by analysis of the important, difficult and hinge points of teaching and learning: "the important points" refer to the fundamental but crucial parts in the knowledge system that teaching aims to help students learn; "the difficult points" are based on what pupils have trouble learning but vary with pupils and schools; and "the hinge points" are specific and key pieces of understanding, strategies or solution methods adopted to help pupils overcome the difficulty in learning the important points (Paine, Fang, & Wilson, 2003, p. 51). Students practice with variation to master the important points, in the process of which the difficult points are surfaced and overcome through teacher feedback and student correction. Therefore, the important points are not necessarily the most difficult to understand but are central and fundamental. Difficult points are not necessarily the most important and they include procedures and forming the desired habits of mind. Such design traditions and institutional arrangement of teaching create instructional resources and tool-mediated interactions that connect students and teachers and teaching and learning to create multi-layered resources (Cohen, Raudenbush, & Ball, 2003).

Fourth, to understand teaching as a system, studying teaching through classroom observation alone would only capture the visible part of the enacted curriculum during the interactive phase of teaching. The pre-active and post-active of teaching as spelt out by Jackson (1990) have to be carefully observed too by shadowing a teacher's work inside and outside the classroom in order to build a holistic view of mathematics teaching in China (Fang & Gopinathan, 2009). What happens in the backstage of teaching, in teachers' offices (different settings from teacher staffrooms in the Western schools) are dynamic interactions between colleagues and between teachers and students involving a lot of informal conversations around homework issues and tutoring of individual students on their homework issues (Fang, 2010). The mediating role of homework as a pedagogical resource cannot be underestimated in a mathematics teacher's work in China. Teachers' homework practice mediates their pedagogical reasoning and action to build comprehension of student learning and different representations of student misconceptions, deepen their mathematics teaching knowledge through analyzing, discussing and addressing such misconceptions through marking, talking, tutoring and classroom explaining.

Finally, as reported earlier, the ways with which such homework practice capitalizes on errors through marking, tutoring, explaining and talking with colleagues speak to the importance that the culture of Chinese mathematics education attaches to student errors in both teaching and learning. Yet, in the current educational reform in Chinese classrooms, even though Tr. Wang's homework feedback was well tailored to students' needs and offered on a timely and systematic basis, some scholars would argue that it was given as if students were led to the water to drink rather than they were let to look for and drink the water on their own. What would it be like if the homework activity system involved the students taking into their own hands the responsibility to spot, analyze, correct, and learn from errors? How would homework's mediating role change in the curriculum and pedagogical reform?

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Chapter 7

Teaching Calculation of Time Intervals: Comparing Mathematics Competence of Students in Macau, Hong Kong and the Netherlands

LI Titus Siu Pang

An investigation is reported of the ability of 106 students in their final year of primary schooling in the Netherlands, 358 in Hong Kong and 389 in Macau to execute a subtraction problem involving using time units in 24-hour clock format. The test item was derived from a test used in PISA 2003 to gauge students' mathematics proficiency. Teachers were asked about the level of difficulty in the time item and to suggest errors that students might make in reaching an answer. Even though the Dutch children outperformed the Macau and Hong Kong children, common errors and misconceptions were being made across all three locations. Many Macau and Hong Kong students used the decimal rather than the 60 unit hour arrangement to answer the question. Interviews and lesson observations revealed how time interval calculations are taught in each location. Less than one-third of the Macau and Hong Kong teachers anticipated that students might use the decimal system, teachers in both locations very regularly drilling students in the mechanics of working out time intervals, while teachers in Dutch habitually use the 'time-line' to help students calculate time intervals. The implications for practice are explored and recommendations are offered.

Keywords: primary mathematics, comparative education, teaching of time intervals, mathematics teaching in Macau

1. Introduction

Regardless of the geographical location, mathematics has for years been the focus of searching educational research. Mathematics educators, curriculum researchers and planners are constantly looking at how to facilitate students' learning of the subject and how to teach it more effectively. There is also a common belief that ethnic Chinese students are particularly good at mathematics, particularly since students from mainland China, Hong Kong, Taiwan, Singapore and Macau have performed excellently in recent years in large scale international mathematics studies, for example in the Second International Mathematics Study (SIMS), the Trends of International Mathematics and Science Study (TIMSS) conducted by the International Association for Evaluation of Educational Achievement (IEA), and the Programme for International Student Assessment (PISA) administrated by the Organisation for Economic Cooperation and Development (OECD) (Beaton, et al., 1996; Mullis, et al., 1997, 2000a, 2000b, 2000c, 2008, 2012; OECD, 2001, 2004, 2007, 2009, 2010; Robitaille & Garden, 1989). However, Fan and Zhu (2004) caution that, even though ethnic Chinese students are good at mathematics, there are "blind spots" in their grasp of some concepts and mental operations. Hence, the study reported in this chapter is an element of the author's doctoral study. It focuses on the test item of time intervals calculation as it was the most serious "blind spot" of ethnic Chinese students being identified in PISA 2003 and the pilot study conducted in 2007 (OECD, 2009; Li & Westerman, 2008).

2. The Need to Adopt a Holistic Approach When Examining the Teaching and Learning of Time Interval Calculations

TIMSS and PISA looked at why children in some parts of the world seem to be more proficient learners of mathematics than children in other parts of the world, and identified concepts that children in particular locations find especially hard to understand. For instance, when PISA experts conducted a study (OECD, 2009) of patterns of students' mathematics performance in order to identify strengths and weaknesses in their reasoning in PISA test domains, analysis of students' proficiency on a Level 5 question based on the 24-hour clock system revealed that less than 40% of Macau 15-year-olds could produce the correct answer. These students' proficiency on the time-interval question was in the bottom 7 of the 41 participating countries and economies, and was lower than that of students in European countries and in neighboring Hong Kong.

As a person closely involved in mathematics education in Macau and Hong Kong in different roles for several decades, the author was curious to learn why Macau students are so poor at calculating time interval subtraction problems and how this topic is taught in Macau. This led him to research this and other aspects of mathematics attainment in primary schools in the East and West as part of his doctoral study (Li, in press). The international comparative study conducted by the author examined similarities and differences in primary school mathematics teaching and learning in part via a test covering major domains of the primary mathematics curriculum. It looked in particular at the mathematical ability of final year primary students from the Netherlands, Hong Kong and Macau, a key aim of the study being to investigate whether final year primary students of ethnic Chinese origin outperform European students in mathematics. It also sought evidence about whether ethnic Chinese and European primary students share similar misconceptions when trying to solve mathematics problems and whether these are linked to the way mathematics has been taught.

The effectiveness of students' learning is related to the effectiveness of teachers' teaching. It is a common belief among educationists that professionally trained and experienced mathematics teachers will have more professional knowledge about mathematics than will have generalist trained colleagues, be able to teach students to learn mathematics more swiftly and to help them grasp mathematical concepts more securely. An important aim of this study was to investigate whether teachers from the East have more professional mathematical knowledge than teachers from the West. If this is the case, it may partly explain why ethnic Chinese students so regularly outperform Western students in PISA, TIMSS and other international studies.

TIMSS and PISA organisers are adamant that simply being aware of the theoretical framework underpinning the mathematics testing they carry out is insufficient for helping teachers improve their performance. TIMSS and PISA researchers used questionnaire data from students, schools and parents to obtain a more complete picture of what makes a successful mathematics curriculum. From a comparative education perspective, is it the case that students will reach their academic potential anywhere in any part of the world given sound teaching, or is it the case that the quality of environmental and pedagogic input and provision is the key to success? Education involves interpersonal and intrapersonal issues: the teacher who teaches and the student who learns. Teaching and learning are interlinked and students' performance mirrors how teachers operate in the classroom. The present study looked at how students learn time interval calculations; how they have been taught the mechanics of the operational procedures; and how well they have really understood the rationale for the procedural steps involved.

3. Mathematics Teaching in Macau Schools in the 21st Century

Unlike Hong Kong and China, there have been few, in-depth scholarly studies of mathematics teaching in Macau. To help explain why such Asian countries as Hong Kong, Japan and Singapore performed so well in TIMSS 1995 and TIMSS 1999, Clarke studied mathematics teaching in a number of Western and Asian countries, including Hong Kong, and Huang (2002) investigated how mathematic lessons are taught in Chinese secondary schools. Huang identified several distinct features of how Chinese teachers teach mathematics in secondary classrooms (Huang, 2002, 2006; Huang & Leung, 2004). In a later study of Macau secondary schools, Huang and Wong (2007) found that Macau secondary teachers share many common points of belief and procedural practices with teachers from Shanghai and Hong Kong. However, they talked much more; gave fewer opportunities for students to explore content independently; and gave less scope for students to discover mathematics principles for themselves. Sadly, Huang's studies did not extend to primary school classrooms.

The Macau Government is presently initiating various projects to improve the standard of mathematics teaching in primary and secondary classrooms. For instance, aware that the small-scale "Mathematics Educators Supporting Schools Project" run by Beijing Normal University from 2000 to 2004 had limited impact, the Education and Youth Affairs Bureau (DSEJ) introduced in 2008 the "Mainland Excellent Teachers in Macau Exchange Scheme". Every year, distinguished Mainland teachers in Chinese Language, mathematics and preschool education have been sent to work in Macau schools, their mission being to coach local teachers in the teaching of mathematics and Chinese (Sou, 2009). In addition, the DSEJ has launched two journals especially for Macau teachers: the "Teacher Magazine" and the "Collaboration of Mainland Excellent Teachers in Macau Exchange Scheme". One aim is to attract and invite mathematics teachers and teacher-coaches to write papers about ways by which teachers can help primary students learn mathematics more effectively. Several articles have been produced about how to teach key topics in mathematics, how to plan lessons and how to teach so that all in the class learn effectively (Chen, 2011; Chen & Lung, 2010; Ching & Poon, 2010; Ng & Ching, 2011; Wu & Chen, 2011). Although these publications identify strengths and weaknesses in how Macau primary mathematics teachers were delivering topics on the mathematics curriculum prior to the Mainland intervention, they do not comprehensively report how primary mathematics is being taught today in the diverse range of Macau primary classrooms. Evaluations of the successfulness of the range of the above interventions have been based on subjective and descriptive observations of classroom teaching, records of students' learning successes and the identification of mathematics failings in need of attention. Success has been measured in terms of whether students are able to perform a range of mathematical procedures, not by how well they understand the rationale of these procedures, for example working out the area of triangles and calculating profit and loss. They give no clues about whether students are confident in their handling of time calculation problems, nor do they offer explanations for why many Macau secondary students have performed so miserably on certain topics in PISA cycles in the past.

4. Major Factors Being Examined in this Study

4.1 *Misconceptions and Students' Mathematics Learning*

The first issue addressed in the study was to identify common mistakes and misconceptions made by primary students when calculating time intervals. From a psychological perspective, Piaget's stage theory of cognitive development has demonstrated that young children bring with them to mathematics lessons concepts that are at odds with many of those universally held by adults (Piaget, 1961). For example, their ability to understand conservation concepts is swayed by the appearance of features in situations. If two rods of identical length are presented so that one sticks out further than the other, then it will be judged to be "longer" than the other. Students do not come to the classroom as "blank slates" but with concepts that, for the time being at least, explain the world around them. Their grasp of these concepts often conflicts with mathematical concepts accepted by their teachers, such "misconceptions" impeding students' constructions of knowledge and trapping them into producing "errors" when completing tasks set by the teacher (Smith et al., 1993). Unless learners have a concrete grasp of principles and concepts, they will resort to learning routines that result in the "right" answer, with no questioning of the rationale of the algorithms employed.

Ryan and Williams (2007) divide misconceptions into four categories: "modelling", "prototyping", "over-generalizing" and "process-object" linking. Modeling refers to how mathematics is connected with the "real", everyday world. Prototyping refers to mistakes resulting from a culturally "typical example" of a concept, such thinking often being intuitive. Overgeneralization results from personal, incomplete insights of a concept taught at a younger age. For instance, misconceptions about "taking the smaller digit from the larger" in subtraction sums, the belief that "multiplication always makes bigger" and "division always makes smaller" are well documented (Vergnaud, 1979, 1983; Bell et al., 1984; Graeber & Baker, 1988; Graeber & Campell, 1993). Process-object conceptions occur when children are very young and in an early stage of learning mathematics (Gray & Tall, 1994). For instance, young children may cope with a $3 + 5 = ?$ question, but if faced by the question "$? - 128 = 200$", may give 72 as the answer.

Misconceptions often arise as a consequence of students' earlier learning, either in the classroom or from their interactions with their physical and social world (Smith et al., 1993). In elementary mathematics, misconceptions frequently originate in prior instruction as students incorrectly generalize earlier learning to solve new problems

(Nesher, 1987; Resnick et al., 1989). Misconceptions are commonly found in students' responses in various areas of the mathematics syllabus, are often widespread and similar in nature and may be resistant to change. They continue to surface even after corrections have been made, so much so that Clement (1982) proposes that misconceptions often "happily co-exist" alongside correct approaches.

Misconceptions are best corrected if the teacher actually understands why they have occurred and whether they have their roots in faulty teaching, teaching that is too complex for the learner or if the learner lacks the intelligence and subject grounding to appreciate the nature of the misconceptions (Ryan & Williams, 2007). To overcome and remove misconceptions, teachers will usually confront students with the disparity between their misconceptions and the correct procedures or answers. In fact, misconceptions are commonly embedded in many learners' grasp of basic arithmetic operations, and the elimination of errors in mathematics through repeated drill and practice should not be taken to signal that teachers have dealt positively with the mathematical misconceptions possessed by their students. In fact, it is not unusual for students to learn algorithms for arriving at correct answers without ever having really comprehended the rationale of the procedures used.

4.2 *Teachers' Knowledge of Students' Strengths and Weaknesses*

The second issue of this study is whether Chinese origins teachers have more professional knowledge in mathematics teaching compared to teachers in the Western. An interesting phenomenon is that mathematical geniuses with a deep understanding of mathematics do not always make good mathematics teachers. Nevertheless, it is widely accepted that good mathematics teachers will benefit from having a clear understanding of mathematics, a willingness to keep up-to-date with methodological advances and the possession of insights into difficulties learners face in grasping concepts (Grossman et al., 1989; Ma, 1999; Silverman & Thompson, 2008). Research has shown that teacher inexperience, limited mathematics knowledge, unclear explanations of procedures, lowly expectations and inability to see where students have gone wrong contribute to errors in students' mathematical performance (Begle, 1972, 1979; Monk, 1994). Interestingly, Romberg and Carpenter (1986) argue that many studies of the relationship between teacher knowledge and student attainment share a common drawback: they look at global measures of teacher knowledge rather than at the instructional skill of teachers in the classroom, and at global performance rather than at the ability to apply specific algorithms and procedures with understanding.

In most countries a rich knowledge of mathematics is a criterion for being a good mathematics teacher, and the ways that teachers behave inside the classroom is a most important factor influencing teaching and learning (Pietilä, 2003). In order to identify what teachers need to know (besides knowledge of their school subject) to make teaching and learning successful, Shulman (1986a) argues that knowledge of subject content alone is insufficient to support teachers' efforts to teach for understanding, and he coined the term "pedagogical content knowledge" (PCK). PCK refers to "ways of presenting a subject which make it comprehensible to others … [it] also includes an understanding of what makes the learning of specific topics easy or difficult; and the conceptions and preconceptions that students of different ages and backgrounds bring with them. If these preconceptions are misconceptions, which they often are, teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners" (Shulman, 1986b, p.9). Shulman (1987) later claimed that subject matter knowledge (SMK) (including mathematics knowledge), pedagogical content knowledge (PCK) and curriculum knowledge (CK) are closely related. SMK is more than knowledge of facts or concepts: it requires knowledge both of the substantive structure (organizing principles and explanatory frameworks) and the syntactic structure (nature of enquiry in the discipline, and how new knowledge is introduced and accepted). In order to be an effective teacher, a teacher needs to transform SMK and CK into PCK as this "conceptualises the link between knowing something for oneself and being able to enable others to know it" (Huckstep, Rowland, & Thwaites, 2002, p. 1).

Shulman's PCK theory has been applied to mathematics teaching by a number of mathematics educators. Some Chinese mathematics educators claim that PCK is composed of three components: the form of knowledge of content, knowledge of the curriculum, and knowledge of teaching (An et al., 2004). Muir (2007) suggests that PCK also involves observable instructional acts such as interacting effectively with students through questions and probes, answers and reactions, praise and criticism. In short, PCK is the combination of content knowledge, knowledge of students' thinking and knowledge of mathematics and pedagogy (Silverman & Thompson, 2008). A successful mathematics teacher needs to have knowledge of content, knowledge of the curriculum, knowledge of teaching and knowledge of students' thinking.

Shulman's PCK has been accepted and developed by many mathematics educators. For instance, Hill, Ball and Schilling (2008) introduced a framework to modify the theory of PCK (see Figure 1).

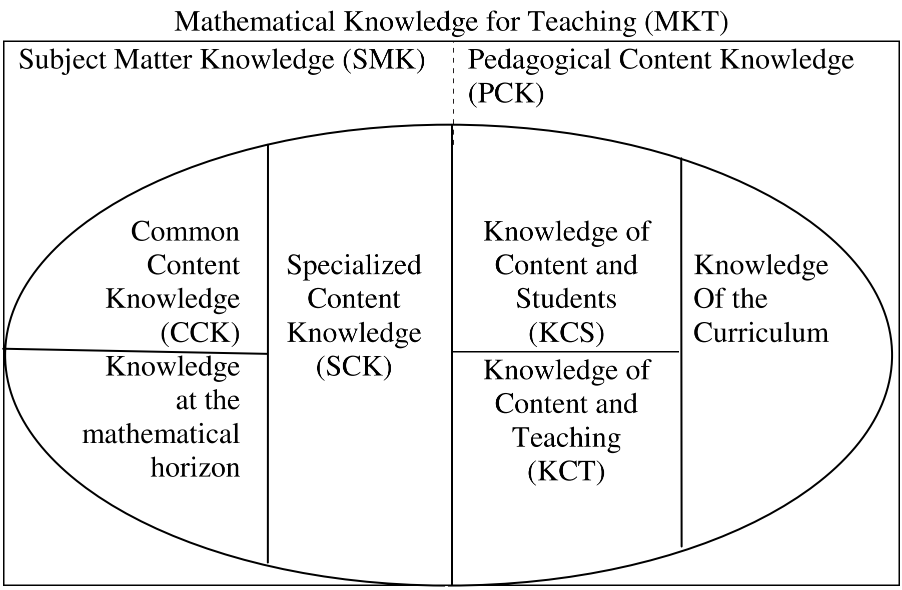


Figure 1. Domain map of mathematical knowledge for teaching (Hill et al., 2008)

Hill et al. argue that mathematical knowledge for teaching (MKT) refers to the entire body of knowledge that a mathematics teacher needs to master for successful teaching in the classroom. MKT is made up of subject matter knowledge (SMK) and pedagogical content knowledge (PCK). SMK is made up of three elements: common content knowledge (CCK), specialized content knowledge (SCK) and knowledge at the mathematical horizon. CCK is equivalent to Shulman's idea of subject matter knowledge which includes the knowledge used in the work of teaching in ways in common with how it is used in other professions or occupations that also use mathematics. Specialized content knowledge (SCK) refers to the mathematical knowledge that enables teachers to engage in particular tasks, such as representing mathematical ideas accurately, providing mathematical explanations for common rules and procedures, and examining and understanding unusual solution methods to problems. PCK is also made up of three elements: knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of the curriculum. KCS is a subset for both PCK and MKT and it focuses on teachers' understanding of how students learn particular content. The separation of KCS from KCT and knowledge of curriculum emphasizes that teachers need to know how students learn, as well as knowledge of the subject itself and the curriculum.

Through introducing KCS, Hill et al. (2008) suggest that teachers need to be familiar with common errors that students make when developing proficiency in areas of mathematics, and in procedures associated with tasks. They also need good knowledge of how to diagnose students' errors, students' understanding of content, developmental sequences in students' mastery of subject matter and common computational strategies that students need to acquire. In other words, teachers need to possess the knowledge and skills to identify and provide explanations of errors, to have insights into which errors arise, in which content areas they surface and why. Teachers also need appropriate knowledge to make correct interpretations of students' answers and solutions to mathematical problems, and to be able to judge which solutions indicate sophisticated thinking and which reflect immature and faulty mathematical thinking. In order to do all of this, teachers need to have rich knowledge of categories of mathematical problems and topics, and mathematical activities that are easy or very difficult for children at particular ages and stages. They need to know what students are expected to be able to do, and to be aware of the demands that will be made in the next phase of learning up the school.

5. The Rationale for the Comparative Study

Based on the argument above that teaching and learning are interrelated, the present study of the teaching of aspects of time intervals in primary schools was informed by evidence from three sources. A specially constructed test was administered; a teacher questionnaire seeking their views on how time and other mathematical concepts are learned and should be taught was distributed; and observations of lessons were carried out and discussions held with teachers. As a key intention of the author was to find out how well Macau primary teachers teach the topic of time and the calculation of time intervals, there was a need for some means of highlighting the effectiveness of the mathematics teaching in Macau and to set this against the outcomes of approaches used by teachers elsewhere. Hence a comparative study was conducted of students in Macau and in neighbouring Hong Kong in the East, and in the Netherlands in the West.

A major consideration for choosing Hong Kong, Macau and Dutch students as subjects was based on the philosophy of fair comparison (Postlethwaite & Leung, 2007). For instance, the PISA 2006 mathematics mean scores gained by 15-year-old students in the three locations, Hong Kong 547, the Netherlands 531, and Macau 525, were quite close and they were all in the top ten of all participating countries, an indication that the mathematics education standards in the three educational jurisdictions were very similar (OECD, 2007).

Another determining consideration was a fundamental principle of comparative education: the familiarity of the researcher to avoid attaching disproportionate significance to chance or surface comparisons (Bray et al., 2007). The author was brought up and educated in Hong Kong where he taught mathematics in secondary and primary schools for over 10 years. He also served in a Macau university for 8 years preparing pre-service and in-service primary mathematics teachers. Even though he presently works in Macau, he is still a Standing Committee member of the Hong Kong Primary School Mathematics Competition. He has also acted as a consultant for a Chinese school and a Chinese Women's Centre in Amsterdam since 1985 following a comparative education field trip. When helping with homework tutorial classes during visits to

Amsterdam, he saw at first hand that Chinese Dutch students are quite smart at mathematics. Hence, he feels confident about conducting a comparative study in mathematics teaching and learning in the Netherlands, Hong Kong and Macau in the hope of identifying issues and contrasts in the teaching of time concepts in these three locations.

In the light of the above discussion, the following research questions guided the execution of the study:

a) To what extent do primary school students in Macau, Hong Kong and the Netherlands differ in terms of their global mathematics performance and in how they respond to a question exploring their grasp of procedures for subtracting time in a 24-hour clock system?
b) What errors are Grade 6[1] primary school students prone to make in executing a 24-hour clock time interval subtraction problem?
c) To what extent are teachers in the three locations able to predict the difficulty level of a 24-hour clock time interval question set for students?
d) To what extent are teachers in the three locations able to anticipate likely errors that students might make in executing a time interval calculation problem?

6. Design of the Study

6.1 *The Mathematics Test*

A mathematics test made up of 12 questions was constructed, the choice of items being based upon public assessment test scripts of the type regularly encountered in the three locations. All of the items were given to students in their own mother tongues. One question was particularly designed to measure the ability of final year primary students to calculate a period of time taken to complete a journey on the basis of 24-hour clock information. The test item is a common area of study in the three regions, particularly the procedure for carrying out four rules of number calculations when the base unit is 60 minutes.

[1] The final year of primary education in the Dutch education system is Grade 8 while it is Grade 6 for both Hong Kong and Macau schools.

6.2 *Student Subjects*

Schools with differing religious/governing bodies were invited to join the study, proportionate to the incidence of these schools in the systems in the three regions. To narrow the diversity of learning ability, most of the Dutch students were caucasians while the Hong Kong and Macau students were locally born Chinese or Mainland China immigrants. The Dutch cohort of students also included a group of children of Chinese extraction, and the Macau and Hong Kong cohorts included a number of students in "elite" streams. Unlike the situation in the author's doctoral study, a group of students from normal classes in Hong Kong and Macau schools were invited in order to strengthen the validity of any comparisons. In the event, there were 6 classes in 5 schools in the Netherlands; 10 classes from 6 schools in Hong Kong; and 8 classes from 8 schools from Macau. In total, 853 final year primary school students took the test: 106 from the Netherlands, 358 from Hong Kong and 389 from Macau (see Table 1). The imbalance of subjects from the three locations to an extent reflects differences in "normal" class sizes in primary schools in each place. Whereas most classes in primary schools in the Netherlands are of mixed ability, some division of classes into "normal" and "elite" occurs in Hong Kong and Macau. As mentioned above, out of interest, a small group of ethnic Chinese students was focused on in classes in the Netherlands (see Table 1).

Table 1. Distribution of students taking the test in the three locations

	Netherlands	Hong Kong	Macau	Total
Elite Class	--	186	214	400
Normal Class	--	172	175	347
Chinese Dutch	9	--	--	9
Dutch	97	--	--	97
Total	106	358	389	853

6.3 *Conduct of the Test*

Testing was conducted in the Netherlands in 2008 by the author and a Dutch university lecturer in the presence of the class teachers of each class, students being given 30 minutes to finish the test. The test in Macau was also conducted in 2008 in eight schools: two Catholic, two other Christian denomination and four other associated schools in the Macau peninsula. Mathematics teachers in these schools supervised the test with their own Primary 6 class, students being given 30 minutes to finish the test. The test in Hong Kong was also conducted in 2008 in six schools: three Roman Catholic, two other Christian and one Taoist in East Kowloon and in the New Territories. The test was conducted in exactly the same way as in Macau.

6.4 *Data Encoding and Analysis of Test Responses*

SPSS was used to analyze the data, with between-group variance analyses being used to examine overall performance. "Percent correct" figures were also used as this was a relatively simple comparison strategy for gauging success on each test item. It also allowed the author to compare, for example, the success rates of students from high-ability and normal classes in the three jurisdictions.

6.5 *Teacher Questionnaire Data*

A teacher questionnaire was distributed asking teachers (a) to judge the level of difficulty of each of the 12 test items and (b) to suggest up to three common mistakes that students might commit for each item. The teacher questionnaire, based on the content of the test given to students, first gathered demographic data then looked into the relationship between teachers' knowledge of students' common misconceptions, their judgment of the ability of their students and predictions of success rates. The first task for teachers was to rate the level of difficulty of each question on a five-point Likert-type scale, 1 standing for very easy and 5 standing for very difficult. The second task was to foretell any mistakes that might be committed by their students on each question.

6.6 *Subjects in the Teacher Survey*

The mathematics teacher of each selected class was invited to answer the teacher questionnaire after students had completed the test. In addition, other mathematics teachers in the subjects' schools teaching the same form were invited to answer the questionnaire in order to enlarge the sample size and obtain a more representative sample. In total, 66 primary mathematics teachers completed questionnaires; 6 from the 3 Dutch schools, 29 from the 6 Hong Kong schools and 31 from the 8 Macau schools. The teacher surveys were carried out in the Netherlands by the author and a Dutch university lecturer, translation of the responses into English being made by the Dutch university lecturer in preparation for data coding and analysis. The teacher survey in Macau was conducted at the same period as the students were tested. The teacher survey in Hong Kong was conducted in exactly the same way as in Macau.

6.7 *Classroom Teaching Observations and Follow-Up Interviews*

After analyses of the student test answers and teacher questionnaire responses, the author carried out classroom teaching observations and interviews in the Netherlands, Hong Kong and Macau primary schools. The aim was to obtain an accurate and up-to-date synopsis of the mathematics teaching approaches used by teachers in the three locations. In particular, the focus was on uncovering differences in the teaching of time intervals in the three locations and to try to explain any major differences in the attainment profiles across the regions.

7. Results

7.1 *Students' Performance on the Time Interval Question in the Attainment Test*

The question on the students' test was as follows:

> *"Lily went to Disney Park yesterday: Take off time: 8:57, Arrival time: 10:41. How long did it take her to go to the park?"*

Students were asked to calculate the time it took for Lily to go to the Disney Park, and allowed to use subtraction or any other method to arrive at the correct answer. They were required to write down the whole calculation process in the test paper. The correct answer rates for students in the Netherlands, Hong Kong and Macau were 73.6%, 46.9% and 31.6% respectively. The performance of the small group of Dutch Chinese students was the best with a "correct" rate of 77.8%, while the group of white Dutch students came second with a correct rate of 73.2%. Students in the "elite" groups from Hong Kong and Macau had correct rates of 62.9% and 41.6%. The performance of the "normal" classes of Hong Kong and Macau students was the worst, with 29.7% and 19.4% (see Table 2). Hence the prediction that Hong Kong would outperform their counterparts was rejected, the Netherlands students outperforming national counterparts ($p<0.001$). The performance of the Macau students was the worst among students in the three locations (see Table 3).

Table 2. Students' performance in solving the time interval question

		Groups									
		Elite Class		Normal Class		Dutch Chinese		Dutch student		Total	
REGION		n	%	n	%	n	%	n	%	n	%
Netherlands	wrong					2	22.2	26	26.8	28	26.4
	correct					7	77.8	71	73.2	78	73.6
	Total					9	100	97	100	106	100
Hong Kong	wrong	69	37.1	121	70.3					190	53.1
	correct	117	62.9	51	29.7					168	46.9
	Total	186	100	172	100					358	100
Macau	wrong	125	58.4	141	80.6					266	68.4
	correct	89	41.6	34	19.4					123	31.6
	Total	214	100	175	100					389	100

Table 3. *Students' performance in solving the time interval question*

No of subjects		Correct %		df	Mean Square	F	Sig.
Netherlands	106	73.6**	Between-Group	2	7.750	33.979	0.000
Hong Kong	358	46.9	Within-Group	850	0.228		
Macau	389	31.6	Total	852			

If the comparison is narrowed down to elite classes with the Dutch students, then the correct answer rates are 73.6%, 62.9% and 41.6% for students of the Netherlands, Hong Kong and Macau respectively. It is clear that the performance gap between the Netherlands and Hong Kong elite class students is large, whilst the correct rate gap is even larger between the Netherlands and Macau student cohorts. Statistically significantly, the Macau students were outperformed by their Netherlands and Hong Kong counterparts (p<0.001, see Table 4).

Table 4. Bonferroni Post Hoc One-Way ANOVA of students' performance in solving the time question according to region and elite class

(I) REGION	(J) REGION	Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Netherlands	Hong Kong	0.11	0.058	0.205	-0.03	0.25
	Macau	0.32(*)	0.057	0.000	0.18	0.46
Hong Kong	Netherlands	-0.11	0.058	0.205	-0.25	0.03
	Macau	0.21(*)	0.048	0.000	0.10	0.33
Macau	Netherlands	-0.32(*)	0.057	0.000	-0.46	-0.18
	Hong Kong	-0.21(*)	0.048	0.000	-0.33	-0.10

* The mean difference is significant at p<0.05 level.

If the comparison focuses on normal classes of students with the Dutch students, then the correct answer rates are 73.2%, 29.7% and 19.4% for students of the Netherlands (excluding the Dutch Chinese students), Hong Kong and Macau respectively. Statistically significantly (p<0.001), both the Hong Kong and Macau normal class students were outperformed by their Dutch counterparts. Even though the Hong Kong normal class students did not outperform the Macau normal class students (as was the case in the elite class comparisons, see Table 5), the performance of the Macau normal class students was the lowest among all of the sub-groups.

Table 5. Bonferroni Post Hoc One-Way ANOVA of students' performance in solving the time question according to region in normal classes

(I) REGION	(J) REGION	Mean Dif. (I-J)	Std. Error	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Netherlands	Hong Kong	0.44(*)	0.053	0.000	0.31	0.57
	Macau	0.54(*)	0.053	0.000	0.41	0.67
Hong Kong	Netherlands	-0.44(*)	0.053	0.000	-0.57	-0.31
	Macau	0.10	0.046	0.084	-0.01	0.21
Macau	Netherlands	-0.54(*)	0.053	0.000	-0.67	-0.41
	Hong Kong	-0.10	0.046	0.084	-0.21	0.01

* The mean difference is significant at p<0.05 level.

7.2 *Teachers' Responses about the Time Interval Item on the Questionnaire*

Among the 66 teacher questionnaires collected, 6 were not included for analysis as there was too much missing data. Consequently, the questionnaires of 6 Dutch teachers, 25 Hong Kong teachers and 29 Macau teachers were selected for analysis. In terms of judging the difficulty of the time interval question, the levels of difficulty for teachers of the Netherlands, Hong Kong and Macau were 1.40, 2.55 and 2.39 respectively. The Netherlands teachers made an accurate estimate that the time test item was an easy question for their students as the "correct" rate was the highest at 73.6%. It is interesting that Macau teachers did not consider the time question to be a difficult question for students in Macau, who in the event had the lowest correct rate of 31.6%.

On the task of making predictions about students' likely mistakes when trying to solve the time question, even though each teacher was asked to write down up to three mistakes their students might commit, few anticipated that students would have any problems. Only 3 Dutch teachers provided 3 predictions, 15 Hong Kong teachers provided 17 predictions in all and 20 Macau teachers offered 22 predictions. The numbers of successful predictions of students' mistakes were 1, 10 and 13 for the Netherlands, Hong Kong and Macau teacher subjects respectively. Hence the correct prediction rate of students' committing mistakes was 33.3%, 58.8% and 59.1% and the "teachers' rate of successful prediction" was 16.7%, 40.0% and 44.8% for the Netherlands (Dutch), Hong Kong and Macau teachers respectively (see Table 6).

With reference to the author's anticipation that Hong Kong teachers would have comparatively greater awareness of students' likely mistakes, the results show that Hong Kong teachers rated the time question as posing only a moderate level of difficulty for their students. In fact, the Hong Kong "teachers' rate of successful prediction" was lower than that of the Macau teachers. Against the expectation of the author, the Macau teachers were best at predicting students' errors in solving the time problem! The results tempt one to suggest that Macau and Hong Kong teachers had similar levels of knowledge of students' common mistakes, while the Netherlands teachers had comparatively less knowledge of students' likely common mistakes in dealing with the time interval problem. In fact, a number of probable explanations may clarify why Macau teacher subjects were able to achieve a higher prediction rate of students' mistakes on the time interval question. For

Table 6. Teachers' prediction of students' common mistakes of the time interval question

Type of students' mistakes expected by the teachers		Dutch N=6		Hong Kong N=25		Macau N=29	
		n	%	n	%	n	%
1	used decimal system –mistakenly converting 1 hr into 100 min rather than into 60 min	0	0.0	6	24.0	9	31.0
2	calculation mistake	0	0.0	3	7.5	3	10.3
3	time conversion mistake	0	0.0	0	0.0	3	10.3
4	mistake in borrowing and place value	0	0.0	0	0.0	3	10.3
5	one hour is 60 minutes	1	16.7	0	0.0	0	0.0
6	mistaken 8:57 to 8:51 to get 1:40	0	0.0	1	2.5	0	0.0
7	mistake when doing 57-41	0	0.0	1	2.5	0	0.0
8	forgot how to convert an hour into minutes	0	0.0	0	0.0	1	3.4
9	borrowing and adding mistake in hour and minute conversion	0	0.0	1	2.5	0	0.0
10	forget to change 1 hour into 60 minutes	0	0.0	1	2.5	0	0.0
11	change back to hour	0	0.0	1	2.5	0	0.0
12	forget to add 57 min	0	0.0	1	2.5	0	0.0
13	convert one hour into 60 minutes and subtract directly	0	0.0	1	2.5	0	0.0
14	don't know how to calculate time	0	0.0	0	0.0	1	3.4
15	don't use deduction	0	0.0	0	0.0	1	3.4
16	used to pointing out the time directly from the clock	0	0.0	0	0.0	1	3.4
17	general error: 8 : 57 + 2 = 10 …	1	16.7	0	0.0	0	0.0
18	errors / carelessness in counting	1	16.7	0	0.0	0	0.0
19	mistake in calculate hours	0	0.0	1	2.5	0	0.0
Sub-total of teachers' guesses**		3	50.0	17	60.0	22	75.7
Success prediction of students' committed mistakes		1	33.3	10	58.8	13	59.1
Teachers' rate of successful predication (n/N)		1/6	16.7	10/25	40.0	13/29	44.8

*Does not include other mistakes not expected by the teachers;
**Sub-total of teachers' expected student mistakes could be more than the number of teachers as one teacher gave three answers. Highlighted expected mistakes were rarely committed by the students.

instance, around 70% of the Macau students failed to give the right answer, while the failure rates for Hong Kong and Dutch students were 56.1% and 26.4%. Proportionately, the correct prediction rate for the

Macau teachers showed that many did not correctly anticipate the failure rate for their students. It would seem that further research is needed to uncover the true strengths and weaknesses of Macau teachers on the issue of teaching time intervals with reference to the 24-hour clock system and in predicting likely errors of computation.

8. Discussion

8.1 *Reasons Why the Macau Students Were Poor at Solving the Time Interval Question*

The results reveal that many Macau students were poor at solving the time interval question. Even though the elite class performed better than the normal class, the correct rate was still below 50%. It is striking that fewer than 1 out of 5 students in the normal class managed to solve the time interval question, indicating that most students in Macau have difficulty in solving time interval calculations, whether or not they are in elite or normal classes. As the students were required to present their calculations on the test paper, it was possible to look into the errors they commonly committed. It is clear that the two most commonly offered wrong answers by the Macau students were 2 hours and 24 minutes (22.9%) and 1 hour 84 minutes (16.7%). Only 0.9% and 1.9% of the Netherlands students committed these two errors (see Table 7). For all the students in the three regions, the third most common erroneous answer was 2 hours 16 minutes (3.6%), and the fourth most common mistake was 2 hours and 44 minutes (2.4%). These errors suggest that students in all three locations were suffering from different misconceptions or had some difficulty in making accurate calculations regarding time differences in hours and minutes.

It is not the author's intention here to discuss all the mistakes made by students. Even though the Dutch students performed much better than students from Hong Kong and Macau, they had similar misconceptions and weaknesses, but to a smaller degree of seriousness. The most common mistake committed by the Dutch students was to apply a common misconception held by many students about subtraction: that

Table 7. Common mistaken answers given by students on the time interval question

	Common mistakes committed by students	Dutch (106)		Hong Kong (358)		Macau (389)		Sub-Total (853)	
		N	%	N	%	N	%	N	%
1	Not attempted	1	0.9	16	4.5	13	3.3	30	3.5
2	1 hr 16 min	1	0.9	4	1.1	9	2.3	14	1.6
3	1 hr 24 min	0	0.0	2	0.6	9	2.3	11	1.3
4	1 hr 34 min	0	0.0	5	1.4	2	0.5	7	0.8
5	1 hr 54 min	0	0.0	7	2.0	3	0.8	10	1.2
6	1 hr 84 min	1	0.9	28	7.8	63	16.2	92	10.8
7	2 hr 24 min	2	1.9	74	20.7	89	22.9	165	19.3
8	2 hr 16 min	9	8.5	13	3.6	9	2.3	31	3.6
9	2 hr 34 min/1 hr 94 min	0	0	3	0.8	9	2.3	12	1.4
10	2 hr 44 min	2	1.9	9	2.5	8	2.1	19	2.2
11	2 hr 84 min	0	0.0	0	0.0	3	0.8	3	0.4
12	Other mistakes	12	11.3	29	8.1	49	12.6	90	10.6
	Sub-total	28	26.4	190	53.1	266	68.4	484	56.7

one always has to take the smaller number away from the larger number. The most common error made by students of Macau and Hong Kong was a "prototyping" misconception in that the students applied the decimal number system when dealing with the 60-unit time system. Hence the most common erroneous response made by the Dutch students was 2 hr 16 minutes, while the most common mistake made by students of Macau and Hong Kong was 1 hr 84 minutes (2 hr 24 min was evolved from 1 hr84 minutes and it will be discussed later). In other words, the most serious mistake committed by students of Macau and Hong Kong was to overlook the 60 (minutes) unit system of time and to apply the decimal number system. In other words, the students converted 1 hour into 100 instead of 60 minutes. In fact, many students treated the subtraction of time as if they were manipulating the more familiar decimal number system.

$$\begin{array}{r} 10.41 \\ -\ \ 8.57 \\ \hline 1.84 \end{array}$$

Some 92 out of the 853 subjects gave 1 hr 84 min as their answer. Some gave 1hr 94 min when answering as they committed an additional mistake when "borrowing". It is interesting to report that 165 "clever" students (most from elite classes) remembered to do a unit transformation from minutes into hours using the 60 unit system correctly after the subtraction of decimal numbers. They refined 1 hr 84 min into 2 hr 24 min, and the calculation thus became:

1.84 hr = 1 hr+60 min+24 min = 1 hr + 1 hr +24 min = 2 hr 24 min

Some students gave 2 hr 34 min or 2 hr 14 min as their answer due to mistakes in borrowing. Table 8 lists four of the most common mistakes that teachers need to be aware of when teaching the time interval topic. As well as mistakes of borrowing in subtraction, the most striking misconception (that of always subtracting the smaller from the larger number) persisted with the final year students in the primary school. Even though the total number of Macau students writing down 2 hr 16 min (2.3%) or 1 hr 16 min (2.3%) was the lowest when compared to Hong Kong (3.6% + 1.1%) and Dutch students (8.5% + 0.9%), such erroneous thinking reflects the fact that many Macau students, particular students in "normal" classes, suffered from misconceptions when subtracting time intervals when hours and minutes units were involved.

8.2 *How Time Issues are Taught in Macau, Hong Kong and Netherlands Primary Schools*

The great regional differences in students' performance in the calculation of time prompted the author to observe classroom practice and talk to teachers from the three locations about erroneous answers on the test paper. The main explanation given by teachers about why the Dutch students were able to solve the time problem more successfully was that Dutch teachers teach learners to apply quite different strategies to tackle time problems than do counterparts in Macau and Hong Kong. Most of the Macau and Hong Kong teachers considered the best way to calculate a period of time is direct addition or subtraction. In addition, they were mindful of the special method or format when subtracting time in the lower primary year classrooms. For instance, teachers in Macau instruct the students to write down " (hour)" at the head of the hour column and "分 (minute)" on the head of the minutes column. They may even write " (second)" at the head of a column for seconds to add or

Table 8. Different mistakes made by students when answering the time interval question

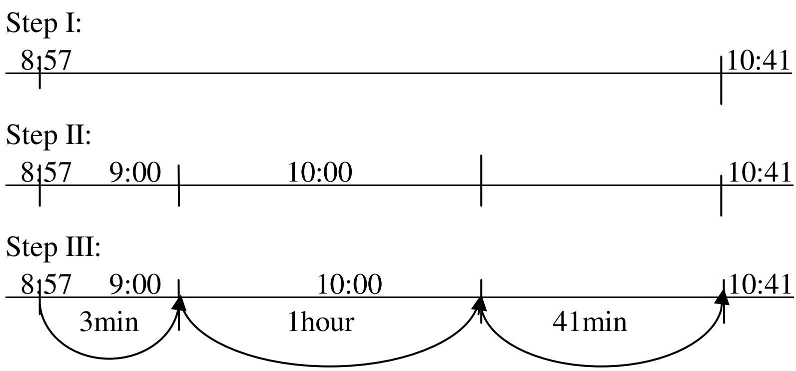
Type of mistake	Example of mistake	Explanation of mistake
Type I: Confusion of 60 unit system of time with decimal system	(a) The basic form of mistake 10:41 − 8:57 1:84	Not using appropriate unit quantities when calculating time interval and applying decimal quantities 10 41 − 8 57 1 84
	(b) For higher ability students 1hr 84min = 2 hr 24min	In trying to round up the answer, it seemed logical to round up 84 min to 1 hr 24 min: 1 hr 84 min = 1 hr+60 min+24 min = 1 hr + 1 hr +24 min = 2 hr 24 min
Type II: Combination of confusion of 60 unit system of time with decimal system and borrowing mistakes in subtraction.	(a) basic mistakes 10.41 (forgot the − 8.57 borrowing 1.94 process)	First, confusion of the 60 unit system with the decimal unit system. Then, students committed the common mistake in subtraction of forgetting the borrowing procedure.
	(b) 1:94 = 2 hr 34 min	Some “clever” students remembered to round up the answer using the 60 unit system of time. Then they transformed 1:94 into1+1 x60 + 34=2hr34min
Type III: Misconceptions of subtraction: always subtract smaller from larger number	(a) basic mistake 2 hr 16 min	Misconception of always subtracting from the large number. 10 is larger than 8, so 10-8 = 2 10:41 (1 − 7 = 7 − 1 = 6) − 8:57 (4 − 5 = 5 − 4 = 1) 2:16
	(a) for “smarter” students 1 hr 16 min	There is a borrowing process for 4-5, so they deducted 1 from 10 into 9 to carry out the subtraction as 9-8 to get 1 10:41 (1 − 7 = 7 − 1 = 6) − 8:57 (4 − 5 = 5 − 4 = 1) 1:16 (10 − 1 − 8=1)
Type IV: Careless mistake of forgetting the process of borrowing	2 hr 44 min	Forgetting to subtract 1 from 10 to make it 9 for subtraction of the hour unit as the borrowing process was carried out on the minute unit: 10 : 41 − 8 : 57 2 : 44
	1 hr 54min or 1 hr 34 min	Forgot the subtraction of 1-7, borrowing was taking place for “4” and failed to reduce 4 into 3. During the second borrowing step from the hour unit, kept 4 + 6 = 10 then subtracted 5 to get 5. 10 : 41 10 : 41 − 8 : 57 or − 8 : 57 1 : 54 1 : 34

subtract as appropriate as follows:

$$\begin{array}{rrrr} & \text{hr} & \text{min} & \text{s} \\ & 10 & 41 & 0 \\ - & 8 & 57 & 0 \\ \hline \end{array}$$

Using this format in the calculation of time intervals, Macau teachers believed that they were adequately emphasizing the distinction between the 60 unit system of time and the 10 unit system used with decimal numbers. Students are taught that they should remember the importance of using the 60 unit system rather than the decimal system when solving time problems. Quite a number of Macau teachers mentioned on their questionnaire that some students might be confused by the fact that the decimal system is not fully operating in all time calculations. Misconceptions about simple subtraction are not a vital focus in the teaching of time, so none of the Macau teachers expected that their students would commit such a low level mistake. In fact, inspection of the papers suggests that a significant number of Macau students in "normal" classes made unexpected errors.

Turning to the finding that how time is taught in Dutch classrooms leads to superior performance among the Dutch students over the Macau students, the author investigated at first-hand how Dutch teachers teach time. He discovered that Dutch teachers teach students to solve time interval problems by making use of the concept of the "Time Line" to help them calculate time intervals. The Dutch teachers said that this approach is more meaningful for students and gives learners concrete, easy-to-understand ways to visualize time. For example, taking the question used in the present study, Dutch teachers would not ask students to calculate through direct subtraction. Rather, they would ask them to draw a "time line" and to calculate the time interval on a minutes and hours gap basis. When teaching the whole class, Dutch teachers will draw a time line on the blackboard or computer as this helps students to visualize and count up how time passes. The first step is marking down the starting time 8:57 at the left end of the line and the arrival time at the right end. The second step is putting down the 9:00 and 10:00 o'clock time points between the two ends of the time line. The third step is to mark out the three time intervals, 3 minutes between 8:57 and 9:00, 1 hour between 9:00 and 10:00, and 41 minutes between 10:00 and 10:41. Finally the problem is solved by adding up the three intervals of time: "3 min + 1 hr + 41 min". Hence the calculation is much easier for students as it is based on concrete concepts of how time passes.



<u>Step IV:</u>
3min + 1 hr + 41 min = 1hr 44min

The counting of time interval periods and units in Dutch classrooms is very obviously linked with a real-life, everyday context. In contrast, in Macau and Hong Kong classrooms the time interval calculation is chiefly arrived at by the subtraction of numbers. This is inviting students to fall into the trap of using a "prototype" misconception, that of overlooking the 60 unit (minutes in one hour) system and using the 100 unit decimal system.

9. Implications of the Study

The study showed that the way students have been taught mathematics in all three locations was predominantly via mastering routines or algorithms. This leads students to learn how to arrive at the right answer but does not guarantee a clear understanding of the mathematical rationale for performing sequences of steps that produce a correct response. The message seems to be: "Learn how to produce the right answer and the reason for taking the steps needed for producing the right answer will become apparent later!" Taught through repeated practice in working out common time problems, the Macau primary students committed a number of mistakes when faced with a problem presented in an unfamiliar format. Whereas many of them suffered a "prototyping" misconception (confusing of 60 unit and 100 unit systems), only 3 out of 106 Dutch students fell into this trap. Clearly, students' performance is a reflection of how they have been taught, not principally the way they understand or internally visualize algorithms. Macau teachers do teach the time topic and they are also very skilled in teaching basic calculations such as addition, subtraction, multiplication and division. On the other hand, there is a widespread belief among Macau mathematics teachers that mathematics requires both speed and accuracy. When students do not have to think out a problem from first principles and, instead, only need repeated practice in working out a tried and tested procedure to produce the right answer, they are assured by teachers and parents that insight will follow. This led students taught by such teachers in the present study to go straight to applying previously learned subtraction and addition operations to calculate time intervals. They were not used to having to think up their own logical approach to solving an unfamiliar mathematical problems presented by the teacher.

Although the Hong Kong students too were taught algorithms, to be perfected in homework and revision lessons, more students had been encouraged to think out a solution by themselves. Parents will pressurize schools to teach all procedures in the class textbook "to the letter" and will often pay for private tuition for their children. Many Hong Kong parents do not agree with the notion that young children should be taught how to think for themselves, and ask: "What is the point, when teachers can short-cut 'thinking out for oneself procedures and go straight to the most efficient and rapid way of solving known problems?"

In the Netherlands, where mathematics procedures are taught as tools for students to use to arrive at solutions to problems that they themselves can visualize, large amounts of repetitive drill and practice are not esteemed, for Dutch teachers believe in giving students strategies for dealing with real-life problems that arise in and out of school. However, Dutch teachers need to realize that some students acquire confidence and self-belief when mathematical "facts" automatically spring to mind or are "at their finger tips". Dutch teachers also need to improve their teaching in other topics as their students did not always perform well in the rest of the 12 question test paper compared to Macau and Hong Kong students. Similarly, Macau teachers should consider the value of teaching topics and routines that make sense to the students, rather than having every student in the class complete exercises that eventually may result in a real understanding of skills taught in school. For example, they need to stray from the textbook from time to time and to use time lines and other ways of visualizing problems to promote genuine understanding of mathematics concepts by children. Teachers need to remember that not all children learn well by using prescribed routines, and that different methods may be used equally validly to help some students arrive at solutions rather than using one method alone to teach students to get answers to problems.

10. Conclusion

A major lesson outcome of this research is that teachers can learn a lot from paying attention to misconceptions that occur in children's minds, and to the "wrong" answers children give in class and in homework assignments. The answers students offer often reveal how they have not yet mastered conceptual problems, and they also give clues about the nature and roots of misconceptions. Teachers from Macau, Hong Kong and the Netherlands can become more professional mathematics teachers through looking closely at what students can do and the reasons why some students cannot do tasks which others in the class find easy. The sharing of teaching experience internationally is also one way for teachers around the world to expand their knowledge of pedagogy and to learn about how counterparts in other countries approach and tackle problems that they themselves find daunting.

Summing up, the present study revealed that teachers from Macau, Hong Kong and the Netherlands need to improve their pedagogical content knowledge (PCK) and mathematical knowledge for teaching (MKT) in order (a) to enhance their teaching of time interval calculations and (b) to encourage students' learning using easy-to-visualise time concepts. The teachers seemed to be concentrating on teaching students strategies that they themselves used, rather than looking at the problem from the child's point of view. There are also improvements that should be kept in mind if the present study is to be replicated or extended. For instance, classroom observations and scrutiny of students actually working out problems should be conducted; the researcher should know the language spoken by the teachers and students; greater effort should be made to increase the number of subjects in some of the design cells; and more examples and variations of time interval questions should be examined and explored.

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Chapter 8

Teaching Number Sense via Interactive Multimedia in a Primary School in Taiwan

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This paper describes how number sense was taught with interactive multimedia in a primary classroom in Taiwan and reports the differences of interviewed students' use of number sense strategies before and after the instruction. One sixth grade class which contains 32 students (18 boys and 14 girls) was selected to join the teaching experiment in a technology-based environment. The results show that the teaching of number sense can be implemented through the appropriate use of interactive multimedia into the mathematics class and a well-designed learning environment created by the teacher. The results also show that interactive multimedia can both be an effective tool in helping children develop number sense and promote children's motivation for learning.

Keywords: teaching of number sense, teaching with technology, primary education in Taiwan

1. Introduction

What is number sense? Number sense refers to a person's general understanding of numbers and operations and the ability to handle daily-life situations that include numbers. This ability is used to develop practical, flexible, and efficient strategies (including mental computation or estimation) for handling numerical problems (McIntosh, Reys, & Reys, 1992; Reys & Yang, 1998; Yang, 2006; Yang & Li, 2008; Yang & Wu, 2010).

Number sense has been considered a key topic for developing formal mathematical concepts and skills in the primary schools (Jordan, Glutting, &

Ramineni, 2010; Jordan, Kaplan, Locuniak & Ramineni, 2007; Yang, Li, & Lin, 2008). Having good number sense is important for higher level mathematical thinking and learning (Geary, Bow-Thomas , & Yao, 1992); however, a lack of number sense will likely result in mathematical learning difficulties (Gersten, Jordan, & Flojo, 2005; Mazzocco & Thompson, 2005). Therefore, the international mathematics education highly emphasizes the development of students' number sense, and regards it as an important topic that should be taught in primary schools (Anghileri, 2000; Faulkner, 2009; Jordan, Kaplan, Oláh, & Locuniak, 2006; Yang, Reys, & Reys, 2009; Yang & Wu, 2010).

However, Jordan, Hanich, and Kaplan (2003) found that students in primary grades had difficulties with arithmetic combinations, counting strategies, and number sense in their longitudinal studies. Similar results have been widely found in the Huaren Region due to the emphasis on written computation in this area without conceptual understanding (Reys, Reys, McIntosh, Emanuelsson, Johansson, & Yang, 1999; Reys & Yang, 1998; Yang et al., 2008; Yang & Li, 2008). Hence, international mathematics evaluations, such as TIMSS 2003, 2008 (Mullis, Martin, Gonzalez, & Chrostowski, 2004; Mullis, Martin, & Foy, 2008), and PISA 2006, 2009 (Lin, 2008; Ministry of Education in Taiwan, 2003) have reported that students in Taiwan were ranked at the top, whereas the study of Reys and Yang (1998) has suggested that students who have skills in written computation do not necessarily possess well-developed number sense. In addition, several studies have further indicated that students in Taiwan preferred to use the written method rather than the use of number sense (Yang, 2006; Yang & Li, 2008). Moreover, the studies of Markovits and Sowder (1994), Menon (2004), and Yang (2005) showed that students tended to use standard written algorithms to answer questions related to number sense. This implies that over emphasis on memorization of mathematical principles and written computation will limit conceptual understanding and result in the lack of number sense (Burns, 1994, 2007; Markovits & Sowder, 1994; Reys & Yang, 1998). This result is also consistent with the finding of Cai (2000) that Chinese students had better performance on a process-constrained task than U.S. students, but the U.S. students had better performance on process-open tasks than the Chinese students. In fact, a tight focus on written algorithms may narrow students' use of flexible strategies and cause some misconceptions.

Due to the importance of number sense and the potential effect of focusing on written computation, mathematics educators internationally have emphasized the teaching of number sense in the hope of helping children develop this capacity (Anghileri, 2000; Markovits & Sowder, 1994; Menon, 2004; Yang & Li, 2008; Yang & Tsai, 2010; Yang & Wu, 2010). Even though teaching number sense in primary school has been considered a key issue (National Council of Teachers of Mathematics [NCTM], 2000; Yang & Li, 2008) and the Guidelines of Nine-Year Integrated Mathematics Curriculum for Grade 1 to 9 in Taiwan (Ministry of Education in Taiwan, 2003) values the importance of number sense, the mathematics textbooks in Taiwan include few activities which relate to number sense (Yang & Li, 2008). Moreover, most of the teachers at primary schools in Taiwan prefer to use traditional teaching methods and are accustomed to emphasizing written computation to find an exact answer. This results in the poor development of number sense for students in Taiwan (Reys & Yang, 1998; Yang, 2005; Yang et al., 2008; Yang & Li, 2008). It is reasonable to believe that students perform poorly on number sense due to the lack of opportunities to learn number sense. In this study, we report on ways to help children develop number sense. The research questions are as follows:

1) How can number sense be taught via interactive multimedia in a primary classroom?

2) What are the differences of the use on number sense methods among the students who being interviewed before and after instruction on number sense via interactive multimedia?

2. Background

2.1 *Number Sense Framework*

What are the components of number sense? Even though different researchers have defined the components of number sense from different perspectives (Berch, 2005), several research studies and documents have included common number sense components, such as understanding the basic meaning of numbers, recognizing number size, judging the reasonableness of a computational result, and so on (Markovits & Sowder, 1994; Reys et al., 1999; Verschaffel, Greer, & De Corte, 2007; Yang & Tsai, 2010). Having reviewed the literature related to number sense (Markovits & Sowder, 1994; McIntosh, Reys, & Reys, 1992; Reys et al., 1999; Reys & Yang, 1998; Verschaffel et al., 2007; Yang & Tsai, 2010), we define five components of number sense for this study as follows:

1. Understanding the basic meaning of numbers: This includes understanding of the number system, including whole numbers, fractions and decimals, and their relationships (McIntosh et al., 1992; Yang & Tsai, 2010).
2. Recognizing the magnitude of numbers: This implies that an individual can recognize the size of numbers. For example, when comparing the fractions $\frac{5}{11}$ and $\frac{8}{15}$, children do not need to depend on a standard written computation (e g., $\frac{5}{11} = \frac{5\times15}{11\times15}$ and $\frac{8}{15} = \frac{8\times11}{15\times11}$, which is taught in mathematics textbooks). Rather, children are able to use an efficient strategy, such as using a benchmark to compare the fractions (e.g., $\frac{5}{11} < \frac{1}{2}$ and $\frac{8}{15} > \frac{1}{2}$, so $\frac{8}{15} > \frac{5}{11}$).
3. Being able to use multiple representations: This implies that an individual can use different forms of representations, such as pictorial representations, symbolic representations, and others, to solve problems efficiently under different numerical situations (Yang & Tsai, 2010).
4. Recognizing the relative effect of an operation on numbers: This implies that children can make sense of how the four basic operations affect a computational result (McIntosh et al., 1992). For example, when children are asked to decide the best estimate for 998×0.98, they do not need to rely on a written computation to find an exact answer. Rather, they should know the answer is about 1000 and, in fact, should be less than 1000 due to 0.98 being slightly less than 1.
5. Being able to judge the reasonableness of a computational result: This implies that children can develop and use an efficient strategy (e g., mental computation or estimation) to solve problems and judge the reasonableness of the computational result (McIntosh et al., 1992; Yang & Tsai, 2010).

2.2 *Studies of Interactive Multimedia and Number Sense*

Due to the convenience and efficiency of technology, there is currently great interest in learning how to use technology, such as interactive multimedia in mathematics education (Aliasgari, Riahinia, & Mojdehavar, 2010; NCTM, 2000; Shiong, Aris, Ahmad, Ali, Harun, & Tasir, 2008). For example, the study of Shiong et al. (2008) showed that interactive multimedia material is user-friendly and many students are preferred learning via interactive multimedia material than traditional learning method. In particular, technology via interactive multimedia can change teaching and learning styles, including fostering whole class engagement and the creation of assessments (Cooper & Brna, 2002; Eseryel, Guo & Law, 2012; Godwin & Sutherland, 2004; Sadik, 2008; Yang & Tsai, 2010). Previous studies have shown that appropriate use of technology, including interactive multimedia, can not only promote students' mathematics performance (Dick, 2007; Eseryel et al., 2012; Inamdar & Kulkarni, 2007; Ruthven, 2007; Shion, 2008; Vulis & Small, 2007; Zbiek, Heid, Blume, & Dick, 2007), but also foster positive attitudes toward mathematics learning (Aliasgari et al, 2010; Chan, Tsai, & Huang, 2006; Eseryel et al., 2012; Lan, Sung, Tan, Lin, & Chang, 2010; Lin, 2008; Olkun, Altun, & Smith, 2005; Yang & Tsai, 2010). In addition, interactive multimedia can connect and bridge the gap between the concrete and abstract mathematical concepts for all mathematics grade levels (Shion, 2008; Yang & Tsai, 2010; Zbiek et al., 2007). Therefore, it has the potential to greatly assist young students in better understanding mathematics and number sense (Lyublinskaya, 2009; Su, Marinas, & Furner, 2010; Yang & Tsai, 2010). Indeed, several studies have found that classroom use of interactive multimedia-based practice for the teaching of estimation concepts can promote students' number sense (Lyublinskaya, 2009; Shion, 2008; Yang & Tsai, 2010). Moreover, an effective problem-based estimation instruction using mobile devices can help students develop computational estimation skills (Lan et al., 2010). This shows that emerging interactive multimedia can supply teachers and students with new opportunities to grasp number sense through creating visual images and practical understanding of mathematical concepts and relationships.

Moreover, the results of Trends in International Mathematics and Science Study [TIMSS] (Mullis, Martin, Foy, & Arora, 2012) show that mathematics self-efficacy beliefs and attitudes toward mathematics learning are quite low.

Under test-driven teaching situation, many mathematics educators emphasize that highly emphasizing on written computation without meaningful understanding will hinder students' thinking and exploring. To promote students' learning interest and understanding, we believed that the technology integrated into mathematics learning is a appropriate approach. In addition, cognitive load theory which reduces cognitive load focuses on considering instructional design and adjusting teaching strategies to enhance quality of teaching and learning. Hence, this study applied the Cognitive Theory of Multimedia Learning (Mayer, 2009) to integrate interactive multimedia into teaching and learning number sense. Cognitive Theory of Multimedia Learning, based on limited working memory and actively processing information with human being, proposed many design principles to improve multimedia teaching. The theory suggests that visual search and attention are highly connected. Therefore, the features of visual search and nature of attention are essential knowledge for interactive multimedia.

3. Method

3.1 *Sample*

The target school was located in a suburb in which parents' social-economic statuses were quite consistent. Most parents were public officials, and took their children's school work very seriously. One sixth grade class of 32 students (18 boys and 14 girls) was selected for the teaching experiment. These students learned number sense in a technology-based environment.

All the 32 students took a web-based two-tier test of number sense (pretest) for sixth graders before the experimental instruction. Based on the pretest, the students' performance was divided into three levels: high level (top 20%), middle level (middle 40%-60%), and low level (bottom 20%). Two students were randomly selected from each level for interviews. They are coded as H1, H2 (high level); M1, M2 (middle level), and L1, L2 (low level).

3.2 *Instruments*

Three instruments were used in this study.

Web-based two-tier test of number sense. This test, designed by Yang and Li (2007), served as pretest and posttest in the study. Two-tier test implies that the first tier test assesses children's responses to number sense related questions and the second tier test examines children's reasons for their related choice made in the first tiered-test (Yang, 2013). It included five number sense components (as defined above) with each component composed of 10 items. The test included 50 items and was divided into two subtests. Each subtest included 25 items and required about 40 minutes of class time. The Cronbach's α coefficient of the test was .881 and its construct reliability was .987.

Interview questions. Twenty questions were selected from the web-based two-tiered test. Four questions from each of the five number sense components were included. These questions were reviewed by researchers and two elementary teachers who are familiar with number sense. They all agreed that these questions could be used to examine the sixth graders' number sense.

Teaching and learning tools. 11 teaching and learning tools (T1 – T11) were used in the study. T1 – T9 were selected from the web-based learning resources http://nlvm.usu.edu, http://www.interactive-resources.co.uk/, and http://120.126.129.40/flashmath/. T10 (cif_01.exe) and T11 (cif_02.exe) were designed by authors and a computer specialist using Visual Basic computer programs to support the development of number sense (Yang & Tsai, 2010). The contents of T1 to T11 were briefly described as follows:

T1 focuses on developing concept of equivalent fractions. The web site is as follows:
http://nlvm.usu.edu/en/nav/frames_asid_105_g_2_t_1.html.
T2 focuses on developing fraction concepts. The web site is as follows:
http://www.interactive-resources.co.uk/mathspack1/equivfrac/fracdrag.html.
T3 focuses on developing the concept of comparing fractional size with circle picture, from:
http://nlvm.usu.edu/en/nav/frames_asid_274_g_2_t_1.html?open=activities.
T4 focuses on developing the concept of comparing fractional size. It was selected from
http://nlvm.usu.edu/en/nav/frames_asid_159_g_2_t_1.html.
T5 focuses on developing concept of equivalent fractions. It was from the following web site:
http://163.30.150.88/lii/flashMath/Games/arithmetic/ 分數_Length.swf.
T6 focuses on developing decimal concepts. It was from the following web site:
http://www.coolmath.com/decimals/03-decimals-expanded-notation.html.
T7 focuses on developing decimal meanings. It was from the following web site:
http://www.interactive-resources.co.uk/mathspack1/placeval/decimal.html.
T8 focuses on developing the concept of comparing three different decimals. It was from:
http://nlvm.usu.edu/en/nav/frames_asid_264_g_3_t_1.html.

T9 focuses on developing the concept of prime factors. It was from the following web site:
http://nlvm.usu.edu/en/nav/frames_asid_202_g_3_t_1.html.
T10 and T11 focus on developing fraction number sense.

These tools were used because 1) they can be used to implement the key purpose of the study, i.e. these tools can be used to help children develop number sense; 2) These tools are very convenient for students to manipulate; 3) they are free to download and use for students in mathematics classes; 4) the researchers do not need to spend a lot of time to design new interactive tools.

3.3 *Treatment*

We designed eight technology-integrated number sense activities for the teaching experiment. Each activity required two class periods of teaching. The activities were taught for four class periods (40 minutes per period) per week and continued for 4 weeks. The total engaged time was 16 class periods. Table 1 reports the instructional process for teaching the experimental class.

Table 1. The teaching activities, tools, and schedule for the Experimental Class

Schedule	Class periods	Teaching tools used in the class
Week 1		
Activity 1: Fraction concepts	2	T1 and T2
Activity 2: Comparing fractional size	2	T2, T3, T10, and T11
Week 2		
Activity 3: Residual strategy	2	T2, T10, and T11
Activity 4: Use the benchmark	2	T1, T2, T4, T10, and T11
Week 3		
Activity 5: Number line	2	T5
Activity 6: Decimal concepts	2	T6, T7, and T8
Week 4		
Activity 7: Comparing decimals	2	T9
Activity 8: Effect of operation on number	2	T2, T10, and T11

During the teaching, students were divided into 8 small groups and each group included 4 students with heterogeneous ability on mathematics. Students were encouraged to do small-group collaboration. Each group had a computer for students to access the materials. At the same time, the materials were also projected from teachers' computer as demonstration.

4. Data Collection and Analysis

Three different types of data were collected and analyzed in this study. They were as follows:

Web-based two-tiered test of number sense. The test was used as a pretest (before teaching) and a posttest (after teaching). Data were collected on-line. The test included 50 items. The scoring rules were: (1) If the answer was correct, then 1 point was given; if the selected reason was a number sense-based method, then 1 more point was given; if the selected reason was a rule-based method, then 0.5 points were given; if the selected reason was a misconception, then 0.25 points were given (children probably own part of knowledge even though he/she has misconceptions); if the selected reason was a guess, then 0 points were given. (2) If the answer was incorrect, then 0 points were given. The maximum score was 100 for each pretest and posttest based on these scoring rules.

Pre- and post-interviews. During the interviews, a number of specific probes were used, such as "Why did you do it that way?", "Can you tell me your reasons?", "Can you do it another way?" to help the interviewer investigate each student's number sense, as well as his or her thinking processes. Each interview lasted about 50 minutes and was conducted in a private schoolroom; the interviews were video taped and later transcribed.

The participants' responses were reviewed and sorted based on earlier studies (Markovits & Sowder, 1994; Yang, 2005). To identify the different strategies used by participants, each response was coded (as correct or incorrect) according to one of the following categories:

1. Number sense (NS)-based method: The participants' strategies fit definition of number sense.
2. Rule-based method: Participants could only use the written or formula method to solve a problem.
3. Wrong explanation: Participants did not understand the concept clearly, explained the strategies vaguely, or guessed the answer.

The participants' responses were reviewed by two raters independently. These initial reviews produced over 90% categorization agreement of the participants' responses. The remaining differences were reexamined and discussed by both raters until complete categorization agreement was reached.

Teaching artifacts. During the integration of technology into teaching, three different artifacts were collected and analyzed. (1) Video and audio of the teaching process was recorded to collect real teaching situations, the interactions between teacher and students, and students' responses. After these data were transcribed, we took them as references for subsequent teaching and analysis. (2) Students were asked to write on worksheets and in mathematics learning diaries. By analyzing students' worksheets and learning diaries, we could examine students' learning situations and teaching problems. (3) After every teaching activity, researchers wrote down teaching reflections, including students' responses and the appropriateness of the teaching materials.

5. Results

To help sixth graders develop number sense, this study embarks on integrating technology into number sense activities designed by the researchers. The interactive multimedia can easily transfer abstract concepts via pictorial representations which help children to understand mathematics concepts and engage students' learning motivation. Here, we report the process of teaching with these number sense activities to help children develop number sense via interactive multimedia. Based on the order of teaching events and the chronological order of data collection, we report the process according to three aspects: teaching process, mathematics learning diary feedback, and teaching retrospectives. Through describing the process of integrating technology into number sense activities, we can observe how students develop their number sense. We begin by describing the teaching process of one of the activities.

5.1 *Teaching Process (Activity Four: Use the Benchmark)*

Teaching goals:

1. Students can observe the existence of a benchmark.
2. Students can use benchmarks to solve fractional size problems.
3. Students can develop and use fractional benchmarks.

Several relevant websites and teaching tools, such as T1, T2, T4, T10, and T11, were used. For example, teacher can use the T5 (Figure 1) to help children make sense that $\frac{6}{13}$ is not greater than $\frac{1}{2}$.

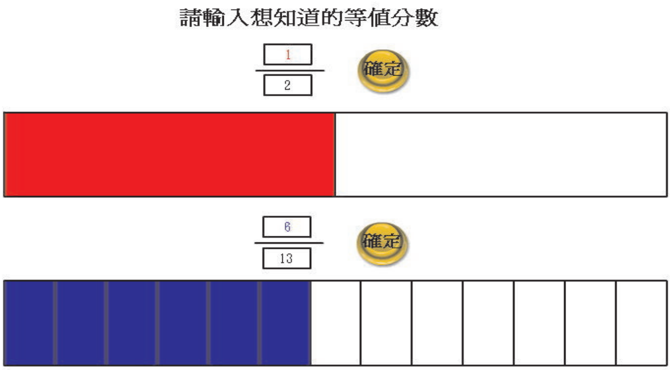

Figure 1. T5

Through manipulation of T5, students can compare the two fractions through pictorial representations. This will help children understand that $\frac{1}{2}$ can be used as an efficient benchmark to compare with the other fractions. The teacher also encourages the children to find out what half of 13 is, and helps the children develop their use of benchmarks abstractly.

5.1.1 *Introducing the Tools and the Problem*

The teacher began by showing the children how to manipulate the animation panels, and encouraged the children to use this software, engaging their learning motivation. Through the teacher's guidance and during the discussion with students, students could use the animation panels to get the answer. At the same time we could determine whether the students were interested in manipulating the animation panels. They were able to find out the results of comparisons quickly by using pictorial representations, which could enhance their conceptual understanding and confidence. While the students were manipulating the animation panels, the teacher asked, "What did you discover?" to encourage them to think and share ideas. In this way, the teachers could transform semi-concrete representations to abstract symbolic representation and build students' number sense.

5.1.2 *Launching a Worthwhile Mathematical Problem*

After the students were familiar with the manipulation of the animation panels, the teacher began the activity by posing the following problem:

> T: I am going to show you several fraction cards. Please order $\frac{6}{8}$, $\frac{12}{11}$, $\frac{15}{16}$, and 1, without using a written method and write down your reasons. You can use the animation panels.

After the teacher posed the problem, she asked if the students had any questions in order to make sure they understood the problem. If no questions were asked by the students, the teacher asked them to begin small-group cooperation and discussion.

5.1.3 *Small-group Cooperation and Discussion*

When each group started its discussion, the teacher asked the students to record their answers, strategies, and thoughts. While the groups were in discussion, the teacher could observe their behavior, explain unclear parts for them, and keep an eye on their discussion.

During the process of problem-solving, members of all the groups provided different thoughts to inspire each other. In the intense debate, students could find the best answer cooperatively. Below, we describe the discussions of some of the groups.

> The first group used figures to express the size of $\frac{15}{16}$ and $\frac{6}{8}$.
>
> 4S5: We can use animation panel 2 to find the answer.
>
> 4S24: This figure is too obvious!
>
> 4S13: You are better at drawing so you do it. (see Figure 2)

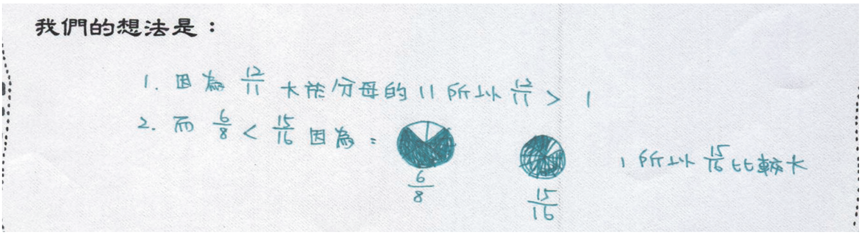

Figure 2. Problem-solving record of the first group

Students in the first group compared the fraction sizes by using animation panel 2. They were able to draw what they saw on the computer and use it to compare the fraction size. We can also see that the students were able to take 1 as a benchmark. As for the proper fractions, they chose to use figures to solve the problem.

In the third group, the students thought that according to what the teacher said they could not use a written method.

4 S25: The teacher asked us not to find a common denominator, so if we did, we are wrong.

4 S7: Our previous teacher always taught us how to find a denominator for this kind of problem. I think this one can be solved by the same way.

4 S6: The point is the teacher forbids us to find a common denominator.

4 S25: Actually, I know the teacher's intention, because the answer to this one can be easily found without finding a common denominator.

4 S8: Fine, you go ahead.

4 S25: $\frac{12}{11}$ $\frac{15}{16}$ $\frac{6}{8}$, this is easy.

4 S8: How do you know?

4 S25: $\frac{12}{11}$ 1 is the biggest; $\frac{15}{16}$ and $\frac{6}{8}$ are smaller then 1. $\frac{15}{16}$ only plus $\frac{1}{16}$ equals 1, but $\frac{6}{8}$ plus $\frac{2}{8}$ equals 1, and $\frac{1}{16}$ $\frac{2}{8}$. That's why $\frac{12}{11}$ $\frac{15}{16}$ $\frac{6}{8}$. We can use the animation panel 2 and panel 3 to show the above results.

4 S28: I got it! $\frac{2}{8}$ $\frac{1}{16}$, $\frac{1}{16}$ $\frac{4}{16}$, this one is really easy.

At first, students in the third group discussed whether they should find a common denominator or not. Because the teacher encouraged them to work without using paper-and-pencil, they started to resolve the problem without finding a common denominator. S25 used 1 as a benchmark, stated that $\frac{12}{11} \quad 1$ is the biggest, and used the distances between $\frac{15}{16}$ and 1 as well as $\frac{6}{8}$ and 1 to judge the fraction sizes.

The fifth group compared the fraction sizes by finding a common denominator.

> 4S27: I can solve this problem without using common denominator.
>
> 4S11: I think that the teacher looks down on us.
>
> 4S27: Just use mental computation! $\frac{12}{11} \quad 1$, $\frac{6}{8} = \frac{6\times2}{8\times2} \quad \frac{12}{16}$, so $\frac{15}{16} \quad \frac{12}{16} = \frac{6}{8}$.
>
> 4S12: You still found the common denominator.
>
> 4S27: Oh – never mind, the answer is correct.

The members in the fifth group only used the written method to find a common denominator. The number was not big and they had multiple relations, so it was easy to compare size by common denominator. Therefore, the teacher' original idea to let students take 1 as a benchmark could be missed. According to these observations, it is important to ensure that the teachers' design of the problem is carefully planned to influence the students' thinking. If we change $\frac{15}{16}$ and $\frac{6}{8}$ to $\frac{15}{16}$ and $\frac{12}{13}$, students will feel the numbers are too big to find a common denominator, which can inspire them to seek different problem-solving strategies.

5.1.4 *Whole-class Discussion*

After small-group cooperation, the teacher leads the whole class to share results with each other and let each group present their answer and problem-solving strategy. We report a portion of the whole-class discussions.

T: Which group would like to share your ideas?

T: (Several groups would like to share their answers): Fourth group please!

4S12: We can solve this problem without using a common denominator.

4S26: Just use the animation panel! $\frac{12}{11}$ 1, because $\frac{12}{11}$ is over 1. It is easy to see $\frac{15}{16}$ $\frac{6}{8}$ by the animation panel. (see Figure 3).

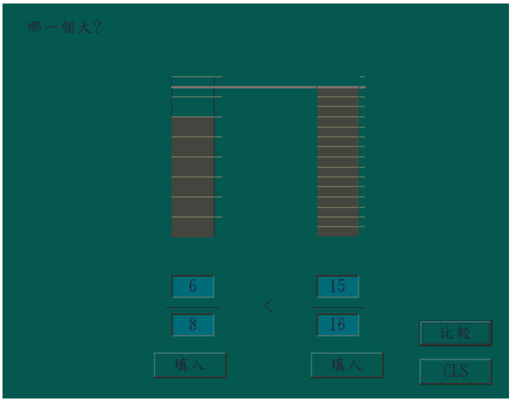

Figure 3. The fourth group used the animation panel 4 to solve problem

T: Very good! Do you understand?

S: Yes!

T: Who has a different method to solve this problem? Third group please!

S25: $\frac{12}{11}$ 1 is the biggest; $\frac{15}{16}$ and $\frac{6}{8}$ are smaller than 1. $\frac{15}{16}+\frac{1}{16}=1$, $\frac{6}{8}+\frac{2}{8}=1$.

However, $\frac{1}{16}$ $\frac{2}{8}$. Therefore, $\frac{12}{11}$ $\frac{15}{16}$ $\frac{6}{8}$.

The members in the fourth group used the animation panel to decide the answer. The pictorial representations helped them to compare fraction sizes easily. The students in the third group were also able to transfer concepts from the pictorial representation to a symbolic representation, including the use of 1 as a benchmark and a residual strategy to solve the problem.

5.1.5 *Posing and Discussing a Second Problem*

To check the students' understanding, the teacher posed a different problem to evaluate the students' responses: "Please write down a fraction which has to be smaller than $\frac{1}{2}$ but very close to it. At the same time, explain why your fraction is smaller than $\frac{1}{2}$."

Students in the fourth group took $\frac{1}{2}$ as a benchmark using the computer animation panel, and they listed the fraction that was smaller than $\frac{1}{2}$.

4S15: Everybody look at the figures A, B, C, D, and E on the webpage. Isn't it easy? (see Figure 4)

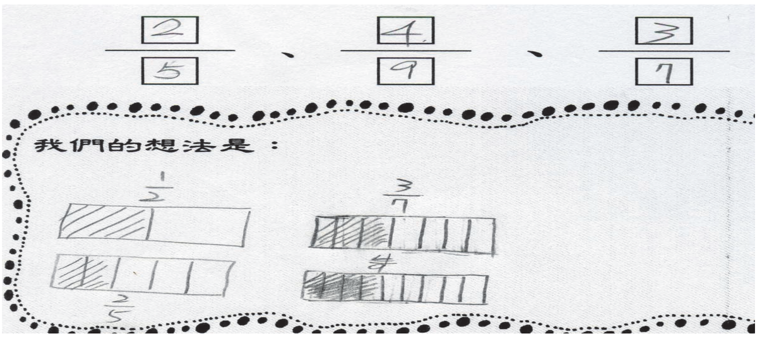

Figure 4. Problem-solving record of the fourth group

T: The answer of the fourth group is correct. It is clear that they used pictures to explain for everybody. What else can you discover with these figures? Who can share ideas with us; I will give a bonus point. (Many students raised their hands.)

The teacher tried to help students make sense of the meaning of $\frac{1}{2}$ from the pictorial representation to the symbolic representation and expected that they could build up their ability to use $\frac{1}{2}$ as a benchmark.

T: S30, can you tell us your idea?

S30: We discussed $\frac{24}{50}$ $\frac{1}{2}$ because half of 50 is close to 24 (see Figure 5).

We can see that the fourth group said half of 7 is 3.5 so they wrote down 3, which is $\frac{3}{7}$ $\frac{1}{2}$. Half of 9 is 4.5, so they wrote down 4, which is $\frac{4}{9}$ $\frac{1}{2}$.

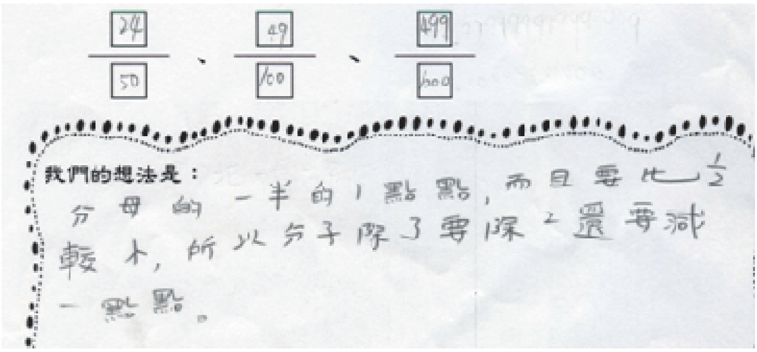

Figure 5. Problem-solving record of the third group

T: Very good! Can everybody understand what S30 means? What other thoughts do you have, except S30?

The teacher encouraged the students to develop different problem-solving strategies, and hoped they could develop another thinking method.

S13: Teacher! We have a different solution. We take 27–13=14 of $\frac{13}{27}$, so 14 13 (see Figure 6). That is why $\frac{13}{27}$ $\frac{1}{2}$. Just like $\frac{3}{7}$ of the first group, and they used 7–3 = 4, 4 3, $\frac{3}{7}$ $\frac{1}{2}$. We are better than them because we don't have to use division.

T: S13, can you explain why 7–3 4, 4 3, $\frac{3}{7}$ $\frac{1}{2}$?

S13: That is the solution I created.

T: Great! Do you know why you can make that decision?
S13: It is my own idea, even if I tell everybody, you won't understand.

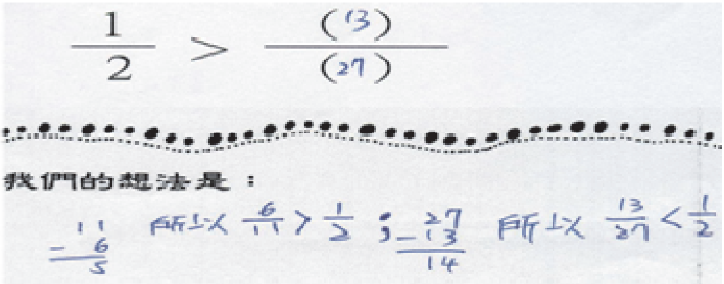

Figure 6. Problem-solving record of the first group

The teacher encouraged the students to present their own ideas, no matter if they were right or wrong, because the teacher could assess students' thinking by their sharing, and moreover could discover if they had any new conceptions or misconceptions. The teacher then summarized the discussion and complimented the students' on their bravery in expressing their problem-solving strategies. Meanwhile, the teacher was looking forward to cultivating the students' individual thinking abilities.

5.1.6 *Posing another Problem to Evaluate Students' Understanding*

The teacher hoped that the students could use $\frac{1}{2}$ as a benchmark to solve problems. Therefore, the teacher used another problem to test whether the students could apply the concept fluently. Moreover, the teacher wanted to strengthen the students' number sense and detect any misunderstandings. After the teacher assigned the problem, each group started to discuss it with the expectation that they would then have to express their ideas. The teacher hoped the students would find the answer through critical thinking and discussion.

T: Please compare $\frac{4}{7}$ $\frac{1}{2}$ $\frac{2}{5}$ 1, without finding the common denominator.

We describe the discussions of the first and second groups.

Second group

T: Who would like to share your idea? … S9.

S9: Using division is not convenient for us to solve this problem. It is easy for us to use the difference of numerator and denominator to find the answer.

T: Can you be more specific?

S9: The difference between $\frac{4}{7}$ and 1 is $\frac{3}{7}$; the difference between $\frac{1}{2}$ and 1 is $\frac{1}{2}$; and the difference between $\frac{2}{5}$ and 1 is $\frac{3}{5}$. $\frac{3}{7}<\frac{1}{2}<\frac{3}{5}$, so $\frac{4}{7}>\frac{1}{2}>\frac{2}{5}$. (see Figure 7)

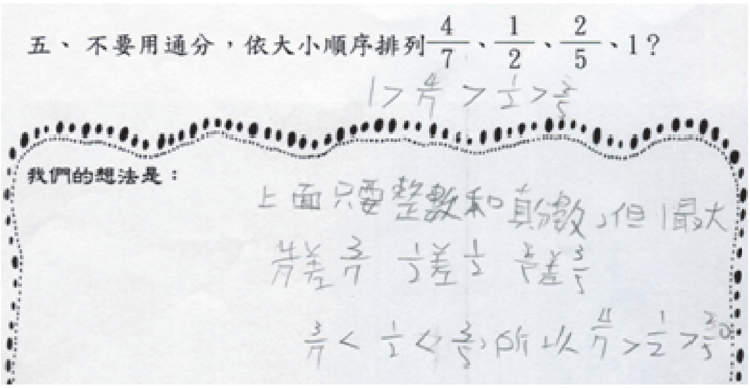

Figure 7. S9's explanation

T: How do you know $\frac{3}{7}$ $\frac{1}{2}$ $\frac{3}{5}$?

S32: Last class, we learned the concept, "When the numerators are the same, the bigger the denominator is, the smaller the fraction is".

T: How do you know when the numerators are the same, the bigger the denominator is, the smaller the fraction is?

S32: It's very easy to use the animation panel to show it. As you can see in the pictures, when the numerator is same, the bigger the denominator is, the smaller each piece is. Therefore, $\frac{3}{7}$ $\frac{3}{5}$.

T: Alright! Now we know $\frac{3}{7} \quad \frac{3}{5}$, but how about $\frac{1}{2}$?

S32: We know that $\frac{1}{2} = \frac{3}{6}$, so $\frac{3}{7} \quad \frac{3}{6} \quad \frac{3}{5}$.

S9: 3 is not over a half of 7 and 3 is over a half of 5, therefore $\frac{3}{7} < \frac{1}{2} < \frac{3}{5}$.

T: Very good! Any questions?

First group

S13: We have a different idea.

T: You can share it with us.

S13: By judging whether the numerator is the half of denominator, we can know that 4 3.5, which is half of 7, so $\frac{4}{7} \quad \frac{1}{2}$ and 2 2.5, which is half of 5, so $\frac{2}{5} \quad \frac{1}{2}$.

T: The first group explained it very clearly. Does anyone have a question?

The teacher and students listened to the first group's problem-solving strategy together, and the teacher encouraged other students to ask questions. Students in the first group were able to take half as a benchmark to compare fractional size, and they used the pictures in their presentation, which shows that these students were able to transform their thinking from a pictorial representation and oral representation to a symbolic representation.

Based on the above, we can see that the students can make sense of fraction size by using pictorial representations via animation panels, and then efficiently compare fraction sizes by different approaches. Also, we see that students have various ways of thinking, and sometimes there are unexpected responses.

5.2 *Feedback from the Mathematics Learning Diaries*

In the learning diaries, students were able to reflect their feelings and reactions directly. Here we describe some results from the learning feedback students provided in their diaries. We can see that many students learned how to use one and half as benchmarks to compare fraction sizes. Of the 32 students, 31 responded in their diaries that they knew how to use the benchmark to compare fraction size. For example:

> S25: When comparing fraction size, I do not use the common denominator all the time, I can use whether numerator is the half of denominator or not to compare fraction size.
>
> S31: Now I know when comparing size, I can cut fractions in half. When the fraction is 0.5 more than the half one, it is bigger and when the fraction is 0.5 less than the half one, it is smaller.

All the students liked the teaching method of integrating information technology into the mathematics class and responded in their diaries that they liked using the computer in mathematics class. For example,

> S11: Every time before math class, I will expect its [computer] coming because the way the teacher explains makes me understand quickly and clearly.
>
> S26: Using the computer for math class is better than using a blackboard.
>
> S1: It was fun today, and I hope the teacher can always use the same way [use the computer] to teach math.
>
> S30: It is really interesting because we can see the pictures and manipulate the software.

Thus, by looking at the students' learning diaries, we observed that many students learned to take half as a benchmark to compare fraction size. In addition, all the students liked the teaching method of integrating information technology because they liked to operate the program by themselves. It is clear that integrating information technology into number sense teaching not only promotes students' number sense but increases their interest in mathematics.

5.3 *Reflections and Improvement of Teaching*

5.3.1 *A Review of the Teaching Material*

Most of the students were able to accept 1 and $\frac{1}{2}$ as benchmarks to compare fraction sizes and they were able to judge the correctness of the answer quickly. However, there were difficulties in the problem posed by the teacher. First, the denominator was not big enough, as in $\frac{6}{8}$ and $\frac{15}{16}$. Second, 8 and 16 are trivially related by multiplication, which allowed students to easily find the common denominator. Both these factors did not encourage students to use the benchmark. Clearly, the appropriateness of the problem is key to the effectiveness of teaching.

For those students who had a low level of mathematics performance, dividing an odd number by 2, such as 13÷2, 17÷2, was hard. So it was a little difficult for them to learn to use $\frac{1}{2}$ as a benchmark. However, S13's problem-solving strategy (subtracting the numerator from the denominator to judge the number size, such as $11-6\quad 5\quad 6\quad 5$ and then $\frac{6}{11}\quad \frac{1}{2}$ or $\frac{6}{11}\times 2$ $\frac{12}{11}\quad 1$ which means $\frac{6}{11}$ is bigger than half) is a different approach. Although this kind of strategy cannot help students detect what a benchmark is, for those students with poor mental computation it provides an alternate way to learn how to compare fraction sizes.

5.3.2 *Reflection on Teaching Skill*

Students can literally see the difference in fraction sizes by using a pictorial representation. However, although it may be easy for the teacher to see this, the connection may be vague for some students because they do not understand the basic concept of fractions and thus find it hard to transform a pictorial representation to symbolic representation and thinking. For example, S23 only remembered that "the bigger the denominator is, the smaller the fraction is" without knowing the meaning.

Students were very excited about using computer animation, but aside from the excitement, did students learn from it? Did they think more when they saw the result? Or was it only interesting for them to play with computer animation? The teacher has to ask questions to engage students' thinking if students are to build up the concept of benchmarks after their experiences manipulating the computer animation. Therefore, the teacher can give hints and help students to raise their thinking ability appropriately.

5.4 *Differences for the Interviewed Students on the Use of Number Sense Strategies before and after Instruction.*

Table 2 reports the interview results for the students in the three different levels when responding to number sense related questions. Five of the students made progress on correctness and use of a number sense-based method (L2 was the exception): 11 times to 16 times for H1, 10 times to 15 times for H2, 3 times to 7 times for M1, 5 times to 12 times for M2, and 2 times to 6 times for L1. However, the use of NS-based methods did not change after the instruction for the L2.

Table 2. Strategies used by students before and after instruction

Strategy	High-level				Middle-level				Low-level			
	（H1		（H2		（M1		（M2		（L1		（L2	
	Pre-	Post-	Pre-	Post-	Pre-	Post-	Pre-	Post-	Pre-	Post-	Pre-	Post-
Correct NS-based	11	16	10	15	3	7	5	12	2	6	3	3
Rule-based	4	2	2	2	3	2	3	4	0	4	2	2
Wrong	1	0	2	0	3	3	2	2	3	4	1	2
Incorrect	4	2	6	3	11	8	10	2	15	6	14	13

In an effort to better understand the changes in the strategies and the thinking used by these students, we analyzed their responses; these responses are discussed below.

5.4.1 *The Changes of H1 after Instruction*

The use of NS-based methods increased from 11 times to 16 times, the use of rule-based method decreased from 4 times to 2 times, and the incorrect responses decreased from 4 times to 2 times for H1. This indicates that H1 made progress on number sense after the instruction. For example, on question 5-5 H1 tried to make a guess to solve the problem before the instruction:

R: Q5-5: Without using paper and pencil, which answer of the following is greater than 1?

(a) $\frac{7}{15}\quad\frac{8}{17}$ (b) $\frac{19}{14}-\frac{6}{11}$ (c) $\frac{16}{17}\quad\frac{1}{16}$ (d) $\frac{14}{13}-\frac{1}{12}$

H1: I guess the answer is $\frac{7}{15}\quad\frac{8}{17}$.

R: Why?

H1: I don't know the answer without using paper-and-pencil. Therefore, I guessed.

After instruction, H1 applied 1 and $\frac{1}{2}$ as benchmarks to solve problem:

H1: The answer is $\frac{16}{17}\quad\frac{1}{16}$.

R: Why?

H1: Because $\frac{16}{17}$ plus $\frac{1}{17}$ is equal to 1 and $\frac{1}{16} > \frac{1}{17}$. Therefore, $\frac{16}{17}\quad\frac{1}{16} > 1$.

R: Why is the answer not $\frac{7}{15}\quad\frac{8}{17}$?

