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Mohammad Ahsanullah

Characterizations of Univariate Continuous Distributions

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Mohammad Ahsanullah

Characterizations of Univariate Continuous Distributions

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Mohammad Ahsanullah
Department of Management Sciences
Rider University
Lawrenceville, NJ
USA

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To my grand children, Zakir, Samil, Amil and Julian.

Preface

Characterization of distributions plays an important role in statistical science. Using the basic properties of data, characterizations provide the type of distributions of that data set. Significant findings in this area have been published over the last several decades, and this book serves to be an extensive compilation of many important characterizations of univariate continuous distributions. Chapter 1 presents basic properties common to all univariate continuous distributions, while Chap. 2 discusses the properties of some select important distributions. Chapter 3 discusses ways to use independent copies of random variables to characterize distributions. Chapters 4–6 characterize distributions using order statistics, record values, and generalized order statistics, respectively.

I would like to thank Prof. Chris Tsokos for his encouragement to publish a book on characterization of distributions and Zeger Karssen of Atlantis Press for his support of this publication. I would also like to thank my wife Masuda for all her support. Finally, I would like to thank Rider University for a summer grant and a sabbatical leave that provided resources for me to complete this book.

Lawrenceville, NJ, USA
December 2016

Mohammad Ahsanullah

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Chapter 1
Introduction

In this chapter some basic materials will be presented which will be used in the book. We will restrict ourselves to continuous univariate probability distributions.

1.1 Distribution of Univariate Continuous Distribution

Let X be an absolutely continuous random variable with cumulative distribution function (cdf) F(x) and probability density function (pdf) f(x). We define

$F(x) = P(X \leq x)$ for all x, $-\infty < x < \infty$ and $f(x) = \frac{d}{dx} F(x)$. F(x) has the following properties

(i) $0 \leq F(x) \leq 1$

$$\lim_{x \to -\infty} F(x) = 0 \text{ and } \lim_{x \to \infty} F(x) = 1$$

(ii) F(x) is non decreasing

(iii) F(x) is right continuous, $F(x) = F(x + 0)$ for all x.

1.2 Moment Generating and Characteristic Functions

The moment generating function $M_X(t)$ of the random variable X with pdf $f_X(x)$ is defined as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx, \quad -\infty < t < \infty$$

provided the integral converge absolutely. $M_X(0)$ always exists and equal to 1.

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The characteristic function $\varphi_X(t)$ of a random variable with pdf f(x) always exits and it is given by

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx} f_X(x)dx, \quad -\infty < t < \infty.$$

The characteristic function has the following properties:

(i) A characteristic function is uniformly continuous on the entire real line,
(ii) It is non vanishing around zero and $\varphi_X(0) = 1$,
(iii) It is bounded, $|\varphi_X(t)| \leq 1$,
(iv) It is Hermitian,

$$\varphi_X(-t) = \overline{\varphi}_X(t),$$

(v) If a random variable has kth moment, then $\varphi_X(t)$ is k times differentiable on the entire real line,
(vi) If the characteristic function $\varphi_X(t)$ of a random variable X has k-th derivative at t = 0, then the random variable X has all moments up to k if k is even and k − 1 if k is odd.

A necessary and sufficient condition for two random variables X_1 and X_2 to have identical cdf is that their characteristic functions be identical.

There is a one to one correspondence between the cumulative distribution function and characteristic function.

Theorem 1.2.1 *If characteristic function φ_X (t) is integrable, then $F_X(x)$ is absolutely continuous, and X has the probability density function $f_X(x)$ that is given by*

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi_X(t)dt$$

1.3 Some Reliability Properties

Hazard Rate

The hazard rate (r(t)) of a positive random variable random variable with F(0) = 0 is defined as follows.

$r(t) = \frac{f(t)}{\overline{F}(t)}$, $\overline{F}(t) = 1 - F(t)$, provided $\overline{F}(x)$ is not zero.

By integrating both sides of the above equation, we obtain

$$\overline{F}(x) = \exp(-\int_0^x r(t)dt).$$

An alternative representation is

$$1-F(x)=e^{-R(x)}.\ R(x)=-\ln(1-F(x)).$$

We will say that the random variable X belongs to class C_1 if the hazard rate is monotonically increasing or decreasing.

New Better (Worse) Than Used (NBU(NWU))
A cumulative distribution function F(x) is NBU(NWU) if

$$\overline{F}(x+y)\le(\ge)\overline{F}(x)\overline{F}(y),\ \text{for}\ x\ge 0, y\ge 0.$$

We will say the random variable X whose cdf F(x) belongs to the class C_2 if it is NBU or NWU.

Memoryless Property
Suppose the random variable X has the property
P(X > t + s|X > t) = P(X > s) for all s, t ≥ 0, then we say that X has memory less property.

The exponential distribution with F(x) = 1 − $e^{-(x-\mu)/\sigma}$ for $\sigma>0, -\infty<x<\mu<\infty$. is the only continuous distribution that has this memoryless property.

1.4 Cauchy Functional Equations

We will consider the following three Cauchy functional equations for a non zero continuous function g(x).

(i) $g(x+y)=g(x)+g(y),\ x\ge 0,\ y\ge 0$

(ii) $g(xy)=g(x)+g(y), x\ge 0,\ y\ge 0$

(iii) $g(xy)=g(x)g(y), x\ge 0,\ y\ge 0$

We will take the solutions as of the functional equations as $g(x)=e^{cx}$, $g(x)=c\ln(x)$ and $g(x)=x^c$, where c is a constant respectively. For details about the solutions see Aczel (1966).

1.5 Order Statistics

Let X_1, X_2,…,X_n be independent and identically distributed (i.i.d.) absolutely continuous random variables. Suppose that F(x) be their cumulative distribution function (cdf) and f(x) be the their probability density function (pdf). Let $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ be the corresponding order statistics. We denote $F_{k,n}(x)$ and $f_{k,n}(x)$ as the cdf and pdf respectively of $X_{k,n}$, k = 1,2,…,n. We can write

$$f_{k,n}(x) = \frac{n!}{(k-1)!(n-k)!} (F(x))^{k-1}(1-F(x))^{n-k}f(x),$$

The joint probability density function of order statistics $X_{1,n}$, $X_{2,n}$,…,$X_{n,n}$ has the form

$$f_{1,2,\ldots,n,n}(x_1, x_2, \ldots, x_n) = n! \prod_{k=1}^{n} f(x_k), \quad -\infty < x_1 < x_2 < \cdots < x_n < \infty$$

and

$$= 0, \textit{otherwise}$$

There are some simple formulae for pdf's of the maximum ($X_{n,n}$) and the minimum ($X_{1,n}$) of the n random variables._

The pdfs of the smallest and largest order statistics are given respectively as

$$f_{1,n}(x) = n(1-F(x))^{n-1}f(x)$$

and

$$f_{n,n}(x) = n(F(x))^{n-1}f(x)$$

The joint pdf $f_{1,n,n}$ (x, y) of $X_{1,n}$ and $X_{n.n}$ is given by

$$f_{1,n}(x,y) = n(n-1)(F(y)-F(x))^{n-2}\, f(x)f(y),$$

$$-\infty < x < y < \infty.$$

Example 1.5.1. Exponential distribution.

Suppose that X_1, X_2,…,X_n are n i.i.d. random variables with cdf F(x) as

$$F(x) = 1 - e^{-x}, \ x \ge 0$$

The pdfs $f_{1,n}(x)$ of $X_{1,n}$ and $f_{n.n}$ (x) are respectively

$$f_{1,n}(x) = ne^{-nx}, x \geq 0.$$

and

$$f_{n,n}(x) = n(1 - e^{-x})^{n-1} e^{-x}, x \geq 0.$$

It can be seen that $nX_{1,n}$ has the exponential distribution.,

1.6 Record Values

Chandler (1952) introduced the record values, record times and inter record times. Suppose that X_1, X_2,... be a sequence of independent and identically distributed random variables with cumulative distribution function F(x). Let Y_n = max (min) $\{X_1, X_2, \ldots, X_n\}$ for $n \geq 2$. We say X_j is an upper (lower) record value of $\{X_n, n \geq 1\}$, if $Y_j > (<) Y_{j-1}$, $j > 2$. By definition X_1 is an upper as well as a lower record value. One can transform the upper records to lower records by replacing the original sequence of $\{X_j\}$ by $\{-X_j, j \geq 1\}$ or (if $P(X_i > 0) = 1$ for all i) by $\{1/X_i, i \geq 1\}$; the lower record values of this sequence will correspond to the upper record values of the original sequence.

The indices at which the upper record values occur are given by the record times $\{U(n)\}$, $n > 0$, where $U(n) = \min\{j|j > U(n-1), X_j > X_{U(n-1)}, n > 1\}$ and $U(1) = 1$. The record times of the sequence $\{X_n\ n \geq 1\}$ are the same as those for the sequence $\{F(X_n), n \geq 1\}$. Since F(X) has an uniform distribution, it follows that the distribution of U(n), $n \geq 1$ does not depend on F. We will denote L(n) as the indices where the lower record values occur. By our assumption U(1) = L(1) = 1. The distribution of L (n) also does not depend on F.

