Premadhis Das · Ganesh Dutta
Nripes Kumar Mandal
Bikas Kumar Sinha

Optimal Covariate Designs

Theory and Applications

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The greatest gifts we ever had were the gifts from god we call them parents

In memory of My Father Late Jadunandan Das and My Mother Late Sankari Das

Premadhis Das

Dedicating to My Father Shri Amarendra Prasad Dutta and to My Mother Shrimati Sankari Dutta for their Love, Affection & Blessings

Ganesh Dutta

In Living Memory of My Parents Late Jatindranath Mandal and Late Bidyutlata Mandal

Nripes Kumar Mandal

Remembering My Parents Late Birendra Nath Sinha and Late Jogmaya Sinha for their Love, Affection & Blessings

Bikas Kumar Sinha

Foreword

It is a standard classroom exercise to assert that in a simple linear regression model involving only one regressor [or, covariate] x, viz., $y = \alpha + \beta x + \text{error}$, the covariate-values ($x$), assumed to be continuous and to lie in a finite nondegenerate interval $a \leq x \leq b$, should allow for maximum dispersion in order that the regression parameters can be estimated with the highest efficiency. This suggests a 50–50 split of the total number of observations, i.e., the set of observations are to be generated by setting the covariate (x) at the two extreme values, viz., $x = a$ and $x = b$, equally often.

Going beyond this, there are basic results, when more than one covariate like this are involved. On the other hand, in the absence of any such covariates, we have available standard ANOVA models involving 'design parameters'.

The ANCOVA models introduced in the textbooks and in the literature are based on the study of models in situations wherein regression parameters and design parameters are both present.

Naturally the question of the most efficient estimation of the regression parameter(s) in the presence of design parameters needs to be studied in very general terms, and also under very specialized experimental settings.

Lopes Troya initiated this study and BKS (Bikas Kumar Sinha) followed it up with his research collaborators [Kalyan Das (KD), Nripes Kumar Mandal (NKM), Premadhis Das (PD), Ganesh Dutta (GD), S.B. Rao, P.S.S.N.V.P. Rao, G.M. Saha (GMS)]. It is amazing to note that so much was hidden in this topic of research, and that their successful collaboration over these years had culminated in a Research Monograph.

I had an opportunity to collaborate with BKS several years back and I am thankful to the authors for approaching me to write this Foreword.

This monograph aims at providing an up-to-date account of the research findings in various experimental settings. As the authors describe and admit, mostly they confine to ‘idealistic scenarios’ in order to develop and apply tools and techniques for the study of optimal estimation of covariates’ parameters. In the introductory chapter as also in Chapter 9, they discuss about ‘real life’ examples and provide a detailed study of optimality.

The authors have taken up a thorough study of the problems associated with this area of research. I personally thank them for their tremendous efforts and congratulate them for this remarkable achievement.

June 2015

Gour Mohan Saha
Retired Professor of Statistics
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Preface

Three of us are ‘Senior Citizens’ in the context of ‘Statistics Learning’ and we are ever-grateful to our revered postgraduate teachers for highlighting the fundamental and basic contributions of R.A. Fisher and Frank Yates in such areas as Design of Experiments [DoE]. We had the opportunity to read their books, so much so that we went through Fisher’s original book published in the 1930s. These are indeed ‘Treasured Collections’! Our fascination for DoE started from that point of time and it has continued to be intriguing for more than 40 years! We thoroughly enjoy reading, learning and discussing all aspects of DoE—theory and applications.

There are two incidences to be told in real-time experience underlying this project.

First, around 2002 one of the co-authors was trying to make a ‘dent’ into a paper on Optimal Covariates Designs [OCDs] with a colleague of him with very little success primarily because the notations were difficult to follow. Fortunately for the rest of us and for the optimal design community at large, they did not give up altogether. Instead, at the earliest opportunity they approached one of the other co-authors for looking into this paper. That was one positive development indeed and together, they could digest the paper and go forward as a ‘high speed jet’! On another occasion around 2003, again one of the co-authors was struggling with a constructional problem involving OCDs and this time he was accompanied by one enthusiastic graph-theorist and one matrix-specialist. While they were in ‘seemingly deep’ trouble and in a ‘confused state of mind’, one of their colleagues—a design specialist—suddenly ‘peeped in’ and made a very casual observation, ‘it seems … you are discussing some aspects of Mixed Orthogonal Arrays’ and that was it to give again another big push to this work.

In a nutshell, these two incidences gave a boost to our group and we did not have to look back any more! We have enjoyed working on this project. We have derived much pleasure working in a group discussing, arguing and counter-arguing, till the time that we thought we came to understand enough of this fascinating topic of research to prepare a Research Monograph.

We must hasten to add that the youngest member of our group [GD] kept the others in toe with his frequent ‘claims’ and ‘counter-claims’ and ‘proofs’ and ‘counter-examples’! Working with him was a matter of great pleasure for us. His enthusiastic and provocative statements/claims frequently served as ‘make-belief’ prophecies which were to be verified by the other three; it was not easy all the time anyway.

Finally, we are here with a comprehensive account of what we believe to be a Treatise on OCDs, more from the viewpoint of ‘Idealistic Scenarios’ in different experimental situations. The emphasis all through is about ‘optimal’ choice of what are called ‘controllable covariates’ in continuous domain(s). Only in the last chapter, we dwell on ‘realistic experimental situations’ and provide solutions to some well-posed problems.

Confusion continued to follow us and it gave us a scope for generating arguments and counter-arguments till we reached *Clarity* with our own understanding of the findings.

Kolkata, West Bengal, India
June 2015

Premadhis Das
Ganesh Dutta
Nripes Kumar Mandal
Bikas Kumar Sinha

Acknowledgment

During the course of our work on this fascinating topic, we had the opportunity to interact with three enthusiastic researchers: Prof. Kalyan Das [Calcutta University], Profs. S.B. Rao and Prasad Rao [both from Indian Statistical Institute, Kolkata]. Their collaboration enriched our thoughts and we are grateful to them for their enthusiasm and encouragement.

During this period, we had the opportunity to be invited to some conferences and make technical presentations of our papers before learned gatherings. We were benefited by the remarks/comments/criticisms received therefrom.

We are thankful to the authorities at Indian Statistical Institute, Kolkata [BKS], Department of Statistics, Calcutta University [NKM], Department of Statistics, Kalyani University [PD] and Basanti Devi College, Kolkata [GD] for providing excellent research opportunities for our team.

We are thankful to Prof. Gour Mohan Saha of the Indian Statistical Institute, Kolkata for his insightful comments during a critical phase of this study and for complying with our request for writing the Foreword of this monograph.

We have freely consulted available published literature and books and journals and benefited immensely from the collective wisdom of researchers worldwide on topics such as Hadamard Matrices, Mutually Orthogonal Latin Squares [MOLS], Orthogonal Arrays [OAs], Mixed Orthogonal Arrays [MOAs], Linear Models, ANOVA Models, Regression designs and ANCOVA models.

We fondly hope this monograph will be received enthusiastically by the statistical design theorists in general and combinatorial design theorists in particular and they will identify open problems in the broad area of Optimal Covariates' Designs.

Kolkata
June 2015

Premadhis Das
Ganesh Dutta
Nripes Kumar Mandal
Bikas Kumar Sinha

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About the Authors

Prof. Premadhis Das is Senior Professor in the Department of Statistics, University of Kalyani, India. He has been working in the area of Design of Experiments for more than 30 years, and has published research articles in many national and international journals of repute. Professor Das has co-authored a Springer-Verlag Lecture Notes Series in Statistics: Monograph on Optimal mixture experiments, Vol. 1028, 2014.

Dr. Ganesh Dutta is Assistant Professor, Department of Statistics, Basanti Devi College affiliated to the University of Calcutta, India. He completed his Ph.D. degree in the area of Design of Experiments from the University of Calcutta in 2009 and has published research articles in the area of Optimum Covariate Designs in reputed peer-reviewed journals.

Prof. Nripes Kumar Mandal is a senior faculty member of the Department of Statistics, University of Calcutta, India. He has visited many countries for collaborative research, and has published about 70 research articles in peer-reviewed journals. He has been associated with many international statistical journals of repute as a reviewer. Professor Mandal has co-authored two Springer-Verlag Lecture Notes Series in Statistics: Monograph on Optimal Designs, Vol. 163, 2002 and Monograph on Optimal mixture experiments, Vol. 1028, 2014.

Prof. Bikas Kumar Sinha was attached to the Indian Statistical Institute [ISI], Kolkata, India for more than 30 years until his retirement on March 31, 2011. He has traveled extensively within USA and Europe for collaborative research and with teaching assignments. He has more than 120 research articles published in peer-reviewed journals and has acted as a referee for many international journals. Professor Sinha has served on the Editorial Board of statistical journals including *Sankhya, Journal of Statistical Planning and Inference* and *Calcutta Statistical Association Bulletin*. He has co-authored three Springer-Verlag Lecture Notes Series in Statistics: Monographs on Optimal Designs [Vol. 54, 1989, Vol. 163, 2002 and Vol. 1028, 2014].

Chapter 1
Optimal Covariate Designs (OCDs): Scope of the Monograph

1.1 Preamble: A Reflection on the Choice of Covariates

Most standard textbooks in the area of linear models and design of experiments provide discussions on what are known as analysis of covariance models applied to completely randomized designs (CRD), randomized block designs (RBD) and latin square designs (LSD). It is a well-accepted practice in experimental design contexts to use one or more available and meaningful covariates together with *local control* to reduce the experimental error. Such a model comprises three components: local control parameter(s) (if any), 'treatment' parameters, and the covariate parameter(s), apart from the error. This generates a family of 'covariate models'—serving as a 'blend' of 'regression models' (in the absence of treatment parameters) and 'varietal design models' (in the absence of covariates). These are the so-called analysis of covariance (ANCOVA) Models. Generally, for such models, emphasis is given on analysis of the data. Inference-related procedures are fairly routine exercises and are well discussed in the texts.

At times there lies a (possibly huge) potential for improving the experimental results by suitably classifying/reclassifying the existing experimental units through a study of the associated covariate values or by first suitably choosing the covariate values from a larger lot and then, hopefully, identifying the associated experimental units from a larger pool.

Here we cite a motivating example from Snedecor and Cochran (1989, p. 377), suitably presented to explain our point. There are 30 patients for a study of leprosy and there are three drugs (two antibiotics A and D, and one control F) to be compared—each to be applied to 10 patients. For each patient we have available a pretreatment score (count of bacilli) which may be used as a covariate. Table 1.1 shows the allocation of the three treatments covering all the 30 patients as against their covariate values. There is nothing wrong with this and the data analysis is fairly routine using an ANCOVA Model, once the CRD is implemented.

We now ask an intriguing question: How was the allocation of treatments (A, D and F) across the pool of 30 patients decided? Was it purely 'ad hoc'? Could

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Table 1.1 Original allocation of patients based on covariate values (patient serial number, covariate value)

1	Treatment A	(P1, 3), (P2, 5), (P3, 6), (P4, 6), (P5, 8), (P6, 10), (P7, 11), (P8, 11), (P9, 14), (P10, 19)
2	Treatment D	(P11, 5), (P12, 6), (P13, 6), (P14, 7), (P15, 8), (P16, 8), (P17, 8), (P18, 15), (P19, 18), (P20, 19)
3	Control F	(P21, 7), (P22, 9), (P23, 11), (P24, 12), (P25, 12), (P26, 12), (P27, 13), (P28, 16), (P29, 16), (P30, 21)

Table 1.2 'Improved' allocation of patients based on covariate values

1	Treatment A	(P1, 3), (P3, 6), (P4, 6), (P22, 9), (P6, 10), (P7, 11), (P24, 12), (P9, 14), (P19, 18), (P10, 19)
2	Treatment D	(P2, 5), (P12, 6), (P14, 7), (P5, 8), (P15, 8), (P8, 11), (P25, 12), (P18, 15), (P28,16), (P20, 19)
3	Control F	(P11, 5), (P13, 6), (P21, 7), (P16, 8), (P17, 8), (P23, 11), (P26, 12), (P27, 13), (P29, 16), (P30, 21)

we do anything 'better'? It would be an interesting exercise to compare different conceivable allocations for say, 'most efficient estimation' of the covariate parameter in the ANCOVA model underlying the CRD. A trial and error solution is given in Table 1.2 and it turns out that we can achieve 12.28 % gain in efficiency by following this plan.

There is much more to it. If we had a larger pool of patients to choose from, what would have been our strategy for most efficient estimation of the covariate parameter? It turns out that our optimal choice would necessarily accumulate all those patients having equal split between the smallest and the largest pretreatment scores and that would be needed for each treatment! If it turns out that in the pool, 3 and 21 are the lowest and highest pretreatment counts of bacilli, then we would recruit five patients with the lowest and five patients with the highest count for each of the three treatments A, D and F. By doing so, we would have gained access over a group of 15 patients—each with the lowest count and another group of 15 patients—each with the highest count! Then it would be a matter of dividing each group of 15 equally into three so that there are five patients for each treatment from each group. This would have resulted in the best possible allocation design with 309.78 % efficiency as against the original allocation in Table 1.1 above and 264.97 % efficiency as against the allocation indicated in Table 1.2 above. We revisit this example in Chap. 9.

We take up a second example now. The data in Table 1.3 are from an experimental piggery arranged for individual feeding of six pigs in each of five pens. From each of five litters, six young pigs, three males (M) and three females (F), were selected and allotted to one of the pens. Three feeding treatments denoted by A, B, C, containing increasing proportions ($p_A < p_B < p_C$) of protein, were used and each was given to one male and one female in each pen. The pigs were individually weighed each

Table 1.3 Data for analysis

Pen	Treatment	Sex	Initial weight	Growth rate in pounds per week
I	A	F	48	9.94
	B	F	48	10.00
	C	F	48	9.75
	C	M	48	9.11
	B	M	39	8.51
	A	M	38	9.52
II	B	F	32	9.24
	C	F	28	8.66
	A	F	32	9.48
	C	M	37	8.50
	A	M	35	8.21
	B	M	38	9.95
III	C	F	33	7.63
	A	F	35	9.32
	B	F	41	9.34
	B	M	46	8.43
	C	M	42	8.90
	A	M	41	9.32
IV	C	F	50	10.37
	A	M	48	10.56
	B	F	46	9.68
	A	F	46	10.90
	B	M	40	8.86
	C	M	42	9.51
V	B	F	37	9.67
	A	F	32	8.82
	C	F	30	8.57
	B	M	40	9.20
	C	M	40	8.76
	A	M	43	10.42

Data Source: Rao (1973), p. 291 and Scheffé (1999), p. 217

week for 16 weeks. For each pig the growth rate in pounds per week was calculated. The weight at the beginning of the experiment is also given in Table 1.3.

There are 15 female pigs and 15 male pigs available for this study and we arrange their initial weights separately into two 5×3 arrays. The arrangements are shown in Table 1.4a, b respectively.

Now one may consider the standard covariate model (ANCOVA) for two-way RBD Pen$\times$Treatment layout with a single covariate (here covariate is initial weights of pigs) separately for females and males. The notations are standard and we use

Table 1.4 Initial weight distribution as per allocation of pigs

Pen	Treatment			Totals
	A	B	C	
(a) Female				
1	48	48	48	144
2	32	32	28	92
3	35	41	33	109
4	46	46	50	142
5	32	37	30	99
Totals	193	204	189	586
(b) Male				
1	38	39	48	125
2	35	38	37	110
3	41	46	42	129
4	48	40	42	130
5	43	40	40	123
Totals	205	203	209	617

γ_F and γ_M to, respectively, denote the covariate effect for female and male pigs. These are routine computations and for the given allocation design in Table 1.4, to be denoted by d_0, $I_{d_0}(\gamma_F) = 57.8667$ and $I_{d_0}(\gamma_M) = 116.2667$.

Again, we ask an intriguing question: For the given collection of 15 female/male pigs, is it possible to identify an improved reallocation plan across the two-way Pen $\times$ Treatment table in the sense of increased precision in the estimation of the covariate parameters? Another related question also makes some sense: If the experimenter is given a 'free choice' of the 15 pigs (both female and male) from a larger pool, what would have been an 'optimal choice', given that initial weight distribution is perfectly known for the pool of pigs? Note that this question has embedded in it (i) selection of pigs with suitable initial weights and (ii) their distribution across the two-way table.

Below we provide answers to the two questions raised above. In Table 1.5a, b we provide improved allocation designs (based on the given collection of pigs), separately for female and male pigs with respective percent gain in efficiency given by 1375.345 and 76.4908 %. In Table 1.6a, b, we provide optimal allocation designs based on free choice of the experimenter, assuming that the initial weight distribution for female pigs lies in the closed interval [28 lbs, 50 lbs] and for males it is in the closed interval [35 lbs, 48 lbs].

We skip the details and will take up this example again in Chap. 9.

Remark 1.1.1 Our purpose in this monograph is to give the readers a taste of such comparative results in diverse experimental contexts and with one or more covariates being encountered simultaneously.

Table 1.5 Improved allocation designs

Pen	Treatment			Totals
	A	B	C	
(a) Improved allocation for female pigs				
	46	28	48	122
	30	37	50	117
	48	35	32	115
	41	46	32	119
	32	48	33	113
Totals	197	194	195	586
(b) Improved allocation for male pigs				
	A	B	C	Totals
	37	38	48	123
	38	46	40	124
	40	42	41	124
	43	39	42	123
	48	40	35	123
Totals	206	205	206	617

Table 1.6 Optimal initial weights for female and male pigs

	Treatment			Totals
	A	B	C	
(a) Female				
	28	50	50	128
	50	28	50	128
	50	50	28	128
	28	50	50	128
	50	28	28	106
Totals	206	206	206	618 = G
(b) Male				
	35	48	48	131
	48	35	48	131
	48	48	35	131
	35	48	48	131
	48	35	35	118
Totals	214	214	214	642 = G

Much of the theory of OCDs has grown out of the 'convenient proposition/supposition' that the experimenters have a 'free' choice in the selection of the experimental units with any preassigned covariate values whatsoever! Notwithstanding the fact that such a situation rarely arises in practice, the published literature is vast and varied in respect of all kinds of experimental design settings, with the proviso that the optimal design theorists/statisticians are the masterminds in the whole business and they have the ‘ultimate say’ in the choice of the experimental units from a conceivably larger ‘pool’ with designated covariate values.

We will dwell on the developments toward characterization and construction of the OCDs as we have witnessed in the published literature, in the contexts of what are identified as ‘ideal’ scenarios. This study will be taken up systematically in Chaps. 2, 3, 4, 5, 6, 7 and 8. In Chap. 9, we will consider application areas and discuss some examples. Our understanding of the OCDs in the so-called ideal scenarios will guide us towards identification of optimal/nearly optimal covariate designs in real-life applications and some applications are discussed in Chap. 9.

We briefly trace the history of development of OCDs below.

The choice of experimental units possessing suitably defined/chosen values of the covariates for a given experimental set-up so as to attain minimum variance/maximum precision for estimation of the regression parameters has attracted the attention of researchers only in recent times. In the context of ANCOVA models where both qualitative and quantitative factors are present, the problem of inference on varietal contrasts corresponding to qualitative factors was studied by Harville (1974, 1975), Haggstrom (1975) and Wu (1981). The problem of determining optimum designs for the estimation of regression parameters corresponding to controllable covariates was first considered by [1] Troya Lopes (1982a, b). She restricted investigations in the set-up of completely randomized design (CRD). Das et al. (2003) extended it to the block design set-up, viz. randomized block design (RBD) and some series of balanced incomplete block designs (BIBDs) and constructed OCDs for the estimation of covariate parameters. Rao et al. (2003) revisited the problem in CRD and RBD set-ups and identified the solutions as mixed orthogonal arrays (MOAs), thereby providing further insights and some new solutions. Dutta (2004, 2009) and Dutta et al. (2007, 2009b, 2010a, c) considered optimal estimation of the regression coefficients under different experimental set-ups where the analysis of variance (ANOVA) effects are non-orthogonally estimable. Dutta et al. (2009a) also considered optimal estimation of the regression coefficients in the set-ups of split-plot and strip-plot designs where the ANOVA effects are orthogonally estimable. These were subsequently generalized in Dutta and Das (2013a) to multi-factor set-up. For one-way set-up, D-optimal designs were proposed by Dey and Mukerjee (2006) and, these were further studied in Dutta et al. (2014). Dutta et al. (2010b) also considered D-optimal covariate designs for estimation of regression coefficients in incomplete block design set-up when global optimal designs do not exist. The other related

[1] Late Professor Jack Kiefer pioneered the study of optimal experimental designs in standard ANOVA models as well as in regression designs. He guided Lopes Troya for her Doctoral Dissertation in a topic which was to bridge ANOVA and regression designs into what are known as ANCOVA models. The unfortunate premature death of Professor Kiefer was a blow to the design theorists in general. His expertise and insightful contributions could have gone a long way in this direction.

references are Wierich (1984), Kurotschka and Wierich (1984), Chadjiconstantinidis and Moyssiadis (1991), Chadjiconstantinidis and Chadjipadelis (1996), Liski et al. (2002), Dutta (2009), Sinha (2009) and Das (2011).

1.2 Basic Set-Up and Optimality Conditions

Let the following covariate model be considered:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\theta} + \mathbf{Z}\gamma + \mathbf{e} \tag{1.2.1}$$

where $\mathbf{Y}^{n\times 1}$ denotes the observation vector, $\mathbf{X}^{n\times p}$ denotes the coefficient matrix for the ANOVA effects parameters $\boldsymbol{\theta}' = (\theta_1, \theta_2, \ldots, \theta_p)$ and $\mathbf{Z}^{n\times c}$ denotes the matrix of the values given to c covariates, viz. $\mathbf{Z} = \left(\mathbf{z}^1, \mathbf{z}^2, \ldots, \mathbf{z}^c\right)$. In the above, $\mathbf{Z}$ is also called the covariate design matrix of the vector of covariate effects $\boldsymbol{\gamma} = (\gamma_1,\ \gamma_2, \ldots, \gamma_c)'$. As usual, $\mathbf{e}$ is the random error component with $\mathrm{E}(\mathbf{e}) = \mathbf{0}$, $Disp(\mathbf{e}) = \sigma^2\mathbf{I}_n$, where $\mathbf{I}_n$ is the identity matrix of order n. We represent the above set-up by the triplet:

$$\left(\mathbf{Y},\ \mathbf{X}\boldsymbol{\theta} + \mathbf{Z}\gamma,\ \sigma^2\mathbf{I}_n\right). \tag{1.2.2}$$

Here the observations are uncorrelated and variances of each of the observations are equal to σ^2. In addition to the comparison of ANOVA effects and in particular, of the underlying treatment effects, interest lies in accommodating as many covariates as possible, subject to these being optimally estimated. Situations where the covariates are *not* under the control of the experimenter, were discussed by Harville (1974, 1975), Haggstrom (1975) and Wu (1981) in the context of comparison of treatment effects. These are also briefly discussed in Shah and Sinha (1989). Traditionally, in a study of linear regression design involving non-stochastic regressors, we tacitly call for homogeneous experimental units so that the assumed model for the $n \times 1$ observation vector $\mathbf{Y}$ is of the form

$$\left(\mathbf{Y},\ \mu\mathbf{1}_n + \mathbf{Z}\gamma,\ \sigma^2\mathbf{I}_n\right) \tag{1.2.3}$$

where μ represents the intercept term, γ is the vector of covariate effects, $\mathbf{Z}$ is, as before, the design matrix of covariate values and $\mathbf{1}_n$ is a vector of order n with all elements unity. Understandably, the homogeneous nature of the experimental units safeguards the same intercept term as indicated in the model (1.2.3) for every expectation.

Here Z's are assumed to be controllable/given non-stochastic covariates. The n values $z_{i1},\ z_{i2}, \ldots, z_{in}$, assumed by the ith covariate Z_i are such that they belong to a finite interval $[a_i,\ b_i]$ for each i and j, i.e.

$$\begin{aligned} a_i \le z_{ij} \le b_i \\ \text{i.e. } z_{ij} = \tfrac{a_i+b_i}{2} + \tfrac{b_i-a_i}{2} z^*_{ij} \end{aligned} \tag{1.2.4}$$

so that z^*_{ij} lies in $[-1, 1]$ for each $i,\ j$. Then replacing z_{ij} by z^*_{ij}'s we get the same covariate model in a reparameterized scenario, i.e. the regression coefficients can be suitably adjusted and the constant part will be adjusted with μ. Thus this transformation does not hamper our optimality study. So, without loss of generality, we can assume in (1.2.3), the covariate values z_{ij}'s to vary within $[-1, 1]$. It is well known that the experimental domain of the regressors being a c-dimensional cube of the form: $[-1, 1]^c$, the most efficient design for estimation of the regression coefficients (i.e. the γ-parameters) is derived from a Hadamard matrix (defined in Chap. 2), whenever the latter exists. When $n > c$ and $n \equiv 0 \pmod 4$, it is enough to start with a Hadamard matrix $\mathbf{H}_n$ of order n (in its standard form) and select any c of its columns for the $\mathbf{Z}$-matrix, leaving the first column which contains 1's only. This yields an optimum design for the (joint) estimation of μ and γ on the basis of n observations. *Optimality here, refers to attaining the least possible value* $\frac{\sigma^2}{n}$ *of the individual variances simultaneously for all the covariate parameter estimates.* It is known that the maximum number of covariates (i.e. $c_{\max}$) cannot exceed the error degrees of freedom (d.f.) of a given set-up. Therefore $c_{\max} = (n-1)$ under the model in (1.2.3); $c_{\max} = n - v$ for a CRD set-up and for a block design set-up $c_{\max} = n - b - \text{Rank}(\mathbf{C})$, where $\mathbf{C}$ is the characteristic matrix of a block design.

In general, the experimental set-ups are much more complicated and so are the models much different from (1.2.3). Use of Hadamard matrices and other tools and techniques has to be introduced in a systematic manner. The points to be noted are:

(i) We want optimal estimation of the covariates parameters.
(ii) We want to know how many covariates can be optimally accommodated.

We mostly confine to the 'idealistic' situations wherein there exist conceivably larger pools of experimental units with experimenter's choice of the covariates' values. This should serve as a basis and a guideline for actual experimental situations.

1.3 Chapter-Wise Summary

In Chap. 2, we study the choice of optimum covariate design in CRD set-up. Troya Lopes (1982a, b) first studied the problem of choice of the $\mathbf{Z}$-matrix in a CRD model when the treatment allocation matrix $\mathbf{X}$ corresponds to an equal allocation number, i.e. when n is a multiple of v. We will write as $n = vb$ so that b is the common allocation number of the v treatments under investigation. Here we discussed some results from Troya Lopes (1982a) with reference to the $\mathbf{W}$-matrices. If n is not an integral multiple of v, this allows us to study situations where no equireplicate design exists. In this situation, it is not possible to find designs attaining minimum variance for the estimated covariate parameters. This problem has been considered by Dey and Mukerjee (2006) and Dutta et al. (2014). They provided optimum designs with respect to ANOVA effect and covariate effects using D-optimality criterion. We also deal with this issue in this chapter.

In Chap. 3, we discuss optimum covariate design in RBD set-up. For an RBD set-up, (Das et al. 2003) studied for the first time, the problem of OCDs. They exploited mutually orthogonal latin squares (MOLS) and Hadamard matrices to construct such designs which attain the upper bound for the number of covariates which can be incorporated in the covariate model for RBD. Rao et al. (2003) re-visited the problem in CRD and RBD set-ups and identified the solutions as mixed orthogonal arrays (MOAs) (defined in Chap. 2), thereby providing further insights.

For BIBD set-up, Das et al. (2003) also initiated the construction of optimal designs for covariates in some series of symmetric balanced incomplete block designs (SBIBD) constructed through Bose's difference technique and some BIBDs with repeated blocks. Dutta (2004) dealt with the problem of constructing OCDs in some other classes of BIBDs which may or may not have cyclic structure. However, he dealt with the problem with the restriction $n \equiv 0 \pmod 4$. But such designs cannot always be obtained because of the restriction $n \equiv 0 \pmod 4$. Dutta et al. (2010b) found optimum designs with respect to covariate effects using D-optimality criterion retaining orthogonality with the treatment and block effect contrasts, where $n \equiv 2 \pmod 4$. Results given by Das et al. (2003), Dutta (2004) and Dutta et al. (2010b) are included in Chap. 4.

In a BIBD set-up, we have noticed that the scope of construction of OCDs becomes limited as the parametric relations do not always permit the existence of Hadamard matrices. Also, the stringency of equal occurrence of each pair of treatments limits the scope of OCDs. For this, Dutta et al. (2009b) extended their research to the partially balanced incomplete block design (PBIBD) set-up. Moreover, PBIBDs are popular among practitioners and OCDs in this set-up will be of help to them. However, in Chap. 5, we restrict to an important subclass of PBIBDs viz., the group divisible designs (GDDs) and discuss about existence and constructional aspects of OCDs. We have given a catalogue at the end of Chap. 5 which shows that the method covers a large number of GDDs obtained from Clatworthy (1973).

Binary proper equireplicate block designs (BPEBDs) form a rich class of block designs and this class encompasses designs beyond those considered in the previous chapters. In Chap. 6, we venture into the constructional aspects of OCDs for such designs. General cyclic and non-cyclic BPEBDs as also t-fold BPEBDs having OCD structure have been studied and a catalogue has also been provided at the end.

In Chap. 7, following Dutta and Das (2013b), we discuss the OCD problem in balanced treatment incomplete block (BTIB) design set-up using Hadamard matrices and other techniques described in previous chapters.

In Chap. 8, we start with a discussion of the OCD problems in crossover design set-up and multi-factor set-up. For these designs, key references are Dutta and SahaRay (2013) and Dutta and Das (2013a). Mixed orthogonal arrays and their generalized version are very useful to construct OCDs in multi-factor set-ups.

In all the above cases so far discussed in various chapters, the observations are naturally uncorrelated. But if they are not, then difficulty arises for the choice of OCDs.

It becomes even more difficult for arbitrary variance-covariance matrix. However, if the variance-covariance matrix has a nice structure, it is possible to construct OCDs. In particular, Dutta et al. (2009a) considered the set-ups of the split-plot and strip-plot designs where the correlations among the observations follow a definite pattern. Further, they have seen that a generalized version of the mixed orthogonal array has a close relationship with the OCDs for such set-ups. They have exploited it to construct OCDs for such experimental contexts. In this Chap. 8, we also discuss this aspect at length.

In the concluding chapter (Chap. 9), we turn back to the questions raised in Chap. 1 and deal with a number of application areas wherein optimality study in the context of uses of covariates has a natural scope for enhancing the experimental results. We rework on the two motivating examples and provide details of the computations. We also take up four other examples arising in experiments involving covariates in natural sciences.

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Chapter 2
OCDs in Completely Randomized Design Set-Up

2.1 Introduction

We consider in this chapter the one-way linear model with v treatments, c covariates and a total of n experimental units. We work under the linear model

$$y_{ij} = \tau_i + \sum_{t=1}^{c} \gamma_t z_{ij}^{(t)} + e_{ij}, \quad 1 \le j \le n_i, \ 1 \le i \le v. \tag{2.1.1}$$

where $n_i (> 1)$ is the number of times the ith treatment is replicated; clearly

$$\sum_{i=1}^{v} n_i = n. \tag{2.1.2}$$

For $1 \le j \le n_i, 1 \le i \le v$, here y_{ij} is the observation arising from the jth replication of the ith treatment, τ_i effect due to the ith treatment.

In matrix notation the above model can be represented as

$$\left(\mathbf{Y},\ \mathbf{X}\boldsymbol{\tau} + \mathbf{Z}\gamma,\ \sigma^2\mathbf{I}_n\right), \tag{2.1.3}$$

where, $\mathbf{Y}$ is an observation vector and $\mathbf{X}$ is the design matrix corresponding to vector of treatment effects $\boldsymbol{\tau}^{v\times 1}$ and $\mathbf{Z} = ((z_{ij}^{(t)}))$ is the design matrix corresponding to vector of covariate effects $\boldsymbol{\gamma}^{c\times 1} = (\gamma_1,\ \gamma_2, \ldots, \gamma_c)'$. This is referred to as *one-way model with covariates* (*without the general mean*).

Troya Lopes (1982a, b) studied the nature of optimal allocation of treatments and covariates in the above set-up for simultaneous estimation of the (fixed) treatment effects (in the absence of the general effect) and the covariate effects with maximum efficiency in the sense of minimum generalized variance. This is to note that the information matrix with respect to model (2.1.3) is given by $\sigma^{-2}\mathbf{I}(\boldsymbol{\eta})$, where

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$$\mathbf{I}(\boldsymbol{\eta}) = \begin{pmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{Z} \\ \mathbf{Z}'\mathbf{X} & \mathbf{Z}'\mathbf{Z} \end{pmatrix} \tag{2.1.4}$$

and $\boldsymbol{\eta}' = \left(\boldsymbol{\tau}', \boldsymbol{\gamma}'\right)$.

The problem is to suggest an optimal allocation scheme (for given design parameters $n,\ v,\ c$) for efficient estimation of the treatment effects as well as the covariate effects by ascertaining the values of the covariates for each one of them, assuming that each one is controllable and quantitative within a stipulated finite closed interval.

The information matrix of $\boldsymbol{\gamma}$ is given by

$$\sigma^{-2} I(\gamma) = \mathbf{Z}'\mathbf{Z} - \mathbf{Z}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Z} \tag{2.1.5}$$

where $(\mathbf{X}'\mathbf{X})^{-}$ is a generalised inverse of $\mathbf{X}'\mathbf{X}$ satisfying

$$\mathbf{X}'\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-}\mathbf{X}'\mathbf{X} = \mathbf{X}'\mathbf{X}$$

(cf. Rao 1973, p. 24). It is evident that $\mathbf{Z}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Z}$ is a positive semi-definite matrix. So from (2.1.5), it follows that

$$\sigma^{-2} I(\gamma) \leq \mathbf{Z}'\mathbf{Z} \tag{2.1.6}$$

in the Loewner order sense (vide Pukelsheim 1993) where for two non-negative definite matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A}$ is said to dominate $\mathbf{B}$ in the Lowener order sense if $\mathbf{A} - \mathbf{B}$ is a non-negative definite matrix.

Equality in (2.1.6) is attained whenever

$$\mathbf{X}'\mathbf{Z} = \mathbf{0}. \tag{2.1.7}$$

If $\mathbf{Z}$ satisfies (2.1.7), then treatment effects and covariate effects are orthogonally estimated. Again under condition (2.1.7), the information matrix $\mathbf{I}(\gamma)$ reduces to $\mathbf{I}(\gamma) = \mathbf{Z}'\mathbf{Z}$. The z-values are so chosen that $\mathbf{Z}'\mathbf{Z}$ is positive definite so that from (2.1.6)

$$Var(\widehat{\gamma_t}) \geq \frac{\sigma^2}{\displaystyle\sum_{i=1}^{v}\sum_{j=1}^{n_i} z_{ij}^{(t)2}} \geq \frac{\sigma^2}{n} \tag{2.1.8}$$

as $z_{ij}^{(t)} \in [-1, 1];\ \ \forall\, i, j, t$.

Now equality in (2.1.8) holds for all i if and only if the $\mathbf{Z}$-matrix is such that

$$\mathbf{z}^{(s)\prime}\mathbf{z}^{(t)} = 0\ \ \forall\, s \neq t. \tag{2.1.9}$$

and

$$z_{ij}^{(t)} = \pm 1 \tag{2.1.10}$$

Condition (2.1.7) implies that the estimators of ANOVA effects parameters or parametric contrasts do not interfere with those of the covariate effects and conditions (2.1.9) and (2.1.10) imply that the estimators of each of the covariate effects are such that these are pairwise uncorrelated, attaining the minimum possible variance.

Thus the covariate effects are estimated with the maximum efficiency if and only if

$$\mathbf{Z}'\mathbf{Z} = n\mathbf{I}_c \tag{2.1.11}$$

along with (2.1.7). The designs allowing the estimators with the minimum variance are called *globally optimal designs* (cf. Shah and Sinha 1989, p. 143). Henceforth, we shall only be concerned with such optimal estimation of regression parameters and by optimal covariate design, *to be abbreviated as OCD hereafter*, we shall only mean *globally optimal design*, unless otherwise mentioned.

It is clear that conditions (2.1.7) and (2.1.11) hold simultaneously if and only if z_{ij}'s are necessarily +1 or −1 and that condition (2.1.7) is satisfied.

It is difficult to visualize the **Z**-matrix satisfying conditions (2.1.7) and (2.1.11). In the set-up of the model (2.1.3), it transpires from Troya Lopes (1982a) that optimal estimation of the treatment effects and the covariates effects is possible when the treatment replications are all necessarily equal, assuming that n is a multiple of v, the number of treatments. We set $n = bv$, where b is the common replication of treatments, henceforth. Das et al. (2003) had represented each column of the **Z**-matrix by a $v \times b$ matrix **W** with elements of ± 1, where the rows of **W** correspond to the v treatments and the columns of **W** correspond to different replication numbers. Condition (2.1.7) implies that the sum of each row of **W** should vanish. Again, condition (2.1.11) implies that the sum of products of the corresponding elements, i.e. the Hadamard product of $\mathbf{W}^{(s)}$ and $\mathbf{W}^{(t)}$, defined in (2.1.13) should also vanish, $1 \leq s < t \leq c$. The above two facts can be represented in the following schematic forms through the row totals and Hadamard product.

Row Totals:

$$\begin{array}{cc|c|c} & \text{Tr.} & \text{Repl. no.} \rightarrow & \text{Row} \\ & \downarrow & 1 \quad 2 \quad \ldots \quad b & \text{Totals} \\ & 1 & & 0 \\ \mathbf{W}^{(s)} = & 2 & & 0 \\ & \vdots & (\pm 1) & \vdots \\ & v & & 0 \end{array} \tag{2.1.12}$$

Hadamard product of $\mathbf{W}^{(s)}$ and $\mathbf{W}^{(t)}$ (cf. Rao 1973, p. 30):

$$\begin{array}{cc|c|} & \text{Tr.} & \text{Repl. no.} \rightarrow \\ & \downarrow & 1 \quad 2 \quad \ldots \quad b \\ & 1 & \\ \mathbf{W}^{(s)} * \mathbf{W}^{(t)} = & 2 & \\ & \vdots & (w_{ij}^{(s)} w_{ij}^{(t)}) \\ & v & \end{array} \tag{2.1.13}$$

where '*' denotes Hadamard product. For orthogonality of sth and tth columns of $\mathbf{Z}$, it is required that $\sum_{i=1}^{v}\sum_{j=1}^{b} w_{ij}^{(s)} w_{ij}^{(t)} = 0$.

The schematic representation (2.1.12), (2.1.13) of Das et al. (2003) is a breakthrough in the sense that handling of $\mathbf{Z}$-matrix has been made much easier and it has been followed throughout the monograph.

Troya Lopes (1982a) first studied the nature of optimal allocation of treatments and covariates in the above set-up when $\frac{n}{v}$ is an integer. It may be noted that whenever condition (2.1.7) is ensured, presence of the covariates in model (2.1.3) does not pose any threat to the usual "optimal treatment allocation" problem. In Sect. 2.2, following Troya Lopes, we intend to discuss about the availability of $\mathbf{Z}$-matrices satisfying (2.1.7) and (2.1.11) when the treatment allocation matrix $\mathbf{X}$ corresponds to equal allocation number, i.e. in situations where n is a multiple of v. We will write $n = vb$ so that b is the common allocation number of the v treatments under investigation. The situations where (2.1.7), (2.1.11) and $b = \frac{n}{v}$ = integer are satisfied, are identified as *regular* cases. Otherwise it is called a *non-regular* case. If the situation is non-regular, then it is not possible to allocate simultaneously the treatments and covariates optimally. For non-regular situation, efficient allocation of treatments and covariates simultaneously can be done by using other specific optimality criteria. Dey and Mukerjee (2006) and Dutta et al. (2014) considered this problem in non-regular situations and found D-optimal designs in this context. Details are presented in Sect. 2.3.

It has been seen that Hadamard matrix plays a key role for constructing OCDs. Definition of Hadamard matrix (cf. Hedayat et al. 1999, p. 145) is given below:

Definition 2.1.1 A Hadamard matrix $\mathbf{H}_t$ of order t is a $t \times t$ matrix with elements ± 1 satisfying

$$\mathbf{H}_t \mathbf{H}_t' = t\mathbf{I}_t.$$

2.2 Covariate Designs Under Regular Cases

Consider the case when n is a multiple of v, that is $n = vb$ where b is such that $\mathbf{H}_b$, Hadamard matrix of order b, exists. We shall also consider some cases where b is even. Then ANOVA parameters as well as the covariate effect-parameters can be estimated orthgonally and/or most efficiently. This holds simultaneously for c covariates and one can deduce maximum possible value of c for this to happen. As already mentioned, the most efficient estimation of γ-components is possible when (2.1.7) and (2.1.11) are simultaneously satisfied and these conditions reduce, in terms of $\mathbf{W}$-matrices defined in above, to C_1, C_2 where

$$\left.\begin{array}{ll} C_1. & \text{Each of the } c \ \mathbf{W}\text{-matrices has all row-sums equal to zero;} \\ C_2. & \text{The grand total of all the entries in the Hadamard product} \\ & \text{of any two distinct } \mathbf{W}\text{-matrices reduces to zero.} \end{array}\right\} \tag{2.2.1}$$

Now we define optimum **W**-matrices for covariate designs in CRD set-up.

Definition 2.2.1 With respect to model (2.1.3), the c **W**-matrices corresponding to the c covariates are said to be optimum if they satisfy conditions C_1 and C_2 of (2.2.1).

In this context, the following results were deduced in Troya Lopes (1982a).

Theorem 2.2.1 *Let c^* be the maximum number of covariates that can be optimally accommodated. Then a lower bound to c^* is given by*

(a) *b–1 when v = odd, $\mathbf{H}_b$ exists*;
(b) *2(b–1) when $v \equiv 2 \pmod 4$, $\mathbf{H}_b$ exists*;
(c) *4(b–1) when $v \equiv 0 \pmod 4$, $\mathbf{H}_b$ exists*;
(d) *3v when $b \equiv 0 \pmod 4$, $\mathbf{H}_v$ exists*;
(e) *v when $b \equiv 2 \pmod 4$, $\mathbf{H}_v$ exists.*

Proof Hadamard matrix $\mathbf{H}_b$ is given to exist and we write it as

$$\mathbf{H}_b = (\mathbf{h}_1,\ \mathbf{h}_2, \ldots, \mathbf{h}_{b-1}, \mathbf{1})\,. \tag{2.2.2}$$

The choice of optimum **W**-matrices is indicated below one by one. The verification of (2.2.1) is immediate and we leave it to the reader. The Kronecker product of two matrices is formally defined in Chap. 5 (Definition 5.1.1) and it is used in the constructions below.

(a)

$$\mathbf{W}^{(j)\ v\times b} = \mathbf{1}_v \otimes \mathbf{h}'_j, \quad 1 \le j \le b-1; \tag{2.2.3}$$

(b)

$$\left.\begin{aligned} \mathbf{W}^{(j)\ v\times b} &= (1,\ \ 1)' \otimes \mathbf{1}_{\frac{v}{2}} \otimes \mathbf{h}'_j, \quad 1 \le j \le b-1; \\ \mathbf{W}^{(b-1+j)\ v\times b} &= (1,\ \ -1)' \otimes \mathbf{1}_{\frac{v}{2}} \otimes \mathbf{h}'_j, \quad 1 \le j \le b-1. \end{aligned}\right\} \tag{2.2.4}$$

(c)

$$\left.\begin{aligned} \mathbf{W}^{(j)\ v\times b} &= (1,\ \ 1,\ \ 1,\ \ 1)' \otimes \mathbf{1}_{\frac{v}{4}} \otimes \mathbf{h}'_j, \quad 1 \le j \le b-1; \\ \mathbf{W}^{(b-1+j)\ v\times b} &= (1,\ \ -1,\ \ 1,\ \ -1)' \otimes \mathbf{1}_{\frac{v}{4}} \otimes \mathbf{h}'_j, \quad 1 \le j \le b-1; \\ \mathbf{W}^{(2(b-1)+j)\ v\times b} &= (1,\ \ -1,\ \ -1,\ \ 1)' \otimes \mathbf{1}_{\frac{v}{4}} \otimes \mathbf{h}'_j, \quad 1 \le j \le b-1; \\ \mathbf{W}^{(3(b-1)+j)\ v\times b} &= (1,\ \ 1,\ \ -1,\ \ -1)' \otimes \mathbf{1}_{\frac{v}{4}} \otimes \mathbf{h}'_j, \quad 1 \le j \le b-1. \end{aligned}\right\} \tag{2.2.5}$$

(d) Let us represent a Hadamard matrix $\mathbf{H}_v$ of order v as

$$\mathbf{H}_v = \left(\mathbf{h}^*_1,\ \mathbf{h}^*_2, \ldots, \mathbf{h}^*_v\right). \tag{2.2.6}$$

$$\left.\begin{aligned} \mathbf{W}^{(j)\ v\times b} &= (1,\ \ -1,\ \ 1,\ \ -1)\otimes \mathbf{1}'_{\frac{b}{4}}\otimes \mathbf{h}^*_j,\ \ 1\le j\le v;\\ \mathbf{W}^{(v+j)\ v\times b} &= (1,\ \ -1,\ \ -1,\ \ 1)\otimes \mathbf{1}'_{\frac{b}{4}}\otimes \mathbf{h}^*_j,\ \ 1\le j\le v;\\ \mathbf{W}^{(2v+j)\ v\times b} &= (1,\ \ 1,\ \ -1,\ \ -1)\otimes \mathbf{1}'_{\frac{b}{4}}\otimes \mathbf{h}^*_j,\ \ 1\le j\le v. \end{aligned}\right\} \tag{2.2.7}$$

(e)

$$\mathbf{W}^{(j)\ v\times b} = (1,\ \ -1)\otimes \mathbf{1}'_{\frac{b}{2}}\otimes \mathbf{h}^*_j,\ \ 1\le j\le v. \tag{2.2.8}$$

□

Remark 2.2.1 In case (c), we can assume existence of $\mathbf{H}_v$ for all practical purposes as $v\equiv 0 \pmod 4$. So in this case, an optimal design for maximum possible $v(b-1)$ optimum **W**-matrices can easily be constructed as

$$\mathbf{W}^{((b-1)(i-1)+j)} = \mathbf{h}^*_i\otimes \mathbf{h}'_j,\ \ i=1,2,\ldots,v,\ \ j=1,2,\ldots,b-1. \tag{2.2.9}$$

This was obtained in (Rao et al. 2003) where it was observed that OCDs in CRD and RBD have one to one correspondences with mixed orthogonal array (MOA) (definition given in Chap. 3). This fact will be discussed in Sect. 3.3 of Chap. 3 in some further details.

2.3 Covariate Designs Under Non-regular Cases

Now we examine the situations where at least any one of the conditions (2.1.7), (2.1.11) and $b=\frac{n}{v}=$ integer is violated. In that case, it is not possible to estimate simultaneously ANOVA parameters and γ-parameters orthogonally and/or most efficiently. Thus we consider D-optimality criterion to give an efficient allocation of treatments and covariates in Set-up (2.1.1). Dey and Mukerjee (2006) and Dutta et al. (2014) have considered this situation and found D-optimal design. Here we discuss their contributions in this direction in details.

The vector of parameters $\boldsymbol{\theta}$, where

$$\boldsymbol{\theta} = (\mu_1,\ \mu_2,\ldots,\mu_v,\gamma_1,\ldots,\gamma_c)' \tag{2.3.1}$$

is assumed to be estimable.

The information matrix for $\boldsymbol{\theta}$ is given by $\sigma^{-2}\mathbf{I}(\boldsymbol{\theta})$, where

$$\mathbf{I}(\boldsymbol{\theta}) = \begin{pmatrix} \mathbf{N} & \mathbf{T}\\ \mathbf{T}' & \mathbf{Z}'\mathbf{Z}\end{pmatrix}, \tag{2.3.2}$$

$$\mathbf{N} = \mathrm{Diag}(n_1,n_2,\ldots,n_v), \tag{2.3.3}$$

$$\mathbf{T} = (\mathbf{T}'_1,\mathbf{T}'_2,\ldots,\mathbf{T}'_v)',\ \ \mathbf{T}_i = \mathbf{1}'_{n_i}\mathbf{Z}_i, \tag{2.3.4}$$

$$\mathbf{Z}^{n\times c} = (\mathbf{Z}_1', \mathbf{Z}_2', \ldots, \mathbf{Z}_v')' \tag{2.3.5}$$

and

$$\mathbf{Z}_i^{n_i\times c} = \begin{pmatrix} z_{i1}^{(1)} & z_{i1}^{(2)} & \cdots & z_{i1}^{(c)} \\ z_{i2}^{(1)} & z_{i2}^{(2)} & \cdots & z_{i2}^{(c)} \\ \vdots & \vdots & \ddots & \vdots \\ z_{in_i}^{(1)} & z_{in_i}^{(2)} & \cdots & z_{in_i}^{(c)} \end{pmatrix}. \tag{2.3.6}$$

For D-optimality, we have to maximize the determinant of $\mathbf{I}(\boldsymbol{\theta})$, denoted as $\det(\mathbf{I}(\boldsymbol{\theta}))$, with respect to the design variables $\{z_{ij}^{(t)}\}$ satisfying $z_{ij}^{(t)} \in [-1, 1], \quad 1 \le j \le n_i, \ 1 \le i \le v$ and n_i's satisfying (2.1.2).

From (2.3.2) it is easy to see that

$$\begin{aligned} \det(\mathbf{I}(\boldsymbol{\theta})) &= \left(\prod_{i=1}^{v} n_i\right)\det(\mathbf{Z}'\mathbf{Z} - \mathbf{T}'\mathbf{N}^{-1}\mathbf{T}) \\ &= \left(\prod_{i=1}^{v} n_i\right)\det(\mathbf{Z}'\mathbf{Z} - \sum_i n_i^{-1}\mathbf{T}_i'\mathbf{T}_i) \\ &= \det(\mathbf{N})\det(\mathbf{C}), \end{aligned} \tag{2.3.7}$$

where

$$\mathbf{C} = \mathbf{Z}'\mathbf{Z} - \sum_i n_i^{-1}\mathbf{T}_i'\mathbf{T}_i. \tag{2.3.8}$$

Note that $\mathbf{C}$ is the information matrix for the regression coefficients $\gamma_1, \gamma_2, \ldots, \gamma_c$. The maximization of $\det(\mathbf{I}(\boldsymbol{\theta}))$ is done in two stages. In the first stage, the maximization is done for varying z-values for fixed n_i's. This leads to an upper bound for $\det(\mathbf{I}(\boldsymbol{\theta}))$ obtained through completely symmetric $\mathbf{C}$-matrices. At the second stage, maximization is done for varying n_i's subject to $\sum_i n_i = n$, and this leads to a sufficiently small class $\mathcal{N}$ of contending $\mathbf{n} = (n_1, n_2, \ldots, n_v)$'s wherein the overall upper bound to $\det(\mathbf{I}(\boldsymbol{\theta}))$ belongs.

2.3.1 First Stage of Maximization

Maximisation of $\mathbf{I}(\boldsymbol{\theta})$ with respect to $z_{ij}^{(t)} \in [-1, 1]$ is based on the following lemma.

Lemma 2.3.1 *A necessary condition for maximization of det(*$\mathbf{C}$*) of* (2.3.8) *with respect to* $z_{ij}^{(t)} \in [-1, 1]$*, for fixed* n_i*'s is that* $z_{ij}^{(t)} = \pm 1 \ \forall i, j$ *and* t*.*

Proof From (2.3.8), $\mathbf{C}$ can be expressed as

$$\mathbf{C} = \mathbf{Z}'\mathbf{M}\mathbf{Z} = \mathbf{Z}^{*\prime}\mathbf{Z}^* \tag{2.3.9}$$

where

$$\mathbf{Z}^* = \mathbf{MZ}, \quad \mathbf{M} = \text{diag}(\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_v), \quad \mathbf{M}_i = (\mathbf{I}_{n_i} - n_i^{-1}\mathbf{1}_{n_i}\mathbf{1}'_{n_i}). \tag{2.3.10}$$

It is known that (cf. Galil and Kiefer 1980; Wojtas 1964), det($\mathbf{Z}^{*\prime}\mathbf{Z}^*$) is maximum at the extreme entries of $\mathbf{Z}^*$. Again, as $z_{ij}^{(t)*}$'s are linear in $z_{ij}^{(t)}$'s, the determinant is maximum at the extreme values of $z_{ij}^{(t)}$'s for all i, j and t. Hence the lemma follows. □

Theorem 2.3.1 *For fixed* $\{n_i\}$*'s satisfying* (2.1.2),

$$det(\mathbf{I}(\boldsymbol{\theta})) \leq \left(\prod_{i=1}^{v} n_i\right)\{a + (c-1)b\}(a-b)^{c-1} \tag{2.3.11}$$

where

$$a = n - \delta, \quad b = |\xi - \delta| \tag{2.3.12}$$

$$\delta = \sum_{i=1}^{v} n_i^{-1}\delta_i, \quad \delta_i = 1(0) \text{ if } n_i = \text{odd(even)} \tag{2.3.13}$$

$$\xi = \xi(n, \delta) = \begin{cases} \lfloor\delta\rfloor & \text{if both of } n,\ \lfloor\delta\rfloor \text{ are odd or even} \\ \lfloor\delta\rfloor + 1 & \text{if } n = \text{odd},\ \lfloor\delta\rfloor = \text{even or n=even},\ \lfloor\delta\rfloor = \text{odd} \end{cases} \tag{2.3.14}$$

$\lfloor\delta\rfloor =$ *greatest integer less than equal to* δ.

Proof Because of Lemma 2.3.1, we restrict $z_{ij}^{(t)}$ to the class $\boldsymbol{\chi} = \{z_{ij}^{(t)} : z_{ij}^{(t)} = \pm 1\}$. From the Eq. (2.3.8), we note that, $c_{t,t'}$, the $(t, t')^{\text{th}}$ element of the $\mathbf{C}$-matrix is given by

$$c_{t,t'} = \sum_i \left\{ \sum_j z_{ij}^{(t)} z_{ij}^{(t')} - \frac{\left(\sum_j z_{ij}^{(t)}\right)\left(\sum_j z_{ij}^{(t')}\right)}{n_i} \right\}, \quad 1 \leq t, t' \leq c. \tag{2.3.15}$$

It follows from Wojtas (1964) that det($\mathbf{C}$) is maximum when $\mathbf{C}$ is completely symmetric with all the diagonal elements equal to a and all off-diagonal elements equal to b where a and b are given by $\max\limits_{1\leq t\leq c} c_{tt}$ and $\min\limits_{1\leq t\neq t'\leq c} |c_{tt'}|$ respectively. Again as

$z_{ij}^{(t)} = \pm 1 \; \forall i, j$ and t, for fixed n_i's, it can be deduced that

$$\max_{1 \le t \le c} c_{tt} = n - \delta = a, \quad \min_{1 \le t \ne t' \le c} |c_{tt'}| = |\xi - \delta| = b \tag{2.3.16}$$

where δ and ξ are given in (2.3.13) and (2.3.14) respectively. Therefore the theorem follows. □

2.3.2 *Second Stage of Maximization*

In view of Theorem 2.3.1, we now consider the problem of maximizing

$$g(\mathbf{n}) = g(n_1, \; n_2, \ldots, n_v) = \left(\prod_{i=1}^{v} n_i \right) \{a + (c-1)b\}(a-b)^{c-1} \tag{2.3.17}$$

with respect to n_i's subject to $\sum_{i=1}^{v} n_i = n$, where a and b are given by (2.3.12)–(2.3.14), so as to find the overall upper bound of $\det(\mathbf{I}(\boldsymbol{\theta}))$. The following lemma helps to reduce the class $\mathcal{N}$ of $\mathbf{n}$'s where $\mathbf{n} = (n_1, \; n_2, \ldots, n_v)$, satisfying (2.1.2), to a subclass in which maximum of $g(\mathbf{n})$ lies.

Lemma 2.3.2 *Let* $\mathbf{n}^* = \left(n_1^*, \; n_2^*, \ldots, n_v^*\right)$ *be a maximizer of* $g(\mathbf{n})$ *of* (2.3.17) *subject to the condition* (2.1.2). *Then* $\mathbf{n}^*$ *cannot have*

(i) *two unequal odd elements*;
(ii) *two even elements that differ by more than 2*;
(iii) *an even and an odd element that differ by more than 1.* □

Proof (i) Without loss of generality it is assumed that n_1^* and n_2^* be odd and $n_1^* \le n_2^* - 2$. Define $\widetilde{\mathbf{n}} = (\widetilde{n}_1, \; \widetilde{n}_2, \ldots, \widetilde{n}_v)$, where $\widetilde{n}_1 = n_1^* + 1, \; \widetilde{n}_2 = n_2^* - 1$ and $\widetilde{n}_i = n_i^* \; \forall i \ne 1, 2$. Note that $\widetilde{n}_i$'s satisfy condition (2.1.2). Then by Eq. (2.3.17),

$$\begin{aligned} \tfrac{g(\widetilde{\mathbf{n}})}{g(\mathbf{n}^*)} &= \left(\prod_{i=1}^{v} \widetilde{n}_i \right) \Big/ \left(\prod_{i=1}^{v} n_i^* \right) \left(\tfrac{\{\widetilde{a} + (c-1)\widetilde{b}\}(\widetilde{a} - \widetilde{b})^{c-1}}{\{a^* + (c-1)b^*\}(a^* - b^*)^{c-1}} \right) \\ &= \tfrac{(n_1^*+1)(n_2^*-1)}{n_1^* n_2^*} \, \tfrac{\{\widetilde{a} + (c-1)\widetilde{b}\}(\widetilde{a} - \widetilde{b})^{c-1}}{\{a^* + (c-1)b^*\}(a^* - b^*)^{c-1}}, \end{aligned} \tag{2.3.18}$$

where,

$$\widetilde{a} = n - \sum_{i=1}^{v} \widetilde{n}_i^{-1} \widetilde{\delta}_i = n - \sum_{i=1}^{v} n_i^{*-1} \delta_i^* + \frac{1}{n_1^*} + \frac{1}{n_2^*} = a^* + \frac{1}{n_1^*} + \frac{1}{n_2^*} \tag{2.3.19}$$

Again,

$$\widetilde{b} = \left|\widetilde{\xi} - \sum_{i=1}^{v} \widetilde{n}_i^{-1}\widetilde{\delta}_i\right| \leq \left|\xi^* - \sum_{i=1}^{v} n_i^{*-1}\delta_i^* + \left(\frac{1}{n_1^*} + \frac{1}{n_2^*}\right)\right| \leq b^* + \left(\frac{1}{n_1^*} + \frac{1}{n_2^*}\right). \quad (2.3.20)$$

We consider the two cases $\widetilde{b} \leq b^*$ and $\widetilde{b} > b^*$ separately.
(a) Let $\widetilde{b} \leq b^*$. Then, as by (2.3.19), $\widetilde{a} > a^*$, it follows that $g(\widetilde{\mathbf{n}}) > g(\mathbf{n}^*)$, which is impossible.
(b) Let $\widetilde{b} > b^*$ and let $\widetilde{b}$ assume the highest possible value given in (2.3.20). Then from (2.3.18)–(2.3.20), it is seen that

$$\frac{g(\widetilde{\mathbf{n}})}{g(\mathbf{n}^*)} > \frac{(n_1^* + 1)(n_2^* - 1)}{n_1^* n_2^*} > 1 \quad (2.3.21)$$

which is again a contradiction. As the inequality (2.3.21) is true for the highest value of $\widetilde{b}$, it will be true for all values of $\widetilde{b}$ in $[b^*,\ b^* + \frac{1}{n_1^*} + \frac{1}{n_2^*}]$ as $\widetilde{a} > a^*$.
(ii) If possible, let $\mathbf{n}^*$ have two even elements, say $n_1^* < n_2^*$ which differ by more than 2. Then as in (i) above, we reach at a contradiction by increasing n_1^* by two and decreasing n_2^* by two.
(iii) If possible, let $\mathbf{n}^*$ have an even element n_1^* and an odd element n_2^* which differ by more than 1.

Case A: Let $n_1^* > n_2^*$. Satisfying (2.1.2), define $\widetilde{\mathbf{n}} = (\widetilde{n}_1,\ \widetilde{n}_2, \ldots, \widetilde{n}_v)$, where $\widetilde{n}_1 = n_1^* - 2,\ \widetilde{n}_2 = n_2^* + 2$ and $\widetilde{n}_i = n_i^*\ \forall i \neq 1, 2$. Then by Eq. (2.3.17), we have

$$\frac{g(\widetilde{\mathbf{n}})}{g(\mathbf{n}^*)} = \frac{(n_1^* - 2)(n_2^* + 2)\{\widetilde{a} + (c-1)\widetilde{b}\}(\widetilde{a} - \widetilde{b})^{c-1}}{(n_1^* n_2^*)\{a^* + (c-1)b^*\}(a^* - b^*)^{c-1}} \quad (2.3.22)$$

where,

$$\widetilde{a} = n - \sum_i \widetilde{n}_i^{-1}\widetilde{\delta}_i = \left(n - \sum_i n_i^{*-1}\delta_i^*\right) + \left(\frac{1}{n_2^*} - \frac{1}{n_2^* + 2}\right) = a^* + \left(\frac{1}{n_2^*} - \frac{1}{n_2^* + 2}\right) \quad (2.3.23)$$

$$\widetilde{b} = |\widetilde{\xi} - \widetilde{\delta}| \leq |(\xi^* - \delta^*) + \left(\frac{1}{n_2^*} - \frac{1}{n_2^* - 2}\right)| \leq b^* + \left(\frac{1}{n_2^*} - \frac{1}{n_2^* - 2}\right). \quad (2.3.24)$$

We consider two cases when $\widetilde{b} \leq b^*$ and $\widetilde{b} > b^*$. For $\widetilde{b} \leq b^*$, it follows, from (2.3.22) that $g(\widetilde{\mathbf{n}}) > g(\mathbf{n}^*)$ as $\widetilde{a} > a^*$. Again, for $\widetilde{b} > b^*$, we assume its highest value viz. $b^* + (\frac{1}{n_2^*} - \frac{1}{n_2^* - 2})$ from (2.3.24) and use it in (2.3.22). It is seen that $g(\widetilde{\mathbf{n}}) > g(\mathbf{n}^*)$, which obviously holds for all other values of $\widetilde{b} > b^*$ as $\widetilde{a} > a^*$.

So we reach at a contradiction that $\mathbf{n}^*$ is a maximizer of $g(\mathbf{n})$.

Case B: Let $n_1^* < n_2^*$ (i.e. $n_1^* \leq n_2^* - 3$), then we have the following two cases:

(a) n_2^* is not the only odd element of $\mathbf{n}^*$.
(b) n_2^* is the only odd element of $\mathbf{n}^*$.

For (a), let $\mathbf{n}^*$ have another odd element n_3^*. Then by part (i) of this lemma, $n_2^*{=}n_3^*$. Define $\widetilde{\mathbf{n}} = (\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_v)$, where $\widetilde{n}_1 = n_1^* + 2$, $\widetilde{n}_2 = \widetilde{n}_3 = n_2^* - 1$ and $\widetilde{n}_i = n_i^* \ \forall\, i \neq 1, 2, 3$. Then by (2.3.17)

$$\frac{g(\widetilde{\mathbf{n}})}{g(\mathbf{n}^*)} = \frac{(n_1^*+2)(n_2^*-1)^2}{(n_1^* n_2^* n_3^*)} \frac{\{\widetilde{a} + (c-1)\widetilde{b}\}(\widetilde{a} - \widetilde{b})^{c-1}}{\{a^* + (c-1)b^*\}(a^* - b^*)^{c-1}}. \tag{2.3.25}$$

where,

$$\widetilde{a} = n - \sum_i \widetilde{n}_i^{-1}\widetilde{\delta}_i = \left(n - \sum_i n_i^{*-1}\delta_i\right) + \frac{2}{n_2^*} = a^* + \frac{2}{n_2^*} \tag{2.3.26}$$

$$\widetilde{b} = |\widetilde{\xi} - \widetilde{\delta}| \leq |(\xi^* - \delta^*) + \frac{2}{n_2^*}| \leq b^* + \frac{2}{n_2^*}. \tag{2.3.27}$$

If $\widetilde{b} \leq b^*$, then from (2.3.25) and (2.3.26) $g(\widetilde{\mathbf{n}}) > g(\mathbf{n}^*)$ which is a contradiction.

If $\widetilde{b} > b^*$, the above contradiction also holds by the same reasons as given in Case A.

For (b), let us define $\widetilde{\mathbf{n}} = (\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_v)$ satisfying (1.2) where $\widetilde{n}_1 = n_1^* + 2$, $\widetilde{n}_2 = n_2^* - 2$ and $\widetilde{n}_i = n_i^* \ \forall\, i \neq 1, 2$. Proceeding as before, it can be proved that

$$\widetilde{a} = a^* + \left(\frac{1}{n_2^*} - \frac{1}{n_2^* - 2}\right), \quad \widetilde{b} = \left(1 - \frac{1}{n_2^* - 2}\right). \tag{2.3.28}$$

Using (2.3.28) in (2.3.17), it is seen that

$$\frac{g(\widetilde{\mathbf{n}})}{g(\mathbf{n}^*)} = \frac{(n_1^*+2)(n_2^*-2)}{n_1^* n_2^*} \frac{\left(n + c - 1 - \frac{c}{n_2^* - 2}\right)}{\left(n + c - 1 - \frac{c}{n_2^*}\right)} > 1 \quad \text{as } (n_2^* - n_1^*) \geq 3.$$

This is again a contradiction. Therefore the lemma follows. □

From Lemma 2.3.2, we get the following theorem whose proof is immediate.

Theorem 2.3.2 *Let $\overline{o}$ be an odd integer, where $\overline{o} = \lfloor \frac{n}{v} \rfloor$ or $\lfloor \frac{n}{v} \rfloor + 1$ according as $\lfloor \frac{n}{v} \rfloor$ is odd or even and $\mathbf{n}^* = \left(n_1^*, n_2^*, \ldots, n_v^*\right)$ be a maximizer of $g(\mathbf{n})$ of (2.3.17) subject to $\sum_i n_i = n$. Then $n_i^* \in \{\overline{o} - 1, \overline{o}, \overline{o} + 1\}$.*

Lemma 2.3.3 *If f, f^- and f^+ be the frequencies of $\overline{o}$, $\overline{o}-1$ and $\overline{o}+1$ respectively, then the following relations*

$$f + f^- + f^+ = v; \quad \overline{o} f + (\overline{o} - 1) f^- + (\overline{o} + 1) f^+ = n, \tag{2.3.29}$$

minimize considerably the search for optimum $\mathbf{n}$*, for which* $g(\mathbf{n})$ *is a maximum.*

Let $\mathcal{N}^*$ $(\subset \mathcal{N})$ denote the class of $\mathbf{n}$'s satisfying Theorem 2.3.2 and Lemma 2.3.2.

Remark 2.3.1 For given n, v and c, let $g(\mathbf{n}^*)$ be the maximum of $g(\mathbf{n})$ of (2.3.17) over $\mathbf{n} = (n_1, n_2, \ldots, n_v)$ subject to $\sum_i n_i = n$. Then by Theorem 2.3.1

$$\det(\mathbf{I}(\boldsymbol{\theta}) \leq g(\mathbf{n}^*). \tag{2.3.30}$$

If a choice of $\{z_{ij}^{(t)}\}$ exists corresponding to $\mathbf{n}^*$, such that equality in (2.3.30) holds, then $\mathbf{n}^*$ together with $\{z_{ij}^{(t)}\}$ gives a D-optimal design.

Remark 2.3.2 If all n_i's are even, so that all the $\mathbf{T}_i$'s of (2.3.4) may be made equal to zero, then it is possible to estimate the regression parameters $\boldsymbol{\gamma}$'s orthogonally to the μ_i's. In that case, $\boldsymbol{\gamma}$'s are estimated most efficiently with the minimum possible variance when $\mathbf{Z}'\mathbf{Z} = n\mathbf{I}_c$.

Remark 2.3.3 If $n_i = \frac{n}{v}$ = an even integer for all i, the situation reduces to regular case and then Remark 2.3.1 is in full agreement with Troya Lopes (1982a) and in that case $\boldsymbol{\gamma}$'s can be estimated most efficiently so that each estimator has minimum possible variance when $\mathbf{Z}'\mathbf{Z} = n\mathbf{I}_c$.

Remark 2.3.4 If the v levels of the single factor set-up are assumed to be the v level combinations of m factors $\mathrm{F}_1, \ldots, \mathrm{F}_m$ having $s_1, \ldots, s_m$ levels, respectively ($v = \prod s_i$), then the optimum design for the single factor set-up is also optimum for the estimation of $\boldsymbol{\gamma}$ and all effects up to m-factor interactions which can be obtained through an orthogonal transformation of $\boldsymbol{\gamma}$ and the mean vector $\boldsymbol{\mu}$ corresponding to the v level combinations.

2.3.3 Examples

Now we consider following examples to illustrate the above method.

Example 2.3.1 Let us consider the one-way set-up with $n = 12$, $v = 4$. It follows that $\mathcal{N}^* = \{(3, 3, 3, 3), (2, 3, 3, 4), (2, 2, 4, 4)\} \equiv \{(3^4),\ (2, 3^2, 4),\ (2^2, 4^2)\}$.

(a) For $c = 1$, $\mathbf{n}^* = (3^4)$ is the unique maximizer of $g(\mathbf{n})$ and this $\mathbf{n}^*$ together with $\mathbf{Z}_1' = (1,\ 1,\ -1)$, $\mathbf{Z}_2' = (1,\ 1,\ -1)$, $\mathbf{Z}_3' = (1,\ 1,\ -1)$, $\mathbf{Z}_4' = (1,\ 1,\ -1)$ gives a D-optimal design.

(b) For $c = 2$ both $\mathbf{n}^* = (2, 3^2, 4)$ and $(2^2, 4^2)$ are maximizers of $g(\mathbf{n})$.

(i) $\mathbf{n}^* = (2^2, 4^2)$ and

$$\mathbf{Z}_1 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \mathbf{Z}_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \mathbf{Z}_3 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{pmatrix}, \mathbf{Z}_4 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix},$$

give a D-optimal design.

(ii) $\mathbf{n}^* = (2, 3^2, 4)$ and

$$\mathbf{Z}_1 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \mathbf{Z}_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}, \mathbf{Z}_3 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{pmatrix}, \mathbf{Z}_4 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}, \text{ also}$$

give a D-optimal design.

(c) For $c = 3,\ 4$, $\mathbf{n}^* = (2^2, 4^2)$ is the unique maximizer of $g(\mathbf{n})$.

Example 2.3.2 In one-way set-up with $n = 9,\ v = 3,\ c = 3$, D-optimal design should be searched within the set $\{(2, 3, 4),\ (3^3)\}$ of $\mathbf{n}$. It is seen that for $\mathbf{n} = (3^3)$ and

$$\mathrm{D}_1:\quad \mathbf{Z}^{(1)} = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},\ \mathbf{Z}^{(2)} = \begin{pmatrix} -1 & + & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \text{ and } \mathbf{Z}^{(3)} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix},$$

$\mathbf{C} = \mathrm{diag}(8, 8, 8)$ and $g(\mathbf{n}) = 3^3.8^3$. But for $\mathbf{n} = (2, 3, 4)$, and

$$\mathrm{D}_2:\quad \mathbf{Z}^{(1)} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix},\ \mathbf{Z}^{(2)} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } \mathbf{Z}^{(3)} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

It can be seen that $\mathbf{C} = 8\mathbf{I}_3 + \frac{2}{3}\mathbf{J}_3$, where $\mathbf{J}_3$ is a 3×3 matrix containing elements one only. Also $g(2, 3, 4)$ which is equal to 15360, attains the upper bound in (2.3.14) and $g(2, 3, 4) > g(3, 3, 3)$ implying that D_2 is D-optimal.

Again for $n = 9$, $v = 3$, $c = 4$, it is noted that $\mathbf{n}^* = (2, 3, 4)$ together with

$$\mathrm{D}_3:\quad \mathbf{Z}^{(1)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix},\ \mathbf{Z}^{(2)} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and

$$\mathbf{Z}^{(3)} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

maximizes $g(\mathbf{n})$ of (2.3.17) and hence gives a D-optimal design.

Remark 2.3.5 It is seen from the examples that the choice of optimum $\mathbf{n}$ depends on the number of the covariates used apart from the number of cells v in the set-up. Again it is noted from (2.3.7) that $\det(\mathbf{I}(\boldsymbol{\theta}))$ depends on two factors viz. $\det(\mathbf{N})(=\prod_i n_i)$ and $\det(\mathbf{C})$. Determinant of $\mathbf{N}$ increases as the homogeneity between the n_i's increases subject to $\sum_i n_i = n$. On the other hand $\det(\mathbf{C})$ increases, apart from c, with the largeness of a and the smallness of b, which again are achieved by inclusion of maximum number of even n_i's closed to $\lfloor \frac{n}{v} \rfloor$. The number of odd n_i's subject to $\sum_i n_i = n$, in between the even ones with proper homogeneity, actually strikes a balance between $\det(\mathbf{N})$ and $\det(\mathbf{C})$. It is also seen that, when c is small, $\det(\mathbf{N})$ is the dominant factor, while, if c is large $\det(\mathbf{C})$ becomes the dominant factor.

Incidentally, the above analysis is based on the work in Dutta et al. (2014) and it improves over what was achieved in Dey and Mukerjee (2006).

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Chapter 3
OCDs in Randomized Block Design Set-Up

3.1 Introduction

For two-way layout, the set-up can be written as

$$\left(\mathbf{Y},\ \mu\mathbf{1}+\mathbf{X}_1\boldsymbol{\tau}+\mathbf{X}_2\boldsymbol{\beta}+\mathbf{Z}\boldsymbol{\gamma},\ \sigma^2\mathbf{I}_n\right) \tag{3.1.1}$$

where μ, as usual, stands for the general effect, $\boldsymbol{\tau}^{v\times 1}$, $\boldsymbol{\beta}^{b\times 1}$ represent vectors of treatment and block effects, respectively, and $\mathbf{X}_1^{n\times b}$ and $\mathbf{X}_2^{n\times v}$ are, respectively, the corresponding incidence matrices. $\mathbf{Y}$, $\mathbf{Z}$ as usual, represent an observation vector of order $n\times 1$ and the design matrix of order $n\times c$ corresponding to vector of covariate effects $\boldsymbol{\gamma}^{c\times 1}$ respectively. It should be noted that each column of $\mathbf{Z}$-matrix has a natural interpretation in terms of the correspondence of the covariate values with the experimental units in the RBD set-up we start with.