H1: Because $\frac{7}{15} < \frac{1}{2}$ and $\frac{8}{17} < \frac{1}{2}$. Therefore, $\frac{7}{15}\quad\frac{8}{17}$ should be less than 1.

Thus, H1 could not use the NS-based method to solve the problem before the instruction. However, H1 was able to apply the NS-based method to solve the problem after the instruction. This implies that the technology integrated into the instruction had a positive effect on the number sense development of H1.

5.4.2 *The Changes of H2 after the Instruction*

The use of NS-based methods increased from 10 times to 15 times, and the incorrect responses decreased from 6 times to 3 times for H2. This indicates that H1 made progress on number sense after the instruction. For example, consider H2's responses to question 2-2. H2 used guessing to solve the problem before the instruction:

R: <u>Without using paper and pencil</u>, determine which of the following is the best estimate of the product of 246×0.512?

(a) Greater than 123 (b) Less than 123 (c) Greater than 246 (d) Can't tell

H2: Less than 123.

R : Why

H2: Without using paper-and-pencil, I don't know how to find the answer. Therefore, I guessed.

Before the instruction, H2 could not find the answer without using paper-and-pencil. Therefore, he randomly guessed the answer. However, after the instruction, H2 was able to apply a benchmark to reasonably decide the answer:

H2: Greater than 123.

R : Why?

H2: Because 0.5 is a half, a half times 246 is equal to 123. 0.512   0.5, therefore the product of 246×0.512 should be greater than 123.

After the instruction, H2 applied 0.5 as a benchmark to solve the problem. This shows the improvement of H2 on the use of NS-based methods after the instruction.

5.4.3 *The Changes of M2 after the Instruction*

The use of NS-based methods increased from 5 times to 12 times for M1 and incorrect responses decreased from 12 times to 2 times after the instruction. M1 originally used a rule-based method to solve the problem before the instruction:

R: Can you tell me your answer for the following question:

Q1-2: Which of the following descriptions about decimals is correct?

Arti: There are more decimals between 2.1 and 2.4 than between 3.7 and 3.9.

Bei: The four basic operations of two decimals will also produce an answer with a decimal.

M2: The answer is Arti.

R: Why?

M2: Because there are two decimals between 2.1~2.4. They are 2.2 and 2.3. However, there is only one decimal 3.8 between 3.7 – 3.9. In addition, 0.5 + 0.5 = 1, so Bei is wrong.

Before the instruction, M2 did not make sense of the density of decimal numbers, however, he could show an example to prove that the statement of Bei is wrong. After instruction, the response of M2 to Q 1-2 was different:

R: Can you tell me your answer about Q1-2?

S: Both Arti and Bei are wrong.

R: Why?

S: There are many decimals between 2.1 and 2.4, such as 2.11, 2.12, …. ; 2.21, 2.22….. There are many decimals between 3.7 and 3.9, such as 3.71, 3.72, … 3.711, 3.712,….

This evidence shows that M2 has developed a good understanding of the concept of density about decimals. It is apparent that teaching has changed the thinking of M2 and affected his understanding of the concept of decimal.

5.4.4 *The Changes of L1 after the Instruction*

The use of NS-based methods increased from 2 times to 6 times for L1 and incorrect responses decreased from 15 times to 6 times after the instruction. L1 used the rule-based method to solve problems before the instruction:

R: Can you tell me the answer of Q4-1: Which of the following four options produces the largest answer? Please answer without using paper and pencil?

(a) 167×0.8 (b) 167÷0.8 (c) 167+0.8 (d)167-0.8

L1: I need to use paper-and-pencil to find the answers.

R: Without using paper and pencil, can you estimate the answer?

L1: 167+0.8. Because 0.8 is less than 1, the result of 167×0.8 should get smaller. The division usually makes the result smaller, so the result of 167 ÷ 0.8 should be smaller.

Before the instruction, L1 knew that when one multiplies a number times another number which is less than 1, the result should be less than the original number. However, L1 had a misconception about division. Li believed that division usually makes the answer smaller. However, after the instruction, L1 was able to make sense of the meaning of division:

R: Can you tell me the answer of Q 4-1?

L1: The answer is 167÷0.8.

R: Why?

L1: Because the result of 167×0.8 becomes smaller, 167+0.8 167.8 increases a little, the result of 167-0.8 becomes smaller.

R: Why the answer is 167÷0.8?

L1: 167 divided by 0.8 is larger than 167 a lot due to 0.8 is less than 1.

This shows that L1 was able to make sense of the meanings of the four operations and the effects of operations on numbers. This implies that the teaching was helpful to L1.

6. Discussion and Conclusion

The results we have described of integrating interactive multimedia into number sense teaching shows that the computer animation panels not only are effective tools to help children make sense of the meanings of fraction and decimal, but also can promote children's use of number sense-based methods efficiently. This supports earlier studies and documents that technology has a positive effect on helping children learn mathematics (Dick, 2007; Inamdar & Kulkarni, 2007; NCTM, 2000; Ruthven, 2007; Vulis & Small, 2007; Zbiek et al., 2007). In addition, the results also show that children are interested in and feel comfortable using the computer animation panels to help them solve problems in mathematics class: "It is fun. I feel that it not only can let us operate animated program but also let us know other students' thinking." (S23); "It is funny and interesting. I hope we can have every math class like this." (S22); and "I like the teacher using the computer rather than the blackboard." (S26). This finding is consistent with earlier studies that have shown that technology has a positive impact on mathematics learning (Lin, 2008; Isiksal & Askar, 2005; Olkun, Altun, & Smith, 2005; Yang & Tsai, 2010).

Why was teaching number sense via interactive multimedia in this Taiwanese primary classroom effective? There are several reasons that contributed to this result. First, the teacher played several key roles in helping children develop number sense. As noted in the study of Yang (2006), "the teacher plays an important role in the creation of good learning environment, which encourages exploration, communication, and reasoning" (p. 109). The teacher in this study knew how to create a good learning environment to help children develop number sense. The teacher in this study knew when and how to appropriately integrate the technology into mathematics teaching and learning. She not only knew how to encourage her students to have small-group cooperation and whole-class discussion, but also knew how to lead her students to communicate and share their problem-solving ideas during whole class discussion. Teachers and students will have a richer teaching and learning environment by using technology. Through group discussion and appropriate guidance from the teacher, students can be encouraged to think more deeply about problems which will help them develop number sense ability. Furthermore, the teacher asked her students to write diaries after each class. This helped the teacher to understand the students' learning problems and their thinking about the integration of technology into the mathematics class. Diary-writing can be a useful tool in helping teachers to revise their teaching methods (Yang, 2005). Again, this finding supports the results of earlier studies that showed the important role that teachers play in helping children develop number sense (Yang, 2006; Yang, Hsu & Huang, 2004). Moreover, the teacher in this study also knew how to lead children to learn the transition from pictorial representations to symbolic representations. For example, children used the computer animation panels to order $\frac{4}{7}$, $\frac{1}{2}$, and $\frac{2}{5}$ easily. However, the teacher knew that it was important to lead the children to make sense of the relationship between the pictorial representation and the symbolic representation. Therefore, she encouraged her students to solve problems by using different approaches than the computer and the rule-based method. With this encouragement from the teacher, some students were able to discover the relationship between the numerator and denominator and apply the benchmark to solve problems, as in the case of S13:

> S13: By judging whether the numerator is the half of denominator, we can know that 4 3.5, which is half of 7, so $\frac{4}{7}$ $\frac{1}{2}$ and 2 2.5, which is half of 5, so $\frac{2}{5}$ $\frac{1}{2}$.

Thus, the students in this study were able to make sense of the relationship between pictorial representations and symbolic representations, which formed a good basis for helping them develop number sense. This finding lends support to the recommendations from several earlier studies and documents that the flexible use of multiple representations is an important indicator of mathematical understanding and that flexible transfer between different representations can help students to develop advanced mathematical understanding (Brenner, Herman, Ho, & Zimmer, 1999; Dreyfus & Eisenberg, 1996; NCTM, 2000; Yang & Huang, 2004). Indeed, representations are powerful tools that can enhance mathematical thinking, understanding and learning which is widely recognized (Fennell & Rowan, 2001; NCTM, 2000; Yang & Huang, 2004).

A second factor in the effectiveness of the number sense instruction in this study was the use of the computer animation panels. The power of the animation panels lies in their ability to allow children to easily manipulate pictorial representations which can help them understand the mathematical concepts. In particular, the pictorial representations made by the students via the computer not only allow them to more easily make sense of the meanings of fractions and decimals, but also help them to see one and half as benchmarks for comparing fraction sizes. Most students can accept this method. It can enhance students' impressions with pictorial representations which can be used to judge the fraction size easily. This is consistent with the study of Olive and Lobato (2008), which found that technological tools which support children to enact their own mathematical thinking can be a very powerful instrument to support children's mathematics learning, and with the assertion that technology provides dynamic visualization, immediate feedback, and interactivity that can help children learn mathematics effectively (NCTM, 2000). In the traditional mathematics classroom, the teacher usually teaches students knowledge directly from the textbook, for example asking them to memorize the formulas and operations. However, textbooks often focus on rules and procedures, telling students to find the common denominator to compare fraction size, or that when multiplying decimals, to locate the decimal point in the product by counting the multiplicand's decimal place and the multiplier's decimal place. A great deal of research has demonstrated the drawbacks of memorizing rules for developing students' understanding and number sense (Cai, 2001; Yang, Hsu, & Huang, 2004). The integration of technology into number sense teaching in this study emphasizes that various animation panels from multimedia can help children make sense of mathematical concepts effectively. This further supports the findings of earlier studies that technology can provide dynamic visualization, immediate feedback, and interactivity that can help children learn mathematics effectively (Moyer, Niezgoda, & Stanley, 2005; Olive & Lobato, 2008; Suh & Moyer- Packenham, 2007).

Third, the use of computer animation panels in this study supported the development of number sense by promoting positive student mathematics learning attitudes. Almost all of the students in this study liked using the computer and enjoyed manipulating the computer animation panels in mathematics class, making statements such as "Using the computer for math class is better than using the blackboard", "It is really interesting because we can see the pictures and manipulate the software", and "It is funny and interesting. I hope we can have every math class like this." This kind of learning environment strongly promotes positive learning attitudes, and is very different from the traditional teaching method and learning environment. This is consistent with the earlier studies showing that integrating technology with mathematics teaching and learning can result in better attitudes towards mathematics learning than traditional teaching without using technology (Isiksal & Askar, 2005; Lin, 2008; Olkun, Altun, & Smith, 2005).

The interview results from this study show that the students tended to use rule-based methods, memorizing the rules to solve problems or giving incorrect answers before the technology was integrated into the number sense instruction. This echoes previous studies around the world that have found that elementary school children perform poorly on number sense and are weak in the use of number sense-based methods to solve problems (Menon, 2004; Reys et al., 1999; Reys & Yang, 1998; Yang, 2005; Yang & Li, 2008; Yang, Li, & Lin, 2008). However, the students interviewed for this study, with the exception of L2, greatly increased their use of number sense-based methods (e.g., understanding the basic meaning of numbers, recognizing the magnitude of numbers, being able to use multiple representations, recognizing the relative effect of an operation on numbers, and judging the reasonableness of computational results) and decreased the use of rule-based methods and inaccuracy after experiences the technology-integrated number sense instruction. This is consistent with earlier studies that found that integrating technology into the mathematics class can help children develop better understanding of mathematical concepts (Bennison & Goos, 2010; Dick, 2007; Inamdar & Kulkarni, 2007; NCTM, 2000; Ruthven, 2007; Vulis & Small, 2007; Zbiek et al., 2007) and number sense (Yang & Tsai, 2010). In addition, our results also show that students' responses after the experimental instruction tend to be more flexible with respect to thinking about the concepts of fraction and decimal and using benchmarks (e.g., 1 and 1/2) efficiently. However, one of the low level students (L2) did not make any progress on the use of number sense-based methods and inaccuracy after the instruction. This reflects the findings of earlier studies that some low performers in mathematics do not improve under instruction without the use of technology as well (Yang, Hsu, & Huang, 2004). The lack of progress shown by L2 is likely due to a weak understanding of the basic meanings of fraction and decimal. Even though L2 can use the computer animation panels to make fractions, it is still very difficult for him to transfer from pictorial representations to symbolic representations. Students who do not have a profound understanding of basic concepts and meanings will almost certainly be unable to develop good number sense. L2 likely needs more remedial instruction and learning opportunities to build his understanding of mathematics.

In conclusion, there are three major contributions of this study to mathematics education research worldwide. Firstly, the integration of interactive multimedia into mathematics classroom can effectively promote students' number sense. In addition, number sense not only has been internationally considered to be a key topic in mathematics education (NCTM, 2000; Verschaffel et al., 2007; Yang, 2013), but also it is a key predictor of students' mathematics performance in the future learning (Jordan et al., 2007; Jordan et al., 2010; Yang et al., 2008). Especially, few practical studies show that children's number sense can be promoted via interactive multimedia materials. It will encourage more studies to integrate interactive multimedia into the learning of number sense. Secondly, this study shows that children's learning interest and motivation can be promoted via the manipulation of interactive materials. They can freely access the computer to discuss and share their ideas via the manipulation of interactive multimedia. Thirdly, this study shows that the interactive multimedia is a powerful tool to help teacher's instruction. Especially, the teacher can demonstrate mathematical concepts via pictorial representations which is the most powerful tool of computer as compared with the traditional board cards.

In addition, there are three elements in this work that would distinguish it from research done in the west. First, in Taiwan, teachers need to follow the competence indicators of national mathematics curriculum guideline when teach mathematics in the classrooms. Therefore, the goals of teaching materials used in this study have to follow the competence indicators of national mathematics curriculum guideline for six graders. Second, parents usually believe that enhancing their children's ability on written computation is very important in Taiwan. Therefore, teacher in the study need to persuade the parents to agree the use of interactive multimedia materials in the class before the teaching experiment. Teacher need to tell children's parents that the use of technology will help their children's mathematics learning. Third, the web-based two-tier testing system and two of the tools (T10 and T11) were designed by the researchers in this study. The testing system and tools are designed in Chinese contexts. Therefore, they can be friendly used by students in Taiwan.

Future research should include a longitudinal study, with different grade levels, more students, and different teachers and researchers invited to join the teaching experiment. Some questions to be considered in future studies are: a) How can we help children develop number sense efficiently through technology?; b) Do students retain what they have learned regarding number sense 1–2 years after their experience using computer animation panels? c) Do students perform better on mathematics achievement or standardized tests after using computer animation panels to learn number sense?

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Chapter 9

Teaching Geometrical Theorems in Grade 8 Using the "*Shen Tou*" Method: A Case Study in Shanghai

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The teaching of geometry, especially the teaching of proof in geometry, is central to mathematics education in China at the lower secondary school level. This chapter uses a case study of how an expert mathematics teacher in Shanghai taught geometrical theorems to a class of Grade 8 students to illustrate the "*Shen Tou*" ("permeation") method of teaching the initial stages of plane geometry. Comprising a set of teaching strategies, the "*Shen Tou*" method aims gradually to develop the multiple layers of reasoning skills required in geometry, especially the skills to use geometrical language in writing proofs.

Keywords: teaching of geometry, "*Shen Tou*" method, mathematics classroom in Shanghai

1. Introduction

Proof and proving are central to geometry, just as they are central to mathematics. However, there are many reports of teachers finding proof in geometry to be a challenging subject to teach well, and of students having significant difficulties in learning (e.g., Arsac, Balacheff, & Mante, 1992; Herbst, 2002; Herbst & Brach, 2006; Knuth, 2002; Lampert, 1993; Schoenfeld, 1988, 1989, 1994; Senk, 1985, 1989; Usiskin, 1982). For instance, Lampert (1993) shows that in doing geometrical proof, students conceive proof as a procedure whereby they must fit together—like a puzzle—the pieces of knowledge they have about the concepts involved to generate the desired sequence of steps. Jones (2000) points out that students are unable to distinguish between different forms of mathematical reasoning, such as explanation, argument, verification and proof as the curriculum emphasizes not on the wider reasons for and forms of proof, but on the format of the result. Herbst et al. (2009) shows that it is hard for students to learn to distinguish between what appears to be true and what they can justify as true based on reasons.

There is a growing research focus on the culturally- and socially-situated mathematics classroom, in particular on the significant role of teachers in developing students' views on proofs and their capabilities in proving. For instance, Heinze and Reiss (2004) show that the typical proof process in the mathematics classroom is planned and controlled by the teacher (e.g., by asking short questions and giving hints). Heinze et al. (2008) show that two quite different instructional approaches practiced by teachers in different cultures (one being Taiwan, the other being Germany) could have different advantages and limits on learners with different levels of achievement on constructing a multi-step proof. Martin et al. (2005) point out that the teacher can draw students into the action of class-negotiated conjecture development and proof construction, and thereby provide the students with the opportunity to learn the rules of the 'game' (of formal proof development) by playing the game, rather than by watching others play. Jones and Herbst (2012) urge to make explicit the role of the teacher in teaching proof and proving in the mathematics classroom, especially in terms of the teacher's part in the teacher-student interactions in the context of mathematics teachers' day-to-day instructional practice.

In our early studies of the Shanghai (SH) classroom (Ding & Jones, 2007, 2009), we noted how the SH expert teacher focused carefully on leading students to experience the systematical network of theorems in constructing a proof through a sequence of well-designed, though demanding, multi-stepped exercises. Based on our study of the SH teachers' classes, we proposed a pedagogical framework (see Ding, 2008) to account for alternative instructions to the van Hiele-based instructional model of teaching in geometry (for the latter, see Geddes, Fuys, & Tischler, 1984). In this chapter, we focus on analyzing and interpreting how the SH expert teacher taught geometrical theorems to a class of Grade 8 students (aged 13-14 years old) from the perspective of Chinese teachers' day-to-day instructional practice. In particular, we address the following question:

How does the teacher's use of the "*Shen Tou*" (渗) ("permeation") method help students improve the multiple layers of reasoning skills required in geometry, especially the skills to use appropriate geometrical language in writing proofs?

The data analysis is in the form of a case study of a sequence of twelve lessons given over three weeks by an expert teacher on the topic of the family of quadrilaterals (parallelogram, rectangle, rhombus and square). In the sequence of lessons, the teacher carefully employed the "*Shen Tou"* method gradually to develop the multiple layers of reasoning skills in geometry, especially the skills to use geometrical language in writing proofs. Before doing so we provide some background to the teaching and learning of plane geometry in China.

2. Teaching and Learning of Plane Geometry in China

Xu Guangqi and Matteo Ricci first introduced *Euclid's Elements* into China in 1607. Since then, this has enabled people in the country to gain insight into the features of logic, rigor, and abstractness of the axiomatic system of western mathematics (Tian, 2001; Yang, 2000; Zhang, 2005a). Plane geometry, which retains the axiomatic system of the first six chapters of *Euclid's Elements* (e.g., the concepts of point, line, plane and angle, triangles, the position relationship of straight lines (e.g., parallel, vertical), quadrilaterals, circle, similarity and the areas of various shapes, etc.), has long been the core of Chinese school mathematics curricula and textbooks (Zhang, 2006).

This approach to plane geometry was initially introduced to secondary schools in the country in 1905 through a Chinese translation of Japanese textbooks. These plane geometry textbooks (known as 3S after the three American mathematicians Schultz, Sevenoak and Schuyler who authored *Plane and Solid Geometry* in 1901) became used in schools from 1930 to 1950. During the 1950s, the axiomatic system of geometry, together with the rigor of proof, was, emphasized in school textbooks due to the heavy influence from the former Soviet Union.

After the 1950s, the amount of geometrical content in the textbooks was generally reduced from what was common prior to then (Zhang, 2006).

A significant change in the amount of plane geometry was made in the school mathematics curriculum syllabus (SMCS) published in 1963. The degree of difficulty of the examples and exercises in the textbooks was largely reduced. The SMCS in 1963 highlights that

> "…geometry at the secondary school level is different from Euclidean geometry as a science. It should not and it is impossible to teach geometry in terms of the rigorous axiomatic system of Euclidean geometry. However, in order to help students more systematically to understand geometrical knowledge, and to cultivate their capability to prove, the rigor of logic should be stressed as much as students are able to appreciate." (Zhang, 2006, p. 7)

Up to 2001, apart from the Cultural Revolution (during the period 1966-1976), the school geometry curriculum and textbooks remained rooted in the 1963 SMCS.

In 2001, another significant change of plane geometry was made in the Mathematics Curriculum Standards (MCS) (Ministry of Education [MOE], 2001). The teaching of geometry, and, in particular, the teaching of geometrical proof, received reduced emphasis. The term 'Geometry', for instance, was replaced by the term 'Space and Shape', which chiefly concerned "the shapes, sizes, position relationship and transformation of objects in the practical world and in geometry" (MOE, 2001, p. 11). However, the 2001 MCS was the subject of considerable criticism from a range of leading mathematicians and mathematics educators, as well as teachers, from across the country (Zhang, 2005b). Zhao (2005, p. 219) summarised the criticisms:

> "The New Math Curriculum has been sharply criticized for betraying an excellent educational tradition, sacrificing mathematical thinking and reasoning for experiential learning, giving up disciplinary coherence in the name of inquiry learning, lowering expectations in the name of reducing students' burden, and causing confusion among teachers and students".

Apart from the criticism from schools and from academics, the MCS in 2001 also drew serious concerns from the wider society in the country. For instance, one of the leading mathematicians in the country, Professor Jiang Boju of Beijing University, who at the time was also a member of

Chinese People's Political Consultative Committee of the country, was interviewed by the *Guangming Daily* (one of the largest official media of China). Jiang gave the following opinion about the usefulness of plane geometry as a means of developing students' mathematical thinking.

> "Many concepts in plane geometry may look simple, yet they must be taught thoroughly. … The simplest concepts are usually the most essential basics in mathematics. Mathematical thinking is scientific spirit. That is, in mathematics, the amazing outcome is usually attained by first mastering the simple concepts and then by establishing thinking system and finally by proving. …" (quoted in Cai, 2005)

Similarly, in an interview with the influential Chinese mathematician Shiing-Shen Chern, he expressed the view that:

> "Students should learn Euclidean (plane) geometry. … because they could then experience the power of reasoning even in a simple situation. … Ordinary students get use to do calculation, but not used to such a way of reasoning in geometry. However, Euclidean geometry should not be deleted just because students have difficulty in learning. On the contrary, it needs to be taught well in order to help students overcome such difficulty…" (Li, 2005, p. 2).

In 2011 the term 'Space and Shape, was changed again in the MCS; this time to 'Shape and Geometry' (MOE, 2011). Thus, an attempt was made to retain the basic axiom-theorem-deductive feature of plane geometry in the 2011 MCS.

Indeed, for many mathematicians, mathematics educators and teachers in the country, plane geometry, taught well, is considered as an effective means of developing students' capability in important mathematical reasoning skills. Over several decades, mathematics educators in China have been engaged in conducting classroom experimental studies for improving geometry teaching and learning (e.g., Gu, 1981; Qingpu County Teaching Reform Experiment, 1991; The Editorial Board, 1992; Yang, 1988). Such studies show that well-designed instruction could enhance students' learning interests and help them to overcome difficulties in plane geometry. In the next section, we discuss further some of the relevant studies by researchers who mostly live in the country.

At this point it is worth noting some interesting findings of students' learning performances in plane geometry in two large-scale national surveys conducted before the MCS in 2001. One finding comes from the national survey led by East China Normal University in 1987 (Tian, 1990). The study sample included 605 secondary schools with 49, 603 Grade 9-10 students (students aged 14-16) from 250 counties across the country. It was found that students' learning performance in geometry was better than that in algebra in the general mathematics test (in geometry, the rate of students' correct answers was 70%, whilst in algebra the rate was 65%). Another finding comes from the first national survey in China of the quality of the compulsory education over the period from 1992 to 1994 (Xie & Tan, 1997). The study sample included 201 schools with 12, 888 Grade 9 students across eleven provinces and cities in the country. They reported that the number of students who attained the full marks in geometry (in total 2, 003 students) was more than those in algebra (in total 1, 424 students), though the rate of students' correct answers in algebra was higher than that in geometry. Even so, for nearly 10% of students their learning attainment was fairly low. In view of such findings, Zhang (2005) drew educators and teachers' attention to the three reasons: 1) there was a large learning gap between the best and the weakest students in plane geometry; 2) plane geometry is more difficult than algebra for average students at the lower secondary school level; 3) the best students generally have good attainment in plane geometry.

In addition, it is important to note the intensive examination-driven culture of classroom teaching and learning in China. As a matter of fact, even though the degree of difficulty of the examples and exercises in the textbooks has been reduced over the recent curriculum reforms, the degree of difficulty in the standard examinations has remained consistently high. Thus, students have to do much more difficult exercises than those in the textbooks to achieve well in the standard examinations. In Figure 1 is an example from the collection of the geometry test items from a lower secondary school final year city standard examinations in Shanghai. This test item requires a long multi-step proof, as it is not valid directly to prove congruent triangles *ABD* and *CDB* according to the givens. Students are expected to prove congruent triangles twice (first to prove congruent triangles *ABE* and *CDE* and then congruent triangles *ABD* and *CDB*).

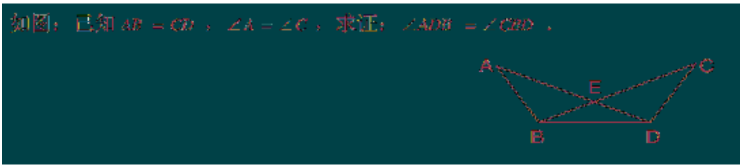


Figure 1. A test item on geometrical proof. In the diagram, the givens are *AB* = *CD*, angle *A* = angle *C*. To prove: angle *ADB* = angle *CBD*

Source: http://wenku.baidu.com/view/4b00e273a417866fb84a8e31.html

In the next section, we outline the "Shen Tou" teaching method at the initial stages of plane geometry based on the work of Yang (1988), with a particular focus on teaching strategies to develop the skills to use geometrical language in writing proofs.

3. The "*Shen Tou*" Teaching Method in the Initial Stages of Plane Geometry

Yang (1988) points out that the "*Shen Tou*" teaching method should be emphasized at the initial stages of plane geometry (ISofPG). For Yang, the "*Shen Tou*" method entails establishing a particular relationship between the teacher's purposeful instruction and students' gradual learning progress from being unfamiliar at the beginning to eventually acquiring some skills or rough understanding of a method in certain area of mathematics. In Chinese, the "*Shen Tou*" method is compared to the civil engineer work of establishing a system of drainage (Chinese saying "*Shui Dao Qu Cheng*" or 水). It is not a single effort, but the diligent repetition and accumulation of practices, with a well-designed plan in hand, that make the system of drainage work eventually.

Some researchers have tried to distinguish the subtle difference between the step-by-step learning process in the Chinese classroom and the 'rote drill' learning supposed by observers from outside the country. For instance, Paine (1990) noted that while the Chinese teacher dominated the classroom talk, knowledge was transmitted progressively to students in a precise and elegant manner. Zhang, Li and Tang (2004) pointed out that a teacher's whole-class approach in the Chinese classroom is by no means a form of spoon-fed teaching. Rather, to teach "Two Basics" (basic knowledge and basic skills) effectively, the teacher commonly applies what Zhang et al. (2004) called the 'small-step' teaching approach in the classroom. That is, the teacher uses a sequence of questions to guide students to reach the learning objectives step-by-step rather than leaving the students to struggle for discovery. Gu, Huang and Marton (2004) highlighted that Chinese teachers also apply the "*Pu Dian*" teaching approach (e.g., procedural variation) to enable students to gain the hierarchical system of experiences and processes in mathematical activities.

Shao et al. (2013) identified a number of ancient Chinese education philosophies. One of them is "emphasizing the cumulative process of learning and the importance of basic knowledge" (p. 11). Shao et al. (2013) further argued that a teaching principle, called "taking a progressive approach" (*Xun Xu Jian Jin,* , in Chinese), particularly emphasizes a learner's foundation and progress in learning. Indeed, the teaching principle of "*Xun Xu Jian Jin*" has been widely practiced in almost every aspect of the Chinese mathematics classroom. Apart from the studies discussed above, there are other teaching practices that underlie the "*Xun Xu Jian Jin*" principle. These include, for example, the multiple perspectives and longitudinal coherence of profound understanding of fundamental mathematics (Ma, 1999), the implicit variation in SH teachers' classroom practice (changes from the prototype of problems to their variations that have to be discerned by abstract and logical analysis by learners are considered as 'implicit variation'; see Huang, Mok, & Leung, 2006, p. 265), the heuristic nature of teaching (Zheng, 2006), and the 'indigenous' variation practice (Sun, 2011).

We consider that the "*Shen Tou*" method also underlies the "*Xun Xu Jian Jin*" principle. For the purpose of our research question of the multi-layered skills of using geometrical language (GL) in writing proofs, we use Yang's (1988) work on the "*Shen Tou*" method to help us to analyze our classroom data. In particular, we consider that the "*Shen Tou*" method enable us to gain insight into the repetition and accumulation of practices in the SH expert teacher's sophisticated instruction. Here, we make two points clear from the book by Yang (1988). First, this book was based on five years of empirical study (from 1982-1987) on the specific issue of plane geometry teaching at the lower secondary schools in the country at the time. The research group consisted of over twenty researchers and a considerable number of teachers from more than thirty local schools in Changzhou (a city in Jiangsu province). Thus, the book should be considered as an educational effort made by a group of leading Chinese educators, researchers and teachers in that region. Secondly, the value of the book is chiefly on its inquiry and exploration of the instructional rules of ISofPG on the ground; namely the research group's instructional experiments and practices with teachers on students' learning difficulties in the classrooms. Thus, the book has been regarded as an important reference by teachers, educators and researchers in China, though it lacks description of the research framework and methodology.

In terms of teaching the skills of using geometrical language (GL) in writing proofs, Yang (1988) argued that correctly understanding and representing GL plays an important role in mastering concepts, identifying figures, and correctly and smoothly writing proofs. However, in classroom practice, Yang (1988) noted that students have great difficulty with the GL at the ISofPG. As a result, language can seem an obstacle for students when learning plane geometry.

In the first place, Yang (1988) identified four factors that relate to students' difficulties with GL.

1) The change in mathematical language resulting from the shift of teaching/learning content from numbers to diagrams; this is the change from algebraic language to geometry language. Yang (1988) argues that in geometry it is difficult to convert word language into symbol language according to a model of generalization. That is, although word language is used to describe geometrical diagrams and their orientation and quantitative relationships, its use is varied due to the varied diagrams and different letters for which there is no agreed standard way of representing the same kind of geometrical diagrams. Consequently, some students may separate the two forms of language and learn geometrical concepts and theorems solely in a rote manner (only reciting word language and making no connection to symbols and diagrams). In that way they are unable to make the translation between the two forms of

language in the use of concepts and theorems which are essential for proving and writing proofs in geometry.

2) When teaching the ISofPG, a large amount of geometrical word language is introduced and used in a concise and rigorous manner in the classroom. It takes time for students to learn to use such language in a concise and rigorous way. For instance, there are some frequently-used terms that students often cannot correctly understand such as “every two points”, “draw/take any”, “any one”, and “have and only have”, etc. It also takes time for students to learn the words that represent the orientation and quantitative relationship of geometrical diagrams and the sentences that show the action of drawing/making such diagrams.

3) Students’ everyday natural language can lead them to make negative transformation in GL learning. For instance, students cannot fully understand the meaning of the basic property of a straight line that “there is one and only one straight line that crosses two points on a plane”. They often find that the word “and only one” is not necessary according to their daily language.

4) Grade 8 students (aged 13-14 years) do not immediately adapt to learning deductive reasoning in geometry due to the limits of their language skills and grammar knowledge. Yang (1988) distinguishes two aspects of student difficulties. First, some Grade 8 students cannot distinguish the essential part and the unessential part of a considerably long simple sentence. Thus, they can neither draw a correct diagram according the meaning of the sentence nor distinguish the “givens” and the “to prove” from the sentence; let alone write the proof. Secondly, Grade 8 students have not yet systematically acquired the grammar knowledge of how to transfer a simple sentence to a compound sentence. Yet, in plane geometry, such instances of word-to-word transformation are highly required. Such a lack of connection across geometric topics can bring difficulty to students when they are asked to transfer a proposition.

Based on observations of such student difficulties, Yang (1988) suggests a number of teaching strategies to help students to overcome these language difficulties. As the data we present later in this chapter focuses on the teaching of geometrical theorems at Grade 8, we concentrate on four teaching strategies for developing skills of mutual-translation of GL that are particularly applicable; these teaching strategies are laid out in Table 1.

Table 1. Four teaching strategies (TS) for developing the mutual-translation of GL

Teaching strategy	Key explanation
1. The W-S strategy: Translate the word language of concepts, definitions and proposition into the geometrical symbol language with diagrams (p. 73).	The W-S strategy is to help students smoothly to draw a proof diagram according to the proposition in words and to use geometrical symbols to distinguish between 'the givens' and 'to prove' when writing a proof.
2. The R-D and O-D strategy: Read the word statements and then draw the diagrams (R-D), and draw the diagram by listening to the teacher's oral statements (O-D) (p. 73).	The R-D strategy is similar to that of W-S. Yet the O-D strategy can be much more flexible and more highly cognitively-demanding for students, as they must be familiar with the commonly-used geometrical terms and be able to understand a number of relative concepts, together with considerable listening and language translation skills.
3. The O-T strategy: Talk about the properties of the diagram according to the observation (O-T) (p. 74).	The O-T strategy is to develop students' skills to use word language to generalize the geometrical properties of diagrams.
4. The W-F-S strategy: Based on the O-T, to use word language correctly and as concisely as possible to generalize geometrical facts (e.g., propositions and theorems) according to the diagram and symbol language (W-F-S) (p. 74).	The W-F-S strategy is to pave the path for students by a multiple-layered instructional procedure.

Yang (1988) points out that the first two teaching strategies in Table 1 are intended to focus on developing students' translation skills from word language to diagram and symbol language. Such kinds of translation is analogue to the process of synthetic thinking (e.g., logic reasoning from the "givens" and the "to prove" in proof). The last two strategies are to stress the translation skills from diagram and symbol language to word language. It is analogue to the process of analytic thinking (e.g., analytic reasoning from "to prove" and the "givens").

4. The Case of Geometry Teaching in Teacher Lily's Grade 8 Class

In this section, we select and describe four teaching episodes from a sequence of twelve lessons (each lesson was 40 minutes long) in teacher Lily's Grade 8 class (all names are pseudonyms in this chapter). By analyzing and interpreting these four episodes, we illustrate the features of the repetition and accumulation of practices in this expert teacher's use of the "*Shen Tou*" method of teaching geometrical theorems across a sequence of lessons. We use "L" to code the lessons observed in Lily's class. "L1" represents the first observed lesson, "L2" the second observed lesson, and so on. As the teacher taught a number of geometrical theorems during the observed lessons, "L1-T1" meant the first theorem observed as a whole that took place in the first observed lesson given by Lily, and so on. We also used four students' voices to represent other students' similar learning responses in the mathematics class (we call the boys Linlin and Liuliu, and the girls Beibei and Youyou).

In our study, we observed the Grade 8 geometry classes at the later stage of the second school term (there are two school terms in each school year in SH). Thus, students were on the transition from experimental geometry to proof geometry. In SH, plane geometry has been divided mainly into two phases at the lower secondary mathematics curriculum: experimental geometry (Grades 7 and 8) and proof geometry (Grade 9) since the regional curriculum reform took place in the late 1980s (SH has its own curriculum and textbooks reforms which are independent to the national curriculum and textbooks reforms). At Grade 8, students acquire geometrical knowledge mainly by observations and experiments. At Grade 9, they are expected to understand the learned geometrical knowledge by rigor proof in geometry (Zhang, 2005a).

4.1 *Teaching Episode One: Teaching T2 in L1*

When teaching L1-T2 (T2: if two pairs of opposite sides of a quadrilateral are equal, then it is a parallelogram), Lily first led students to review one of the properties of a parallelogram (two pairs of opposite sides of a parallelogram are equal). Students were then engaged in working on a sequence of tasks set by Lily. They were first asked to transfer this proposition to its inverse proposition. They were then requested to draw a diagram of T2 and write down 'the givens' and 'to prove' on their notebook. Finally, they were engaged in proving T2. One girl student was invited by Lily to write her proof on the blackboard while the rest of students in the class were asked to write their proof in their notebook.

We observed that there were a certain number of students who had difficulty in transferring the proposition to its inverse. Some of them confused the proposition with the definition of parallelogram (D1: two pairs of opposite sides of a parallelogram are parallel). Some of them had difficulty in distinguishing 'the givens' and "to prove" of the T2. These are evident in the classroom interactions as follows:

Lily: (moved around the class and talked to one student to assist his learning) How could this (proposition) be the definition? Two pairs of opposite sides (of a parallelogram) are equal. The definition is about two pairs of parallel opposite sides. You would not be requested to prove the definition for it is already given (as in the textbook).

S: (one boy sitting near the researcher asked his classmate) What is the given (of this proposition)?

Lily: (moved around the class and talked to another student to assist his learning) How could it be a parallelogram? You are asked to prove a parallelogram.

After Lily had taken a look around the students' work in the class for a while, she drew the students' attention to the use of definition and theorem in proving a parallelogram in the classroom as follows:

Lily: Did we learn how to prove a parallelogram? (Lily stopped for a moment for students to think about the question.) We haven't yet learned the theorems to verify a parallelogram. We only know the definition and the properties of a parallelogram. As you know, we cannot use the properties to verify a parallelogram.

Beibei: (responded to Lily's talk above) Is it a parallelogram if its two pairs of opposite sides are equal?

Linlin: (responded to Lily's talk above) We need to prove congruent triangles. … That's the only way to prove it.

The above classroom interactions show that while the teacher intended to draw students' attention to the definition (D1) which was a means to develop the analytic thinking from 'to prove' to 'the givens', some students (exemplified by Linlin) thought about how to write the proof by first proving congruent triangles. Responses from students like Beibei also indicate that some students did not understand the relationship between definition and theorems in the axiomatic system of plane geometry. Noticeably, it was the teacher Lily who emphasized the use of the definition of a parallelogram to prove T2. A number of students appeared to be quite surprised and puzzled by the teacher's instructional emphasis. This is evident from their responses as follows:

Linlin: Ah? (in a surprised tone)

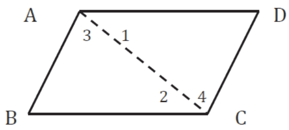
Liuliu: (talked to Beibei) See, the teacher said that we should use the definition to prove it.

Beibei: (responded to Liuliu) What is the definition?

Liuliu: (replied to Beibei) I don't know. What is the definition? (asked others in the class)

Lily then went on to address the theorem of verifying parallel lines, which was again to lead students to develop analytic thinking from 'what is the conclusion' (parallel lines) to 'what are the conditions'. Following this attention to developing students' analytic thinking for writing the proof, Lily led the students to consider a girl's proof on the blackboard, the main part of which (namely a proof that $\angle 1=\angle 2$, $\angle 3=\angle 4$) is shown in Figure 2.

In sum, our analysis of the teaching episode of L1-T2 indicates that in using the "*Shen Tou*" method, Lily purposefully conducted three teaching steps to support students to practice skills of mutual-translation of GL and analytic thinking in writing a proof, as shown in Table 2. The



Prove: Linking A and C.

In $\triangle ABC$ and $\triangle DCA$,

$$\begin{cases} AB=DC \text{ (given)} \\ BC=AD \text{ (given)} \\ AC=AC \text{ (common side)} \end{cases}$$

$\therefore \triangle ABC \cong \triangle DCA$ (SSS)

$\therefore \angle 1=\angle 2$, $\angle 3=\angle 4$ (the corresponding angles of congruent triangles are equal)

Figure 2. The main body of the proof of L1-T2 written by a student on the blackboard

three steps shown in Table 2 are the word-symbol (W-S) strategy, the use of definition and theorems, and the word-figure-symbol (W-F-S) strategy.

Table 2. An analysis of the "*Shen Tou*" method in L1-T2

Teacher's instruction in each teaching steps	Students' difficulties and feedbacks
Step 1. used the W-S strategy to enable students to practice skills of making the translation from word language to diagram and symbol language.	• some students were unable to transfer the proposition to its inverse. • some confused the proposition with the definition. • some were unable to distinguish 'the givens' and "to prove" of the theorem.
Step 2. addressed the use of definition and theorems to train the analytic thinking from 'what to prove' to 'what are givens'.	• students did not understand the relationship between definition and theorems in the axiomatic system of plane geometry. • students were unawareness of the function of definition and theorem in proving. • students' thinking naturally focused on moving from 'what are givens' to 'what to prove'.
Step 3. used the W-F-S strategy to enable students to talk freely about the properties of the diagram and then to use geometrical symbols and language to prove the theorem.	

4.2 *Teaching Episode Two: Reviewing T2 in L2*

At the beginning of the next lesson (L2), the approach used by teacher Lily could appear to be a repeat of her teaching of T2 in L1. Yet a closer analysis of her teaching illustrates the important role of the teacher in using the "*Shen Tou*" method to enable students simultaneously to practice the skills of the mutual-translation of GL and two key forms of thinking (analytic and logical) in proof writing. In so teaching, Lily first guided students to translate the simple sentence of the proposition (T2) into a compound sentence in the form of "*if* … *then* …". The teacher's instructional intention in using the word-to-word (W-W) translation was to deepen the students' understanding of the structure of the proposition and to help them to distinguish "the givens" and "to prove" for writing the proof. Noticeably, Lily corrected students' W-W translation by highlighting the key word "two" that some students missed in representing the group of opposite sides of a parallelogram. Lily then directly drew a parallelogram on the blackboard and used letters and geometrical symbols to highlight 'the givens' (in quadrilateral *ABCD*, *AB*=*CD*, *AD*=*BC*) on the diagram (as illustrated in Figure 3(1)) and to represent 'to prove' (prove that quadrilateral *ABCD* is a parallelogram) on the blackboard (see Figure 3(4)). For convenience the whole teaching procedure of L2-T2 is illustrated in Figure 3. For the teacher's work on the blackboard, see the left part of Figure 4.

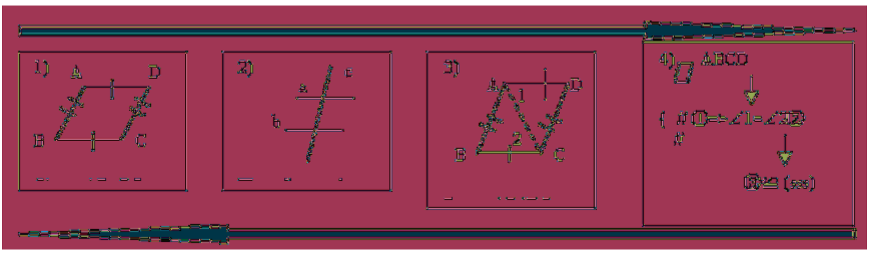


Figure 3. The teaching procedure of L2-T2

Noticeably, in the teaching process of L2-T2, Lily repeated the role of definition in proving the theorem. This is evident in the classroom interaction as follows:

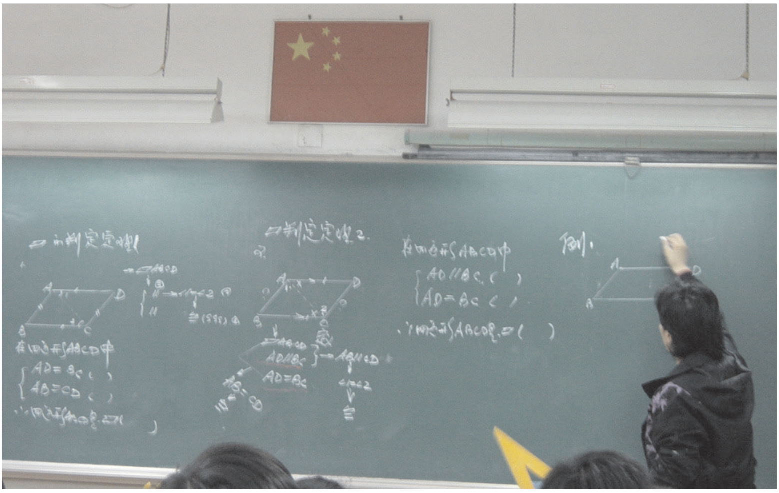


Figure 4. Lily's writing on the blackboard in L2-T2

Lily: So far, we have only known one way to prove a shape that is a parallelogram. (The teacher then waited students to respond to her talk.)

Some students: Use the definition (D1).

Such learning responses from the students show that they learned to think of the definition for proving this theorem. Lily further used two symbols "//" to represent the two pairs of parallel lines on the blackboard (see Figure 3(4)). Lily then led students to develop analytic thinking of the theorems from 'what to prove' to 'what are given' in proving the parallel lines.

Lily: To prove two lines are parallel, what methods did we learn early at Grade 7?

Some students: The alternate interior angles are equal.

Lily: The 'three lines and eight angles' (TLEA), right? (Three lines mean that two parallel lines are intersected by a transversal; eight angles mean that eight angles, less than 180 , are formed by the three lines.

Teachers and students commonly use the TLEA in Chinese to represent the theorems of parallel lines; see Figure 3(2).

Figure 3(2) was not drawn on the blackboard. It was a basic diagram that students learned at Grade 7 to prove parallel lines and the property of parallel lines. Some students then discussed about linking *A* and *C*, while others suggested *B* and *D*. The teacher listened to the class discussion and chose to link *A* and *C* as one of the ways to prove the problem (see Figure 3(3)).

Our analysis of the classroom interaction above further shows the small but important difference between Lily's and her students' thinking. While Lily drew students' attention to think first of the theorem which required the analytic thinking from 'what to prove' to 'what are given', students tended to think from 'what are givens' to 'what to prove' (e.g., if the alternate interior angles are equal, then two lines are parallel). As well as the development of analytic thinking, Lily also purposefully used the "*Shen Tou*" method to develop students' logic thinking. Lily's instructional intention is evident in the following classroom interactions.

> Lily: If to prove 1= 2, what should we turn to prove first? (Some students answered congruent triangles. Then the teacher briefly guided students to prove two congruent triangles *ABC* and *ACD* in Figure 3(3))

While the teacher asked students these questions above, she gradually wrote down an analytic path of the proof on the blackboard (see Figure 3(4)). The emphasis on writing such an analytic path is shown in Lily's instruction as follows:

> Lily: So, from now on, we need gradually to cultivate a thinking habit in (plane) geometry. We have now learned (plane) geometry for over one year. As far as we are asked to prove a problem, we should first seek for an analytic path for solving it. Now good students start to learn to write the analytic path before writing the proof. An ordinary student still focuses on writing the logical paragraphs of a proof. You can certainly write so according to the analytic path.

Lily's instructional statement above shows that the teacher simultaneously developed the analytic thinking for writing the analytic path and the logic thinking for writing the multi-step proof. We also note that Lily took into account the need of different individual students in her instructional effort. Moreover, after the proof was discussed according to the analytic path on the blackboard, Lily led students to represent the theorem (T2) in two types of language: one is the word language, the other the geometrical symbol language (see the diagram and the geometrical symbols on the left part of Fig. 4).

In sum, the analysis of the teaching episode of L2-T2 shows that Lily not only demonstrated to students about how to translate the multiple types of GL (e.g., the W-S and the W-F-S translations), but also addressed the practice of using both the analytic and the logic thinking for writing the proof. In so teaching, Lily skilfully used the "*Shen Tou*" method through the three teaching steps: word-to-word (W-W) strategy, questioning strategy, and word-to-symbol (W-S) strategy (see Table 3).

Table 3. An analysis of the "Shen Tou" method in L2-T2

Teacher's instruction in each teaching steps	Students' difficulties and feedbacks
Step 1. used the W-W strategy to deepen students understanding of the structure of the proposition and to develop their skills to distinguish 'the givens' and 'to prove' for writing the proof.	• students did not pay attention to the structure of the proposition. • some were unable to distinguish 'the givens' and 'to prove' for writing the proof.
Step 2. • used questioning strategy to enable students to practice analytic thinking. For instance, "*we have only known one way (definition/theorem) to prove …*", "*To prove …, what methods did we learn …?*", "*If to prove …, what should we turn to prove first?*", etc. • wrote the analytic path to develop logic thinking for writing the proof.	• students did not understand the relationship between definition and theorems in the axiomatic system of plane geometry. • students were unaware of the function of definition and theorem in proving. • students' thinking naturally focused not on moving from 'what to prove' to 'what are givens', but on moving from 'what are givens' to 'what to prove'.
Step 3. used both the W-W and the W-S strategies to develop students' skills to represent the newly learned theorem.	• students learned to represent the new theorem in two types of language: word language and geometry symbol language.

4.3 *Teaching Episode Three: Analyzing the Structure of D1, T2 and T3 in L4*

When Lily began her instruction at the start of lesson L4, she began by providing the students with feedback on two common problems with proof writing that was evident in their homework. First, in writing a proof, a few students were not able to use directly the theorem of parallel lines (the distances between two parallel lines are always equal) but reproved this theorem. Secondly, a considerable number of students did not prove a true proposition (when two triangles are on the same side with a common height, the rate of the areas of these two triangles is the rate of their two bottom sides) that was requested to prove, but used it as a theorem. Such a feedback shows that these students had not yet understood the relation between theorems and propositions in the axiomatic system of plane geometry.

According to Lily's feedback of students' homework, some students also mixed the theorems of verifying a parallelogram with those of describing its properties. Thus, she used questions to draw their attention to the role of definition and theorems in proving a parallelogram.

Lily: Yesterday, we talked about how to verify a parallelogram. Up to now, how many methods are available?

Youyou: (responded to the teacher with some other students in the class) Three methods.

Lily: Which three?

Youyou: (responded to the teacher with some other students in the class) The definition (D1), the first theorem (of verifying a parallelogram) (T2) and the second theorem (of verifying a parallelogram) (T3: if one group of opposite side is parallel and equal, then it is a parallelogram).

Here, students like Youyou were able to think about the definition and theorems (D1, T2, T3) they had learned as the methods to prove a parallelogram.

Moreover, Lily's feedback about the students' homework indicates that many students wrote only a brief statement to describe T2 in their proving. Noticeably, it was the teacher who addressed the rigor of the word language of the definition and the theorems, and the need to add the words "two group" to represent the theorem T2 precisely. She further drew students' awareness to the rigor of the word "two" by a question of whether one group of equal opposite sides would ensure a parallelogram.

Next, Lily led students to develop further their understanding of the structure of the definition and theorems of verifying a parallelogram by questions as follows.

Lily: Usually, how many conditions are requested for proving a parallelogram? (Lily invited one student to answer the question. Yet it appeared hard for the student to answer. Then, Lily invited another student to talk about her idea.)
S: Two.
Lily: Two. There are two conditions in each of the theorems (T2 and T3). There are two conditions in the definition (D1) as well. It does not like the theorems of verifying congruent triangles. To prove congruent triangles, three conditions are needed. To prove a parallelogram, two conditions are needed.
Lily: So, to prove a parallelogram, what should we actually turn to prove? (Students' voices were low in the audio recorder. According to their responses, they were engaged in thinking of Lily's questions.)
Lily: To turn to prove equal or parallel line segments. (Students' voices like "En" indicated that they agreed with Lily's thought.)
Lily: We already learned the general methods to prove equal line segments at Grade 7. To prove parallel lines, we learned the TLEA. In fact, if we know how to prove parallel or equal line segments, we know how to prove a parallelogram.

The classroom interactions above show the important role of the teacher in ensuring students develop an insight into the role of theorems in writing proofs. It is unlikely that students would automatically think of the connections of the learned theorems and compare the structure of these theorems when writing proofs. Moreover, it was Lily who purposefully drew students' attention to the function of the theorems of verifying a parallelogram.

Lily: It (meant the theorem like T2 and T3) provides an alternative for proving parallel or equal line segments. Early, we had to turn to prove congruent triangles or an isosceles triangle in order to get equal line segments. Now if two pairs of straight lines are parallel or equal, and they are in a quadrilateral, we can try to firstly prove a parallelogram.

In sum, Lily carefully used the "*Shen Tou*" method through three teaching steps over this teaching process of L4-D1, T2 & T3, as shown in Table 4.

Table 4. An analysis of the "Shen Tou" method in L4-D1, T2&T3

Teacher's instruction in each teaching step	Students' difficulties and feedbacks
Step 1. used questioning strategy to draw students' attention to the role of definition and theorems in proving.	• in writing a proof, a few students were not able to use directly a theorem but reproved the theorem. • many did not prove a proposition, but used it as a theorem. • these students had not yet understood the relation between theorems and propositions in the axiomatic system of plane geometry. • some also mixed the theorems of verifying a parallelogram with those of describing its properties. • many did not use the rigor of the word language when stating the definition and the theorems. • some were able to think about the definition and theorems learned as the methods to prove a parallelogram.
Step 2. • used the W-F-S strategy at a higher level to deepen understanding of the structure of the theorems, and to use geometrical symbols to represent the sub-structure of the conditions of the theorems. • made an analogical thinking between the theorems of verifying a parallelogram with those of congruent triangles when comparing the structure of these theorems. It was to deepen students' insights into the relation of theorems. • used questioning strategy like "*If to prove …, what should we turn to prove first?*" to simultaneously train students' analytic and logic thinking in writing proofs.	• although some students in the class could recite the theorems (e.g., T2, T3) as requested by the teacher, they did not go any further to think of the structure of a theorem. It was not natural for students to analyze the structure of a theorem when proving a parallelogram. • It was not natural for students to think the connections of theorems in their proving.
Step 3. drew students' attention to the function of learning the theorems of verifying a parallelogram.	• same to the difficulties and feedbacks in *Step* 1.

4.4 *Teaching Episode Four: Teaching T8 in L5*

In L5 Lily's focus when teaching a new theorem of rectangle (T8) was not so much on the development of students' skills of translation from word language to figure and symbol language (the W-S form). She only briefly drew the proof diagram on the blackboard and orally demonstrated about 'the given' (rectangle *ABCD*) and 'to prove' (*AC*=*BD*) in the class, as shown in Figure 5.

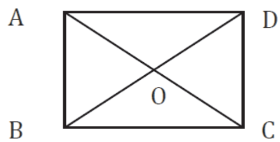


Figure 5. L5-T8

Students almost simultaneously discussed in the class of the key ideas of proving this theorem. Noticeably different from their early learning responses (e.g., see L1-T2), some students (such as Linlin, Youyou and Liuliu) learned to use theorems to demonstrate their own logic and analytic thinking of the proof.

Linlin, Youyou and Liuliu: We can use congruent triangles to prove it.

Liuliu: SAS (meant the theorem of two sides with an included angle to prove congruent triangles).

Lily then invited one of these students to stand up in the class to demonstrate his thinking of the proof.

Lily: The given is a rectangle *ABCD*. We need to prove *AC*=*BD*. Who can prove it?

The boy: (invited by Lily and stood up in the class) Because this quadrilateral is a rectangle. … (interrupted by Lily as follows)

Lily: Please tell us what should we turn to prove first, to prove these two equal sides?

The boy: To prove congruent triangles.

Lily: Which two (triangles)?

The boy: Triangles *ABC* and *ABD*.

Lily: To prove these two congruent triangles, what conditions are already given?

The boy: The common side (meant *AB*), the opposite sides (meant *AD*, *BC*), and two angles 90 (while the student talked about the three conditions, Lily wrote *AB*=*AB*, *AD*=*BC*, angle *DAB* = angle *ABC* as the analytic path on the blackboard).

Lily's interventions in this boy's presentation of the proof above show that her instructional intention was tightly-focused on leading the students in the class firstly to think of the analytic path of the proof and then to prove it. Lily then used questions to develop the logic thinking of students in the class as follows:

> Lily: *AB*=*AB*. It's the common side. We do not need to prove it. How to prove *AD*=*BC*? What reason will you give? How do you write the proof here? (Students' voices were low in the audio-recorder. As Lily appeared to address some of the students' talk, we use Lily's voice to show the classroom interactions as follows.)
>
> Lily: You can write that in rectangle *ABCD*, *AD*=*BC*. You can write the reason that the opposite sides of a rectangle are equal.
>
> Lily: Next, angle *DAB* = angle *ABC*. What reason will you give?
>
> The boy: Each angle of a rectangle is 90 .
>
> Lily: Next, we need to write the congruent triangles and then get *AC*=*BD*. So we get the property that the diagonals of a rectangle are equal.

In this teaching episode, we note that some students learned to think of the proof according to the theorems. We consider it as an outcome of Lily's sophisticated teaching in the use of the "*Shen Tou*" method over through her early instructions (e.g., L1-T2, L2-T2, L4-D1, T2&T3).

In sum, our analysis on Lily's instruction in this episode of L5-T8 indicates that Lily made a shift of emphasis from the W-S form to the W-F-S form of GL. Moreover, in the W-F-S strategy, Lily paved the path for developing students' deeper thinking through two steps of teaching. This is shown in Table 5.

5. The Multiple-Layered Teaching of Geometrical Theorems in the "Shen Tou" Method

At first glance, many of the Grade 8 students in Lily's class appeared to have no problem in making a one-step proof by thinking from 'what is given' to 'what to prove'. What is more, some capable students showed no problem in writing a relatively complex multiple-step proof (as captured in Figure 2). Yet a closer analysis of teacher Lily's homework feedback and instructional interventions in the class shows that many of these students encountered various sorts of difficulties in the process of

Table 5. An analysis of the "Shen Tou" method in L5-T8

Teacher's instruction in each teaching step	Students' difficulties and feedbacks
Step 1. made a shift of emphasis from the W-S strategy to the W-F-S strategy.	• students almost simultaneously discussed in the class of the key ideas of proving the theorem.
Step 2. • used the questioning strategy like "*If to prove …, what should we turn to prove first?*" to simultaneously develop students' analytic and logic thinking in writing proofs. • used the W-F-S strategy to emphasize on using geometrical symbols to write the analytic path for the proof. • addressed the analytic thinking of a particular kind, namely "to make a reasoning according to the givens (e.g., the premise, the definitions, the axioms, and the theorems)" (Yang, 1988, p. 92).	• some students learned to use theorems to demonstrate their own logic and analytic thinking of the proof.
Step 3. drew students' attention to the rigorous form of proof writing (e.g., condition-conclusion-theorem/definition).	

teaching/learning geometrical theorems in the classroom. Here, we list some of these from our data analysis:

1) unable to represent precisely the theorems in rigorous word language;
2) unable to transfer correctly the proposition to its inverse proposition;
3) unable to recognize the structure of the definitions and theorems even if able to recite theorems/definitions;
4) unable to understand the role and relation of definitions and theorems in the axiomatic system of deductive geometry.

Yang (1988) points out that a small-step teaching procedure is essential to enable students to gradually make progress at the ISofPG. Our overall analysis of the teacher's use of the "*Shen Tou*" method demonstrates that Lily carefully paved the teaching steps for individual students to overcome their difficulties and to lead them constantly to practice various kinds of skills in geometry to a higher level. Here, we identify two types of the multiple-layered instruction over these teaching steps: one is the longitudinal multiple-layered teaching of a single theorem (e.g., L1-T2, L2-T2, L4-D1, T2&T3), the other the transverse multiple-layered teaching of a set of theorems (e.g., L1-T2, L5-T8).

The longitudinal multiple-layered teaching of a single theorem (e.g., T2) can be seen as follows:

The first teaching layer: When teaching a new theorem (e.g., L1-T2), the W-S strategy (Yang, 1988) can be used to engage students in practising how to correctly translate the word language of the theorem into the symbol language with diagrams. Students could also be encouraged to prove the theorem by correctly use geometrical diagrams/symbol language.

The second teaching layer: After the new theorem is taught (e.g., L2-T2), teacher could address the translation from the simple sentence of the proposition (e.g., T2) into the compound sentence in the form of "if … then …". This W-W form of translation is to deepen students' understanding of the structure of the theorem and to enhance their skills to distinguish "the conditions" and "the conclusion" of the theorem. Teacher also needs to use the W-F-S strategy (Yang, 1988) to help students correctly to represent the newly learned theorem (e.g., T2) in both the rigorous word language and geometrical diagram/symbol language.

The third teaching layer: In analyzing the structure of the new theorem in writing proof, the teacher could help students to see the connection of the new theorem with other learned definition/theorems (e.g., L4-D1, T2&T3). Through comparing and understanding the connections of the different theorems/definition, students are given opportunity to develop an insight into the relation of definitions and theorems in the axiomatic system of plane geometry.

The transverse multiple-layered teaching of a set of theorems (e.g., L1-T2, L5-T8) is mainly to emphasize the mutual development of various kinds of geometrical thinking, apart from the development of GL skills. Such transverse multiple-layered teaching for the development of the analytic and logical thinking can be summarized as follows:

The first teaching layer: in teaching the first a few theorems of a set of theorems, teacher needs to purposefully address the development of analytic thinking from 'what to prove' to 'what are given' in proving the theorem. Teacher could use a sequence of questions to assist students to write down an analytic path of the proof. For instance, "*we have only known one way (definition/theorem) to prove …*", "*To prove …, what methods did we learn …?*", "*If to prove …, what should we turn to prove first?*", etc.

The second teaching layer: gradually, teacher needs to develop students' insights into the role of theorems in writing proofs. In particular, students need to be guided "to make a reasoning according to the givens (e.g., the premise, the definitions, the axioms, and the theorems)" (Yang, 1988, p. 92). Teacher could also draw students' awareness to the role of the theorems and the connections between the theorems/definitions in writing proofs.

The third teaching layer: in addition to the emphasis on the use of geometrical symbol language to write the analytic path for the proof, the teacher needs to draw students' attention to the rigorous form of proof writing (e.g., condition-conclusion-theorem/definition).

We conclude from the analysis of Lily's Grade 8 class that the teacher purposefully set up two types of instructional procedure when using the "*Shen Tou*" method to teach geometrical theorems: one is of the longitudinal multiple-layered teaching of a single theorem for training the GL, the other the transverse multiple-layered teaching of a set of theorems for enhancing the mutual development of various kinds of geometrical thinking.

It is also noted that the longitudinal and the transverse multiple-layered teaching procedures correspond to the hierarchical ordered skills and the systematical connections of knowledge in plane geometry. For instance, across the longitudinal layers of teaching of the single theorem (e.g., T2), the skills were ordered as follows: translating the word language of the proposition into the symbol language with diagrams and proving the proposition ➔ analyzing the structure of the theorem and representing the theorem in both the rigor word language and geometrical diagram/symbol language ➔ seeing the relation of definitions and theorems in the axiomatic system of deductive geometry. Across the transverse layers of teaching of a set of theorems (e.g., T2, T3, and T8), the skills were ordered as follows: developing the analytic thinking from 'what to prove' to 'what are given' in proving the theorem ➔ seeing the role of the theorems and the connections between the theorems/definitions in writing proofs ➔ seeing the rigor form of proof writing.

6. Conclusion

In this chapter, our aim has been made to provide insight into the repetition and accumulation of practices in the SH expert teacher's use of the '*Shen Tou*" method in Grade 8 geometry class. We identify two key features of this expert teacher's instruction on geometrical theorems: one is the complex learning support structure established by the longitudinal and the transverse multiple-layered teaching procedures, the other the repetition and accumulation of practices of the hierarchically-ordered skills and gradual understanding the systematical connections of knowledge within the multiple-layered teaching procedures. At the ISofPG, Yang (1988) suggests that teachers use the "*Shen Tou*" method to enhance a range of skills such as to appreciate, draw, and make geometrical diagrams; to understand, represent and translate geometrical language; and to write proofs. Furthermore, when using the "*Shen Tou*" method, Yang suggests that a teacher should first establish a whole structure in which the different skills to be developed are carefully ordered. On this basis the teacher can then develop students' skills through well-designed and hierarchical layers of instruction. The two key features of this expert teacher's instructional practice on geometrical theorems in our study substantiates Yang's (1988) didactical ideas in an authentic classroom setting. Our study further supports the hypothesis by Martin et al. (2005, p. 122) that students can, with the appropriate instructional strategies, become more skilled in how to construct proofs on a *multi-tiered procedure*.

The data analysis presented in this chapter further substantiates our recent work with other researchers on the role of expert teacher and teaching in Chinese mathematics classrooms. For instance, Mok and Ding (in press) show that the Chinese expert teacher is able to see ahead of the cognitive capacity of the students and then provide expert scaffolding (Holton & Clarke, 2006) for helping the students to reach a higher level of development in mathematics. Ding, Jones and Pepin (2013) use the term "Hypothetical Learning Structure" (HLS) to distinguish the Chinese expert teacher's concept of "Hypothetical Learning Trajectory" (HLT) from Simon's (1995) notion of HLT. That is, in the Chinese expert teacher's view, pupil learning could be more 'efficient' (in the sense of 'whole class learning'), if they are engaged in the teacher's well-designed mathematical tasks. Cao, He and Ding (in press) show that in the two LPS (Learner's Perspective Study) Shanghai classes, classroom interactions were largely initiated by the teachers. Interaction between a teacher and a class is intertwined with teacher and individual student interaction. Our analysis of the teacher-student interactions in this SH expert teachers' day-to-day instructional practice, in particular the teacher's part in the teacher-student interactions highlights the significant role of the teacher in designing and conducting the hierarchical and sophisticated instructional procedures to develop rigorous GL, to develop students' analytic thinking, and to help them to see the structure and the role of theorems, as well as the relation of theorems/definitions in axiomatic system of plane geometry. In the on-going studies, we go further into examining the relationship of the hypothetical scheme of the Chinese expert teachers' instructional path/structure with the most fundamental cognitive conflicts of students in the process of solving the geometrical proof problems in axiomatic system.

As this chapter focuses on the teacher, we are not able to delve further into understanding the development of students' independent and creative thinking in plane geometry within the multiple-layered instructional procedures. Nevertheless, we consider that the two multiple-layered teaching procedures identified from the micro perspective of our study reveal some part of the whole picture of the "*Shen Tou*" method from the macro perspective of teaching plane geometry across the different grade curricula. We also consider that the two multiple-layered teaching procedures underly the fundamental Chinese teaching principle of "*Xun Xu Jian Jin*", which addresses on gradually deepening learning through an orderly-layered teaching procedure.

In analyzing this SH expert teacher's dynamic approach to leading students to make certain explorations and discoveries in proof problem solving activities, we (Ding, Jones, & Zheng, 2009) highlighted what Zheng called the "indoor flying" approach (in Chinese " 小 ") — it is like flying, but it is like flying indoors as it is constrained in certain ways. We (Ding, Jones, & Zheng, 2009) did not consider that the 'indoor flying' approach as a truly heuristic approach because students' freedom to explore and think through the problem remained under the control of the teacher's instruction. In the next step of our work, we aim to pursue an understanding of the distinction between the "*Shen Tou*" method and the 'indoor flying' approach in teaching proofs in geometry. In particular, we wish to develop understanding of the relation between the "*Shen Tou*" method practiced in this expert teacher's mathematics classes and Gu's (2012) recent lecture on the "potential distance" in "*Pu Dian*" (scaffolding). In so doing, we wish to develop a new insight into the fundamental Chinese teaching principle of "*Xun Xu Jian Jin*", together with a deeper understanding of the complex classroom learning space varied across the multiple-layered teaching procedures towards the goal of developing students' independent and creative thinking in mathematics.

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Chapter 10

Implementation of Objectives Based on the Curriculum Standards: A Case of Teaching Using Letter to Represent Number at a Chinese Primary School in Chinese Mainland

HUANG Xingfeng YANG Jinglei LI Shiqi

To study the implementation of new mathematics curriculum in primary schools, we focused on how teachers use letter to represent number, an important content in algebra. Three experienced teachers were selected from three primary schools in China (mainland). Four lessons of each teacher were observed, and interviews with teachers and students were recorded after each lesson. Based on the objectives of curriculum standards, classroom teaching was examined in four strands (knowledge and skills, mathematical thinking, solving problem, and affect and attitude). It was found that these teachers' teaching showed great similarities in the four strands. They intended to enhance students' abilities of using letter to represent quantitative relation in realistic situations, and improve students' deductive reasoning, and encourage them to explore and explain different ways in solving problem. However, some aspects in the four strands were overlooked in their classroom practice.

Keywords: curriculum innovation, classroom teaching, use of letter to represent number

1. Introduction

Teachers' classroom teaching reflects their understanding of the curriculum and shows how they implement it (Huang & Fan, 2009). Goodlad and Su (1982) claimed that classroom teaching is an internalized and personalized curriculum by students, and also is the ultimate examination of curriculum innovation. Therefore, in order to explore the gain and loss of curriculum implementation, it is necessary for researchers and educators to focus on classrooms in practice.

In the past ten years, many researchers conducted detailed observation and analysis on Chinese mathematics teaching. For example, Gu, Huang, and Marton (2004) summarized that variance and foreshadowing are characteristics of Chinese mathematics teaching. Huang and Leung (2004) pointed that Chinese teachers highlight students' exploring, and encourage students to participate in mathematics classroom. Mok (2006) in learner's perspective (LPS) found that the mathematics teachers in China (mainland) often promote students to explore, and encourage student to learn at groups. Lopez et al. (2004) identified a teaching model which can describe Chinese teaching style.

Grounded on the new curriculum innovation, what features did Chinese mathematics teaching have? What happened and changed in Chinese classrooms? Xu, Kong, and Su (2009) claimed that mathematics classroom teaching in China kept the traditional features, such as emphasis on review and introduction, advance gradually in due order, analysis of typical examples, consolidated exercises and classroom feedback. On the other hand, classroom teaching has made some changes, such as use of information technology, setup scenarios for students' exploration and so on. We have launched a series of studies on Chinese mathematics classroom. For example, a comparative study was conducted on mathematics classrooms across three decades (Huang, Pang, & Li, 2009). It revealed that teacher questioning, classroom organization were different in different eras. In several excellent mathematics lessons, some common features of classroom activities were found, such as fast rhythm, frequent change, and abundant content (Huang, 2009).

The Mathematics Curriculum Standards (later referred to as *the Standards*) indicated that developing students' symbol sense is an important content for learning at the stage of compulsory education. *The Standards* also claimed that making use of letters to represent quantitative relations and patterns abstracted from the specific contexts is a main way to develop students' symbol sense (Ministry of Education, 2001). In the content of Number and Algebra, *The Standards* put forward specific objectives as following: "At the second school stage, students can use letter to represent number in specific situation. At the third stage, students can further understand the significance of using letter to represent number in real situation. They can use algebraic expression to represent quantitative relation in the real world."

In fact, the usage of letter has multiple meanings including: (a) Representing unknown number, which can be operated directly. For example, the letter x in the equation of $x+3=5$ represents a certain but unknown number. (b) Representing general number, which is a generalized number. For example, letter can be used to represent the commutative law of addition ($a+b=b+a$), in which the letter can represent a real number. (c) Representing variation, which can describe the relation between two quantities. For example, the letters of x and y in the linear function $y=2x+1$ represent the relation between the two quantities. (d) Representing parameter. In the elliptic parameter equation $x=3cost$, $y=4sint$, the letter t represents a parameter (Kieran, 2006). Many studies showed that students had many difficulties in understanding use of letter to represent number (Bills, 1997, 2001; Fujii, 1993 Trigueros & Ursini, 1999; Ursini, 1990; Ursini & Trigueros, 1997). For instance, the well-known CSMS (Concept in Secondary Mathematics and Science) study found that students endowed letters with six different meanings at four levels, which are correspond to four development stages described in Piaget's theory (Kuchemann, 1981).

Usiskin (1988) argued that teachers how to teach use of letter to represent number is originated from their understanding of algebra. If algebra is regarded as the generalization of arithmetic, then letter is the generalization of number. If algebra is regarded as a tool of solving problem, then letter stands for an unknown or constant. If algebra is regarded to exploring quantitative relation, then letter stands for a variation or parameter. If algebra is regarded to studying structure, then letter is a conventional sign. Wagner (1983) argued that teachers should be aware of various use of letter and its meaning in different contexts, and tell students the difference and connection among letter, number, and word, if they want to help students understand the meaning of letter. Especially, he suggested that if they want to help their students have comprehensive understanding of the meaning of letter, they should permeate it gradually in mathematics curriculum, rather than depend on tedious sermon. Ursini and Trigueros (2001) proposed a theoretical model for teaching usage of letter in algebra. They believed usage of letter mainly including three dimensions (as unknown, general number, and variation) in elementary algebra course, and established teaching goals for each dimension. Through the teaching experiment, they found that students' understanding of usage of letter would be promoted if three-dimension teaching goals were integrated in the course (Ursini & Trigueros, 2002). However, few empirical studies in China explored how to teach usage of letter in algebra.

Grounded on Chinese new curriculum innovation, two research questions were posed: (a) How did teachers implement *The Standards* when they taught usage of letter to represent number at primary mathematics classrooms? (b) What was difference or similarity when *The Standards* were implemented in classrooms by different teachers?

2. Methodology

2.1 *Conceptual Framework*

The Standards set up four strands curriculum objectives including knowledge and skills, mathematical thinking, solving problem and affect and attitude. It also gave brief explanation for the four strands in each school stage. The concept framework in this study is constructed on the four strands of *The Standards*. *The Standards* claimed that knowledge and skills is important for students learning mathematics. The stand is a four-level hierarchy involving knowing, understanding, mastering and flexible applying (Table 1). Knowing refers to give relative examples for interpreting a concept, and identify an object in different contexts. Understanding means describing difference and relationship between different objects. Mastering refers to transferring current knowledge to new contexts. Flexible applying means synthesizing knowledge and selecting reasonable methods to complete specific mathematics tasks (Ministry of Education, 2001). According to the four-level hierarchy definition, the sub-objectives on the topic of using number to represent number were listed in Table 1. The second strand is mathematical thinking, which refers to students could discover mathematical phenomenon within various problems which they are face with, and could apply mathematics knowledge and method to solve it (Ministry of Education, 2002). *The Standards* further constructed sub-objectives for the strand in the second school stage. Specially, in the topic of using letter to represent number, three sub-objectives were set up in this study, including use of number and letter to describe real problem, plausible reasoning, and deductive reasoning. The third strand is solving problem. Problem should not be defined as pure mathematics problem. Solving problem is not an action as recognizing problem type, recalling problem solution, or imitating closed example. It also should be distinguished from practicing exercise. Solving problem requires student mathematics thinking as conjecture, discourse, reasoning, and so on (Ministry of Education, 2002). *The Standards* set up sub-objectives based on Pólya (1945) how to solving problem. Five sub-objectives of this topic are shown in Table 1. *The Standards* stated that student's affect and attitude could be fostered by mathematics teaching. For instance, teacher could encourage students to see world on the perspective of mathematics so as to foster their intellectual curiosity and thirst for knowledge. Teacher could improve student's engagement through designing various classroom activities. Teacher could set some obstacles for students when they solve problems, then encourage them to overcome these obstacles, so that students could gain successful experience after getting over difficulties. In classroom, teacher could encourage students to communicate ideas, and pose questions so as to improve their independent and creative thinking (Ministry of Education, 2002). Based on the above mentioned causes, six sub-objectives were set up in this study according to curriculum objectives (Table 1).

2.2 *Participants*

Because of limited research funding, classrooms were just selected from three primary schools in Changshu, located in south-east China, and affiliated to Suzhou. Changshu has 1, 266 square kilometres, 1.04 million population, and 11 towns. According to the location, teaching quality, and school size, three primary schools S, Y, and L were chosen. School S is the best local primary school, located in the city centre. The school history can date back to Qing Dynasty (1636-1912), and it has formed a unique educational tradition. School Y was established before 1949, and has just moved to a new campus at eastern city three years ago because of school expansion. School L is a town primary school at 20 kilometres west of the city, which was founded in the 1950s. As the study was to understand the implementation of the curriculum innovation, each teacher, who has witnessed the whole curriculum innovation, was chosen from each school.

2.3 *Data Collection and Analysis*

In May 2011, three teachers' lessons were observed and videoed. Each teacher taught the same topic of using letter to represent number. They used the same version textbook and teaching reference book published by Jiangsu Education Publishing House. The textbook divides this topic into three sections. In the first section, the textbook content includes use of letter to represent quantitative relation with single operational sign (e.g., $5+a$, $3a$), and writing rules of algebraic expression with multiplication sign. The section content includes use of letter to represent quantitative relation with two operational signs (e.g., $5+2a$), and finding the value of an expression. The third section requires students to use letter to represent a sum of two products with same factor (e.g., $5a+2a=7a$). Each teacher spent same lessons (four lessons) on teach this topic. Each lesson spent about 40 minutes. In each classroom, two video cameras were prepared. One aimed at teacher, the other one focused on a group of students. After lessons, teacher and students in that group were semi-structured interviewed. The video-lessons and audio-interviews were transcribed into scripts.

In classroom teaching study, each lesson could be divided into several episodes or units according to the change of classroom elements, such as classroom organization, teaching content, and student activity and so on (Doyle, 1986). The intention of this study is to explore how curriculum

Table 1. The framework for episode analysis

Knowledge and skills	Knowing	Knowing different means of letter in life and mathematics. Knowing use of letter to represent number. Knowing expression means Distinguishing multiplication from addition
	Understanding	Understanding use of letter to represent number Understanding square meaning Understanding operational law Understanding the basis of combining like terms
	Mastering	Using letter to represent quantitative relation in realistic situation Using letter to represent pattern Finding expression value Mastering the rule of writing algebraic expression Mastering the rule of combining like term Mastering the rule of expression multiplication
	Flexible applying	Flexible use of letter to represent quantitative relation in realistic situation
Mathematics thinking	Use of number and letter to describe quantity in the real world Plausible reasoning Deductive reasoning	
Solving problem	Finding and posing simple problems from real life Exploring effective ways to solve problem, and try to find other way Learning to cooperate with peers in problem solving Expressing problem solving process, and explaining result Reviewing and analyzing the process of problem solving	
Affect & Attitude	Engaging to discuss mathematical problem perceiving connection between mathematics and life Experiencing successfulness after overcoming difficulties in problem solving Experiencing exploration and challenge in problem solving Perceiving logicality of mathematical thinking, and certainty of mathematical result Having consciousness of questioning	

objectives were carried out in classrooms. In fact, curriculum objectives were realized in the classroom teaching steps (e.g., review & introduction, teacher-students interaction, classroom practice, group discussion). In general, each teaching step involves several activities. For example, the introduction of SL1 (lesson 1 at S primary school) could be divided into three episodes by the change of student activity. The first one that students played a game named counting 24 points with cards could be encoded as SL1-1. The second one that teacher required students to look for letters in real life could be encoded as SL1-2. The last one that students played magic box game could be encoded as SL1-2. Like this, 96 episodes in 12 lessons were obtained (Table 2). Then the framework was employed to encode lesson episodes. Each lesson episode was encoded by two researchers. The consistencies of coding episode were between70% and 85%. The code system was constructed for the classroom analysis.

Table 2. Episodes in lessons from three primary schools

Primary School	L1	L2	L3	L4	Total
S	8	7	10	7	32
Y	11	8	7	6	32
L	10	6	7	9	32

3. Results

3.1 *Knowledge and Skills*

(1) Knowing use of letter to represent number. Using letter to represent number has four hierarchy levels in classroom teaching. Knowing use of letter to represent number is the primary level. In order to provide opportunities for their students to appreciate letter used to represent number, the three teachers set up different situations at the beginning of the classrooms. S school teacher designed a game named counting 24 points with playing cards (SL1-1). She showed “5, 2, 8 and A” four cards, and ask her students to count with 24 points. When they calculated with numbers, students naturally used the letter A to

represent the number 1. She then asked her students: "which letter can represent number in playing cards?" Differently, the other two teachers posed mathematical problems in their classrooms (YL1-2, LL1-2). For example, which number does the letter represent in the pattern 2, 4, 6, *m*, 10? It is well know that using letter to represent specific number is a foundation for student learning equation.

(2) Understanding use of letter to represent variation. Using letter to represent variation should be understood by students. S school teacher designed a magic box game (SL1-3). She said: "This is a magic box. If I put a number into the box, another number will come out. The number 2 enters, and then the number 12 comes out. …Who can find the secret of the magic box? " After several minutes, Students used words to generalize the pattern that coming-out number plus 10 equals entering number. Then, they tried to use letter *n* to represent the entering number, and using n+10 to represent the coming-out number. In fact, using letter to represent variation is a preparation for students to learn important mathematical concept, such as function.

(3) Mastering using letter to represent quantitative relation in realistic situations. Students should master how to use letters to represent quantity relations in realistic situations. For example, the three teachers employed the same problem from their textbook (SL4-5，YL4-6，LL4-5): A theatre has rows of seats upstairs, and each row has 22 seats. There are *b* seats totally downstairs. How many seats are there at this theatre?

(4) Flexible use of letter to represent quantitative relation. Teacher required students to use letter flexibly to represent quantitative relation. For example, S school teacher and L school teacher implemented the task from their textbook: In the following shapes (Figure 1), select two or more and make up a rectangle. Can you use letters to present length, width and area of the rectangle?

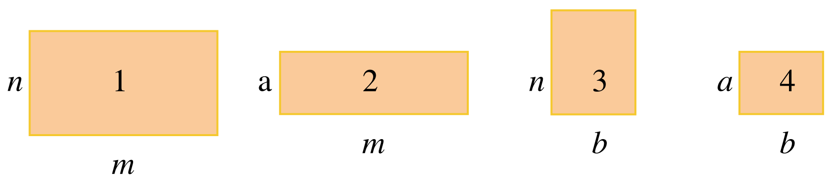


Figure 1. Rectangle problem

Table 3 shows the high frequencies occurred at the first three hierarchy levels. When the three teachers carried out objectives on knowledge and skills, there was no significant difference in their classroom (2(6)=9.341, *p*=0.155). It indicates that teachers emphasized on developing students' knowledge and skills in their classrooms. Moreover, each teacher consistently preferred to teach at the mastering level (2(3) = 42.925, *p*<0.001; 2(2)=73.524, *p*<0.001; 2(3)=63.090, *p*<0.001). It means that teachers focused on high cognitive level when they implemented objectives on knowledge and skills in mathematics classrooms. Specifically, teachers intended to enhance student's ability to use letter to represent quantitative relation in realistic situations. In contrast with *The Standards*, teachers' requirement was higher than curriculum objectives.

Table 3. Knowledge and skills: curriculum objectives carried out in three schools

Knowledge and skills		S(32)	Y(32)	L(32)
Knowing	Knowing different means of letter in life and mathematic	1	1	1
	Knowing use of letter to represent number	1	1	1
	Knowing expression means	2	1	2
	Distinguishing multiplication from addition	5	0	7
Understanding	Understanding use of letter to represent number	3	2	4
	Understanding square meaning	1	1	3
	Understanding operational law	1	1	2
	Understanding the basis of combining like terms	5	3	2
Mastering	Using letter to represent quantitative relation in realistic situation	13	21	20
	Using letter to represent pattern	3	3	2
	Finding expression value	2	5	4
	Mastering the rule of writing algebraic expression	6	14	11
	Mastering the rule of combining like term	8	10	7
	Mastering the rule of expression multiplication	1	0	0
Flexible applying	Flexible use of letter to represent quantitative relation in realistic situation	1	0	1

In China, that the principle of "basic knowledge and basic skills" was explicitly put forward for the teaching of mathematics has a long tradition. Most Chinese teachers believe that knowledge and skills are a foundation for learning mathematics. Without a solid foundation it is impossible to realize student's creativity, and ultimately let alone students' individual development (Zhang, Li, & Tang, 2004).

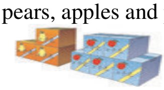
In China, examination is an important part of school mathematics teaching and learning. In fact, examination achievement is one of important evaluative indexes for teachers and students' work. In order to help their students overcome different items and gain high performance in examinations, teachers usually increased requirement for students learning when they implemented *The Standards* in their classrooms. Teachers believe that difficult and deep content is beneficial for students' learning. Even if they could not reach the level, students also could develop their inspiration from the learning process. Teacher' belief maybe originated from ancient philosophy. Ancient Chinese philosophers believed that only when a high goal was set, a satisfying achievement could be gained (取 ，得 取 ，得 ; 取 ，则 得).

3.2 *Mathematical Thinking*

(1) Using number and letter to describe quantity in the real world. Mathematical thinking involves three aspects. The first one is using number and letter to describe quantities in the real world. The textbook highlights this aspect and supplies a large number of exercises for student practice. These textbook exercises were totally employed by the three teachers in their classrooms (Table 4).

Furthermore, the L school teacher selected a problem from other material in her classroom (LL1-5). She said: "Let us play a game now. Look at this picture. There are three figures. There are three expressions of $y-25$, y, and $y+21$. If we use them to express ages, which expression can present grandmother, mother, or girl's age?" When students solved the problem, they should connect figure's age and appearance, and compare the quantities represented by expressions (see the pictures shown in Figure 2).

Table 4. Textbook exercises on using number and letter to describe quantity in the real world

<table>
<tr><th>Column in the textbook</th><th>Page</th><th>No. of exercises on using number and letter to describe quantity in the real world</th><th>Example</th></tr>
<tr><td>Thinking and doing</td><td>107</td><td>4(80%)</td><td>4. Fill in formula containing the letters in parentheses.
(1) A coat costs x dollar(s), a pair of trousers is 12 dollars cheaper than the coat. The pair of trousers is ( ) dollar(s).
(2) Xiao Gang reads 15 pages of book every day, ( ) pages are read totally in x days.
(3) There were 35 people in a bus, x people got off the bus at Xinjie station while y people got on. Now there are ( ) people in the bus.</td></tr>
<tr><td>Thinking and doing</td><td>109</td><td>3(60%)</td><td>3. Fill in formula containing the letters in parentheses.
(1) Xiao Ling bought 1 pen and 4 notebooks in a shop, a pen was 7 dollars, a notebook was x dollar(s). She paid ( ) dollars in total.
(2) Canteen imported x bags of rice, one bag of rice weighed 50kg. It had been eaten y kg, there were still ( ) kg remaining.
(3) Young Pioneers play mass games. There are x boys and y girls in a line, standing 8 lines, ( ) people take part in it totally.</td></tr>
<tr><td>Thinking and doing</td><td>110</td><td>4(80%)</td><td>1. There are ( ) kg of apples and ( ) kg of pears, apples and pears are ( ) kg in total.

apple is ( ) kg heavier than pear.</td></tr>
<tr><td>Exercise 9</td><td>112</td><td>3(50%)</td><td>3. Fill in the appropriate formula of the blanks in the table below.
<table><tr><th>Speed (km/h)</th><th>Time (h)</th><th>Distance (km)</th></tr><tr><td>80</td><td>t</td><td></td></tr><tr><td>v</td><td></td><td>s</td></tr><tr><td></td><td>t</td><td>s</td></tr></table></td></tr>
</table>

grandmother mother girl

Figure 2. Pictures of grandmother, mother, or girl

(2) Plausible reasoning. Developing student plausible reasoning through experiment, induction, and analogy is the second aspect of this strand. In the textbook, three exemplars (totally six exemplars) were designed for students exploring geometric patterns. The textbook authors believed that geometric patterns are valuable materials for improving students' plausible reasoning. Teachers should require students to make generalization based on numerical sentences in list when they employed these exemplars in practice (Sun & Wang, 2011). The following pattern is the first exemplar in the textbook (Figure 3).

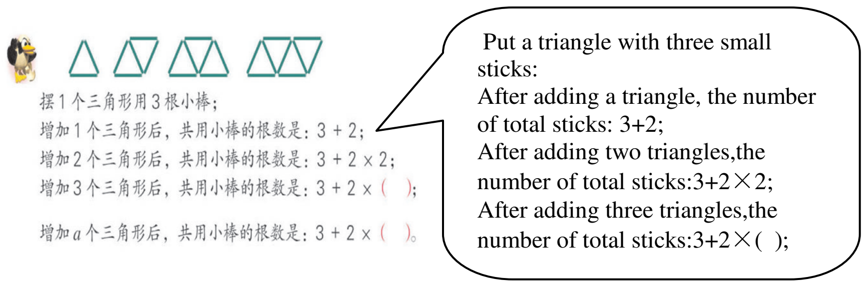


Figure 3. Triangle pattern

(3) Deductive reasoning. Developing students' deductive reasoning in problem solving is the third one. One case is from S school teacher classroom (SL3-1). She required students to explain why *3a +4a* equal to *7a*. She asked: "why do you say the two expressions are equivalent?" One student answered: "*7a* is the result of distributive multiplication. You know 3 plus 4 multiplied by *a*." In this episode, the teacher required students to not only know how to operate, but also know the rule of operation.

Another case is from L school classroom (LL4-1). In the textbook, there is a question whether $2x$ equals x^2. It's an easy question. Students answered correctly. The teacher asked: "can you let the two expressions equal?" She let the easy question become an interesting question. The equal in the former means the two expressions are identically equal, but the meaning of the equal in the latter means is different. It means that the values of the two expressions could be equal if special number was selected. When students answered the question, the teacher asked: "what's the relation between them?"

> *T*：which one is bigger? Let us try!
> *S23*：$x=1$，$2x=2$，$x^2=1$，$2x>x^2$.
> *S24*：$x=3$，$2x=6$，$x^2=9$，$2x<x^2$.
> ……

In fact, there exists logical relation among the three questions. When the teacher elicited her students to think step by step, they should consider the condition of statement, and make it classification. Therefore, the teacher provided a good opportunity for students to improving their deductive reasoning.

Table 5. Mathematical thinking: curriculum objectives carried out in three schools

Mathematical thinking	S(32)	Y(32)	L(32)
Use of number and letter to describe quantity in the real world	9	15	13
Plausible reasoning	3	2	3
Deductive reasoning	6	6	5

Table 5 data shows that there was no significant difference among the three teachers, when they implemented the curriculum objective on mathematical thinking (2(4)=1.307, p=0.860). Table 5 also shows that the frequencies occurred in three aspects are different. It was found that Y and L school teachers emphasized that students should learn to use number and symbol to describe quantity in the real world (2(2)=11.565, p=0.002; 2(2)=8.000，p=0.018). Although there were no significant difference, the frequencies varied widely, when the S school teacher implemented the three aspects of the strand (2(2)=3.000, p=0.223).

It indicated that when the three teachers carried out the objectives on mathematical thinking, their attention on the three aspects maybe was different. It was also found that each teacher paid more attention to use of letter to represent quantity in real life, but overlook improvement of plausible reasoning. Although, *The Standards* did not explain how to deal with the three aspects in classroom practice, the used textbook gave teachers a hint in their practice, because a large number of exercises on the first aspect were designed in the textbook. As was natural, under the guidance of the textbook, teachers spent more time on using letter to represent quantity in the real world when they implemented *The Standards*. In fact, using number and letter to describe quantity in the real world could be seem as a preliminary modeling, which was list as one of core concepts in revised *Standards* published in 2012 (later referred to as *The Standards* 2011, Ministry of Education, 2012).

However, to a surprising extent, teachers ignored the second aspect, even if the textbook provided 50% examples on plausible reasoning. In contrast, teachers did not overlook improving student's deductive reasoning, even if the textbook gives only one example (Figure 4). It was well known that the two kinds of reasoning have distinctive roles in solving problem. They complete each other. In plausible reasoning the principle thing is to explore solution and discovery conclusion. In deductive reasoning the principle thing is to proof conclusion (Ministry of Education, 2012). Furthermore, it was state clearly that in the first and second school stage plausible reasoning students should be provided more opportunities to learning plausible reasoning and in the third school stage deductive reasoning should be highlighted (Ministry of Education, 2012). Therefore, in primary school teachers should pay more attention to plausible reasoning than the other in their classroom practice. Unfortunately, they did nott do so.

In the teachers' interview, they presented mathematics as a well-considered and close-knit science. In their eyes, deductive reasoning had more important role than plausible reasoning for students learning mathematics. In fact, Chinese traditional mathematics education emphasizes on "logical and formal expressions of mathematical concepts, an awareness of logical accuracy of categorization and mathematics propositions and conformity of reasoning in solution process to rigorous logical rules with sufficient reason and being expressed in a clear and formal way" (Zhang, Li, & Tang, 2004).

Xiaohua put *a* triangle(s) with small sticks, Xiaofang put *a* square(s) with small sticks,how many sticks they take in all?

Xiaohua takes $3a$ sticks, Xiaofang takes $4a$ sticks, the number of total small sticks are (**$3a+4a$**).	Seven sticks are taken to put a triangle and a square, thus they take $7a$ sticks in all.
$3a+4a=7a$	What operation has applied in fact ?

Figure 4. Why is the equation $3a+4a=7a$ true?

3.3 *Solving Problem*

There were five aspects in the curriculum objectives of solving problem. However, Table 6 shows S school teacher just carried out two of the five aspects in this strand. Y school teacher and L school teacher carried out three aspects of it. However they just emphasized on the two aspects.

(1) Exploring effective ways to solve problem, and try to find other way. They usually required their students to explore effective ways to solve problems. For example, the three teachers carried out a same task provided by their textbook: Hua constructed a triangle with toothpicks, and Fang constructed a square with toothpicks. How many toothpicks did Fang use more than Hua? Two ways were explored and explained by students in the three schools. When students used $4a-3a$ to present the answer, teachers encouraged them to employ $(4-3)\times a=a$ to present it again, and elicit them to construct the equation $4a-3a=(4-3)\times a=a$.

(2) Expressing problem solving process, and explaining the result. Sometimes, when students got a correct result, teachers would give some wrong answers, and ask students to judge and explain. One case was observed from Y school when the teacher asked her students to use expression to represent an angle (∠3) degree in a triangle. The two angles in the triangle were given, which are ∠1=a°, and ∠2=b°. After a student answered correctly, the teacher said: "I also find two other answers from the class. One is $180°–a°b°$, and the other is $180°– a°+b°$. What do you think?"