Many properties of the upper record value sequence can be expressed in terms of the function R(x), where $R(x) = -\ln \overline{F}(x)$, $0 < \bar{F}(x) < 1$ and $\bar{F}(x) = 1 - F(x)$. Here 'ln' is used for the natural logarithm. If we define $F_n(x)$ as the cdf of $X_{U(n)}$ for $n \geq 1$, then we have

$$F_1(x) = P[X_{U(1)} \leq x] = F(x) \tag{1.6.1}$$

$$\begin{aligned} F_2(x) &= P[X_{U(2)} \leq x\,] \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} \sum_{i=1}^{\infty} (F(u))^{i-1}\, dF(u)\, dF(y) \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{dF(u)}{1 - F(u)}\, dF(y) \\ &= \int_{-\infty}^{x} R(y)\, dF(y) \end{aligned} \tag{1.6.2}$$

If F(x) has a density f(x), then the probability density function (pdf) of $X_{U(2)}$ is

$$f_2(x) = R(x)\ f(x) \tag{1.6.3}$$

The cdf F_3 (x) of $X_{U(3)}$ is given by

$$\begin{aligned} F_3(x) &= P(X_{U(3)} \le x) \\ &= \int_{-\infty}^{x}\int_{-\infty}^{y}\sum_{i=0}^{\infty}(F(u))^i\, R(u)\, dF(u)\, dF(y) \\ &= \int_{-\infty}^{x}\int_{-\infty}^{y}\frac{R(u)}{1-F(u)}\, dF(u)\, dF(y) \\ &= \int_{-\infty}^{x}\frac{(R(u))^2}{2!}\, dF(u). \end{aligned} \tag{1.6.4}$$

The pdf $f_3(x)$ of $X_{U(3)}$ is

$$f_3(x) = \frac{(R(x))^2}{2!}\ f(x), \quad -\infty < x < \infty \tag{1.6.5}$$

It can similarly be shown that the cdf $F_n(x)$ of $X_{U(n)}$ is

$$\begin{aligned} F_n(x) &= P(X_{U(n)} \le x) \\ &= \int_{-\infty}^{x} f(u_n)du_n \int_{-\infty}^{u_n}\frac{f(u_{n-1})}{1-F(u_{n-1})}du_{n-1} \ldots \int_{-\infty}^{u_2}\frac{f(u_1)}{1-F(u_1)}du_1. \\ &= \int_{-\infty}^{x}\frac{R^{n-1}(u)}{(n-1)!}\, dF(u),\ -\infty < x < \infty \end{aligned} \tag{1.6.6}$$

This can be expressed as

$$\bar{F}_n(x) = \int_{-\infty}^{R(x)}\frac{u^{n-1}}{(n-1)!}e^{-u}\, du,\ -\infty < x < \infty,$$

$$\begin{aligned} \bar{F}_n(x) &= 1 - F_n(x) \\ &= \bar{F}(x)\sum_{j=0}^{n-1}\frac{(R(x))^j}{j!} \\ &= e^{-R(x)}\sum_{j=0}^{n-1}\frac{(R(x))^j}{j!} \end{aligned}$$

The corresponding pdf $f_n(x)$ of $X_{U(n)}$ is

$$f_n(x) = \frac{R^{n-1}(x)}{(n-1)!} f(x), \ -\infty < x < \infty. \tag{1.6.7}$$

.

The joint pdf f(x_1, x_2,…, x_n) of the n record values $X_{U(1)}$, $X_{U(2)}$,…, $X_{U(n)}$) is given by

$$\mathbf{f}(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}) = \mathbf{r}(\mathbf{x_1})\mathbf{r}(\mathbf{x_2})\ldots\mathbf{r}(\mathbf{x_{n-1}})\mathbf{f}(\mathbf{x_n}) \tag{1.6.8}$$

for $-\infty < \mathbf{x_1} < \mathbf{x_2} < \cdots < \mathbf{x_n} < \infty$
where $\mathbf{r}(\mathbf{x}) = \frac{f(x)}{1-F(x)}$.

The function r(x) is known as hazard rate.

The joint pdf of $X_{U(i)}$ and $X_{U(j)}$ is

$$f(x_i, x_j) = \frac{(R(x_i))^{i-1}}{(i-1)!} r(x_i) \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} f(x_j), \ 1 \le i < j \le n, \tag{1.6.9}$$

for $-\infty < x_i < x_j < \infty$.

Suppose we use the transformation $Y_1 = R(X_{U(i)})$ and $Y_2 = R(X_{U(i)})/R(X_{U(j)})$, i < j, then it can be shown that the pdf $f_2^*(y)$ of Y_2 is as follows:

$$f_2^*(y) = \frac{\Gamma(j)}{\Gamma(i)} \frac{1}{\Gamma(j-i)} y^{i-1}(1-y)^{j-i-1} \ 0 < y < 1 \tag{1.6.10}$$

Thus Y_2 is distributed as Beta distribution with parameters i and j (i.e. B(i, j − i)). The mean and variance of Y2 are

$E(Y_2) = \frac{i}{j}$ and $Var(Y_2) = \frac{ij}{(j+1)j^2}$.

If we use the transformation $V_i = R(X_{U(i)})$, then the joint pdf of V_i, i = 1,2,…,n, is

$$f(v_1, v_2, \ldots, v_n) = e^{-v_n}, 0 < v_1 < v_2 < \cdots < v_n < \infty. \tag{1.6.11}$$

The joint distribution of V_m and V_r, r > m, is

$$f(v_m, v_r) = \frac{1}{\Gamma(m)} \cdot \frac{(v_r - v_m)^{r-m-1}}{\Gamma(r-m)} \cdot e^{-v_r} \ 0 < v_m < v_r < \infty$$
$$= 0, \text{ otherwise.}$$

$$E(V_k^l) = \int_0^\infty t^l \frac{1}{\Gamma(k)} t^{k-1} e^{-t} dt = \frac{\Gamma(k+l)}{\Gamma(k)}.$$

Thus $E(V_k) = k$ and $Var(V_k) = k$. The conditional pdf of

$$X_{U(j)} \mid X_{U(i)} = x_i \text{ if } (x_j \mid X_{U(i)} = x_i) = \frac{f_{ij}(x_i, x_j)}{f_i(x_i)} = \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} \frac{f(x_j)}{1 - F(x_i)} \quad (1.6.12)$$

for $-\infty < x_i < x_j < \infty$.

For j = i + 1

$$f(x_{i+1} | X_{U(i)} = x_i) = \frac{f(x_{i+1})}{1 - F(x_i)} \quad (1.6.13)$$

for $-\infty < x_i < x_{i+1} < \infty$.

The marginal pdf of the nth lower record value can be derived by using the same procedure as that of the nth upper record value. Let

H(u) = −ln F(u), 0 < F(u) < 1 and $h(u) = -\frac{d}{du} H(u)$, then

$$P(X_{L(n)} \le x) = \int_{-\infty}^{x} \frac{\{H(u)\}^{n-1}}{(n-1)!} dF(u) \quad (1.6.14)$$

and the corresponding the pdf $f_{(n)}$ can be written as

$$f_{(n)}(x) = \frac{(H(x))^{n-1}}{(n-1)!} f(x) \quad (1.6.15)$$

.

The joint pdf of $X_{L(1)}, X_{L(2)}, \ldots, X_{L(m)}$ can be written as

$$f_{(1),(2),\ldots,(m)}(x_1, x_2, \ldots, x_m) = h(x_1)\, h(x_2) \ldots h(x_{m-1})\, f(x_m) \quad -\infty < x_m < x_{m-1} < \cdots < x_1 < \infty \quad (1.6.16)$$
$$= 0, \text{ otherwise}$$

The joint pdf of $X_{L(r)}$ and $X_{L(s)}$ is

$$f_{(r),(s)}(x, y) = \frac{(H(x))^{r-1}}{(r-1)!} \frac{[H(y) - H(x)]^{s-r-1}}{(s-r-1)!} h(x)\, f(y) \quad (1.6.17)$$
$$\text{for } s > r \text{ and } -\infty < y < x < \infty$$

Using the transformations U = H(y) and W = H(x)/H(y) it can be shown easily that W is distributed as B(r, s-r).

Proceeding as in the case of upper record values, we can obtain the conditional pdfs of the lower record values.

Example 1.6.1 Let us consider the exponential distribution with pdf f(x) as

$$f(x) = e^{-x},\ 0 \le x < \infty$$

and the cumulative distribution function (cdf) F(x) as

$$F(x) = 1 - e^{-x},\ 0 \le x < \infty$$

Then R(x) = x and

$$\begin{aligned} f_n(x) &= \frac{x^{n-1}}{\Gamma(n)} e^{-x},\ x \ge 0 \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

The joint pdf of $X_{U(m)}$ and $X_{U(n)}$, n > m is

$$\begin{aligned} f_{m,n}(x,y) &= \frac{x^{m-1}}{\Gamma(m)\Gamma(n-m)} (y-x)^{n-m-1} e^{-y}, \\ &\quad \textit{for } 0 \le x < y < \infty, \\ &= 0, \text{ otherwise.} \end{aligned}$$

The conditional pdf of $X_{U(n)} \mid X_{U(m)} = x$) is

$$\begin{aligned} f(y|X_{U(m)} = x) &= \frac{(y-x)^{n-m-1}}{\Gamma(n-m)} e^{-(y-x)} \\ &\quad 0 \le x < y < \infty \\ &= 0, \text{ otherwise} \end{aligned}$$

Thus the conditional distribution of $X_{U(n)} - X_{U(m)}$ given $X_{U(m)}$ is the same as the unconditional distribution of $X_{U(n-m)}$ for n > m.