We straightway compute the form of the information matrix for the whole set of parameters $\boldsymbol{\eta}=\left(\mu,\ \boldsymbol{\beta}',\ \boldsymbol{\tau}',\ \boldsymbol{\gamma}'\right)'$ underlying a design d with $\mathbf{X}_{1d}$, $\mathbf{X}_{2d}$ and $\mathbf{Z}_d$ as the versions of $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{Z}$ in (3.1.1):

$$\mathbf{I}_d(\boldsymbol{\eta})=\begin{pmatrix} n & \mathbf{1}'\mathbf{X}_{1d} & \mathbf{1}'\mathbf{X}_{2d} & \mathbf{1}'\mathbf{Z}_d \\ & \mathbf{X}_{1d}'\mathbf{X}_{1d} & \mathbf{X}_{1d}'\mathbf{X}_{2d} & \mathbf{X}_{1d}'\mathbf{Z}_d \\ & & \mathbf{X}_{2d}'\mathbf{X}_{2d} & \mathbf{X}_{2d}'\mathbf{Z}_d \\ & & & \mathbf{Z}_d'\mathbf{Z}_d \end{pmatrix}. \tag{3.1.2}$$

For the covariates, as before, we assume, without loss of generality, the (location-scale)-transformed version: $|z_{ij}^{(t)}|\le 1;\ i,\ j,\ t$.

It is evident from (3.1.2) that orthogonal estimation of treatment and block effect contrasts on one hand and covariate effects on the other is possible when the conditions

$$\mathbf{X}_{1d}'\mathbf{Z}_d=\mathbf{0},\qquad \mathbf{X}_{2d}'\mathbf{Z}_d=\mathbf{0} \tag{3.1.3}$$

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are satisfied. It is to be noted that under (3.1.3), $\mathbf{1}'\mathbf{Z}_d = \mathbf{0}'$ also holds. Further, as before, most efficient estimation of γ-components is possible whenever, in addition to (3.1.3), we can also ascertain

$$\mathbf{Z}_d'\mathbf{Z}_d = n\mathbf{I}_n. \tag{3.1.4}$$

It is also true that, whenever (3.1.3) is ensured, presence of the covariates in (3.1.1) does not pose any threat to the usual optimal design problem in a block design set-up as the covariate parameters and the block design parameters are orthogonally estimable.

As before for an RBD set-up, following Das et al. (2003), we recast each column of the $\mathbf{Z}^{n\times c} = (\pm 1)$ matrix by a $\mathbf{W}$-matrix of order $v \times b$. Corresponding to the treatment $\times$ block classifications, conditions (3.1.3) and (3.1.4) reduce, in terms of $\mathbf{W}$-matrices, to $C_1 - C_3$ where

$$\left.\begin{array}{l}C_1.\ \text{Each } \mathbf{W}\text{-matrix has all column-sums equal to zero;}\\ C_2.\ \text{Each } \mathbf{W}\text{-matrix has all row-sums equal to zero;}\\ C_3.\ \text{The grand total of all the entries in the Hadamard product}\\ \quad\ \ \text{of any two distinct } \mathbf{W}\text{-matrices reduces to zero.}\end{array}\right\} \tag{3.1.5}$$

Now we define optimum $\mathbf{W}$-matrix for covariate design, in an RBD set-up.

Definition 3.1.1 With respect to model (3.1.1), the c $\mathbf{W}$-matrices corresponding to the c covariates are said to be optimum if they satisfy the conditions $C_1 - C_3$ of (3.1.5).

We arrange the remaining sections of this chapter as follows. In Sect. 3.2, we consider the constructional methods of optimum $\mathbf{W}$-matrices given by Das et al. (2003) and in Sects. 3.3 and 3.4, we discuss the relationships between OCDs and MOAs and construction of optimum $\mathbf{Z}$s given in Rao et al. (2003).

3.2 Construction of Optimum W-Matrices

Here we consider the following method for constructing optimum $\mathbf{W}$-matrices given in Das et al. (2003). They used mutually orthogonal latin squares (MOLS) for construction of optimum $\mathbf{W}$s. The method is given in the following theorem.

Theorem 3.2.1 *Suppose $\mathbf{H}_v$ and m MOLS of order v exist. Then $m(v-1)$ optimum $\mathbf{W}$-matrices can be constructed for an RBD with $b = v$ blocks and v treatments.*

Proof For the construction of the optimum $\mathbf{W}$-matrices, we will proceed as follows:

Step 1 We set the Hadamard Matrix $\mathbf{H}_v$ in the following form:

$$\mathbf{H}_v = \left(\mathbf{h}_1,\ \mathbf{h}_2,\ \ldots,\ \mathbf{h}_{v-1},\ \mathbf{1}\right) \tag{3.2.1}$$

where $\mathbf{h}_j$ denotes the jth column of $\mathbf{H}_v$.
Step 2 We can construct the ith member L_i of the set of m MOLS of order v by using the symbols

$$a_{i1},\ a_{i2}, \ldots, a_{iv}; \quad 1 \le i \le m. \tag{3.2.2}$$

Step 3 Take L_i and replace the symbols $a_{i1},\ a_{i2}, \ldots, a_{iv}$ by the elements of $\mathbf{h}_j$ successively and we get a $\mathbf{W}$-matrix. By varying $i,\ j$ we get $m(v-1)$ $\mathbf{W}$-matrices. We can easily check from the properties of MOLS and Hadamard matrices that these are optimum $\mathbf{W}$s. □

Remark 3.2.1 When $b = v = 2^p$, $p =$ integer, we have a complete set of MOLS of order v. Then we can construct $(b-1)(v-1)$ optimum $\mathbf{W}$-matrices. In this situation, it exhausts the error degrees of freedom in RBD model.

Example 3.2.1 We illustrate the above method of construction by citing an example. Take $b = v = 2^2$ and replacing a_{ij} by other suitable symbols, we write down the MOLS of order 4 as follows:

$$L_1 = \begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{pmatrix}, \quad L_2 = \begin{pmatrix} \alpha & \delta & \beta & \gamma \\ \beta & \gamma & \alpha & \delta \\ \gamma & \beta & \delta & \alpha \\ \delta & \alpha & \gamma & \beta \end{pmatrix}, \quad L_3 = \begin{pmatrix} p & s & r & q \\ q & r & s & p \\ s & p & q & r \\ r & q & p & s \end{pmatrix}.$$

We write $\mathbf{H}_4$ as

$$\mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} = (\mathbf{h}_1,\ \mathbf{h}_2,\ \mathbf{h}_3,\ \mathbf{1})\,. \tag{3.2.3}$$

Using $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{h}_3$ in L_1, we get the following three optimum $\mathbf{W}$-matrices:

$$\mathbf{W}^{(1)} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \ \mathbf{W}^{(2)} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \ \mathbf{W}^{(3)} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

Similarly, using $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{h}_3$ in L_2 and L_3 respectively, we get six more optimum $\mathbf{W}$-matrices as

$$\mathbf{W}^{(4)} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \ \mathbf{W}^{(5)} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}, \ \mathbf{W}^{(6)} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix},$$

$$\mathbf{W}^{(7)} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \ \mathbf{W}^{(8)} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}, \ \mathbf{W}^{(9)} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

Remark 3.2.2 When $b = pv$, $v = 0 \pmod 4$, $p \geq 1$, $\mathbf{H}_v$ and m MOLS of order v exist, then by writing the $\mathbf{W}$-matrices of order $v \times b$ side by side p times, we can get $m(v-1)$ optimum $\mathbf{W}$-matrices. If in addition $\mathbf{H}_p$ exists, then we can construct $pm(v-1)$ optimum $\mathbf{W}$-matrices. Below in Theorem 3.3.1 we provide non-trivial generalization of these results using mixed orthogonal arrays.

3.3 Relationship Between OCDs and MOAs

Orthogonal arrays (OA) introduced by Rao (1947) were generalized by Rao (1973) to Mixed orthogonal arrays (MOA) which have wide applications specially in the construction of designs. There are various results on constructions of OAs and MOAs. We refer to the books of Hedayat et al. (1999) and Dey and Mukerjee (1999) for details. Also a website of Sloane is available for ready reference and we also have a catalogue of potential sources on OAs and MOAs (cf. http://neilsloane.com/oadir/index.html). Definition of MOA (cf. Hedayat et al. (1999), p. 200) is given below:

Definition 3.3.1 An MOA$(N, s_1^{k_1} s_2^{k_2} \ldots s_v^{k_v}, t)$ is an array of size $k \times N$, where $k = \sum_{i=1}^{v} k_i$ is the total number of factors, in which the first k_1 rows have symbols from $\{0, 1, \ldots, s_1 - 1\}$, the next k_2 rows have symbols from $\{0, 1, \ldots, s_2 - 1\}$, and so on, with the property that in any $t \times N$ sub-array every t-tuple occurs an equal number of times as a column.

Rao et al. (2003) identifies the construction of OCDs to that of MOAs. In this chapter, we will discuss the relationship between the OCDs in the set-ups of CRD, RBD and MOAs. This was established in Rao et al. (2003).

We consider the following theorem given in Rao et al. (2003).

Theorem 3.3.1 *A set of c optimum $\mathbf{W}$-matrices of order $v \times b$ under the RBD set-up co-exist with an MOA$(vb, v \times b \times 2^c, 2)$.*

Proof For $i = 1, 2, \ldots, c$, let $\mathbf{W}^{(i)}$ matrix be written as

$$\mathbf{W}^{(i)\ v \times b} = \left(\mathbf{w}_1^{(i)}, \ \mathbf{w}_2^{(i)}, \ldots, \mathbf{w}_b^{(i)}\right)$$

where $\mathbf{w}_j^{(i)}$ is the jth column of $\mathbf{W}^{(i)}$. Now we consider an array $\mathbf{A}$ with $2 + c$ rows and vb columns where the first two rows of $\mathbf{A}$ form the following $2 \times vb$ sub-array

$$\begin{matrix} 1\ 2 \ldots v\ 1\ 2 \ldots v \ldots 1\ 2 \ldots v \\ 1\ 1 \ldots 1\ 2\ 2 \ldots 2 \ldots b\ b \ldots b \end{matrix} \tag{3.3.1}$$

corresponding to the vb level combinations of the treatment and block factors and the $(2+i)$th row of $\mathbf{A}$ is given by $\left(\mathbf{w}_1^{(i)\prime},\ \mathbf{w}_2^{(i)\prime}, \ldots, \mathbf{w}_b^{(i)\prime}\right)$, $i = 1, 2, \ldots, c$. Note that first row and second row of $\mathbf{A}$ have v and b symbols respectively and the remaining rows have two symbols $+1$ and -1. From the properties of optimum $\mathbf{W}$-matrices it can be easily proved that $\mathbf{A}$ is an MOA$(vb,\ v \times b \times 2^c,\ 2)$.

Conversely, given any MOA $(vb,\ v \times b \times 2^c,\ 2)$, we can take, without loss of generality, the first two rows in the form (3.3.1).

We construct $\mathbf{W}^{(i)}$-matrix by using the elements of $(i+2)$th row of $\mathbf{A}$, where $w_{m,m'}^{(i)}$, the (m, m')th element of $\mathbf{W}^{(i)}$ = the element in the $(i+2)$th row of $\mathbf{A}$ corresponding to the ordered pair (m, m') in the first and second rows of $\mathbf{A}$, $m \neq m' = 1, 2, \ldots, c$. □

Corollary 3.3.1 *A set of c optimum $\mathbf{W}$-matrices of order $v \times b$ under the CRD set-up co-exist with an MOA$(vb, v \times 2^c, 2)$.*

Proof Given a set of c optimum $\mathbf{W}$-matrices $\mathbf{W}^{(1)}, \mathbf{W}^{(2)}, \ldots, \mathbf{W}^{(c)}$ of order $v \times b$ under the CRD set-up, observe that in this situation, the column sums, corresponding to the blocks, of the $\mathbf{W}$-matrices need not be zero. Hence the array $\mathbf{A}$ in the above result without the second row can be shown to be a mixed orthogonal array MOA$(vb, v \times 2^c, 2)$. □

Remark 3.3.1 Theorem 3.3.1 and Corollary 3.3.1 help us to construct OCDs for the set-ups of CRDs and RBDs from the list of suitable orthogonal arrays.

3.4 Some Further Constructions of Optimum W-Matrices

In this subsection we exploit the properties of Hadamard matrices and conference matrices to construct OCDs in CRD and RBD set-ups.

Theorem 3.4.1 *If there exist $\mathbf{H}_b$ and $\mathbf{H}_v$, then $(b-1)(v-1)$ optimum $\mathbf{W}$-matrices can be constructed for an RBD with b blocks and v treatments.*

Proof Write

$$\mathbf{H}_v = \left(\mathbf{h}_1,\ \mathbf{h}_2,\ \ldots,\ \mathbf{h}_{v-1},\ \mathbf{1}\right) \tag{3.4.1}$$

and

$$\mathbf{H}_b = \left(\mathbf{h}_1^*,\ \mathbf{h}_2^*,\ \ldots,\ \mathbf{h}_{b-1}^*,\ \mathbf{1}\right). \tag{3.4.2}$$

Let us write

$$\mathbf{W}^{((b-1)(i-1)+j)} = \mathbf{h}_i \otimes \mathbf{h}_j^{*\prime},\quad i = 1, 2, \ldots, v-1,\ \ j = 1, 2, \ldots, b-1. \tag{3.4.3}$$

We can easily check that these **W**-matrices satisfy conditions $C_1 - C_3$ of (3.1.5) giving $c = (b-1)(v-1)$ OCDs. These **W**s exhaust the error degrees of freedom of the RBD. □

Remark 3.4.1 We note that, by Theorems 3.3.1 and 3.4.1 we can construct an MOA $(vb,\ v \times b \times 2^{(v-1)(b-1)},\ 2)$ from these $(v-1)(b-1)$ optimum **W**-matrices and conversely for given this MOA$(vb,\ v \times b \times 2^{(v-1)(b-1)},\ 2)$, we can also construct $(b-1)(v-1)$ optimum **W**-matrices, $\mathbf{H}_v$ and $\mathbf{H}_b$.

Corollary 3.4.1 *If there exist* $\mathbf{H}_b$ *and* $\mathbf{H}_v$*, then* $v(b-1)$ *optimum* **W**-*matrices can be constructed for an CRD with v treatments, each being replicated b times.*

Proof The matrices defined in Eq. (3.4.3) can also be treated as optimum **W**-matrices of CRD set-up considered in Corollary 3.4.1. In this situation, we can construct an additional number of $(b-1)$ optimum **W**-matrices for this CRD set-up given by

$$\mathbf{W}^{((b-1)(v-1)+j)} = \mathbf{1}_v \otimes \mathbf{h}_j^{*\prime}, \quad j = 1, 2, \ldots, b-1. \tag{3.4.4}$$

Thus in total, we get $v(b-1)$ optimum **W**-matrices for this CRD set-up. These exhaust the error degrees of freedom of the CRD. As stated Corollary 3.3.1 an MOA$(vb,\ v \times 2^c,\ 2)$ can be constructed from the above **W**-matrices in usual way. □

Example 3.4.1 Let $b = v = 4$. Consider $\mathbf{H}_4$ of (3.2.3). From (3.4.3), we can construct optimum **W**-matrices as follows:

$$\mathbf{W}^{(1)} = \mathbf{h}_1 \otimes \mathbf{h}_1' = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}; \quad \mathbf{W}^{(2)} = \mathbf{h}_1 \otimes \mathbf{h}_2' = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix};$$

$$\mathbf{W}^{(3)} = \mathbf{h}_1 \otimes \mathbf{h}_3' = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}; \quad \mathbf{W}^{(4)} = \mathbf{h}_2 \otimes \mathbf{h}_1' = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix};$$

$$\mathbf{W}^{(5)} = \mathbf{h}_2 \otimes \mathbf{h}_2' = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}; \quad \mathbf{W}^{(6)} = \mathbf{h}_2 \otimes \mathbf{h}_3' = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix};$$

$$\mathbf{W}^{(7)} = \mathbf{h}_3 \otimes \mathbf{h}_1' = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}; \quad \mathbf{W}^{(8)} = \mathbf{h}_3 \otimes \mathbf{h}_2' = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix};$$

$$\mathbf{W}^{(9)} = \mathbf{h}_3 \otimes \mathbf{h}_3' = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

Note that here $h_j^* = h_j$ for all j=1, 2, 3. Therefore, MOA(16, $4 \times 4 \times 2^9$, 2) can be constructed in the lines of Theorem 3.4.1:

$$\mathbf{A} = \begin{pmatrix}
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1
\end{pmatrix}.$$

The above **W**-matrices are also optimum in CRD set-up with 4 treatments each being replicated 4 times. However as mentioned in Corollary 3.4.1 three additional **W**-matrices can be constructed and these are given below:

$$\mathbf{W}^{(10)} = \mathbf{1}_3 \otimes \mathbf{h}_1' = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}; \quad \mathbf{W}^{(11)} = \mathbf{1}_3 \otimes \mathbf{h}_2' = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix};$$

$$\mathbf{W}^{(12)} = \mathbf{1}_3 \otimes \mathbf{h}_2' = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

The corresponding MOA(16, 4×2^{12}, 2) for the CRD set-up is given below:

$$\begin{pmatrix}
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}.$$

Theorem 3.4.2

(a) If $\mathbf{H}_{2b}$ *and* $\mathbf{H}_{\frac{v}{2}}$ *both exist, then* $v(b-1)$ *optimum* $\mathbf{W}$*-matrices can be constructed for a CRD with* v *treatments and* b *replications.*

(b) If $\mathbf{H}_{2b}$ *and* $\mathbf{H}_{\frac{v}{2}}$ *both exist, then* $(b-1)(v-1)-(b-2)$ *optimum* $\mathbf{W}$*-matrices can be constructed for an RBD with* b *blocks and* v *treatments.*

Proof of (a) Write $\mathbf{H}_{2b}$ as a $2b \times 2b$ matrix with the last column as $(1, 1, \ldots, 1)'$ and the last but one column as $(\mathbf{1}'_b, -\mathbf{1}'_b)'$. Further write $\mathbf{H}_{\frac{v}{2}}$ as a matrix with the last column as $\mathbf{1}'_{\frac{v}{2}}$. Let $\mathbf{H}^{**}_{2b}$ be a matrix of order $2b \times 2(b-1)$ obtained from $\mathbf{H}_{2b}$ by deleting the last two columns. It follows that in each column of $\mathbf{H}^{**}_{2b}$ both the top b elements and the bottom b elements have equal number of 1's and -1's. Now we construct a matrix $\mathbf{A}_1$ of order $v(b-1) \times vb$ as:

$$\mathbf{A}_1 = \mathbf{H}^{**\prime}_{2b} \otimes \mathbf{H}_{\frac{v}{2}}.$$

We convert $\mathbf{A}_1$ into an MOA$(vb,\ v \times 2^{v(b-1)},\ 2)$ by appending the row: $(1, 2, \ldots, b, 1, 2, \ldots, b, \ldots, 1, 2,\ \ldots, b)$ of length vb. This establishes the result via Corollary 3.3.1. □

Proof of (b) Let $\mathbf{H}^{*}_{\frac{v}{2}}$ be a matrix of order $\frac{v}{2} \times (\frac{v}{2}-1)$ obtained from $\mathbf{H}_{\frac{v}{2}}$ ignoring the last column consisting of all 1s. Now we construct $\mathbf{A}_2$ of order $(v-2)(b-1) \times vb$ as follows:

$$\mathbf{A}_2 = \mathbf{H}^{**\prime}_{2b} \otimes \mathbf{H}^{*\prime}_{\frac{v}{2}}.$$

From $\mathbf{A}_2$, we can construct an MOA$(vb,\ v \times b \times 2^{(v-2)(b-1)+1},\ 2)$ by adjoining three more rows; the first two rows are used for coordinatisation and third row is $\mathbf{1}'_{\frac{v}{2}} \otimes (1,\ -1) \otimes (-\mathbf{1}'_{\frac{b}{2}},\ \mathbf{1}'_{\frac{b}{2}})$. The proof follows from the method of construction of Theorem 3.3.1. □

Remark 3.4.2 Theorem 3.4.2(b) strengthens and generalises Theorem 4.3.4, p. 54 in Dey and Mukerjee (1999).

Example 3.4.2 Let $b = 6, v = 4$. We take $\mathbf{H}_{12}$ in accordance with the proof of Theorem 3.4.2:

$$\mathbf{H}_{12} = \begin{pmatrix} 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \end{pmatrix} = \left(\mathbf{H}^{**}_{12}, \mathbf{h}_{11}, \mathbf{1}\right),$$

and

$$\mathbf{H}_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = (\mathbf{H}_2^*, \mathbf{1}).$$

Therefore using $\mathbf{A}_1 = \mathbf{H}_{12}^{**\prime} \otimes \mathbf{H}_2$, we can construct MOA$(24,\ 4 \times 2^{20},\ 2)$ as described in Corollary 3.3.1. Using $\mathbf{A}_2 = \mathbf{H}_{12}^{**\prime} \otimes \mathbf{H}_2^{*\prime}$ and the row $(1, 1) \otimes (1, -1) \otimes (-1, -1, -1, 1, 1, 1) = (-1, -1, -1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, 1, 1, 1, 1, 1, 1, -1, -1, -1)$, we can construct MOA$(24, 4 \times 4 \times 2^{11},\ 2)$ as described in Theorem 3.3.1. Now it is routine task to construct optimum **W**-matrices for CRD and RBD from these MOAs.

For further construction of MOAs we need the concept of Conference Matrices which is introduced below (cf. Hedayat et al. (1999), p. 152).

Definition 3.4.1 A symmetric matrix $\mathbf{S}$ of order n with elements $+1$, -1 and 0 is said to be a conference matrix (CM) if it can expressed in the form

$$\mathbf{S} = \begin{pmatrix} 0 & \mathbf{1}_{n-1}' \\ \mathbf{1}_{n-1} & \mathbf{A} \end{pmatrix} \tag{3.4.5}$$

satisfying $\mathbf{S}\mathbf{S}' = (n-1)\mathbf{I}_n$.

In such a representation of $\mathbf{S}$ the matrix $\mathbf{A}$ in (3.4.5) is called the core matrix of the CM. It can be easily checked that this $\mathbf{A}$ satisfies the conditions

$$\mathbf{A}\mathbf{A}' = (n-1)\mathbf{I}_{n-1} - \mathbf{1}_{n-1}\mathbf{1}_{n-1}', \quad \mathbf{A} = \mathbf{A}' \text{ and } \mathbf{A}\mathbf{1}_{n-1} = \mathbf{0}.$$

Note that CMs are known to exist for the following values of n (cf. Wallis et al. (1972)):

(1) $n = p^s + 1$ where p is a prime and s is a positive integer such that $p^s \equiv 1 \pmod 4$.
(2) $n = (h-1)^2 + 1$ where h is the order of a Skew-Hadamard matrix.
(3) $n = (h-1)^\mu + 1$ where h is the order of a CM and $\mu > 0$ is an odd integer.

Set $n - 1 = p$ and let $\mathbf{A}$ be as in (3.4.5) and let $\mathbf{A}^*$ be the matrix of order $p^2 \times p^2$ obtained by taking the Kronecker product of $\mathbf{A}$ with itself. Define a matrix $\mathbf{X}$ as

$$\mathbf{X} = \mathbf{A}^* + \begin{pmatrix} \mathbf{J}_p - \mathbf{I}_p & -\mathbf{I}_p & \dots & -\mathbf{I}_p \\ -\mathbf{I}_p & \mathbf{J}_p - \mathbf{I}_p & \dots & -\mathbf{I}_p \\ \vdots & \vdots & \vdots & \vdots \\ -\mathbf{I}_p & -\mathbf{I}_p & \dots & \mathbf{J}_p - \mathbf{I}_p \end{pmatrix}$$

where each block is of order $p \times p$ and $\mathbf{J}_p = \mathbf{1}_p\mathbf{1}_p'$.

Theorem 3.4.3 $\mathbf{X}$ *is a core matrix of a CM of order* p^2.

The proof is given in the Appendix. In the following theorem we give a method of constructing OCDs from CMs.

Theorem 3.4.4

(a) *If $b \equiv 2$ (mod 4), $(b-1)$ is a prime or a prime power and $\mathbf{H}_v$ exists, then $c = v(b-1)$ optimum $\mathbf{W}$-matrices can be constructed for a CRD with v treatments and b replications.*
(b) *If $b \equiv 2$ (mod 4), $(b-1)$ is a prime or a prime power and $\mathbf{H}_v$ exists, then $c = (b-1)(v-1) - (b-2)$ optimum $\mathbf{W}$-matrices can be constructed for an RBD with b blocks and v treatments.*

Proof We will construct OCDs through the following steps.

Step I: We start with $\mathbf{S}$, a CM of order $(p+1) \times (p+1)$where $\mathbf{A} = (a_{ij})$ be the core matrix of $\mathbf{S}$ of order $p \times p$.
Step II: Define a matrix $\mathbf{B}$ of order $(p+1) \times p$ such that the (i, j)th element b_{ij} is given by

$$\begin{aligned} b_{ii} &= -\beta \ \text{ for } i = 1, 2, \ldots, p, \\ b_{ij} &= a_{ij}\alpha \ \text{ for } i, j = 1, 2, \ldots, p, \text{ and } i \neq j \\ b_{(p+1),j} &= \beta \quad \text{ for } j = 1, 2, \ldots, p, \end{aligned}$$

where α and β are elements satisfying $1.\alpha = \alpha;\ -1.\alpha = -\alpha;\ 1.\beta = \beta;\ -1.\beta = -\beta;\ \alpha.\alpha = (-\alpha).(-\alpha) = \beta.\beta = (-\beta).(-\beta) = c$ a constant; $\alpha.\beta = \beta.\alpha = -(\alpha).\beta = (-\beta).\alpha = \alpha.(-\beta) = \beta.(-\alpha) = 0$, $\alpha.(-\alpha) = (-\alpha).\alpha = \beta.(-\beta) = (-\beta).\beta = -c$.

Define another a matrix $\mathbf{C}$ of order $(p+1) \times p$ such that the (i, j)th element c_{ij} is given by

$$\begin{aligned} c_{ii} &= \alpha \quad \text{ for } i = 1, 2, \ldots, p, \\ c_{ij} &= a_{ij}\beta \ \text{ for } i, j = 1, 2, \ldots, p, \text{ and } i \neq j \\ c_{(p+1),j} &= -\alpha \ \text{ for } j = 1, 2, \ldots, p. \end{aligned}$$

Now we construct a matrix $\mathbf{D}$ of order $(p+1) \times 2p$ as

$$\mathbf{D} = (\mathbf{B} : \mathbf{C}).$$

It is observed that the columns of the matrix $\mathbf{D}$ of order $(p+1) \times 2p$ are orthogonal (cf. Theorem 3.4.5 in the Appendix).

Example 3.4.3 For $n = 6$, let the core matrix $\mathbf{A}$ be

$$\begin{pmatrix} 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 & -1 \\ -1 & 1 & 0 & 1 & -1 \\ -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix}.$$

From the definitions of **B**, **C** and **D**, we have

$$\mathbf{D} = (\mathbf{B} : \mathbf{C}) = \begin{pmatrix} -\beta & \alpha & -\alpha & -\alpha & \alpha & \alpha & \beta & -\beta & -\beta & \beta \\ \alpha & -\beta & \alpha & -\alpha & -\alpha & \beta & \alpha & \beta & -\beta & -\beta \\ -\alpha & \alpha & -\beta & \alpha & -\alpha & -\beta & \beta & \alpha & \beta & -\beta \\ -\alpha & -\alpha & \alpha & -\beta & \alpha & -\beta & -\beta & \beta & \alpha & \beta \\ \alpha & -\alpha & -\alpha & \alpha & -\beta & \beta & -\beta & -\beta & \beta & \alpha \\ \beta & \beta & \beta & \beta & \beta & -\alpha & -\alpha & -\alpha & -\alpha & -\alpha \end{pmatrix}. \tag{3.4.6}$$

Step III: By assumption, $\mathbf{H}_v$ exists and we write it as

$$\mathbf{H}_v = (\mathbf{h}_1,\ \mathbf{h}_2, \ldots, \mathbf{h}_{v-1},\ \mathbf{h}_v = \mathbf{1}_v)\,. \tag{3.4.7}$$

Take one pair, say $(\mathbf{h}_i, \mathbf{h}_j)$ and replace the two symbols α, β by $\mathbf{h}'_i, \mathbf{h}'_j$ respectively in the matrix **D** of order $b \times 2(b-1)$. Then each column of **D** will give a matrix of order $b \times v$ and so we can get $2(b-1)$ matrices using all the columns of **D** for the fixed pair $(\mathbf{h}_i, \mathbf{h}_j)$. Now using ith column of **D** and jth pair $(\mathbf{h}_{2j-1}, \mathbf{h}_{2j})$ of columns of $\mathbf{H}_v$, we get a matrix of order $b \times v$ which is denoted by $\mathbf{U}^{((i-1)v/2+j)}$. Now varying i over $1,\ 2, \ldots, 2(b-1)$ and j over $1,\ 2, \ldots, \frac{v}{2}$, we get $v(b-1)$ matrices $\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \ldots, \mathbf{U}^{(v(b-1))}$. We can easily check that $\mathbf{W}^{(1)}=\mathbf{U}^{(1)\,\prime}$, $\mathbf{W}^{(2)}=\mathbf{U}^{(2)\,\prime}, \ldots,$ $\mathbf{W}^{(v(b-1))}=\mathbf{U}^{(v(b-1))\,\prime}$ are $v(b-1)$ optimum **W**-matrices for CRD set-up.

However in RBD set-up, we cannot use the last column $\mathbf{h}_v$ as the sum of elements of the last column is not zero. So leaving it out we have only $\frac{v-2}{2}$ distinct pairs of columns $(\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{v-2})$ and an extra column, $\mathbf{h}_{v-1}$. By using these distinct pairs of columns we can construct $(b-1)(v-2)$ **W**-matrices from **D** in the same manner as described in above. Here we can also construct one more optimum **W**-matrix using the residual column $\mathbf{h}_{v-1}$ as

$$\mathbf{W}^{((b-1)(v-2)+1)\,\prime} = \begin{pmatrix} \mathbf{1}_{\frac{b}{2}} \\ -\mathbf{1}_{\frac{b}{2}} \end{pmatrix} \otimes \mathbf{h}'_{v-1}.$$

Therefore for RBD set-up, we can construct $(b-1)(v-2)+1$ optimum **W**-matrices in all. □

Now we illustrate the above method by considering the following example.

Example 3.4.4 Let $b = 6$ and $v = 4$. Then $\mathbf{H}_4$ is

$$\mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} = (\mathbf{h}_1,\ \mathbf{h}_2,\ \mathbf{h}_3,\ \mathbf{h}_4 = \mathbf{1}_4)\,.$$

Then take $\alpha = \mathbf{h}'_1$ and $\beta = \mathbf{h}'_2$ and using the first column of **D** of (3.4.6), we get the following **W**-matrix:

$$\begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Similarly, using the above methods we get other **W**-matrices for CRD and RBD.

Appendix

Proof of Theorem 3.4.3 It is observed that **X** is a symmetric matrix. Now (i, i)th block matrix of $\mathbf{XX}'$ is

$$\begin{aligned}
(\mathbf{XX}')_{ii} &= \sum_{k=1,\ k\neq i}^{p} (a_{ik}\mathbf{A} - \mathbf{I}_p)(a_{ki}\mathbf{A} - \mathbf{I}_p) + (a_{ii}\mathbf{A} + \mathbf{J}_p - \mathbf{I}_p)(a_{ii}\mathbf{A} + \mathbf{J}_p - \mathbf{I}_p) \\
&= \sum_{k=1\ k\neq i}^{p} (a_{ik}^2\mathbf{A}^2 - 2a_{ik}\mathbf{A} + \mathbf{I}) + (\mathbf{J}_p - \mathbf{I}_p)(\mathbf{J}_p - \mathbf{I}_p) \text{ since } a_{ij} = a_{ji} \text{ and } a_{ii} = 0\ \forall i \\
&= (p\mathbf{I}_p - \mathbf{J}_p)\sum_{k=1}^{p} a_{ik}^2 - 2\mathbf{A}\sum_{k=1}^{p} a_{ik} + (p-1)\mathbf{I}_p + p\mathbf{J}_p - 2\mathbf{J}_p + \mathbf{I}_p \\
&= (p\mathbf{I}_p - \mathbf{J}_p)(p-1) - 2\mathbf{A}.0 + (p-1)\mathbf{I}_p + (p-2)\mathbf{J}_p + \mathbf{I}_p \\
&= (p(p-1) + p)\mathbf{I}_p + (p-2)\mathbf{J}_p - (p-1)\mathbf{J}_p \\
&= p^2\mathbf{I}_p - \mathbf{J}_p
\end{aligned}$$

(i, j)th block matrix of $\mathbf{XX}'$ is

$$\begin{aligned}
(\mathbf{XX}')_{ij} &= \sum_{k=1,\ k\neq i,j}^{p} (a_{ik}\mathbf{A} - \mathbf{I}_p)(a_{kj}\mathbf{A} - \mathbf{I}_p) + (a_{ii}\mathbf{A} + \mathbf{J}_p - \mathbf{I}_p)(a_{ji}\mathbf{A} - \mathbf{I}_p) \\
&\qquad + (a_{ij}\mathbf{A} - \mathbf{I}_p)(a_{jj}\mathbf{A} + \mathbf{J}_p - \mathbf{I}_p) \\
&= \sum_{k=1,\ k\neq i,j}^{p} (a_{ik}a_{jk}\mathbf{A}^2 - a_{ik}\mathbf{A} - a_{jk}\mathbf{A} + \mathbf{I}_p + (\mathbf{J}_p - \mathbf{I}_p)(a_{ji}\mathbf{A} - \mathbf{I}_p) \\
&\quad + (a_{ij}\mathbf{A} - \mathbf{I}_p)(\mathbf{J}_p - \mathbf{I}_p) \\
&= -(p\mathbf{I}_p - \mathbf{J}_p) + a_{ij}\mathbf{A} + a_{ij}\mathbf{A} + (p-2)\mathbf{I}_p - \mathbf{J}_p - a_{ij}\mathbf{A} - \mathbf{I}_p - \mathbf{J}_p - a_{ij}\mathbf{A} + \mathbf{I}_p \\
&= -\mathbf{J}_p
\end{aligned}$$

Thus $\mathbf{XX}' = p^2\mathbf{I}_{p^2} - \mathbf{J}_{p^2}$. □

Theorem 3.4.5 *The columns of the matrix* **D** *are orthogonal.*

Proof The cross product of ith and jth elements of **B**,

$$\begin{aligned}
\sum_{k=1}^{p} b_{ki}b_{kj} &= \sum_{k=1,\ k\neq i,j}^{p} b_{ki}b_{kj} + b_{ii}b_{ij} + b_{ji}b_{jj} + b_{(p+1),i}b_{(p+1),j} \\
&= \sum_{k=1,\ k\neq i,j}^{p} (a_{ki}\alpha)(a_{kj}\alpha) + (-\beta)(a_{ij}\alpha) + (a_{ji}\alpha)(-\beta) + \beta.\beta \\
&= \sum_{k=1,\ k\neq i,j}^{p} a_{ki}a_{kj}(\alpha.\alpha) + a_{ij}((-\beta).\alpha) + a_{ji}(\alpha.(-\beta)) + c \\
&= c\sum_{k=1}^{p} a_{ki}a_{kj} + 0 + 0 + c \\
&= -c + c = 0.
\end{aligned}$$

Similarly, it can be shown that the columns of **C** are also orthogonal. Now we want to show that any column of **B** is orthogonal to any column of **C**. For this, we consider the cross product of ith column of **B** and jth column of **C**:

$$\begin{aligned}
\sum_{k=1}^{p} b_{ki}c_{kj} &= \sum_{k=1,\ k\neq i,j}^{p} b_{ki}c_{kj} + b_{ii}c_{ij} + b_{ji}c_{jj} + b_{(p+1),i}c_{(p+1),j} \\
&= \sum_{k=1,\ k\neq i,j}^{p} (a_{ki}\alpha)(a_{kj}\beta) + (-\beta)(a_{ij}\beta) + (a_{ji}\alpha)(\alpha) + \beta.(-\alpha) \\
&= \sum_{k=1,\ k\neq i,j}^{p} a_{ki}b_{kj}(\alpha.\beta) + a_{ij}((-\beta).\beta) + a_{ji}(\alpha.(-\alpha)) + 0 = 0 - c + c + 0 = 0
\end{aligned}$$

□

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Chapter 4
OCDs in Balanced Incomplete Block Design Set-Up

4.1 Introduction

A balanced incomplete block design (BIBD) as an arrangement of v treatments into b blocks each of k $(< v)$ treatments, satisfying the conditions:

1. Every symbol occurs at most once in each block.
2. Every treatment occurs in exactly r blocks.
3. Every pair of symbols occurs together in exactly λ blocks.

Let us consider a BIBD (b, v, r, k, lambda) satisfying (3.1.3) and (3.1.4) where $\mathbf{X}_{2d}$ has a similar structure as in RBD. But now the structure of $\mathbf{X}_{1d}$ is somewhat different. In the **W**-matrix corresponding to the incidence matrix of the said design the non-zero elements (± 1) appear only in the r positions in every row and the k positions in every column. So the situation is more complex than before in the sense that in the case of an RBD, we were to place ± 1's in all the vb cells of the **W**-matrices while here, we have to place ± 1 in the non-zero cells of the incidence matrix $\mathbf{N}^{v \times b}$. Thus, the construction of optimum **W**-matrix or equivalently the **Z**-matrix depends on the method of construction of the corresponding BIBD.

The elements of optimum **W**-matrices for a BIBD set-up should satisfy following conditions

$$\left.\begin{array}{l} \displaystyle\sum_{i=1}^{v} w_{ij}^{(s)} = 0 \;\forall j; \qquad\qquad \sum_{j=1}^{b} w_{ij}^{(s)} = 0 \;\forall i \\ \text{and} \\ \displaystyle\sum_{i=1}^{v}\sum_{j=1}^{b} w_{ij}^{(s)} w_{ij}^{(s')} = 0 \;\forall s \neq s. \end{array}\right\} \tag{4.1.1}$$

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Condition (4.1.1) can be presented schematically as

$$\mathbf{W}^{(s)} = \begin{array}{c|c|c} \text{Tr.} & \text{bl. no.} \rightarrow & \text{Row} \\ \downarrow & 1\ 2 \ldots b & \text{Totals} \\ 1 & & 0 \\ 2 & & 0 \\ \vdots & (\pm n_{ij}) & \vdots \\ v & & 0 \\ \hline \text{Column} & & \\ \text{Total} & 0\ 0 \ldots 0 & \end{array} \quad 1 \leq s \leq c \tag{4.1.2}$$

and

$$\mathbf{W}^{(s)v \times b} * \mathbf{W}^{(t)v \times b} = \begin{array}{c|c|l} \text{Tr.} & \text{bl. no.} \rightarrow & \\ \downarrow & 1\ 2 \ldots b & \\ 1 & & \\ 2 & & \\ \vdots & (\pm n_{ij}) & \\ v & & \\ \hline & & 0 = \sum_{i=1}^{b} \sum_{j=1}^{v} w_{ij}^{(s)} w_{ij}^{(s')} \end{array} \tag{4.1.3}$$

In the set-up of a binary design, we use the notations n_{ij}s to indicate the incidence pattern of the treatments across different blocks. Naturally, $n_{ij} = 1$ or 0 according as the $(i,\ j)$ combination is present or absent. When $n_{ij} = 1$, we need to ascertain the value of w_{ij} (+1 or –1). It is observed that the condition (4.1.1) does not require any other property of BIBD other than that it is proper, binary and equireplicate. So it follows that the above principle of constructing optimum **W**-matrices equally applies for any binary proper equireplicate block design (BPEBD) not necessarily for the BIBD set-up only. So in this context we shall consider also BPEBD set-up whenever necessary.

In a BIBD set-up, **W**-matrices of order $v \times b$ can be constructed from the incidence matrix of the BIBD by placing ± 1's in the non-zero r positions in every row and in the non-zero k positions in every column such that **W**-matrices satisfy condition (4.1.1). For the BIBD set-up, Das et al. (2003) initiated the construction of OCDs in the following experimental designs:

(i) some series of SBIBDs constructed through Bose's difference technique (cf. Bose 1939),
(ii) some BIBDs with repeated blocks.

Later Dutta (2004) dealt with a number of classical series of BIBDs having incidence matrices derived essentially through Bose's difference technique which was not covered in Das et al. (2003). With this, he covered a large class of existing

BIBDs. Again Dutta et al. (2007) considered the problem of OCDs for a series of complements of SBIBDs obtained through projective geometry.

It may be mentioned that in the series considered in Sect. 4.2, the layouts have cyclical pattern which simplified the choice of $\mathbf{W}$-matrices. But the series of SBIBDs considered in Sect. 4.3 does not possess the above cyclical property.

When $n \neq 0 \pmod 4$, it is impossible to find designs attaining minimum variance for estimated covariate parameters. Dutta et al. (2010) considered this problem and instead of using the criterion of attaining the lower bound (viz. $\frac{\sigma^2}{n}$) to the variance of each of the estimated covariate parameters γ, they found optimum designs with respect to covariate effects using D-optimality criterion retaining orthogonality with respect to treatment and block effect contrasts, where $n \equiv 2 \pmod 4$. We consider their work in Sect. 4.4.

4.2 BIBDs Through Bose's Difference Technique

In this section, we consider some series of BIBDs constructed by applying Bose's difference technique (Bose 1939) and present construction procedures given by Das et al. (2003) and Dutta (2004) for $\mathbf{W}$-matrices satisfying (4.1.2) and (4.1.3).

Theorem 4.2.1 *Suppose a SBIBD ($v = b,\ r = k,\ \lambda$) is obtained by applying Bose's difference technique and a Hadamard matrix $\mathbf{H}_k$ of order k exists. Then $(k-1)$ optimum $\mathbf{W}$-matrices can be constructed.*

Proof $\mathbf{H}_k$ exists by assumption and it can be represented as

$$\mathbf{H}_k = (\mathbf{h}_1,\ \mathbf{h}_2, \ldots, \mathbf{h}_{k-1},\ \mathbf{1})\,. \tag{4.2.1}$$

Without loss of generality take the initial block of SBIBD as the first block and transform it into the form of the first column vector of the incidence matrix. Then we replace the non-zero positions of this column vector successively by the elements of $\mathbf{h}_t$. This gives the first column vector of $\mathbf{W}^{(t)}$. Now we develop this column into the full form of $\mathbf{W}^{(t)}$ cyclically. If the above method is carried out for each of the vectors $\mathbf{h}_1,\ \mathbf{h}_2, \ldots, \mathbf{h}_{k-1}$, then we get $(k-1)$ $\mathbf{W}$-matrices. We can easily check that these $\mathbf{W}$-matrices satisfy condition (4.1.1) and are optimum. □

Example 4.2.1 Consider SBIBD (7, 4, 2) obtained by cyclical development of the initial block (0, 3, 5, 6) mod 7. Note that the first column of the incidence matrix is given by $\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}'$ and others are obtained by cyclic permutations of this column. As block size is 4 we consider the 3 columns of $\mathbf{H}_4$ viz. $\mathbf{h}_1' = \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}$, $\mathbf{h}_2' = \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}$ and $\mathbf{h}_3' = \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix}$ excluding $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}'$. Let us consider $\mathbf{h}_1$ and construct $\mathbf{W}^{(1)}$ by replacing the non-zero elements of the first column of $\mathbf{N}$ by the elements of $\mathbf{h}_1$ in that order and permute cyclically. $\mathbf{W}^{(1)}$ is given by

$$\mathbf{W}^{(1)} = \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}^{7\times 7}$$

and the corresponding column of $\mathbf{Z}$ is (1, −1, 1, −1, −1, 1, −1, 1, 1, −1, 1, −1, 1, −1, 1, −1, −1, 1, −1, 1, −1, 1, −1, 1, −1, 1, −1, 1).

Similarly, construct $\mathbf{W}^{(2)}$ and $\mathbf{W}^{(3)}$ by using $\mathbf{h}_2$ and $\mathbf{h}_3$ respectively and the corresponding columns of $\mathbf{Z}$ accordingly. It can be seen that all the conditions in (4.1.1) are satisfied by $\mathbf{W}$s. $\mathbf{Z}$ gives the OCD in the design format. Thus an OCD for three covariates is obtained.

If the blocks of such BIBD is repeated m times each where $\mathbf{H}_m$ exists then we can increase the number of covariates in the new BIBD with repeated blocks and the result is represented in following corollary.

Corollary 4.2.1 *Suppose an SBIBD* (b, r, λ) *is available as per the description in Theorem 4.2.1. Suppose further that* $\mathbf{H}_m$ *exists for some* m*. Then for the BIBD* $(v,\ B = mb,\ R = mr,\ k,\ \Lambda = m\lambda)$ *obtained by repeating the blocks of the SBIBD, we can construct* $c^* = m(k-1)$ *optimum* $\mathbf{W}$*-matrices.*

Proof Let us write $\mathbf{H}_m$ as

$$\mathbf{H}_m = (\mathbf{h}_1^*, \mathbf{h}_2^*, \ldots, \mathbf{h}_m^*) = (h_{rt}^*).$$

Denote the $\mathbf{W}^{(t)}$-matrices of Theorem 4.2.1 by $\mathbf{W}^{(t)}_{v\times b}$ and the required $\mathbf{W}$-matrices by $\mathbf{G}_{v\times B}$-matrices as follows:

$$\mathbf{G}^{(t,r)}_{v\times B} = \left(h^*_{1t}\mathbf{W}^{(t)}_{v\times b},\ h^*_{2t}\mathbf{W}^{(t)}_{v\times b}, \ldots, h^*_{mt}\mathbf{W}^{(t)}_{v\times b}\right) = \mathbf{h}_r^{*\prime} \otimes \mathbf{W}^{(t)}_{v\times b}. \tag{4.2.2}$$

It is now a routine task to verify the claim of the corollary. □

Example 4.2.2 Consider BIBD (7, 28, 16, 4, 8) obtained by repeating 4 times each of 7 blocks of SBIBD (7, 4, 2) of Example 4.2.1. $\mathbf{H}_4$ can be written as

$$\mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} = \left(\mathbf{h}_1^*,\ \mathbf{h}_2^*,\ \mathbf{h}_3^*,\ \mathbf{h}_4^*\right) = (h_{rt}^*).$$

Take $\mathbf{W}^{(1)}$ of Example 4.2.1 and the corresponding $\mathbf{G}$-matrices are as follows:

$$\mathbf{G}^{(1,1)}_{v\times B} = \left(\mathbf{W}^{(1)}_{7\times 7}, -\mathbf{W}^{(1)}_{7\times 7},\ \mathbf{W}^{(1)}_{7\times 7}, -\mathbf{W}^{(1)}_{7\times 7}\right);$$
$$\mathbf{G}^{(1,2)}_{v\times B} = \left(\mathbf{W}^{(1)}_{7\times 7}, -\mathbf{W}^{(1)}_{7\times 7}, -\mathbf{W}^{(1)}_{7\times 7},\ \mathbf{W}^{(1)}_{7\times 7}\right);$$
$$\mathbf{G}^{(1,3)}_{v\times B} = \left(\mathbf{W}^{(1)}_{7\times 7},\ \mathbf{W}^{(1)}_{7\times 7}, -\mathbf{W}^{(1)}_{7\times 7}, -\mathbf{W}^{(1)}_{7\times 7}\right);$$
$$\mathbf{G}^{(1,4)}_{v\times B} = \left(\mathbf{W}^{(1)}_{7\times 7},\ \mathbf{W}^{(1)}_{7\times 7},\ \mathbf{W}^{(1)}_{7\times 7},\ \mathbf{W}^{(1)}_{7\times 7}\right).$$

Similarly, we construct other $\mathbf{G}$-matrices using other $\mathbf{W}$-matrices of Example 4.2.1 and the columns of $\mathbf{H}_4$.

Remark 4.2.1 If a BIBD $(v,\ mv,\ mk,\ k,\ \lambda)$ is formed by developing m initial blocks each of size k, then $m(k-1)$ optimum $\mathbf{W}$-matrices can be constructed whenever $\mathbf{H}_m$ and $\mathbf{H}_k$ exist. The result follows by noting that the above principle may be applied when the blocks are not repeated but are obtained by developing m initial blocks.

Remark 4.2.2 Let for a BIBD (v, b, r, k, λ) t optimum $\mathbf{W}$-matrices be available. Then for the BIBD $(V = v, B = mb, R = mr, K = k, \Lambda = m\lambda)$ obtained by repeating each block m times, mt optimum $\mathbf{W}$-matrices can be constructed whenever $\mathbf{H}_m$ exists. A similar but a more general result is discussed in Chap. 6.

When a BIBD is not necessarily cyclic, we can always accommodate $c^* = k-1$ covariates optimally if each block of the design is repeated twice and $\mathbf{H}_k$ exists.

Theorem 4.2.2 *Suppose a BIBD* (v, b, r, k, λ) *exists which is not necessarily cyclic. Then if* $\mathbf{H}_k$ *exists, we can construct* $c^* = k-1$ *optimum* $\mathbf{W}$*-matrices for the BIBD* $(V = v, B = 2b, R = 2r, K = k, \Lambda = 2\lambda)$.

Proof Let $\mathbf{N}^{v\times b}$ denote the incidence matrix of the former BIBD. Let $\mathbf{H}_k = (\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{k-1}, \mathbf{1})$. In order to construct $\mathbf{W}^{(t)}_{V\times B}$-matrix, we fill up the non-empty positions in $\mathbf{N}^{v\times b}$, the incidence matrix, by placing the elements of $\mathbf{h}_t$ successively in each column and in the order the positions appear. We denote the resultant matrix as $\mathbf{W}_t^{v\times b}$. Then

$$\mathbf{W}^{(t)}_{V\times B} = (\mathbf{W}_t^{v\times b}, -\mathbf{W}_t^{v\times b}).$$

It is now easy to assert the claim. □

Now we consider some other series of BIBDs which are not necessarily symmetric but are constructed by Bose's difference technique and give the constructional method of OCDs as given in Dutta (2004). At first we consider the complementary designs of the Steiner's triple system (cf. Bose 1939) obtained by difference technique

$$\mathrm{BIBD}(v = 3(2t+1), b = (3t+1)(2t+1), r = 3t+1, k = 3, \lambda = 1). \quad (4.2.3)$$

Theorem 4.2.3 *Let* t *be an even positive integer such that* $2t+1$ *be a prime number or a prime power and further let* $\mathbf{H}_{2t}$ *and* $\mathbf{H}_{6t}$ *exist. Then we can construct* $(2t-1)$ *optimum* $\mathbf{W}$*-matrices for the following complementary design of (4.2.3)*

$$\text{BIBD}(v' = 3(2t + 1),\ b' = (3t + 1)(2t + 1),\ r' = 2t(3t + 1),\ k' = 6t,$$
$$\lambda' = (3t + 1)(2t - 1) + 1). \tag{4.2.4}$$

Proof Let 0, 1,...,2t be the elements of GF $(2t + 1)$. To each element a of GF $(2t+1)$, we associate three symbols 1, 2, 3 to have three treatments $a_1,\ a_2,\ a_3$. It is well known that the initial blocks for the series (4.2.3) are given by (cf. Bose 1939, p. 373)

$$S_1' = \{(1_1, (2t)_1, 0_2),\ (2_1, (2t-1)_1, 0_2), \ldots, (t_1, (t+1)_1, 0_2)\};$$
$$S_2' = \{(1_2, (2t)_2, 0_3),\ (2_2, (2t-1)_2, 0_3), \ldots, (t_2, (t+1)_2, 0_3)\};$$
$$S_3' = \{(1_3, (2t)_3, 0_1),\ (2_3, (2t-1)_3, 0_1), \ldots, (t_3, (t+1)_3, 0_1)\};$$
$$S_4' = (0_1, 0_2, 0_3).$$

We divide the initial blocks of the design (4.2.4) which is the complementary design of (4.2.3) into the following four sets:

$$S_1 = \{(0_1, 2_1, 3_1, \ldots, (2t-1)_1, 1_2, 2_2, \ldots, (2t)_2, 0_3, 1_3, \ldots, (2t)_3),\ (0_1, 1_1, 3_1, \ldots, (2t-2)_1, (2t)_1, 1_2, 2_2, \ldots, (2t)_2, 0_3, 1_3, \ldots, (2t)_3), \ldots, (0_1, 1_1, \ldots, (t-1)_1, (t+2)_1, \ldots, (2t)_1, 1_2, 2_2, \ldots, (2t)_2, 0_3, 1_3, \ldots, (2t)_3)\};$$
$$S_2 = \{(0_2, 2_2, 3_2, \ldots, (2t-1)_2, 1_3, 2_3, \ldots, (2t)_3, 0_1, 1_1, \ldots, (2t)_1),\ (0_2, 1_2, 3_2, \ldots, (2t-2)_2, (2t)_2, 1_3, 2_3, \ldots, (2t)_3, 0_1, 1_1, \ldots, (2t)_1), \ldots, (0_2, 1_2, \ldots, (t-1)_2, (t+2)_2, \ldots, (2t)_2, 1_3, 2_3, \ldots, (2t)_3, 0_1, 1_1, \ldots, (2t)_1)\};$$
$$S_3 = \{(0_3, 2_3, 3_3, \ldots, (2t-1)_3, 1_1, 2_1, \ldots, (2t)_1, 0_2, 1_2, \ldots, (2t)_2),\ (0_3, 1_3, 3_3, \ldots, (2t-2)_3, (2t)_3, 1_1, 2_1, \ldots, (2t)_1, 0_2, 1_2, \ldots, (2t)_2), \ldots, (0_3, 1_3, \ldots, (t-1)_3, (t+2)_3, \ldots, (2t)_3, 1_2, 2_2, \ldots, (2t)_1, 0_2, 1_2, \ldots, (2t)_2)\};$$
$$S_4 = \{(1_1, 2_1, \ldots, (2t)_1, 1_2, 2_2, \ldots, (2t)_2, 1_3, 2_3, \ldots, (2t)_3)\}.$$

Let us assume the existence of $\mathbf{H}_{k'}$, where $k' = 6t$ and write it as

$$\mathbf{H}_{k'} = (\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{k'-1}, \mathbf{1}). \tag{4.2.5}$$

Consider the first $\frac{t}{2}$ initial blocks of the set $S_i (i = 1, 2, 3)$ and display them in the form of column vectors of the incidence matrix. Let us replace the non-zero elements of the jth column by the elements of $\mathbf{h}_s$, where $\mathbf{h}_s$ is any one of the first $(2t - 1)$ columns of $\mathbf{H}_{k'}$. We develop this initial block by cyclically permuting the elements to form a matrix $\mathbf{U}_{is}^{j}$ of order $v' \times (2t+1)$, $j = 1, 2, \ldots, \frac{t}{2}$. Using the same procedure we transform $(\frac{t}{2}+j)$th block of S_i by -$\mathbf{h}_s$ and develop in the same manner. We denote this matrix of order $v' \times (2t+1)$ by $\mathbf{U}_{is}^{\frac{t}{2}+j}$ $(i = 1, 2, 3,\ j = 1, 2, \ldots, \frac{t}{2})$. In this way we can construct $\mathbf{U}_{is}^{j}$ and $\mathbf{U}_{is}^{\frac{t}{2}+j}$ $(j = 1, 2, \ldots, \frac{t}{2})$ for different s $(s = 1, 2, \ldots, (2t-1))$. Again, for an even integer t, we assume that $\mathbf{H}_{2t}$ exists and write it as

$$\mathbf{H}_{2t} = (\mathbf{h}_1^*, \mathbf{h}_2^*, \ldots, \mathbf{h}_{2t-1}^*, \mathbf{1}). \tag{4.2.6}$$

Consider the single initial block S_4. Note that the $2t$ elements except zero of S_4 correspond to each of the symbols 1, 2 and 3. Transform the elements of this block into the form of a column vector of the incidence matrix. Then we replace the non-zero elements of each class of this column by the elements of $\mathbf{h}_s^*$ $(s = 1, 2, \ldots, (2t - 1))$ and develop this column into full form of $\mathbf{U}_s^{(4)}$ as

$$\mathbf{U}_s^{(4)\prime} = (\mathbf{V}_{1s}^{(4)\prime},\ \mathbf{V}_{2s}^{(4)\prime},\ \mathbf{V}_{3s}^{(4)\prime}), \tag{4.2.7}$$

where, for $i = 1, 2, 3$, $\mathbf{V}_{is}^{(4)}$ $(i = 1, 2, 3)$ is a matrix of order $(2t + 1) \times (2t + 1)$ obtained by cyclical permutation of the elements of the column vector after replacing the non-zero elements of ith class of the initial block of S_4 by $\mathbf{h}_s^*$.

Schematically, the form of the $\mathbf{W}^{(s)}$-matrix, $s = 1, 2, \dots, 2t - 1$, can be written as

$$\mathbf{W}^{(s)\prime} = \left[\begin{array}{c}
\left.\begin{array}{l}
\left.\begin{array}{l} \mathbf{U}_{1s}^{(1)\prime\ (2t+1)\times v'} \\ \vdots \\ \mathbf{U}_{1s}^{(\frac{t}{2})\prime\ (2t+1)\times v'} \end{array}\right\} \text{Using } \mathbf{h}_s \\
\left.\begin{array}{l} \mathbf{U}_{1s}^{(\frac{t}{2}+1)\prime\ (2t+1)\times v'} \\ \vdots \\ \mathbf{U}_{1s}^{(t)\prime\ (2t+1)\times v'} \end{array}\right\} \text{Using } -\mathbf{h}_s
\end{array}\right] = S_1 \\ \hline
\left.\begin{array}{l}
\left.\begin{array}{l} \mathbf{U}_{2s}^{(1)\prime\ (2t+1)\times v'} \\ \vdots \\ \mathbf{U}_{2s}^{(\frac{t}{2})\prime\ (2t+1)\times v'} \end{array}\right\} \text{Using } \mathbf{h}_s \\
\left.\begin{array}{l} \mathbf{U}_{2s}^{(\frac{t}{2}+1)\prime\ (2t+1)\times v'} \\ \vdots \\ \mathbf{U}_{2s}^{(t)\prime\ (2t+1)\times v'} \end{array}\right\} \text{Using } -\mathbf{h}_s
\end{array}\right] = S_2 \\ \hline
\left.\begin{array}{l}
\left.\begin{array}{l} \mathbf{U}_{3s}^{(1)\prime\ (2t+1)\times v'} \\ \vdots \\ \mathbf{U}_{3s}^{(\frac{t}{2})\prime\ (2t+1)\times v'} \end{array}\right\} \text{Using } \mathbf{h}_s \\
\left.\begin{array}{l} \mathbf{U}_{3s}^{(\frac{t}{2}+1)\prime\ (2t+1)\times v'} \\ \vdots \\ \mathbf{U}_{3s}^{(t)\prime\ (2t+1)\times v'} \end{array}\right\} \text{Using } -\mathbf{h}_s
\end{array}\right] = S_3 \\ \hline
\mathbf{U}_s^{(4)\prime} = \left[\begin{array}{ccc} \mathbf{V}_{1s}^{(4)\prime} & \mathbf{V}_{2s}^{(4)\prime} & \mathbf{V}_{3s}^{(4)\prime} \\ \uparrow & \uparrow & \uparrow \\ \text{Using } \mathbf{h}_s^* & \text{Using } \mathbf{h}_s^* & \text{Using } \mathbf{h}_s^* \end{array}\right]
\end{array}\right]$$

So varying $s = 1, 2, \dots, 2t - 1$, we can construct $(2t - 1)$ optimum $\mathbf{W}$-matrices. This establishes the claim. □

We shall illustrate the construction through the following example.

Example 4.2.3 For $t = 2$, BIBD (15, 35, 28, 12, 22) is the complementary design of BIBD (15, 35, 7, 3, 1). Thus we have three optimum $\mathbf{W}$-matrices, each of order 15×35. We exhibit the construction in detail.

Note that the four sets are:

$S_1 = \{(0_1, 2_1, 3_1, 1_2, 2_2, 3_2, 4_2, 0_3, 1_3, 2_3, 3_3, 4_3),\ (0_1, 1_1, 4_1, 1_2, 2_2, 3_2, 4_2, 0_3, 1_3, 2_3, 3_3, 4_3)\}$;
$S_2 = \{(0_2, 2_2, 3_2, 1_3, 2_3, 3_3, 4_3, 0_1, 1_1, 2_1, 3_1, 4_1),\ (0_2, 1_2, 4_2, 1_3, 2_3, 3_3, 4_3, 0_1, 1_1, 2_1, 3_1, 4_1)\}$;
$S_3 = \{(0_3, 2_3, 3_3, 1_1, 2_1, 3_1, 4_1, 0_2, 1_2, 2_2, 3_2, 4_2),\ (0_3, 1_3, 4_3, 1_1, 2_1, 3_1, 4_1, 0_2, 1_2, 2_2, 3_2, 4_2)\}$;
$S_4 = \{(1_1, 2_1, 3_1, 4_1, 1_2, 2_2, 3_2, 4_2, 1_3, 2_3, 3_3, 4_3)\}$;

$\mathbf{H}_{12}$ is available in standard literature (cf. Hedayat et al. 1999, p. 151). Without loss of generality, we take

$$\mathbf{h}_1 = (-1, 1, 1, -1, 1, 1, 1, -1, -1, -1, 1, -1)';$$

$$\mathbf{h}_2 = (-1, -1, 1, 1, -1, 1, 1, 1, -1, -1, -1, 1)';$$

$$\mathbf{h}_3 = (-1, 1, -1, 1, 1, -1, 1, 1, 1, -1, -1, -1)'.$$

$\mathbf{H}_{2t} = \mathbf{H}_4$ can be written as

$$\mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} = (\mathbf{h}_1^*, \mathbf{h}_2^*, \mathbf{h}_3^*, \mathbf{1}).$$

Then proceeding along the steps described in Theorem 4.2.3, we obtain $\mathbf{U}_{11}^{(1)}$, $\mathbf{U}_{11}^{(2)}$, $\mathbf{U}_{21}^{(1)}$, $\mathbf{U}_{21}^{(2)}$, $\mathbf{U}_{31}^{(1)}$ and $\mathbf{U}_{31}^{(1)}$ each of order 15×5, where

$$\mathbf{U}_{11}^{(1)\prime} = \begin{array}{c} \begin{array}{ccccccccccccccc} 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 0_3 & 1_3 & 2_3 & 3_3 & 4_3 \end{array} \\ \begin{pmatrix} -1 & 0 & 1 & 1 & 0 & 0 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 0 & -1 & 0 & 1 & 1 & 1 & 0 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 & 1 & 1 & 0 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 & 1 & 1 & 1 & 0 & -1 & -1 & 1 & -1 & -1 \end{pmatrix} \end{array},$$

$$\mathbf{U}_{11}^{(2)\prime} = \begin{array}{c} \begin{array}{ccccccccccccccc} 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 0_3 & 1_3 & 2_3 & 3_3 & 4_3 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 0 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 & 0 & -1 & 0 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & -1 & 0 & -1 & -1 & 0 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & -1 & -1 & -1 & -1 & 0 & 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & -1 & 0 & 1 & 1 & -1 & 1 & 1 \end{pmatrix} \end{array},$$

$$\mathbf{U}_{21}^{(1)\prime} = \begin{pmatrix} -1 & 1 & 1 & -1 & 1 & 1 & 0 & 1 & -1 & 0 & 0 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & 1 & 1 & -1 & 0 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & -1 \\ 1 & 1 & -1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 1 & -1 & 0 \end{pmatrix},$$

$$
\mathbf{U}_{21}^{(2)\prime} = \begin{pmatrix}
1 & -1 & -1 & 1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 1 & 1 & -1 & 1 \\
-1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & 0 & 1 & -1 & -1 & 0 & -1 & 1 & 0 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 1 \\
-1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 1 & -1 & 1 & 0
\end{pmatrix},
$$

$$
\mathbf{U}_{31}^{(1)\prime} = \begin{pmatrix}
0 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 1 & -1 & 0 \\
-1 & 0 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 0 & -1 & 0 & 1 & -1 \\
1 & -1 & 0 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 0 & -1 & 0 & 1 \\
1 & 1 & -1 & 0 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 0 & -1 & 0 \\
-1 & 1 & 1 & -1 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & 1 & -1 & 0 & -1
\end{pmatrix}
$$

and

$$
\mathbf{U}_{31}^{(2)\prime} = \begin{array}{c}
\begin{array}{ccccccccccccccc} 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 0_3 & 1_3 & 2_3 & 3_3 & 4_3 \end{array} \\
\begin{pmatrix}
0 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 0 & 0 & 1 \\
1 & 0 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 0 & 1 & 1 & -1 & 0 \\
-1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & 0 & -1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 1
\end{pmatrix}
\end{array}
$$

Using $\mathbf{h}_1^*$, the matrices $\mathbf{V}_{11}^{(4)}$, $\mathbf{V}_{21}^{(4)}$ and $\mathbf{V}_{31}^{(4)}$ each of order 5×5 are obtained as

$$
\mathbf{V}_{11}^{(4)\prime} = \begin{array}{c}
\begin{array}{ccccc} 0_1 & 1_1 & 2_1 & 3_1 & 4_1 \end{array} \\
\begin{pmatrix}
0 & 1 & 1 & -1 & -1 \\
-1 & 0 & 1 & 1 & -1 \\
-1 & -1 & 0 & 1 & 1 \\
1 & -1 & -1 & 0 & 1 \\
1 & 1 & -1 & -1 & 0
\end{pmatrix}
\end{array}, \quad
\mathbf{V}_{21}^{(4)\prime} = \begin{array}{c}
\begin{array}{ccccc} 0_2 & 1_2 & 2_2 & 3_2 & 4_2 \end{array} \\
\begin{pmatrix}
0 & 1 & 1 & -1 & -1 \\
-1 & 0 & 1 & 1 & -1 \\
-1 & -1 & 0 & 1 & 1 \\
1 & -1 & -1 & 0 & 1 \\
1 & 1 & -1 & -1 & 0
\end{pmatrix}
\end{array},
$$

$$
\mathbf{V}_{31}^{(4)\prime} = \begin{array}{c}
\begin{array}{ccccc} 0_3 & 1_3 & 2_3 & 3_3 & 4_3 \end{array} \\
\begin{pmatrix}
0 & 1 & 1 & -1 & -1 \\
-1 & 0 & 1 & 1 & -1 \\
-1 & -1 & 0 & 1 & 1 \\
1 & -1 & -1 & 0 & 1 \\
1 & 1 & -1 & -1 & 0
\end{pmatrix}
\end{array}.
$$

Thus $\mathbf{W}^{(1)}$ is obtained by suitably arranging the $\mathbf{U}$ and $\mathbf{V}$-matrices as

$$
\mathbf{W}^{(1)\prime\, 35\times 15} = (\mathbf{U}_{11}^{(1)\prime}, \mathbf{U}_{11}^{(2)\prime}, \mathbf{U}_{21}^{(1)\prime}, \mathbf{U}_{21}^{(2)\prime}, \mathbf{U}_{31}^{(1)\prime}, \mathbf{U}_{31}^{(2)\prime}, \mathbf{U}_{1}^{(4)\prime}),
$$

where

$$
\mathbf{U}_{1}^{(4)\prime} = (\mathbf{V}_{11}^{(4)\prime}, \mathbf{V}_{21}^{(4)\prime}, \mathbf{V}_{31}^{(4)\prime}).
$$

Similarly, $\mathbf{W}^{(2)}$ and $\mathbf{W}^{(3)}$ can be constructed by using $(\mathbf{h}_2, \mathbf{h}_2^*)$ and $(\mathbf{h}_3, \mathbf{h}_3^*)$ respectively.

Dutta (2004) also constructed OCDs for the following series of BIBD. For detailed discussion, readers are referred to the original paper.

$$v' = 5(4t+1),\ b' = (5t+1)(4t+1),\ r' = 4t(5t+1),\ k' = 20t,\ \lambda' = (5t+1)(4t-1)+1, \tag{4.2.8}$$

and

$$v' = 4(3t+1),\ b' = (4t+1)(3t+1),\ r' = 3t(4t+1),\ k' = 12t,\ \lambda' = (4t+1)(3t-1)+1. \tag{4.2.9}$$

4.3 BIBDs Through Projective Geometry

As mentioned earlier in the series considered in Sect. 2.2, the layouts had cyclical patterns which simplified the choice of optimum $\mathbf{W}$-matrices. Now we consider complementary designs of the SBIBDs obtained through projective geometry. However, by suitable partition of the blocks into different sets, and by judicious choice of the covariate values, it is possible to construct OCDs for the series with parameters $v' = b' = s^N + s^{N-1} + \cdots + s + 1$, $r' = k' = s^N$, $\lambda' = s^N - s^{N-1}$.

4.3.1 Partitioning of the Blocks

With the help of the Galois field GF (s), we can construct the finite projective geometry of N dimensions, to be written as PG $(N,\ s)$, where, $s = p^n$, p is a prime number and n is any positive integer. Any ordered set of $(N+1)$ elements $(x_0, x_1, \ldots, x_N)$ where the x_i's belong to GF (s) and are not simultaneously zero, is called a point of the projective geometry PG (N, s). $(x_0, x_1, \ldots, x_N) = \mathbf{x}'$ and $\rho\mathbf{x}'$ represent the same point, where $\rho(\neq 0) \in$GF (s). It is known that the number of points in PG (N, s) is equal to $\phi(N, m, s)$, where $\phi(N, m, s) = \frac{(s^{N+1}-1)(s^N-1)\ldots(s^{N-m+1}-1)}{(s^{m+1}-1)(s^m-1)\ldots(s-1)}$. For more detailed discussions in this respect one is referred to Bose (1939).

By making a correspondence between the points and the m-flats of PG (N, s) with the varieties and the blocks respectively, we get a BIBD with parameters (cf. Bose 1939, p. 362): $v = \phi(N, 0, s)$, $b = \phi(N, m, s)$, $r = \phi(N-1, m-1, s)$, $k = \phi(m, 0, s)$, $\lambda = \phi(N-2, m-2, s)$. For $m = N-1$, the following SBIBD is obtained:

$$\begin{aligned} v = b = s^N + s^{N-1} + \cdots + s + 1, \quad r = k = s^{N-1} + s^{N-2} + \cdots + s + 1 \\ \lambda = s^{N-1} + s^{N-2} + \cdots + s + 1 \end{aligned} \tag{4.3.1}$$

We consider the complementary design given in (4.3.1) which is also an SBIBD with the following parameters:

$$v' = b' = s^N + s^{N-1} + \cdots + s + 1,\ r' = k' = s^N,\ \lambda = s^N - s^{N-1}. \quad (4.3.2)$$

It was mentioned earlier that the choice of the levels of the covariates in a BIBD set-up depends on the method of construction of the BIBD and the maximum number of covariates satisfying condition (3.1.5) varies from series to series. The blocks of the SBIBD with parameters given in (4.3.2) are partitioned into $(s^{N-1} + s^{N-3} + \cdots + s^2 + 1)(=t$, say) disjoint sets; each set contains $(s + 1)$ blocks such that the portion of the incidence matrix of the complementary design corresponding to these $(s + 1)$ sets conforms to that of the incidence matrix of an RBD with suitable parameters. This fact has been used in the choice of the **Z**-matrix.

We note that the number of $(N-1)$-flats passing through a particular $(N-2)$-flat is given by $\phi(1, 0, s) = s+1$. Such $(s+1)$, $(N-1)$-flats passing through a particular $(N-2)$-flat can be obtained as follows:

Consider an $(N-2)$-flat, given by

$$\mathbf{a}'\mathbf{x} = 0,\quad \mathbf{b}'\mathbf{x} = 0 \quad (4.3.3)$$

where, **a** and **b** are two column vectors with elements from GF (s) such that rank $(\mathbf{A}') = \text{rank}(\mathbf{a}, \mathbf{b}) = 2$.

The $(s+1)$, $(N-1)$-flats containing the $(N-2)$-flat given in (4.3.3) are given by $(\lambda_1\mathbf{a}' + \lambda_2\mathbf{b}')\mathbf{x} = 0$; $(\lambda_1,\ \lambda_2) \neq (0, 0)$ and $(\lambda_1,\ \lambda_2) \equiv \rho(\lambda_1,\ \lambda_2)$ where, ρ is a non-zero element of GF (s). If N is odd, then the full set of $\phi(N, N-1, s)$, $(N$-1)-flats can be partitioned into $\frac{s^{N+1}-1}{(s-1)(s+1)} = \frac{\phi(N,N-1,s)}{s+1} = (s^{N-1} + s^{N-3} + \cdots + s^2 + 1)$ sets each containing $(s+1)$, $(N-1)$-flats having a common $(N-2)$-flat, are disjoint. As the blocks correspond to $(N-1)$-flats, so through one to one correspondence, we can partition the blocks into $(s^{N-1} + s^{N-3} + \cdots + s^2 + 1)$ disjoint sets each containing $(s + 1)$ blocks. It will be clear from the following two examples from Dutta et al. (2007) covering both the situations where s is a prime number and a prime power.

Example 4.3.1 $N = 3,\ m = 2,\ s = 2$. There are 15 blocks which can be partitioned into five sets each of size 3:

$$S_1 : \begin{array}{l} x_0 = 0 \\ x_1 + x_2 = 0 \\ x_0 + x_1 + x_2 = 0 \end{array} \qquad S_2 : \begin{array}{l} x_1 = 0 \\ x_0 + x_3 = 0 \\ x_0 + x_1 + x_3 = 0 \end{array} \qquad S_3 : \begin{array}{l} x_2 = 0 \\ x_1 + x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{array}$$

$$S_4 : \begin{array}{l} x_3 = 0 \\ x_0 + x_2 = 0 \\ x_0 + x_2 + x_3 = 0 \end{array} \qquad S_5 : \begin{array}{l} x_0 + x_1 = 0 \\ x_2 + x_3 = 0 \\ x_0 + x_1 + x_2 + x_3 = 0. \end{array}$$

It is to be noted that only two equations in each set S_i are independent and these can conveniently be represented as $\mathbf{Ax} = \mathbf{0}$. It is clear that the choice of $\mathbf{A}$-matrix in S_1 is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

The choice of $\mathbf{A}$-matrices for other S's are obvious.