Table 6. Solving problem: curriculum objectives carried out in three schools

Solving Problem	S(32)	Y(32)	L(32)
Finding and posing simple problems from real life	0	0	0
Exploring effective ways to solve problem, and try to find other way	5	8	6
Learning to cooperate with peers in problem solving	0	1	2
Expressing problem solving process, and explaining result	9	7	8
Reviewing and analyzing the process of problem solving	0	0	0

It indicated that in classrooms students lacked opportunities to pose mathematics problems from real life, and review in problem solving. Students had a few opportunities to cooperate with peers in classrooms. On the other hand, students were provided so many opportunities to exploration, explanation, and discussions in classrooms. In other words, the classroom formed a special accountability structure (Schoenfeld, 2007). Teachers posed all problems for students, elicited students to explore multiple ways to solve problems, and encourage them to provide explanation in public. Students solved all problems followed their teachers. It means that teachers controlled content which was taught in classroom, but they opened solving problem space for students. This phenomenon reflected a special classroom culture that student learning is an autonomic learning under teacher's leading. However, the classroom environment should be seemed as a productive learning environment. Engle and Conant argued that highly productive learning environments have substantial consistencies. Common characteristics of those environments are that students are encouraged to take on intellectual problems, and given authority in addressing such problems (Engle & Conant, 2002).

However, *The Standards* also claim that classroom practice should emphasize on students posing problem, peer cooperation, and solving review (Ministry of Education, 2001). Why were these aspects overlooked in classrooms? In fact, in China, the textbook used by teacher is unified by local educational bureau. In order to supervise teaching, local Educational Bureau set up Teaching and Research Department to carry out academic activities, teaching evaluation, and achievement examination. The education system leads local different schools to keep substantial consistencies on content taught, learning requirement, and teaching schedule. In each school, Lesson Preparation Group of grade (LPG) also is set up. The mathematics LPG consists of all mathematics teachers in a grade. Teachers in group discuss and determine term teaching plan, unit teaching plan, and week teaching plan. They also unified teaching objectives, homework, and examination in the group. In this environment, any teacher has to keep consistency with other colleagues. The effective way to do this was to control what is taught in classroom, and determine teaching pace by prepared lesson plan. In fact, these classroom activities, such as students posing problem, peer cooperation, and solving review can easily disarrange lesson plan unless a teacher have good pedagogical content knowledge. Moreover, in China, there are about forty students in each classroom more than in western developed county. In such classroom environment, it is difficult for teachers to organize and manage students to discuss and cooperate in groups. These factors lead to the result that teachers did not want to run any risk in their classrooms.

3.4 *Affect and Attitude*

Table 7 indicates that there was no significant difference in the three schools, when the three teachers carried out the objectives on affect and attitude (2(10)=7.334, p=0.694). However, there existed significant differences among the six aspects in the strand (2(5)=74.889, p<0.001; 2(4)=59.179, p<0.001; 2(3)=39.604, p<0.001). Table 7 shows that classroom teaching in the three schools improved students to engage in classroom activities, and encourage them to discourse in public. In students' interviews, we also can feel it. For example, Lin (a boy from S school) said: "my teacher always gives us so much time to explore and discuss difficult problems." Jun (a girl from L school) said: "we like to discuss with my teacher and classmates."

Table 7. Affect and attitude: curriculum objectives carried out in three schools

Affect & Attitude	S(32)	Y(32)	L(32)
Engaging to discuss mathematical problem	32	32	32
Perceiving connection between mathematics and life	9	15	13
Experiencing successfulness after overcoming difficulties in problem solving	1	0	0
Experiencing exploration and challenge in problem solving	3	2	3
Perceiving logicality of mathematical thinking, and certainty of mathematical result	6	6	5
Having consciousness of questioning	3	1	0

However, students had few opportunities to experience successfulness after overcoming difficulties in classrooms. In fact, when a student gave a wrong answer, teachers usually asked other students to resolve it other than gave him or her one more opportunity to correct it.

Moreover, Teacher should provide adequate opportunities for students to question peer ideas. In the following episode (YL1-3), we can find that student's question is valuable for mathematics teaching.

T: There are 3a toothpicks. Do you know what the letter a represents?
S9: Any number.
T: Any number?
S10: Except zero.
S11: Zero is Ok! Construct zero triangles with zero toothpicks.
S12: Any natural number. Because a can not represent decimal fraction, such as 1.3, 4.5.

In fact, as *The Standards* claim, the four strands of curriculum objectives are integrated into an organic whole. Curriculum objectives could be achieved in diverse mathematics activities. The four strands complement and reinforce each other (Ministry of Education, 2001). For example, as above mentioned, teachers emphasized on how to use number and letter to represent quantitative relation in real life, therefore, students could perceive connection between mathematics and life.

As the three teachers highlighted some aspects of other strands, students could have more opportunities to foster related affect and attitude.

4. Conclusion

In the context of curriculum innovation, the three teachers' lessons showed great similarities. They developed students' knowledge and skills of using letter to represent quantitative relations beyond suggested in *The Standards*. In the strand of mathematical thinking, classroom focused on using of number and letter to describe quantity in the real world. Teachers made light of developing students' deductive reasoning in comparison with plausible reasoning. When students solved problem, teachers improved students to use multiple-way and provide explanations. However, they provided students few opportunities to pose questions from realistic context. Students lack opportunities to reflect on problem solving. These implemented objectives influenced on students' affect and attitude. It indicated that curriculum objectives were unbalanced carried out in classrooms, although curriculum innovation had carried out for about ten years. Moreover, the implemented unbalance is a common characteristic in mathematics classrooms.

In fact, characteristics of curriculum innovation itself will affect the process of implementation (Fullan & Stiegelbauer, 1991). Curriculum innovation is complexity which reflects the amount of new skills, altered beliefs and different materials and so on required by innovation. A crucial factor is the innovation's clarity (about goals and means). Even if teachers have unanimously agreed to implement an innovation, and intend to make changes in their practice, they are not clear about what they are expected to do differently when they face with curriculum research unearthed examples. Teachers expect that teaching strategies are clearly described, and material is well-thought of. Perhaps, *The Standards* lacked adequate examples and cases, and was short of specific suggestions and strategies for practice for teachers' use. Fortunately, some changes have been made in *The Standards 2011*.

Any national curriculum innovation is immersed in its cultural tradition and national policies. In Chinese tradition, teachers are required to shoulder main responsibility for students' learning. It was written in an ancient classic that to teach without severity is the teacher's laziness (，). Therefore, it is no wonder that teachers' requirements are often beyond *The Standards*, and they usually teach for test, in order to help their students get good grades in unified examinations. They believe that preparation for examination is more important than teaching reform. Moreover, in teaching practice, teachers are often required to keep unified schedule, and complete the textbook contents. In such circumstance constraints, few teachers dare to open up their own classroom. They intend to take control of their classrooms, so as to ensure to carry out lesson plans successfully. On the other hand, however classroom discourse and cooperation in groups were highlighted by *The Standards*. In fact, these ideas originated from western curriculum innovation (see, e.g., *Principles and Standards for School Mathematics* by the National Council of Teachers of Mathematics (2000) and *Common Core State Standards* by the National Governors Association Center for Best Practices and the Council of Chief State School Officers (2010)). Now it seems difficult to actualize them in classrooms. Recently, a large-scale assessment found that teacher demonstration is also a main teaching approach in classrooms (Ren, 2012). In China, a story is very popular. It said that oranges grown south of the Huaihe River are true oranges; once transplanted to the north of the river, they become trifoliate oranges (橘 则 橘， 则). Although they resemble the shape of leaves, yet they differ widely in taste. What accounts for it? The difference is in water and soil. It means that the same thing, because of different factors, will show different results. From this point of view, curriculum innovation can not be separated from particular cultural tradition and national policies. Otherwise it is meaningless.

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Chapter 11

Chinese Project-Based Classroom Practices: Promoting Students' Engagement in Mathematical Activities

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We explore two cases of project-based instruction in Chinese Mainland which reflect the trend of change in current mathematics classroom teaching. The first case is to guide high school students in conducting statistical investigations. The second involves the mathematical activity of searching linear functions in real life. We evaluate students' engagement in the project-based classroom practices through multiple aspects. By analyzing the classroom practice reports, we find that teachers can guide students to incorporate their previous experience as well as innovative ideas in mathematical activities. The findings also suggest that Chinese teachers emphasize the collaborative learning process, which intensively involves the teamwork in the classroom practices. We also administer quizzes to the secondary school students participating in the classroom practice. The result indicates that the teachers can apply project-based learning activities to help students master the fundamental mathematical concepts and principles. Further survey feedback shows that these project-based classroom practices can improve students' engagement in mathematical activities.

Keywords: mathematics classroom practices, mathematical activities, students' engagement in mathematics classroom

1. Introduction

In the last decades, many studies have explored students' engagement in mathematics activities from different perspectives. For instance, Filloy, Puig and Rojano (2008) studied the nature of mathematical activities from the perspective of mathematical phenomena. They recognized that "mathematical objects are incorporated into the world of our experience, which they enter as phenomena in a new relation of phenomena/means of organization in which new mathematical concepts are created, and this process is repeated again and again" (p. 42). The authors emphasized that mathematics developed as the phenomena growing with each product of mathematical activities. Štech (2008) pointed out that mathematics activities in school mathematics should contribute to the development of thinking, motivation and identity from novice to expert. In order to clarify the contribution of mathematical activities, the author used a theoretical framework to analyze the features of school mathematics activities which have great potential for further development.

Some researchers explored the components of mathematical activities in classroom, such as classroom communication, collaboration, re-creation, learning motivation and appreciation of mathematics. Alrø and Skovsmose (2003) investigated the relation between the qualities of communication in the classroom and the qualities of learning mathematics. They stated that "qualities of communication can be expressed in terms of interpersonal relationships, learning is rooted in the act of communicating itself, not just in the information conveyed from one part to another." (pp.1-2). Steinbring (2005) investigated how new mathematical knowledge was interactively constructed in a typical instructional communication between students and teachers. The author claimed that true mathematical communications should be emphasized in classroom and "require the maintenance of a balance between situatedness and intended generality in the instruction, as well as the communication of given facts and properties with the intention of an interpretation and construction of relations" (p. 220). Middleton and Spanias (1999) reviewed researches about motivation for achievement in mathematics. They concluded that achievement motivation in mathematics can be affected through careful instructional design.

There are amount of studies which focused on designing effective teaching and learning environment to promote students' engagement in mathematical activities. For example, Bransford and his collaborators (1988) created problems situated with the useful applications of mathematics to stimulate students' imaginations. However, Štech (2008) discussed some limitation of such situated approach to mathematics learning. Hoek and Gravemeijer (2011) investigated the influence on students' mathematical activity manner due to teachers' instructional skills such as the group-oriented coaching style. Students gradually developed a more exploratory way of collaboration, which indicated improved collaborative learning outcome. Focusing on project-based classroom practices, Meyer, Turner and Spencer (1997) studied students' challenge during project-based mathematics instruction and illustrated how to support students in challenging academic work, such as a project-based learning. Boaler (1998) reported that students who learned mathematics in an open, project-based environment developed a conceptual understanding and project students had been "apprenticed" into a system of thinking and using mathematics in both school and non-school settings. Puntambekar (2005) discussed the important features of scaffolding theory such as on-going diagnosis, calibrated support and fading, etc. He also investigated how to apply scaffolding tools to support student learning in project-based and design-based classrooms.

Since the year of 2000, Chinese researchers started to pay more attention to mathematical activities. A major research area is to investigate the characteristics of mathematical activities (e.g., Deng, 2009; Huang & Tong, 2008; Li & Shi, 2012; Wang & Xia, 2012). Deng (2009), for example, pointed out that mathematics activities can help students to recreate mathematical concepts and solve mathematical problems based on their previous knowledge and experience. Wang (2012) emphasized that mathematics activity in classroom should be based on "designing problem clusters" which facilitates students to experience and engage in problem solving process. Such mathematical activities can also enhance students' self-motivation on learning mathematics.

The mathematics education researchers in China also focus on how to design effective mathematical activities from different angles (e.g., Li, 2004; RGCCT, 2012; Zhong, 2009). For instance, Zhong (2009) conducted case studies on the teaching process of several expertise mathematics teachers and analyzed how the carefully designed learning environment in classroom encouraged students to gain experiences of inquiring, reflecting and concluding mathematical problems in challenging mathematical activities. The Research Group of Core Contents and the Teaching in Primary and Secondary Schools (RGCCT) (2012) investigated the significance of information technology as a learning platform and pointed out that IT-based tools can be used as a basic mathematical activity.

Comparing with the research from other countries or economics, the Chinese researchers basically explore the ideas, features and roles of mathematics activities. However, the Chinese mathematics curriculum standards have been developed and modified since 2000, which affected the Chinese mathematics teaching and learning to a certain extent. The mathematics classroom has gradually changed with more emphasis on mathematics activities. In this chapter, we conduct two case studies to investigate whether these project-based classroom practices are effective in promoting students' engagement in mathematical activities. We also analyze the teachers' instructional method and students' feedbacks to probe which pedagogical strategies are effective in this project-based classroom practices.

2. General Characteristics of Mathematics Classroom Teaching

In the last decades, some studies explored mathematics classroom teaching in China and described several basic characteristics of mathematics classroom practices. Leung (2001) pointed out that the emphasis in Chinese mathematics classroom was on the mathematics content and the procedures or skills in dealing with the content. Moreover, Chinese teachers devoted great efforts in designing exercises with variation to develop students' mathematical abilities or enhance their understanding of mathematics knowledge (Gu, Huang & Marton, 2004; Huang & Leung, 2004; Ma, 1999; Xu et al., 2012). In addition, some studies pointed out that Chinese teachers also emphasized building up new knowledge on the stock of prior knowledge. An (2004) has investigated a Chinese teaching procedure in which the prior knowledge was reviewed and then connected to new knowledge in order to reinforce understanding of mathematics concepts. Also, Mok and Morris (2001) studied that the major form of interaction in classroom between individual pupils and the teacher was explication of text upon the teacher's request. The teachers raised questions to review materials, inspected students' comprehension and improved their presentation skills. However, these questions did not directly target at determining how the pupil came to a particular understanding or misunderstanding of certain mathematical concepts.

Such mathematics classroom teaching plays an important role in laying a solid foundation and building fundamental skills for the Chinese students. Li (1999, 2006) argued that the motto "practice makes perfect" was one underlying belief in mathematics class since many math teachers believed that through repeated imitation and practice, students would become highly skilled in solving mathematical problems. Some international assessments of mathematics education such as IAEP 1992, TIMSS and IMO exhibited high achievements of Chinese students who had taken large amount of routine practice and frequent tests. Li (2000) pointed out that the mechanism of routine practice should not be simply interpreted as mechanical imitation and memorization of mathematical rules and skills. He mentioned that the routine practice provided students with a necessary condition of concept formation and formed the first step of mathematical comprehension (Li, 2006).

The traditional Chinese culture also emphasizes on students' basic knowledge and basic skills of mathematics. Zhang, Li and Tang (2004) pointed out that the strong cultural backgrounds of mathematics education include the long-lasting agricultural culture, Confucianism learning culture, and the strict and unified examination systems. Base on such cultural traditions, the following general characteristics of mathematics classroom teaching can be identified (Xu et al., 2013):

- Emphasis on introducing new knowledge step-by-step
- Emphasis on the analysis of sample questions
- Emphasis on consolidation of knowledge by revision
- Emphasis on exercises and feedbacks

The general characteristics above indicate that the Chinese mathematics classroom practices are well structured. Teachers often use several mathematical teaching models in the mathematics classroom, e.g., the trial-instruction and feedback-regulation to achieve overall improvement in teaching quality (Gu, 1997a; Gu, 1997b), or the GX (*Gao Xiao*, Chinese phonetic symbol for "High Effectiveness") mathematical classroom teaching experiment to increase the effectiveness and reduce learning difficulties (Li & Wei, 1999).

3. Current Changes of Mathematics Classroom Teaching in China

Several international studies revealed the relation between the mathematical achievements and the Chinese culture (Fan & Zhu, 2004). When entering the 21st century, the Chinese government endowed education with tasks to deepen education reform, optimize education structure and push forward the implementation of quality education (Pan, 2005). Hence, the mathematics curriculum in China needs innovation to face the new challenges such as how to meet the needs for young talents who should not only build a solid foundation but also be capable in resolving complex situations. The new Chinese mathematics curriculum standards (MOE, 2001; MOE, 2012) were published for the first time, instead of revising the previous version of Chinese mathematics syllabus, which indicated an essential policy change. Unlike the traditional classes that focus on course content, the reformed Chinese mathematics curriculum emphasizes on student individual development. This standards-based reform has been studied from international points of view (e.g., Kulm & Li, 2009; Li et al., 2009) as well as Chinese perspectives (e.g., Cao, 2005; Jiang, 2005; Sun, 2005).

One debate among the Chinese mathematics education researchers and teachers is about the value and goals of the mathematics curriculum. The trial version of mathematics curriculum standards (MOE, 2001) announced a goal of "mathematics for all". The debate argued that the original mechanism of "mathematics for the elite" should be adjusted (Sun, 2005). The revised standard states that "every student must receive well-grounded mathematics education, while different students should have the opportunity to develop themselves in mathematics differently" (MOE, 2012, p. 2). The revised standard (MOE, 2012) promotes the general objective of mathematics education to be basic knowledge, basic skills, basic thoughts and basic experiences. It outlines students' skills as discovering, raising, analyzing and solving problem and emphasizes on students' ability to raise questions and find solutions.

Different voices have emerged regarding the structure and revision of curriculum content. For example, a heated dispute arose on the reform of geometry. The trial version (MOE, 2001) changed the traditional geometrical reasoning system by combining plausible reasoning with deductive reasoning. Jiang, a mathematician, expressed alternate views, "Plane geometry is thought to be too difficult for junior secondary school students. However, without including it in the junior secondary school curriculum, students will lack scientific spirit and competencies in generalizing, summarizing, and abstracting …" (Jiang, 2005).

Based on the standards, the inquiry learning mode of "understanding situations, creating models, finding solutions, making applications, and invoking reflection and generalization" was advocated. But in reality, teachers found that although the classrooms became more active with the new method, teaching results were often less satisfactory. Sometimes, students rarely learned anything in one teaching period as they took part in more activities but thought less about the knowledge behind these activities. Some students joined in activities for fun and experience little intellectual involvement (Cao, 2005). It seemed that teachers wanted to give up using new teaching methods. In fact, the reform of teaching methods brought changes into the classroom. Students are encouraged to participate in more activities to be more active and independent in the learning process (Ma et al., 2013). Meanwhile, teachers are facing more challenges. They need to think about how to engage students in the exploration activities, and how to balance the independent exploration and mastery of knowledge.

Currently there are few research-based works which investigate the characteristics of mathematics classrooms other than the general features (Huang, 2008). In the following section, we use case study to exhibit how teacher design classroom practice to promote engagement in mathematics activities. The goal of these mathematical activities is to develop effective teaching models within which students can actively work, think, understand, and reflect on mathematics. In addition, students are motivated in these reformed classroom practices to communicate more with their classmates. We will also analyze the limitation and difficulties in the classroom practice.

4. Project-based Classroom Practices Promoting Students' Engagement in Mathematical Activities

Below we describe teachers' pedagogical methods for two project-based mathematics classroom practices that attempt to promote students' engagement in mathematical activities. One practice was administered in high school and the other was conducted in middle school. Both practices were integrated in the teachers' syllabus as regular projects for their curriculum. To investigate the effectiveness of the project-based classroom practice in promoting students' engagement in mathematical activities, we first recorded and analyzed the design of the classroom practices. Then we explored the learning outcomes of the students who have participated into the project-based classroom practice. We also summarized the common difficulties that the teacher and students may encounter in the teaching and learning process of mathematical activities. Qualitative and quantitative evaluations of students' engagement were administered at the end of the project-based classroom practice.

4.1 *Project-Based Classroom Practice Case 1: Engaging Statistical Investigation in Class*

4.1.1 *Design of the Project-Based Classroom Practice*

The first project-based classroom practice was administered to the first-year high school students (10th grade) in the spring semester. When teaching the content about statistics (a mandatory topic of 10th grade mathematics), the teacher required the students to form several study groups. A study group usually consisted of four to six members and the students can freely choose their group partners. The teacher guided each group to first brainstorm together and pick out the investigation topic of their group. In the period of project-based classroom practice, the teacher taught students the statistical concepts and principles through traditional lectures. At the same time, the teacher monitored each study group to apply the knowledge they learnt in class and conduct in-depth investigation of the topic they selected. Students needed to complete a studying report and submit it to the teacher. At the end of the regular lecturing period of this chapter about statistics, the teacher would observe and evaluate students' presentation of their projects. In Table 1 below, we use the timeline of one class carrying out the classroom practice in 2012 to illustrate a typical investigating process.

Table 1. The timeline of the classroom practice for the study groups investigating the topics about statistics

May 10th – May 13th (one 45-min class)	Brainstorm and confirm the investigating topic of each group. Design the corresponding surveys and questionnaires (before learning the chapter of statistics).
May 14th – May 20th (two 45-min classes)	Collect data and conduct preliminary analysis (when learning the topics about random sampling).
May 21st – May 25th (two 45-min classes)	Draw statistical conclusion based on the data analysis (when learning the topics of estimating population behavior from sample data).
May 26th – May 28th (weekend, no class)	Complete the report and make posters of the investigating topics (when learning the topics of correlation of variables and reviewing the chapter of statistics).
May 29th (one 45-min class)	Present the outcome of each study group.

The group members collaborated to complete the tasks of selecting topics, making surveys, collecting data, conducting statistical analysis and presenting their works. This project-based classroom practice can be divided into three stages as described below.

Stage 1. Design survey questions and make investigation plan. At the beginning of the classroom practice, students should clarify the research questions in their investigation and write the corresponding surveys or questionnaires. Each group needed to send the draft of their questionnaire to the teacher and modify the survey questions according to the teacher's feedback. Some groups would collect data from medium such as books or internet instead of distributed surveys. Then for these groups, they needed to submit a detailed plan about the original source of their data, the method of collecting data and the principle of filtering data.

In this stage, the teacher's major role is to guide the students in brainstorming and help them organize ideas in a statistical viewpoint. The teacher would give specific suggestions about each group's investigation plan to make sure they started the classroom practice on the right track. The students selected the following topics to conduct statistical investigation:

- the nutritional components and target users for different drinks
- the market size of the low-investment movies in local Chinese movie industry
- the factors influencing the tendency of studying abroad for high school students in Beijing
- the factors influencing the attendance of after-school academy for high school students in Beijing
- the evaluation of different teams attending the 2012 Euro Cup and their supporting rate from Chinese high school students
- the coverage and targeted users of WIFI hot-spots in Zhong Guan Cun area
- the factors influencing the studying motivation for high school students
- the factors influencing the choice between science and social science majors for freshmen in high school
- the statistical analysis and prediction of NBA (National Basketball Association) final champions
- the average lifetime for people in different occupations
- the statistical analysis and prediction of gold medals won by China in 2012 London Olympics
- Chinese citizen's awareness of the history and political situation about Huang Yan Island
- the factors influencing the attendance of different sports for high school students
- students' expectation of "happiness" and the analysis of "happiness index"
- statistics about the academic degree received by basketball players and the influence of academic degree on players' career development in NBA
- the factors influencing the reading situation of high school students
- the situation about students' network usage and the corresponding guidance of public voice
- the trend of change in cell phones used by students
- the market size of different lotteries and students' opinions about lottery
- etc.

Stage 2. Data collection and analysis. The teacher first needed to help each group to decide how to select the samples from population corresponding to their investigation questions. Then the students would distribute the questionnaires and collect data from the selected samples. The teacher would also provide materials about how to use Microsoft Excel or SPSS for the students to analyze the raw data and generate the statistical diagrams. Based on the statistical diagrams, the teacher would examine the preliminary conclusion on the investigation of each group.

Stage 2 is the most challenging part in this investigating project. Students may encounter various difficulties when distributing surveys, organizing data or sketching diagrams. In this stage, the teacher would recommend suitable materials for students to learn and discuss. The teacher would also provide necessary guidance and suggestions according to the practical situation of each group.

Stage 3. Write analysis reports and present works of the investigation. The investigating works should be presented in three ways: reports, posters and powerpoint slides. Each group would write report based on the statistical result of their investigation. Students also need to make posters to introduce their investigating process. In this editing process, the teacher offered the students with specific requirement on the format of reports and posters. The teacher also provided suggestions for each group to improve and finalize their manuscript.

In the presentation day, the teacher would raise questions and comments according to students' presentation. After all groups had presented their works, the teacher would summarize and evaluate the effect of the classroom practice based on the overall performance of different groups.

This practice is a student-centered learning process. In this process, the teacher's major role is to instruct, participate in and coordinate the group activities. The relation between the students' activities and the teacher's responsibility in different stages are shown in Figure 1.

To facilitate the engagement in the classroom practice, the teacher needs to design suitable learning activities based on the target mathematical concepts and principles. The learning goals and requirements should be clarified to the students at the beginning of the project so that they can integrate these learning goals into their investigation appropriately. During the learning process, the teacher scaffolds the related mathematical knowledge to match the students' zone of proximal development. The teacher's duties also include participating in each group's investigation to monitor the project progress and providing necessary support to help students overcome their learning difficulties. At the end of the classroom practice, the teacher should

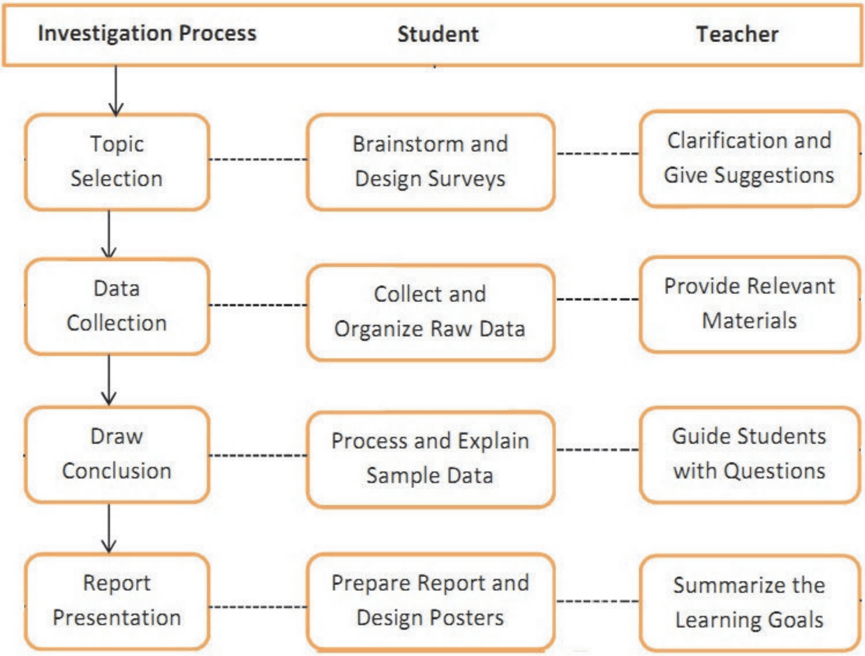

Figure 1. The relation between the students' activities and the teacher's responsibility in different stages of the classroom activity

summarize learning goals and comment on the outcome of each group in order to improve students' understanding of mathematics.

4.1.2 *Learning Outcome in the Project-based Classroom Practice*

The classroom practice of statistical investigation covers the major statistics contents required in the Chinese mathematics curriculum standards. The teacher would differentiate the statistical features of the concepts such as mean, median and mode. The teacher also guided the students in retrieving information from statistical diagrams (bar graph, line graph, pie graph, etc.) and denoting the frequency distribution with histogram. Moreover, students were expected to understand the characteristic of discrete and continuous data distribution. They also need to learn the appropriate sampling method to test different hypothesis of the population. These learning goals were carefully monitored by the teacher through the whole process of classroom practice.

While participating in the classroom practice about statistics, students were engaged with the learning process from multiple aspects. Below we selected several excerpts from students' investigation reports in 2012 to illustrate how the teacher conducted the project-based classroom practices that involved students' life experience, teamwork, and creativity.

A. Life experience reflected in the statistical investigation

At the beginning of the project-based classroom practice, the teacher guided the students to pay more attention to the topics which were direct reflections of the hotspots or the issues existed in their lives. For example, since the students are going to decide whether to choose science or social science as their high school major, the teacher encouraged a group of freshmen to investigate "the factors influencing the choice between science and social science majors for freshmen in high school". Another group of students carried out an investigation about "the factors influencing the attendance of after-school academy for high school students in Beijing" since many students and parents had the demand of tutoring. Through such investigations, students not only learned the statistical knowledge but also acquired a better interpretation of the issues they concerned about in life.

Moreover, the investigating project can even help some students to make important decisions in their life. There was one boy in the class whose parents wanted him to apply to the universities overseas. However, when discussing with the teacher, this boy expressed unwillingness to leave Beijing and go abroad. Therefore, in this project-based classroom practice, the teacher suggested him to conduct an investigation about "the factors influencing the tendency of studying abroad for high school students in Beijing". By processing and analyzing the collected data, the boy wrote the following conclusion in the investigation report:

> "… by surveying the intention of studying abroad, we conclude that the majority of the high school students in Beijing Haidian district prefer to study oversea. The main factors affecting their choice include the occupation and oversea experience of their parents, the strong need of self-enhancement of the students, the academic level and

> educational quality of the universities in North America and Europe, etc. Meanwhile, we also find that there are some difficulties impeding the plan of studying abroad, such as the conflict between studying English and other subjects, the deprived sleeping time, etc."

From the result of the investigation, this student realized that studying in an abroad university might be a promising development path. He started to seriously reconsider the suggestion of his parents. One week after his group submitted the investigation report, this boy came to the teacher and said he was going to study TOEFL (the Test of English as Foreign Language, required for applying the universities in the United States) in the summer and try to apply the universities abroad in the future. The statistical investigation helped him to plan his future academic and career path from a new perspective, which reflected the positive influence of various mathematical studying activities on students' life experiences. Hence, teachers can keep an eye on students' life experience and use the project-based classroom practice as a chance to help students make decisions about their doubts or confusions.

B. Teamwork involved in the collaborative study

In the collaborative studying process, the teamwork of the group is of great significance. To complete the project, the teacher must illustrate the students about how to coordinate their works in an organized way. Also, the teacher needed to discuss with the students to make sure the characteristic and personality of each group member were respected. Otherwise, any uncooperative behavior may cause large obstacle for the whole project. The team members also need to communicate well with each other and properly utilize everyone's specialty to accomplish the project efficiently. One group recorded their experience about teamwork in the investigation report as below.

> "… In this statistical investigation, we have learned a lot and experienced a lot. We find that there is uncertainty in statistics since the result somehow depends on our subjective ideas (e.g., the choices listed in the survey). We have also acquired new skills such as how to use the software (e.g., Excel and SPSS). Furthermore, the most important thing we learned from this project is teamwork and responsibility. Our team had frequent discussion and everyone contributed their ideas about how to improve our project. We tried to divide the works according to

> each member's specialty. For instance, Charlie (pseudonym of the student) has good logical thoughts so he wrote the draft about our survey questions based on the group discussion. The enthusiastic Ella distributed our questionnaires to many students in the school. The duty of making poster was assigned to Alice and Bob who are talent in art and design. The collaboration between our team members improves gradually through the process of investigation and presentation. We now understand that we must take responsibility to ourselves, to others, and to the decisions we made. We also appreciate all the help and support provided by our teacher and classmates. "

Such valuable learning experience about teamwork and collaboration can be hardly achieved from traditional lectures in the mathematics class. However, some groups did not performed well in the investigation and presentation although the group members had talents and capabilities. These groups failed to conduct a perfect project due to the unorganized collaboration and the lack of responsibilities. The teacher would use this chance to guide the students to realize the problems in their team and help them to behave more collaborative and responsible.

C. Innovation appeared in the learning process

In the project-based classroom practice, the teacher can elicit students' innovation and creativity while they were engaged in their learning process. Specifically, the teacher provided the students with broad views about the topic selection as well as sharp opinions in some of the investigating questions. For example, based on the survey to 150 middle school and high school students, a study group developed a mathematical model to describe the "happiness index" of secondary school students. They concluded that in general the girls were happier than boys and the middle school students were happier than high school students. Another group of students provided feasible advice about how to choose after-school academy for the different levels of students after they had analyzed the data of relevant factors. These inspiring performances of the students can rarely be observed by the teacher from traditional lectures and homework.

Moreover, through the project-based classroom practice, the teacher can promote students' epistemology and help them realize the significance of general learning activities. A study group described their experience in the investigation process as follows.

> "We tried to use computer to process our data and group the data into different categories. However, certain difficulties arose when we were doing statistics in Excel. There was a mismatch when we input the data into Excel, which caused a chaos in the output diagram. To solve this problem, we sought help from the teacher and searched the relevant forum on internet. Finally we fixed the problem and created the correct diagram. From this experience, we learned that deep thoughts would raise questions. Once a question was answered, new questions would arise. However, in this seemingly repeated cycle, these questions evoked us to think independently and to learn from each other. Such cycle of "thinking→questioning→learning→answering" reveals the goal and significance of the general learning activities."

4.1.3 *Difficulties in the Project-based Classroom Practice*

Although the teachers carefully designed the project-based classroom practice so that students were actively engaged and endeavored to improve their investigation and presentations, the students may still encounter certain difficulties in the learning process. Below we describe the common difficulties which students have experienced in the statistical investigation. The teachers can emphasize on these aspects in the project-based classroom practice to facilitate students' learning process.

(a) Difficulties with designing survey questions. The textbooks used in Chinese middle school and high school had little discussion about how to design surveys. Therefore, there were many problems in the questionnaires created by the students. For example, many questionnaires just started asking questions in the first line without collecting the respondents' information. Some questionnaires did not even have a title.

Some problems also existed in the structure of the survey questions. Specifically, some questionnaires had scattered questions and the theme of the survey appeared vague; some surveys only reflected facial feature of the investigating topic and differentiation rate of the questions were low; the options in the questions were incomplete so that certain situations were not covered in the survey, etc.

To improve the quality of the survey questions, the teacher could first give a brief introduction of the standards for good surveys. After the introduction, the teacher can provide the students with several sample questionnaires and lead discussion about the defects of each question. Hence students would be aware of the elements that should be included in their surveys. They would also be able to avoid the potential misunderstanding of the investigation questions.

(b) Difficulties with data procession. Students had common difficulties about how to process the collected data. When organizing the raw data, students needed to categorize the data into appropriate groups and count the frequency of the data falling in each group. Although some groups created multiple choice questions to quantitatively categorize the survey data, the interval of the options may not be properly set up. Hence the frequency distribution cannot properly reflect the nature of the investigated topic. Moreover, in the high school textbook of mathematics, the chapter of statistics did not introduce the application of statistical software. Hence some groups with little experience in using Excel or SPSS encountered great difficulties in processing data, especially after they had collected a large amount of survey feedback.

Since high school students are usually proficient in using multimedia and internet, the teacher can search the related multimedia materials online about the basic operation of Excel and SPSS. Students can refer to these online materials when they are learning how to use the statistical software. Moreover, the teacher can also prepare manuals or video tutorials with concrete examples selected from students' projects. Hence students can follow the steps of processing the sample data when they conduct statistical analysis of their own data with Excel or SPSS.

(c) Difficulties with investigation report. In the traditional learning process, the students usually submit written materials as homework or test papers. They seldom had chance to present their interpretation of mathematics in the form of academic report. Therefore, there were many problems existed in the statistical reports written by the students. For example, a draft report from a study group cited a poem (shown below in underlined italic) as the introduction part and then directly started discussing the statistical result.

> "Sewing threads in my kind Mother's hand was shuttling through a coat for me a wayward boy. She sewed them before seeing me off neatly closed for fearing that I might return much too lately. Oh , how can I believe a human being's gifts can repay the blessings of the god in any way. A meaningful investigation successfully

ends and we obtained the results as expected…"

There was no description in the report about the background of the investigation, how to select the sample and how to analyze the collected data. Such problems of missing necessary introduction and elaboration existed in the reports from many groups (though most groups did not cite an irrelevant poem as above).

Besides the introduction part, students also had difficulties with deriving reasonable conclusion from their statistical results. Some groups' conclusions were weakly or hardly supported by the investigation data. For example, a group conducted a survey about the use of cellphones in high school campus. The survey included ten questions about the cellphones such as the price, color, function, brand, usage, etc. After the group summarized the statistical result of these ten questions, they wrote the following conclusion in the report.

> "Through the investigation, we conclude that high school students' purchasing behavior depends on the financial support from their family and most of the purchasing behavior is rational consuming. In the trend of economic globalization and political multi-polarization, we high school students are endowed with new responsibility. We should actively absorb knowledge and cultivate correct consuming habit. We can pursue distinct personality but not blindly follow the fashion. We should also oppose extravagance and waste…"

In the report above, there was little statistical data that supported the conclusion. Their comments and discussion about the investigation went off-topic about the theme of the survey. To help students write reports with acceptable academic standards, the teacher can ask students to read several research papers published in academic journals. The teacher can further lead group discussions about the common features in the format of academic papers so that the students could follow the similar structures in writing their own investigation reports.

4.1.4 *Evaluation of Students' Engagement in the Classroom Practice of Statistical Investigation*

We administered a quiz to the 34 students engaged in the classroom practice of statistical investigation. The quiz was designed based on the framework of statistical literacy (Gal, 2002). Unlike the traditional tests that focused on quantitative calculation, the quiz used for evaluation of the classroom practice emphasized on students' qualitative understanding of statistics. The two questions in this quiz were adapted from real-life situations. The full score for this quiz was 50 points. All of the sub-questions in the quiz required the students to explicitly explain their answers or choices and partial credits were assigned to incomplete answers. In this chapter we elaborate two questions to illustrate students' understanding of statistics after their classroom practice.

Question 1 in the quiz required the students to analyze an investigation plan as shown below in italic.

- *To investigate the satisfaction of tourists visiting the twelve free parks in the city, the surveyor makes the following plan: He is going to interview tourists randomly in the three parks near his home from 9am to 5pm through Monday to Wednesday every week. The sample will contain 300 interviewees with a 50-50 gender ratio. The satisfaction rate can be estimated from the data of the 300 tourists.*

(a) What is the sampling method used in this investigation?
(b) Is there any deficiency with the investigation plan? Explain.
(c) Suppose you are going to take charge of this investigation. Please make an outline of your investigation plan and create a questionnaire for distribution.

The first sub-question (a) asked about the sampling method used in this investigation. Only 10 out of 34 students provided the correct answer of stratified sampling (full score of 5 points) while other students wrote incomplete answers such as random sampling (partial credits assigned). The next sub-question (b) asked the students to find out the possible faults in this investigation plan, e.g., location, time, gender, sample size, etc. Among the 34 students taking the quiz, 7 of them found three faults (full score of 6 points), 14 students found two faults (4 points), 8 students found only one fault (2 points) and 5 students cannot find any potential faults in the plan. Then students needed to create surveying questions and write their own plan to investigate the satisfaction rate of tourists visiting the twelve free parks. The full score of this plan-designing sub-question (c) was 25 and students' performance was graded from multiple aspects such as the validity of survey questions, feasibility of data collection, time arrangement and workload distribution, etc. About 1/3 of the students got a score above 20 (out of 25) in this sub-question and half of the students had a score around 15. The result of this question indicated that the project-based classroom practice were effective in teaching students how to apply their statistical knowledge in solving practical problems.

Question 2 in the quiz showed two graphs to the students and tested their ability of interpreting statistical diagrams. A line graph (Figure 2) represented the pollution of a river in a town through 1990 to 2000. Students needed to judge whether the river had been polluted and explain their answer. Two points were assigned to the students who gave both correct answers and reasonable explanations. 33 out of 34 students got two points in this sub-question and only one student had one point deduced for incomplete explanation.

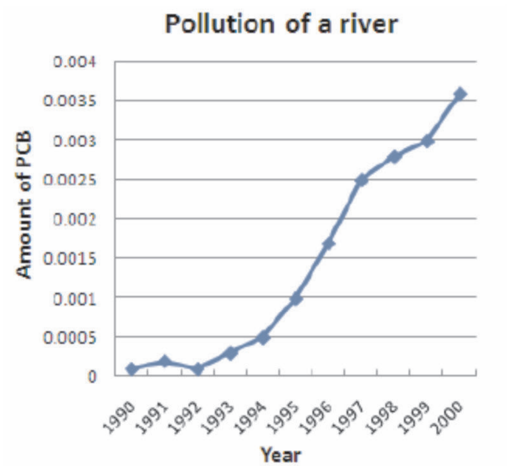

Figure 2. Line graph used in the quiz to probe students' understanding of statistics after classroom practice

Another bar graph (Figure 3) in question 2 displayed the traffic flow of the town through 1990 to 2000 and students needed to describe and explain the change of traffic flow. Similar to the previous sub-question of line graph, most students obtained full credits and only 2 out of 34 students got 1 point in this sub-question for incomplete answers.

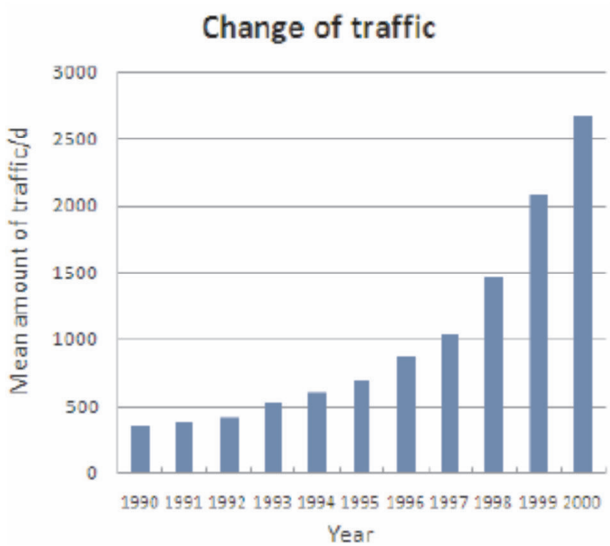

Figure 3. Bar graph used in the quiz to probe students' understanding of statistics after classroom practice

After the students had answered the two sub-questions based on analysis of the two graphs, a third question challenged them to draw a conclusion about whether the town was urbanized through 1990 to 2000. Students should realize that although there was correlation between urbanization and increased pollution or heavy traffic, pollution and traffic are not causal factors of urbanization processes. Hence we cannot assert that the town was urbanized simply from the given graphs. Compared with the previous two sub-questions (each had a full score of 2 points), the third sub-question weighed more credits in evaluating students' understanding of statistics so the students were awarded 10 points if they made correct conclusion and proposed potential factors that may indicate urbanization. Partial credits were assigned for incomplete answers or insufficient explanations. The rubric for this sub-question and the distribution of students' scores are listed in Table 2. Twenty out of thirty-four students made correct conclusion. But fifteen of them only claimed that the factors of pollution and traffic were not sufficient evidence for urbanization without further explain the possible standard of urbanization. The average score for 34 students in this sub-question is 6 (out of 10). The result indicated that further instruction on interpreting statistical data was still required in the project-based classroom practice to help students build a hierarchical knowledge structure of statistics.

Table 2. Rubric for the question of urbanization and distribution of students' scores.

Type of Answers	Score	Number of Students
Give correct conclusion and provide other reasonable factors that may indicate urbanization	10	5
Give correct conclusion without providing other urbanization factors	7.5	15
Give wrong conclusion but provide a logical discussion about the factors of pollution and traffic flow	5	2
Give wrong conclusion and provide some discussions irrelevant to the graphs in this question	2.5	12

4.2 *Classroom Practice Case 2: Search Linear Functions in Life*

4.2.1 *Design of the Classroom Practice*

The second classroom practice was administered to the 8th grade middle school students. When teaching the content about linear functions, the teacher organized students into several study groups to investigate the application of linear functions in real life. The students were encouraged to brainstorm and select their own investigation topic though the teacher provided sample topics that were manageable by middle school students. The topics used include:

- What is the relation between the weight of an object and the reading on the beam scale?
- What factors determines the taxi fare? What is the relation between the taxi fare and these factors?
- What factors determines the monthly cost of utilities such as

electricity, water and gas? How much money does your family pay for these bills?

- Which shopping mall should the customers choose if the rules of discount are different?

The teacher guided students to propose an investigation after they had selected their investigation topics. The teacher would clarify the requirements of the classroom practice and discuss the investigation plan with each group. Then students started to (1) organize the concepts and principles of linear functions in the textbook, (2) search for the relevant data and materials online or from library, (3) observe or measure the practical application of linear functions in real life, and (4) discuss and create the corresponding mathematical model. At the end of the project-based classroom practice, the teacher needed to evaluate students' performance based on their written reports and PowerPoint presentation. The entire classroom practice took about 3 to 4 class hours in a 10-day period. Part of the classroom practice was arranged at the extracurricular time, e.g., math and science clubs in the afternoon, instead of the regular lecture time in the morning. Table 3 illustrated the schedule arrangement in the investigation plan submitted by one study group in 2012.

Table 3. The timeline of the classroom practice for the study groups investigating the application of linear functions in real life

Apr. 3rd – Apr. 4th	Brainstorm and confirm the investigating topic of each group. Divide the work for each group member.
Apr. 5th – Apr. 9th	Search evidence and materials about the application of linear functions in real life. Share and discuss the materials in group.
Apr. 10th – Apr. 11th	Complete the report and make PowerPoint slides of the investigating topics
Apr. 12th	Present the outcomes of each study group.

In this classroom practice, the most challenging part is to observe the application of linear functions in real life and construct the corresponding mathematical model. On the one hand, students should explicitly express the linear function describing a practical situation and evaluate whether the linear function they found is reasonable. On the other hand, students need to analyze the advantages and disadvantages of different mathematical models and propose possible methods of solving practical problems. Hence the teacher participated in each group's investigation to help students realize the relation between the abstract mathematical concepts and the concrete real-life themes. At the end of the classroom practice, the teacher also summarized and evaluated the learning outcomes of each group so that the students can better understand the contents about linear functions.

4.2.2 *Learning Outcome in the Classroom Practice*

In the classroom practice of "searching linear functions in life", the teacher helped the students to understand the similarity and difference between direct proportion functions and linear functions. For example, students learned the properties of linear functions such as increasing and decreasing trend through the investigation of the linear relation in daily life. The teacher also used the classroom practice as an opportunity to review the previous knowledge about the corresponding relation of functions, i.e., the independent variables and dependent variables. During the investigation process, students in some group also had chance to discover new contents, e.g., piecewise functions, constant functions or inverse proportions, and the teacher would guide the students to study the connection between these new functions and the linear function.

The middle school students were actively engaged in this project-based classroom practice. The teacher led a class discussion so that each group can exchange ideas about what type of real-life situation they can observe, how to conduct investigation, and where to find the required materials, etc. In the previous case about statistical investigation, we illustrated high school students' engagement from the aspects of life experience, teamwork and creativity (Section 4.1.2). Below we would describe the middle school students' engagement in their classroom activity from the same perspectives in order to compare the effect of the project-based classroom practice in different grades of secondary school.

1. Life experience reflected in the investigation. Since the theme of this middle school project-based classroom practice is to "search linear functions in life", students had greatly enriched their life experience in the investigation process. The teacher helped a group of students to probe the linear relation between the mass of an object and the reading on a beam scale. Before the classroom practice, students had rarely used any scales other than bathroom scale and none of them knew how to read a beam scale. In the investigation process, the teacher provided students with a real beam scale and demonstrated how to use it. Hence, students observed that the weight on the scale needed to be moved further when heavier object was measured. Through discussion with the group members, the teacher helped the students to create the mathematical model of real beam scale in a graphical representation as shown in Figure 4.

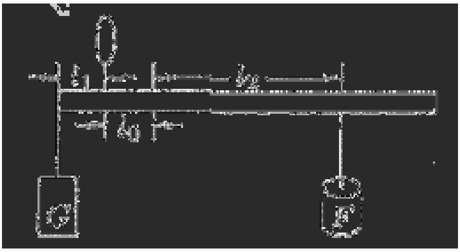

Figure 4. Graphical representation of a beam scale

Students realized that reading on the scale depend on the distance between the zero line and the ring (l_0). The linear relation between the mass of the object (G) and the reading on the scale (l_2) can be expressed as $Gl_1=F(l_0+l_2)$, where F is the mass of the balance weight and l_1 is the distance between the ring and the measured object. The teacher provided clues to the students so that they can figure out the secret of some vendors who gave short measurement by using lighter balance weight. When F decreases, the length of l_2 would increase to keep the equation balanced. Hence the reading on the beam scale appeared larger than the actual mass of the measured object.

The teacher helped another group of students to investigate the purchasing strategies at different shopping malls with different sale discounts. For instance, suppose the shopping mall A had 100 yuan discount for any purchase above 199 and another shopping mall B had 40% off on any purchase. The discount rate with the model of linear functions can be described as below.

$$Mall\ A: y_A = x - 100\ (x > 199)$$

$$Mall\ B: y_B = 0.6x$$

When $y_A < y_B$, the values of x satisfy that $x - 100 < 0.6x$, i.e., $x < 250$. Hence, students realized that it was worth purchasing from the mall A only if the origin price was between 199 and 249 yuan. While analyzing the investigation result, the teacher suggested the students to further discuss about rational consumption or extend their mathematical models of discounting sale to other purchasing behaviors such as installment payment and cashback.

2. Teamwork involved in the collaborative study. The middle school students previously had limited experience in collaborative study. Hence the teacher can enhance students' collaboration and teamwork in their learning process of the project-based classroom practice. At the beginning of the classroom practice, each group first needed to divide the investigating work appropriately among the group members. Here we take the study group of the topic about "linear function in public transportation" as an example. In their investigation plan, the students described the task of each group member as follows (Table 4).

Table 4. Division of the work for each group member in the collaborative study of the topic "linear function in public transportation"

Name (pseudonym)	Task
Alice	Search the information online about ticket price of public transportation (bus, train, taxi) in the city of Wenzhou
Bob	Collect the receipts of taxi and analyze the relation between the distance travelled and the price of the taxi
Charles	Collect the tickets of bus and trains and analyze the relation between the distance travelled and the price of the bus or the train
David	Organize the materials collected by others and write the draft of the investigation report

The teacher asked the members in the group not only finished their individual tasks but also helped each other to collect relevant information and materials. Students exchanged these materials in their group meeting and discussed the collected information together. Through the discussion, the teacher guided the students in the study group to form integrated understanding about the investigation topic from different perspectives. For example, in the investigation of "linear function in public transportation", the student Bob took charge of collecting taxi receipts. His major collection was made at the city of Wenzhou. However, another student in that group just travelled to a nearby city, Taizhou, on the weekend during the investigation process. So he brought back some taxi receipts from the Taizhou city. In the group meeting, the teacher reminded the students to notice that the taxi fare in different cities were different. In Wenzhou, the initial flat rate was 10 yuan within 4 km, 2 yuan/km for extra distance between 4 to 10 km, and 2.5 yuan/km for extra distance above 10 km. The additional gas charge is 2 yuan for each service. In Taizhou, the initial flat rate was 6 yuan within 1.5km and each additional kilometer cost 1.9 yuan. While investigating the different taxi charge in different cities, students proposed a question about whether it is cheaper to take taxi in Taizhou than in Wenzhou. The teacher discussed this question together with the students and analyzed the elements that can be described by mathematical models. Hence, students modified this question as "within what distance is the taxi fare in Taizhou cheaper than in Wenzhou", which reflected an improved understanding of applying the knowledge of linear functions.

3. Innovation appeared in the learning process. In the project-based classroom practice, the teacher observed students' creativities in extending their knowledge of linear function to explain the real life situations. Although the teacher had not taught the chapter of piecewise function in class, the students searched and self-studied the related knowledge when investigating the mathematical model of taxi fare. They successfully applied their new knowledge about piecewise function to build a mathematical model of taxi fare in Wenzhou city as below (the taxi fare rate was discussed in section 4.2.2.B).

$$y = \begin{cases} 10 + 2 & (0 < x \leq 4) \\ 10 + 2 + 2(x - 4) & (4 < x \leq 10) \\ 10 + 2 + 2 \times 6 + 2.5(x - 10) & (x > 10) \end{cases}$$

In the function, y is the taxi fare and x is the distance travelled (in the unit of kilometer). The piecewise function can be simplified with combined constant, i.e.,

$$y = \begin{cases} 12 & (0 < x \leq 4) \\ 2x + 4 & (4 < x \leq 10) \\ 2.5x - 1 & (x > 10) \end{cases}$$

The first part of the piecewise function is a constant 12. To convert the constant function into the standard format of linear function $y=kx+b$, the teacher led the group members to discuss about the meaning of the slope of a linear function. Such discussion helped students to better understand the nature of linear functions so they expressed the function in the linear form as $y=0x+12$. They further used the zero slope to describe the bus ticket price as $y=0x+2$ (fixed price of 2 yuan for buses).

The teacher also helped the students to relate the mathematical knowledge with other scientific subjects. When the students were investigating the linear relation of the beam scale, the teacher recommended them to search for relevant information from textbooks of other disciplines. As a result, students found that the application of beam scale can be explained by the lever principle introduced in the physics textbooks. Hence the students successfully build the mathematical model of beam scales by using linear functions to describe the lever principle. With the material provided by the teacher, the students can make a simple beam scale to illustrate their understanding of the lever principle in the presentation day. Through this project-based classroom practice, the teacher not only enhanced students' understanding of mathematical theories but also improved their practical skills of applying math and science knowledge in real life.

4.2.3 *Difficulties in the Classroom Practice*

As described in case study 1 (Section 4.1.3), high school students engaged in the project-based classroom practice of statistical investigation may encounter difficulties in designing survey questions, using software to process data, generating statistical diagram and writing report with suitable structure. In case 2, since the theme of the classroom practice was about linear functions which did not explicitly involved the statistical survey, the middle school students did not experience difficulties in designing survey questions or processing a large sample of data. However, they showed similar problems as the high school students when dealing with diagrams and reports. Besides, the teacher noticed that the middle school students had additional difficulties in their learning process such as interpreting and applying the graphs of linear functions.

When the students were preparing their investigation reports, the teacher asked them to describe the mathematical model through various representations, i.e., algebraic expression, word expression or graphical expression. The students showed a good grasp of handling algebra equations as well as verbally expressing the mathematical models. However, they had certain difficulties with graphical representation of linear functions. Many students did not realize the connection between the algebraic and graphical representation, e.g., the relationship between the coefficient of the independent variable and the slope of the graph, and that between the constant item in the function and the intercept on the y-axis, etc. Hence when sketching the graph of linear functions, the students preferred to first mark down four or five points in the coordinate by plugging values of x and then connect these dots into a straight line. Therefore, to help students improve their capability of applying mathematical knowledge about linear functions, the teacher can demonstrate how to draw the straight line by observing the slope and intercepts of the linear function when leading the project-based classroom practice.

4.2.4 *Evaluation of Students' Engagement in the Project-based Classroom Practice*

To evaluate the middle school students' engagement in the project-based classroom practice, we distributed a questionnaire to 24 students who had finished their project of "searching linear functions in life". The survey questions were designed in a five-point Likert scale (5-strongly agree, 4-agree, 3-neutral, 2-disagree, 1-strongly disagree). Three questions in this survey probed how students thought about the relation between the classroom practice and the corresponding mathematical knowledge.

- Q1. The classroom practice strengthened my impression on the related mathematical contents.
- Q4. The classroom practice increased my interest in learning mathematics.
- Q7. The project I have done in the classroom practice can improve my score in mathematics exams.

Twenty-three out of the twenty-four students agreed or strongly agreed the statement in Q1 that the classroom practice strengthened their impression about linear functions. Twenty out of twenty-four students agreed that they had more interest in learning mathematics as stated in Q4. The average scores for Q1 and Q4 were 4.6 and 4.3 respectively. However, there was an even distribution for students' answers to Q7. One third of the students believed that the project-based classroom practice would improve their score in the math tests. One third of the students thought that the project had no effects on their test scores while the other students' responses were neutral. Such neutral and negative response to Q7 was majorly due to the fact that the students had rarely seen project-based questions in their quizzes or exams. The average score for Q7 was only 3.1 out of 5. Hence, the teacher can add contents related to the mathematical projects into the quizzes and exams in order to emphasize the significance of the project-based classroom practice.

Two questions in this survey examined students' achievement from the classroom practice besides mathematical knowledge.

- Q3. The project-based classroom practice provided me more opportunities to discuss the relevant topics with my classmates.
- Q5. The classroom practice was beneficial to other subjects such as sciences or arts.

All twenty-four students agreed or strongly agreed that they had more chance to discuss with their classmates while working on the project together. About 70% of the students believed that the mathematical project also helped them to practice the related subjects of sciences or arts. Two out of twenty-four students disagreed that the classroom practice was beneficial to other subjects while five students hold a neutral position about this statement in Q5. The average scores for Q3 and Q5 were 4.6 and 4.0 respectively.

Another two questions in this questionnaire asked about students' overall experience with the project-based classroom practice.

- Q2. I am NOT used to studying math through the project-based classroom practice.
- Q6.I feel the project-based classroom practice is a waste of time and unnecessary.

Only one student chose neutral and all other twenty-three students disagreed or strongly disagreed with that statement in Q2. No student considered the classroom practice was a waste of time as claimed in Q6. These results showed that students can accommodate to the project-based learning and had an overall positive opinion about their experience in the classroom practice. The average scores for Q2 and Q6 were 1.5 and 1.3 respectively (disagree with the negative attitude toward the project-based classroom practice). The results showed that the project-based classroom practice was an effective teaching method that can improve students' engagement in mathematical activities.

In the survey we listed five potential improvements that the students can achieve from the classroom practice.

A. improved the understanding of mathematical concepts

B. changed the attitude about mathematics

C. elicited interest in learning mathematics

D. enhanced thinking skills in mathematics

E. understood the relation between mathematics and other subjects

Students need to list all the achievements they obtained in the project-based classroom activities and arrange these possible improvements in the sequence from the most important (5) to the least (1). If a student did not write down a particular factor in his/her list, that questions would be assigned zero point. According to the feedback of 24 students, we calculated the total points for each of the factors (Table 5).

Table 5. The sum of students' evaluation of the importance for different factors they may achieved from the classroom practice

Factor	A. concepts	B. attitude	C. interest	D. skills	E. other subjects
Total Points	71	59	83	86	42

Two most important factors were C (83 points) and D (86 points) while the least important factor was E (42 points). Therefore, when conducting the project-based classroom practice, the teacher could emphasize on the cognitive function of the mathematical project to enhance students' interest and skills in mathematics.

We further interviewed the middle school students about their general comments for the classroom practice. Students felt that the mathematical projects were interesting and they enjoyed the fun of thinking in mathematics. When investigating the topic of "searching linear function in life", students improved their ability of observation and found that mathematics was widely applied in their daily life. Some students said that they experienced certain pressure in the classroom practice when approaching the deadline and they felt nervous when giving presentation in front of the whole class. However, these students were also excited about the fact that they complete the project on time and presented their works successfully.

Students provided valuable suggestions to their teacher on improving the classroom practice. They hoped to participate in developing the schema of classroom practice and collaborate with the teacher in selecting the research project. They also wanted to have some instruction manuals to guide them in designing investigation plan and distributing workload for team members. Some also suggested creating an effective rubric to rank the difficulty level of different projects and fairly evaluate the work for each group. The winning group should be rewarded so that they would be more enthusiastic about this classroom practice. Most students were willing to have more project-based study when learning mathematics and they suggested the teacher to use five or ten minutes in each math lesson to discuss the interesting application of mathematics.

5. Summary and Conclusion

In traditional classrooms, the teachers act as information providers and lead the educational process in the format of lectures and presentations. Students receive information in classroom and review the relevant knowledge through recitation and homework. However, in the project-based classroom practice, students need to gather information by themselves and select useful information through group discussion. Then the teacher helps the students to organize the selected information into a hierarchical knowledge structure when they are preparing the investigation report. The students lead their learning process while the instructor's role is to assist the learners achieving their goals. Hence, the learning outcome of the project-based classroom practice largely depends on whether the teacher can improve students' engagement in their learning process.

The evaluation results indicate that the teacher properly conducted the project-based classroom practice so that the students were actively engaged in the mathematical activities. The teacher carefully incorporated the course requirement of the national mathematics curriculum standards into the project-based classroom practice. Hence, students had a solid grasp of the fundamental concepts and principles in middle school or high school mathematics. Meanwhile, the teacher helped the students to generate innovative ideas in the learning process of applying mathematical knowledge to solve practical problems. During the investigation, students closely collaborated with each other and learned the importance of teamwork in collaborative study. In the surveys and interviews, most students appreciated the learning experience in mathematical activities and expressed interest in having more opportunities to study mathematics through classroom practice.

Acknowledgments

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Chapter 12

A Large-Scale Video Survey on Taiwanese Fourth-Grade Classrooms of Mathematical Teaching Behaviors

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This study is to investigate the fourth grade teachers' mathematics teaching behaviors in Taiwan. A large scale of two-staged stratified cluster sampling of fourth-grade math teacher teaching in the whole Taiwan was conducted. Video survey was applied, and the videos were taped within one semester. Usually, it took one day to tape one teacher's instruction in one school; altogether, 60 teachers' instructions in the participating schools were recorded. Instructional videos were recorded and digitized, coded and analyzed. The results showed that in the fourth-grade mathematics classroom teaching, concept statements, problem statements and solving questions each took about 30%, 20% and 50% of the time, respectively. In Content strand, the mathematics problems teachers taught were mostly of low complexity and repetition problems. In Mathematical ability, stating concepts and using procedures were the majority. In Mathematical power, connections with real-life were the majority.

Keywords: Taiwanese classrooms, national sampling, video study, mathematics teaching

1. Introduction

In recent years, the results of Trends in International Mathematics and Science Study (TIMSS) (Olson, Martin, Mullis, 2008) and Programme for International Student Assessment (PISA) (Organisation for Economic Co-operation and Development (OECD), 2007) showed that in East Asian countries, the fourth grade and eighth grade students performed well in mathematics, ranking top throughout the world. Many scholars tried to analyze the factors that affect students' achievement. Lin (2010) states that the Ambiguity of Factors Contributing to Students Performance can be expressed as aX + bY + cZ, where X represents school lessons, Y represents out-of-school lessons, and Z represents the others, e.g., Societal, Cultural ..., and a, b, c is their weight. In the variable X, the teacher's teaching is a very important factor. Therefore, we are interested in mathematics teachers' teaching methods, as the basis for the characterization of Students Performance.

In 1999, seven countries including the United States and Japan participated in Third International Mathematics and Science Study 1999 Video Study (TIMSS VS 1999) (Jacobs et al., 2003a). The aim was to compare eighth-grade teachers' mathematics teaching behavior of various countries. Taiwan did not participate in the study. With the technique of TIMSS VS 1999, Lee et al. (2010) has studied Taiwan's fourth-grade teachers in their general behaviors of mathematics teaching.

Although the TIMSS VS 1999 (Jacobs et al., 2003a) reported and compared the mathematical teaching behaviors of eighth-grade teachers in various countries, it did not offer a good analytical framework. Thus, we will probe into literatures and propose a framework on teachers' mathematical teaching behaviors as the basis for analysis. The purpose of this study is to analyze Taiwanese fourth-grade teachers' mathematical teaching behaviors. We took analyzing the teaching behaviors of Taiwanese mathematics teachers as a basis to find out what X represents in aX + bY + cZ and also to understand the possible causes of Taiwanese students' internationally top-ranking mathematical performance.

2. Theoretical Framework

Taiwan's national mathematical curriculum guidelines (Ministry of Education, 2003) stress not only students' learning of mathematical content knowledge, but also that students should acquire the ability to use what they have learned, so it advocates that students should be the pivot of teachers' teaching, keeping the students' development of mathematical abilities in mind. Students' capabilities of problem solving, reasoning, communication, connection were emphasized in learning.

Literature reviews found that the NAEP (NAGB, 2002) assessments of mathematics from 1996 to the years 2000 and 2003 came up with a three dimensional assessment framework of Content strands, Mathematical ability, Mathematical power. This content of assessment framework is consistent with the Taiwanese national mathematical curriculum guidelines structure and the concept of NCTM (2000). In NAEP (NAGB, 2002), Content strands are divided into number sense, properties and operations, measurement, geometry and spatial sense, data analysis, statistics and probability, and algebra and functions; mathematical ability is divided into conceptual understanding, procedural knowledge and problem solving; mathematical power is divided into reasoning, connections, and communication. Chinese literary reformer and philosopher Hu Shi once said, “As you sow, so shall you reap.” If we wish to cultivate the inner ability of students on some dimensions, we must first make sure that the teacher is capable of teaching those dimensions. Suppose a teacher can’t pass certain abilities on to his/her students, but the students can still acquire such capabilities, then the importance of School Teaching (X) will decline, and the impact of Out of School Lessons (Y) and Social Factors (Z) will increase on students’ achievements. Therefore, we find that this assessment framework is fairly suitable to be used to describe the teachers’ mathematical teaching behaviors and can be used as a basis for analyzing School Teaching. In this paper, we will modify this framework to analyze the teachers’ mathematical teaching behaviors.

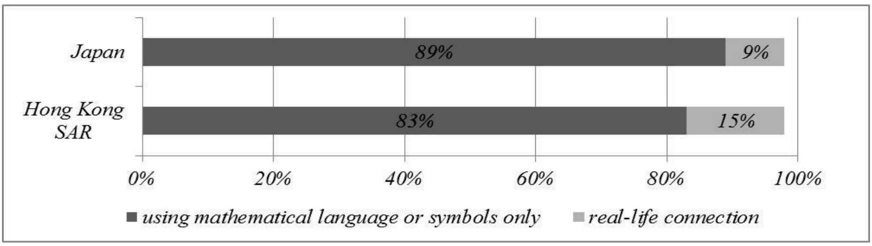
3. Related Research

About the teachers’ mathematics teaching research, we found that there are two similar studies, TIMSS video study and Learner’s Perspective Study (LPS) (Clarke & Novotná, 2008). We first analyzed the two related studies for this research.

3.1 *TIMSS Video Study*

TIMSS VS 1995 (Stigler, Gonzales, Kawanaka, Knoll, & Serrano,1999) is a video study of teachers’ mathematics teaching aimed at eighth-grade mathematics classrooms in the United States, Japan and Germany, it is actually the pilot study of TIMSS VS 1999. Participating countries increased to seven countries including the United States, the Netherlands, Japan, Australia, Czech Republic, Hong Kong, and Switzerland in TIMSS VS 1999 (Jacobs et al., 2003a Jacobs, et al., 2003b; Jacobs et al., 2003c). TIMSS VS 1999 sampled in a similar way as the TIMSS achievement test, adopting two-staged stratified cluster sampling design, and sampled representative teachers. It collects and analyzes data including sampled classroom teaching videos and teacher questionnaires. In the video analysis of classroom teachers, TIMSS VS 1999 developed two sets of coding systems; the first set of encoding is more of a general teaching behavior, the second set of coding systems mainly focuses on the mathematical teaching behavior. Nevertheless, no relevant theoretical frameworks were pointed out. "Teaching Mathematics in Seven Countries: Results From the TIMSS 1999 Video Study" (Hiebert et al., 2003) has a complete report on TIMSS VS 1999 results.

Due to the fact that Hong Kong and Japanese students' achievements are similar to Taiwanese students' achievement, for example in 1999, their 8 grade students' came in third and fifth place respectively, which is not remarkably different from Taiwanese students, therefore we were interested in their teaching, and we gave two examples of results. Figure 1 describes the number of percentage that national math teachers use connections from real-life when presenting problems.



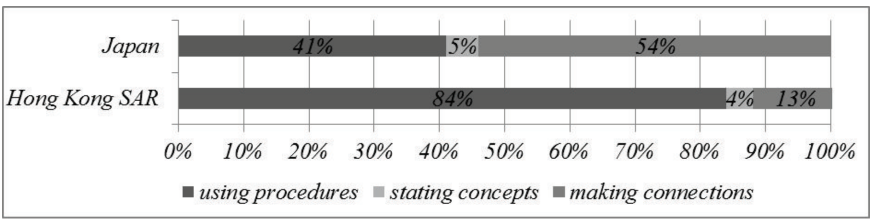


Figure 1. TIMSS video study of problem statement in frequency coding

Japan and Hong Kong each have the ratio of up to 89% and 83% using of mathematical language or symbolic representation to problem statements. Next graph presents the percentage of teachers who hope that students using procedures, stating concepts, or making connections to answer problems, when presenting problems. Japan has the highest need for making connections, at 54%, while in Hong Kong, requiring the procedural knowledge is the highest, at 84%. Interestingly, students from Japan and Hong Kong have similar achievements, but their teachers have great differences when using procedures and making connections in their teaching. This means that teachers' teaching has great cultural differences. What about the teaching of Taiwanese teachers? We are very interested about the answer.

Since the TIMSS VS 1999 results were published, as a follow-up, some scholars have explored the beliefs, knowledge and professional development of mathematics teachers, and compared the characteristics of the textbooks to see if it is similar to actual teaching. Kuntze and Reiss (2005) used two tapes of video of TIMSS VS 1999, one which was identified as having discussions and arguments between teacher and students, another was a teacher-centered geometry proof video led by teachers. The survey investigated beliefs and professional knowledge of 42 Swiss and German in-service teachers. Kuntze and Reiss (2006) use the video of TIMSS VS 1999 and worked on 32 high school math teachers for in-service training program. Vincent and Stacey (2008) discovered that in the TIMSS VS 1999 research, Australian eighth-grade mathematics teaching had a high ratio of low-complexity and repetitive problems, and a very low ratio of making connections. They analyzed nine eighth-grade textbooks used in 2006 in four states and found that the characteristics of the questions in textbook are very similar to the results of TIMSS VS 1999.

If such a national study were available in Taiwan, then we would have sufficient data to conduct similar follow-up study, for instance research on teachers' professional development, the relation between textbooks and teachers; teaching, etc. Therefore, it is necessary for us to conduct a national video study.

3.2 *LPS Project*

LPS is another important international classroom teaching video research program. It is initiated by Clarke (Clarke & Novotná, 2008) from Australia, and the program is useful to help us understand the eighth-grade mathematics classrooms and the classroom context of teaching and other practical situations. Participating countries include Australia, Germany, Japan, the United States, China, Czech Republic, Israel, Korea, Philippines, Singapore, South Africa, and Sweden, 16 countries or regions in total. LPS required each country or region to select at least three excellent eighth-grade mathematics teachers, the excellent teachers were identified by local education authorities and the research team, who are considered capable teachers of mathematics teaching. The LPS project has two distinct characteristics; one is using three cameras on each teacher for 10~15 classes for recording mathematics teaching materials, rather than a single classroom. The other is that they study from the viewpoint of the learner. LPS thinks that the student activities, attitudes, beliefs and knowledge may constitute a specific teaching culture. Therefore, after school they play videos to re-construct teaching, in order to interview teachers and students.

LPS (Clarke & Novotná, 2008) mainly focuses on qualitative comparison on mathematics teaching in different countries (or regions), not on defining the teachings of the country or culture, nor to quantify inter-country comparison of teachers teaching methods. For example, the Singapore team to focus on the use of textbooks and homework; the China Team (including Shanghai, Hong Kong and Macau) placed the focus on whether teaching is teacher-centered or student-centered; and the Japan team stressed on seatwork.

In recent years, Taiwanese curriculum has undergone many reforms, from being subject-oriented in 1975 to becoming student-oriented in 1995, and to finally being ability-oriented in 2005 (Zhong, 2005). What are Taiwanese teachers' main focuses on now? We can get some answers from research of teachers across the nation.

Some follow-up research is still being done for the LPS plan, such as analysis of important moments in the classroom or mathematics lessons which students consider to be good by using LPS data. Fujii (2009) used video stimulated interviews to let the Japanese students and teachers become involved in LPS, they watched their own videos of the classrooms, to identify students' discussions and important moments in the classroom. Shimizu (2009) studied 60 eighth-grade Japanese students involved in the LPS project, using the stimulation interview method after the video-taping of the classroom, trying to build the essence of a good mathematics classroom from the student's eyes. Ding, Anthony, and Walshaw (2009) used LPS method to explore the teaching experiment of eighth-grade equivalent fractions in New Zealand, to see how teachers chose and used examples to develop students' mathematical thinking. This teaching experiment is the Secondary Numeracy Project (SDP) in New Zealand secondary schools.

The researchers found that TIMSS VS 1999 aims to analyze teaching activities which affect students' learning. LPS research focuses on exploring mathematics teaching of outstanding teachers, not on comparing similarities and differences between teaching methods of participating countries. Because the aim of the TIMSS 1999 video study is more in line with the aim of this study, which is to describe the teaching appearance of mathematics teachers in Taiwan, therefore, we used the TIMSS 1999 video study as a basis of analyzing teachers' mathematical teaching behavior.

4. Methods

4.1 *Methodology*

This study uses the video survey (Hiebert et al., 2003a) and the questionnaire method. Traditionally, if you want to measure a large sample of classroom teaching, it is usually done through questionnaires. However, classroom teaching is very complex, because it is hard for teachers to reflect and narrate incidents that happen or that are said instantly. So in addition to teacher questionnaires, we should be looking for other research methods. Video survey (Jacobs et al., 2003a) allow researchers to examine complex teaching and learning activities from different perspectives. The classroom activities preserved through videos can allow many people, even professionals of various fields, to repeatedly observe teaching at a low speed, giving us a chance to make an in-depth study of classroom activity, allowing us to describe teaching with internal consistency, allowing us to use new methods of analysis to analyze teaching, at the same time enabling us to use more convenient ways to communicate results from the analysis. Therefore, video survey (Jacobs et al., 2003a) enable researchers to repeatedly examine and analyze classroom teaching giving us a chance to make an in-depth study of the teaching and learning inside the classroom.

4.2 *Research Process*

The first step of the research process is to explore relevant literature, and then begin a national systematic sampling, determining the sampling of schools and classes. A group of 3 filming crew were trained to certify good video quality, and another crew started designing surveys and the video coding system of classroom videos. During the first semester in 2007, video taping and collection of related data of sampled classroom teaching officially took place, and when it was complete, digitizing of videos, transcribing classroom teaching, analysis of surveys and coding analysis of classroom videotaping were done.

4.3 *Subject of Study*

The national fourth grade teacher population was studied. When determining the number of classrooms to be video-taped, the team consulted with researchers responsible for the TIMSS project in Taiwan, thinking that since only 50 classes in Japan participated in TIMSS VS (50 classes, one lesson each class) and that since Taiwan has a much lower population than that of Japan, so sampling 50 to 60 classes would be sufficient to characterize the teaching appearance of the country. Therefore, for this study, we decided to sample 60 classes.

The sampling method is based on the TIMSS two-staged stratified cluster design, using the data of national primary schools provided by the Department of Statistics, Ministry of Education (2007), on Feb. 9, 2007. Using probabilities-proportional-to-size (PPS) systematic sampling method, and arranging classes according to school districts, student population, and the size of classes. The first phase of sampling is determining (the number of) the schools. First of all, Taiwan is divided into five regions: the North, Midst, South, East and Islands. Take the North as an example: the total number of students is 127287 (44.3% of all students), the original classroom number for sampling is 60, so 44.3% of 60 should be sampled, rounded up to 27 classes. 127287÷27 4714.3, so the range is 4714 students. We select a random number between 1 and 4714, then every 4714 people as a range select the students of number to select schools for sampling (the school to which the randomly picked student attends will be sampled). Also, simultaneously selecting replacement schools (the successive school of the sampled school) in case the sampled school is unwilling to be video-taped. The numbers of participating schools in the North, Midst, South, and East of Taiwan are 27, 16, 16, and 1. The numbers of sampling large-sized schools, medium-sized schools, and small-sized schools are shown in Table 1.

Table 1. Distribution of 60 participating schools from nationwide sampling

	Large-sized school	Medium-sized school	Small-sized school	Subtotal
North	17	8	2	27
Midst	6	7	3	16
South	7	6	3	16
East	0	0	1	1
Subtotal	30	21	9	60

*According to the definition of TIMSS 2007: Large-sized school contains at least 8 fourth-grade classes, medium-sized school contains 3 to 7 fourth-grade classes, and small-sized school contain no more than 2 fourth-grade classes.

The second phase of the sampling unit is the classrooms within the sampled school. After determining the schools to be sampled, a random number is used to randomly select the classes from sampled schools. After the two stages of sampling have been finished, we then contacted to schools and teachers. If a school does not agree to be video-taped, then we replace it with a replacement school. After the school chosen agrees to be video-taped, we ask teachers of the chosen class for approval. When the teacher does not agree to be video-taped, we ask the school to select another teacher who had similar teaching methods, teaching experience and qualifications. This research completed 60 classroom video-tapings in total, with 18 of them being from replacement schools or classrooms in total, two of them due to school factors, the reason being that those schools had never allowed visitors visiting the school, and does not agree to be video-taped. 14 classrooms did not agree to be video-taped due to teachers' factors including that they felt pressured, nervous, being close to retirement, job transfers, or they thought their teaching methods were not special. Two classrooms did not agree due to students' factor, for there were students requiring special education in the classes, and parents wanted to protect the students' right to privacy.

The average teaching experience of 60 teachers is about 10 years, including 9 years of mathematics teaching. In terms of qualification, all of them are university graduates and have received teacher training courses, with 75% graduating from Normal systems, about 25% graduating from post-baccalaureate teacher courses, and 15% with a Master degree. Around 50% of the teachers majored in education in university, which is a non-mathematics subject. Around 15% of participating teachers majored in mathematics or mathematics education, about 35% majored in subjects other than the two previously-mentioned (they majored in subjects from classes like Postgraduate teacher courses). In a 40 minute video lesson, about 38 minutes 36 seconds were spent on teaching, which includes 37 min. 16 sec. of work related to mathematics. Only a short period of time was spent on work unrelated to mathematics, for instance classroom management or wiping the board, etc.

All classes of sampling were video-taped in the first semester of 2007 school year, videoing a class every day from Monday to Friday in principle, until all assignments were completed at the end of the semester, which totals to about 4 months of videoing time. Because we did a two-stage stratified cluster national sampling, we video-taped a class from all classes every day during the whole semester, as shown in Figure 2. We can imagine our approach is that we found a theoretical national representative teacher to represent teachers nationwide, and video-taped this representative teacher's classroom teaching almost every day.

4.4 *Data Collection*

When we were collecting data of classroom videoing, to ensure better quality of recording and analysis, we used three video recorders (similar

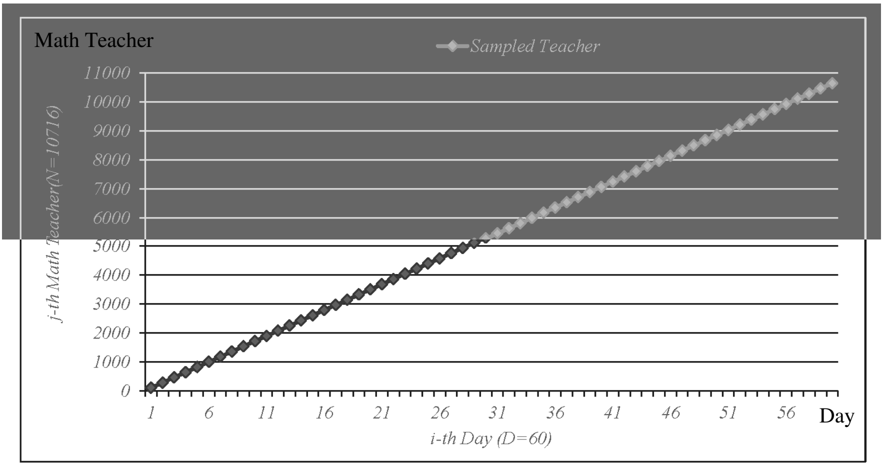


Figure 2. Representation of theoretical national representative teacher

to LPS method). The first camera focused on the whole students; the second focused on the teachers' teachings, the students in front of the blackboard and all contents of the blackboard; and the third focused on a group (usually the group that performs ordinarily) or the students' micro sounds and actions. Viewing the three screens simultaneously and selecting suitable scenes (The screen will be switched to the one that can capture whoever the main character was at that point. For instance, when a student is speaking, the scene will be adjusted so that it can be captured, or when a teacher writes on the board, then the focus will be altered to view the things written on the board.), We store the teaching in two machines, one is the three video screens simultaneously and the other is selecting suitable video scenes, to enable analysis of teaching.

To enable recording results show the real status of the teachers' teaching of mathematics, we required teachers to teach in the normal way thrice, and also gave teachers and students to time adapt to classroom recording. First, in the second semester of the 2006 school year, at the same time the teachers were invited to participate in the study, teachers were informed verbally and in paper that they were required to teach in accordance with the daily progress of teaching, and not to make any alterations. Second, before the beginning of the first semester of the 2007 school year, after teachers had confirmed school time tables, researcher contacted teachers once again, on the one hand to confirm the time of video-taping, on the other hand to request again that the teacher should teach with usual teaching methods, based on the usual progress of teaching. Third, one week before the video-taping, reconfirmed the time and place of shooting with the teacher, and asked the teacher to teach in accordance with the daily progress of teaching and with usual teaching methods, and not to make any alterations. In addition, one lesson before the formal recording of a class, camera personnel entered the teaching field to simulate videoing to let teachers and students become familiar to shooting conditions, and began formal shooting at the next lesson time. Our experience with the survey results (Lee et al., 2010) tells us that after the first warm-up class with simulation of videoing, teachers and students to adapt to it and can behave as they would do originally.

4.5 *Data Analysis*

For the classroom video coding, we mainly referred to NAEP 2003 (NAGB, 2002) mathematics assessment framework and divided it into three dimensions, Content strands, Mathematical ability and Mathematical power, as shown in Figure 3.

Before analyzing the three dimensions, we referred to TIMSS VS 1999 (Jacobs et al., 2003a) coding system and divided the classroom instructional activities into segments of Concept Statement (CS), Problem Statement (PS), and Solving Problem (SP). CS referred to teachers' explaining of mathematic concepts, including introducing new concepts and reviewing already-learned ones. PS referred to the process of teachers' posing problems. SP meant the time that teachers spent on answering some questions. The reason we divided teachers' classroom teaching activities into 3 segments is mainly that conceptual teaching segments and problem solving segments are different. Problem statement segments is mainly used to understand the properties of the questions asked by the teachers, similar to when we assess students' learning through giving tests.

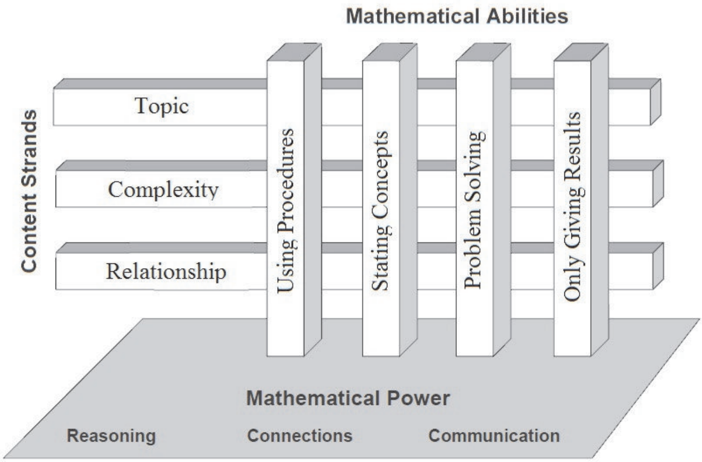


Figure 3. Analysis framework of mathematical teaching behaviors

The first dimension is Content strands, referring to TIMSS VS 1999 (Jacobs et al., 2003a) and the NAEP 2003 coding systems, it can be divided into topic, complexity and relationship. The topic is taken from a questionnaire survey and divided into five themes, number, measurement, geometry, statistics and algebra. Complexity and relationship mainly analyze the properties and relationships of questions in PS. Complexity divided mathematic problems into Low Complexity (LC), Moderate Complexity (MC) and High Complexity (HC). Relationship mainly divides the relationship between problems into Repetition (R), Dependent (D), Extension (EX), Simplification (SI), Elaboration (E), and Unrelated (U).

The second dimension is Mathematical ability, referring to TIMSS VS 1999 (Jacobs et al., 2003a) and the NAEP 2003 (NAGB, 2002) coding systems and contents, we divide Concept Statement into Using Procedures (CUP) and Stating Concepts (CSC). Problem Statement is divided into Using Procedures (PUP), Stating Concepts (PSC) and Problem Solving (PPS). Solving Problem is divided into Only Giving Results (SOR), Using Procedures (SUP), Stating Concepts (SSC) and Problem Solving (SPS).

The third dimension is Mathematical power, referring to NAEP 2003 (NAGB, 2002) and NCTM (2000) contents, analyzes teachers to see if in

CS, PS and SP, they underwent Reasoning (RS), Communication (CU) and Connection (CN).

As for content and structure of the three dimensions, see Table 2. For its definitions and examples, please see analysis of the results.

When coding took place, we referred to the coding method of TIMSS VS 1999, dividing it into time coding, frequency coding and class coding. Time coding uses time as the unit, each range of coding mutually exclusive to one another. Frequency coding records the number of specific events that happened in the classroom. Class coding records whether a particular event occurred in classrooms. All events are coded according to frequency coding as a basis, and the number of occurrences is recorded. In addition, complexity coding, relationship coding and mathematic ability coding occur in every class, therefore we only conduct time coding and frequency coding.

In terms of coding of Mathematical power, it doesn't occur in every class, therefore class coding is meaningful; at the same time not every teacher conducts reasoning, connection and communication in every event, therefore comparing the time code is not needed.

Analysis and weight of data was entered into Excel spreadsheets for quantitative analysis after we had completed coding of classroom video-taping and teaching surveys. In order to explain the sample design which resulted in different probabilities of the selection of samples, we must calculate the sampling weight of the entire sampling process: the first step is to calculate the weight of the school, and the second step is to calculate the weight of the class, to reflect the correct information of the sampling population. The Base Weight (BW*i*) of each classroom video is the reciprocal number of the product of the School Selection Probabilities (P*i*) multiply the probability that a class might be picked (C_i, representing the number of classes). The formula is as follows:

$$\mathrm{BW}i \quad \frac{C_i}{P_i}$$

The final weight of classroom videos (FW*i)* is the product of multiplying BW*i* by NRF*i*. NRF*i* representing the no-response adjustment factor of the location of school *i*.

$$\mathrm{FW}i = \mathrm{BW}i \times \mathrm{NRF}i$$

Table 2. Classroom video coding system

	Concept Statement (CS)	Problem Statement (PS)	Solving Problem (SP)
Content Strands	Topic (T): Number, Measurement, Geometry, Statistics, Algebra		
		Complexity (C) • Low Complexity (LC) • Moderate Complexity (MC) • High Complexity (HC) Relationship (RE) • Repetition (R) • Dependent (D) • Extension (EX) • Simplification (SI) • Elaboration (E) • Unrelated (U)	
Math Abilities	Using Procedures (CUP) Stating Concepts (CSC)	Using Procedures (PUP) Stating Concepts (PSC) Problem Solving (PPS)	Only Giving Results (SOR) Using Procedures (SUP) Stating Concepts (SSC) Problem Solving (SPS)
Math Power	Reasoning (RS) Communication (CU) Connection (CN) • Connection of Real Life Situation (CRLS) • Connection Within Mathematics (CWM) • Connection Among Other Disciplines (CAOD)	Reasoning (RS) Communication (CU) Connection (CN) • Connection of Real Life Situation (CRLS) • Connection Within Mathematics (CWM) • Connection Among Other Disciplines (CAOD)	Reasoning (RS) Communication (CU) Connection (CN) • Connection of Real Life Situation (CRLS) • Connection Within Mathematics (CWM) • Connection Among Other Disciplines (CAOD)

As all sampled schools were successfully videotaped, the no-response adjustment factor NRFi =1. The weight for each classroom video is:

$$\mathrm{FW}i=\frac{C_i}{P_i}$$

4.6 *Reliability and Validity*

The reliability and validity is based on the sampling methods of TIMSS VS 1999 (Jacobs et al., 2003a) and TIMSS 2007 (Olson, Martin, Mullis, 2008), sampling was conducted, therefore this there has nationally representative. For the method and coding of data analysis, we referenced the definitions of school mathematics principles and contents of TIMSS VS 1999, NAEP 2003 (NAGB, 2002) and NCTM (2000), they have undergone discussion and coding done by multi-country experts and scholars in mathematics education, so there is content validity and expert validity. Four members of the research team were responsible for the classification of classroom video coding. At first these four members used the same coding for four video tapes and did in-depth discussion for different items to establish concurring perceptions of classroom coding. If one of the members did not concur with others, the whole group would discuss with an expert and the expert would make decision. After that, members of the research team paired up and each pair took 14 videos (28 in total) for coding, each group comparing results after coding to establish reliability. The calculation of reliability is as follows:

Reliability of Time Coding = The sum of time with same coding ÷ (sum of time with same coding + sum of time with different coding)

Reliability of Frequency coding = Total frequency of same coding ÷ (total frequency of same coding + total frequency of different coding)

Reliability of Class coding = Total Class of same coding ÷ (total class of same coding + total class of different coding)

This research aggregated reliability of coding, and results were between 89%~100%, which meets the requirements of TIMSS VS 1999 that states the reliability should be above 85%.

After the reliability of coding measured up, four members proceeded to code the remaining 28 tapes individually.

5. Analysis of Results

In this study, the three dimensions Content strands, Mathematical ability and Mathematical power were used to analyze fourth-grade teachers' mathematical teaching behaviors.

5.1 Classroom Segments

The average teaching time of a math class is 37 minutes 16 seconds, deducting 6 minutes and 30 seconds which involved individual or group activities for students that cannot be properly classified, leaves 30 minutes and 46 seconds in remaining. This can be divided into three teaching segments: Concept Statement (CS), Problem Statement (PS) and Solving Problem (SP). From Figure 4, that the total teaching time of the three segments of accounted for 33%, 15% and 52% respectively; and the frequency of them accounted for 10% (2.11 concepts), 47% (10.08 problems) and 44% (9.46 solving Problems). This shows that teachers spend about 3/10 of the time explaining concepts, the average time describing a concept is 4 minutes 49 seconds; the teachers spends about 1/10 of the time on problem statement, spending 27 seconds on each question in average; the teachers spend about 1/2 of the time on Solving Problems, each problems taking about 1 minute 43 seconds. Study shows that the ratio of time spent on problem teaching and non-problem teaching are 2/3:1/3, respectively, demonstrating the fact that teachers' teaching mainly focus on problem teaching.

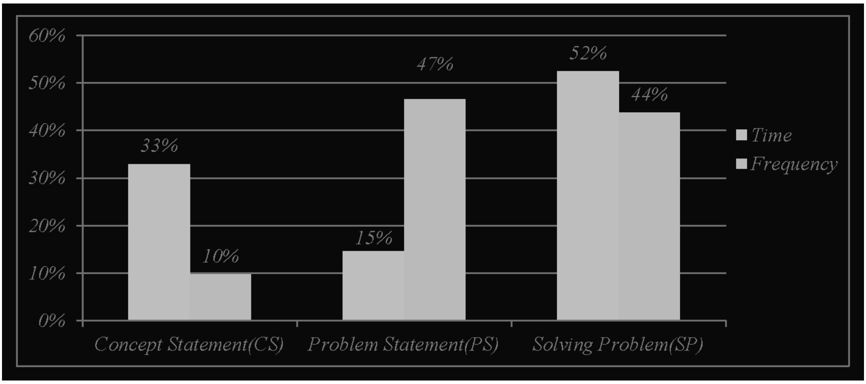


Figure 4. Ratio of mathematical abilities

5.2 *Content Strands*

Content strands were divided into 3 stages, topic, complexity and relationship. In the Content strands of fourth-grade mathematics classrooms, the number and measurement are covered the most. Research on classroom videos reveals that the percentages of time spent in number, measurement, geometry, statistics and algebra were 43%, 39%, 15%, 3%, 1%, respectively. This study compared teaching units of 3 main textbooks of the year of the video study, and found that the number of units found were 13 (41%), 14 (44%), 4 (13%), 1 (3%), 0 (0%) units, respectively.

Complexity coding is mainly used to understand the complexity of the mathematical problems presented in segments of Problem Statement which is sorted into Low Complexity, Moderate Complexity and High Complexity. Low Complexity problems expect students to recall or recognize concepts or procedures specified in the framework. Items typically specify what the student is to do, which is often to carry out some procedure that can be performed mechanically. For example, "Teachers asked students to measure the length of red or blue line segments, and then fill in the brackets." The Moderate Complexity problems required involve more flexibility of thinking and choice among alternatives than do those in the low-complexity category. The student is expected to decide what to do and how to do it, bringing together concepts and processes from various domains. For instance, "Teacher asked for students to read high-speed rail fare table, and calculated how much will be left if taking two thousand dollars to buy two tickets from Taoyuan to Chiayi?" High Complexity problems were heavy demands on students, because they are expected to use reasoning, planning, analysis, judgment, and creative thought. Students may be expected to justify mathematical statements or construct a mathematical argument. In Figure 5, Low Complexity problems (LC) and Moderate Complexity problems (MC) take up 79% and 21% of teaching time, respectively, and the ratio of the number of problems (i.e., the proportion of events) were 88% and 12% respectively. High Complexity problems (HC) were not found in this research.