Example 1.6.2 Suppose that the random variable X has the Gumbel distribution with pdf $f(x) = e^{-x} e^{-e^{-x}}$, $-\infty < x < \infty$. Let $F_{(n)}$ and $f_{(n)}$ be the cdf and pdf of $X_{L(n)}$. It is easy to see that

$$F_{(n)}(x) = \int_{-\infty}^{x} \frac{e^{-nu}}{\Gamma(n)} e^{-e^{-u}}\, du$$

and $f_{(n)}(x) = \frac{e^{-nx}}{\Gamma(n)} e^{-e^{-x}}, \quad -\infty < x < \infty.$

Let $f_{(m,n)}(x, y)$ be the joint pdf of $X_{L(m)}$ and $X_{L(n)}$, m < n. Using (1.6.17) we get for the Gumbel distribution

$$f_{(m,n)}(x,y) = \frac{(e^{-y} - e^{-x})^{n-m-1}}{\Gamma(n-m)} \frac{e^{-mx}}{\Gamma(m)} e^{-y} e^{-e-y},$$

$$-\infty < y < x < \infty$$

Thus the conditional pdf $f_{(n|m)}(y|x)$ of $X_{L(n)}| X_{L(m)} = x$ is given by

$$f_{(n|m)}(y|x) = \frac{(e^{-y} - e^{-x})^{n-m-1}}{\Gamma(n-m)} e^{-y} e^{-(e^{-y} - e^{-x})}.$$

For simplicity we will denote X(m) and $X_{(m)}$ respectively for $X_{U(m)}$ and $X_{L(m)}$.

1.7 Generalized Order Statistics

Kamps (1995) introduced the generalized order statistics. The order statistics, record values and sequential order statistics are special cases of this generalized order statistics Suppose X(1, n, m, k),…, X(n, n, m, k), ($k \geq 1$, m is a real number), are n generalized order statistics. Then their joint pdf $f_{1,\ldots,n}(x_1,\ldots,x_n)$ can be written as

$$\begin{aligned} f_{1,\ldots,n}(x_1, \ldots, x_n) &= k \prod_{j=1}^{n-1} \gamma_j \prod_{i=1}^{n-1} (1 - F(x_i))^m f(x_i)(1 - F(x_n))^{k-1} f(x_n), \\ &\text{for } F^{-1}(0) < x_1 < \cdots < x_n < F^{-1}(1). \\ &= 0, \text{ otherwise,} \end{aligned} \tag{1.7.1}$$

where $\gamma_j = k + (n - j)(m + 1)$ and $f(x) = \frac{dF(x)}{dx}$.

If m = 0 and k = 1, then X(r, n, m, k) reduces to the ordinary rth order statistic and (1.7.1) is the joint pdf of the n order statistics $X_{1,n} \leq \cdots \leq X_{n,n}$. If k = 1 and m = −1, then (1.7.1) is the joint pdf of the first n upper record values of the independent and identically distributed random variables with cdf F(x) and the corresponding probability density function f(x). Let $F_{r,n,m,k}(x)$ and $f_{r,m,n,k}(x)$ be the cdf and pdf of X(r, n, m, k).

$$F_{r,n,m,k}(x) = I_{\alpha(x)}\left(r, \frac{\gamma_r}{m+1}\right), \text{ if } m > -1 \tag{1.7.2}$$

and

$$F_{r,n,m,k}(x) = \Gamma_{\beta(x)}(r), \text{ if } m = -1, \tag{1.7.3}$$

where

$$\alpha(\mathrm{x}) = 1 - (\overline{F}(x))^{m+1}, \overline{F}(x) = 1 - F(x)$$

$$\beta(\mathrm{x}) = -k \ln \overline{F}(x)),$$

and

$$\Gamma_x(r) = \int_0^x \frac{1}{\Gamma(r)} u^{r-1} e^{-u} du$$

Proof For m > −1, from (21.1)

$$\mathrm{F}_{\mathrm{r,n,m,k}}(\mathrm{x}) = \int_{F^{-1}(0)}^{x} \frac{\mathrm{c_r}}{(\mathrm{r}-1)!} (1 - \mathrm{F(u)})^{\mathrm{k}+(\mathrm{n}-\mathrm{r})(\mathrm{m}+1)-1} \mathrm{g_m^{r-1}}(\mathrm{F(u)})\, \mathrm{f(u)}\, \mathrm{du}$$

Using the relation

$B(r, \frac{\gamma_r}{m+1}) = \frac{\Gamma(r)(m+1)^r}{c_r}$ and substituting $t = 1 - (\bar{F}(x))^{m+1}$., we get on simplification

$$\mathrm{F}_{\mathrm{r,n,m,k}}(\mathrm{x}) = \frac{1}{B(r, \frac{\gamma_r}{m+1})} \int_0^{1-(\bar{F}(x))^{m+1}} (1-u))^{\gamma_{\mathrm{r}}-1} (1-u)^{r-1} du$$
$$= I_{\alpha(x)}(r, \frac{\gamma_r}{m+1}).$$

For m = −1

$$\mathrm{F}_{\mathrm{r,n,m,k}}(\mathrm{x}) = \int_{F^{-1}(0)}^{x} \frac{\mathrm{k^r}}{(\mathrm{r}-1)!} (1 - \mathrm{F(u)})^{\mathrm{k}-1} (-\ln(1 - F(u))^{r-1} \mathrm{f(u)du}$$
$$= \int_0^{-k \ln \bar{F}(x)} \frac{1}{(\mathrm{r}-1)!} t^{r-1} \mathrm{e}^{-\mathrm{t}} dt$$
$$= \Gamma_{\beta(x)}(r), \quad \beta(x) = -k \ln F(x)$$

$$F_{r,n,m,k}(r, \frac{\lambda_r}{m+1}) - F_{r,n,m,k}(r+1, \frac{\lambda_{r+1}}{m+1}) = \frac{1}{\gamma_{r+1}} \frac{\overline{F}(x)}{f(x)} f_{r+1,n,m,k}$$

$$\mathrm{f}(\mathrm{x}, \theta) = \sigma^{-1} \exp(-\sigma^{-1}\mathrm{x}), \text{for x} > 0,\ \sigma > 0,$$
$$= 0, \text{ otherwise.} \tag{1.7.4}$$

1.8 Lower Generalized Order Statistics (Lgos)

Suppose $X^*(1,n,m,k), X^*(2,n,m,k), \ldots, X^*(n,n,m,k)$ are n lower generalized order statistics from an absolutely continuous cumulative distribution function (cdf) $F(x)$ with the corresponding probability density function (pdf) f(x). Their joint pdf $f^*_{12\ldots n}(x_1, x_2, \ldots, x_n)$ is given by

$$f^*_{12\ldots n}(x_1, x_2, \ldots, x_n) = k \mathop{\Pi}_{j=1}^{n-1} \gamma_j \mathop{\Pi}_{i=1}^{n-1} (F(x_i))^m (F(x_n))^{k-1} f(\mathrm{x})$$

for $\overset{-1}{F}(1) \geq x_1 \geq x_2 \geq \cdots \geq \overset{-1}{F}(0), m \geq -1,$

$\gamma_r = k + (n-r)(m+1), r = 1, 2, \ldots, n-1,\ k \geq 1$ and n is a positive integer. The marginal pdf of the rth lower generalized order (lgos) statistics is

$$f^*_{r,n,m,k}(x) = \frac{c_{r-1}}{\Gamma(r)} (F(x))^{\gamma_r - 1} (g_m(F(x)))^{r-1} f(x), \tag{1.7.5}$$

where

$$c_{r-1} = \mathop{\Pi}_{i=1}^{r} \gamma_i,$$

$$g_m(x) = \frac{1}{m+1}\left(1 - x^{m+1}\right), \text{ for } m \neq -1$$
$$= -\ln x, \text{ for } m = -1.$$

Since $\lim_{m \to -1} g_m(x) = -\ln x$, we will take $g_m(x) = \frac{1}{m+1}(1 - x^{m+1})$ for all m with $g_{-1}(x) = -\ln x$. For $m = 0$, $k = 1$, $X^*(r,n,m,k)$ reduces to the order statistics $X_{n-r+1,n}$ from the sample $X_1, X_2, \ldots, X_n$, while $m = -1$, $X^*(r,n,m,k)$ reduces to the *rth* lower *k*-record value.

If F(x) is absolutely continuous, then

$$\bar{F}^*_{r,n,m,k}(x) = 1 - F^*_{r,n,m,k}(x) = I_{\alpha(x)}\left(r, \frac{\gamma_r}{m+1}\right), \text{ if } m > -1,$$
$$= \Gamma_{\beta(x)}(r), \text{ if } m = -1,$$

where

$$\alpha(x) = 1 - (F(x))^{m+1},\ I_x(p,q) = \frac{1}{B(p,q)} \int_0^x u^{p-1}(1-u)^{q-1} du, \ \mathrm{x} \leq 1$$

$\beta(x) = -k\, 1n\, F(x),\ \Gamma_x(r) = \frac{1}{\Gamma(r)} \int\limits_0^x u^{r-1} e^{-u} du.$ and $\mathrm{B(p,q)} = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$.