Example 4.3.2 $N = 3$, $m = 2$ and $s = 2^2$. There are 85 blocks which can be partitioned into 17 sets each of size 5. Let the elements of GF (2^2) be $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = x$, $\alpha_3 = 1 + x$, where x is a primitive root of GF (2^2). Then the 17 sets are:

$$S_1 : \begin{array}{l} x_0 = 0 \\ x_1 = 0 \\ x_0 + x_1 = 0 \\ x_0 + \alpha_2 x_1 = 0 \\ x_0 + \alpha_3 x_1 = 0 \end{array} \quad S_2 : \begin{array}{l} x_2 = 0 \\ x_3 = 0 \\ x_2 + x_3 = 0 \\ x_2 + \alpha_2 x_3 = 0 \\ x_2 + \alpha_3 x_3 = 0 \end{array} \quad S_3 : \begin{array}{l} x_0 + x_2 = 0 \\ x_1 + x_3 = 0 \\ x_0 + x_1 + x_2 + x_3 = 0 \\ x_0 + \alpha_2 x_1 + x_2 + \alpha_2 x_3 = 0 \\ x_0 + \alpha_3 x_1 + x_2 + \alpha_3 x_3 = 0 \end{array}$$

$$S_4 : \begin{array}{l} x_0 + \alpha_2 x_2 = 0 \\ x_1 + \alpha_3 x_3 = 0 \\ x_0 + x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \\ x_0 + \alpha_2 x_1 + \alpha_2 x_2 + x_3 = 0 \\ x_0 + \alpha_3 x_1 + \alpha_2 x_2 + \alpha_2 x_3 = 0 \end{array} \quad S_5 : \begin{array}{l} x_0 + \alpha_3 x_2 = 0 \\ x_1 + \alpha_2 x_3 = 0 \\ x_0 + x_1 + \alpha_3 x_2 + \alpha_2 x_3 = 0 \\ x_0 + \alpha_2 x_1 + \alpha_3 x_2 + \alpha_3 x_3 = 0 \\ x_0 + \alpha_3 x_1 + \alpha_3 x_2 + x_3 = 0 \end{array}$$

$$S_6 : \begin{array}{l} x_0 + x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_0 + x_1 + x_2 = 0 \\ x_0 + \alpha_2 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \\ x_0 + \alpha_3 x_1 + \alpha_3 x_2 + \alpha_2 x_3 = 0 \end{array} \quad S_7 : \begin{array}{l} x_0 + \alpha_2 x_3 = 0 \\ x_1 + \alpha_3 x_2 + \alpha_2 x_3 = 0 \\ x_0 + x_1 + \alpha_3 x_2 = 0 \\ x_0 + \alpha_2 x_1 + x_2 + x_3 = 0 \\ x_0 + \alpha_3 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \end{array}$$

$$S_8 : \begin{array}{l} x_0 + \alpha_3 x_3 = 0 \\ x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \\ x_0 + x_1 + \alpha_2 x_2 = 0 \\ x_0 + \alpha_2 x_1 + \alpha_3 x_2 + \alpha_2 x_3 = 0 \\ x_0 + \alpha_3 x_1 + x_2 + x_3 = 0 \end{array} \quad S_9 : \begin{array}{l} x_0 + x_2 + x_3 = 0 \\ x_1 + x_2 = 0 \\ x_0 + x_1 + x_3 = 0 \\ x_0 + \alpha_2 x_1 + \alpha_3 x_2 + x_3 = 0 \\ x_0 + \alpha_3 x_1 + \alpha_2 x_2 + x_3 = 0 \end{array}$$

$$S_{10} : \begin{array}{l} x_0 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \\ x_1 + \alpha_2 x_2 = 0 \\ x_0 + x_1 + \alpha_3 x_3 = 0 \\ x_0 + \alpha_2 x_1 + x_2 + \alpha_3 x_3 = 0 \\ x_0 + \alpha_3 x_1 + \alpha_3 x_2 + \alpha_3 x_3 = 0 \end{array} \quad S_{11} : \begin{array}{l} x_0 + \alpha_3 x_2 + \alpha_2 x_3 = 0 \\ x_1 + \alpha_3 x_2 = 0 \\ x_0 + x_1 + \alpha_2 x_3 = 0 \\ x_0 + \alpha_2 x_1 + \alpha_2 x_2 + \alpha_2 x_3 = 0 \\ x_0 + \alpha_3 x_1 + x_2 + \alpha_2 x_3 = 0 \end{array}$$

$$S_{12}: \begin{array}{l} x_0+\alpha_3x_2+\alpha_3x_3=0\\ x_1+\alpha_2x_2+x_3=0\\ x_0+x_1+x_2+\alpha_2x_3=0\\ x_0+\alpha_2x_1+x_3=0\\ x_0+\alpha_3x_1+\alpha_2x_2=0 \end{array} \qquad S_{13}: \begin{array}{l} x_0+\alpha_2x_2+\alpha_2x_3=0\\ x_1+\alpha_3x_2+x_3=0\\ x_0+x_1+x_2+\alpha_3x_3=0\\ x_0+\alpha_2x_1+\alpha_3x_2=0\\ x_0+\alpha_3x_1+x_3=0 \end{array}$$

$$S_{14}: \begin{array}{l} x_0+x_2+\alpha_3x_3=0\\ x_1+\alpha_2x_2+\alpha_2x_3=0\\ x_0+x_1+\alpha_3x_2+x_3=0\\ x_0+\alpha_2x_1+\alpha_2x_2=0\\ x_0+\alpha_3x_1+\alpha_2x_3=0 \end{array} \qquad S_{15}: \begin{array}{l} x_0+x_2+\alpha_2x_3=0\\ x_1+\alpha_3x_2+\alpha_3x_3=0\\ x_0+x_1+\alpha_2x_2+x_3=0\\ x_0+\alpha_2x_1+\alpha_3x_3=0\\ x_0+\alpha_3x_1+\alpha_3x_2=0 \end{array}$$

$$S_{16}: \begin{array}{l} x_0+\alpha_2x_2+x_3=0\\ x_1+x_2+\alpha_2x_3=0\\ x_0+x_1+\alpha_3x_2+\alpha_3x_3=0\\ x_0+\alpha_2x_1+\alpha_2x_3=0\\ x_0+\alpha_3x_1+x_2=0 \end{array} \qquad S_{17}: \begin{array}{l} x_0+\alpha_3x_2+x_3=0\\ x_1+x_2+\alpha_3x_3=0\\ x_0+x_1+\alpha_2x_2+\alpha_2x_3=0\\ x_0+\alpha_2x_1+x_2=0\\ x_0+\alpha_3x_1+\alpha_3x_3=0 \end{array}$$

where, (x_0, x_1, x_2, x_3) is a point of PG $(3, 2^2)$.

As an illustration, the choice of **A**-matrix corresponding to S_1 and S_4 are given, respectively, by

$$\begin{pmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \alpha_2 & 0 \\ 0 & 1 & 0 & \alpha_3 \end{pmatrix}.$$

Similarly, **A**-matrices for other S_i's can be written.

4.3.2 Optimum Covariate Designs

From (4.3.1), we see that any block of the design contains $k = (s^{N-1} + \lambda)$ treatments and any two blocks have exactly λ treatments in common. As any two blocks of the set S_i $(i = 1, 2, \ldots, t;\ t = (s^{N-1} + s^{N-3} + \cdots + s^2 + 1))$, have the same λ treatments in common, without loss of any generality, we can write the portion $\mathbf{N}_i$, the incidence matrix corresponding to the blocks in S_i $(i = 1, 2, \ldots, t)$ in the following form (with some rearrangement of blocks if necessary):

$$\mathbf{N}_i' = \begin{pmatrix} \mathbf{1}'_{s^{N-1}} & \mathbf{0}' & \ldots & \mathbf{0}' & \mathbf{1}'_\lambda \\ \mathbf{0}' & \mathbf{1}'_{s^{N-1}} & \ldots & \mathbf{0}' & \mathbf{1}'_\lambda \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}' & \mathbf{0}' & \ldots & \mathbf{1}'_{s^{N-1}} & \mathbf{1}'_\lambda \end{pmatrix}^{(s+1)\times v}. \tag{4.3.4}$$

The part of the incidence matrix of the design with parameters in (4.3.2) corresponding to the part $\mathbf{N}_i$ of the design with parameters in (4.3.1) is obtained by replacing ones by zeros and zeros by ones in (4.3.4) and is given by:

$$\mathbf{N}_i^{c\,\prime} = \begin{pmatrix} \mathbf{0}' & \mathbf{1}'_{s^{N-1}} & \dots & \mathbf{1}'_{s^{N-1}} & \mathbf{0}'_\lambda \\ \mathbf{1}'_{s^{N-1}} & \mathbf{1}'_{s^{N-1}} & \dots & \mathbf{1}'_{s^{N-1}} & \mathbf{0}'_\lambda \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{1}'_{s^{N-1}} & \mathbf{1}'_{s^{N-1}} & \dots & \mathbf{0}' & \mathbf{0}'_\lambda \end{pmatrix}^{(s+1)\times v} . \tag{4.3.5}$$

Using the structure (4.3.5) above, we develop a method for choosing covariates optimally for the series of complementary designs of (4.3.1). The precise statement follows.

Theorem 4.3.1 *If $s = 2^p$ where p is any positive integer, then $(s^{N-1} - 1)(s-1) + (s-1)$ optimum $\mathbf{W}$-matrices can be constructed for the design with parameters in (4.3.2), where N is an odd integer.*

Proof Since s is a power of 2, $\mathbf{H}_{s^{N-1}}$ and $\mathbf{H}_s$ exist and we write them as

$$\mathbf{H}_{s^{N-1}} = \left(\mathbf{h}_1, \dots, \mathbf{h}_{s^{N-1}-1}, \mathbf{1}\right)$$

$$\mathbf{H}_s = \left(\mathbf{h}_1^*, \dots, \mathbf{h}_{s-1}^*, \mathbf{1}\right).$$

Again, the matrix (4.3.5) can be written as

$$\mathbf{N}_i^{c\,\prime} = \left(\mathbf{A}_{1i}, \mathbf{A}_{2i}, \dots, \mathbf{A}_{ji}, \dots, \mathbf{A}_{(s+1)i}, \mathbf{0}_i\right)$$

where $\mathbf{A}_{ji}$ is the matrix in the jth column block of $\mathbf{N}_i^{c\,\prime}$, $j = 1, 2, \dots, (s+1)$. We replace kth non-null row of $\mathbf{A}_{ji}$ by the kth row of $\mathbf{h}_m^*\mathbf{h}_n'$; $k = 1, 2, \dots, s$, $m = 1, 2, \dots, (s-1)$ and $n = 1, 2, \dots, (s^{N-1} - 1)$ and denote the resultant matrix by $\mathbf{A}_{ji}^*$. We repeat the procedure for each $\mathbf{A}_{ji}$ with the same m, n. This leads to a matrix $\mathbf{W}_{i;m,n}^*$ with elements ± 1 satisfying the properties $\mathbf{C}_1$ and $\mathbf{C}_2$ of condition (3.1.5). Using the same $\mathbf{h}_m$ and $\mathbf{h}_n^*$ we get different $\mathbf{W}_{i;m,n}^*$'s corresponding to different $\mathbf{N}_i^{c\prime}$'s. Therefore, for fixed m, n

$$\mathbf{W}_{m,n}^* = \left(\mathbf{W}_{1;m,n}^{*\prime}, \mathbf{W}_{2;m,n}^{*\prime}, \dots, \mathbf{W}_{t;m,n}^{*\prime}\right)$$

satisfies the properties C_1 and C_2 of condition (3.1.5). For different choices of $\mathbf{h}_m$ and $\mathbf{h}_n^*$ we get $(s^{N-1} - 1)(s-1)$, $\mathbf{W}_{m,n}^*$-matrices which satisfy condition (3.1.5). The transformation required to be applied on (4.3.5) to get back the corresponding portion of the incidence matrix of the design may also be applied on the elements of the above $\mathbf{W}^*$-matrices to get the original $\mathbf{W}$-matrices.

Again, note that the number of unit vectors in the rows of $\mathbf{N}_i^{c\,\prime}$ is s which is the same as that of the elements of $\mathbf{h}_m^*$. We replace the qth vector $\mathbf{1}'_{s^{N-1}}$ in the first column block matrix of $\mathbf{N}_i^{c\,\prime}$ by $+\mathbf{1}'_{s^{N-1}}$ or by $-\mathbf{1}'_{s^{N-1}}$ according as the qth element of $\mathbf{h}_m^*$ is +1 or –1, respectively, to get $\mathbf{A}_1^{**}$. Now we permute $+\mathbf{1}'_{s^{N-1}}$, $-\mathbf{1}'_{s^{N-1}}$ and $\mathbf{0}'_{s^{N-1}}$

in the rows of $\mathbf{A}_1^{**}$ cyclically to get $\mathbf{A}_2^{**}, \mathbf{A}_3^{**}, \ldots, \mathbf{A}_{s+1}^{**}$ and hence can construct a new $\mathbf{W}$-matrix viz. $\mathbf{W}_m^{**}$. By taking different $\mathbf{h}_m^*$, we can construct $(s-1)$, $\mathbf{W}_m^{**}$-matrices. It is easy to show that these $\mathbf{W}_m^{**}$-matrices together with the $\mathbf{W}_{m,n}^*$-matrices, $m = 1, 2, \ldots, (s-1)$, $n = 1, 2, \ldots, (s^{N-1}-1)$ satisfy condition (3.1.5). Thus in all, we get $(s^{N-1}-1)(s-1) + (s-1)$ optimum $\mathbf{W}$-matrices. □

Example 4.3.3 We consider the SBIBD whose blocks are the 2-flats of PG (3,2), so that the parameters of the SBIBD are $v = b = 15$, $r = k = 7$, $\lambda = 3$. Now for the complementary design, the parameters are: $v' = b' = 15,\ r' = k' = 8, \lambda' = 4$.

According to Example 4.3.1, the sets of blocks of the complementary design, where the treatment corresponding to the point (x_0, x_1, x_2, x_3) is indexed by $2^3 x_0 + 2^2 x_1 + 2x_2 + x_3$, are:

$$
\begin{aligned}
S_1 &= [(8, 9, 10, 11, 12, 13, 14, 15), (2, 3, 4, 5, 10, 11, 12, 13), (2, 3, 4, 5, 8, 9, 14, 15)]\\
S_2 &= [(4, 5, 6, 7, 12, 13, 14, 15), (1, 3, 5, 7, 8, 10, 12, 14), (1, 3, 4, 6, 8, 10, 13, 15)]\\
S_3 &= [(2, 3, 6, 7, 10, 11, 14, 15), (1, 3, 4, 6, 9, 11, 12, 14), (1, 2, 4, 7, 9, 10, 12, 15)]\\
S_4 &= [(1, 3, 5, 7, 9, 11, 13, 15), (2, 3, 6, 7, 8, 9, 12, 13), (1, 2, 5, 6, 8, 11, 12, 15)]\\
S_5 &= [(4, 5, 6, 7, 8, 9, 10, 11), (1, 2, 5, 6, 9, 10, 13, 14), (1, 2, 4, 7, 8, 11, 13, 14)]
\end{aligned}
$$

We write $\mathbf{H}_2$ and $\mathbf{H}_4$ as

$$
\mathbf{H}_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = (\mathbf{h}_1,\ \mathbf{1}) \ \text{ and } \ \mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} = (\mathbf{h}_1^*, \mathbf{h}_2^*, \mathbf{h}_3^*, \mathbf{1}).
$$

Using $\mathbf{h}_1$ and $\mathbf{h}_i^*$ $(i = 1, 2, 3)$ and proceeding according to Theorem 4.3.1 we can construct three optimum $\mathbf{W}$-matrices. Below we give $\mathbf{W}_{1,1}^*$-matrix which is constructed by using $\mathbf{h}_1$ and $\mathbf{h}_1^*$.

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\
0 & -1 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\
-1 & 0 & 1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\
1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\
-1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\
0 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\
-1 & -1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 \\
-1 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0
\end{pmatrix}'.
$$

Similarly, by taking the combinations $(\mathbf{h}_1, \mathbf{h}_2^*)$ and $(\mathbf{h}_1, \mathbf{h}_3^*)$ we can construct $\mathbf{W}_{1,2}^*$ and $\mathbf{W}_{1,3}^*$ respectively. Using $\mathbf{h}_1$, we can get another matrix $\mathbf{W}_1^{**}$ which is given below:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}'.$$

Thus four optimum $\mathbf{W}$-matrices are constructed.

Remark 4.3.1 In Sects. 4.2 and 4.3, OCDs have been constructed for BIBD set-ups. The series of BIBDs considered here are either constructed through Bose's method of difference (cf. Bose 1939) or through projective geometry. As mentioned earlier, it is very difficult to find OCDs for arbitrary BIBDs. But for the particular case when $b = mv$, where m is any positive integer, OCDs can be constructed for arbitrary BIBDs. More generally, in such situation, OCDs can be constructed for any BPEBD which will be considered in Chap. 6. The class of BPEBDs contains cyclic designs which also contain a number of BIBDs. Though the method described in Chap. 6 covers a large class of BIBDs, but the methods applied in these sections are illustrative and important in their own merit.

4.4 D-Optimal Covariate Designs in Block Design Set-Up

The optimal designs considered in previous sections of this chapter are necessarily D-optimal. But such designs cannot always be obtained because of the restriction $n \equiv 0 \pmod 4$. When $n \not\equiv 0 \pmod 4$, finding optimal design is very difficult. Dutta et al. (2010) consider D-optimal design in this set-up when $n \equiv 2 \pmod 4$. In this case of a block design for given b and v, the reduced normal equation for estimation of γ is given by

$$(\mathbf{Z}'\mathbf{Q}\mathbf{Z})\gamma = \mathbf{Z}'\mathbf{Q}\mathbf{y}$$

which yields

$$\hat{\gamma} = (\mathbf{Z}'\mathbf{Q}\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Q}\mathbf{y}$$

where

$$\mathbf{Q} = (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'), \quad \mathbf{X} = (\mathbf{X}_1,\ \mathbf{X}_2).$$

Hence, the information matrix for γ is given by $\mathbf{I}(\gamma) = \mathbf{Z}'\mathbf{Q}\mathbf{Z}$. Since $\mathbf{Q}$ is non-negative definite, it follows that

$$\mathbf{Z}'\mathbf{Q}\mathbf{Z} \le \mathbf{Z}'\mathbf{Z} \text{ (in Lowener order sense; Pukelsheim 1993)}$$

'=' if and only if $\mathbf{Z}'\mathbf{X} = \mathbf{0}$, i.e., if and only if

$$\mathbf{Z}'\mathbf{X}_1 = \mathbf{0}, \ \ \mathbf{Z}'\mathbf{X}_2 = \mathbf{0}. \tag{4.4.1}$$

Thus the problem is that of selecting $\mathbf{Z}$-matrix with $|z_{ij}^{(t)}| \le 1$ satisfying (4.4.1) such that the covariate design is D-optimal, i.e., $det(\mathbf{Z}'\mathbf{Z})$ is maximum when $\mathbf{Z} \in \mathcal{Z}$, $\mathcal{Z} = \{\mathbf{Z} : z_{ij}^{(t)} \in [-1, 1]\ \forall\, i, j\}$.

4.4.1 Conditions for D-Optimality

We have already observed that when $n \equiv 2 \pmod 4$, it is impossible to estimate γ-components most efficiently in the sense of attaining the lower bound $\frac{\sigma^2}{n}$ to the variance of the estimated covariate parameters. Thus, in the case $n \equiv 2 \pmod 4$, the problem is that of choosing a matrix $\mathbf{Z}^{n\times c} = (z_{ij}^{(t)})$ with $z_{ij}^{(t)} \in [-1, 1]\ \forall\, i, j$ such that $det(\mathbf{Z}'\mathbf{Z})$ is a maximum subject to the orthogonality condition (4.4.1). Towards this, we state the following lemma giving a necessary condition for maximization of $det(\mathbf{Z}'\mathbf{Z})$, $\mathbf{Z} \in \mathcal{Z}$ (cf. Galil and Kiefer 1980; Wojtas 1964).

Lemma 4.4.1 *A necessary condition for maximization of $det(\mathbf{Z}'\mathbf{Z})$ where $\mathbf{Z} \in \mathcal{Z}$, is that $z_{ij}^{(t)} = \pm 1\ \forall\, i, j, t$.*

From the above lemma, it is clear that we can restrict to the class $\mathcal{Z}^* = \{\mathbf{Z} : z_{ij}^{(t)} = \pm 1\ \forall\, i, j, t\}$ for finding the D-optimum design. In this direction, we have the following theorem.

Theorem 4.4.1 *A covariate design $\mathbf{Z}^* \in \mathcal{Z}^*$ is D-optimal in the sense of maximizing $det(\mathbf{Z}'\mathbf{Z})$ subject to the condition (4.4.1), if it satisfies*

$$\mathbf{Z}^{*\prime}\mathbf{Z}^* = (n-2)\mathbf{I}_c + 2\mathbf{J}_c \tag{4.4.2}$$

where $\mathbf{I}_c$ *is the identity matrix of order c and* $\mathbf{J}_c$ *is the matrix of order c with all elements equal to unity.*

Proof Because of Lemma 4.4.1, we can restrict to the class $\mathcal{Z}^*$ for maximization of $det(\mathbf{Z}'\mathbf{Z})$. For any $\mathbf{Z} \in \mathcal{Z}^*$, we can write

$$det(\mathbf{Z}'\mathbf{Z}) = det \begin{pmatrix} n & s_{12} & \dots & s_{1c} \\ s_{12} & n & \dots & s_{2c} \\ \vdots & \vdots & \vdots & \vdots \\ s_{1c} & s_{2c} & \dots & n \end{pmatrix}, \tag{4.4.3}$$

where $s_{tt'} = \sum_i \sum_j z_{ij}^{(t)} z_{ij}^{(t')}$, $t \neq t' = 1, 2, \dots, c$. Because of (4.4.1), each column of $\mathbf{Z}$ is orthogonal to $\mathbf{1}_n$, and hence orthogonality of any pair of columns of $\mathbf{Z}$ implies that $n \equiv 0 \pmod 4$ which violates our assumption that $n \equiv 2 \pmod 4$. So, no off-diagonal element of $\mathbf{Z}'\mathbf{Z}$ can be zero. From Wojtas (1964) the determinant in (4.4.2) is maximum if all s_{ij}'s are equal to s, where

$$0 \leq s \leq \min_{i \neq j} |s_{ij}|. \tag{4.4.4}$$

As $z_{ij}^{(t)} = \pm 1$ and $n \equiv 2 \pmod 4$, $|s_{ij}|$ can not be equal to 0 or 1 $\forall i \neq j$. Therefore, the minimum value of $|s_{ij}|$ is 2. So the theorem is proved. $\square$

Now we can represent any column of $\mathbf{Z}^*$ (which is a column vector of order $n \times 1$) in the form of a matrix $\mathbf{U}^{v \times b}$ corresponding to the $v \times b$ incidence matrix of the block design.

With the conditions (4.4.1) and (4.4.2) in terms of $\mathbf{U}$-matrix, the conditions reduce to:

$$\left.\begin{array}{ll} C_1. & \text{Each } \mathbf{U}\text{-matrix has all column-sums equal to zero;} \\ C_2. & \text{Each } \mathbf{U}\text{-matrix has all row-sums equal to zero;} \\ C_3. & \text{The grand total of all the entries in the Hadamard product} \\ & \text{of any two distinct } \mathbf{U}\text{-matrices reduces to 2.} \end{array}\right\} \tag{4.4.5}$$

4.4.2 Construction of the D-Optimal Covariate Design in a SBIBD Set-Up

In Sect. 4.4.1, we have established that a $\mathbf{Z}$-matrix is D-optimal subject to condition (4.4.1) if it satisfies (4.4.2). Now in a BIBD set-up, the $\mathbf{U}$-matrices defined in Sect. 4.4.1 can be constructed by suitably replacing the non-zero elements of the incidence matrix of BIBD by ± 1 such that the conditions in (4.4.5) are satisfied. Here, we consider the series of irreducible SBIBD (cf. Raghavarao 1971) with parameters

$v = b,\ r = k = v - 1,\ \lambda = v - 2$, where $k \equiv 2 \pmod 4$, b is an odd integer. To start with, we consider the following lemma which gives a method of construction for particular value of the parameters viz. $v = b = 7,\ r = k = 6$ and $\lambda = 5$. This will help understand the method for the general case.

Lemma 4.4.2 *Three* $\mathbf{U}$*-matrices can be constructed for the irreducible SBIBD with parameters* $v = b = 7,\ r = k = 6,\ \lambda = 5$.

Proof Without loss of generality the incidence matrix $\mathbf{N}^{7\times 7}$ can be written in the following partitioned form:

$$\mathbf{N} = \begin{pmatrix} 0&1&1&1&1&1&1\\ 1&0&1&1&1&1&1\\ 1&1&0&1&1&1&1\\ 1&1&1&0&1&1&1\\ 1&1&1&1&\boxed{0}&1&1\\ 1&1&1&1&1&0&1\\ 1&1&1&1&1&1&0 \end{pmatrix}. \tag{4.4.6}$$

Let us denote the 5×5 top left-hand matrix by $\mathbf{N}_{11}$; the 4×2 top right-hand matrix by $\mathbf{N}_{12}$; the 2×4 bottom left-hand matrix by $\mathbf{N}_{21}$; the 3×3 bottom right-hand matrix by $\mathbf{N}_{22}$. Then, we can write

$$\mathbf{N}_{11} = \begin{pmatrix} 0&1&1&1&1\\ 1&0&1&1&1\\ 1&1&0&1&1\\ 1&1&1&0&1\\ 1&1&1&1&0 \end{pmatrix},\quad \mathbf{N}_{21} = \mathbf{N}'_{12} = \begin{pmatrix} 1&1&1&1\\ 1&1&1&1 \end{pmatrix},\quad \mathbf{N}_{22} = \begin{pmatrix} 0&1&1\\ 1&0&1\\ 1&1&0 \end{pmatrix}. \tag{4.4.7}$$

We see that '0' the element of 5th row and 5th column is common to both $\mathbf{N}_{11}$ and $\mathbf{N}_{22}$. We shall see later on that this particular element always remains static in this position during the process of construction. Such bordering of an element which is common both in $\mathbf{N}_{11}$ and $\mathbf{N}_{22}$ does not, in any way, hamper the construction of optimum $\mathbf{Z}$-matrix. Consider a Hadamard matrix $\mathbf{H}_4$ of order 4, where the first two columns are $\mathbf{h}_1 = (1, -1, 1, -1)'$ and $\mathbf{h}_2 = (1, 1, -1, -1)'$. Now we replace the non-zero elements of the first column of $\mathbf{N}_{11}$ by the elements of $\mathbf{h}_1$ and through cyclical development of this column we generate $\mathbf{U}_{11}^{(1)}$ of order 5×5 as

$$\mathbf{U}_{11}^{(1)} = \begin{pmatrix} 0&-1&1&-1&1\\ 1&0&-1&1&-1\\ -1&1&0&-1&1\\ 1&-1&1&0&-1\\ -1&1&-1&1&0 \end{pmatrix}. \tag{4.4.8}$$

Again with $\mathbf{h}_2$, we generate another matrix $\mathbf{U}_{11}^{(2)}$ in the same way. It can be checked that the row sums and column sums of each of $\mathbf{U}_{11}^{(1)}$ and $\mathbf{U}_{11}^{(2)}$ are equal to zero and the sum of all elements of the Hadamard product of these two matrices also vanishes. Next, by replacing the non-zero elements in the first column of $\mathbf{N}_{22}$ in (4.4.7) by $(1, -1)$, we get a column vector $(0, 1, -1)'$. By cyclically permutation of this column, we generate a 3×3 matrix $\mathbf{U}_{22}$ where

$$\mathbf{U}_{22} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}. \tag{4.4.9}$$

Finally, we construct three 7×7 matrices $\mathbf{U}_1$, $\mathbf{U}_2$ and $\mathbf{U}_3$ corresponding to the incidence matrix $\mathbf{N}$, by replacing the matrices $\mathbf{N}_{11}$, $\mathbf{N}_{12}$, $\mathbf{N}_{21}$ and $\mathbf{N}_{22}$ in (4.4.7), respectively, by:

(a) $\mathbf{U}_{11}^{(1)}, \mathbf{U}_{12}^{(1)}, \mathbf{U}_{12}^{(1)\prime}, \mathbf{U}_{22}$;
(b) $\mathbf{U}_{11}^{(2)}, \mathbf{U}_{12}^{(2)}, \mathbf{U}_{12}^{(1)\prime}$, -$\mathbf{U}_{22}$; and
(c) -$\mathbf{U}_{11}^{(2)}, \mathbf{U}_{12}^{(2)}, \mathbf{U}_{12}^{(1)\prime}$, -$\mathbf{U}_{22}$.

Thus, finally, corresponding to (a)–(c) above, we have the following three $\mathbf{U}$-matrices:

$$\mathbf{U}_1 = \left(\begin{array}{ccccc|cc} 0 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 0 & -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 0 & -1 & 1 \\ \hline 1 & -1 & 1 & -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 0 \end{array}\right), \tag{4.4.10}$$

$$\mathbf{U}_2 = \left(\begin{array}{ccccc|cc} 0 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 0 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 0 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 0 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 1 & 1 & -1 & 0 \end{array}\right), \tag{4.4.11}$$

$$\mathbf{U}_3 = \left(\begin{array}{ccccc|cc} 0 & 1 & 1 & -1 & -1 & 1 & -1 \\ -1 & 0 & 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 0 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 1 & 1 & -1 & 0 \end{array}\right), \tag{4.4.12}$$

It can be easily checked that $\mathbf{U}_1$, $\mathbf{U}_2$ and $\mathbf{U}_3$ satisfy all of condition (4.4.5) and these constitute the required D-optimal covariate design. □

Theorem 4.4.2 If a Hadamard matrix of order $(v-7)$ exists, then we can construct three **U**-matrices for an irreducible SBIBD $(v = b,\ r = k = v-1,\ \lambda = v-2)$ where k is 2 (mod 4), $k > 6$.

Proof As in (4.4.6), we partition the incidence matrix **N** as

$$\mathbf{N} = \left(\begin{array}{ccccc|cccccc} 0 & 1 & \ldots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \ldots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \ldots & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cline{5-11} 1 & 1 & \ldots & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & \vdots & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \vdots & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & \vdots & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & \vdots & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & \vdots & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & \vdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right). \tag{4.4.13}$$

As in Lemma 4.4.2, we denote the $(v-6) \times (v-6)$ top left-hand matrix by $\mathbf{N}^*_{11}$; the $(v-7) \times 6$ top right-hand matrix by $\mathbf{N}^*_{12}$; the $6 \times (v-7)$ bottom left-hand matrix by $\mathbf{N}^*_{21}$; the 7×7 bottom right-hand matrix by $\mathbf{N}^*_{22}$. Then we can write

$$\mathbf{N}^*_{11} = \mathbf{J}_{v-6} - \mathbf{I}_{v-6},\ \mathbf{N}^*_{12} = \mathbf{J}_{(v-7)\times 6} = \mathbf{N}^{*\prime}_{21}, \mathbf{N}^*_{22} = \mathbf{J}_7 - \mathbf{I}_7, \tag{4.4.14}$$

where $\mathbf{I}_*$ is the identity matrix of order (*), $\mathbf{J}_*$ is the matrix of order (*) with all elements equal to unity.

Let the first three columns of a Hadamard matrix of order $(v-7)$ be $\mathbf{h}^*_1$, $\mathbf{h}^*_2$ and $\mathbf{h}^*_3$. Following the same steps as in Lemma 4.4.1, we construct three matrices $\mathbf{U}^{(1)*}_{11}$, $\mathbf{U}^{(2)*}_{11}$ and $\mathbf{U}^{(3)*}_{11}$ each of order $(v-6) \times (v-6)$ corresponding to the matrix $\mathbf{N}^*_{11}$ of (4.4.14) with the help of $\mathbf{h}^*_1$, $\mathbf{h}^*_2$ and $\mathbf{h}^*_3$, respectively. Again for $\mathbf{N}^*_{12}$, we construct three matrices $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ each of order $(v-7) \times 6$ as

$$\mathbf{V}_1 = \mathbf{h}^*_1 \otimes \mathbf{a}',\ \mathbf{V}_2 = \mathbf{h}^*_2 \otimes \mathbf{a}' \text{ and } \mathbf{V}_3 = \mathbf{h}^*_3 \otimes \mathbf{a}', \text{ where } \mathbf{a}' = (1, -1, 1, -1, 1, -1).$$

Now using $\mathbf{U}_i$ from (4.4.10) to (4.4.12) for $\mathbf{N}^*_{22}$ and $\mathbf{U}^{(i)*}_{11}$, $\mathbf{V}_i$, $\mathbf{V}'_i$ for $\mathbf{N}^*_{11}$, $\mathbf{N}^*_{12}$ and $\mathbf{N}^*_{21}$ of (4.4.14), respectively, $i = 1, 2, 3$, we get the D-optimal design. □

Example 4.4.1 Let us consider a SBIBD with parameters $v = b = 11,\ r = k = 10,\ \lambda = 9$ where the initial block is (1, 2, 3, 4, 5, 6, 7, 8, 9, 10) mod 11. The incidence matrix can be displayed as

$$\mathbf{N} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

The first **U**-matrix is given by:

$$\mathbf{U}_1 = \begin{pmatrix} 0 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 0 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 0 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 0 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 0 & -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 \end{pmatrix}.$$

Similarly, we can construct the other two.

Remark 4.4.1 The proposed design is also optimal with respect to any Type I criteria in the class of $\mathcal{Z}^* = \{\mathbf{Z}^{*n\times c} : z_{ij}^{(t)} = \pm 1\ \forall\, i, j, t\}$, rank$(\mathbf{Z}^*) = c$ (cf. Cheng 1980).

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Chapter 5
OCDs in Group Divisible Design Set-Up

5.1 Introduction

It is observed that the BIBDs are restrictive as every pair of treatments should occur equal number of times. As a result the availability of OCDs in this set-up becomes limited. In this context, it is observed that PBIBDs are less restrictive and at the same time are popular among practitioners. So it is desirable to have OCDs involving these set-ups. Dutta et al. (2009) have considered the problem of construction of OCDs in the series of PBIBDs which are obtained not only through the method of differences but also are obtained by other methods as described by Bose et al. (1953), Zelen (1954) and Vartak (1954). In this chapter, we will only confine to GDDs and discuss methods of construction of the OCDs based on GDDs.

To construct OCDs we have often applied two matrix-products, viz. Kronecker product and Khatri-Rao product. The definitions of the matrix-products can be found in Rao (1973), p. 29–30, where the Khatri-Rao product has been termed as 'New Product'. For completeness we reproduce the two definitions below:

Definition 5.1.1 (*Kronecker-Product*) Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be two matrices of orders $m \times n$ and $p \times q$ respectively. Then the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \otimes \mathbf{B}$, is defined to be an $mp \times nq$ matrix expressible as a partitioned matrix with $a_{ij}\mathbf{B}$ as the (i, j)th partition, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, i.e.

$$\mathbf{A} \otimes \mathbf{B} = (a_{ij}\mathbf{B}). \tag{5.1.1}$$

Definition 5.1.2 (*Khatri-Rao Product*) Let $\mathbf{A} = (\mathbf{A}_1, \ldots, \mathbf{A}_k)$ and $\mathbf{B} = (\mathbf{B}_1, \ldots, \mathbf{B}_k)$ be two partitioned matrices with the same number of partitions. Then the Khatri-Rao product of $\mathbf{A}$ and $\mathbf{B}$, denoted as $\mathbf{A} \odot \mathbf{B}$, is defined by

$$\mathbf{A} \odot \mathbf{B} = (\mathbf{A}_1 \otimes \mathbf{B}_1, \ldots, \mathbf{A}_k \otimes \mathbf{B}_k). \tag{5.1.2}$$

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5.2 Optimum Covariate Designs

In this section, we mainly confine to the work in Dutta et al. (2009) and describe the construction of OCDs described therein for different series of GDDs.

These designs are based on the concept of association scheme with respect to PBIBDs, which is defined below for the sake of completeness.

Definition 5.2.1 Given v symbols $1, 2, \ldots, v$, a relation satisfying the following conditions is said to be an association scheme with m classes:

1. Any two treatments are either 1st, 2nd, ..., or mth associates, the relation of association being symmetrical; that is, if the symbol α is the ith associate of the symbol β, then β is the ith associate of α.
2. Each treatment α has n_i ith associates, the number n_i being independent of α.
3. If any two treatments α and β are ith associates, then the number of symbols that are jth associates of α, and kth associates of β, is p^i_{jk} and is independent of the pair of ith associates α and β.

The numbers v, n_i $(i = 1, 2, \ldots, m)$ and p^i_{jk} $(i, j, k = 1, 2, \ldots, m)$ are called the parameters of the association scheme.

Given an association scheme for the v treatments, we define a PBIBD as follows:

Definition 5.2.2 Given an association scheme with m classes and given parameters as above, we get a PBIBD with m associate classes if the v symbols are arranged into b blocks of size k $(< v)$ such that

1. Every symbol occurs at most once in a block.
2. Every symbol occurs in exactly r blocks.
3. If two symbols α and β are ith associates, then they occur together in λ_i blocks, the number λ_i being independent of the particular pair of ith associates α and β.

The numbers v, b, r, k, λ_i $(i = 1, 2, \ldots, m)$ are called the parameters of the design. Two-associate class PBIBDs were classified by Bose and Shimamoto (1952) in the following types depending on the association schemes:

1. Group divisible (GD)
2. Simple (SI)
3. Triangular (T)
4. Latin-square type (L_i)
5. Cyclic (C).

In the context of cyclic design, more refined definition has been suggested by Nandi and Adhikari (1966). However, our consideration of OCDs will be based only on the GDDs. For the other types, we refer to Dutta et al. (2009).

Definition 5.2.3 (*GD association scheme and design*) For integers $m \geq 2$ and $n \geq 2$, consider $v = mn$ treatments, which are divided in an m groups is containing n treatments. Any two treatments of the same group are called first associate and any two treatments for different groups are called 2nd associate. The parameters of the GD association scheme are as follows:

$$v = mn,\ n_1 = n - 1, n_2 = n(m - 1),$$

$$\mathbf{P}_1 = (p^1_{ij}) = \begin{pmatrix} n-2 & 0 \\ 0 & n(m-1) \end{pmatrix},\ \ \mathbf{P}_2 = (p^2_{ij}) = \begin{pmatrix} 0 & n-1 \\ n-1 & n(m-1) \end{pmatrix}. \quad (5.2.1)$$

A PBIBD is said to be group-divisible if it is based on the GD association scheme.

If $\mathbf{N}$ be the incidence matrix of GD design then the characteristic roots θ_i of the $\mathbf{NN}'$ matrix and the respective multiplicity $\alpha_i,\ \ i = 0, 1, 2$ are given by

$$\begin{aligned} \theta_0 &= rk,\ \ \alpha_0 = 1 \\ \theta_1 &= r - \lambda_1,\ \ \alpha_1 = m(n-1), \\ \theta_2 &= rk - v\lambda_2,\ \ \alpha_2 = m - 1. \end{aligned} \quad (5.2.2)$$

A GD design is called

(a) singular if $r = \lambda_1$;
(b) semi-regular, if $r > \lambda_1$ and $rk = v\lambda_2$;
(c) regular, if $r > \lambda_1$ and $rk > v\lambda_2$.

Note 5.2.1 In what follows, the incidence matrices of the relevant designs are represented in terms of their transposes, keeping the same style as in the case of BIBDs followed in earlier chapters.

5.2.1 Singular Group Divisible Design (SGDD) Set-Up

It had been shown in Bose et al. (1953) that if in a BIBD with parameters $v^*,\ b^*,\ r^*,\ k^*$ and λ^* each treatment is replaced by a group of n treatments, an SGDD can be obtained with parameters

$$v = nv^*,\ b = b^*,\ r = r^*,\ k = nk^*, \lambda_1 = r^*,\ \lambda_2 = \lambda^*,\ m = v^*,\ n = n. \quad (5.2.3)$$

Here m stands for the number of groups in the corresponding association scheme. It will be seen that $\mathbf{W}$-matrices for such an SGDD with parameters in (5.2.3) can be constructed and the construction of $\mathbf{W}$ in this case does not depend on the method of construction of the corresponding BIBD.

Theorem 5.2.1 *A set of t optimum $\mathbf{W}$-matrices can be constructed for the SGDD with parameters in (5.2.3), where*

(i) *$t = c$, if c optimum $\mathbf{W}$-matrices exist for an RBD with n treatments and r blocks;*
(ii) *$t = v^*(n-1)(r-1)$, if $\mathbf{H}_{v^*}$, $\mathbf{H}_n$ and $\mathbf{H}_r$ exist;*

(iii) $t = v^*((n-1)(r-1) - (n-2))$,

(a) *if* $n \equiv 2\,(mod\,4)$, $(n-1)$ *is a prime or a prime power and* $\mathbf{H}_{v^*}$ *and* $\mathbf{H}_r$ *exist;* or

(b) *if* $\mathbf{H}_{v^*}$, $\mathbf{H}_{2n}$ *and* $\mathbf{H}_{\frac{r}{2}}$ *exist;*

(iv) $t = v^*$ *if* $n = even$, $r = even$ *and* $\mathbf{H}_{v^*}$ exists.

Proof Consider the SGDD with parameters in (5.2.3) obtained by replacing each treatment of the BIBD(v, b, r, k, λ) by a group of n treatments. Let the n treatments of the SGDD corresponding to the treatment θ_i $(i = 1, 2, \ldots, v^*)$ of the BIBD be denoted by $(\theta_{1i}, \theta_{2i}, \ldots, \theta_{ni})$ and the transpose of the partitioned incidence matrix of the SGDD be denoted by

$$\mathbf{N}' = \left(\mathbf{N}_1', \mathbf{N}_2', \ldots, \mathbf{N}_i', \ldots, \mathbf{N}_{v^*}'\right) \tag{5.2.4}$$

where $\mathbf{N}_i$ is the incidence matrix corresponding to $(\theta_{1i}, \theta_{2i}, \ldots, \theta_{ni})$; $i = 1, 2, \ldots, v^*$. If the rows of $\mathbf{N}_i$ containing the null elements only are omitted, then the reduced matrix corresponds to the incidence matrix of an RBD with n treatments arranged in r blocks. We denote an RBD with r blocks and n treatments by RBD(n, r). This is true for all i. For the time being, let it be assumed that c optimum $\mathbf{W}$-matrices for an RBD(n, r) exist and let them be denoted by $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_c$. Putting the elements of $\mathbf{W}_j$ of RBD(n, r) in the corresponding non-zero positions of each $\mathbf{N}_i$, a matrix $\mathbf{W}_j^*$ is obtained and let its transpose be written as

$$\mathbf{W}_j^{*\prime} = (\mathbf{W}_{1j}^{*\prime}, \mathbf{W}_{2j}^{*\prime}, \ldots, \mathbf{W}_{v^*j}^{*\prime}). \tag{5.2.5}$$

It is easy to verify that each of $\mathbf{W}_1^*, \mathbf{W}_2^*, \ldots, \mathbf{W}_c^*$ give optimum $\mathbf{W}$-matrices for the SGDD (5.2.3) and thus (i) of the theorem follows.

Again if $\mathbf{H}_{v^*}$ exists then the number of optimum $\mathbf{W}$-matrices can be increased by application of Khatri-Rao product. Let $\mathbf{H}_{v^*}$ be written as

$$\mathbf{H}_{v^*} = (h_{lm}), \text{ where } h_{lm} \text{ is the } (l, m)\text{th element of } \mathbf{H}_{v^*}.$$

For $l = 1, 2, \ldots, v^*$, a matrix $\mathbf{W}_{lj}^{**}$ is constructed by Khatri-Rao product where the transpose of $\mathbf{W}_{lj}^{**}$ is

$$\mathbf{W}_{lj}^{**\prime} = \mathbf{h}_l \odot \mathbf{W}_j^{*\prime} = (h_{l1}\mathbf{W}_{1j}^{*\prime}, h_{l2}\mathbf{W}_{2j}^{*\prime}, \ldots, h_{li}\mathbf{W}_{ij}^{*\prime}, \ldots, h_{lv^*}\mathbf{W}_{v^*j}^{*\prime}), \tag{5.2.6}$$

where $\mathbf{h}_l$ is the lth row of $\mathbf{H}_{v^*}$. Now varying l and j, v^*c optimum $\mathbf{W}_{lj}^{**}$-matrices can be constructed and it can be easily checked that these matrices satisfy the condition (3.1.5).

It is proved in Chap. 3 that the values of c are (a$_1$) $(n-1)(r-1)$, if $\mathbf{H}_n$ and $\mathbf{H}_r$ exist; (a$_2$) $(n-1)(r-1) - (n-2)$, if $n \equiv 2$ (mod 4), $(n-1)$ is a prime or a prime power and $\mathbf{H}_r$ exists and (a$_3$) $(n-1)(r-1) - (n-2)$, if $\mathbf{H}_{2n}$ and $\mathbf{H}_{\frac{r}{2}}$ exist. These values imply, respectively, (ii), (iii) of the theorem when $\mathbf{H}_{v^*}$ exists.

Again if n and r are even, we can write a $n \times r$ matrix $\mathbf{W}_1$ as

$$\mathbf{W}_1 = \begin{pmatrix} \mathbf{J} & -\mathbf{J} \\ -\mathbf{J} & \mathbf{J} \end{pmatrix}$$

where $\mathbf{J}$ is a $\frac{n}{2} \times \frac{r}{2}$ matrix with all elements unity. It is easy to see that $\mathbf{W}_1$ gives an optimum $\mathbf{W}$-matrix for an RBD$(n,\ r)$. Thus (iv) of the theorem follows. $\square$

Remark 5.2.1 Exchanging the roles of r and n in (iii) of Theorem 5.2.1, we may get $t = v^*((n-1)(r-1) - (r-2))$ optimum $\mathbf{W}$-matrices for the SGDD with parameters in (5.2.3) if $r \equiv 2 \pmod 4$, $(r-1)$ is a prime or a prime power, $\mathbf{H}_{v^*}$ and $\mathbf{H}_n$ exist or $\mathbf{H}_{v^*}$, $\mathbf{H}_{2r}$ and $\mathbf{H}_{\frac{n}{2}}$ exist.

Remark 5.2.2 If v^* is an even integer, then a set of t optimum $\mathbf{W}$-matrices can be constructed for the SGDD with parameters (5.2.3) by using $\mathbf{1}'_{v^*}$ and $(\mathbf{1}'_{\frac{v^*}{2}}, -\mathbf{1}'_{\frac{v^*}{2}})$ respectively in place of the rows of $\mathbf{H}_{v^*}$ in (5.2.6). Again from (ii)–(iii) of Theorem 5.2.1 it follows that

(i) $t = 2(n-1)(r-1)$ if $\mathbf{H}_n$ and $\mathbf{H}_r$ exist;
(ii) $t = 2((n-1)(r-1) - (n-2))$ If $n \equiv 2 \pmod 4$, $(n-1)$ is a prime or a prime power and $\mathbf{H}_r$ exists or if $\mathbf{H}_{2n}$ and $\mathbf{H}_{\frac{r}{2}}$ exist;

respectively.

Remark 5.2.3 It is easily seen that for the construction of optimum $\mathbf{W}$-matrices for RBD$(n,\ r)$, it is necessary that r and n must be even. If r, n and v^* are even but none of them are multiple of 4, then 2 optimum $\mathbf{W}$-matrices can always be constructed for the SGDD with parameters (5.2.3) by using two orthogonal rows as in Remark 5.2.2.

Remark 5.2.4 Suppose t_1 optimum $\mathbf{W}$-matrices exist for the BIBD$(v^*, b^*, r^*, k^*, \lambda^*)$; then additional t_1 optimum $\mathbf{W}$-matrices, orthogonal to the previous ones, can be constructed for a SGDD with parameters given in (5.2.3).

We give some examples illustrating (i), (ii) and (iv) of Theorem 5.2.1 and Remark 5.2.4.

Example 5.2.1 Consider a BIBD with parameters $v^* = b^* = 3,\ r^* = k^* = 2,$ $\lambda^* = 1$ with the incidence matrix

$$\mathbf{N}^* = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Now for $n = 2$, the SGDD with parameters $v = 6,\ b = 3,\ r = 2,\ k = 4,$ $\lambda_1 = 2,\ \lambda_2 = 1,\ m = 3,\ n = 2$ has the transpose of the incidence matrix,

$$\mathbf{N}' = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1}'_2 & \mathbf{1}'_2 & \mathbf{0}' \\ \mathbf{0}' & \mathbf{1}'_2 & \mathbf{1}'_2 \\ \mathbf{1}'_2 & \mathbf{0}' & \mathbf{1}'_2 \end{pmatrix}.$$

$\mathbf{H}_2$ is written as

$$\mathbf{H}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \left(\mathbf{h}_1^*, \mathbf{h}_2^* \right).$$

Applying the method described in Theorem 3.4.1 and using $\mathbf{h}_2^*$, only one $\mathbf{W}$-matrix for RBD(2,2) can be constructed and it is given by

$$\mathbf{W}_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{h}_2^{*\prime} \\ -\mathbf{h}_2^{*\prime} \end{pmatrix}.$$

Using $\mathbf{W}_1$, only one $\mathbf{W}$-matrix for above SGDD can be constructed (vide Eq. (5.2.6)) and its transpose is given by

$$\mathbf{W}_1^{*\prime} = \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ -1 & 1 & 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{h}_2^{*\prime} & \mathbf{h}_2^{*\prime} & \mathbf{0}' \\ \mathbf{0}' & -\mathbf{h}_2^{*\prime} & \mathbf{h}_2^{*\prime} \\ -\mathbf{h}_2^{*\prime} & \mathbf{0}' & -\mathbf{h}_2^{*\prime} \end{pmatrix}.$$

Again, there exists a $\mathbf{W}$-matrix for the BIBD (cf. Chap. 4) which is given by

$$\mathbf{W}_{(1)} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Using $\mathbf{W}_{(1)}$, one more $\mathbf{W}$-matrix for the SGDD can be constructed through Kronecker product and its transpose is given by

$$\mathbf{W}_{(1)}^{*\prime} = \mathbf{W}_{(1)}' \otimes \mathbf{1}_2' = \begin{pmatrix} \mathbf{1}_2' & -\mathbf{1}_2' & \mathbf{0}' \\ \mathbf{0}' & \mathbf{1}_2' & -\mathbf{1}_2' \\ -\mathbf{1}_2' & \mathbf{0}' & \mathbf{1}_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ -1 & -1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

It is easy to check that $\mathbf{W}_{(1)}^*$ is orthogonal to $\mathbf{W}_1^*$.

Example 5.2.2 Consider a BIBD with parameters $v^* = 4,\ b^* = 24,\ r^* = 12,\ k^* = 2,\ \lambda^* = 4$ (this is obtained by repeating BIBD(4, 6, 3, 2, 1) 4 times) with the transpose of the incidence matrix

$$\mathbf{N}^{*\prime} = \mathbf{1}_4 \otimes \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

The SGDD with parameters $v = 16,\ b = 24,\ r = 12,\ k = 8,\ \lambda_1 = 12,\ \lambda_2 = 4,\ m = 4,\ n = 4$ is obtained by replacing each treatment of the BIBD with $n = 4$ treatments. The transpose of the incidence matrix $\mathbf{N}^{16\times 24}$ of SGDD can be written as

$$\mathbf{N}' = (\mathbf{N}^{*\prime} \otimes \mathbf{1}_4') = \mathbf{1}_4 \otimes \begin{pmatrix} \begin{array}{c|c|c|c} \mathbf{1}_4' & \mathbf{1}_4' & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0}' & \mathbf{1}_4' & \mathbf{1}_4' & \mathbf{0}' \\ \mathbf{0}' & \mathbf{0}' & \mathbf{1}_4' & \mathbf{1}_4' \\ \mathbf{1}_4' & \mathbf{0}' & \mathbf{0}' & \mathbf{1}_4' \\ \mathbf{1}_4' & \mathbf{0}' & \mathbf{1}_4' & \mathbf{0}' \\ \mathbf{0}' & \mathbf{1}_4' & \mathbf{0}' & \mathbf{1}_4' \end{array} \end{pmatrix} = \left(\mathbf{N}_1'^{\,24\times 4}, \mathbf{N}_2'^{\,24\times 4}, \mathbf{N}_3'^{\,24\times 4}, \mathbf{N}_4'^{\,24\times 4}\right).$$

$\mathbf{H}_4$ and $\mathbf{H}_{12}$ are written as follows:

$$\mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \\ \mathbf{h}_4 \end{pmatrix} = \left(\mathbf{h}_1^*,\ \mathbf{h}_2^*,\ \mathbf{h}_3^*,\ \mathbf{h}_4^*\right), \tag{5.2.7}$$

$$\mathbf{H}_{12} = \begin{pmatrix} 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} = \left(\mathbf{h}_1^{**}, \mathbf{h}_2^{**}, \ldots, \mathbf{h}_{11}^{**}, \mathbf{h}_{12}^{**}\right),$$

Now we define the matrix: $\mathbf{U}_{i,j}^{(12\times 4)} = \mathbf{h}_i^* \otimes \mathbf{h}_j^{**\prime};\ \forall i = 2, 3, 4;\ \ j = 2, 3, \ldots, 12.$

It can easily be checked that these 33 $\mathbf{U}_{i,j}$'s give the optimum $\mathbf{W}$-matrices for an RBD(4, 12). We write $\mathbf{U}_{2,1} = \mathbf{W}^{(1)}$, $\mathbf{U}_{2,2} = \mathbf{W}^{(2)}, \ldots, \mathbf{U}_{4,11} = \mathbf{W}^{(33)}$ respectively. Let us consider

$$\begin{aligned} \mathbf{W}^{(1)} &= \mathbf{h}_2^* \otimes \mathbf{h}_2^{**\prime} \\ &= (1, -1, 1, -1)' \otimes (1, -1, 1, 1, -1, -1, -1, 1, 1, -1, -1, 1) \\ &= \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix} \\ &= (\mathbf{a}, -\mathbf{a}, \mathbf{a}, \mathbf{a}, -\mathbf{a}, -\mathbf{a}, -\mathbf{a}, \mathbf{a}, \mathbf{a}, -\mathbf{a}, -\mathbf{a}, \mathbf{a}), \end{aligned}$$

where $\mathbf{a} = \mathbf{h}_2^*$ is of order 4×1 (vide (5.2.7)).

By putting the elements of $\mathbf{W}^{(1)}$ in the non-zero positions of each $\mathbf{N}_i$ $(i = 1, 2, 3, 4)$, $\mathbf{W}_1^*$ is obtained and its transpose is written as

$$\mathbf{W}_1^{*\prime} = \left(\mathbf{W}_{11}^{*\prime}, \mathbf{W}_{21}^{*\prime}, \mathbf{W}_{31}^{*\prime}, \mathbf{W}_{41}^{*\prime}\right), \tag{5.2.8}$$

where

$$\begin{aligned}
\mathbf{W}_{11}^* &= (\mathbf{a}, \mathbf{0}, \mathbf{0}, -\mathbf{a}, \mathbf{a}, \mathbf{0}, \mathbf{a}, \mathbf{0}, \mathbf{0}, -\mathbf{a}, -\mathbf{a}, \mathbf{0}, -\mathbf{a}, \mathbf{0}, \mathbf{0}, \mathbf{a}, \mathbf{a}, \mathbf{0}, -\mathbf{a}, \mathbf{0}, \mathbf{0}, -\mathbf{a}, \mathbf{a}, \mathbf{0})\\
\mathbf{W}_{21}^* &= (\mathbf{a}, -\mathbf{a}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{a}, \mathbf{a}, -\mathbf{a}, \mathbf{0}, \mathbf{0}, \mathbf{0}, -\mathbf{a}, -\mathbf{a}, \mathbf{a}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{a}, -\mathbf{a}, -\mathbf{a}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{a})\\
\mathbf{W}_{31}^* &= (\mathbf{0}, \mathbf{a}, -\mathbf{a}, \mathbf{0}, \mathbf{a}, \mathbf{0}, \mathbf{0}, \mathbf{a}, -\mathbf{a}, \mathbf{0}, -\mathbf{a}, \mathbf{0}, \mathbf{0}, -\mathbf{a}, \mathbf{a}, \mathbf{0}, \mathbf{a}, \mathbf{0}, \mathbf{0}, -\mathbf{a}, -\mathbf{a}, \mathbf{0}, \mathbf{a}, \mathbf{0})\\
\mathbf{W}_{41}^* &= (\mathbf{0}, \mathbf{0}, \mathbf{a}, -\mathbf{a}, \mathbf{0}, \mathbf{a}, \mathbf{0}, \mathbf{0}, \mathbf{a}, -\mathbf{a}, \mathbf{0}, -\mathbf{a}, \mathbf{0}, \mathbf{0}, -\mathbf{a}, \mathbf{a}, \mathbf{0}, \mathbf{a}, \mathbf{0}, \mathbf{0}, -\mathbf{a}, -\mathbf{a}, \mathbf{0}, \mathbf{a}).
\end{aligned}$$

In this way, by using the remaining 32 **W**-matrices of RBD(4, 12), we get another 32 $\mathbf{W}_j^*$'s, $j = 2, 3, \ldots, 33$.

Now by taking the Khatri-Rao product of the $\mathbf{h}_2$ of (5.2.7) and the matrix $\mathbf{W}_1^*$ in (5.2.8), we get $\mathbf{W}_{2,1}^{**}$ whose transpose is

$$\mathbf{W}_{2,1}^{**\prime} = \mathbf{h}_2 \otimes \mathbf{W}_1^{*\prime} = \left(\mathbf{W}_{11}^{*\prime}, -\mathbf{W}_{21}^{*\prime}, \mathbf{W}_{31}^{*\prime}, -\mathbf{W}_{41}^{*\prime}\right).$$

Similarly, by taking the Khatri-Rao product of $(\mathbf{h}_1, \mathbf{W}_1^*)$, $(\mathbf{h}_3, \mathbf{W}_1^*)$ and $(\mathbf{h}_4, \mathbf{W}_1^*)$, respectively, three other optimum **W**-matrices, i.e. $\mathbf{W}_{1,1}^{**}$, $\mathbf{W}_{3,1}^{**}$, $\mathbf{W}_{4,1}^{**}$ can be constructed. As before, by using different rows of $\mathbf{H}_4$ and other 32 $\mathbf{W}_j^*$'s, we get additional 128 optimum **W**-matrices for the said SGDD.

Moreover, there exist three optimum **W**-matrices for the BIBD and these are constructed by the method described in Chap. 4 and are given by

$$(\mathbf{h}_1 \otimes \mathbf{U}^*)'; \ (\mathbf{h}_2 \otimes \mathbf{U}^*)'; \ (\mathbf{h}_3 \otimes \mathbf{U}^*)';$$

where

$$\mathbf{U}^* = \begin{pmatrix} 1 & -1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 1 & -1\\ 1 & 0 & 0 & -1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Using these three **W**-matrices of BIBD, we can construct three more optimum **W**-matrices for the SGDD as described in Remark 5.2.4 and it is easy to see that these three are orthogonal to previous 132 optimum **W**-matrices. So we get 135 optimum **W**-matrices in all for the said SGDD.

Example 5.2.3 Consider a BIBD with parameters $v^* = 4,\ b^* = 8,\ r^* = 6,\ k^* = 3,\ \lambda^* = 4$ with the transpose of the incidence matrix

$$\mathbf{N}^{*\prime} = \begin{pmatrix} 1\\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & 1 & 0\\ 0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

An SGDD with parameters $v = 8,\ b = 8,\ r = 6,\ k = 6,\ \lambda_1 = 6,\ \lambda_2 = 4,\ m = 4,\ n = 2$ is obtained by replacing each treatment of the BIBD with $n = 2$ treatments. The transpose of the incidence matrix $\mathbf{N}$ of SGDD can be written as

$$\mathbf{N}' = \mathbf{N}^{*\prime} \otimes (1,\ 1).$$

$\mathbf{H}_2$ is written as

$$\mathbf{H}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \left(\mathbf{h}_1^* \ \mathbf{h}_2^*\right).$$

It follows that the matrix $\mathbf{W}_1$ given below is the transpose of a $\mathbf{W}$-matrix for an RBD(2, 6):

$$\mathbf{W}_1' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \\ -1 & 1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{h}_2^{*\prime} \\ \mathbf{h}_2^{*\prime} \\ \mathbf{h}_2^{*\prime} \\ -\mathbf{h}_2^{*\prime} \\ -\mathbf{h}_2^{*\prime} \\ -\mathbf{h}_2^{*\prime} \end{pmatrix}.$$

Proceeding in the lines of Theorem 5.2.1, we construct

$$\mathbf{W}_1^{*\prime} = (\mathbf{W}_{11}^{*\prime}, \mathbf{W}_{21}^{*\prime}, \mathbf{W}_{31}^{*\prime}, \mathbf{W}_{41}^{*\prime}),$$

where

$$\mathbf{W}_{11}^{*\prime} = \begin{pmatrix} \mathbf{h}_2^{*\prime} \\ \mathbf{0}' \\ \mathbf{h}_2^{*\prime} \\ \mathbf{h}_2^{*\prime} \\ -\mathbf{h}_2^{*\prime} \\ \mathbf{0}' \\ -\mathbf{h}_2^{*\prime} \\ -\mathbf{h}_2^{*\prime} \end{pmatrix};\ \mathbf{W}_{21}^{*\prime} = \begin{pmatrix} \mathbf{h}_2^{*\prime} \\ \mathbf{h}_2^{*\prime} \\ \mathbf{0}' \\ \mathbf{h}_2^{*\prime} \\ -\mathbf{h}_2^{*\prime} \\ -\mathbf{h}_2^{*\prime} \\ \mathbf{0}' \\ -\mathbf{h}_2^{*\prime} \end{pmatrix};\ \mathbf{W}_{31}^{*\prime} = \begin{pmatrix} \mathbf{h}_2^{*\prime} \\ \mathbf{h}_2^{*\prime} \\ \mathbf{h}_2^{*\prime} \\ \mathbf{0}' \\ -\mathbf{h}_2^{*\prime} \\ -\mathbf{h}_2^{*\prime} \\ -\mathbf{h}_2^{*\prime} \\ \mathbf{0}' \end{pmatrix};\ \mathbf{W}_{41}^{*\prime} = \begin{pmatrix} \mathbf{0}' \\ \mathbf{h}_2^{*\prime} \\ \mathbf{h}_2^{*\prime} \\ \mathbf{h}_2^{*\prime} \\ \mathbf{0}' \\ -\mathbf{h}_2^{*\prime} \\ -\mathbf{h}_2^{*\prime} \\ -\mathbf{h}_2^{*\prime} \end{pmatrix}.$$

and using $\mathbf{W}_1^*$ and the second row of $\mathbf{H}_4$ of (5.2.7), one optimum $\mathbf{W}$-matrix for the SGDD can be constructed where its transpose is

$$\mathbf{W}_{11}^{**\prime} = \mathbf{h}_2 \odot \mathbf{W}_1^{*\prime} = (\mathbf{W}_{11}^{*\prime}, -\mathbf{W}_{21}^{*\prime}, \mathbf{W}_{31}^{*\prime}, -\mathbf{W}_{41}^{*\prime}),$$

Similarly by taking the Khatri-Rao product of $(\mathbf{W}_1^*, \mathbf{h}_1)$, $(\mathbf{W}_1^*, \mathbf{h}_3)$ and $(\mathbf{W}_1^*, \mathbf{h}_4)$, three other optimum $\mathbf{W}$-matrices can be constructed for the above SGDD, where $\mathbf{h}_1$, $\mathbf{h}_3$ and $\mathbf{h}_4$ are, respectively, the first row, third row and 4th row of $\mathbf{H}_4$ in (5.2.7).

5.2.2 Semi-Regular Group Divisible Design (SRGDD) Set-Up

According to Bose et al. (1953), it is known that the existence of an SRGDD with parameters $v = mn,\ b = n^2\lambda_2,\ r = n\lambda_2,\ k = m,\ \lambda_1 = 0,\ \lambda_2,\ m,\ n$ implies the existence of an orthogonal array, OA$(n^2\lambda_2,\ m,\ n,\ 2)$ and conversely. The definition of an orthogonal array (cf. Raghavarao 1971, p. 10) is given below:

Definition 5.2.4 A $k \times N$ matrix $\mathbf{A}$ with entries from a set of s (≥ 2) elements is called an orthogonal array of size N, k constraints, s levels, strength t, and index λ if any $t \times N$ sub-matrix of $\mathbf{A}$ contains all possible $t \times 1$ column vectors with same frequency λ. Such an array is denoted by OA$(N,\ k,\ s,\ t)$.

In this case, using the properties of orthogonal array (cf. Raghavarao 1971) one can find the optimum covariate designs which is stated in the following theorem.

Theorem 5.2.2 *Let the existence of an OA$(n^2\lambda_2,\ m,\ n,\ 2)$ and the existence of Hadamard matrices of order n and $m_1 = k(2 \leq m_1 < m)$ be assumed. Then $(n-1)(k-1)m_2$ optimum $\mathbf{W}$-matrices can be constructed for an SRGDD with parameters $v = m_1 n,\ b = n^2\lambda_2,\ r = n\lambda_2,\ k = m_1,\ \lambda_1 = 0,\ \lambda_2,\ m_1,\ n$ where $m_1 + m_2 = m$ and $m_2 > 2$.*

Proof Let the orthogonal array OA$(n^2\lambda_2,\ m,\ n,\ 2)$ be denoted by the matrix $\mathbf{A}$ with $n^2\lambda_2$ columns and m rows. The n symbols in the pth row of the orthogonal array are denoted as $(p-1)n+1,\ (p-1)n+2, \ldots, pn;\ p = 1, 2, \ldots, m$. Let it be partitioned into two sub-matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, i.e. $\begin{pmatrix}\mathbf{A}_1\\ \mathbf{A}_2\end{pmatrix}$ where $\mathbf{A}_1$ corresponds to first m_1 rows and $\mathbf{A}_2$ corresponds to last m_2 $(m_2 = m - m_1)$ rows of $\mathbf{A}$. Using $\mathbf{A}_1$, an SRGDD with parameters $v = m_1 n,\ b = n^2\lambda_2,\ r = n\lambda_2,\ k = m_1,\ m_1,\ n,\ \lambda_1 = 0,\ \lambda_2$, where $m_1 + m_2 = m$ and $m_2 > 2$ can be constructed, where the $n^2\lambda_2$ columns of $\mathbf{A}_1$ give the $b = n^2\lambda_2$ blocks of the SRGDD. Let a Hadamard matrix of order n be written as

$$\mathbf{H}_n = [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-1}, \mathbf{1}]. \tag{5.2.9}$$

Again let the n symbols in each row of $\mathbf{A}_2$ be replaced by $(h_{j1}, h_{j2}, \ldots, h_{jn})$, where h_{ji}'s are the elements of $\mathbf{h}_j$, the jth column of $\mathbf{H}_n$, $j = 1, 2, \ldots, (n-1)$ and the new array $\mathbf{A}_2^{*\prime}(j) = (\mathbf{a}_1^{*\prime}(j), \mathbf{a}_2^{*\prime}(j), \ldots, \mathbf{a}_{m_2}^{*\prime}(j))$ thus obtained is still an orthogonal array of strength 2, but with the two symbols $+1$ and -1 in each row. Let the incidence matrix of the SRGDD corresponding to the orthogonal array $\mathbf{A}_1$ be denoted as $\mathbf{N}^{v\times b}$ with the jth column as, $\mathbf{n}_j = (n_{1j}, n_{2j}, \ldots, n_{vj})'$, $n_{ij} = 0$ or $1;\ 1 \leq i \leq v,\ 1 \leq j \leq b$. The non-zero elements of each column of $\mathbf{N}$ (containing k non-zero elements) are replaced by the k elements (± 1) of $\mathbf{h}_u^*$, the uth column of $\mathbf{H}_k$, $u = 1, 2, \ldots, (k-1)$ in that order and thus $\mathbf{N}_u^*$ is obtained with the jth column as $\mathbf{n}_j^*(u) = (n_{1j}^*(u), n_{2j}^*(u), \ldots, n_{vj}^*(u))'$, $j = 1, 2, \ldots, b$. Obviously the element $n_{ij}^*(u)$ assumes one of the three distinct values $+1$ or -1 or 0. Now, a matrix $\mathbf{W}(j, u, q)$ is obtained by taking the Khatri-Rao product of $\mathbf{a}_q^*(j)$ and $\mathbf{N}_u^*$. A matrix $\mathbf{W}(j, u, q)$ is written as

$$\mathbf{W}(j,u,q) = \mathbf{a}_q^*(j) \odot \mathbf{N}_u^* = \begin{pmatrix} a_{1q}^*(j) \\ \vdots \\ a_{bq}^*(j) \end{pmatrix} \odot \left(\mathbf{n}_1^*(u) \ldots \mathbf{n}_b^*(u)\right) \tag{5.2.10}$$

It is easy to see that the $\mathbf{W}(j,u,q)$, $q = 1, 2, \ldots, m_2$, $u = 1, 2, \ldots, (k-1)$, $j = 1, 2, \ldots, (n-1)$ matrices given by (5.2.10) satisfy the condition (3.1.5). Thus the theorem follows. □

Example 5.2.4 Consider SRGDD with parameters $v = 8$, $b = 8$, $r = 4$, $k = 4$, $m_1 = 4$, $n = 2$, $\lambda_1 = 0$, $\lambda_2 = 1$ which is obtained from OA(8, 7, 2, 2) as follows:

Let $\mathbf{A} = \text{OA}(8, 7, 2, 2)$ where

$$\mathbf{A}' = \left(\begin{array}{cccc|ccc} 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ 2 & 3 & 6 & 7 & 10 & 11 & 14 \\ 1 & 4 & 6 & 7 & 9 & 12 & 14 \\ 2 & 4 & 5 & 7 & 10 & 12 & 13 \\ 1 & 3 & 5 & 8 & 10 & 12 & 14 \\ 2 & 3 & 6 & 8 & 9 & 12 & 13 \\ 1 & 4 & 6 & 8 & 10 & 11 & 13 \\ 2 & 4 & 5 & 8 & 9 & 11 & 14 \end{array}\right) = (\mathbf{A}_1|\mathbf{A}_2).$$

Using $\mathbf{A}_1$, the SRGDD with above parameters is obtained and the incidence matrix $\mathbf{N}$ corresponding to the design is written as in the form of its transpose

$$\mathbf{N}' = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathbf{H}_2$ and $\mathbf{H}_4$ are written as

$$\mathbf{H}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = (\mathbf{h}_1, \mathbf{1}); \quad \mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} = (\mathbf{h}_1^*, \mathbf{h}_2^*, \mathbf{h}_3^*, \mathbf{1}).$$

Replacing the elements in the columns of $\mathbf{A}_2$ by those of $\mathbf{h}_1$, $\mathbf{A}_2^*(1)$ can be written as

$$\mathbf{A}_2^*(1) = \begin{pmatrix} 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} = (\mathbf{a}_1^*(1), \mathbf{a}_2^*(1), \mathbf{a}_3^*(1)).$$

If the non-zero elements of each row of $\mathbf{N}$ are replaced by the four elements (± 1) of first column $\mathbf{h}_1^*$ of $\mathbf{H}_4$ in that order, then the transpose of $\mathbf{N}_1^*$ is obtained as

$$\mathbf{N}_1^{*\prime} = \begin{array}{c} \text{Block} \\ \downarrow \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \text{Treatment} \rightarrow \\ \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

Now using the Khatri-Rao product between the first column $\mathbf{a}_1^*(1)$ of $\mathbf{A}_2^*(1)$ and $\mathbf{N}_1^*$, the following the transpose of the $\mathbf{W}$-matrix can be constructed as

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \odot \mathbf{N}_1^{*\prime} = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & -1 \end{pmatrix} = \mathbf{W}'(1, 1, 1).$$

Note that $\mathbf{W}'(1, 1, 1)$ matches with $\mathbf{N}_1^{*\prime}$. In this way, altogether 9 optimum $\mathbf{W}$-matrices can be constructed for different choices of columns of $\mathbf{A}_2^*(1)$ and first three columns of $\mathbf{H}_4$ (excluding $\mathbf{h_1}$).

Remark 5.2.5 It follows from Theorem 5.2.2 that the maximum number of $\mathbf{W}$-matrices that can be constructed depends on the maximum value of $m_2(m_1-1)(n-1)$ where $m_1 > 0$, $m_2 > 0$, $m_1 + m_2 = m$ and each of m_1, n is such that $\mathbf{H}_{m_1}$ and $\mathbf{H}_n$ exist.

Remark 5.2.6

(a) If n is even but $\mathbf{H}_n$ does not exist, then it is possible to construct $(k-1)m_2$ optimum $\mathbf{W}$-matrices for the SRGD with the above parameters by using a vector of the form $(\mathbf{1}'_{\frac{n}{2}}, -\mathbf{1}'_{\frac{n}{2}})'$ in place of the columns of $\mathbf{H}_n$.

(b) Similarly if k is even but $\mathbf{H}_k$ does not exist, then it is possible to construct $(n-1)m_2$ optimum $\mathbf{W}$-matrices for the SRGD with the above parameters by using a vector of the form $(\mathbf{1}'_{\frac{k}{2}}, -\mathbf{1}'_{\frac{k}{2}})'$ in place of the columns of $\mathbf{H}_k$.

(c) Again if n, k are both even but $\mathbf{H}_n$ and $\mathbf{H}_k$ do not exist, then it is possible to construct m_2 optimum $\mathbf{W}$-matrices for the SRGD with the above parameters by using two vectors of the form $(\mathbf{1}'_{\frac{n}{2}}, -\mathbf{1}'_{\frac{n}{2}})'$ and $(\mathbf{1}'_{\frac{k}{2}}, -\mathbf{1}'_{\frac{k}{2}})'$ in place of the columns of $\mathbf{H}_n$ and $\mathbf{H}_k$.