Relationship coding is to understand the relationship between mathematical problems taught in segments of Problem Statement, divided into Repetition (R), Dependent (D), Extension (EX), Simplification (SI), Elaboration (E) and Unrelated U). According to the definition in TIMSS VS 1999 (Jacobs et al., 2003a), Repetition referred to the problem was exactly or mostly the same as the preceding problem. The numbers or algebraic expressions may have been different, but the procedures were the same. For example, "Teachers has already taught that how much cake one could get if there're four pies given to three people. So, how much cake could one get if there're five pies given to

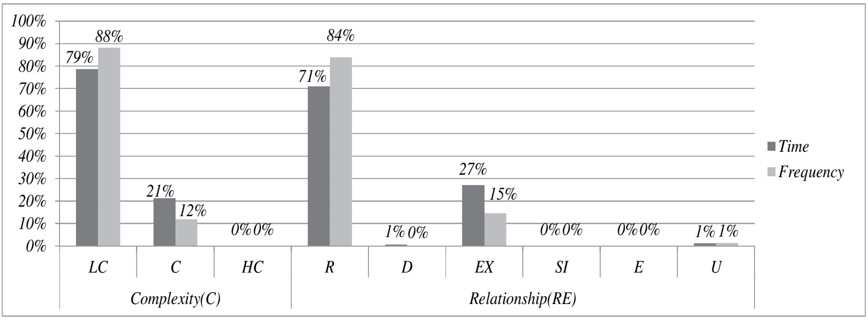


Figure 5. Proportion of types of mathematical problems

three individuals?" Dependent referred to the solution to one of the previous problems in the lessons was necessary to solve the current problem. For example, "Teachers had the children measure the bulletin and works boards to know how big a column plate was needed. So, which board would be bigger?" Extension referred to the problem required many of the same operations as the preceding problem plus some important additional operations. It also includes cases where the problem was a generalization of previous problems. For instance, "Teachers had asked children to calculate three days and six hours are equivalent to how many hours? (To fill in: greater, equal or less) 6 days and 18 hours (   ) 200 hours." unrelated meant the problem required operations much different than other problems in the lessons and neither of the thematic codes applied. For example, "Teachers were teaching time unit conversion. Please select the wrong options: (1) lunch time at school is about 40 minutes; (2) climbing the stairs to the second floor from the first takes about 30 seconds; (3) the sleeping time of pupils is about 8 minutes; (4) a boy rides a bicycle every day for about 30 minutes." Simplification referred to was assigned when the problem illustrated a simpler example of the previous problem or was used to provide emphasis. Elaboration were similar to the previous problem but used a different set of operations (e.g., solving the problem another way). Simplified and Elaboration problems were not found in this study. As seen in Figure 5, We found that Repetition, Dependent, Extension, Simplification, Elaboration, as well as Unrelated accounted for 71%, 1%, 27%, 0%, 0%, 1%, respectively, of the teaching time in math problems; moreover, for 84%, 0%, 15%, 0%, 0%, 1% of the frequency of them.

5.3 *Mathematical Ability*

Mathematical abilities were divided into Concept Statement, Problem Statement, and Solving Problem. Concept Statement was divided into Using Procedures (CUP) and Stating Concepts (CSC). Using Procedures referred to the procedural knowledge used to let students learn mathematical concepts by teachers. Routine operation, symbol manipulation or formula introduction was all included to be applied to problem-solving by students. For example, "Teachers asked what the area of a rectangle is. Students answered that its length multiplied by width, and without asking what the area formula's origin is." Stating Concepts referred to the basic concept-explaining by teachers in the classroom to make students get conceptual understanding. For instance, "Teachers used two straight lines intersecting at right angles (90°) to introduce the concept of being perpendicular to one another." In Figure 6, the results showed that the teaching time of Using Procedures and Stating Concepts accounted for 26% and 74% of Concept Statement, and the frequency of them were 26% and 74%, respectively. This shows the teachers spent about 3/4 of time and frequency on describing concepts during conceptual teaching, this is far more than time and frequency spent when conducting procedural teaching only. This is reasonable that the teachers use more conceptual knowledge when teaching concepts.

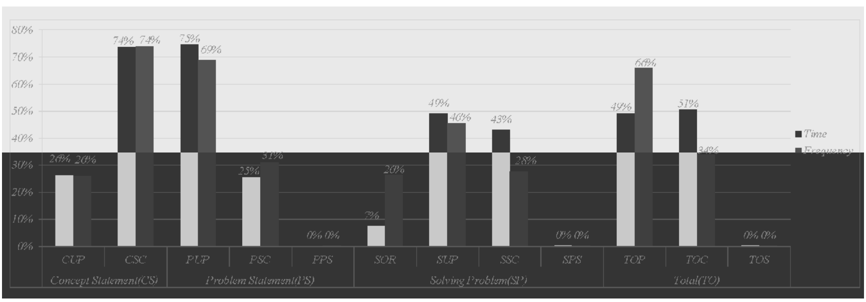


Figure 6. Breakdown of the percentage of mathematics ability

Problem Statement was divided into Using Procedures (PUP), Stating Concepts (PSC) and Problem Solving (PPS). Using Procedures was those typically associated with routine algorithms such as calculations, symbol manipulation, and practicing of formulae. These problems are generally associated with following a routine process or set of "steps". This category did not imply that there were no mathematical decisions to be made, but rather that the decisions assumed a set path - such as in a computer decision-making scheme. For example, "How many hours and minutes will be equal to 110 minutes?" Stating Concepts asked students to recall information regarding a mathematical definition, formula, or property. These problems typically had one step in which the recall of such information was needed to fit the example to a definition or property. For example, "Teacher asked students to classify some plane figures and explain the reasons." Problem Solving meant students were asked to do some mathematical reasoning, such as deduction, induction and proving. This kind of problems wanted students to think about mathematical concepts, to develop mathematical ideas or to expand those concepts and ideas. For example, "Students were asked to compare and find the law of $16 \div 8 = 2$, $160 \div 8 = 20$, $1600 \div 8 = 200$." As seen in Figure 6, Using Procedures, Stating Concepts and Problem Solving accounted for 75%, 25% and 0% of Problem Statement, while the frequency of them accounted for 69%, 31% and 0%, respectively.

Solving Problem was divided into Giving Results Only (SRO), Using Procedures (SUP), Stating Concepts (SSC) and Problem Solving (SPS). Giving Results Only means the public talk about the problem centered solely on the statement of the final result. Using Procedures mean the routine execution of an algorithm was used to work on and complete a problem. Generally speaking, in this type of problem students and teacher talked only about how to progress to find the answer, such as stating the steps taken along the way. For example, "Teachers simply demonstrated the process of a three-digit number divided by two-digit number without explaining the concept of them." Stating Concepts mean the class alluded to a mathematical concept but did not provide any descriptions of mathematical relationships or note why the concept was appropriate for the given situation. For example, "Teachers asked how many millimeters is equal to five-tenths centimeters? Students answered not only five millimeters but telling the concept of five-tenths just being the fifth grid among the ten grids from zero to one." Problem Solving referred to recognizing and formulating problems; determining the sufficiency and consistency of data; using strategies, data, models, and relevant mathematics; generating, extending, and modifying procedures; using reasoning in new settings; and judging the reasonableness and correctness of solutions. For example, "Students were asked to compare $16 \div 8 = 2$, $160 \div 8 = 20$, $1600 \div 8 = 200$ and then to discuss the results and reasons." As seen in Figure 6, Giving Results Only, Using Procedures, Stating Concepts and Problem Solving accounted for 7%, 49%, 43% and 0% of teaching time, while the frequency of them was 26%, 46%, 28% and 0%, respectively.

We integrated them and divided them into Using Procedures (TOP), Stating Concepts (TOC) and Problem Solving (TOS) and found that the ratio of time is about 5:5:0, showing that fourth grade teachers spent half the time in their mathematics classrooms each on using procedures and stating concepts.

5.4 *Mathematical Power*

5.4.1 *Reasoning*

About the Reasoning, according to the definition in NCTM (2000), it means that the students can make and investigate mathematical conjectures, can develop and evaluate mathematical arguments and proofs. For example, "Teachers encourage students to discuss if the weight of pineapple and watermelon is equal, and ask them to use different ways to explain why." As seen in Figure 7, the frequency of Reasoning among Concept Statement, Problem Statement and Solving Problem are 23%, 4%, and 14% respectively, while taking up 32%, 18% and 60% of total classes.

We integrated the data and discovered that about 1/10 of frequency was spent on mathematics reasoning, and about 7/10 of classes had ever done reasoning in teaching.

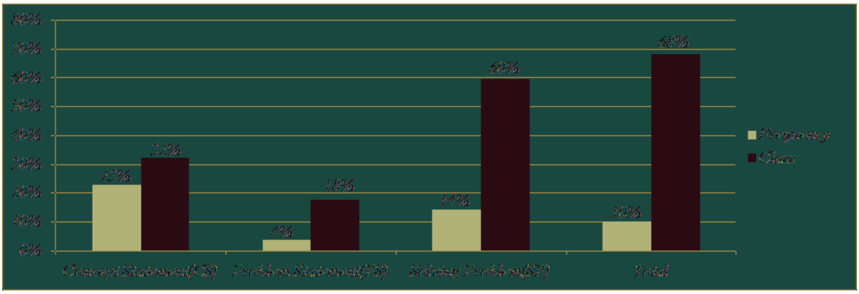


Figure 7. Proportion of reasoning in teaching

Note: Only showing the percentage of classes actually conducting Reasoning, so the sum will not be 100%.

5.4.2 *Communication*

Communication means students can communicate about the mathematics they are studying–to justify their reasoning to a classmate or to formulate a question about something that is puzzling–they gain insights into their thinking. In order to communicate their thinking to others, students naturally reflect on their learning and organize and consolidate their thinking about mathematics. For example: "Teachers encourage students to explain the definition of fraction, and ask students to give some examples to illustrate it." As seen in Figure 8, the frequency of Communication among teachers in Concept Statement, Problem Statement and Solving Problem are 22%, 0% and 19%, while taking up 29%, 0% and 64% of total classes.

We integrated the data and discovered that similar to whether teachers have conducted reasoning, the ratio of communication during teaching was only about 1/10 of frequency, and less than 7/10 of classes had ever conducted communication in teaching.

5.4.3 *Connection*

Connection means teachers can see and experience the rich interplay among mathematical topics, between mathematics and other subjects, and between mathematics and their own interests. As seen in Figure 9, the frequency of Connection among teachers in Concept Statement,

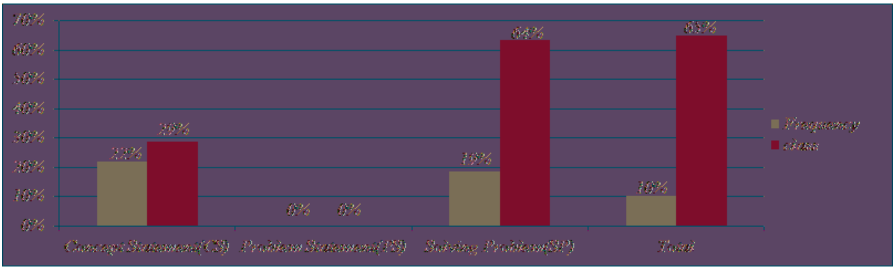


Figure 8. Proportion of communication in teaching

Note: Only showing the percentage of classes actually conducting Communication, so the sum will not be 100%.

Problem Statement and Solving Problem are 63%, 44% and 37%, which is 75%, 84% and 76% of total classes, respectively.

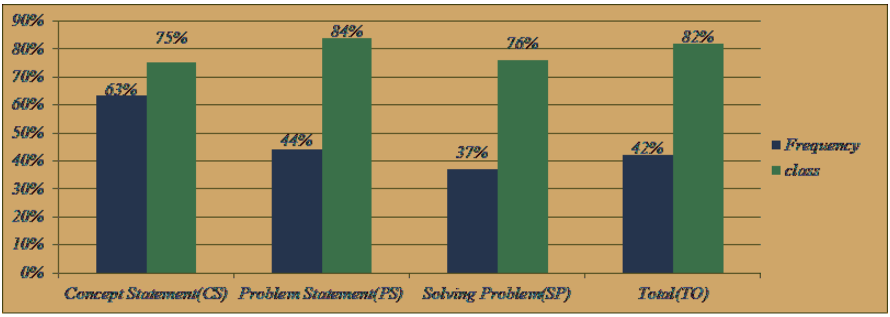


Figure 9. Proportion of connection in teaching

Note: Only showing the percentage of classes actually conducting Connection, so the sum will not be 100%.

We can relate the coding of connections with Life Situation, Inner Mathematics and Other Subject. When we say Life Situations, we mean the teaching activity that can connect to what a student may or have experienced in life which teachers perform. For example, "The teacher introduces the concept of length-unit a few days after measuring the students' height in class." Inner Mathematics means the mathematical concepts or questions which the teachers have taught is connected to the mathematical concepts the students have already learned. For example, "Teacher reviewed the meaning of hour taught in second grade, and introduces the content of how many minutes and hour." Other Subject means students can recognize and apply mathematics in other subject. For example, “The teacher tells the students the pole-vaulting record of the world and the record of Taiwan, to teach about both pole-vaulting and compare with lengths at the same time.” In Concept Statement, as seen in Figure 10, the frequency of connections with Life Situation, Inner Mathematics and Other Subject are 46 %, 22% and 2%, which is 68%, 37% and 3% of total classes, respectively. In Problem Statement, the frequency of connections with Life Situation, Inner Mathematics and Other Subject are 44%, 1% and 0%, which is 84%, 4% and 0% of total classes. In Solving Problem, the frequency of connections with Life Situation, Inner Mathematics and Other Subject are 35%, 2% and 0%, which is 72%, 18% and 5% of all classes, respectively.

We integrated the data and found that most of the connections were on Life Situation, which is about 2/5 of the frequency, and 19/20 of the classes. Inner Mathematics connections were scarce, but about, 1/2 of the classes had conducted inner mathematics connections. Connections with Other Subjects were fairly scarce.

Because the connections of Life Situation mentioned above include connections with students’ life experience, it has a broader definition. Therefore, this study sorts connections with Life Situation into 1) the situations designed from textbooks, 2) the situations the teachers designed and 3) the situations the students experience in real life. As seen in Figure 11, it was found the frequencies are 68%, 12% and 19%, which is 76%, 33% and 49% of total classes that have connections with daily situation, respectively.

6. Discussion and Recommendations

Taiwanese teachers’ mathematics teaching in fourth-grade classrooms spent 3.5:1.5:5 of the time on concept statement (CS), problem statement (PS) and solving problem (PS), each lesson teaches about 2 concepts, states about 10 problems and solves about 9.5 problems. This shows that teachers spends more time on setting and solving problems, asking more questions, and spending less time on conceptual teaching, which means that teachers’ teaching tends to be more focused on problem solving.

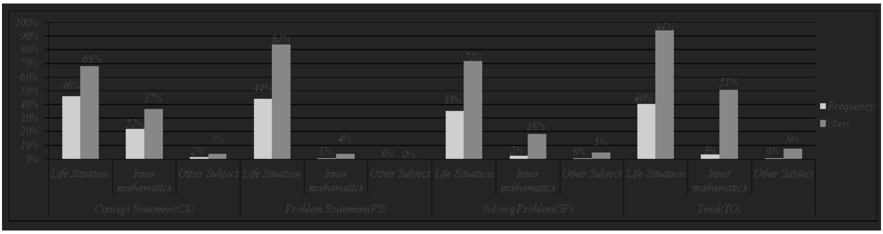


Figure 10. Breakdown of proportion of connection in teaching

Note: The sum is over 100% because connections of life situation, inner mathematics and other subject connections may happen at the same time during teaching.

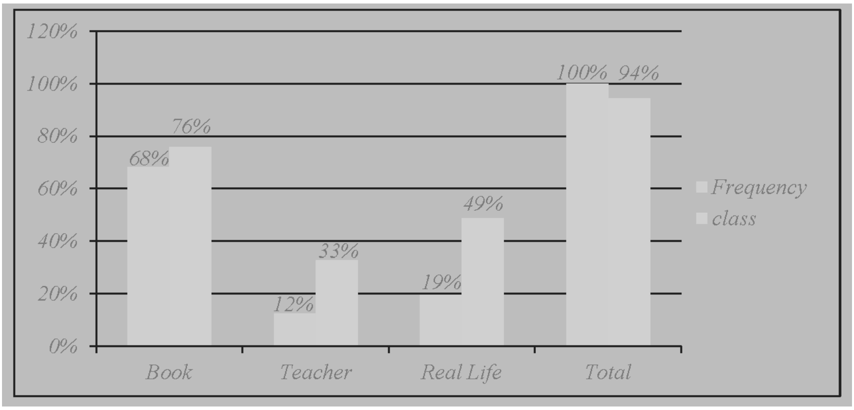


Figure 11. Proportion of connections with life situations.

In terms of Content strands, fourth-grade mathematics teaching is more number and measurement based, with about 40% each, following by geometry with about 15%, which is consistent with features of Taiwan's elementary mathematics curriculum. Low complexity problems and moderate complexity problems were 9:1. Repetition and extension problems were about 8.5:1.5 It shows that fourth grade teachers' teaching tend to be more focused on low-complexity and repetitive problems.

Researchers think that fourth-graders are only beginning to make the transition from Piaget's concrete operation stage into the formal operation stage, therefore it is reasonable that low-complexity and repetitive problems are used more often, but the appropriateness of this ratio needs further investigation. Also, it is a pity that high-complexity and elaboration problems did not appear in classrooms. If by the end of a lesson, a teacher occasionally poses one or two high complexity problems for the students to think about at home, to increase students' opportunities of learning outside the classroom, it may help to enhance students' achievements. Therefore, whether Taiwanese fourth-grade teachers have the ability to pose elaboration and high complexity problems, or whether they have the ability but presume that students are not suitable for learning in this way, are questions worthy of exploration.

In terms of Mathematic ability, the time spent on concept statement by using procedures and stating concepts is about 2.5:7.5. In problem statement, the frequency the students spent on using procedures, stating concepts and problem solving is 7:3:0. Integrating the data, we see that the ratio of time spent on using procedures, stating concepts and problem solving is about 5:5:0. About solving problems, the frequency that teachers are giving results only, using procedures, stating concepts and problem solving is 2.5:4.5:3:0. It showed that teachers spent about the same time on concept statement and using procedures, and when fourth-grade teachers are teaching, they can describe the concept fairly well, but when teaching problems, teachers tend to be using problems of using procedures and giving procedural solutions.

Teachers' teaching should first allow a certain percentage of students to understand the conceptual knowledge, and then transfer into teaching procedural problems. What the actual percentage of fourth grade students that already knew about conceptual knowledge when it is presented in teachers' teaching, and in scientific theory should be the proportion of students who understand conceptual knowledge when it is presented in teachers' teachings. Only after clarifying these two issues, will we have a way to answer whether such a proportion of the teaching of mathematical ability is appropriate. In addition, the researchers believe that teachers in the classroom should pose some problem-solving questions, this way, students will have the chance to learn how to answer some non-routine problems, but this seldom appeared in Taiwanese mathematical classrooms. The researchers also found through practical experience that some of the of teachers think that as long as students can solve problems, the establishment of mathematical concepts can wait until the students grow up, emphasize it when they have better comprehension ability, after they go to junior high school. But whether this view is correct remains to be elucidated. Do teachers actually not have time to pose problem-solving questions? Do they not have the ability to pose problem-solving questions? Or do they think that it isn't necessary? Maybe there are other reasons? It is worthy to make in-depth discussion on this topic.

About the Mathematical power, the frequency of reasoning appearing among concept statement, problem statement, and solving problem is about 2.5:0.5:1.5, the proportion of school classes is about 3:2:6. In total, only 10% of the frequency and 70% on the classes have conducted reasoning. The frequency of communication is 2:0:2, the proportion of school classes is 3:0:6.5, respectively. In total, only 10% of the frequency and 70% on the classes have conducted communicating. The frequency of connection is about 6.5:4.5:3.5, the proportion of school classes is 7.5:8.5:7.5, respectively. In total, only 40% of the frequency and 80% on the classes have made connection. The majority of connection is with life situation, inner mathematics connections were only found more in concept statement, which was 20%. It shows that fourth-grade students are exposed to connection with daily situations, and that reasoning, communication and internal mathematic connections were scarcely found.

Students in primary school, they are in the concrete operation stage. The fourth graders should gradually be exposed to reasoning, communication and connections within their mathematics learning. It is still required that we do further relevant studies to see if the proportion and frequency of such classes can help provide optimal opportunity of learning. About connections, more connections with daily situation can provide students with a more concrete sense, but if we wish that students can understand the inner-connections of mathematics, have a more mathematical sense, then we must conduct internal-mathematics connections. In this respect, whether inner-connections of mathematics are enough to fourth-grade students also requires further study. Also, when teachers do not use reasoning, communication and connections in teaching, is it because they think it is not necessary, or is it a matter of professional knowledge? This is worth exploring. Integrating analysis of all dimensions, we find that teachers' teaching of fourth-grade mathematics is partial to using low-complexity and repetitive problems, using procedure to solve problems, at the same time lacking in communication and reasoning. To understand why the math performance of Taiwanese students rank top internationally, we did further preliminary study on the contents of TIMSS 2007 assessment on fourth-graders, and discovered that knowing, applying, and reasoning items were 40%, 40% and 20% of questions, respectively. Therefore, in Taiwanese fourth-grade teachers' teaching, the frequency of reasoning in concept statement is 25%, the frequency of reasoning in problem solving is 15%, it seems to be in accordance with TIMSS 2007 assessment. Only the proportion of problems needing students' reasoning is relatively low, at 5%. Teachers' teaching perhaps has something to do with Taiwanese students ranking top internationally, but the real relation between teachers' teaching and students' performances still need further investigation, we still need to consider students' out-of-school learning (X) and others (Z).

7. Implications for Further Studies

Because fourth-grade and eighth-grade mathematics content varies greatly, and students' way of thinking has great characteristic differences, we did not compare the results with TIMSS VS 1999 assessment of eighth-graders. Readers interested may make comparisons on your own. Nevertheless, fourth-grade mathematics teaching classroom video analysis let us see the mathematical teaching behavior of teachers, at the same time triggering issues worthy of further investigation.

7.1 *Research on Influencing Factors of Student Achievement*

The frequency of Taiwanese fourth-grader teachers using reasoning when stating concepts and solving problems seems to be consistent with TIMSS reasoning assessment problems, but it is somewhat lower in problem statement. Therefore, it is still not directly evident to explain Taiwanese students' achievements being among the best, because out-of-school lessons (Y) and societal, cultural, and other factors (Z) will also affect students' achievements. Perhaps we may use national sampling, focusing on sample student's school learning, out-of-school learning and other factors, conform to carry out a systematic large-scale research, perhaps we can find the direct evidence that explains Taiwan student achievement being among the best, and to find the weight of a, b and c.

7.2 *Video Study on Real Classroom Teaching of Teachers at Each Grade*

The fourth-grade mathematics classroom video study found that teachers' teaching seems to be contrary to the Taiwanese curriculum guideline which requires teachers to emphasize communication, reasoning, connecting and problem solving. Whether it conforms to the learning characteristics of fourth-graders making the transition from the concrete operation stage into the formal operation stage still requires further study. Thus, we can try mathematics classroom video study on second, sixth and eighth grade students to understand general teaching behavior and mathematical teaching behavior, from the diversification we can make appropriate inferences on whether the proportion of fourth-grade teachers' teaching is suitable, at the same time see whether the results correspond to TIMSS VS 1999 eighth-grade assessment. And only be doing so can we truly understand the meaning of X (school lessons) of $aX+bY+cZ$ (Lin, 2010). We need to find out Y (out-of-school lessons) and Z (others, e.g., Societal, Cultural…) to find the weight of a, b and c.

7.3 *Study on Teachers' Problem-posing and Teaching Ability, Belief and Knowledge*

Kuntze and Reiss (2005, 2006) inquired into teacher's belief and knowledge using the TIMSS VS material. In Taiwan, the mathematics problems fourth-grade mathematics teachers taught were mostly of low complexity, repetitive issues and using procedure to find solutions. Moderate complexity, high complexity, extension, and elaborate problems were seldom or almost never found. Whether Taiwanese fourth-grade teachers have the ability to pose elaboration and high complexity problems, or do they think that it isn't necessary? Reasoning, communicating, internal and external links were seldom found in concept statement and problem solving. Is this due to the teachers' lack in teaching ability? Or are the teachers capable, but believe that fourth-graders do not need to be taught too much, or even that they will fail to learn so much? We may also refer to Kuntze and Reiss (2005, 2006) research, conducting a study on teachers' problem-posing and teaching ability, belief and knowledge; it may even be worthy of conducting a nationwide investigation.

7.4 *Research on Relationship of Teachers' Teaching and Students' Interest and Confidence*

The fourth-grade teacher's mathematics problems tend to require students' usage of procedure in order to respond, so the probability that the solution itself is answered by simply giving the solution or using procedures is also very high, the teaching proportion of reasoning, communication and internal-mathematics connections is very low. The mathematics rankings of fourth grade students in Taiwan in TIMSS 2003 (Mullis, Martin, Gonzalez, & Chrostowski, 2004) and in TIMSS 2007 (Mullis et al., 2008) are fourth place and third place respectively. However, their attitude of learning mathematics and their self-confidence are both almost at the bottom of the rankings. Are they connected? Perhaps we may aim a study at student's learning interest and their self-confidence and whether the teachers emphasized conceptual understanding, reasoning, communication, and internal-mathematics connections.

7.5 *Research on Comparison of Textbooks and Teachers' Teaching*

Vincent and Stacey's (2008) comparative study on the TIMSS VS issues found that there are many similar places with the topics of local textbooks, in addition, our survey also found that teachers are very dependent on textbooks. But aside from similar topics, textbooks basically are inanimate while the teaching is dynamic, thus whether teachers' interpretation of textbooks and actual teaching are the same, and if they are not the same then the reason of this, is also a topic worthy of a national study.

7.6 *Study on Expert Teachers' Problem Posing and Teaching Abilities*

The main purpose of the LPS (Clarke & Novotná, 2008) plan is to investigate good mathematics teaching. So in scientific theory when teachers are teaching the fourth-grade mathematics, what should the proportion of Complexity problems, the proportion of Relationship problems, the proportions of conceptual problem and problem solving questions be posing? In problem teaching, to how many percentages should the teacher conduct reasoning, communication, and the internal connections, to enable the student to obtain suitable development, to fully maximize the students' zone of Proximal Development (ZPD), which is what we all consider as good mathematics teaching. Perhaps in this aspect, we can refer to the LPS plan, pursue a group of expert teachers of high teaching-quality who can help students to develop fully, carry out a one-semester or one-year video study, then analyze their teachings to understand what general teachers must achieve to have expert-teacher quality, understand what kind of teaching a teacher must carry out in scientific theory. In addition we may also refer to the Shimizu (2009) research, see from the students' angle what characteristics they consider to be good teaching.

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Chapter 13

Features of Exemplary Lessons under the Curriculum Reform in Chinese Mainland: A Study of Thirteen Elementary Mathematics Lessons

MA Yunpeng ZHAO Dongchen

Dramatic changes in mathematics education in Chinese mainland have taken place since the new mathematics curriculum standard was implemented in 2001. What new features do exemplary lessons appear under the context of the curriculum reform? This chapter will answer this question by presenting a case study of 13 elementary mathematics lessons that were evaluated as excellent exemplary lessons by mathematics educators in China (mainland). It finds that, consistent with the ideas advocated by the new curriculum, the selected lessons demonstrated the features of emphasizing on student's overall development, connecting mathematics to real-life, providing students the opportunities for inquiring and collaborating, and teachers' exploiting various resources for teaching. Meanwhile, the selected lessons also shared other common features in the lesson structure, interaction between the teacher and students, and classroom discourse. The results reveal that the exemplary lessons have practiced the advocated ideas of the current reform, while they also embodied some elements that might be the stable characteristics of Chinese mathematics education.

Keywords: Chinese mathematics classroom, teaching practice reform, exemplary lesson, elementary mathematics

1. Introduction

In the past decades, investigating and understanding Chinese mathematics education, especially the mathematics classroom in China, has been of interest to many educators and researchers (e.g., Gu, Huang, & Marton, 2004; Huang & Leung, 2004; Huang, Mok, & Leung, 2006; Leung, 1995; Ma, Zhao, & Tuo, 2004; Stevenson & Stigler, 1992). Recently, efforts to improve the quality of classroom instruction have led to ever-increased interests in research on excellent lessons (e.g., Huang, Pang, & Li, 2009; Li & Shimizu, 2009; Li & Yang, 2003; Zhao & Ma, 2012). From different perspectives, the existent studies have deepened the understanding of Chinese mathematics classroom. Yet, the picture of Chinese mathematics classroom is not clear enough. More investigations and studies on Chinese mathematics classroom are needed.

In pursuit of knowing and understanding the characteristics of Chinese mathematics classroom, we should be aware that some changes might be taking place in Chinese mathematics classroom with the global change and the development of Chinese society. At the turn of the 21st century, with the aim of preparing younger generations for an age in which the economy is globalized, and the society is information-rich and "knowledge-based", mathematics curriculum in many education systems around the world have undergone dramatic changes (Wong, Han, & Lee, 2004). In such a situation, the mathematics curriculum in China is no exception. In September, 2001, China initiated and implemented the new round of curriculum reform of compulsory education (Ministry of Education, 2001). According to some reports, changes have taken place in the classroom as a result of the current curriculum reform (Li, 2002; Song, 2003). If the reform could be implemented deeply and continuously, we may expect that the practice of China's mathematics classroom will demonstrate many differences from that of the classroom in past decades. While as a cultural activity, teaching has its relative stability. In some comparative studies, both differences and similarities were found in the exemplary mathematics lessons in different decades (Huang, Pang, & Li, 2009; Li & Yang, 2003; Zhao & Ma, 2012).

What characteristics do the "excellent" lessons have in the current curriculum reform? Or, how mathematics is taught and learned in the exemplary lessons in the context of reform? How are the ideas advocated by the new curriculum embodied in these exemplary lessons? Are there any other common features shown in these lessons? Much remains unknown about these questions. This chapter will attempt to answer these questions by presenting a study, in which 13 elementary mathematics lessons valued as the excellent exemplary lessons under the new curriculum reform were analysed.

This chapter is structured with four parts: firstly, the background and main changes of the mathematics curriculum will be briefly introduced; secondly, the background of the lessons analysed in this study and the analysis method will be described; thirdly, we will present the results of this study in two aspects: one is the lessons' features that were consistent with the ideas advocated by the curriculum reform; the other one is other common features embodied in the lessons; and at last, we will give a short summary and discussion on the results. Based on the discussion, some implications are drawn from this study.

2. Background: Current Curriculum Reform in China

Mathematics curriculum in China has experienced several waves of changes since the founding of People's Republic of China in 1949 (Su & Xie, 2007). The current new round of mathematics curriculum reform of compulsory education[1] was initiated and implemented under the guidance of *Mathematics Curriculum Standard for Full-time Compulsory Education (Manuscript for consultation)* (hereafter *Standards*) in September, 2001 (Ministry of Education, 2001). Since September, 2005, all the students in the first academic year of primary school and junior high school have used the new curriculum. Now, the new curriculum has spread out nationwide.

Before the implementation of the new curriculum, the latest mathematics curriculum was developed under the guidance of *Mathematics Syllabus for Elementary School of Nine-year Compulsory Education* and *Mathematics Syllabus for Junior High School of Nine-year*

Compulsory Education issued in 1992. The mathematics curriculum guided by the above-mentioned two syllabi was suitable for social development at that time, but there are still some problems left to be solved. For example, the syllabi issued in 1992 over-emphasizes "two basics" (basic knowledge and basic skill) and did not take into account students' development of affection and attitude, and this resulted in student's unbalanced development. Some contents of the curriculum were too difficult and narrow, and were not related to the students' real life (Ma, 2001; Zhang, 2002). Furthermore, the teaching method was monotonous; teachers used textbooks as the only reference for their teaching, and perceived teaching as transferring knowledge from textbook to students (Ma, 2001). In order to solve these problems and make the mathematics curriculum more responsive to the need of the development of both students and society, the Ministry of Education (MOE) of China initiated the mathematics curriculum reform.

The new mathematics curriculum at the stage of compulsory education aims at providing a solid foundation for students' full, sustainable and harmonious development, and to provide mathematics education for all students (MOE, 2001, p. 1). Students' over-all development has been the most important goal of China's education especially because quality-oriented education was advocated by Chinese government since the 1990s (CCCPC & the State Council, 1999). The Standards takes student's affection and attitude as one important dimension of their over-all development, and takes students' learning "process" as important as "outcome". For example, the Standards emphasize students' full development by focusing curriculum objectives on four aspects: knowledge and skill, mathematical thinking, problem solving, as well as affection and attitude. The curriculum contents consist of four dimensions: Number & Algebra, Space & Graph, Statistics & Probability, and Integration & Practice. Nine-year Compulsory education is divided into three phases: the first is for Grade 1 to 3; the second is for Grade 4 to 6; and the third is for Grade 7 to 9. For each phase, objectives for knowledge and skills, mathematical thinking, problem solving, and affect & attitude are elaborated in the Standards. Some contents in the former mathematics curriculum was trimmed down, meanwhile, some new contents was added to the new curriculum. Calculating and solving problem in multiple ways and strategies are encouraged.

The new curriculum also proposes some new ideas for improving mathematics classroom practice. It suggests that teaching should be closely related to students' daily life so that students can connect mathematics with real world (MOE, 2001, p. 51). It emphasizes that mathematics teaching and learning should "begin from student's primary experience of real life, and encourage student to experience the process of abstracting mathematics model from real-life problem, and the process of interpreting and applying." (MOE, 2001, p. 1) "Contents of mathematics learning for school children ought to be realistic, meaningful and challenging. These contents should facilitate school children to engage actively in mathematical activities, such as observation, experimentation, guessing, hypothesis testing, inference making, and communication." (MOE, 2001, p. 2) It is also claimed that "effective mathematics learning activities cannot simply rely on imitation and memorization. Instead, hands-on practical work, autonomous investigation and cooperative exchanges are important modes of mathematics learning." (MOE, 2001, p. 2) Besides, the mathematics standards also encourage teachers to design and enact their lessons creatively rather than to perceive teaching as transferring knowledge from textbook to students mechanically (MOE, 2001, p.51).

In a word, dramatic changes have taken place in the mathematics curriculum in China since 2001. The ideas advocated by the new curriculum bring both opportunities and challenges for mathematics teachers. How to implement the new ideas in the classroom? And what should an excellent mathematics lesson be like? Mathematics educators have been thinking about these questions and putting their understanding into their classroom practice. It is also of interest to researchers to identify and examine the features of the excellent lessons in this reform context.

3. Methodology

3.1 *Research Questions*

The analyzed lessons in this study were the prized exemplary lessons at the national level in the current context of curriculum reform. This study aims at answering the following two questions:

1) How the ideas advocated by the *Standards* are implemented in the exemplary lessons?
2) What other common features could be found in the exemplary lessons?

3.2 *The Selected Lessons*

In China, the institutions responsible for administrating educational research at the national or provincial levels often organize teaching contests and teaching exhibitions (see Li & Li, 2009). In 2008, the NCCT (National Centre for School Curriculum and Textbook Development) of the Ministry of Education organized the 1st National Contest in Exemplary Lessons of Elementary Mathematics in the new curriculum reform context. Elementary mathematics teachers were encouraged to design and implement mathematics lessons to show how the new curriculum was taught and learned in their classrooms. The teachers had many choices in the teaching topic, grades, mathematics content fields, and lesson types. They had their lessons video-taped and submitted the lessons to the NCCT. About 820 video-taped lessons were called up from each province (municipality and autonomous region) in China, which covered grade 1-6, four fields of content (*Number & Algebra, Space & Graph, Statistics & Probability,* and *Integration & Practice*), and 3 types of lesson (*XinShou Ke*–Teaching and learning new content, *FuXi Ke*–Reviewing the previously learned content, and *ZongHe ShiJian Ke*–Integrated using knowledge to solve problems). Then, these lessons were evaluated by an Expert Evaluating Group which was constituted of Mathematics educators and researchers with prepared evaluation criteria.

The criteria include (a) lesson plan, (b) enacted lesson (i.e., the process of teaching and learning), (c) outcome of the lesson, and (d) teacher's reflection on the lesson. Under each aspect, sub-criteria and its descriptions were also provided. For example, the enacted lesson were evaluated on the ways of teaching and learning, classroom management, interaction between teacher and student, assessment and feedback, usage of facility or media, and teacher's personal style of teaching. For the "usage of facility or media", it was described as "appropriately use a variety of media and resources to facilitate teaching and learning". Based on the Experts Group's judgement and grading, 55 lessons were finally selected and honoured as the First Prize.

In this study, we focused on the lessons in type of "*Number & Algebra*" and "*XinShou Ke*", and selected lessons only from those in grade 3 or 4. Thirteen lessons in total were selected for analysis. Their topics of teaching and learning, grades of students', and their codes in this study are shown in Table 1.

Table 1. General background information about the selected lessons

Lessons taught in grade 3		Lessons taught in grade 4	
Code	Topic of teaching and learning	Code	Topic of teaching and learning
A	Knowing and understanding second (time unit)	H	Countermeasure
B	Knowing and understanding fractions	I	Multiplication: 3-digit by 2-digit
C	Division with remainder	J	Using letters to present numbers
D	Year, month, and day	K	Multiples and factors
E	Year, month, and day	L	Solving the problems of planting trees
F	Year, month, and day	M	Solving the problems of planting trees
		N	Solving the problems of planting trees

Note: Lesson D, E, and F focused on the same topic, and lesson L, M, and N focused on another same topic. This is coincidental.

In addition, the textbooks used or referred by these lessons and the lesson plans of eleven lessons were also collected for analysis.

3.3 *Method of Analyzing*

The lesson analysis in this study was mainly based on the videotapes and the plans of the selected 13 lessons. The method of analysing are decided according to the research questions. Two different analysis strategies were used for answering the two questions.

3.3.1 *"Up-Down" Strategy*

For the first research question, an "up-down" strategy was used. We extracted the advocated ideas relevant to mathematics teaching and learning from the *Standards* and selected four ideas that were suitable for video analysis and lesson plan analysis. Then the analysis focused on whether these ideas embodied in the lessons videos or lesson plans and how they were implemented.

As introduced in the part of "Background" in this chapter, many ideas relevant to mathematics teaching and learning were proposed in the *Standards*. However, some of them are difficult to be examined and identified in a lesson by video analysis. For example, in the "Suggestion for teaching" in the *Standards*, it is suggested that teachers should create contexts and guide students learning in the contexts (MOE of China, 2001, p. 51, p. 64). What is context? The *Standards* does not give a definition. Instead, it gives some suggestions for creating contexts for the phase one (Grade 1-3) and phase two (Grade 4-6) as following.

> Design the lively, interesting, and visual mathematical activities, such as the use of storytelling, games, visual demonstration, and scenario performance, to stimulate students' interest in learning, so that it can help the students know and understand the mathematics knowledge in a vivid and specific context (p. 51). (Suggestions for the phase one)
>
> Create the contexts relevant to students' living situation and knowledge background in which the students are interested (p. 64). (Suggestions for the phase one)

From the suggestions we can infer that the purpose of creating contexts is to make mathematics lively, interesting, and relevant to real-life, so that it can provide motivation and experience foundation for students' learning. Even so, only from researcher's perspective without consulting the concerning students' opinions, it is difficult to judge whether teaching and learning are lively and interesting. By contrast, it is practicable to judge whether the teaching and learning are related to real-life.

Due to the limitation of the method used in the study, finally, only four ideas about teaching were identified for examining the selected lessons. They are: (1) taking students' over-all development into consideration; (2) connecting mathematics to real-life; (3) providing opportunity for student to inquire and collaborate; and (4) exploiting resources for teaching rather than just following the textbook. The first idea was examined by analyzing the objectives listed in the lesson plans. The second and the third were examined by analyzing the lesson videos. And the fourth was examined by contrasting the actual taught content and the content in the textbook.

3.3.2 *"Down-Up" Strategy*

For the second research question, we adopted an open method for analysing rather than determined any analytical framework. The "constant comparison method" (Glaser & Strauss, 1967) was used for analyzing the selected lessons. We watched the lesson videos and read the transcripts of the lessons several times until some themes came to our attention. Then these themes were further examined until they were found to represent the common features of all the 13 selected lessons. In other words, common features were gradually summarized. Finally, six common features were found in the lessons. More details will be reported in the next section.

All the selected lessons were analyzed by two researchers. The results were tested and discussed to obtain consistency between the researchers

4. Results: Features of Exemplary Lessons

4.1 *Features Consistent with the Ideas Advocated by Curriculum Reform*

4.1.1 *Students' Overall Development Was Emphasized*

By analyzing the instructional objectives in the lesson plans, we found that students' overall development was paid attention to in all the lessons. Both objectives about the results of learning and the process of learning were all shown in the lesson plans. Students' mathematical development and the non-mathematical-relevant development all could be found in the lesson plans. The objectives of two lessons were shown as follows:

> Help students (1) estimate the range of the product of 2-digit by 3 digit multiplication in a specific context, and calculate the 2-digit by 3 digit multiplication by listing vertical formula; (2) explore the methods of 2-digit by 3 digit multiplication, compute correctly, and be willing to exchange the methods with others; (3) develop the interests in calculating and a good habit, and improve the ability to use multiplication to solve practical problems; and stimulate students enthusiasm to love science by introducing current events. (Extracted from the plan of lesson I "2-digit by 3 digit multiplication")

> Help students (1) construct the preliminary concept of fractions based on their exploring and discussing the things in their real-life and geometric figures, correctly read and write simple fractions, and explain the meaning of a fraction by using geometric figures; (2) compare two fractions whose numerators are 1 by using geometric figures; (3) develop students' awareness of collaboration with others, and their ability of observation and analysis, hands-on skills and language skills, and develop students' mathematics thinking. (Extracted from the plan of lesson B "Knowing and understanding fractions")

The traditional mathematics teaching has been criticized for its over-emphasis on the results of learning (mathematical knowledge and skills) and neglecting the learning process. According to the objectives listed in the lesson plans, we found the learning results as long as the learning process was taken into consideration by teachers. Furthermore, some non-mathematics-relevant skills, such as the awareness of cooperation, communication, and interests, also were covered in the instruction objectives. It should be noted that the teaching plans do not necessarily become the reality of classrooms. However, the broader scope of the instructional objectives outlined in the lesson plans did indicate that students' over-all development was considered by teachers.

4.1.2 *Mathematics Was Connected with Real-Life*

By analyzing the lesson videos, it was found that all the lessons contained the real-life contexts during which mathematics was taught and learned. Three strategies were found in these lessons to create such a context. One is to begin a lesson with a real-life event or problem. All the lessons used this kind of strategy. The contexts created in these lessons were summarized as shown in Table 2.

The second strategy is to use real-life tasks or problems during teaching and learning the new content. The third strategy is to provide opportunity for students to apply the learned new content to the real-life.

Table 2. The context of teaching and learning at the beginning of each lesson

Lesson	The context
A	Watched video: Opening ceremony of Olympic Games. Felt the scene of countdown. Led to the time unit “Second”. Then students gave examples that they used “Second” in daily lives.
B	Students allotted several types of learning tools equally with their deskmates, and recorded the numbers of each type of learning tool that each student received. They finally found that a half could not expressed by any whole number. So 1/2 was introduced.
C	Students played the game of splicing flowers with 12 petals. Two results emerged: one is all the petals were used; the other is one or several petals was/were left. These lead to the "divisible division" and the “division with divisor”.
D	Watched video: Opening ceremony of Olympic Games. Felt the scene at the time, recalled the date of the Olympic Games, lead to the topic of “Year, Month, and Day”.
E	Watched the pictures of history events and holidays, students answered the dates of these events and holidays, and then the topic of “Year, Month, and Day” was introduced.
F	Students interchanged the memories about the Olympic Games, introduced the topic of “Year, Month, and Day” from the date of Olympic Games
H	Teacher played cards with the class. The teacher always won the game by using countermeasure. Students felt curious. Then the topic of “countermeasure” was introduced.
I	Students watched a simulative animation in which a satellite was running around the Earth. After having known the circumference of the orbit, students were asked to raise mathematical problems from this event.
J	Students sorted 13 pieces of playing cards (2 to 10, and J to A). Students looked J, Q, K, and A as the number of 11, 12, 13, and 14. Then the topic “Using letters to present numbers” was introduced.

K	Students made up a big rectangle with 12 small squares, and then expressed the length and width of the rectangle with a multiplication formula. They found, with the different splicing method, the multiplication formula was different. This led to multiples and factors.
L	Students observed their finger spacing, gave examples of spacing in daily life, and then raised the problem of planting trees.
M	Appreciated the picture of the urban landscape, led to the topic of urban greening, and then raised the problem of planting trees.
N	Began from a riddle: "Two trees have 10 branches, but they have not leaves and do not flower" (The answer is two hands). Students observed finger spacing, and then gave examples of spacing in daily life, which led to the problem of planting trees.

4.1.3 *Inquiry Learning and Collaborative Learning during Lessons*

It was found that inquiry learning and collaborative learning existed in all of the lessons. The students had the opportunities of exploring knowledge and methods by themselves and communicating or discussing their opinions or findings with desk-mates or group members. The inquiry learning and collaborative learning in these lessons were summarized as shown in Table 3.

Table 3. Overview of the inquiry learning and collaborative learning in selected lessons

Lesson	Summary of the inquiry learning or collaborative learning
A	Groups studied how to prove one minute was equal to 60 seconds.
B	Students communicated how they got the fractions that they wanted to learn by folding square papers. Students divided a square paper into 8 equal parts with different methods. Then they were asked to discuss in pairs whether two parts in different shapes were in the same size.
C	Groups played the game of splicing 5-petal flowers to investigate the relationship between the remainder and the dividend. Students discussed the problem of planning a schedule for cleaning.
D	Students observed the calendar independently, and then shared their findings with deskmates.
E	Students observed calendar in groups and collected the data about year, month, and day. Then the whole class compared and analysed the data to investigate the relationship between year, month, and day. Students discussed the methods of calculating the days of a common year. Students communicated the methods of calculating the days of a leap year.
F	Students observed the calendar independently. Then they found there were 12 months in a year, and the number of days varied in the 12 months.
H	Groups designed the program of horse racing with the method of countermeasure.

	Pairs played a game to apply the method of countermeasure.
I	Pairs communicated the methods of estimating. Groups compared two different methods of calculation.
J	Groups discussed how to present the relationship of two persons' ages that were increasing simultaneously.
K	Groups communicated the methods of enumerating the factors of a number.
L	Groups studied the different methods of planting trees and recorded the number of trees and corresponding spacing. Then students analysed the data to find the relationship between the numbers of trees and spacing.
M	Groups discussed why the number of planted trees was one more than the number of spacing.
N	Groups studied the relationship between the number of spacing and the number of trees. Groups discussed the relationship between the number of spacing and the number of trees in two different situations (planting trees from one end to another end of a line, and planting trees in a line without planting at the two ends.

4.1.4 *Teacher Adapted the Textbook and Used Other Resources for Teaching*

By contrasting the curriculum resources used in the lessons with the resources listed in the corresponding textbook, it was found that none of the 13 lessons completely conform to the textbook. In these lessons, the teachers selected some resources, such as the pictures, examples and exercises, from the textbook for their teaching and also exploited various resources by themselves. These results reveal that the teachers have made their adaption and creation while they designed and implemented their lessons. This is consistent with the ideas advocated by the new curriculum that the teachers should actively utilize various teaching resources and creatively use the textbook. However, a further analysis showed that, although the adaptations on the textbooks and the development of new resources were made in all of the lessons, the content of teaching and learning in the lessons do not show differences from the content in the textbooks regarding of the coverage on mathematical knowledge and skills. Therefore, from the ways in which teachers used textbooks, we can see the teachers in the selected lessons did not depend on the resources in the textbook, but intended to follow the mathematical objectives embodied in the textbook.

4.2 *Other Common Features*

In addition to the features reported above, some other common features were also found existing in the selected lessons.

4.2.1 *Features of Lesson Structure*

Seven types of teaching activities with different purposes for student's learning were found in the selected lessons. The purposes of each type of activities and their frequencies in the total 13 lessons are presented in Table 4.

Table 4. Seven types of teaching activities and their purposes

Type	Purpose	Frequency
Introducing topic	To arouse students' interests or to activate students' previous experience relevant to the topic of current lesson (including reviewing previous lesson), and then introduce the topic.	13
Teaching and learning new content	To acquire knowledge, concepts, skills, or procedures that have not been learned in earlier lessons.	63
Practicing the new content	To consolidate the new content or to apply it in a new situation, including solving routine exercise and non-routine problems.	56
Summary	To help students get an overall view on the previously learned new content or previous teaching activity in the current lesson	17
Homework assignment	To give students assignment for them to accomplish at home.	3
Extended learning on non-mathematical content	To have a relaxation or celebration, or to introduce a current social event. The content or activity is irrelevant to mathematics.	3
Proposing problems for future study	To invite students raise questions or problems for studying in future lessons.	1

The "Extended learning on non-mathematical content" only existed in three lessons. For example, in the lesson N "solving the problem of planting trees", students sung a "*Tree-planting Song*" to celebrate their accomplishment of previous learning. Taking another example, at the end of the lesson I (Multiplication: 3-digit by 2-digit), the teacher introduced a current affair of the manned space rocket. Only lesson L had the "Proposing problems for future study". Lesson D, E, and J had "Homework assignment". By contrast with these three types of activities, the other four types of activities were very popular in all of the 13 lessons.

Figure 1 shows a picture of the lesson structure, in which each type of activity was presented according to its location in the teaching process and the percentage of its duration to the whole lesson. As it is shown in this figure, in all of the lessons, it started with introducing the topic of current lesson, during which the students' interests were aroused and their previous experience was activated. After introducing the topic, the new content was taught and learned. The new content was divided into several parts and each part was taught and learned gradually. Practices were set following some parts (not all) of the new content or were set after all of the new contents were finished. One or more summaries were given during the lesson or at the end of the lesson. Overall, all the lessons showed three features as following: (1) introducing topic; (2) teaching and learning new content accompanied by practicing; (3) summarizing during the lesson or near to the end of the lesson.

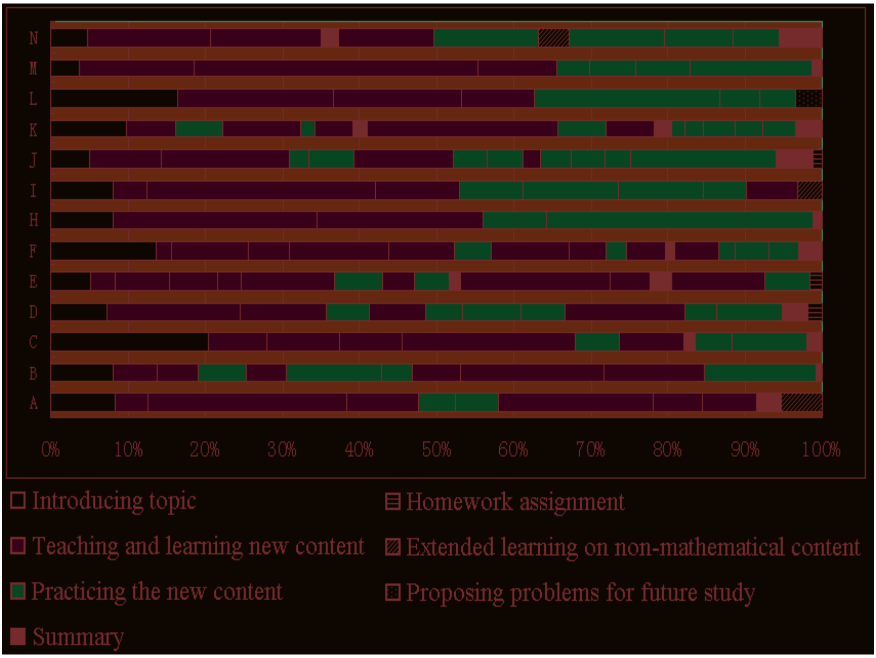

Figure 1. Lesson structure

4.2.2 *Teaching and Learning New Content Accompanied by Practicing*

Practices were found in all of the lessons. The type of these lessons is "*Xinshou Ke*", which means that teaching and learning the new content is the main purpose of these lessons. However, most of the lesson time was not only spent on teaching and learning new content but on both teaching and learning new content and practicing the new content. The percentages of time spent on teaching and learning new content and practicing in each lesson are shown in Figure 2. As far as the percentages of time spent on "practicing" is concerned, the highest one is lesson J (48.3%), and the lowest one is lesson A (10.5%). Nine lessons spent more than 30% of their lesson time on practicing. In some lessons (H, J, and N), the total time spent on practicing was nearly as much as the time spent on teaching and learning new content. Therefore, the selected lessons showed an obvious feature that teaching and learning new content was accompanied by practicing.

A further analysis found that two strategies were used for accompanying practicing with teaching and learning new content. One is to arrange the practice after all of the new content was taught and learned (e.g., lesson H, L, M, and N, see Figure 1). Another one is to place one or more practice between the sections of teaching and learning new content (e.g., lesson B, D, K, et al.; see Figure 1).

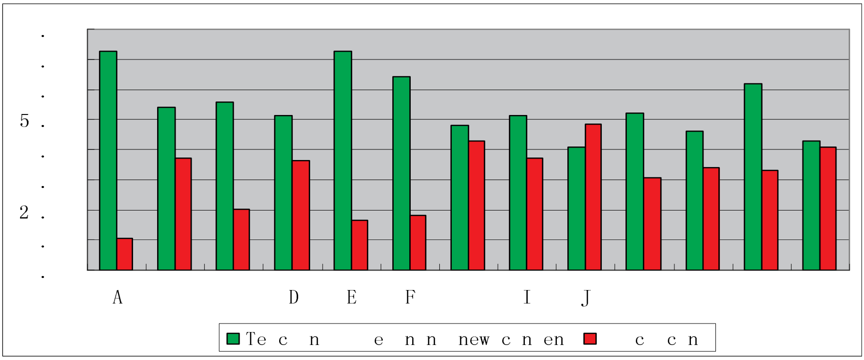

Figure 2. The percentages of time spent on teaching and learning new content and practicing

4.2.3 *Most Lessons Included Summary, and Some Were Made By Students*

Except lesson I and lesson L, the other 11 lessons all contained at least one section of summary. There were two types of summary: (1) it took place during the process of teaching and learning and intended to review the key points of the just learned content or the just accomplished activity; (2) it occurred near to the end of a lesson and intended to review the whole lessons. From Figure 1, we can conclude that lessons A, B, D, H, J, and M used the first type of summary, while lessons C, E, F, K, and N used both types of summary. A further analysis found that the teachers always invited students to give summary during the second type of summary. In this occasion, several students reflect what they had acquired in the lesson, including the knowledge and skills, as well as their experience and affection.

4.2.4 *Public Interaction Dominated the Lessons*

It was found that, although the students had opportunities to explore mathematics knowledge by themselves or to discuss and cooperate with their classmates, most of lesson time was spent on the whole-class work. By referring to the TIMSS 1999 Video Study (Hiebert et al., 2003), two categories of classroom work patterns were used in this study. One is public interaction, in which the teacher and students interact publicly, with the intent that all students give their attention to the presentation by the teacher or one or more students. Another category is private interaction, in which students complete assignments either individually, in pairs, or in small groups. An analysis of the different types of interaction showed that more than 70% of lesson time in the selected lessons was spent in public interaction. The percentage of private interaction in most of lessons was not more than 20%. It was under 10% in seven lessons. This indicates that all these lessons were dominated by public interaction (more details see Figure 3).

The private interaction in the 13 lessons included the students discussing or communicating in pairs or groups, working with tasks individually or collaboratively, doing exercise at seat, and reading textbook. The public interaction included presenting information by teachers (such as explaining, questioning, and blackboard writing) and by students (such as answering questions, reporting findings, demonstrating personal or group work). Most of the public interaction took place in the form of dialogue between teacher and students.

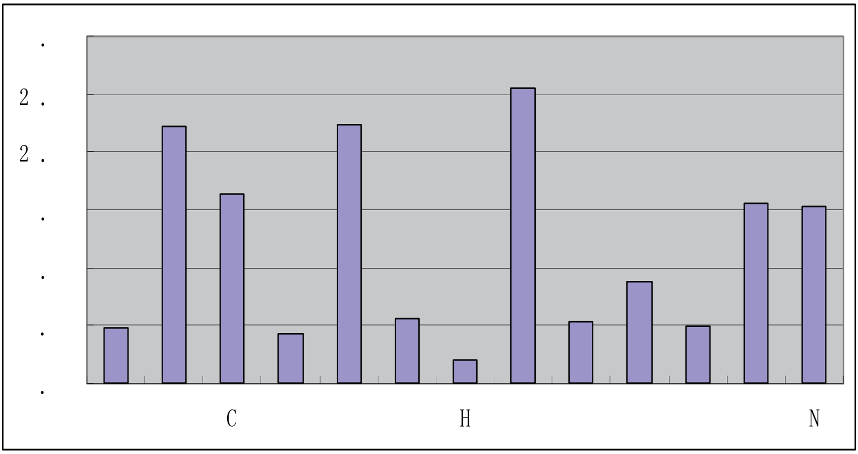

Figure 3. Percentage of time duration of private interaction

4.2.5 *Teacher Had More Opportunities to Talk, and Students' Talking in Chorus Was Evident*

It has found that most of the public interactions were in form of dialogue between teacher and students. The discourse of teacher's and students' was analyzed for further examining the dialogue between teacher and students. By watching lesson videos, we found there were four types of talking during the public interaction. They were: (1) the teacher talking individually; (2) single student talking; (3) students talking in chorus without teacher's participation, in which two or more students talked together; and (4) teacher and students talking in chorus, in which the talk made by two or more students accompanied their teacher. In this study, the teacher talking includes both (1) and (4), while the student talking refers to all of the (2), (3), and (4). The examples of each type of talking are shown in the episode bellow. In this chapter, the "talking in chorus" is defined as more than one person talking simultaneously, such as the talking numbered 4, 6 and 8 in the following episode.

No.	Discourse	Type of talking
1	Teacher: *What is twenty multiplied by four? You answer.*	(1)
2	One student: *Eighty.*	(2)
3	Teacher: *Is this answer right?*	(1)
4	Most of students: *Yes, right.*	(3)
5	Teacher: *Then we add seven to eighty. We get—*	(1)
6	Teacher and most of students: *Eighty-seven.*	(4)
7	Teacher: *It this question easy?*	(1)
8	All of students: *Easy.*	(3)

All of the discourses during the public interaction in the 13 lessons were transcribed verbatim. Then the frequency of teacher talking (FT), frequency of student talking (FS), ratio of FT to FS, number of teacher's words (TW), number of students' words (SW), ratio of TW to SW, frequency of students talking in chorus (FSC), and the ratio of FSC to FS were analysed quantitatively.

Table 5. The results of quantitative analysis in dialogue during public interaction

	A	B	C	D	E	F	H	I	J	K	L	M	N
FT	214	215	239	184	197	245	147	158	259	396	149	137	154
FS	223	164	217	155	181	210	149	135	205	325	139	118	142
FT:FS	1.0	1.3	1.1	1.2	1.1	1.2	1.0	1.2	1.3	1.2	1.1	1.2	1.1
TW	3336	3921	5007	3596	3915	5739	3525	3456	4030	5238	4080	3218	3985
SW	1753	2015	1629	1842	2130	1621	1451	2125	1834	2114	1587	1656	1433
TW:SW	1.9	1.9	3.1	2.0	1.8	3.5	2.4	1.6	2.2	2.5	2.6	1.9	2.8
FSC	79	86	93	64	119	109	45	76	119	180	56	42	75
FSC:FS	35%	52%	43%	41%	66%	52%	30%	56%	58%	55%	40%	36%	53%

As shown in Table 5, the ratios of FT to FS ranged from 1.0 to 1.3, indicating the FT is not much more than the FS. Moreover, the frequencies of student talking in the 13 lessons are all more than 100. This reveals that the students in these lessons were not the passive, quiet listeners.

Regarding the spoken words during the public interaction, it could be seen in Table 5 that the ratios of TW to SW in the 13 lessons are different to some extent, among which the highest one is 3.5:1 and the lowest one is 1.6:1. However, the feature that the teachers talked more than their students was shared by all of the lessons. The ratio of average TW to average SW is 2.3:1.

A further analysis in the amount of student talking found that the student talking in chorus (including the chorus with teacher's participation) was frequent. As shown in Table 5, at least 30% of student talking was in chorus in each of the selected lessons. The highest percentage of FSC to ST was found in lesson E, which reached 66%.

4.2.6 *Questioning-responding Occurred Frequently, but Students Rarely Asked Questions*

We found that many dialogues between the teacher and students appeared in the way of teacher's asking questions and the students' responding. The frequency of mathematical questioning (not including the questioning for lesson management) during each lesson is all more than 40. The questioning and responding took place not only during teaching and learning new content, but also occurred in other sections of a lesson.

However, nearly all mathematical questions were raised by the teachers. Students' asking question on their own initiative (not including the questioning motivated by teacher's invitation) was found only in three lessons (once in lesson B, once in lesson J, 5 times in lesson A). No student presented any questions in any of the other ten lessons.

5. Conclusion and Discussion

5.1 *Conclusion*

By analyzing 13 elementary lessons, we found some features of the exemplary lessons under the curriculum reform in China. On the one hand, some of the features were consistent with the ideas advocated by the new curriculum, such as emphasizing on student's overall development, connecting mathematics to real-life, providing students opportunities of inquiring and collaborating, and exploiting various resources for teaching. These features have demonstrated an attempt to reform the practice of mathematics classroom in China. If the current curriculum reform could be implemented efficiently and continuously, we may expect that China's mathematics classroom will show many differences from that of the classroom in past decades.

On the other hand, the selected lessons in this study also shared many other common features. All of the exemplary lessons in this study began with introducing topic, during which the teacher aroused the students' interests or activated students' previous experience. After introducing topic, new contents were taught and learned gradually. Some of the new contents were accompanied by practice. Most of lessons (9 of 13) spent more than 30% of their lesson time on practicing the new content. In addition, in most lessons, at least one summary was set during or at the end of the lessons. It also found that all the selected lessons were dominated by public interaction and most of the public interaction took place in the form of dialogue between teacher and students. By contrast, the percentage of private interaction in most lessons was no more than 20%. In all the lessons, teachers talked more than their students. Student talking in chorus was very frequent. Many dialogues between the teacher and students appeared in the way of teacher's asking questions and students' responding. However, nearly all mathematical questions were raised by the teachers. Students rarely asked questions on their own initiative.

5.2 *Discussion*

The lessons in this study were all selected from a national contest in exemplary lessons in the new curriculum reform context, so it is not surprisingly that these lessons have practiced the advocated ideas of the current reform. Relatively speaking, other common features found in these lessons are more interesting and worth discussing. Some of these common features are consistent with the findings of other studies on Chinese mathematics classroom.

Regarding the lesson structure, all the selected lessons began with introducing topic, accompanied teaching and learning new content by practicing, and summarized during the lesson or near to the end of the lesson. This is consistent with the findings of Zhao and Ma (2012)'s comparative analysis of four exemplary lessons in different decades in China. Chen and Li (2010)'s study on a Chinese competent teacher's four consecutive lessons also found that the teacher tended to structure the lesson into reviewing previous lesson, teaching and learning new content, and summary, which resulted in an instructional coherence. Other studies reveal the instructional coherence seems be a characteristics of Asian mathematics classroom. For example, Shimizu (2007) found the summary also played an important role in Japanese mathematics classroom.

The selected lessons in this study also showed a feature that teaching and learning new content was accompanied by practicing. This feature also was found in Zhao and Ma (2012)'s study. It is well known that the Chinese mathematics classroom in the last half of 20th century, had been predominated by the belief that "students should have sufficient exercises in order to consolidate the learned knowledge" (Zhang, Li, & Tang, 2004) and that "practice makes perfect" (Li, 2006). We could not conclude whether the practice in the lessons in this study was emphasized as much as it was in traditional classrooms, but it is evident that none of the lessons in this study neglected the role of practice.

It was also found that all the lessons in this study were dominated by public interaction, and teachers talked more than their students. These two features are consistent with the findings of TIMSS 1999 Video Study (Hiebert et al., 2003) on eighth-grade mathematics lessons in seven countries. However, it should be noted that the ratio of teacher's words to students' words in this study is 2.3:1, which is much less than the ratio found in TIMSS 1999 Video Study[2]. Based on the data of Learner's Perspective Study (LPS) (Clarke, Emanuelsson, Jablonka, & Mok, 2006), Cao, Wang and Wang (2008) analysed the discourse in Chinese competent mathematics teachers' lessons. It was found the ratio of average teacher words to average students' words is 6.6:1. Taking these findings as a whole, we may hypothesize that the differences might exist in the discourse between classrooms in different stage of schooling.

In addition, the phenomenon of students' talking in chorus found in this study is similar to the findings of Wang (2010)'s study on two elementary mathematics classrooms in China. And the feature of frequent questioning-responding in the lessons in this study also was found in other studies on Chinese exemplary mathematics lessons in different decades (Huang, Pang, & Li, 2009; Zhao & Ma, 2012).

By reviewing the existing studies and comparing their findings with the features found in this study, we may find the lessons in this study embodied some elements that might be the stable characteristics of Chinese mathematics education. Stigler and Hiebert (1999, p. 86) pointed out that the teaching is a cultural activity. As a cultural activity, teaching has its relative stability. Therefore, it is understandable that both differences and similarities exist in mathematics lessons of different periods of time.

In this chapter, we have reported the features of 13 exemplary lessons under the curriculum reform in China. We hope our findings could help you understand the current changes in elementary mathematics classroom. In addition, as noted above, the classroom under the reform not only reveals the new ideas advocated by the reform, it also contains some stable elements that might be inherited from the traditional classroom. This reminds us that the classroom under the reform and traditional classroom are not completely conflicting and exclusive. We should not ignore reflecting upon the tradition while implementing the new ideas. The traditional mathematics classroom may contain the asset that is worth preserving and carrying forward, and may also hide the drawbacks to be discovered. From this point, the teaching reform is a successive and gradual changing process, during which the reflection on present and history is always needed. This is the implication drawn from a case study. Perhaps it also could be a reference for mathematics educators in a reform.

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Note:

[1] Compulsory education at the present time in China includes 9-year schooling from elementary school to junior high school.

[2] In TIMSS 1999 Video Study, the average number of teacher words to every on student word per eighth-grade mathematics lesson in six countries/regions was reported. They were as following: Australia, 9:1; Czech Republic, 9:1; Hong Kong SAR, 16:1; Japan, 13:1; Netherlands, 10:1; and United States, 8:1. (Hiebert et al., 2003, pp. 109-110).

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Chapter 14

Qingpu Mathematics Teaching Reform and Its Impact on Student Learning

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This chapter highlights findings from the recent progress of the latest version of the Qingpu Experiment (New Actions of the Qingpu Experiment in the 21st Century) which investigates the impacts of Qingpu mathematics teaching reform on eighth-grade students' cognitive level and learning ability using factor analysis. We conducted different levels of tests for all grade eight students in the Qingpu school district of Shanghai, China. Our study revealed the limitation of Bloom's taxonomy to describe cognitive levels; we propose a four-level taxonomy based on our empirical results. Furthermore, comparison of the cognitive levels of students based on data from 1990 and 2007 indicated a gradual increase in the "analysis" level." Other details such as differences between male and female, and between urban and rural areas, are also discussed.

Keywords: mathematics education objectives, Bloom's taxonomy, Qingpu Experiment, cognitive levels, comparative study

1. Introduction

The Qingpu Experiment in Qingpu school district of Shanghai, China is a landmark in Chinese mathematics teaching reform; starting in the late 1970s, the Qingpu Experiment tested ideas about how to improve mathematics teaching quality locally and then to share the ideas globally (Gu, 1996). The successful experiences of Qingpu mathematics teaching reform were first promoted in Shanghai and then eventually to the whole

China (mainland) from 1986 to 1992. In the last 12 years, the basic education quality of the Qingpu school district continuously developed to reach the top level of the nineteen districts in Shanghai. In 2001, the Qingpu Experiment Research Institute of Shanghai cooperated with Teachers Development Research Center of Shanghai Academy of Educational Sciences in starting a new project called the New Actions of the Qingpu Experiment in the 21st Century. In 2002, as a result of the first stage of this project, an innovative model of promoting teacher professional development was widely used in school-based teaching research activities in mainland China. Same as with other parts of China, curriculum reform has been running in Shanghai for more than 10 years. The second stage of this project focused on the changes of classroom teaching reform and the behavior of teachers and students at the end of 2006. Later, in order to know whether there was any change in students' learning ability, a third stage of the project started. A large-scale research on objectives of students' learning was finished in 2007, which was compared with the performance of the 8th grade students in 1990. This chapter discusses an empirical study on the mathematics educational objectives of this project, and some important conclusions of the third stage.

In the modern theoretical research of education, the role of educational objectives becomes more and more important. And mathematics educational objectives become the basis of all instructional design. The attempts at establishing a taxonomy of educational objectives were proposed at the Annual Conference of the American Psychological Association in Boston in 1948. Bloom and his colleagues published a taxonomy of the cognitive domain and affective domain in 1956 and 1964, respectively. Later, other researchers published a taxonomy of psychomotor domain. The 1956 version—the Taxonomy of Educational Objectives, or colloquially known as Bloom's Taxonomy — defined six levels of cognitive domain: knowledge, comprehension, application, analysis, synthesis and evaluation, each with several subcategories (Bloom et al., 1956). In 1989, Wilson proposed four categories of cognitive objectives in mathematics education, including calculation, comprehension, application and analysis. Wilson's calculation level incorporated Bloom's knowledge level, and Wilson's analysis level incorporated Bloom's analysis, synthesis and evaluation levels.

For decades, several researchers pointed out that there were some limitations in Bloom's Taxonomy of cognitive objectives (Wang, 2000; Han, 2001; Pi & Cai, 2006). In 2001, Anderson and other experts revised Bloom's taxonomy (Anderson et al., 2001). The new taxonomy has two dimensions: knowledge and cognition processes. The knowledge dimension includes four categories: factual knowledge, conceptual knowledge, procedural knowledge, and metacognitive knowledge, each with several subcategories. The cognitive process dimension includes (according to their complexity) six categories: remember, understand, apply, analyze, evaluate, and create, each with their subcategories.

Much related taxonomy research following Bloom's framework has been carried out in China. In the Qingpu Experiment, two large-scale studies attempting to measurement the impact on students learning were carried out in 1990 and 2007. In this chapter, we introduce the results of the large-scale measurement in 2007 of the Qingpu Experiment and explore the students' performance of cognitive level in mathematical learning based on the revised Bloom taxonomy theory (Anderson et al., 2001).

2. Background of Qingpu Experiment

In 1990, we began the data analysis of the Qingpu Experiment. We spent one year in formulating the assessment and assessment items, and the next year testing 3,000 students in Grade 8. The research was conducted utilizing the framework of Bloom's taxonomy. Considering the fact that in mathematics education knowledge and calculation are treated as two domains in China, Wilson's notion of "calculation including knowledge" was not adopted. Wilson's calculation was added in the Qingpu Experiment test (Wilson, 1989). Therefore, there were seven tests in the overall experiment, namely, knowledge, calculation, comprehension, application, analysis, synthesis and evaluation. There were fifty assessment items total, a hundred and six assessment points, 56% of which were algebra and 44% were geometry (Qingpu Experimental Group of Mathematics Teaching Reform, 1991).

Factor analysis was used as the statistical method to deal with the multivariable data. From this method we summarized and concluded a few common unobservable variables (recessive variables or common factors), which revealed the inner link and key role of dominance variables by observing a great number of original variables. After the test and data analysis, we obtained the common factors load matrix (see Table 1).The result of the factor analysis showed that instructional objectives of the cognitive field were determined by the even more basic latent factors inside, which (listed sequentially) are: F1—mainly as memorization, F2—mainly as comprehension, F3—mainly as judgment and criticism. The proportion of variance of these three factors was 56.17%, 3.49% and 1.42% (together totaling 61.08% of the total variance).

Table 1. Load matrix of common factors in the 1990 tests

Tests	F1	F2	F3	h^2
1	0.788	0.204	0.154	0.686
2	0.792	0.166	0.178	0.687
3	0.671	0.453	0.048	0.658
4	0.641	0.474	0.104	0.647
5	0.472	0.533	0.161	0.533
6	0.487	0.558	0.065	0.553
7	0.428	0.410	0.400	0.512

Note: Tests 1, 2, 3, 4, 5, 6, 7 measure in turn: *knowledge*, *calculation*, *comprehension*, *application*, *analysis*, *synthesis* and *evaluation*.

Drawing the six assessment variables (*knowledge*, *calculation*, *comprehension*, *application*, *analysis* and *synthesis*) in a two-dimensional space defined by F1 and F2 (Figure 1), suggests that *synthesis* and *analysis* belong to the same level of cognition, despite the difference between these two tests. Similarly, *application* and *comprehension* can be combined. *Calculation* and *knowledge* (because memorization of the textbook content is emphasized) can also be put together as a similar objective. Thus, we believe that these original six mathematics educational objectives could be summed up with only three levels: *memorization* (originally called *knowledge* and *calculation*), *explaining comprehension* (originally called *application* and *comprehension*), and *exploring comprehension* (originally called *synthesis* and *analysis*). The results indicate a positive experience of the Qingpu mathematics teaching reform on students' learning and a refinement of the relevant mathematics learning theory.

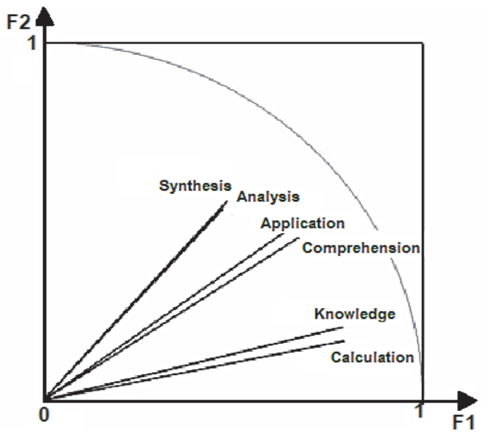

Figure 1. Six assessment variables in the two-factor vector plane

3. Research Method

In the beginning of 2007, we decided to conduct a large sample study of the mathematics educational objectives with the eighth grade students for a second time in the Qingpu district. In consideration of the comparability with the study in 1990, we did some revision of the research framework. *Knowledge* in the 1990 study was changed to *knowing-about*, which was listed after *calculation*. And then *comprehension* and *application* were kept in their sequence since there was no controversy. Actually, we distinguished Wilson's calculation from Bloom's knowledge in 1990 study. The former meant just by memorizing the examples in textbooks or procedural step of exercises, students could give the answer to a question which was at the operational level, and there was difference from knowledge in cognitive level. For the higher cognitive levels (*analysis*, *synthesis*, and *evaluation* in Bloom's 1956 version, though they were changed to *analysis*, *evaluation*, and *creation* in later versions), we still concluded them in one level: *analysis*. The result in 1990 study showed that *analysis* and *synthesis* belonged to same cognitive level, which was consistent with the research of Wilson (1989). Our study in 2007 emphasized non-routine problem solving, and in the process of problem solving, there was no separation between *analysis* and sy*n*thesis, and the *evaluation* of the solving process was included.

Based on the taxonomy of objectives shown in Table 2, we designed the research questions and problems.

Table 2. Mathematics educational objectives and explanations

Objective	Explanation
A. Calculation	Simple exercises including memorization, requiring students to do basic calculation according to the requirements from the textbooks and assessing their routine operation of basic elements on the items, including basic construction of geometric figures.
B. Knowing-about	Students can recall or recite definitions, concepts, propositions, rules, formulas and facts about mathematics, or can present facts, terms, and basic concepts of mathematics. B1 Knowledge of facts and terms (ability to memorize or present) B2 Knowing about basic concepts (ability to recall or present)
C. Compre-hension	Referring to the understanding of concepts, principles, rules and the meanings of mathematical structures beyond mere recitation; the ability to convert between different forms, to understand the logical reasoning, and to understand the meanings of assessment items. C1 Understanding of concepts, principles, rules and the meanings of mathematical structures (ability to understand rather than recite meanings) C2 Conversion of problems (ability to convert problems from one form to another) C3 Continuation of reasoning (ability to understand and continue the reasoning process) C4 Read and interpret problems (ability to understand and interpret problems)

Table 2. (*Continued*)

D. Application	Referring to solving of routine problems following the examples in the textbooks, including comparing the difference between problems' conditions and types. D1 Solving routine problems following examples (ability to follow examples) D2 Identifying types (ability to solve a problem by matching it to the type practiced) D3 Making comparison (ability to solve a problem by comparing it with the type practiced)
E. Analysis	Referring to overall analysis of non-routine problems which the students have no prior experience with the solving process/methods of the problem. E1 Finding the mathematical factors and relationships (ability to analyze the relationship between conditions and conclusions and the major steps of problem-solving) E2 Synthesizing the process of analysis (ability to synthesize a complete solution to the problem) E3 Finding and proving general rules (ability to come to and prove general rules) E4 Judging according to logical reasoning (ability to find the truth), judging according to criteria (ability to decide better and simpler solutions to problems), evaluating (ability to reflect on the contents and solving process of the problem)

Given that this study involved all the eighth grade students in the Qingpu district, in which some schools were using new mathematical materials and others were not, assessment items of the study covered only the common parts of these two different materials. Therefore, items in this test mainly contained algebra and geometry. This would facilitate comparison with the 1990 study. The distributions of assessment items within the categories and subcategories of our 2007 objectives are shown in Table 3.

Three principles were followed while we were setting and revising the assessment items. First, the assessment items would be as consistent with the objectives as much possible. In fact, successful solving of any mathematics problem was the combination of several abilities and thus each item should reflect mainly the primary objective that played the key role in the solution process. Second, all the assessment items on the same test should be of identical quality, i.e. they should be at the same level of difficulty. Third, the grading criteria for a certain item was also set according to the degree to which it reflected the objective that set the item itself. Traditionally, the grading was in accordance with the difficulty of the item, and sometimes certain items were weighted more than others for the sake of getting higher passing percentages. Though the objectives of the assessment were quite specific, the assessment items were set in the forms where test-takers were familiar with multiple choice, fill-in-the-blank, simple answer, constructing graphs, etc. to avoid interference with the test results. Examples of assessment items of the five tests are as follows.

Table 3. Distribution of assessment items in objectives and contents

		A Calculation	B Knowing-about		C Understanding				D Application			E Analysis			
			B1	B2	C1	C2	C3	C4	D1	D2	D3	E1	E2	E3	E4
ALGEBRA	Real number	2	3	2	2					1	1	1			
	Algebraic Expression	5	2	1					1		1				
	Factoriza-tion	1						1							
	Radical Expression	1													
	Inequalities	3		1			1								
	Equation	4			2										1
	Function	3	1	1				1	1		1				
GEOMETRY	Symmetry			2					1	1		1			
	Lines	1	2	1											
	Triangles			2		1	1			1				1	1
	Circles												1		
	Coordinates		1		1										

Test One consisted of twenty items. A sample problem was: "Calculate $(3-\sqrt{5})^2 =$ _____"; the primary objective would be *calculation*, i.e., the item is a routine problem requiring the test-taker to calculate.

Test Two also consisted of twenty items. A sample problem was: "The relationship between the absolute value of x and absolute value of $-x$ is ____" with multiple choice options: (a) they are equal, (b) they are opposite numbers, (d) the former is bigger than the latter, and (d) the former is smaller than the latter. The objective was "*Knowing-about*: B2 Knowing about the basic concepts," that is, to evaluate the test-taker's ability to recall the concept of absolute value.

Test Three consisted of ten items. A sample problem was: "Of the following statements, the correct one is ____", with options: (a) The square root of a positive number is also a positive number, (b) Only positive numbers have a square root, (c) The cube root of a negative number is also a negative number, and (d) Only negative numbers have a cube root. The objective was "*Comprehension*: C1 Understanding of concepts, principles, rules and the meanings of mathematical structures." The question requires test-takerss to understand the relationship between different concepts, which is more complex than knowing *about* the concepts.

Test Four consisted of nine items. A sample problem was: In $Rt\Delta ABC$, if $\angle C = 90°$, $\angle A = 20°$, D is the mid-point of AB, then the measure of $\angle BCD$ is _____. The objective was "Application: D2 Identifying types," that is, the test-takers should be able to connect the problem with what they have practiced (which in the given example means to have dealt with the central line in a right-angle triangle and to correctly solve the problem according to the characteristics of the central line on the hypotenuse).

Test Five, different from Tests Three and Four, consisted of six non-routine problems which the test-takers had never encountered before. A sample problem was: "Ferry I and Ferry II started from Island A for Island B at noon. The speed of Ferry I was 10 km per hour and that of Ferry II was 8 km per hour. The two ferries traveled directly from Island A to Island B and returned to Island A at the same time, noon the next day. Find the distance between Island A and Island B." The objective was "Analysis: E1 Finding the mathematical factors/relationships," that is, the key to the problems lies in the realization of the relationship between the distance Ferry I traveled more than Ferry II and the distance between Island A and Island B, which is of a much higher level than the questions on the previous four tests.

On 13 April 2007, 4,349 students in eighth grade from the Qingpu district were paper-tested. Tests of *calculation*, *knowing-about*, *comprehension* and *application* lasted for thirty minutes each and were held in two sessions in the morning with a twenty-minute break in between. Tests of *analysis* lasted for one hundred minutes each and were held in the afternoon in one session without any break.

We used factor analysis to analyze the data; factor analysis is a statistical method to deal with multivariable data. From this method we could summarize a few common unobservable variables (recessive variables or common factors), which would reveal the inner link between variables and the key role of dominant variables by factor analyzing the greater number of original variables. The purpose of factor analysis was to determine hidden factors, which were the keys of students' performance on mathematics. Further, a developed framework of these hidden factors could be used to evaluate students' learning and analyze educational progress.

In order to compare the results of 1990 with those of 2007, the seven cognitive levels at the early stage in 1990 had been merged into five levels by 2007. One third of the original assessment items remained, while the level of difficulty had been increased for the remaining two-thirds of the items. But by linking items, we could statically adjust to compare scores.

4. Main Results and Discussion

4.1 *Outcome of Common Factors Analysis*

After testing and data analysis, we obtained the matrix of the correlated coefficient of the testing results for the five levels (see Table 4). With factor analysis we obtained the common factors and the load matrix of the common factors (see Table 5). Similar to our summarization and inference of the 1990's factor analysis experiment, F1 signified the factor of *memorization* (with 75.78% of the total variance), and F2 signified the factor of *comprehension* (with 9.37% of the total variance). Their combined total was 85.14% of the total variance. The common factors became two, while the ratio of the load of common factors raised to 24%.

Table 4. The matrix of the correlated coefficient of the testing results of the five tests

Tests	1	2	3	4	5
1	1				
2	0.7555	1			
3	0.6722	0.7390	1		
4	0.6892	0.7409	0.7390	1	
5	0.5597	0.6515	0.7001	0.7165	1

Note: Tests 1, 2, 3, 4, 5 measure in turn: *calculation, knowing-about, comprehension, application* and *analysis*.

Table 5. Load matrix of common factors in the 2007 tests

Tests	F1	F2	h^2
1	0.9099	0.2753	0.9037
2	0.7795	0.4826	0.8405
3	0.5738	0.6809	0.7929
4	0.5800	0.6866	0.8078
5	0.2661	0.9173	0.9123

Note: Tests 1, 2, 3, 4, 5 measure in turn: *calculation, knowing-about, comprehension, application* and *analysis*.

In the two-dimensional system of coordinates (*memorization-comprehension*), the load vector of the five assessment variables becomes spread out as illustrated in Figure 2.

The results of factor analysis revealed the limitation of Bloom's Taxonomy, especially the problems of its continuity and equidistance.

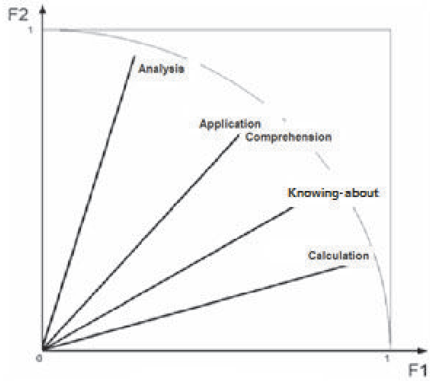

Figure 2. Five assessment variables in two-factor vector plane

The objectives of *comprehension* and *application* could be merged together for their similar cognitive level, despite their apparent superficial characteristics.

The objective of mathematical knowledge mentioned in Bloom's Taxonomy, which was defined as "knowing about the concept", manifest a disparity between "knowing about the concept" and "knowing how tocalculate" at the cognitive level, because the later was operational memorization-oriented, while the former had the factors of *comprehension*.

These two points mentioned above revealed the connections and disparities between the cognitive levels and the superficial characteristics of mathematics educational objectives. So these two common factors could be explained as F1: *memorization* and F2: *comprehension*. The taxonomy of educational objectives could be divided into a four-level framework with equal distance in the memorization-comprehension two-dimensional plane (see Table 6).

Table 6. Four-level framework of mathematics educational objectives

Relatively Low Cognitive Level	Relatively High Cognitive Level
1. Calculation: operational memorization	3. Comprehension: explaining comprehension
2. Knowing-about: conceptual memorization	4. Analysis: exploring comprehension

4.2 *Significant Changes to Students' Learning from 1990 to 2007*

The original assessment results of the 1990 and 2007 tests are displayed in Table 7. Although the overall difficulty of the tests had increased from 1990 to 2007, student performance on each of the assessment items and their total overall score had also increased, suggesting educational reform progress over the past two decades. For further comparison, in the framework of the four levels mentioned above, we combined *comprehension* and *application* together, *analysis*, *synthesis* and *evaluation* together, and then converted these combined objectives into a percentile system (Figure 3), on which the estimated score of 2007 was adjusted according to the difficulty of the assessment items of for *evaluation*.

Table 7. The comparison of the original results in the 1990 and 2007 assessments

	N	Cal'n	Knowledge: Knowing about	Compreh'n.	Appl'n.	Analysis	Synthesis	Eval'n	Scoring Rate(%)
1990	3000	67.19	63.96	47.11	41.33	23.84	44.28	29.17	45.27
2007	4349	84.07	75.28	54.82	51.00		28.96		58.83

Note: The total possible is 100 points. The numbers shown indicate average points. The scoring rates of 1990 and 2007 reflect the total for seven or five assessment levels, respectively.

The profound changes of the mathematics teaching philosophy had rolled out among the teachers in the Qingpu district. Although the tests' difficulties increased, we can clearly see that the classroom teaching efficiency had been improving greatly from a total mean score of 45.27% in 1990 in 58.83% in 2007.

From the framework using only four levels of mathematics educational objectives, the memorization-oriented level as *calculation* and *knowing-about* had been improving greatly since 1990. The objective of the *explaining comprehension* level (*comprehension* and *application*) had been reached. But the *exploring comprehension* level (especially the ability of *analysis*) stayed the same as 17 years previous.

This phenomenon was probably due to the excessive pressure students felt for the senior high school entrance examinations. Because of this, we recommend that how to enhance students' *exploring comprehension* level should become a major focus of mathematics teaching reform for the future.

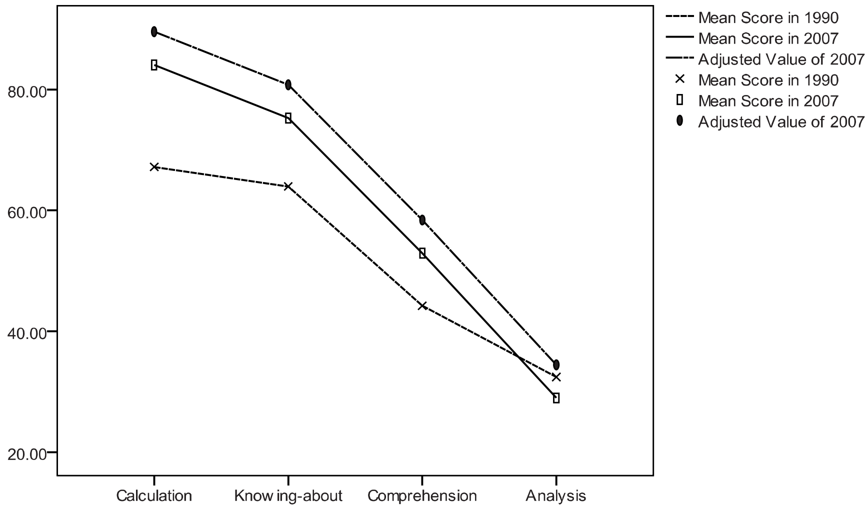

Figure 3. The comparison of the mathematics educational objectives in the 1990 and 2007 assessments.

4.3 *Difference Comparisons in terms of Area, School and Gender*

In order to compare the location changes (such as area differences or school differences) from 1990 to 2007, two different approaches were used for data analysis. The first way was to compute the original scores under the framework of the four levels. The other way was to divide students into three groups according to their aptitude inclination by using factor analysis.

Before 1990, most schools in the Qingpu district were terrible in mathematics education except a few schools that were participating in educational reform. Therefore, a big gap existed between the experimental classes and normal classes (see Table 8). By 2007, all of the schools in Qingpu had taken part in mathematics teaching reform. In order to compare with 1990, the schools were divided into two parts. Some schools were called experimental schools, which had joined the educational reform in 1990. The others were called normal schools. A comparison of these schools, based on their length of time using the reform, is given in Table 9. Obviously, the difference between schools became smaller.

Table 8. The comparison between experimental schools and normal schools in 1990

	Calculation	Knowing-about	Comprehension	Analysis
Experimental Schools (249 students)	77.36	69.14	56.29	45.56
Normal Schools (2751 students)	66.27	63.49	43.13	31.24
Mean Score Difference	11.09	5.65	13.16	14.32

Note: The total score of each assessment is 100 points and the data in the table is the mean score.

Table 9. The comparison between experimental schools and normal schools in 2007

	Calculation	Knowing-about	Comprehension	Analysis
Experimental Schools (1645 students)	84.61	76.38	54.93	30.79
Normal Schools (2704 students)	83.72	74.53	51.64	27.85
Mean Score Difference	0.89	1.86	3.30	2.94

Note: The total score of each assessment is 100 points and the data in the table is the mean score.

The comparison of the students' aptitude inclination is shown in Table 10. The distributions of students' aptitude inclination were quite different between experimental schools and normal schools, both in 1990—χ^2 (*2, N=3000*) *=80.550, p<.001*—and in 2007—χ^2 (*2, N=4349*) *=47.716, p<.001*. According to the data in Table 10, the changes in the normal schools on the structural ratio of the students' inclination types was not significant—χ^2(*2, N=5455*)*=1.793, p=.408*—during the intervening ten years. Then the tendency of the students' aptitude inclination in the experimental schools showed the shift from *exploration* to *memorization*—χ^2 (*2, N=1894*) *=34.997, p<.001*, which meant a higher proportion of students belonged to *memorization* category while a lower proportion of students belonged to the *exploration* category.

Table 10. Percent of students' aptitude inclination (%)

	1990			2007		
	Memorizat'n	Compreh'n	Explorat 'n	Memorizat'n	Compreh'n	Explorat'n
Experimental Schools	12.05	47.39	40.65	26.81	47.05	26.14
Normal Schools	32.13	48.75	19.12	32.69	49.59	17.71

Note: The total score of each assessment is 100 points and the data in the table is the mean score.

We also investigated the difference of students' performance between males and females. In the 1990 assessment, the mean scores of all four levels of the males were higher than the females except for the level of *calculation* (see Table 11). The situation changed in the 2007 assessment. The result showed that all the mean scores of the females were higher than the males (see Table 12).

The aptitude inclination analysis (see Table 13) showed that the males' proportion of *comprehension* and *exploration* were higher than the females', while the females' proportion of *memorization* was higher than the males' in the 1990 assessment—χ^2(*2, N=3000*)*=33.111, p<.001*). However, the females made significant progress over seventeen years. The distributions of the aptitude inclination between males and females were not significantly difference—$\chi^2(2, N=4349)=3.150, p=.207$—on the 2007 assessment.

Table 11. The comparison between males and females in 1990

	Calculation	Knowing-about	Comprehension	Analysis
Male students N=1416	67.04	64.75	46.69	33.71
Female students N=1584	67.33	63.25	42.02	31.33
Mean Score Difference	-0.29	1.50	4.67	2.38

Note: The total score of each assessment is 100 points and the data in the table is the mean score.

Table 12. The comparison between boys and girls in 2007

	Calculation	Knowing-about	Comprehension	Analysis
Male students N=2213	82.50	74.32	51.54	27.49
Female students N=2136	85.66	76.17	54.27	30.48
Mean Score Difference	-3.16	-1.85	-2.73	-2.99

Note: The total score of each assessment is 100 points and the data is the mean score.

Table 13. Percent of aptitude inclination of males and females (%)

	1990			2007		
	Memorizat'n	Compreh'n	Explorat'n	Memorizat'n	Compreh'n	Explorat'n
Male students	25.44	51.37	23.19	29.53	50.17	20.30
Female students	34.98	46.17	18.85	28.01	49.63	22.37

Note: The total possible score is 100 points for each assessment. The data is the mean score.

5. Summary

Two large-scale experiments were conducted and factor analysis was used to analyze data in 1990 and 2007 in the Qingpu school district. This research showed that there were problems on the continuity and the equidistance inherent in the framework of Bloom's Taxonomy. Based on this, we proposed and experimented with a four-level framework of mathematics educational objectives, i.e., *calculation* (operational memorization and knowing-about), *conceptual memorization*, *comprehension* (explaining comprehension and analysis), and *exploring comprehension*. The first two levels were categorized as lower-level cognition, and the latter two were classified as higher-level cognition.

Comparisons between the eighth grade students in 1990 and the same grade-level students in 2007 on their academic achievements were also made. The results of the level tests showed that despite the increase in assessment items' difficulty, the scores were much higher. Among them, the students' levels of *calculation* and *knowing-about* increased substantially. The students' level of *comprehension* reached the basic requirements. But the students' level of *analysis* (the ability of problem analyzing and problem solving), remained the same. So we recommended that mathematics teaching reform should focus on improving students' ability to analyze in the future.