Proof For $m > -1$,

$$1 - F^*_{r,n,m,k}(x) = \frac{c_{r-1}}{\Gamma(r)} \int\limits_x^\infty (F(u))^{\gamma_{r-1}} (g_m(F(u)))^{r-1} f(u) du$$

$$= \frac{c_{r-1}}{\Gamma(r)} \int\limits_x^\infty (F(u))^{\gamma_{r-1}} \left[\frac{1-(F(u))^{m+1}}{m+1}\right]^{r-1} f(u) du$$

$$= \frac{c_{r-1}}{\Gamma(r)} \frac{1}{(m+1)^r} \int\limits_0^{1-(F(x))^{m+1}} t^{r-1}(1-t)^{(\gamma_{r+1}/(m+1))-1} dt$$

$$= I_{\alpha(x)}\left(r, \frac{\gamma_r}{m+1}\right)$$

For $m = -1,\ \gamma_j = k,\ j = 1, 2, \ldots, n$

$$1 - F^*_{r,n,m,k}(x) = \int\limits_x^\infty \frac{k^r}{\Gamma(r)} (F(u))^{k-1} (-1nF(u))^{r-1} f(u) du$$

$$= \int\limits_0^{-k\ln F(x)} \frac{t^{r-1} e^{-t}}{\Gamma(r)} dt$$

$$= \Gamma_{\beta(x)}(r), \quad \beta(x) = -k \ln F(x).$$

Example 1.7.1 Suppose that X is an absolutely continuous random variable with cdf F(X) with pdf f(x).

For $m > -1$

$$\gamma_{r+1}\left(F^*_{r+1,n,m,k}(x) - F^*_{r,n,m,k}(x)\right) = \frac{F(x)}{f(x)} f^*_{r+1,n,m,k}(x)$$

and for $m = -1$

$$k\left(F^*_{r+1,n,m,k}(x) - F^*_{r,n,m,k}(x)\right) = \frac{F(x)}{f(x)} f^*_{r+1,n,m,k}(x)$$

Proof: For $m > -1$

$$F^*_{r+1,n,m,k}(x) - F^*_{r,n,m,k}(x) = I_{\alpha(x)}\left(r, \frac{\gamma_r}{m+1}\right) - I_{\alpha(x)}\left(r+1, \frac{\gamma_{r+1}}{m+1}\right)$$
$$= I_{\alpha(x)}\left(r, \frac{\gamma_r}{m+1}\right) - I_{\alpha(x)}\left(r+1, \frac{\gamma_r}{m+1} - 1\right)$$

We know that

$$I_x(a,b) - I_x(a+1,b-1) = \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} x^a (1-b)^{b-1}$$

Thus

$$F^*_{r+1,n,m,k}(x) - F^*_{r,n,m,k}(x) = \frac{\Gamma(r+\frac{\gamma_r}{m+1})}{\Gamma(r+1)\Gamma(\frac{\gamma_r}{m+1})}\ \left(1-(F(x))^{m+1}\right)^r \left(F(x)^{m+1}\right)^{\frac{\gamma_r}{m+1}-1}$$
$$= \frac{\gamma_1 \ldots . \gamma_r}{\Gamma(r+1)}\ \left(\frac{1-(F(x))^{m+1}}{m+1}\right)^r\ (F(x))^{\gamma_{r+1}}$$
$$= \frac{F(x)}{\gamma_{r+1} f(x)}\ f^*_{r+1,n,m,k}(x).$$

Thus

$$\gamma_{r+1}\left(F^*_{r+1,n,m,k}(x) - F^*_{r,n,m,k}(x)\right) = f^*_{r+1,n,m,k}(x)\frac{F(x)}{f(x)}$$

For $m = -1$,

$$F^*_{r+1,n,m,k}(x) - F^*_{r,n,m,k}(x) = \Gamma_{\beta(x)}(r) - \Gamma_{\beta(x)}(r+1),\ \beta(x) = -k\ln F(x)$$
$$= (\beta(x))^r e^{-\beta(x)}\frac{1}{\Gamma(r+1)}$$
$$= \frac{(F(x))^{\ k}}{\Gamma(r+1)}\ \ (-k\ln F(x))^r$$

Thus

$$k\left[F^*_{r+1,n,m,k}(x) - F^*_{r,n,m,k}(x)\right] = \frac{F(x)}{f(x)} f^*_{r+1,n,m,k}(x).$$

1.9 Some Useful Functions

Beta function B(m, n)

$$\mathrm{B}(\mathrm{m},\mathrm{n}) = \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{dx},\ \mathrm{m} > 0,\ \mathrm{n} > 0.$$

Incomplete Beta function $B_x(m,n)$

$$B_x(m,n) = \int_0^x u^{m-1}(1-u)^{n-1}\mathrm{du}$$

Gamma function $\Gamma(n)$

$$\mathrm{B}(\mathrm{m},\mathrm{n}) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \mathrm{B}(\mathrm{n},\mathrm{m}).$$

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx,\ \mathrm{n} > 0.$$

If n is an integer, then $\Gamma(n) = (n-1)!$
Incomplete gamma function $\gamma(n,x)$

$$\gamma(n,x) = \int_0^x u^{n-1}e^{-u}du$$

Psi(Digamma) function $\psi(n)$

$$\psi(n) = \frac{d}{dz}\ln\Gamma(\mathrm{n}).$$

$\psi(1) = -\gamma$, The Euler's constant.

$$\gamma = 0.577216$$

Chapter 2
Some Continuous Distributions

In this chapter several basic properties of some useful univariate distributions will be discussed. These properties will be useful for our characterization problems.

2.1 Beta Distribution

A random variable X is said to have a BE(m, n) distribution if its pdf $f_{m,\ n}(x)$ is of the following form.

$$f_{m \cdot n}(X) = \frac{1}{B(m,n)} x^{m-1}(1-x)^{n-1}, 0 < x < 1, m > 0, n > 0. \qquad (2.1.1)$$

Mean $= \frac{m}{m+n}$ and variance $= \frac{mn}{(m+n)^2(m+n+1)}$.
The moment generating function $M_{m,\ n}(t)$ is
$M_{m,\ n}(t)$ = F(m, m + n, t), where

$$F(a, b, x) = 1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)}\frac{x^2}{2!} + \cdots$$

The characteristic function $\phi_{m.n}(t) = F(m, m+n, it)$
The pdfs of BE(3, 3), BE(4, 6) and BE(4, 9) are given in Fig. 2.1.
If m = 1/2 and n = 1/2, then BE(1/2, 1.2) is the arcsine distribution.
If X is distributed as BE(m, n), then 1-x is distributed as BE(n, m).

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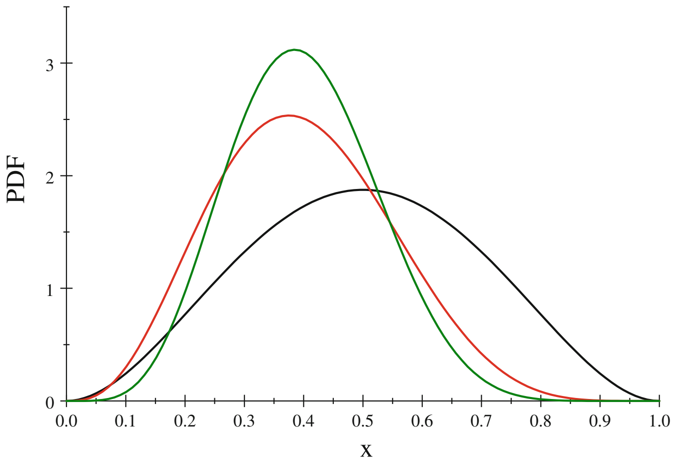

Fig. 2.1 BE(3, 3) *Black*, BE(4, 6) *Red* and BE(6, 9) *Green*

2.2 Cauchy Distribution

A random variable X is said to have a Cauchy $(\mathrm{CA}(\mu, \sigma))$ distribution with location parameter μ and scale parameter σ if the pdf $(\mathrm{f_c}(\mathrm{x}, \mu, \sigma))$ is of the following form.

$$\mathrm{f_c}(\mathrm{x}, \mu, \sigma) = \frac{1}{\pi\sigma\left(1 + \left(\frac{x-\mu}{\sigma}\right)^2\right)}, -\infty < x < \mu < \infty, \sigma > 0. \quad (2.1.2)$$

The Fig. 2.2 gives the pdfs of CA(0, 1), CA(0, 2) and CA(0, 5).

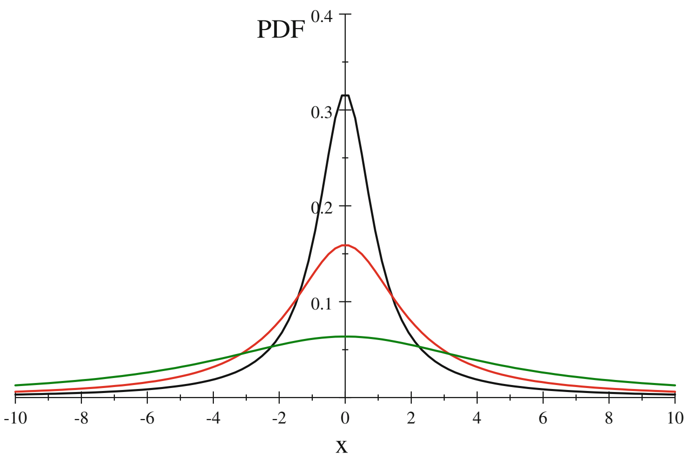

Fig. 2.2 CA(0, 1) *Black*, CA(0, 2) *Red* and CA(0, 5) *Green*

The mean of CA(μ, σ) does not exist. The median and the mode are equal to μ. The cdf $F_c(x, \mu, \sigma)$ is

$$F_c(x, \mu, \sigma) = \frac{1}{2} + tan^{-1}(\frac{x-\mu}{\sigma}) \tag{2.1.3}$$

If $X_1, X_2, \ldots, X_n$ are n independent CA(μ, σ), then $S_n = X_1 + X_2 + \cdots + X_n$ is distributed as CA$(n\mu, n\sigma)$..