5.2.3 Regular Group Divisible (RGD) Design Set-Up

It is known that if from a BIBD with parameters $v^*, b^*, r^*, k^*, \lambda^* = 1$, all the r^* blocks in which a particular treatment occurs are deleted, then a RGD design with parameters $v = v^* - 1, b = b^* - r^*, r = r^* - 1, k = k^*, \lambda_1 = 0, \lambda_2 = 1, m = r^*, n = k^* - 1$ can be obtained (Bose et al. (1953); also see Raghavarao (1971)). It is difficult to construct covariate design optimally for such GD design obtained from arbitrary BIBD with the parameters $v^*, b^*, r^*, k^*, \lambda^* = 1$. However, for some series of BIBDs, it is possible to provide optimum covariate designs. Let the series of BIBD designs with parameters:

$$b^* = (4t+1)(3t+1), v^* = 4(3t+1), r^* = 4t+1, k^* = 4, \lambda^* = 1 \tag{5.2.11}$$

be considered with the initial blocks:

$$\begin{aligned} &\left(x_1^{2i}, x_1^{2t+2i}, x_2^{\alpha+2i}, x_2^{\alpha+2t+2i}\right) \\ &\left(x_2^{2i}, x_2^{2t+2i}, x_3^{\alpha+2i}, x_3^{\alpha+2t+2i}\right) \\ &\left(x_3^{2i}, x_3^{2t+2i}, x_1^{\alpha+2i}, x_1^{\alpha+2t+2i}\right) \\ &(0_1, 0_2, 0_3, \infty); \quad i = 0, 1, \ldots, t-1, \end{aligned} \tag{5.2.12}$$

where $4t+1$ is prime or prime power and x is a primitive root of GF$(4t+1)$; 1, 2, 3 are the three symbols attached to x, α is an odd integer and ∞ the invariant treatment symbol (cf. Bose 1939). If the initial block containing treatment symbol ∞ in (5.2.12) is deleted and others are developed, then an RGD design with parameters:

$$b = 3t(4t+1), v = 3(4t+1), r = 4t, k = 4, \lambda_1 = 0, \lambda_2 = 1, m = 4t+1, n = 3 \tag{5.2.13}$$

is obtained. The $(4t+1)$ groups obtained by developing $(0_1, 0_2, 0_3)$ over GF$(4t+1)$ give the association scheme for the above RGD design. The following theorem provides optimum covariate designs for the series with parameters given in (5.2.13).

Theorem 5.2.3 *If* $\mathbf{H}_t$ *exists, then* $3t$ *optimum* $\mathbf{W}$*-matrices can be constructed for the RGD design with parameters given in (5.2.13).*

Proof Let the $3t$ initial blocks other than $(0_1, 0_2, 0_3, \infty)$ of (5.2.12) be divided into t sets of 3 blocks each, the ith set being

$$S_{i+1} = \Big(\Big(x_1^{2i}, x_1^{2t+2i}, x_2^{\alpha+2i}, x_2^{\alpha+2t+2i}\Big), \ \Big(x_2^{2i}, x_2^{2t+2i}, x_3^{\alpha+2i}, x_3^{\alpha+2t+2i}\Big)$$
$$\Big(x_3^{2i}, x_3^{2t+2i}, x_1^{\alpha+2i}, x_1^{\alpha+2t+2i}\Big)\Big), \quad i = 0, 1, \ldots, t-1.$$

Also, let each of the initial blocks of S_{i+1} be displayed in the form of column vectors of the incidence matrix and development of these initial blocks will give rise to the sub-incidence matrix $\mathbf{N}_i$ of order $3(4t+1) \times 3(4t+1)$, once more we restrict to the transpose matrix where

$$\mathbf{N}_i' = \begin{pmatrix} \mathbf{N}_1^{(i)\prime} & \mathbf{N}_2^{(i)\prime} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_1^{(i)\prime} & \mathbf{N}_2^{(i)\prime} \\ \mathbf{N}_2^{(i)\prime} & \mathbf{0} & \mathbf{N}_1^{(i)\prime} \end{pmatrix}.$$

It is easy to see that $\mathbf{N}_1^{(i)}$ and $\mathbf{N}_2^{(i)}$ matrices corresponding to two portions of the initial blocks of S_i, are obtained by cyclically permuting the column vectors of each of the matrices. For $j = 1, 2$, the two non-zero positions of the first column of $\mathbf{N}_j^{(i)}$ is replaced by $+1$ and -1 successively and then this column is permuted cyclically in the same way as $\mathbf{N}_j^{(i)}$ was obtained. The resultant matrix is denoted by $\mathbf{W}_j^{(i)}$. By replacing the $\mathbf{N}_j^{(i)}$ by $\mathbf{W}_j^{(i)}$ in $\mathbf{N}_i$'s one would get a matrix $\mathbf{W}_{i1}$ of order $3(4t+1) \times 3(4t+1)$ whose transpose can be displayed as

$$\mathbf{W}_{i1}' = \begin{pmatrix} \mathbf{W}_1^{(i)\prime} & \mathbf{W}_2^{(i)\prime} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_1^{(i)\prime} & \mathbf{W}_2^{(i)\prime} \\ \mathbf{W}_2^{(i)\prime} & \mathbf{0} & \mathbf{W}_1^{(i)\prime} \end{pmatrix}.$$

Then two other matrices, viz. $\mathbf{W}_{i2}$ and $\mathbf{W}_{i3}$ are constructed from $\mathbf{W}_{i1}$ and $\mathbf{N}_i$, respectively, where their transpose matrices are respectively

$$\mathbf{W}_{i2}' = \begin{pmatrix} \mathbf{W}_1^{(i)\prime} & -\mathbf{W}_2^{(i)\prime} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_1^{(i)\prime} & -\mathbf{W}_2^{(i)\prime} \\ -\mathbf{W}_2^{(i)\prime} & \mathbf{0} & \mathbf{W}_1^{(i)\prime} \end{pmatrix}, \quad \mathbf{W}_{i3}' = \begin{pmatrix} \mathbf{N}_1^{(i)\prime} & -\mathbf{N}_2^{(i)\prime} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_1^{(i)\prime} & -\mathbf{N}_2^{(i)\prime} \\ -\mathbf{N}_2^{(i)\prime} & \mathbf{0} & \mathbf{N}_1^{(i)\prime} \end{pmatrix}.$$

It can be easily checked that these three matrices satisfy the condition (3.1.5) for each $i, i = 1, 2, \ldots, t$. If Hadamard matrix $\mathbf{H}_t = (h_{ml})$ exists, the number of $\mathbf{W}$-matrices can be increased t times. The $3t$ optimum $\mathbf{W}$-matrices can be constructed and their transpose can be displayed as

$$\mathbf{W}'_j(m) = \begin{pmatrix} h_{m1} \\ h_{m2} \\ \vdots \\ h_{mt} \end{pmatrix} \odot \begin{pmatrix} \mathbf{W}'_{1j} \\ \mathbf{W}'_{2j} \\ \vdots \\ \mathbf{W}'_{tj} \end{pmatrix} \quad \forall m = 1, 2, \ldots, t; \;\; j = 1, 2, 3. \tag{5.2.14}$$

It can be easily seen that these $3t$ matrices given in (5.2.14) satisfy the condition (3.1.5) and give optimum **W**-matrices. □

Example 5.2.5 With $t = 1$ the RGD design with parameters $v = 15, b = 15, r = 4, k = 4, \lambda_1 = 0, \lambda_2 = 1, m = 5, n = 3$ is considered and the initial blocks forming the single set are $\{(1_1, 4_1, 2_2, 3_2), (1_2, 4_2, 2_3, 3_3),\ (1_3, 4_3, 2_1, 3_1)\}$ and the groups of the association scheme are generated from $(0_1, 0_2, 0_3)$. The transpose of the incidence matrix of this design

$$\mathbf{N}' = \begin{array}{c} \begin{array}{ccccc|ccccc|ccccc} 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 0_3 & 1_3 & 2_3 & 3_3 & 4_3 \end{array} \\ \left(\begin{array}{ccccc|ccccc|ccccc}
0&1&0&0&1&0&0&1&1&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&1&1&0&0&0&0&0\\
0&1&0&1&0&1&0&0&0&1&0&0&0&0&0\\
0&0&1&0&1&1&1&0&0&0&0&0&0&0&0\\
1&0&0&1&0&0&1&1&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&1&0&0&1&0&0&1&1&0\\
0&0&0&0&0&1&0&1&0&0&0&0&0&1&1\\
0&0&0&0&0&0&1&0&1&0&1&0&0&0&1\\
0&0&0&0&0&0&0&1&0&1&1&1&0&0&0\\
0&0&0&0&0&1&0&0&1&0&0&1&1&0&0\\
\hline
0&0&1&1&0&0&0&0&0&0&0&1&0&0&1\\
0&0&0&1&1&0&0&0&0&0&1&0&1&0&0\\
1&0&0&0&1&0&0&0&0&0&0&1&0&1&0\\
1&1&0&0&0&0&0&0&0&0&0&0&1&0&1\\
0&1&1&0&0&0&0&0&0&0&1&0&0&1&0
\end{array}\right) \end{array} = \begin{pmatrix} \mathbf{N}_1^{(i)\prime} & \mathbf{N}_2^{(i)\prime} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_1^{(i)\prime} & \mathbf{N}_2^{(i)\prime} \\ \mathbf{N}_2^{(i)\prime} & \mathbf{0} & \mathbf{N}_1^{(i)\prime} \end{pmatrix}$$

with

$$\mathbf{N}_i^{(1)\prime} = \begin{pmatrix} 0&1&0&0&1 \\ 1&0&1&0&0 \\ 0&1&0&1&0 \\ 0&0&1&0&1 \\ 1&0&0&1&0 \end{pmatrix}, \quad \mathbf{N}_2^{(2)\prime} = \begin{pmatrix} 0&0&1&1&0 \\ 0&0&0&1&1 \\ 1&0&0&0&1 \\ 1&1&0&0&0 \\ 0&1&1&0&0 \end{pmatrix}.$$

The transpose matrices of the three optimum **W**-matrices are respectively

$$\mathbf{W}'_{11} = \begin{pmatrix} \mathbf{W}_1^{(1)\prime} & \mathbf{W}_2^{(1)\prime} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_1^{(1)\prime} & \mathbf{W}_2^{(1)\prime} \\ \mathbf{W}_2^{(1)\prime} & \mathbf{0} & \mathbf{W}_1^{(1)\prime} \end{pmatrix}.$$

$$\mathbf{W}'_{12} = \begin{pmatrix} \mathbf{W}_1^{(1)\prime} & -\mathbf{W}_2^{(1)\prime} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_1^{(1)\prime} & -\mathbf{W}_2^{(1)\prime} \\ -\mathbf{W}_2^{(1)\prime} & \mathbf{0} & \mathbf{W}_1^{(1)\prime} \end{pmatrix}, \quad \mathbf{W}'_{13} = \begin{pmatrix} \mathbf{N}_1^{(1)\prime} & -\mathbf{N}_2^{(1)\prime} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_1^{(1)\prime} & -\mathbf{N}_2^{(1)\prime} \\ -\mathbf{N}_2^{(1)\prime} & \mathbf{0} & \mathbf{N}_1^{(1)\prime} \end{pmatrix}.$$

where,

$$\mathbf{W}_1^{(1)\prime} = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathbf{W}_2^{(1)\prime} = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix}$$

and $\mathbf{N}_1^{(1)}$ and $\mathbf{N}_2^{(1)}$ as above.

Appendix

A list of OCDs for suitable subclasses of GDDs, viz. SGDDs, SRGDDs and RGDDs divided as singular (S), semi-regular (SR), regular (R) is given below. These are extracted from the catalogue prepared by Clatworthy (1973) and amenable to construction of OCDs. See Dutta et al. (2009, 2010) in this context. In the constructional method column, T stands for Theorem and R for Remark (Tables 5.1, 5.2 and 5.3).

Table 5.1 OCDs in SGDDs

Sl. no.	Design no.	v	b	r	k	λ_1	λ_2	m	n	v^*	b^*	r^*	k^*	λ^*	t	$t+t_1$	Method of construction
1	S1	6	3	2	4	2	1	3	2	3	3	2	2	1	1	2	T 5.2.1(i), R 5.2.4
2	S2	6	6	4	4	4	2	3	2	3	6	4	2	2	2	5	T 5.2.1(i), R 5.2.4
3	S3	6	9	6	4	6	3	3	2	3	9	6	2	3	1	2	T 5.2.1(i), R 5.2.4
4	S4	6	12	8	4	8	4	3	2	3	12	8	2	4	4	11	T 5.2.1(i), R 5.2.4
5	S5	6	15	10	4	10	5	3	2	3	15	10	2	5	1	2	T 5.2.1(i), R 5.2.4
6	S7	8	12	6	4	6	2	4	2	4	12	6	2	2	1	5	T 5.2.1(iv), R 5.2.4
7	S9	10	10	4	4	4	1	5	2	5	10	4	2	1	2	5	T 5.2.1(i), R 5.2.4
8	S10	10	20	8	4	8	2	5	2	5	20	8	2	2	4	11	T 5.2.1(i), R 5.2.4
9	S12	12	30	10	4	10	2	6	2	6	30	10	2	2	1	3	R 5.2.3, R 5.2.4
10	S13	14	21	6	4	6	1	7	2	7	21	6	2	1	1	2	T 5.2.1(i), R 5.2.4
11	S15	18	36	8	4	8	1	9	2	9	36	8	2	1	4	11	T 5.2.1(i), R 5.2.4
12	S17	22	55	10	4	10	1	11	2	11	55	10	2	1	1	2	T 5.2.1(i), R 5.2.4
13	S19	8	8	6	6	6	4	4	2	4	8	6	3	4	0	4	T 5.2.1(iv), R 5.2.4

(continued)

Table 5.1 (continued)

Sl. no.	Design no.	v	b	r	k	λ_1	λ_2	m	n	v^*	b^*	r^*	k^*	λ^*	t	$t+t_1$	Method of construction
14	S21	9	3	2	6	2	1	3	3	3	3	2	2	1	1	1	R 5.2.4
15	S22	9	6	4	6	4	2	3	3	3	6	4	2	2	2	2	R 5.2.4
16	S23	9	9	6	6	6	3	3	3	3	9	6	2	3	1	1	R 5.2.4
17	S24	9	12	8	6	8	4	3	3	3	12	8	2	4	4	4	R 5.2.4
18	S25	9	15	10	6	10	5	3	3	3	15	10	2	5	1	1	R 5.2.4
19	S26	10	10	6	6	6	3	5	2	5	10	6	3	3	0	1	T 5.2.1(i)
20	S29	12	12	6	6	6	2	4	3	4	12	6	2	2	1	1	R 5.2.4
21	S31	12	20	10	6	10	4	6	2	6	20	10	3	4	0	2	R 5.2.3
22	S33	14	14	6	6	6	2	7	2	7	14	6	3	2	0	1	T 5.2.1(i)
23	S35	15	10	4	6	4	1	5	3	5	10	4	2	1	2	2	R 5.2.4
24	S36	15	20	8	6	8	2	5	3	5	20	8	2	2	4	4	R 5.2.4
25	S37	18	12	4	6	4	1	9	2	9	12	4	3	1	0	3	T 5.2.1(i)
26	S39	18	24	8	6	8	2	9	2	9	24	8	3	2	0	7	T 5.2.1(i)
27	S40	18	30	10	6	10	2	6	3	6	30	10	2	2	1	1	R 5.2.4
28	S42	21	21	6	6	6	1	7	3	7	21	6	2	1	1	1	R 5.2.4
29	S44	26	26	6	6	6	1	13	2	13	26	6	3	1	0	1	T 5.2.1(i)
30	S45	27	36	8	6	8	1	9	3	9	36	8	2	1	4	4	R 5.2.4
31	S48	33	55	10	6	10	1	11	3	11	55	10	2	1	1	1	R 5.2.4
32	S50	42	70	10	6	10	1	21	2	21	70	10	3	1	0	1	T 5.2.1(i)

(continued)

Table 5.1 (continued)

Sl. no.	Design no.	v	b	r	k	λ_1	λ_2	m	n	v^*	b^*	r^*	k^*	λ^*	t	$t+t_1$	Method of construction
33	S51	10	5	4	8	4	3	5	2	5	5	4	4	3	3	6	T 5.2.1(i), R 5.2.4
34	S52	10	10	8	8	8	6	5	2	5	10	8	4	6	6	13	T 5.2.1(i), R 5.2.4
35	S53	12	3	2	8	2	1	3	4	3	3	2	2	1	1	4	T 5.2.1(i), R 5.2.4
36	S54	12	6	4	8	4	2	3	4	3	6	4	2	2	2	11	T 5.2.1(i), R 5.2.4
37	S55	12	9	6	8	6	3	3	4	3	9	6	2	3	1	12	T 5.2.1(i), R 5.2.4
38	S56	12	12	8	8	8	4	3	4	3	12	8	2	4	4	25	T 5.2.1(i), R 5.2.4
39	S57	12	15	10	8	10	5	3	4	3	15	10	2	5	1	20	T 5.2.1(i), R 5.2.4
40	S58	12	15	10	8	10	6	6	2	6	15	10	4	6	1	3	R 5.2.3
41	S59	14	7	4	8	4	2	7	2	7	7	4	4	2	3	6	T 5.2.1(i), R 5.2.4
42	S60	14	14	8	8	8	4	7	2	7	14	8	4	4	6	13	T 5.2.1(i), R 5.2.4
43	S62	16	12	6	8	6	2	4	4	4	12	6	2	2	1	45	T 5.2.1(iii), R 5.2.4
44	S65	18	18	8	8	8	3	9	2	9	18	8	4	3	6	13	T 5.2.1(i), R 5.2.4
45	S66	20	10	4	8	4	1	5	4	5	10	4	2	1	2	11	T 5.2.1(i), R 5.2.4

(continued)

Table 5.1 (continued)

Sl. no.	Design no.	v	b	r	k	λ_1	λ_2	m	n	v^*	b^*	r^*	k^*	λ^*	t	$t+t_1$	Method of construction
46	S67	20	15	6	8	6	2	10	2	10	15	6	4	2	0	2	R 5.2.3
47	S68	20	20	8	8	8	2	5	4	5	20	8	2	2	4	25	T 5.2.1(i), R 5.2.4
48	S70	24	30	10	8	10	2	6	4	6	30	10	2	2	1	39	R 5.2.2(ii)
49	S71	26	13	4	8	4	1	13	2	13	13	4	4	1	3	6	T 5.2.1(i), R 5.2.4
50	S72	26	26	8	8	8	2	13	2	13	26	8	4	2	6	13	T 5.2.1(i), R 5.2.4
51	S73	28	21	6	8	6	1	7	4	7	21	6	2	1	1	12	T 5.2.1(i), R 5.2.4
52	S76	32	40	10	8	10	2	16	2	16	40	10	4	2	1	17	T 5.2.1(iv), R 5.2.4
53	S77	36	36	8	8	8	1	9	4	9	36	8	2	1	4	25	T 5.2.1(i), R 5.2.4
54	S79	44	55	10	8	10	1	11	4	11	55	10	2	1	1	20	T 5.2.1(i), R 5.2.4
55	S80	50	50	8	8	8	1	25	2	25	50	8	4	1	6	13	T 5.2.1(i), R 5.2.4
56	S99	12	12	10	10	10	8	6	2	6	12	10	5	8	0	2	R 5.2.3
57	S100	15	3	2	10	2	1	3	5	3	3	2	2	1	1	1	R 5.2.4
58	S101	15	6	4	10	4	2	3	5	3	6	4	2	2	2	2	R 5.2.4
59	S102	15	9	6	10	6	3	3	5	3	9	6	2	3	1	1	R 5.2.4

(continued)

Table 5.1 (continued)

Sl. no.	Design no.	v	b	r	k	λ_1	λ_2	m	n	v^*	b^*	r^*	k^*	λ^*	t	$t+t_1$	Method of construction
60	S103	15	12	8	10	8	4	3	5	3	12	8	2	4	4	4	R 5.2.4
61	S104	15	15	10	10	10	5	3	5	3	15	10	2	5	1	1	R 5.2.4
62	S105	18	18	10	10	10	5	9	2	9	18	10	5	5	0	1	T 5.2.1(i)
63	S107	20	12	6	10	6	2	4	5	4	12	6	2	2	1	1	R 5.2.4
64	S111	22	22	10	10	10	4	11	2	11	22	10	5	4	0	1	T 5.2.1(i)
65	S112	25	10	4	10	4	1	5	5	5	10	4	2	1	2	2	R 5.2.4
66	S113	25	20	8	10	8	2	5	5	5	20	8	2	2	4	4	R 5.2.4
67	S115	30	30	10	10	10	2	6	5	6	30	10	2	2	1	1	R 5.2.4
68	S116	35	21	6	10	6	1	7	5	7	21	6	2	1	1	1	R 5.2.4
69	S119	42	42	10	10	10	2	21	2	21	42	10	5	2	0	1	T 5.2.1(i)
70	S120	45	36	8	10	8	1	9	5	9	36	8	2	1	4	4	R 5.2.4
71	S121	50	30	6	10	6	1	25	2	25	30	6	5	1	0	1	T 5.2.1(i)
72	S123	55	55	10	10	10	1	11	5	11	55	10	2	1	1	1	R 5.2.4
73	S124	82	82	10	10	10	1	41	2	41	82	10	5	1	0	1	T 5.2.1(i)

$(v, b, r, k, \lambda_1, \lambda_2, m, n)$ denote the parameters of the PBIBD and $(v^*, b^*, r^*, k^*, \lambda^*)$ denote the parameters of the BIBD used to construct the PBIBD. t denotes the number of optimum **W**-matrices and t_1 denotes the additional number of **W**-matrices mentioned in Remark 5.2.4

Table 5.2 OCDs in SRGDDs

		v	b	r	k	λ_1	λ_2	m	n	$OA(n^2\lambda_2, m, n, 2)$	m_2	$c = (n-1)(k-1)m_2$ (cf. T 5.2.2)
74	SR1	4	4	2	2	0	1	2	2	$OA(4, 3, 2, 2)$	1	1
75	SR2	4	8	4	2	0	2	2	2	$OA(8, 7, 2, 2)$	5	5
76	SR3	4	12	6	2	0	3	2	2	$OA(12, 11, 2, 2)$	9	9
77	SR4	4	16	8	2	0	4	2	2	$OA(16, 15, 2, 2)$	13	13
78	SR5	4	20	10	2	0	5	2	2	$OA(20, 19, 2, 2)$	17	17
79	SR9	8	16	4	2	0	1	2	4	$OA(16, 5, 4, 2)$	1	3
80	SR10	8	32	8	2	0	2	2	4	$OA(32, 10, 4, 2)$	8	24
81	SR13	12	36	6	2	0	1	2	6	$OA(36, 7, 6, 2)$	5	5
82	SR15	16	64	8	2	0	1	2	8	$OA(64, 9, 8, 2)$	6	42
83	SR17	20	100	10	2	0	1	2	10	$OA(100, 11, 10, 2)$	9	9
84	SR36	8	8	4	4	0	2	4	2	$OA(8, 7, 2, 2)$	3	9
85	SR37	8	12	6	4	0	3	4	2	$OA(12, 11, 2, 2)$	7	21
86	SR39	8	16	8	4	0	4	4	2	$OA(16, 15, 2, 2)$	11	33
87	SR40	8	20	10	4	0	5	4	2	$OA(20, 19, 2, 2)$	15	45

(continued)

Table 5.2 (continued)

		v	b	r	k	λ_1	λ_2	m	n	$OA(n^2\lambda_2, m, n, 2)$	m_2	$c = (n-1)(k-1)m_2$ (cf. T 5.2.2)
88	SR44	16	16	4	4	0	1	4	4	OA(16, 5, 4, 2)	1	9
89	SR45	16	32	8	4	0	2	4	4	OA(32, 10, 4, 2)	6	54
90	SR49	32	64	8	4	0	1	4	8	OA(64, 9, 8, 2)	5	105
91	SR51	40	100	10	4	0	1	4	10	OA(100, 11, 10, 2)	7	21
92	SR66	12	8	4	6	0	2	6	2	OA(8, 7, 2, 2)	1	1
93	SR67	12	12	6	6	0	3	6	2	OA(12, 11, 2, 2)	5	5
94	SR69	12	16	8	6	0	4	6	2	OA(16, 15, 2, 2)	9	9
95	SR70	12	20	10	6	0	5	6	2	OA(20, 19, 2, 2)	13	13
96	SR74	24	32	8	6	0	2	6	4	OA(32, 10, 4, 2)	4	12
97	SR78	48	64	8	6	0	1	6	8	OA(64, 9, 8, 2)	3	21
98	SR91	16	12	6	8	0	3	8	2	OA(12, 11, 2, 2)	3	21
99	SR92	16	16	8	8	0	4	8	2	OA(16, 15, 2, 2)	7	49
100	SR93	16	20	10	8	0	2	8	4	OA(32, 19, 2, 2)	11	77
101	SR95	32	32	8	8	0	2	8	4	OA(32, 10, 4, 2)	2	42
102	SR97	64	64	8	8	0	1	8	8	OA(64, 9, 8, 2)	1	49
103	SR106	20	12	6	10	0	3	10	2	OA(12, 11, 2, 2)	1	1
104	SR107	20	16	8	10	0	4	10	2	OA(16, 15, 2, 2)	5	5
105	SR108	20	20	10	10	0	5	10	2	OA(20, 19, 2, 2)	9	9

Table 5.3 OCDs in RGDDs

		v	b	r	k	λ_1	λ_2	m	n	t	$3t$	Method of construction
106	R114	15	15	4	4	0	1	5	3	1	3	Example 5.2.5
107	R129	27	54	8	4	0	1	9	3	2	6	T 5.2.3

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Chapter 6
OCDs in Binary Proper Equireplicate Block Design Set-Up

6.1 Introduction

In Chaps. 4 and 5, we have considered OCDs in the set-ups of BIBDs and PBIBDs, which belong to the class of BPEBDs. It was observed earlier that the constructions of OCDs on BIBDs and PBIBDs depend heavily on the method of constructions of these designs and also that the designs having cyclic nature were more suitable for constructing OCDs. Dutta et al. (2010) investigated the problem of construction of OCDs for the general class of BPEBDs where b is a multiple of v. The cyclic designs with 'full sets', a number of BIBDS, PBIBDs and a host of other designs belong to this class of PBEBDs and consequently the construction of OCDs on these set-ups will follow from the general method. The only restriction that the designs have to follow is that b should be a multiple of v.

The cyclic designs with 'partial sets' do not have b as a multiple of v but as these are BPEBDs we have considered the set-ups as a related discussion. In this chapter, we mainly concentrate on Dutta et al. (2010) and describe the construction of OCDs described therein.

6.2 BPEBDs with $b = mv$

It can be noticed that it is difficult to construct OCDs for any arbitrary block design. The procedures depend heavily on the methods of construction of the corresponding block designs and often optimum **W**-matrices are searched for designs which are mainly constructed through the method of differences. But now we shall describe a technique for constructing OCDs in BPEBDs with $b = mv$, m = positive integer, which does not depend on the method of construction and hence can be widely applied to a large class of commonly used block designs. The following lemma and theorem will help us in the construction of OCDs in such set-ups.

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Lemma 6.2.1 *Let* $\mathbf{C}$ *be a* $k \times b$ *matrix with* v *elements* $t_1, t_2, \ldots, t_v$ *where* $b = mv$, $m=$ *a positive integer, such that each element occurs at most once in each column and an equal number of times in the whole matrix* $\mathbf{C}$. *Then from* $\mathbf{C}$ *we can construct a* $v \times b$ *matrix* $\mathbf{A}$ *with* $(k+1)$ *symbols* $a_1, a_2, \ldots, a_k$ *and 0 such that each of the non-null symbols occurs once and only once in each of the* b *columns and* m *times in each of the* v *rows of* $\mathbf{A}$.

Proof From the properties of the matrix $\mathbf{C}$ it can be easily seen that the columns can be identified with the b blocks of a BPEBD d with constant block size k and with v treatments $t_1, t_2, \ldots, t_v$. We know from Agrawal (1966) that for a BPEBD with $b = mv$, the k treatments in the b blocks of d can always be arranged such that each treatment occurs m times in each of the k positions in the blocks. We denote such an arrangement by a $k \times b$ matrix $\mathbf{B}$. From the above matrix $\mathbf{B}$, we can construct a $v \times b$ matrix $\mathbf{A}$ by putting the element a_l in its (i, j)th cell if t_i occurs in the lth row and jth column of $\mathbf{B}$, $l = 1, 2, \ldots, k,\ i = 1, 2, \ldots, b,\ j = 1, 2, \ldots, v$. Other positions are filled in with zeros. Obviously it follows from the property of $\mathbf{B}$ that each of $a_1, a_2, \ldots, a_k$ occurs once and only once in each of the b columns of $\mathbf{A}$. As every treatment occurs m times in each of the k rows of $\mathbf{B}$, it is evident that each of the symbols $a_1, a_2, \ldots, a_k$ occurs m times in each row of $\mathbf{A}$. Thus the lemma is proved. □

Remark 6.2.1 It may sometimes be challenging to construct a $\mathbf{B}$ mentioned above. But if a BPEBD with $b = mv$ has a cyclic solution, it is very straightforward to construct the $\mathbf{B}$-matrix. When the block design with $b = mv$ does not have a cyclic solution, the construction of $\mathbf{B}$ seems to be difficult and a trial and error method is used to get the desired configuration, whose existence is guaranteed by Lemma 6.2.1.

Now we prove the main theorem.

Theorem 6.2.1 *For any BPEBD* $d(v, b, r, k)$ *with* $b = mv$; m (≥ 1) *a positive integer,* $(k-1)$ *optimum* $\mathbf{W}$*-matrices can be constructed provided* $\mathbf{H}_k$, *a Hadamard matrix of order* k, *exists.*

Proof We write the matrix $\mathbf{H}_k$ as

$$\mathbf{H}_k = (\mathbf{1}, \mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{k-1}) \tag{6.2.1}$$

From a BPEBD $d(v, b, r, k)$, we can always, by Lemma 6.2.1, construct a $v \times b$ matrix $\mathbf{A}$ where each of $a_1, a_2, \ldots, a_k$ occurs m times in each row and once in each column. We identify the k elements of $\mathbf{h}_i$ with the symbols $a_1, a_2, \ldots, a_k$ and replace these symbols in $\mathbf{A}$ with their identified elements of $\mathbf{h}_i$; $i = 1, 2, \ldots, k$. Thus we get $(k-1)$ matrices $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_{k-1}$ corresponding to $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{k-1}$ respectively. From the properties of the matrix $\mathbf{A}$ and those of $\mathbf{H}_k$, it easily follows that the $\mathbf{W}_i$'s satisfy the optimality condition (3.1.5). □

Example 6.2.1 Let us consider the symmetric BIBD with parameters $v = b = 7$, $r = k = 4, \lambda = 2$ constructed heuristically by Nandi (1946). The blocks are: (1, 2, 3, 4), (1, 2, 5, 6), (1, 3, 6, 7), (1, 4, 5, 7), (2, 3, 5, 7), (2, 4, 6, 7), (3, 4, 5, 6). The **B**- and **A**-matrices of Lemma 6.2.1 can respectively be written as

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 7 & 6 \\ 2 & 1 & 6 & 5 & 7 & 4 & 3 \\ 3 & 5 & 1 & 7 & 2 & 6 & 4 \\ 4 & 6 & 7 & 1 & 3 & 2 & 5 \end{pmatrix} \text{ and } \mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 \\ a_2 & a_1 & 0 & 0 & a_3 & a_4 & 0 \\ a_3 & 0 & a_1 & 0 & a_4 & 0 & a_2 \\ a_4 & 0 & 0 & a_1 & 0 & a_2 & a_3 \\ 0 & a_3 & 0 & a_2 & a_1 & 0 & a_4 \\ 0 & a_4 & a_2 & 0 & 0 & a_3 & a_1 \\ 0 & 0 & a_4 & a_3 & a_2 & a_1 & 0 \end{pmatrix}.$$

Consider

$$\mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} = (\mathbf{1}, \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3).$$

Using Lemma 6.2.1, we construct the following three **W**-matrices by using the identification $\mathbf{a} = \mathbf{h}_1$, $\mathbf{a} = \mathbf{h}_2$ and $\mathbf{a} = \mathbf{h}_3$ respectively, where $\mathbf{a}' = (a_1, a_2, a_3, a_4)$ and they are as follows:

$$\mathbf{W}_1 = \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 \end{pmatrix}, \mathbf{W}_2 = \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 \end{pmatrix},$$

$$\mathbf{W}_3 = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

It is easy to observe that $\mathbf{W}_i$-matrices satisfy the optimality condition (3.1.5).

Remark 6.2.2 If k is even, then it follows from Theorem 6.2.1 that at least one optimum **W**-matrix can always be constructed by identifying the **a**'s with $(\mathbf{1}'_{\frac{k}{2}}, -\mathbf{1}'_{\frac{k}{2}})$.

In the following theorem, we shall see that the number of optimum **W**-matrices can be increased substantially if the BPEBD obeys an additional condition of k-resolvability. Now we give the definition of α-resolvability of a design (Raghavarao 1971, p. 59).

Definition 6.2.1 A BPEBD with number of treatments $= v$, number of blocks $= b$, number of replications of each treatment $= r$ and block size $= k$ is said to be α-resolvable if the sets can be grouped into t classes $S_1, S_2, \ldots, S_t$, each with β sets, such that in each class every symbol is replicated α times.

We then have

$$v\alpha = k\beta, \quad b = t\beta, \quad r = t\alpha.$$

Thus a k-resolvable BPEBD with $b = mv$ requires that the $b = mv$ blocks can be partitioned into m sets $S_1, S_2, \ldots, S_m$ each of which contains v blocks such that each of the v treatments occurs k times in each S_i, $i = 1, 2, \ldots, m$.

Theorem 6.2.2 *For a k-resolvable BPEBD with $b = mv$, it is possible to construct $m(k-1)$ optimum $\mathbf{W}$-matrices, provided $\mathbf{H}_k$ and $\mathbf{H}_m$ exist.*

Proof As the design is k-resolvable, then Lemma 6.2.1 is applicable to the blocks of each S_i and from these v blocks a matrix $\mathbf{A}_i^{v \times v}$ can be constructed where $\mathbf{A}_i$ contains each of the symbols $a_1, a_2, \ldots, a_k$ once and only once in each row and in each column, $i = 1, 2, \ldots, m$. It is also to be noted that

$$\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m) \tag{6.2.2}$$

is the $\mathbf{A}$-matrix of Lemma 6.2.1 corresponding to the $b = mv$ blocks of BPEBD where each of the symbols occurs m times in each row and just once in each column of $\mathbf{A}$.

According to the method described in Theorem 6.2.1, we can construct a matrix $\mathbf{W}_{ji}$ from $\mathbf{A}_j$ by identifying $a_1, a_2, \ldots, a_k$ with the elements of $\mathbf{h}_i$, the ith column of $\mathbf{H}_k$ in (6.2.1). By juxtaposing $\mathbf{W}_{ji}$, $j = 1, 2, \ldots, m$, for fixed i, we obtain a matrix $\mathbf{W}_i$, where

$$\mathbf{W}_i = (\mathbf{W}_{1i}, \mathbf{W}_{2i}, \ldots, \mathbf{W}_{mi}), \quad i = 1, 2, \ldots, (k-1). \tag{6.2.3}$$

Varying i in (6.2.3), we get $(k-1)$ matrices $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_{k-1}$ which are optimum $\mathbf{W}$-matrices for the BPEBD. As the BPEBD is k-resolvable and $\mathbf{H}_m$ exists, we can increase the number of optimal $\mathbf{W}$-matrices. By taking the Khatri-Rao product among $\mathbf{h}_j^* = (h_{j1}^*, h_{j2}^*, \ldots, h_{jm}^*)'$, the jth column of $\mathbf{H}_m$ and $\mathbf{W}_i$ of (6.2.3), $m(k-1)$ matrices $\mathbf{W}_{ji}^*$ can be constructed, where

$$\begin{aligned} \mathbf{W}_{ji}^* &= \mathbf{h}_j^{*\prime} \odot \mathbf{W}_i = \left(h_{j1}^*\mathbf{W}_{1i},\ h_{j2}^*\mathbf{W}_{2i},\ \ldots,\ h_{jm}^*\mathbf{W}_{mi}\right), \\ &\forall i = 1, 2, \ldots, (k-1);\ j = 1, 2, \ldots, m. \end{aligned} \tag{6.2.4}$$

It is easy to verify that $\mathbf{W}_{ji}^*$'s satisfy the condition (3.1.5) and hence give $m(k-1)$ optimum $\mathbf{W}$-matrices for the k-resolvable BPEBD. □

Example 6.2.2 Let us consider the following 2-resolvable BIBD with parameters $v = 5, b = 10, r = 4, k = 2, \lambda = 1$ where the blocks can be represented in the form of a matrix $\mathbf{B}$ of order 2×10 as

$$\mathbf{B} = \left(\begin{array}{ccccc|ccccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 & 3 & 4 & 5 & 1 & 2 \end{array}\right) = (\mathbf{B}_1, \mathbf{B}_2)$$

Now the **A**-matrix of order 10×5 can be constructed as

$$\begin{array}{c} Tr. \\ \downarrow \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \text{Bl.} \rightarrow \\ \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \end{array}$$

$$\mathbf{A} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \left(\begin{array}{ccccc|ccccc} a_1 & 0 & 0 & 0 & a_2 & a_1 & 0 & 0 & a_2 & 0 \\ a_2 & a_1 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & a_2 \\ 0 & a_2 & a_1 & 0 & 0 & a_2 & 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & a_1 & 0 & 0 & a_2 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 & a_1 & 0 & 0 & a_2 & 0 & a_1 \end{array}\right) = (\mathbf{A}_1, \mathbf{A}_2)$$

Considering the column $(1, -1)'$ of $\mathbf{H}_2$ and identifying 1 with a_1 and -1 with a_2, one **W**-matrix can be constructed by using Theorem 6.2.1 as

$$\mathbf{W}_1' = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left(\begin{array}{ccccc} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{array}\right) \end{array} = \begin{pmatrix} \mathbf{W}_{11}' \\ \mathbf{W}_{21}' \end{pmatrix}.$$

Since this is a resolvable design and $m = 2$, when $\mathbf{H}_2$ exists, two optimum **W**-matrices can be constructed by using Theorem 6.2.2 as

$$\mathbf{W}_{11}^* = \mathbf{1}_2' \odot \left(\mathbf{W}_{11}, \mathbf{W}_{21}\right); \quad \mathbf{W}_{21}^* = \left(1, -1\right) \odot \left(\mathbf{W}_{11}, \mathbf{W}_{21}\right) = \left(\mathbf{W}_{11} \; -\mathbf{W}_{21}\right).$$

Remark 6.2.3 Let $\mathbf{H}_k$ exist and m (>2) be even, then $2(k-1)$ optimum **W**-matrices can be obtained for a resolvable BPEBD by using $(k-1)$ columns (except the column of all 1's) of $\mathbf{H}_k$ and using $\mathbf{1}_m$ and $(\mathbf{1}'_{\frac{m}{2}}, -\mathbf{1}'_{\frac{m}{2}})'$ for the two choices of orthogonal vectors in the Khatri-Rao product in Theorem 6.2.2.

Remark 6.2.4 If both of k (>2) and m (>2) are even, then two optimum **W**-matrices can be constructed for a resolvable BPEBD by using the two pairs of vectors $((\mathbf{1}'_{\frac{k}{2}}, -\mathbf{1}'_{\frac{k}{2}})', \mathbf{1}_m)$ and $((\mathbf{1}'_{\frac{k}{2}}, -\mathbf{1}'_{\frac{k}{2}})', (\mathbf{1}'_{\frac{m}{2}}, -\mathbf{1}'_{\frac{m}{2}})')$ for the columns of $\mathbf{H}_k$ and $\mathbf{H}_m$ respectively in Theorem 6.2.2.

Remark 6.2.5 Let $\mathbf{H}_m$ exist and $k(>2)$ be even. Then following Theorem 6.2.2, m optimum **W**-matrices can be constructed for a resolvable BPEBD by using m columns of $\mathbf{H}_m$ and $(\mathbf{1}'_{\frac{k}{2}}, -\mathbf{1}'_{\frac{k}{2}})'$ as a column of fictitious $\mathbf{H}_k$.

Remark 6.2.6 If t optimum **W**-matrices exist for any BPEBD, then the same number of **W**-matrices exist for the dual of that design where the blocks of the original design play the role of the treatments. This is because an optimum **W**-matrix for the dual design is also optimum for the original design.

6.3 Cyclic Designs

Cyclic designs are BPEBDs obtained by developing $m(\geq 1)$ initial blocks where the v treatments are the elements of a module M. All cyclic designs belong to the class of PBIBDs with at most $\frac{v}{2}$ associate classes. Many incomplete block designs may be set out as cyclic designs.

If there are v treatments denoted by $0, 1, \ldots, v-1$ which are elements of a module M, and are arranged in blocks of size k so that each treatment is replicated r times, then the cyclic design with these parameters is denoted by $\mathrm{C}(v, k, r)$. Given any initial block, another block is generated by adding $\alpha \pmod v$ to each treatment of the initial block where $\alpha \in M$. If all the v blocks thus obtained from the given initial block are all distinct then this set of blocks is said to form a 'full set'. If v and k are relatively prime to each other then the v blocks generated from an initial block always give a full set with parameters $(v, k = r)$. On the other hand if v and k have a common divisor d, then for every value of d, there always exists at least one initial block where all the v blocks generated from an initial block are not distinct; only $\frac{v}{d}$ of them are distinct. This set of blocks forms a 'partial set' with parameters $(v, k, r = \frac{v}{d})$. Full or partial sets can be used singly or in combination to construct cyclic designs. For a detailed study, one is referred to John et al. (1972) and John (1987).

According to John (1987), for given v and k, the $\binom{v}{k}$ distinct blocks can be set out in a number of cyclic sets where the sets are either 'full sets' consisting of v blocks each or are 'partial sets'. If v and k are relatively prime, then all the sets are 'full sets'. On the other hand, if v and k are not relatively prime then 'partial sets' consisting of $\frac{v}{d}$ distinct blocks arise, where d is any common divisor of v and k. For example, for $v = 7$ and $k = 3$ the $35 = \binom{7}{3}$ all possible distinct blocks can be set out in five cyclic 'full sets' each of 7 blocks and the five initial blocks can be taken as (0, 1, 2), (0, 1, 3), (0, 1, 4), (0, 1, 5), (0, 2, 4) mod 7. On the other hand, for $v = 8$ and $k = 4$, the $70 = \binom{8}{4}$ all possible blocks can not be divided into all 'full sets' as 'full sets' because 8 does not divide 70. Moreover as 4 and 2 are common factors of $v = 8$ and $k = 4$, there should be 'partial sets', one containing 4 blocks and another containing 2 blocks. The seventy distinct blocks can be set out in 8 'full sets' of 8 blocks each; one half-set of four blocks viz. (0, 1, 4, 5), (1, 2, 5, 6), (2, 3, 6, 7), (3, 4, 7, 0) and one quarter-set of two blocks given by (0, 2, 4, 6), (1, 3, 5, 7).

If there exists a partial set consisting of $\frac{k}{d}$ blocks in a cyclic design, then we see that each treatment is replicated $\frac{k}{d}$ times in these blocks. So the number of covariates to be accommodated in a cyclic design depends on whether the sets are full or partial and also on the number of sets. When cyclic designs consist of 'full sets' only, then a systematic way for assigning values to the covariates can be developed. However, when a design contains 'partial sets', it is difficult to specify the number of covariates to be accommodated beforehand. Some examples of cyclic designs containing 'partial sets' are considered in Dutta et al. (2010) where they have provided a solution for OCDs through an ad hoc method.

It is to be noted that Das et al. (2003) and Dutta (2004) proposed OCDs on some series of BIBD's which belonged to the class of cyclic designs. Moreover, all irreducible BIBDs can also be obtained by cyclically developing some sets of initial blocks. So we can cover all these designs and a lot of other designs under a general technique described in the following section.

6.3.1 Cyclic Designs Containing 'Full Sets' Only

It is proved in Theorem 6.2.2 that, for resolvable BPEBDs the number of covariates can be increased over the number of covariates for ordinary BPEBD with the same parameters. It can be easily noted that cyclic designs with m initial blocks giving m full sets of blocks always give resolvable BPEBDs. The particular case when the resolvable BPEBDs are cyclic designs, construction can be done more easily by exploiting the circular nature of the blocks. The precise statement follows.

Theorem 6.3.1 *Let a cyclic design with parameters* $v, b = mv, r = mk, k$ *be obtained by developing* m *initial blocks each of size* k *and also let* $\mathbf{H}_k$ *and* $\mathbf{H}_m$ *exist. Then* $m(k-1)$ *optimum* $\mathbf{W}$*-matrices can be constructed.*

Proof Let $\mathbf{H}_k$ and $\mathbf{H}_m$ be written respectively in the following form:

$$\mathbf{H}_k = (\mathbf{1}, \mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{k-1}) \text{ and } \mathbf{H}_m = (\mathbf{h}_1^*, \mathbf{h}_2^*, \ldots, \mathbf{h}_m^*)$$

where

$$\mathbf{h}_i^{*\prime} = (h_{i1}^*, h_{i2}^*, \ldots, h_{im}^*); \quad i = 1, 2, \ldots, m.$$

Also let the m initial blocks of the design be displayed in the form of column vectors in the incidence matrix of that design. The k non-zero elements of the qth initial block are replaced by the k elements of $\mathbf{h}_j$ in that order and are cyclically permuted to get a $v \times v$ matrix $\mathbf{W}_{jq}$. Again from $\mathbf{h}_i^*$ and $\mathbf{W}_{jq}, q = 1, 2, \ldots, m,$ we construct a $v \times b$ matrix $\mathbf{W}_{ji}^*$ by applying Khatri-Rao product, i.e.

$$\mathbf{W}_{ji}^* = \mathbf{h}_i^{*\prime} \odot \mathbf{W}_j = \left(h_{i1}^*\mathbf{W}_{j1}, h_{i2}^*\mathbf{W}_{j2}, \ldots, h_{im}^*\mathbf{W}_{jm}\right); \tag{6.3.1}$$

where

$$\mathbf{W}_j = (\mathbf{W}_{j1}, \mathbf{W}_{j2}, \ldots, \mathbf{W}_{jm}), \quad j = 1, 2, \ldots, (k-1). \tag{6.3.2}$$

By varying j and i, $m(k-1)$ such $\mathbf{W}^*_{ji}$-matrices can be obtained. It can easily be seen that these matrices satisfy the condition (3.1.5) and are optimum. Thus the theorem follows. □

Note 6.3.1 It is to be noted that the cyclic design with 'full sets' is k-resolvable. So $m(k-1)$ optimum **W**-matrices could have been constructed following Theorem 6.2.2. But as the design possesses cyclic nature, OCDs can be constructed more easily through Theorem 6.3.1.

Example 6.3.1 Let a cyclic design with parameters $v = 13, b = 26, r = 8, k = 4$ with the initial blocks as (1, 4, 12, 13), (1, 4, 10, 13) mod 13, be constructed. Let $\mathbf{H}_4$ and $\mathbf{H}_2$ be written as

$$\mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} = (\mathbf{1}, \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3); \quad \mathbf{H}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = (\mathbf{h}^*_1, \mathbf{h}^*_2).$$

We identify the elements of $\mathbf{h}_1$ with the non-zero elements of the first column of $\mathbf{N}_1$, the part of the incidence matrix corresponding to the first initial block. Then we permute cyclically this column in the same way as $\mathbf{N}_1$ was obtained and get matrix $\mathbf{W}_{11}$. In the same way, by identifying the elements of $\mathbf{h}_1$ with non-zero elements of the first column of $\mathbf{N}_2$, the part of the incidence matrix corresponding to the second initial block and cyclically permuting it, we get $\mathbf{W}_{12}$. These matrices can be visualized as

$$\begin{array}{c} \text{Treatments} \rightarrow \\ \begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \end{array}$$
$$\mathbf{W}'_{11} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ \multicolumn{13}{c}{\text{and cyclic permutations}} \end{pmatrix};$$

$$\begin{array}{c} \text{Treatments} \rightarrow \\ \begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \end{array}$$
$$\mathbf{W}'_{12} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ \multicolumn{13}{c}{\text{and cyclic permutations}} \end{pmatrix}.$$

Then by Theorem 6.3.1, six optimum **W**-matrices for this design can be constructed by using $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$ and the two columns of $\mathbf{H}_2$. For instance, if $\mathbf{h}_1$ and $\mathbf{h}^*_1$ are used, then from (6.3.1), the two **W**-matrices $\mathbf{W}^*_{11}, \mathbf{W}^*_{12}$ are given by

$$\mathbf{W}^*_{11} = \mathbf{h}^{*\prime}_1 \odot \mathbf{W}_1 = \begin{pmatrix} 1, & 1 \end{pmatrix} \odot \begin{pmatrix} \mathbf{W}_{11}, & \mathbf{W}_{12} \end{pmatrix} = \begin{pmatrix} \mathbf{W}_{11}, & \mathbf{W}_{12} \end{pmatrix}$$

and

$$\mathbf{W}_{12}^{*} = \mathbf{h}_2' * \odot \mathbf{W}_1 = \left(1, \ -1\right) \odot \left(\mathbf{W}_{11}, \ \mathbf{W}_{12}\right) = \left(\mathbf{W}_{11}, \ -\mathbf{W}_{12}\right)$$

Incidentally, it is seen that the cyclic design, discussed in Chap. 5, is a two-associate class PBIBD with parameters $v = 13, b = 26, r = 8, k = 4, \lambda_1 = 1, \lambda_2 = 3, n_1 = n_2 = 6$. The first associates of the treatment i are $(i+2, i+5, i+6, i+7, i+8, i+11)$ (mod 13).

Below we make some remarks regarding methods of construction of optimum $\mathbf{W}$-matrices where at least one of $\mathbf{H}_k$ and $\mathbf{H}_m$ does not exist so that Theorem 6.3.1 can not be applied.

Remark 6.3.1 If $\mathbf{H}_k$ exists but m is an odd integer, so that $\mathbf{H}_m$ does not exist, then $(k-1)$ optimum $\mathbf{W}$-matrices can be obtained for the said cyclic design and they are given by $\mathbf{1}_m \odot \mathbf{W}_j$, $j = 1, 2, \ldots, k-1$, where $\mathbf{W}_j$ is obtained from (6.3.2).

Remark 6.3.2 If none of $\mathbf{H}_k$ and $\mathbf{H}_m$ exists and $k \equiv 2$ (mod 4) and m is an odd integer, then one optimum $\mathbf{W}$-matrix can be constructed as $\mathbf{1}_m \odot \mathbf{W}^*$, where $\mathbf{W}^*$ is a matrix analogous to $\mathbf{W}_j$ of equation (6.3.2) obtained by using $(\mathbf{1}'_{\frac{k}{2}}, -\mathbf{1}'_{\frac{k}{2}})'$ for the cyclical permutation in the incidence matrix.

Remark 6.3.3 Let $\mathbf{H}_k$ exist and $m \equiv 2\,(\mathrm{mod}\,4), m > 2$. In this case, $2(k-1)$ optimum $\mathbf{W}$-matrices given by $\mathbf{h}^{**} \odot \mathbf{W}_j$ can be obtained, where $\mathbf{h}^{**}$ is $\mathbf{1}_m$ or $(\mathbf{1}'_{\frac{m}{2}}, -\mathbf{1}'_{\frac{m}{2}})'$.

Remark 6.3.4 If each of k, m is of the form 2 (mod 4) so that none of $\mathbf{H}_k$ and $\mathbf{H}_m$ exists, then 2 optimum $\mathbf{W}$-matrices can be constructed as $\mathbf{h}^{**} \odot \mathbf{W}^*$ where $\mathbf{h}^{**}$ is $\mathbf{1}_m$ or $(\mathbf{1}'_{\frac{m}{2}}, -\mathbf{1}'_{\frac{m}{2}})'$ and $\mathbf{W}^*$ is the same as in Remark 6.3.2.

Remark 6.3.5 If $\mathbf{H}_m$ exists but $\mathbf{H}_k$ does not where $k \equiv 2$ (mod 4), $k > 2$, then m optimum $\mathbf{W}$-matrices can be constructed as $\mathbf{h}_i^* \odot \mathbf{W}^*$, $i = 1, 2, \ldots, m$ where $\mathbf{W}^*$ is the same as Remark 6.3.2.

6.3.2 Cyclic Designs Containing Some Partial Sets

It was mentioned earlier that it is difficult to propose a systematic method for finding OCDs for cyclic designs containing 'partial sets'. It should be noted that the number of optimum $\mathbf{W}$-matrices depends on the properties of the 'partial sets' and consequently on the nature of the columns of $\mathbf{H}_k$ whose elements are used to replace the non-zero elements in the blocks of the incidence matrix. We consider the following example illustrating an ad hoc method which depends on the nature of the partial set.

Example 6.3.2 Consider the irreducible BIBD with parameters $v = 6, b = \binom{6}{4} = 15$; $r = \binom{5}{3} = 10, k = 4, \lambda = \binom{4}{2} = 6$. The design can be obtained from the three initial blocks: [(0, 1, 2, 3), (0, 2, 3, 4), (0, 2, 3, 5)] mod 6, where the first two give 'full sets' containing six distinct blocks each and the last one gives a 'partial set' containing only three distinct blocks. We consider $\mathbf{H}_4$ as

$$\mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} = (\mathbf{1}, \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3). \tag{6.3.3}$$

It is to be noted that, as in Theorem 6.3.1, all the three columns $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{h}_3$ of (6.3.3) cannot be used in each of the three subsets of blocks obtained by developing cyclically the three initial blocks. The last three blocks obtained from the third initial block are 'partially cyclic'; only $\mathbf{h}_1$, $\mathbf{h}_2$ can be used to construct $\mathbf{W}$-matrices but $\mathbf{h}_3$ cannot be used as it will not lead to zero column-sums. Using $\mathbf{h}_1$ and $\mathbf{h}_2$, we get two $\mathbf{W}$-matrices, namely $\mathbf{W}'_{(1)}$ and $\mathbf{W}'_{(2)}$ respectively by applying the method described in Theorem 6.3.1:

$$\begin{array}{c} \text{Treatments} \rightarrow \\ \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \end{array} \qquad \begin{array}{c} \text{Treatments} \rightarrow \\ \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \end{array}$$

$$\mathbf{W}'_{(1)} = \left(\begin{array}{rrrrrr} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ -1 & 0 & 0 & 1 & -1 & 1 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ -1 & 1 & -1 & 0 & 0 & 1 \\ \hline 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ -1 & 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \\ \hline 1 & 0 & -1 & \multicolumn{1}{|r}{1} & 0 & -1 \\ -1 & 1 & 0 & \multicolumn{1}{|r}{-1} & 1 & 0 \\ 0 & -1 & 1 & \multicolumn{1}{|r}{0} & -1 & 1 \end{array}\right); \quad \mathbf{W}'_{(2)} = \left(\begin{array}{rrrrrr} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ -1 & -1 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 1 & 0 & 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \\ \hline 1 & 0 & -1 & \multicolumn{1}{|r}{-1} & 0 & 1 \\ -1 & 1 & 0 & \multicolumn{1}{|r}{1} & -1 & 0 \\ 0 & -1 & 1 & \multicolumn{1}{|r}{0} & 1 & -1 \end{array}\right)$$

6.3.3 Cyclic Designs Where Each Element Corresponds to a Number of Symbols

Here any treatment is denoted by α_j where α is any element of the module $M = (0, 1, \ldots, m)$ and j is one of the n symbols $1, 2, \ldots, n$. The following is an example of a design which is obtained by the classical method of difference (cf. Bose 1939) where each symbol of the module $M = (0, 1, 2, 3, 4)$ corresponds to two symbols

1 and 2. If the blocks can be grouped into sets which have cycles, then optimum **W**-matrices can be constructed by exploiting this property. The method is illustrated through the following example.

Example 6.3.3 Consider the GD design with parameters $v = 10, b = 20, r = 8, k = 4, \lambda_1 = 0, \lambda_2 = 3, m = 5, n = 2$ with the initial group $(0_1, 0_2)$ mod 5. The initial blocks are $(0_1, 1_2, 2_2, 4_2), (0_2, 1_1, 2_1, 4_1), (0_1, 2_2, 3_2, 4_2), (0_2, 2_1, 3_1, 4_1)$ mod 5. We divide the four initial blocks into two sets viz. $S_1 = \{(0_1, 1_2, 2_2, 4_2), (0_1, 2_2, 3_2, 4_2)\}$ and $S_2 = \{(0_2, 1_1, 2_1, 4_1), (0_2, 2_1, 3_1, 4_1)\}$. In the first five columns of the incidence matrix corresponding to the initial block $(0_1, 1_2, 2_2, 4_2)$ of S_1, the non-zero elements are replaced by the elements of $\mathbf{h}_1$ of (6.3.3) and in the last five columns corresponding to $(0_1, 2_2, 3_2, 4_2)$ of S_1, the non-zero elements are replaced by those of $-\mathbf{h}_1$. We denote this matrix, of order 10×10, by $\mathbf{U}_1^{(1)}$. In the same way by using $\mathbf{h}_1$ and $-\mathbf{h}_1$ in the two initial blocks of S_2 we get a matrix $\mathbf{U}_1^{(2)}$. $\mathbf{U}_1^{(1)}$ and $\mathbf{U}_1^{(2)}$ are given by

$$\mathbf{U}_1^{(1)} = \begin{array}{rr} & 0_1 \\ & 1_1 \\ & 2_1 \\ & 3_1 \\ Tr. & 4_1 \\ \downarrow & 0_2 \\ & 1_2 \\ & 2_2 \\ & 3_2 \\ & 4_2 \end{array} \left(\begin{array}{rrrrr|rrrrr} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & -1 & 1 \\ 1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & 1 & 0 & 0 \end{array}\right),$$

$$\mathbf{U}_1^{(2)} = \begin{array}{rr} & 0_1 \\ & 1_1 \\ & 2_1 \\ & 3_1 \\ Tr. & 4_1 \\ \downarrow & 0_2 \\ & 1_2 \\ & 2_2 \\ & 3_2 \\ & 4_2 \end{array} \left(\begin{array}{rrrrr|rrrrr} 0 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1 & -1 \\ -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

From $\mathbf{U}_1^{(1)}$ and $\mathbf{U}_2^{(2)}$, we construct two optimum **W**-matrices as

$$\mathbf{W}^{(1,1)} = (\mathbf{U}_1^{(1)}, \mathbf{U}_1^{(2)}), \quad \mathbf{W}^{(1,2)} = (\mathbf{U}_1^{(1)}, -\mathbf{U}_1^{(2)}).$$

Similarly we can get four more optimum **W**-matrices viz. $\mathbf{W}^{(2,1)}, \mathbf{W}^{(2,2)}, \mathbf{W}^{(3,1)}$ and $\mathbf{W}^{(3,2)}$ by using $\mathbf{h}_2$ and $\mathbf{h}_3$.

6.4 t-Fold BPEBDs

In Das et al. (2003) it was been seen that optimum **W**-matrices could be constructed for BIBDs with repeated blocks. In this section, we propose to extend the result to any BPEBD with repeated blocks. Thus the method will be applicable to BIBDs, PBIBDs and a lot of others with repeated blocks. Also it has been seen that in our method the number of optimum **W**-matrices can substantially be increased. Precise statement follows.

Theorem 6.4.1 *Let t repetitions of the blocks of a BPEBD $d(v, b, r, k)$ be considered where $b = mv$. Also let $\mathbf{H}_k$ and $\mathbf{H}_t$ exist. Then we can construct $t(k-1)$ optimum $\mathbf{W}$-matrices for the t-fold design BPEBD $d(v, bt, rt, k)$.*

Proof As $\mathbf{H}_k$ exists, according to the method described in Theorem 6.2.1, it is possible to construct $(k-1)$ optimum **W**-matrices for the lth BPEBD $d(v, b, r, k)$ with $b = mv$, m, a positive integer, where the nth optimum **W**-matrix for the lth BPEBD be denoted as $\mathbf{U}_l^{(n)}$, $l = 1, 2, \ldots, t$, $n = 1, 2, \ldots, k-1$.

For fixed n, the optimum **W**-matrix for the whole design considering all the folds together is given by

$$\mathbf{U}^{(n)} = (\mathbf{U}_1^{(n)}, \mathbf{U}_2^{(n)}, \ldots, \mathbf{U}_t^{(n)}).$$

By assumption, $\mathbf{H}_t$ exists and it is written as

$$\mathbf{H}_t = (\mathbf{h}_1^*, \mathbf{h}_2^*, \ldots, \mathbf{h}_{t-1}^*, \mathbf{h}_t^*) \tag{6.4.1}$$

where $\mathbf{h}_t^* = (1, 1, \ldots, 1)'$.

Then as the t-fold BPEBD is resolvable, then by Theorem 6.2.2

$$\mathbf{U}_{jn} = \mathbf{h}_j^{*\prime} \odot (\mathbf{U}_1^{(n)}, \mathbf{U}_2^{(n)}, \ldots, \mathbf{U}_t^{(n)}), \quad j = 1, 2, \ldots, t; \; n = 1, 2, \ldots, k-1 \tag{6.4.2}$$

gives $t(k-1)$ optimum **W**-matrices for the t-fold BPEBD $d(v, tb, tr, k)$. □

Corollary 6.4.1 *Let t repetitions of the blocks of a BPEBD $d(v, b, r, k)$ be considered where $b = mv$ where $b = mv = p^h + 1$, m, h are positive integers, $p^h \equiv 1$ (mod 4) and p is a prime odd number. Also let $\mathbf{H}_k$ and $\mathbf{H}_t$ exist. Then we can construct $(t-1)(b-1)(k-2)$ optimum $\mathbf{W}$-matrices in addition to $t(k-1)$ optimum $\mathbf{W}$-matrices constructed earlier.*

Proof Additional $(t-1)(b-1)(k-2)$ optimum **W**-matrices for the BPEBD $d(v, bt, rt, k)$ can be constructed through the following steps:

Step 1:
Let, s be an odd prime power and let $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$ denote the elements of GF(s). Consider, an $s \times s$ matrix $\mathbf{Q} = (q_{ij})$, where $q_{ij} = \chi(\alpha_i - \alpha_j)$, i,j $= 0, 1, \ldots, (s-1)$ and χ is the Legendre function satisfying

$$\begin{aligned}\chi(\beta) &= 1 \quad \text{if } \beta \text{ is a quadratic residue in GF(s)}\\ &= 0 \quad \text{if } \beta = 0\\ &= -1 \text{ otherwise.}\end{aligned}$$

This map satisfies $\chi(\beta_1\beta_2) = \chi(\beta_1)\chi(\beta_2)$. It is well known that $\mathbf{Q}$ satisfies the following properties (cf. Hedayat et al. (1999), p. 150):

a. $\mathbf{Q}\mathbf{1}_s = \mathbf{0}, \;\; \mathbf{1}_s'\mathbf{Q} = \mathbf{0}'$
b. $\mathbf{Q}\mathbf{Q}' = s\mathbf{I}_s - \mathbf{J}_s$
c. $\mathbf{Q}$ is symmetric if $s \equiv 1 \pmod 4$, skew$-$symmetric if $s \equiv 3 \pmod 4$.

Under condition of the theorem it follows that $b \equiv 2 \pmod 4$ and $b-1$ is an odd prime power and so a symmetric matrix $\mathbf{Q} = (q_{ij})$ of order $(b-1) \times (b-1)$ exists.

Let $\mathbf{u}_{li}^{(n)}$ be the ith row of the nth optimum $\mathbf{W}$-matrix $\mathbf{U}_l^{(n)}$ for the lth BPEBD $d(v, b, r, k)$, $n = 1, 2, \ldots, k-1$, $l = 1, 2, \ldots, t$. Using the rows of $\mathbf{U}_l^{(n)}$ and the elements of $\mathbf{Q}^{(b-1)\times(b-1)}$, we define a matrix $\mathbf{A}_l(n, n')$ of order $b \times (b-1)v$ where the ith row is the partitioned into $(b-1)$ sub-vectors $\mathbf{a}_{ij}^{(l)}(n, n')$ of order $1 \times v$, as

$$\begin{aligned}\mathbf{a}_{ij}^{(l)}(n, n') &= \mathbf{u}_{lj}^{(n)} \qquad \text{if } i = j = 1, 2, \ldots, b-1\\ &= q_{ij}\mathbf{u}_{lj}^{(n')} \;\; \text{if } i \neq j = 1, 2, \ldots, b-1, \;\; n \neq n' = 1, 2, \ldots, k-1\\ &= -\mathbf{u}_{lb}^{(n)} \quad \text{if } i = b, \;\; j = 1, 2, \ldots, (b-1).\end{aligned}$$

Similarly we can define another matrix $\mathbf{B}_l(n, n')$ of order $b \times (b-1)v$ where sub-vector $\mathbf{b}_{ij}^{(l)}(n, n')$ in $\mathbf{B}_l(n, n')$ stands for $\mathbf{a}_{ij}^{(l)}(n, n')$ of the $\mathbf{A}_l(n, n')$-matrix. Actually

$$\begin{aligned}\mathbf{b}_{ij}^{(l)}(n, n') &= -\mathbf{u}_{lj}^{(n')} \;\; \text{if } i = j = 1, 2, \ldots, b-1\\ &= q_{ij}\mathbf{u}_{lj}^{(n)} \;\; \text{if } i \neq j = 1, 2, \ldots, b-1, \;\; n \neq n' = 1, 2, \ldots, k-1\\ &= \mathbf{u}_{lb}^{(n')} \qquad \text{if } i = b, \;\; j = 1, 2, \ldots, (b-1).\end{aligned}$$

This is to be noted that $\mathbf{u}_{li}^{(n)}\mathbf{u}_{li}^{(n)\prime} = k$ =block size and $\mathbf{u}_{li}^{(n)}\mathbf{u}_{li}^{(n')\prime} = 0$ for all $l, \; i, \; n \neq n'$. Now we construct the matrix $\mathbf{C}_l(n, n')$ of order $b \times 2(b-1)v$ as follows:

$$\mathbf{C}_l(n, n') = (\mathbf{A}_l(n, n') : \; \mathbf{B}_l(n, n')).$$

Let $\mathbf{C}_{lj}(n, n')$ denote the jth set of v columns of $\mathbf{C}_l(n, n')$, $j = 1, 2, \ldots, 2(b-1)$, i.e.

$$\mathbf{C}_{lj}(n, n') = \begin{pmatrix}\mathbf{a}_{1j}^{(l)}(n, n')\\ \mathbf{a}_{2j}^{(l)}(n, n')\\ \vdots\\ \mathbf{a}_{bj}^{(l)}(n, n')\end{pmatrix}, \quad \mathbf{C}_{l,b-1+j}(n, n') = \begin{pmatrix}\mathbf{b}_{1j}^{(l)}(n, n')\\ \mathbf{b}_{2j}^{(l)}(n, n')\\ \vdots\\ \mathbf{b}_{bj}^{(l)}(n, n')\end{pmatrix}, \quad j = 1, 2, \ldots, b-1. \tag{6.4.3}$$

Using the properties (b) and (c) of the $\mathbf{Q}$-matrix, we can easily check that $\mathbf{1}'_b[\mathbf{C}_{lj}(n, n') * \mathbf{C}_{lj'}(n, n')]\mathbf{1}_v = 0 \; \forall \; j \neq j' = 1, 2, \ldots, 2(b-1)$, where '*' denotes the Hadamard product.

Step 3:
For fixed $n \neq n'$, let us define the following $v \times bt$ matrix as

$$\mathbf{W}_{pi}(n, n') = \mathbf{h}_p^{*\prime} \odot \left(\mathbf{C}'_{1i}(n, n'),\ \mathbf{C}'_{2i}(n, n'),\ \ldots,\ \mathbf{C}'_{ti}(n, n')\right);$$
$$p = 1, 2, \ldots, (t-1),\ i = 1, 2, \ldots, 2(b-1). \tag{6.4.4}$$

We show below that $\mathbf{W}'_{lpi}(n, n')$ gives an optimum $\mathbf{W}$-matrix for a BPEBD $d(v, bt, rt, k)$.

From the properties of $\mathbf{C}_{li}(n, n')$ and $\mathbf{h}_p^*$ it can be proved that

(i) $\mathbf{1}'_{tb}\mathbf{W}_{pi}(n, n') = \mathbf{0}' \;\; \forall i,\ p$ as $\mathbf{1}_{tb} = \mathbf{1}_t \otimes \mathbf{1}_b$
(ii) $\mathbf{u}_{li}^{(n)\prime}\mathbf{1}_v$=0 $\forall i = 1, 2, \ldots, b, n = 1, 2, \ldots, (k-1), l = 1, 2, \ldots, t$,
(iii) $\mathbf{1}'_{mb}[\mathbf{W}_{pi}(n, n') * \mathbf{W}_{pj}(n, n')]\mathbf{1}_v$=0 $\forall l = 1, 2, \ldots, (t-1); i \neq j = 1, 2, \ldots, 2(b-1)$,

which imply conditions $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}_3$ of (3.1.5) respectively.

Again for $(n, n') \neq (n'', n''')$, $\mathbf{W}_{pi}(n, n')$ and $\mathbf{W}_{pi}(n'', n''')$ are orthogonal in the sense that sum of the elements of the Hadamard product of the above matrices is zero. Thus we have only $\frac{k-2}{2}$ such distinct pairs of (n, n'). So using these $\frac{k-2}{2}$ distinct pairs for each l and i, j, we can generate $\frac{2(t-1)(b-1)(k-2)}{2}$ i.e. $(t-1)(b-1)(k-2)$ optimum $\mathbf{W}$-matrices for the BPEBD $d(v, bt, rt, k)$. We can also easily check that these $\mathbf{W}$-matrices are orthogonal to the $t(k-1)$ optimum $\mathbf{W}$-matrices of (6.4.2).

So in all, we get $t(k-1) + (t-1)(b-1)(k-2)$ optimum $\mathbf{W}$-matrices for the BPEBD $d(v, bt, rt, k)$. $\square$

Example 6.4.1 We consider 2-fold of the BIBD($v = 9,\ b = 18,\ r = 8,\ k = 4,\ \lambda = 3$) with the initial blocks $(x^0,\ x^2,\ x^4,\ x^6)$ and $(x,\ x^3,\ x^5,\ x^7)$, where x is a primitive root of GF(3^2).

We write $\mathbf{H}_4$ as

$$\mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} = (\mathbf{1}, \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3).$$

Applying Theorem 6.2.1 we construct $\mathbf{U}_l^{(1)}$ for the lth fold of the design by using $\mathbf{h}_1$ as

$$
\mathbf{U}_l^{(1)\ 18\times 9} = \begin{array}{c}
\begin{array}{ccccccccc} 0 & 1 & x & 2x+1 & 2x+2 & 2 & 2x & x+2 & x+1 \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\
1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\
-1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\
0 & -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & -1 \\
-1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 \\
1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \\
0 & 0 & -1 & -1 & 1 & 0 & 0 & 1 & 0 \\
1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
= \begin{pmatrix}
\mathbf{u}_{l,1}^{(1)} \\ \mathbf{u}_{l,2}^{(1)} \\ \mathbf{u}_{l,3}^{(1)} \\ \mathbf{u}_{l,4}^{(1)} \\ \mathbf{u}_{l,5}^{(1)} \\ \mathbf{u}_{l,6}^{(1)} \\ \mathbf{u}_{l,7}^{(1)} \\ \mathbf{u}_{l,8}^{(1)} \\ \mathbf{u}_{l,9}^{(1)} \\ \mathbf{u}_{l,10}^{(1)} \\ \mathbf{u}_{l,11}^{(1)} \\ \mathbf{u}_{l,12}^{(1)} \\ \mathbf{u}_{l,13}^{(1)} \\ \mathbf{u}_{l,14}^{(1)} \\ \mathbf{u}_{l,15}^{(1)} \\ \mathbf{u}_{l,16}^{(1)} \\ \mathbf{u}_{l,17}^{(1)} \\ \mathbf{u}_{l,18}^{(1)}
\end{pmatrix} ;\ l = 1, 2.
$$

Similarly using $\mathbf{h}_2$ we get $\mathbf{U}_l^{(2)}$ as

$$
\mathbf{U}_l^{(2)\ 18\times 9} = \begin{array}{c}
\begin{array}{ccccccccc} 0 & 1 & x & 2x+1 & 2x+2 & 2 & 2x & x+2 & x+1 \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\
-1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\
0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & -1 \\
-1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 \\
-1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\
0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 \\
1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\
-1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
= \begin{pmatrix}
\mathbf{u}_{l,1}^{(2)} \\ \mathbf{u}_{l,2}^{(2)} \\ \mathbf{u}_{l,3}^{(2)} \\ \mathbf{u}_{l,4}^{(2)} \\ \mathbf{u}_{l,5}^{(2)} \\ \mathbf{u}_{l,6}^{(2)} \\ \mathbf{u}_{l,7}^{(2)} \\ \mathbf{u}_{l,8}^{(2)} \\ \mathbf{u}_{l,9}^{(2)} \\ \mathbf{u}_{l,10}^{(2)} \\ \mathbf{u}_{l,11}^{(2)} \\ \mathbf{u}_{l,12}^{(2)} \\ \mathbf{u}_{l,13}^{(2)} \\ \mathbf{u}_{l,14}^{(2)} \\ \mathbf{u}_{l,15}^{(2)} \\ \mathbf{u}_{l,16}^{(2)} \\ \mathbf{u}_{l,17}^{(2)} \\ \mathbf{u}_{l,18}^{(2)}
\end{pmatrix} ;\ l = 1, 2.
$$

In the same way, using $\mathbf{h}_3$ we can construct $\mathbf{U}_l^{(3)}$. Since $b-1=17$, so we can construct $\mathbf{Q}$-matrix of order 17×17. From (6.4.3) we write the $\mathbf{C}_l(n, n')$-matrix by using the appropriate elements of $\mathbf{Q}$. To save space, below we show $\mathbf{C}_{l,1}(1, 2)$, $\mathbf{C}_{l,2}$, $\mathbf{C}_{l,18}(1, 2)$ and $\mathbf{C}_{l,19}(1, 2)$ only.