An equitable and balanced education across locales was also an important topic in the Qingpu district because of its regional characteristic. Results of this research indicated that the gap among different schools became smaller and many more students obtained a balanced education. By using factor analysis, we depicted the ability types of students' mathematical thinking. The distribution of students' aptitude inclination type had significant changes. More students belonged to the memorization type while less students belonged to the exploration type after 17 years. The improvement of females was also more significant than that of males. The mean scores of females were lower than that of males in 1990. On the contrary, the mean scores of females were higher than that of males after 17 years. Changes like these will be worth studying further in the future.

Acknowledgments

Shanghai Academy of Educational Sciences jointly with Shanghai Institute of Qingpu Experiment launched the survey in 2007. Many colleagues have been involved in the study and we feel sorry that we could not list their names one by one here, but we would like to express our deep appreciation here for their great contribution to the project. We would also like to express our special appreciation to Dr. Thomas E. Ricks from Louisiana State University, USA for offering excellent language help, which makes the paper possible to be published in English version.

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Section 3

CHINESE MATHEMATICS TEACHERS, TEACHER EDUCATION AND TEACHER PROFESSIONAL DEVELOPMENT

Chapter 15

Beliefs, Knowledge and Teaching: A Series of Studies About Chinese Mathematics Teachers

ZHANG Qiaoping WONG Ngai-Ying

In the past few decades, research on teaching has significantly shifted its focus from seeking effective teaching behavior to exploring teachers' knowledge, beliefs, and thinking behind their actions. Numerous empirical studies have revealed that teachers' beliefs about mathematics and mathematics learning are strong influences on their views of effective mathematics teaching, which in turn influence students' learning. These beliefs come into play when teachers shape students' learning experience and affect their learning outcomes. Although extensive research studies have been conducted on how Chinese learn and teach mathematics, much more needs to be done to explore how beliefs and knowledge influence teaching among mathematics teachers in the Chinese context. This chapter first reviews the literature that concerns how beliefs and knowledge influence teaching among mathematics teachers, with a particular focus on studies conducted in the Chinese regions. We then describe a series of our studies on how beliefs and knowledge affect teaching. Finally we suggest inspirations and directions for future research.

Keywords: beliefs about mathematics, mathematical knowledge, pedagogical content knowledge, mathematics teaching

1. Introduction

The Chinese learner phenomenon has, especially in the subject of mathematics, attracted a lot of attention in the past few decades (Watkins & Biggs, 1996, 2001; Wong, 2013). This has been extensively investigated in our previous book *How Chinese learn mathematics* (Fan, Wong, Cai, & Li, 2004). To enhance mathematics learning, the quality of the teacher naturally plays a crucial role. This is precisely the theme of the present book *How Chinese teach mathematics*. In the past few decades, research on teaching has significantly shifted in focus from seeking effective teaching behavior to exploring teachers' knowledge, beliefs, and thinking behind their actions. How mathematics should be effectively taught does not only concern teachers' professional knowledge, but also their beliefs about mathematics.

Knowledge and beliefs are not segregated (Bromme, 1994; Holm & Kajander, 2012) and some even see beliefs as a kind of knowledge (Furinghetti, 1998; Furinghetti & Pehkonen, 2002). Together, they have many influences on teachers' teaching (Fennema & Franke, 1992). While further empirical studies are expected on the investigations on how these influences actually occur in the Chinese regions, over the past fifteen years, Wong and his colleagues have conducted a series of studies based on the notion of *lived space* which arose from phenomenography. In brief, students' *outcome space* can be seen as a result of the *space* (learning environment) they *live* in and it is the teacher who shapes the *lived space*, and teacher's beliefs are inevitably one of the major driving forces behind this process (Wong, Marton, Wong, & Lam, 2002). In its initial phase, a number of studies were conducted investigating the beliefs among students and teachers about mathematics and mathematics learning. The conceptual framework is depicted in Figure 1.

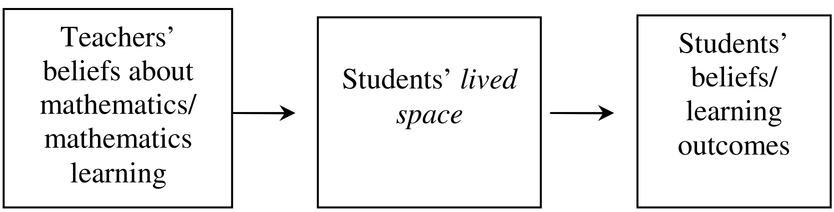

Figure 1. The *lived space* of mathematics learning

We began by examining beliefs held by both students and teachers in Hong Kong. We later extended the study to the other Chinese regions. The team then turned their focus on students' and teachers' beliefs about **effective** mathematics teaching. Admittedly, the teacher's belief is but one factor that shapes students' *lived space*. The teacher's knowledge would also affect how the shape the *lived space* (Phillip, 2007). As such, we included teachers' knowledge into our framework (Figure 2). Before reporting details of these studies (in Section 4), we will first discuss the notions behind our studies and review related studies.

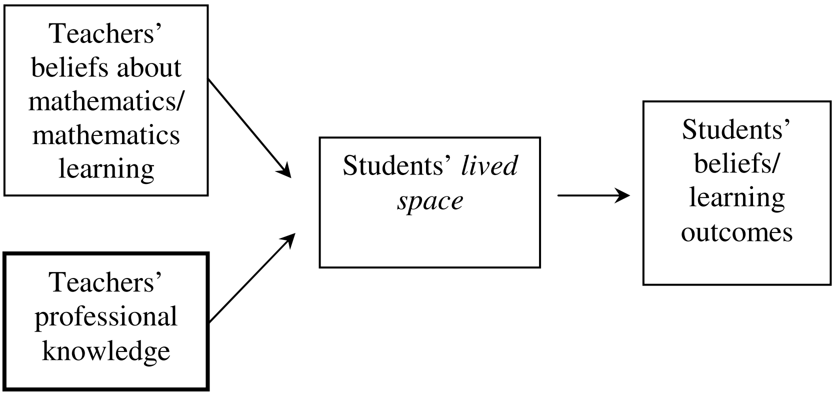

Figure 2. Teachers' beliefs, knowledge and the *lived space* of mathematics learning

2. Teachers' Beliefs and Teaching

2.1 *Beliefs and Beliefs about Mathematics*

Before a teacher enters the classroom, s/he already possesses a set of beliefs about teaching and learning. On top of these, they may also harbour some common myths regarding mathematics (Kogelman & Warren, 1978; Paulos, 1992). These beliefs may come from different sources and Richardson (1996) identified three major ones. They are personal experience, experience with schooling and instruction, and experience with formal knowledge — both school subjects and pedagogical knowledge. By the use of meta-analysis, Kagan (1992) concluded that pre-service teachers entered teacher education programmes with personal beliefs about images of good teachers, images of themselves as teachers, and memories of themselves as students. Indeed, teachers' beliefs represented a complex concept internally associated with their attitudes, expectations and personal experience.

Although beliefs, including those of mathematics teachers, have become a central theme in mathematics education (Leder, Pehkonen, & Törner, 2002; Pehkonen & Törner, 1996), the notion remains not well defined. As Thompson (1992) remarked "for the most part, researchers have assumed that readers know what beliefs are" (p. 129). Terms like 'conception,' 'belief,' 'view,' 'image,' were used quite loosely in literature (Furinghetti, 1994; Pehkonen, 1998a, 1998b; Philip, 2007). Furinghetti (1997) attempted a comprehensive review of all such terms used by scholars but the intention to come to a unified definition cannot be considered as successful. In this chapter, we regard beliefs as "psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs might be thought of as lenses that could affect one's view of some aspect of the world or as dispositions toward action" (Phillip, 2007, p.259).

What we find as more important is the identification of various dimensions of mathematics related beliefs. Such a categorization initiated from McLeod (1992), where the following four dimensions were proposed: beliefs about mathematics, beliefs about mathematics teaching, beliefs about self, and beliefs about the contexts in which mathematics education takes place. Among them, beliefs about self mainly refer to self-concept, confidence, causal attribution and motivation of **students**. Hannula (2012) updated McLeod's framework above. However, with reference to mathematics-related beliefs among **teachers**, how teachers see mathematics and how they view mathematics should be learnt and taught are together considered as teachers' belief system on mathematics education (Beswick, 2005; Thompson, 1992).

Literature repeatedly confirms that teachers' views or conceptions play a role in their teaching but how these views appear in practice may not be apparent. Furinghetti (1997) even likened this as a 'ghost' which creeps in 'quietly'. Individuals are not always conscious of their beliefs, since some are conscious and some unconscious which nevertheless generate subtle effects on classroom teaching (Furinghetti & Pehkonen, 2002, p. 53). Having thus said, much effort was paid to delineate the relationships between teachers' beliefs, in the subject of mathematics in particular, and their classroom practices (e.g., Beswick, 2012; Raymond, 1997), though the justified suspicion (that the two are associated) remains not entirely certain. Both consistency (e.g., Stipek, Givvin, Salmon, & MacGyvers, 2001; Thompson, 1984; Wilkins, 2008) and inconsistency (e.g., Cooney, 1985; Cross, 2009; Shield, 1999) between teachers' beliefs and practice were found in empirical studies. Some began to turn to other (mediating) factors influencing how teachers perceive and enact their roles in the classroom. These factors included internal psychological constructs, such as goals, emotions, teacher identity, and teacher efficacy and also external factors, such as school and department culture, curriculum mandate and class size (Clarke & Hollingsworth, 2002; Raymond, 1997). Schoenfeld (2002) further reminded us that much more needed to be taken into account in order to establish reasonable links between belief systems and behavior. Before we proceed to discuss the impact of teacher's knowledge, we will give a brief account of the two major facets on teachers' mathematics related beliefs: beliefs about the nature of mathematics as well as the beliefs about mathematics teaching and learning.

2.1.1 *Teachers' Beliefs about the Nature Of Mathematics*

In simplistic terms, the belief about the nature of mathematics is the answer to the question "mathematics is …". In his seminal work, Ernest (1989) established the classification of teachers holding different beliefs about this question on different philosophies of mathematics. They are the instrumentalist, the Platonist and the problem-solving. Although there are other categorizations in literature, in general, teachers' beliefs about the nature of mathematics range from viewing mathematics as a static, procedure-driven body of facts and formulas, to a dynamic domain of knowledge based on sense-making and pattern-seeking (Cross, 2009). As mentioned in Liljedahl (2009), though different literature might carry different labels, there are in fact correspondences among them. This is summarized in Table 1.

Table 1. Beliefs about the nature of mathematics

	Dionne (1984)	Ernest (1989)	Törner and Grigutsch (1994)
Beliefs about the nature of mathematics	traditional perspective	instrumentalist	toolbox aspect
	formalist perspective	Platonist	system aspect
	constructivist perspective	problem-solving	process aspect

2.1.2 *Teachers' Beliefs about Mathematics Learning and Teaching*

Similarly, there is a number of categorizations of teachers' beliefs about mathematics learning and teaching in literature. Some see mathematics teaching as a transmission of knowledge while others see it as the facilitation of students' construction of their own mathematical knowledge (Burton, 1993). In addition to the transmission–constructivism dimension, Perry, Tracey, and Howard (1998) identified yet another dimension of child–teacher centeredness, inferring that teacher centeredness and constructivism may not be conflicting. Besides the major distinctions between the transmitter and facilitator, there are other mathematics specific categorizations by Beswick (2005), Kuhs and Ball (1986), Van Zoest, Jones, and Thorntor (1994) and others. The resulting categorizations that emerged from empirical studies are basically in line with Ernest (1989)'s theoretical framework. The variations are probably due to difference in context. The situation under the Chinese context will be discussed in Section 4.

2.2 *Teachers' Beliefs and Teaching Behavior*

As mentioned above, it is the teacher who shapes students' *lived space* and teacher's beliefs would inevitably have much impact in the process of shaping such lived space. For instance, a teacher with a problem-solving view will incorporate progressive constructivist teaching methodologies into their teaching in order to have their students experience the 'doing' of mathematics, and s/he will be a facilitator to students' learning. A mathematics teacher with a Platonist view will make extensive use of definitions and proofs both as a pedagogical strategy and as content to be acquired, and her/his teaching will be expository. Further, Liljedahl (2009) proposed that a teacher with a view of mathematics as a toolbox (instrumentalist) will teach with an emphasis on rules, formula, and procedures with an abundance of practice to enforce memorization and mastery, and s/he guides the students to follow her/his procedures.

However, the relationship between teachers' beliefs and their behaviours, those on teaching in particular, has never been simple (Schoenfeld, 2002). On the one hand, we cannot simply conclude that teacher's beliefs influence his/her teaching behavior. Sometimes change in beliefs either precedes or occurs simultaneously with changes in teacher behaviour/practice (Cooney, 2001; Jaworski, 1998). Conversely, teachers' instructional practice might influence his/her beliefs (Buzeika, 1996). In Guesky's (1986) model, changes in teachers' classroom practice may result in changes in student learning outcomes, and subsequently bring about changes in teachers' beliefs. The relationship between beliefs and practice appears to be a more reciprocal or circular one with each influencing the other (Cobb, Wood, & Yackel, 1990). On the other hand, as mentioned above, both consistencies and inconsistencies were found between teachers' beliefs and their practices (Cross, 2009; Raymond 1997; Thompson, 1992; Wilkins, 2008). As Fang (1996) suggested, the inconsistencies might be due to the complexities of classroom life, which may constrain teachers' abilities to follow their personal beliefs and provide instruction that is aligned with their theoretical beliefs. Teachers' theoretical beliefs could be situational and manifested in instructional practices only in relation to the complexities of the classroom.

So far, researches on teachers' beliefs are mainly conducted in the Western regions (Leder, Pehkonen, & Törner, 2002; Philipp, 2007; Pohkonen & Törner, 1998). Other non-Chinese Eastern regions are not the focus of this book and we expect more studies on such research among Chinese communities. Since beliefs are cultural dependent, it would be of interest to know what beliefs about the nature of mathematics and mathematics teaching the Chinese hold and how these beliefs influence their teaching? Investigations into such issues would also contribute to unfolding the Chinese learner phenomenon.

In this, there are a few empirical studies in literature. By the use of a questionnaire, Chang (2001) found that Taiwanese in-service junior high school mathematics teachers are more inclined to Platonism, while Chen and Chou (1999) and Juang (2002) revealed that most elementary school teachers who participated in the studies hold a constructivist view. Whether such a difference is attributable to the difference in the grade/level is subject to further investigations (see below). Nevertheless, the majority of the teachers in Juang's (2002) study believed that mathematics instruction should be organized to facilitate children's construction of mathematics knowledge and those who held a traditional view opted for direct instruction (Juang, 2002). This echoed with another study of Chen and Chou (1999) in which it was found that those teachers holding a constructivist belief preferred to pose more open-ended problems, were more willing to apply different teaching approaches, and preferred to use group discussions to assist students learning mathematics (Chen & Chou, 1999).

The beliefs among three levels (elementary, junior high and senior high school) of mathematics teachers were compared in Leu and Wen (2001) in Taiwan. It showed that the constructivist perspective was more accepted by elementary mathematics teachers. The teachers believed that the purpose of learning mathematics was to enable students to solve problems in daily life. They believed mathematics learning activities should be aligned with real life experience. On the contrary, junior and senior high school mathematical teachers believed that the purpose of students' mathematical learning was to develop their logic thinking.

Besides questionnaires, qualitative methods like interviews were also employed in other studies. In the study of Chin and Lin (1998), three socially shared views of mathematics teaching were identified among pre-service teachers, which concern learning, instruction, and the classroom atmosphere. The data also revealed that pre-service teachers' preconceptions of mathematics teaching were strongly related to and enhanced by their previous school mathematics learning experience, which is consistent with the earlier findings of Kagan (1992) and Richardson (1996) among others.

These studies focused mainly on teachers' beliefs alone but Su and Chan (2005) moved a step forward to investigate how such beliefs influenced teacher behaviour. Wen and Leu (2004) further explored the consistencies and inconsistencies between beliefs and teaching practice among mathematics teachers. It was found that a crucial factor leading to consistency is teachers' strong will. A strong view would reinforce the teacher's belief in implementing his/her teaching strategies.

Besides the intrinsic nature of the issue, the lack of sensitive measurements also affects these empirical results. After all, beliefs are a sensitive issue and social desirability could affect the accuracy of these measurements (Pajares, 1992). A number of methods, including quantitative and qualitative ones, were used to measure beliefs by researchers, which included questionnaires, interviews, and observations, but these tools have faced scepticism from researchers (Lester, 2002; Schoenfeld, 2002). Indeed, Mason (1997) has stated that an absence of evidence of behavior does not necessarily translate to an absence of evidence of conception. In our previous studies, we adapted the idea from Kouba and McDonald (1991) in using hypothetical situations. The details will be discussed in Section 4.

The studies reviewed above are based in Taiwan. As for studies conducted in the Chinese mainland, Zhang, Wong, and Lam (2009) reviewed the literature related to beliefs among mathematics teachers in the Chinese mainland in the past twenty years and found that most of the publications were only descriptive discussions in nature. Although some studies concerned mathematics related beliefs of either pre-service or in-service mathematics teachers, no empirical researches on the relationship between teachers' beliefs and practice have come to our attention. In sum, a number of studies on the relationship between beliefs and their teaching behavior among mathematics teachers in the Chinese communities were conducted, especially in Hong Kong and Taiwan. The picture needs further examination when teachers' knowledge is put into consideration.

3. Research on Teachers' Knowledge and Teaching

3.1 *Teacher's Professional Knowledge*

Teachers' beliefs undoubtedly impact their teaching. However, it is not so much what they say but what they do that demonstrates teachers' beliefs in their teaching (Schoenfeld, 2011, p. 458). Besides beliefs, teachers' knowledge is also a focus in educational studies (Schoenfeld, 2002). There are many studies on teachers' professional knowledge and quite a few on mathematics teachers in particular.

However, most previous studies concerned the kinds of knowledge teachers had and should have, as well as the source of such professional knowledge (Ball, 1991; Ball, Thames, & Phelps, 2008; Grossman, 1990; Shulman, 1986, 1987). Numerous studies have consistently revealed that many mathematics teachers do not possess sufficient knowledge for effective mathematics teaching (e.g., Carpenter, Fennema, Peterson, & Carey, 1988; Even, 1990; Fan, 1998, 2014; Ma, 1999). Obviously, knowledge of mathematical content for teachers goes beyond simply being able to solve mathematical problems (Holm & Kajander, 2012). Naturally, this would depend on what aspects of problem solving one is referring to. Ma (1999) noted that teachers need to have a "profound understanding of fundamental mathematics [which] goes beyond being able to compute correctly and to give a rationale for computational algorithms" (p. xxiv). This brought about different approaches of defining teacher's knowledge (e.g., Begle, 1979; Brown & Borko, 1992; Hill, Rowan, & Ball, 2005; Monk, 1994). There are quite a number of related terminologies found in the literature (Ball, Thames, & Phelps, 2008; Ma, 1999; Rowland, 2005; Rowland, Huckstep, & Thwaites, 2003; Rowland, Turner, Thwaites, & Huckstep, 2009), the most famous of which are the notions of pedagogical knowledge, subject content knowledge (SK)[a], pedagogical content knowledge (PCK) laid down in

[a] There are many terminologies like CK (Content Knowledge), SK (Subject-matter Knowledge), SMK (Subject Matter knowledge) and MCK (Mathematical Content Knowledge) used by different authors, carrying slightly different meanings. In this chapter, we will treat them as interchangeable and use SK or (Mathematics) SK whenever appropriate.

the seminal work by Shulman and his colleagues (Grossman,1990; Shulman, 1986, 1987). Subsequently, different scholars have updated these terms (Ball, Thames, & Phelps, 2008; Grossman, 1990; Marks, 1990; Rowland, Huckstep, & Thwaites, 2003).

Decades of research reveal that SK, PCK and beliefs are the three most crucial factors that directly affect the teaching of teachers (Bromme, 1994; Tatto, Schwille, Senk, Ingvarson, Peck, & Rowley, 2008). Research also showed a close relationship between teachers' SK and PCK (Ball, 2000; Bromme, 1994; Even, 1993; Krauss, Baumert, & Blum, 2008; Li, 2004). The lack of teacher's SK might result in the lack of teacher's PCK. Teachers with a solid SK would demonstrate strong PCK, which allowed them to flexibly adapt the reform-oriented teaching methods (Rollnick, Bennett, Rhemtula, Dharsey, & Ndlovu, 2008). In the Chinese context, investigations of teacher knowledge have been attracting increasing interest, especially in the Chinese mainland. In her doctoral study, Han (2005) found that one of the distinguishing features in SK among expert teachers is their problem schema. Subsequently, three kinds of practical knowledge were identified, which were practical experiential knowledge in using propositions, knowledge on problem solving strategies and problem schema. Together with other collaborators, Han proceeded to conduct a number of investigations in the North-eastern part of China by means of questionnaires. These studies included the sources of teachers' knowledge (Han, Ma, Zhao, & Wong, 2011) and knowledge structures among mathematics teachers (Han, Wong, Ma, & Lo, 2011). The SK among mathematics teachers in Hong Kong and Changchun (a large city in Northeast China) was also studied. It was found that both groups of teachers tended to simply search and apply routines when solving mathematics problems (Wong, Rowland, Chan, Cheung, & Han, 2010).

Another cluster of studies have been on PCK in the Chinese mainland, which mainly appears in several unpublished doctoral theses (Dong, 2008; Jing, 2006; Liu, 2011; Ma, 2011; Tong, 2008), in addition to Han's (2005) mentioned above. They investigated the characteristics and development of PCK among secondary school mathematics teachers. Expert-novice comparison was a methodology commonly used in these studies. By contrasting between expert and novice teachers, Li (2004) found significant differences in both SK and PCK among these two groups. In addition, expert teachers tended to hold a problem-solving view of mathematics and mathematics learning, while non-expert teachers tended to hold an instrumental view (Li, Ni, & Siu, 2005, 2006, 2007).

Besides expert-novice differences in professional knowledge, there also exists regional differences. There are differences in five aspects of SK and PCK between elementary mathematics teachers in Hong Kong and the Chinese mainland (Shanghai in this case) (Lao, 2008). They are correctness of concepts, connecters of concepts, understanding of basic principles, handling with mathematics representations, handling with feedback, which in turn reflect teachers' different beliefs about mathematics, mathematics teaching and learning in these two regions. Similarity was also found in these five aspects. In addition, East-West comparisons were conducted. By researching on mathematics teachers in the Chinese mainland and the United States, An, Kulm, and Wu (2004) found teachers' PCK did have a deep impact on their teaching in both regions, although there were different demands on PCK. The Chinese teachers emphasized developing procedure and conceptual knowledge through reliance on rational, more rigid practices, while the American teachers emphasized a variety of activities designed to promote creativity and inquiry to develop students' concept mastery.

A newly emerging area is the notion of Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008). Some related studies in the Chinese context include Li, Wan, and Yang (2012), Pang (2011), and Tong (2010). By the use of questionnaire, Pang (2011) investigated nearly 400 pre-service mathematics teachers in the Chinese mainland on their Mathematical Knowledge for Teaching. Results showed that pre-service teachers, in general, did not have a strong mathematical knowledge for teaching. In particular, their specialized content knowledge, knowledge of content and student, and knowledge of content and teaching were very limited. In contrast, the level of their (Mathematics) SK was significantly higher than their PCK (Pang, 2011).

The topic was also researched in Taiwan. Both Huang (2000a) and Lin (2002) found that teachers' competency was restricted by a weak understanding of mathematics, a limited understanding of students' thinking and learning processes, and a limited knowledge of pedagogical alternatives in mathematics classrooms. In order to enhance teachers' professional knowledge for teaching, PCK in particular, Lin (2002) employed case analyses to stimulate teachers' reflection on their practice, arriving at promising results.

The above studies focussed basically on the knowledge teachers possess, yet how teachers' knowledge influences their teaching and consequently student learning should be a greater concern. In Li's (2004) research, results showed the difference in SK and PCK among expert teachers and non-expert teachers both had an effect on their teaching behavior. Compared to non-expert teachers, expert teachers had profound understanding of mathematical knowledge with an explicit and well-organized knowledge package. Expert teachers could maintain high-level cognitive demands of tasks in their teaching, while non-expert teachers tended to reduce the tasks to the mere utility for procedure acquisition without making use of the tasks to enhance students' understanding. Different classroom discourses were also found between these two groups. In expert teachers' classrooms, the discourse showed a pattern of student statement – teacher questioning – student explaining (student – teacher – student). In contrast, in the non-expert teachers' classrooms, the discourse was typically showed to be a pattern of teacher initiation-student response-teacher evaluation (teacher-student-teacher). While we have studies on the relationship between teachers' knowledge and students' learning in the West (Hill, Rowan, & Ball, 2005), the relationships between teachers' SK, PCK and the knowledge of children's cognition in mathematics were also investigated in Taiwan. Huang (2000b) identified three types of teachers' PCK which involves the oral explanation and demonstration by teachers, providing children with mathematical exercise to master learning, and the collaborative learning and discussion in mathematical problem solving. However, although teachers' PCK and knowledge of children's cognition in mathematics were positively correlated, relationships between teachers' SK and the children's cognition did not reach statistical significance.

In sum, professional knowledge among mathematics teachers has attracted increasing attention in the Chinese communities. However, more investigation is needed on the relationships between knowledge and teaching. Before we elaborate on our studies in this aspect (Section

4), we will discuss how knowledge potentially interacts with beliefs and affects teacher's teaching.

3.2 *The Beliefs, Knowledge and Teaching 'Triad'*

It is generally accepted that both beliefs and knowledge influence teaching but how they translate into practice can be further scrutinized (Cooney & Wilson, 1993). Ernest (1989) suggested that beliefs are a primary regulator between knowledge and behavior. Furthermore, he suggested that there is a cycle between subjective and objective knowledge, in which each contributes to the renewal of the other (Ernest, 1991). Several empirical studies were conducted under such a premise that beliefs and knowledge influence teaching interactively (An, Kulm, & Wu, 2004; Fennema & Franke, 1992). There is also a recent trend of including both teachers' professional knowledge and their beliefs as parts of teachers' professional competence (Blömeke, Felbrich, Müller, Kaiser, & Lehmann, 2008; Weiner, 2001).

The interaction among beliefs, knowledge and teaching can be understood in the following way: what teachers believe as crucial about teaching or learning would guide their classroom practices (or, the shaping of students' lived space), and how teachers enacted these beliefs would be constrained by their professional knowledge (Fennema & Franke, 1992).

Aside from these theoretical considerations, a number of empirical studies have been conducted. For instance, Wilkins (2008) found that teachers with higher levels of mathematical content knowledge tended to use less inquiry-based methods in their classroom. These teachers thought that conventional methods worked for them, they did not see the need to try something new. Wilkins further argued that "increasing the level of mathematical content knowledge without also helping teachers develop positive beliefs and attitudes related to mathematics within the context of teaching and learning will in the end limit the value of learning the content" (p. 157). Holm and Kajander (2012) found that by enhancing the mathematical knowledge of pre-service teachers, they became more confident in themselves as an effective mathematics teacher, especially with the new teaching methods. They concluded that addressing either the beliefs or the knowledge of prospective teachers is not enough to support their professional growth. Both beliefs and knowledge need to be targeted together in order to potentially support lasting changes in classroom practices.

There are but a few such empirical studies at present, and much more investigation is needed not just to confirm such an interaction exists to influence teaching behaviour but how beliefs, knowledge and teaching interact in practice. As far as we are aware, our study (reported below) is the only one of this nature in the Chinese region.

4. *Series of Studies on Teachers' Beliefs, Knowledge and Their Teaching*

As mentioned above, there have been only a few empirical studies conducted among the Chinese community on how teachers' beliefs or knowledge affect their teaching. Even fewer investigations have been conducted taking into account teachers' beliefs and knowledge together. In this section, we will report on what we have researched in this area so far. The series of studies is listed below:

1) Initial attempts: classroom environment and perception of mathematical understanding
2) Students' beliefs about mathematics
3) Teachers' beliefs about mathematics
4) Teachers' beliefs about effective mathematics teaching.
5) Teachers' beliefs and teaching
6) Teachers' beliefs, knowledge and teaching

Although this book focuses on the teacher, when we launched our studies in the mid-1990s, we began from the students' perspectives. The results among the students inspired our methods of research into the teachers' perspectives. As such, from a chronological viewpoint, we deem it appropriate to present our initial research among the students.

4.1 *Classroom Environment and Perception of Mathematical Understanding*

How students considered themselves to understanding (some) mathematics and their perceptions of the classroom environment were investigated. First, it was found that students generally see the ability to solve mathematical problems as being able to understand the mathematics (Wong & Watkins, 2001). As for the psychosocial environments of the mathematics classroom, how both students and teachers preferred as well as perceived classroom atmospheres were investigated in depth, both in Hong Kong and in Changchun. Generally, the classroom atmosphere appeared more favorable to the teachers than to the students. Details are referred to in Ding and Wong (2012).

4.2 *Students' and Teachers' Beliefs about Mathematics*

As mentioned above, since beliefs pertain to personal opinions or convictions, a variety of methods were used in the above studies such as open-ended questions ('Mathematics is …'; 'Mathematics classroom is …') and episode writing. It was also the first time we employed hypothetical situations in our studies on students' and teachers' beliefs about mathematics. Some examples are,

Are they considered as doing mathematics in the following situations?

- One day the classmate sitting next to you took out a ruler and measured his/her desk;
- One day Siu Wan made a Valentine card in the shape of a heart by paper folding,

Fruitful results were obtained. Briefly, the students generally identified mathematics with its terminology and content, and that mathematics was often perceived as a set of rules. To them, mathematics is a subject of *calculables*. With such a perception, they tended to tackle mathematical problems by the search for routines. In order to do so, they would look for clues embedded in the questions including the given information, what is being asked, the context (or, the topic) and the format of the question (Wong, Marton, Wong, & Lam, 2002).

For the teachers, we altered the questions slightly by asking, "What would you expect your students' reactions when they are asked whether the following situations are considered as doing mathematics and what would your responses be to your students?" Although the beliefs among teachers appeared more refined than those of the students, the conceptions of mathematics among the teachers basically resemble those of the students. Nevertheless, it was found that the conception of mathematics among the teachers is broader, among which, 'mathematics involves thinking' was unanimously agreed. Other facets of mathematics, as reflected by the teachers, include 'Mathematics is a subject of number and shapes', 'Mathematics is closely related to manipulation', 'Mathematics is precise and rigorous', 'Mathematics is beautiful' and 'Mathematics is applicable' (see, e.g., Lam et al., 1999; Wong, 2000, 2002; Wong et al., 2002; Wong, Lam, & Wong, 1998; Wong, Lam, Wong, Ma, & Han, 2002).

4.3 *Teachers' Beliefs about Effective Mathematics Teaching*

The above focused on the nature of mathematics rather than beliefs about mathematics teaching. We then proceeded to investigate views on effective mathematics teaching among expert teachers (Wong, 2007). A cross-regional comparison was conducted in this study in which Australia, the Chinese mainland, Hong Kong and the United States were involved (Cai, Kaiser, Perry, & Wong, 2009). We found that the Chinese teachers' (Hong Kong and Chinese mainland inclusive)views on a teachers with effective mathematics teaching is one who can set a path of mathematization for the students that goes from the concrete to the abstract, that enhances understanding and that students can acquire a flexible use of rules. To this end, well-organized practices (repetition with variation) may serve as a scaffolding that leads from the basics to higher-order thinking skills. To actualize such a 'teacher-led, student-centred' teaching, in which the teachers are believed to be most effective, teacher professionalism comes first place. Teachers must have a strong professional knowledge, including the mastery of teaching skills and the ability to understand the students.

In this connection, there has been a recent project entitled the Third Wave Study in which both students' and teachers' beliefs about effective teaching were considered in a single study. Please refer to Seah and Wong (2012) for details.

4.4 *Teachers' Beliefs and Teaching*

The systematic study of Wong (2003) is yet another empirical study that involved Chinese mathematics teachers in Hong Kong. In it, questionnaires, interviews and classroom observations were involved. In addition to the above-mentioned hypothetical situations, hypothetical situations which involved the teaching context were used. Here are some examples of these situations:

- Consider the following question: "Siu Fong spent \$75 on dolls and snacks. Each doll sells for \$59. How much did Siu Fong spend on snacks?" Which of the following expressions are acceptable to you? (a) $x = 75 - 59$; (b) $75 - x = 59$; (c) $59 + x = 75$; (d) $75 - 59 = x$.
- Each apple costs \$3. The total price of 2 apples should be expressed as \$3 × 2, \$ (3 × 2) or \$ (2 × 3)?

Wong, Wong, Lam, and Zhang (2009) provide details of both the study and of these hypothetical situations in particular. In this study, teachers' beliefs about effective mathematics teaching in relation to their beliefs about mathematics and mathematics learning were investigated. Seven elementary mathematics teachers in Hong Kong were interviewed through two interview guides, which were based on Thompson (1991) and Wong, Lam and Wong (1998). The former included a set of hypothetical situations that helped to focus the interview on what mathematics is (see the examples above). The latter covered not only what mathematics is but also what mathematics learning and what effective mathematics teaching is. Some sample interview questions are:

1) What does it mean to teach/learn mathematics?
2) What should be taught/learned when you are teaching/learning mathematics?

3) What is the aim of teaching/learning mathematics?
4) If you were observing a mathematics lesson, what rules would you use to evaluate it?
5) What should the roles of the teacher and the students be?
6) What are the characteristics of an effective mathematics teacher?
7) What do you think an ideal mathematics teacher should be?

Results showed that teachers' beliefs did make a difference in their teaching. Three types of teachers were further identified, namely the pragmatic-oriented, the understanding-oriented, and the thinking-development-oriented. Different types of teachers exhibit different views on effective mathematics teaching. For the pragmatic-oriented mathematics teacher, they possess, to a certain extent, an absolutistic view of mathematics. Conformity is emphasized and thus learning mathematics is essentially 'copying' what the teacher does and (re-) producing what is 'correct' mathematically. For the understanding-oriented mathematics teacher, mathematics is seen as a way of thought. Thus, their view of effective mathematics teaching is that understanding, instead of getting the correct answer, is the main learning outcome, and every means should be employed to enhance students' understanding. The thinking-development-oriented category may overlap with the understanding-oriented category. However, for the thinking-development-oriented category, the knowledge structure of mathematics and mathematical rigor are repeatedly stressed. Their picture of the ultimate outcome of mathematics teaching is the acquisition of a 'mathematical way of thought' and the construction of an 'objective' mathematical knowledge structure in the students' mind.

Our series of studies mentioned above are basically the main one involving empirical data. Ding and Wong (2006) conducted another empirical study on how beliefs affect teaching among Chinese mathematics teachers. Their results revealed that, among elementary teachers in the Chinese mainland, those who emphasized the precision and rigidity of mathematics resulted in their students focusing on the superficial features of mathematics like the accuracy of the answers. For instance, though the teacher might touch upon notions like estimation, due to such a confined lived space set by the teacher, the students only focused on the calculable aspect of it, disregarding non-paper-and-pencil methods like estimation by eyeballing or body measurement. In contrast, for teachers holding a broader conception of mathematics, the teaching context is much richer, incorporating more realistic examples and their students were found to be more interested in mathematics.

4.5 *Teachers' Beliefs, Knowledge and Teaching*

From the literature reviews above, we see that with strong mathematics knowledge and the skills for effectively presenting the knowledge to students, together with professional beliefs that were conducive to student learning, the mathematics teacher would approach a lesson in a professional way. This would result in effective mathematics teaching. Desirable learning outcomes may not come about otherwise. However, not many research studies have drawn attention to how teachers' beliefs translate into their teaching practice. As far as we are aware, there are no such studies carried out in the Chinese context. As mentioned above, this worth pursuing since beliefs are culturally dependent. Zhang (2010) fills the research gap by extending the above research model (Figure 2) by including teachers' approach to teaching (Figure 3). Professional knowledge in this context is referred to as both (Mathematics) SK and PCK. In the study, the relationship between beliefs and knowledge was deliberately left open.

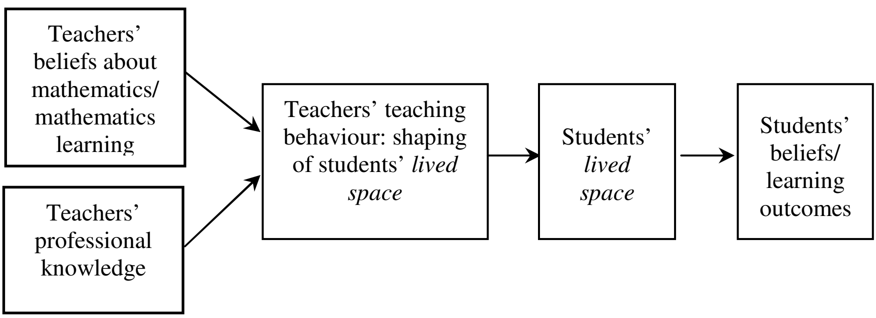

Figure 3. Teachers' professional knowledge, beliefs and their teaching

Both quantitative and qualitative methods were used in the study, which also included the use of hypothetical situations. The study had two phases. In phase one, ninety two mathematics secondary teachers from Wuhan, a large industrial city located in central China, were invited to participate in a questionnaire survey. The survey aimed at getting a general picture of the teachers' beliefs about mathematics, their SK and PCK. The second phase was case study, which aimed at delineating how beliefs and knowledge influenced teachers' teaching approach.

The questionnaire survey in the first phase comprised two parts: *Beliefs about Mathematics Scale* and *The Mathematical Knowledge Questionnaire*. The former was adopted from Tatto et al. (2008) and Lam, Wong and Wong (1999). It contained 40 items put across a 5-point scale. There are three subscales: *Beliefs about the Nature of Mathematics subscale* (10 items), *Beliefs about Mathematics Learning* (14 items), and *Beliefs about Mathematics Teaching* (16 items). *The Mathematical Knowledge Questionnaire* assessed teachers' SK and PCK under the context of mathematics function. There were 5 items on SK and 5 items on PCK. Items for both types of knowledge were based on students' misconceptions or mistakes in the area of mathematics function as described in the literature or known from teacher's teaching experiences (Even,1993; Even & Markovits, 1993; Llinares, 2000). The questionnaire was based on Zhang and Wong (2010).

Results revealed that the participating teachers fell into the three categories: instrumentalist, Platonist, and problem-solver, consistent with Ernest's (1989) classification. With the help of a mathematical knowledge test, six secondary mathematics teachers (2 females and 4 males) were identified for further classroom observations and in depth semi-structured interviews in the second phase. For each of the three categories, one participant possessed limited knowledge while the other had strong knowledge. The notion of function was used as the underlying topic of investigation. Hypothetical situations that concerned teaching in general as well as the teaching of that particular topic of function were used in the interview (Wong et al., 2009; Zhang & Wong, 2010).

Analyses of the case study revealed that beliefs and knowledge impacted teachers' teaching strategies. Teachers with different depths of professional knowledge would have different teaching arrangements and different understandings of students and their mathematics learning. Also, teachers who hold similar professional knowledge might have different interpretations of students' mathematics learning due to differences in their beliefs about mathematics.

Those teachers holding instrumental beliefs on mathematics tended to design their teaching to help students memorize, understand the contents and avoid making mistakes. To them, the ultimate teaching goal was to help students solve problems. Teachers possessing such a belief, but with limited professional knowledge, would follow the textbook when organizing their teaching. They focused on understanding mathematical concepts. Consequently, these teachers emphasized the clear definitions of concepts, and the grasping the key words and core elements of the concepts. Further, with weak PCK, these teachers lacked variations in their teaching methods. In contrast, teachers with stronger professional knowledge were able to use textbooks flexibly according on students' situations. Their teaching design were based on practical considerations, and focused on the operation without special emphasis on the understanding of the concept. When coupled with a rich PCK, these teachers tended to use a lot of metaphors and analogies in their classroom teaching.

Similar to the 'instrumental teacher', Platonic teachers also focused on content in their lesson. However, their teaching aimed to promote students' understanding of mathematics. Here, the scope of understanding included facts, rules, procedures and principles in mathematics. They thought that students' understanding was an ever-deepening process. Teachers with strong professional knowledge organized their teaching by the spiral approach. Much attention was given to present the content progressively. Those teachers with limited professional knowledge focused on the systematic structure of mathematics knowledge. To them, each lesson should be self-contained and standardized. Specifically, to teach a concept, teachers had to help students understand the context of the particular concept and the connection between this concept and others in order to give students a clear impression. These teachers would then consolidate students' understanding through exercises.

As for the problem-solving oriented teachers, their classroom teaching design was more student-centered. They emphasized students' exploration in the classroom. Specifically, teachers with rich professional knowledge used realistic examples to design their teaching. They encouraged students to engage in classroom activities, thus allowing them to present their thinking and to explore new knowledge. These teachers would give counter-examples based on students' own mistakes, so as to deepen students' understanding. As for teachers with limited professional knowledge, they too organized their teaching from the students' perspectives. New knowledge was introduced through students' problem-solving experiences and the application of mathematics in real life. The students were requested to come up with the concept and with the pattern behind the concept. However, due to limited professional knowledge, such teachers were unable to provide adequate examples to illustrate important issues. In such cases, the teachers eventually resorted to instructing students directly on mathematical results or principles.

While Zhang (2010) provides further details in this regard, it is clear from these results that both beliefs and knowledge impact how teachers approach a lesson.

5. Discussions

We have reviewed research studies involving beliefs and knowledge among mathematics teachers, and in particular empirical studies conducted in the Chinese communities. With this backdrop, we presented our series of studies on this topic. Through our series, the conceptions of mathematics among both students and teachers were first identified. In general, both groups hold a relatively narrow belief about mathematics which, from the students' perspective, affects their problem solving performances. Such a confined *outcome space* might be the result of the *lived space* shaped by teachers. In this aspect, both teachers' beliefs and knowledge contributed to their teaching methods.

Although the above may generally be taken as obvious, the picture painted by Chinese teachers appears to be slightly different. The Eastern educational regions, in particular the Chinese ones, have a long tradition of having a centrally designed curriculum and textbooks (Wong, Han, &

Lee, 2004). With such well-designed curriculum documents, together with rigorous teacher preparation systems, teachers would normally deliver the curriculum as designed. Indeed, generic skills and process abilities are very much emphasized in the current mathematics curriculum reform taking place in the Chinese regions (Lam, Wong, Ding, Li, & Ma, in press). It is only in situations where teachers are faced with the choice of gearing students for public examinations or to cultivate students' interest in mathematics via activities that we see the effects of teachers' beliefs. In terms of research, it justifies our use of hypothetical situations, which we believe have a high research potential. These include hypothetical situations on mathematics and mathematics teaching. In this, the hypothetical situations on mathematics teaching mentioned above are mainly derived from actual 'frequently asked questions' in the classroom. Present day classrooms can be quite unlike the 'one-way' teaching of the past, where students listen as teachers talk. Where teachers attempt to address students' queries during a lesson, which is important to ensure students' understanding, the teachers' beliefs clearly affect their response to students. In this regard, it is also imperative that teachers have a strong mathematical background (SK) to respond to such questions and indeed a strong PCK to transform SKs into something that is comprehensible to students.

During discussions in the Topic Study Group on pre-service teachers at the 12th Congress on Mathematics Education, we concluded that PCK is student dependent. In other words, for a single SK, we may need several PCKs to address different target student groups. That makes the training of PCK particularly difficult. Some have suggested asking student teachers or pre-service teachers whether, when faced with mathematical queries from students, they are equipped with alternative methods/ answers (see Berliner, 1986; Cai & Wong, 2012). By having thus, they become reflective practitioners (Shön, 1983). This is one of the research foci in the project 'Knowledge competency among Hong Kong mathematics teachers: Their readiness, strength, and weakness during the reform of New Senior Secondary school curriculum' (with Prof I. K. C. Leung as principal investigator). This project recently earned support from the Hong Kong Research Grant Council.

It is found that mathematics teachers in the Chinese regions generally possess a strong foundation in subject knowledge (Ma, 1999). However, if they hold relatively narrow beliefs (on mathematics, mathematics learning and teaching), these beliefs would not only affect their teaching but would impose their beliefs onto their students. The similarities on beliefs among students and teachers have already been shown in previous studies. This could constitute a vicious circle. We realize that students' beliefs are not only influenced by the teachers. Both are potentially affected by the 'collective Anshuaang' (collective worldview) of the public (Lam & Ernest, 2000; Siu, 1995; Zheng, 1994). In this regard, the teacher is an important contributor to such a 'collective Anshuaang' as, is it not true that one of the core goals of education is to effect positive change in society? To reverse such a circle, it is suggested that a new breed of teachers, or 'scholar teachers', is needed (Siu, Siu, & Wong, 1993).

As for methodology issues for future research, 'expert-novice' contrasts used in earlier research yielded fruitful results. Mixed methods (quantitative and qualitative) were utilized. The use of hypothetical situations proved promising and possessed high potential. These situations included those on what is mathematics, on mathematics teaching and how one approaches teaching. As it has already been established that teachers' beliefs and knowledge have an impact on their teaching, we can proceed to ask how the impact takes effect. Furthermore, there has been criticism that research on teaching approaches had often relied on participants' stated response (Gregory & Jones, 2009; Kane, Sandretto, & Heath, 2002). Indeed, it is suggested that one should go beyond how teachers approach a lesson (planning stage) to how teachers actually deliver their lesson. Classroom observation should be the most direct and valid way to tap into how a teacher is in practice. How beliefs, SK and PCK interplay and how other mediating factors like self-willingness, gender, and teaching experiences enter into the scene are all worth further investigation. To answer the 'how question' above, conventional large scale quantitative studies can be conducted and sophisticated statistical methods like structural equation modeling, or, alternatively, in-depth ethnographic investigations might reveal how such influences take place and how deficiency of knowledge and confined beliefs would restrict their teaching in mathematics.

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Chapter 16

What Makes a Master Teacher? A Study of Thirty-One Mathematics Master Teachers in Chinese Mainland

FAN Lianghuo ZHU Yan TANG Caibin

This chapter presents a study investigating the reasons behind the success of mathematics master teachers in their acclaimed teaching career in the Chinese mainland. Data were collected from 31 mathematics master teachers in four provinces and three municipalities through questionnaires and interviews. The results revealed that the master teachers valued internal factors more than external ones in their professional growth. In particular, dedication to education, inner quality, and true professional care towards students appear to be three most important factors. In contrast, capability in dealing with interpersonal relationship was rated as less crucial by the master teachers. The chapter also documented those master teachers' experiences, reflections, and suggestions concerning teachers' professional development. Implications and interpretations of the findings are discussed.

Keywords: mathematics teachers, master teachers, teacher education, teacher professional development, primary education

1. Background and Introduction

'Master teacher' as an honorary title has been used to recognize teachers' outstanding performance in the Chinese mainland since 1978 when the system was initiated. This title does not belong to the official career rank system for teachers, which was not established until the mid-1980s. The official teacher career rank system, which is rather unique in the Chinese mainland, consists of three grade levels, i.e., "Second Grade" (or junior grade), "First Grade" (or middle grade), and "Senior Grade" (or higher grade). Although it is only an honorary title, "master teacher" is widely believed to be above the level of senior grade teacher, as virtually all master teachers are also senior grade teachers.

Master teachers in the Chinese mainland, as in some other countries, are usually regarded as models for other teachers, experts in teaching, and having great reputations and accomplishment in their subject domains. Regarding their teaching, master teachers are expected to have "a distinctive personal style in classroom teaching", which exemplifies "practicality, innovation, flexibility, and teaching as an art". Furthermore, they are expected to craft their own ideas about teaching materials and teaching strategies. Specifically in the subject of mathematics, master teachers are believed to have a systematic understanding of mathematics, know how to integrate mathematics education theories and psychology into classroom teaching, pay attention to mathematics as a culture, and be able to analyze textbooks with deep understanding (e.g., see Ferreras, Olson, & Sztein, 2010).

Generally speaking, the evaluation and selection process of master teachers in the Chinese mainland is very strict. Consequently, the number of master teachers is very low. In fact, according to the regulation issued by the central government in 1993, the ratio of master teachers to all the teachers should be controlled at no more than 0.15% (Source: the official website of the Ministry of Education at http://www.moe.edu.cn).

Wu and Kong's (2010) analysis of the group of master teachers in the city of Tianjin provides another look at the situation of master teachers in the Chinese mainland. According to them, the city has run the selection of master teachers seven times during the period from 1978 to 2009. There were 11 teachers selected in 1978 and 121 in 2009, and a total of 609 teachers were awarded with this honorable title during the whole period. Wu and Kong found that the gender distribution tended to be more balanced nowadays. More and more younger teachers were awarded the title. However, there was also a tendency that a great proportion of selected teachers were at the secondary school level. For example, in 2009 only 15% of the master teachers selected were from primary schools and 5% from kindergartens.

In fact, master teachers as a recognized group of school teachers of highest quality is a term that is also used in some other countries, such as Singapore and the USA. Increasing attention from education policymakers, researchers, and school practitioners has been paid to the group of teachers, which is particularly evident in the last 10 years (e.g., see Fan & Shen, 2008; Ferreras, Olson, & Sztein, 2010; Gong, 2008; Li, 2010; Lim, 2010).

In Singapore, since the government established the system of “master teachers” in 2001, there have been a total of 5 master teachers in mathematics, 20 in languages, 5 in science, and 4 in other school subjects (Source: the official website of the Academy of Singapore Teachers at http://www.academyofsingaporeteachers.moe.gov.sg). These teachers have been recognized on the basis of their invaluable contribution in their field of teaching, and are deemed to be at the pinnacle of their career. They are providing leadership in the teaching and learning of corresponding subjects across schools in different zones (Fan & Shen, 2008; Lim, 2010).

In the United States, many important aspects and issues have been attached to the concept of “master teachers”, such as what its advantages and disadvantages are, whether it should be tied to merit pay, what “master” should mean, who should be master teachers, and how they should be selected, among others (Klein, 1985). Recognizing master teachers’ significant role, in 2012 the US government announced a $1 billion program to support up to 10,000 master teachers in science, technology, engineering and mathematics (STEM) education within the next four years. These teachers are expected to be an elite group of teachers leading their communities, professional development, and mentorship activities, and contributing new lesson plans and strategies to transform and improve science and mathematics teaching (Koebler, 2012).

Related to the small size and the strict selection process of master teachers, it seems reasonable to assume that master teachers provide not only exemplary teaching practice, but also a role model for teachers’ professional development. In this regard, the study presented herein is intended to investigate the reasons behind the successful stories of mathematics master teachers in their acclaimed teaching career in the

Chinese mainland, and by doing so, to shed light on how to effectively seek teachers' professional development.

2. Review of Relevant Literature

Although the master teacher system was established in the Chinese mainland in the late 1970s, research with clear research questions and methods on master teachers has been few and scattered for a long period of time. The situation has gradually changed since around 2000, and over the recent years, researchers have shown increasing interest in the study of master teachers, from different angles with main focus on their characteristics, beliefs and perceptions, and their teaching practices.

Many studies about master teachers were carried out in comparison with other non-master teachers or, in a more convenient term, ordinary teachers. It appears that questionnaire survey is a most commonly used method in these comparisons.

A notable study in this area was completed by Zhang and his colleagues, who conducted a province-wide survey study to investigate the differences between 111 master teachers and 160 ordinary teachers, covering primary, junior secondary, senior secondary levels and all school subjects, from a variety of perspectives, including teachers' job psychograph (Zhang, 2009a), job pressure and working condition (Zhang, 2009b), growth environment (Zhang, 2009c), self-evaluation about their own teaching ability (Zhang, 2009d), didactical reflection (Zhang, 2009e), conducting education research (Zhang, 2010), and career identity (Zhang, 2011).

There are also some other aspects about the differences between master teachers and ordinary teachers being investigated through questionnaire surveys. For instance, Xu, Cao, and Lan (2010) did their comparison about teachers' lesson preparations. Wang and Zhang (2010) looked into the issues related to teachers' reading. Zhao, Tao and Zhou (2010) focused on teachers' observing others' lessons and conducting their own lessons.

Most of these survey studies revealed significant differences between the two groups of teachers. In most cases, the responses received from master teachers are closer to the ideal ones, such as being less stressful, demonstrating more positive working attitudes, showing more confidence, being more self-motivated in participating in research activities, etc. There are also some aspects in which more similarities were revealed between master teachers and ordinary teachers. For instance, Zhang (2009c) found that there were no significant differences between the two groups of teachers in terms of school environment and society environment. If all teachers generally grew in a similar environment, what made some teachers become master teachers? It is the central question for the present study. While no differences were found between master teachers and ordinary teachers in their understanding of observing others' lessons and the following sharing and evaluation of the lessons observed, Zhao et al. (2010) found that the two groups of teachers had different purposes for observing others' lessons to some extent and further master teachers spent significantly more time in the relevant activities. One question here is what made these differences occur while both groups hold a similar understanding.

There have also been studies exclusively on the master teachers, without a comparison with other teachers. For instance, Li (2010) surveyed 72 school master teachers followed by interviews with 10 selected ones in Guangxi Province, with the purpose to identify master teachers' needs for their professional training in six dimensions, including knowledge, capacity, instructional ability, training methods, training types, and trainers. The study found that the master teachers had strong passion for further learning and development, and particularly for research of teaching and learning, and they prefer specific training over general training, and have diverse needs for further training. Also focusing on teacher training, Cai (2011) studied the impact of in-service training on master teachers' professional development. The results showed that the biggest impact were on the changing of their ideas, innovation of their methods, uplifting of their experiences, and the gaining of opportunities.

While the majority of the studies about master teachers are not subject-specific, there have been a small number of studies particularly focusing on mathematics master teachers, with some being as master dissertation studies, which were often single case studies, focusing on a particular master teacher (e.g., Xu, 2010; Yu, 2012; Zhang, 2007).

Using video-stimulated interviews, Huang and Li (2009) investigated Chinese mathematics master teachers' beliefs about effective mathematics teaching. Through analyzing 10 master teachers' evaluation of video-taped lessons, the researchers revealed that Chinese master teachers focused on the following five aspects in their evaluating mathematics lessons: instructional objectives, instructional design, teaching procedure, learning environment and the teacher's quality, though the emphasis placed on different aspects are varied. The study indicated that the ways in which these teachers evaluate specific lessons might not be closely related to their beliefs about effective lessons in general.

Focusing on how to improve mathematics teachers' expertise, Li, Cai and Gong (2011) studied master teachers' perception about instruction through a different angle. The researchers investigated a master teacher's work station about its focuses and approaches used to help in teachers' professional development. It is found that the master teacher emphasized on a deeper understanding of mathematics content as well as its structure through intensive studies of textbooks. Furthermore, the trainees' thinking and instruction showed dramatic changes through the learning procedures.

Fan and Shen (2008) conducted a comparative study of one Chinese and one Singapore master teachers in the primary schools. Mainly using interview, they found that the two master teachers showed more similarities and fewer differences in their perceptions of effective mathematics learning and their designing of teaching specific mathematics topics. The researchers believe that the similarities might reflect the nature of mathematics and mathematics education, while the differences might reflect the cultural and teaching philosophy between the two countries and personal experiences of the two master teachers.

In a review of research about master teachers, Qiao, Zhang, Cui and Liu (2009) summarized five main research themes, including (1) case studies of master teachers, for example, evaluation and analysis of teaching, interviews on special topics, or experience-sharing, (2) studies of master teachers' characteristics, focusing on, for example, their social class, social expectation, self-requirement, interpersonal relationship, and teaching style, (3) studies of master teachers' professional development in areas of such as academic, instruction, practice, and management, (4) studies of reasonable deployment of master teachers, and (5) studies of master teacher selection system, including factors such as the ratio of master teachers being selected, their qualification, the transparency of selection, etc.

A similar review was carried out by Wu (2010), who provided a summary of the features of research on master teachers in different periods in the Chinese mainland. According to her, the relevant research starting from 1977 mainly targeted at publicizing the exemplary deeds of master teachers in the 1970s. Research during the period from the end of 1980s to early 1990s focused on the patterns and commonalities of master teachers' successful experiences. From 2001, individual master teachers, master teacher groups, and master teacher systems became three main research topics. Regarding the research themes, Wu identified three major types, including personal introduction with exemplary deeds, descriptions and analyses of master teachers' teaching methods, and master teachers' looking-back at and reflection of their own experiences.

Based on the review, Qiao et al. (2009) pointed out the problems existing in the available studies. These problems include that the scope of the available research was limited, the research methods were oversimplified, the quality of research was low, the research findings were superficial, and the number of researchers was small, among others. Consistently, Wang and Cai's (2005) earlier report also suggested that the main problems in the existing research in the Chinese mainland as insufficient attention being paid and the domain of research being too narrow. Meng (2008) criticized that the prevailing research about master teachers' professional growth was usually simply mixed with some experience-sharing or articulation. Strictly speaking, due to being lacking in theoretical support, such research cannot be counted as real research. Moreover, the issue on how master teachers become master teachers was seldom studied.

It appears to us that the problems in the available studies on general master teachers as revealed above also exist in the studies on mathematics master teachers, and it is clear that overall, both the scope and methods of research in this area needs to be further expanded and improved. The present study tends to delve into the causes.

3. Research Design and Procedures

As mentioned earlier, this study aims to investigate the reasons behind the successful stories of mathematics master teachers in their acclaimed teaching career, or in other words, to examine, from the perspectives of mathematics master teachers themselves, what are the important factors that make them master teachers.

More specifically, the investigation intended to mainly address the following questions:

(1) What are mathematics master teachers' views about the importance of various factors to their own professional development?
(2) What are mathematics master teachers' views about the importance of various factors to general teachers' professional development?
(3) What are mathematics master teachers' views about the importance of various conditions or traits for being a master teacher?
(4) What are mathematics master teachers' views about the importance of various pathways for teachers' professional development?

Furthermore, the researchers are also interested in examining whether these views would vary among master teachers with different background characteristics. This section is devoted to the methodological matters of the study.

3.1 *Participants*

Thirty-one mathematics master teachers participated in this study. All but two are teaching at the primary school level. They were from different schools in four provinces: Guangxi, Hubei, Jiangsu, and Zhejiang, and three municipalities: Beijing, Shanghai, and Tianjin. Most of the participants were selected from Jiangsu Province (18) and Zhejiang Province (5) due to practical reasons, as it was more feasible for the researchers to reach and gather information about the master teachers in the two provinces, which obviously is a limitation of this study due to its scope.

Table 1 presents the background information about these 31 participating teachers, including their gender, highest education level, and experience of mathematics teaching in terms of the length of their teaching (years). All the information was gathered from the first six questions in the questionnaire.

Table 1. Profile of participating teachers

	Provinces (4)	Municipalities (3)	Total
Gender of teachers			
Male	17	5	22
Female	8	1	9
Age			
35-40	5	0	5
40-45	17	3	20
45-50	0	0	0
50-55	2	2	4
55-60	1	1	2
Experience of teaching mathematics			
15-20	2	0	2
20-25	13	1	14
25-30	7	3	10
Above 30	3	2	5
Highest level of education			
Junior College	1	0	1
Bachelor	20	4	24
Master	4	2	6
School Location			
Cities	25	5	30
Villages and Towns	0	1	1
Rural areas	0	0	0
School level of teaching			
Primary	24	5	29
Junior Secondary	0	0	0
Others	1	1	2

As we can notice from the table, most participants were aged from 40 to 45 with about 20-25 years of mathematics teaching experiences. All but one of the teachers obtained at least a bachelor degree for their formal education. All but one school where these teachers were working were located in cities rather than towns or rural areas.

3.2 *Instruments and Data Collection*

To collect data, two instruments were designed for the study: self-designed questionnaire and follow-up interview.

3.2.1 *Questionnaire*

The questionnaire consists of six parts. Questions 1 to 6 in Part One are set to collect teachers' background information, which is reported in Table 1. The information is helpful to understand and analyze teachers' responses to the questions in the questionnaire.

Questions in Part Two are about teachers' views on the importance of various factors to their own professional development. They were first conceptualized into two broad categories: internal (or personal) factors and external (or non-personal) factors. The latter is further classified into two sub-categories: school factors and beyond-the-school factors. Under this framework, there are five personal factors (e.g., knowledge level and desire for improvement), five school factors (e.g., school support for professional development and school working environment), and five beyond-the-school factors (e.g., societal respect for outstanding teachers and the government policy about teachers). Open spaces are also provided in the questionnaire for the master teachers to list factors not identified in the questionnaire but they think are important, and to tell how important these factors are.

Part Three is about these mathematics master teachers' views about the importance of various factors to general teachers' professional development. Eight factors, such as teachers' own quality and professional background and teachers' sustained efforts, were listed.

Part Four focuses on these teachers' views about the importance of various conditions or traits for being a master teacher. Eight items were listed, and they include, for instance, professional dedication and good personality.

Different from the previous parts, Part Five of the questionnaire looks into the importance of various pathways for teachers' professional development. The pathways listed in this part include, for example, self-learning and reflections, attending specialists' talks, and exchanges with peers.

All the questions in Part Two to Part Five require the participating teachers to evaluate the importance of various factors to the corresponding domains with a 4-point Likert scale from the highest "very important" (4) to the lowest "not important" (1). In addition, an "N" is given as a choice if teachers feel "difficult to answer" regarding an item. Like Part Two, besides these multiple selections on the scale of importance for the listed factors, open spaces are also provided in Part Three to Part Five for the master teachers to list factors that they think are relevant but are not listed in these parts, and to tell the corresponding scale of importance.

The last part, Part Six, consists of three multiple-choice questions and two open-ended ones. Question 6.1 asks the teachers to identify the key developmental period for a teacher to become a master teacher after he/she starts the teaching career, whereas Question 6.2 asks about a period in which they needed help most based on their own professional growth experience. Question 6.3 asks the teachers to evaluate the importance of teacher training to their success, using a 5-point Likert scale from "very important" to "not importance at all". Finally, the last two questions in Part Six ask the teachers to write down up to three areas in which they want to seek further improvement in their profession and up to three advices if a novice teacher aims to become a master teacher in future.

3.2.2 *Interviews*

The study also employed structured-interviews for data collection. Besides the purpose of data triangulation, the interviews with a selected group of master teachers is also intended to get more in-depth information about the professional development of master teachers.

Eleven questions were pre-designed to guide the actual interviews. The construction of these interview questions was mainly based on the structure of the questionnaire, therefore all the interview questions, except for Question 6, can find their corresponding ones in the questionnaire. These questions further ask teachers to explain their responses in the questionnaire with one or two examples being required to illustrate and/or supplement their views. Question 6 asks teachers to reflect on their own development experience and identify favorable factors and adverse factors in promoting teachers' professional development as well as the most efficient ways and training forms for this purpose.

Eight teachers were first selected after the review of all the participating teachers' responses to the questionnaire. The selection was intended to reflect teachers' demographic characteristics and geographical location.

After we selected the eight teachers, six teachers responded positively and therefore participated in the interview. The other two did not participate due to unforeseen reasons. The six master teachers who accepted our interview are from both provinces (3) and municipalities (3), different age groups and genders (5 male, 1 female), and teaching experiences of different years (3 less than 25 years and 3 having 25 or more years). As the six teachers are working in different parts of the Chinese mainland, all the interviews were conducted through telephone conversations, which were audio-recorded and took about 40-45 minutes, or written responses via emails.

The questionnaire survey took place in March of 2013, with 100% response rate from the participating teachers, which is likely related to the fact that researchers know these participating teachers, communicated with them effectively via phone or email, and the responses were through online methods. The follow-up interviews with six master teachers took place from April to May of 2013.

3.3 *Data Processing and Analysis*

The data in tape-recorded form obtained from interviews were first transcribed verbatim. Together the transcripts (as collected data) were translated from Chinese into English for the purpose of analysis. The data from the questionnaire were stored, processed, and analyzed using SPSS mainly by quantitative methods. The analysis is intended to get an overall portrait about how the mathematics master teachers view the roles of various factors in their professional development. Mean rating for each factor has been calculated followed by a series of Wilcoxon signed ranks tests with all factors within respective dimensions to examine whether there are significant differences among factors in terms of their importance to the relevant issues.

To detect the factors that might affect the master teachers' beliefs about their own growth experience, the participating teachers were further classified into the different groups for comparison:

1. Region: Provincial cities vs. Municipalities
2. Gender: Male vs. Female
3. Age: Young vs. Mid-aged
4. Teaching experience: Experienced vs. Senior

In the study, "Young teachers" refer to the teachers who are 45-year-old or younger and the remaining teachers are defined as "mid-aged teachers". Regarding teaching experience, researchers usually used "ten years" as a criterion to classify teachers into novice group or experienced group of teachers, which is however not applicable in the study, for it is expected that all the master teachers have many years of experiences in teaching mathematics and as Table 1 shows, all the participating teachers have at least 15 years of teaching experience. After reviewing the participating teachers' profiles, we used "25 years" as a classification criterion for the analysis, mainly because that it separated nicely the 31 participants into two groups (Experienced vs. Senior) with nearly an even of number in each.

As listed in Table 1, there are six pieces of personal information collected in the first part of the questionnaire, about which the researchers initially tended to explore their possible influences on the master teachers' views about professional development. However, the data showed that these 31 teachers were quite homogenous in some dimensions. In particular, all but one were university graduates or master degree holders, [1] all but one came from city schools, and all but two were teaching at the primary school level. [2] As a result, it is difficult for

[1] To be noted that most school mathematics teachers at both the primary and secondary levels in the Chinese mainland are subject specialists. It implies, in most cases, that for their undergraduate majors they specialized in mathematics.

[2] Though there were two teachers not teaching at primary schools when the study was

us to detect whether these demographic characteristics would have impacts on teachers' views so that the three variables were not used for classification. Correspondingly, four sets of Mann-Whitney U tests were used to identify the possible differences related to teachers' background characteristics within each dimension.

The data from the interviews were analyzed mainly by qualitative methods. It is used for the researchers to understand better the interviewees' responses in the questionnaire and to get more in-depth information about the professional development of master teachers. For both anonymity and convenience, the six teachers interviewed are denoted by T1, T2, T3, T4, T5 and T6 respectively.

3.4 *Limitations of Research Methods*

Due to the fact that the main purpose of this study is to understand master teachers' perceptions about teachers' professional development, questionnaire surveys and follow-up interviews are believed to be appropriate. However, the researchers are also aware of the limitations of the two instruments, particularly in investigating teachers' actual practices. In fact, researchers have found that the ways teachers did certain things might not be closely related to their beliefs on the same issues (e.g., Huang & Li, 2009). Therefore, more research efforts with a variety of research methods are needed to gain a better and more intensive understanding about what actually occurred in the domain.

4. Results and Discussions

The main results of the study are reported in the following sequence: important factors in master teachers' professional development, important factors in general teachers' professional development, important conditions (or traits) for being a master teacher, important pathways for professional development, and some other issues (e.g., key stage in professional development and professional advices for novice

conducted, they both earned the title "master teacher" because of their excellent teaching performance as primary school mathematics teachers earlier. In this sense, their data were equivalently valuable as those from the other master teachers in the present study.

teachers). The sequence is the same as that of the questions provided in the questionnaires as well as in the follow-up interviews.

4.1 *Important Factors in Master Teachers' Professional Development*

As mentioned earlier, to explore the importance of various factors that contribute to master teachers' professional development, the study classified them into three general types, including personal factors, school factors, and beyond-the-school factors.

4.1.1 *Personal Factors*

Among the five personal factors suggested in the questionnaire, the participating master teachers gave the highest rating to the factor "A3: actively engaging in self-reflection and looking-back" about their own work (M = 3.97). In fact, 30 out of 31 teachers viewed this factor as "very important" while the remaining one considered it "relatively important".

The factor "A2: strong desire for improvement" also received very high rating (M = 3.94), with 29 teachers viewing it "very important". A Wilcoxon signed ranks test showed that there was no significant difference in these mastter teachers' views about the importance of the two factors in their professional development, $Z = 0.577$, $p = .564$.

The third most important factor is "A1: having high level of knowledge". All the teachers viewed the role of the factor in their professional development at least "relatively important", with the average rating being M= 3.35.

In comparsion, the two factors receiving the lowest ratings are related to getting helps from others, that is, "A4: receiving helps and guidances from leaders" (M =3.16) and "A5: learning from colleagues" (M = 3.29), though the absolute value of the ratings are still both larger than 3, which is relatively important. While the majority of the teachers (58.1%) viewed the two factors in their own professional development as "relatively important", some regarded them as "not so important" (A4: 12.9%; A5: 6.5%).

Wilcoxon signed ranks tests revealed that factors A2 and A3 were significantly more important than the other three factors, A1, A4, and A5, at .001 level, while the participants' views about the importance of these three factors showed no significant difference.

It should be pointed out that, to detect the possible differences in the views of the master teachers in different groups about different factors, we conducted a series of Mann-Whitney *U* tests on each paired groups of teachers in terms of teachers' geographical locations (provincial cities vs. municipalities), genders (male vs. female), ages (young vs. mid-aged), and teaching experience (experienced vs. senior). The results showed that there are overall no significant differences in these master teachers' views about the importance of the various personal factors to their own professional development. The only exception is that male teachers valued more about "A2: strong desire for improvement" than their female colleagues ($U = 77.0$, $p < .05$, $r = .40$)[3]. The teachers from municipalities tended to have considerably more positive perspective about "A1: having high level of knowledge" than those from provincial cities ($U = 46.0$, $p = .080$). In general, the results indicate that all the master teachers have an overall similar view about this issue. To us, this finding is to a degree not only surprising and but also enlightening, indicating a strong communality of master teachers.

The results of interview with all the six teachers are consistent with the findings from the questionnaire survey. They were further asked during the interviews to elaborate their views about the five factors with concrete examples. T1 gave a synoptic description about the functions of these factors in his professional growth; that is, "having high level of knowledge" makes teachers feel confident, having "strong desire for improvement" greatly motivates teachers, "actively engaging in self-reflection and looking-back" enables teachers to get a great sense of accomplishment, and "receiving help and guidance from leaders" may shorten the pathways of professional development. About his own practices on actively engaging himself in self-reflection and looking-back, the teacher pointed out that he has published three monographs on primary school mathematics teaching based on his regular reflections and research and that he had also worked with colleagues and had jointly published more than twenty teaching guidebooks. Both T2 and T5 highlighted the importance of strong passion for the profession. T2 emphasized that as long as one regards education as a career for life, the teacher will then possess strong desire for improvement, pay close attention to his/her own behaviors, and consistently make necessary adjustment so as not to departure from the right pathway. Also because of the same reason, one would be humble and willing to accept all kinds of help from various parties. T5 emphasized that that without passion, a teacher's effort in seeking improvement on the job will not be sustained for long.

[3] Effect sizes for Wilcoxon signed rank tests and Mann-Whitney U tests are expressed by r with 0.1 being small, 0.3 being medium, and 0.5 being large.

Besides the five factors given in the questionnaires, the participants were also invited to suggest other personal factors which they believed have certain importance in their growth to become a master teacher. As a result, these teachers proposed 30 different factors from a variety of perspectives. Some are related to their learning habits (e.g., learning from students, the ability in persistent learning, and professional reading), some are about their personalities (e.g., personal interests, personal characters, and rich in practical experiences), and others referred to views about the profession (e.g., educational ideals, sense of professional responsibility, and enthusiasim to the profession). The result suggests that besides the commonality about the five factors reported above, the teachers have also their own unique personal factors that have played an important role in their professional development.

Interestingly, both T2 and T4 also pointed out the their family environment is also an important factor in their teaching career. T4 mentioned that he has 10 family members and close relatives who are teachers, and among them his father has the biggest influence on himself in becoming a master teacher.

T6 in the interview further commented her gains from consistent academic writing. From her growth experience, she summarized three writing-related factors which made her feel successful, that is, self-sustained passion in writing, adequate motivations and attitudes toward academic writing. T3 also mentioned that writing is a good way of promoting self-reflection, though he recalled that his self-reflection and looking back was more passive at the early stage in his career.

4.1.2 *School Factors*

Compared to the importance of personal factors in the master teachers' professional development, the ratings on the suggested school factors were generally lower. The teachers gave the highest rating to the factor "B1: School support for professional development" (M = 3.39, higher than "relatively important) and lowest to the factor "B3: Help gained from the school-wide mentoring program" (M = 2.83, lower than "relatively important), with two teachers considering factor B3 "not important" to their professional growth. In addition, on all the five factors indicated, there were a number of teachers giving "not so important" ratings. On factors B3 and B4 ("Help received from school colleagues"), one teacher indicated that he/she is unable to rate them.

Wilcoxon Signed Ranks Tests showed that factor B1 was viewed significantly more important than both factors B3 and B4 (for B3: Z = 3.532, $p < .001$, $r = .64$, and for B4: $Z = 2.828$, $p < .005$, $r = .52$). Furthermore, factors B3 and B4 were also found to be significantly less important than the factor "B2: Good working environment in school" (M = 3.29; for B3: $Z = -2.977$, $p < .005$, $r = .54$ and for B4: $Z = -2.179$, $p < .05$, $r = .40$) and factor B3 was further viewed significantly less important than the factor "B5: School leaders offering clear helps and encouragements" ($M = 3.20$; $Z = -2.598$, $p < .01$, $r = .47$).

Similar to the findings on teachers' perceptions about the importance of different personal factors, these master teachers demonstrated a similar view on the importance of school factors. No significant difference was found between different groups as classified earlier except that male teachers tended to value considerably more about "B3: Help gained from the school-wide mentoring program" than female teachers ($U = 54.5$, $p < .050$).

Based on the responses provided in the questionnaires, the interviewees were asked to elaborate on their ratings. T1 commented on the importance of school support, as he said "it is difficult for teachers, even more capable ones, to get proper development if their school leaders did not offer the teachers opportunities". He shared in detail with the researcher an example of a school principal he knows. He observed that the principal was willing to appoint her teachers based on their merits, value teachers' talents, spend money to train novice teachers (e.g., sending young teachers to attend various national conferences), and provide platforms for teachers to show their abilities and achievements. As a result, many teachers in the school have become excellent ones. In contrast, he also observed that another school's principal was envious of her teachers' ability, only loved words of praise from teachers, expected teachers to be docile and obedient, and could not permit the existence of teachers who had better abilities than herself. Eventually, many outstanding teachers had to leave for other schools. T4 explained that the trust his principal gave him in his earlier days as a teacher was very helpful in his professional development. Similarly, T6 expressed her appreciation about what her principal offered her. She was impressed from her principal's in-depth understanding about teaching materials, distinctive views about instruction, and subtle and refined warmth in dealing with people. She benefitted greatly from her principal's support, understanding and encouragement, which she described as "humanistic power", along the way of her professional growth.

T2 shared in detail with the researcher his understanding of the importance of school working environment and colleagues' help. He stressed that the collegiality he enjoyed in the first school he worked was very helpful for him to form the value of "helping others". He also commented that school's nurturing excellent characters for teachers is much more important than providing good physical facilities. T5 offered a similar view and emphasized that a supportive and understanding environment is very important, and schools should also make necessary arrangement for young teachers to learn from experienced teachers. On the other hand, T3 candidly explained in the interview that he missed one developmental opportunity in the early 1990s because a leading teacher who was already a master teacher did not provide an equal opportunity for all junior teachers in favor of these who were his own students.

Beyond the five suggested school factors, the participating teachers listed another fifteen relevant factors which they believed having contributed to their professional development. While some were an extension of those listed in the questionnaires (e.g., specialists' guidance, good interpersonal relationship, and relaxing working environment), others were related to school leaders' philosophy for running a school, school culture and tradition, and working atmosphere, as well as school policy for teachers' development, personnel evaluation, and recognition of teachers' personalities, etc. It seems clear that school factors played a reasonably important role, though not as important as personal factors, in these master teachers' professional development.

4.1.3 *Beyond-the-school Factors*

Regarding the importance of beyond-the-school factors, the 31 master teachers viewed all the five factors listed in the questionnaire as important ones, to a greater or lesser degree. The highest rating was on "C4: Family members' understanding and support" (M = 3.48), whereas the factors "C1: Societal respect for outstanding teachers" (M = 3.07) and “C2: Government’s policies on teachers” (M = 3.03) were given the lowest rating. Furthermore, one teacher noted that he had difficulty in rating the importance of factor C1 (hence not rated) and another female teacher viewed factor C2 as “not important” in her professional growth.

In the questionnaire, T1 offered a low rating on factor C1 and C2. During the interview, T1 explained that the motivation for professional development should be from inside. In his view, the present society has not paid enough attention to education and the respect given to the teachers were merely on words but not in substance. However, he believed that many outstanding teachers were not affected too much by such a situation, as they viewed the profession as their career and not merely a job. Though not highly paid, these teachers could find it joy amid challenges. When a teacher regarded the job as a pleasure, external factors would then not have a great impact on him. T3 and T5 also indicated that external factors only have a certain level of importance.

In contrast, T2 and T4 viewed society-related and government-related factors important to their personal development. T2 also emphasized the importance of the family-related and friend-related factors. In his words, the first two factors would play a role for positive direction of development. Regarding the support from family, T2 commented that with family members’ understanding and support, teachers could devote themselves more to teaching. On the contrary, family members’ complains will affect one’s mood as well as passion for the career. He described friends’ influence as “one takes on the color of one’s company”, and believed that friends’ value has direct impact on one’s mentality. On the other hand, T4 highlighted the support he received from the education administration when he was a young teacher, and believed it was important.

Similarly, T6 also appreciated the values of friend-related and government-related factors in her professional growth experience. She particularly described how a teaching and research fellow has influenced her professional development. According to her, the teaching and research fellow’s rigorous scholarship and profound understanding about mathematics helped improve her ways of thinking. Furthermore, the fellow also offered many development opportunities for her, such as working together in developing resource booklets and encouraging her to take part in open evaluations about others teachers' classroom teaching and to conduct lessons of high quality for others to observe. In short, T6 regarded the teaching and research fellow in the government as an "important factor" in her career life.

The comparison of the relative importance of the suggested beyond-the-school factors in these master teachers' professional growth showed that "understanding and support from family" (C4) was significantly more important than "government's policy on teachers" (C2: $Z = 2.568$, $p < .05$, $r = .46$) and "societal respect for outstanding teachers" (C1: $Z = 3.153$, $p < .001$, $r = .58$). Further, this factor was also considerably more important than "colleagues' and friends' influences" (C5: $Z = 1.767$, $p = .077$). The second highest rated beyond-the-school factor (C3: "education administrative department's encouragement and support", $M = 3.29$) was found significantly more important than factors C1 and C2 at the .05 level (for C1: $Z = -2.111$, $p < .05$, $r = .39$ and for C2: $Z = -2.309$, $p < .05$, $r = .41$). No significant differences were found among the remaining three factors.

It was again noted that all the master teachers' views about the importance of beyond-the-school factors in their professional growth experiences were generally the same. Except compared with the mid-aged teachers, young ones tended to value colleagues' and friends' influences (C5) considerably more important ($U = 41.0$, $p < .067$). Further, female teachers seems to value about government's policy on teachers (C2) considerably more important ($U = 62.0$, $p < .082$).

Compared to personal factors and school factors, the number of additional beyond-the-school factors suggested by the participating teachers was smaller, with six in total. Two of them were related to students' parents. There was one teacher particularly mentioning about the importance of participating in teaching competition at various levels.

4.1.4 *Comparisons of the Importance of Personal Factors, School Factors, and Beyond-The-School Factors in Professional Growth*

Table 2 listed the descriptive statistics of teachers' rating on all the fifteen factors by types of factors. Personal factors are found to be overall more important than the other two types of factors and the corresponding differences were statistically significant at the .001 level, whereas the teachers' perceptions about the importance of both school and beyond-the-school factors were not much different. Moreover, it is worth noticing that the ratings of factors B1 (school supports) and C4 (colleagues' helps) were particularly high within each type of factors, while that of factor A4 (school leaders' guidance) was particularly low compared to the other four personal factors.

Table 2. Teachers' perceptions about the importance of various factors in their master teachers' growth experiences

Level of Importance		*M*	*SD*	Rank
Personal factors (M = 3.54, SD = .26)	A1	3.35	.49	5
	A2	3.94	.25	2
	A3	3.97	.18	1
	A4	3.16	.64	11
	A5	3.29	.59	6
School factors (M = 3.14, SD = .57)	B1	3.39	.67	4
	B2	3.29	.78	8
	B3	2.83	.83	15
	B4	3.00	.64	14
	B5	3.19	.65	10
Beyond-the-school factors (M = 3.21, SD = .49)	C1	3.07	.64	12
	C2	3.03	.80	13
	C3	3.29	.64	7
	C4	3.48	.57	3
	C5	3.19	.75	9

4.2 *Important Factors for General Teachers' Professional Development*

As said earlier, Part Three of the questionnaire asks the participants to evaluate the importance of eight factors for general teachers' professional development, based on their own experience. The result showed that these master teachers all believe that the factor "3b: Teachers' sustained efforts" (M =3.97) is the most important factor, followed by the factor "3a: Teachers' own quality and professional background" (M = 3.73). In fact, all but one teacher believed that factor 3b is "very important" to teachers' professional development. Comparatively, the factor "3e: mutual helps from colleagues" received the lowest rating (M = 3.00) with only four teachers viewing it "very important".

In the questionnaire, T1 gave a rating of 4 ("very important") for both factors 3b and 3d (expert's professional guidance). During interview, he enumerated several important people at various levels in his professional development, including school principal, director of teaching, and teaching and research fellows. Furthermore, he gave a specific comment on one teaching and research fellow, who guided him to take part in a variety of teaching competitions, eventually at the national level. As T1 said, "the influence was profound, and it was the preparation procedure for those competitions that made me to set my affection on mathematics teaching, get onto the road of research, and eventually become a master teacher". In the interview, T3 also agreed that real experts' professional guidance is indeed important, but he also cautioned that sometimes the so-called "experts" might only be able to offer some general principles and not very practical theories. In this case, teachers' own experience and reflection is more important.

Besides factors 3a and 3b, T2 also highlighted the important role of mentors (3c) and family members (3h). He indicated that it is not easy and also not necessary to find out the solutions to all problems by oneself. One can always learn from his predecessors' well-established experiences. In this sense, getting a wise mentor, who does not necessarily need to be a famous one, is important, as a wise mentor's advices can help in avoiding many detours.

T6 also emphasized the significant impact of experts on her own professional development. She elaborated what she learnt from an invited master teacher's lesson demonstration, including integrating daily life examples into teaching (getting out of classrooms), enlightening students in mathematics ideas and methods (getting out of knowledge), and self-developing teaching materials creatively (getting out of textbooks). From that lesson, T6 had a sudden feeling of being enlightened and refreshed. After that lesson, she searched for many materials about that master teacher and started to "imitate" what that master teacher did and later tried to form her own way of teaching. She said she experienced the procedure from being similar in form to being similar in spirit.

Table 3 displays a comparison of these master teachers' perception about the eight factors in terms of their importance to teachers' professional development. It can be observed that the importance of factor 3b is significantly higher than all the other factors. In the opposite direction, the participants viewed factor 3e significantly less important than all the other seven factors.

Table 3. Wilcoxon signed ranks test results on master teachers' perceptions of the importance of various factors for teachers' professional development

	3a	3b	3c	3d	3e	3f	3g	3h
3a (2)	-	-2.333*	3.638***	2.840**	4.234***	3.357***	3.771***	2.138*
3b (1)		-	4.413***	4.359***	4.817***	4.264***	4.707***	3.873***
3c (6)			-	-1.414	2.111*	-.905	.302	-2.000*
3d (4)				-	2.668**	.302	1.897†	-.943
3e (8)					-	-3.162**	-2.121*	-3.441***
3f (5)						-	1.134	-1.291
3g (7)							-	-2.183*
3h (3)								-

Note: 3a: Teachers' inner quality; 3b: Teachers' persistent efforts; 3c: Mentor's guidance on instruction; 3d: experts' professional guidance; 3e: Colleagues' mutual help; 3f: Colleagues' influence and encouragement; 3g: Leaders' caring and encouragements; 3h: Family members' understanding and support; † $p < .1$, * $p < .05$, ** $p < .01$, *** $p < .001$; The numbers in the brackets represent the ranking of the factors in terms of importance in ordinary teachers' professional development.

Besides the eight factors listed in the questionnaire, the participants further suggested another eight factors that they believe have importance in general teachers' professional development. These suggested factors, such as obtaining school's recognition, getting students' support and trust, teachers' motivations to work, and pleasant working atmosphere, reflect a variety of these teachers' perspectives. Also there was one teacher highlighting the importance of getting parents' trust and another teacher appreciating the value of opportunities. Interestingly, one teacher proposed a unique point of view, that is, accumulating (knowledge) from reading books.

A series of comparisons of the views of the master teachers in different groups, as aforementioned, revealed that, in general, the 31 master teachers held similar views about the importance of various factors for mathematics teacher's professional development, though some significant differences were found. More specifically, female teachers rated the importance of leaders' caring and encouragements (3g: $U = 49.0$, $p < .01$, $r = .49$) as well as colleagues' mutual help (3e: $U =$

58.5, $p < .05$, $r = .41$) significantly higher than their male colleagues. Furthermore, the female group also gave considerably higher rating to the factors of teachers' own quality and professional background (3a: $U = 56.0$, $p < .050$) and persistent efforts and colleagues' influence and encouragements (3f: $U = 66.5$, $p < .081$). In addition, the analysis showed that the teachers from provincial cities gave significantly higher rating to teachers' persistent effort (3b), with $U = 62.5$, $p < .05$, $r = .37$, and considerably higher rating to mentor's guidance on instruction (3c) than their peers from the municipalities, $U = 46.5$, $p < .090$.

4.3 *Teachers' Own Conditions for Being a Master Teacher*

To explore how important teachers' own conditions or traits are for them to be a master teacher, Part Four of the questionnaire lists eight related factors.

The result shows that all the 31 master teachers consistently considered the factor "4.1, professional dedication" to be "very important" (i.e., $M = 4.00$), followed by the factor "4.5, good classroom teaching ability", with 27 out of the 31 teachers' ratings being "very important" ($M = 3.87$). Factor "4.7, strong leadership and coordination ability" was viewed to be least important ($M = 2.96$) with one perceiving it "not important" and two "not so important". Factor "4.8, strong communication ability" is another one on which there is one teacher giving a rating as "not so important" ($M = 3.36$). In fact, all the three factors related to abilities in dealing with interpersonal relationships received significantly lower ratings than the other factors (see Table 4). On the contrary, teachers' "professional dedication" (4.1) was viewed significantly more important than all the other seven factors. In addition, little difference was found among teachers' personal attainments (i.e., factor 4.2 to factor 4.5) in terms of their importance for being a master teacher. Table 4 summarizes the results.

Significant differences were observed among teachers with different comparison groups as mentioned earlier in their views about the importance of factor "4.7, strong leadership and coordination abilities" for being a master teacher. One case occurred between the teachers from

Table 4. Wilcoxon signed ranks test results on teachers' perceptions of the importance of teachers' own conditions for being a master teacher

	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8
4.1(1)	-	2.449*	2.449*	2.646**	2.000*	4.359***	4.772***	4.025***
4.2(4)		-	.000	-.447	1.000	3.606***	4.234***	3.207**
4.3(3)			-	.577	-1.414	3.357***	4.234***	3.207**
4.4(5)				-	-1.732†	3.207**	4.119***	3.207**
4.5(2)					-	3.638***	4.456***	3.500**
4.6(6)						-	2.972***	.816
4.7(8)							-	-2.887**
4.8(7)								-

Note: 4.1: Professional dedication; 4.2: Personality charisma; 4.3: Pedagogical knowledge; 4.4: Subject knowledge; 4.5: Classroom teaching ability; 4.6: Handling interpersonal relationship ability; 4.7: Leadership and coordination ability; 4.8: Communication ability; † $p < .1$, * $p < .05$, ** $p < .01$, *** $p < .001$; The numbers in the brackets represent the ranking of the factors in terms of importance in ordinary teachers' professional development.

municipalities and those from provincial cities with the former group giving significantly higher rating, $U = 28.5$, $p < .005$, $r = .55$. The other difference was found between young and mid-aged master teachers which was more appreciated by the mid-aged teachers, $U = -32.5$, $p < .05$, $r = .40$.

During the interview, most of the teachers expressed their agreement with the results collated in the questionnaire. Two teachers (T4 and T5) particularly elaborated the importance of personality charisma of teachers. According to T5, the influence of teachers on students as individuals is a most important thing, and mathematics teachers are first teachers, and then are teachers of "mathematics"; teachers need to win students' hearts so students can like them, to establish equal and friendly relationship with students (not always use the so-called "absolute authority of the teachers"), to build up trust with students, and from time to time, to show a young heart to mingle with them.

By the way, in the questionnaire, the participants also put forward another five abilities/attainments which they perceived to be important. They include passion and ability of self-learning, ability in doing research on teaching, good thinking ability and methods, good ability in expressing, and good background in philosophy. The result shows both the width and depth of these master teachers' thinking about the issues.

4.4 *Pathways for Teachers' Professional Development*

Items in Part Five of the questionnaire focus on the pathways for teachers' professional development. Six types of pathways were provided for these master teachers to share their views about the importance of those different pathways.

The result revealed that 28 out of the 31 master teachers considered pathway "5.1, self-learning and reflection" to be "very important" (M = 3.90), whose rating is significantly higher than the other five pathways in these teachers' views at the .001 level. In contrast, there are six and thirteen teachers considering pathway "5.6, various training for degree or formal certificate" to be "not important" and "not so important", respectively, and only one chose "very important" one (M = 2.23). The importance of this pathway was rated significantly lower than all the other pathways at the .005 or .001 level.

Consistently, another training-related pathway "5.5, various short-term training" was the next lowest rated one (M = 2.87). Nevertheless, no teacher claimed that it is "not important", while eight considered it "not so important" and four "very important". For the other three types of pathways, master teachers' average ratings were all above 3.00 but lower than 3.30 (5.2, experts' lectures: M = 3.07; 5.3, seniors' (mentors') guidance: M = 3.29; 5.4, communications and discussions with colleagues: M = 3.23).

T1 was one of the respondents who perceived pathway 5.6 as "not important" one in the questionnaire survey. During the interview, he mentioned that the main purpose of such training was often merely for earning money and too commercial. He also elaborated other problems in these training programs, which were echoed by T3, T4 and T5, including that the participants were more for getting the certification so it could be used for future promotion, the contents of training were often alienated from the present classroom realities and irrelevant to the curriculum reform, and the materials provided were often out of date and in some cases, the examples provided were used 30 years ago, which cannot meet present teachers' needs. Both T1 and T4 explicitly pointed out that the current teacher education lagged far behind the curriculum reform and needs change and improvement, a challenge to teacher educators.

T2 rated the both types of trainings (pathway 5.5 and 5.6) "not so important". In the interview, he illustrated that the effectiveness of such trainings was determined by the learner's intrinsic needs. If one has a strong need about the training contents, the effect of the relevant lectures and trainings would be good. Otherwise, it will go to the opposite. On the other hand, T2 also stressed the importance of practices for "digesting" what have been learnt.

During the interview, T2 also pointed out that learners would encounter many difficulties during their practices; and while short-term trainings cannot do much on these, exchange with more experienced colleagues can offer good help. Consequently, T2 rated both pathways "self-learning and reflection" and "seniors' (mentors') guidance" as "very important" ones. Similarly, T6 also rated highly the pathway "self-learning and reflection". She quoted that "the master teaches the trade, but apprentice's skill is self-made".

A relevant question in Part Six of the questionnaire asked the master teachers to evaluate the relationship between their career success and teacher training they have received, according to their own professional growth experience. With a 5-point Likert scale, the result showed that, while no teacher claimed there is "no relation" (scale: 0) between the two, one teacher viewed it as "not so important" (scale: 1) and eight rated it "relatively important" (scale: 4) with an average rating being 3.97. Overall, the result shows that teacher training only has moderately importance, which is lower than we expected. No significant differences between different groups of teachers were identified, indicating the consistency of teachers' views on this issue.

All the six interviewees were asked how their career success is related to the training they have received. T1 maintained that some high-level professional conferences and activities were important to his development, such as competitions on teaching skills and annual meetings of teaching. T6 believed that what an individual can reach in his own ability is limited so that teacher training was "utmost important". According to her, the training can help teachers broaden their vision, make friends with other teachers, and sometimes give on-stage demonstrations of teaching personally. From the training received, T6 gained encouragement from the group she belonged to and received help from her master (mentor). As she mentioned, when one gets tired, friends will provide reminders; more importantly, there will be a group of people sharing the same ambitions and purposes and moving ahead together, "if one wants to travel fast, he/she should move alone; if one wants to travel far, he/she should move in a group". T3 offered another angle to view this issue. He reported that training was very important to him at the beginning, then became not so important after a period of time, and then again became very important in a later stage; the key factor here is that teachers need to have a period of accumulation and reflection on their own, and only when they have questions before they participate in the training, can then training be really important. In addition, T5 mentioned that the training is not very important for master teachers, though it could have importance for general teachers due to the different background and needs of these teachers.

During the interview, T1 also pointed out that the institutionalization of teacher training is a most effective training form for teachers' professional development, as outstanding teachers can be released from their regular work for additional studies in institutions of education or normal universities. On the other hand, T2 believed that the form of apprentice training was a most effective training form, and he also highlighted the important role of being engaged in self-reflection in one's development and further commented that everyone has great inner potential ability.

Regarding the importance of various pathways for teachers' professional development, the 31 participating master teachers gave an overall consistent view. An exception is that the female teachers valued the pathway "various short-term training" significantly more important than male teachers, $U = 56.5$ $p < .05$, $r = .36$, which might be related to Chinese cultural and societal factors, as females often play more direct and essential roles in taking care of their children and families, and hence participating in short-term training is easier for them. Nevertheless, further research evidence is needed about this issue, which is beyond the scope of this study.