If X_1 and X_2 are distributed as normal with mean = 0 and variance = 1, the X/Y is distributed as CA(0, 1).

If X is CA(0, 1), then $2X/(1-X^2)$ is distributed as CA(0, 1).

The pdf $f_{gc}(x, \mu, \sigma)$ of generalized Cauchy (GCA(μ, σ)) is given by

$$f_{gc}(x, \mu, \sigma) = \frac{\Gamma(n)}{\sigma\sqrt{\pi}\Gamma(n-\frac{1}{2})} \frac{1}{(1+(\frac{x-\mu}{\sigma})^2)^n}, n \geq 1, -\infty < \mu < x < \infty, \sigma > 0. \tag{2.1.4}$$

For n = 1, the mean does not exist.

For n > 1, the mean = μ, the median = μ and the odd moments are zero.

For n > 1,

$$E(X^m) = \frac{\Gamma(\frac{m+1}{2})\Gamma(n - \frac{m+1}{2})}{\Gamma(\frac{1}{m})\Gamma(n-\frac{1}{m})} \text{for m even, m} < 2n-1, m > 1.$$

2.3 Chi-Squared Distribution

A random variable X is said to have a Chi-squared (CH(μ, σ, n)) distribution with location parameter μ and scale parameter σ if the pdf ($f_{ch}(x, \mu, \sigma, n)$) is of the following form.

$$f_{ch}(x, \mu, \sigma, n) = \frac{1}{2^{n/2}\Gamma(\frac{n}{2})}(\frac{x-\mu}{\sigma})^{\frac{n}{2}-1} e^{-(\frac{x-\mu}{2\sigma})}, n > 1, -\infty < \mu < x < \infty, \sigma > 0.$$

The parameter n is known as degrees of freedom.

For n > 1, Mean = $\mu + n\sigma$, *and variance* $= 2n\sigma^2$.

The moment generating function M_{CH} (t) is

$$M_{CH}(t) = e^{\mu t}(1-2\sigma t)^{-n/2}, t < \frac{1}{2\sigma}.$$

The Fig. 2.3 gives the pdfs of The CH(0, 1, 4), CH(0, 1, 10) and CH(0, 1, 20).

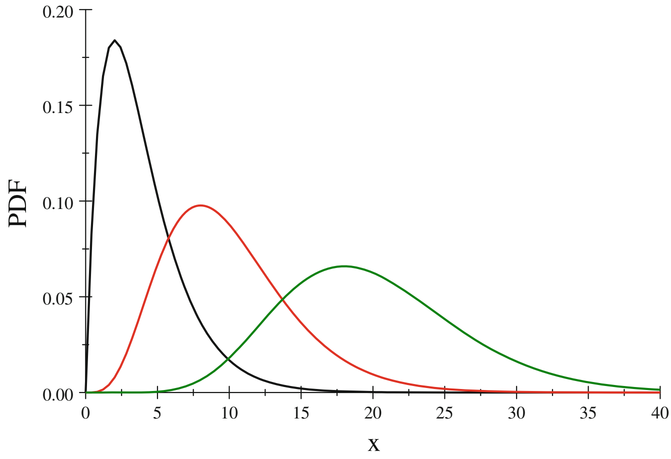

Fig. 2.3 The CH(0, 1, 4)-Black, CH(0, 1, 10)-red and CH(0, 1, 20)-green

If X_i, i = 1, 2, …, n are n independent CH(0, 1, n_i). i = 1, 2, …, n, random variables then $S_k = X_1 + X_2 + \cdots + X_k$, then S_k is distributed as CH(0, 1, m), where $m = n_1 + n_2 + \cdots + n_K$. If X is standard normal (N(0, 1)), then X^2 is distributed as CH(0, 1, 1).

2.4 Exponential Distribution

A random variable X is said to have a exponential $(E(\mu, \sigma))$ distribution with location parameter μ and scale parameter σ if the pdf $(f_e(x, \mu, \sigma))$ is of the following form.

$$f_e(x, \mu, \sigma) = \frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)}, \ -\infty < \mu < x < \infty.$$

The exponential distribution E(0, 1) is known as standard exponential.

The Fig. 2.4 gives the pdfs of E(0, 1), E(0, 2) and E(0, 5).

The cdf $F_e(x, \mu, \sigma)$ is given by

$$F_e(x, \mu, \sigma) = 1 - e^{-\left(\frac{x-\mu}{\sigma}\right)}, \ -\infty < \mu < x < \infty.$$

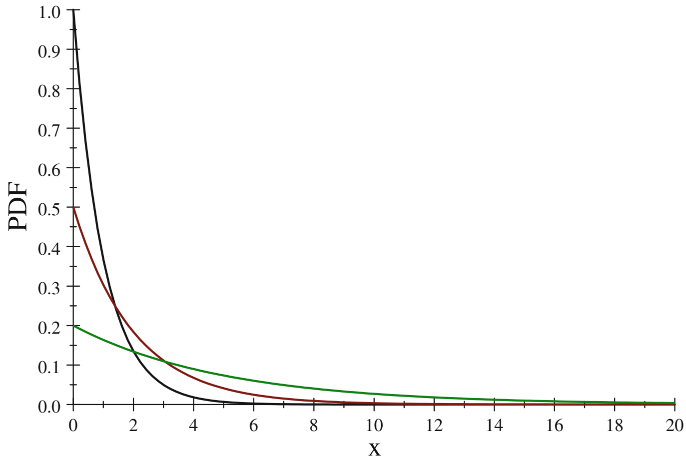

Fig. 2.4 E(0, 1) *Black*, E(0, 2) *Red* and E(0, 5) *Green*

The moment generating function $M_{ex}(t)$

$$M_{ex}(t) = (1 - \sigma t)^{-1} e^{-\mu t}$$

Mean = $\mu + \sigma$ and Variance = σ^2.

If X_i, i = 1, 2, …, n are i.i.d. exponential with $F(x) = 1 - e^{-x/\sigma}, x \geq 0, \sigma > 0$, and $S(n) = X_1 + X_2 + \ldots + X_n$, then pdf $f_{S(n)}(x)$ of S(n) is

$$f_{S(n)}(x) = \frac{1}{\sigma} e^{-x/\sigma} \frac{(x/\sigma)^{n-1}}{\Gamma(n)}, x \geq 0, \sigma > 0.$$

This is a gamma distribution with parameters n and σ.

If X_1 and X_2 are independent exponential random variables with scale parameters σ_1 and σ_2, then $P(X_1 < X_2) = \frac{\sigma_2}{\sigma_1 + \sigma_2}$.

If X_i, i = 1, 2, …, n are n independent exponential random variable with $F(x) = 1 - e^{-x/\sigma}, x \geq 0, \sigma > 0$. Let m(n) = min $\{X_1, \ldots, X_n\}$ and

M(n) = max$\{X_1, \ldots, X_n\}$, $F_{(m)}$ be the cdf of m(n) and $F_{(M)}$ be the cdf of M(n), then

$$\begin{aligned} 1 - F_{(m)}(x) &= P(X_1 > x, X_2 > x, \ldots X_n > x) \\ &= e^{-nx/\sigma} \end{aligned}$$

$$F_{(M)}(x) = P(X_1 < x. X_2 < x, \ldots, X_n < x) = (1 - e^{-x/\sigma})^n.$$

Memoryless Property. P(X > s+t|X > t) P(X > s).

$$\begin{aligned} \mathrm{P(X > s + t|X > t)} &= \frac{P(X > s + t, X > t}{P(X > t)}) \\ &= \frac{e^{-(s+t-2\mu)/\sigma}}{e^{-(t-\mu)/\sigma}} \\ &= e^{-(s-\mu)/\sigma} \\ &= P(X > s) \end{aligned}$$

2.5 F-Distribution

A random variable X is said to have F distribution F(m, n) with numerator degree of degrees of freedom m and numerator degrees of freedom n if its pdf f_F (x, m, n) is given by

$$\mathrm{f_F(x, m, n)} = \frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{\frac{m}{2}} x^{(m-2)/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})(1 + \frac{mx}{n})^{(m+n)/2}}, \mathrm{x > 0, m\ > 0, n > 0.}$$

The cdf F_F(x, m, m) is given by

$$\mathrm{F_F(x, m, n)} = I_{\frac{mx}{m+n}}(\frac{m}{2}, \frac{n}{2}),$$

where = $I_x(a, b) = \int_0^x u^{a-1}(1-u)^{b-1}du$ is the incomplete beta function.

The Fig. 2.5 gives the pdfs of F(5, 5), F(10, 1)) and F(10, 20).

Mean = $\frac{n}{n-2}$, n > 2 and variance = $\frac{2n^2(m+n-2)}{m(m-2)^2(n-4)}$, n > 4.

The characteristic function ϕ_F (m · n) = $\frac{\Gamma(\frac{m+n)}{2}}{\Gamma(\frac{n}{2})}\mathrm{U}(\frac{m}{2}, 1 - \frac{n}{2}, -\frac{m}{n}it)$, where U(a, b, x) is the confluent hypergeometric function of the second kind.