$\mathbf{C}_{l,1}(1, 2) = (\mathbf{u}_{l,1}^{(1)\prime}, \mathbf{u}_{l,2}^{(2)\prime}, \text{-}\mathbf{u}_{l,3}^{(2)\prime}, \mathbf{u}_{l,4}^{(2)\prime}, -\mathbf{u}_{l,5}^{(2)\prime}, \mathbf{u}_{l,6}^{(2)\prime}, -\mathbf{u}_{l,7}^{(2)\prime}, \mathbf{u}_{l,8}^{(2)\prime}, -\mathbf{u}_{l,9}^{(2)\prime}, \mathbf{u}_{l,10}^{(2)\prime}, -\mathbf{u}_{l,11}^{(2)\prime}, \mathbf{u}_{l,12}^{(2)\prime}, -\mathbf{u}_{l,13}^{(2)\prime}, \mathbf{u}_{l,14}^{(2)\prime}, -\mathbf{u}_{l,15}^{(2)\prime}, \mathbf{u}_{l,16}^{(2)\prime}, -\mathbf{u}_{l,17}^{(2)\prime}, -\mathbf{u}_{l,18}^{(1)\prime})'$,

$\mathbf{C}_{l,2}(1, 2) = (\mathbf{u}_{l,1}^{(2)\prime}, \mathbf{u}_{l,2}^{(1)\prime}, \mathbf{u}_{l,3}^{(2)\prime}, \mathbf{u}_{l,4}^{(2)\prime}, \mathbf{u}_{l,5}^{(2)\prime}, -\mathbf{u}_{l,6}^{(2)\prime}, \mathbf{u}_{l,7}^{(2)\prime}, -\mathbf{u}_{l,8}^{(2)\prime}, -\mathbf{u}_{l,9}^{(2)\prime}, \mathbf{u}_{l,10}^{(2)\prime}, \mathbf{u}_{l,11}^{(2)\prime}, -\mathbf{u}_{l,12}^{(2)\prime}, -\mathbf{u}_{l,13}^{(2)\prime}, -\mathbf{u}_{l,14}^{(2)\prime}, -\mathbf{u}_{l,15}^{(2)\prime}, \mathbf{u}_{l,16}^{(2)\prime}, -\mathbf{u}_{l,17}^{(2)\prime}, -\mathbf{u}_{l,18}^{(1)\prime})'$,

$\mathbf{C}_{l,18}(1, 2) = (-\mathbf{u}_{l,1}^{(2)\prime}, \mathbf{u}_{l,2}^{(1)\prime}, -\mathbf{u}_{l,3}^{(1)\prime}, \mathbf{u}_{l,4}^{(1)\prime}, -\mathbf{u}_{l,5}^{(1)\prime}, -\mathbf{u}_{l,6}^{(1)\prime}, -\mathbf{u}_{l,7}^{(1)\prime}, \mathbf{u}_{l,8}^{(1)\prime}, -\mathbf{u}_{l,9}^{(1)\prime}, \mathbf{u}_{l,10}^{(1)\prime}, -\mathbf{u}_{l,11}^{(1)\prime}, \mathbf{u}_{l,12}^{(1)\prime}, -\mathbf{u}_{l,13}^{(1)\prime}, \mathbf{u}_{l,14}^{(1)\prime}, -\mathbf{u}_{l,15}^{(1)\prime}, \mathbf{u}_{l,16}^{(1)\prime}, -\mathbf{u}_{l,17}^{(1)\prime}, \mathbf{u}_{l,18}^{(2)\prime})'$,

$\mathbf{C}_{l,19}(1, 2) = (\mathbf{u}_{l,1}^{(1)\prime}, -\mathbf{u}_{l,2}^{(2)\prime}, \mathbf{u}_{l,3}^{(1)\prime}, \mathbf{u}_{l,4}^{(1)\prime}, \mathbf{u}_{l,5}^{(1)\prime}, -\mathbf{u}_{l,6}^{(1)\prime}, \mathbf{u}_{l,7}^{(1)\prime}, -\mathbf{u}_{l,8}^{(1)\prime}, -\mathbf{u}_{l,9}^{(1)\prime}, \mathbf{u}_{l,10}^{(1)\prime}, \mathbf{u}_{l,11}^{(1)\prime}, -\mathbf{u}_{l,12}^{(1)\prime}, -\mathbf{u}_{l,13}^{(1)\prime}, -\mathbf{u}_{l,14}^{(1)\prime}, -\mathbf{u}_{l,15}^{(1)\prime}, \mathbf{u}_{l,16}^{(1)\prime}, -\mathbf{u}_{l,17}^{(1)\prime}, \mathbf{u}_{l,18}^{(2)\prime})'; l = 1, 2.$

As $t = 2$, $\mathbf{H}_2$ exists and is written as

$$\mathbf{H}_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = (\mathbf{h}_1^*, \mathbf{h}_2^*).$$

From (6.4.4),

$$\mathbf{W}_{1,1}(1, 2)^{36\times 9} = \mathbf{h}_1^* \odot \begin{pmatrix} \mathbf{C}_{l,1}(1, 2) \\ \mathbf{C}_{2,1}(1, 2) \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{l,1}(1, 2) \\ -\mathbf{C}_{2,1}(1, 2) \end{pmatrix}$$

can be constructed.

Similarly other $\mathbf{W}$-matrices such as $\mathbf{W}_{1,2}(1, 2)$, $\mathbf{W}_{1,18}(1, 2)$, and $\mathbf{W}_{1,19}(1, 2)$ can be constructed. In this way we can construct $(t-1)(b-1)(k-2) = 34$ $\mathbf{W}$-matrices for BIBD(9, 36, 16, 4, 6) which is a 2-fold of the BIBD(9, 18, 8, 4, 3). Again from (6.4.2) 6 additional optimum $\mathbf{W}$-matrices can be obtained. So in all, we get 40 optimum $\mathbf{W}$-matrices for BIBD(9, 36, 16, 4, 6).

So far we have we assume $b = mv$ for the BPEBDs. Now we try to construct optimum $\mathbf{W}$-matrices for t-fold BPEBD $d(v, b, r, k)$, where $b \neq mv$, $m(\geq 1)$, a positive integer.

Theorem 6.4.2 *Suppose a cyclic BPEBD $d(v = b, r = k)$ exists. Again, if $\mathbf{H}_k$, $\mathbf{H}_t$ and $\mathbf{Q}$ of order $(b-1) \times (b-1)$ exist, then we can construct $(t-1)(b-1)(k-2) + (t-1)(k-1)$ optimum $\mathbf{W}$-matrices for the t-fold of these BPEBD $d(v = b, r = k)$.*

Proof As $\mathbf{H}_k$ and $\mathbf{H}_t$ exist, $(t-1)(k-1)$ optimum $\mathbf{W}$-matrices can be constructed using the cyclic property of the incidence matrix as in Theorem 6.3.1 and the additional number $(t-1)(b-1)(k-2)$ optimum $\mathbf{W}$-matrices can be constructed by using properties of $\mathbf{Q}$-matrix as in proof of Corollary 6.4.1. □

Appendix

As mentioned earlier, BIBDs and PBIBDs form an important sub-class of BPEBDs and the lists of BIBDs and PBIBDs are readily available (Tables 6.1 and 6.2). So, for OCDs in BPEBD set-up with $b = mv$ we have considered BIBDs with $b = mv$ where $v, b \leq 100,\ r, k \leq 15$ from the list given in Raghavarao (1971) and the GDDs with $b = mv$ and $r \leq 10,\ k \leq 10$ from the same catalogue of Clatworthy (1973). In this connection, we have to mention that in Chap. 5, we have also prepared tables for OCDs in GDDs set-up. However the readers need not get confused between the tables of the two respective chapters. Here we can construct some more OCDs in GDDs set-up and these are separated by using a '$*$' mark for additional designs, which are not in Tables 5.1, 5.2 and 5.3 of Chap. 5, in the 11th column in Table 6.2. A number of BIBDs and GDDs with $b = mv$ are cyclic designs (Table 6.3).

These designs have not been considered separately in cyclic design class. A separate list for the cyclic BIBDs having partial cycles has only been considered. Here c denotes the number of optimum **W**'s for BPEBD. Other parameters have usual significance.

Table 6.1 OCDs in BIBD with $b = mv$

Sl. no.	Parameters					c	Design no.	Method of construction
	v	b	r	k	λ			
1	5	10	4	2	1	2	3	Theorem 6.2.2
2	5	5	4	4	3	3	4	Theorem 6.2.1
3	7	7	4	4	2	3	11	Theorem 6.2.1
4	7	21	6	2	1	1	12	Theorem 6.2.1
5	7	7	6	6	5	1	13	Theorem 6.2.1
6	9	36	8	2	1	4	18	Theorem 6.2.2
7	9	18	8	4	3	6	19	Theorem 6.2.2
8	9	9	8	8	7	7	21	Theorem 6.2.1
9	11	11	6	6	3	1	30	Theorem 6.2.1
10	11	55	10	2	1	1	31	Theorem 6.2.1
11	11	11	10	10	9	1	32	Theorem 6.2.1
12	13	13	4	4	1	3	37	Theorem 6.2.1
13	13	26	12	6	5	2	40	Remark 6.2.5
14	15	15	8	8	4	7	44	Theorem 6.2.1
15	16	16	6	6	2	1	47	Remark 6.2.2
16	16	16	10	10	6	1	49	Remark 6.2.2
17	19	19	10	10	5	1	56	Remark 6.2.2
18	19	57	12	4	2	3	57	Theorem 6.2.1
19	21	42	12	6	3	2	61	Remark 6.2.5

(continued)

Table 6.1 (continued)

Sl. no.	Parameters					c	Design no.	Method of construction
	v	b	r	k	λ			
20	23	23	12	12	6	11	Dual of 64	Theorem 6.2.1
21	23	50	8	4	1	6	66	Theorem 6.2.2
22	27	27	14	14	7	1	Dual of 71	Remark 6.2.2
23	31	31	6	6	1	1	75	Remark 6.2.2
24	31	31	10	10	3	1	76	Remark 6.2.2
25	45	45	12	12	3	11	85	Theorem 6.2.1
26	57	57	8	8	1	7	87	Theorem 6.2.1
27	91	91	10	10	1	1	91	Remark 6.2.2

Table 6.2 OCDs in GDDs

Sl. no.	v	b	r	k	λ_1	λ_2	m	n	c	Design no.	Method of construction
1	6	3	2	4	2	1	3	2	2	S1	Remark 6.2.6
2	6	6	4	4	4	2	3	2	3	S2	Theorem 6.2.1
3	6	12	8	4	8	4	3	2	6	S4	Theorem 6.2.2
4	10	10	4	4	4	1	5	2	3	S9	Theorem 6.2.1
5	10	20	8	4	8	2	5	2	6	S10	Theorem 6.2.2
6	18	36	8	4	8	1	9	2	6	S15	Theorem 6.2.2
7	8	8	6	6	6	4	4	2	1	S19	Remark 6.2.2
8	9	3	2	6	2	1	3	3	1	S21	Remark 6.2.2
9	9	9	6	6	6	3	3	3	1	S23	Remark 6.2.2
10	10	10	6	6	6	3	5	2	1	S26	Remark 6.2.2
11	12	12	6	6	6	2	4	3	1	S29	Remark 6.2.2
12	14	14	6	6	6	2	7	2	1	S33	Remark 6.2.2
13	21	21	6	6	6	1	7	3	1	S42	Remark 6.2.2
14	26	26	6	6	6	1	13	2	1	S44	Remark 6.2.2
15	10	5	4	8	4	3	5	2	6	S51	Remark 6.2.6
16	10	10	8	8	8	6	5	2	7	S52	Theorem 6.2.2
17	12	3	2	8	2	1	3	4	4	S53	Remark 6.2.6
18	12	6	4	8	4	2	3	4	6	S54	Remark 6.2.6
19	12	12	8	8	8	4	3	4	7	S56	Theorem 6.2.2
20	14	7	4	8	4	2	7	2	6	S59	Remark 6.2.6
21	14	14	8	8	8	4	7	2	7	S60	Theorem 6.2.2
22	18	18	8	8	8	3	9	2	7	S65	Theorem 6.2.2
23	20	10	4	8	4	1	5	4	6	S66	Remark 6.2.6

(continued)

Table 6.2 (continued)

Sl. no.	v	b	r	k	λ_1	λ_2	m	n	c	Design no.	Method of construction
24	20	20	8	8	8	2	5	4	7	S68	Theorem 6.2.2
25	26	13	4	8	4	1	13	2	6	S71	Remark 6.2.6
26	26	26	8	8	8	2	13	2	7	S72	Theorem 6.2.2
27	36	36	8	8	8	1	9	4	7	S77	Theorem 6.2.2
28	50	50	8	8	8	1	25	2	7	S80	Theorem 6.2.2
29	12	12	10	10	10	8	6	2	1	S99	Remark 6.2.2
30	15	3	2	10	2	1	3	5	1	S100	Remark 6.2.6
31	15	15	10	10	10	5	3	5	1	S104	Remark 6.2.2
32	18	18	10	10	10	5	9	2	1	S105	Remark 6.2.2
33	22	22	10	10	10	4	11	2	1	S111	Remark 6.2.2
34	30	30	10	10	10	2	6	5	1	S115	Remark 6.2.2
35	42	42	10	10	10	2	21	2	1	S119	Remark 6.2.2
36	55	55	10	10	10	1	11	5	1	S123	Remark 6.2.2
37	82	82	10	10	10	1	41	2	1	S124	Remark 6.2.2
38	4	4	2	2	0	1	2	2	1	SR1	Theorem 6.2.1
39	4	8	4	2	0	2	2	2	2	SR2	Theorem 6.2.2
40	4	12	6	2	0	3	2	2	1	SR3	Theorem 6.2.1
41	4	16	8	2	0	4	2	2	4	SR4	Theorem 6.2.2
42	4	20	10	2	0	5	2	2	1	SR5	Theorem 6.2.1
43	6	18	6	2	0	2	2	3	1	SR7*	Theorem 6.2.1
44	8	16	4	2	0	1	2	4	2	SR9	Theorem 6.2.2
45	8	32	8	2	0	2	2	4	4	SR10	Theorem 6.2.2
46	10	50	10	2	0	2	2	5	1	SR12*	Theorem 6.2.1
47	12	36	6	2	0	1	2	6	1	SR13	Theorem 6.2.1
48	16	64	8	2	0	1	2	8	4	SR15	Theorem 6.2.2
49	20	100	10	2	0	1	2	10	1	SR17	Theorem 6.2.1
50	8	8	4	4	0	2	4	2	3	SR36	Theorem 6.2.1
51	8	16	8	4	0	4	4	2	6	SR39	Theorem 6.2.2
52	16	16	4	4	0	1	4	4	3	SR44	Theorem 6.2.1
53	16	32	8	4	0	2	4	4	6	SR45	Theorem 6.2.2
54	32	64	8	4	0	1	4	8	6	SR49	Theorem 6.2.2
55	12	12	6	6	0	3	6	2	1	SR67	Remark 6.2.2
56	12	12	6	6	2	3	3	4	1	SR68*	Remark 6.2.2
57	18	18	6	6	0	2	6	3	1	SR72*	Remark 6.2.2
58	16	16	8	8	0	4	8	2	7	SR92	Theorem 6.2.1
59	32	32	8	8	0	2	8	4	7	SR95	Theorem 6.2.1

(continued)

Table 6.2 (continued)

Sl. no.	v	b	r	k	λ_1	λ_2	m	n	c	Design no.	Method of construction
60	64	64	8	8	0	1	8	8	7	SR97	Theorem 6.2.1
61	20	20	10	10	0	5	10	2	1	SR108	Remark 6.2.2
62	4	8	4	2	2	1	2	2	2	R1*	Theorem 6.2.2
62	4	8	4	2	2	1	2	2	2	R1*	Theorem 6.2.2
63	4	12	6	2	4	1	2	2	1	R4*	Theorem 6.2.1
64	4	16	8	2	6	1	2	2	4	R8*	Theorem 6.2.2
65	4	16	8	2	4	2	2	2	4	R9*	Theorem 6.2.2
66	4	16	8	2	2	3	2	2	4	R10*	Theorem 6.2.2
67	4	20	10	2	8	1	2	2	1	R14*	Theorem 6.2.1
68	4	20	10	2	6	2	2	2	1	R15*	Theorem 6.2.1
69	4	20	10	2	4	3	2	2	1	R16*	Theorem 6.2.1
70	4	20	10	2	2	4	2	2	1	R17*	Theorem 6.2.1
71	6	12	4	2	0	1	3	2	2	R18*	Theorem 6.2.2
72	6	18	6	2	2	1	3	2	1	R19*	Theorem 6.2.1
73	2	24	8	2	4	1	3	2	4	R22*	Theorem 6.2.2
74	6	24	8	2	0	2	3	2	4	R23*	Theorem 6.2.2
75	6	24	8	2	1	2	2	3	4	R24*	Theorem 6.2.2
76	6	30	10	2	6	1	3	2	1	R28*	Theorem 6.2.1
77	8	24	6	2	0	1	4	2	1	R29*	Theorem 6.2.1
78	8	32	8	2	2	1	4	2	4	R30*	Theorem 6.2.2
79	8	40	10	2	2	1	2	4	1	R32*	Theorem 6.2.1
80	8	40	10	2	4	1	4	2	1	R33*	Theorem 6.2.1
81	9	27	6	2	0	1	3	3	1	R34*	Theorem 6.2.1
82	9	45	10	2	2	1	3	3	1	R35*	Theorem 6.2.1
83	10	40	8	2	0	1	5	2	4	R36*	Theorem 6.2.2
84	10	50	10	2	2	1	5	2		R37*	Theorem 6.2.1
85	12	48	8	2	0	1	3	4	4	R38*	Theorem 6.2.2
86	12	16	10	2	0	1	6	2	1	R40*	Theorem 6.2.1
87	15	75	10	2	0	1	3	5	1	R41*	Theorem 6.2.1
88	6	6	4	4	3	2	2	3	3	R94*	Theorem 6.2.1
89	6	12	8	4	6	4	2	3	6	R95*	Theorem 6.2.2
90	6	12	8	4	4	5	3	2	6	R96*	Theorem 6.2.2
91	8	16	8	4	4	3	2	4	6	R98*	Theorem 6.2.2
92	8	16	8	4	6	3	4	2	6	R99*	Theorem 6.2.2
93	9	9	4	4	3	1	3	3	3	R104*	Theorem 6.2.1
94	9	18	8	4	6	2	3	3	6	R105*	Theorem 6.2.2

(continued)

Table 6.2 (continued)

Sl. no.	v	b	r	k	λ_1	λ_2	m	n	c	Design no.	Method of construction
95	10	20	8	4	0	3	5	2	6	R106	Theorem 6.2.2
96	12	12	4	4	2	1	6	2	3	R109*	Theorem 6.2.1
97	12	24	8	4	4	2	6	2	6	R110*	Theorem 6.2.2
98	14	14	4	4	0	1	7	2	3	R112*	Theorem 6.2.1
99	14	28	8	4	0	2	7	2	6	R113*	Theorem 6.2.2
100	15	15	4	4	0	1	5	3	3	R114*	Theorem 6.2.1
101	15	30	8	4	6	1	5	3	6	R115*	Theorem 6.2.2
102	15	30	8	4	0	2	5	3	6	R116*	Theorem 6.2.2
103	15	30	8	4	1	2	3	5	6	R117*	Theorem 6.2.2
104	16	32	8	4	4	1	4	4	6	R120*	Theorem 6.2.2
105	26	52	8	4	0	1	13	2	6	R128*	Theorem 6.2.2
106	27	54	8	4	0	1	9	3	6	R129*	Theorem 6.2.2
107	28	56	8	4	0	1	7	4	6	R130*	Theorem 6.2.2
108	10	10	6	6	5	2	2	5	1	R166*	Remark 6.2.2
109	15	15	6	6	5	1	3	5	1	R168*	Remark 6.2.2
110	27	27	6	6	3	1	9	3	1	R170*	Remark 6.2.2
111	28	28	6	6	2	1	7	4	1	R171*	Remark 6.2.2
112	12	12	8	8	6	5	6	2	7	R186*	Theorem 6.2.1
113	14	14	8	8	7	2	2	7	7	R187*	Theorem 6.2.1
114	21	21	8	8	7	1	3	7	7	R188*	Theorem 6.2.1
115	24	24	8	8	4	2	4	6	7	R189*	Theorem 6.2.1
116	48	48	8	8	4	1	12	4	7	R190*	Theorem 6.2.1
117	63	63	8	8	0	1	9	7	7	R191*	Theorem 6.2.1
118	12	12	10	10	9	8	4	3	1	R203*	Remark 6.2.2
119	14	14	10	10	8	6	2	7	1	R204*	Remark 6.2.2
120	14	14	10	10	6	7	7	2	1	R205*	Remark 6.2.2
121	18	18	10	10	9	2	2	9	1	R206*	Remark 6.2.2
122	27	27	10	10	9	1	3	9	1	R207*	Remark 6.2.2
123	32	32	10	10	6	2	4	8	1	R208*	Remark 6.2.2
124	75	75	10	10	5	1	15	5	1	R209*	Remark 6.2.2

Table 6.3 OCDs in cyclic designs

Sl. no.	Parameters					Solution	c	Method of construction
	v	b	r	k	λ			
1	6	15	10	4	6	Two ‘full sets’ of blocks each and the initial blocks: [(0, 1, 2, 3), (0, 2, 3, 4)] mod 6; the initial blocks (0, 2, 3, 5) mod 6	2	Analogous to Example 6.3.2
2	19	37	12	4	2	Difference set: $(0, x^0, x^6, x^{12})$; $(0, x^1, x^7, x^13)$; $(0, x^2, x^8, x^{14})$; x is a primitive root of GF(19)	3	Remark 6.3.1
3	22	77	14	4	2	Difference set: $(x_1^0, x_1^3, x_2^\alpha, x_2^{\alpha+3})$; $(x_1^1, x_1^4, x_2^{\alpha+1}, x_2^{\alpha+4})$; $(x_1^2, x_1^5, x_2^{\alpha+2}, x_2^{\alpha+5})$; $(x_2^0, x_2^3, x_3^\alpha, x_3^{\alpha+3})$; $(x_2^1, x_2^4, x_3^{\alpha+1}, x_3^{\alpha+4})$ $(x_2^2, x_2^5, x_3^{\alpha+2}, x_3^{\alpha+5})$; $(x_3^0, x_3^3, x_1^\alpha, x_1^{\alpha+3})$; $(x_3^1, x_3^4, x_1^{\alpha+1}, x_1^{\alpha+4})$; $(x_3^2, x_3^5, x_1^{\alpha+2}, x_1^{\alpha+5})$; $(\infty, 0_1, 0_2, 0_3)$; $(\infty, 0_1, 0_2, 0_3)$; x is a primitive root of GF(7)	2	Analogous to Example 6.3.2 and Example 6.3.3

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Chapter 7
OCDs in Balanced Treatment Incomplete Block Design Set-Up

7.1 Introduction

Here we deal with the balanced treatment incomplete block (BTIB) design set-up with $p+1$ treatments and c covariates and investigate the problem of most efficient estimation of the covariate parameters in BTIB design set-ups. As a useful class of designs for testing test treatments against control, Bechhofer and Tamhane (1981) introduced BTIB design. Suppose in a test-control design d, the treatments are indexed by $c_0, 1, \ldots, v$ with c_0 denoting the control treatment and $1, 2, \ldots, v$ denoting the $v(\geq 2)$ test treatments. Let k denote the common size of each block, and b denote the number of blocks available for experimentation. Thus $n = kb$ is the total number of experimental units. According to Bechhofer and Tamhane (1981), the design d is called a BTIB design if

(a) d is incomplete, i.e. no block contains all the $v+1$ treatments,
(b) $\lambda_{c_0 i} = \lambda_{c_0},\ i = 1, 2, \ldots, v$ and $\lambda_{i_1 i_2} = \lambda,\ i_1 \neq i_2 = 1, 2, \ldots, v$, where $\lambda_{uu'} = \sum_{j=1}^{b} n_{uj} n_{u'j},\ u \neq u' = c_0, 1, \ldots, v$ and n_{ij} denotes the number of times the ith treatment appears in the jth block, $i = c_0, 1, \ldots, v,\ j = 1, 2, \ldots, b$.

According to Bechhofer and Tamhane (1981), a BTIB design neither needs to satisfy the condition that $r_i = \sum_{j=1}^{b} n_{ij}, (1 \leq i \leq v)$, the number of replications of the ith test treatment are all equal nor, does it require to be binary in the test/control treatments. But as mentioned earlier Dutta and Das (2013) considered only those BTIB designs which were constructed in Bechhofer and Tamhane (1981) and Das et al. (2005) where the designs had all $r_i = r$.

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So here the discussion is restricted to the set-up of BTIB design with parameters $v,\ b,\ k,\ r,\ r_{c_0},\ \lambda,\ \lambda_{c_0}$ which is denoted by BTIB$(v,\ b,\ k,\ r,\ r_{c_0},\ \lambda,\ \lambda_{c_0})$, where $r_{c_0} = \sum_{j=1}^{b} n_{c_0 j}$ is the replication of the control treatment.

Let y_{ijl} be the response and $z_{ijl}^{(t)}$ be the value of the tth covariate when the treatment i is applied to the unit l of block j, $i = c_0, 1, \ldots, v$, $j = 1, 2, \ldots, b$, $l = 1, 2, \ldots, n_{ij} (n_{ij} = 0, 1, 2, \ldots)$, $t = 1, 2, \ldots, c$. The model which we work with is

$$y_{ijl} = \mu + \tau_i + \beta_j + \sum_{t=1}^{c} \gamma_t z_{ijl}^{(t)} + e_{ijl}, \tag{7.1.1}$$

where μ is the general mean, τ_i is the effect of treatment i, β_j the effect of block j, $\gamma_1, \gamma_2, \ldots, \gamma_c$ are the regression coefficients associated with the c covariates Z_1, $Z_2, \ldots, Z_c$ respectively and e_{ijl} is the observational error corresponding to y_{ijl}. As usual, the random errors $\{e_{ijl}\}$ are assumed to be uncorrelated and homoscedastic with common variance σ^2. As in other chapters it is assumed that the values of each covariate are in the interval $[-1, 1]$, i.e.

$$z_{ijl}^{(t)} \in [-1, 1],\ i = c_0, 1, \ldots, v;\ j = 1, 2, \ldots, b;\ l = 1, 2, \ldots, n_{ij};\ t = 1, 2, \ldots, c. \tag{7.1.2}$$

In matrix notation Model (7.1.1) can be represented as

$$(\mathbf{Y},\ \mu\mathbf{1}_n + \mathbf{X}_1\boldsymbol{\tau} + \mathbf{X}_2\boldsymbol{\beta} + \mathbf{Z}\gamma,\ \mathbf{I}_n\sigma^2) \tag{7.1.3}$$

where $\mathbf{Y} = (\ldots, y_{ijl}, \ldots)'$ is the vector of observations of order $n \times 1$, $\boldsymbol{\tau}$, $\boldsymbol{\beta}$ and γ correspond, respectively, to the vectors of treatment effects, block effects and the covariate effects; $\mathbf{X}_1$ and $\mathbf{X}_2$ are, respectively, the design matrices of treatment effects and block effects and $\mathbf{Z} = ((z_{ijl}^{(t)}))$ is the design matrix corresponding to the covariate effects. $\mathbf{1}_n$ is a vector of order n with all elements unity and $\mathbf{I}_n$ is the identity matrix of order n.

With reference to the model (7.1.3), it is evident that for the estimation of the covariate effects orthogonally to the treatment and block effects we will impose the conditions as stated in (3.1.3) and the regression parameters are estimated with maximum efficiency if additionally, (3.1.4) holds.

7.2 OCDs and W-Matrices

With respect to Model (7.1.3), γ is estimated most efficiently if $\mathbf{Z}$-matrix satisfies conditions (3.1.3) and (3.1.4). The choice of the $\mathbf{Z}$-matrix is usually difficult under the most general block design set-up. As mentioned in Chap. 4 that in the binary design set-up Das et al. (2003) had represented each column of the $\mathbf{Z}$-matrix by a matrix $\mathbf{W}$, where the rows of $\mathbf{W}$ corresponded to the treatments and the columns of $\mathbf{W}$ corresponded to the blocks. This brought in some ease in construction of the $\mathbf{Z}$-matrix. But a BTIB design need not be binary; however, in the BTIB designs which we are considering here, the portion for the test treatments is binary, but the control treatment may occur more than once in a block. So to represent the columns of an optimum $\mathbf{Z}$-matrix by $\mathbf{W}$-matrices, we convert the incidence matrix of a BTIB design to one which is binary.

$$\text{Let } \mathbf{N} = \left(\mathbf{n}'_C,\, \mathbf{N}'_T\right)'$$

be the incidence matrix of the BTIB design, where $\mathbf{n}_C^{1\times b}$ indicates the incidence vector of the control treatment and $\mathbf{N}_T$ indicates the incidence matrix of the test treatments. For our convenience in construction of OCDs, we replace the incidence vector $\mathbf{n}_C$ of the control treatment by a $n_{c_0} \times b$ matrix $\mathbf{N}_C^*$ with elements 0 and 1 where the jth column of $\mathbf{N}_C^*$ contains $n_{c_{0j}}$ unities in some order and $(n_{c_0} - n_{c_0 j})$ zeros in other places, $j = 1, 2, \ldots, b$, n_{c_0} being the maximum of $n_{c_{01}},\ n_{c_{02}}, \ldots, n_{c_{0b}}$. To fix ideas and to illustrate the technique we will use a definite order of 1s and 0s where 1's are followed by 0's. Therefore, the incidence matrix $\mathbf{N}$ can be written in a transformed form with the elements 0 and 1 as

$$\mathbf{N}^{*(n_{c_0}+v)\times b} = \left(\mathbf{N}_C^{*'},\, \mathbf{N}'_T\right)' \tag{7.2.1}$$

where $\mathbf{N}_C^{*(n_{c_0}\times b)}$ is actually the incidence matrix corresponding to the control treatment in the binary form.

Now we can make a correspondence between the elements of any column of $\mathbf{Z}$ with the positive entries of $\mathbf{N}^*$. Also, as the other entries of $\mathbf{N}^*$ are zeros and the z-values are ± 1, we can get a matrix $\mathbf{W}^{(t)}$ from $\mathbf{N}^*$ by replacing n_{ij}^*'s by $\pm n_{ij}^*$ according to the values of tth column of $\mathbf{Z}$. The $\mathbf{W}^{(t)}$-matrix precisely represents the tth column of $\mathbf{Z}$. Note from (7.2.1) that $\mathbf{W}^{(t)}$ can be accordingly partitioned as

$$\mathbf{W}^{(t)\ (n_{c_0}+v)\times b} = \left(\mathbf{W}_C^{(t)\prime},\, \mathbf{W}_T^{(t)\prime}\right)'. \tag{7.2.2}$$

Here optimum $\mathbf{W}$-matrices are being constructed from $\mathbf{N}^*$, the incidence matrix of a BTIB design, by putting $+1$ or -1 in the non-zero positions of every row and every column of $\mathbf{N}^*$. From the definition of the $\mathbf{W}$-matrix it follows that the conditions (3.1.3) and (3.1.4) change to the following:

$$\left.\begin{array}{ll} C_1. & \mathbf{W}^{(t)}\text{-matrix has all column sums equal to zero;} \\ C_2. & \mathbf{W}_T^{(t)}\text{-matrix has all row sums equal to zero;} \\ C_3. & \text{The grand total of all the entries in the Hadamard product} \\ & \text{of } \mathbf{W}^{(t)} \text{ and } \mathbf{W}^{(t')}\ is\ equal\ to\ n\delta_{tt'},\ 1 \le t \ne t' \le c. \end{array}\right\} \tag{7.2.3}$$

We may note in passing that (i) $\mathbf{W}_C^{(t)} = (\pm 1, 0)$ does not posses any such property of its row total (ii) however, it is trivially true that $\sum\sum w_C^{(t)}(i, j)$ is equal to zero.

Definition 7.2.1 With respect to model (7.1.3), the **W**-matrices corresponding to the c covariates are said to be optimum if they satisfy the condition (7.2.3).

Remark 7.2.1 It is to be noted that if $c = 1$, only the conditions C_1 and C_2 are to be satisfied by the **W**-matrix to be optimum.

7.3 Optimum Covariate Designs

As it has already been mentioned earlier that the construction of OCDs depends on the methods of construction of the corresponding BTIB designs, we divide this section into three subsections according to the methods of construction.

7.3.1 BTIB Design Obtained from Generator Designs

Following (Bechhofer and Tamhane 1981), we define *generator* designs which are BTIB designs with v test treatments and b blocks of size k each such that no proper subsets of blocks can give rise to a BTIB design. Suppose that there are s_0 generator designs $\mathrm{D}_1, \mathrm{D}_2, \ldots, \mathrm{D}_{s_0}$ (say) and let $\lambda^{(i)}, \lambda_{c_0}^{(i)}$ be the frequency parameters associated with D_i and let b_i be the number of blocks required by D_i $(i = 1, 2, \ldots, s_0)$. Then a BTIB design $\mathrm{D} = \bigcup_{i=1}^{s_0} f_i \mathrm{D}_i$ obtained by forming unions of $f_i > 0$ replications of D_i has the frequency parameters $\lambda = \sum_{i=1}^{s_0} f_i \lambda^{(i)}$, $\lambda_{c_0} = \sum_{i=1}^{s_0} f_i \lambda_{c_0}^{(i)}$ and $b = \sum_{i=1}^{s_0} f_i b_i$ blocks cf. Bechhofer and Tamhane (1981). We consider the generator designs constructed by Bechhofer and Tamhane (1981) and construct OCDs for these BTIB designs.

(i) For each $v \geq 2$, $k = 2$ there are exactly two generator designs and these are

$$\mathrm{D}_1 = \begin{Bmatrix} c_0 & c_0 & \ldots & c_0 \\ 1 & 2 & \ldots & v \end{Bmatrix}, \qquad \mathrm{D}_2 = \begin{Bmatrix} 1 & 1 & \ldots & v-1 \\ 2 & 3 & \ldots & v \end{Bmatrix}. \tag{7.3.1}$$

From these generator designs, implementable BTIB designs of the type $\mathrm{D} = f_1 D_1 \cup f_2 \mathrm{D}_2$ can be constructed for $f_1, f_2 > 0$. When $f_1 = f_2 = f$ (say), the corresponding design parameters for D are

$$v,\ b = f(v(v+1)/2),\ k = 2,\ r = r_{c_0} = fv,\ \lambda_{c_0} = f,\ \lambda = f. \tag{7.3.2}$$

For the construction of OCDs for BTIB design with v even and $f = 1$ the following lemma will be helpful.

Lemma 7.3.1 *Let v (≥ 2) be even and $f = 1$ in the parameters (7.3.2). Then a* $\mathbf{W}$*-matrix for the BTIB design with parameters $v,\ b = v(v+1)/2,\ k = 2,\ r = r_{c_0} = v,\ \lambda = 1,\ \lambda_{c_0} = 1$ obtained by $\mathrm{D} = \mathrm{D}_1 \cup \mathrm{D}_2$, can always be constructed.*

Proof The incidence matrix $\mathbf{N}^*$ of the design D can be written as

$$\mathbf{N}^* = \begin{pmatrix} \mathbf{1}_v' & \mathbf{0}' & \mathbf{0}' & \ldots & \mathbf{0}' & \mathbf{0}' \\ \mathbf{I}_v & \mathbf{A}_1^* & \mathbf{A}_2^* & \ldots & \mathbf{A}_{(v-2)/2}^* & \mathbf{B} \end{pmatrix} \tag{7.3.3}$$

where $\mathbf{A}_i^{*\prime}$ of order $v \times v$ is obtained from cyclic permutation of the following row

$$\begin{array}{c} 1\ 2 \ldots i\ \ i+1\ \ i+2 \ldots v \\ \big(\,1\ 0 \ldots 0 \quad 1 \qquad 0 \quad \ldots 0\,\big); \quad i = 1, 2, \ldots, (v-2)/2\,\cdot \end{array} \tag{7.3.4}$$

and

$$\mathbf{B} = \big(\mathbf{I}_{v/2},\ \mathbf{I}_{v/2}\big)'. \tag{7.3.5}$$

Corresponding to $\mathbf{A}_i'$ we construct a matrix $\mathbf{W}_{2,i}^{*\prime}$ by cyclical permutation of the vector

$$\begin{array}{c} 1\ 2 \ldots i\ \ i+1\ \ i+2 \ldots v \\ \big(\,1\ 0 \ldots 0\ \ -1 \qquad 0 \quad \ldots 0\,\big); \quad i = 1, 2, \ldots, (v-2)/2 \end{array} \tag{7.3.6}$$

obtained from (7.3.4) by replacing the non-null elements by 1 and -1 respectively, $i = 1, 2, \ldots, (v-2)/2$. Thus we get the following matrix

$$\mathbf{W}_2^{*\ v \times v(v-2)/2} = \big(\mathbf{W}_{2,1}^*,\ \mathbf{W}_{2,2}^*, \ldots, \mathbf{W}_{2,(v-2)/2}^*\big) \tag{7.3.7}$$

after juxtaposition of $\mathbf{W}_{2,i}^*$'s. Again corresponding to the matrices $\mathbf{I}_v$ of (7.3.3) and $\mathbf{B}$ of (7.3.5) we define the two following matrices

$$\mathbf{W}_1^{*\ v \times v} = \begin{pmatrix} -\mathbf{I}_{v/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{v/2} \end{pmatrix}, \quad \mathbf{W}_3^{*\ v \times v/2} = \begin{pmatrix} \mathbf{I}_{v/2} \\ -\mathbf{I}_{v/2} \end{pmatrix}. \tag{7.3.8}$$

Using $\mathbf{W}_1^*$, $\mathbf{W}_2^*$ and $\mathbf{W}_3^*$ in $\mathbf{N}^*$ of (7.3.3), the following $\mathbf{W}$-matrix of order $(v+1) \times v(v+1)/2$

$$\mathbf{W}^* = \left(\begin{array}{cc|c|c} \mathbf{1}_{v/2}' & -\mathbf{1}_{v/2}' & \mathbf{0}' & \mathbf{0}' \\ \hline \multicolumn{2}{c|}{\mathbf{W}_1^*} & \mathbf{W}_2^* & \mathbf{W}_3^* \end{array}\right) \tag{7.3.9}$$

can be seen to satisfy conditions C_1–C_3 of (7.2.3). □

Example 7.3.1 Take $v = 4$. The incidence matrix $\mathbf{N}^*$ of the design when $v = 4$ looks like

$$\mathbf{N}^{*} = \left(\begin{array}{cccc|cccc|cc} 1&1&1&1&0&0&0&0&0&0 \\ \hline 1&0&0&0&1&0&0&1&1&0 \\ 0&1&0&0&1&1&0&0&0&1 \\ 0&0&1&0&0&1&1&0&1&0 \\ 0&0&0&1&0&0&1&1&0&1 \end{array}\right) = \left(\begin{array}{c|c|c} \mathbf{1}_4' & \mathbf{0}' & \mathbf{0}' \\ \hline \mathbf{I}_4 & \mathbf{A}_1^{*} & \mathbf{B} \end{array}\right) \tag{7.3.10}$$

From (7.3.6), $\mathbf{W}^{*\prime}_{2,1}$ can be written as follows:

$$\mathbf{W}^{*\prime}_{2,1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore from (7.3.9), $\mathbf{W}^{*}$-matrix is given by

$$\mathbf{W}^{*} = \left(\begin{array}{cc|cc|cccc|cc} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & -1 \end{array}\right). \tag{7.3.11}$$

The following theorem gives method of construction of OCDs for the general form of the above designs.

Theorem 7.3.1 *Let v be even and a Hadamard matrix of order f exist. Then f optimum $\mathbf{W}$-matrices for the series of BTIB designs with parameters given in (7.3.2) can be constructed.*

Proof Incidence matrix of the design $(f\mathrm{D}_1 \cup f\mathrm{D}_2)$ is actually f replications of $\mathbf{N}^{*}$ of (7.3.3) and hence can be written as

$$\mathbf{N}^{**\ (v+1)\times fv(v+1)/2} = \mathbf{1}_f' \bigotimes \mathbf{N}^{*} \tag{7.3.12}$$

where $\mathbf{N}^{*}$ is defined in (7.3.3) and $\bigotimes$ denotes Kornecker product. By assumption, Hadamard matrix of order f exists and is written as

$$\mathbf{H}_f = (\mathbf{h}_1,\ \mathbf{h}_2, ..., \mathbf{h}_f). \tag{7.3.13}$$

Now we construct the matrix $\mathbf{W}_i^{**}$ as follows

$$\mathbf{W}_i^{**} = \mathbf{h}_i \bigotimes \mathbf{W}^{*};\quad i = 1, 2, ..., f \tag{7.3.14}$$

where $\mathbf{W}^{*}$ is defined in (7.3.9). We can easily check that $\mathbf{W}_i^{**}$'s satisfy all properties of optimum $\mathbf{W}$-matrices. □

Example 7.3.2 Let $v = 4,\ f = 4$. Considering $\mathbf{N}^*$ from (7.3.10)

$$\mathbf{N}^{**} = \mathbf{1}'_f \bigotimes \left(\begin{array}{cccc|cccccc} 1&1&1&1&0&0&0&0&0&0 \\ 1&0&0&0&1&0&0&1&1&0 \\ 0&1&0&0&1&1&0&0&0&1 \\ 0&0&1&0&0&1&1&0&1&0 \\ 0&0&0&1&0&0&1&1&0&1 \end{array}\right) \tag{7.3.15}$$

$\mathbf{H}_4$, a Hadamard matrix of order 4, is written as

$$\mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} = (\mathbf{h}_1,\ \mathbf{h}_2,\ \mathbf{h}_3,\ \mathbf{h}_4) \tag{7.3.16}$$

where $\mathbf{h}_l$ is the lth column of $\mathbf{H}_4$. Four optimum $\mathbf{W}$-matrices can be constructed as follows:

$$\mathbf{W}_1^{**} = \mathbf{h}'_1 \bigotimes \mathbf{W}^* = \left(\mathbf{W}^*,\ \mathbf{W}^*,\ \mathbf{W}^*,\ \mathbf{W}^*\right);\ \ \mathbf{W}_2^{**} = \mathbf{h}'_2 \bigotimes \mathbf{W}^*$$
$$= \left(\mathbf{W}^*, -\mathbf{W}^*,\ \mathbf{W}^*, -\mathbf{W}^*\right);$$

$$\mathbf{W}_3^{**} = \mathbf{h}'_3 \bigotimes \mathbf{W}^* = \left(\mathbf{W}^*, -\mathbf{W}^*, -\mathbf{W}^*,\ \mathbf{W}^*\right);\ \ \mathbf{W}_4^{**} = \mathbf{h}'_4 \bigotimes \mathbf{W}^*$$
$$= \left(\mathbf{W}^*,\ \mathbf{W}^*, -\mathbf{W}^*, -\mathbf{W}^*\right);$$

where $\mathbf{W}^*$ is given in (7.3.11).

(ii) Following Bechhofer and Tamhane (1981), for given $(v,\ k)$ and $k \geq 3$, let a generator design D_m have $m + 1$ plots assigned to the control treatment in each block; the v test treatments be assigned to the remaining $k_m = (k - m - 1)$-plots of the b_m blocks $(0 \leq m \leq k - 2)$ of the design in such away that they form a BIBD. The incidence matrix of D_m can be transformed into $\mathbf{N}^*$ of expression (7.2.1), where $\mathbf{N}^*_C$ looks like the incidence matrix of a RBD with $(m+1)$ treatments arranged in b_m blocks. This is denoted by RBD$(m + 1,\ b_m)$. $\mathbf{N}_T$ is incidence matrix of the BIBD with parameters $v,\ b_m,\ r_m,\ k_m = k - m - 1,\ \lambda_m$ which is denoted by BIBD$(v,\ b_m,\ r_m,\ k_m,\ \lambda_m)$. Here three cases have been considered viz $m = 0,\ m =$ even and $m =$ odd and in each of the three cases, OCDs can be constructed for generator designs.

Case 1: When $m = 0$ one plot in each block is assigned to the control treatment and the test treatments in the blocks each of size $k_0 = (k - 1)$ form a BIBD$(v, b_0, r_0, k_0, \lambda_0)$. Let $\mathbf{N}$ be the incidence matrix of the BTIB design D_0 with parameters $v,\ b = b_0,\ r = r_0,\ k = k_0 + 1,\ \lambda = \lambda_0,\ \lambda_{c_0} = r$ and it can be written as,

$$\mathbf{N} = (\mathbf{1}_{b_0},\ \ \mathbf{N}'_T)' \tag{7.3.17}$$

where $\mathbf{N}_T$ is the incidence matrix of a BIBD(v, b_0, r_0, k_0, λ_0). It is convenient for the construction of OCD for D_0, if $\mathbf{N}_T$ is the incidence matrix of a k_0-resolvable BIBD with $b_0 = sv$, $s \geq 1$ being an integer. This requires that the $b_0 = sv$ blocks can be partitioned into s sets $T_1, T_2, \ldots, T_s$ each of which contains v blocks such that each of the v treatments occurs k_0 times in each T_i, $i = 1, 2, \ldots, s$. By exploiting the properties of k_0-resolvable BIBD, it is possible to construct OCD for D_0. Precise statement follows:

Theorem 7.3.2 *If a k_0-resolvable BIBD(v, $b_0 = sv$, $r_0 = sk_0$, $k_0 = k - 1$, λ_0) exists, then it is possible to construct $sk_0/2$ optimum $\mathbf{W}$-matrices for the generator design D_0 with parameters v, $b = sv = b_0$, k, $r = sk_0 = r_0$, $r_{c_0} = sv$, $\lambda = \lambda_0$, $\lambda_{c_0} = sk_0$, provided $\mathbf{H}_{k_0+1}$ and $\mathbf{H}_{s/2}$ exist.*

Proof As the BIBD is k_0-resolvable, then Lemma 6.2.1 of Chap. 6 is applicable to the blocks of each T_i. According to Lemma 6.2.1 of Chap. 6, the k_0 treatments in the v blocks of T_i can always be arranged such that each treatment occurs exactly once in each of the k_0 positions in the blocks and this arrangement is denoted by a $k_0 \times v$ matrix $\mathbf{B}_i$ and from the matrix $\mathbf{B}_i$, it is possible to construct a $v \times v$ matrix $\mathbf{A}_i^{v \times v}$ by putting an element a_l in its (m, q) th cell if mth treatment occurs in the lth row and q th column of $\mathbf{B}_i$, $l = 1, 2, \ldots, k_0$, m, $q = 1, 2, \ldots, v$. Other positions are filled in with zeros. It is easily seen that $\mathbf{A}_i$ contains each of the symbols $a_1, a_2, \ldots, a_{k_0}$ once and only once in each row and in each column, $i = 1, 2, \ldots, s$. Now a matrix $\mathbf{A}_i^*$ of order $(v + 1) \times 2v$ is defined by pairing the $\mathbf{A}_i$'s as follows

$$\mathbf{A}_i^* = \begin{pmatrix} \mathbf{1}_v' & \mathbf{1}_v' \\ \mathbf{A}_{2i-1} & \mathbf{A}_{2i} \end{pmatrix}; \quad \forall i = 1, 2, \ldots, s/2. \tag{7.3.18}$$

It is given that $\mathbf{H}_k$, a Hadamard matrix of order k, exists. Let it be written as:

$$\mathbf{H}_{k_0+1} = \begin{pmatrix} 1 & \mathbf{1}_{k_0}' \\ \mathbf{1}_{k_0} & \mathbf{H}_{k_0}^* \end{pmatrix}. \tag{7.3.19}$$

where

$$\mathbf{H}_{k_0}^* = (\mathbf{h}_1^*, \mathbf{h}_2^*, \ldots, \mathbf{h}_{k_0}^*) = \text{core matrix of } \mathbf{H}_{k_0+1}, \tag{7.3.20}$$

$\mathbf{h}_j^*$ is the jth column of $\mathbf{H}_{k_0}^*$. Now a matrix $\mathbf{W}_{j,i}$ can be constructed from $\mathbf{A}_i^*$ by identifying the symbols a_1, $a_2, \ldots, a_{k_0}$ of $\mathbf{A}_{2i-1}$ and $\mathbf{A}_{2i}$ with the elements of $\mathbf{h}_j^*$ and $-\mathbf{h}_j^*$ respectively and also replacing first row of $\mathbf{A}_i^*$ by $(\mathbf{1}_v', -\mathbf{1}_v')$. By juxtaposing $\mathbf{W}_{j,i}$, $i = 1, 2, \ldots, s/2$, for fixed j, we obtain a matrix $\mathbf{W}_j$, where

$$\mathbf{W}_j = (\mathbf{W}_{j,1}, \mathbf{W}_{j,2}, \ldots, \mathbf{W}_{j,s/2}); \quad j = 1, 2, \ldots, k_0. \tag{7.3.21}$$

Varying j in (7.3.21), k_0 optimum $\mathbf{W}$-matrices, $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_{k_0}$, are obtained. Again as $\mathbf{H}_{s/2}$ exists, it is possible to increase the number of optimum $\mathbf{W}$-matrices. By taking the Khatri–Rao product of $\mathbf{h}_i = (h_{i1}, h_{i2}, \ldots, h_{i,s/2})$, the ith row of $\mathbf{H}_{s/2}$, with $\mathbf{W}_j$ of (7.3.21), $sk_0/2$ matrices $\mathbf{W}_{ji}$ can be constructed, where

$$\mathbf{W}_{ji} = \mathbf{h}_i \bigodot \mathbf{W}_j = (h_{i1}\mathbf{W}_{j,1},\ h_{i2}\mathbf{W}_{j,2}, \ldots, h_{i,s/2}\mathbf{W}_{j,s/2}),$$
$$\forall i = 1, 2, \ldots, s/2;\ \ j = 1, 2, \ldots, k_0. \tag{7.3.22}$$

It is easy to verify that $\mathbf{W}_{ji}$'s satisfy condition (7.2.3). □

Remark 7.3.1 If $s = 2$, then k_0 optimum $\mathbf{W}$-matrices, $\mathbf{W}_1$, $\mathbf{W}_2$,...,$\mathbf{W}_{k_0}$, can be constructed whenever $\mathbf{H}_k$ exists.

This is illustrated by considering the following example:

Example 7.3.3 Let us consider the following 3-resolvable BIBD(5, 10, 6, 3, 3) obtained by cyclical development of two initial blocks (0, 1, 2) and (0, 1, 3) constructed from the module $M = (0,\ 1,\ 2,\ 3,\ 4)$. The blocks with the treatments renamed as (1, 2, 3, 4, 5) can be represented in the form of a matrix $\mathbf{B}$ of order 3×10 as

$$\mathbf{B} = \left(\begin{array}{ccccc|ccccc} 1&2&3&4&5&1&2&3&4&5 \\ 2&3&4&5&1&2&3&4&5&1 \\ 3&4&5&1&2&4&5&1&2&3 \end{array}\right) = (\mathbf{B}_1,\ \mathbf{B}_2).$$

Now the $\mathbf{A}$-matrix of order 5×10 can be constructed from $\mathbf{B}$ as

$$\mathbf{A} = \left(\begin{array}{ccccc|ccccc} a_1&0&0&a_3&a_2&a_1&0&a_3&0&a_2 \\ a_2&a_1&0&0&a_3&a_2&a_1&0&a_3&0 \\ a_3&a_2&a_1&0&0&0&a_2&a_1&0&a_3 \\ 0&a_3&a_2&a_1&0&a_3&0&a_2&a_1&0 \\ 0&0&a_3&a_2&a_1&0&a_3&0&a_2&a_1 \end{array}\right) = (\mathbf{A}_1,\ \mathbf{A}_2).$$

The core matrix $\mathbf{H}_3^*$ obtained from Hadamard matrix $\mathbf{H}_4$ of (7.3.16) is given

$$\mathbf{H}_3^* = \begin{pmatrix} -1&-1&1 \\ 1&-1&-1 \\ -1&1&-1 \end{pmatrix} = (\mathbf{h}_1^*,\ \mathbf{h}_2^*,\ \mathbf{h}_3^*). \tag{7.3.23}$$

Considering $\mathbf{h}_1^*$ for $\mathbf{A}_1$ and then identifying -1 with a_1, 1 with a_2 and -1 with a_3 of $\mathbf{A}_1$ and similarly identifying the elements of $-\mathbf{h}_1^*$ with those of $\mathbf{A}_2$, $\mathbf{W}_1$ can be constructed by using Theorem 7.3.2 as

$$\mathbf{W}_1 = \left(\begin{array}{ccccc|ccccc} 1&1&1&1&1&-1&-1&-1&-1&-1 \\ -1&0&0&-1&1&1&0&1&0&-1 \\ 1&-1&0&0&-1&-1&1&0&1&0 \\ -1&1&-1&0&0&0&-1&1&0&1 \\ 0&-1&1&-1&0&1&0&-1&1&0 \\ 0&0&-1&1&-1&0&1&0&-1&1 \end{array}\right). \tag{7.3.24}$$

Similarly $\mathbf{W}_2$ and $\mathbf{W}_3$ can be constructed by using $\mathbf{h}_2^*$ and $\mathbf{h}_3^*$ respectively. It is easy to see that $\mathbf{W}_1$, $\mathbf{W}_2$ and $\mathbf{W}_3$ satisfy condition (7.2.3).

Remark 7.3.2 If $c^* = sk_0/2$ optimum $\mathbf{W}$-matrices for the generator design D_0 with parameters $v,\ b = sv,\ k,\ r = sk_0,\ r_{c0} = sv,\ \lambda = \lambda_0,\ \lambda_{c0} = sk_0$ exist, then it is possible to construct c^*u optimum $\mathbf{W}$-matrices for the BTIB($v,\ b = svu,\ k,\ r = suk_0,\ r_{c0} = suv,\ \lambda = u\lambda_0,\ \lambda_{c0} = suk_0$), which is obtained by repeating D_0 u-times, provided $\mathbf{H}_u$ exists.

Case 2: m=even and $\mathbf{N}^*$, the incidence matrix of D_m which is a BTIB design with parameters $v,\ b = b_m,\ k,\ r = r_m,\ r_{c0} = (m+1)b_m,\ \lambda = \lambda_m,\ \lambda_{c0} = (m+1)r_m$ can be written as

$$\mathbf{N}^* = (\mathbf{J}'^{(m+1)\times b_m},\ \ \mathbf{N}_T')' = (\mathbf{J}^{b_m\times m},\ \ \mathbf{N}_0^{*\prime})' \tag{7.3.25}$$

where $\mathbf{N}_0^*$ is the incidence matrix of D_0, a BTIB design with parameters $v,\ b = b_m,\ k-m,\ r = r_m,\ r_c = b_m,\ \lambda = \lambda_m,\ \lambda_{c0} = r_m$. $\mathbf{J}^{m\times b_m}$, a matrix of order $m \times b_m$ with all elements unity, can be considered as the incidence matrix of an RBD($m,\ b_m$). From Theorem 7.3.2 it follows that if the BIBD in D_0 is k_m-resolvable, then OCDs can be constructed for D_0. By combining the optimum $\mathbf{W}$-matrices for D_0 with those for RBD($m,\ b_m$) (cf. Chap. 3), it is possible to construct OCDs for D_m. The results are stated in the following theorem.

Theorem 7.3.3 *If u_1 optimum $\mathbf{W}$-matrices exist for D_0 with parameters $v,\ b = b_m,\ k_m + 1 = k - m,\ r = r_m,\ r_{c0} = b_m,\ \lambda = \lambda_m,\ \lambda_{c0} = r_m$ and u_2 optimum $\mathbf{W}$-matrices exist for RBD($m,\ b_m$), then $u = \min\{u_1, u_2\}$ optimum $\mathbf{W}$s for D_m with parameters $v,\ b = b_m,\ k,\ r = r_m,\ r_{c0} = (m+1)b_m,\ \lambda = \lambda_m,\ \lambda_{c0} = (m+1)r_m$ can be obtained.*

Proof Let the u_1 optimum $\mathbf{W}$-matrices for D_0 and u_2 optimum $\mathbf{W}$-matrices for RBD($m,\ b_m$) be denoted as $\mathbf{W}_{0,1},\ \mathbf{W}_{0,2},\ldots,\mathbf{W}_{0,u_1}$ and $\mathbf{W}^{(1)}_{m,RBD}, \mathbf{W}^{(2)}_{m,RBD}, \ldots, \mathbf{W}^{(u_2)}_{m,RBD}$ respectively. It can be seen that the following u matrices

$$\mathbf{W}_{m,i} = \left(\mathbf{W}^{(i)\prime}_{m,RBD},\ \mathbf{W}'_{0,i}\right)',\quad i = 1, 2, \ldots, u \tag{7.3.26}$$

satisfy the condition C and hence u optimum $\mathbf{W}$-matrices are obtained for D_m. □

Example 7.3.4 Consider D_2 with parameters $v = 5,\ b_2 = 10,\ k = 6,\ r = 6,\ r_{c0} = 30,\ \lambda = 3,\ \lambda_{c0} = 18$ and the incidence matrix of D_2 is

$$\mathbf{N} = \left(\begin{array}{ccccc|ccccc}
3&3&3&3&3&3&3&3&3&3\\
1&0&0&1&1&1&0&1&0&1\\
1&1&0&0&1&1&1&0&1&0\\
1&1&1&0&0&0&1&1&0&1\\
0&1&1&1&0&1&0&1&1&0\\
0&0&1&1&1&0&1&0&1&1
\end{array}\right).$$

Here 3 optimum $\mathbf{W}$s exist for D_0 (see Example 7.3.3) but one optimum $\mathbf{W}$-matrix can be constructed for RBD(2, 10) which is given by

$$\mathbf{W}_{2,RBD}^{(1)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Therefore one optimum $\mathbf{W}$ can be constructed for D_2 by using any one of 3 optimum $\mathbf{W}$s for D_0 given in (7.3.24) and $\mathbf{W}_{2,RBD}^{(1)}$ as follows:

$$\left(\begin{array}{rrrrr|rrrrr} 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 0 & -1 & 1 & 1 & 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & -1 & 1 \end{array}\right).$$

Case 3: When $m =$ odd, then optimum $\mathbf{W}$-matrices for D_m are obtained in the same manner pairing the optimum $\mathbf{W}$-matrices.

Theorem 7.3.4 *Suppose u_1 and u_2 optimum $\mathbf{W}$-matrices exist for RBD$((m+1),\ b_m)$ and BIBD$(v,\ b_m,\ r_m,\ k_m = k - m - 1,\ \lambda_m)$ respectively. Then u=min$\{u_1,\ u_2\}$ optimum $\mathbf{W}$-matrices for D_m with parameters $v,\ b = b_m,\ k,\ r = r_m,\ r_{c_0} = (m+1)b_m,\ \lambda = \lambda_m,\ \lambda_{c_0} = (m+1)r_m$ can be constructed.*

Proof Let the $\mathbf{W}_{RBD}^{(1)}, \mathbf{W}_{RBD}^{(2)}, ..., \mathbf{W}_{RBD}^{(u_1)}$ be u_1 optimum $\mathbf{W}$-matrices of RBD and the $\mathbf{W}_{BIBD}^{(1)}, \mathbf{W}_{BIBD}^{(2)}, ..., \mathbf{W}_{BIBD}^{(u_2)}$ be u_2 optimum $\mathbf{W}$-matrices of BIBD. Then u optimum $\mathbf{W}$ matrices of D_m can be constructed as follows:

$$\mathbf{W}_i = (\mathbf{W}_{RBD}^{(i)\prime},\ \mathbf{W}_{BIBD}^{(i)\prime})^\prime,\ \ i = 1, 2, ..., u. \tag{7.3.27}$$

It can be easily checked that $\mathbf{W}_i$s satisfy the condition (7.2.3). □

Remark 7.3.3 For given $(v,\ k)$, it is possible to construct optimum $\mathbf{W}$s for D_0, D_1,..., D_{k-2} respectively by using Theorems 7.3.2–7.3.4 and by imposing suitable conditions. The generator design D_{k-1} contains no control treatment; it is an RBD or a BIBD with the v test treatments if $v = k$ or $v > k$ respectively. Optimum $\mathbf{W}$s for D_{k-1} are the same as the optimum $\mathbf{W}$s for the corresponding RBD or BIBD as the case may be. Hence the optimum $\mathbf{W}$s for the combined BTIB design, $D = \bigcup_{i=1}^{k-1} f_i D_i$ with at least one $f_i > 0$ $(i = 1, 2, ..., k-2)$ can be obtained by suitably using the $\mathbf{W}$s of the generator designs D_m $(0 \le m \le k-1)$. But it is difficult to say beforehand how many $\mathbf{W}$s exist for D since the number of optimum $\mathbf{W}$-matrices depends on the parameters of the generator designs and the number of generator designs used.

Remark 7.3.4 Below we describe the construction of OCDs for a BTIB design which looks similar to that described in Remark 7.3.3, but the constructional method described therein is not applicable since the irreducible BIBD used here is not necessarily resolvable. Let G_i be the set of $\binom{v}{i}$ blocks formed by choosing all possible i

treatments from a set of v test treatments and then by augmenting with $(v-i)$ repetitions of the control treatment c_0, $i = 1, 2, \ldots, v$. It easily follows that $\bigcup_{i=1}^{v} \mathrm{G}_i = \mathrm{G}$ forms a BTIB design with parameters $v,\ b = 2^v - 1,\ k = v,\ r = 2^{v-1},\ r_{c_0} = p(2^{v-1} - 1),\ \lambda = 2^{v-2}$ and $\lambda_{c_0} = (v-1)2^{v-2}$. For $v = 4,\ k = 4$, the construction of OCDs for such BTIB design is illustrated bellow.

$$G_1 = \begin{Bmatrix} c_0 & c_0 & c_0 & c_0 \\ c_0 & c_0 & c_0 & c_0 \\ c_0 & c_0 & c_0 & c_0 \\ 3 & 4 & 1 & 2 \end{Bmatrix}, \quad G_2 = \begin{Bmatrix} c_0 & c_0 & c_0 & c_0 & c_0 & c_0 \\ c_0 & c_0 & c_0 & c_0 & c_0 & c_0 \\ 1 & 2 & 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 & 3 & 4 \end{Bmatrix},$$

$$G_3 = \begin{Bmatrix} c_0 & c_0 & c_0 & c_0 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{Bmatrix}, \quad G_4 = \begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{Bmatrix}.$$

Using $\mathbf{h}_2$, $\mathbf{h}_3$ and $\mathbf{h}_4$ of (7.3.15), 3 optimum $\mathbf{W}$s can be constructed by exploiting the inherent cyclic nature of the G_i's where

$$\mathbf{W}_1 = \left(\mathbf{W}_C^{(1)\prime}, \mathbf{W}_T^{(1)\prime}\right)' =$$

$$\begin{array}{c} \\ \text{Control} \\ \text{Tr.} \\ \downarrow \\ \hline \text{Test} \\ \text{Tr.} \\ \downarrow \end{array} \begin{array}{c} \qquad G_1 \qquad \Big| \qquad\qquad G_2 \qquad\qquad \Big| \qquad G_3 \qquad \Big| G_4 \\ \left(\begin{array}{cccc|cccccc|cccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 0 \\ \hline -1 & 0 & -1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \end{array}\right) \end{array},$$

$$\mathbf{W}_2 = \left(\mathbf{W}_C^{(2)\prime}, \mathbf{W}_T^{(2)\prime}\right)' =$$

$$\begin{array}{c} \\ \text{Control} \\ \text{Tr.} \\ \downarrow \\ \hline \text{Test} \\ \text{Tr.} \\ \downarrow \end{array} \begin{array}{c} \qquad G_1 \qquad \Big| \qquad\qquad G_2 \qquad\qquad \Big| \qquad G_3 \qquad \Big| G_4 \\ \left(\begin{array}{cccc|cccccc|cccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ \hline -1 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \end{array}\right) \end{array},$$

$$\mathbf{W}_3 = \left(\mathbf{W}_C^{(3)\prime}, \mathbf{W}_T^{(3)\prime}\right)' =$$

$$\begin{array}{c} \\ \text{Control} \\ \text{Tr.} \\ \downarrow \\ \hline \text{Test} \\ \text{Tr.} \\ \downarrow \end{array}
\begin{array}{c} \begin{array}{cccc} \quad G_1 \quad & \Big| \quad G_2 \quad & \Big| \quad G_3 \quad & \Big| \; G_4 \end{array} \\
\left(\begin{array}{rrrr|rrrrrr|rrrr|r}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 0 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 0 \\
\hline
1 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 \\
-1 & 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 \\
-1 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & -1 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1
\end{array}\right) \end{array}.$$

7.3.2 BTIB Designs Obtained from BIBDs

In this section, we consider two constructional methods of BTIB designs. Method (i) was given by Bechhofer and Tamhane (1981) and Method (ii) was described in Das et al. (2005). Both are based on BIBDs.

(i) From Bechhofer and Tamhane (1981), it is known that from a BIBD(v^*, b, r, k, λ) where $v^* > v$, a BTIB design with parameters v, b, k, r, $r_{c_0} = (v^* - v)r$, λ, $\lambda_{c_0} = (v^* - v)\lambda$ can be obtained by replacing the treatments $v+1, v+2, \ldots, v^*$ by the control treatment. Here v should be such that each of the new blocks after replacement contains at least one test treatment. The optimum **W**-matrices for BTIB design can be constructed by using optimum **W**s for the corresponding BIBD. The following theorem gives the results precisely:

Theorem 7.3.5 *If c^* optimum **W**-matrices exist for BIBD(v^*, b, r, k, λ), then an equal number of optimum **W** matrices for BTIB design with parameters v, b, k, r, $r_{c_0} = (v^* - v)r$, λ, $\lambda_{c_0} = (v^* - v)\lambda$ can be constructed.*

Proof of the theorem follows from the fact that any optimum **W**-matrix for the BIBD remains optimum for the corresponding BTIB design as the latter is obtained from the first one by just renaming of the $(v^* - v)$ treatments.

The method will be clear from an example. Consider a symmetric BIBD ($v^* = b = 7, r = k = 4, \lambda = 2$). Here 3 optimum **W**s for the BIBD can be constructed (cf. Chap. 4) and these are denoted by $\mathbf{W}_{1,BIBD}$, $\mathbf{W}_{2,BIBD}$ and $\mathbf{W}_{3,BIBD}$. These **W**s provide three OCDs for the BTIB design. Take $v = 5$ and Treatments 6 and 7 of the BIBD are relabeled by the control treatment c of BTIB design. Using $\mathbf{W}_{1,BIBD}$, an optimum **W**-matrix for the BTIB design can be constructed as

$$\mathbf{W}'_{1,BIBD} = \begin{array}{c} \text{Treatment} \longrightarrow \\ \begin{array}{ccccc|cc} \text{Test} & & & & & \text{Control} & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left(\begin{array}{rrrrr|rr} 1 & 0 & 0 & -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 \end{array}\right) \end{array} = \mathbf{W}'_{1,BTIBD}.$$

Similarly from $\mathbf{W}_{2,BIBD}$ and $\mathbf{W}_{3,BIBD}$, two more optimum $\mathbf{W}$s for BTIB design can be obtained.

Remark 7.3.5 It is to be noted that the number of optimum $\mathbf{W}$s for the BTIB design considered above does not depend on the numbers of either the test treatments or the control treatments used but depends only on the existence of optimum $\mathbf{W}$-matrices for the corresponding BIBD as the same $\mathbf{W}$-matrixe for BIBD is used for the BTIB design.

(ii) Consider a BIBD, d_0, with the parameters $v^*,\ b^*,\ r^*,\ k^*,\ \lambda^*$. Replace a given set of i $(0 \leq i \leq v^* - 2)$ of the treatments in d_0 by the control treatment and call the resultant design $\text{BIB}_i(v^*,\ b^*,\ k^*)$. Finally, each block of the design $\text{BIB}_i(v^*,\ b^*,\ k^*)$ is augmented by $u \geq 0$ replications of the control treatment, such that $(i,\ u) \neq (0,\ 0)$. Denote this design by d. Then, it is easy to see that d is a BTIB design with parameters $v = v^* - i,\ b = b^*,\ k = k^* + u,\ r = r^*,\ r_{c_0} = ir^* + b^*u,\ \lambda = \lambda^*,\ \lambda_{c_0} = i\lambda^* + r^*u,\ 0 \leq i \leq v^* - 2,\quad u \geq 0$. For convenience, the design d is denoted by $\text{BIB}_i(v^*,\ b^*,\ k^*,\ u)$. Note that a $\text{BIB}_0(v^*,\ b^*,\ k^*,\ u)$ is a BTIB of the R-type while a $\text{BIB}_1(v^*,\ b^*,\ k^*,\ u)$ is a BTIB of the S-type. For a definition of R- and S-type BTIB designs see Hedayat and Majumdar (1984). OCDs for such BTIB design can be obtained by the following theorem:

Theorem 7.3.6 *Suppose that c_1^* and c_2^* optimum $\mathbf{W}$-matrices exist for RBD($b^*,\ u$) and BIBD($v^*,\ b^*,\ r^*,\ k^*,\ \lambda^*$) respectively. Then $c^{**} = min\{c_1^*,\ c_2^*\}$ optimum $\mathbf{W}$-matrices exist for BTIB($v = v^* - i,\ b = b^*,\ k = k^* + u,\ r = r^*,\ r_{c_0} = ir^* + b^*u,\ \lambda = \lambda^*,\ \lambda_{c_0} = i\lambda^* + r^*u$) for $i = 0, 1, \ldots, v^*-2$.*

Proof The proof is simple and follows along the lines of Theorems 7.3.4 and 7.3.5. □

7.3.3 BTIB Designs Obtained from Group Divisible (GD) Designs

In this section, again we deal with two methods of construction of BTIB designs from GD designs where the first method was illustrated in Bechhofer and Tamhane (1981) and the lastone in Das et al. (2005).

(i) According to Bechhofer and Tamhane (1981), BTIB design can be constructed from group divisible (GD) design with u treatments and b blocks of size k. The association scheme of such a GD design can be obtained by representing the treatments in the form of an $m \times w$ array (with $mw = u$). Any two treatments in the same row of the array are first associates, and those in the different rows are second associates. For OCDs in GD design set-up one is referred to Chap. 5. Suppose that $m \geq k$. One can take $v = m$ and relabel the entries in each of $n_1 > 0$ columns of the array by $1, 2, \ldots, v$ and entries in the remaining $n_2 = w - n_1 > 0$ columns by the control treatment c, thus obtaining a BTIB design. We shall only consider those BTIB designs, no block of which contains the control treatment entirely. OCDs can be constructed for such BTIB design and precise statement follows:

Theorem 7.3.7 *If c^{***} optimum* **W***-matrices exist for a GD design then an equal number of optimum* **W***-matrices exists for the BTIB design obtained from the GD-PBIBD.*

Proof of the theorem is simple. The method is explained through an example by considering the singular group divisible (SGD) design S2 ($u = 6,\ b = 6,\ r = 4,\ k = 4,\ \lambda_1 = 4,\ \lambda_2 = 2$) in Clatworthy's table (1973), page 83, given by

$$blocks \longrightarrow$$

1	2	3	4	5	6
4	5	6	1	2	3
2	3	1	5	6	4
5	6	4	2	3	1

with the following association scheme:

$$\left\{\begin{matrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{matrix}\right\}.$$

By relabeling the treatments 4, 5 and 6 by c_0's, a BTIB design can be obtained where no block contains the control treatment entirely. The blocks are:

$$blocks \longrightarrow$$

b_1	b_2	b_3	b_4	b_5	b_6
1	2	1	1	2	1
2	3	3	2	3	3
c_0	c_0	c_0	c_0	c_0	c_0
c_0	c_0	c_0	c_0	c_0	c_0

For the given SGD design, 5 optimum **W**-matrices (cf. Chap. 5) can be constructed and using these **W**-matrices 5 optimum **W**s for the BTIB design can also be constructed by identifying the control treatment in different blocks with the original treaments 4, 5, and 6. One of the **W**-matrix is given by

$$\mathbf{W}_1 = \begin{array}{c} \text{control Tr.} \\ \downarrow \\ \\ \text{Test Tr.} \\ \downarrow \\ \end{array} \begin{pmatrix} 0 & -1 & 1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ \hline 0 & 1 & -1 & 0 & -1 & 1 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{pmatrix} = (\mathbf{W}_C^{(1)\prime}, \ \mathbf{W}_T^{(1)\prime})'.$$

Remark 7.3.6 In Chap. 5, we provide a list of optimum **W**s for a large number of GD designs. These optimum **W**s may be used to generate the optimum **W**s for the BTIB designs, obtainable from these GD designs.

(ii) Consider two GD designs d_1 and d_2. Suppose that the parameters of d$_i$ are $v,\ b_i,\ r_i,\ k_i,\ \lambda_{1i},\ \lambda_{2i}, i$ =1, 2. Assume that $k_2 > k_1$ and $\lambda_{11}+\lambda_{12} = \lambda_{21}+\lambda_{22} = \lambda$. Then the design obtained by taking the union of the blocks of d_1 and d_2, after adding the control treatment $k_2 - k_1$ times to the blocks of d_1, is a BTIB design (cf. Das et al. 2005) with the following parameters:

$$v,\ b = b_1 + b_2,\ k = k_2,\ r = r_1 + r_2,\ r_{c_0} = b(k_2 - k_1),\ \lambda,\ \lambda_{c_0} = r_1(k_2 - k_1). \tag{7.3.28}$$

Following theorem describes the construction of OCDs for such BTIB designs.

Theorem 7.3.8 *Suppose c_i^{**} and c_3^{**} optimum* **W***-matrices exist for the GD designs with parameters $v,\ b_i,\ r_i,\ k_i,\ \lambda_{1i},\ \lambda_{2i},\ i = 1, 2,\ k_2 > k_1$ and $\lambda_{11} + \lambda_{12} = \lambda_{21} + \lambda_{22} = \lambda$ and the RBD($k_2 - k_1,\ b$) respectively, then it is possible to construct $c^{****} = min\{c_1^{**},\ c_2^{**},\ c_3^{**}\}$ optimum* **W***-matrices for the BTIB design with the parameters given in (7.3.28).*

Proof The proof is simple and hence omitted. □

Remark 7.3.7 Theorem 7.3.8 also holds for any two 2-associate PBIB designs with same association scheme. For details, we refer the original paper of Dutta and Das (2013).

Remark 7.3.8 In this chapter, construction of OCDs in BTIB design set-up has been considered and a large number of commonly used BTIB designs is covered. It is expected that these designs will serve practical purposes to a large extent. As the results are of varied nature, a summary of the BTIB design set-ups and the conditions of existence of OCDs, etc. is presented in the following table which may be helpful for ready reference (Tables 7.1, 7.2 and 7.3).