By the way, four additional pathways were also suggested by the respondents concerning teachers' professional development, which include teachers' conducting educational research project with the guidance from experts, first-hand practical experiences and training opportunities, demonstrations of lesson study, and systematic study on textbooks.

4.5 ***Some Other Issues***

Finally, we report and discuss findings from questionnaire and/or interview on some other issues related to teachers' professional development.

4.5.1 *Key Periods of Time in the Development of Master Teachers*

In the last part of questionnaire, two questions were designed to ask about the key period of time in the professional development of these master teachers. One is about the most important stage during their professional growth and other is about the key period when these teachers felt that they needed help most.

The result showed that about three fourths of the teachers considered "5-15 years (after they started their teaching career)" as the most important stage in their own development. More specifically, 48.4% of the teachers considered "5-10 years" and another 26.7% of teachers chose "10-15 years" as the most important stage.

Different from the majority of the teachers, in the interviews T1, T4 and T6 all stressed the importance of the first five years. T1 argued if one can start his/her teaching career in a school with good teaching and research environment and have an excellent teacher as his/her mentor, the teacher could professionally grow up in a fast pace. T4 added that the first two years is a period for teachers to be familiar with the classroom teaching, and hence the next 3 years is actually more important. T6 suggested that the period of the first five years may determine the working direction for a novice teacher, which, in her words, will "set the tone" (either "being full of enthusiasm" or "being slack in one's work"). Interestingly, T2 did not pick any particular period of time but commented that only when one knows clearly about his/her responsibilities and missions, he/she would then really start to grow. From his point of view, different people would need different length of time to reach this stage. In fact, the questionnaire survey showed that there were two other teachers having similar responses. One of them believed that the any period of time is important, while the other claimed that different time periods had different focuses. Overall, no statistically significant difference was found among different groups except that teachers from municipalities valued significantly more important about the role of an earlier period in a master teacher's development than those from provincial cities, $U = -36.5$, $p < .05$, $r = .36$.

Regarding which period these teachers hoped to get help most, more than 80% of the master teachers selected either "first 5 years" (45.2%) or "5-10 years" (35.5%). T1 in the interview stressed the importance of both the periods of time, while T2, T3 and T5 all selected the period of "5-10 years", as they believe that first five years of teaching is more for teachers to gain initial experience and get familiar with the school and classroom realities. The statistical analyses on group comparisons showed that the teachers in the young group suggested a significantly later period on this item than their mid-aged colleagues, $z = 24.0$, $p < .005$. $r = .49$, which, in our view, might be related to the fact that younger teachers received better pre-service teacher training. A similar difference was also observed between experienced teachers and those senior ones, $U = 53.0$, $p < .005$, $r = .51$. No significant difference was found for other group comparisons.

4.5.2 *What Do Master Teachers Hope to Develop Further*

In both the questionnaire and interview, these master teachers were asked what they hoped to further improve as a master teacher.

In the questionnaire, each teacher was requested to list at most three aspects they hoped for further improvement. As a result, 18 teachers listed three desired aspects and 8 listed two. Those responses could be summarized into the following five areas:

- To establish their own distinctive classroom teaching style/model/viewpoint;
- To further improve their knowledge in wider areas such as philosophy, subject knowledge, psychology, etc.;
- To study further, and in a more systematic way, to deepen their understanding of education theories. Relevant to this aspect, some master teachers asked for

high-end advanced studies; in particular, they hope to have training programs specifically designed for master teachers.
- To set up research teams including young teachers;
- To have more communications with colleagues locally, nationally and abroad.

In the interviews, T1 expressed his strong desire to have chance to take a three-year university program, as he felt regretful that he did not have a chance to study in a Normal (i.e. teacher education) university and he hopes that he could have opportunities to do so with his rich experience. T2 responded that it is important to plan one's professional development based on what his/her students need, and what he felt lacking most in mathematics classrooms was the guidance by a sense of worth. He believed that without establishing a good sense of worth, it cannot bring children a happy life no matter how much knowledge and skill they have learnt and how many abilities they have developed. T3 hoped to "keep (his) profession but go beyond profession", that is, to further expand his perspectives and establish a wider and interdisciplinary knowledge base. Both T4 and T5 expressed their desire to learn more about students and children's psychology (T5 also hope to learn more about brain science), while T6 hoped to seek further improvement in relation to the role of master teacher's working studio as well as the publication of her own articles and monographs. It is evident that these master teachers have diverse, but clear visions about what they hoped to seek further in their own professional development.

4.5.3 *Advice for Novice Teachers*

In the study, we also invited the master teachers to provide at most three advices to novice teachers for professional development. The result was somehow beyond our expectation. As many as 82 pieces of advice were noted down in the questionnaire from various perspectives. Several keywords can be summarized from these teachers' responses, including persistence, dedication, determination, earnest, entreprenant, diligence, endurance, enthusiasm and passion, being creative and fearless, individuation, observant, cooperation, extensive reading, being planned and organized, good at seizing opportunities, and strong research ability.

The master teachers interviewed further illustrated their advices. T1 highlighted the importance of professional care for students and teaching as a profession from one's heart. He believes that teachers should think of how to make students like their mathematics classroom every day. T2 stressed that teachers should give true professional care to students, as it could produce endless power. T3 suggested that a teacher should view all experiences and difficulties as opportunities for learning and development, and he/she must read many books, and learn from different books. T4 emphasized that a teacher must devote himself/herself to teaching wholeheartedly and be humble to learn unknowns. Both T5 and T6 also emphasized the importance of loving students heartily, and T5 further suggested that a teacher should always actively identify problems and look for solutions to solve them in their career, while T6 further remarked about the role of writing, the significance of having friends who shared same ambitions and objectives, the importance of having good personality and mentality, and the belief of life-long learning. Overall, we think these advices are highly relevant and valuable.

5. Summary and Conclusions

As mentioned earlier, "master teachers" as a special group of teachers have received increasing attention in international education community over the last decade or so. The purpose of this study was mainly to investigate the reasons behind the success of master teachers in their acclaimed teaching career in the Chinese mainland, and by doing so, we also hope to shed light on relevant issues concerning teachers' professional development. The data were collected from 31 mathematics master teachers in four provinces and three municipalities through questionnaires and interviews. It should also be noted that almost all of the participants are master teachers at the primary school level.

From the results discussed earlier, we can see that, although the participating teachers came from different regions and have different background, their views on teachers' professional development were generally consistent with only a few significant differences found between teachers of different genders and different age groups. Overall, from the study, we can obtain the following main conclusions.

Firstly, a most important finding of the study is that, compared to external factors including both school and beyond-the-school factors, teachers' internal or personal factors are the most important ones to the master teachers' professional growth. In particular, wholehearted dedication to education, one's inner quality, and true professional care to students appear to be the most important factors. The result suggests the importance of recruiting people of high internal quality (such as having genuine passion for education) into teaching forces.

Secondly, although these master teachers in general appreciated greatly school support for their professional development, it was found that school mentoring programs did not receive much attention from them. The result is somehow surprising and indicates a need for us to rethink the issues concerning mentoring programs.

Thirdly, it is interesting but can be easily understood that, regarding beyond-the-school factors, the master teachers perceived family's understanding and support as the most important factor to their professional growth. Other factors such as societal respect for outstanding teachers and governmental policies are also viewed positively, but to a lesser degree.

Fourthly, concerning general teachers' professional development, the master teachers emphasized the importance of teachers' own sustained efforts and their own quality and professional background, while the role of getting help from colleagues was considered much less significant.

Fifthly, from their own successful experience, the master teachers suggested that professional dedication and good teaching ability are two most important requisites for them to be master teachers. On the other hand, they valued less about the ability in dealing with interpersonal relationships. Consistently, they believe that self-learning and reflection were the most important pathways for teachers to seek professional development, while various formal training and short-term training was considered to be less effective. The result is consistent with what many other studies have found on teachers' professional development (e.g., see Fan, 2014).

Finally, while these master teachers think that current teacher training programs are often not so effective and hence need change and improvement, they also expressed a strong desire to receive advanced or so-called "high-end" teacher training that is specifically designed to cater to their needs in further professional development, which is now largely not available, an issue meriting more attention for policy makers and teacher educators, and researchers.

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Chapter 17

Chinese Teachers' Mathematics Beliefs in the Context of Curriculum Reform

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This chapter reports case studies of three Chinese teachers' mathematics beliefs, espoused as well as enacted, in the context of a constructivism-oriented curriculum reform. Ernest's and Kuhs and Ball's theoretical frameworks of mathematics beliefs were used to guide the design of the instruments and subsequent characterization of the teachers' mathematics beliefs. Semi-structured interview and videotaped classroom observation were used to collect the data. The interview transcripts were analysed through content analysis, and the classroom videos were examined according to a scheme covering three dimensions: mathematical tasks, learning environment and classroom discourse. Based on the data analysis, it was found that among the three teachers, two teachers' mathematics beliefs were quite traditional, but the third teacher's beliefs were reform-oriented or constructivist. Discussions of findings and implications are presented at the end of this chapter.

Keywords: Chinese curriculum reform, Chinese mathematics teachers, teachers' mathematics belief

1. Introduction

Teachers' beliefs play a crucial role in their classroom practices, and thus affect their students' learning processes as well as outcomes (Beswick, 2007; Thompson, 1992). In the past few decades, teachers' beliefs have gained much attention from many researchers and educators. However, in the broad research literature, empirical studies on Chinese teachers' mathematics beliefs are limited. Most of the existing studies (e.g. Wong, 2002; Wong, Lam, Wong, Ma, & Han, 2002) are confined to analysis of teachers' espoused beliefs, with their enacted beliefs largely neglected. Moreover, since the constructivism-oriented curriculum reform was implemented in China in 2001, only a few studies have examined teachers' mathematics beliefs, typically espoused beliefs. In view of these, a more comprehensive investigation into Chinese teacher' mathematics beliefs is deemed essential and significant. The authors believe that such investigation would help answer questions like how Chinese teach mathematics, and hopefully would provide valuable information for the reform advocates as well. The two major research questions for this study can be described as the following:

1) What are the Chinese teachers' mathematics beliefs in the context of constructivism-oriented curriculum reform?
2) What are the implications for teacher educators and reform advocates in China?

2. Literature Review

2.1 *Mathematics Beliefs*

The term 'beliefs' and 'mathematics beliefs' have been defined in various ways by researchers (Leder & Forgasz, 2002; Pajares, 1992). This study adopts Raymond's (1997) definition of 'mathematics beliefs' as "personal judgments about mathematics formulated from experiences in mathematics, including beliefs about the nature of mathematics, learning mathematics, and teaching mathematics" (p. 552). Therefore, teacher's mathematics beliefs, including beliefs about the nature of mathematics, learning mathematics, and teaching mathematics, are his or her personal judgments about mathematics formulated from experiences in mathematics. It is noteworthy that the three dimensions contained in this definition have been generally regarded as the core of belief systems of mathematics teachers and thus received great attention (Ernest, 1989a; Thompson, 1992).

Mathematics beliefs have been categorized in different ways (Ernest, 1989a; Kuhs & Ball, 1986; Lerman, 1990). Among these schemes, the work of Ernest (1989a) and Kuhs and Ball (1986) are particularly influential. Ernest (1989a) distinguished three conceptions of the nature of mathematics.

> First of all, there is the instrumentalist view that mathematics is an accumulation of facts, rules and skills to be used in the pursuance of some external end. Thus mathematics is a set of unrelated but utilitarian rules and facts. Secondly, there is the Platonist view of mathematics as a static but unified body of certain knowledge. Mathematics is discovered, not created. Thirdly, there is the problem-solving view of mathematics as a dynamic, continually expanding field of human creation and invention, a cultural product. Mathematics is a process of inquiry and coming to know, not a finished product, for its results remain open to revision (p. 250).

The problem-solving view is also referred to as social constructivist view by Ernest (1998) elsewhere. Thompson (1992) points out that Ernest's Platonist and problem-solving view parallel Lerman's (1990) absolutist and fallibilist view respectively. Roulet (1998) agrees with Thompson and further argues that the Platonist and instrumentalist views are within the domain of absolutism whereas the problem-solving view is in accordance with fallibilist position.

In terms of mathematics learning, Ernest (1989a) proposed three views, i.e. learning as reception of knowledge, as mastery of skills and as active (social) construction of understanding, which correspond to the Platonist, instrumentalist and problem-solving (social constructivist) view respectively.

With regard to how mathematics should be taught, Kuhs and Ball (1986) identified the following four dominant and distinctive views:

1) Learner-focused: mathematics teaching that focuses on the learner's personal construction of mathematical knowledge;
2) Content-focused with an emphasis on conceptual understanding: mathematics teaching that is driven by the content itself but emphasizes conceptual understanding;
3) Content-focused with an emphasis on performance: mathematics teaching that emphasizes student performance and mastery of mathematical rules and procedures; and
4) Classroom focused: mathematics teaching based on knowledge about effective classrooms (p. 2).

Among these, the first three models have been allied to Ernest's (1989a) three conceptions of mathematics, i.e. the problem-solving (social constructivist), the Platonist and the instrumentalist views respectively whilst the fourth assumes that the content to be covered is outside the control of the teacher whose only task is to present the material in ways found to be effective by process-product research studies (Thompson, 1992).

It is important to note that Ernest's problem-solving (social constructivist) view and Kuhs and Ball's learner-focused view are the ones which have been strongly advocated by the current education reformers and researchers, as opposed to the Platonist, instrumentalist and content-focused (or teacher-centred) views (Chen, 2010). In view of the wide use of the frameworks by Ernest (1989a), Kuhs and Ball (1986), as well as the close relationships between them and other frameworks (e.g. Lerman, 1990), this study used their frameworks to characterize the teachers' mathematics beliefs.

2.2 *Relationship between Beliefs and Practices*

Based on an extensive review of studies on teachers' beliefs, Thompson (1992) concludes that the relationship between beliefs and practice is not a simple cause-and-effect one, but dialectical in nature. Many models have been proposed to examine the relationship between teachers' beliefs and practice (e.g. Ernest, 1989a; Raymond, 1997), these models have illustrated that this relationship is mediated by numerous factors.

Studies on the relationship between beliefs and instructional behaviours have continuously reported different degrees of consistency (Handal, 2003; Thompson, 1992). On the one hand, high degrees of consistency have been found by some researchers (e.g. Stipek, Givvin, Salmon, & MacGyvers, 2001). On the other hand, the inconsistencies between teachers' beliefs and their practices have been found by more researchers (e.g. Raymond, 1997).

Researchers have explored the possible reasons for the inconsistencies between teachers' professed beliefs and observed practice, such as the social context of teaching situation, teachers' level of thought processes and reflection (Ernest, 1989b), mismatches between the contexts in which teachers provide beliefs data and the contexts in which their practices are observed or described (Beswick, 2005, 2007), teachers' lack of skills and knowledge necessary to implement espoused teaching ideals (Thompson, 1992), etc. Besides, it is suggested that in order to accurately characterize a teacher's beliefs, researchers should examine the teacher's both verbal and observational data (Thompson, 1992).

2.3 *Studies on Chinese Teachers' Mathematics Beliefs*

Since 1990s, Chinese teachers' mathematics beliefs have aroused growing attention. Interestingly, most of research findings are revealed in cross-cultural or cross-country comparative studies where Chinese teachers are compared with teachers in other contexts (e.g. Leung, 1992). One major concern of these researchers is the cultural differences in teachers' beliefs.

Leung (1992) adopted a quantitative approach to investigate teachers' attitudes towards mathematics and mathematics teaching and learning in Beijing, Hong Kong and London. It was indicated that the teachers in Beijing tended to treat mathematics as a rule-oriented and fixed discipline, while the teachers in London regarded mathematics as more heuristic and changing. The teachers in Hong Kong fell between the two extremes. In addition, the teachers in Hong Kong and Beijing regarded mathematics as a tool for training the mind while the teachers in London treated mathematics as a tool for communication and application.

Starting from 1996, Wong and his colleagues (Wong, 2002; Wong et al., 2002) carried out a series of studies on conceptions of mathematics, quantitatively as well as qualitatively, involving students and teachers at primary and secondary levels in different places. According to Wong (2002), in a quantitative study, 369 primary and 275 secondary teachers in Hong Kong, 156 primary teachers in Taiwan and 105 in Changchun, a city in Northeast China were surveyed by a questionnaire developed by Australian researchers based on two factors—'child-centeredness' and 'transmission' (Perry, Tracey, & Howard, 1998). It was found that Taiwan teachers were least "transmission-oriented" and most "student-centered", in contrast, Hong Kong secondary teachers were most "transmission-oriented" and Changchun teachers were least "student-centered".

On the other hand, in a qualitative study reported by Wong et al. (2002), data analysis on the interviews with 15 secondary teachers in Changchun revealed the following conceptions of mathematics: 1) Mathematics is a subject of numbers and shapes; 2) Mathematics is closely related to computation; 3) Mathematics is precise and rigorous; 4) Mathematics is beautiful; 5) Mathematics is applicable; 6) Mathematics involves thinking. It was further concluded that the secondary school teachers in the Chinese mainland held Ernest's Platonist view of mathematics.

More recently, a series of studies (e.g. Cai, 2007; Wang & Cai, 2007) investigated the views of effective mathematics teaching and learning held by teachers in the Chinese mainland, Hong Kong, Australia and U.S. through semi-structured interviews. In particular, based on the interviews with 9 experienced mathematics teachers (ranging from 19 to 30 years of teaching experience), Wang and Cai (2007) found that in general, Chinese (mainland) teachers tended to view mathematics as an abstract and coherent knowledge system that is refined from real life mathematical problems. It was concluded that Chinese (mainland) teachers' beliefs about the nature of mathematics were close to the Platonist view, which is in accord with the argument by previous researchers (Wong et al., 2002). Moreover, it was found that consistent with their view on the nature of mathematics, Chinese (mainland) teachers saw constructing a coherent knowledge system as the key to mathematics understanding. They emphasized that both learning and teaching should help students to understand abstract mathematics knowledge in a rational and coherent way. They were also found to believe that practice and memorizing are indispensable for mathematics learning. On the other hand, it was found that Chinese (mainland) teachers realized the importance of "student-centered" teaching, but they often considered the general needs of students instead of particular needs of individual students, due to the large class size and broad coverage of content required by the national curriculum.

To sum up, the available empirical studies have consistently provided evidence that mathematics teachers in the Chinese mainland, even in the climate of curriculum reforms, often hold traditional mathematics beliefs. These evidences seem to imply that the reform has not brought about changes in teachers' beliefs. The review of these studies provides a backdrop for the present investigation into mathematics beliefs of teachers in the Chinese mainland. Through comparing the findings of these studies with those to be revealed in the present study, the extent of change in teachers' mathematics beliefs during the implementation of the reform-oriented curriculum can be measured to a certain degree.

Nevertheless, it must be pointed out that the existing studies can only provide a partial picture of Chinese teachers' mathematics beliefs, because in these studies, the analysis of teachers' beliefs was basically limited to their self-report in questionnaire survey and/or interview. Beliefs elicited in such way are defined as espoused beliefs, and they are differentiated from enacted beliefs which are inferred based on observation of teachers' classroom practice (Lerman, 2002). The existing studies on Chinese teachers' mathematics beliefs have been largely concerned with teachers' espoused beliefs, and teachers' enacted beliefs have not been paid due attention. It is cautioned that in order to gain a full understanding of teachers' beliefs, both what they say (espoused beliefs) and what they do (enacted beliefs) should be examined (Thompson, 1992). Therefore, a more comprehensive understanding of Chinese teachers' mathematics beliefs deserves to be achieved, particularly in the context of curriculum reform given that teachers may easily talk about the rhetoric of reform, but meanwhile think and teach in a traditional way (Cohen & Ball, 1990). Besides, it is noteworthy that some researchers have used Ernest's (1989a) three views of the nature of mathematics as a framework to characterize the (espoused) beliefs of Chinese mathematics teachers, which provided an important reference to the present study.

3. Methodology

3.1 *Participants*

Three teachers who taught junior secondary mathematics (Grade 7 to 9) in Chongqing, the Chinese mainland were studied as cases, including Anna, Simon and Mandy (pseudonyms). Their mathematics beliefs were elicited through videotaped classroom observation (enacted beliefs) and semi-structured interview (espoused beliefs). The three case teachers were selected based on three criteria: (1) use of the reform-oriented mathematics curriculum; (2) diversity of demographic characteristics, such as gender, teaching experience, etc.; (3) their willingness to participate in this research. The details about the three case teachers are described later.

3.2 *Data Collection*

Semi-structured interview and videotaped classroom observation were used to collect the data regarding teachers' mathematics beliefs. An interview schedule was framed based on relevant literature and *National Mathematics Curriculum Standards at the Compulsory Education Level (Draft for Consultation)* (National Ministry of Education, 2001). The interview questions were mainly divided into two categories: (1) mathematics beliefs, aimed at eliciting the teachers' mathematics beliefs, including beliefs about the nature of mathematics, beliefs about learning mathematics, and beliefs about teaching mathematics; and (2) teachers' opinions about the reform-oriented mathematics curriculum and its implementation, aimed at inferring the extent to which the teachers were receptive to the reform ideas and suggestions. The interview schedule was piloted for improvement before its use in the formal data collection.

During the formal data collection, the three case teachers were required to arrange their schedule so that videotaped classroom observation and interview can be conducted on the same day. It was basically up to the teachers themselves to select the lesson topic, time and venue. Nevertheless, two basic requirements were conveyed to them. Firstly, the lesson(s) to be observed should deal with a completely new mathematical topic, instead of being a review or exercise-oriented. It was believed that through analysing how a teacher developed teaching around a new topic, his or her beliefs were easier to be captured. Furthermore, the interviews need to be carried out as soon as possible after the classroom observation. It was expected that both the teacher and the researcher could have a vivid memory of the videotaped lesson so that they could make points based on the events just happened in the classroom teaching, if necessary. All the interviews were audiotaped with the permission of the teachers.

3.3 *Data Analysis*

All the audiotaped data from the interviews were transcribed verbatim. The three teachers' espoused mathematics beliefs were inferred from the transcripts and then characterized in accordance with the frameworks by Ernest (1989a), Kuhs and Ball (1986). All the videos were also transcribed verbatim. During the process of data analysis, the lesson transcripts composed the major data, although the videos were also referred back to from time to time to ensure that the description represented the reality as closely as possible. The teaching suggestions in *National Mathematics Curriculum Standards at the Compulsory Education Level (Draft for Consultation)* (National Ministry of Education, 2001) were used as important reference for data analysis, but they were too general. In order to analyse the teachers' behaviours and then deduce their enacted mathematics beliefs, a more operable analytical framework was developed based on relevant research literature (Artzt & Armour-Thomas, 2002; Grant & Kline, 2001; Hiebert et al., 1997; Stein & Smith, 1998). The analytical framework consisted of three key dimensions: (1) mathematical tasks, (2) learning environment, and (3) classroom discourse. Each of the three dimensions included several aspects respectively. Overall, if a teacher's mathematics beliefs are highly consistent with the reform ideas, then his or her mathematics classroom could be characterized by the framework in Table 1.

Table 1. Lesson dimensions and dimension indicators of a reform-oriented classroom

Dimension	Indicator	Description of Dimension Indicators
Mathematical Tasks	Task features	*Contexts*: The tasks have real-life contexts. *Solution strategies*: The tasks are solved in multiple ways. *Representations*: The tasks include the use of multiple representations, e.g. words, diagrams, manipulatives, computers, or calculators. *Communication*: The tasks enable students to produce mathematical explanations or justifications. *Collaboration*: The tasks enable students work in a collaborative way.
	Cognitive demand	The tasks require the students to engage in high-level cognitive process—either the active "doing of mathematics" or the use of procedures with connection to concepts, meaning, or understanding.
Learning Environment	Social and intellectual climate	The teacher establishes and maintains a positive rapport with and among students by showing respect for and valuing students' ideas and ways of thinking. The teacher does not play a role of authority of knowledge.
	Modes of instruction	The teacher uses instructional strategies that encourage and support student involvement and facilitate goal attainment. He or she provides time necessary for students to express themselves and explore mathematical ideas and problems.
Classroom Discourse	Teacher-student interactions	The teacher communicates with students in a nonjudgmental manner and encourages the participation of each student. He or she requires students to give full explanations and justifications or demonstrations orally and/or in writing. He or she listens carefully to students' ideas and makes appropriate decisions regarding when to offer information, provide clarification, model, lead, and let students grapple with difficulties.
	Student-student interactions	The teacher encourages students to listen to, respond to, and question each other so that they can evaluate and, if necessary, discard or revise ideas and take full responsibility for arriving at mathematical conjectures and/or conclusions.
	Questioning	The teacher poses a variety of levels of questions that elicit, engage, and challenge students' thinking, and the students give a variety of types of responses.

Note: Adapted from the framework by Artzt & Armour-Thomas (2002).

Specifically, in analyzing the teaching videos, both quantitative (e.g. number of kinds of instructional strategies) and qualitative (e.g. description of teacher's role) measures were used in the hope that the analysis could best reflect the classroom reality in a meaningful way.

4. Chinese Teachers' Mathematics Beliefs

This section presents case studies of the three Chongqing teachers. Due to limit of page space and great similarities between two of the three teachers, their stories are described together. This section is composed of three parts: (1) background of the three teachers, their schools and students; (2) their espoused mathematics beliefs inferred from the interview transcripts; (3) their enacted mathematics beliefs inferred from classroom practices.

4.1 *Background*

Anna obtained a Bachelor degree in mathematics from a normal university in Chongqing more than two decades ago. After graduation, she taught mathematics in a middle school attached to a factory for more than ten years during which she attended a graduate-level advanced program in the same normal university. Seven years ago, she joined the current school through an open-recruitment examination. It was her sixth year of implementing the new mathematics curriculum when she joined this research. She had attended some in-service training programs regarding the new curriculum. Anna's school was coeducational. It was one of most prestigious secondary schools in Chongqing due to its success in producing high achievers in a variety of exams and competitions. Students there generally had relatively good family background. Particularly, Anna told the researcher that her students were quite active and cooperative in classroom teaching.

Simon also graduated from a normal university in Chongqing with a Bachelor degree in mathematics. He had been teaching mathematics for six years. It was his fourth year of implementing the new mathematics curriculum when he was involved in this research. He had attended a few in-service training programs regarding the new curriculum. Simon's school was coeducational. It was one of the key schools in Chongqing. The students there had diverse levels of achievement and family background. According to Simon, his students generally took studies seriously.

Mandy graduated from an education college in Chongqing with a Bachelor degree in mathematics education. She had been teaching mathematics in a junior secondary school for sixteen years, and assumed the Head of Mathematics Department for three years. It was her seventh year of implementing the new mathematics curriculum when she took part in this research. She had attended various in-service training programs regarding the new curriculum. Mandy's school was coeducational. It was one of the key schools in Chongqing, and the students there had a variety of levels of achievement and family background. She told the researcher that generally, her students were attentive and interested in mathematics lessons.

4.2 *Espoused Mathematics Beliefs*

4.2.1 *Beliefs about the Nature of Mathematics*

All the three case teachers held that mathematics is a fixed body of certain knowledge. For example, Anna said:

> I like mathematics, particularly its rigorous thinking and reasoning. Furthermore, I think that mathematics is not subject to people's opinions…. It is good that in mathematics, one is one, two is two.

Particularly, concerning the 'absolutely true' property of mathematical knowledge, Simon and Mandy expressed similar opinions:

> Mathematics is not necessarily absolute truth. However, it is impossible to disprove something well-accepted. (Simon)

> The content is basically true. It seems unlikely to disprove what we accept as true today, because many people have been researching mathematics for long. (Mandy)

In addition, all the three teachers recognized that *mathematical knowledge is usually interrelated* and that *mathematics is closely related to real life*. Finally, they all agreed that *mathematical problems often can be solved through multiple approaches.*

4.2.2 *Beliefs about Learning Mathematics*

Anna and Simon both emphasized that students should learn mathematics through listening to teacher's teaching and doing enough exercises. For example, Anna said:

> I think that classroom teaching is most important to students' mathematics learning. Take today's lesson as an example, I repeatedly explained the definition of linear function, because they cannot master if you only teach once. If the students can grasp the main points and do enough exercises, then the goals of this lesson are achieved.

Anna's words also suggested that teacher should play a role of explainer in classroom teaching.

Unlike Anna and Simon, Mandy seemed to believe that students should be active constructors rather than passive recipients of knowledge. She said:

> There are many ways of learning mathematics. Listening attentively to teacher is not necessarily the most effective one. I think that a more effective way is to let students explore and conclude by themselves.

With respect to learning approaches, all the three teachers claimed that *mathematics cannot be learnt by rote*. Particularly, both Anna and Simon stressed the importance of memorization and imitation to mathematics learning. For example, Simon said:

> Memorization and imitation are important to mathematics learning, I require my students to memorize some formulae and theorems to solve problems. Understanding and memorization are related to each other, teacher should teach the students diverse ways of memorizing things more rapidly, for instance, associative memorization….I also demand my students to imitate some solutions used by the textbooks or myself, because they need to cope with the exams. Many formats or solutions are not introduced in the textbooks, as a result, teacher should teach the students so that they can write correctly. Otherwise, they will lose points in exams.

Simon pointed out that teaching the students how to memorize things should be the teacher's responsibility. It was also suggested that his teaching was heavily influenced by the exams. In order to cope with the exams, his students had to imitate what he did in the classroom.

On the other hand, Mandy argued that in learning mathematics, understanding is more important than memorization. Agreeing with

Anna and Simon on the importance of imitation, Mandy considered that imitation is the basis of learning, and understanding may follow imitation:

> Imitation is important! At the beginning, learning is actually a kind of imitation, especially in the primary school. When the teacher gives you a formula, you imitate it first and then understand why. In fact, when students learn something new, whether I require them to imitate or not, the first thing is to imitate. Students cannot understand, especially when they are in lower grades, so I often write the procedures [of solving problems] clearly on the blackboard, then they do exercises step by step according to my format. In doing so, they can gradually understand why they need to do in that way.

Furthermore, all three teachers expressed their views about the reform-oriented approaches, i.e. independent inquiry, mathematical communication and collaborative learning. Generally, they all acknowledged that these approaches are important to learning mathematics. However, Anna and Simon considered them impractical, and even admitted that they rarely adopted these approaches in their classrooms. For example, Simon said:

> Independent inquiry is important, but it is only suitable for able students and difficult topics. … Mathematical communication among students is certainly helpful to their mathematics learning, but I don't dare to say that everyone can benefit from the discussions, because all students are not at the same level after all. … The idea about collaborative learning is reasonable, but it is unrealistic. I think it can be only performed at public lesson.

In contrast, Mandy showed more acceptances of these approaches and claimed that independent inquiry and mathematical communication were common in her classroom. She stated:

> Independent inquiry is important. The students can make some discoveries when they explore mathematics themselves.… I often ask my students to make some inquiries. I think they are interested to do this.… Mathematical communication is important. For example, when I teach lessons, especially when I revisit their exam papers, I always let the students explain, that is actually a kind of communication. Sometimes when students explain a problem, they are more likely to think about the problem from their peer's perspective, which is quite different from the teacher's. They may get the right answer through communication, discussion even debate. I would like to allow them to communicate, because it is very helpful to their mathematics learning. I think it is very necessary to enable students to learn how to communicate.… Collaborative learning enables students' joint involvement. It is important

> because anyone in our society will have to cooperate with others in future. [Furthermore] It is helpful to mathematics learning itself because through cooperation, you can realize the difference between you and others in terms of thinking and consequently improve your thinking.

In spite of her recognition of the merits of collaborative learning, Mandy did admit that she seldom allowed students to collaborate in her teaching.

> [Collaborative learning] is not common practice [in my classroom]. I don't really know when cooperation is needed. In some topics, like probability and statistics, many activities can be carried out. … but I don't know about collaborative learning, Is communication a kind of collaborative learning?

Her statements suggested that she did not know what collaborative learning is, and she lacked the necessary knowledge and skills to implement it in her classroom.

4.2.3 *Beliefs about Teaching Mathematics*

Both Anna and Simon seemed to believe that mathematics should be taught by direct instruction, and teacher should play a role of explainer in classroom teaching. For example, Simon said:

> [To me], the most important thing in a lesson is to enable the students to clearly know what the essential points and difficult points of this lesson are, and understand these points through [my] explanation.

His statements showed that identifying the essential points and difficult points of a lesson, and explaining these points to the students were his duties as a teacher, whereas his students tended to be the listeners of his lecture. In a word, Simon was unlikely to believe that students should be active knowledge constructors.

Unlike Anna and Simon, Mandy argued that *teacher should not teach mathematics directly*. She further talked about the role of teacher in mathematics instruction:

> Students don't like a teacher who tells them all knowledge in the textbook. Teacher only plays a role of guiding the discussion. As a matter of fact, teacher is the one who asks questions. When new problem emerges, teacher points it out and guides students to think in certain direction, just like scaffolding. Whatever mathematics I teach, I won't tell the answer directly. Instead, I will let the students to discuss and argue if they can present powerful proof.

According to her statements, in the teaching process, the teacher should play the role of guide instead of transmitter of knowledge, and students should be active constructors rather than passive recipients of knowledge.

In particular, Mandy stressed two aspects in the instruction. Firstly, she pointed out that the teacher should teach new knowledge based on what students already know. Secondly, consistent with her view that *mathematical knowledge is usually interrelated,* she suggested that the teacher should make the (implicit) inter-relationships of knowledge explicit to the students.

4.2.4 *Summary*

Anna and Simon seemed to hold quite similar espoused beliefs about the nature of mathematics, learning mathematics and teaching mathematics. Their beliefs can be summarized as follows: 1) Mathematics is a fixed, coherent body of certain knowledge; 2) Mathematics is closely related to real life; 3) There are often multiple ways to solve mathematical problems; 4) Listening attentively to the teacher and doing enough exercises are most important for students to learn mathematics; 5) Mathematics cannot be learnt by rote, but memorization and imitation are important approaches to learning mathematics; 6) The reform-oriented learning approaches, including independent inquiry, mathematical communication and collaborative learning are important but impractical; 7) Mathematics should be taught by direct instruction. In teaching, Teacher should play a role of explainer, and students should be recipients rather than constructors of knowledge.

Generally, Anna's and Simon's espoused beliefs about the nature of mathematics seemed to be mostly close to the Platonist view. Their espoused beliefs about learning mathematics seemed to be close to the reception view of learning, and espoused beliefs about teaching mathematics seemed to incline towards the teacher-centred view of teaching.

As compared to Anna and Simon, Mandy's espoused beliefs about the nature of mathematics seemed to be quite similar to theirs, and can be summarized as the following: 1) Mathematical knowledge is basically fixed, infallible, and interrelated; 2) Mathematics is closely related to real life; 3) Mathematical problem solving allows for multiple approaches.

Nevertheless, Mandy's espoused beliefs about learning mathematics and teaching mathematics were markedly different from Anna's and Simon's, despite a few similarities. Her beliefs could be summarized as follows: 1) Listening attentively to the teacher is not necessarily the most effective way of learning mathematics; 2) Mathematics cannot be learnt by rote, understanding is more important than memorization. Imitation is the basis of learning, and understanding may follow imitation; 3) The reform-oriented learning approaches are important and practical; 4) Mathematics should not be taught by direct instruction. In the teaching process, teacher should mainly play a role of guide rather than knowledge transmitter, and students should be active constructors rather than passive recipients of knowledge; 5) Teacher should teach new knowledge based on students' prior knowledge and experience; 6) Teacher should make the (implicit) interrelationships among the knowledge explicit to the students.

Overall, Mandy's espoused beliefs about the nature of mathematics seemed to be relatively similar to the Platonist view. Her espoused beliefs about learning mathematics seemed to be greatly consistent with the social constructivist view of learning, and her beliefs about teaching mathematics seemed to be very consistent with the learner-focused view of teaching.

4.3 *Enacted Mathematics Beliefs*

Anna's and Simon's classes were both at grade 8, and the class sizes were 56 and 53 respectively. Anna's lesson was about linear function and direct proportional function, and Simon's lesson was about squares. Mandy's class was at grade 9, and the class size was 36. The topic of her lesson was the positional relationships between two circles. All the three lessons lasted about 40 minutes, and all the time were used for teaching.

4.3.1 *Mathematical Tasks*

The mathematical tasks used in the three teachers' lessons are summarized in Tables 2 to 4.

Table 2. Mathematical tasks in Anna's lesson

<u>Task A</u> Determine if the following algebraic expressions are linear function. In linear functions, find k and b . (1) $y \equiv 2x+3$ (2) $y \equiv -x-1$ (3) $y \equiv 3x$ (4) $y \equiv kx$ (5) $y \equiv -\frac{3}{2x}$ (6) $y \equiv x^2-(x-1)(x+3)$ (7) $y \equiv \sqrt{5}-\sqrt{2}x$ (8) $y \equiv \frac{5}{x}+4$ (9) $y = kx+b$, k, b is constant (10) $y \equiv 3\sqrt{x}+2$.
<u>Task B</u> Find the unknowns. (1) Given that $y \equiv (3m+2)x-m+1$ is direct proportional function, find m . (2) Given that $y = (k-2)x+k^2-4$ is direct proportional function, find k . (3) Given that $y \equiv (k-4)x^{\lvert k \rvert -3}+2$ is linear function, find k . (4) If $y = (m+2)x^{m^2-3}+m-2$ is linear function, find m . (5) If $y = 2x^{k-2}+k^2-9$ is direct proportional function, find k . (6) If $y = (m+3)x^{2m+1}+4x-5$ is linear function, find m .

Table 3. Mathematical tasks in Simon's lesson

<u>Task A</u> Fill in the following blanks: (1) Rectangle that ___________ is square. (2) Rhombus that ___________ is square. (3) Parallelogram that ________is square.
<u>Task B</u> (1) Summarize the properties of square in terms of side, angle and diagonal; (2) Describe these properties in geometric language.

Task C

(1) Fill in the table with 'Yes' to indicate whether parallelogram, rectangle, rhombus, and square have each of the properties respectively.

	Parallelogram	Rectangle	Rhombus	Square
Both pairs of opposite sides are parallel and equal in length.				
The four sides are equal in length.				
Opposite angles are congruent				
The four angles are all right angle.				
The diagonals bisect each other.				
The diagonals are Perpendicular.				
The diagonals are equal in length.				
The diagonals bisect a pair of opposite angle.				

(2) Based on table above, draw a diagram to represent the relationships among parallelogram, rectangle, rhombus, and square.

Task D

Prove that the two diagonals cut a square into four congruent isosceles right triangles.

Task E

Quadrilateral $ABCD$ is square, and the two diagonals intersect at point O.

(1) Find $\angle AOB, \angle OAB$.

(2) If $AC = 4$, then the length of side is ______, the area is_______.

(3) If the area of the square is 64 cm^2, then the distance from point O to one side of the square is _______ cm.

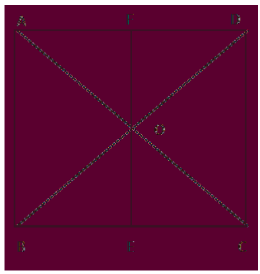

Table 4. Mathematical tasks in Mandy's lesson

Task A

Use physical objects with circle shape to explore the possible relationships between two circles, and draw the conclusion based on demonstration and discussion.

Task B

Construct the graphs representing the five positional relationships between two circles.

Task C Explore the methods of determining the positional relationship between two circles.
Task D (Textbook) Given that the radius of two circles is 3 and 4 respectively, determine the positional relationship between the two circles according to the distance between the two centers of circles O_1O_2: (1) O_1O_2=8; (2) O_1O_2=7; (3) O_1O_2=5; (4) O_1O_2=1; (5) O_1O_2=0.5; (6) O_1O_2=0.
Task E (Workbook) Given that the radius of two circles is r_1 and r_2 respectively and the distance between the two centers of circles is d, fill in the table below.

Positional relationship between Two Circles	Formula for Determining the Relationship
Separation	$d > r_1 + r_2$
Externally tangent	
Intersection	
Internally tangent	
Inclusion	

(1) *Task features*

Context

All the mathematical tasks had no real-life contexts, and they were situated completely in the abstract world of mathematics.

Solution strategies

In Anna's and Simon's class, all the tasks were solved by using single-solution strategy. For example, both Task A and B in Anna's lessons were solved by applying the rule 'exponent should be 1 and coefficient should not be 0', or 'exponent should be 1, coefficient should not be 0, and constant is 0'. Differently, in Mandy's class, two of the five tasks, i.e. Task A and C involved use of multiple solution strategies. There was no obvious solution to the two tasks. The students were allowed to use a variety of solution strategies to complete the two tasks. The remaining three tasks, i.e. Task B, D and E were solved by using single-solution strategy.

Representation

In Anna's lesson, only single representation, i.e. mathematical symbols was used in the two tasks. In Simon's lesson, only single representation, i.e. words was used for Task A while multiple representations, including mathematical symbols and diagrams were used for other four tasks, i.e. Task B, C, D and E. In Mandy's lesson, multiple representations were used for all the five tasks. Task A required the students to make use of their physical objects with circle shape (manipulatives) to explore the possible positional relationships between two circles, and then use the circle models prepared by the teacher to demonstrate and discuss their findings. Task B, C, D and E involved the use of mathematical symbols, diagrams and words.

Communication

In Anna's and Simon's lesson, all the tasks did not enable the students to produce any explanations or justifications. In contrast, in Mandy's lesson, four out of the five tasks, i.e. Task A, C, D and E enabled the students to produce explanations and justifications.

Collaboration

All the tasks in the three teachers' lesson did not enable the students to work in a collaborative way.

(2) *Cognitive demand*

In Anna's class, all the two tasks were procedures with connection tasks (high level). They focused the students' attention on the use of procedures for the purpose of developing deeper level of understanding of the definition of linear function and direct proportional function. Starting from the two definitions, Anna summarized two rules: 'exponent should be 1 and coefficient should not be 0' and 'exponent should be 1, coefficient should not be 0, and constant should be 0'. The two tasks required the students to apply the procedures based on their understanding of the two rules. Specifically, Task A required the students to determine if algebraic expressions are linear function according to the first rule and then identify the coefficient and constant. Task B required the students to apply the two rules to find the unknown so that algebraic expressions can be linear function or direct proportional function. The two tasks demanded certain degree of cognitive effort. In order to successfully complete the two tasks, the students had to engage with the conceptual ideas that underlie the procedures, particularly the concept 'exponent', 'coefficient' and 'constant' in function. Particularly, in Task B, the students had to make connection to their prior knowledge about equation to find the unknowns.

In Simon's class, among the five tasks, Task A and B were memorization tasks (low level) while Task C, D and E were procedures with connection tasks (high level). Task A and B only required the students to reproduce the newly learned knowledge, i.e. the three definitions and the properties of square. The other three tasks required some degree of cognitive effort. In Task C, in order to do Question (1), the students just need to recall their prior knowledge, i.e. the properties of parallelogram, rectangle, and rhombus and new knowledge, i.e. the properties of square; in order to do Question (2), the students must make use of the information in the table to identify the relationships among the four concepts: parallelogram, rectangle, rhombus, and square, and then construct the diagram based on their accurate understanding of these relationships. Task D required the students to connect their new knowledge, i.e. the properties of square with previous knowledge, i.e. method of determining congruent triangles to write out the proof. Task E required the students to apply the properties of a square to find the unknowns. The students needed to engage with the conceptual ideas that underline the procedures in order to successfully complete this Task.

In Mandy's class, all the five tasks involved high-level cognitive demand, among these, Task A, B and C were doing mathematics tasks while Task D and E were procedures with connection tasks. The five tasks were connected in a coherent way. Task A required the students to explore and make conclusion about the possible positional relationships between two circles through a variety of activities, including hands-on experience, observation, complex thinking and discussion. It required considerable cognitive effort and may involve some level of anxiety for the students due to unpredictable nature of the solution process required. The students needed to figure out the appropriate method of classifying the relationships. Task B required the students to use geometric graphs to represent the five relationships found in Task A based on their understanding of the method. Task C required the students to access relevant knowledge, i.e. algebraic expressions and then make connections between the algebraic and geometrical representations of the five positional relationships between two circles. Task D required the students to apply the algebraic expressions obtained in Task C to determine the positional relationships between two known circles so that they could better understand and master the algebraic decision method. Task E required the students to formulate the algebraic method of determining the five positional relationships between two circles. In order to complete this task successfully, the students needed to make use of the relevant knowledge, such as absolute value etc.

4.3.2 *Learning Environment*

(1) *Social and intellectual climate*

Anna established and maintained a positive rapport with students. Generally speaking, the students were very supportive and cooperative by answering her questions actively. However, students' contributions of ideas were fairly limited. Anna dominated in the classroom and acted as the authority of mathematical knowledge. She almost told the students everything about the topic. The whole lesson flowed as a chain of teacher's questions and students' short answers. When dealing with the mathematical tasks, she told the students how to analyze and solve the problems. She was also the one who judged the correctness of students' work.

Simon did not establish a positive rapport with and among the students by showing respect for and valuing students' ideas and ways of thinking. As a matter of fact, he provided few opportunities for the students to express their ideas. He seemed to press for only right answers. Besides, for some students, the learning environment did not seem to be psychologically safe. When a student failed to supply a right response, Simon allowed other students to laugh at him or her. Like Anna, Simon dominated in the classroom and acted as the authority of mathematical knowledge. He almost taught the students everything about this topic, including the definitions of square, the meanings of the definitions, the properties of square and relevant problem solving skills. He also required the students to strictly follow his instruction to describe the properties of square in mathematical language in order of side, angle and diagonal.

Mandy established and maintained a very positive rapport with and among the students by showing respect for and valuing students' ideas and ways of thinking. Generally, the students were very supportive and cooperative by actively doing the tasks, voluntarily contributing their ideas for public discussion, etc. Different from Anna's and Simon's students, in the instructional process, Mandy's students behaved like active knowledge constructors while she played a role of organizer, guide and facilitator. She organized the students to carry out independent inquiry, posed some significant questions to elicit, engage and challenge their thinking, provided them with adequate opportunities to express, communicate even debate, and facilitated their activities by offering suggestions, providing models etc. In a word, Mandy did not act as the authority of knowledge, like Anna and Simon did.

(2) *Modes of instruction*

Anna and Simon both used two kinds of instructional strategies in the class, i.e. student individual seatwork and teacher lecture/demonstration. Mandy employed five kinds of instructional strategies, i.e. student individual seatwork, teacher lecture/demonstration, student inquiry, student demonstration and discussion. The percentages of instructional time devoted to each kind of instructional strategy in the three teachers' class are shown in Figure 1.

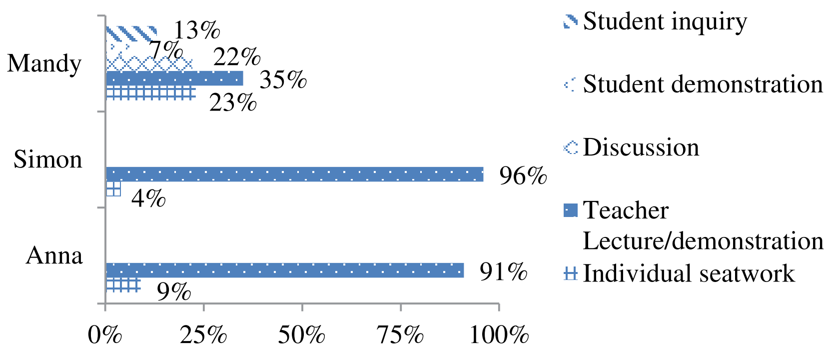

Figure 1. Percentages of instructional time devoted to different teaching strategies

As indicated by Figure 1, both Anna and Simon spent a considerable amount of the instructional time on giving lectures and demonstrations to the class. Also, a very small amount of time was provided for the students to do the tasks individually. Little time was allowed for the students to express themselves and explore the mathematical ideas. Generally, teacher-centered direct instruction was the dominant mode of instruction in Anna's and Simon's class. In contrast, Mandy spent less than half of the instructional time on giving lectures and demonstrations to the class. She provided certain amount of time for the students to express themselves and explore mathematical ideas and problems. The students took main responsibilities for their mathematics learning. Generally, teacher-centered direct instruction was not the main mode of instruction. Rather, student-centered teaching approach dominated in the classroom.

4.3.3 *Classroom Discourse*

(1) *Teacher-student interaction*

Anna and Simon communicated with the students mostly in a judgmental manner. When the students provided right answers, they tended to give positive feedback and then move on. However, when the students supplied wrong answers, they reacted in different ways. Anna did not directly point out that the answer was wrong. She seemed to encourage the students to explain or justify their thinking, but she could hardly wait to give a clear and right direction. Unlike Anna, when some student failed to provide desirable answer, Simon immediately turned to another one. The student who failed to give right response seemed to feel ashamed and sat down quickly while the classmates were laughing at him or her.

Throughout the lesson, Anna and Simon did not really require the students to give full explanations or justifications. They provided little time for the students to express their ideas. Therefore, they did not have opportunity to listen carefully to the students' ideas. Take the following episode of Simon's lesson as an example:

(The teacher was going to teach the symmetrical property of the square.)

T: How many axes of symmetry does rhombus have?
Whole class: Two!
T: The axes of symmetry of rhombus are the lines where the diagonals lie, right?
Whole class: Yes!
T: How many axes of symmetry does rectangle have?
Whole class: Two!
T: The axes of symmetry of rectangle are also the lines where the diagonals lie, right?
Whole class: No!
T: Then are the perpendicular bisectors of the sides, right?
Whole class: Yes!
T: Rhombus has two, and rectangle has two. How about square?
Whole class: Four!
T: Which four?
Whole class: The lines where the diagonals lie and the perpendicular bisectors of the sides.
T: First, is square rhombus?
Whole class: Yes!
T: So the axes of symmetry of rhombus are also the axes of symmetry of square. Second, is square rectangle?
Whole class: Yes!
T: The axes of symmetry of rectangle are the perpendicular bisectors of the sides. Therefore, if we put all the axes of symmetry of the rhombus and those of the rectangle into one figure, we can get the axes of symmetry of square. How many?

Whole class: Four!
T: Yes, two for rhombus plus two for rectangle is equal to four!
…

It is clear from above that what the students did was just supplying short answers to Simon's simple questions, mostly 'Yes' or 'No'.

Different from Anna and Simon, Mandy communicated with the students in a non-judgmental manner and encouraged the participation of each student. Look at the following episode:

(Task A was presented to the class. After the majority of the students had finished their exploration of the possible positional relationships between two circles by using their physical objects, the teacher encouraged the students to voluntarily report their findings.)

T: Well, who is willing to tell us your finding? How many kinds of relationships between two circles have you found?
A: The two circles can be intersecting, and …
T: I have two circles (models) here, please use them to show us what you have found.
(A went up to the platform, stood close to the teacher and in front of the rest of class)
A (demonstrating): This is being separated, and this is being intersecting, another situation is that the smaller circle is inside the bigger one, being contained.
T: Well, he found three kinds of relationships. Is there more?
B (stood up): When there is only one point of intersection.
T: When there is only one point of intersection? Could you come up and demonstrate to us?
(B went up to the platform and showed the situation of two circles being externally tangent to each other.)
T: Now we have four kinds, is there more?
(Several seconds of wait)
T: Only four types? Have you found any other different?
C (stood up): Yes!
T: What else have you found? Show us!
(C went up to the platform and showed the situation of two circles being internally tangent to each other.)
T: C found another kind. Now we have five kinds, right? Do you have any other?
(No response was given.)
T: Let us summarize their findings. A found three kinds, B added one kind, and then C added one more kind. Is there more?
(A went up to the platform again and showed the situation of two circles being concentric circles.)
T (facing A): Do you think it is different from what you just said?
A: Yes.
T: Why?

A: Because in other cases, the centres do not coincide.
T: So you mean that there are two situations, one is coincide and the other is not coincide?
A: Yes.
T: What is your criterion for judgment? Is that whether the centres coincide or not?
A: Yes.
T: Then I want to ask, are being intersecting and internally tangent the same situation?
A: No.
T: But according to what you said just now, they should be the same situation.
A: No. One is inside the circle, but the other is outside the circle.
T: Oh, you mean, one situation is that the centre (of one circle) is inside the (second) circle, the other is that the centre of (one circle) is outside the (second) circle?
A: Yes.
T (demonstrating): Then, according to what you just said, being intersecting and internally tangent should be the same situation, right?
A (puzzled): Oh, almost…
(At this moment, A realized that he was probably wrong and smiled with a little shame, the teacher also smiled and the rest of class laughed.)
T: Well, now we encounter a problem. That is about how to classify the relationships. If we follow what A just said, i.e. according to the position of the centers to classify, he got three kinds of situations: outside, inside and coincide. Furthermore, he felt that being coincident and inside may be the same. Do you agree to his method?
Some students: No.
T: Why?
Some students: Because it is not clear.
T: Do you have other better method?
…

As the above episode indicated, the students were very active to display their findings, and the teacher was also very democratic and supportive. She knew that student A's solution was problematic, but she did not hasten to give her own judgment by simply saying right or wrong. Instead, she engaged and challenged the students' thinking by asking probing questions. Later on, she summarized student A's solution in a neutral tone and encouraged the rest of class to give their opinions. Clearly, the students' ideas and ways of thinking were respected and valued.

Throughout the lesson, Mandy required the students to give full explanations and justifications five times, and give demonstrations six times. The above episode involved two full explanations and four demonstrations. Mandy listened carefully to the students' ideas. Taking the above episode as an example, she paid attention to student A's thoughts, and then identified the flaws of his method and accordingly challenged his thinking through questioning.

Particularly, Anna and Simon both required their students to memorize the newly learned knowledge, but Mandy did not do so. Anna required the students to memorize the two rules: 'exponent should be 1 and coefficient should not be 0' and 'exponent should be 1, coefficient should not be 0, and constant should be 0'. This happened four times in total. Simon required the students to memorize mathematical knowledge twice. One time was for the relationships among parallelogram, rectangle, rhombus and square, and the other was for the formula of area of square: $S=l2\times1/2$ (l denotes diagonal of square). Besides, Anna required the class to imitate her solution once, but Simon and Mandy did not do so.

(2) *Student-student interaction*

Anna and Simon did not encourage their students to listen to, and respond to each other's ideas. When the students responded to her questions, Anna gave feedback, but other students did not give any opinions. In Simon's class, when some student gave wrong answer to the teacher's question, other students laughed at him or her, but did not give other feedback. Generally, there was no student-student interaction in Anna's and Simon's classroom.

Unlike Anna and Simon, Mandy encouraged students to listen to, respond to and even question each other so that they could evaluate and if necessary, discard or revise the ideas and take full responsibility for arriving at conclusions. Take the above episode as one example, she encouraged the students to listen to student A and give opinions about his method. Generally, student-student interaction was quite active during the lesson.

(3) *Questioning*

Throughout the lesson, Anna asked a total of 221 questions, Simon asked a total of 117 questions, and Mandy asked a total of 95 questions. The questions included both closed questions inviting short answers like basic facts, Yes or No, and open questions inviting explanations from the students, as shown by Figure 2.

From Figure 2, it can be seen that the vast majority of the questions asked by the three teachers were closed questions. These questions demanded low-level cognitive efforts, thus could not challenge the students' thinking. In Anna's lesson, some open questions were aimed at eliciting the students' explanations, but they were answered by Anna herself. In Simon's class, a few open questions were asked. However, as shown by the earlier episode, Simon greatly reduced the level of difficulty of all potential complex questions, even some simple factual questions. For example, he changed a simple question "what are the axes of symmetry of the rhombus?" to "the axes of symmetry of rhombus are the lines where the diagonals lie, right?" so that the students just gave answer 'Yes' or 'No'. In Mandy's class, a larger amount of open questions were posed and they did challenge the students' thinking.

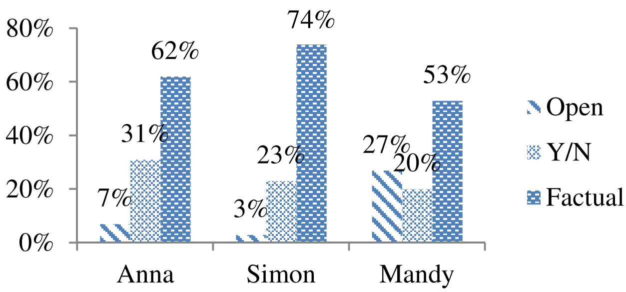

Figure 2. Percentages of different types of questions

On the other hand, a total of 202, 102 and 81 responses were elicited from the students in Anna's, Simon's and Mandy's class respectively, these responses could be classified into three categories. The details are shown in Figure 3.

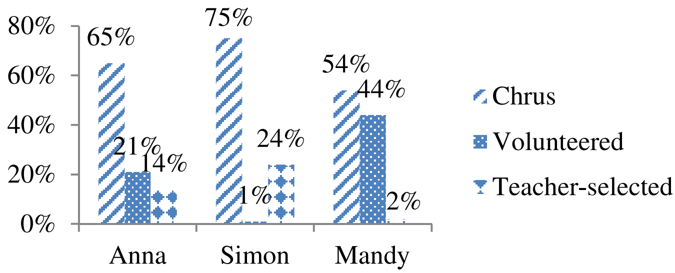

Figure 3. Percentages of different types of responses

It is noticeable from Figure 3 that in all the three teachers' class, chorus responses accounted for more than 50%; particularly in Mandy's class, volunteered responses took up 44%, much more than those in Anna's and Simon's class. That is, the students in all the three class mainly answered questions collectively rather than individually. However, Mandy's students were far more willing to contribute their responses voluntarily, as compared to Anna's and Simon's.

Furthermore, in Anna's and Simon's class, the students' responses were limited to basic facts, or Yes or No, few explanations were actually sought. In contrast, Mandy's students gave more explanations and justifications.

4.3.4 *Summary*

As in the case with espoused mathematics beliefs, Anna's enacted mathematics beliefs seemed to be quite similar to Simon's (despite some small differences), but substantially different from Mandy's. According to their teaching practices, both Anna and Simon seemed to believe in the following views: 1) Mathematics is independent of real life; 2) Mathematics is a static united body of knowledge, and mathematical knowledge is interrelated; 3) Mathematics is not just a collection of rules, formulas and procedures, but it stresses using certain rules or procedures to solve problems; 4) Mathematical problems can only be solved by single approach; 5) Students should learn mathematics mainly through listening attentively to teacher's lecture and doing enough exercises; 6) Memorization and/or imitation are important approaches to learning mathematics; 7) Students do not need to engage in independent inquiry to learn mathematics. In other words, independent inquiry is not important approach to learning mathematics; 8) Students do not need to communicate their mathematical ideas; In other words, mathematical communication is not important approach to learning mathematics; 9) Students do not need to learn mathematics in a collaborative way. In other words, collaborative learning is not important approach to learning mathematics; 10) Mathematics should be mainly taught by direct instruction. In teaching process, students should be recipients of knowledge while teacher should be transmitter of knowledge; 11) Teacher should play a role of the authority of knowledge.

Overall, Anna's and Simon's enacted beliefs about the nature of mathematics seemed to be basically similar to the Platonist view. Their enacted beliefs about learning mathematics seemed to be relatively close to the reception view of learning, and enacted beliefs about teaching mathematics seemed to be relatively close to the teacher-centered, content-focused with an emphasis on conceptual understanding view of teaching.

On the other hand, judging from her classroom teaching, Mandy seemed to believe in the following views: 1) Mathematics is independent of real life; 2) Mathematics is dynamic and expanding; 3) Mathematical knowledge is interrelated; 4) Mathematical problem solving allows for multiple approaches; 5) Memorization and imitation are not important approach to learning mathematics; 6) Students need to engage in independent inquiry to learn mathematics; in other words, independent inquiry is important approach to learning mathematics; 7) Students need to express, explain, justify and communicate their ideas; in other words, mathematical communication is important approach to learning mathematics; 8) Student do not need to learn mathematics in collaborative way; in other words, collaborative learning is not important approach to learning mathematics; 9) Mathematics should not be taught by direct instruction; rather, mathematics teaching should be student-centered; 10) In teaching process, teacher should be a facilitator of students' learning rather than a knowledge deliver; students should and can be active constructors of knowledge; 11) Teachers should not play a role of the authority of knowledge.

Overall, Mandy's enacted beliefs about the nature of mathematics seemed to be basically similar to the social constructivist view. Her enacted beliefs about learning mathematics seemed to be relatively close to the social constructivist view of learning, and enacted beliefs about teaching mathematics seemed to be close to the learner-focused view of teaching.

5. Findings and Discussion

In this study, three Chinese teachers' espoused as well as enacted mathematics beliefs were investigated in depth. The findings from the case studies are summarized in Table 5. According to the table, among the three teachers, two i.e. Anna and Simon seemed to hold traditional mathematics beliefs while the third one, Mandy seemed to have mathematics beliefs that were basically consistent with the reform ideas.

In the case studies, great similarities in terms of both espoused and enacted mathematics beliefs were identified between Anna and Simon, and the differences between them and Mandy were found to be evident. Particularly, the traditional Platonist view of the nature of mathematics, the reception view of learning and the teacher-centered view of teaching seemed to be shared by Anna and Simon. On the other hand, Mandy seemed to hold the reform-oriented social constructivist view of the nature of mathematics, the social constructivist view of learning, and the learner-focused view of teaching. Therefore, this finding is not fully consistent with those from previous studies, in which teachers in the Chinese mainland unanimously held the Platonist view of the nature of mathematics (e.g. Wong et al., 2002) and emphasized the teacher-centered teaching method (e.g. An, Kulm, Wu, Ma, & Wang, 2006). It seemed that among the three teachers, substantial changes had only occurred to the mathematics beliefs of Mandy, during the implementation of the reform-oriented mathematics curriculum.

Table 5. Summary of the Mathematics Beliefs of the Three Teachers

		Beliefs about the Nature of Mathematics	Beliefs about Learning Mathematics	Beliefs about Teaching Mathematics
Anna	Espoused	Platonist	Reception	Teacher-centered
	Enacted	Platonist	Reception	Teacher-centered, content-focused with an emphasis on conceptual understanding
Simon	Espoused	Platonist	Reception	Teacher-centered
	Enacted	Platonist	Reception	Teacher-centered, content-focused with an emphasis on conceptual understanding
Mandy	Espoused	Platonist	Social constructivist*	Learner-focused*
	Enacted	Social constructivist*	Social constructivist*	Learner-focused*

Note: * means that the beliefs were consistent with the reform ideas.

Furthermore, both consistencies and inconsistencies between the teachers' espoused beliefs and enacted beliefs were found. Specifically, the former were reflected in the teachers' beliefs about learning and teaching mathematics while the latter were in their beliefs about the nature of mathematics. Therefore, these findings do not completely agree with those from previous studies that only identified either consistencies (e.g. Stipek, et al., 2001) or inconsistencies (e.g. Raymond, 1997) between teachers' beliefs and practice (enacted beliefs). Research literature suggests that teachers' beliefs about mathematics teaching and learning appear to be related to their beliefs about mathematics (Thompson, 1992). It is also suggested that Kuhs and Ball's (1986) three views of teaching (Content-focused with an emphasis on conceptual understanding, Content-focused with an emphasis on performance, Learner-focused) are underpinned by or logically follow from Ernest's (1989a) three views of the nature of mathematics (Platonist, Instrumentalist, Social constructivist) respectively (Thompson, 1992). According to Table 5, such logical relationship was only established in the teachers' enacted beliefs. It seems that there is no logical necessity between teachers' espoused beliefs about the nature of mathematics and their espoused beliefs about learning and teaching mathematics. However, the close connections between teachers' beliefs about learning mathematics and their beliefs about teaching mathematics are confirmed.

Generally, this study has contributed to the body of research literature on teacher beliefs, particularly on Chinese teachers' mathematics beliefs in two ways. Firstly, unlike the conventional studies where only teachers' espoused beliefs are examined, this study has provided us a more comprehensive understanding of three Chinese teachers' mathematics beliefs by focusing on their espoused as well as enacted beliefs. Secondly, this study has added a new perspective to the extant research on Chinese teachers' mathematics beliefs through investigating the three teachers who responded differently to China's recent constructivism-oriented curriculum reform. Besides, this study sheds some light on the query about how Chinese teach mathematics. It seems encouraging to observe that some Chinese teachers have adopted the reform-oriented, learner-focused (student-centered) teaching approaches in their mathematics classrooms. However, it is noted that this study has two limitations. Firstly, only three teachers were involved in this study, thus the findings have limited generalizability. Secondly, due to the teachers' busy schedule, only one lesson was observed for each teacher so as to infer his or her enacted beliefs, which may threaten the validity of research to a certain extent. Extending the period of data collection could help reduce this threat.

6. Implication

A few specific findings from this study provide important implications for teacher educators and reform advocates in China. Firstly, two of the three teachers, i.e. Anna and Simon showed strong commitment to the traditional ideas and approaches (e.g. memorization and imitation, drill and practice, direct instruction, the role of teacher as the authority of knowledge) which are rooted in the cultural values of Chinese societies for long (Fan, Wong, Cai, & Li, 2004). To challenge and transform these deep-rooted ideas and practices, revolutionary interventions by teacher educators are indispensable. Such interventions should provide the teachers with opportunities to reflect on their taken-for-granted mindset and behaviors from a critical perspective, and thereby appreciate the need for reform-oriented change. On the other hand, Anna and Simon both contended that the reform-oriented approaches are important but impractical. Therefore, if teachers like them are expected to teach in a reform-oriented way, they need to get convinced of the practical values of these approaches. To prove such values is an important task for teachers, teacher educators and reform advocates.

Secondly, all the three teachers argued that mathematics is closely related to real life. However, none of them used mathematical tasks with real life context in classroom teaching. This is probably because that they had difficulty in designing appropriate tasks to illustrate the connections between real life and mathematics. Besides, Mandy showed more acceptances to the reform ideas and approaches, in words as well as actions. However, she did not have accurate understanding of the 'collaborative learning' approach, she also seldom used this approach in her class. All these indicated that teachers, even those who support the reform, may lack of necessary knowledge and skills to implement the reform ideas and suggestions. Therefore, teacher educators should equip the teachers with essential knowledge and skills to actualize their reform ideals.

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Chapter 18

"Same Content Different Designs" Activities and Their Impact on Prospective Mathematics Teachers' Professional Development: The Case of Nadine

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As a new form of Teaching Research Group (TRG) activity, Same Content Different Designs (SCDD, or *Tong Ke Yi Gou* in Chinese) activities have gradually become popular among teacher preparation and professional development programs in Chinese mainland at the beginning of the 21st century. To explore their impact on the professional development of prospective mathematics teachers, we organized two SCDD activities before and during the educational field work of a group of prospective teachers in China. Data was collected through clinic interviews, classroom teaching observations and videotaping, and cross-referenced with teaching related documents and artifacts. A case analysis reveals the SCDD activities' impact on Nadine, one of the prospective teachers' professional development. Based on the activities and the results, we propose and discuss a model that characterizes the key components and stages of SCDD activities.

Keywords: Same Content Different Designs (SCDD), prospective mathematics teachers, professional development, logarithmic functions

1. Introduction

Since the early 1950s, *Teaching Research Group* (TRG) activities have been a mandatory component of continued professional development for practicing mathematics teachers in China. Teachers in the same TRG collaboratively examine curriculum materials, design lesson plans, share teaching experiences, and observe and comment on each other's lessons.

Among other factors, researchers have attributed Chinese mathematics teachers' profound mathematical knowledge for teaching and effective instructional practices to teachers' constant involvement in TRG activities (Ma, 1999; Paine & Ma, 1993; Sun, 2009; Yang, 2009; Yang, Li, Gao, & Xu, 2009; Yang & Ricks, 2012). The objectives, structure, and content of TRG activities have evolved considerably in the past decade to meet the ever-changing demands from mathematics curriculum reforms (Yang et al., 2009). Same Content Different Designs (SCDD, or *Tong Ke Yi Gou* in Chinese) activities emerged as one of the new and popular types of mathematics TRG activities in schools all across China (Yuan, 2012; Yuan & Liu, 2011; Yuan & Ricks, 2011).

During a typical SCDD activity, two or more teachers teach a common topic to different groups of students with distinct lesson designs, while their fellow teacher participants observe each of these lessons. After all lessons are completed, all teachers involved gather to discuss the lesson designs and classroom teaching practices, make comments and suggestions for future revisions and improvements. An alternative process begins with a teacher designing and teaching a lesson on a certain topic. Based on her/his own reflections on the classroom teaching practices, results of formative assessment of student learning, and feedbacks from fellow teacher observers in the same TRG, the teacher then redesigns the lesson and teaches it again to a different group of students. Afterward all teachers involved go through a second cycle of reflection, comparison, and discussion.

With the keyword "*Tong Ke Yi Gou*" (in Chinese), the authors searched in one of the largest online database for academic periodicals in China, *Chongqing WeiPu Information* (www.cqvip.com), and found more than 700 articles on SCDD activities that were published between 2005 and 2013. Most of these articles appear as an observation and description of two lessons taught by two teachers on the same topic during an SCDD activity, followed by the article author(s)' reflections and comments specifically on the lessons, or on SCDD activities in general. Almost all of these SCDD activities involve only practicing mathematics teachers, with only one exception which is a research conducted by the first author of this chapter and his collaborator on an SCDD activity involving both practicing and prospective teachers. Some of these articles are general discussions on the nature, roles, goals, principles, processes and strategies of SCDD activities. They either briefly mention two SCDD lessons as examples, or do not refer to any specific lessons or relevant data at all. To better understand the nature and dynamics of SCDD activities as well as their impact on mathematics teachers' professional development, we believe it is necessary to extend the existing studies in two ways: (1) carrying out more systematic studies based on multiple points and sources of data and more rigorous and in-depth analysis; (2) including teacher participants at various stages of their teaching careers, especially, prospective mathematics teachers.

In designing the current research study, we are interested in finding out how prospective mathematics teachers learn and grow as novice educators through their SCDD activities experiences. Specifically, we tried to investigate the following research question through a case study: When prospective mathematics teachers are engaged in SCDD activities together with experienced teachers, how would such activities influence these prospective teachers' professional development, mainly, their classroom teaching practices?

Underlying the above research question is our main assumption that prospective teachers would potentially benefit from well-designed SCDD activities which also involve experienced practicing teachers. When prospective teachers first design and practice teaching SCDD lessons among themselves, they develop initial experiences with different lesson designs for the same topic. Comments from fellow student teachers and supervising teachers help them see the strengths and weaknesses of the various designs and practices, and improve those of their own. Later when prospective teachers and practicing teachers teach SCDD lessons simultaneously in authentic classroom environment, prospective teachers learn directly from practicing teachers, not only through the comments and suggestions they make, but also by observing and discussing lessons taught by practicing teachers on the same topic. Throughout this process, prospective teachers are exposed to multiple ways of designing and teaching the same topic, understand why and how each way works well or not through reflections, comparisons, and discussions, which may lay both broad and solid foundation for the growth in prospective teachers' practices. We hope to verify these assumptions through this research.

2. Theoretical Framework

SCDD activities groups, or TRGs in general, can be organized or analyzed through the notion of teachers' learning communities (Jaworski, 2004; Lin & Ponte, 2008; Roth & Lee, 2006) which are essentially a special form of communities of practice (Lave & Wenger, 1991; Wenger, 1998). A learning community demonstrates four main characteristics:

(1) The community is built within general and specific teaching and learning contexts, such as cultural, social, and school environments, national and local educational policy, curriculum and assessment standards, and teacher and student characteristics (Stigler & Hiebert, 1998, 1999).

(2) Members of the learning community follow common norms and routines and share common language and sensibilities. All of these common features are shaped by the contextual settings of the community. Teachers' practices in such a community are bounded by both the contexts and the established norms and routines, whereas growths in teachers' knowledge and changes in teachers' beliefs and practices would in turn reshape the interactions between teachers and those contexts and norms.

These imply that the dynamics of a learning community consisting mainly of prospective teachers would be in many ways different from those of a practicing teachers' learning community, and both would also somewhat differ from those of a learning community that includes both prospective and practicing teachers. Hence, researchers must explicitly pay attention to and address the membership of each community when designing, analyzing, and reporting results from studies related to learning communities.

(3) Members of the community learn from each other. Lin and Ponte (2008) believe that "the most important feature of a learning community is that its members learn from one another" (p.112). A team of teachers can form a learning community only if all teachers in the team realize and value the fact that they can learn from each other (Jaworski, 2004). Jaworski (2005, 2008) also characterized teachers' learning communities as inquiry communities. In an inquiry community, teachers are not satisfied with the status quo. They frequently question and examine their

daily, normal practices, and look for alternatives ways of teaching (Jaworski, 2005, 2008).

(4) Members of the community are reflective practitioners, which echo the reflective nature of a community of practice (Lave & Wenger, 1991; Wenger, 1998). Teachers in the community must constantly dwell on their current and past practices in order to fully understand their context, norms, and complexities, and to potentially improve future practices. Reflections can be in many forms, e.g., journal writing, watching and discussing videos of teaching, or discussing observations on lessons taught by fellow teachers.

The above four features determine an individual-collective duality of a teachers' learning community. On the one hand, individual members of the community have distinct knowledge, experiences, viewpoints, beliefs, goals, expertise, and preferences, which may make it difficult for members to communicate and collaborate. On the other hand, all teachers' practices are framed by the same contexts and norms, and reframed through daily reflections, sharing, comparisons, and co-learning among members of the community. Through such a process, an individual teacher's knowledge and expertise could be significantly improved, and also be integrated into the collective knowledge and expertise shared by the community. Such a duality seems to be well reflected in prospective teachers' SCDD activities which start with individual teachers designing and implementing lessons relatively independently then followed by collective sharing of observations, comments, and suggestions.

We use the above characterization of SCDD activities in designing our research and in data analysis.

3. Methods

3.1 *Participants*

Twelve (3 males, 9 females) prospective mathematics teachers and one practicing teacher with 5-year teaching experience participated in this study on SCDD activities. The prospective mathematics teachers came from a mathematics teacher preparation program at a teachers' university on the south-east coast of China, and chose to fulfill their educational field work requirement as a team in the same local high school, Gulou High School (pseudonym). The practicing teacher, Linda, has been teaching at Gulou High School for 5 years. The first author of this chapter was appointed field work supervisor of this team. Two of these prospective teachers, Nadine and Howard, previously participated in a SCDD activity in a teacher development experiment (Yuan, 2012). For that SCDD activity, Howard taught a lesson in an authentic high school classroom environment. All other prospective teachers (for example, Yolanda) didn't have any formal classroom teaching experience prior to the first SCDD activity. Even before the second SCDD activity, these prospective teachers only had taught 3 to 5 lessons during their 8-week educational field work. The leader of the team, Nadine, was chosen as the focus participant since she was very reliable and agreed to try her best to finish all research procedures.

3.2 *A Six-step Model of SCDD Activities*

Based on the theoretical framework introduced earlier, the first author's previous study (Yuan, 2012), and a few related discussions (He, 2007; Li, 2010; Tong, 2010), we propose a generic model for SCDD activities for prospective teachers. During a typical SCDD activity, prospective teachers are expected to undertake six major steps: (1) preparing for and writing lesson plans individually; (2) having open lessons (i.e., lessons observed by other teachers) and peer observations; (3) explaining and evaluating lessons; (4) interviewing students; (5) observing self-videotapes and writing reflective journals; (6) revising lesson plans.

3.2.1 *Preparing for and Writing Lesson Plans Individually*

During teaching preparation, each teacher who will teach an open lesson ponders on and prepares for the lesson carefully. Such a teacher is expected to write the first draft of a lesson plan independently without discussing it with anyone else, but can refer to any existing curriculum resources, including curriculum standards, textbooks, teachers' guide books, lesson plans and courseware developed by other teachers, and videotapes of classroom teaching. After the lesson plan is completed, the teacher can discuss it with a brain trust (such as a mentor and/or a TRG leader). He or she can also have a simulation teaching (i.e., in front of other student teachers) or authentic teaching in front of a group of school students. Based on these discussions and experimentations, the teacher will generate a new draft of the lesson plan together with a written explanation on which and why certain revisions are made.

3.2.2 *Having Open Lessons and Peers Observations*

In implementing the lessons, two or more teachers teach the same topic to different groups of students with distinct lesson designs, while other teacher participants observe each of these lessons. The teachers who teach one of the open lessons are encouraged to observe the lessons taught by others and be willing to tell honestly any changes they plan to make to their own lesson plans as the result of observing the other teachers' teaching. The other teacher participants should document the noticeable events happened in the open lessons so that they can later evaluate the lessons based on solid evidence.

3.2.3 *Explaining and Evaluating Lessons*

On-site explanations and evaluations will be held immediately after all open lessons are completed. Teachers who taught these lessons will take turns to explain their lesson designs and thinking and actions during classroom teaching. The main topics for explaining the lessons include the preparation process, the lesson design, reflections on teaching, and reflections on the peer observations. After lesson explaining, the other participants will provide their evaluations of the lessons one by one.

3.2.4 *Interviewing Students*

Through teaching and explaining open lessons as well as peer evaluations, those teachers who taught the open lessons are supposed to share experiences with their fellow observers and receive many valuable suggestions. However, what are the effects of the lessons on students learning? Among other possible measures, we believe conducting clinical interviews with students is an effective way to qualitatively evaluate student learning outcomes. Especially, teachers can receive more detailed feedbacks from the students if a good interview protocol is prepared ahead of time.

3.2.5 *Observing Self-Videotapes and Writing Reflective Journals*

John Dewey (1933) ever said that "Experience plus reflection equals growth". After communicating with the other participants, the teacher who taught an open lesson need reflect thoroughly. Watching videotapes of classroom teaching and discussion provide good opportunities for reflection. The teacher can replay the videotapes repeatedly to examine his or her classroom teaching practices. A formal reflective journal helps to consolidate and document the reflections.

3.2.6 *Revising Lesson Plans*

After going through the five steps described above, teacher's knowledge, beliefs and practices are expected to undergo considerable changes. One way to cumulate evidence for such changes and for the teachers to apply everything they learned in the five steps is to further revise the lesson plans. The new lesson plans can be used in future teaching.

3.3 ***Process***

This study included two SCDD activities organized in two phases. The first SCDD activity was held in the phase of simulation teaching training for prospective teachers before their educational field work. Twelve prospective mathematics teachers participated in this activity. The second SCDD activity was held in Gulou High School classrooms towards the end of their educational field work. Two of these prospective teachers, Nadine and Howard, and a practicing teacher from Gulou High School, Linda, each taught an open lesson observed by practicing teachers and other prospective teachers. All of the other 10 prospective teachers observed these open lessons.

3.3.1 *First SCDD Activity*

As a tradition, prospective mathematics teachers from the teacher's university would form teams and practice simulation teaching before their actual educational field work. As the supervisor of one group of prospective teachers, the first author of this chapter was in charge of this simulation teaching training. The following training procedure was used. Each prospective teacher had simulation teaching twice in a week. The first simulation teaching was held in the form of a SCDD activity in August 2011. The same topic, Logarithmic Functions and Their Properties, was taught by all 12 prospective teachers. This lesson topic was adopted from the first volume of the textbooks for senior high schools, which was a required topic for educational field work. In the process of this SCDD activity, each prospective teacher went through five steps: (1) preparing for and writing lesson plans individually; (2) simulation teaching and peer observations; (3) explaining and evaluating lessons; (4) observing self-videotapes and writing reflective journals; (5) revising lesson plans. In the second simulation teaching, different lesson topics (such as the concept of functions, the monotonic property of functions, exponential functions and their properties) from the same textbook were taught by different participants.

3.3.2 *Second SCDD Activity*

The second SCDD activity was held towards the end of the prospective teachers' educational field work, on October 20-21, 2011, at Gulou High School. Two of the prospective teachers, Nadine and Howard, and the practicing teacher, Linda, each taught an open lesson one after another in the morning of October 20. A lesson explaining and evaluating activity was held immediately after the last lesson ended. In the afternoon of October 21, the first author interviewed these three teachers one by one. Then Nadine and Howard, each interviewed three students selected from the class she or he is student teaching. Since the focus of this study was prospective teachers' professional development, we didn't arrange for Linda to interview her own students. Instead, another prospective teacher, Yolanda, interviewed three students from Linda's class.

The mathematics topic of the second SCDD activity was the same as that of the first activity, Logarithmic Functions and Their Properties. But there are several differences between these two activities: (1) the first activity was held in a simulated school classroom environment where peer student teachers acted as "students", but the second activity was held in an actual high school classroom with more than fifty students; (2) twelve prospective teachers participated in the first SCDD activity, but only two of these prospective teachers and one practicing teacher took part in the second activity because of the restriction of conditions; (3) the first SCDD activity only involved prospective teachers and their supervisor, but the second SCDD activity involved many "outsiders", such as the college administrator who are in charge of educational field work, some practicing teachers from other local high schools and 16 junior college students; (4) More interviews were conducted after the second SCDD activity compared to the first SCDD activity. Detailed information is shown in Table 1.

Table 1. Schedule for the second SCDD activity

Date	Time	Content	Key Person	Observer
Thur., Oct. 20, 2011	7:55-8:40	open lesson	Nadine	1st author, prospective & school teachers
	8:50-9:35	open lesson	Howard	
	10:00-10:45	open lesson	Linda	
	10:55-12:00	assessing lessons	1st author	
Fri., Oct. 21, 2011	13:40-16:30	Interview with Nadine	1st author	No
		Interview with Howard		
		Interview with Linda		
	16:50-17:50	Interview with student S1, S2, S3	Nadine	
		Interview with student S4, S5, S6	Howard	
		Interview with student S7, S8, S9	Yolanda	

3.4 *Data Collection*

Multiple types of data were collected through in-depth interviews, classroom observations and videotaping, and artifact collection (See Table 2).

Table 2. Data collection during the SCDD activities

Process	Data Sources	Data Types	Date
First SCDD activity	Nadine, Howard and other 10 prospective teachers	Classroom teaching videotapes post-lesson interview videotapes; teaching reflective journals; lesson plans and courseware	2011/08/25 2011/08/26
Second SCDD activity	Nadine, Howard, and Linda	Classroom teaching videotapes post-lesson interview videotapes; teaching reflective journals; lesson plans and courseware	2011/10/20 2011/10/21
	9 senior high school students S1-S9	Interview videotapes or audiotapes	2011/10/21
Writing bachelor thesis	Nadine	Thesis	2012/03

In the first SCDD activity, we used one camera to videotape the prospective teachers' entire simulation teaching lessons as well as the follow-up explanation and evaluation activities. In the second SCDD activity, we used three cameras to videotape the entire open lessons: the first camera focused on the teacher, the second one focused on the students, and the third one videotaped the whole class. Then two cameras were used in the explanation and evaluation sessions: one of them was used for videotaping the whole meeting room and the other one focused on the teacher who was speaking. One camera was used in interviewing teachers and students. Besides those videotapes, we also collected a lot of teaching related documents and artifacts, including lesson plans, courseware, and reflective journals. We also used Nadine's bachelor thesis as one of data sources since her thesis was related to this study. In the process of data collection, we followed three principles for a case study: (1) using multiple, not just single, sources of evidence; (2)

creating a case study database; and (3) maintaining a chain of evidence (Yin, 2003, p.85).

3.5 *Data Analysis*

Since this study focused on the professional development of one of the prospective mathematics teachers, Nadine, we mainly analyzed her data. At the initial stage of data analysis, we put together and looked through all data collected from Nadine. We wrote down all noticeable events in this step for further detailed analysis. Then, we divided each lesson taught by Nadine in the two SCDD activities into seven episodes. The main instructional strategies were identified from each episode. By analyzing the critical events and critical persons during the SCDD activities, we identified evidence for the influences that the SCDD activities had on her classroom teaching practices.

4. Results and Analysis

Nadine was still a prospective teacher whose educational field work was just over a month. We wouldn't expect her to teach a perfect lesson in the second SCDD activity. However, we do hope SCDD activities could facilitate her professional development to a certain extent. The changes in Nadine's classroom teaching practices in the two SCDD activities are described below. Some possible explanations for these changes are also discussed.

4.1 *Changes in Nadine's Teaching Practice in the Two SCDD Activities*

By analyzing Nadine's lessons in the two SCDD activities, we see that the instructional structures of the two lessons are identical. Both lessons can be divided into the following seven consecutive episodes: (1) introducing the definition of logarithmic functions; (2) graphing the functions $y = \log_2 x$ and $y = \log_{1/2} x$; (3) exploring the graphs and properties of logarithmic functions $y = \log_a x$; (4) giving Type I examples: finding the domains of functions; (5) giving Type II examples: comparing two logarithmic quantities; (6) summarizing; (7) assigning homework. The time spent in each episode was shown in Table 3.

Table 3. Time spent in each episode in the two SCDD activities

Episode	First		Second	
	Time second	Percent %	Time second	Percent %
1	204	9.4	590	21.4
2	216	9.9	683	24.8
3	1050	48.2	732	26.6
4	130	6	159	5.8
5	505	23.2	321	11.7
6	58	2.7	221	8
7	15	0.7	48	1.7
Total	2178	100	2754	100

By comparing the percentages of time spent on each episode in the two activities (Table 3), we can see that Nadine spent greater portions of time on the 1st, 2nd, 6th and 7th episodes and spent less portions of time on the 3rd and 5th episodes in the second SCDD activity. She spent almost equal percentages of time on the 4th episode in the two activities. Why did she change her classroom time allocations? More fine-grained, detailed analysis is needed to understand the changes of time spent in each episode in the two SCDD activities. For this purpose, we identified the instructional strategies used in each episode (See Table 4).

Table 4. Instructional strategies used in each episode

Teaching Episode	Instructional Strategies	
	First SCDD Activity	Second SCDD Activity
1	Directly show the definition of logarithmic functions $y=\log_a x$, then analyze its structure and three core elements: domain, range and corresponding rule.	Review the relationship between the number of times a cell divides (x) and the total number of cells (y) after x divisions, $y=2^x (x\in N^*)$, then ask a question: If we know the total number of cells y, how can we know the number of divisions x? Based on this real life problem, show and analyze the definition of logarithmic functions $y=\log_a x$.

Table 4. (*Continued*)

2	Show two functions $y=\log_2 x$ and $y=\log_{1/2} x$ on the screen, then tell the students how to select a set of points on the graph and show the coordinates of these points in a table on the screen. Ask students to graph these two functions in the same coordinate system. After a while, show answers using Geometer's Sketchpad (GSP).	Show two functions $y=\log_2 x$ and $y=\log_{1/2} x$ on the screen, then spend 4.5 minutes to explain how to select a set of points on the graph. Then divide students into two groups and ask them to graph one of two functions on their coordinate paper. After a while, ask two students to graph these two functions respectively on the blackboard. Finally, summarize the trend of change of graphs of two functions.
3	Explore the following four properties of logarithmic functions $y=\log_a x$ using Geometer's Sketchpad: (1) all graphs pass through the point (1,0); (2) the monotonic properties of logarithmic functions; (3) the graphs of the two functions $y=\log_a x$ and $y=\log_{1/a} x$ have *x*-axis symmetry; (4) the effects of changes in the base *a* on the graphs of the logarithmic functions $y=\log_a x$. Different GSP documents were used in exploring these properties. A main line to explore the properties of functions was lack.	GSP was used only in showing the impact of base *a* to the graph of logarithmic functions $y=\log_a x$. After all students observed GSP demonstration, a student was asked to come to the front of the classroom and summarize the properties of logarithmic functions. In this process, the students were reminded to recall the method for exploring functions: (1) observing their structure, domains, ranges and so on; (2) exploring their common properties, for example, properties of monotonic functions, properties of even and odd functions (3) exploring their special properties, for example, passing through the same point (1,0). Finally, show these properties in a table on the screen.
4	Give three problems on finding the domains of functions and ask students to answer orally. 1 $y=\log_a(3-x)$ 2 $y=\log_a(x^2+1)$ 3 $y=\log_{(x-1/2)}(3-x)$.	Give two problems on finding the domains of functions and ask students to answer orally. 1 $y=\log_2(3-x)$ 2 $y=\log_{(x-2)} 3$.

Table 4. (*Continued*)

5	Give five problems on comparing two numbers and ask students to answer orally, there is no blackboard writing. 1 $\log_2 3.4$ and $\log_2 8.5$; 2 $\log_{0.3} 1.8$ and $\log_{0.3} 2.7$; 3 $\log_a 5.1$ and $\log_a 5.9$; 4 $\log_6 8$ and $\log_7 8$; 5 $\log_3 5$ and $\log_6 4$.	Give one problem on comparing two numbers and ask students to answer orally, and write the solution on the blackboard. Finally, suggest the students solve the problem from the point of view of functions. 1 $\log_2 3.4$ and $\log_2 8.5$
6	Summarize the knowledge points according to the blackboard writing.	Summarize the knowledge points and mathematical thoughts and methods using PPT.
7	Give a series of problems using PPT	Write a problem on the blackboard.

According to Table 4, we argue that Nadine did much better in her second SCDD classroom teaching activity compared to her teaching in the first activity. In the second activity, she paid much more attention to creating real life contexts and facilitating student initiated inquiries. She used a clearer framework to guide her students in exploring the graphs and properties of logarithmic functions. She also selected more appropriate mathematics problems for her students. The use of GSP was more reasonable because she integrated all explorations into one GSP document and demonstrated it more clearly.

To determine whether the instructional strategies adopted by Nadine in the two SCDD activities are appropriate, we referred to some official documents. According to the Standards for Senior High School Mathematics Curriculum (Ministry of Education of China, 2003), the expected student learning outcomes in studying logarithmic functions include: (1) through concrete examples, visually understanding the numerical relationship a logarithmic function describes, preliminarily understanding the concept of logarithmic functions, and experiencing logarithmic functions as an important type of function; (2) graphing specific logarithmic functions with calculators or computers, exploring and understanding the monotonic property and special points of logarithmic functions (p.15). From the first episode in Table 4, we can see that instructional strategies used in the second SCDD activity were more reasonable than that of in the first SCDD activity from the Standards' perspective. Especially, we noticed that the specific example cell division was mentioned in the second SCDD activity.

A popular teachers' guide book (Institute of Curriculum and Textbook Research, 2004) makes the following suggestions on teaching logarithmic functions: (1) the graphs and properties of logarithmic functions are the important and difficult knowledge points. Teachers should engage students in actively exploring the properties, especially, recognizing the effects of changes in the base a on the logarithmic functions $y = \log_a x$; (2) Teachers should use information technology to explore the properties of logarithmic functions as much as they can. Since this popular teacher' guide book was written by a group of expert teachers, we assumed the above two suggestions are indeed effective instructional strategies. From the second and third episodes in Table 4, we can see that instructional strategies used in the second SCDD activity were more reasonable than that of in the first SCDD activity.

This teachers' guide book also makes suggestions on the use of sample problems: (1) the purpose of demonstrating sample problems on finding the domains of functions is to make students better understand logarithmic functions. Teacher should not give students too many difficult problems; (2) the purpose of giving sample problems on comparing two quantities in logarithmic forms is for students to fully understand the monotonic properties of logarithmic functions and apply them to solving problems. Teacher should remind students to solve problems from the point of view of functions. From the fourth and fifth episode in Table 4, we can see that instructional strategies used in the second SCDD activity were more reasonable than that of in the first SCDD activity.

Overall, we believe Nadine improved her classroom teaching practices in her second SCDD activity. However, we also notice that not all instructional strategies are necessarily appropriate in her second SCDD activity. For example, in the second episode, she spent 4.5 minutes in explaining how to select a set of points on the graph before students plotted the functions. She could have spent less time and be more efficient.

When comparing Nadine's instructional strategies in the two SCDD activities, we do notice that the classroom environments were different. The first SCDD activity was in a simulation classroom where the other prospective teachers in Nadine's team acted as students. The second SCDD activity was in an actual classroom where she interacted with a group of high school students. Nonetheless, since the instructional strategies listed in Table 4 were prepared by Nadine before the classes, we assumed they were independent of whether real students were in the classroom. Hence we conclude that it must be the changes in Nadine's knowledge and beliefs that led to the changes in her classroom teaching practices. Next we will analyze what events have enabled changes in Nadine's teaching.

4.2 *Factors Influencing Nadine's Classroom Teaching Practices*

4.2.1 *Learning from the First SCDD Activity*

Nadine was a prospective teacher who never taught in an actual classroom environment before her first SCDD activity. Undoubtedly, it was very difficult for her to prepare for and implement a simulation lesson. But it provided a good opportunity for her to thoroughly analyze the lesson and carefully prepare for it.

In her first post-lesson interview (2011/08/25), Nadine mentioned that she spent a lot of time in studying textbooks, curriculum standards, and lesson plans and courseware created by some experienced teachers. She practiced at least five times before her first SCDD activity.

However, Nadine still had a lot of perplexities. She felt that: (1) the textbook content was too easy for students; (2) one lesson would be enough to cover all the content that she would teach in three lessons; (3) it was too complicated to introduce the concept of logarithmic functions by using real-world problems; (4) the requirements in the curriculum standards were too low. We argue that the perplexities that Nadine faced in preparing for teaching were typical for most prospective teachers.

Through the first SCDD activity, prospective teachers developed a deeper understanding of this lesson by communicating with peers and the supervisor. For example, in evaluating Nadine's lesson (2011/08/25), one prospective teacher commented, "it was disorganized when you discussed the properties of the graphs". Another prospective teacher commented that Nadine showed the table "too quickly" before asking students to graph the functions $y = \log_2 x$ and $y = \log_{1/2} x$. These comments helped Nadine to make revisions in designing the lesson for the second SCDD activity.