If U_1 and U_2 are independently distributed as chi-squared distribution with m and n degrees of freedom, then X = (n/m) (U_1/U_2) is distributed as F with cdf F_F (m, n).

If X is distributed as Beta (m/2, n/2), then $\frac{nX}{m(1-X)}$ is distributed as F with cdf F_F (m, n).

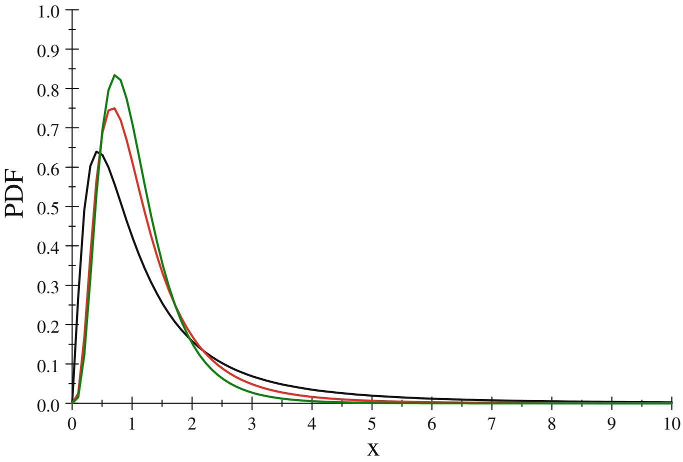

Fig. 2.5 F(5, 5) *Black*, F(10, 10) *Red* and F(10, 20) *Green*

2.6 Gamma Distribution

A random variable X is said to have gamma distribution GA (a, b) if its pdf $f_{ga}(a, b, x)$ is of the following form.

$$f_{ga}(a, b, x) = \frac{1}{\Gamma(a) b^a} x^{a-1} e^{-x/b}, \mathrm{x} \geq 0, a > 0, b > 0.$$

Mean = ab and variance = ab^2.

The Fig. 2.6 give the pdfs of GA(2, 1), GA(5, 1) and GA(10, 1).

The moment generating function M(t) is

$$\mathrm{M(t)} = (1 - bt)^{-a}, \mathrm{t} < 1/\mathrm{b}.$$

The characteristic function $\phi_{ga}(t) is\ \phi_{ga}(t) = (1 - ibt)^{-a}$.

If a = 1, b = 1 then we GA(1, 1) is an exponential distribution and if a is a positive integer, then GA(a, b) is an Erlang distribution.

If b = 1, then we call GA (a, b) as the standard gamma distribution.

If X_1 and X_2 are independent gamma random variables then the random variables $X_1 + X_2$ and $\frac{X_1}{X_1 + X_2}$ are mutually independent.

If X_1, X_2, .., X_n are n independently distributed as GA(a, b), then S(n) = $X_1 + X_2 + \cdots + X_n$ is distributed as GA(na, b).

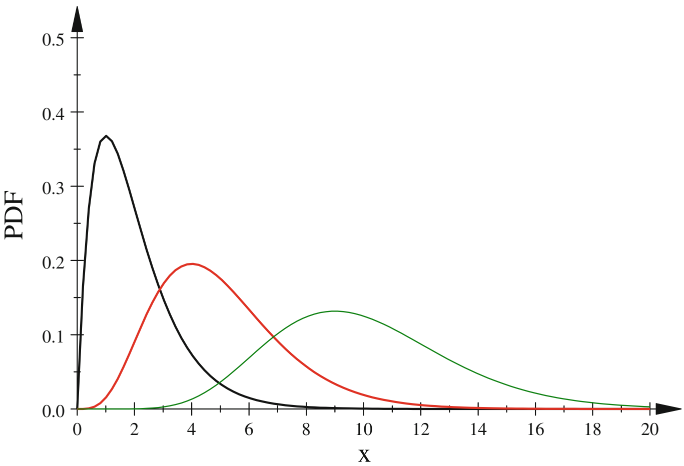

Fig. 2.6 GA(2, 1) *Black*, GA(5, 1) *Red* and GA(10, 1) *Green*

2.7 Gumbel Distribution

A random variable X is said to have Gumbel $(\mathrm{GU}(\mu, \sigma))$ distribution with location parameter μ and scale parameter σ if its pdf, $\mathrm{f_{ig}}\ (\mathrm{x}, \mu, \sigma)$ is of the following form

$$\mathrm{f_{gu}}(\mathrm{x}, \mu, \sigma) = \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}} e^{-e^{-\frac{x-\mu}{\sigma}}}, -\infty < \mu < x < \infty, \sigma > 0.$$

Gumbel distribution is also known as Type I extreme (maximum) value distribution. The cdf $\mathrm{F_{gu}}(\mathrm{x}, \mu, \sigma)$ is of the following form

$$\mathrm{F_{gu}}(\mathrm{x}, \mu, \sigma) = e^{-e^{-\frac{x-\mu}{\sigma}}}, -\infty < \mu < x < \infty, \sigma > 0.$$

Mean = $\mu + \gamma\sigma$, where γ is Euler's constant.
Median = $\mu - \ln(ln2)\sigma$.
Variance = $\frac{\pi^2\sigma^2}{6}$.
The Fig. 2.7 gives the pdfs of GU(0, 1/2), GU(0, 1) and GU(0, 2).
If X is distributed as E(0, 1), then $\mu - \sigma lnX$ is distributed as $\mathrm{GU}(\mu, \sigma)$).
If X is distributed as GU(0, 1), then $Y = e^{-X}$ is distributed as E(0, 1).

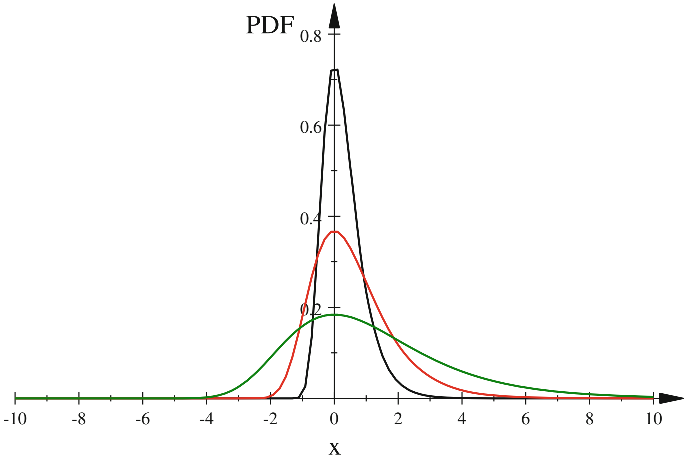

Fig. 2.7 GU(0, 1/2) *Black*, GU(0, 1) *Red* and GU(0, 2) *Green*

2.8 Inverse Gaussian (Wald) Distribution

A random variable X is said to have Inverse Gaussian $(\mathrm{IG}(\mu, \sigma))$ distribution with parameters μ and λ if its pdf

$f_{ig}(x, \mu, \lambda)$ is of the following form

$$f_{ig}(x, \mu, \lambda) = \left(\frac{\lambda}{2\lambda x^3}\right)^{\frac{1}{2}} e^{-\frac{\lambda}{2x}\left(\frac{x-\mu}{\mu}\right)^2}, 0 < \mu < x < \infty, \lambda > 0.$$

Mean $= \mu$ and variance $= \frac{\mu^3}{\lambda}$.
The Fig. 2.8 gives the pdfs of IG(1, 10), IG(1, 2) ans IG(1, 3).

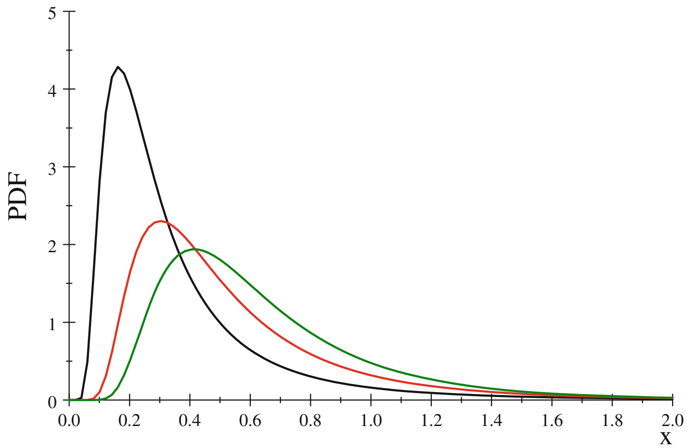

Fig. 2.8 IG(1, 1) *Black*, IG(1, 2) *Red* and IG(1, 3) *Green*

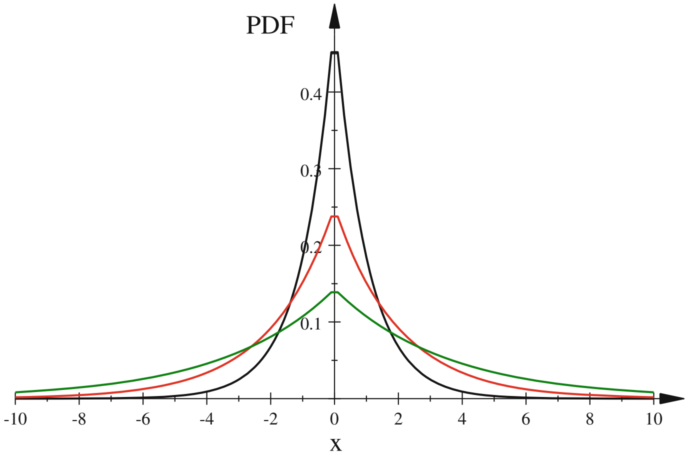

Fig. 2.9 LP(0, 1) *Black*, LP(0, 2) *Red* and LP(0, 3.5) *Green*

If X is distributed as IG (μ, λ) then αX is distributed as IG $(\alpha\mu.\, \alpha\lambda)$.