Table 7.1 BTIB designs obtained from generator designs

Design	Conditions	No. of optimum **W**-matrices
$D_0 \cup D_1$ (Lemma 7.3.1)	$m = 0,\ k = 2,\ v =$ even	1
$f(D_0 \cup D_1)$ (Theorem 7.3.1)	$m = 0,\ k = 2,\ v =$ even and $\mathbf{H}_f$ exists	f
D_0 (Theorem 7.3.2)	$k > 2,\ m = 0,\ (k-1)-$ resolvable BIBD$(v,\ sv,\ s(k-1),\ k-1,\ \lambda_0)$ $\mathbf{H}_k$ and $\mathbf{H}_{s/2}$ exist	$s(k-1)/2$
D_m (Theorem 7.3.3)	$k > 2,\ m(>0) =$ even, u_2 OCDs for RBD$(m,\ b_m)$ and u_1 OCDs for $D_0(v,\ b_m,\ k-m,\ r_m,\ b_m,\ \lambda_m,$ $\lambda_c = r_m)$ exist	u = $\min\{u_1,\ u_2\}$
D_m (Theorem 7.3.4)	$k > 2,\ m(>0) =$ odd, u_1 OCDs for RBD$(m+1,\ b_m)$ and u_2 OCDs for BIBD$(v,\ b_m,\ r_m,\ k-m-1,\ \lambda_m)$ exist	u = $\min\{u_1,\ u_2\}$

Table 7.2 BTIB designs obtained from BIBDs

Design	Conditions	No. of optimum **W**-matrices
BTIB design mentioned in Theorem 7.3.5	Existence of c^* OCDs for BIBD$(v^*,\ b^*,\ r^*,\ k^*,\ \lambda^*)$	c^*
BTIB design mentioned in Theorem 7.3.6	Existence of c_1^* OCDs for RBD$(b^*,\ u)$ and c_2^* OCDs for BIBD$(v^*,\ b^*,\ r^*,\ k^*,\ \lambda^*)$	$c^{**} = \min\{c_1^*, c_2^*\}$

Table 7.3 BTIB designs obtained from GD designs

Design	Conditions	No. of optimum **W**-matrices
BTIB design mentioned in Theorem 7.3.7	Existence of c^{***} OCDs for GD design	c^{***}
BTIB design mentioned in Theorem 7.3.8	Existence of c_i^{**} OCDs for GDDs$(v,\ b_i,\ r_i,\ k_i,\ \lambda_{1i},\ \lambda_{2i})$, $i = 1,\ 2, k_2 > k_1, \lambda_{11} + \lambda_{12} = \lambda_{21}$ $+ \lambda_{22}$ and existence of c_3^{**} OCDs for RBD$(k_2 - k_1,\ b)$	$c^{****} = \min\{c_1^{**}, c_2^{**}, c_3^{**}\}$

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Chapter 8
Miscellaneous Other Topics and Issues

8.1 OCDs in the Crossover Designs

The problem of optimal choice of covariates in the set-up of crossover design or repeated measurement design (RMD) has been considered by Dutta and SahaRay (2013). A crossover design is used in an experiment in which a unit is exposed to various treatments over different periods. In such an experiment, t treatments are assigned to n experimental units each of which receives one treatment during each of the p periods. Such designs are very often used in many industrial, agricultural and biological experiments. Under the traditional model, it is assumed that each treatment assigned to an experimental unit (e.u.) has a direct effect on the e.u. in the same period and also carryover effects (residual effects) in the subsequent periods. Efficient estimation and testing of the direct effects as well as residual effects are of interest to practitioners from an application point of view. The reader is referred to Stufken (1996) and Bose and Dey (2009) for a review on this topic. In practice, situations arise when controllable covariates are used conveniently in this set-up to control the experimental error. For example, in treating arthritis pain or prevention of heart disease, the duration of daily exercise or walking plays a role, besides the effects of medicines. Thus the duration of exercise or walking can be viewed as a controllable covariate when formulating an appropriate model for the study of the effects of different medicines in such cases. So the problem arises to propose appropriate designs which will allow most efficient estimation of these covariate effects on the response. The aim is to address this issue dealing with c covariates for some classes of strongly balanced and balanced crossover designs which are known to be universally optimal for the estimation of direct treatment effects and residual treatment effects in an appropriate class of competing designs.

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8.1.1 Preliminary Definitions and Notations

We assume t treatments, denoted by $1, 2, \ldots, t$ are to be compared using n e.u.s over p periods. Let $\Omega_{t,n,p}$ denote the class of such crossover designs. A design $d \in \Omega_{t,n,p}$ is *uniform on the periods* if each treatment is assigned to an equal number of subjects in each period. A design $d \in \Omega_{t,n,p}$ is *uniform on the subjects* if each treatment is assigned equally often to each subject. A design is said to be *uniform* if it is uniform on the periods and uniform on the subjects. A crossover design is said to be *balanced*, if no treatment is immediately preceded by itself and each treatment is immediately preceded by every other treatment equally often. A crossover design is called *strongly balanced* if each treatment is immediately preceded by every treatment (including itself) equally often.

Here, we deal with a covariate model allowing c covariates under the crossover design set-up. Let $d(i, j)$ denote the treatment assigned by $d \in \Omega_{t,n,p}$ in the ith period to the jth e.u.; $i = 1, 2, \ldots, p,\ \ j = 1, 2, \ldots, n$. The model of response for the observation y_{ij} with $z_{ij}^{(l)}$, the value of the lth covariate Z_l received in the ith period on the jth experimental unit is given by

$$y_{ij} = \mu + \alpha_i + \beta_j + \tau_{d(i,j)} + \rho_{d(i-1,j)} + \sum_{l=1}^{c} \gamma_l z_{ij}^{(l)} + e_{ij}, \tag{8.1.1}$$

where μ is the general mean, α_i is the ith period effect, β_j is the jth experimental unit effect, $\tau_{d(i,j)}$ is the direct effect due to treatment $d(i, j)$, $\rho_{d(i-1,j)}$ is the first order residual effect of treatment $d(i-1, j)$ with $\rho_{d(0,j)} = 0$ for all $j = 1, 2, \ldots, n$; γ_l is the regression coefficient associated with the lth covariate, $l = 1, 2, \ldots, c$. As usual, the random errors $\{e_{ij}\}'s$ are assumed to be uncorrelated and homoscedastic with the common variance σ^2.

Writing the observations unit wise, in matrix notation the above model can be represented as

$$(\mathbf{Y},\ \mu\mathbf{1}_{np} + \mathbf{X}_1\boldsymbol{\alpha} + \mathbf{X}_2\boldsymbol{\beta} + \mathbf{X}_3\boldsymbol{\tau} + \mathbf{X}_4\boldsymbol{\rho} + \mathbf{Z}\boldsymbol{\gamma},\ \mathbf{I}_{np}\sigma^2) \tag{8.1.2}$$

where $\mathbf{Y}$ is the observation vector of order $np \times 1$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\tau}$, $\boldsymbol{\rho}$ and $\boldsymbol{\gamma}$ correspond, respectively, to the vectors of period effects, experimental unit effects, direct effects, first-order residual effects and the covariate effects; $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{X}_4$ and $\mathbf{Z}$ denote, respectively, the part of the design matrix corresponding to the period effects, experimental unit effects, direct effects, first-order residual effects and covariate effects, $\mathbf{1}_{np}$ is a vector of all ones of order np and $\mathbf{I}_{np}$ is the identity matrix of order np.

In model (8.1.2) each of the covariates Z_l's, $l = 1, 2, \ldots, c$ is assumed to be a controllable non-stochastic variable. Applying a location scale transformation of the original limits of the values of the covariates, without loss of generality, it is assumed that the np values $z_{ij}^{(l)}$'s taken by the lth covariate Z_l can vary within the interval $[-1, 1]$, i.e.

$$z_{ij}^{(l)} \in [-1, 1],\ i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, n;\ l = 1, 2, \ldots, c. \tag{8.1.3}$$

With reference to model (8.1.2), it is evident that orthogonal estimation of the ANOVA effects and the covariate effects is possible whenever the following conditions:

$$\mathbf{X}_i'\mathbf{Z} = \mathbf{0},\quad \forall i = 1, 2, 3, 4 \tag{8.1.4}$$

are satisfied. Further, the covariate effects are estimated with the maximum efficiency if and only if (cf. Pukelsheim 1993)

$$\mathbf{Z}'\mathbf{Z} = np\mathbf{I}_c. \tag{8.1.5}$$

Therefore, optimal estimation of each of the covariate effects is possible while the estimates of the ANOVA effects remain unaltered, if and only if $\mathbf{Z}$ satisfies the conditions (8.1.4) and (8.1.5) simultaneously. In the sequel, any Hadamard matrix of order R is written as

$$\mathbf{H}_R = (\mathbf{h}_1^{(R)}, \ldots, \mathbf{h}_R^{(R)}). \tag{8.1.6}$$

For a Hadamard matrix in the *seminormal* form we assume, without loss of generality, $\mathbf{h}_R^{(R)}$ to be $\mathbf{1}$.

Note that under model (8.1.2) for any $d \in \Omega_{t,n,p}$, $\mathbf{X}_1 = \mathbf{I}_p \otimes \mathbf{1}_n$ and $\mathbf{X}_2 = \mathbf{1}_p \otimes \mathbf{I}_n$. Thus for d, conditions (8.1.4) and (8.1.5) are equivalent to the following conditions:

$$\left.\begin{array}{lll}
(i) & z_{ij}^{(l)} & = \pm 1\ \forall\, i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, n;\ l = 1, 2, \ldots, c, \\
(ii) & \sum_{i=1}^{p} z_{ij}^{(l)} & = 0\ \forall\, j = 1, 2, \ldots, n;\ l = 1, 2, \ldots, c, \\
(iii) & \sum_{j=1}^{n} z_{ij}^{(l)} & = 0\ \forall\, i = 1, 2, \ldots, p;\ l = 1, 2, \ldots, c, \\
(iv) & \sum_{(i,j):d(i,j)=k} z_{ij}^{(l)} & = 0\ \forall\, k = 1, 2, \ldots, t;\ l = 1, 2, \ldots, c, \\
(v) & \sum_{(i,j):d(i-1,j)=k} z_{ij}^{(l)} & = 0\ \forall\, k = 1, 2, \ldots, t;\ l = 1, 2, \ldots, c, \\
(vi) & \sum_{i=1}^{p}\sum_{j=1}^{n} z_{ij}^{(l)} z_{ij}^{(l')} & = 0\ \forall\, l \neq l' = 1, 2, \ldots, c.
\end{array}\right\} \tag{8.1.7}$$

Thus to obtain an OCD for any $d \in \Omega_{t,n,p}$ it is required to construct the $\mathbf{Z}$-matrix satisfying the conditions laid down in (8.1.7). In general, for any arbitrary d this problem of construction is combinatorially intractable. Dutta and SahaRay (2013) handled this issue of construction by adopting the technique used by Das et al. (2003) where *each column* of the $\mathbf{Z}$-matrix can be recast to a $\mathbf{W}$-matrix. Using this idea, the lth column of $\mathbf{Z}$-matrix, a vector of order $np \times 1$ is represented in the form of the matrix $\mathbf{W}^{(l)}$ of order $p \times n$, where the columns correspond to the experimental units in the order $1, 2, \ldots, n$ and the rows correspond to the periods in the order $1, 2, \ldots, p$.

To elucidate the idea, the lth column of $\mathbf{Z}$-matrix is written as $\mathbf{W}^{(l)}$-matrix in the following way:

$$\mathbf{W}^{(l)} = \begin{pmatrix} z_{11}^{(l)} & z_{12}^{(l)} & \ldots & z_{1n}^{(l)} \\ z_{21}^{(l)} & z_{22}^{(l)} & \ldots & z_{2n}^{(l)} \\ & & \vdots & \\ z_{p1}^{(l)} & z_{p2}^{(l)} & \ldots & z_{pn}^{(l)} \end{pmatrix}, \; l = 1, 2, \ldots, c. \tag{8.1.8}$$

The requirement of the $\mathbf{Z}$-matrix satisfying the conditions (*ii*) and (*iii*) of (8.1.7) is equivalent to having zero row sums and zero column sums for each row and each column of $\mathbf{W}^{(l)}$, $l = 1, 2, \ldots, c$. To visualize the conditions (*iv*) and (*v*) of (8.1.7) in terms of the $\mathbf{W}$-matrix we define two more matrices of order $p \times n$ as follows:

$$\mathbf{V}_1 = \begin{pmatrix} d(1,1) & d(1,2) & \ldots & d(1,n) \\ d(2,1) & d(2,2) & \ldots & d(2,n) \\ & & \vdots & \\ d(p,1) & d(p,2) & \ldots & d(p,n) \end{pmatrix}, \; \mathbf{V}_2 = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ d(1,1) & d(1,2) & \ldots & d(1,n) \\ & & \vdots & \\ d(p-1,1) & d(p-1,2) & \ldots & d(p-1,n) \end{pmatrix}. \tag{8.1.9}$$

Recalling that $d(i, j)$ denotes the treatment assigned to the jth unit in the ith period of $d \in \Omega_{t,n,p}$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, n$, it is now easy to verify that the requirement of the lth column of the $\mathbf{Z}$-matrix satisfying the conditions (*iv*) and (*v*) of (8.1.7) is equivalent to the requirement of the sums of $z_{ij}^{(l)}$'s corresponding to the same treatment to be equal to zero after superimposition of $\mathbf{W}^{(l)}$ on $\mathbf{V}_1$ and $\mathbf{V}_2$ respectively, $l = 1, 2, \ldots, c$.

Thus the necessary and sufficient conditions in terms of the elements of $\mathbf{W}^{(l)}$, $l = 1, 2, \ldots, c$ for the existence of an OCD are summed up as follows:

$$\left.\begin{array}{ll} (C_1) & \text{Each of the elements of } \mathbf{W}^{(l)} \text{ is either } +1 \text{ or } -1; \\ (C_2) & \mathbf{W}^{(l)}\text{-matrix has all row sums equal to zero;} \\ (C_3) & \mathbf{W}^{(l)}\text{-matrix has all column sums equal to zero;} \\ (C_4) & \text{After superimposing } \mathbf{W}^{(l)} \textit{ on } \mathbf{V}_1,\ \text{for every treatment as specified in } \mathbf{V}_1, \\ & \text{the sum of the elements of } \mathbf{W}^{(l)} \text{corresponding to the same treatment} \\ & \text{is equal to zero;} \\ (C_5) & \text{After superimposing } \mathbf{W}^{(l)} \textit{ on } \mathbf{V}_2,\ \text{for every treatment as specified in } \mathbf{V}_2 \\ & \text{the sum of the elements of } \mathbf{W}^{(l)} \text{ corresponding to the same treatment} \\ & \text{is equal to zero;} \\ (C_6) & \text{The grand total of all entries in the Hadamard product of } \mathbf{W}^{(l)} \text{ and} \\ & \mathbf{W}^{(l')} \text{ is equal to } np \text{ or zero depending on } l = l' \textit{ or } l \neq l' \text{ respectively.} \end{array}\right\} \tag{8.1.10}$$

It is worthwhile to note that a covariate design **Z** for c covariates is equivalent to c **W**-matrices which are convenient to work with.

Definition 8.1.1 With respect to model (8.1.2), the c **W**-matrices corresponding to the c covariates are said to be optimum if they satisfy the conditions laid down in (8.1.10).

Remark 8.1.1 It is to be noted that if $c = 1$, only the conditions C_1–C_5 of (8.1.10) are to be satisfied by the **W**-matrix for an OCD to exist.

Definition 8.1.2 The maximum number of covariates cannot exceed the error degrees of freedom for the ANOVA part of a given set-up.

Here, our aim is to construct an OCD. In other words optimum **W**-matrices, with as many **W**-matrices as possible for a crossover design which is uniform strongly balanced or strongly balanced, uniform on the periods and uniform on the units in the first $p - 1$ periods or uniform balanced. The construction of **W**-matrices is much dependent on the particular method of construction of the underlying basic crossover design. We will denote by c^* the maximum value of c, the number of covariates in a given context as attained by a given method of construction. In reality, a limited number of covariates turn out to be useful. Thus given the choice of c^* optimum **W**-matrices, the experimenter has the flexibility of selecting the optimum values of the required number of covariates from a large pool of possible options, appropriate to the experimental situation and availability of the resources.

8.1.2 Main Results

Here the construction of **W**-matrices satisfying (8.1.10) for different series of strongly balanced and balanced crossover designs obtained through different constructional methods are given. We briefly discuss the method of construction of the underlying basic crossover design to understand the construction of optimum **W**-matrices as their interdependency has already been pointed out.

Strongly Balanced Crossover Design Set-up in $\Omega_{t,\lambda_1 t^2,\lambda_2 t}$
It has been shown in Stufken (1996) that a uniform strongly balanced crossover design d^* in $\Omega_{t,n,p}$ is universally optimal for the estimation of direct treatment effects and residual treatment effects and can always be constructed using latin squares and orthogonal arrays whenever $n = \lambda_1 t^2$ and $p = \lambda_2 t$ for integers $\lambda_1 \geq 1$ and $\lambda_2 \geq 2$. We start with this particular method of construction of d^* assuming $\lambda_1 = 1$ and obtain an OCD. The construction of OCD with $n = \lambda_1 t^2$, $\lambda_1 > 1$ will be taken up later.

Let $\mathbf{A}_t$ be an orthogonal array, denoted by OA $(t^2, 3, t, 2)$ with entries from $S = \{1, 2, \ldots, t\}$. Such an orthogonal array can easily be obtained from a latin square $L = ((l_{ij}))$ of order t, $t \geq 2$ as follows:

$$\mathbf{A}_t : \begin{array}{cccc} \overbrace{1\ 1\ \ldots\ 1}^{t \text{ times}} & \overbrace{2\ 2\ \ldots\ 2}^{t \text{ times}} & \ldots & \overbrace{t\ t\ \ldots\ t}^{t \text{ times}} \\ 1\ 2\ \ldots\ t & 1\ 2\ \ldots\ t & \ldots & 1\ 2\ \ldots t \\ l_{11}l_{12}\ldots l_{1t} & l_{21}l_{22}\ldots l_{2t} & \ldots & l_{t1}l_{t2}\ldots l_{tt} \end{array}. \tag{8.1.11}$$

Let $\mathbf{B}_t$ be an orthogonal array OA $(t^2, 2, t, 2)$, obtained from $\mathbf{A}_t$ by deleting the third row in $\mathbf{A}_t$. For $i \in \{1, 2, \ldots, t-1\}$ let $\mathbf{A}_i = \mathbf{A}_t + i$ and $\mathbf{B}_i = \mathbf{B}_t + i$, where i is added to each element of $\mathbf{A}_t$ or $\mathbf{B}_t$, modulo t. Let the two arrays $\mathbf{A}$ and $\mathbf{B}$ of order $3t \times t^2$ and $2t \times t^2$, respectively, be defined as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_t \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \vdots \\ \mathbf{B}_t \end{pmatrix}. \tag{8.1.12}$$

With $\lambda_2 \geq 2$, writing $\lambda_2 = 3\delta_1 + 2\delta_2$ for non-negative integers δ_1 and δ_2, the $p \times t^2$ array d^* defined by

$$d^* = (\mathbf{A}', \ldots, \mathbf{A}', \ \mathbf{B}', \ldots, \mathbf{B}')' \tag{8.1.13}$$

consisting of δ_1 copies of $\mathbf{A}$ and δ_2 copies of $\mathbf{B}$ is a uniform strongly balanced crossover design in $\Omega_{t,n,p}$.

We now present the actual construction of OCD, in other words optimum $\mathbf{W}$-matrices for d^* in $\Omega_{t,t^2,p}$ under a variety of choices of t accommodating the maximum number of covariates as attained by the given method of construction.

***Case* 1: $t = 0$(*mod* 4)**

The following theorem relates to an OCD for d^* in $\Omega_{t,t^2,3t}$.

Theorem 8.1.1 *Suppose $\mathbf{H}_t$, $\mathbf{H}_{3t}$ and further $s(\geq 2)$ mutually orthogonal latin squares (MOLS) of order t exist. Let d^* in $\Omega_{t,t^2,3t}$ be constructed as described in (8.1.13). Then there exists a set of $(3t-1)(t-1)(s-1)$ optimum $\mathbf{W}$-matrices $d^* \in \Omega_{t,t^2,3t}$.*

Proof Without loss of generality we assume that $\mathbf{H}_t$ and $\mathbf{H}_{3t}$ are in the *seminormal* form. Let $L_1, L_2, \ldots, L_s$ be s MOLS of order t, based on the symbols $1, 2, \ldots, t$. Suppose L_s is used for constructing $\mathbf{A}_t$ in (8.1.11) and $L_s^{(q)} = L_s + q$, where q is added to each element of L_s modulo t, is used to develop the third row of $\mathbf{A}_q$, $q = 1, 2, \ldots, t-1$ in (8.1.12) to give rise to d^* in $\Omega_{t,t^2,3t}$ as described in (8.1.13). Now we proceed to construct the optimum $\mathbf{W}$-matrices for d^* in $\Omega_{t,t^2,3t}$ as follows.

In each of the L_i, $i = 1, 2, \ldots, s-1$, replace the symbols $1, 2, \ldots, t$ by the elements of $\mathbf{h}_j^{(t)}$ in order, for $j = 1, 2, \ldots, t-1$. Let $\mathbf{d}_m^{ij\prime}$ denote the replaced mth row of L_i, $m = 1, 2, \ldots, t$ written with the symbols of $\mathbf{h}_j^{(t)}$. Now juxtaposing side by side these t rows, we obtain a row vector $\mathbf{D}'_{ij}$ of order t^2 given by

$$\mathbf{D}'_{ij} = \left(\mathbf{d}_1^{ij\prime} : \mathbf{d}_2^{ij\prime} : \ldots : \mathbf{d}_t^{ij\prime}\right). \tag{8.1.14}$$

Now we construct $\mathbf{W}_{ijf}$ of order $3t \times t^2$ as follows:

$$\begin{gathered}\mathbf{W}^{(l)} = \mathbf{W}_{ijf} = \mathbf{h}_f^{(3t)} \otimes \mathbf{D}'_{ij};\\ i = 1, 2, \ldots, s-1,\ j = 1, \ldots, t-1,\ f = 1, \ldots, 3t-1,\\ l = (i-1)(t-1)(3t-1) + (j-1)(3t-1) + f.\end{gathered} \tag{8.1.15}$$

Using the properties of latin square, Hadamard matrices and the fact that L_i, $i = 1, 2, \ldots, s-1$ is orthogonal with $L_s^{(q)}$, $q = 1, 2, \ldots, t-1$, defined above, it is not hard to see that $\mathbf{W}^{(l)}$'s satisfy the conditions of (8.1.10) and the maximum number of covariates in the given context attained by the method of construction is $c^* = (3t-1)(t-1)(s-1)$. □

An illustration of the above method of construction with $t = 4$ follows.

Example 8.1.1 $t = 4$, $d^* \in \Omega_{4,16,12}$

$$L_1 = \begin{pmatrix} 1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\\ 3 & 4 & 1 & 2\\ 4 & 3 & 2 & 1 \end{pmatrix},\ L_2 = \begin{pmatrix} 1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\\ 4 & 3 & 2 & 1\\ 2 & 1 & 4 & 3 \end{pmatrix},\ L_3 = \begin{pmatrix} 1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\\ 2 & 1 & 4 & 3\\ 3 & 4 & 1 & 2 \end{pmatrix}$$

and

$$\mathbf{A} = \begin{pmatrix} 2&2&2&2&3&3&3&3&4&4&4&4&1&1&1&1\\ 2&3&4&1&2&3&4&1&2&3&4&1&2&3&4&1\\ 2&3&4&1&1&4&3&2&3&2&1&4&4&1&2&3\\ 3&3&3&3&4&4&4&4&1&1&1&1&2&2&2&2\\ 3&4&1&2&3&4&1&2&3&4&1&2&3&4&1&2\\ 3&4&1&2&2&1&4&3&4&3&2&1&1&2&3&4\\ 4&4&4&4&1&1&1&1&2&2&2&2&3&3&3&3\\ 4&1&2&3&4&1&2&3&4&1&2&3&4&1&2&3\\ 4&1&2&3&3&2&1&4&1&4&3&2&2&3&4&1\\ 1&1&1&1&2&2&2&2&3&3&3&3&4&4&4&4\\ 1&2&3&4&1&2&3&4&1&2&3&4&1&2&3&4\\ 1&2&3&4&4&3&2&1&2&1&4&3&3&4&1&2 \end{pmatrix}.$$

The forms of $\mathbf{H}_4$ and $\mathbf{H}_{12}$ for our use are

$$\mathbf{H}_4 = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{8.1.16}$$

$$\mathbf{H}_{12} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\ -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \end{pmatrix}. \tag{8.1.17}$$

Now using $\mathbf{h}_1^{(4)}$, the first column of $\mathbf{H}_4$ and L_1, we construct $\mathbf{D}'_{11}$ as

$$\mathbf{D}'_{11} = \begin{pmatrix} -1 & 1 & -1 & 1 : & 1 & -1 & 1 & -1 : & -1 & 1 & -1 & 1 : & 1 & -1 & 1 & -1 \end{pmatrix}.$$

Hence using $\mathbf{h}_1^{(12)}$, the first column of $\mathbf{H}_{12}$, $\mathbf{W}^{(1)} = \mathbf{W}_{111} = \mathbf{h}_1^{(12)} \otimes \left(\mathbf{d}_1^{11},\ \mathbf{d}_2^{11},\ \mathbf{d}_3^{11},\ \mathbf{d}_4^{11}\right)$ takes the form

$$\mathbf{W}^{(1)} = \begin{pmatrix} -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

Similarly 65 more choices of the optimum **W**-matrices can be constructed.

Remark 8.1.2 In practice in the above situation the experimenter has the flexibility to choose the values of the required number of optimum covariates from the set of 66 possible optimum choices.

Remark 8.1.3 In particular for $t = 4$, three more optimum $\mathbf{W}^{(l)}$ for d^* in $\Omega_{4,16,12}$ can be constructed by trial and error method as follows:

$$\mathbf{W}^{(67)} = \begin{pmatrix} \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \end{pmatrix}, \quad \mathbf{W}^{(68)} = \begin{pmatrix} \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \end{pmatrix},$$

$$\mathbf{W}^{(69)} = \begin{pmatrix} \mathbf{1}'_4 & -\mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ \mathbf{1}'_4 & -\mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ \mathbf{1}'_4 & -\mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ \mathbf{1}'_4 & -\mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ \mathbf{1}'_4 & -\mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ \mathbf{1}'_4 & -\mathbf{1}'_4 & -\mathbf{1}'_4 & \mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \\ -\mathbf{1}'_4 & \mathbf{1}'_4 & \mathbf{1}'_4 & -\mathbf{1}'_4 \end{pmatrix}.$$

Theorem 8.1.2 *Suppose $\mathbf{H}_t$, and further $s(\geq 2)$ mutually orthogonal latin squares (MOLS) of order t exist. Let d^* in $\Omega_{t,t^2,2t}$ be constructed as described in (8.1.13). Then there exists a set of $(2t-1)(t-1)s$ optimum $\mathbf{W}$-matrices $d^* \in \Omega_{t,t^2,2t}$.*

Proof The proof is along the similar lines of the proof of Theorem 3.1. Note that $d^* \in \Omega_{t,t^2,2t}$ as described in (8.1.13) can be constructed without requiring to use L_s. So L_s can also be used to construct the row vector $\mathbf{D}'_{ij}$ (8.1.14) of order t^2 as before, $i = 1, 2, \ldots, s$; $j = 1, 2, \ldots, t-1$. Since $\mathbf{H}_t$ and hence $\mathbf{H}_{2t}$ exist, assuming both of these in the *seminormal* form, we construct $\mathbf{W}^{(l)}$ of order $2t \times t^2$ as follows:

$$\mathbf{W}^{(l)} = \mathbf{W}_{ijf} = \mathbf{h}_f^{(2t)} \otimes \left(\mathbf{d}_1^{ij'} : \mathbf{d}_2^{ij'} : \ldots : \mathbf{d}_t^{ij'}\right);$$
$$i = 1, 2, \ldots, s, \ j = 1, \ldots, t-1, \ f = 1, \ldots, 2t-1,$$
$$l = (i-1)(t-1)(2t-1) + (j-1)(2t-1) + f$$

with $c^* = (2t-1)(t-1)s$ in the given context. □

Remark 8.1.4 For $t = 4$, four more optimum $\mathbf{W}^{(l)}$ for d^* in $\Omega_{4,16,8}$ can be constructed by trial and error method as described below:

$$\mathbf{W}^{(64)} = \begin{pmatrix} \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \\ \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \\ \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \\ \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \end{pmatrix}, \quad \mathbf{W}^{(65)} = \begin{pmatrix} \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \\ \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \\ \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \\ \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \end{pmatrix},$$

$$\mathbf{W}^{(66)} = \begin{pmatrix} \mathbf{1}_4' & -\mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \\ \mathbf{1}_4' & -\mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \\ \mathbf{1}_4' & -\mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \\ \mathbf{1}_4' & -\mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \end{pmatrix}, \quad \mathbf{W}^{(67)} = \begin{pmatrix} \mathbf{1}_4' & -\mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \\ \mathbf{1}_4' & -\mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \\ \mathbf{1}_4' & -\mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \\ -\mathbf{1}_4' & \mathbf{1}_4' & \mathbf{1}_4' & -\mathbf{1}_4' \\ \mathbf{1}_4' & -\mathbf{1}_4' & -\mathbf{1}_4' & \mathbf{1}_4' \end{pmatrix}.$$

***Case* 2: $t = 2$ (*mod* 4), $t \neq 2, 6$**

It is clear that $\mathbf{H}_t$ does not exist but if s MOLS of order t exist then $(s-1)$ optimum $\mathbf{W}$-matrices can be constructed for $d^* \in \Omega_{t,t^2,3t}$ (vide 8.1.13) using the same steps followed in the proof of Theorem 3.1 and the vector $\mathbf{a}_1 = \left(\mathbf{1}'_{\frac{t}{2}}, -\mathbf{1}'_{\frac{t}{2}}\right)'$ and $\mathbf{a}_2 = \left(\mathbf{1}'_{\frac{3t}{2}}, -\mathbf{1}'_{\frac{3t}{2}}\right)'$ instead of the columns of $\mathbf{H}_t$ and $\mathbf{H}_{3t}$ respectively. Similarly if $\mathbf{H}_{2t}$ exists, $(2t-1)s$ optimum $\mathbf{W}$-matrices can be constructed for $d^* \in \Omega_{t,t^2,2t}$ (vide 8.1.13) following the same steps of Theorem 8.1.2 using the vector $\mathbf{a}_1 = \left(\mathbf{1}'_{\frac{t}{2}}, -\mathbf{1}'_{\frac{t}{2}}\right)'$ instead of the columns of $\mathbf{H}_t$.

***Case* 3: $t = 2$**

Since a pair of MOLS does not exist for $t = 2$, the methods discussed in earlier cases do not apply here to construct an OCD. We adopt trial and error method to construct optimum $\mathbf{W}$-matrices.

Theorem 8.1.3 *Let d_1^* in $\Omega_{2,4,6}$ and d_2^* in $\Omega_{2,4,4}$ be constructed as described in (8.1.13). Then there exist 2 optimum* $\mathbf{W}$*-matrices for each of d_1^* and d_2^*.*

Proof Recalling (8.1.12) it is easy to see that d_1^* and d_2^* given below represent the strongly balanced design in $\Omega_{2,4,6}$ and $\Omega_{2,4,4}$ respectively.

$$d_1^* : \begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{pmatrix}, \quad d_2^* = \begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix}. \tag{8.1.18}$$

Optimum $\mathbf{W}$-matrices denoted by $\mathbf{W}_1^*$ and $\mathbf{W}_2^*$ for d_1^* and $\mathbf{W}_1^{**}$ and $\mathbf{W}_2^{**}$ for d_2^*, respectively, can be constructed as

$$\mathbf{W}_1^* = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad \mathbf{W}_2^* = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

$$\mathbf{W}_1^{**} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad \mathbf{W}_2^{**} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

□

***Case* 4: $t = 6$**

It is known that for $t = 6$ a pair of MOLS does not exist and hence we take up the construction of OCD in this case separately.

We start with a uniform strongly balanced crossover design $d^* \in \Omega_{6,36,18}$ constructed (vide 8.1.13) using the latin square L (say) given by

$$L = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \\ 6 & 5 & 1 & 2 & 3 & 4 \\ 5 & 6 & 2 & 1 & 4 & 3 \\ 4 & 3 & 6 & 5 & 2 & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}. \tag{8.1.19}$$

Theorem 8.1.4 *Let d_1^* in $\Omega_{6,36,18}$ and d_2^* in $\Omega_{6,36,12}$ be constructed (vide 8.1.13) using L of (8.1.19). Then there exist an optimum $\mathbf{W}$-matrix for d_1^* and 11 optimum $\mathbf{W}$ matrices for d_2^*.*

Proof Let $\mathbf{D}$ be a matrix of order 6×6 with elements ± 1 as follows:

$$\mathbf{D} = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} \mathbf{d}_1' \\ \mathbf{d}_2' \\ \mathbf{d}_3' \\ \mathbf{d}_4' \\ \mathbf{d}_5' \\ \mathbf{d}_6' \end{pmatrix}. \tag{8.1.20}$$

It is to be noted that the row sums and column sums of $\mathbf{D}$ are zero. Moreover superimposing $\mathbf{D}$ on L, it can be seen that for each symbol in L, the sum of the corresponding elements of $\mathbf{D}$ is also zero. Thus an optimum $\mathbf{W}$-matrix for d_1^* in $\Omega_{6,36,18}$ (vide 8.1.13) using L of (8.1.19) can be formed taking $\mathbf{a} = \left(\mathbf{1}_9', \ -\mathbf{1}_9'\right)'$ and the rows of matrix $\mathbf{D}$ as

$$\mathbf{W}^{(1)} = \mathbf{a} \otimes \left(\mathbf{d}_1' : \mathbf{d}_2' :, \mathbf{d}_3' : \mathbf{d}_4' : \mathbf{d}_5' : \mathbf{d}_6'\right).$$

But for d_2^* in $\Omega_{6,36,12}$ (vide 8.1.12), 11 optimum $\mathbf{W}$-matrices can be formed using $\mathbf{H}_{12}$ of (8.1.17) as follows:

$$\mathbf{W}^{(l)} = \mathbf{h}_l^{(12)} \otimes \left(\mathbf{d}_1' : \mathbf{d}_2' : \mathbf{d}_3' : \mathbf{d}_4' : \mathbf{d}_5' : \mathbf{d}_6'\right), \ l = 1, 2, \ldots, 11.$$

□

So far we have discussed the construction of optimum $\mathbf{W}$-matrices for uniform strongly balanced crossover design d_1^* in $\Omega_{t,t^2,3t}$ and d_2^* in $\Omega_{t,t^2,2t}$ separately. Let c_1^* and c_2^* denote the maximum number of optimum $\mathbf{W}$-matrices for d_1^* and d_2^* respectively in the given context. Now we will consider the construction of optimum $\mathbf{W}$-matrices for a strongly balanced crossover design d^* in $\Omega_{t,t^2,p}$ (vide 8.1.13) where $p = (3\delta_1 + 2\delta_2)t$ for non-negative integers δ_1 and δ_2. Write

$$d^* = \left[\, d_1^{*\prime}, \ldots, d_1^{*\prime}, d_2^{*\prime}, \ldots, d_2^{*\prime} \,\right]' \tag{8.1.21}$$

taking δ_1 copies of d_1^* and δ_2 copies of d_2^*.

Define

$$\delta_0 = \min\{\delta_1, \ \delta_2\} \text{ and } c_0 = \min\{c_1^*, c_2^*\}. \tag{8.1.22}$$

Corollary 8.1.1 *Suppose $\mathbf{H}_{\delta_1}$ and $\mathbf{H}_{\delta_2}$ exist. Let d^* in $\Omega_{t,t^2,p}$ be constructed as described in (8.1.21) for $p = (3\delta_1 + 2\delta_2)t$, $\delta_1, \ \delta_2 \geq 0$, non-negative integers. Then there exists a set of $\delta_0 c_0$ optimum $\mathbf{W}$-matrices for d^* where δ_0 and c_0 are defined in (8.1.22).*

Proof Let the c_1^* optimum $\mathbf{W}$-matrices for d_1^* be denoted by $\mathbf{W}_1^*, \ldots, \mathbf{W}_{c_1^*}^*$ and the c_2^* optimum $\mathbf{W}$-matrices for d_2^* be denoted by $\mathbf{W}_1^{**}, \ldots, \mathbf{W}_{c_2^*}^{**}$. Then it can be easily seen that $\mathbf{W}^{(l)}$ defined as

$$\mathbf{W}^{(l)} = \mathbf{W}_{ij} = \begin{pmatrix} \mathbf{W}^{*}_{ij} \\ \mathbf{W}^{**}_{ij} \end{pmatrix}; \text{ where } \mathbf{W}^{*}_{ij} = \mathbf{h}^{(\delta_1)}_i \otimes \mathbf{W}^{*}_j \text{ and } \mathbf{W}^{**}_{ij} = \mathbf{h}^{(\delta_2)}_i \otimes \mathbf{W}^{**}_j \tag{8.1.23}$$

$i = 1, 2, \ldots, \delta_0,\ j = 1, 2, \ldots, c_0,\ l = c_0(i-1) + j$, are the required **W**-matrices for d^*. □

Remark 8.1.5 Note that $\mathbf{H}_{\delta_1}$ and $\mathbf{H}_{\delta_2}$ are not necessarily assumed to be in the seminormal form. Thus $\mathbf{h}^{(\delta_1)}_i$ and $\mathbf{h}^{(\delta_2)}_i$ can as well be of the form of a vector all ones.

Remark 8.1.6 It is not hard to see that the set of $\delta_0 c_0$ **W**-matrices in Corollary 8.1.1 is not unique.

Remark 8.1.7 The construction of optimum **W**-matrices for a strongly balanced design d^* in $\Omega_{t,\lambda_1 t^2,p}$ for $\lambda_1 > 1$ can easily be obtained by taking the Kronecker product of the rows of $\mathbf{H}_{\lambda_1}$ and the corresponding optimum **W**-matrix of $\Omega_{t,t^2,p}$ whenever $\mathbf{H}_{\lambda_1}$ exists. In case of non-existence of $\mathbf{H}_{\lambda_1}$ for λ_1 even, the role of the rows of $\mathbf{H}_{\lambda_1}$ above can be taken by the vectors $\mathbf{1}'_{\lambda_1}$ and $(\mathbf{1}'_{\frac{\lambda_1}{2}}, -\mathbf{1}'_{\frac{\lambda_1}{2}})'$. In case of λ_1 odd, the vector of all ones serves the purpose.

Case* 5: *t odd

Whenever t is odd, it is easy to verify that an OCD for a uniform strongly balanced crossover design d^* in $\Omega_{t,t^2,p}$ as described in (8.1.13) does not exist as Condition C_2 of (8.1.10) is not attainable. Let a uniform strongly balanced crossover design $d^{**} \in \Omega_{t,\lambda_1 t^2,p}$ be defined as

$$d^{**} = \mathbf{1}'_{\lambda_1} \otimes d^* \tag{8.1.24}$$

for some positive integer λ_1. The following theorem relates to the construction of OCD for d^{**}.

Theorem 8.1.5 *Suppose $\mathbf{H}_{\lambda_1 t}$, $\mathbf{H}_p$ and a pair of mutually orthogonal latin squares of order t exist. Let d^{**} be defined as in (8.1.24). Then there exists a set of $(\lambda_1 t - 1)(p-1)$ optimum **W**-matrices for d^{**}.*

Proof Suppose L_1 and L_2 are pairwise orthogonal latin squares of order t and L_2 has been used in (8.1.12) and (8.1.13) to construct a uniform strongly balanced crossover design d^* in $\Omega_{t,t^2,p}$. Now we proceed to construct the optimum **W**-matrices for d^{**}. Assuming $\mathbf{H}_{\lambda_1 t}$ and $\mathbf{H}_p$ in the *seminormal* form, for each $i = 1, 2, \ldots, \lambda_1 t - 1$, partitioning $\mathbf{h}^{(\lambda_1 t)}_i$ into λ_1 parts as

$$\mathbf{h}^{(\lambda_1 t)}_i = \left(\mathbf{h}^{(\lambda_1 t)\prime}_{i1}, \ldots, \mathbf{h}^{(\lambda_1 t)\prime}_{ij}, \ldots, \mathbf{h}^{(\lambda_1 t)\prime}_{i\lambda_1}\right)' \tag{8.1.25}$$

we construct a row vector $\mathbf{D}^{*\prime}_{ij}$ of order t^2 considering L_1 and $\mathbf{h}^{(\lambda_1 t)}_{ij}$, for every fixed $j \in \{1, 2, \ldots, \lambda_1\}$, following the steps as described in Theorem 3.1. Thus

$$\mathbf{D}_{ij}^{*\prime} = \left(\mathbf{d}_1^{*ij\prime},\ \mathbf{d}_2^{*ij\prime}, \ldots, \mathbf{d}_t^{*ij\prime}\right). \tag{8.1.26}$$

Now we construct $\mathbf{W}_{if}^{(j)}$ of order $p \times t^2$ as follows:

$$\mathbf{W}_{if}^{(j)} = \mathbf{h}_f^{(p)} \otimes \left(\mathbf{d}_1^{*ij\prime},\ \mathbf{d}_2^{*ij\prime}, \ldots, \mathbf{d}_t^{*ij\prime}\right)';\ \ i = 1, 2, \ldots, \lambda_1 t - 1,\ \ f = 1, 2, \ldots, p-1. \tag{8.1.27}$$

Finally $\mathbf{W}^{(l)}$ matrix of order $p \times \lambda_1 t^2$ is given by:

$$\mathbf{W}^{(l)} = (\mathbf{W}_{if}^{(1)}, \ldots, \mathbf{W}_{if}^{(j)}, \ldots, \mathbf{W}_{if}^{(\lambda_1)}),$$
$$i = 1, 2, \ldots, \lambda_1 t - 1,\ \ f = 1, 2, \ldots, p-1,\ \ l = (i-1)(p-1) + f.$$

It can be easily checked that these $\mathbf{W}^{(l)}$'s are the required optimum $\mathbf{W}$-matrices for d^{**} in $\Omega_{t,\lambda_1 t^2,p}$ and $c^* = (\lambda_1 t - 1)(p-1)$ in this given context. □

Remark 8.1.8 If for p even, $\mathbf{H}_p$ does not exist, then $\mathbf{a} = \left(\mathbf{1}'_{\frac{p}{2}}, -\mathbf{1}'_{\frac{p}{2}}\right)'$ can be used instead of $\mathbf{h}_f^{(p)}$ in the above theorem.

Strongly Balanced Crossover Design Set-Up in $\Omega_{t,\lambda_1 t,\lambda_2 t+1}$

It has been shown in Stufken (1996) that a strongly balanced crossover design that is uniform on the periods and uniform on the units in the first $p-1$ periods is universally optimal for the estimation of direct treatment effects as well as residual treatment effects in $\Omega_{t,n,p}$. We now take up the construction of OCD for such design whenever t is odd and λ_1 is even, as otherwise an OCD fails to exist.

Whenever t is odd, a uniform balanced design d_0^* exists in $\Omega_{t,2t,t}$, which is obtained by juxtaposing two special latin squares of order t side by side (cf. Bose and Dey 2009; Williams 1949). A strongly balanced design $\tilde{d}^{**}$ obtained by repeating the last period of d_0^* is uniform on the periods and uniform on the units in the first t periods (cf. Cheng and Wu 1980). Now for some positive integer λ, taking λ copies of this design let a strongly balanced design $\tilde{d}^*$ in $\Omega_{t,2\lambda t,t+1}$ be constructed as

$$\tilde{d}^* = \mathbf{1}'_\lambda \otimes \tilde{d}^{**} \tag{8.1.28}$$

Theorem 8.1.6 *Suppose* $\mathbf{H}_{2\lambda}$ *exists. Let* $\tilde{d}^*$ *be defined as in (8.1.28). Then there exists a set of* $2\lambda - 1$ *optimum* $\mathbf{W}$*-matrices for* $\tilde{d}^*$.

Proof Assuming $\mathbf{H}_{2\lambda}$ in the *seminormal* form, the optimum $\mathbf{W}^{(l)}$-matrix for $\tilde{d}^*$ in $\Omega_{t,2\lambda t,t+1}$ can be constructed as:

$$\mathbf{W}^{(l)} = \mathbf{a}^* \otimes \mathbf{h}_l^{(2\lambda)} \otimes \mathbf{1}'_t,\ \ l = 1, 2, \ldots, 2\lambda - 1,$$

where $\mathbf{a}^* = \left(\mathbf{1}'_{\frac{t+1}{2}}, -\mathbf{1}'_{\frac{t+1}{2}}\right)'$. □

It has been shown in Stufken (1996) that the above idea of (Cheng and Wu 1980) to construct a strongly balanced design from a uniform balanced design can be extended to cover $p = \lambda_2 t + 1$. The required uniform balanced design d_0^* in $\Omega_{t,\lambda_1 t,\lambda_2 t}$ is a $\lambda_2 \times \lambda_1$ array of special latin square of order t. We refer to Stufken (1996) and Bose and Dey (2009) for the details of the construction. Now repeating the last period of this uniformly balanced design, we get a strongly balanced design $\tilde{d}^*$ in $\Omega_{t,\lambda_1 t,\lambda_2 t+1}$ which is uniform on the periods and uniform on the units in the first $p-1$ periods. The following theorem deals with the construction of OCD for this $\tilde{d}^*$.

Corollary 8.1.2 *Suppose* $\mathbf{H}_{\lambda_2 t+1}$ *and* $\mathbf{H}_{\lambda_1}$ *exist. Then there exists a set of* $\lambda_2 t(\lambda_1 - 1)$ *optimum* $\mathbf{W}$*-matrices for a strongly balanced* $\tilde{d}^*$ *in* $\Omega_{t,\lambda_1 t,\lambda_2 t+1}$.

Proof It is readily verified that assuming $\mathbf{H}_p$ and $\mathbf{H}_{\lambda_1}$ in the *seminormal* form,

$$\mathbf{W}^{(l)} = \mathbf{W}_{ij} = \mathbf{h}_i^{(\lambda_2 t+1)} \otimes \mathbf{h}_j^{(\lambda_1)\prime} \otimes \mathbf{1}_t^{\prime},$$
$$i = 1, 2, \ldots, \lambda_2 t,\ \ j = 1, 2, \ldots, \lambda_1 - 1,\ \ l = (\lambda_1 - 1)(i - 1) + j \quad (8.1.29)$$

are the required optimum $\mathbf{W}$-matrices. □

Balanced Crossover Design Set-Up

In this section we consider the construction of OCD for Williams square (1949) and Patterson (1952) designs as the basic designs which are uniform balanced crossover design with appropriate parameters.

It is known that for all even values of t, a uniform balanced design d_0^* in $\Omega_{t,t,t}$ exists which is a balanced latin square and is referred to as a Williams Square in the literature. There does not exist any optimum $\mathbf{W}$-matrix for d_0^* in $\Omega_{t,t,t}$ as $t-1$ being odd, Condition C_5 is not attainable. Let for some positive integer λ, a uniform balanced crossover design be constructed as

$$d_0^{**} = \mathbf{1}_\lambda^{\prime} \otimes d_0^*. \quad (8.1.30)$$

We next deal with the construction of optimum $\mathbf{W}$-matrices for d_0^{**} in $\Omega_{t,\lambda t,t}$.

Theorem 8.1.7 *Suppose* $\mathbf{H}_t$ *and* $\mathbf{H}_\lambda$ *exist. Then there exist* $(t-1)^2(\lambda - 1)$ *optimum* $\mathbf{W}$*-matrices for* d_0^{**} *in* $\Omega_{t,\lambda t,t}$ *as defined in (8.1.30).*

Proof Assuming $\mathbf{H}_t$ and $\mathbf{H}_\lambda$ in the *seminormal* form

$$\mathbf{W}^{(l)} = \mathbf{W}_{ijf} = \mathbf{h}_f^{(\lambda)\prime} \otimes \mathbf{h}_i^{(t)} \otimes \mathbf{h}_j^{(t)\prime};\ \ i, j = 1, 2, \ldots, t-1,\ \ f = 1, 2, \ldots, \lambda - 1,$$
$$l = (i-1)(\lambda - 1)(t-1) + (j-1)(\lambda - 1) + f \quad (8.1.31)$$

are the required optimum $\mathbf{W}$-matrices for d_0^{**} in $\Omega_{t,\lambda t,t}$. □

Remark 8.1.9 If $\mathbf{H}_t$ does not exist but $\mathbf{H}_\lambda$ exists then a set of $\lambda - 1$ optimum $\mathbf{W}$-matrices for d_0^* can be constructed as

$$\mathbf{W}_l^* = \mathbf{h}_l^{(\lambda)\prime} \otimes \mathbf{a}^* \otimes \mathbf{a}^{*\prime},\ l = 1, 2, \ldots, \lambda$$

where $\mathbf{a}^* = \left(\mathbf{1}'_{t/2}, -\mathbf{1}'_{t/2}\right)'$.

Remark 8.1.10 An OCD for a uniform balanced crossover design in $\Omega_{t,t,t}$ or $\Omega_{t,2t,t}$ cannot be constructed for t odd.

A popular choice of balanced crossover design is the one given by Patterson (1952) for $p \leq t$, as this often involves a moderate number of subjects while keeping the number of periods small. For t a prime or prime power, consider $\{L_i\}, i = 1, 2, \ldots, t-1$, a complete set of MOLS of order t where L_{i+1} can be obtained by cyclically permuting the last $t-1$ rows of L_i. Then the $t \times t(t-1)$ array P given by

$$P = (L_1,\ L_2, \ldots, L_{t-1})\,. \tag{8.1.32}$$

yields a Patterson design in $\Omega_{t,t(t-1),t}$. Now, on deleting any $t-p$ rows of P one gets a Patterson design in $\Omega_{t,t(t-1),p}$ with $p < t$ (cf. Bose and Dey 2009; Patterson 1952). The construction of optimum $\mathbf{W}$-matrices for a Petterson design in $\Omega_{t,t(t-1),p}$ is very much dependent on the existence of the optimum $\mathbf{W}$-matrices for a randomized block design (RBD) (cf. Chap. 3).

Now we consider the following theorem which gives the optimum $\mathbf{W}$-matrices for a Patterson design.

Theorem 8.1.8 *If there exists a set of c* $\mathbf{W}$*-matrices of order* $p \times (t-1)$ *for an RBD* $(p, t-1)$*, then there exists a set of c optimum* $\mathbf{W}$*-matrices for a Patterson design in* $\Omega_{t,t(t-1),p}$.

Proof The optimum $\mathbf{W}$-matrices for the Patterson design in $\Omega_{t,t(t-1),p}$ can be obtained by replacing 1 by $\mathbf{1}'_t$ and –1 by $-\mathbf{1}'_t$ in the $\mathbf{W}$-matrices of RBD $(p, (t-1))$. □

For t prime of the form $4u + 3$, where u is a positive integer, a Patterson design exists in $\Omega_{t,2t,(t+1)/2}$ which is formed by juxtaposing two RBDs($(t+1)/2, t$) side by side. For details of the method of construction we refer to Patterson (1952).

Theorem 8.1.9 *Suppose* $\mathbf{H}_{(t+1)/2}$ *exists. Then there exists a set of* $(t-1)/2$ *optimum* $\mathbf{W}$*-matrices for a Patterson design in* $\Omega_{t,2t,(t+1)/2}$.

Proof Assuming $\mathbf{H}_{(t+1)/2}$ in the *seminormal* form,

$$\mathbf{W}^{(l)} = \mathbf{h}_l^{(t+1)/2)} \otimes (1,\ -1) \otimes \mathbf{1}'_t;\ \ l = 1, 2, \ldots, (t-1)/2 \tag{8.1.33}$$

□

8.2 OCDs in Multi-factor Set-Ups

Rao et al. (2003) proposed optimum covariate designs (OCD) through mixed orthogonal arrays for set-ups involving at most two factors where the effects for the qualitative factors and those of the quantitative controllable covariates were orthogonally estimable. In essence, completely randomised designs and randomized block designs were studied in Rao et al. (2003). Dutta and Das (2013) extended these results and proposed OCDs for the m-factor set-ups where the factorial effects involving at most t ($\leq m$) factors and those of the covariates are orthogonally estimable. It is seen that for such model specifications, optimum designs can be obtained through extended mixed orthogonal arrays (EMOA, Dutta et al. 2009) which reduce to mixed orthogonal arrays for the particular set-ups of Rao et al. (2003).

In this section, we will introduce extended mixed orthogonal arrays and cite some applications. In the process, we will deal with the following simple illustrative examples:

(i) RBD with $b = v = 4$ and two observations per cell; (ii) LSD of order 4; (iii) Graeco LSD of order 4; (iv) LSD with 2 observations per cell; (v) LSD of order 6.

8.2.1 Model and Optimality Conditions

Let $F_1, F_2, \ldots, F_m$ be m factors with $s_1, s_2, \ldots, s_m$ levels, respectively ($s_i \geq 2$, $1 \leq i \leq m$), and $Z^{(1)}, Z^{(2)}, \ldots, Z^{(c)}$ denote c covariates. Also, let n combinations be chosen from all possible $v = \prod_{\alpha=1}^{m} s_\alpha$ level combinations and Ω denote the set of n chosen level combinations. For a level combination $(j_1, j_2, \ldots, j_m)$ of Ω, let $(y_{j_1 j_2 \ldots j_m}, z^{(1)}_{j_1 j_2 \ldots j_m}, z^{(2)}_{j_1 j_2 \ldots j_m}, \ldots, z^{(c)}_{j_1 j_2 \ldots j_m})$ denote the vector of observation and the values assumed by the covariates. As mentioned earlier, we assume the location-scale transformed version of the covariate values, viz $|z^{(l)}_{j_1 j_2 \cdots j_m}| \leq 1$ for all $(j_1, j_2, \ldots, j_m) \in \Omega$ and $l = 1, 2, \ldots, c$. We also assume that the level combinations in Ω are so chosen that the interactions, involving at most t factors ($1 \leq t \leq m$), are orthogonally estimable and all the effects involving $(t + 1)$ and higher order interactions are negligible and contribute to the error.

The reader may note that we are now in the framework of a factorial design of a very general nature. In the above we are referring to asymmetric factorial design. The definition of main effects and interaction effects are very standard and excellent expository article of Bose (1947) provides all the basic results in this direction (also see Gupta and Mukerjee 1989 and Kshirsagar 1983).

The following linear model is assumed (cf. Kshirsagar 1983)

$$y_{j_1 j_2 \ldots j_m} = \mu + \sum_{1 \leq i_1 \leq m} \theta^{i_1}_{j_{i_1}} + \sum_{1 \leq i_1 < i_2 \leq m} \theta^{i_1 i_2}_{j_{i_1} j_{i_2}} + \cdots + \sum_{1 \leq i_1 < i_2 < \ldots < i_t \leq m} \theta^{i_1 i_2 \ldots i_t}_{j_{i_1} j_{i_2} \cdots j_{i_t}}$$

$$+\sum_{l=1}^{c}\gamma_l z^{(l)}_{j_1 j_2\ldots j_m} + e_{j_1 j_2 \ldots j_m}, \tag{8.2.1}$$

where, $(j_1, j_2, \ldots, j_m) \in \Omega$, $\theta^{i_1}_{j_{i_1}}$ is the effect due to j_{i_1}th level of F_{i_1}, $\theta^{i_1 i_2}_{j_{i_1} j_{i_2}}$ is the interaction between j_{i_1}th level of F_{i_1} and j_{i_2}th level of F_{i_2} and so on, $1 \le t \le m$. Further, γ_l is the regression coefficient for the lth concomitant variable $Z^{(l)}$, $l = 1, 2, \ldots, c$. The restrictions on the factorial effects are the following:

$$\sum_{j_{i_\alpha}=1}^{s_{i_\alpha}} \theta^{i_1 i_2 \ldots i_u}_{j_{i_1} j_{i_2} \cdots j_{i_u}} = 0 \;\; \forall j_{i_1},\ j_{i_2}, \ldots, j_{i_u}\ (\ne j_{i_\alpha}),\ \ 1 \le i_1 < i_2 < \ldots < i_u \le m,\ \ 1 \le u \le t.$$

In matrix notations, the model (8.2.1) can be rewritten as

$$(\mathbf{Y},\ \mathbf{X}\boldsymbol{\theta} + \mathbf{Z}\boldsymbol{\gamma},\ \sigma^2 \mathbf{I}_n), \tag{8.2.2}$$

where $\mathbf{X}$ and $\mathbf{Z}$ are suitably defined.

Therefore, for the model (8.2.2), the condition (3.1.3) for estimating the γ-components orthogonally to the ANOVA effects reduces to

$$\mathbf{Z}'\mathbf{X} = \mathbf{0}. \tag{8.2.3}$$

Further, from (3.1.3) and (3.1.4) it follows that the most efficient estimation of $\boldsymbol{\gamma}$-components independently of the ANOVA effects is possible whenever, in addition to (8.2.3), we can also ascertain

$$\mathbf{Z}'\mathbf{Z} = n\mathbf{I}_c. \tag{8.2.4}$$

The condition (8.2.4) implies that $z^{(l)}_{j_1 j_2 \cdots j_m} = \pm 1\ \forall l$ and $\sum_{(j_1, j_2, \ldots, j_m)\in\Omega} z^{(l)}_{j_1 j_2 \cdots j_m} z^{(l')}_{j_1 j_2 \cdots j_m} = n\delta_{ll'}\ \forall\ 1 \le l \ne l' \le c$, where $\delta_{ll'} = 1(0)$ when $l = l'$ $(l \ne l')$.

Let us consider a fixed set of t factors viz. $F_1, F_2, \ldots, F_t$. Also, let $\mathbf{X}^{12\ldots t}$ denote the coefficient matrix of order $n \times (\prod_{i=1}^{t} s_i)$, corresponding to the factorial effects of the factors $F_1, F_2, \ldots, F_t$. It is not difficult to verify that $\mathbf{X}^{12\ldots t}$ is a (0, 1)-matrix and the condition $\mathbf{X}^{12\ldots t\prime}\mathbf{Z} = \mathbf{0}$ implies that $\mathbf{Z}$ is also orthogonal to any design matrix corresponding to any sub-set of the factors $F_1, F_2, \ldots, F_t$. Again, it must be noted that conditions such as above need to be satisfied for any choice of t factors out of m factors.

Thus we get the following theorem.

Theorem 8.2.1 *With respect to the linear model (8.2.2) for an m-factor set-up, the following conditions:*

(i) $z^{(l)}_{j_1 j_2 \ldots j_m} = \pm 1 \; \forall \, (j_1, j_2, \ldots, j_m) \in \Omega, \; l = 1, 2, \ldots, c;$

(ii) $\sum'' z^{(l)}_{j_1 j_2 \ldots j_m} = 0$; *the summation* $\sum''$ *is taken over all those level combinations in* Ω *which contain any given level combination for the* t *factors* $F_{i_1}, F_{i_2}, \ldots, F_{i_t}, \; 1 \le i_1 < i_2 < \ldots < i_t \le m$;

(iii) $\sum_{(j_1, j_2, \ldots, j_m) \in \Omega} z^{(l)}_{j_1 j_2 \cdots j_m} z^{(l')}_{j_1 j_2 \cdots j_m} = n\delta_{ll'} \; \forall \, l, \; l' = 1, 2, \ldots, c$, *where* $\delta_{ll'} = 1(0)$ *when* $l = l' (l \neq l')$,

are necessary and sufficient for the optimal estimation of each of the covariate effects γ_l*'s, with the minimum variance Var* $(\widehat{\gamma_l}) = \frac{\sigma^2}{n} \; \forall \, l = 1, 2, \ldots, c$.

From data analysis point view, to attain simplicity and optimality, it is desirable that a fractional factorial design should be such that all t and less factor effects would be orthogonally estimable with balance. This requires that the fraction denoted by **A**, should be an MOA $(n, \; s_1 \times s_2 \times \cdots \times s_t, \; u), u = \min\{2t, \; m\}$ (cf. Dutta et al. 2009). To construct an OCD on this set-up, we should search for **z** vectors with elements ± 1 such that condition (8.2.3) for orthogonality to the design matrix is satisfied. This, in effect, implies that the elements, viz. ± 1 of any **z** vector should occur orthogonally to any choice of t rows of **A**, i.e. all the level combinations for the choice of any t rows from **A** and any one row of **Z** should occur an equal number of times.

It thus transpires that a systematic study of OA, MOA and EMOA introduced below, can be profitably utilized for construction of OCDs in factorial design contexts.

8.2.2 *Extended Mixed Orthogonal Array (EMOA) and Construction of OCDs*

We describe a new type of array, called extended mixed orthogonal arrays (EMOA) introduced in Dutta et al. (2009) in connection with OCDs in the set-ups of split- and strip-plot designs. The definition of EMOA is as follows.

Definition 8.2.1 Let us consider a $k \times n$ array where the k rows corresponding to the k factors be divided into p sets $\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_p$. The ith set S_i contains $k_i \; (\ge 2)$ factors $\mathrm{F}_{i1}, \mathrm{F}_{i2}, \ldots, \mathrm{F}_{ik_i}$, with $\sum_{i=1}^{p} k_i = k$, where F_{ij} has $s_{ij} \; (\ge 2)$ levels. The array is said to be an extended mixed orthogonal array (EMOA) if

(i) for the choice of any $d_i \; (\ge 2)$ factors from S_i, all possible level combinations of these d_i factors occur equally often (say λ; λ may depend on the selected factors), $i = 1, \; 2, \ldots, p$;

(ii) for the choice of any d sets ($d \ge 2$), say $\mathrm{S}_{i_1}, \mathrm{S}_{i_2}, \ldots, \mathrm{S}_{i_d}$, the level combinations arising out of any t_{i_1} factors from S_{i_1}, any t_{i_2} factors from $\mathrm{S}_{i_2}, \ldots$, any t_{i_d} factors from S_{i_d}, where $1 \le t_{i_j} \le \; d_{i_j}, 1 \le i_1 < i_2 < \cdots < i_d \le p$, occur equally often (say μ times; μ may depend on the selected factors).

Such an array is denoted by EMOA $[n,\ k,\ \prod_{i=1}^{p}\prod_{j=1}^{k_i} s_{ij}, (d_1,\ d_2, \ldots, d_p), (d;\ t_1,\ t_2, \ldots, t_p)]$. The frequency parameters λ and μ can be obtained from the parameters already included in the notation above. An EMOA $[n,\ k,\ \prod_{i=1}^{p}\prod_{j=1}^{k_i} s_{ij}, (d_1,\ d_2, \ldots, d_p),\ (d;\ t_1,\ t_2, \ldots, t_p)]$ is also an EMOA $[n,\ k,$
$\prod_{i=1}^{p}\prod_{j=1}^{k_i} s_{ij},\ (d'_1,\ d'_2, \ldots, d'_p),\ (d';\ t'_1,\ t'_2, \ldots, t'_p)]$, where $d'_i < d_i$, $t'_i < t_i$, $\forall i$ and $d' < d$. It is to be noted that a compound orthogonal array (cf. Hedayat et al. 1999, p. 230) and the array proposed by Chakravarti (1956) can also be seen as EMOAs with particular parameters.

Remark 8.2.1 If $d_i = t$ and u_i be a non-negative integer such that $1 \le u_i \le t_i,\ 1 \le i \le p$ satisfying $\sum_{j=1}^{q} u_{i_j} = t$, where $2 \le q \le d$ and $1 \le i_1 < i_2 < \ldots < i_q \le p$, then the EMOA $[n,\ k,$
$\prod_{i=1}^{p}\prod_{j=1}^{k_i} s_{ij},\ (d_1,\ d_2, \ldots, d_p),\ (d;\ t_1,\ t_2, \ldots, t_p)]$ is an MOA $(n,\ \prod_{i=1}^{p}\prod_{j=1}^{k_i} s_{ij},\ t)$.
Also it follows that an EMOA can always be looked upon as a MOA of strength 2.

Remark 8.2.2 From the above discussions it follows that the OCD on the factorial set-up under consideration can be displayed in the form of an array; the chosen n level combinations of the m factors form n columns of a $m \times n$ matrix denoted by $\mathbf{A}$ and the z-values of the c covariates form c rows of a $c \times n$ matrix denoted by $\mathbf{B}$. This $(m + c) \times n$ array $\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$ is such that $\mathbf{A}$ forms an MOA$(n,\ s_1 \times s_2 \times \cdots \times s_m,\ u)$, $u = \min\{2t,\ m\}$, with elements in the ith row as the levels of F_i, $i = 1, 2, \ldots, m$ and $\mathbf{B}$ forms an OA $(n,\ c,\ 2,\ 2)$ with elements +1 or –1 in each row. From the discussion after Theorem 8.2.1, it follows that all the level combinations for the choice of any t rows from $\mathbf{A}$ and any one row from $\mathbf{B}$ occur an equal number of times. This array is actually an EMOA $[n,\ m + c,\ s_1 \times s_2 \times \cdots \times s_m \times 2^c,\ (u,\ 2),\ (2;\ t,\ 1)]$.

Thus we get the following theorem.

Theorem 8.2.2 *The existence of an EMOA* $[n,\ m + c,\ s_1 \times s_2 \times \cdots \times s_m \times 2^c,\ (u,\ 2),\ (2;\ t,\ 1)]$ *implies the existence of an OCD in a multi-factor set-up where all the main effects and interactions up to t-factors are orthogonally estimable,* $u = \min\{2t, m\}$.

Below we cite an example of an EMOA for clear understanding of the concepts and definitions.

Example 8.2.1 Let us consider the following orthogonal array **D**, with parameters (16, 5, 4, 2)

$$\begin{pmatrix} 1&1&1&1&2&2&2&2&3&3&3&3&4&4&4&4\\ 1&2&3&4&1&2&3&4&1&2&3&4&1&2&3&4\\ 1&2&3&4&2&1&4&3&3&4&1&2&4&3&2&1\\ 1&2&3&4&3&4&1&2&4&3&2&1&2&1&4&3\\ 1&2&3&4&4&3&2&1&2&1&4&3&3&4&1&2 \end{pmatrix} \tag{8.2.5}$$

and another orthogonal array **B** with parameters (4, 3, 2, 2)

$$\mathbf{B} = \begin{pmatrix} 1&-1&1&-1\\ 1&-1&-1&1\\ 1&1&-1&-1 \end{pmatrix}. \tag{8.2.6}$$

Replacing the level j in the first row of **D** by the jth column of **B**, $j = 1, 2, 3$ and 4, we construct an array $\mathbf{B}_1$ of order 3×16. Now let $\mathbf{D}_1$ be the 4×16 array obtained from **D** after ignoring the first row. Then the 7×32 array **C** obtained as

$$\mathbf{C} = \left(\begin{array}{c|c} \mathbf{B}_1 & \overline{\mathbf{B}}_1 \\ \hline \mathbf{D}_1 & \mathbf{D}_1 \end{array}\right)$$

is an EMOA [32, 7, $2^3 \times 4^4$, (3, 2), (2; 3, 1)], where $\overline{\mathbf{B}}_1$ is the array obtained from $\mathbf{B}_1$ by interchanging –1 and 1. Let, S_1 denote the set of three rows corresponding to the B's and S_2 denote the set of four rows corresponding to $\mathbf{D}_1$'s. Then see that each level combination arising out of three rows of S_1 occurs four times, while any level combination arising out of any two rows of S_2 occurs twice. So $\lambda_{123} = 4$ while $\lambda_{i_1 i_2} = 2, 4 \le i_1, i_2 \le 7$. Again, for the choice of the three rows of S_1 and any row from S_2 all possible level combinations occur just once. So $\mu_{123 i_1} = 1, 4 \le i_1 \le 7$.

Some modified versions of this array have been used in the construction of OCDs in the examples considered below.

8.2.3 *Examples of OCDs*

We undertake several examples for construction of OCDs in simple experimental set-ups. Subsequently, in the sections to follow, we develop general results.

Example 8.2.2 RBD with $b = v = 4$ and 2 observations per cell.

Consider $\mathbf{H}_{16}$ and denote the columns of $\mathbf{H}_{16}$ by the vectors $\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \ldots, \mathbf{z}^{(16)}$.

Let $\mathbf{y}^{(1)}$ denote the 16×1 observation vector arising out of the RBD involving the first observation in each cell. Similarly, we also have $\mathbf{y}^{(2)}$ available as the second observation vector across the 16 cells. It does not matter if the observations are laid down row-wise or column-wise.

Let $\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \ldots, \mathbf{z}^{(16)}$ be associated with $\mathbf{y}^{(1)}$ vector and let their 'negations' occupy the respective positions in $\mathbf{y}^{(2)}$.

It is noted that $\mathbf{U} =$ vector of average of the two observations in each cell has, for its expectation, exclusively the terms involving the general mean μ, the block effect parameter(s) and the treatment parameter(s) and these are free from the covariate parameter(s) represented by the $\mathbf{z}$-components. On the other hand, $\mathbf{V}$ = vector of differences [divided by 2] has, for its expectation, terms involving only the covariate parameters and these are free from the 'design parameters'. These covariate parameters have associated with them the corresponding $\mathbf{z}$ vectors. Since the $\mathbf{z}$ vectors are mutually orthogonal with elements (± 1), we are in a position to optimally accommodate 16 covariates.

Remark 8.2.3 This approach is definitely very transparent and one can see how orthogonalization of the two observations within each cell has resulted into separation of the two sets of parameters: design parameters and covariates parameter.

Remark 8.2.4 It is not clear if this approach might lead to the possibility of including any more covariates optimally. Towards an affirmative answer for this we take a look at the RBD with $v = b = 4$ with one observation per cell. From Das et al. (2003) and Rao et al. (2003) it is known that for this set-up with single observation per cell there are nine $\mathbf{z}$ vectors that can be accommodated optimally. Denote these vectors of order 16×1 by $\mathbf{z}_1^*, \mathbf{z}_2^*, \ldots, \mathbf{z}_9^*$. It is now enough to repeat these $\mathbf{z}^*$ at both the positions in each cell. These provide additional 9 covariates, besides the 16 outlined above, thereby giving a total of 25 covariates, the maximum number that can be achieved with 32 observations and $v = b = 4$.

Below we demonstrate an equivalent but unified method of arriving at the same result by means of EMOA to ascertain the existence of an OCD with 25 covariates optimally included.

Consider the array $\mathbf{D}$ defined in (8.2.5). By replacing the levels 1, 2, 3, 4 in the kth row of $\mathbf{D}$ by the elements of the jth row of $\mathbf{B}$ defined in (8.2.6) successively, $j = 1, 2, 3$, we can construct an array $\mathbf{C}_k$ of order 3×16, $k = 1, 2, 3, 4, 5$. Now let $\mathbf{A}_1$ be the 2×16 array obtained from $\mathbf{D}$ after ignoring the last three rows. $\mathbf{A}_1$ provides the set-up for the RBD. Then the 27×32 array $\mathbf{E}$ is obtained as

$$\mathbf{E} = \left(\begin{array}{c|c} \mathbf{A}_1 & \mathbf{A}_1 \\ \hline \mathbf{1}'_{16} & -\mathbf{1}'_{16} \\ \mathbf{C}_1 & -\mathbf{C}_1 \\ \mathbf{C}_2 & -\mathbf{C}_2 \\ \mathbf{C}_3 & -\mathbf{C}_3 \\ \mathbf{C}_4 & -\mathbf{C}_4 \\ \mathbf{C}_5 & -\mathbf{C}_5 \\ \mathbf{C}_3 & \mathbf{C}_3 \\ \mathbf{C}_4 & \mathbf{C}_4 \\ \mathbf{C}_5 & \mathbf{C}_5 \end{array} \right) = (\mathbf{E}^{R_1}, \mathbf{E}^{R_2}).$$

It is readily verified that **E** is an EMOA [32, 27, $4^2 \times 2^{25}$, (2, 2), (2; 1, 1)] which provides 25 optimal covariates. The matrix **E** is displayed below in the partitioned form with the observations separately shown in two cells.

Cell position 1

$$
\mathbf{E}^{R_1} = \begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1
\end{array}
$$

Cell position 2

$$
\mathbf{E}^{R_2} = \begin{array}{rrrrrrrrrrrrrrrr}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
\hline
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
-1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1
\end{array}
$$

Remark 8.2.5 In the structure of the EMOA we can readily identify the two sets of $\mathbf{z}$-vectors arising out of the first Method. In $\mathbf{D}_1$, the rows of the three components $[\mathbf{C}_3,\ \mathbf{C}_3];\ [\mathbf{C}_4,\ \mathbf{C}_4];\ [\mathbf{C}_5,\ \mathbf{C}_5]$ represent the nine $\mathbf{z}^*$s vectors of Remark 8.2.4 while the rest are identified as the $\mathbf{z}$ vectors.

Remark 8.2.6 Here we note that

$$
\mathbf{E}_1^{R_1} = \begin{pmatrix} \mathbf{A}_1 \\ \hline \mathbf{C}_3 \\ \mathbf{C}_4 \\ \mathbf{C}_5 \end{pmatrix}
$$

is an EMOA $[16, 11, 4^2 \times 2^9, (2, 2), (2; 1, 1)]$ and thus from Theorem 8.2.2 it follows that this EMOA provides an OCD for RBD set-up with 4 blocks and 4 treatments with single observation per cell, i.e. a standard RBD with $b = v = 4$ (this in agreement with Dutta et al. 2009; Rao et al. 2003). Here we accommodate the maximum possible number of 9 covariates optimally.

Example 8.2.3 LSD of order 4 with provision for formation of six z vectors
Define

$$\mathbf{E}^{\mathrm{LSD}} = \begin{pmatrix} \mathbf{A}_2 \\ \hline \mathbf{C}_4 \\ \mathbf{C}_5 \end{pmatrix}.$$

It is observed that $\mathbf{E}^{\mathrm{LSD}}$ is an EMOA $[16, 9, 4^3 \times 2^6, (2, 2), (2; 1, 1)]$, where $\mathbf{A}_2$ is the 3×16 array obtained from $\mathbf{D}$ after ignoring the last two rows. $\mathbf{A}_2$ gives the set-up the 4×4 LSD. We can easily infer from Theorem 8.2.2 that this EMOA gives an OCD for LSD set-up with 4 rows, 4 columns and 4 treatments. $\mathbf{E}^{\mathrm{LSD}}$ is displayed as follows:

$$\mathbf{E}^{\mathrm{LSD}} = \begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 2 & 1 & 4 & 3 & 3 & 4 & 1 & 2 & 4 & 3 & 2 & 1 \\
\hline
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1
\end{array}$$

Thus we construct OCD with 6 covariates which is the maximum possible number of covariates.