After observing and discussing the other prospective teachers' simulation teaching, Nadine wrote in her reflective journal (2011/08/25): "When I was a senior high school student, I was impressed by how changes in the base *a* of logarithmic functions would affect their graphs. Although this property is not mentioned in textbooks, it is a key knowledge point for students. So I spent a lot of time in exploring this property. However, most prospective teachers didn't mention this property in their lessons. "

After noticing the above situation, Nadine decided to not explore this property in her second SCDD activity. It's right for her to make this change since there is too much content in the first lesson of Logarithmic Functions and Their Properties.

In Nadine's bachelor thesis (2012/03), she commented on her first SCDD activity: "By participating in this SCDD activity,…, I gradually changed my conceptions of teaching. I recognized that student leaning was a step-by-step process. The purpose of classroom teaching was not to transfer all knowledge to students like force-feeding ducks in a lesson, but to teach students how to learn through classroom interactions." We will see that these conceptions of teaching were clearly reflected later in her second SCDD activity.

4.2.2 *Learning from the Educational Field Work*

Educational field work also had major influences on Nadine's classroom teaching practices. During the second post-lesson interview (2011/10/21), Nadine said she had already taught three mathematics lessons in her educational field work. The topics of the lessons were representations of functions and exponential functions and their properties. Besides teaching these three lessons, Nadine also observed several practicing teachers' classroom teaching and evaluated student homework. Consequently, two critical events happened before the second SCDD activity may have also influenced Nadine's classroom teaching practices. The first event was collectively preparing for the lessons three days before the second SCDD activity (2011/10/17). During this TRG activity, one experienced teacher introduced a few key points for teaching logarithm and logarithmic functions. Several teachers gave some complements. Nadine stated some teaching principles (such as, seeking common ground while reserving differences) as the result of this activity.

The second critical event was communicating with her school mentor one day before the second SCDD activity. Nadine mentioned the mentor "took a look at my lesson plan, suggested several minor revisions, and basically approved my lesson plan" (post-lesson interview, 2011/10/21). On the one hand, we see that the educational field work provided possibility for Nadine to improve classroom teaching practices. On the other hand, because she basically designed the lesson for the second SCDD activity all by herself without major influence from her mentor, any changes in her teaching would be the result of her own reflections and realizations.

4.2.3 *Learning from the Second SCDD Activity*

Nadine was the first teacher who taught in the second SCDD activity; therefore she didn't have opportunity to revise her instructional contents and strategies by observing the other teachers' teaching (especially the practicing teacher, Linda). However, a series of events in the second SCDD activity may have influenced profoundly on Nadine's professional development. Although we didn't measure such influence directly, we did find some evidence from the post-lesson interview. For example, when the first author questioned one of her instructional strategies (spending 4.5 minutes explaining how to select a set of points on the graph before students plotted the functions in the second episode), Nadine said, "I have reflected on this strategy, especially after I observed Linda's lesson. I felt it was not necessary to teach students how to graph before they tried to do this" (post-lesson interview, 2011/10/21). Since Nadine had thought carefully and taught this lesson before she observed Linda's lesson, she was able to observe it more purposefully. In the process of collectively evaluating the lessons, Nadine received some comments from teachers and experts, which were further confirmed by her interviews with students. These events may help to reinforce her beliefs and preferences on certain instructional strategies.

5. Discussion and Conclusion

We analyzed the influence of SCDD activities on the professional development of a prospective mathematics teacher, Nadine. By comparing two lessons she taught in the first and second SCDD activities, we found that she was able to improve her classroom teaching practices. Some factors that have possibly influenced her lesson designs and implementations were discussed.

We argue that the Teaching Research Group consisting of Nadine and her prospective mathematics teacher peers, supervisor, school mentor, and other practicing teachers formed a learning community. This community existed in very positive teaching and learning contexts: both the teacher preparation program Nadine and her peers enrolled in and the administrators, practicing teachers, and students at Gulou High School were very supportive of the prospective teachers' educational field work; the new mathematics curriculum and assessment brought both opportunities and challenges to the field work. Members of this community followed common norms and routines, such as university and school policies regulating prospective teachers' field work, the calendar and organization of the field work, and the six-step model of SCDD activities that prospective teachers must go through. The prospective teachers learned from their mentor teachers and other practicing teachers by observing the classroom teaching and explaining and evaluating lessons while participating in SCDD activities, and they kept on reflecting by observing self-videotapes and writing reflective journals.

We believe that prospective teachers' learning in such a community is "both an individual and a social process. People learn as they interact with the physical and social world and as they reflect on what they do." (Lin & Ponte, p. 111). Based on this assumption as well as our characterization of learning communities, we proposed a six-step model of SCDD activity: (1) preparing for and writing lesson plans individually; (2) having open lessons and peer observations; (3) explaining and evaluating lessons; (4) interviewing students; (5) observing self-videotapes and writing reflective journals; and (6) revising lesson plans. As an individual learning process, prospective teachers must "prepare for and writing lesson plans individually" in the first step of the entire SCDD activities. They have to adapt existing lesson plans, and refer to curriculum standards and teachers' guide books. As a social learning process, prospective teachers need to interact with their peers, supervisor, school mentor, other practicing teachers and school students. Individual preferences and differences are revealed and discussed, which becomes a natural motivation for teachers to reflect on their prior lesson design and implementation, and seeks realistic and effective ways to improve. The second, third and fourth steps provide the opportunity to do so. The fifth step makes prospective teachers "reflect on what they do", while the last step is a great opportunity for prospective teachers to cumulate all their achievements during SCDCL activities.

In an empirical study aiming at developing prospective mathematics teachers' technological pedagogical content knowledge, the first author (Yuan, 2012) designed and carried out SCDD activities based on the above six-step model. Many positive changes occurred. For instance, three prospective teachers' overarching conceptions about the purposes of integrating information technology into mathematics teaching changed significantly. Teacher 1 changed from focusing on interest to understanding. Teacher 2 changed from focusing on teachers to both teachers and students. Teacher 3 changed from vaguely paying attention to interest and understanding to clearly focusing on interest and understanding.

We believe that SCDD activities are effective ways to promote prospective teachers' professional development. Especially, the six-step model of SCDD activities involving both prospective and practicing teachers is quite effective since it embodies that learning is both an individual and a social process. It also guarantees that teacher learning in and through this model is both collaborative and reflective, which has been proven to be essential features of productive professional development for teachers (Chazan et al., 1998; Roth & Tobin, 2004).

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Chapter 19

Exploration into Chinese Mathematics Teachers' Perceptions of Concept Map

JIN Haiyue LU Jun ZHONG Zhihua

Concept map has been advocated as an effective tool for teaching, learning, and assessment of conceptual understanding in science education. Although the term "concept map" might not be new to many Chinese mathematics teachers, few of them are familiar with its uses and fewer apply it in educational environments. How Chinese mathematics teachers feel about concept map and how they would use and incorporate it in teaching are issues of interest. This study investigated Chinese mathematics teachers' perceptions of concept map, and their incorporation of concept map in mathematics teaching through lesson plans and practical teaching. With brief introduction, both the prospective and in-service teachers were positive with the use of concept map and indicated a willingness to try it in their classroom teaching. But to actually apply concept map in practice, the teachers, especially the in-service teachers, were to a degree hesitant due to some practical reasons, such as curriculum schedule and pressure from exam. They needed more operational training on concept map to guide their trial and solid evidences to convince them the effectiveness of concept map. Successful initial experience seemed to be a key for teachers' willingness to use concept map in future teaching.

Keywords: Chinese prospective and in-service mathematics teachers, teacher perception, concept map, survey, mathematics lesson plan

1. Introduction

Concept maps (Novak & Gowin, 1984) are graphic representations that use nodes representing concepts and labeled links denoting relationships among nodes. Over the last three decades, educational researchers have devoted significant interest to the use of concept map as instructional tools (see Jegede, Alayemola, & Okebukola, 1990; Hortons, etc, 1993). Concept map has also been used by school teacher as advance organizers, as aids for lesson display and course development, and as a means to integrate information (e.g., Ollerton, 2001; Malone & Dekkers, 1984). In China, however, not much attention has been given to the uses of concept map in school settings as well as in research. There are literature reviews concerning the applications of concept maps in general (e.g., Zhu, 2002) and analyses of the feasibilities (e.g., Wang & Tang, 2004). But we can hardly find published works on the practical uses of concept map especially in the domain of mathematics education. The term *concept map* might not be new to many Chinese mathematics teachers, but few of them are familiar with its uses and fewer apply it in practice.

For many years, Chinese mathematics teachers have their own ways of organizing lessons and teaching. For example, for the teaching of new concepts or knowledge, the "standard" steps are *review relevant knowledge*, *specify goals of the lesson*, *present the new content*, and *foster understanding of the new content*. In classroom teaching, Chinese teachers are used to clearly and concisely write the focal points, difficult points, and hinges of the lessons on blackboard line by line. Such teaching writing on blackboard is called "banshu (板)", which is recognized as one of the basic teaching skills required for teachers in China (Nan & Yin, 2003). On the other hand, Chinese students were among the top performers in a number of international comparative studies such as the Third International Mathematics and Science Study (TIMSS) and the International Assessment of Educational Progress (IAEP) studies (Fan & Zhu, 2004). Without external pressure, it does not seem necessary for Chinese teachers to adopt unfamiliar techniques since their traditional ways of teaching have shown to be "successful" in some sense.

The traditional teaching methods have limitations as well as strengths. Taking conceptual variation, which has proved to be an effective way of promoting mathematics learning (Gu, Huang, & Marton, 2004), as an example. It is an important method through which students can learn mathematical concepts from multiple perspectives. Traditionally, it employs varying instances to highlight the essence of a mathematical concept and clarify its invariant and variant features. However, it usually deals with individual concepts or at most a very limited number of concepts at a time. Students may have difficulty seeing the big picture about how the concepts are organized together. Such difficulty might be addressed by using concept map. Concept map deals with a relatively large number of concepts by presenting the relations among the concepts. In other words, if conceptual variation takes care of the depth of knowledge of individual concepts, concept map contributes to the breadth of knowledge of concepts by showing how they connected with other concepts within a domain. Seeing connections among pieces of information is essential to the ability to use knowledge flexibly and appropriately in different settings (e.g., Bransford, Brown & Cocking, 1999; Kilpatrick, Swafford & Findell, 2001). By specifying the connections using concept map, teachers can help students organize the knowledge learned in a more effective manner.

Another reason that may account for the unpopularization of concept map is teachers' lack of access to concept map. We cannot assume that school teachers can read research papers as researchers did, nor can we expect teachers to be enthusiastic about trying new techniques by themselves in practice. One way to systematically introduce concept map to school teachers is through educational authorities. However, this is not presently possible for our project. Besides, it is often the case in practice that teachers, due to time and energy constraints, are only likely to learn new technique in a brief manner. That is why the present study decided to introduce concept map briefly rather than comprehensively to the prospective and in-service mathematics teachers.

Researchers have reported their investigations into students' perceptions toward the use of concept map in science education (e.g., Kankkunen, 2001; Mohamed, 1992; Wang, 2005). Their findings were mostly positive. The students, including primary students, middle school students, and university students, expressed moderate to high levels of agreement on the usefulness of concept map as a teaching method, as a learning strategy, and/or an assessment device. Some students even indicated their preference for their teachers to use concept map in classroom (Wang, 2005). Few studies have examined teachers', especially mathematics teachers' perceptions of concept map. How mathematics teachers appreciate the uses of concept map and whether they are willing to use it for educational purposes in the future is a subject worthy of study. As an exploration, in this chapter, we set out to answer two fundamental key research questions:

- What is mathematics teachers' perception of concept map after a brief introduction?
- How do mathematics teachers incorporate concept map in teaching?

2. Methods

In order to answer the above research questions, a survey study and a case study were conducted. In the survey study, both prospective and in-service teachers' perceptions of concept map were examined. These participants were first prepared with basic knowledge about what a concept map is, how to construct a concept map, and its uses as a teaching/learning strategy and/or an assessment device. Their perceptions of concept map were then collected through a questionnaire which was self-designed by referring to the existing literature.

After the questionnaire, randomly selected prospective and in-service teachers were interviewed so as to obtain detailed information about their views on the use of concept map in mathematics in general. The interviews were one-to-one and were tape-recorded.

Concerning on the second research question, eleven randomly selected prospective teachers were required to select a topic from the primary, secondary, or high-school mathematics textbooks and design a lesson plan to incorporate concept map. The questions used in the interview were assigned to these eleven prospective teachers as open-ended questions. With the experience in lesson plan design, they would have more insightful views toward concept map. Two in-service teachers were asked to use concept map in their classroom teaching and, after the trial, write a journal to describe their uses of concept map and their feelings with the experience.

2.1 *Participants*

The participants of the survey study consisted of 173 normal school students, 176 master students who enrolled in mathematics education courses, and 55 in-service mathematics teachers from different schools in Jiangsu, China. Three normal school students and 10 master students did not respond to item 9 to 28, which were located on the back side of the paper, in the questionnaire. These 13 participants' responses to the questionnaire were removed from analysis. A few participants missed one or two items. Since the number of missing values was small, the missing values were replaced with the item means so that the sample size of the data would not be reduced (Little & Rubin, 1987). The normal school students and the master students are categorized as prospective teachers since they have declared to be a mathematics teacher. The in-service teachers were receiving summer training courses from July to August 2012.

Table 1 presents the background information of the participants, including their gender, age group, grade (for prospective teachers), and school (for in-service teachers).

As shown in Table 1, the number of female prospective and in-service teachers is more than twice the number of male prospective and in-service teachers. This is consistent with the general situation in China that there are obviously more female teachers than male teachers especially in kindergarten and primary schools (e.g., Wang & Tang, 2006; Li, 2005). The years of teaching experience of the in-service teachers ranged from 1 year to 24 years. Nearly half of the in-service teachers had 6 to 10 years teaching experience.

The two in-service teachers of the classroom trial did not participate in the survey study. One of the teachers was a secondary school mathematics teacher, female, aged 31. She was taking master courses in a normal university during the data collection period. Her master thesis proposal was about the use of concept map as an assessment technique in mathematics. She had some background knowledge about concept map. The other in-service teacher was the head of mathematics department of a high school, male, aged 43. He helped with the data collection of one of the first author's studies on concept map. He had a general idea about

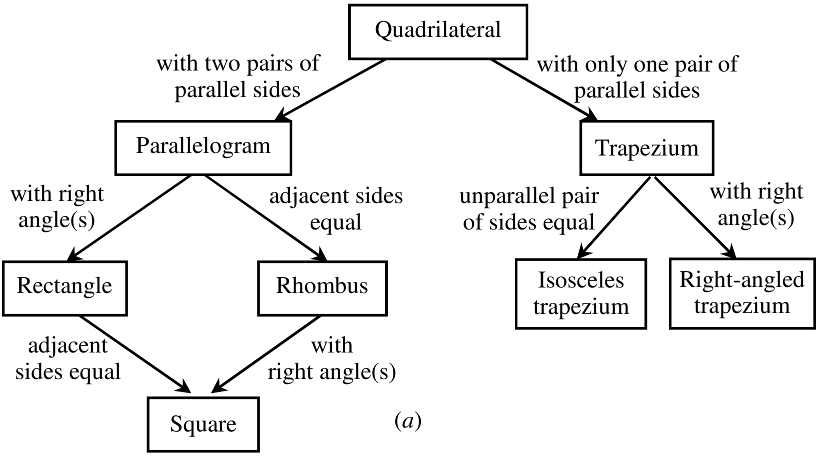
Table 1. Background information of the participants

<table>
<tr><th></th><th></th><th></th><th>No. of participants</th><th>Percent</th></tr>
<tr><td rowspan="5">Prospective Teachers (university students) (n=173)</td><td rowspan="2">Gender</td><td>Male</td><td>68</td><td>39.3%</td></tr>
<tr><td>Female</td><td>105</td><td>60.7%</td></tr>
<tr><td rowspan="2">Grade</td><td>Year 2</td><td>36</td><td>20.8%</td></tr>
<tr><td>Year 3</td><td>137</td><td>79.2%</td></tr>
<tr><td>Age group</td><td>Under 25 years old</td><td>173</td><td>100%</td></tr>
<tr><td rowspan="7">Prospective Teachers (master students) (n=176)</td><td rowspan="2">Gender</td><td>Male</td><td>40</td><td>22.7%</td></tr>
<tr><td>Female</td><td>136</td><td>77.3%</td></tr>
<tr><td rowspan="3">Grade</td><td>Year 1</td><td>72</td><td>40.9%</td></tr>
<tr><td>Year 2</td><td>65</td><td>36.9%</td></tr>
<tr><td>Year 3</td><td>39</td><td>22.2%</td></tr>
<tr><td rowspan="2">Age group</td><td>Under 25 years old</td><td>135</td><td>76.7%</td></tr>
<tr><td>26 to 30 years old</td><td>41</td><td>23.3%</td></tr>
<tr><td rowspan="9">In-service Teachers</td><td rowspan="2">Gender</td><td>Male</td><td>13</td><td>23.6%</td></tr>
<tr><td>Female</td><td>42</td><td>76.4%</td></tr>
<tr><td rowspan="4">School</td><td>Primary</td><td>24</td><td>43.6%</td></tr>
<tr><td>Secondary</td><td>13</td><td>23.6%</td></tr>
<tr><td>High school</td><td>12</td><td>21.8%</td></tr>
<tr><td>College or University</td><td>6</td><td>10.9%</td></tr>
<tr><td rowspan="3">Age group</td><td>Under 25 years old</td><td>6</td><td>10.9%</td></tr>
<tr><td>26 to 30 years old</td><td>40</td><td>72.7%</td></tr>
<tr><td>31 to 35 years old</td><td>9</td><td>16.4%</td></tr>
</table>

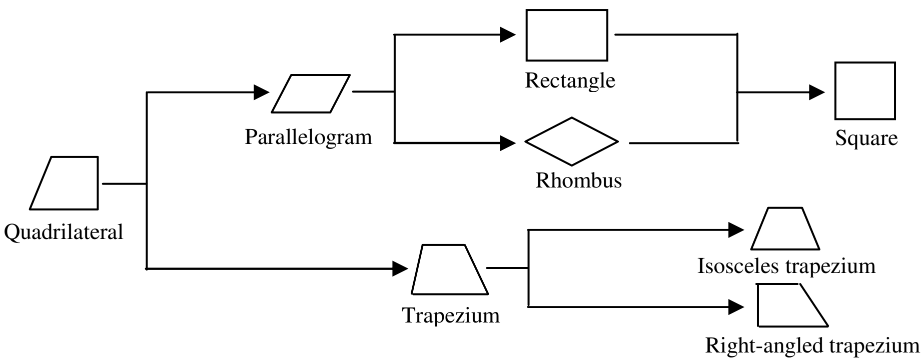
the definition and applications of concept map. He was positive with the uses of concept map in mathematics before the trial. Both teachers never used concept map in teaching. They were given one week time to apply concept map in at least one of their lessons. They decided on their own when and how to use concept map in class.

2.2 *Introduction of Concept Map*

Concept map is a two-dimensional map which using nodes representing concepts and lines denoting the relations between pairs of nodes. In Novak and Gowin's (1984) definition, the nodes should be hierarchically arranged and the lines should be unidirectional and labelled with linking phrases. Their definition is later on modified by other researchers for different purposes. For example, researchers from the semantic-network tradition prefer non-hierarchical concept maps such as "spider maps" (e.g., Harnisch, Sato, Zheng, Yamagi & Connell, 1994); the lines of the concept maps in Barenholz and Tamir's study (1992) are non-directional. In the present study, concept map does not limit to its original definition given by Novak and Gowin (1984). The nodes can be mathematical concepts, examples and non-examples of the concepts, diagrams, symbols, and formulas; the lines can be labeled or unlabeled, directional or non-directional; and the structure can be either hierarchical or non-hierarchical. In the introduction, we provided two examples of concept map (see Figure 1, translated from Chinese).



(*a*)



(*b*)

Figure 1. Examples of concept map (translated from Chinese)

The first example is a concept map with top-down hierarchy and labeled links. The second example is similar to a knowledge framework given at the end of the chapter of quadrilaterals in the Chinese Secondary 2 mathematics textbook (People's Education Press) summarizing the relations between the key concepts taught in the chapter. It has a hierarchy that goes from left to right and its links are unlabeled. This second example helps the participants to understand this western term concept map by linking it with what they are familiar with. Broadly defined, the framework in the textbook can be classified as a type of concept map.

Concept map can be used for teaching/learning and/or assessment purposes. Before the implementation of the questionnaire, we briefly introduced the uses of concept map as follows:

- When used for teaching, concept map can serve as an advance organizer providing students with a context to incorporate new knowledge; it can be used as aids for lesson display; it can be used to summarize the newly learned knowledge at the end of a lesson; and it can also be used as an aid in review to organize what have been learned and to optimize students' knowledge structure.
- When used as a learning strategy, concept map can make students to build connections between prior and new knowledge; it can provide an overview of knowledge learned; and it can also serve as a meta-cognitive strategy to help with students' self-learning and self-reflection.
- When used as an assessment technique, concept map can serve as an external representation of students' knowledge structure; it can specialize students' understanding of connections among concepts of a certain topic; and students can even learn in the concept mapping process.

The introduction took about 10 minutes. It tried to provide the participants a brief view of the attributes and uses of concept map. The introduction did not specify the steps for constructing a concept map and allow much time for practice. Instead, it illustrated how to incorporate the concept *kite* into the quadrilateral concept maps in Figure 1. This can provide the participants with a general impression on how concept map is constructed.

In summary, the participants may have different thoughts about concept map before this study. This introduction would prepare the participants with a unified conception of concept map before they answered the questionnaire.

2.3 *Perceptions of Concept Map Questionnaire*

Information about the participants' prior experiences with concept map was gathered through the following three multiple choice questions. Such information could help us better understand the participants' responses.

- Do you know concept map before this instruction?
  Yes/No
- Have you ever tried concept map in your teaching/learning?
  Never/Once/Sometimes/Often
- Have you ever tried similar knowledge structure(s) in your teaching/learning?
  Never/Once/Sometimes/Often

Since there is no validated questionnaire in the literature concerning mathematics teachers' perceptions of concept map, the perceptions of concept map questionnaire in this study was designed by referring to Mohamed's (1992) attitude toward concept map questionnaire in science and Kankkunen's study (2001) which collected students' opinions about concept map through inquiry and interviews. The questionnaire consisted of three subscales: *interest on concept map, appreciation of the usefulness of concept map, and willingness to use concept map*. The face validity was checked by three mathematics educators in normal universities. It included 22 items among which 6 were negatively worded. It used a 5-point Likert Scale ranging from 1 (Strongly Disagree) to 5 (Strongly Agree). Reverse scoring was applied to the negatively stated items. Table 2 provides an overview of the subscales and their corresponding Cronbach's α (based on standardized items).

Table 2. Aspects of the attitudes toward concept map questionnaire and its corresponding item numbers

No.	Aspects	Cronbach's α
1	Interest on concept map	0.751
2	Appreciation of the usefulness of concept map	0.713
3	Willingness to use concept map	0.765

The Cronbach's alphas of the aspects *interest on concept map*, *appreciation of the usefulness of concept map,* and *willingness to use concept map* show that these aspects have acceptable internal consistency (Cronbach's α >.70, see Hair, Anderson, Tatham & Black, 1998, p. 730).

2.4 *Informal Interview and Open-ended Questions*

After the questionnaire, four prospective teachers and four in-service teachers were randomly selected and interviewed informally. The interview concerned on the following six questions. These questions dealt with the teachers' perceptions of concept map in general. The first three are about the feasibility of using concept map in school settings; the fourth and fifth questions are relevant to the teachers' willingness to use concept map in classroom; and the sixth question is about students' feelings about concept map from teachers' perspective.

(1) How do you prefer to introduce to you concept map in details? Concentrated training or providing reference materials for self-study?
(2) How do you find the feasibility of using concept map for mathematics education?
(3) Which area do you think concept map is more suitable, for teaching, learning, or assessment?
(4) Would you use concept map in teaching?
(5) Would you like to introduce concept map to your students and encourage them to use it for learning mathematics?
(6) From your point of view, how will the students think about concept map?

The same questions were assigned to another eleven prospective teachers as open-ended task after they developed a lesson plan incorporating concept map in different stages of teaching. Their lesson plans and responses to the questions were sent back to the researchers through emails.

3. Results

3.1 *Perceptions of Concept Map Questionnaire*

The participants' experience with concept map before the introduction in this study is described in Table 3. A few participants did not respond to some of the questions. Hence, the sum of the percents for each question is not necessarily 100% in Table 3.

Table 3. Participants' experience with concept map before introduction

			Number of participants	Percent
Prospective Teachers (university students) (n=173)	Do you know concept map before this introduction?	Yes	143	82.7%
		No	30	17.3%
	Have you ever tried concept map in your teaching/ learning?	Never	95	54.9%
		Once	41	23.7%
		Sometimes	32	18.5%
		Often	4	2.3%
	Have you ever tried similar framework(s) in your teaching/learning?	Never	64	37.0%
		Once	55	31.8%
		Sometimes	46	26.6%
		Often	7	4.0%
Prospective Teachers (master students) (n=176)	Do you know concept map before this introduction?	Yes	127	72.2%
		No	47	26.7%
	Have you ever tried concept map in your teaching/ learning?	Never	70	39.8%
		Once	34	19.3%
		Sometimes	59	33.5%
		Often	12	6.8%
	Have you ever tried similar framework(s) in your teaching/learning?	Never	34	19.3%
		Once	52	29.5%
		Sometimes	70	39.8%
		Often	19	10.8%
In-service Teachers	Do you know concept map before this introduction?	Yes	49	89.1%
		No	6	10.9%
	Have you ever tried concept map in your teaching/ learning?	Never	7	12.7%
		Once	12	21.8%
		Sometimes	27	49.1%
		Often	9	16.4%
	Have you ever tried similar framework(s) in your teaching/learning?	Never	1	1.8%
		Once	6	10.9%
		Sometimes	38	69.1%
		Often	10	18.2%
All Participants	Do you know concept map before this introduction?	Yes	319	79.0%
		No	83	20.5%
	Have you ever tried concept map in your teaching/ learning?	Never	172	42.6%
		Once	87	21.5%
		Sometimes	118	29.2%
		Often	25	6.2%
	Have you ever tried similar framework(s) in your teaching/learning?	Never	99	24.5%
		Once	113	28.0%
		Sometimes	154	38.1%
		Often	36	8.9%

The result shows that most of the participants have heard of *concept map* before this study. The majority of the in-service teachers (over 70%), however, have once or sometimes used concept map or similar frameworks in their teaching. Over 15% even indicated that they used concept map or similar knowledge structure often. The participants' familiarity with concept map was beyond our expectation. The finding suggests that concept map is not a brand new technique for the participants although it was not formally introduced in schools. This might be because, broadly defined, the knowledge frameworks (such as Figure 1(*b*)), which was called 结 in Chinese, is a type of concept map. On this basis, the participants may realize that they actually know concept map though they might hear this term for the first time.

The master students' experience with concept map is similar to that of the normal university students, except that they seemed to have more experience trying concept map in teaching/learning. Almost all the master students did part time job in some tutorial schools or worked as private tutor. They had more teaching experience than the university students. Some of the master students indicated that they once drew diagrams similar to concept map to help their students to clarify relationships among concepts, formulas, and theorems.

Since the participants had difference teaching experience and also different experience with concept map (see Table 3), the analysis considered the difference of the perception of concept map between pre-service teachers and in-service teachers.

Since the three aspects, i.e., *interest on concept map*, *appreciation of the usefulness of concept map*, and *willingness to use concept map*, had Cronbach's alpha greater than 0.7 (see Table 2), a mean score was calculated by averaging the means of the items, with the scores of the negatively worded items being reversed. The mean scores and standard deviations of the three aspects for the prospective teachers (university students), prospective teachers (master students), and in-service teachers are shown separately in Table 4. Since a 5, 4, 3, 2, 1 scoring mode was adopted, the mean score of 3.0 was taken as a benchmark above which the participants is said to be in favor of concept map and below which the participants is categorized as not being in favor.

The means reported in Table 4 indicated that the participants were generally positive with concept map. They were interested in concept map and agreed with the usefulness of concept map. They also indicated

Table 4. Mean scores and standard deviations of three aspects of attitudes toward concept map.

Aspects	All participants (n=391)	Prospective teachers (university students) (n=173)	Prospective teachers (master students) (n=163)	In-service teachers (n=55)
Interest on concept map	3.80 (0.43)	3.75 (0.45)	3.83 (0.42)	3.85 (0.38)
Appreciation of the usefulness of concept map	3.95 (0.43)	3.98 (0.44)	3.91 (0.44)	3.97 (0.38)
Willingness to use concept map	3.76 (0.39)	3.75 (0.42)	3.75 (0.38)	3.80 (0.32)

their willingness to use concept map in the future. The in-service teachers showed slightly higher interest on concept map and stronger willingness to use concept than the prospective teachers, though the differences were not statistically significant as shown by one-way ANOVA. A possible reason for the differences might be that the in-service teachers in this study were those who took training courses during summer vacations. They had shown a motivation to learning more for their teaching. Likewise, they were more willing to accept concept map than others. Besides, the in-service teachers were more experienced in teaching than the prospective teachers; they may know better about how concept map could contribute to mathematics education.

3.2 *Interview and Open-ended Questions*

The participants' responses to the interview and the open-ended questions were analyzed together since the same questions were used and the findings were similar. The participants were encouraged to express their ideas in accordance with, but not restricted to, the questions. For example, for the question "would you use concept map in teaching?" they were told that it would be valuable if they could identify the factors that motivated them to adopt concept map or prevented them from using concept map. Their responses to the interview and open-ended questions provided additional information to the questionnaire. The findings were summarized below.

First, concerning on the introduction, both the prospective teachers and in-service teachers indicated that the 10-minute introduction did not give them substantial ideas about how to apply concept map in practice. They hoped that further training on concept map can show them model lessons or concrete examples of its applications. Both concentrated training and reference materials were welcomed to help them learn more about concept map.

Second, the participants seemed to find using concept map as a teaching method more feasible than using it as a learning strategy or as an assessment technique. They indicated that, if teachers can use concept map appropriately to clarify the relationships among concepts in classroom teaching, their students would benefit from it in organizing knowledge in a more effective manner. At the same time, they admitted that concept map may be useful for teaching some, but definitely not all, mathematics topics. They were not yet sure which topics were more suitable for adopting concept map.

Third, the participants listed following four factors that may prevent teachers from adopting concept map for mathematics education.

(1) *Access to concept map*. They seldom had access to concept map systematically. Few teachers around used concept map in schools. They had no examples to follow. Comparing with trying this new technique, they preferred to use their traditional teaching methods. This is consistent with their claims that they needed model lessons or concrete examples to help them get started with concept map.

(2) *Research-based evidence*. There is a lack of research-based evidence to convince them that concept map is useful for Chinese students as well as for students in western countries. Though there are experimental studies supporting the use of concept map for mathematics education, the studies are mostly conducted in western countries. They were not sure whether concept map is suitable for the situation in China.

(3) *Current evaluation system*. The evaluation systems in China pay much attention to results rather than processes. The strength of concept map is to display a process of building connections; on the other hand, concept map is not a task type in large-scale examinations, e.g., the university entrance examination in China. Therefore, they could hardly see how concept map would contribute to students' performance in examinations.

(4) *Tight curriculum schedule*. The strength of concept map is mainly embodied in students' active involvement. Hence, presenting directly teacher-constructed concept map may not contribute much to students' learning. But it would be time-consuming if asked students to construct concept map individual or cooperatively in class. The tight curriculum schedule does not allow much time for incorporating such activities in classroom teaching.

Fourth, the participants were not sure whether they would like to introduce concept map to their students and encourage students to use it for learning mathematics since they themselves were not yet familiar with the applications of concept map. They mentioned that whether students would accept concept map may mainly depend on how they appreciate the usefulness of concept map at the initial stage. Hence, teachers needed to be careful when they introduced concept map in class.

3.3 *Lesson Plans*

Eleven prospective teachers were asked to develop a lesson plan to illustrate how he/she would incorporate concept map in the course of instruction. The topics they selected for the lesson plans and their grade levels are shown in Table 5, together with a brief description of the incorporation of concept map.

For *quadrilateral*, three prospective teachers' teaching procedure generally covered the following five steps: (1) review relevant concepts and construct concept map, (2) class/group discussion and polish concept map, (3) exercises and return to concept map, (4) knowledge transfer and expand concept map, and (5) summarize and review concept map. The three lesson plans were all well-organized and made good uses of concept map. On one hand, quadrilateral is a typical mathematical topic where clarifying relations among the special types of quadrilaterals and their properties is key for the study of this topic. Many studies (e.g., Mansfield & Happs, 1991) in the literature also used quadrilaterals as a carrier for the applications of concept map. On the other hand, in the introduction, we mentioned quadrilaterals as an example for illustrating the features of concept map. The participants may find it easier to start with this topic. In fact, their concept maps were quite similar to the one

Table 5. Incorporation of concept map in the lesson plans

	Topic	Incorporation of concept map
1 2 3	Quadrilateral (Grade 8)	Three prospective teachers chose the same topic *quadrilateral.* They used concept map in the same manner for review lessons. Concept map was used as a classroom activity. They planned to ask students to work either individually or cooperatively to construct a concept map and then polish their concept maps after discussion.
4	Logarithm (Grade 10)	Two concept maps were constructed in the lesson preparation. One was used as an aid to clarify the position of the section *logarithm* in the chapter; the other was used as a flow chart displaying the sequence of the lesson. The concept maps were not presented to students in class.
5		Concept map was used at the end of the lesson summarizing the new knowledge taught in class and clarifying the relationships between logarithm and other relevant concepts.
6	Judgment of parallelogram (Grade 8)	Concept map was used at the end of the lesson helping students to clarify the connections between the properties of special types of parallelograms and the theorems. It was planned to be constructed by cooperation of teacher and students.
7	Rational numbers and irrational numbers (Grade 7)	Concept map was used at the beginning of the lesson summarizing students' prior knowledge of numbers. It was then presented at the end of the lesson assimilating the newly taught knowledge. New connections were added.
8	Four kinds of propositions (Grade 10)	Concept map was used at the end of the lesson. It was assigned to students as a fill-in-the-blank task, helping them to clarifying the relations among the four kinds of propositions.
9	Positions between line and plane (Grade 11)	The first concept map was teacher-constructed and was presented at the beginning for the introduction of the new lesson. The newly taught knowledge was assimilated into the concept map at the end of the lesson.
10	Plane vectors (Grade 11)	This lesson plan was too general. The prospective teacher mentioned only that concept map was used to build connections among relevant concepts.
11	Area of circle (Grade 5)	Concept map was used at the beginning of the lesson helping students to build connections among the area formulas of rectangle, square, parallelogram, triangle, and trapezium. With the concept map, the prospective teacher expected students to guess the derivation of the area formula of circle. The area formula of circle was assimilated into the concept map at the end of the lesson.

presented in Figure 1(*b*).

Two prospective teachers chose the topic *logarithm*. One constructed concept maps for her own use in lesson preparation. One of her concept maps is presented in Figure 2. The nodes in the map were not limited to concepts; they included phrases or even sentences. Some of the links were labeled while some others were not. The prospective teacher even used different shapes to represent different types of nodes. The concept map severed as a flow chart indicating the sequence of the lesson, including the introduction to the new lesson, definition of the new operation, relation between logarithm and exponent, properties of logarithm, variations and examples. With the concept map, the teacher and the readers could quickly pick up the sequence of the lesson. The other prospective teacher mentioned concept map only at the end of the lesson plan. Her description of the incorporation of concept map was quite general. She repeated the ideas mentioned by the researchers in the introduction that concept map could be used for summarizing the new knowledge taught in class and clarifying the relationships between logarithm and other relevant concepts. No concept map was shown in the lesson plan.

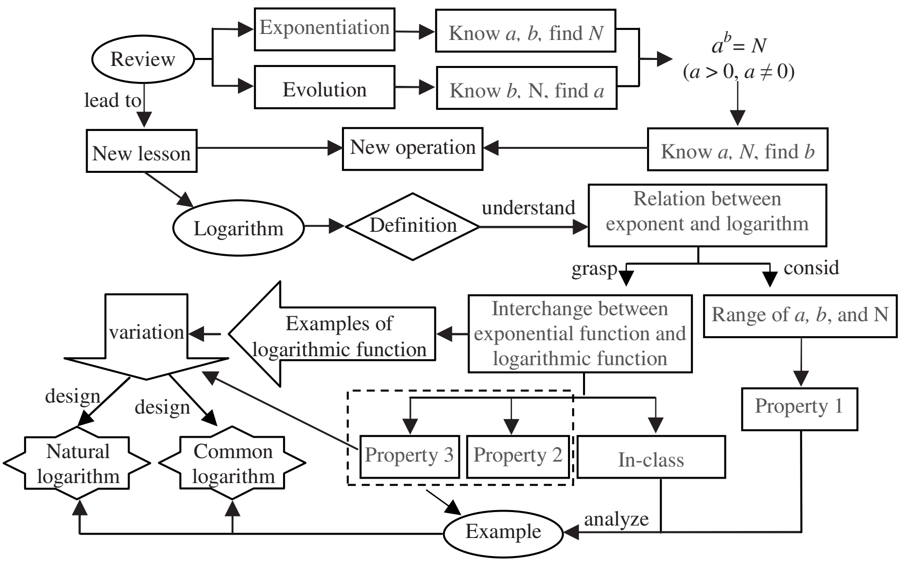


Figure 2. A concept map showing the sequence of a lesson of logarithm (translated from Chinese)

In the lesson plans of *judgment of parallelogram*, *rational numbers and irrational numbers*, *positions between line and plane* and *area of circles*, concept map was used as aids to help students build connections between prior knowledge and the new knowledge. At the beginning of the lesson, the prospective teachers used concept map as an advance organizer to introduce to the new topics. The concept maps were either teacher-constructed or constructed in class by cooperation of teacher and students. These maps were then revised after the new topic was taught to assimilate the new knowledge. New nodes or connections would be added. In this process, students would see the connections between the new knowledge and what they already knew. For example, the revised concept map of *rational numbers and irrational numbers* is shown in Figure 3. Compared with the earlier version, the bi-directional links between *integral numbers, fractions*, and *decimal numbers that can be transformed into fractions* were newly added. The concept map helped students make clear the relations and build upon their existing knowledge. The prospective teachers also paid attention to students' involvement in the construction of the concept maps.

Concept map was used as an assessment task in the lesson plan of *four types of propositions.* The four types of propositions were original proposition, converse proposition, negative proposition, and converse-negative proposition. After the propositions were introduced, the prospective teacher assigned a fill-in-the-blank concept map task to students. The task required students to fill in the blanks in a given concept map (see Figure 4). The prospective teacher pointed out that she sometimes used similar frameworks in her learning of mathematics, especially when she needed to figure out the relations among similar or relevant concepts. But she did not realize that it was what is referred as *concept map*.

One prospective teacher chose the topic *vector*. After introducing the definition of vector, its representation, special vectors, and relations among vectors (e.g., parallel vectors, equal vectors, and collinear vectors), she asked students to work in groups and construct a concept map cooperatively. After that, they discussed the strengths and weaknesses of the student-constructed concept maps. After in-class exercises, she asked students to polish their concept maps and add

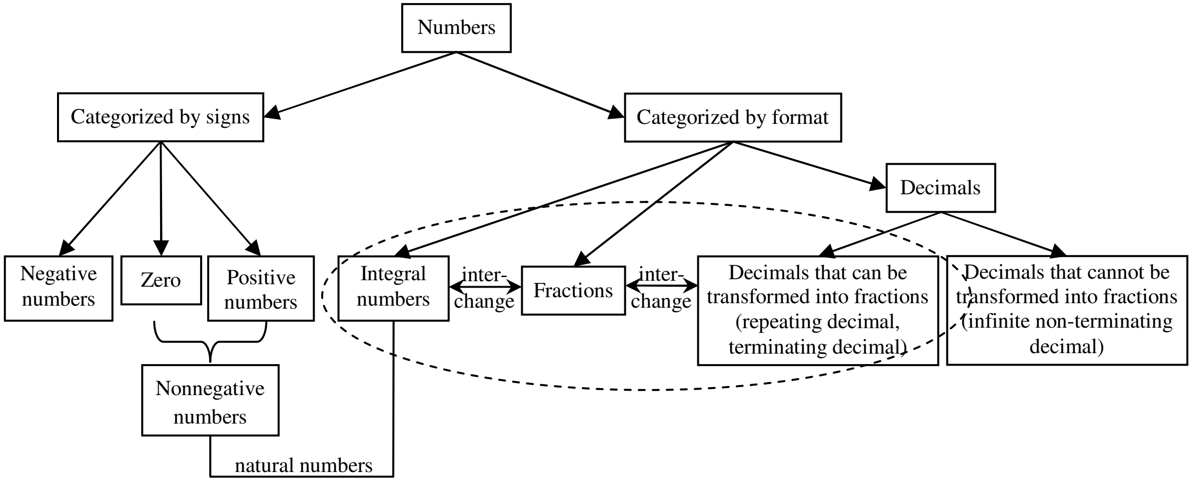


Figure 3. An expanded concept map showing the connections among numbers (translated from Chinese

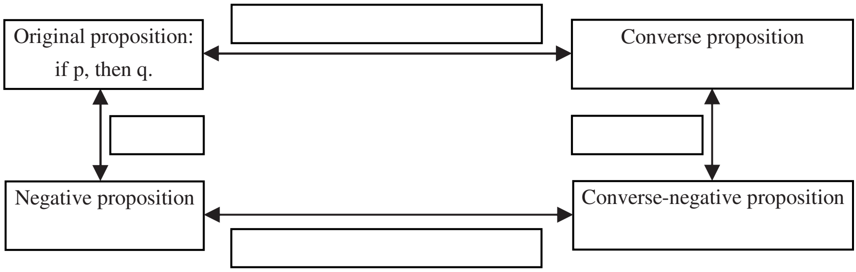


Figure 4. A concept map showing the connections among four types of propositions (translated from Chinese)

connections between the concepts and other relevant concepts. However, the prospective teacher did not provide any concept map in the lesson plans and the descriptions were quite general. It seems that she mentioned concept map merely to meet the requirement of the task.

Compared with the other participants' responses in the interview, the prospective teachers who seriously considered the use of concept map in the lesson plans provided more detailed feedbacks. Particularly, concerning on the usefulness of concept map, they indicated that asking students to construct concept maps would get students more involved in the class. When students worked in groups, cognitive conflict may be caused; this could encourage students to reflect on their learning and learn from each other.

3.4 *Classroom Trials*

The researchers did not observe on their own the two in-service teachers' classroom teaching. The information was mostly gathered through the teachers' journals wrote after the trials and their responses to the open-ended questions. The informal talks with the two teachers also helped to reveal their real thoughts about concept map.

3.4.1 *Case of the Secondary School Mathematics Teacher*

The secondary school teacher used concept map to clarify the relations among a set of statistic concepts, i.e., population, sample, sample size, individual, median, mean, mode, and standard deviation, in class. She stated that concept map was useful to describe the relations; but it was a bit time-consuming to draw the map and describe the meaning of the links in the map. Compared with concept map, she found Venn diagram to be better in showing the relations among population, samples, and individuals since it was clear and straightforward for the students to get the set-subset relations. She preferred to employ formulas and concrete examples to explain the relations among mean, standard deviation, median, and mode. With concrete examples, the students could easily obtained the idea that mean, median, and mode were not the same thing though they may be equal in some special cases. Besides, they could apply the knowledge from the examples more directly in solving problems.

The secondary school teacher indicated that her trial of concept map was not successful as expected; but she was still interested in concept map and agreed that concept map could be very useful for teaching and also helpful for students' learning of some mathematics topics. Especially, since she had read a number of papers on the use of concept map as an assessment tool in science education, she believed that concept map could address students' mathematical understanding that was not easily detected by traditional school tests.

However, this secondary school teacher admitted that her passion on concept map may be mostly because of her master thesis. She needed to try concept map in her teaching to know better about it and also prepared her students with certain concept mapping skills for her data collection. With existing experience, she felt that it might be difficult for teachers and students to recognize its usefulness in the initial stage of using concept map. And since the mathematics curriculum was very tight, they did not have much time to try on concept map. Both teachers and students were more willing to accept a technique that can improve their performance or test scores within a relatively short period. If they felt the technique does not work, they would quickly drop it. From her point of view, concept map did not have such quick effect that can be easily appreciated by teachers and students. Hence, if not demanded, it would be difficult to popularize concept map in schools.

3.4.2 *Case of the High School Mathematics Teacher*

The high school teacher used concept map as an aid in review. His students were preparing for the upcoming university entrance examination. They were at the general review stage which required them to integrate the knowledge they had learned for solving problems. To be specific, the teacher used concept map to make clear the relations among equations, inequalities, linear function, quadratic function, parabola, curves, and their properties. After reviewed the definitions and properties of the concepts separately, he drew on the blackboard a concept map showing the relations among the concepts. As he built each link, he explained its meaning and, sometimes, he employed diagrams and examples to help the students understand why and how the concepts were connected.

The teacher said that he did not try this method for review before. He had once drawn similar links to show relations among three or no more than four concepts. The concept map he constructed this time involved about twelve concepts and also diagrams and examples. It was complex and time-consuming. He spent one hour preparing for the lesson, browsed through the textbooks, thought carefully about the possible connections, and considered how to explain the connections clearly to the students. And he spent another half an hour in class for the map drawing and the interpretation. He concluded that it was a rewarding experience. He himself could see much clearer the connections among the concepts. The concept map provided the students an integral picture about what they had learned about equations and functions; he believed that this teaching method was more efficient than providing the students isolated pieces of information.

The students' feedbacks after the lesson were positive. They seemed to find concept map new and interesting. The teacher indicated that he was willing to use concept map in the future; he would also like to prepare a model lesson to show the use of concept map in the school and recommend it to other mathematics teachers.

4. Discussion and Conclusion

Concept map is similar in form to the Chinese knowledge framework ( ). Many of the participants had heard of concept map before this study but they had little knowledge about its applications in classroom teaching. After the brief introduction to concept map, they realized that the knowledge framework, which they were familiar with, could actually be seen a type of concept map. With the introduction, though brief, they got a more systematic view of the features and applications of concept map.

The participants were generally positive with the use of concept map in mathematics education. This is consistent with the findings by Okebukola (1992) and Wang (2005). Okebukola (1992) investigated 141 Australian teachers' attitudes toward concept map and vee diagram as meta-learning tools in science and mathematics. The teachers responded to a questionnaire after a five-day workshop on strategies for improving teacher effectiveness. They showed favorable attitudes toward concept map in terms of its benefits in facilitating meaningful learning and reducing anxiety levels. Wang (2005) examined the attitudes toward concept map of Chinese normal university students who majored in chemistry. For one semester, the students were asked to constructed concept map after each chapter of their learning of physics. In the present study, the participants had relatively less experience with concept map; but with only brief introduction, they also appreciated that concept map would be a useful technique for teaching, learning, and assessment of mathematics. Comparatively, the prospective teachers seemed to be more positive than in-service teachers with the exploration of concept map in mathematics education. The former had not formally worked as a teacher in schools; hence, it might be easier for them to accept concept map since they are not yet bounded by the traditional ways of teaching, curriculum schedule, and exam pressure. If further study wants to explore the use of concept map in mathematics education, it may be easier to start from prospective teachers since they are less-bounded and have more time to digest the information.

The usefulness of concept map is a necessary but not a sufficient consideration for teachers to decide whether to adopt it in their classroom teaching; they need to take its feasibility and other real issues into account. Though the participants were generally interested in concept map, positive with its uses, and even indicated a wish to try it in the future in the questionnaire, their responses to the informal interview and the open-ended questions seemed to suggest that they hesitated to actually use concept map. They listed a few factors that prevented them from adopting it. The main reason behind might be that they were not yet confident with how concept map can be appropriately used in class and they doubted whether concept map can contribute to students' exam performances. They needed more operational training so as to direct their practice. Besides, few experimental studies on the use of concept map in mathematics education were conducted in China. The teachers were not sure whether concept map was suitable for Chinese students as well as for students in other countries. They needed solid evidences to convince them the effectiveness of concept map.

The 10-minute introduction on concept map might be too brief for some teachers to grasp the ideas about how to use concept map in practice. More concrete examples and evidence could be included to help them get more confident in practice. However, with this brief introduction, the participants seemed to know how to incorporate concept map in their teaching. As indicated by the prospective teachers' lesson plans, concept map had been used as aids for lesson preparation, as advance organizers, and as organizers for summarize or review learned knowledge. Eight out of the eleven lesson plans illustrated in details about when and how to incorporate concept map in classroom teaching. This finding is encouraging for the further popularization of concept map.

The classroom trials suggest that teachers' preliminary experience with concept map is key to whether they would like to use concept map voluntarily in the future. Richardson (1994) indicated that, when teachers try new activities, they normally follow their sense of what students need and what is working. If they feel the activity does not work, they may quickly drop or radically alter it. From this perspective, how concept map is introduced to the teachers plays an important role for the popularization of concept map since, without sufficient preparation, the chance for teachers to successfully practice concept map in classroom teaching may be reduced.

This study has a limitation related to the design of the perception of concept map questionnaire. The questionnaire was designed by the authors particularly for this study. It was not piloted. Though the Cronbach's α reported in Table 2 supported the internal consistency of the subscales, the questionnaire needs to be further modified to be a more reliable instrument for measuring teachers' perception of concept map. For example, some of its items are not suitable for the prospective teachers, especially the university students, since they had no or little teaching experience. As a result, many of them were not sure whether their students will be interested in concept map and whether the students would appreciate their use of concept map in the teaching of mathematics. Besides, more than half of the participants indicated that they were not sure whether they would still prefer their old ways of teaching mathematics, instead of using concept map in classroom teaching,. But as an instrument for exploration, the questionnaire does provide insights about the Chinese mathematics teachers' perceptions of concept map.

In summary, the prospective and in-service mathematics teachers were generally positive with concept map. With limited knowledge of concept map, they seemed to have a general idea about how to incorporate concept map in classroom teaching. Other than the basic knowledge of concept map to get teachers prepared technologically, further introduction of concept map needs to include concrete examples showing them how concept map can be implemented in different stages of classroom teaching and how it can be integrated with mathematics curriculum.

At the same time, it should make clear that different people may have different personalized way of organizing knowledge. The purpose of using concept map in classroom teaching is not to unify to a standard concept map and impose teachers' concept map on the students but to encourage students to think about connections of different concepts and to organize knowledge in an effective manner.

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Chapter 20

Assisting Teachers in Maintaining High-Level Cognitive Demands of Mathematical Tasks in Classroom Practices: A Training Course in Taiwan

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The study focuses on how training courses were designed and implemented by making the reflective use of research-based cases to assist in-service teachers in identifying and maintaining high-level cognitive demands of mathematics tasks in classroom teaching. Eight in-service teachers enrolling in a university course "Theory and Practice of Case Method (TPCM)" in summer M.A. program participated in the study. Data were mainly collected from case analyses, reflective journals and video-tapes of classroom observation. It is found that the use of research-based cases enhanced the teachers' awareness of differentiating levels of cognitive demand of tasks determining students thinking and their ability in maintaining high-level cognitive demands of tasks in classroom instruction. The major factors associated with maintaining the high level of cognitive demands were related to teachers' expertise of mathematics instruction, such as selecting and sequencing the tasks, selecting and sequencing students' various solution for advancing students' high-level thinking, and encouraging students to make mathematical connections between other student responses. The factors associated with the decline in the high level of cognitive demands were related to the tasks, teachers, students, and time to explore in classroom.

Keywords: Taiwan mathematics education, research-based cases, teacher education, cognitive demand, mathematics tasks

1. Introduction

The superior performance of students from Asian countries in international studies in mathematics achievement such as The Trends in

Mathematics and Science Study (TIMSS) (Mullis, Martin, & Foy, 2012) and Programme for International Student Assessment (PISA) (OECD, 2010) has drawn much attention. There are many variables related to students' achievement, but one of the most important and obvious factors which directly contribute to students' achievement is the quality of mathematical instruction. This implies that the quality of mathematical instruction in East Asia might be different from that in other countries in the world. It is important for the high-achieving countries in East Asia to share ideas about teaching strategies and techniques in classroom contexts as well as ways employed for its development.

There have been several published books that mainly focus on students' learning, curriculum, teachers, and mathematical instruction valued in high-achieving education systems in East Asia. For instance, Fan, Wong, Cai and Li (2004) published a book that focuses on the Chinese way of learning mathematics from insiders. Li and Shimizu (2009) organized a special issue in ZDM which focuses on exemplary mathematics instruction in six countries (i.e., Chinese mainland, Hong Kong, Japan, Singapore, South Korea, and Taiwan). Leung and Li (2010) edited a book which contributes to sharing ideas on the changes and practices in mathematics curriculum and teacher education in the six countries in East Asia. Li and Kaiser (2011) edited a book which focuses on the nature of expertise, and how expertise is theoretically conceptualized and empirically measured from Eastern and Western perspectives. The chapters in the books reveal that there are increasing platforms for high-achieving countries in East Asia to share classroom practices with the western countries. However, there has not been adequate research to share the ideas and practices in mathematics education in Taiwan. Much remains to be understood about the ways that are utilized to shape the quality of mathematics instruction for improving students' mathematics achievement in Taiwan. Thus, there is a need to have more studies to share the approaches and practices that are employed to develop teacher quality in mathematics instruction. This chapter focuses on the ideas of how a group of 8 teachers learn from a training course with the case-based method to identify the cognitive levels of instructional tasks and maintaining the high cognitive demands in classroom instruction.

The article begins with a brief introduction of three curricular reforms in Taiwan in the last two decades. It is followed by describing the significance of the use of cases in which supports in-service teachers to align with the tenets of curricular reforms. Third, Stein, Smith, Henningsen and Silver's (2000) Task Analysis Guide and the Mathematics Tasks Framework as the theoretical framework of the study are described in detail. The Task Analysis Guide is utilized as the guide for classifying the levels of cognitive demand required in mathematics tasks, while the Mathematics Tasks Framework (MTF) including three phases is employed as a framework of the study. The MTF was utilized to examine if teachers maintained high cognitive demands when the mathematical tasks were carried out. The research design and research results are displayed in the fourth and the fifth sessions. Finally, the article ends with a session of conclusion and discussion to address the factors in the influence on the levels of cognitive demands when mathematics tasks are carried out.

1.1 *Curricular Reforms of Mathematics in Taiwan*

The school mathematics of Taiwan has been undergone three major reforms in two decades (1990s-update). The first reform was from 1993 to 2000. The reform was a milestone of mathematics education in Taiwan. The drastic changes under the reform were based on socio-cultural and economical changes. Besides, the impact of constructivism in the realm of mathematics education was another reason (MOE, 1993). These changes included: (1) learning mathematics is viewed as an integrated set of intellectual tools of making sense of mathematical situations rather than as accumulating facts and procedures; (2) students are expected to attain mathematical power. Mathematical power involves the ability to explore, conjecture, and reason logically; to communicate about and through mathematics, and to connect ideas within mathematics and between mathematics; (3) instructional approach is student-centered instead of teacher-centered approach; (4) mathematics classroom is cultivated as mathematical communities rather than classroom as a collection of individuals; and (5) teacher is a problem poser instead of the sole authority for right answers (Lin, 2000; Tam, 2010).

Shortly after the implementation of the 1993 version curriculum, a new curriculum under the second reform was launched for grade 1 to 9 school mathematics in 2001. The curriculum under this reform was termed as "The Grades 1-9 Temporary Curriculum Guidelines" (MOE, 2001) highlights acquiring basic mathematical competency rather than only learning mathematics knowledge. The 2001 version of curriculum highlights the transition from primary to junior high school mathematics. The other highlight is to encourage teachers to develop school-based curriculum that meets students' needs in individual classroom (Chung, 2005).

In 1995, mathematicians expressed their worry and dissatisfaction with the curriculum on reducing important mathematics topics and decreasing the difficulty. Hence, they launched an innovation to revise the temporary guidelines by adding deeper and broader mathematics contents. As a result, the third curricular reform started in 2003. The Grade 1-9 Temporary Curriculum Guidelines was replaced by the Grade 1-9 Formal Curriculum Guidelines (MOE, 2003). The 2003 version of curriculum is continuously replaced by 2008 version with minor revision. It is noticeable that through the three curricula reforms, mathematical power remains the highlight of the school mathematics curriculum.

The change of mathematics teaching switches from traditional instruction approach to contemporary view of mathematics teaching. The contemporary view of mathematics teaching emphasizes mathematical power, conceptual understanding, and responding to individual students' experiences and needs instead of treating all students alike. The reform documents define new roles for teachings related to the issues of knowledge constructing and supportive scaffolding. The contemporary mathematics instruction involves high quality of mathematics teaching, such as guiding students to evaluate each other's thinking and promoting building of mathematical content over time.

Helping teachers move toward a contemporary view of teaching seems to require new experience. One of the ways to acquire the new experience is based on others' experiences. The use of cases that reflect aspects of classroom practice is one way to learn from others' real experience. Thus, the use of research-based cases is considered as a learning strategy for in-service teachers to develop their perspectives on identifying the cognitive levels of instructional tasks and improve their skill in maintaining high level of cognitive demands in classroom contexts.

2.1 *The Use of Cases in Teacher Education*

It is well accepted that curriculum reforms require teachers to have long-term support and adequate resources. Preston and Lambdin (1995) suggest that "reformed curricula seem likely to succeed only to the extent that teachers are helped to become knowledgeable about mathematics content, and well supported in their efforts to use new methods of instruction" (p.173). Their suggestion indicates that teachers need to be assisted in understanding the reform-oriented instructional approach, but the assistance needs to involve more than imposed prescriptions. Castle and Aichele (1994) further assert that the knowledge of curricular reform can be constructed by each individual teacher bringing her lived experiences as a learner. One way to provide lived experiences is through the use of cases (Harrington, 1995; Lin, 2002).

With the growing interest in the use of cases, the purposes of the use of case-based pedagogy vary with various teacher education programs and staff development programs in many countries (Dolk & den Hertog, 2001; Lin, 2005; Pang, 2011; Stein et al., 2000). For instance, cases can be dilemma-driven in which the cases portray problematic situations requiring problem identification, analysis, and decision-making (Kleinfeld, 1992). Such kind of cases aims to help teachers (1) to realize that teaching is an inherently dilemma-ridden enterprise and (2) to learn how to think about the trade-offs involved in selecting one course of action over another. Besides, cases can be exemplars to establish the best practice or to make the effective teaching more public and available for others to analyze and review (Merseth, 1996). Such kind of cases aims to assist teachers to develop (1) an understanding how the cognitive demands of mathematical tasks evolve during a lesson and (2) the skill of critical reflection on their own practice guided by reference to the Mathematics Tasks Framework (Stein et al., 2000).

The effects of the use of cases on teacher education include: (1) cases maintaining the cognitive demands and encouraging the generation of multiple pedagogical techniques. Cases are often seen as stimulants for preservice and in-service teachers to examine and discuss alternative instructional strategies and to construct new ones; (2) cases often foster reflection (Richert, 1991). Once teachers begin to view cases of various patterns of instructional tasks, they can begin to reflect on their own practice through the lens of the cognitive demands of tasks; and (3) cases help to present a realistic picture of the complexities of teaching. Cases open a window to illustrate the complicated aspects of teaching. Teachers become sensitive to important cues in teaching episodes (Merseth, 1996; Stein et al., 2000).

Studies on cases used in teacher education revealed that cases can enhance teachers' awareness of students' learning and becoming more reflective practitioners (Dolk & den Hertog, 2001; Lin, 2005; Pang, 2011; Stein et al., 2000). These studies also show that cases are more effective than traditional expository approaches to teaching since cases reflect real situations and pose problems and challenges for teachers (Barnett, 1998). However, these studies do not indicate that how the use of cases increases teachers' awareness of different levels of cognitive demands of mathematical tasks resulting in students' different thinking. Thus, to help teachers learn to differentiate levels of cognitive demand of instructional tasks through the use of case becomes the purpose of the study.

2. Theoretical Framework

Why are cognitive demands of instructional tasks important for student learning? It is not simply creating the opportunity by putting students into groups or by placing teaching aids in front of them. Rather, it is the mathematics tasks in which students engage determines what they will learn. For instance, tasks that require students to perform a memorized procedure lead to low-level thinking, while tasks that stimulate students to make purposeful connections to meaning lead to high-level thinking.

Mathematical tasks are "not only the problems written in a textbook or a teacher's lesson plan, but also the classroom activity that surrounds the way in which those problems are set up and actually carried out by teachers and students" (Stein et al., 2000, p. 24). According to this definition, mathematical task intertwines with the goals, intentions, actions, and interactions of teachers and students. Thus, mathematical tasks play the role in determining what students will learn.

All tasks are not created equal, that is, different tasks require different levels of student thinking. Stein et al. (2000) differentiate four levels of cognitive demand of instructional tasks as memorization, procedures without connection, procedures with connection, and doing mathematics. They also provide task analysis guide served as a scoring rubric for each level of cognitive demand (as seen in Table 1).

Table 1. The task analysis guide (Stein et al., 2000, p.16)

Lower-Level Demands	**Level 1: Memorization Tasks** Involving reproducing previous learned facts, rules, formula, or definitions. Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure. Are not ambiguous -such tasks involve what is to be reproduced is clearly and directly stated. No connection to the meaning that underlie the facts, rules, formula, or definitions being learned.
	Level 2: Procedures Without Connections Tasks Are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction. Require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it. No connection to the meaning that underlie the procedure being used. Require no explanation, or explanations that focus solely on describing the procedure that was used.
Higher-level Demands	**Level 3: Procedures with Connections Tasks** Focus students' attention on the use of procedures for the purpose of developing deeper understanding. Represented in multiple representations. Making connections among multiple representations help to develop meaning. Require some degree of cognitive effort. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.
	Level 4:Doing Mathematics Tasks Require complex thinking (there is not a predictable pathway explicitly suggested by the task or work-out example). Require students to access relevant knowledge and make appropriate use of them in working through the task. Require students to explore and understand the nature of mathematical concepts or relationships. Require considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.

Although it is important to determine the level of cognitive demand of a task, it happens that low-level tasks to be identified as high-level, such as acquiring the use of manipulatives and real-world contexts. It is also possible for tasks to be designated low-level when in fact they should be considered high-level. Being aware of the cognitive demands of tasks is a central role in selecting or creating instructional tasks matching instructional objectives. For example, if a teacher wants students to learn how to justify or explain their solution, she should select a task that is deep and rich enough to afford such opportunities. The cognitive demands of tasks can be changed during a lesson. Although starting with high-level task does not guarantee student engagement at a high-level (Stein, Grover, & Henningsn, 1996).

As suggested in Stein et al.'s Mathematical Tasks Framework (2000), tasks are seen passing through three phases: First, as they appear in curricular or instructional materials or as created by teachers; Next, as they are set up or announced by the teachers in the classroom; and finally, as they are carried out by students. All of these, especially the third phase are viewed as important influences on what students actually learn. This framework indicates that simply selecting and beginning a lesson with a high level task did not guarantee that students would actually think and reason in cognitive complexity ways. Some factors would reduce the level of cognitive demand of a task once it is implemented into classroom. These factors involve a variety of teacher-, student-, and related conditions, actions, and norms (Stein et al., 2000).

Stein et al. (2000) suggest that one way to help teachers learn to differentiate levels of cognitive demand is through the use of task-sorting activity. They recommend that the use of cases not only enable teachers to develop an understanding of mathematics tasks but also how the cognitive demands evolve during a lesson (Stein et al., 2000). Their cases are research-based because the cases are developed from a research project that creates practice-based materials based on the Mathematical Task Framework for mathematics teacher professional development. Thus, each case they created can depict the events that unfold in the classroom as the instructor and students engaged with cognitively challenging mathematical tasks. Once teachers use the research-based cases, they can begin to reflect on their own practice through the lens of the cognitive demands of tasks. This indicates that the use of cases is likely to be a tool for teachers maintaining high cognitive demands of mathematical tasks when they unfold during a lesson. The cases can be used as exemplars or problem situations (Markovits & Smith, 2008).

The use of cases includes reading cases and discussing cases. Cases alone are not efficient as a curriculum material contributing to users' learning. Reading a case does not ensure that the users automatically engage with all the embedded ideas or spontaneously make connections to their own practice. Case discussion creates the users with the opportunity of discussing a case altogether. A facilitator of the case discussion may want users to interpret and analyze various students-generated strategies in a classroom. The users in case discussion share and learn multiple pedagogical perspectives what occurred in a case. However, facilitating case discussion itself is a skill that facilitators need to learn. For instance, facilitators must listen intently to the users and learn how to steer the conversation in useful direction (Engle & Conant, 2002; Stein, Engle, Smith, & Hughes, 2008). Toward this aim, it is important for the facilitators to have specific learning goals in mind for the case discussion. Given this importance, good facilitators should accept training about the cases to be used. It seems more likely that one who has been involved in the construction of the cases would be a good facilitator, since one knows well about the case background and the goals in a specific case.

This is not to suggest that all tasks used by a teacher should engage students in high-level cognitive demand. However, to meet the need of innovative curricula that emphasized on reasoning, problem solving, connection, and mathematical communication (MOE, 2003), students need to have the opportunities to engage with tasks that lead to deeper, more generative understanding with respect to the mathematical concepts. However, mathematics teachers often have the difficulty not only with creating high cognitive demand mathematical tasks but also with maintaining high cognitive demand of the tasks during instruction (Stigler & Hiebert, 2004). Therefore, teacher educators need to help in-service teachers to provide such an opportunity to students engaging in instructional tasks that are indeed implemented in such a way that students thought in complex and meaningful ways. The use of cases is considered to be used in a training course designed in the study to help teachers in identifying and maintaining high level cognitive demands of mathematical tasks in mathematics instruction.

The purpose of the study was to design and enact a training course by using research-based cases to assist in-service teachers to identify and maintain high-level cognitive demands of mathematics tasks in classroom teaching. Thus, there are two research questions to be answered. First, how the use of cases would increase teachers' awareness of different levels of cognitive demands of mathematics tasks? Second, how the use of cases would help teachers maintain high-level cognitive demand as the tasks were carried out in classroom? The research-based cases referred to the study were successively created by a group of teachers with the authors of the articles participating in professional development programs that were designed to implement the three reform-oriented curricula from 1997 to 2011. The cases were research-based due to the construction of the cases by going through the cycles of Japanese Lesson Study including working in a small group, teachers collaborated with one another, meeting to discuss learning goals, to plan an actual classroom lesson, to observe how it works in practice, and then to revise and report on the results so that other teachers can benefit from it (Isoda, 2007). The research-based cases under consideration were scenarios of problematic situations actually encountered by teachers as they worked toward implementing the new curricula.

The research-based are featured as follows: (1) they are authentic teaching, (2) they are constructed by classroom teachers and the researchers, (3) they are able to provide vicarious experience, (4) the instructor in each case can be invited to participate the case discussion for articulating the context of the case teaching, (5) they are based on valid research; and (6) they are potential to initiate critical discussion by users. The tasks adopted by this study are not only the problems written in a textbook or a teacher's lesson plan, but also the classroom activity that surround the way in which those problems are set up and actually carried out by teachers and students (Stein et al., 2000).

3. Methodology

3.1 *Participants*

Eight teachers, selected from 15 teachers enrolling in a course called "Theory and Practices of Case-based Method (TPCM)" in summer Master Degree program at the University, participated in this study. All participants were in the first year study in the program. The selection of the teachers of the study was based on two reasons. First, they needed to teach mathematics subject matter in the following school years immediately after the TPCM course. Second, the location of the school where the teachers were teaching was not allowed to be long distance away from the university. Otherwise, it was inconvenient for the teachers to observe classroom teaching.

Three of the 15 teachers were administrators, and they did not teach mathematics in the following school year. The 3 teachers either became the consultants of mathematics teaching or the leaders of professional development group of mathematics in their own schools. The rest of the 12 teachers were the room teachers who were teaching mathematics. However, four of the 12 teachers were too far away from the university. There were only 8 teachers matching the two requirements for participation in the study.

Three teachers (T1, T2, T3) had at least 10 years of teaching experience. Three teachers (T4, T5, T6) had 5 to 10 years of teaching experience and two teachers (T7, T8) had less than 5 years of teaching experience. T3 and T5 were teaching grade 3 and grade 4 respectively. T6 and T8 were teaching grade 6. Four teachers (T1, T2, T4, T7) were teaching at grade 5 during the school year. T1 was first time to teach fifth-grade mathematics, although she has taught for 12 years in lower grade level.

3.2 *Settings*

The cases were presented in a video form in the TPCM course. The weekly two 3-hour TPCM course continuing for 48 hours consists of two parts. Part I consisting of 12 hours began by introducing the use of reflective mathematics journals, the Mathematical Tasks Framework and the Task Analysis Guides (Stein et al., 2000), and empirical papers relevant with case-based pedagogy, to enrich teachers' knowledge about cases. Part II contains 36 hours designed to help teachers learn to identify and provide students with increasing opportunities for constructing high-order mathematical thinking. During this part, the teachers were offered with six research-based cases. After viewing a video case, each case was immediately discussed in small groups and immediately followed by whole-class discussion. Finally, eight teachers from the TPCM course took turns observing each other's instruction during the school year after they ended up the summer course. Every teacher who enrolled in the TPCM course could voluntarily choose to participate in the final part, because some of the teachers did not teach mathematics during the school year.

These classroom observations aimed to examine how the use of research-based cases improved the teachers' ability in designing high-level instructional tasks and how the tasks were carried out in classrooms. Besides, the context of observation was for inspecting the eight teachers' perspective of levels of cognitive demands for mathematical tasks implemented in classrooms.

3.3 *Video Cases*

Twelve video cases were utilized to share the discussion in the course, since videotapes allowed teachers to watch and re-watch a segment, to discern exactly what was going on as students working on a particular task. Six video cases were related to fractions and the rest of the cases were with respect to other mathematics topics, such as decimals, proportion, and measurements.

3.4 *Case Discussion Session*

Each case had its own focus in terms of different instructional approaches on a same topic, teacher's questioning, students' various strategies of solving a given problem, and etc. For the purpose of this article, six cases related to fractions were described here. The main concept corresponding to each video case was summarized in Table 2.

Table 2. The main concept of each case related to fractions

Case #	Main mathematics concept of each case
1	2 units of 1/7 *vs.* 2 out of 7 parts.
2	Distinction of fractions parts with one piece from more than one pieces
3	Constructing improper and mix fractions through iterating of unit fraction
4	Why should whole units be same as comparing fractions?
5	Equivalent fraction by reducing form and extending form.
6	3/4 of a box containing 20 bottles of drink vs. 3/4 box with 20 bottles of drink.

In one of the six videotapes, "Is 2 units of $\frac{1}{7}$ equals $\frac{2}{7}$?", was excerpted as an example to illustrate what a case looks like. The case has to do with third graders' difficulty in understanding about "2 units of $\frac{1}{7}$ equals $\frac{2}{7}$ ". The video case contains a fragment with 12 minutes in length. The focuses zoomed at the tasks, students' various solutions, and dialogues between students and teachers, as shown in Figure 1.

The class regularly began by watching a video and was immediately followed by a one-hour case discussion with small groups of 4. It ended with a one-and-half hours whole-class discussion. T1, T2, T4, and T7 sit in the same group, while T3, T5, T6, and T8 sit in the same group. One of the authors of the article was the instructor of the TPCM and was the facilitator of the case discussion. The author was one of the members who generated the video cases. Thus, the instructor knew very well about the background of the case. However, the instructor did not provide the participants extra information such as guiding question, even though they asked about the students' preconception or the case-teacher's goals. The questions they asked in a small group became the central issues of the whole-class discussion.

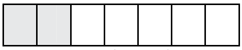
The intention of each video case was to encourage the participants to identify how mathematical tasks differ with respect to levels of cognitive demand. In the case discussion, the participants were asked to answer the following questions: (1) what is the main mathematical idea in the case?

Case: Is 2 units of $\frac{1}{7}$ equal to $\frac{2}{7}$?

Objectives: To represent a proper fraction in which its denominator is no more than 10.

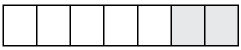
Tasks: Each student was given a strip paper that has been marked into seven equal parts. Students were asked to shade $\frac{2}{7}$ of the strip and explain it.

The following three drawings were given by three students in the class.



Wein-Wei

Uei-Shang



Ling-Wein

Dialogues between the teacher and students:

….

T: What is the fraction of the shaded by Uei-Shang?

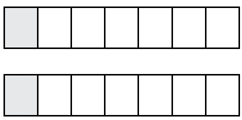
S: $\frac{2}{7}$

T: Is $\frac{2}{7}$ equal to 2 units of $\frac{1}{7}$?

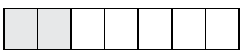
Uei-Shang: 2 units of $\frac{1}{7}$ is not equal to $\frac{2}{7}$, since $\frac{1}{7}$ adds $\frac{1}{7}$ is $\frac{2}{14}$.

Sue-Ling: 2 units of $\frac{1}{7}$ is not equal to $\frac{2}{7}$, since their representations are distinct.

2 units of $\frac{1}{7}$ is represented as



, and $\frac{2}{7}$ is represented as



(The discussing is continuous)

Figure 1. An excerpt of a video case

(2) Which level of cognitive demand of the instruction task would you like to place in? Why did you say so? (3) What evidence is there that the students learn these ideas or that the difficulties students encountered in this case? (4) What pedagogical issues would you like to address for sharing with your colleagues? To answer these questions, the participants learned to identify different tasks resulting in different levels of and kinds of students' thinking. The answers to these questions will reveal the cognitive processes required in the complex tasks. Only responses to question (1) and (2) were used for the purpose of the study.

3.5 *Data Collection*

Data collected from Part II for this study included case analysis of the video cases and teachers' weekly reflective journals. The cases discussion was audio-taped and transcribed verbally. Teachers' weekly reflective journals were one of the assignments of the course. The reflective journal was to help the participants to draw their attentions to the level of cognitive demand of the tasks used by the case-teachers and what case-students are actually doing. The four questions described previously discussed in the cases discussion and the evaluation of the level of cognitive demands was required to include in the reflective journals.

The data collected from Part III in school semester included classroom observations and the participants' evaluation sheets of the mathematical tasks carried out in the classrooms. The classroom observations provided a measure how their ability in maintaining the level of cognitive demand in implementation phase has been changed as a result of the video case discussion. Besides, the evaluation was to improve and to measure the participants' perspectives of identifying the cognitive levels of tasks. The components of the evaluation sheet for each task consisted of level of cognitive demand to be assigned, explanation of categorization, and features of the tasks.

Each lesson of the four teachers (T1, T2, T4, T7) who taught fifth grade was observed by the participants, since they taught at the same topic of unlike fraction comparison. The lesson was the first hour the unit of equivalent fractions. There were 3-4 tasks in each lesson designed by each of the four teachers and were conducted in each lesson.

In addition, eight teachers' perspectives of the cognitive level of mathematical tasks to the four teachers' instruction were collected. Due to the space limitation, only one (T1) of the four teachers was analyzed to document how she maintained the level of cognitive demands of students thinking evolved in a classroom. T1 was selected to report here based on two reasons: (1) T1 is one of the four teachers teaching in the same grade; and (2) T1 is teaching mathematics at the fifth grade for the first time; there is no confounding factor of affecting T1 teaching except the effect of the course. The other three teachers (T2, T4, and T7) have taught fifth grade for at least one year. The classroom observations were videotaped and audio-taped and transcribed verbally.

In a nutshell, the various data collected for answering the research questions of the study were summarized in Table 3. The various data were triangulated together for better address research questions.

Table 3. Various data for answering research questions

Data collected from	The tasks sourced from	Evaluators	Answering research questions
During TPCM course	Six video cases	eight teachers	RQ1: Understanding the eight teachers' perspective of identifying the cognitive level of mathematics tasks
After TPCM course	T1's, T2's, T4's, T7's lesson	eight teachers	
	T1 teaching		RQ2: Understanding the skill of maintaining the high level of cognitive demands of the tasks implemented in the classroom

3.5 *Data Analysis*

This study employed case analyses and cross-case analyses to examine how the teachers learned about the cognitive demands from video research-based cases carried out in classrooms. Cross-case analyses for the data collected from case discussion were conducted to identify similarities across cases, differences among them, and overall patterns.

Four levels of cognitive demand of instructional tasks suggested by Stein et al.'s (2000) Task Analysis Guide, was the framework of the data analysis for the eight teachers' perspectives. The four levels consist of memorization, procedures without connection, procedures with connection, and doing mathematics. The analysis of implementation of mathematical tasks was rooted in the three phases of Mathematical Tasks Framework suggested by Stein et al. (2000). The three phases were composed of the curricular phase, the setup phase, and the implementation phase.

The factors of maintaining and declining the level of cognitive demands required the task in the implementation stage from where the setup stage were analyzed. The eight teachers were data analyzers for the level of cognitive demands in each task.

4. Results

The first part of the results included the responses the teachers made to the questions raised in the video discussion according to the tasks in the setup phase. The responses to the four teachers' lessons of fraction comparisons according to the tasks in the setup phase and the implementation stage were the second part of the results. How a teacher maintained or declined the level of cognitive demands required in instructional tasks evolved in a lesson was the third part of the results.

4.1 *Teachers' Perspectives of Cognitive Demands of the Tasks*

4.1.1 *Under the Context of Video-cases Discussion during TPCM Course*

After engaging in each video case, the teachers working in small groups responded to the question "which level of cognitive demand would you place the instruction task in? Why did you say so?" Their responses to each video case collected from the eight teachers' mathematical journals were summarized in Table 4.

Table 4. Teachers' perspective of cognitive demands (CD) and features of the tasks under the context of video-cases discussion

Case #	Level of CD	Explanation of categorization	Features	Teachers
Case 1	4	The focus is on the connection between of two constructs: part-whole and iterating units.	requires an explanation involves multiple representations requires complex thinking	Eight teachers
		The task displays students' unexpected misconception.	activates students' misconception makes connections between fraction meanings	T3.T5.T6 T8
Case 2	4	The tasks distinguish the size of each fraction part with more than one piece from exact one piece involving in discrete quantities.	activate students' misconception emphasize the importance of language "parts" emphasize inappropriate fraction.	Eight teachers
Case 3	4	The task focuses on the need of using proper and improper fractions.	has real-world context uses manipulative makes connection between proper and improper fractions.	Eight teachers
			emphasizes the importance of iterating units of fraction.	T1.T2.T4 T7
Case 4	4	The tasks focus on the requirement of same whole unit as comparing two fractions.	have real-world context use fraction boards display students' misconceptions. distinguish discrete from continuous quantity	Eight teachers
Case 5	4	The task focuses the differences between reducing form and extending form of equivalent fractions though partition and quotient divisions.	requires an explanation has real-world context uses a pictorial representation. requires to access relevant knowledge requires considerable cognitive effort	T1.T2.T4 T7
	3	The task focuses attention on the procedure for finding equivalent fractions, but in a meaningful context.	has real-world context requires considerable cognitive efforts	T3.T5.T6 T8

Table 4. (*Continued*)

Case #	Level of CD	Explanation of categorization	Features	Teachers
Case 6	4	The tasks require making a distinction between two constructs of fractions: part-whole and operator.	require an explanation involve various constructs of fractions	Eight teachers
		The tasks highlight the importance of the size of multiplier of fraction multiplication	have real-world contexts require to understand the nature of math concepts.	T1.T2.T4 T7
		The tasks focus on the significance of mathematical language.	make connections between representations to help to developing meaning	T3.T5.T6. T8

Through sorting the given tasks, they did not only simply complete the sorting but also have the opportunity for conversation that moved back and forth for putting the level of cognitive demand of the given tasks. They also negotiated the characteristics of each level. It is found that the teachers did not always agree with each other on how tasks should be categorized. However, the agreement and disagreement were productive for them to understand the features of each level of cognitive demands. For instance, eight teachers consistently placed level 4 of cognitive demand with various features for the tasks involved in case 2 and case 4. For other tasks, such as in case 1, 3, and 6, they put the tasks at the level 4 of cognitive demand with several same features and several different features. T1, T2, T4, and T7 in one group placed the task in case 5 at the level 4, while T3, T5, T6, and T8 in another group placed at the level 3. They shared each other with different features for the task.

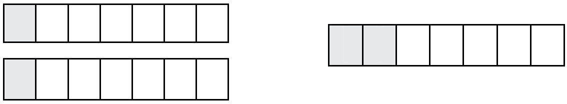
Table 4 shows that the common features of the tasks involved in the video cases addressed by the eight teachers include: (1) the tasks involved in each case highlighted representations; (2) the tasks were contextualized real world, such as case 3, 4, 5, and 6; (3) the tasks activated students' misconceptions, such as case 1, 2, and 4; (4) the tasks required to explore, explain, and understand the meaning of the mathematics concepts, such as case 1, 5, and 6.

Case 1 is an example for illustrating why the eight teachers assigned the tasks at the level 4 of cognitive demand as follows. The answers the teachers responded to case 1 in case discussion in the TPCM course were classified into three categories: conceptualizing the meaning of fraction, transformation between representations of fraction, and linking the relationship between the iteration of unit fraction and part-whole model fraction.

Regarding the category of the meaning of fraction, they elaborated that the students in the case (termed as case-students) did not recognize one construct with iterating unit fraction such as "2 units of 1/7" distinct from the other construct with part-whole model, such as 2 parts of 7 equal-size parts. They learned from case 1 that the third graders had difficulty with making connection between the two constructs.

With respect to transformation between representations of fraction and linking the relationship between the iteration of unit fraction and part-whole model fraction, they had an agreement to case 1 as a doing mathematics task. For example, the four teachers (T1, T2, T4, T7) in one group described that the task was featured as: (1) the task required students to explain why 2 units of 1/7 is equivalent to 2/7, but they had different representations; (2) the task was not textbook-like; (3) the task was involved in multiple representations including the transforms from verbal to manipulatives and to diagram; (4) the task of the case was an incidental rather than a predicable pathway; and (5) the task required students complex thinking. The other four teachers (T3, T5, T6, T8) in one group added two more features. The task activated students' misconception of 1/7+1/7=2/14 and makes connections between iteration of unit fraction and a non-unit fraction with part-whole model (e.g., 2/7 is equal to 1/7+1/7).

The teachers stated that they have learned from the case-teacher in the case in creating the task for provoking students' difficulties and misconceptions. The eight teachers perceived the impact of perceptual distracters, since they were surprised with students' difficulty in deciding the fraction 2/7 while the partitioning line was missing in the diagram. The missing line represented significant perceptual distractors. It was hard for third graders mentally "put in" the partitioning line in the diagram. Besides, it went beyond their expectation that students' difficulty with "2 units of 1/7 is equal to 2/7", since the case-students agreed that "2 units of 1/7 is represented as (a), rather than as (b) below.



(a) (b)

Through case discussion, some of the teachers were aware of the significance of consolidating fraction concept by shading in two separate areas instead of two successive areas in a rectangle. The two teachers (T3 & T5) appreciated that the case-teacher has already partitioned 7 parts in a strip for third-graders. As a result, students were not necessary to attend to partition the 7 parts; they would rather attend to shade 2 parts out of 7 parts. T4 made a reflection on her previous teaching that she spent too much time in partitioning a whole into odd parts.

4.1.2 *Under the Context of Classroom Observations After TPCM Course*

Table 5 and Table 6 display the data collected from the setup phase and the implementation phase of the tasks carried out by four teachers (T1, T2, T4, and T7) during the school year immediately after the TPCM course. The setup phase of the tasks was defined as the instructional tasks announced by teachers during teaching. Excepting one teacher as an instructor, seven teachers observed a classroom teaching altogether. The implementation phase was the tasks carried out in the classroom practices.

4.1.2.1 *The levels of tasks in the setup phase*

The data of Table 5 shows that there was often complete consensus among the eight teachers that the tasks created by T1 and T4 required the highest level of cognitive demand. The task created by T2 was classified at level 3 as procedure with connections and the task created by T7 was classified at level 2 as procedure without connections. The result indicates that the eight teachers improved their ability in classifying a given mathematical task into the level of cognitive demands. The teachers had a deeper analysis of the relationship between the tasks the

Table 5. Teachers' perspective of cognitive demands (CD) and features of the tasks in the setup phase under the context of classroom observations

Tasks (#)	Levels of CD	Explanation of categorization	Features	Evaluators
T1's 3 tasks	4,4,4,4	There is no pathway suggested by the tasks. The focus is on multiple methods to compare two unlike fractions.	activates students' misconception provokes students' multiple solutions requires an explanation	T2, T3, T4, T5, T6, T7, T8
T2's 3 tasks	3,4,3	The task provides a procedure for finding a common denominator of two fractions but connects the procedure to meaning.	requires some degree of cognitive effort suggests procedure to follow implicitly	T1, T2, T4, T5, T6, T7, T8
T4's 3 tasks	4,4,4	There is no predictable pathway suggested by the task and it requires complex thinking.	requires an explanation activates students' misconception requires complex thinking	T1, T2, T3, T5, T6, T7, T8
T7's 4 tasks	2,2,3,2	The task provides a procedure of finding a least common denominator for comparing two unlike fractions but requiring no connection to meaning.	requires limited cognitive effort explains on the procedure that was used is textbook-like are algorithm-oriented	T1, T2, T3, T4, T5, T6, T8

four teachers selected or created and the level of cognitive demand that were required of students.

Table 5 also suggests that some of the teachers did not achieve their tasks in the setup phase up to the high level of cognitive demands, since only the tasks created by T1, T2, and T4 reached to the level 4. The tasks T7 designed in the setup phase were still at the lower level of cognitive demands that did not require complex and deep thinking.

The finding shows that the effect of the research-based cases utilized in the TPCM course contributed to all teachers' identification of the cognitive level of the tasks. The research-based cases improved three of the four teachers in designing the high-level of cognitive demands. However, one of the four teachers frequently designed the tasks at level 2 of cognitive demand.

4.1.2.2 *The levels of tasks in the implementation phase*

Table 6 displays the decline or maintenance of the level of the tasks from the setup phase to implantation phase and its associated factors.

The data of Table 6 shows that there was consistent agreement among the eight teachers on identification of the level of cognitive demands of the tasks unfolded instruction. On the other hand, 3 out of the 14 tasks enacted by the four teachers declined the cognitive levels and 11 tasks maintained the cognitive levels from the setup phase to implementation phase. Four tasks that were set up by the T1 to place at the level 4 of cognitive demand on students' thinking were enacted in such a way that students reasoned in meaningful ways. For instance, T1's students were given to order the first pair of fractions. Students started out by figuring out various solutions based on their prior knowledge. As T1 kept coming up with the following pairs of fractions for advancing students' high level of thinking, the students realized they needed to keep track of the strategies they had already used. Students were invited to report their solutions and explaining the meaning of the solutions. During this time, T1 asked such questions as "How do you know?" "Can you make a distinction of your solution from others'?" "Why do you use the strategy for solving the problem but not for other problems?" This led the students to construct various strategies of ordering pair of fractions. T1 sustained pressure for explaining and reasoning.

When the tasks were enacted, there were usually a large number of support factors affecting the level of cognitive demands. Throughout the data shown in Table 6, the factors associated with the maintenance of the level of cognitive demands from the setup phase to implementation phase included: the tasks built on students' prior knowledge, well anticipating students' solutions, sufficient time to explore, appropriately selecting and ordering students' various solutions, asking various types of questions for different purposes, making connections between mathematical concepts, as well as providing the opportunities of justifications, explanations through asking follow-up questions relied on students' work. Most of the factors associated with the maintenance of the level of cognitive demands from the setup phase to implementation phase were related to teacher's expertise of mathematics instruction.

When the cognitive demands of the tasks declined in the implementation phase from the setup phase, a set of factors were operated in classrooms, as indicated in Table 6.

Table 6. Teachers' perspective of cognitive demands (CD) and factors associated with maintenance and decline of high-level CD from the setup phase to implementation phase

Tasks (#)	Levels of CD	Maintenance or decline	factors	Evaluators
T1's 3 tasks	(4,4,4,4) 4,4,4,4	maintenances	appropriately selecting and ordering students' various solutions providing the opportunities of justifications, explanations through asking follow-up questions relied on students' work sufficient time to explore	T2, T3, T4, T5, T6, T7, T8
T2's 3 tasks	(3,4,3) 3,3,3	maintenances	tasks built on students' prior knowledge making connections between mathematical concepts asking various types of questions for different purposes	T1, T2, T4, T5, T6, T7, T8
		decline	task expectations not clear enough to put students in the right place	
T4's 3 tasks	(4,4,4) 4,4,2	maintenance	providing the opportunities of explanations by questioning and comments. well anticipating students' solutions	T1, T2, T3, T5, T6, T7, T8
		declines	telling students how to do the problem shifting the emphasis from conceptual understanding to the correctness of the answer	
T7's 4 tasks	(2,2,3,2) 2,2,3,1	maintenances	students pressing the teacher to reduce the complexity of the task by specifying the explicit procedures and steps to perform.	T1, T2, T3, T4, T5, T6, T8
		declines	not enough time to explore	

Note: The number in ( ) level of CD of the tasks in the setup phase.

These factors were: task expectations not clear enough to put students in the right place, telling students how to do the problem, shifting the emphasis from conceptual understanding to the correctness of the answer, students pressing the teacher to reduce the complexity of the task by specifying the explicit procedures and steps to perform, and not enough time to explore. The factors associated with the decline of the level of cognitive demands from the setup phase to implementation phase were related to tasks, teacher, students, and time engaged in the classroom.

4.2 *Maintaining High-Level Cognitive Demand of the Tasks*

How mathematical tasks unfolded during classroom instruction was the other way for the study in order to document the effect of the training course on maintaining high level of cognitive demands of mathematical tasks. The implementation of mathematical tasks passed through three phases: the curricular phase, the setup phase, and the implementation phase. Due to the space limitation, the first lesson with respect to fraction comparison taught by T1 in the unit of equivalent fraction, as an expmple is discussed here.

4.2.1 *Phase I: The Curricular Phase*

The tasks at the first phase means the tasks were created by T1 or selected from curricular materials. T1 analyzed and reported the learning objectives of the lesson scheduled in the textbook she used. The objectives of the lesson included: (1) exploring equivalent fractions by ordering fractions, (2) naming a fraction in more than one way, (3) finding an equivalent fraction given its denominator, and (4) ordering general fractions (that is, neither numerators nor denominators are equal).

T1 argued that the size of the set and the denominator and numerator of fraction given in the textbook should be a small number instead of a big number, in order to compare fractions determined by the amount of the given fraction. Thus, she changed the size of the whole set in problem 1 from 36 to 24 and in problem 2 from 60 to 36. The argument she stated in journal was:

–Because the size of the whole set involving in problem 1, 2, and 3 is 36, 60, and 150 respectively, is too big for students to drawing a picture and being partitioned into a big number of parts, it is not only time consuming but also avoiding students of using the size of parts to determine the equivalence of fractions. Thus, I would revise the size of the set and the number of parts (T1, Journal).

To provide students the opportunity of using various strategies to order fractions, T1 replaced the fractions 5/20 and 2/12 presented in problem 2 in the textbook by 5/9 and 2/4, such that one fraction is larger than 1/2 and the other is less than 1/2. T1's rationale of designing tasks was revealed in the following reflective journal.

–Problem 2 involves ordering fractions with unlike denominators. One way is through finding the length of the fraction of the rope and then to decide which of the lengths is longer. The other way as indicated in the textbook is to find the least common denominator. However, students did not learn the algorithm of finding the least common fraction yet. I should provide my students the opportunity of using reference point strategy as suggested in the literature. I purposely compared the fractions (4/9 and 8/12) instead of the given fractions (5/20 and 2/12) [as indicated in Table 7]. In the second pair, 8/12 is bigger than 4/9, since 8/12 is more than 1/2 but 4/9 is less than 1/2 (T1, Journal).

Table 7. The comparison between the tasks in textbook and T1 generated

Objective: The size of a set is a multiple of denominators in which explores equivalent fractions by comparing fractions.

Tasks of textbook	Tasks created by T1
1. A chocolate box contains 36 pieces. David ate 6/18 box, Chris ate 1/3 box, and Joe ate 4/12 box of the chocolates. (1) Who ate more between David and Chris? (2) Who ate more between Chris and Joe?	1. David ate 1/5 of a strawberry pie and Joe ate 1/7 of the same pie. Who ate more? 2. Chris ate 5/9 of a strawberry pie and Sophie ate 5/16 of the same pie. Who ate more?
2. A rope has 60 meters in length. (1) Is 5/20 of the rope longer than 2/12 of the rope? (2) Is 4/15 of the rope longer than 5/20 of the rope?	3. A rope has 36 meters in length. Is 4/9 of the rope longer than 8/12 of the rope?
3. Ben plans to build a house on a land with 150 m^2. Plan I uses 2/3 of the land, plan II uses 4/5 of the land, and plan III uses 7/10 of the land. Which of the plans would use the biggest of the land?	4. A rope has 120 meters in length. Is 11/12of the rope longer than 14/15of the rope?

4.2.2 *Phase II: The Setup Phase*

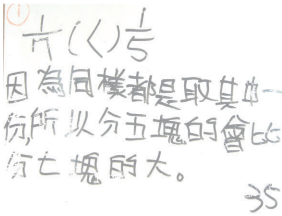
This phase referred to the tasks announced by the instructor T1 in the classroom. It contained T1's communication to her fifth grade students regarding learning objectives. This session was relevant to ordering fractions with unlike denominators. Students had learned ordering fractions with like denominator. In lesson plan, T1 created four pairs of fractions to decide which fraction is greater. The four pairs were sequenced by the following order: (1/5, 1/7), (5/16, 5/9), (4/9, 8/12), and (11/12, 14/15). Here, 1/5 and 1/7 were unit fractions, 5/16 and 5/9 were the fractions with same numerator. 4/9 and 8/12 were different denominators and different numerators, but one is a multiple of the other. 11/12 and 14/15 were the fractions with the difference of a unit fraction (1/12 and 1/15) away from 1.