If X is distributed as IGv $(1, \lambda)$, then X is known as Wald distribution.

2.9 Laplace Distribution

A random variable X is said to have Laplace $(LP(\mu, \sigma))$ distribution with location parameters μ and scale parameter λ if its pdf

$f_{lp}(x, \mu, \lambda)$ is of the following form

$$f_{lp}(x, \mu, \lambda) = \frac{1}{2\sigma} e^{-\left|\frac{x-\mu}{\sigma}\right|}, \infty < x < \mu < \infty, \sigma > 0.$$

Mean $= \mu$ and variance $= 2\sigma^2$.

The Fig. 2.9 gives the pdfs of LP(0, 1), LP(0, 2) and LP(0, 3.5).

The moment generating function is $M_{lp}(t)$ is

$$M_{lp}(t) = \frac{e^{\mu t}}{1 - \sigma^2 t^2}.$$

The characteristic function $\phi_{lp}(t)$ is

$$\phi_{lp}(t) = \frac{e^{i\mu t}}{1+\sigma^2 t^2}.$$

If X and Y are independent E(0, 1), then X-Y is LP(0, 1).
If X is LP (μ, σ), then kX is LP $(k\mu, k\sigma)$.
If X is LP(0, 1), then |X| is E(0, 1).

2.10 Logistic Distribution

A random variable X is said to have Logistic $(\text{LG}(\mu, \sigma))$ distribution with location parameters μ and scale parameter λ if its pdf

$f_{lg}(x, \mu, \sigma)$ is of the following form

$$\mathrm{f}_{\mathrm{lg}}(x, \mu, \sigma) = \frac{1}{\sigma} \frac{e^{-\frac{x-\mu}{\sigma}}}{(1+e^{-\frac{x-\mu}{\sigma}})^2}, \ -\infty < \mu < x < \infty, \sigma > 0.$$

Mean = μ and variance $= \frac{\pi^2\sigma^2}{3}$.
Moment generating function M_{lg} $_{(t)}$ is

$$\mathrm{M}_{\mathrm{lg}}(\mathrm{t}) = e^{\mu t}\Gamma(1+\sigma t)\Gamma(1-\sigma t), t < \frac{1}{\sigma}.$$

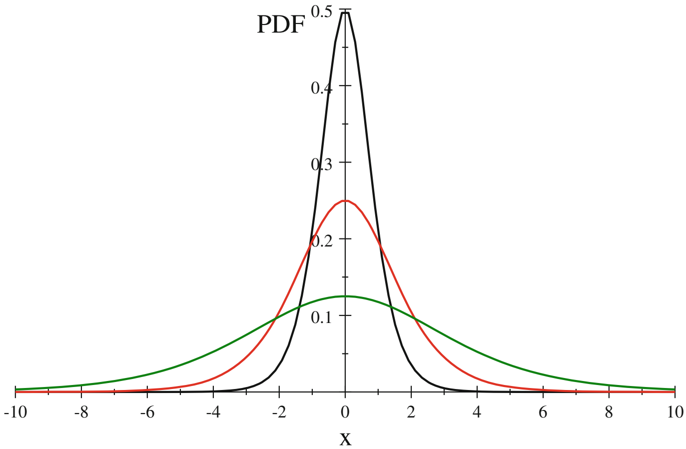

Fig. 2.10 LG(0, 1/2) *Black*, LG(0, 1) *Red* and LG(0, 2) *Green*

The characteristic function $\phi_{\text{lg}}(t) is$

$$\phi_{\text{lg}}(t) = e^{i\mu t}\Gamma(1 + i\sigma t)\Gamma(1 - i\sigma t).$$

The Fig. 2.10 gives the pdfs of LG(0, 1/2), LG(0, 1) and LB(0, 2).

Let $X_{i,}$, i = 1, 2, …, n are independent and identically distributed as LP(0, 1), then Y = $X_1X_2\ldots X_n$ is distributed as LG(0, 1).

If X and Y are independent GU $(\mu, \sigma), then\, X - Y$ is LG(0, 1).

If X is LG((μ, σ) then kX is LG $(\mu k, k\sigma)$.

If X and Y are independent and E(0, 1), then $\mu - \sigma \ln(\frac{X}{Y})$ is LG (μ, σ).

2.11 Lognormal Distribution

A random variable X is said to have Lognormal $(\text{LN}(\mu, \sigma))$ distribution with location parameters μ and scale parameter σ if its pdf

$\text{f}_{\text{ln}}(\text{x}, \mu, \sigma)$ is of the following form

$$\text{f}_{\text{ln}}(\text{x}, \mu, \sigma) = \frac{1}{x\sigma\sqrt{(2\pi)}} e^{-\frac{1}{2}(\frac{\ln(x-\mu)}{\sigma})^2}, x > 0, > 0, \mu > 0, \sigma > 0.$$

Mean = $e^{\mu + \frac{\sigma^2}{2}}$

Variance = $(e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$.

Moment generating function $M_{ln}(t)$ is

$$M_{ln}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} e^{n\mu + \frac{n^2\sigma^2}{2}}.$$

The characteristic function $\phi_{ln}(t) is$

$$\phi_{ln}(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} e^{n\mu + \frac{n^2\sigma^2}{2}}$$

The Fig. 2.11 gives the pdfs of LN(0, 1/2), LN(0, 1) and LN(0, 2).

If X is distributed as normal with location parameter μ and scale parameter σ, *then* e^X is distributed as LN (μ, σ).

If X is distributed as LN $(\mu,\ \sigma)$, *then ln X is distributed as* normal with location parameter μ and scale parameter σ.

If X_i, i = 1, 2, …, n are independent and identically distributed as LN $(\mu,\ \sigma)$, *then* Y = $X_1X_2\ldots X_n$ is distributed as LN $(n\mu, \sigma\sqrt{n})$.

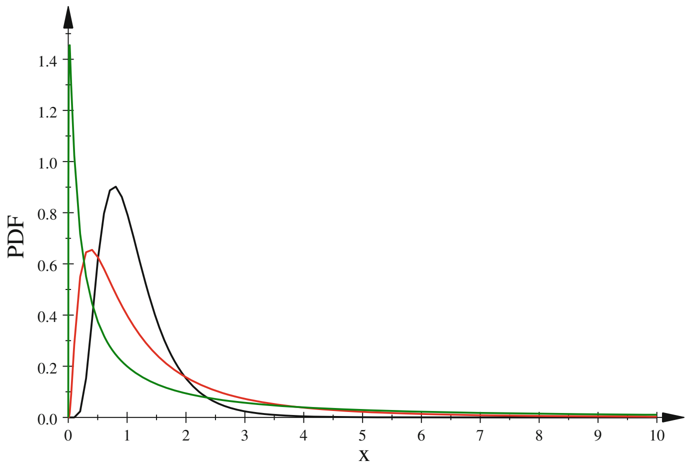

Fig. 2.11 LN(0, 1/2) *Black*, LN(0, 1) *Red* and LN(0, 2) *Green*

2.12 Normal Distribution

A random variable X is said to have normal $(N(\mu, \sigma))$ distribution with location parameters μ and scale parameter σ if its pdf

$f_n(x, \mu, \sigma)$ is of the following form

$$f_n(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < \mu < x < \infty, \sigma > 0.$$

$$Mean = \mu \; and \; variance = \sigma^2.$$

$$The \; moment \; generating \; function \; M_n(t) is$$

$$M_n(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

The characteristic function $\phi_n(t)$ is

$$\phi_n(t) = e^{i\mu t - \frac{\sigma^2 t^2}{2}}.$$

The Fig. 2.12 gives the pdfs of N(0, 1/2), N(0, 1) and N(0, 2).

If X_i is N (μ_i, σ_i), i = 1, 2, …, n and X_i's are independent, then for any α_i, i = 1, 2, …, n, $\sum_{i=1}^n \alpha_i X_i$ is N($\sum_{i=1}^n \alpha_i \mu_I$, $\sqrt{(\sum_{i=1}^n \alpha_i^2 \sigma_i^2)}$).

If X is normal and $X = X_1 + X_2$, where X_1 and X_2 are independent, then both X_1 and X_2 are normal.