Example 8.2.4 Graeco LSD of order 4 with provision for formation of three z vectors.

Define

$$\mathbf{E}^{\mathrm{GLSD}} = \begin{pmatrix} \mathbf{A}_3 \\ \hline \mathbf{C}_5 \end{pmatrix}.$$

It is observed that $\mathbf{E}^{\mathrm{GLSD}}$ is an EMOA $[16, 7, 4^4 \times 2^3, (2, 2), (2; 1, 1)]$, where $\mathbf{A}_3$ is the 4×16 array providing the set-up for Graeco LSD and is obtained from $\mathbf{D}$ after ignoring the last row. It easily follows from Theorem 8.2.2 that this EMOA is the OCD for Graeco LSD set-up. $\mathbf{E}^{\mathrm{GLSD}}$ is displayed as follows:

$$
\mathbf{E}^{\mathrm{GLSD}} = \begin{array}{rrrrrrrrrrrrrrrr}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 2 & 1 & 4 & 3 & 3 & 4 & 1 & 2 & 4 & 3 & 2 & 1 \\
1 & 2 & 3 & 4 & 3 & 4 & 1 & 2 & 4 & 3 & 2 & 1 & 2 & 1 & 4 & 3 \\
\hline
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1
\end{array}
$$

Here we construct OCD with maximum possible number of 3 covariates.

Example 8.2.5 LSD of order 4 with 2 observations per cell.

By mimicking the arguments as in the case of an RBD of Example 8.2.2 with two observations per cell, we can immediately associate 6 covariates in an optimal manner since there are 6 error d.f. in the set-up of a latin square of order 4. These are analogous to the $\mathbf{z}^*$ vectors of Example 8.2.2. The remaining 16 $\mathbf{z}$ components are obtained by referring to $\mathbf{H}_{16}$ in the same way as was done there. The whole analysis can be carried out by referring to EMOA. This is explained below.

Define

$$
\mathbf{F} = \begin{pmatrix}
\mathbf{A}_2 & \mathbf{A}_2 \\
\hline
\mathbf{1}'_{16} & -\mathbf{1}'_{16} \\
\mathbf{C}_1 & -\mathbf{C}_1 \\
\mathbf{C}_2 & -\mathbf{C}_2 \\
\mathbf{C}_3 & -\mathbf{C}_3 \\
\mathbf{C}_4 & -\mathbf{C}_4 \\
\mathbf{C}_5 & -\mathbf{C}_5 \\
\mathbf{C}_4 & \mathbf{C}_4 \\
\mathbf{C}_5 & \mathbf{C}_5
\end{pmatrix} = (\mathbf{F}^{\mathrm{R}_1}, \mathbf{F}^{\mathrm{R}_2}),
$$

which is readily verified to be an EMOA $[32, 25, 4^3 \times 2^{22}, (2, 2), (2; 1, 1)]$. The matrix $\mathbf{F}$ is displayed below in the partitioned form with the observations separately shown in two cell positions.

Cell position 1

$$
\mathbf{F}^{R_1} = \begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 2 & 1 & 4 & 3 & 3 & 4 & 1 & 2 & 4 & 3 & 2 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1
\end{array}
$$

Cell position 2

$$
\mathbf{F}^{\mathrm{R}_2} = \begin{array}{rrrrrrrrrrrrrrrr}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 2 & 1 & 4 & 3 & 3 & 4 & 1 & 2 & 4 & 3 & 2 & 1 \\
\hline
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
-1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1
\end{array}
$$

Example 8.2.6 LSD of order 6.

It is very difficult to construct OCDs for latin square design when MOLS do not exist. However, using some special structure of latin square it is possible to construct at least one OCD in some case. We consider the following latin square (Sinha 2009, p. 224)

$$
L = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \\ 6 & 5 & 1 & 2 & 3 & 4 \\ 5 & 6 & 2 & 1 & 4 & 3 \\ 4 & 3 & 6 & 5 & 2 & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}. \tag{8.2.7}
$$

Using **W**-matrix given in Sinha (2009), we can construct the following EMOA [36, 4, $6 \times 6 \times 6 \times 2^1$, (2, 1), (2; 2, 1)]:

$$\left(\begin{array}{cccccccccccccccccc}
1&1&1&1&1&1&2&2&2&2&2&2&3&3&3&3&3&3\\
1&2&3&4&5&6&1&2&3&4&5&6&1&2&3&4&5&6\\
1&2&3&4&5&6&2&1&4&3&6&5&6&5&1&2&3&4\\
1&1&1&-1&-1&-1&1&1&-1&-1&-1&1&1&-1&-1&-1&1&1
\end{array}\right.$$

$$\left.\begin{array}{cccccccccccccccccc}
4&4&4&4&4&4&5&5&5&5&5&5&6&6&6&6&6&6\\
1&2&3&4&5&6&1&2&3&4&5&6&1&2&3&4&5&6\\
5&6&2&1&4&3&4&3&6&5&2&1&3&4&5&6&1&2\\
-1&-1&-1&1&1&1&-1&-1&1&1&1&-1&-1&1&1&1&-1&-1
\end{array}\right),$$

which gives an OCD with one covariate.

8.2.4 OCDs on Some General Set-Ups

Following are some examples of general nature and the OCDs thereon follow from direct application of Theorem 8.2.2. In all the results stated above and below, c denotes the number of covariates optimally included. This may be noted once and for all.

Generalization 1 **(main effects plan set-up)**: Let $\mathbf{A}$ be an MOA$(n,\ s_1 \times s_2 \times \cdots \times s_m,\ 2)$ giving an orthogonal main effects plan. Then, according to Theorem 8.2.2, the matrix $\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}$ gives an OCD if $\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}$ is an EMOA $[n,\ m+c,\ s_1 \times s_2 \times \cdots \times s_m \times 2^c,\ (2,\ 2),\ (2;\ 1,\ 1)]$. It follows that this EMOA is an MOA$(n,\ s_1 \times s_2 \times \cdots \times s_m \times 2^c,\ 2)$.

Below we discuss a particular type of main effect plan obtained through hypergraecolatin square.

Hypergraecolatin square set-up: Let $\mathbf{A}$ be an $m \times s^2$ matrix giving an OA $(s^2,\ m,\ s,\ 2)$ obtained from m mutually orthogonal *latin* squares (MOLS) of order s. The columns of $\mathbf{A}$ actually give the set-up of a hypergraecolatin square (cf. Raghavarao 1971). Then $\mathbf{B}$ gives an OCD in the above set-up if $\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} =$ MOA $(s^2,\ s^m \times 2^c,\ 2), c \le\ (s-1)(s+1-m)$.

If $m = 3$, then $\mathbf{B}$ gives an OCD for an $s \times s$ latin square set-up (compare Example 8.2.4).

The following theorem states a method of getting OCDs for this set-up with a compromise on the error d.f. and pushing them to the covariates.

Theorem 8.2.3 *Suppose* $\mathbf{H}_s$ *and* $(m-2)$ *MOLS of order s with symbols* $1, 2, \ldots, s$ *exist,* $m \le s+1$. *Let* $\mathbf{A} = OA\ (s^2,\ m_1+2,\ s,\ 2)$ *be constructed from* m_1 *MOLS out of the* $(m-2)$ $(= m_1 + m_2)$ *MOLS of order* s. *Then an OCD for the estimation of* $c = m_2(s-1)$ *regression coefficients in the set-up of an orthogonal main effects plan involving* (m_1+2) *factors can be constructed from the remaining* m_2 *MOLS.*

Proof First we construct an orthogonal array, OA $(s^2, m, s, 2)$ using the $(m-2)$ MOLS of order s (cf. Hedayat et al. 1999). Let this orthogonal array be denoted by the following matrix $\mathbf{E}$ in a partition form as

$$\mathbf{E} = \begin{pmatrix} \mathbf{A}_{(m_1+2)\times s^2} \\ \mathbf{D}_{m_2\times s^2} \end{pmatrix}.$$

Here $\mathbf{D}$ is a resolvable orthogonal array of strength one (cf. Raghavarao 1971). A Hadamard matrix of order s is written as

$$\mathbf{H}_s = (\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{s-1}, \mathbf{1}). \tag{8.2.8}$$

Let the symbol i of $\mathbf{D}$, be replaced by h_{ji}, where h_{ji} is the ith element of the vector $\mathbf{h}_j$, $i = 1, 2, \ldots, s$, and a new $m_2 \times s^2$ array $\mathbf{B}^{(j)}$ is obtained from $\mathbf{D}$, $j = 1, 2, \ldots, s-1$. Note that $\mathbf{B}^{(j)}$ is an orthogonal array of strength 2 with the two symbols +1 and –1, $j = 1, 2, \ldots, s-1$. Next we construct the $m_2(s-1)\times s^2$ array $\mathbf{B}$ by the juxtaposition of $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}, \ldots, \mathbf{B}^{(s-1)}$ row-wise, as

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}^{(1)} \\ \mathbf{B}^{(2)} \\ \vdots \\ \mathbf{B}^{(s-1)} \end{pmatrix}. \tag{8.2.9}$$

We can easily check that $\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$ is an EMOA $[s^2,\ m_1+2+c,\ s^{m_1+2}\times 2^c,\ (2,\ 2),\ (2;\ 1,\ 1)]$ where $c = m_2(s-1)$. So by Theorem 8.2.2, the result follows. □

Remark 8.2.7 If $m_1 = 1$, then $\mathbf{B}$ gives an OCD for an $s \times s$ latin square design set-up and in this case $c = (s-1)(m-3)$.

Generalization 2 **(Set-up of m-way classification with single observation per cell)**: Let $\mathbf{A}$ be an $m \times v$ array containing all the $v = \prod_{i=1}^{m} s_i$ level combinations of the m factors. $\mathbf{A}$ is actually an MOA of strength m and all the factorial effects (v in number), together with the mean, are orthogonally estimable from the v observations. But as there is no error degrees of freedom left, no covariate can be accommodated. For this, according to the usual practice, we assume that the m-factor interactions are negligible and contribute to error. As $\mathbf{A}$ is an MOA with strength m, then all the factorial effects up to $(m-1)$-factor interactions are orthogonally estimable. So a $c \times v$ matrix $\mathbf{B}$ with elements ± 1, gives an OCD for the estimation of c regression coefficients if $\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$ is an EMOA $[v,\ m+c,\ s_1 \times s_2 \times \cdots \times s_m \times 2^c,\ (m,\ 2),\ (2;\ m-1,\ 1)]$, where $c \le \prod_{i=1}^{m}(s_i - 1)$.

Generalization 3 **(Set-up of m-way classification with $r\ (>1)$ observations per cell)**: Let in the above set-up each level combination be repeated $r\ (>1)$ times in $\mathbf{A}$. Then all the v factorial effects can be included in the model as the replications provide with the error and a matrix $\mathbf{B}$ satisfying $\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} = \text{EMOA}\ [vr,\ m+c,\ s_1 \times s_2 \times \cdots \times s_m \times 2^c,\ (m,\ 2),\ (2;\ m,\ 1)]$ will give an OCD for the estimation of c regression coefficients where $c \le v(r-1)$.

Generalizations 2 and 3 indicate how the OCDs can be obtained for this set-up through EMOAs with suitable parameters. Constructions of such EMOAs can be obtained by suitable adaptation of those for the MOAs given in Rao et al. (2003). The results are stated in the following theorems.

Theorem 8.2.4 *If $r = 1$ and,*

(i) if there exists a Hadamard matrix of order $s_i\,(i = 1, \ldots, m)$, then an EMOA $[v = \prod_{i=1}^{m} s_i,\ m+c,\ s_1 \times s_2 \times \cdots \times s_m \times 2^c,\ (m,\ 2),\ (2;\ m-1, 1)]$ exists, where $c = \prod_{i=1}^{m}(s_i - 1)$;

(ii) if Hadamard matrices of orders $s_1/2$, $2s_2$ and $s_i\,(i = 3, \ldots, m)$ exist, where s_2 is even, then an EMOA $[v = \prod_{i=1}^{m} s_i,\ m+c,\ s_1 \times s_2 \times \cdots \times s_m \times 2^c,\ (m,\ 2),$ $(2;\ m-1, 1)]$ exists, where $c = \{(s_1-1)(s_2-1) - (s_2-2)\}\prod_{i=3}^{m}(s_i - 1)$;

(iii) if Hadamard matrices of orders s_1 and $s_i\,(i = 3, \ldots, m)$ exist and $s_2 = 2$ (mod 4) and $(s_2 - 1)$ is a prime or prime power, then an EMOA $[v = \prod_{i=1}^{m} s_i,\ m+c,\ s_1 \times s_2 \times \cdots \times s_m \times 2^c,\ (m,\ 2),\ (2;\ m-1, 1)]$ exists, where $c = \{(s_1-1)(s_2-1) - (s_2-2)\}\prod_{i=3}^{m}(s_i - 1)$.

Theorem 8.2.5 *If $r > 1$ and,*

(i) if there exist Hadamard matrices of orders $v = \prod_{i=1}^{m} s_i$, r, then an EMOA $[vr,\ m+c,\ s_1 \times s_2 \times \cdots \times s_m \times 2^c,\ (m,\ 2),\ (2;\ m,\ 1)]$ exists, where $c = v(r-1)$;

(ii) if Hadamard matrices of orders $v/2$, $v == \prod_{i=1}^{m} s_i$ and $2r$ exist, where r is even, then an EMOA $[vr,\ m+c,\ s_1 \times s_2 \times \cdots \times s_m \times 2^c,\ (m,\ 2),\ (2;\ m,\ 1)]$ exists, where $c = v(r-1)$;

(iii) if a Hadamard matrix of order $v = \prod_{i=1}^{m} s_i$ *exists and* $r \equiv 2$ *(mod 4) and* $(r-1)$ *is a prime or prime power, then an EMOA* $[vr,\ m+c,\ s_1 \times s_2 \times \cdots \times s_m \times 2^c,\ (m,\ 2),\ (2;\ m,\ 1)]$ *exists, where* $c = v(r-1)$. *Below we cite an example of an EMOA for clear understanding of Theorems 8.2.4 and 8.2.5.*

Example 8.2.7 Let us consider a $4 \times 2 \times 2$ full factorial with one observation per cell. Then EMOA [16, 6, $4 \times 2 \times 2 \times 2^3$, (2, 2), (2, 1, 1)] can be constructed as follows:

$$\left(\begin{array}{cccccccccccccccc} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ \hline 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \end{array}\right) = \left(\frac{\mathbf{A}^{3\times 16}}{\mathbf{Z}'^{\,3\times 16}}\right),$$

where $\mathbf{Z}^{16\times 3} = (\mathbf{z}_1,\ \mathbf{z}_2,\ \mathbf{z}_3)$ and $\mathbf{z}_1' = (1,-1,1,-1) \otimes (1,-1) \otimes (1,-1)$, $\mathbf{z}_2' = (1,-1,-1,1) \otimes (1,-1) \otimes (1,-1)$, $\mathbf{z}_3' = (1,1,-1,-1) \otimes (1,-1) \otimes (1,-1)$. Here we accommodate 3 covariates optimally in the factorial set-up when all the main effects and two-factor interactions are orthogonally estimable.

Again let us consider the $4 \times 2 \times 2$ full factorial with two observations per cell. Then EMOA [32, 22, $4 \times 2 \times 2 \times 2^{19}$, (2, 2), (2, 1, 1)] can be constructed as follows:

$$\left(\begin{array}{c|c} \text{First set of} & \text{Second set of} \\ \text{16 observations} & \text{16 observations} \\ \hline \mathbf{H}_{16} & -\mathbf{H}_{16} \end{array}\right).$$

Here we accommodate 16 covariates optimally in the factorial set-up with two observations per cell when all the main effects and interactions are orthogonally estimable.

Remark 8.2.8 **(RBD set-up as a particular case of Generalization 2)**: Let $\mathbf{A}_{2\times s_1 s_2}$ contain the all possible level combinations of an RBD with s_1 blocks and s_2 treatments. Then by Remark 8.2.1, $\mathbf{B}$, a $c \times s_1 s_2$ matrix with elements ± 1 gives an OCD if $\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$ is an EMOA $[s_1 s_2,\ 2+c,\ s_1 \times s_2 \times 2^c,\ (2,\ 2),\ (2;\ 1,\ 1)]$, which is actually an MOA $(s_1 s_2,\ s_1 \times s_2 \times 2^c;\ 2)$. This is in full agreement with Rao et al. (2003) (compare Example 3.4.1 of Chap. 3).

Remark 8.2.9 **(CRD set-up as a particular case of Generalization 3)**: If in particular, $m = 1$ in the set-up of Remark 8.2.10, then a matrix $\mathbf{B}$, where $\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} =$ MOA $(vr,\ v \times 2^c;\ 2)$ gives an OCD for the estimation of the regression coefficients under the CRD set-up with v treatments. This is also in agreement with Rao et al. (2003) (compare Example 3.4.1 of Chap. 3).

Remark 8.2.10 **(Incomplete block design set-up)**: Let $m = 2$ and the columns of $\mathbf{A}$ give the set-up of an incomplete block design where the block and the treatment effects are non-orthogonally estimable. The same conditions (i)–(iii) of Theorem 8.2.1 apply for an OCD, but no general result similar to Theorem 8.2.2 can be proposed. The OCDs are difficult to construct here unless some patterns in the incidence matrices exist (cf. Chaps. 4, 5 and 6).

8.3 OCDs in Split-Plot and Strip-Plot Design Set-Ups

In the previous chapters we considered set-ups where the errors were assumed to be uncorrelated. In this section, we consider the problem of finding OCDs for the estimation of covariate parameters in the correlated set-ups of standard split-plot and strip-plot designs with the levels of the whole-plot factor laid out in r randomized blocks. An EMOA and Hadamard matrices play the key role for such construction.

8.3.1 Preliminaries

In the earlier chapters, we considered the set-up where the observations are uncorrelated. For the correlated model, the issue of finding the optimal covariate designs was considered by Dutta et al. (2009) which we discuss in the present section. For the general variance-covariance structure, it is difficult to construct the optimum $\mathbf{Z}$-matrix retaining orthogonality with effects related to the ANOVA part. Dutta et al. (2009) dealt with standard split-plot and strip-plot design set-ups (cf. Cochran and Cox 1950) for which variance–covariance matrices have special structures that can be conveniently exploited to find the OCDs.

Consider the following non-stochastic controllable covariates model of a standard split-plot design set-up with the levels of the whole-plot factor (whole-plot treatments) in r randomized blocks (cf. Chakrabarti 1962)

$$(\mathbf{Y},\ \mu\mathbf{1}_{rpq} + \mathbf{X}_1\boldsymbol{\alpha} + \mathbf{X}_2\boldsymbol{\beta} + \mathbf{X}_3\boldsymbol{\tau} + \mathbf{X}_4\boldsymbol{\delta} + \mathbf{Z}\boldsymbol{\gamma},\ \sigma^2\boldsymbol{\Sigma}) \tag{8.3.1}$$

where $\mathbf{Y} = \left(y_{111}, \ldots, y_{ijk}, \ldots, y_{rpq}\right)'$ is the $rpq \times 1$ observation vector corresponding to the rpq level combinations of the three factors, viz. the block (R), the whole plot factor (A), and the sub-plot factor (B) arranged lexicographically; $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{X}_4$, $\mathbf{Z}$ are the design matrices corresponding to the block effects vector $\boldsymbol{\alpha}^{r\times 1}$, the whole-plot effects vector $\boldsymbol{\beta}^{p\times 1}$, the sub-plot effects vector $\boldsymbol{\tau}^{q\times 1}$, the whole-plot$\times$ sub-plot interaction effects vector $\boldsymbol{\delta}^{pq\times 1}$ and the covariate effects $\gamma^{c\times 1}$ respectively. Obviously, $\mathbf{1}_{rpq}$ is the coefficient vector corresponding to the intercept term μ. It may be noted that $\mathbf{X}_{ij}$'s are (0,1) incidence matrices. $\mathbf{Z}$ is the matrix of covariate values. For convenience, we partition $\mathbf{X}_g$ $(g = 1, 2, 3, 4)$ and $\mathbf{Z}$ as follows:

$$\left.\begin{aligned} \mathbf{X}_g^{rpq\times n_g} &= \left(\mathbf{X}_{11}^{(g)\prime\, n_g\times q},\ldots,\mathbf{X}_{1p}^{(g)\prime\, n_g\times q},\ldots,\mathbf{X}_{ij}^{(g)\prime\, n_g\times q},\ldots,\mathbf{X}_{rp}^{(g)\prime\, n_g\times q}\right)' \\ \mathbf{Z}^{rpq\times c} &= \left(\mathbf{Z}_{11}^{\prime\, c\times q},\ldots,\mathbf{Z}_{1p}^{\prime\, c\times q},\ldots,\mathbf{Z}_{ij}^{\prime\, c\times q},\ldots,\mathbf{Z}_{rp}^{\prime\, c\times q}\right)' \end{aligned}\right\} \tag{8.3.2}$$

where n_g stands for the number of parameters in the gth classification corresponding to the block, the whole-plot treatment and the sub-plot treatment, i.e. $n_g = r, p, q, pq$ for $g = 1, 2, 3, 4$ respectively. $\mathbf{X}_{ij}^{(1)\ q\times r}$, $\mathbf{X}_{ij}^{(2)\ q\times p}$, $\mathbf{X}_{ij}^{(3)\ q\times q}$, $\mathbf{X}_{ij}^{(4)\ q\times pq}$ and $\mathbf{Z}_{ij}^{q\times c}$ are the portions of the design matrices $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4$ and $\mathbf{Z}$, respectively, corresponding to the observations of the ith block and the jth whole-plot treatment ($i = 1, 2, \ldots, r$; $j = 1, 2, \ldots, p$). Thus, if the structure of $\mathbf{X}_{ij}^{(1)}$ is investigated it is noted that in the ith column, 1 corresponds to each of the q observations on the q levels of the sub-factor B when R and A are fixed at i and j respectively. Other columns contain 0's only. We write $\mathbf{1}$ as $q \times 1$ vector with all elements unity, $\mathbf{e}_i$ as $q \times 1$ unit vector with 1 at the ith position, $\boldsymbol{\delta}_j = (\delta_{j1},\ \delta_{j2}, \ldots, \delta_{jq}),\quad j = 1, 2, \ldots, p$, jth vector of interactions of jth whole plot treatment with q sub-plot treatments, $j = 1, 2, \ldots, p$. With these notations we write the following $\mathbf{X}_{ij}^{(g)}$ matrices.

$$\begin{array}{cc} & \begin{array}{cccccccc} \alpha_1 & \alpha_2 & \ldots & \alpha_{i-1} & \alpha_i & \alpha_{i+1} & \ldots & \alpha_r \end{array} \\ \mathbf{X}_{ij}^{(1)} = & \left(\begin{array}{cccccccc} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{1} & \mathbf{0} & \ldots & \mathbf{0} \end{array}\right)^{q\times r} \quad \forall j; \end{array} \tag{8.3.3}$$

It is to be noted that the structure of $\mathbf{X}_{ij}^{(1)}$ is independent of j. In this way, we can write the other $\mathbf{X}_{ij}$-matrices as follows:

$$\begin{array}{cc} & \begin{array}{cccccccc} \beta_1 & \beta_2 & \ldots & \beta_{j-1} & \beta_j & \beta_{j+1} & \ldots & \beta_p \end{array} \\ \mathbf{X}_{ij}^{(2)} = & \left(\begin{array}{cccccccc} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{1} & \mathbf{0} & \ldots & \mathbf{0} \end{array}\right)^{q\times p} \quad \forall i; \end{array} \tag{8.3.4}$$

$$\begin{array}{cc} & \begin{array}{cccc} \tau_1 & \tau_2 & \ldots & \tau_q \end{array} \\ \mathbf{X}_{ij}^{(3)} = & \left(\begin{array}{cccc} \mathbf{e}_1 & \mathbf{e}_2 & \ldots & \mathbf{e}_q \end{array}\right)^{q\times q} \quad \forall i,\ j; \end{array} \tag{8.3.5}$$

$$\begin{array}{cc} & \begin{array}{cccccccc} \boldsymbol{\delta}_1 & \boldsymbol{\delta}_2 & \ldots & \boldsymbol{\delta}_{j-1} & \boldsymbol{\delta}_j & \boldsymbol{\delta}_{j+1} & \ldots & \boldsymbol{\delta}_p \end{array} \\ \mathbf{X}_{ij}^{(4)} = & \left(\begin{array}{cccccccc} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{I}_q & \mathbf{0} & \ldots & \mathbf{0} \end{array}\right)^{q\times pq} \quad \forall i; \end{array} \tag{8.3.6}$$

Now we consider the following example which illustrates the above set-up and the representations.

Example 8.3.1 Let us take $r = 2,\ p = 2,\ q = 4$. Then $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ and $\mathbf{X}_4$ are written as follows.

$$\mathbf{X}_1 = \begin{pmatrix} 1\ 0 \\ 1\ 0 \\ 1\ 0 \\ 1\ 0 \\ \hline 1\ 0 \\ 1\ 0 \\ 1\ 0 \\ 1\ 0 \\ \hline 0\ 1 \\ 0\ 1 \\ 0\ 1 \\ 0\ 1 \\ \hline 0\ 1 \\ 0\ 1 \\ 0\ 1 \\ 0\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_{11}^{(1)4\times 2} \\ \mathbf{X}_{12}^{(1)4\times 2} \\ \mathbf{X}_{21}^{(1)4\times 2} \\ \mathbf{X}_{22}^{(1)4\times 2} \end{pmatrix}, \ \mathbf{X}_2 = \begin{pmatrix} 1\ 0 \\ 1\ 0 \\ 1\ 0 \\ 1\ 0 \\ \hline 0\ 1 \\ 0\ 1 \\ 0\ 1 \\ 0\ 1 \\ \hline 1\ 0 \\ 1\ 0 \\ 1\ 0 \\ 1\ 0 \\ \hline 0\ 1 \\ 0\ 1 \\ 0\ 1 \\ 0\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_{11}^{(2)4\times 2} \\ \mathbf{X}_{12}^{(2)4\times 2} \\ \mathbf{X}_{21}^{(2)4\times 2} \\ \mathbf{X}_{22}^{(2)4\times 2} \end{pmatrix},$$

$$\mathbf{X}_3 = \begin{pmatrix} 1\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0 \\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 1 \\ \hline 1\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0 \\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 1 \\ \hline 1\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0 \\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 1 \\ \hline 1\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0 \\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_{11}^{(3)4\times 4} \\ \mathbf{X}_{12}^{(3)4\times 4} \\ \mathbf{X}_{21}^{(3)4\times 4} \\ \mathbf{X}_{22}^{(3)4\times 4} \end{pmatrix}, \ \mathbf{X}_4 = \begin{pmatrix} 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0 \\ \hline 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \\ \hline 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0 \\ \hline 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_{11}^{(4)4\times 8} \\ \mathbf{X}_{12}^{(4)4\times 8} \\ \mathbf{X}_{21}^{(4)4\times 8} \\ \mathbf{X}_{22}^{(4)4\times 8} \end{pmatrix}.$$

For a standard split-plot design, where intra-class correlation structure of the dispersion matrix is assumed, the elements of $\mathbf{\Sigma}$-matrix (cf. Chakrabarti 1962) of (8.3.1) are given by

$$\frac{1}{\sigma^2} Cov(y_{ijk}, y_{i',j',k'}) = \begin{cases} 1 \text{ if } \ i = i', \ j = j', \ k = k' \\ \rho \text{ if } \ i = i', \ j = j', \ k \neq k' \\ 0 \text{ otherwise,} \end{cases} \tag{8.3.7}$$

and it can be expressed as

$$\boldsymbol{\Sigma} = \mathbf{I}_{pr} \bigotimes \boldsymbol{\Sigma}_1; \quad \boldsymbol{\Sigma}_1 = (1-\rho)\mathbf{I}_q + \rho \mathbf{J}_q \tag{8.3.8}$$

where ρ is the common intra-class correlation coefficient among the observations corresponding to the sub-plot treatments within the same whole-plot treatment in a block and $\mathbf{J}_u = \mathbf{1}_u \mathbf{1}'_u$ is the square matrix of order u with all elements unity. Following Cochran and Cox (1950), p. 220, we assume $\rho > 0$ as the observations corresponding to the different levels of the sub-plot treatments under the same level of the whole-plot treatment are expected to be positively correlated.

In this correlated set-up, we are concerned with the optimum choice of $\mathbf{Z}$ for the estimation of each of the regression parameters in the split-plot set-up with maximum accuracy in the sense of minimizing the variance of the best linear unbiased estimators of regression parameters retaining orthogonality with the estimators of the ANOVA effects.

The Optimality Conditions for the Split-Plot Design Set-Up

The information matrix for $\boldsymbol{\eta} = (\mu, \ \boldsymbol{\alpha}', \ \boldsymbol{\beta}', \ \boldsymbol{\tau}', \ \boldsymbol{\delta}', \ \boldsymbol{\gamma}')'$ in the split-plot design set-up (8.3.1) is given by

$$\mathbf{I}(\boldsymbol{\eta}) = (\mathbf{X}, \ \mathbf{Z})' \boldsymbol{\Sigma}^{-1} (\mathbf{X}, \ \mathbf{Z}). \tag{8.3.9}$$

where $\mathbf{X} = (\mathbf{1}, \ \mathbf{X}_1, \ \mathbf{X}_2, \ \mathbf{X}_3, \ \mathbf{X}_4)$. From (8.3.8), $\boldsymbol{\Sigma}^{-1}$ can be written as

$$\left.\begin{aligned} \boldsymbol{\Sigma}^{-1} &= \mathbf{I}_{pr} \otimes \boldsymbol{\Sigma}_1^{-1} \\ \boldsymbol{\Sigma}_1^{-1} &= \tfrac{1}{1-\rho}\left(\mathbf{I}_q - \tfrac{\rho}{1+(q-1)\rho}\mathbf{J}_q\right) \end{aligned}\right\} \tag{8.3.10}$$

It is evident from (8.3.9) that $\boldsymbol{\gamma}$ is estimable orthogonally to the ANOVA effects if and only if

$$\mathbf{X}'_g \boldsymbol{\Sigma}^{-1} \mathbf{Z} = \mathbf{0}, \quad g = 1, \ 2, \ 3, \ 4, \tag{8.3.11}$$

where $\mathbf{X}_g$ is the design matrix of order $rpq \times n_g$ corresponding to the gth ANOVA effect described in (8.3.2), with $n_g = r, \ p, \ q, \ pq$ respectively for $g = 1, \ 2, \ 3, \ 4$. Using (8.3.10), the orthogonality conditions in (8.3.11) can be reduced to

$$\sum_{i=1}^{r}\sum_{j=1}^{p} \mathbf{X}_{ij}^{(g)\prime}\mathbf{Z}_{ij} - \frac{\rho}{1+(q-1)\rho}\sum_{i=1}^{r}\sum_{j=1}^{p} \mathbf{X}_{ij}^{(g)\prime}\mathbf{J}_q\mathbf{Z}_{ij} = \mathbf{0}, \tag{8.3.12}$$

which is satisfied if

$$\sum_{i=1}^{r}\sum_{j=1}^{p}\mathbf{X}_{ij}^{(g)\prime}\mathbf{Z}_{ij} = \mathbf{0}, \tag{8.3.13}$$

and

$$\frac{\rho}{1+(q-1)\rho}\sum_{i=1}^{r}\sum_{j=1}^{p}\mathbf{X}_{ij}^{(g)\prime}\mathbf{J}_q\mathbf{Z}_{ij} = \mathbf{0}. \tag{8.3.14}$$

For $g = 1, 2, 3$ and 4, (8.3.3)–(8.3.6) imply that the left-hand side of (8.3.13) becomes, respectively,
the $r \times c$ matrix

$$\left(\sum_{j=1}^{p}\sum_{l=1}^{q} z_{lm}^{(ij)}\right)_{i=1,2,\ldots,r,\ m=1,2,\ldots,c}, \tag{8.3.15}$$

the $p \times c$ matrix

$$\left(\sum_{i=1}^{r}\sum_{l=1}^{q} z_{lm}^{(ij)}\right)_{j=1,2,\ldots,p,\ m=1,2,\ldots,c}, \tag{8.3.16}$$

the $q \times c$ matrix

$$\left(\sum_{i=1}^{r}\sum_{j=1}^{p} z_{lm}^{(ij)}\right)_{l=1,2,\ldots,q,\ m=1,2,\ldots,c}, \tag{8.3.17}$$

and the $pq \times c$ matrix

$$\left(\sum_{i=1}^{r} z_{lm}^{(ij)}\right)_{j=1,2,\ldots,p,\ l=1,2,\ldots,q,\ m=1,2,\ldots,c}. \tag{8.3.18}$$

Again for $g = 1, 2, 3$ and 4, (8.3.3)–(8.3.6) imply that the left-hand side of (8.3.14) becomes, respectively,
the $r \times c$ matrix

$$q\left(\sum_{j=1}^{p}\sum_{l=1}^{q} z_{lm}^{(ij)}\right)_{i=1,2,\ldots,r,\ m=1,2,\ldots,c}, \tag{8.3.19}$$

the $p \times c$ matrix

$$q\left(\sum_{i=1}^{r}\sum_{l=1}^{q} z_{lm}^{(ij)}\right)_{j=1,2,\ldots,p,\ m=1,2,\ldots,c}, \tag{8.3.20}$$

the $q \times c$ matrix

$$\mathbf{1}_q \otimes \left(\sum_{i=1}^{r}\sum_{j=1}^{p}\sum_{l=1}^{q} z_{l1}^{(ij)}, \ldots, \sum_{i=1}^{r}\sum_{j=1}^{p}\sum_{l=1}^{q} z_{lm}^{(ij)}, \ldots, \sum_{i=1}^{r}\sum_{j=1}^{p}\sum_{l=1}^{q} z_{lc}^{(ij)}\right) \tag{8.3.21}$$

and the $pq \times c$ matrix

$$\mathbf{U} = \left(\mathbf{U}_1', \ldots, \mathbf{U}_j', \ldots, \mathbf{U}_p'\right)', \tag{8.3.22}$$

where

$$\mathbf{U}_j^{q\times c} = \mathbf{1}_q \otimes \left(\sum_{i=1}^{r}\sum_{j=1}^{p}\sum_{l=1}^{q} z_{l1}^{(ij)}, \ldots, \sum_{i=1}^{r}\sum_{j=1}^{p}\sum_{l=1}^{q} z_{lm}^{(ij)}, \ldots, \sum_{i=1}^{r}\sum_{j=1}^{p}\sum_{l=1}^{q} z_{lc}^{(ij)}\right). \tag{8.3.23}$$

Therefore, from (8.3.15)–(8.3.23), a set of sufficient conditions for (8.3.13) to satisfy is

$$\left.\begin{aligned} &\sum_{i=1}^{r} z_{lm}^{(ij)} = 0 \qquad \forall j = 1,2,\ldots,p,\ l = 1,2,\ldots,q,\ m = 1,2,\ldots,c,\\ &\text{and}\\ &\sum_{j=1}^{p}\sum_{l=1}^{q} z_{lm}^{(ij)} = 0\ \forall i = 1,2,\ldots,r,\ m = 1,2,\ldots,c. \end{aligned}\right\} \tag{8.3.24}$$

It is seen from (8.3.9) that the information matrix for $\boldsymbol{\gamma}$ under (8.3.24) when $\mathbf{X}_{ij}^{(g)}$'s follow the structure (8.3.3)–(8.3.6), is proportional to $\mathbf{Z}'\boldsymbol{\Sigma}^{-1}\mathbf{Z}$. Again, by virtue of (8.3.10)

$$\begin{aligned} \mathbf{Z}'\boldsymbol{\Sigma}^{-1}\mathbf{Z} &= \tfrac{1}{1-\rho}\left(\sum_{i=1}^{r}\sum_{j=1}^{p}\mathbf{Z}_{ij}'\mathbf{Z}_{ij} - \frac{\rho}{1+(q-1)\rho}\sum_{i=1}^{r}\sum_{j=1}^{p}\mathbf{Z}_{ij}'\mathbf{J}_q\mathbf{Z}_{ij}\right)\\ &\le \tfrac{1}{1-\rho}\left(\sum_{i=1}^{r}\sum_{j=1}^{p}\mathbf{Z}_{ij}'\mathbf{Z}_{ij}\right) \end{aligned} \tag{8.3.25}$$

in the sense of Partial Loewner Order (PLO) dominance (cf. Pukelsheim (1993)) since by assumption $\rho > 0$ and $\sum_{i=1}^{r}\sum_{j=1}^{p} \mathbf{Z}'_{ij}\mathbf{J}_q\mathbf{Z}_{ij} = \sum_{i=1}^{r}\sum_{j=1}^{p} \mathbf{Z}'_{ij}\mathbf{1}_q\mathbf{1}'_q\mathbf{Z}_{ij}$ is non-negative definite. Equality holds in (8.3.25) if $\mathbf{Z}'_{ij}\mathbf{1}_q = \mathbf{0}\ \forall i, j$ or, equivalently

$$\sum_{l=1}^{q} z_{lm}^{(ij)} = 0\ \forall i = 1, 2, \ldots, r,\ \ j = 1, 2, \ldots, q,\ \ m = 1, 2, \ldots, c. \tag{8.3.26}$$

If, in addition to (8.3.24) and (8.3.26), $\mathbf{Z}_{ij}$ satisfies

$$\sum_{i=1}^{r}\sum_{j=1}^{p}\sum_{l=1}^{q} z_{lm}^{(ij)} z_{lm'}^{(ij)} = 0\ \forall\ m \neq m = 1, 2, \ldots, c, \tag{8.3.27}$$

then γ_m's are estimated orthogonally among themselves and orthogonally to the ANOVA effects. Under the above conditions (8.3.24), (8.3.26) and (8.3.27), γ_m can be estimated with the minimum variance $\frac{(1-\rho)\sigma^2}{rpq}$ for each m if $z_{lm}^{ij} = \pm 1\ \forall\ i, j, l, m$. Hence we get the following theorem given in Dutta et al. (2009).

Theorem 8.3.1 *In the standard split-plot design set-up (8.3.1) the following set of conditions:*

(i) $z_{lm}^{(ij)} = \pm 1 \quad \forall i = 1, 2, \ldots, r,\ \ j = 1, 2, \ldots, p,\ \ l = 1, 2, \ldots, q,\ \ m = 1, 2, \ldots, c$

(ii) $\sum_{l=1}^{q} z_{lm}^{(ij)} = 0 \quad \forall i = 1, 2, \ldots, r,\ \ j = 1, 2, \ldots, p,\ \ m = 1, 2, \ldots, c$

(iii) $\sum_{i=1}^{r} z_{lm}^{(ij)} = 0 \quad \forall j = 1, 2, \ldots, p,\ \ l = 1, 2, \ldots, q,\ \ m = 1, 2, \ldots, c$

(iv) $\sum_{i=1}^{r}\sum_{j=1}^{p}\sum_{l=1}^{q} z_{lm}^{(ij)} z_{lm'}^{(ij)} = rpq\delta_{mm'}$ *where* $\delta_{mm'} = 1$ *if* $m = m'$; *=0 if* $m \neq m'$,

is sufficient for the optimum estimation of each of the covariate effects with the minimum possible variance $Var\,(\widehat{\gamma}_m) = \frac{(1-\rho)\sigma^2}{rpq},\ \ m = 1, 2, \ldots, c.$

Note 8.3.1 It must be noted that the conditions laid down above are independent of the actual value of ρ, assumed to be known and positive.

The Optimality Conditions for the Strip-Plot Design Set-Up

In a standard strip-plot design, as the levels of the sub-plot factor B are arranged in strips, the dispersion matrix of the observation vector $\mathbf{Y}$ gets changed though the mean vector remains the same as in (8.3.1). So the linear model (8.3.1) can be adapted by replacing $\mathbf{\Sigma}$ by $\mathbf{\Sigma}^*$, with the elements of $\mathbf{\Sigma}^*$ as

$$\frac{1}{\sigma^2}Cov(y_{ijk}, y_{i',j',k'}) = \begin{cases} 1 & \text{if } i=i',\ j=j',\ k=k' \\ \rho_1 & \text{if } i=i',\ j=j',\ k\neq k' \\ \rho_2 & \text{if } i=i',\ j\neq j',\ k=k' \\ 0 & \text{otherwise,} \end{cases} \tag{8.3.28}$$

where y_{ijk}, arranged lexicographically, is the yield of the plot belonging to the kth column-strip and the jth row-strip in the ith block ($i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, p;\ k = 1, 2, \ldots, q$). Therefore, we can write (8.3.28) as

$$\left.\begin{aligned} Disp(\mathbf{Y}) &= \sigma^2\mathbf{I}_r \otimes \mathbf{\Sigma}^{**} \\ \Sigma^{**} &= \mathbf{I}_p \otimes \mathbf{\Sigma}_1^* + \mathbf{J}_p \otimes \mathbf{\Sigma}_2^* \\ \mathbf{\Sigma}_1^* &= (1-\rho_1-\rho_2)\mathbf{I}_q + \rho_1\mathbf{J}_q,\quad \mathbf{\Sigma}_2^* = \rho_2\mathbf{I}_q. \end{aligned}\right\} \tag{8.3.29}$$

Following the same arguments as in split-plot design, here it is also assumed that $\rho_1 > 0,\ \rho_2 > 0$. In a standard strip-plot design, for estimation of the covariate effects orthogonally to the ANOVA effects, we, in analogy to (8.3.11), have from (8.3.1) and (8.3.29)

$$\mathbf{X}_g'\mathbf{\Sigma}^{*-1}\mathbf{Z} = \mathbf{0},\quad \forall g = 1,\ 2,\ 3,\ 4, \tag{8.3.30}$$

where $\mathbf{X}_g$'s and $\mathbf{Z}$ are defined in (8.3.2). By virtue of (8.3.29), the conditions in (8.3.30) reduce to

$$\sum_{i=1}^{r}\mathbf{X}_i^{(g)'}\mathbf{\Sigma}^{**-1}\mathbf{Z}_{(i)} = \mathbf{0},\quad \forall g = 1,\ 2,\ 3,\ 4, \tag{8.3.31}$$

where $\mathbf{X}_i^{(g)}$ and $\mathbf{Z}_{(i)}$ are the portions of $\mathbf{X}_g$ and $\mathbf{Z}$ corresponding to the pq observations in the ith block, $i = 1, 2, \ldots, r$.

Again from (8.3.29),

$$\mathbf{\Sigma}_1^{**-1} = \mathbf{I}_p \otimes \mathbf{B}_1 - \mathbf{J}_p \otimes \mathbf{B}_2 \tag{8.3.32}$$

where

$$\left.\begin{aligned} \mathbf{B}_1 &= \mathbf{\Sigma}_1^{*-1} = \tfrac{1}{1-\rho_1-\rho_2}\left(\mathbf{I}_q - \tfrac{1}{1+(q-1)\rho_1-\rho_2}\mathbf{J}_q\right), \\ \mathbf{B}_2 &= \left(\mathbf{\Sigma}_1^* + p\mathbf{\Sigma}_2^*\right)^{-1}\mathbf{\Sigma}_2^*\mathbf{\Sigma}_1^{*-1} \\ &= \tfrac{\rho_2}{(1-\rho_1-\rho_2)(1+\rho_1+(p-1)\rho_2)}\left(\mathbf{I}_q - \tfrac{\rho_1(2+(q-2)\rho_1+(p-2)\rho_2)}{(1+(q-1)\rho_1-\rho_2)(1+(q-1)\rho_1+(p-1)\rho_2)}\mathbf{J}_q\right). \end{aligned}\right\} \tag{8.3.33}$$

By virtue of (8.3.2) and (8.3.32), the condition (8.3.31) reduces to

$$\sum_{i=1}^{r}\sum_{j=1}^{p}\mathbf{X}_{ij}^{(g)'}\mathbf{B}_1\mathbf{Z}_{ij} - \sum_{i=1}^{r}\sum_{j=1}^{p}\mathbf{X}_{ij}^{(g)'}\mathbf{B}_2(\mathbf{Z}_{i1} + \mathbf{Z}_{i2} + \cdots + \mathbf{Z}_{ip}) = \mathbf{0} \tag{8.3.34}$$

which is satisfied if

$$\sum_{i=1}^{r}\sum_{j=1}^{p}\mathbf{X}_{ij}^{(g)\prime}\mathbf{B}_1\mathbf{Z}_{ij} = \mathbf{0} \tag{8.3.35}$$

and

$$\sum_{i=1}^{r}\sum_{j=1}^{p}\mathbf{X}_{ij}^{(g)\prime}\mathbf{B}_2(\mathbf{Z}_{i1}+\mathbf{Z}_{i2}+\cdots+\mathbf{Z}_{ip}) = \mathbf{0}. \tag{8.3.36}$$

Since $\mathbf{B}_1$ is a completely symmetric matrix, it is seen that (8.3.24) is also sufficient for (8.3.35) to hold. Again, using (8.3.33) in (8.3.36), a set of sufficient conditions for (8.3.36) to satisfy is

$$\sum_{i=1}^{r}\sum_{j=1}^{p}\mathbf{X}_{ij}^{(g)\prime}(\mathbf{Z}_{i1}+\mathbf{Z}_{i2}+\cdots+\mathbf{Z}_{ip}) = \mathbf{0}, \tag{8.3.37}$$

and

$$\sum_{i=1}^{r}\sum_{j=1}^{p}\mathbf{X}_{ij}^{(g)\prime}\mathbf{1}_q\left(\sum_{j=1}^{p}\sum_{l=1}^{q}z_{l1}^{(ij)},\ldots,\sum_{j=1}^{p}\sum_{l=1}^{q}z_{lm}^{(ij)},\ldots,\sum_{j=1}^{p}\sum_{l=1}^{q}z_{lc}^{(ij)}\right) = \mathbf{0}. \tag{8.3.38}$$

It is seen that the condition (8.3.38) holds if (8.3.24) holds. Similarly as before, for $g = 1,\ 2,\ 3$ and 4, the left-hand side of (8.3.37) becomes, respectively,

the $r \times c$ matrix

$$p\left(\sum_{j=1}^{p}\sum_{l=1}^{q}z_{lm}^{(ij)}\right)_{i=1,2,\ldots,r,\ m=1,2,\ldots,c}, \tag{8.3.39}$$

the $p \times c$ matrix

$$\mathbf{1}_p \otimes \left(\sum_{i=1}^{r}\sum_{j=1}^{p}\sum_{l=1}^{q}z_{l1}^{(ij)},\ldots,\sum_{i=1}^{r}\sum_{j=1}^{p}\sum_{l=1}^{q}z_{lm}^{(ij)},\ldots,\sum_{i=1}^{r}\sum_{j=1}^{p}\sum_{l=1}^{q}z_{lc}^{(ij)}\right), \tag{8.3.40}$$

the $q \times c$ matrix

$$p\left(\sum_{i=1}^{r}\sum_{j=1}^{p} z_{lm}^{(ij)}\right)_{l=1,2,\ldots,q,\ m=1,2,\ldots,c} \tag{8.3.41}$$

and the $pq \times c$ matrix

$$\mathbf{1}_p \otimes \left(\sum_{i=1}^{r}\sum_{j=1}^{p} z_{lm}^{(ij)}\right)_{l=1,2,\ldots,q,\ m=1,2,\ldots,c}. \tag{8.3.42}$$

So (8.3.37) holds whenever (8.3.24) holds. Using $\mathbf{X}_{ij}^{(g)}$'s from (8.3.3) to (8.3.6), and following similar arguments as in a split-plot design, it can be concluded that a set of sufficient conditions to satisfy (8.3.35) and (8.3.36) is

$$\left.\begin{array}{ll}\sum_{i=1}^{r} z_{lm}^{(ij)} = 0 & \forall j = 1, 2, \ldots, p,\ l = 1, 2, \ldots, q,\ m = 1, 2, \ldots, c, \\ \sum_{j=1}^{p}\sum_{l=1}^{q} z_{lm}^{(ij)} = 0 & \forall i = 1, 2, \ldots, r,\ m = 1, 2, \ldots, c.\end{array}\right\} \tag{8.3.43}$$

These are the same as the conditions in (8.3.24) for orthogonality in a split-plot design. The z-values satisfying (8.3.43) will ensure estimation of γ orthogonally to the estimates of the ANOVA effects. Under (8.3.43), the information matrix for γ in standard strip-plot design set-up will be proportional to $\mathbf{Z}'\boldsymbol{\Sigma}^{*-1}\mathbf{Z}$. Now from (8.3.32) and (8.3.33)

$$\mathbf{Z}'\boldsymbol{\Sigma}^{*-1}\mathbf{Z} = \sum_{i=1}^{r}\mathbf{Z}_i'\boldsymbol{\Sigma}^{**-1}\mathbf{Z}_i = \sum_{i=1}^{r}\sum_{j=1}^{p}\mathbf{Z}_{ij}'\mathbf{B}_1\mathbf{Z}_{ij} - \sum_{i=1}^{r}\sum_{j=1}^{p}\mathbf{Z}_{ij}'\mathbf{B}_2\left(\sum_{j=1}^{p}\mathbf{Z}_{ij}\right). \tag{8.3.44}$$

Here, as the observations in the same row-strip or in the same column-strip, are subject to the influence of the same level of A and the same level of B, respectively, it is expected that $\rho_1 > 0$, $\rho_2 > 0$ (cf. Cochran and Cox 1950) and $\mathbf{B}_2$ is assumed to be a positive definite matrix. Because of this assumption and the structure of $\mathbf{B}_1$, (8.3.44) implies that a design for which

$$\sum_{l=1}^{q} z_{lm}^{(ij)} = 0\ \forall i, j, m;\ \sum_{j=1}^{p} z_{lm}^{(ij)} = 0\ \forall i, l, m \tag{8.3.45}$$

hold, dominates any other design in the sense of PLO. If, in addition,

$$\left.\begin{array}{ll} z_{lm}^{(ij)} = \pm 1 & \forall i, j, l, m \\ \sum_{i=1}^{r}\sum_{j=1}^{p}\sum_{l=1}^{q} z_{lm}^{(ij)} z_{lm'}^{(ij)} = 0 & \forall m \neq m' = 1, 2, \ldots, c. \end{array}\right\} \quad (8.3.46)$$

then

$$\mathbf{Z}'\mathbf{\Sigma}^{*-1}\mathbf{Z} = \frac{rpq}{1-\rho_1-\rho_2}\mathbf{I}_c. \quad (8.3.47)$$

So from (8.3.43), (8.3.45) and (8.3.46), we get the following theorem which gives a set of sufficient conditions for optimum estimation (in the sense of the minimum variance for the estimator of each γ-component) of the covariate effects in a strip-plot design.

Theorem 8.3.2 *With respect to the linear model (8.3.1) for the standard strip-plot design with variance structure (8.3.29), the following set of conditions:*

(i) $z_{lm}^{(ij)} = \pm 1 \quad \forall i = 1, 2, \ldots, r, \ j = 1, 2, \ldots, p, \ l = 1, 2, \ldots, q, \ m = 1, 2, \ldots, c$

(ii) $\sum_{i=1}^{r} z_{lm}^{(ij)} = 0 \quad \forall j = 1, 2, \ldots, p, \ l = 1, 2, \ldots, q, \ m = 1, 2, \ldots, c$

(iii) $\sum_{j=1}^{p} z_{lm}^{(ij)} = 0 \quad \forall i = 1, 2, \ldots, r, \ l = 1, 2, \ldots, q, \ m = 1, 2, \ldots, c$

(iv) $\sum_{l=1}^{q} z_{lm}^{(ij)} = 0 \quad \forall i = 1, 2, \ldots, r, \ j = 1, 2, \ldots, p, \ m = 1, 2, \ldots, c$

(v) $\sum_{i=1}^{r}\sum_{j=1}^{p}\sum_{l=1}^{q} z_{lm}^{(ij)} z_{lm'}^{(ij)} = rpq\delta_{mm'}$

where $\delta_{mm'} = 1$ *if* $m = m'$; *=0 if* $m \neq m'$,

are sufficient for the optimum estimation of each of the covariate effects with the minimum variance Var $(\widehat{\gamma}_m) = \frac{(1-\rho_1-\rho_2)\sigma^2}{rpq}, \quad m = 1, 2, \ldots, c.$

Note 8.3.2 Comparing Var $(\widehat{\gamma}_m)$ in split-plot with that in strip-plot set-up, it is expected that Var $(\widehat{\gamma}_m)$ under strip-plot is less than Var $(\widehat{\gamma}_m)$ under split-plot as ρ is expected to be less than $\rho_1 + \rho_2$. ρ is expected to be equal to ρ_1 if the row-strips are taken to be the strips in split-plot design. The reduction is due to introduction of column strips in strip-plot design.

Note 8.3.3 Condition (iii) of Theorem 8.3.2 for OCDs in strip-plot design is an additional condition with those conditions for OCDs in split-plot design set-up. We can still get an OCD for split-plot design set-up without satisfying this condition. Condition (iii) is called for to meet the condition of orthogonality with respect to row-strip.

8.3.2 Optimum Covariate Designs

We can represent the sufficient conditions of Theorems 8.3.1 and 8.3.2 in terms of a $(3+c) \times rpq$ rectangular array where the first three rows (forming the first group) contain all possible combinations of the levels of the block (R), the whole-plot factor (A) and the sub-plot factor (B), respectively, arranged lexicographically. The $(3+i)$th row of the second group which corresponds to the ith row of $\mathbf{Z}'$ have elements ± 1, $i = 1, 2, \ldots, c$. It is easy to verify that if the array satisfies the following conditions, then both Theorems 8.3.1 and 8.3.2 hold true:

(a_1) $\mathbf{Z}'$ is an orthogonal array of strength 2.
(a_2) in any $3 \times rpq$ sub-array containing any two rows from the first group and any one row from the second group every level combinations occur equally often.

Conditions (a_1)–(a_2) imply that the array $(3+c) \times rpq$ array is obviously an EMOA$[rpq,\ 3+c,\ r \times p \times q \times 2^c, (3, 2), (2; 2, 1)]$. Therefore, we get the following theorem.

Theorem 8.3.3 *The existence of an EMOA $[rpq,\ 3+c,\ r \times p \times q \times 2^c$, (3, 2), (2; 2, 1)] implies the existence of an OCD for both split- and strip-plot set-ups.*

Below we describe some methods of getting an EMOA $[rpq,\ 3+c,\ r \times p \times q \times 2^c$, (3, 2), (2; 2, 1)] which gives an OCD for both split-plot and strip-plot set-ups.

Theorem 8.3.4

(1) If $\mathbf{H}_r$, $\mathbf{H}_p$ and $\mathbf{H}_q$ exist, then an EMOA $[rpq,\ 3+c,\ r \times p \times q \times 2^c$, (3, 2), (2; 2, 1)] can be constructed, where $c = (r-1)(p-1)(q-1)$.
(2) If $\mathbf{H}_{2r}$, $\mathbf{H}_p$ and $\mathbf{H}_{\frac{q}{2}}$ exist, where r is even, then an EMOA $[rpq,\ 3+c,\ r \times p \times q \times 2^c$, (3, 2), (2; 2, 1)] can be constructed, where $c = (r-1)(p-1)(q-1) - (r-2)(p-1)$.
(3) If $r \equiv 2 \pmod 4$, $(r-1)$ is a prime or a prime power and $\mathbf{H}_p$ and $\mathbf{H}_q$ exist, where r is even, then an EMOA $[rpq,\ 3+c,\ r \times p \times q \times 2^c$, (3, 2), (2; 2, 1)] can be constructed, where $c = (r-1)(p-1)(q-1) - (r-2)$.

Example 8.3.2 Let us take $r = 2,\ p = 2,\ q = 4$. $\mathbf{H}_2$ and $\mathbf{H}_4$ can be, respectively, written as

$$\mathbf{H}_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{H}_2^* \\ \mathbf{1}' \end{pmatrix},$$

$$\mathbf{H}_4 = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{H}_4^* \\ \mathbf{1}' \end{pmatrix}$$

For $r = 2,\ p = 2,\ q = 4$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ and $\mathbf{X}_4$ are written in Example 8.3.1.

The optimum $\mathbf{Z}'$-matrix for split- and strip-plot designs with $r = 2,\ p = 2,\ q = 4$ is given by:
$\mathbf{Z}' = \mathbf{H}^{3\times 16} = \mathbf{H}_2^* \otimes \mathbf{H}_2^* \otimes \mathbf{H}_4^*$

$$= \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \end{pmatrix}. \quad (8.3.48)$$

Let us augment the matrix $\mathbf{Z}'$ with a 3×16 matrix $\mathbf{D}$ whose columns denote the coordinates of the cells of the z-values in lexicographic order. Then $(\mathbf{D}', \mathbf{Z}')'$ gives the EMOA[16, 2, $2\times 2 \times 4 \times 2^3$, (3,2), (2; 2,1)] which is as follows:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

Condition (iv) of Theorem 8.3.2 is an additional condition with those conditions for OCDs in split-plot design set-up. We can still get an OCD for split-plot design set-up if we use $\mathbf{H}_2$ instead of instead of $\mathbf{H}_2^*$. Therefore, the optimum $\mathbf{Z}'$-matrix for split-plot design with $r = 2,\ p = 2,\ q = 4$ is given by:
$\mathbf{Z}' = \mathbf{H}^{6\times 16} = \mathbf{H}_2^* \otimes \mathbf{H}_2 \otimes \mathbf{H}_4^*$

$$= \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}. \quad (8.3.49)$$

It can be easily be verified that the above $\mathbf{Z}$-matrices in (8.3.48) and (8.3.49) satisfy all the conditions of Theorems 8.3.1 and 8.3.2 respectively.

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Chapter 9
Applications of the Theory of OCDs

9.1 Introduction: Eye-Openers

In this concluding chapter, we propose to discuss at length several examples from standard textbooks. All of these examples deal with ANCOVA models and related analyses of data. We intend to capitalize on our understanding of OCDs in different ANCOVA models as discussed in Chaps. 2–8 and revisit these examples with a view to suggest optimal/highly efficient designs for estimation of the covariate parameter(s). As we will see, for some examples our task is very much routine but for others, it is indeed a highly non-trivial exercise. Most of the material in this chapter is based on Dutta and Sinha (2015).

Example 9.1.1 We started with this example in Chap. 1. It relates to a leprosy study quoted from Snedecor and Cochran's book (1989, p. 377). The point we made is that there is ample scope of improvement in the efficiency of the estimates for the covariates' parameters if we have a 'free' hand in the recruitment of the patients and if a 'pool' is made available to us. Since the basic design is a CRD and there are three 'treatments' under consideration—with ten patients to be recruited under each treatment—an OCD suggests the following scheme of recruitment of the patients in terms of their possession of original pre-treatment score (count of bacilli)—under the supposition that we have a 'free choice' of the patients from a conceivably larger pool. Table 9.1 shows the scheme.

It was further stated that as against the given patients' ad hoc recruitment scheme in Table 1.1 (Chap. 1), the above scheme provides more than 300 % gain in efficiency towards estimation of the covariate parameter. Even with the 'given' pool of 30 patients, a suitable reallocation of the patients across the three treatments, as indicated in Table 1.2 (Chap. 1), would have provided 12 % gain in efficiency against the 'adhoc' allocation in Table 1.1 (Chap. 1). The OCD given in Table 9.1 is based on the theory developed in Chap. 2 with regard to the CRD. Recall the formation of **W**-matrix with the coded covariate values. In applications, the code -1 (respectively, $+1$) is to be replaced by x_{min} (respectively, x_{max}) which are '3' and '21' in the above example.

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Table 9.1 Recruitment of patients based on pre-treatment score in actual units (patient serial number, covariate value)

1.	Treatment A	(P1, 3), (P2, 3), (P3, 3), (P4, 3), (P5, 3), (P6, 21), (P7, 21), (P8, 21), (P9, 21), (P10, 21)
2.	Treatment D	(P11, 3), (P12, 3), (P13, 3), (P14, 3), (P15, 3), (P16, 21), (P17, 21), (P18, 21), (P19, 21), (P20, 21)
3.	Control F	(P21, 3), (P22, 3), (P23, 3), (P24, 3), (P25, 3), (P26, 21), (P27, 21), (P28, 21), (P29, 21), (P30, 21)

We will now carry out the non-trivial exercise of identifying the design indicated in Table 1.2 (Chap. 1) as obtained through adequate re-allocation of the covariate values of the given pool of 30 patients as in the given design, to be denoted by d_0. For the sake of completeness, we display the allocation of covariate-values over the three treatments as in d_0.

$$\begin{array}{|l|}\hline \mathrm{A}: 3,\ 5,\ 6,\ 6,\ 8,\ 10,\ 11,\ 11,\ 14,\ 19 \\ \hline \mathrm{D}: 5,\ 6,\ 6,\ 7,\ 8,\ 8,\ 8,\ 15,\ 18,\ 19 \\ \hline \mathrm{F}: 7,\ 9,\ 11,\ 12,\ 12,\ 12,\ 13,\ 16,\ 16,\ 21 \\ \hline \end{array} = d_0,\ \text{say}.$$

It follows that, in terms of the Z-scores ranging in $[-1, 1]$,

$$\mathbf{I}(\boldsymbol{\theta}) = \begin{pmatrix} 10 & 0 & 0 & -3.0000 \\ 0 & 10 & 0 & -2.2222 \\ 0 & 0 & 10 & 1.0000 \\ -3.0000 & -2.2222 & 1.0000 & 8.8148 \end{pmatrix}.$$

Routine computation yields: Information for γ, $I_{d_0}(\gamma) = 7.3210$.

Towards an 'improved' allocation, we arrange the data of pre-treatment scores of all the 30 patients in ascending order: 3, 5, 5, 6, 6, 6, 6, 7, 7, 8, 8, 8, 8, 9, 10, 11, 11, 11, 12, 12, 12, 13, 14, 15, 16, 16, 18, 19, 19, 21.

Now by using the following algorithms, we make an attempt to search for a design for which the information of γ is maximum.

Algorithm 1

Step 1: We conveniently divide ordered Z-scores into three blocks. Block 1 consists of the first nine observations of arranged data, i.e. (3, 5, 5, 6, 6, 6, 6, 7, 7); Block 2 consists of the next 12 observations, i.e. (8, 8, 8, 8, 9, 10, 11, 11, 11, 12, 12, 12); Block 3 consists of the last 9 observations (13, 14, 15, 16, 16, 18, 19, 19, 21).

Step 2: In Block 1 we allocate the first 3 observations i.e. (3, 5, 5) under treatment A, the next 3 observations, i.e. (6, 6, 6) under treatment D and the last 3 observations, i.e. (6, 7, 7) under treatment F.
Step 3: In Block 2 we allocate the first 4 observations, i.e. (8, 8, 8, 8) under treatment D, the next 4 observations, i.e. (9, 10, 11, 11) under treatment F and the last 4 observations, i.e. (11, 12, 12, 12) under treatment A.
Step 4: In Block 3 we allocate the first 3 observations i.e. (13, 14, 15) under treatment F, the next 3 observations, i.e. (16, 16, 18) under treatment A and the last 3 observations, i.e. (19, 19, 21) under treatment D.

Hence we get the following arrangement:

	Block 1	Block 2	Block 3
A	3, 5, 5	11, 12, 12, 12	16, 16, 18
D	6, 6, 6	8, 8, 8, 8	19, 19, 21
F	6, 7, 7	9, 10, 11, 11	13, 14, 15

$= d_1$, say.

The information of γ from $d_1 = I_{d_1}(\gamma) = 8.1852$.
Step 5: Start with d_1. Consider the left block, i.e. Block 1. Permute the rows and generate $3! = 6$ options for this block, while keeping the middle and the right block intact. Work out I(γ) for all the 6 options generated from the left block. Identify the best case scenario and hold this intact while passing into the middle block. Here the best design is found to be d_1.
Step 6: For the middle block, i.e. Block 2, we follow a similar step. Here the best design using Step 6 is

	Block 1	Block 2	Block 3
A	3, 5, 5	9, 10, 11, 11	16, 16, 18
D	6, 6, 6	8, 8, 8, 8	19, 19, 21
F	6, 7, 7	11, 12, 12, 12	13, 14, 15

$= d_2$, say.

The information of γ for $d_2 = I_{d_2}(\gamma) = 8.2$.
Step 7: For the last block, i.e. Block 3, we again follow similarly step. Ultimately we get d_2 as the best design.

We now consider other aspects towards improving d_2.

Algorithm 2
Here we consider three allocations:

(I) (ADF—DFA—FAD——ADF—DFA—FAD—ADF——DFA—FAD—ADF)
(II) (ADF—DFA—FAD——ADF—DFA—FAD—ADF——DFA—FAD—DFA)
(III) (ADF—DFA—FAD——ADF—DFA—FAD—ADF——DFA—FAD—FAD)

and the the designs are respectively:

	Block 1	Block 2	Block 3
A	3, 6, 7	8, 10, 11, 12	15, 16, 19
D	5, 6, 7	8, 8, 11, 12	13, 18, 19
F	5, 6, 6	8, 9, 11, 12	14, 16, 21

$= d_{(I)}$, say;

	Block 1	Block 2	Block 3
A	3, 6, 7	8, 10, 11, 12	15, 16, 21
D	5, 6, 7	8, 8, 11, 12	13, 18, 19
F	5, 6, 6	8, 9, 11, 12	14, 16, 19

$= d_{(II)}$, say;

	Block 1	Block 2	Block 3
A	3, 6, 7	8, 10, 11, 12	15, 16, 19
D	5, 6, 7	8, 8, 11, 12	13, 18, 21
F	5, 6, 6	8, 9, 11, 12	14, 16, 19

$= d_{(III)}$, say.

For the above three designs, $\mathrm{I}_{d_{(I)}}(\gamma) = 8.2198$, $\mathrm{I}_{d_{(II)}}(\gamma) = 8.2148$ and $\mathrm{I}_{d_{(III)}}(\gamma) = 8.2148$.

Algorithm 3
We may consider another allocation:

(AFD—FDA—DAF——AFD—FDA—FDA—AFD——DAF—FDA——AFD)

and the corresponding design is

	Block 1	Block 2	Block 3
A	3, 6, 7	8, 10, 11, 12	14, 18, 19
D	5, 6, 6	8, 9, 11, 12	13, 16, 21
F	5, 6, 7	8, 8, 11, 12	15, 16, 19

$= d_3$, say.

Here also $\mathrm{I}(\gamma) = 8.2198$.
Heuristic Search:

	G_1	G_2	G_3
A	3, 6, 6	9, 10, 11, 12	14, 18, 19
D	5, 6, 7	8, 8, 11, 12	15, 16, 19
F	5, 6, 7	8, 8, 11, 12	13, 16, 21

$= d_4$, say.

This yields $I_{d_4}(\gamma) = 8.2198$ and d_3 is equivalent to d_4. Further, these are also equivalent to $d_{(I)}$ in the sense of same information.

In the final analysis, we find that there is substantial gain in efficiency (more than 12 %) in the performance of the design $d_{(I)}$ or d_3, as against the original design d_0. This is the design ($d_{(I)}$ or d_3) displayed in Table 1.2 (Chap. 1).

Example 9.1.2 We now elaborate on the second example discussed in Chap. 1. Recall that this refers to an RBD with $b = 5,\ v = 3$. In Chap. 3, we have discussed at length OCDs under RBD set-ups but mostly the 'regular' cases, viz. both b and v being multiples of 4 so that Hadamard matrices exist. Here is a deviation from that and we take this rare opportunity to discuss the example in quite details. Note that we have already provided solutions to two different aspects of the example: (i) For given covariate-values, improved allocation of the available experimental units across the RBD layout; (ii) For a 'free' choice of the covariate values within certain well-defined closed intervals, identification of the experimental units with covariate values of the experimenter's choice. Below we give detailed derivations of the above results. We refer to Tables 1.3, 1.4a, b, 1.5a, b and 1.6a, b in Chap. 1.

Under an RBD ANCOVA model with a single covariate, recall the standard expression for information on γ, viz.

$$\begin{aligned} I(\gamma) &= \sum_{i=1}^{5}\sum_{j=1}^{3} z_{ij}^2 - \frac{1}{3}\sum_{i=1}^{5} R_i^2 - \frac{1}{5}\sum_{j=1}^{3} C_j^2 + \frac{G^2}{15} \\ &= \sum_{i=1}^{5}\sum_{j=1}^{3} z_{ij}^2 - \frac{1}{3}\sum_{i=1}^{5} R_i^2 - 5\sum_{j=1}^{3} \left(\bar{z}_{0j} - \bar{z}_{00}\right)^2 \end{aligned} \tag{9.1.1}$$

Our aim is to maximize the information of γ given in (9.1.1) by properly allocating the pigs in the two-way RBD layout. This applies to both female and male pigs. Note that towards this, the row totals of the covariate-values should be as close as possible and the same is true of the column totals. We start with the 5×3 table of covariate values for the female pigs and proceed through the following steps:

Step 1: First, we arrange the rows in three sets where the first set consists of the rows where all the covariate values are equal; in the second set, we consider those rows where two of the three covariate values are not equal and the last set consists of the rows where all the covariate values are unequal. The arrangement is shown in Table 9.2.

Step 2: We select the first row of second set (i.e., Pen No. 2) and permute the covariate values keeping the other rows fixed. Next we compute the information of γ_F under each permutation. Then we choose the design in which the information of γ_F will be a maximum. We do the same for the second row of the second set (i.e., Pen No. 4) keeping the other rows of the new design intact. Similarly, we do the same for the third set also (Pen No. 3 and 5).

Step 3: We repeat Step 2 until all C_{Fj}'s are as close as possible to $\frac{G_F}{3}$. Finally, we get the design where the information of γ_F is a maximum with C_{Fj}'s as close as possible to $\frac{G_F}{3}$. We denote it by d_{F1} and $I_{d_{F1}}(\gamma_F) = 81.0667$, where d_{F1} is displayed in Table 9.3.

Table 9.2 Female

Pen	Treatment			Totals
	A	B	C	
1	48	48	48	144
2	32	32	28	92
4	46	46	50	142
3	35	41	33	109
5	32	37	30	99
Totals	193	204	189	586

Table 9.3 d_{F1}

Pen	Treatment			Totals
	A	B	C	
1	48	48	48	144
2	32	28	32	92
4	46	46	50	142
3	41	35	33	109
5	30	37	32	99
Totals	197	194	195	586

Table 9.4 An alternate arrangement

Treatment			Totals
A	B	C	
30	48	*	78
32	46	*	78
41	37	*	78
46	35	*	81
48	28	*	76

Table 9.5 d_{F2}

	Treatment			Totals
	A	B	C	
	30	48	48	126
	32	46	33	111
	41	37	32	110
	46	35	32	113
	48	28	50	126
Totals	197	194	195	586

Step 4: We arrange the initial weights under treatment A in ascending order and the initial weights under treatment B in descending order. The arrangement is shown in Table 9.4.

Since the sum of the two entries in each of 5 rows are 78, 78, 78, 81, 76, we fill the entries under treatment C as 48, 33, 32, 32, 50. Then we get the design d_{F2} and here $I_{d_{F2}}(\gamma_F) = 782.4$. We display the design d_{F2} in Table 9.5.

For another option, we arrange the initial weights under treatment A in ascending order and the initial weights under treatment C in descending order. The arrangement is shown in Table 9.6.

Since the sum of the two entries in each of 5 rows are 80, 80, 74, 78, 80, we fill the entries under treatment B as 37, 35, 48, 46, 28. Then we get the design d_{F3} displayed in Table 9.7 and here $I_{d_{F3}}(\gamma_F) = 817.0667$.

Table 9.6 A second alternative

Treatment			Totals
A	B	C	
30	*	50	80
32	*	48	80
41	*	33	74
46	*	32	78
48	*	32	80

Table 9.7 d_{F3}

	Treatment			Totals
	A	B	C	
	30	37	50	117
	32	35	48	115
	41	48	33	122
	46	46	32	124
	48	28	32	108
Totals	197	194	195	586

Table 9.8 A third alternative

Pen	Treatment			Totals
	A	B	C	
1	*	28	50	78
2	*	35	48	83
4	*	37	33	70
3	*	46	32	78
5	*	48	32	80

Table 9.9 d_{F4}

	Treatment			Totals
	A	B	C	
	41	28	50	119
	30	35	48	113
	48	37	33	118
	46	46	32	124
	32	48	32	112
Totals	197	194	195	586

Lastly, we arrange the initial weights under treatment B in ascending order and the initial weights under treatment C in descending order. The arrangement is shown in Table 9.8.

Since the sum of the two entries in each of 5 rows are 78, 83, 70, 78, 80, we fill the entries under treatment A as 41, 30, 48, 46, 32. Then we get the design d_{F4} shown in Table 9.9 and here $I_{d_{F4}}(\gamma_G) = 838.4$.

Table 9.10 d_{F5}

	Treatment			Totals
	A	B	C	
	41	28	48	117
	30	35	50	115
	48	37	33	118
	46	46	32	124
	32	48	32	112
Totals	197	194	195	586

Table 9.11 d_{F6}

	Treatment			Totals
	A	B	C	
	41	28	48	117
	30	37	50	117
	48	35	33	116
	46	46	32	124
	32	48	32	112
Totals	197	194	195	586

Table 9.12 d_{F7}

	Treatment			Totals
	A	B	C	
	46	28	48	122
	30	37	50	117
	48	35	33	116
	41	46	32	119
	32	48	32	112
Totals	197	194	195	586

Now we start with d_{F4} and proceed with Step 1 and Step 2. Then we observe that d_{F4} is a better design. Next we can improve over d_{F4} by interchanging the first element and the second element under Treatment C and denote the design by d_{F5} shown in Table 9.10. Here $I_{d_{F5}}(\gamma_F) = 843.7333$.

Again we can improve d_{F5} by interchanging the second element and the third element under treatment B and we denote the design by d_{F6} shown in Table 9.11. Here $I_{d_{F6}}(\gamma_F) = 845.0667$.

We can further improve d_{F6} by interchanging the first element and the fourth element under Treatment A and denote the design by d_{F7} shown in Table 9.12. Here $I_{d_{F7}}(\gamma_F) = 851.7333$.

Lastly, we improve d_{F7} by interchanging the third element and the fourth element under Treatment C and denote the design by d_{F8} shown in Table 9.13. Here $I_{d_{F8}}(\gamma_F) = 853.7333$.