In this lesson, each student worked individually and wrote individual solution on each whiteboard. Seven teachers identified the TI's task with four pairs of fractions in setup phase as "doing mathematics". T5 and T3 claimed that the task enable students to displayed students' misconception of fractions. T6 and T8 pointed out the task enable students to present multiple solutions in one pair of fractions. T7 claimed that the four pairs of fractions focused on developing mathematical understanding. Conversely, T4 suggested that T1 purposely changed the tasks with different types of fraction from the textbook for developing students' multiple strategies, so that these tasks required high-level demands. T2 commented that T1 designed intentionally the numerals of numerator and denominator between two fractions for developing students' various strategies rather than emphasizing on the algorithm.

4.2.3 *Phase III: The Implementation Phase*

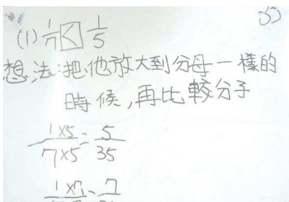
In the study, the implementation phase starts from as the tasks were carried out or worked on by students in classroom. T1 gave students enough time to explore the tasks and think individually. She appropriately selected and put in order students' various solutions after each task was explored. Seven teachers consistently agreed the tasks enacted by T1 maintained at the cognitive level of "doing mathematics", since six different strategies were used by her students for the tasks.

They were valid strategies. For instance, students used two strategies to compare 1/5 and 1/7. One strategy used unit fraction. Students realized that there is an inverse relation between the number of parts into which the whole is divided and the resulting size of each part, so that 1/5 > 1/7 (as Figure 2a). The other strategy was to find out a common denominator of two different denominators. For instance, the common denominator of 5 and 7 was 35, 1/5 =1 7/5 7=7/35, 1/7 =1 5/7 5=5/35 as the first step, since7/35 is greater than 5/35, so that 1/5 >1/7 (as Figure 2b).



English translation:
A size of each part of 5 parts is greater than that of each part of 7 parts.

Figure 2a.

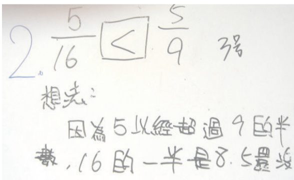


English translation:
Making them have same denominator 35.Then, compare their numerators.

Figure 2b.

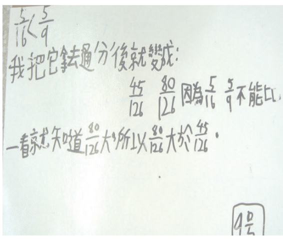
It was followed by the second task" comparing the size of 5/16 and 5/9". Most of the students still used two previous strategies, partitioning and finding a common denominator. They also developed a new strategy that used a reference point (usually 1/2 or 1) and were successfully in order the two given fractions. For instance, one of the students used "half of 16 is 8, bigger than 5 and half of 9 is 4.5, less than 5. Thus 5/16 is less than 1/2 and 5/9 is greater than 1/2, seen as Figure 3a. To this problem, TI attempted to reduce the use of common denominator, since the product of 16x5 is too big to getting correct answer. TI expected students to learn various strategies and each strategy can be applied in a suitable situation. She pointed to Su-Jing's solution (as shown in Figure 3b) and had the following conversation with her.

> T1: How did you change the number $\frac{5}{16}$ into $\frac{45}{126}$?
>
> Su-Jing: I used the fraction $\frac{5}{16}$ with denominator and numerator multiplying 9 and got the answer $\frac{45}{126}$.
>
> T1: Why did 16 change into 126?
>
> Su-Jing: I made a calculation error. It should be 144.
>
> T1: Did all of you think it is a good strategy to find the common denominator?
>
> Students: No.



English translation:
5 is greater than half of 9. Half of 16 is 8. 5 is less than 8.

Figure 3a.



English translation:
Finding a common denominator and their equivalent fractions are $\frac{45}{126}$ and $\frac{80}{126}$.

Figure 3b.

Moving to the third task "Order the pair of 4/8 and 8/12", the two fractions with different numerators and denominators were getting harder for students. In this case, students focused only on the numerator or only on the denominator and as a result made incorrect conclusions.

We found that T1 encouraged students to solve the problem successfully by either using reference point 1/2 (as shown in Figure 4a), or finding a common denominator requires finding $\frac{4x4}{9x4}$ equivalent to $\frac{4}{9}$ and $\frac{8x3}{12x3}$ equivalent to $\frac{8}{12}$ with the same denominator 36 (as Figure 4b) or finding the same numerator 4 requires finding 4/6 equivalent to 8/12 and then ordering 4/6 and 4/9 (as shown in Figure 4c), or finding the same numerator 8 requires finding 8/18 equivalent to 4/9 and then ordering 8/18 and 8/12 (as shown in Figure 4d).

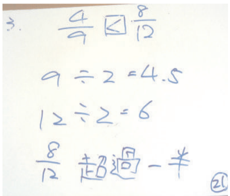


Figure 4a.

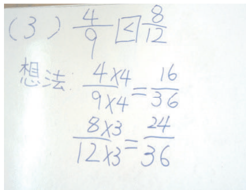


Figure4b.

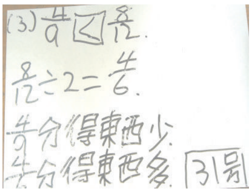


Figure 4c.

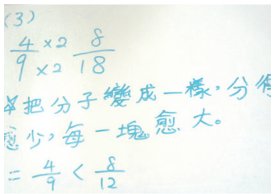


Figure 4d.

These analyses indicated that during the implementation phase, both T1 and her students were viewed as important contributors to how tasks were carried out. T1 questioning to students or asking follow-up questions was relied on what her students worked on the task. This was the factor of the maintenance of the high-level of cognitive demands when the tasks were unfolded in instruction. The seven teachers consistently agree that the ways and the extent to which T1 supported students' thinking was a crucial ingredient of maintaining high-level tasks. In this lesson, TI promoted deeper levels of understanding by consistently asking students to explain how they were doing about the problems. These tasks evolved in the instruction involved multiple strategies, required an explanation, and connected procedures to meaning. Thus, T1 maintained the high cognitive level when the mathematical tasks were carried out in the classroom.

5. Conclusion and Discussion

The study concluded that the training course designed and implemented by the use of research-based cases contributed to the teachers' awareness of the importance of differentiating levels of cognitive demand of tasks determining students thinking. Besides, the use of research-based cases contributed to teachers' skill in maintaining the high cognitive demands of the task enacted in classrooms. However, their awareness was gradually improved throughout the school year. During the case discussion in the TPCM course, eight teachers did not have consistent agreement with the level of cognitive demand to be placed in for a task. It is improved that there was a consensus when they were required to classify the tasks created by the teachers into the level of cognitive demands during the school year. This result indicates that the use of research-based cases created a good opportunity for teachers toward a deeper analysis of the relationship between the tasks they created and the level of cognitive demand determining students' thinking. The result is consistent with Stein et al.'s claim (2000).

The effect of cases on maintaining a high level of cognitive demands of mathematical tasks can be rooted in the following possible explanation.

Firstly, the cases used in the study were research-based, so that the mathematical tasks in the cases had been revised in advance to require high cognitive demand. When engaging in task-sorting activity, they have more opportunities to learn to differentiate levels of cognitive demand and have more opportunities to appreciate the high level mathematical tasks. When classifying the categorization of cognitive demands, the teachers reflected on the lesson and identified the levels of cognitive demand that the task placed on students during the setup and implementation phases. Thus, the classification of tasks made the teachers ultimately become more analytic and reflective about the role of tasks in instruction.

Secondly, the Task Analysis Guide provided explicit standards for viewing the research-based video cases and their own actual mathematical instruction. This guide helped teachers scrutinize what their students were doing in classroom.

Finally, the case discussion initiated the teachers' agreement or disagreement with the level of cognitive demands. Through the negotiation, the teachers began to consider how and why tasks differ and how these differences impacted opportunities for student learning. This made the teachers realize the significance of their role. However, achieving complete consensus on each task was not the intention of the case discussion of the study.

Despite having better understanding of the nature of the mathematical tasks, it was a challenge for some of the teachers to have implemented a task with high level of cognitive demands during instruction. It is noted that the effect of the use of cases on the tasks designed by the teachers at high cognitive levels was not the same as the effect of the research-based cases on maintaining the cognitive levels in the implementation phase. Some of the teachers maintained the same level of cognitive demands from the setup phase to implementation phase, while some of the teachers declined the high level to lower level of cognitive demand.

It is found that there were usually many supportive factors present in the teachers' classrooms. The major factor associated with the maintenance of high level cognitive demands was related to teachers' expertise of mathematics instruction, such as selecting and sequencing the tasks, selecting and sequencing students' various solution for advancing students' high-level thinking. The factors associated with the decline of the high level of cognitive demands were related to the tasks, teachers, and students. Besides, not enough time to explore in classroom was against the maintenance of the high-level cognitive demand.

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Chapter 21

The Relationships Between Religious Beliefs and Teaching among Mathematics Teachers in Chinese Mainland, Taiwan and Hong Kong

LEU Yuh-Chyn CHAN Yip-Cheung WONG Ngai-Ying

Despite the fact that the phenomenon of Confucian Heritage Culture Learners has become an educational focus in past decades, it would appear an over-simplification to think that Confucianism is the dominating Chinese school of thought. The Chinese hold various religious beliefs and philosophical thoughts and these thoughts in turn may have different impacts on mathematics teaching. Examining such links between religious beliefs and teaching is precisely the objective of the present chapter. Our focus includes the major religious beliefs of Confucianism, Buddhism and Christianity, and our scope covers three regions: the Chinese mainland, Taiwan and Hong Kong. We first provide an overview on beliefs about teaching among Chinese mathematics teachers, then briefly describe the religious situations in the three regions, and discuss ideological considerations and empirical studies on possible connections between religious beliefs and education in general and mathematics education in particular. Finally, we report on two empirical studies we conducted. The first study is a survey on beliefs about mathematics and mathematics teaching among mathematics teachers holding different religious beliefs in the Chinese mainland, Taiwan and Hong Kong. The second study is a comparative study on the values held in mathematics teaching by a Buddhist mathematics teacher and a Confucian mathematics teacher. Based on these two studies, how teachers' religious beliefs interplay with beliefs about and values in mathematics teaching will be discussed.

Keywords： Chinese mathematics teachers, religious belief, beliefs about mathematics teaching

1. Beliefs about Teaching among Chinese Mathematics Teachers

1.1 *How the Chinese See Mathematics and Mathematics Teaching*

The traditional way of instructional practices and the outstanding academic success of the Asians, particularly those studying abroad, has caught the attention of sociologists and educationalists worldwide since the early 1990s. The Stevenson group attributed the academic success of Asians to Confucianism (Stevenson & Stigler, 1992). The term "Confucian heritage culture" was thus coined. A vast number of publications on Confucian heritage culture were generated thereafter like that of Stigler and Hiebert (1999), Watkins and Biggs (1996, 2001), and Zhang, Biggs, and Watkins (2010). As it stands, there are quite a number of such publications relating to mathematics education alone, including our earlier book *How Chinese learn mathematics: Perspectives from insiders*. Other publications on this topic include those by Kaiser, Luna, and Huntley (1999) and Leung, Graf and Lopez-Real (2006) (see Wong, 2004 and Wong, 2013 for more details). Besides studying the sociological and cultural antecedents behind the academic success of Asians, academics began examining how beliefs affect such a phenomenon.

It is common knowledge that mathematics is not value-free. Beliefs affect the teaching and learning of mathematics (Bishop, 1976; Bishop & Seah, 2008; Bishop, Seah, & Chin, 2003; Leder, Pehkonen, & Törner, 2002). In particular, teachers' beliefs about mathematics and mathematics teaching play a significant role in their instructional practices (Philipp, 2007; Thompson, 1984, 1992). All these can be conceptualized by the notion of *lived space* (Figure 1): the teachers shape the space students live in day-to-day classroom learning by teachers' own beliefs about mathematics and mathematics teaching, which consequently result in students' *outcome space* (Wong, Marton, Wong, & Lam 2002). This *outcome space* does not only include students' affective and cognitive learning outcomes, but students' beliefs are also developed throughout the entire learning process.

Based on this overall framework, a number of studies on students' and teachers' beliefs about mathematics and mathematics learning were conducted in the three Chinese regions: the Chinese mainland, Taiwan and Hong Kong (Wong, 2002). These empirical studies reinforced the above premise. On the one hand, it was found that students' and teachers' beliefs mirror each other: that they believe that mathematics involves thinking, is useful, but is more or less a subject of *calculables*. On the other hand, operational procedures, practices and memorization are found to be central in teaching and learning. We refer readers to relevant publications or a summary in Zhang and Wong (2015) for details.

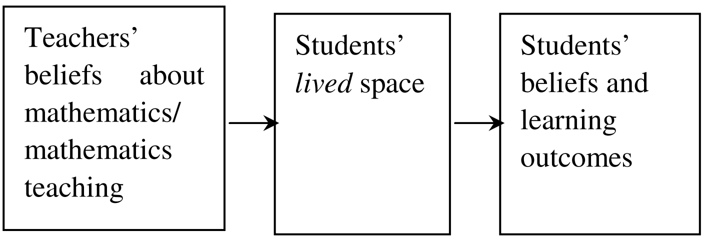


Figure 1. The *lived space* of mathematics learning

The results of these studies (together with the on-going Third Wave project) showed a general picture of effective classroom learning and teaching in the eyes of the Chinese, both for student and teacher. An effective (mathematics) teacher is one who possesses rich mathematics knowledge and teaching experience, prepares well for the lesson and has a reflective and flexible mind. As for teaching, the teacher uses various methods to explain clearly the content and provides well-structured exercises. Though teaching is by and large teacher-led, the teacher will put effort into making lessons student-centered by facilitating teacher-student interactions. Questioning solicit students feedback and not only informs teachers' understanding of students, but also keeps students engaged and the class in order. From the teachers' viewpoint, there are two facets in the goal of teaching: generic and specific. On the one hand, teachers are resolved to meet the requirements of the mathematics curriculum (and examinations). They aim to help students develop their mathematical concepts and problem-solving abilities. On the other hand, they realize that non-mathematics-specific educational goals (e.g. being observant, equipped with analytical skills and creativity) have to be addressed. From the teaching perspective, besides practising mathematics itself, these goals can be accomplished by providing student-teacher interactive classroom activities such as those mentioned above. Students are brought up in such a lived space shaped by the teacher that it generates students' learning outcomes, including students' beliefs about mathematics and mathematics learning. One may refer to Zhang and Wong (2015) for a fuller account.

1.2 *Teachers' Worldviews and Their Beliefs about Mathematics and Mathematics Teaching*

Factors influencing teachers' beliefs about mathematics/mathematics teaching are numerous. Bishop (1996) proposed three (overlapping) kinds of values in mathematics teaching, that is mathematical, mathematics educational and general educational. Apart from these three kinds of values, teachers' worldviews may also influence their beliefs about mathematics/mathematics teaching (and hence how they make decisions in their teaching). For instance, the aim to help students obtain high scores in public examinations may have conflict with students' genuine understanding of mathematical concepts, and teachers' worldviews might influence how they approach this dilemma. For example, in helping mentally challenged students, beliefs or values may support the teachers' untiring efforts (Wong, Wong, & Wong, 2012). Heie (2002) defines *worldview* as "one's comprehensive set of beliefs about the nature of reality and how one should live in the light of those beliefs" (p. 99). He further explains the multiple source of one's worldview:

> As formidable as that sounds, everyone has a worldview, although it is often neither examined nor articulated. The source of one's worldview beliefs are multiple, including knowledge gained from the academic disciplines, starting with one's own disciplinary specialization, and including relevant connections, hopefully, with other academic disciplines. One's worldview also includes beliefs emerging from one's faith commitment, be that religious or secular. (ibid)

In short, as a non-negligible factor of one's worldview, no matter how strong or subtle it may be, teachers' religious beliefs and philosophical thoughts will have an impact (Figure 2). Some scholarly works about the relationship between religious beliefs and school education have been published in recent years. Brown (2008) describes an inspiring story of a teacher who "brings a Buddhist perspective into the classroom to explore the ethical quandaries, lived experiences and intimacy of teaching" (back cover). Wong and Canagarajah (2011) and Wong, Kristjánsson, and Dörnyei (2013) report a series of research studies on the interrelationship of Christian faith and English language teaching. Although religious-related issues are addressed occasionally in mathematics education literatures which will be reviewed in later section, so far there are no systematic research studies specifically focusing on the relationship between teachers' religious beliefs and mathematics teaching. The objective of the present chapter is to address this issue.

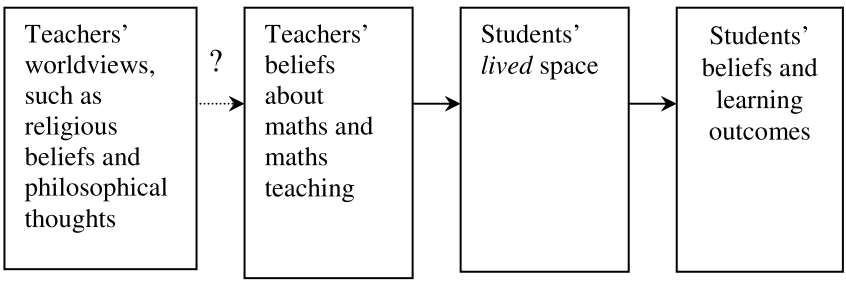


Figure 2. Teachers' worldviews and the *lived space* of mathematics learning

2. Religions in the Chinese Mainland, Taiwan and Hong Kong

As a start, we wish to emphasize to the readers that China is a multicultural country. The Chinese mainland alone is made up of 56 ethnic groups with different cultural traditions. For example, there is a large contingent of Muslims residing in the North-western part (Xinjiang) of the country as well as a large number of Tibetans residing in the Western regions, the culture of the latter being influenced by Buddhism. Even within the 'middle kingdom', Confucianism is not the only school of thought, and the ideology of Confucianism itself has changed over time. What we see as Confucianism today is a blend of ideas from both Confucianism and other schools of thought (and not 'authentic' Confucianism). This blending has also occurred in other schools of thought. There were also discussions whether Confucianism and Buddhism (or even Daoism) are actually religions and whether Buddhism, as imported from India, should be considered as Chinese. Emperors (and their high officials) in history also took part in manipulating (distorting) Confucianism, using it for a means of government control. Even in our own research data reported below, many so-called Confucian notions perceived by the respondents would, strictly speaking, appear to be mistaken Confucian. For the purpose of this study, we have termed this secular-Confucianism. More detailed discussions can be found in Wong, Wong, and Wong (2012). As mentioned above, we refrain from these debates and choose to use the terms 'religion' and 'religious beliefs' loosely and interchangeably, to embrace these concepts in a broader sense. Although some of the religions mentioned here do not have a Chinese origin, we include them because they have a considerable number of followers in China.

Another point of interest is, as stated in Wong, Wong, and Wong (2012), "for the past two centuries, Western values have been largely imported into the Chinese region and traditional Chinese values have been denounced in the late Qing dynasty and the May Fourth movement, as well as in the Cultural Revolution. In these movements, 'down with the Confucian Mansion' was the refrain" (p. 17). It is difficult to judge how much 'Chinese-ness' there is for both mathematics and education. Contemporary Chinese (be they from the Chinese mainland, Taiwan or Hong Kong) are more likely to be learning 'global' mathematics in a Western educational system, though with some ingredients of traditional Chinese values.

2.1 *The Chinese Mainland*

In the Chinese mainland, Marxism, which implies atheism, is the official ideology since the establishment of the People's Republic of China. Yet there are quite a number of followers of Buddhism, Daoism, Islam, Catholicism, and Protestantism (Chinese Academy of Social Sciences, 2007). In this regard, Buddhism is also considered an atheist religion. According to People's Daily Online (http://english.people.com.cn/index.html), there are more than one hundred million believers of various religions in the Chinese mainland.

Buddhism is one of the most influential religions in the Chinese mainland. In simple terms, there are three branches of Buddhism in China, namely Chinese Buddhism, Tibetan Buddhism, and Theravada Buddhism. According to People's Daily Online, there are about 200 thousand Buddhist monks and nuns, more than 13 thousand Buddhist temples, 33 Buddhist colleges, and nearly 50 types of Buddhist publications in the Chinese mainland. The Buddhist Association of China is the largest national Buddhist organization in China.

As an indigenous traditional religion of China, Daoism is another influential religion in the Chinese mainland. According to the Chinese Daoist Association (http://www.taoist.org.cn), there are more than 1600 temples, more than 25,000 Daoist priests and an uncountable number of followers of Daoism in the Chinese mainland. Whereas there are many regional Daoist organizations in China, the Chinese Daoist Association is the largest and leading one.

Despite the fact that many people in the Chinese mainland are influenced by Confucianism, there are no Confucian organizations in China–probably because Confucianism is usually regarded as a philosophy rather than a religion.

2.2 *Taiwan*

In Taiwan, according to a recent survey, around 35% of people are Buddhists, while 33% are Daoists, though quite a proportion of them claim to be followers of both. A small percentage of the Taiwanese are Christian Protestants, Catholics or followers of I-Kwan-Tao[1] (American Institute in Taiwan, 2010). There are six television channels preaching Buddhism, while there is only one channel each for Daoism and Christianity. From the numbers of followers and TV channels, we can

[1] A new religion which is, in simplistic terms, a blend of Buddhism and Daoism

roughly understand the degree of influence from each religion in Taiwan (Hsieh, 2005; National Communications Commission, 2012).

There are many Buddhist groups in Taiwan where the most influential ones include the *Dharma Drum* Mountain Monastery in the North, the *Chung Tai Chan* Monastery in the central region, the *Fo Guang Shan* Monastery in the South and the *Tzu Chi* Foundation in the East. Among them, the *Tzu Chi* Foundation and the *Dharma Drum* Mountain Monastery established their teachers' associations in 1992 and 1996 respectively. Though they are religious bodies, they treasure education heavily. This can be reflected in their goals. For the *Tzu Chi* Foundation, it is said that their goals are "to deliberate the humanity spirits of *Tzu Chi* and integrate them into teaching activities" and "to purify the campus and to harmonize the society" (*Tzu Chi* Foundation, 2009). The goals for the *Dharma Drum* Mountain Monastery are "to integrate Buddhism and education as a devotion to elevate the humane quality" and "to establish Pure Land on Earth" (Teachers' Association of the *Dharma Drum* Mountain Monastery, n.d.). These two Buddhist groups sponsor many meditation camps for teachers, principals and students, creating considerable impact on education in Taiwan. In contrast, even though there are many Daoist groups in Taiwan, they do not have similar teachers' association and/or mediation camps targeted for educators in general.

Although the debate on whether Confucianism is a religion or not still exists in Taiwan, we will not delve into the topic for the purpose of this study. Whether one regards Confucianism as a religion or not, its ideology undoubtedly has its influence in Taiwan and its impact in education. The Ministry of Education in Taiwan included the *Four Sacred Books of Confucianism* into the compulsory teaching materials of Chinese cultures for senior high school students. The curriculum goals include: "to develop consciousness for moral and ethics and to nurture spirits of humane and loving" and "to learn from the life wisdom from the ancestors so that one can distinguish the right and wrong and to practice it in the daily life" (Ministry of Education, 2011). However, there are opponents to the implementation and they argue against this for at least two reasons. First, they doubt that the teaching of the *Four Sacred Books* can really enhance students' moral education. Second, they insist that the government should not prohibit any religion nor advocate any religion (Hsieh, 2011). Nevertheless, Confucianism exhibits some degree of influence in education in Taiwan.

As regards to religious bodies involved in education, there are 4 primary schools, 6 junior high schools, 9 senior high schools, and 6 universities run by Buddhist groups in Taiwan. Although Daoism is the second largest religion in Taiwan, it does not involve itself in education. In contrast, there are 8 primary schools, 21 junior high schools, 33 senior high schools, and 6 universities run by Catholic groups despite their smaller number of followers (List of religious schools in Taiwan, 2012).

2.3 *Hong Kong*

Ninety five percent of Hong Kong's residents are Chinese. However, as Hong Kong has been ruled by Britain for a century, both Catholic and Protestant churches are influential (Wong & Tang, 2012). According to recent government census (Hong Kong Government, 2010), around 14 %, 14%, 7%, 5%, 3% are Buddhists, Daoists, Protestants, Catholics, and Muslims respectively. There are also minority groups of Hindus, Sikhs and Jews.

Buddhism is one of the largest religions in Hong Kong. After the return of Hong Kong to China, the Buddha's birthday has become a public holiday in Hong Kong along with Christmas and Easter which have been public holidays for a long time. There are more than one million Buddhists and hundreds of Buddhist organizations in Hong Kong (Hong Kong Government, 2010). Among these organizations, the Hong Kong Buddhist Association (http://www.hkbuddhist.org/index.html) is the largest and leading one. It has over 10,000 members. It is actively engaged in activities and services such as promotion of Buddhism, medical services, education, child care services, youth activities, services for the elderly, charity welfare services and burial services. According to a database of Hong Kong school lists (Education Bureau, n.d.), there are[2] 11 Buddhist kindergartens, 1 Buddhist nursery, 14 Buddhist primary

[2] Not all the Buddhist schools are organized by the Hong Kong Buddhist Association.

schools, 20 Buddhist secondary schools, and 1 Buddhist private university in Hong Kong.

Daoism is another popular religion in Hong Kong. There are about one million followers of Daoism and more than 300 Daoist abbeys and temples in Hong Kong. The Hong Kong Taoist Association (www.hktaoist.org.hk) organizes a wide range of activities to promote the beliefs of Daoism. The greatest event is the Hong Kong Daoist Festival which is held on the birthday of the Supreme Patriarch of Daoism. Daoism is also actively involved in community services including education. In Hong Kong, there are over 40 schools and kindergartens which have a Daoist background (Hong Kong Government, 2010).

Confucianism is not as popular as Buddhism and Christianity in Hong Kong, Confucian organizations such as the Confucian Academy has put much effort into promoting Confucianism as the nation's major religion in order to enhance the cohesion of the Chinese nation and foster a community with moral betterment. The academy has organized different academic activities and published Confucian books and magazines to spread Confucianism (Hong Kong Government, 2010). According to a database of Hong Kong school lists (Education Bureau, n.d.), the Confucian Academy has set up 1 primary school and 1 secondary school in Hong Kong.

3. Religions and Mathematics Learning and Teaching

In the following sections, we will first discuss on how the religious ideologies view education and mathematics education in particular. Then, we will give an overview on empirical studies which involved religious-related issues that are found in mathematics education literatures.

3.1 *Ideological Considerations*

One's religious beliefs may affect daily routines especially during critical moments. For instance, whether one should order a fish at a wedding party when this would involve killing, or whether a Buddhist should celebrate Christmas involves religious considerations. Literature has raised the possibility that religious beliefs influence teaching and learning. As early as the 1970s, the *Time* cover story 'The New Whiz Kids' (Brand, 1987) distinguished students with Confucian and Buddhist backgrounds. Wong, Wong, and Wong (2012) made an extensive discourse on how the three Chinese religious beliefs of Confucianism, Daoism and Buddhism influence education in general and mathematics education in particular. Although these three religions have major differences in their standpoints (even by what is meant by being 'educated'), they all fundamentally value education. Both Buddhism and Confucianism emphasize the educability of all human beings and Buddhist teachings even extend the course of nurturing beyond the earthly life. Although Daoism (and to some extent, Buddhism) despise worldly training, they believe human beings (sentient beings in the case of Buddhism) possess innate wisdom. Experiencing rather than indoctrination is particularly appreciated in Daoism and Buddhism. Blended with the examination culture, the educability of all human beings implied by these schools of thoughts make the Chinese learner hard working, achievement oriented and attributing success to effort (Watkins & Biggs, 1996, 2001). However the ancient Chinese had many more ways of nurturing rather than just teaching by rote. They have gradually developed a teaching course that goes from 'entering the *way*' to 'transcending the *way*' (Wong, 2004, 2006).

In education, interestingly and first and foremost, mathematics did not earn particularly high regard in ancient China. Secondly, after the Westernisation movement in the post-medieval period, it is difficult to say how much of the mathematics taught in schools uses ancient Chinese methodology. Indeed, the educational model now existing in the Chinese regions appears to be a Western one. One can only argue that Chinese students are presently learning Western (or worldwide) mathematics in a Western educational system whilst having Chinese cultural values such as social-achievement orientation, diligence, attributing success to effort, and collectivism still coming into play (Wong, Wong, & Wong, 2012).

The above concern the three 'Chinese' religions. As previously stated, though Christianity originates in the West, we find a considerable number of Christians in the Chinese regions. The core faith of Christianity lies on the relationship between humans and the creator God.

Christianity believes that God created the world, creatures and humans. As mentioned in *Westminster Shorter Catechism* (1647), “Man’s chief end is to glorify God and to enjoy him forever”. Christians also believe that every human is sinful and needs to restore the relationship with God via Jesus Christ. Therefore, applying these beliefs, the ultimate goal of education (including school education) is to bring students to Jesus and nurture them to have a good relationship with God (Proverbs 22:6). Following these beliefs, school education should be integrated with the Christian faiths as, in Christianity, and there is no distinction between religious knowledge and secular knowledge. All knowledge comes from God and humans’ ability to comprehend knowledge is limited and incomplete. This suggests that Christian teachers would guide their students to know God by means of school subjects, and that they may because of their faith, investigate possible connections between the school subjects and Christian faith, and verify all acquired knowledge with the teachings of the Bible, rejecting those which contradict the Bible.

The Bible does not have explicit statements about God’s purposes for mathematics. However, it may be implied from God’s broader purposes as revealed in Bible, that mathematics is a tool for understanding nature and has a great impact on human culture. A Christian perspective may thus suggest that God has given some people the capacity and interest to study mathematics. “Ultimately mathematics is intended to enable us to serve God and other human beings” (Howell & Bradley, 2001, p. 372). The effective applicability of mathematics “is a marvellous gift of God and a reminder of the stewardship that he has entrusted to us” (Howell & Bradley, 2001, p. 376).

3.2 *Empirical Studies*

There are implications one can draw from various religions or schools of thought on teaching. In this, there are a number of empirical studies about the influences of religious beliefs (especially Christian faith) on education in general. These studies cover a variety of areas such as religious education, cultural conflict between religious values and school culture, and impact of religious beliefs on teachers’ identities. As they are out of the reach of this chapter, interested readers may refer to papers scattered in journals such as *Journal of Beliefs and Values*, *British Journal of Educational Studies*, *Journal of Education and Christian Belief*, *British Journal of Religious Education*, *Science and Education* and many others.

Apart from these, there are empirical studies about possible influences of religious beliefs on specific school subjects. A recent book edited by Wong, Kristjánsson and Dörnyei (2013) collates a series of research studies on the interrelationship of the Christian faith and English language teaching. The studies cover three main areas: (a) faith and language teacher identity; (b) faith and language learner identity; and (c) faith and language acquisition. There are even more empirical studies on the relationship between religious beliefs and science education. As early as the 1990s, Cobern (1991) applies worldview (in particular, religious beliefs) as an interpretative framework for the studies of science teaching and learning. BouJaoude, Wiles, Asghar and Alters (2011) investigated secondary students' conception of evolution among Muslims in Egypt and Lebanon. Brem, Banney and Schindel (2003) investigated the college students' perceived consequences of evolution. A continuum of perspectives ranging from strong creationist to strong evolutionist is identified. Varying opinions on the relationship between religious beliefs and science teaching are discussed. Martin-Hansen (2008) investigated how a group of first-year college students' understanding on theory of evolution and creationism are changed as they progressed through a course on the Nature of Science. All these studies support that religious beliefs do indeed have impact on students' scientific learning. Concerning teachers, Clément, Quessadam, Munoz, Laurent, Valente and Carvalho (2009) conducted an international comparison study on teachers' conceptions of the theories of evolution and creationism. The study involved 19 countries in Europe, Africa and Middle East. Using the qualitative approach, BoudJaoude, Asghar, Wiles, Jaber, Sarieddine and Alters (2009) investigated the views of Christian and Muslim biology secondary teachers and university professors on the theory of evolution, and Mansour (2008) investigated the influence of Muslim teachers' personal religious beliefs on the teaching of controversial scientific issues. All these studies suggest that religion does indeed have some influences on teachers' conceptions of scientific teaching. Lastly, religious priests' views on the relationship between religion and science education may have influence on teachers or students belonging to the religious groups. Dickerson, Dawkins, and Penick (2008) conducted an interesting study on 63 Christian church ministers' views on the relationship between science and the Christian faith.

Fewer studies have been conducted to explore the relationship between religion and mathematics education. Amir and Williams (1999) found that religious belief is the most influential factor on children's probabilistic thinking. Some of the cases in the study of Sharma (2006) reflected that students were unable to predict the probability of the gender of a baby because they believed that gender is determined by God. It has also been shown that religious beliefs also influence mathematics teaching. In a study by Jett (2010), all the four African-American graduate students in mathematics or mathematics education connected their academic success with their personal religious and spiritual experiences. Norton (2002a, 2002b, 2003) found that university mathematics teachers who subscribed to Judaism, Christianity and Buddhism showed very different relationships between their religious beliefs and their mathematics research and teaching practices. More recently, Nokelainen and Tirri (2010) investigated the correlation between motivation and moral and religious judgments among 20 mathematically gifted adolescents. However, they did not explain why people who are mathematically gifted individuals were chosen as their sample subjects.

To date, the two series of studies conducted by the co-authors of this chapter are the only empirical studies focusing on how religious beliefs impact teaching among **mathematics** teachers in the **Chinese** regions. The first series of studies was led by Chan, the second co-author of this chapter. It aimed at extending the scope of religious impact to all the popular religions in the Chinese regions, that is Confucianism, Buddhism, Daoism, Catholicism, Protestantism, as well as those who do not subscribe to any religion. The second series of studies was led by Leu, one of the co-authors. Her (and her collaborators) studies unfolded how mathematics teachers possessing Buddhist and Confucian beliefs might demonstrate subtle differences in how they view learning and teaching mathematics. While some earlier findings were reported in papers like Leu and Wu (2004) and Leu (2005), it is the first time that the value systems of the two teachers of different religious beliefs were compared comprehensively.

In the remaining part of this chapter, we will provide an overview of these two series of empirical studies which are our preliminary attempt to fill the research gap, at least for the Chinese regions.

4. Study One: A Survey on Beliefs about Mathematics Education among Teachers with Chinese Religious Beliefs, Christian Beliefs, and Those Not Subscribing to Any Religions

4.1 *Research Focus*

The focus of the first study was to compare the beliefs about mathematics and mathematics teaching and learning among teachers with different religions, including Chinese religious beliefs (Confucianism, Buddhism and Daoism), Christian beliefs (Catholicism and Protestantism), and those not subscribing to any religion.

4.2 *Participants and Procedures*

Six hundred and thirteen (613) mathematics teachers from the Chinese mainland, Taiwan and Hong Kong were invited to respond to a questionnaire on their religious involvement and beliefs about mathematics and mathematics teaching. The distribution of religions among these teachers is summarized in Table 1. As some religious groups had relatively few respondents, the religious groups were combined into three larger groups, namely, that 186 hold Chinese religious beliefs (Confucianism, Buddhism and Daoism), 137 hold Christian beliefs (Roman Catholicism and Protestantism) and 271 do not subscribe to any religion. The 14 teachers who left religion blank and the 5 teachers who subscribe to other religions (Islam, Sikhism and Hinduism) were excluded from the analysis because there were too few respondents in these religious groups.

Table 1. Distribution of religions among the participants

Religion	Frequency
Confucianism	29
Buddhism	101
Daoism	51
More than one religion among Confucianism, Buddhism and Daoism	5
Roman Catholicism	24
Protestantism	113
Islam	3
Sikhism	1
Hinduism	1
Do not subscribe to any religion	271
Invalid (left religion blank)	14
Total	613

Besides demographic data, the questionnaire consists of two portions. The first portion comprises three parts: religious involvement, beliefs about mathematics, and beliefs about mathematics teaching.

Firstly, the respondents were asked to select one religion that influences them most. Then, they were asked to complete a scale which measures their religious involvement. The scale was constructed by the authors. This scale consists of 9 items put against a 5 point scale (disagree, quite disagree, neutral, quite agree, and agree, with an additional choice of "not applicable" for those who do not possess a religious belief) in which the respondents were asked to indicate how often they attend religious activities, participate religious practices and read religious books.

Teachers' beliefs about mathematics were measured in 5 subscales by a 54 items questionnaire put against a 5 point scale (same options as above, only the option "not applicable" was removed). There were 12 items in the subscale "Mathematics is a subject of *calculables*" (a sample item is "Mathematics is just mechanical computation"). The second subscale "Mathematics involves thinking" comprises 13 items (a sample item is "Mathematics is a subject that uses the brain"). The third subscale "Mathematics is useful" consists of 11 items (a sample item is "There are plenty of daily life applications of mathematics"). The fourth subscale

"Mathematics is precise" consists of 10 items (a sample item is "Answers in mathematics should be very exact") while the last subscale "Mathematics is logical" consists of 8 items (a sample item is "The most essential ingredient of mathematics is the proofs of theorems and formulae"). The first three subscales were developed through ethnographic research and were used several times, yielding satisfactory reliability indices (Wong, Chiu, Wong, & Lam, 2005; Wong, Kong, Lam, & Wong, 2010; Wong, Lam, & Wong, 1998) and the scale was now extended into 5 subscales according to the work of Leu and Wen (2001).

A revised version (Capraro, 2001, 2005) of Fennema et al's *Mathematics Beliefs Scale* (Fennema, Carpenter, & Loef, 1990) was used. It consists of 3 subscales (6 items each) put against the same 5 point scale as above. The first subscale related to how students learn mathematics (a sample item is "Time should be spent practicing computational procedures before students are expected to understand the procedures"). The second related to the role of the teacher in the sequencing of teaching both computational and application skills (a sample item is "Students need explicit instruction on how to solve word problems"). The third is about the relationships between teaching computational skills and problem solving skills (a sample item is "The goals of instruction in mathematics are best achieved when students find their own methods for solving problems"). Capraro (2005) further averaged the scores of all the items to form a score which reflects a constructivist belief.

The second portion consists of five open-ended questions. They are:

In your opinion,

1）Mathematics is…
2）Mathematics learning is…
3）Mathematics teaching is…
4）Education is…
5）Religious belief is…

Similar open-ended questions have been used in Wong (1993, 1996), which yielded fruitful results.

4.3 *Results from the First Portion*

Preliminary analysis of the first portion of the questionnaire has been reported in Chan, Wong, and Leu (2012). Satisfactory internal consistency reliability indices were obtained. The Cronbach alpha of religious involvement was .97. The Cronbach alphas of the five subscales of beliefs about mathematics ranged from .70 to .88. The Cronbach alphas of the three subscales of beliefs about mathematics teaching ranged from .66 to .82.

Beliefs about mathematics among teachers of the three religious groups (Chinese religious beliefs, Christian beliefs, and having no religion) were compared. One-way ANOVA was conducted. The statistics are summarized in Table 2. Results reveal that respondents holding Christian beliefs (Catholicism and Protestantism) see mathematics as precise and as a subject of *calculables* more than their counterparts. Respondents holding Chinese religious beliefs (Confucianism, Buddhism and Daoism) consider that mathematics involve thinking more than those holding other beliefs.

Table 2. Comparison of beliefs about mathematics among Groups

Subscale	Mean (standard deviation)			F	Scheffe Post-hoc comparisons
	Chin	Christ	NoRel		
Mathematics is a Subject of *Calculables*	2.64 (0.71)	2.86 (0.65)	2.57 (0.70)	8.008***	Christ > Chin Christ > NoRel
Mathematics Involves Thinking	4.33 (0.46)	4.20 (0.42)	4.22 (0.48)	4.593*	Chin > Christ Chin > NoRel
Mathematics is Useful	4.34 (0.56)	4.22 (0.53)	4.24 (0.62)	2.384	N.S.
Mathematics is Precise	2.50 (0.72)	2.69 (0.61)	2.49 (0.72)	4.205*	Christ > Chin Christ > NoRel
Mathematics is Logical	3.56 (0.58)	3.54 (0.56)	3.42 (0.57)	3.799*	Chin > NoRel

* $p < .05$, **$p < .01$, *** $p < .001$.
Chin: Chinese religious beliefs
Christ: Christian beliefs
NoRel: No religions
N.S.: difference not reached statistical significance

Results reveal that in general Christians see that mathematics is precise and is a subject of *calculables* more than those holding other religions. Furthermore, those subscribing to the Chinese religions see that mathematics involves thinking more than those holding other religions, and view mathematics as logical more than those who do not subscribe to any religion.

Beliefs about mathematics teaching among teachers of these three religious groups were also compared. One-way ANOVA was conducted. The statistics are summarized in Table 3. Results reveal that Christians possess relatively weaker constructivist view on mathematics teaching than others.

Table 3. Comparison of beliefs about mathematics teaching among groups

Subscale	Mean (standard deviation)			F	Scheffe Post-hoc comparisons
	Chin	Christ	NoRel		
LERN	3.35 (0.85)	2.95 (0.79)	3.28 (0.93)	8.912***	Chin > Christ NoRel > Christ
ROLE	2.99 (0.75)	2.93 (0.72)	3.09 (0.77)	2.045	N.S.
RELT	4.11 (0.60)	3.90 (0.47)	4.03 (0.56)	4.952**	Chin > Christ

* $p < .05$, ** $p < .01$, *** $p < .001$.

Chin: Chinese religious beliefs
Christ: Christian beliefs
NoRel: No religions
LERN: How Students Learn Mathematics
ROLE: The Role of Teacher in Sequencing of Teaching Computational and Application Skills
RELT: Relationships between Teaching Computational Skills and Problem Solving Skills
N.S.: difference not reached statistical significance

Results reveal that the Christians view that understanding should take place before computational practices when compared with mathematics teachers holding other religious beliefs. Yet they also considered that having students find their own methods to solve problems may not necessarily be the best way to achieve the goal of mathematics teaching.

Correlational analyses further reveal that the degree of religious involvement is mildly correlated to some subscales of beliefs about mathematics in the positive direction. The calculation of Pearson correlations reveals that the degree of religious involvement has a slight positive correlation to 'mathematics is a subject of *calculables*' (r =.206, p < .001), 'mathematics is precise' (r =.167, p < .001), and 'mathematics is logical' (r =.166, p < .001). On the other hand, the degree of religious involvement is mildly correlated to the constructivist view about mathematics teaching in the **negative** sense (r = -.222, p < .001).

4.4 *Results from Open-ended Questions*

Twenty six percent of respondents left the open-ended questions blank, and these were thus discarded. Content analyses were performed on valid responses and several themes emerged from these responses. Some of the responses may fall under more than one theme. Themes that contain responses by less than 5% in **all** the three groups were **not** reported because there were too few respondents. The themes were arranged in descending order of the **total** frequency. The responses to these questions are shown in Figures 3-7. In these figures, the percentages are checked over each religious group. Consider the first triple-bar in Figure 3, 16% among those holding a Chinese religious beliefs (and **not** of the entire sample) offered a response in the first theme, 27% among the Christians and 6% among those subscribing to no religions offered responses falling into the first theme.

The response rate to the first question ('mathematics is...') of the groups of Chinese religious beliefs, of Christian beliefs, and of No religion were 74%, 72% and 69% respectively. The percentages of the responses to the first question are shown in Figure 3. Results reveal that 'thinking and inference' was the most popular theme among respondents holding Chinese religious beliefs whereas 'mathematics contents and methods' was the most popular theme among respondents holding Christian beliefs. Furthermore, both the themes 'mathematics contents and methods' and 'real life applications' were relatively more popular among respondents holding Chinese religious beliefs and Christian beliefs. On the other hand, the themes of 'thinking and inference' and 'seeing mathematics as a discipline' were also relatively popular among respondents holding Chinese religious beliefs but less popular among respondents holding Christian beliefs. No obviously popular themes were found among respondents who claimed not to subscribe to any religion.

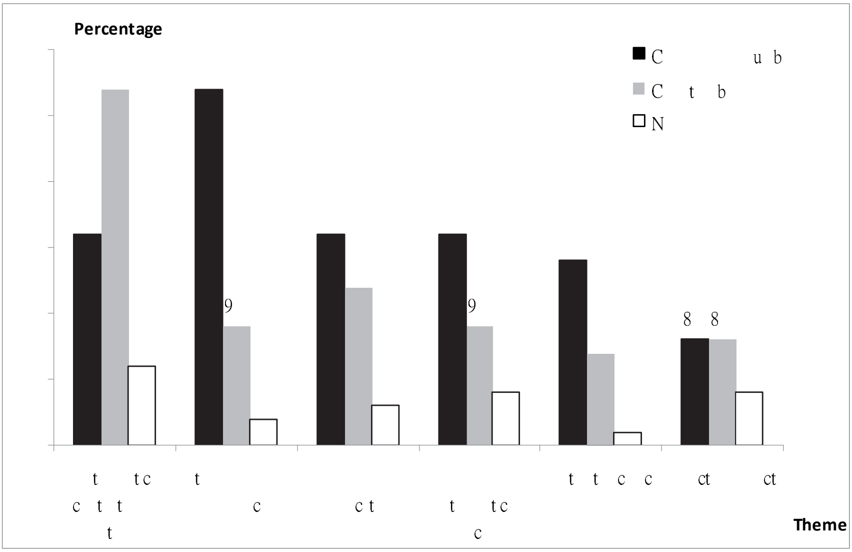


Figure 3. Percentages of various responses to ‘mathematics is…’

The response rate to the second question (‘mathematics learning is...’) of the groups of Chinese religious beliefs, of Christian beliefs, and of No religion were 74%, 72% and 70% respectively. The percentages of the responses to the second question are shown in Figure 4. Results reveal that ‘thinking and inference’ was the most popular theme in all the three groups. Yet, the percentage of respondents who do not have religion for this theme was not very high. ‘Feeling towards learning mathematics’ was also the most popular theme among respondents holding Christian beliefs and the group of respondents who claimed not to subscribe to any religion, and this theme was also relatively popular among respondents holding Chinese religious beliefs. Yet, the percentage of respondents who do not belong to any religion for this theme was not very high. Furthermore, the theme ‘methods to learn mathematics’ was relatively popular among the respondents holding Chinese religious beliefs but less popular among the other two groups.

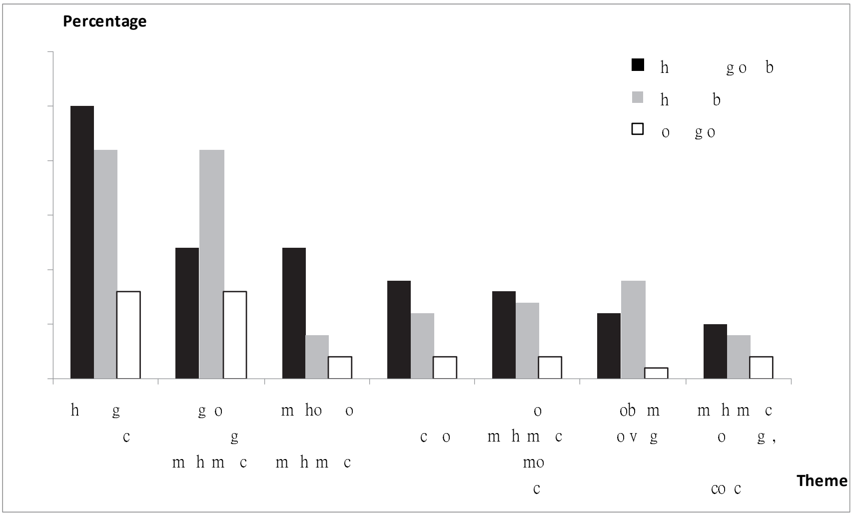


Figure 4. Percentages of various responses to 'mathematics learning is…'

The response rate to the third question ('mathematics teaching is...') of the groups of Chinese religious beliefs, of Christian beliefs, and of not subscribing to any religion were 73%, 69% and 67% respectively. The percentages of the responses to the third question are shown in Figure 5. Results reveal that 'thinking and inference' and 'feeling towards teaching mathematics' were the two most popular themes among all the three groups. However, regarding to the two themes, the percentages of respondents who do not subscribe to any religion were not very high. Furthermore, the theme 'mathematics knowledge, skills and concepts' was also relatively popular among the respondents holding Chinese religious beliefs and Christian beliefs, but less popular among the respondents who do not subscribe to any religion. The theme 'guidance and interaction with students' was also relatively popular among the respondents holding Chinese religious beliefs but less popular among the other two groups.

The response rate to the fourth question ('education is...') of the groups of Chinese religious beliefs, of Christian beliefs, and of those not subscribing to any religion were 72%, 69% and 66% respectively. The

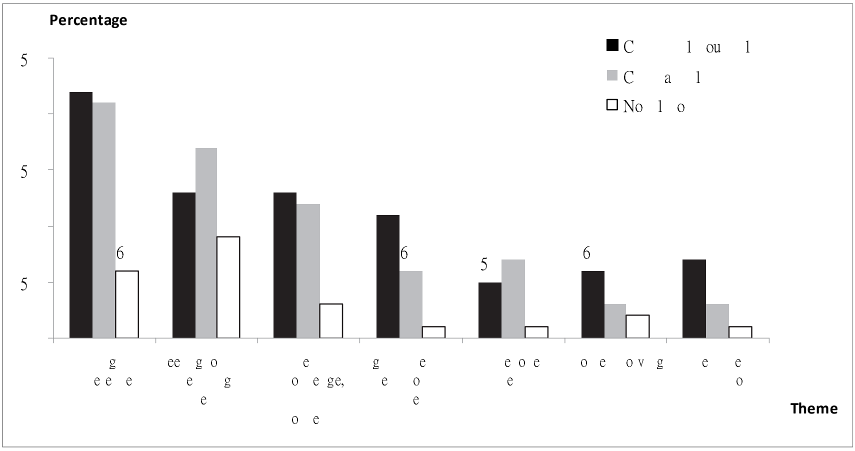


Figure 5. Percentages of various responses to ‘mathematics teaching is…’

percentages of the responses to the fourth question are shown in Figure 6. Results reveal that ‘nurturing the growth of a person’ was the most popular theme among all the three groups, and respondents holding Christian beliefs had the highest percentage for this theme among these three groups. Furthermore, the themes ‘knowledge transfer and heritage’ and ‘mind, spiritual and moral aspects’ were also relatively popular among the respondents holding Chinese religious beliefs and Christian beliefs but less popular among the respondents who claimed not to subscribe to any religion.

The response rate to the fifth question (‘religious belief is...’) of the groups of Chinese religious beliefs, of Christian beliefs, and of not subscribing to any religion were 71%, 69% and 58% respectively. The percentages of the responses to the fifth question are shown in Figure 7.

Results reveal that ‘mind and spiritual aspects’ was the most popular theme among the respondents holding Chinese religions whereas ‘meaning of life’ was the most popular theme among the respondents holding Christian beliefs. Notably, this does not necessarily accord with the general perception that Chinese religions pay more attention to earthly life. The reason(s) behind this occurrence may be worth further exploration in future studies. Nevertheless, the percentages for these two themes were also relatively high among the respondents of these two

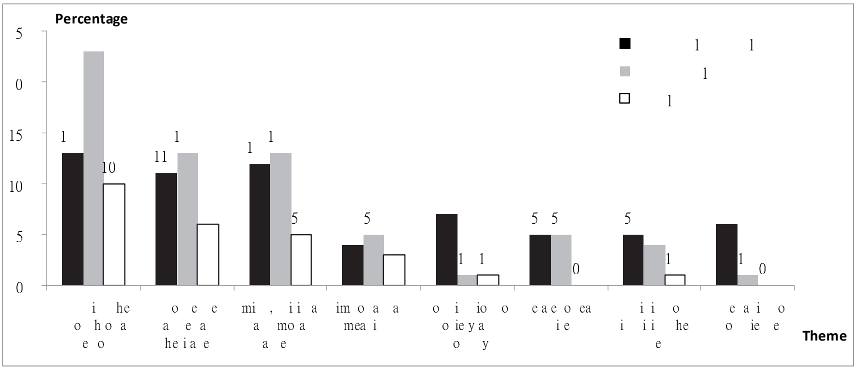


Figure 6. Percentages of various responses to ‘education is…’

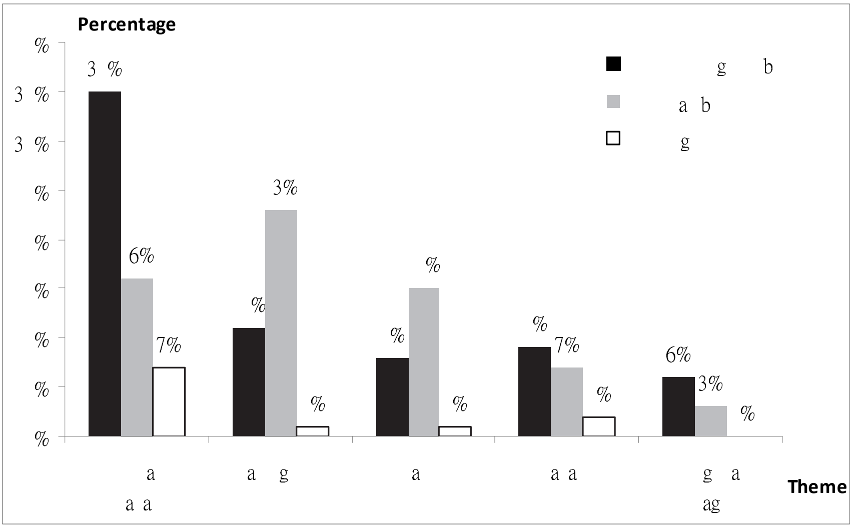


Figure 7. Percentages of various responses to ‘religious belief is…’

groups. Furthermore, ‘faith’ was also a relatively popular theme among the respondents holding Christian beliefs but less popular among the other two groups. No obviously popular theme was found among the respondents who claimed not to subscribe to any religion.

4.5 *Summary*

Before we draw conclusions from our two-part study, we must point out that the study is explorative in nature. However, we assert that this may still be a step forward in research, no matter how small it may be. Our original intention was to compare the beliefs about mathematics and mathematics teaching among Chinese mathematics teachers who hold different religions. As the sample sizes of some religious groups were relatively small, we decided to combine the religious groups more generally into Chinese religious beliefs, Christian beliefs and those not subscribing to any religion. The drawback is, the group holding Chinese religious beliefs is a mixed one and consequently the results are not decidedly conclusive. The same is true for the combined Christian group, Catholicism and Protestantism. However, for the purpose of worthwhile analysis, we maintain the combinations of the three religious groups. The differences in beliefs about mathematics and mathematics teaching among teachers holding Chinese religious beliefs, Christian beliefs and those who do not subscribe to any religion may not be conclusive, yet our primary aim of exploring whether such religious differences exist at all, in their impact on mathematics education, is fundamentally achieved. What precisely these differences are, and how far such differences can be attributed to their religions as well as their religious engagements requires further investigation.

Generally, our study was not lacking respondents. However, some religious beliefs like Confucianism do not have a formal ritual (like baptism in Christianity and taking refuge in Buddhism) for becoming a member of that religious group. In those cases, respondents may find difficulties in labelling themselves as followers of such religions. In this, we believe that advancement in methodology can overcome the issue.

Concerning the beliefs about mathematics, the first portion of the questionnaire found that respondents holding Chinese religious beliefs consider that mathematics involves thinking more than those holding other beliefs. This was consistent with the finding that 'thinking and inference' was the most popular theme for the first open-ended question ('mathematics is...') among respondents holding Chinese religious beliefs. Furthermore, the first portion found that respondents holding

Christian beliefs see mathematics as precise and as a subject of *calculables* more than their counterparts. This may explain why 'mathematics contents and methods' was the most popular theme for the first open-ended question among respondents holding Christian beliefs.

Concerning the beliefs about mathematics teaching and learning, the first portion of the questionnaire found that the respondents holding Christian beliefs possess relatively weaker constructivist view on mathematics teaching than the other two groups of respondents and the degree of religious involvement is mildly correlated to the constructivist view about mathematics teaching in the **negative** sense. Whether this relates to Christian beliefs agrees to some aspects but disagrees to some other aspects of (radical) constructivism (Howell & Bradley, 2001), the mono-theistic nature may be one perspective worth consideration.

The use of open-ended questions in the second portion of the questionnaire aimed to gain a deeper understanding of any differences among the groups. Although there were more similarities than differences between the respondents holding Chinese religious beliefs and Christian beliefs, subtle discrepancies were found. For instance, the most prominent 'Christian theme' on 'what is mathematics' is content and method, while the most prominent 'Chinese theme' is thinking and inference. This can be seen to echo the earlier questionnaire portion that the Christians see mathematics more as a subject of *calculables*. More discrepancies were found when those who do not subscribe to any religion were included in our comparison. In particular, the percentages of the most popular theme of responses among this group in the open-ended question section were not very high compared with their counterparts. It implies that there was no obviously popular theme of responses to the open-ended questions from the group of respondents not subscribing to any religion. Inevitably, such surveys can only solicit respondents' opinions in a snapshot and in a one-sided fashion. In this, more can be revealed through in-depth qualitative studies.

Despite the fact that there is an imbalance between the sample sizes of teachers subscribing to Buddhism and of those subscribing to Confucianism, further analyses focusing on these two groups indicate traces of evidence that there are slight differences between these two groups. The difference shown in our study may be far from salient due to sample size and insensitivity of instrument. In this area, we have conducted more in-depth research which we report below.

5. Study Two: Comparison of Values in Mathematics Teaching between a Buddhist Teacher and a Confucian Teacher

5.1 *Research Focus*

The second study is an in-depth comparative case study which involves two primary mathematics teachers in Taiwan, one a devoted Buddhist and the other influenced by Confucianism. The study aims at investigating their values in mathematics teaching. Part of the results were reported in Leu (2005) and Leu and Wu (2004), though this is the first time that the value systems of the two teachers were combined for a comprehensive comparison.

5.2 *Participants*

Since this research aims at exploring the values in primary school mathematics, we tried to identify subject teachers with relatively stable mathematics pedagogical and personal values. Senior teachers are more likely to have stable mathematical pedagogical values. Therefore, senior teachers were selected for our research. There are two categories of primary teachers in Taiwan. One is the subject-specialized teacher, who provides instruction in a single subject, such as natural sciences, physical education or art. The other category of teachers is the homeroom teacher, who is responsible for a designated class and the instruction in several subjects, such as Mandarin[3], writing, mathematics, social studies, and ethics. More importantly, the homeroom teacher is the primary mentor and counsellor for the students' behaviour and social conduct at school. Therefore, homeroom teachers are selected as our research participants.

It was not easy to find teachers willing to participate in our study because participation was time consuming and involved personal details

[3] Official language in Taiwan, which is equivalent to *Putonghua* in the Chinese mainland.

which not everyone would wish to disclose. Two senior homeroom teachers, Ms. Chen and Ms. Lin, were finally recruited for our study.

Ms. Chen and Ms. Lin have teaching experiences of 21 and 9 years respectively. Ms. Chen teaches at an affiliated primary school of a university. Most of the parents of the students at that school are civil servants or teachers, having middle to upper socioeconomic status. There are 36 students in her class (Grade 5). Ms. Lin teaches at an ordinary primary school. Most of the parents of the students at that school run small businesses or are workers, their socioeconomic status being medium to low. There are 27 students in Ms. Lin's class (Grade 6).

Among the two participants, one is a Buddhist and the other is influenced by Confucism. Ms. Chen, a devoted Buddhist, commented that, "*From Buddhism, I realized many different perspectives. Buddhism is actually an education, not a religion. My life opens up in Buddhism.*" Further, her school principal commented that, "*She [Ms. Chen] is a Buddhist. She likes to talk about Buddhism with her colleagues. […] She uses this power [the power from Buddhism] to pass her days and uses this power to work.*" Ms. Chen stated that she is a follower of the Pureland sect. Ms. Lin however, considers herself as adopting a Confucian worldview as she repeatedly commented that she hopes her children and students will go on to attain higher academic qualifications. Such values are consistent with what was stated in Huang and Yore (2003), that: under the influence of Confucian thoughts, "Chinese parents usually have rather high exceptions for their children and their future success" (p. 427). Indeed, Ms. Lin also pointed out that she was influenced by the notion of gentry (shì dàfū) possessed by her parents.

5.3 *Method*

Bishop (2001) argued that teacher's decision making is influenced by his or her value structure and current teaching situations. When a teacher makes decisions in a consistent way and then executes these decisions, we can observe his or her mathematics teaching behaviours and identify the intentions of these decisions through interviews. By means of identifying a teacher's decision mechanism, we can trace back his or her structure of values and then his or her background culture. In view of this, classroom observations and interviews were employed in this case study to understand the mathematics pedagogical values of two teachers. The objective of classroom observations was to search for recurrent behaviours which occurred in mathematics teaching, critical events, and scenarios of decision making or valuing. Different values may be displayed through the same teaching behaviour. For instance, many primary teachers in Taiwan like to ask students to read classics but their intentions can be very different. Some teachers want to foster the students' attention abilities. Some teachers want to enhance the students' language abilities. Some teachers want to promote moral education. Therefore, interviews were conducted to discover into the values behind such recurrent behaviours, critical events and scenarios.

The respective classrooms of Ms. Chen and Ms. Lin have been observed for about one year. The number of lessons observed for Ms. Chen and Ms. Lin were twelve and ten respectively, while the number of interviews conducted were seventeen and eighteen, respectively.

The data were collected by various methods over a variety of occasions and topics including once-a-week, whole-unit and unscheduled observations, and interviews to allow data triangulation. The regular once-a-week observation aimed at providing a full picture of the teachers' instructional behaviours across different mathematics topics. The topics covered in Ms. Chen's mathematics class include: area of a rectangle, equivalent fractions, multiplication and division of integers by place value, and the inverse relationship between multiplication and division. The topics covered in Ms. Lin's mathematics class include: the area of a circle, direct proportion, inverse proportion, map scales, and the concept of probability. The teachings of entire topics were also observed. The teaching of Ms. Chen on the area of plane figures, and the teaching of Ms. Lin on formulas of circumference were followed.

5.4 *Data Analysis and Interpretation*

The observed lessons were videotaped while the interviews were audio taped and transcribed. They were all analysed under grounded theory and constant comparison (Creswell, 1998), under the following procedures:

Repetitive teaching behaviours were identified and defined.

The rationale for the teaching behaviours was abstracted from the interviews.

Teaching behaviours and rationales were compared and contrasted continuously throughout the analysis, leading to the classification of behaviours and rationales into categories.

Informant checks were used to confirm data, analyses, and interpretations. Any disagreements were discussed, and consensus was reached among the researchers and informants.

The relationships among these mathematics pedagogical values were determined for Ms. Chen/ Ms. Lin's teaching actions and personal beliefs.

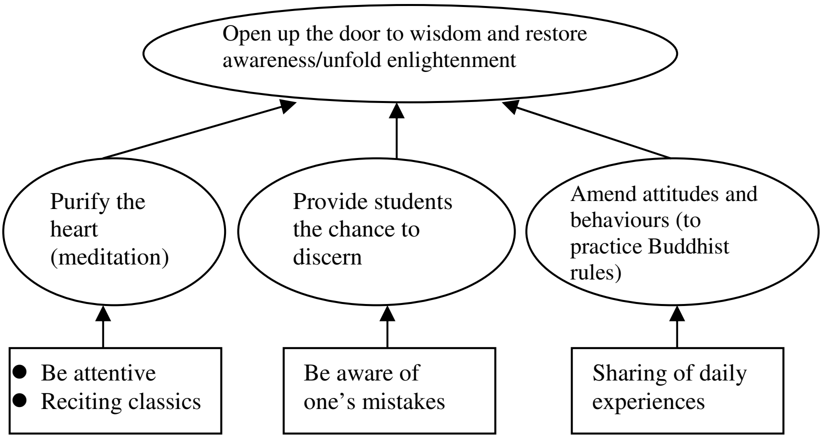
Potential bias was addressed by having one Buddhist and two non-Buddhists as research team members. All three team members participated in the classroom observations and interviews, as well as discussed the classroom observations and interpretations of interview data. The different perspectives were helpful during the classroom observations, follow-up interviews and data analyses, since the chief researcher is a Buddhist and the non-Buddhist members had limited preconceptions about Buddhism. The research team met monthly with an external panel of three researchers with expertise in mathematics pedagogical values of Taiwanese secondary school teachers. Research data and interpretations were shared with the panel and discussed to improve data analysis, interpretations, and research procedures.

5.5 *Results*

Under the above analytical procedure, the value systems of both Ms. Chen and Ms. Lin were revealed, including their views on the goals of education and of mathematics teaching itself. Results revealed that they demonstrated different perspectives on the goals of education and of mathematics teaching, as influenced by their religious beliefs. In the following, quotations from the interview are marked in italics.

5.5.1 *The Goal of Education*

For Ms. Chen, the primary goal of education is to unfold the students' own nature of enlightenment; for Ms. Lin, the goal of education is for achieving success and fame in life. The value systems of Ms. Chen and Ms. Lin on education are depicted in Figure 8 and Figure 9, respectively. In these figures, the two teachers' ideologies in teaching are depicted in ovals. Although their teaching actions are not the focus of the study, they reflect the teachers' beliefs so we have shown these in rectangular boxes.



represents the mathematics teaching behaviours of Ms. Chen

represents the teaching goals or values of Ms. Chen

Figure 8. The value system regarding education of a devoted Buddhist teacher

Although Ms. Chen and Ms. Lin possess different goals in education, their teaching methods have a lot in common. For instance, both require their students to be attentive in the class. However, the two teachers have different reasons for their methods (see below). There are also differences in their teaching behaviours, for instance, how they react when their students make mistakes in solving mathematics problems, in which the reasons behind their reactions also differ. Further, there are teaching behaviours which are particular to a teacher, such as the sharing of daily experiences for Ms. Chen.

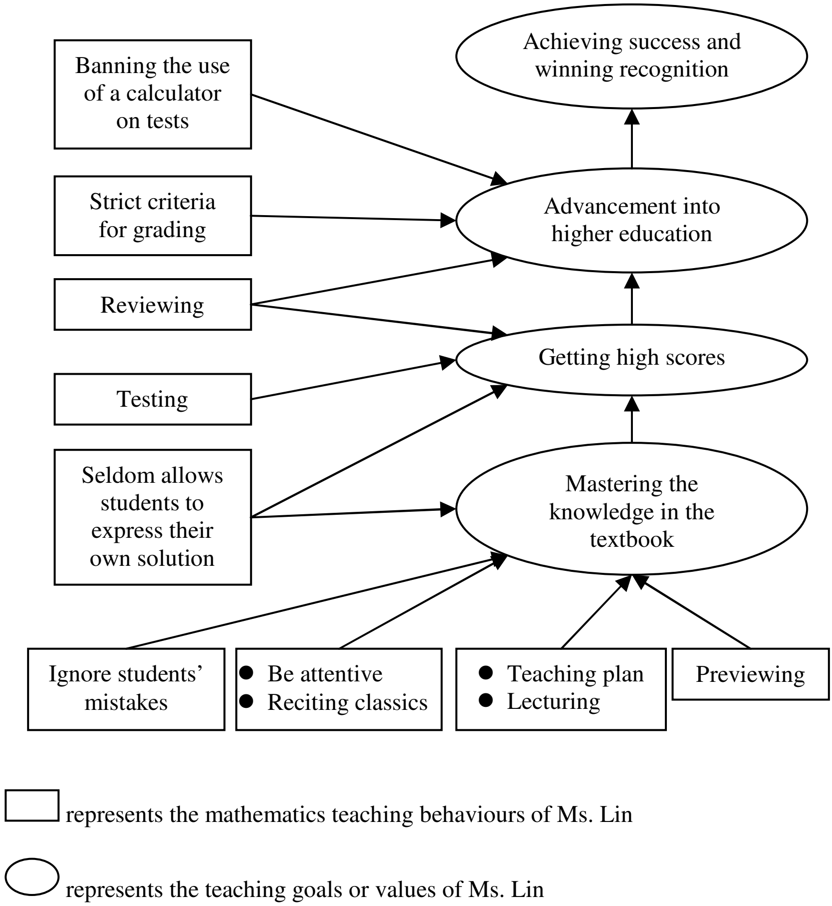


Figure 9. The value system regarding education of a Confucian teacher

As for requesting students to be attentive in class, Ms. Chen often encouraged her students to reiterate the problem and solution. She explained, "*My purpose is to redraw absent-minded students' attention. … I realize that if one can calm down and has a peaceful mind, he/she can learn things well and quickly*." Other than asking her students to be attentive in the class, she also asks them to recite the classics for 3 to 5 minutes before each lesson. She said, "*When you focus on the Classics, you get morality, meditation and wisdom. As long as you can concentrate, your mind won't wander. Then you get meditation. You can realize the meaning of the Classics and you will gain wisdom.*" Ms. Chen's rationale behind these classroom practices was that concentration can lead to calmness and meditation, which will then lead the way to wisdom and enlightenment.

Like Ms. Chen, Ms. Lin would ask students to recite classics before a class and seeks their full attention during the class. She commented that, "*My students recite classics. They would do that for 3 to 5 minutes before the class. I think the 5-minute classic recitation calms them down*" and "*I think if a child has other things on his/her mind and cannot settle down, his/her learning outcome would suffer. So no matter what subject I'm teaching, I would ask them to recite*". As we try to distinguish whether the purpose of Ms. Lin's teaching behaviours is the same as Ms. Chen's, we asked Ms. Lin the question, "*Other than calming students down, is there any other purpose for the recitation?*" She responded "*I think the children nowadays have poor language abilities. I believe that by reciting these classics, they have more exposure to the language as practiced in olden times.*" From her response, we can see that Ms. Lin places a great weight on students' academic achievement. Contrastingly, Ms. Chen emphasized on the unfolding of enlightenment.

When students make mistakes in solving mathematics problems, Ms. Chen would simply point out that they have made mistakes, but she would not explicitly tell students where/what their mistakes were. Ms. Chen's rationale on developing students' own awareness of mistakes is that the awareness of self mistakes would lead to reflection and consequently bring out the awareness of enlightenment.

When Ms. Lin spots incorrect problem-solving strategies of students either during class instruction or in their homework, she never discusses them in front of the class. She explained that, "*(in my Mandarin class) I used to write the wrong characters made by students on the blackboard and point out the mistakes, hoping that other students would not make the same mistakes. But the result was that many students copied the wrong writing.*"

Ms. Lin did not realize the different characteristics in teaching Mandarin and Mathematics and used her experiences in teaching Mandarin to draw an analogy to the teaching of Mathematics. The statement of Ms. Lin is clear that she attaches more importance to the correct solution rather than to the problem-solving processes.

As for the sharing of daily experiences, Ms. Chen would tell her students to "*Write down a simple line in the communication book about what you've learned from what you've seen and heard today. …*" She would ask her students to share daily experiences in the class. Ms. Chen's educational rationale for the sharing of life experiences is that it will trigger students to reflect on the behaviours and attitudes in their life and finally lead students to unfold their own enlightenment.

Ms. Chen and Ms. Lin have dramatically different goals of education. Ms. Chen understands the difference between Confucianism and Buddhism. She explained, "*The teachings of Confucius and Mencius focus only on the present life. But Buddhism tells you about your past, present and future. We all have that innate ability. I want to help my students to restore that feeling, to restore their self-awareness. The word "Buddha" means awareness. Buddhism gives answers to life. Our knowledge (those in the textbooks) can't set one's mind free.*" Therefore, the educational goal of Ms. Chen is to guide her students to unfold their own enlightenment.

Ms. Lin helped her students to master the knowledge in the textbooks by means of previewing and well-organized teaching plans so that the students can achieve high scores (please refer to the next section for further details). She would forbid her students to use calculators in school mathematics examinations because "*calculators are not allowed in the public examination of Taiwan. I require my students to do mathematics without a calculator so that they can adapt to the public examination culture of Taiwan.*" She said, "*There is pressure of moving up the academic ladder in Taiwan. Mathematics, Science and English are determining subjects. And what primary teachers can help with this is the subject of Mathematics.*" Ms. Lin's concern on students' academic performances is consistent with what Huang (2004) has said on the moving up the academic ladder as an educational goal. Huang stated, "Although there are many career choices, the most rewarding path is 'junior secondary – senior secondary – university' which is achieved through examinations" (p. 214). This can be echoed by the comment in International Commission on Mathematical Instruction (2006) that "in East Asian societies, students, teachers, and parents view written tests and examinations as the most important thing in a students' school life and as a key to the success of their future life" (p. 17). Ms. Lin further elaborated this point in her address at the graduation ceremony, "*I wish you achieve success and win recognition (g ng ch ng m ng ji) as early as possible, and come back to visit us*." It reflects that she strongly values achievement and recognition and regards them to be her goal of education. This idea is generally perceived as being influenced by Confucianism[4].

5.5.2 *The Goal of Mathematics Teaching*

Other than different goals of education, Ms. Chen and Ms. Lin also have different values on mathematics teaching. The goal of Ms. Lin's mathematics teaching is to help students to master knowledge acquired through the textbooks and achieve high scores in tests and examinations. In contrasting, other than helping students understand mathematics knowledge, Ms. Chen places more emphasis on how to develop students' problem-solving attitudes in life. Under these different goals of mathematics teaching, the teachers have demonstrated different behaviours in their mathematics teaching. Most of the primary teachers in Taiwan follow closely the contents in mathematics textbooks in their teaching. Ms. Lin requested her students to preview the textbook contents and questions before attending mathematics lessons. The purpose of previewing lessons is to "*make students have a better understanding when [I] explain the solving strategy provided in the textbook*." The teaching behaviours of Ms. Lin's mathematics instruction included designing a well-organized teaching plan and lecturing systematically about the solution provided in the textbook. Ms. Lin

[4] As mentioned above, there is a fine line between Confucianism proper and (secular-) Confucianism as perceived by the public. Again please refer to Wong, Wong, & Wong (2012) for details.

lectured on the textbook solutions most of the time and allowed little or no time for students to present their solutions. Ms. Lin said "*I prefer the textbook solution because it is the solution proposed by the experts and should be the most appropriate solution for their age.*" She further states, "*There is a time limit on the test. If they don't do the problems as quickly as they can, they may not have enough time to finish all the problems. Then their grade will suffer. So, I don't encourage them to use their own solution if it's not a better one.*" From her answer, Ms. Lin's viewpoint on student solutions is related to "how to get good grades on the test."

Besides the above teaching behaviours, Ms. Lin also required students to fully concentrate during her class, providing many review sheets for students after each lesson is complete, giving tests regularly and reviewing students' work constantly. For Ms. Lin, the goal of all these teaching behaviours is to help students master the knowledge acquired in the textbook and attain high scores in tests.

Ms. Lin cares about her students' academic achievement to a great extent. The interviewers asked her, "*Do you think getting higher education is students' only choice? If one student didn't get good grades at school but later achieves success in his/her career, do you feel any regrets for him/her?*" She responded, "*Yes, I do feel sorry for them. If my students can have good grades, their goal is clear: doing well in the college entrance exam and getting into a good university. But for those who don't have good grades, they can't go to the college and they have fewer choices.*" This idea is similar to what is expressed in the popular poem, "All walks of life are inferior, only being an academic is the top rank". Again, this notion is commonly perceived as being influenced by Confucianism. Putting this idea into mathematics instruction, one can postulate that "*One who can learn mathematics well has a better chance of attaining higher education and it will be easier for those who attain higher education to achieve a respected social status and a successful life.*" We clearly see that Ms. Lin attaches great importance on her students achieving high scores in mathematics examinations and acquiring high academic qualifications. This idea is consistent with Liu (1995), that "teenagers' academic pressures come from the urge to move up the academic ladder, which is rooted in the notion of gentry (shì dàfū) which looks down upon the blue collar" (p. 29).

Separately, Ms. Chen would not only ask students to solve the questions on their own but also have them explain their problem-solving process. Ms. Chen would even tell her students' parents, "*In solving mathematics problems, getting the right answer is but one important goal. Nevertheless, understanding is essential.*"

Ms. Chen believes she was emphasizing another purpose of mathematics teaching. She remarked that, "*Just like solving a problem in mathematics, we encounter many problems in our life and we should learn how to solve the problems. The attitudes are important. I think mathematics lessons teach more than mathematics knowledge. We need to teach the students to apply the spirit of solving mathematics problems in their lives. In other words, when encountering a problem in our life, we should try to solve the problem on our own and not get stuck simply because no one has taught us how to solve that life problem. Mathematics is not only in the textbooks but we should practice and integrate the spirit of solving mathematics problems in our life.*" She adds, "*Don't keep on telling yourself that it is impossible to make it and give up, especially when you are solving problems. … Don't be afraid to make mistakes. Just find out what is wrong, so you won't repeat the same mistakes.*"

Further, Ms. Chen supplemented some other reasons to support her emphasis on developing students' problem-solving attitudes. "*No matter which subject you are teaching, I think students can learn the subject easily if their problem solving methods and attitudes can be fostered. If they are not equipped with such methods and attitudes, they would face a lot of learning difficulties.*" She adds, "*I think the purpose of education is to nurture human beings. The human part (the development of problem solving skills and disposition, together with the unfolding of one's own nature of enlightenment) is more important. A teacher should teach both skills and knowledge. But I think if a teacher can set up good moral standards (which encompass problem solving methods and attitudes, as well as self awareness), his/her students won't be far from attaining the knowledge.*" The above two statements show how Ms. Chen's purposes on mathematics teaching and education differ from Ms Lin's above.

5.6 *Summary*

The above in-depth case study using qualitative methods suggests that teachers influenced by Buddhism and Confucianism possess different value systems, including their views on the goals of education and of mathematics teaching. Results revealed that, the goal of education according to Ms. Chen, who is a devoted Buddhist, is to initiate students' self awareness. Consequentially, the goal of mathematics teaching is the development students' problem-solving dispositions for their future adult life. By contrast, the educational goal of Ms. Lin, who is influenced by Confucianism, is for her students to gain success and fame in life. In practice, her goal of mathematics teaching is to help students master the knowledge acquired in textbooks and attain high scores in tests. Though the various teacher behaviours of these teachers may have similarities as well as differences, these behaviours are all grounded on different premises, rooted in their own religious beliefs. Such differences might be attributed to the devotedness of these two teachers in their religions. Whether religious beliefs of more general followers of various religions will have an impact on their teaching would require further investigations.

6. Conclusion

6.1 *Summary of the Studies*

First of all, it was found in Study One that those mathematics teachers who were influenced by Chinese religious beliefs, Christian beliefs and those who did not subscribe to any religions hold different beliefs about mathematics and about mathematics teaching, no matter how subtle these differences may be (Chan, Wong, & Leu, 2012). It is not our present aim to draw definite conclusions of precisely what these differences are and whether these differences can be clearly attributable to religious beliefs. However, careful inspection of the results affirms that the differences are present. Since data collected in Study One simply relies on a questionnaire survey, the results are not entirely conclusive. As such, we supplemented the first study with Study Two in which thick data was collected. In Study Two, ethnographic observations were employed over a lengthy period of one year. Two cases, a Buddhist mathematics teacher and a Confucian mathematics teacher, were portrayed in detail. By comparing these two cases, it is clear that their religious beliefs affect their teaching behaviours. Though some of their teaching behaviours were similar, the rationale behind them were different and by-and-large supported by their religious convictions.

Putting the results of the two studies together, we see a general picture that, although both Christians and those subscribing to the Chinese religions see that mathematics involves thinking, the Christians are relatively less constructivist and more Platonistic in their thinking. Focusing on the Confucians and the Buddhists, the former group appears to place higher attention on academic achievements whereas the latter group appears to place higher regards on the nurturing of students.

We have repeatedly stressed that we should not take the above results as entirely conclusive. Further studies are needed to arrive at an affirmative result. Inevitably, there are a number of limitations in our two studies (which we will discuss below), and the relationship between religious beliefs and mathematics teaching deserves further exploration. However, we believe that we have opened up the door to meaningful research.

6.2 *Theoretical and Pedagogical Implications*

It is well established that both mathematics and mathematics teaching are not value-free (Zhang & Wong, 2015). As an important component of values and beliefs, it seems unlikely that religious beliefs have no influence on teaching, and on mathematics teaching in particular. Religious beliefs appear deeper than general values and we cannot expect to always see these beliefs manifest in day-to-day routines. When it comes to critical moments however, such as when one has to choose between drilling students for a better academic career and development of thinking habits which are not readily assessable in conventional examination, such beliefs, as pointed out by Furinghetti (1994), would creep in like a ‘ghost’, which would otherwise be hidden in a dark corner.

It would not be viable, ethical nor theoretically sound to suggest altering teachers' religious beliefs in order to deliver a prescribed mode of teaching. However, we might suggest that both curriculum developers and advocates of certain instructional theories should be culturally and religiously sensitive to teachers' underlying religious beliefs. Every way of teaching has its cultural assumptions (Wong, 2008), and teaching and learning traditions that work well in one culture may not necessarily work well in another (Biggs & Watkins, 2001). This is particularly true when one faces a multi-cultural and multi-religious setting. For teachers who hold various religious beliefs (including those who claim to possess no traditional religions), whether they are subscribing to rituals, engaging in various religious activities or possess a genuine understanding of the doctrine might be another point of study to examine teachers' own reflections on how their beliefs impact on their teaching (Wong, Wong, & Wong, 2012).

6.3 *Directions of Further Research and a Glimpse of an On-going Research*

The two studies reported in this chapter have released a new research agenda, viz. possible connections between Chinese mathematics teachers' religious beliefs and their teaching. The first study contrasted Chinese religious beliefs (rooted in 'Eastern' traditions) with Christian beliefs (rooted in 'Western' customs). The second study contrasted two Chinese religions, Buddhism and Confucianism whose ideologies are close to each other. Admittedly, the contrast revealed in the second study may not be generally applied. Indeed, throughout the ages, these two religions have evolved. One can expect that, as indicated in the first study, a bigger contrast can be found if we compared 'Western' religions with 'Eastern' ones. There is much room for exploration if one can include various other kinds of religious beliefs like Islam, Hinduism and Judaism.

To facilitate such investigations, both quantitative and qualitative methods can be used. A validated instrument is a pre-requisite for a quantitative approach. To date, there are quite a number of well established instruments on spirituality and religiosity based on Western religions. Hill (2005) provides a summary of measurements in the psychology of religion and spirituality of which most are based on Western religions. There is a need to develop such instruments for Eastern religions as well as for those who claim to belong to no religion (one can be spiritual even though one is not a follower of any religion). Adapting from existing Western instruments can be one possibility. However, there may be an even more earnest need to re-develop such instruments of measurement from grounded studies. With a validated instrument and larger samples, we can then conduct more sophisticated statistical analyses like t-test, correlation and regression analyses.

To initiate a bigger body of related studies, there are several lines we can consider. As mentioned above, the delineation of the characteristics of a culture can often be done by contrasting with other cultures. This can be done by collecting data from other religions like Hinduism and Islam. Hopefully, this can be achieved through cross-regional collaborations. Another challenge we faced, especially in Study One, is that there are religions that do not have a clear ritual of subscribing to it (in contrast with baptism in Christianity). By translating various religions into their religious values and/or worldview, it is possible to identify those who are not formally Confucians (taking Confucianism as an example) but possess a Confucian oriented mindset. The first step appears to be to clarify what 'Confucian orientation' means. If this can be achieved, we would have a clearer delineation of those who had previously said to subscribe to no religion. This group of people may not be homogeneous but may be possessing different religious/philosophical orientations.

Methodologically, the two studies involve two extremes. Study One relies on a one-way questionnaire which has the strength of having a larger sample though what is asked may not be sufficiently in-depth. For Study Two, data collected may be in-depth but it had proved too time consuming to attract volunteers to participate. Questionnaire and follow-up interviews could be a 'middle way', where a mixed method (both quantitative and qualitative) is employed. This is the method of our on-going research in this area.

We acknowledge the complexity of religious worldviews. There are differences (sometimes rather substantial) in interpretations of doctrines (and hence worldviews) within different branches of the same religion. Furthermore, one's personal religious beliefs may have significance different from the orthodox beliefs of the (branch of) religion subscribed by him/her because the former is a result of interaction of the latter with one's socio-cultural context and experiences in life history. (See for instance, Glennan, 2009.) To address this issue, a follow-up research in qualitative approach is conducted by the second and third authors of this chapter. It involves an interview study with fifteen mathematics teachers in Hong Kong (five Buddhists, five Christians, and five claiming not to subscribe to any religion) (Chan & Wong, 2014). The focus of this ongoing study is to investigate the possible similarities of teachers' (personal) religious worldview and philosophical thoughts with their views on effective mathematics teaching and learning. The participants were asked to complete a questionnaire which consists of 7 open-ended questions on how they entered the profession, what kind of calibre they expect to nurture in their students and how their religious beliefs might influence their teaching. Then, an individual semi-structured interview about their views on mathematics education and religions, which lasted for 0.5 to 1 hour, was conducted. The interview questions were modified from the study of Cai, Perry, Wong, and Wang (2009) on teachers' perspective of effective mathematics teaching, with additional questions about religious beliefs and its possible impact on mathematics teaching. A rather promising result has emerged. In brief, some alignments of the teachers' personal religious beliefs and their beliefs about mathematics teaching/learning can be observed at least in some cases. For instance, a Buddhist teacher emphasised on "seeing wider" in order to look for the connection between different mathematics topics. At the same time, she also had a wide perspective in seeing the Buddhist perspective and claimed that "everything is [within] the Buddhist doctrine". Interested readers may refer to a detailed report published in Chan and Wong (2014) and other forthcoming related papers.

Last but not least, there may be discretions between teachers' beliefs about teaching and their actual instructional practices although the former usually plays a strong role in determining the latter. On top of in-depth interviews, class observations may be introduced so that the relationship among teachers' religious beliefs and worldviews, beliefs about mathematics teaching, and instructional practices in mathematics classrooms can be explored. This is precisely an ongoing research project currently conducted by the second and third authors of this chapter. The investigation on the relationship between religious beliefs and mathematics teaching and learning has a long way to go. But as the ancient Chinese philosopher Laozi once said, "a journey of a thousand miles starts from beneath one's feet", we need to investigate bit by bit to explore the potentials of this deep blue sea.

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Epilogue

Why the Interest in the Chinese Learner?

FAN Lianghuo WONG Ngai-Ying CAI Jinfa LI Shiqi

1. Why the Chinese Learner Became a Learning Phenomenon?

There is much that we can learn about and from any country. The Chinese region is a big and up-and-coming one that naturally is a focus of attention worldwide. However, there are reasons beyond its size and history. At a more macro level, the economic success of the four little dragons[1], plus Japan, in the 1970s and 1980s has led to the conjecture that there are some common features among these Asian economies. This was reinforced by the fact that consistently these four little dragons, and Japan as well, occupied the top positions in the Third International Mathematics and Science Study (TIMSS)[2] which commenced, on a four-year circle, in the early 1990s. The continuous stellar performance of the Chinese in various international comparisons such as the 1992 International Assessment of Education Progress (IAEP) mathematics study, the Programme for International Student Assessment (PISA), the International Mathematical Olympiads and even the earlier 2nd IEA Mathematics Study (SIMS) conducted in the early 1980s not only raised the eyebrows of sociologists, psychologists and educationalists internationally but also confused them, as there had been a perception that Chinese learning and teaching relied on rote-memorization, drill and practice (Biggs, 1996). That is why when Western scholars first encountered this Chinese learner phenomenon, terms like 'myth' and 'paradox' were coined (Wong, 2013). It is believed that cracking the

[1] Hong Kong, Taiwan, South Korea, and Singapore.
[2] Later renamed "Trends in International Mathematics and Science Study".

paradox would not only allow us to understand better the Chinese learner phenomenon, but also inform us about how learning and teaching can be improved.

2. The Emergence of the Chinese Mathematics Education Circle

Professor D. Zhang, one of the advisors for this book, raised the question whether an East Asian school of mathematics education has emerged (Zhang, 1992, 2009). Let us first look at criteria for a local or regional academic community to exist. Some have mentioned the following milestones of localization: having expatriates leading local research, local researchers getting doctoral degrees abroad by conducting local studies ["import"], local researchers getting local doctoral degrees, local academics nurturing their own doctoral students, and local academics producing doctoral graduates beyond the local territory ["export"]. These milestones may not be fully applicable to the Chinese region since it has its own long cultural tradition. Moreover, there has been increasing discussion that localization and globalization are not dichotomous. In particular, the notion of "glocalization" was put forth, starting in the business sector (e.g., see Robertson, 1994). Indeed there are several questions we can further pursue here. Do there exist particularly "Chinese" educational practices that are distinct from those elsewhere? Are there variations in practice within the Chinese regions, and are such within-region variations larger than between-region variations (Wong, 2009)? Or, are we just looking for good "Chinese" practices that can benefit the world that were previously overlooked? Moreover, are such practices "transplantable" (Watkins & Biggs, 2001)?

Certainly localization or local awareness does not necessarily infer a closed-door policy or self-centeredness. Otherwise, we would be endlessly searching for features in Chinese mathematics education that are distinct from, say, Korean and Japanese mathematics education, or focusing on whether there is a 'Shanghai-style mathematics education', etc., which does not really make much sense. Even as we search for a Chinese identity, we fully realize that the Chinese culture, like most other cultures, is an open system. It keeps absorbing elements from other cultures while at the same time exhibiting different color spectrums or diversity across the country. As pointed out in Wong, Wong, and Wong (2012), we can even ponder how "Chinese" contemporary Chinese mathematics education really is. Not much traditional Chinese mathematics is still taught nowadays and mathematics is basically taught as it is in Western educational systems. It is precisely this unique blend of globalization that defines localness.

3. The Motives for Publishing These Two Books

As we indicated in the introduction of this book, the present book, *How Chinese Teach Mathematics,* is in a large sense a continuation of the previous one, *How Chinese Learn Mathematics*. The underlying theme for both has been "*perspectives from insiders*". The issues about Chinese learners attracted the public's attention early in the 1980s, when we had the 2nd IEA Mathematics Study (SIMS) which initiated the series of investigations by the Stevenson, Hiebert and Stigler's group (Chen & Stevenson, 1995; Stevenson & Stigler, 1992; Stigler & Hiebert, 1999). At that time, China, such a big country, did not take part because it had only implemented its open door policy in the late 1970s and was not yet participating in too many international affairs. The 1992 IAEP mathematics study was the first international study in which students from Chinese mainland took part, and it placed first with Taiwan and Korea placing second (e.g., see Lapointe, Mead, & Askew, 1992).

The Chinese learner phenomenon also aroused the interest of a group of Western scholars including Biggs, Watkins, Bond and Marton. Their interests were not confined to mathematics. They published a number of books, of which the most frequently cited include *The Chinese Learner* (Watkins & Biggs, 1996), *Teaching the Chinese Learner* (Watkins & Biggs, 2001), *Revisiting the Chinese Learner* (Chan & Rao, 2009), *Psychology of the Chinese People* (Bond, 1986), *The Handbook of Chinese Psychology* (Bond, 1996) and *The Oxford Handbook of Chinese Psychology* (Bond, 2010).

Inspired by the advisors of our book series, and Professor D. Zhang in particular, we saw the need to publish a book focused particularly on Chinese mathematics education research. This is the origin of the two books that are before you.

We see these books serving at least the following unfolding purposes:

(1) letting the researchers on Chinese mathematics education discuss their own views, experiences, and interested issues in mathematics education,
(2) telling the world the ‘Chinese story’ and responding to their queries, and
(3) having Chinese and non-Chinese, who also have interest and passion in Chinese mathematics education, join hands to discuss issues in mathematics education worldwide.

Besides shifting the focus from learning to teaching (they are in fact two faces of the same coin), we incorporated more authors from the younger generation in the present book, to share with readers more recent developments concerning Chinese mathematics education. Most likely this will be the same spirit if one day we have a third book. Indeed, we will likely have a next generation to act as editors for such a book since the current editorial board sees itself as having accomplished its historical mission.

4. Further Reflections and Outlook

Coming back to the questions that the academic circle originally asked, are Chinese students really so smart in mathematics, or, do they just work harder (Wong, 1998)? A number of explanations have been offered in recent years (Morrison, 2006). The use of deep procedures (Star, 2005) as exemplified by *bianshi* teaching is one (Gu, Huang, & Marton, 2004; Huang & Leung, 2006; Wong, Lam, & Chan, 2012). Certainly the notion of deep procedures in relation to the Chinese learner is worth further exploration (Cai & Wong, 2012). With the strength of academic performance in recent years, Zhang once remarked that we should not get contented too fast. There are both strengths and limitations. First, too much focus has been put on traditional subjects like mathematics, obscuring the need for broad-based learning in the new era. Second, the examination culture is hampering creativity. Third, mathematics is presented in its abstract form without showing its connections with both the real world and other sciences (Zhang, 1993, p. 94). These words were said over 2 decades ago, and they ring as true today as then, if not more so.

Each educational region or system, no matter in the East or West, should build on its own strength and search for its own path of learning and teaching. The Chinese maxim that "Those stones from other hills can be used to polish the jade"[3] suggests that practices in other countries can serve as food for the improvement of one's own practice. There is no need to label such practices as "good" ones (let alone "best ones"), since whether it is good or not depends on how one uses it, and also on cultural backgrounds. By reflecting on the practices of these regions, one reflects on one's own culture, understands oneself more, and forms a basis of moving forward in one's own way. By doing so, it is possible to not just get the stones from the other hill, but to use these stones to polish our own jade. We hope that, through reading these two books, readers will not only know more about the Chinese mathematics education, but also gain and come up with insights (which are basically their own) about how to improve the teaching and learning of mathematics in their classrooms.

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