Table 9.13 d_{F8}

	Treatment			Totals
	A	B	C	
	46	28	48	122
	30	37	50	117
	48	35	32	115
	41	46	32	119
	32	48	33	113
Totals	197	194	195	586

Table 9.14 d_{F9}

	Treatment			Totals
	A	B	C	
	46	28	48	122
	30	37	50	117
	46	35	32	113
	41	48	32	121
	32	48	33	113
Totals	195	196	195	586

Table 9.15 d_{M1}

Pen	Treatment			Totals
	A	B	C	
5	43	40	40	125
1	38	39	48	110
2	37	38	35	129
3	41	46	42	130
4	48	42	40	123
Totals	207	205	205	617

Now we construct design d_{F9} shown in Table 9.14 by interchanging the third element under Treatment A and the fourth element under Treatment B of d_{F8}. Here $I_{d_{F9}}(\gamma_G) = 846.5333$ which is less than $I_{d_{F8}}(\gamma_F)$ even though column sums are more or less equal. We stop here and recommend the design d_{F8} for use.

As is indicated in the above, the designs d_{F6} to d_{F9} are displayed in Tables 9.11, 9.12, 9.13 and 9.14 respectively. Similarly, for increasing the information of γ_M, we follow the Steps 1, 2, 3 and get the design denoted by d_{M1} and shown in Table 9.15.

Here $I_{d_{M1}}(\gamma_H) = 119.4667$.

In the same way as mentioned above for female pigs, to get three designs we follow Step 4 and find d_{M2} shown in Table 9.16 with $I_{d_{M2}}(\gamma_M) = 195.4667$, d_{M3} shown in Table 9.17 with $I_{d_{M3}}(\gamma_H) = 202.1333$ and d_{M4} displayed in Table 9.18 with $I_{d_{M4}}(\gamma_M) = 193.4667$. Therefore, d_{M3} is a better design. Now we start with d_{M3} and

Table 9.16 d_{M2}

	Treatment			Totals
	A	B	C	
	37	46	40	123
	38	42	48	128
	41	40	42	123
	43	39	40	122
	48	38	35	121
Totals	207	205	205	617

Table 9.17 d_{M3}

	Treatment			Totals
	A	B	C	
	37	38	48	123
	38	46	42	126
	41	42	40	123
	43	40	40	123
	48	39	35	122
Totals	207	205	205	617

Table 9.18 d_{M4}

	Treatment			Totals
	A	B	C	
	37	38	48	123
	43	39	42	124
	48	40	40	128
	38	42	40	120
	41	46	35	122
Totals	207	205	205	617

we improve it further and denote the design by d_{M5} with $I_{d_{M5}}(\gamma_M) = 202.5333$ using Steps 1, 2, 3. This design d_{M5} is displayed in Table 9.19. Again we improve d_{M5} by interchanging the second element and the fourth element under Treatment C and denote it by d_{M6} which is displayed in Table 9.20. Here $I_{d_{M6}}(\gamma_M) = 203.8667$. Next we improve d_{M6} by interchanging the third element and the fourth element under treatment C and denote it by d_{M7}. Here $I_{d_{M7}}(\gamma_M) = 205.2$. Lastly there is another possibility to improve d_{M7} by interchanging the 4th element and the 5th element under Treatment B. We denote it by d_{M8}. Here $I_{d_{M8}}(\gamma_M) = 204.5333$.

We stop here and accept the design d_{M7} for the use of male pigs. It is a routine task to compute the percent gain. The designs d_{M7} and d_{M8} are displayed in Tables 9.21 and 9.22 respectively.

Now we consider the 'hypothetical' situation for female data wherein the row totals are equal or nearly equal to each other and also the column totals are equal or nearly equal to each other. For female data, that would amount to the row sums being

Table 9.19 d_{M5}

	Treatment			Totals
	A	B	C	
	37	38	48	123
	38	46	42	126
	40	42	41	123
	43	40	40	123
	48	39	35	122
Totals	206	205	206	617

Table 9.20 d_{M6}

	Treatment			Totals
	A	B	C	
	37	38	48	123
	38	46	40	124
	40	42	41	124
	43	40	42	124
	48	39	35	122
Totals	206	205	206	617

Table 9.21 d_{M7}

	Treatment			Totals
	A	B	C	
	37	38	48	123
	38	46	40	124
	40	42	42	124
	43	39	41	123
	48	40	35	123
Totals	206	205	206	617

Table 9.22 d_{M8}

	Treatment			Totals
	A	B	C	
	37	38	48	123
	38	46	40	124
	40	42	42	124
	43	40	41	124
	48	39	35	122
Totals	206	205	206	617

117, 117, 117, 117 and 118 and column sums being 195, 195, 196 since the total is fixed at $G_F = 586$. In this hypothetical situation $I(\gamma_F) = 870.5333$. Therefore, the relative efficiency of $d_{F8} = 98.0702\,\%$ whereas the relative efficiency of γ_F under design d_0 is $6.6473\,\%$. In other words, relative gain in efficiency by use of d_{F8} as against d_0 is more than 1300 %. Similarly, we consider the hypothetical situation for male data where the row sums are meant to be 123, 123, 123, 124, 124 and column sums are also meant to be 206, 206, 205. Here $I(\gamma_M) = 205.2$ which is equal to d_{M7}. Therefore, the relative efficiency of $d_{M7} = 100\,\%$ whereas the relative efficiency of γ_M under design d_0 is 56.6602 %. In other words, relative gain in efficiency by the use of d_{M7} as against d_0 is more than 75 %. Hence we can improve the information of covariate parameter by properly allocating the covariate values in the rows and columns separately.

Recall the expression for I(γ) given in (9.1.1) above. It is readily seen that

$$I(\gamma) \leq \sum_{i=1}^{5}\sum_{j=1}^{3} z_{ij}^2 - \frac{1}{3}\sum_{i=1}^{5} R_i^2 \tag{9.1.2}$$

equality holds if $\bar{z}_{0j} = \bar{z}_{00}$ for all j, where $\bar{z}_{0j} = \frac{C_j}{5}$ and $\bar{z}_{00} = \frac{G}{15}$ and since we fix z_{ij} at ± 1, $I(\gamma) \leq \sum_{i=1}^{5}\sum_{j=1}^{3} z_{ij}^2 - \frac{1}{3}\sum_{i=1}^{5} R_i^2 \leq 15 - \frac{5}{3} = \frac{40}{3} = 13.33$.

Again,

$$I(\gamma) \leq \sum_{i=1}^{5}\sum_{j=1}^{3} z_{ij}^2 - \frac{1}{5}\sum_{j=1}^{3} C_j^2 = I_2(\gamma) \text{ (say)} \tag{9.1.3}$$

equality holds if $\bar{z}_{i0} = \bar{z}_{00}$ for all i, where $\bar{z}_{i0} = \frac{R_i}{3}$ and $\bar{z}_{00} = \frac{G}{15}$ and $I_2(\gamma) \leq 15 - \frac{3}{5} = \frac{72}{5} = 14.4$. Therefore, combining the two inequalities, we deduce that $I(\gamma) \leq 13.33$. We display a design in Table 9.23 for which $I(\gamma)$ attains the bound 13.33.

The above exercise suggests that if we have such flexibility to choose the initial weights for males and females separately, then our suggestion is the design shown in Table 9.23 in terms of z-values and this applies to both males and females. For female and male pigs, the arrangement of weights for OCD are already shown in Table 1.6a, b in Chap. 1 in terms of original weights.

Table 9.23 Design where $I_1(\gamma)$ is attained

	A	B	C	Totals
	−1	1	1	1
	1	−1	1	1
	1	1	−1	1
	−1	1	1	1
	1	−1	−1	−1
Totals	1	1	1	3 = G

9.2 Other Application Areas

In this section, we undertake four different types of examples of application of OCD.

Example 9.2.1 We consider the observations and the design (see Goos and Jones (2011), p. 79) in Table 9.24.

The response variable is peel strength, which measures the amount of force required to open the package. Raw material from three suppliers (S_1, S_2 and S_3) is used in the sealing process. We consider temperature (Z_1), pressure (Z_2) and speed (Z_3) on the peel strength as covariates. These three covariates are controllable. The range and unit of each covariate is given in Table 9.25.

Note that the authors present a very general ANCOVA model in the book and discuss some aspects of D-optimal designs. Instead, we will consider the simplest model:

Table 9.24 Data for the robustness experiment

Run number	Temperature	Pressure	Speed	Supplier	Peel strength
1	211.5	2.2	32	1	4.36
2	193.0	2.7	32	1	5.20
3	230.0	3.2	32	1	4.75
4	230.0	2.2	41	1	5.73
5	193.0	3.2	41	1	4.49
6	193.0	2.2	50	1	6.38
7	230.0	2.7	50	1	5.59
8	211.5	3.2	50	2	5.40
9	193.0	2.2	32	2	5.78
10	230.0	2.7	32	2	4.80
11	193.0	3.2	32	2	4.93
12	211.5	2.7	41	2	5.96
13	211.5	2.7	41	2	6.55
14	230.0	2.2	50	2	6.92
15	193.0	2.7	50	2	6.18
16	230.0	3.2	50	2	6.55
17	230.0	2.2	32	3	5.44
18	193.0	2.7	32	3	4.57
19	211.5	3.2	32	3	4.48
20	193.0	2.2	41	3	4.78
21	211.5	2.7	41	3	5.03
22	230.0	3.2	41	3	3.98
23	211.5	2.2	50	3	4.73
24	193.0	3.2	50	3	4.70

Table 9.25 Range and unit of covariates

Covariate	Range	Unit
Temperature	193–230	°C
Pressure	2.2–3.2	bar
Speed	32–50	cpm

$$\mathrm{E}(y_{ij}) = s_i + \sum_{l=1}^{3} \gamma_l z_{ij}^{(l)},$$

where y_{ij} is the jth observation corresponding to ith supplier and s_i is the effect due to ith supplier and γ_l is the lth covariate effect, $z_{ij}^{(l)}$ is the lth covariate value corresponding the (i, j)th observation and $|z_{ij}^{(l)}| \leq 1$ for all $i,\ j,\ l$. Further, we will take up the problem of estimating the three covariate parameters most efficiently by selecting the design. With the replication numbers each equal to 8, it would have been a trivial exercise in a CRD model with three covariates. Vide Chap. 2. However, with (7, 8, 9) as the replication numbers, standard theory breaks down and we run into what has been termed as 'non-regular case'. As usual, the covariates are all coded to lie inside the closed interval $[-1, 1]$. From Theorem 2.3.1 in Chap. 2, for fixed $\{r_i\}$, i.e. $r_1 = 7,\ r_2 = 9,\ r_3 = 8$,

$$det(\mathbf{I}(\boldsymbol{\theta})) \leq (7 \times 9 \times 8)\,(a + 2b)\,(a - b)^2$$

where $a = 24 - \delta,\quad b = \delta = \frac{1}{7} + \frac{1}{9}$ and r_i is the number of times the ith supplier replicated.

Therefore, $det(\mathbf{I}(\boldsymbol{\theta})) \leq 7 \times 9 \times 8 \times 13385.21$. But it is very difficult to construct $\mathbf{Z}$-matrix where $det(\mathbf{I}(\boldsymbol{\theta}))$ attains the upper bound. Here we construct the design (say d^*) whose $det(\mathbf{I}(\boldsymbol{\theta}))$ is very close to the upper bound and the $\mathbf{Z}$-matrix is written as:

$$\mathbf{Z}' = \left(\mathbf{Z}'(\mathrm{S}_1), \mathbf{Z}'(\mathrm{S}_2), \mathbf{Z}'(\mathrm{S}_3)\right)$$

where

$$\mathbf{Z}'(\mathrm{S}_1) = \begin{pmatrix} 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}; \quad \mathbf{Z}'(\mathrm{S}_2) = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix};$$

$$\mathbf{Z}'(\mathrm{S}_3) = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

The information matrix for $\boldsymbol{\theta} = (s_1, s_2, s_3, \gamma_1, \gamma_2, \gamma_3)'$ is given below:

$$\mathbf{I}_{d^*}(\boldsymbol{\theta}) = \begin{pmatrix} 7 & 0 & 0 & -1 & -1 & -1 \\ 0 & 9 & 0 & 1 & 1 & 1 \\ 0 & 0 & 8 & 0 & 0 & 0 \\ -1 & 1 & 0 & 24 & 0 & 0 \\ -1 & 1 & 0 & 0 & 24 & 0 \\ -1 & 1 & 0 & 0 & 0 & 24 \end{pmatrix}$$

whence the information matrix for $\boldsymbol{\gamma}$ is

$$\mathbf{I}_{d^*}(\gamma) = \begin{pmatrix} 23.746 & -0.254 & -0.254 \\ -0.254 & 23.746 & -0.254 \\ -0.254 & -0.254 & 23.746 \end{pmatrix}.$$

Here $det(\mathbf{I}_{d^*}(\boldsymbol{\theta})) = 7 \times 9 \times 8 \times 13385.14$ and the relative D-efficiency $= \frac{13385.14}{13385.21} \times 100 = 99.9995\,\%$. It is interesting to note that the three parts of the design d^* are derived essentially from $\mathbf{H}_8$.

Example 9.2.2 Here we consider the data given in van Belle et al. (2004) and reproduced below in Tables 9.26 and 9.27. It relates to 'exercise' data for healthy active males (44) and females (43). There are four covariates, viz. Heart Rate, Age, Height and Weight. The response variable is the $V0_2$ Max, measured in a suitable unit.

As usual, we introduce coded covariates Z_1, Z_2, Z_3, Z_4, each in the range $[-1, 1]$. We reproduce the above tables only in terms of the four covariates in coded forms, skipping the responses (vide Tables 9.28 and 9.29).

It is a routine task to carry out data analysis using a 4-variate linear regression model for each data set involving the four covariates. We skip that part. Instead, we ask a non-trivial problem. If an experimenter is to design the survey to accommodate 10 males and 10 females out of the above 'pool' and to maximize information-content for the joint estimation of the covariate parameters, what would have been our recommendation? Again, if we had a 'larger' pool of healthy males and females with specified Z_{min} and Z_{max} for each covariate, would our recommendation be far better off? We propose to address both the problems below. We assume that the covariate parameters are the same for both males and females for each of the characteristics. In a way, we set $\gamma_{1,M} = \gamma_{1,F}$ and so on.

For the first problem, we may use the following selection methods.

Method 1: **Use of Heart Rate Data** We start with the above data set and arrange the data set separately for males and females both where the values of heart rate Z_1 are in the ascending order. Then select 10 persons (5 males and 5 females) from the smallest values of the Z_1-scores and 10 persons (5 males and 5 females) from the largest values of the Z_1-scores and denote this design by d_1. The data on covariates of the selected persons are shown in Table 9.30.

Table 9.26 Exercise data for healthy active males

Sl. no.	(VO_2 max)/Duration	Heart rate	Age	Height	Weight
1	0.0588	192	46	165	57
2	0.0627	190	25	193	95
3	0.0694	190	25	187	82
4	0.0670	174	31	191	84
5	0.0670	194	30	171	67
6	0.0662	168	36	177	78
7	0.0579	185	29	174	70
8	0.0618	200	27	185	76
9	0.0670	164	56	180	78
10	0.0652	175	47	180	80
11	0.0604	175	46	180	81
12	0.0609	162	55	180	79
13	0.0656	190	50	161	66
14	0.0688	175	52	174	76
15	0.0603	164	46	173	84
16	0.0547	156	60	169	69
17	0.0655	174	49	178	78
18	0.0545	166	54	181	101
19	0.0615	184	57	179	74
20	0.0695	160	50	170	66
21	0.0621	186	41	175	75
22	0.0713	175	58	173	79
23	0.0615	175	55	160	79
24	0.0579	175	46	164	65
25	0.0632	174	47	180	81
26	0.0500	174	56	183	100
27	0.0695	168	82	183	82
28	0.0549	164	48	181	77
29	0.0490	146	68	166	65
30	0.0606	156	54	177	80
31	0.0566	180	56	179	82
32	0.0638	164	50	182	87
33	0.0628	166	48	174	72
34	0.0626	184	56	176	75
35	0.0619	186	45	179	75
36	0.0619	174	45	179	79
37	0.0717	188	43	179	73
38	0.0413	180	54	180	75
39	0.0642	168	55	172	71
40	0.0702	174	41	187	84
41	0.0712	166	44	185	81
42	0.0753	174	41	186	83
43	0.0672	180	50	175	78
44	0.0746	182	42	176	85

Data source van Belle et al. (2004), p. 294

Table 9.27 Exercise data for healthy active females

Sl. no.	$(VO_2$ max)/Duration	Heart rate	Age	Height	Weight
1	0.0577	184	23	177	83
2	0.0611	183	21	163	52
3	0.0655	200	21	174	61
4	0.0583	170	42	160	50
5	0.0485	188	34	170	68
6	0.0398	190	43	171	68
7	0.0527	190	30	172	63
8	0.0552	180	49	157	53
9	0.0615	184	30	178	63
10	0.0566	162	57	161	63
11	0.0578	188	58	159	54
12	0.0597	170	51	162	55
13	0.0649	184	32	165	57
14	0.0594	175	42	170	53
15	0.0564	180	51	158	47
16	0.0590	200	46	161	60
17	0.0558	190	37	173	56
18	0.0488	170	50	161	62
19	0.0614	158	65	165	58
20	0.0552	186	40	154	69
21	0.0511	166	52	166	67
22	0.0495	170	40	160	58
23	0.0662	188	52	162	64
24	0.0480	190	47	161	72
25	0.0556	194	43	164	56
26	0.0540	190	48	176	82
27	0.0509	190	43	165	61
28	0.0520	188	45	166	62
29	0.0497	184	52	167	62
30	0.0560	170	52	168	62
31	0.0661	168	56	162	66
32	0.0634	175	56	159	56
33	0.0560	156	51	161	61
34	0.0525	184	44	154	56
35	0.0544	180	56	167	79
36	0.0555	184	40	165	56
37	0.0654	156	53	157	52
38	0.0508	194	52	161	65
39	0.0564	190	40	178	64
40	0.0587	188	55	162	61
41	0.0647	164	39	166	59
42	0.0575	185	57	168	68
43	0.0542	178	46	156	53

Data source van Belle et al. (2004), p. 341

Table 9.28 Coded data for males

Sl. no.	Heart rate	Age	Height	Weight
1	0.7037	−0.2632	−0.6970	−1.0000
2	0.6296	−1.0000	1.0000	0.7273
3	0.6296	−1.0000	0.6364	0.1364
4	0.0370	−0.7895	0.8788	0.2273
5	0.7778	−0.8246	−0.3333	−0.5455
6	−0.1852	−0.6140	0.0303	−0.0455
7	0.4444	−0.8596	−0.1515	−0.4091
8	1.0000	−0.9298	0.5152	−0.1364
9	−0.3333	0.0877	0.2121	−0.0455
10	0.0741	−0.2281	0.2121	0.0455
11	0.0741	−0.2632	0.2121	0.0909
12	−0.4074	0.0526	0.2121	0.0000
13	0.6296	−0.1228	−0.9394	−0.5909
14	0.0741	−0.0526	−0.1515	−0.1364
15	−0.3333	−0.2632	−0.2121	0.2273
16	−0.6296	0.2281	−0.4545	−0.4545
17	0.0370	−0.1579	0.0909	−0.0455
18	−0.2593	0.0175	0.2727	1.0000
19	0.4074	0.1228	0.1515	−0.2273
20	−0.4815	−0.1228	−0.3939	−0.5909
21	0.4815	−0.4386	−0.0909	−0.1818
22	0.0741	0.1579	−0.2121	0.0000
23	0.0741	0.0526	−1.0000	0.0000
24	0.0741	−0.2632	−0.7576	−0.6364
25	0.0370	−0.2281	0.2121	0.0909
26	0.0370	0.0877	0.3939	0.9545
27	−0.1852	1.0000	0.3939	0.1364
28	−0.3333	−0.1930	0.2727	−0.0909
29	−1.0000	0.5088	−0.6364	−0.6364
30	−0.6296	0.0175	0.0303	0.0455
31	0.2593	0.0877	0.1515	0.1364
32	−0.3333	−0.1228	0.3333	0.3636
33	−0.2593	−0.1930	−0.1515	−0.3182
34	0.4074	0.0877	−0.0303	−0.1818
35	0.4815	−0.2982	0.1515	−0.1818
36	0.0370	−0.2982	0.1515	0.0000
37	0.5556	−0.3684	0.1515	−0.2727
38	0.2593	0.0175	0.2121	−0.1818
39	−0.1852	0.0526	−0.2727	−0.3636
40	0.0370	−0.4386	0.6364	0.2273
41	−0.2593	−0.3333	0.5152	0.0909
42	0.0370	−0.4386	0.5758	0.1818
43	0.2593	−0.1228	−0.0909	−0.0455
44	0.3333	−0.4035	−0.0303	0.2727

Table 9.29 Coded data for females

Sl. no.	Heart rate	Age	Height	Weight
1	0.2727	−0.9091	0.9167	1.0000
2	0.2273	−1.0000	−0.2500	−0.7222
3	1.0000	−1.0000	0.6667	−0.2222
4	−0.3636	−0.0455	−0.5000	−0.8333
5	0.4545	−0.4091	0.3333	0.1667
6	0.5455	0.0000	0.4167	0.1667
7	0.5455	−0.5909	0.5000	−0.1111
8	0.0909	0.2727	−0.7500	−0.6667
9	0.2727	−0.5909	1.0000	−0.1111
10	−0.7273	0.6364	−0.4167	−0.1111
11	0.4545	0.6818	−0.5833	−0.6111
12	−0.3636	0.3636	−0.3333	−0.5556
13	0.2727	−0.5000	−0.0833	−0.4444
14	−0.1364	−0.0455	0.3333	−0.6667
15	0.0909	0.3636	−0.6667	−1.0000
16	1.0000	0.1364	−0.4167	−0.2778
17	0.5455	−0.2727	0.5833	−0.5000
18	−0.3636	0.3182	−0.4167	−0.1667
19	−0.9091	1.0000	−0.0833	−0.3889
20	0.3636	−0.1364	−1.0000	0.2222
21	−0.5455	0.4091	0.0000	0.1111
22	−0.3636	−0.1364	−0.5000	−0.3889
23	0.4545	0.4091	−0.3333	−0.0556
24	0.5455	0.1818	−0.4167	0.3889
25	0.7273	0.0000	−0.1667	−0.5000
26	0.5455	0.2273	0.8333	0.9444
27	0.5455	0.0000	−0.0833	−0.2222
28	0.4545	0.0909	0.0000	−0.1667
29	0.2727	0.4091	0.0833	−0.1667
30	−0.3636	0.4091	0.1667	−0.1667
31	−0.4545	0.5909	−0.3333	0.0556
32	−0.1364	0.5909	−0.5833	−0.5000
33	−1.0000	0.3636	−0.4167	−0.2222
34	0.2727	0.0455	−1.0000	−0.5000
35	0.0909	0.5909	0.0833	0.7778
36	0.2727	−0.1364	−0.0833	−0.5000
37	−1.0000	0.4545	−0.7500	−0.7222
38	0.7273	0.4091	−0.4167	0.0000
39	0.5455	−0.1364	1.0000	−0.0556
40	0.4545	0.5455	−0.3333	−0.2222
41	−0.6364	−0.1818	0.0000	−0.3333
42	0.3182	0.6364	0.1667	0.1667
43	0.0000	0.1364	−0.8333	−0.6667

Table 9.30 Choice of males and females based on heart rate data

Sl. no.	Heart rate	Age	Height	Weight	Sex	
1	−1.0000	0.5088	−0.6364	−0.6364	M	
2	−0.6296	0.2281	−0.4545	−0.4545	M	
3	−0.6296	0.0175	0.0303	0.0455	M	
4	−0.4815	−0.1228	−0.3939	−0.5909	M	
5	−0.4074	0.0526	0.2121	0.0000	M	
6	0.6296	−1.0000	1.0000	0.7273	M	
7	0.6296	−1.0000	0.6364	0.1364	M	
8	0.7037	−0.2632	−0.6970	−1.0000	M	
9	0.7778	−0.8246	−0.3333	−0.5455	M	
10	1.0000	−0.9298	0.5152	−0.1364	M	$= d_1$, say
1	−1.0000	0.3636	−0.4167	−0.2222	F	
2	−1.0000	0.4545	−0.7500	−0.7222	F	
3	−0.9091	1.0000	−0.0833	−0.3889	F	
4	−0.7273	0.6364	−0.4167	−0.1111	F	
5	−0.6364	−0.1818	0.0000	−0.3333	F	
6	0.5455	0.0000	0.4167	0.1667	F	
7	0.7273	0.0000	−0.1667	−0.5000	F	
8	0.7273	0.4091	−0.4167	0.0000	F	
9	1.0000	−1.0000	0.6667	−0.2222	F	
10	1.0000	0.1364	−0.4167	−0.2778	F	

The information matrix of γ under d_1 is

$$\mathbf{I}_{d_1}(\gamma) = \mathbf{I}_{Md_1}(\gamma) + \mathbf{I}_{Fd_1}(\gamma) = \begin{pmatrix} 5.0485 & -3.4438 & 1.8097 & 0.6033 \\ -3.4438 & 2.8317 & -2.0649 & -1.1962 \\ 1.8097 & -2.0649 & 3.0785 & 2.4386 \\ 0.6033 & -1.1962 & 2.4386 & 2.2241 \end{pmatrix}$$

$$+ \begin{pmatrix} 7.1085 & -2.5908 & 1.5553 & 0.7469 \\ -2.5908 & 2.6322 & -1.4470 & -0.1490 \\ 1.5553 & -1.4470 & 1.6592 & 0.4199 \\ 0.7469 & -0.1490 & 0.4199 & 0.5682 \end{pmatrix} = \begin{pmatrix} 12.1570 & -6.0346 & 3.3650 & 1.3502 \\ -6.0346 & 5.4639 & -3.5119 & -1.3452 \\ 3.3650 & -3.5119 & 4.7377 & 2.8585 \\ 1.3502 & -1.3452 & 2.8585 & 2.7923 \end{pmatrix},$$

$det(\mathbf{I}_{d_1}) = 61.6096$, $20^{-4} \times det(\mathbf{I}_{d_1}) = 0.0004$.

Method 2: **Use of Age Data** Similarly as in Selection 1, we select 20 persons (10 males and 10 females) by arranging the available reported information on age Z_2 in ascending order. The selected data set is shown in Table 9.31.

Table 9.31 Choice of males and females based on reported data on age

Sl. no.	Heart rate	Age	Height	Weight	Sex	
1	0.6296	−1.0000	1.0000	0.7273	M	
2	0.6296	−1.0000	0.6364	0.1364	M	
3	1.0000	−0.9298	0.5152	−0.1364	M	
4	0.4444	−0.8596	−0.1515	−0.4091	M	
5	0.7778	−0.8246	−0.3333	−0.5455	M	
6	0.4074	0.1228	0.1515	−0.2273	M	
7	0.0741	0.1579	−0.2121	0.0000	M	
8	−0.6296	0.2281	−0.4545	−0.4545	M	
9	−1.0000	0.5088	−0.6364	−0.6364	M	
10	−0.1852	1.0000	0.3939	0.1364	M	$= d_2$, say
1	0.2273	−1.0000	−0.2500	−0.7222	F	
2	1.0000	−1.0000	0.6667	−0.2222	F	
3	0.2727	−0.9091	0.9167	1.0000	F	
4	0.5455	−0.5909	0.5000	−0.1111	F	
5	0.2727	−0.5909	1.0000	−0.1111	F	
6	−0.4545	0.5909	−0.3333	0.0556	F	
7	0.3182	0.6364	0.1667	0.1667	F	
8	−0.7273	0.6364	−0.4167	−0.1111	F	
9	0.4545	0.6818	−0.5833	−0.6111	F	
10	−0.9091	1.0000	−0.0833	−0.3889	F	

The information matrix of γ under d_2 is

$$\mathbf{I}_{d_2}(\gamma) = \mathbf{I}_{Md_2}(\gamma) + \mathbf{I}_{Fd_2}(\gamma) = \begin{pmatrix} 3.7360 & -3.4305 & 1.9192 & 0.9087 \\ -3.4305 & 4.9602 & -1.5228 & -0.6202 \\ 1.9192 & -1.5228 & 2.5565 & 1.7466 \\ 0.9087 & -0.6202 & 1.7466 & 1.5144 \end{pmatrix}$$

$$+ \begin{pmatrix} 3.2678 & -3.0323 & 1.5652 & 0.0854 \\ -3.0323 & 6.1191 & -2.8871 & -0.6283 \\ 1.5652 & -2.8871 & 3.0063 & 1.3940 \\ 0.0854 & -0.6283 & 1.3940 & 2.0522 \end{pmatrix} = \begin{pmatrix} 7.0038 & -6.4628 & 3.4844 & 0.9941 \\ -6.4628 & 11.0793 & -4.4099 & -1.2485 \\ 3.4844 & -4.4099 & 5.5628 & 3.1406 \\ 0.9941 & -1.2485 & 3.1406 & 3.5666 \end{pmatrix},$$

$det(\mathbf{I}_{d_2}) = 196.608,\ 20^{-4} \times det(\mathbf{I}_{d_2}) = 0.0012.$

Method 3: **Use of Height Data** Similarly for height Z_3, we show the arranged data in Table 9.32.

Table 9.32 Choice of males and females based on height data

Sl. no.	Heart rate	Age	Height	Weight	Sex	
1	0.0741	0.0526	−1.0000	0.0000	M	
2	0.6296	−0.1228	−0.9394	−0.5909	M	
3	0.0741	−0.2632	−0.7576	−0.6364	M	
4	0.7037	−0.2632	−0.6970	−1.0000	M	
5	0.0370	−0.4386	0.5758	0.1818	M	
6	−1.0000	0.5088	−0.6364	−0.6364	M	
7	0.6296	−1.0000	0.6364	0.1364	M	
8	0.0370	−0.4386	0.6364	0.2273	M	
9	0.0370	−0.7895	0.8788	0.2273	M	
10	0.6296	−1.0000	1.0000	0.7273	M	$= d_3$, say
1	0.3636	0.1364	−1.0000	0.2222	F	
2	0.2727	0.0455	−1.0000	−0.5000	F	
3	0.0000	0.1364	−0.8333	−0.6667	F	
4	0.0909	0.2727	−0.7500	−0.6667	F	
5	−1.0000	0.4545	−0.7500	−0.7222	F	
6	1.0000	−1.0000	0.6667	−0.2222	F	
7	0.5455	0.2273	0.8333	0.9444	F	
8	0.2727	−0.9091	0.9167	1.0000	F	
9	0.2727	−0.5909	1.0000	−0.1111	F	
10	0.5455	−0.1364	1.0000	−0.0556	F	

The information matrix of γ under d_3 is

$$\mathbf{I}_{d_3}(\gamma) = \mathbf{I}_{Md_3}(\gamma) + \mathbf{I}_{Fd_3}(\gamma) = \begin{pmatrix} 2.3566 & -1.4126 & 0.5880 & 0.3333 \\ -1.4126 & 2.0137 & -2.8538 & -1.5551 \\ 0.5880 & -2.8538 & 6.2519 & 3.3611 \\ 0.3333 & -1.5551 & 3.3611 & 2.6572 \end{pmatrix}$$

$$+ \begin{pmatrix} 2.4000 & -1.4396 & 2.2153 & 1.2949 \\ -1.4396 & 2.2983 & -2.5925 & -1.1803 \\ 2.2153 & -2.5925 & 7.7979 & 3.2704 \\ 1.2949 & -1.1803 & 3.2704 & 3.6061 \end{pmatrix} = \begin{pmatrix} 4.7566 & -2.8522 & 2.8033 & 1.6282 \\ -2.8522 & 4.3120 & -5.4463 & -2.7354 \\ 2.8033 & -5.4463 & 14.0498 & 6.6315 \\ 1.6282 & -2.7354 & 6.6315 & 6.2633 \end{pmatrix}$$

$det(\mathbf{I}_{d_3}) = 267.4351,\ 20^{-4} \times det(\mathbf{I}_{d_3}) = 0.0017.$

Method 4: **Use of Weight Data** Lastly for weight Z_4, we show the arranged data in Table 9.33.

Table 9.33 Choice of males and females based on reported weight data

Sl. no.	Heart rate	Age	Height	Weight	Sex	
1	0.7037	−0.2632	−0.6970	−1.0000	M	
2	0.0741	−0.2632	−0.7576	−0.6364	M	
3	0.6296	−0.1228	−0.9394	−0.5909	M	
4	−0.4815	−0.1228	−0.3939	−0.5909	M	
5	−1.0000	0.5088	−0.6364	−0.6364	M	
6	0.3333	−0.4035	−0.0303	0.2727	M	
7	−0.3333	−0.1228	0.3333	0.3636	M	
8	0.6296	−1.0000	1.0000	0.7273	M	
9	0.0370	0.0877	0.3939	0.9545	M	
10	−0.2593	0.0175	0.2727	1.0000	M	$= d_4$, say
1	0.0909	0.3636	−0.6667	−1.0000	F	
2	−0.3636	−0.0455	−0.5000	−0.8333	F	
3	0.2273	−1.0000	−0.2500	−0.7222	F	
4	−1.0000	0.4545	−0.7500	−0.7222	F	
5	0.0000	0.1364	−0.8333	−0.6667	F	
6	0.3636	−0.1364	−1.0000	0.2222	F	
7	0.5455	0.1818	−0.4167	0.3889	F	
8	0.0909	0.5909	0.0833	0.7778	F	
9	0.5455	0.2273	0.8333	0.9444	F	
10	0.2727	−0.9091	0.9167	1.0000	F	

The information matrix of γ under d_4 is

$$\mathbf{I}_{d_4}(\gamma) = \mathbf{I}_{Md_4}(\gamma) + \mathbf{I}_{Fd_4}(\gamma) = \begin{pmatrix} 2.8050 & -1.4000 & 0.1887 & 0.0062 \\ -1.4000 & 1.3298 & -1.0116 & -0.5517 \\ 0.1887 & -1.0116 & 3.6323 & 3.8410 \\ 0.0062 & -0.5517 & 3.8410 & 5.1531 \end{pmatrix}$$

$$+ \begin{pmatrix} 1.9424 & -0.6423 & 1.1352 & 1.9689 \\ -0.6423 & 2.6365 & -0.9936 & -0.2254 \\ 1.1352 & -0.9936 & 4.0618 & 3.5874 \\ 1.9689 & -0.2254 & 3.5874 & 5.8422 \end{pmatrix} = \begin{pmatrix} 4.7474 & -2.0423 & 1.3239 & 1.9751 \\ -2.0423 & 3.9663 & -2.0052 & -0.7771 \\ 1.3239 & -2.0052 & 7.6941 & 7.4284 \\ 1.9751 & -0.7771 & 7.4284 & 10.9953 \end{pmatrix}$$

$det(\mathbf{I}_{d_4}) = 292.4578,\ 20^{-4} \times det(\mathbf{I}_{d_4}) = 0.00183.$

Selection Method Based on Principal Component

Selection of 10 Males

A PCA is concerned with explaining the variance–covariance structure of a set of responses through a few linear combinations of these responses. In this study, only two eigenvalues were larger (14.6743 and 11.3163, respectively); so two components

Table 9.34 Details for PCA for Male Data

	Comp.1	Comp.2	Comp.3	Comp.4
Standard deviation	0.5594	0.4865	0.2595	0.2102
Proportion of variance	0.4733	0.3580	0.1018	0.0668
Cumulative proportion	0.4733	0.8313	0.9332	1.0000

were extracted, based on Kaiser principle (cf. Kaiser 1960). The first component (PC_{M1}) accounts for about 47.33 % and the second component (PC_{M2}) accounts for about 35.80 % of the total variance in the data set. Therefore the first two components account for 83.13 % of the variance (vide Table 9.34).

The eigenvalues of $\mathbf{Z}'_M\mathbf{Z}_M$ are $14.6743 = \lambda_{M1}$, $11.3163 = \lambda_{M2}$, $3.4597 = \lambda_{M3}$, $1.9446 = \lambda_{M4}$; where $\mathbf{Z}_M^{44\times 4} = (z_{Mj}^{(l)})$ is design matrix for covariate parameters for male data and the corresponding eigenvectors $(\xi_{M1}, \xi_{M2}, \xi_{M3}, \xi_{M4})$ are:

$$\begin{pmatrix} \xi_{M1} & \xi_{M2} & \xi_{M3} & \xi_{M4} \\ -0.4520 & 0.5225 & 0.7182 & -0.0828 \\ 0.5955 & -0.3808 & 0.6107 & -0.3569 \\ -0.5887 & -0.4745 & -0.0998 & -0.6468 \\ -0.3074 & -0.5973 & 0.3182 & 0.6689 \end{pmatrix}$$

Therefore

$$\begin{aligned} PC_{M1} &= -0.4520Z_{M1} + 0.5955Z_{M2} - 0.5887Z_{M3} - 0.3074Z_{M4} \\ PC_{M2} &= 0.5225Z_{M1} - 0.3808Z_{M2} - 0.4745Z_{M3} - 0.5973Z_{M4} \end{aligned}$$

Now we take a convex combination of PC_{M1} and PC_{M2} and get a new score and denote it by $P_M = \frac{\lambda_{M1}}{\lambda_{M1}+\lambda_{M2}} PC_{M1} + \frac{\lambda_{M2}}{\lambda_{M1}+\lambda_{M2}} PC_{M2} = -0.0277Z_{M1} + 0.1704Z_{M2} - 0.5390Z_{M3} - 0.4336Z_{M4}$. Now we arrange P_M score in ascending order and select 10 males where 5 are from the top and 5 are from the bottom. The data of the selected males are shown in Table 9.35.

Selection of 10 Females

A similar PCA is carried out for female data. In this study, three eigenvalues were larger (21.3051, 10.9512 and 7.2187, respectively); so three components were extracted, based on Kaiser principle. The first component (PC_{F1}) accounts for about 51.29 %, the second component (PC_{F2}) accounts for about 22.92 % and the third component (PC_{F3}) accounts for about 17.65 % of the total variance in the data set. Therefore, the first three components account for 91.86 % of the variance (vide Table 9.36).

The eigenvalues of $\mathbf{Z}'_F\mathbf{Z}_F$ are $21.3051 = \lambda_{F1}$, $10.9512 = \lambda_{F2}$, $7.2187 = \lambda_{F3}$, $3.8364 = \lambda_{F4}$; where $\mathbf{Z}_F^{44\times 4} = (z_{Fj}^{(l)})$ is design matrix for covariate parameters for

Table 9.35 Male data

Heart rate	Age	Height	Weight	
0.6296	−1.0000	1.0000	0.7273	
0.6296	−1.0000	0.6364	0.1364	
0.0370	−0.7895	0.8788	0.2273	
−0.2593	0.0175	0.2727	1.0000	
0.0370	0.0877	0.3939	0.9545	$= d_{M5}$, say
0.7037	−0.2632	−0.6970	−1.0000	
0.6296	−0.1228	−0.9394	−0.5909	
0.0741	0.0526	−1.0000	0.0000	
0.0741	−0.2632	−0.7576	−0.6364	
−1.0000	0.5088	−0.6364	−0.6364	

Table 9.36 Details of PCA for Female Data

	Comp.1	Comp.2	Comp.3	Comp.4
Standard deviation	0.6910	0.4618	0.4054	0.2753
Proportion of variance	0.5129	0.2292	0.1765	0.0814
Cumulative proportion	0.5129	0.7421	0.9186	1.0000

female data and the corresponding eigenvectors $(\xi_{F1}, \xi_{F2}, \xi_{F3}, \xi_{F4})$ are:

$$\begin{pmatrix} \xi_{F1} & \xi_{F2} & \xi_{F3} & \xi_{F4} \\ 0.4400 & 0.7258 & -0.5234 & -0.0753 \\ -0.4615 & -0.2532 & -0.6623 & -0.5333 \\ 0.6504 & -0.3185 & 0.2000 & -0.6599 \\ 0.4128 & -0.5547 & -0.4975 & 0.5239 \end{pmatrix}$$

Therefore

$$\begin{aligned} \mathrm{PC}_{F1} &= 0.4400\mathrm{Z}_{F1} - 0.4615\mathrm{Z}_{F2} + 0.6504\mathrm{Z}_{F3} + 0.4128\mathrm{Z}_{F4} \\ \mathrm{PC}_{F2} &= 0.7258\mathrm{Z}_{F1} - 0.2532\mathrm{Z}_{F2} - 0.3185\mathrm{Z}_{F3} - 0.5547\mathrm{Z}_{F4} \\ \mathrm{PC}_{F3} &= -0.5234\mathrm{Z}_{F1} - 0.6623\mathrm{Z}_{F2} + 0.2000\mathrm{Z}_{F3} - 0.4975\mathrm{Z}_{F4} \end{aligned}$$

Now we take convex combination of PC_{F1}, PC_{F2} and PC_{F3} and get a new score and denote it by $\mathrm{P}_F = \frac{\lambda_{F1}}{\lambda_{F1}+\lambda_{F2}+\lambda_{F3}}\mathrm{PC}_{F1} + \frac{\lambda_{F2}}{\lambda_{F1}+\lambda_{F2}+\lambda_{F3}}\mathrm{PC}_{F2} + \frac{\lambda_{F3}}{\lambda_{F1}+\lambda_{F2}+\lambda_{F3}}$ $\mathrm{PC}_{F3} = 0.3431\mathrm{Z}_{F1} - 0.4404\mathrm{Z}_{F2} + 0.2992\mathrm{Z}_{F3} - 0.0220\mathrm{Z}_{F4}$. Now we arrange P_F score in ascending order and select 10 females where 5 are from top and 5 are from bottom. The data of the selected females are shown in Table 9.37.

Table 9.37 Female data

Heart rate	Age	Height	Weight	
−0.7273	0.6364	−0.4167	−0.1111	
−0.9091	1.0000	−0.0833	−0.3889	
−0.4545	0.5909	−0.3333	0.0556	
−1.0000	0.3636	−0.4167	−0.2222	
−1.0000	0.4545	−0.7500	−0.7222	$= d_{F5}$, say
0.2727	−0.9091	0.9167	1.0000	
1.0000	−1.0000	0.6667	−0.2222	
0.5455	−0.5909	0.5000	−0.1111	
0.2727	−0.5909	1.0000	−0.1111	
0.5455	−0.1364	1.0000	−0.0556	

Table 9.38 Selection of 10 males based on total scoring

Heart rate	Age	Height	Weight	
0.7037	−0.2632	−0.6970	−1.0000	
−0.6296	0.2281	−0.4545	−0.4545	
−0.4815	−0.1228	−0.3939	−0.5909	
0.0741	−0.2632	−0.7576	−0.6364	
−1.0000	0.5088	−0.6364	−0.6364	$= d_{M6}$, say
0.6296	−1.0000	1.0000	0.7273	
−0.2593	0.0175	0.2727	1.0000	
0.0370	0.0877	0.3939	0.9545	
−0.1852	1.0000	0.3939	0.1364	
0.2593	0.0877	0.1515	0.1364	

Therefore, $d_5 = \begin{pmatrix} d_{M5} \\ d_{F5} \end{pmatrix}$.

Therefore $det(\mathbf{I}_{d_5}(\gamma)) = det(\mathbf{I}_{Md_5}(\gamma) + \mathbf{I}_{Fd_5}(\gamma)) = 255.4019$ and $20^{-4} \times det(\mathbf{I}_{d_5}(\gamma)) = 0.00160$.

Selection Method Based on Total Scoring

We may also adopt one more ad hoc method of selection. This time we select males or females by use of total Z-scores from all the covariates. For males, we base on the $Z_M = Z_{M1} + Z_{M2} + Z_{M3} + Z_{M4}$ values. We arrange Z_M in ascending order and select 10 males where 5 are from the top and 5 are from the bottom. The data of the selected males are shown in Table 9.38.

Likewise, we select 10 females based on $Z_F = Z_{F1} + Z_{F2} + Z_{F3} + Z_{F4}$ values. We arrange Z_F in ascending order and select 10 females where 5 are from the top and 5 are from the bottom. The data of the selected females are shown in Table 9.39.

Table 9.39 Selection of 10 females based on total scoring

Heart rate	Age	Height	Weight	
0.2273	−1.0000	−0.2500	−0.7222	
−0.3636	−0.0455	−0.5000	−0.8333	
−0.3636	−0.1364	−0.5000	−0.3889	
−1.0000	0.4545	−0.7500	−0.7222	
0.0000	0.1364	−0.8333	−0.6667	$= d_{F6}$, say
0.2727	−0.9091	0.9167	1.0000	
0.5455	0.2273	0.8333	0.9444	
0.0909	0.5909	0.0833	0.7778	
0.5455	−0.1364	1.0000	−0.0556	
0.3182	0.6364	0.1667	0.1667	

Therefore, $d_6 = \begin{pmatrix} d_{M6} \\ d_{F6} \end{pmatrix}$. Consequently, $det(\mathbf{I}_{d_6}(\gamma)) = det(\mathbf{I}_{Md_6}(\gamma) + \mathbf{I}_{Fd_6}(\gamma)) = 294.0333$ and $20^{-4} \times det(\mathbf{I}_{d_6}(\gamma)) = 0.00184$.

At the end, we conclude that $d_1 \prec d_2 \prec d_5 \prec d_3 \prec d_4 \prec d_6$, since $20^{-4} \times det(\mathbf{I}_{d_1}(\gamma)) = 0.0004$, $20^{-4} \times det(\mathbf{I}_{d_2}(\gamma)) = 0.0012$, $20^{-4} \times det(\mathbf{I}_{d_3}(\gamma)) = 0.0017$, $20^{-4} \times det(\mathbf{I}_{d_4}(\gamma)) = 0.00183$, $20^{-4} \times det(\mathbf{I}_{d_5}(\gamma)) = 0.00160$, $20^{-4} \times det(\mathbf{I}_{d_6}(\gamma)) = 0.00184$ and $87^{-4} \times det(\mathbf{I}_d(\gamma)) = 0.0006$, where d is a design based on whole data set.

It thus transpires that the criterion of selection (d_6) based on total scoring of the participating males or females fares much better than the rest for efficient estimation of the covariate parameters. It may be noted that implicitly we have exploited our understanding of OCDs in carrying out this exercise.

Example 9.2.3 The data in Table 9.40 are from a study to evaluate the effect of roll gap and variety of wheat on the amount of flour produced (percent of total wheat ground) during a run of a pilot flour mill (cf. Milliken and Johnson (2001), p. 406). Three batches of wheat from each of three varieties were used in the study with enough raw material for all where a single batch of wheat was used for all four roll gap setting. The experiment was conducted on three different days.

For each variety type, there are three batches of wheat with fixed but varying moisture contents and the batches have been assigned across the 3 days of the experiment as shown in Table 9.40.

Our purpose is to revisit this data set and examine the possibility of improving the design for increased information on the covariate parameter. We will examine the possibility of re-allocation of the batches of wheats across different days of the experiment. Towards this, we start with a model description.

Model for 3 way layout (Day × Variety × Roll Gap):

$$y_{ijl} = \mu + \beta_i + \tau_j + \delta_l + z_{ij}\gamma + e_{ijl}; \quad i = 1, 2, \ldots, r; \; j = 1, 2, \ldots, p; \; l = 1, 2, \ldots, q; \tag{9.2.1}$$

Table 9.40 Data for amount of flour milled (percent) during the first break of a flour mill operation from three varieties of wheat using four roll gaps

Day	Variety	Moist	Roll gap			
			0.02in	0.04in	0.06in	0.08in
1	A	12.4	18.3	14.6	12.2	9.0
1	B	12.8	18.8	14.9	11.7	8.3
1	C	12.1	18.7	15.5	12.7	9.2
2	A	14.4	19.1	15.4	12.4	8.7
2	B	12.4	17.5	14.4	11.3	8.3
2	C	13.2	18.9	15.4	12.5	8.2
3	A	13.1	18.2	15.0	12.1	8.4
3	B	14.0	20.4	16.2	12.9	9.1
3	C	13.4	19.7	16.9	13.4	9.5

Source Milliken and Johnson (2001). Analysis of Messy Data: Volume III: Analysis of covariance, Chapman and Hall/CRC

where μ is general effect, β_i is effect due to ith day, τ_j is effect due to jth variety, δ_l is effect due to lth roll gap, γ is the covariate effect and z_{ij} is the covariate value corresponding to ith day and jth variety. Without loss of generality, we assume $|z_{ij}| \leq 1$ for all $i,\ j$. Note that z_{ij}'s are to be computed by using the standard transformation of the original covariate values x_{ij}'s and it is given by $z_{ij} = \frac{2(x_{ij}-x_{min})}{x_{max}-x_{min}} - 1$. These x_{min} and x_{max} are to be based on the entire collection of x_{ij} values. In (9.2.1), e_{ijl}s are random errors with

$$\begin{aligned} V(e_{ijl}) &= \sigma^2 && \forall i, j, l \\ Cov(e_{ijl}, e_{ijl'}) &= \rho\sigma^2 && \forall i, j, l \neq l' \\ Cov(e_{ijl}, e_{ij'l}) &= 0 && \forall i, j \neq j', l \\ Cov(e_{ijl}, e_{i'jl}) &= 0 && \forall i \neq i', j, l \end{aligned}$$

Therefore, $Disp(\mathbf{y}_{ij}) = \sigma^2((1-\rho)\mathbf{I}_q + \rho\mathbf{J}_q),\ \ \forall i, j$ where $\mathbf{I}_q$ is an identity matrix of order q and J_q is a matrix of order q with all elements unity.

Since for each day $\times$ variety combination, there are four correlated observations, observational contrasts would provide information on the contrasts involving effects of roll gaps, eliminating the effects of all other parameters, including the covariate parameter, viz. the effect of moisture content. The average of the four observations, on the other hand, will provide information on all the three components, viz. day effect, variety effect and covariate effect, as in a standard RBD set-up.

We will start from there and proceed to examine the possibilities of rearrangement of the moist batches of wheat for extracting maximum information on the covariate effect.

In a general RBD set-up involving q correlated observations, we would proceed as follows:

$$\mathbf{L}=\begin{pmatrix} \frac{1}{\sqrt{q}} & \frac{1}{\sqrt{q}} & \frac{1}{\sqrt{q}} & \frac{1}{\sqrt{q}} & \dots & \frac{1}{\sqrt{q}} & \frac{1}{\sqrt{q}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{q}} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{q(q-1)}} & \frac{1}{\sqrt{q(q-1)}} & \frac{1}{\sqrt{q(q-1)}} & \frac{1}{\sqrt{q(q-1)}} & \dots & \frac{1}{\sqrt{q(q-1)}} & -\frac{q-1}{\sqrt{q(q-1)}} \end{pmatrix}$$

such that $\mathbf{LL}' = \mathbf{I}_q$.

Then

$$Disp(\mathbf{Ly}_{ij}) = \sigma^2\mathbf{L}((1-\rho)\mathbf{I}_q + \rho\mathbf{J}_q)\mathbf{L}' = \sigma^2\begin{pmatrix} 1+(q-1)\rho & 0 & 0 & \dots & 0 \\ 0 & 1-\rho & 0 & \dots & 0 \\ 0 & 0 & 1-\rho & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1-\rho \end{pmatrix}$$

Now we take

$$\mathbf{M}=\begin{pmatrix} \frac{1}{\sqrt{1+(q-1)\rho}} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{1-\rho}} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-\rho}} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{\sqrt{1-\rho}} \end{pmatrix}$$

Then $Disp(\mathbf{MLy}_{ij}) = \sigma^2\mathbf{I}_q$. Let $\mathbf{MLy}_{ij} = \mathbf{y}^*_{ij}$. Now our Model (9.2.1) can be written as:

$$\frac{1}{\sqrt{q(1+(q-1)\rho)}}\sum_{l=1}^{q} y_{ijl} = \frac{1}{\sqrt{q(1+(q-1)\rho)}}\left(q(\mu+\beta_i+\tau_j+z_{ij}\gamma)+\sum_{l=1}^{q}\delta_l+\sum_{l=1}^{q}e_{ijl}\right)$$
$$\frac{1}{\sqrt{2(1-\rho)}}(y_{ij1}-y_{ij2}) = \frac{1}{\sqrt{2(1-\rho)}}(\delta_1-\delta_2)+\frac{1}{\sqrt{2(1-\rho)}}(e_{ij1}-e_{ij2})$$
$$\frac{1}{\sqrt{6(1-\rho)}}(y_{ij1}+y_{ij2}-2y_{ij3}) = \frac{1}{\sqrt{6(1-\rho)}}(\delta_1+\delta_2-2\delta_3)+\frac{1}{\sqrt{6(1-\rho)}}(e_{ij1}+e_{ij2}-2e_{ij3})$$
$$\vdots \quad \vdots$$

$$\frac{1}{\sqrt{q(q-1)(1-\rho)}}(y_{ij1}+y_{ij2}+\cdots+y_{ijq-1}-(q-1)y_{ijq})$$

$$= \frac{1}{\sqrt{q(q-1)(1-\rho)}}\left((\delta_1+\delta_2+\cdots+\delta_{q-1}-(q-1)\delta_q)+(e_{ij1}+e_{ij2}+\cdots+e_{ijq-1}-(q-1)e_{ijq})\right) \tag{9.2.2}$$

As explained above, the only source of information on the covariate parameter γ is the set of rp subtotals or means on Day $\times$ Treatment with the roll gap observations averaged out. Hence the model is:

$$\bar{y}_{ij0} = \mu^* + \beta_i + \tau_j + z_{ij}\gamma + \bar{e}_{ij0} \tag{9.2.3}$$

where $\bar{y}_{ij0} = \frac{1}{q}\sum_{l=1}^{q} y_{ijl}$, $\bar{e}_{ij0} = \frac{1}{q}\sum_{l=1}^{q} e_{ijl}$, $\mu^* = \mu + \frac{1}{q}\sum_{l=1}^{q}\delta_l$ with $V(\bar{y}_{ij0}) = \frac{(1+(q-1)\rho)}{q}\sigma^2$ and $\bar{y}_{ij0}$'s are uncorrelated. This is simply covariate model for RBD with single covariate. The information of γ is

$$I(\gamma) = \sum_{i=1}^{r}\sum_{j=1}^{p} z_{ij}^2 - \frac{1}{p}\sum_{i=1}^{r} R_i^2 - \frac{1}{r}\sum_{j=1}^{p} C_j^2 + \frac{G^2}{rp}, \tag{9.2.4}$$

where $R_i = \sum_{j=1}^{p} z_{ij}$, $i = 1, 2, \ldots, r$ and $C_j = \sum_{i=1}^{r} z_{ij}$, $j = 1, 2, \ldots, p$, and $G = \sum_{i=1}^{r}\sum_{j=1}^{p} z_{ij} = \sum_{i=1}^{r} R_i = \sum_{j=1}^{p} C_j$. Now we transform the moisture values of Table 9.40 into z-values and the coded covariate values are shown in Table 9.41.

Information of $\boldsymbol{\theta} = (\mu, \boldsymbol{\beta}', \boldsymbol{\tau}', \gamma)$ is

$$I_{d_1}\boldsymbol{\theta}) = \begin{pmatrix} 9 & 3 & 3 & 3 & 3 & 3 & 3 & -1.261 \\ 3 & 3 & 0 & 0 & 1 & 1 & 1 & -2.130 \\ 3 & 0 & 3 & 0 & 1 & 1 & 1 & 0.217 \\ 3 & 0 & 0 & 3 & 1 & 1 & 1 & 0.652 \\ 3 & 1 & 1 & 1 & 3 & 0 & 0 & 0.13 \\ 3 & 1 & 1 & 1 & 0 & 3 & 0 & -0.478 \\ 3 & 1 & 1 & 1 & 0 & 0 & 3 & -0.913 \\ -1.261 & -2.13 & 0.217 & 0.652 & 0.13 & -0.478 & -0.913 & 3.707 \end{pmatrix}$$

$I_{d_1}(\gamma) = 1.8533.$

Table 9.41 Coded z-values

Day	Variety	Moist	
1	A	−0.7391	
1	B	−0.3913	
1	C	−1.0000	
2	A	1.0000	$= d_1$, say
2	B	−0.7391	
2	C	−0.0435	
3	A	−0.1304	
3	B	0.6522	
3	C	0.1304	

Now we consider the design given in Table 9.42.
Information of $\boldsymbol{\theta} = \left(\mu, \boldsymbol{\beta}', \boldsymbol{\tau}', \gamma\right)$ is

$$I_{d_2}\boldsymbol{\theta}) = \begin{pmatrix} 9 & 3 & 3 & 3 & 3 & 3 & 3 & -3 \\ 3 & 3 & 0 & 0 & 1 & 1 & 1 & -1 \\ 3 & 0 & 3 & 0 & 1 & 1 & 1 & -1 \\ 3 & 0 & 0 & 3 & 1 & 1 & 1 & -1 \\ 3 & 1 & 1 & 1 & 3 & 0 & 0 & -1 \\ 3 & 1 & 1 & 1 & 0 & 3 & 0 & -1 \\ 3 & 1 & 1 & 1 & 0 & 0 & 3 & -1 \\ -3 & -1 & -1 & -1 & -1 & -1 & -1 & 9 \end{pmatrix}$$

$I_{d_2}(\gamma) = 8$. The D-efficiency of d_1 with respect to d_2 is 23.1674 %. Hence if we use d_2 instead of d_1, then we improve the information of γ a lot. The moisture values of d_2 are given in Table 9.43.

Here we conclude that the design which is optimum under uncorrelated model is also optimum under correlated model.

Table 9.42 Coded z-values

Day	Variety	Moist	
1	A	−1	
1	B	1	
1	C	−1	
2	A	−1	$= d_2$, say
2	B	−1	
2	C	1	
3	A	1	
3	B	−1	
3	C	−1	

Table 9.43 Uncoded z-values

Day	Variety	Moist	
1	A	12.1	
1	B	14.4	
1	C	12.1	
2	A	12.1	$= d_2$, say
2	B	12.1	
2	C	14.4	
3	A	14.4	
3	B	12.1	
3	C	12.1	

Remark 9.2.1 The reader may note that the allocation design d_2 relates to a hypothetical scenario under the assumption that there are indeed batches of wheat varieties available with the stipulated moisture contents coded ± 1's. The uncoded values are shown in Table 9.43. Naturally, d_2 does not address the reality which is governed by the given values of the moisture contents of nine bundles of the wheat. Therefore, we turn back to the original collection of z-values as in d_1 and try to suggest improvements over the given allocation.

Now we want to improve d_1 by reallocating the batches of wheat of each type across the 3 days so that SSDays $(=\frac{1}{p}\sum_{i=1}^{r} R_i^2 - \frac{G^2}{rp})$ is the least i.e. R_i's are as equal as possible. So we have three treatments A, B, C and each has three bundles of input material with naturally given day specific moisture contents. Consider Treatment A so that we have $3! = 6$ ways of distributing the bundles across the days. Likewise $3! = 6$ for those of B and $3! = 6$ for those of C and all are independent. So in effect given the experimental material and no further input, there are $6 \times 6 \times 6 = 216$ ways of distribution of the bundles across 3 days and the given design is just one of these 216 possible allocations. Now we rewrite d_1 as in Table 9.44 and follow certain steps for reallocation of coded z-values in each column.

Step 1: We select the first column and permute the covariate values keeping the other columns fixed. Next we compute the information of γ under each permutation. Then we choose the design in which the information of γ will be a maximum. We denote the design by d_3 (Table 9.45).

The information of γ is $I_{d_3}(\gamma) = 3.1942$.

Step 2: Now replace the second column by $(-0.3913, 0.6522, -0.7391)$ and keep the third column intact, and then permute the first column in all possible ways. Then we get the design d_4 which improves the information of γ and is equal to 3.2849 (Table 9.46).

Step 3: Now replace the second column by $(-0.7391, -0.3913, 0.6522)$ and keep the third column intact, and then permute the first column in all possible ways. But

Table 9.44 d_1

Day	A	B	C	Total
1	−0.7391	−0.3913	−1.0000	−2.1304
2	1.0000	−0.7391	−0.0435	0.2174
3	−0.1304	0.6522	0.1304	0.6522

Table 9.45 d_3

Day	A	B	C	Total
1	1.0000	−0.3913	−1.0000	−0.3913
2	−0.1304	−0.7391	−0.0435	−0.913
3	−0.7391	0.6522	0.1304	0.0435

Table 9.46 d_4

Day	A	B	C	Total
1	1.0000	−0.3913	−1.0000	−0.3913
2	−0.7391	0.6522	−0.0435	−0.1304
3	−0.1304	−0.7391	0.1304	−0.7391

Table 9.47 d_5

Day	A	B	C	Total
1	1.0000	−0.7391	−1.0000	−0.7391
2	−0.7391	0.6522	−0.0435	−0.1304
3	−0.1304	−0.3913	0.1304	−0.3913

Table 9.48 d_6

Day	A	B	C	Total
1	1.0000	−0.3913	−1.0000	−0.3913
2	−0.1304	−0.7391	0.1304	−0.7391
3	−0.7391	0.6522	−0.0435	−0.1304

Table 9.49 d_7

Day	A	B	C	Total
1	1.0000	−0.7391	−1.0000	−0.7391
2	−0.1304	−0.3913	0.1304	−0.3913
3	−0.7391	0.6522	−0.0435	−0.1304

we cannot improve the information of γ than d_4 (since here we get the design where the information of γ is 3.2345 and the design is best among the designs obtained from all six permutations of the first column).

Step 4: Replace the second column by (−0.7391, 0.6522, −0.3913) and keep the third column intact, and then permute the first column in all possible ways. But we cannot improve the information of γ than d_4 (since here we get the design where the information of γ is 3.2849 and the design is best among the designs obtained from all six permutations of the first column). We denote this design by d_5 (Table 9.47).

Step 5: Replace the second column by (0.6522, −0.3913, −0.7391) and keep the third column intact, and then permute the first column in all possible ways. But we cannot improve the information of γ than d_4.

Now replace the third column of d_1 by (−1.000, 0.1304, −0.0435) and by following all the steps mentioned above we get the designs d_6 and d_7 where the information of γ is the same as d_4 (Tables 9.48 and 9.49).

Table 9.50 d_8

Day	A	B	C	Total
1	−0.7391	0.6522	−0.0435	−0.1304
2	1.0000	−0.3913	−1.0000	−0.3913
3	−0.1304	−0.7391	0.1304	−0.1304

Table 9.51 d_9

Day	A	B	C	Total
1	−0.7391	0.6522	−0.0435	−0.1304
2	1.0000	−0.7391	−1.0000	−0.7391
3	−0.1304	−0.3913	0.1304	−0.3913

Table 9.52 d_{10}

Day	A	B	C	Total
1	−0.7391	0.6522	−0.0435	−0.1304
2	−0.1304	−0.3913	0.1304	−0.3913
3	1.0000	−0.7391	−1.0000	−0.7391

Table 9.53 d_{11}

Day	A	B	C	Total
1	−0.7391	0.6522	−0.0435	−0.1304
2	−0.1304	−0.7391	0.1304	−0.7391
3	1.0000	−0.3913	−1.0000	−0.3913

Now replace the third column of d_1 by $(-0.0435, -1.000, 0.1304)$ and by following all the steps mentioned above we get the designs d_8, d_9 where the information of γ is the same as d_4 (Tables 9.50 and 9.51).

Now replace the third column of d_1 by $(-0.0435, 0.1304, -1.000)$ and by following all the steps mentioned above we get the designs d_{10}, d_{11} where the information of γ is same as d_4 (Tables 9.52 and 9.53).

Now replace the third column of d_1 by $(0.1304, -1.000, -0.0435)$ and by following all the steps mentioned above we get the designs d_{12}, d_{13} where the information of γ is same as d_4 (Tables 9.54 and 9.55).

Now replace the third column of d_1 by $(0.1304, -0.0435, -1.000)$ and by following all the steps mentioned above we get the designs d_{14}, d_{15} where the information of γ is same as d_4 (Tables 9.56 and 9.57).

Therefore, d_4 is the best design when the covariate values are given and the relative efficiency of d_1 with respect to d_4 is 56.42 %. Therefore, we are in a position to improve the information of γ in the design d_1 almost twice. But if we have such

Table 9.54 d_{12}

Day	A	B	C	Total
1	−0.1304	−0.3913	0.1304	−0.3913
2	1.0000	−0.7391	−1.0000	−0.7391
3	−0.7391	0.6522	−0.0435	−0.1304

Table 9.55 d_{13}

Day	A	B	C	Total
1	−0.1304	−0.7391	0.1304	−0.7391
2	1.0000	−0.3913	−1.0000	−0.3913
3	−0.7391	0.6522	−0.0435	−0.1304

Table 9.56 d_{14}

Day	A	B	C	Total
1	−0.1304	−0.3913	0.1304	−0.3913
2	−0.7391	0.6522	−0.0435	−0.1304
3	1.0000	−0.7391	−1.0000	−0.7391

Table 9.57 d_{15}

Day	A	B	C	Total
1	−0.1304	−0.7391	0.1304	−0.7391
2	−0.7391	0.6522	−0.0435	−0.1304
3	1.0000	−0.3913	−1.0000	−0.3913

flexibility to choose covariate values then d_2 is the best design. Here we note that we continued our search till all R_i's are same or almost the same.

Example 9.2.4 The data in Table 9.58 are from an experiment on the effects of two drugs on Mental Activity (MA). The mental activity score is the sum of the scores on seven items in a questionnaire given to each of 24 volunteer subjects. The treatments are Morphine, Heroin and a Placebo (an inert substance) given in subcutaneous injections. On different occasions, each subject received each drug in turn. The mental activity is measured before taking the drug (Z) and at $\frac{1}{2}$, 2, 3 and 4 h after. The response data (Y) in Table 9.58 are those at 2 h after. As a common precaution in these experiments, eight subjects took Morphine first, eight took Heroin first and eight took the Placebo first, and similarly on the second and third occasions. These data show no apparent effect of the order in which drugs were given, and the order is ignored here.

Since each subject gets three treatments in three different time points in some predetermined order, we have a flexibility to allocate subjects to the three treatments at

Table 9.58 Mental Activity scores before (Z) and 2 h after (Y) a drug

	Subject	Morphine (M)		Heroin (H)		Placebo (P)	
		Z	Y	Z	Y	Z	Y
	1	7	4	0	2	0	7
	2	2	2	4	0	2	1
	3	14	14	14	13	14	10
	4	14	0	10	0	5	10
	5	1	2	4	0	5	6
	6	2	0	5	0	4	2
	7	5	6	6	1	8	7
	8	6	0	6	2	6	5
	9	5	1	4	0	6	6
	10	6	6	10	0	8	6
	11	7	5	7	2	6	3
	12	1	3	4	1	3	8
	13	0	0	1	0	1	0
	14	8	10	9	1	10	11
	15	8	0	4	13	10	10
	16	0	0	0	0	0	0
	17	11	1	11	0	10	8
	18	6	2	6	4	6	6
	19	7	9	0	0	8	7
	20	5	0	6	1	5	1
	21	4	2	11	5	10	8
	22	7	7	7	7	6	5
	23	0	2	0	0	0	1
	24	12	12	12	0	11	5
Total		138		141		144	

the first time point only. Thereafter, we repeat the allocation of the subjects of Time point 1 for the next two time points. We analyse the data for each of the three time points and employ CRD model for respective time points separately.

Although RBD analysis was carried out in the book, it is more appropriate to treat this as an exercise in repeated CRD analysis and we may assume that the covariate effect is the same across all the three time points. The point to be noted is that the experimental subjects may be classified into three groups based on their covariate values at Time Point 1 only. Subsequently, there is no scope to alter their classifications to other groups.

We now proceed to objectively look into the given data on the covariates.

Towards this, we rearrange Z-values in Table 9.59a–c corresponding to time point 1, time point 2 and time point 3, respectively, and denote it by d_0.

Now we consider the CRD model for single covariate for each Time point.

Table 9.59 Arrangement of the Z-values of the 24 subjects in the three different time points

(a) Time point 1

	M	H	P
	7	4	10
	2	10	6
	14	7	8
	14	4	5
	1	1	10
	2	9	6
	5	4	0
	6	0	11
Totals	51	39	56

(b) Time point 2

	H	P	M
	0	6	11
	4	8	6
	14	6	7
	10	3	5
	4	1	4
	5	10	7
	6	10	0
	6	0	12
Totals	49	44	52

(c) Time point 3

	P	M	H
	0	5	11
	2	6	6
	14	7	0
	5	1	6
	5	0	11
	4	8	7
	8	8	0
	6	0	12
Totals	44	35	53

For Table 9.59a–c, routine computations for a CRD yield, $I_{d_0}(\gamma_1) = 364.75$, $I_{d_0}(\gamma_2) = 330.875$ and $I_{d_0}(\gamma_3) = 365.75$. We assume that $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ (say), i.e. there is no effect of time on the mental activity within the subject. Therefore $I_{d_0}(\gamma) = I_{d_0}(\gamma_1) + I_{d_0}(\gamma_2) + I_{d_0}(\gamma_3) = 1061.375$.

It is evident that we have some flexibilities in suggesting an OCD for a CRD model based on the covariate-values at Time Point 1 and, naturally, we should utilize that provision at least to maximize $I(\gamma_1)$. We have considered all possible permutations of Z-values at Time point 1 (TP 1) and by computer search we have found

$I(\gamma_1) = 383.75$, which is the maximum among all and we get 162 OCDs for Time point 1 where $I(\gamma_1) = 383.75$. These constitute one 'Equivalence Group' for the subjects. This exercise is very special. We recommend one member of this family in the absence of any other information and that presumably is the end of our task. What is the impact of our choice on subsequent TPs 2 and 3? For various choices of the family members, we obtain a range of $I(\gamma)$ values combining all the three TPs. We should look at the entire spectrum of $I(\gamma)$ and see if there is wide variation among them. One acceptable criterion may be $I(\gamma)_{\text{min}}$ is within 2 % of $I(\gamma)_{\text{max}}$. If that has been the case in this example as assessed by the covariate values at TP 2 and TP 3, we need not worry about our specific recommendation. Otherwise, there is a scope of introspection and dig into 'possible relations' among the covariate values for each subject over the TPs.

Since the size of EG is very large, therefore we have considered one member of this group and then found out the impact of our choice on subsequent Time Points 2 and 3.

We fix the same allocation of the subjects, shown in Table 9.60, for Time point 2 and Time point 3. In Table 9.61 we show the allocation of Z-values under each treatment for Time point 1, Time point 2 and Time point 3. We denote the design by d_1.

If we look at Table 9.60, then there is still chance to improve d_1, i.e. if we permute the allocation of the subjects where same covariate values appear in the rows of Table 9.60, then there is a possibility to improve the value of $I_2(\gamma)$ and $I_3(\gamma)$. Here the possible rows are R2, R3, R4, R5, R7, R8. We have considered all $2! \times 3! \times 2! \times 2! \times 3! \times 2! = 576$ permutations. By computer search, we have found $I(\gamma)_{\text{max}} = 1102.875$ and $I(\gamma)_{\text{min}} = 1018.625$ and the improvement of the design with $I(\gamma)_{\text{max}}$ (=1102.875) over the design with $I(\gamma)_{\text{min}}$ (=1018.625) is 8.27 % $(=\frac{1102.875-1018.625}{1018.625} \times 100\,\%)$ only. In Table 9.62, we show the design with maximum $I(\gamma)$ and denote the design by d_2.

Now we consider the design with $I_{\text{min}}(\gamma) = 1018.625$ and denote it by d_3. This design is shown in Table 9.63.

Table 9.60 Allocation of subjects based on OCD for Time Point 1 [TP1]

	M	H	P	Allocation of subjects		
				M	H	P
	0	0	1	16	23	5
	2	2	1	2	6	13
	4	4	4	9	12	15
	6	5	5	8	7	20
	6	6	7	18	22	1
	7	8	9	11	19	14
	10	10	10	10	17	21
	14	14	11	3	4	24
Totals	49	49	48			

Table 9.61 Arrangement of the Z-values of the 24 subjects for Time point 1, Time point 2 and Time point 3

(a) Time point 1

	M	H	P
	0	0	1
	2	2	1
	4	4	4
	6	5	5
	6	6	7
	7	8	9
	10	10	10
	14	14	11
Totals	49	49	48

(b) Time point 2

	H	P	M
	0	0	4
	4	5	1
	6	3	10
	6	6	5
	6	7	0
	6	7	10
	8	11	4
	14	10	12
Totals	50	49	46

(c) Time point 3

	P	M	H
	0	0	5
	2	4	0
	5	1	8
	6	8	6
	6	7	0
	7	0	8
	6	11	11
	14	5	12
Totals	46	36	50

Here $I_{d_2}(\gamma_2) = 333.875$, $I_{d_2}(\gamma_3) = 385.25$, $I_{d_3}(\gamma_2) = 313.625$ and $I_{d_3}(\gamma_3) = 321.25$. It follows that the relative efficiencies of d_2 and d_3 with respect to d_0 are 103.91 % and 95.97 %. Therefore, though we start with optimal design for Time point 1, yet there might be a chance of getting less efficient design (viz., d_3) than d_0 at the end of Time point 3. Fortunately, still d_3 has a very high relative efficiency as against d_0. So our decision is not too bad to look for an optimal design with reference to Time point 1 and follow it through.

Table 9.62 Design d_2 with maximum I (γ)

	Time Point 1			Time Point 2			Time Point 3			Allocation		
	M	H	P	M	H	P	M	H	P	M	H	P
	0	0	1	0	0	4	0	0	5	16	23	5
	2	2	1	4	5	1	2	4	0	2	6	13
	4	4	4	6	3	10	5	1	8	9	12	15
	6	5	5	6	6	5	6	8	6	8	7	20
	6	6	7	6	7	0	6	7	0	18	22	1
	7	8	9	6	7	10	7	0	8	11	19	14
	10	10	10	11	4	8	11	11	6	17	21	10
	14	14	11	10	14	12	5	14	12	4	3	24
Totals	49	49	48	49	46	50	42	45	45			

Table 9.63 Design d_3 with minimum I (γ)

	Time Point 1			Time Point 2			Time Point 3			Allocation		
	M	H	P	M	H	P	M	H	P	M	H	P
	0	0	1	0	0	4	0	0	5	16	23	5
	2	2	1	5	4	1	4	2	0	6	2	13
	4	4	4	10	3	6	8	1	5	15	12	9
	6	5	5	6	5	6	6	6	8	8	20	7
	6	6	7	7	6	0	7	6	0	22	18	1
	7	8	9	6	7	10	7	0	8	11	19	14
	10	10	10	11	8	4	11	6	11	17	10	21
	14	14	11	14	10	12	14	5	12	3	4	24
Totals	49	49	48	59	43	43	57	26	49			

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