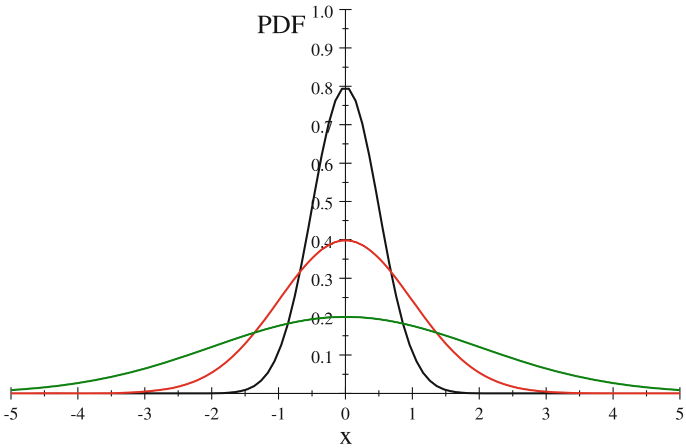

Fig. 2.12 N(0, 0.5) *Black*, N(0, 1) *Red* and N(0, 2) *Green*

2.13 Pareto Distribution

A random variable X is said to have Pareto (PA(α, β)) distribution with parameters α and β if its pdf $f_{pa}(x, \alpha, \beta)$ is of the following form

$$f_{pa}(x, \alpha, \beta) = \frac{\beta\alpha^{\beta}}{x^{\beta+1}}, x > \alpha > 0, \beta > 0.$$

Mean $= \frac{\alpha\beta}{\beta - 1}, \beta > 1$ and variance $= \frac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}. \beta > 2.$
The characteristic function $\phi_{pa}(t)$ is given by

$$\phi_{pa}(t) = \beta(-i\alpha t)^{\beta}\Gamma(-\beta, -i\alpha t).$$

The Fig. 2.13 gives the pdfs of PA(1, 1/2), PA(1, 1) and PA(1, 20).
If $X_1, X_2, \ldots, X_n$ are n independent PA(α, β), then

$2\beta \ln\left(\frac{\prod_{i=1}^{n} X_i}{\alpha^n}\right)$ is distributed as CH(0, 1, n).

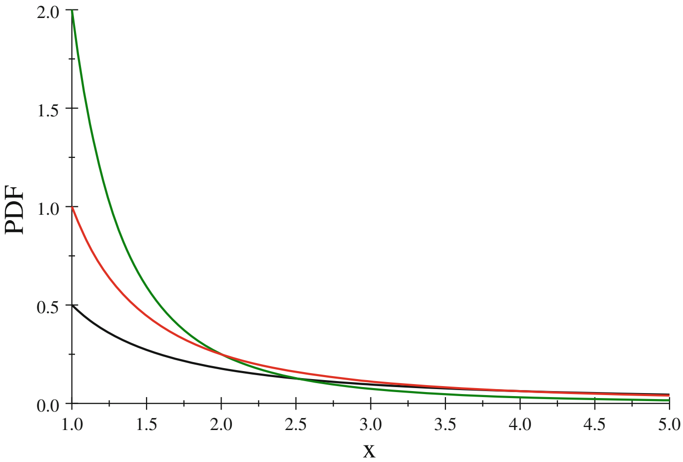

Fig. 2.13 PA(0, 1/2) *Black*, PA(1, 1) *Red* and PA(1, 2) *Green*

2.14 Power Function Distribution

A random variable X is said to have power function (Po(α, β, δ)) if its pdf $f_{po}(\alpha, \beta, \delta)$ is of the following form

$$f_{po}(\alpha,\beta,\delta) = \frac{\delta}{\beta-\alpha}\left(\frac{x-\alpha}{\beta-\alpha}\right)^{\delta-1}, -\infty < \alpha < x < \beta < \infty, \delta > 0$$

Mean = $\alpha + \frac{\delta}{\delta+1}(\beta-\alpha)$ and variance = $\frac{\delta(\beta-\alpha)^2}{(\delta+1)^2(\delta+2)}$.

The Fig. 2.14 gives the pdfs of Po(0, 1, 3), Po(0, 1, 4) and Po(0, 1, 4).

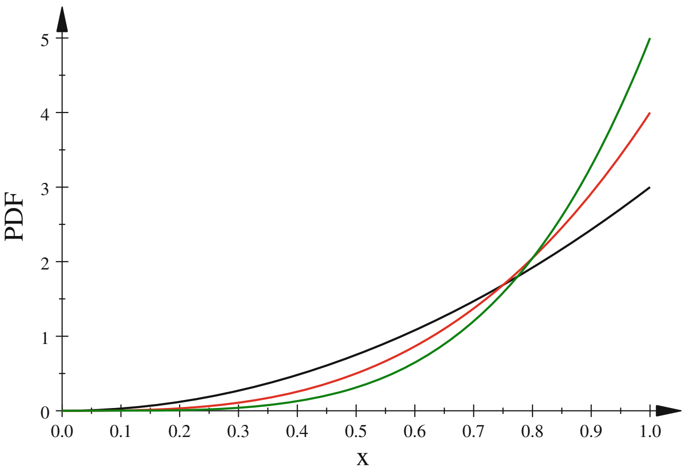

Fig. 2.14 Po(0, 1, 3) *black*, Po(0, 1, 4) *red* and Po(0, 1, 4) *green*

If $\delta = 1, then\, PO(\alpha, \beta, 1)$ becomes a uniform (U(α, β)) with pdf $f_{un}(x, \alpha, \beta)$ as

$$F_{un}(x, \alpha, \beta) = \frac{1}{\beta - \alpha}, -\infty < \alpha < \beta < \infty.$$

2.15 Rayleigh Distribution

A random variable X is said to have Rayleigh (RA(μ, σ)) with location parameter μ and scale parameter σ if its pdf $f_{ra}(x, \mu, \sigma)$ is of the following form

$$f_{ra}(x, \mu, \sigma) = \frac{x-\mu}{\sigma^2} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < \mu < x < \infty, \sigma > 0.$$

Mean $= \mu + \sigma\sqrt{\frac{\pi}{2}}$ and variance $= \frac{4-\pi}{2}\sigma^2$;
Moment generating function $M_{ra}(t)$ is

$$M_{ra}t = e^{\mu t}\left(1 + \sigma t e^{-\frac{\sigma^2 t^2}{2}} \sqrt{\frac{\pi}{2}} \left(\operatorname{erf}\left(\frac{\sigma t}{\sqrt{2}}\right) + 1\right)\right)$$

The Fig. 2.15 gives the pdfs of RA(0, 1/2), RA(0, 1) and RA(0, 2).

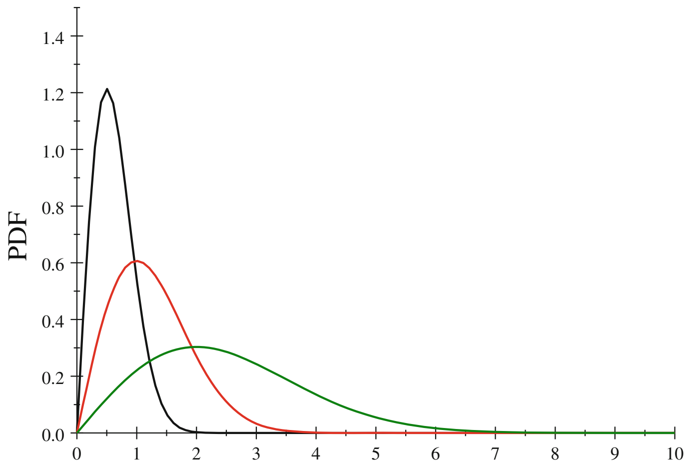

Fig. 2.15 RA(0, 1/2) *Black*, RA (0, 1) *Red* and RA(0, 2) *Green*

2.16 Student's t-Distribution

A random variable X is said to have Students t-distribution ST(n)) with n degrees of freedom, if its pdf f_{st} $(x,\ n)$ is as follows.

$$\mathrm{f_{st}}(\mathrm{x},\mathrm{n}) = \frac{1}{\sqrt{n}}\frac{1}{B(n/2, 1/2)}(1+\frac{x^2}{n})^{-(n+1)/2},\ -\infty < t < \infty, n \geq 1.$$

Mean = 0 if n > 1 and is not defined for n = 1.
Variance = $\frac{n}{n-2}, n > 2.$
The Fig. 2.16 gives the pdf of ST(!), ST(4) and ST(16) (Fig. 2.16).

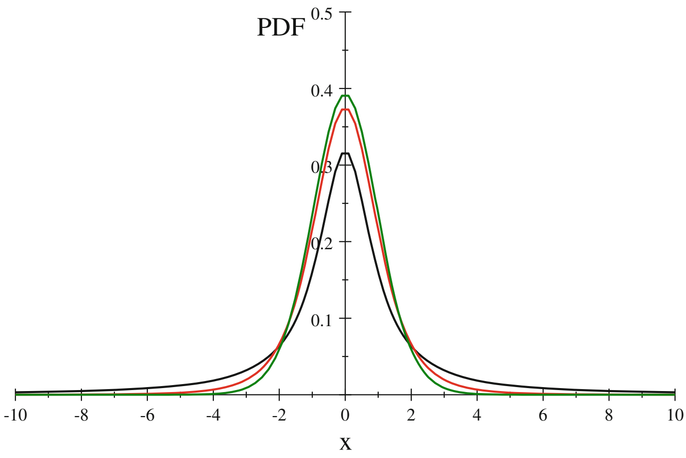

Fig. 2.16 ST(1) *Black*, ST(4) *Red* and ST(16) *Green*

2.17 Weibull Distribution

A random variable is said to have Weibull WB(x, μ, σ, δ) with location parameter μ, scale parameter σ and shape parameter δ if its pdf $\mathrm{f_{Wb}}$(x, μ, σ, δ) is of the following form.

$$\mathrm{f_{Wb}}(\mathrm{x}, \mu, \sigma, \delta) = \frac{\delta}{\sigma}(\frac{x-\mu}{\sigma})^{\delta-1}e^{-(\frac{x-\mu}{\sigma})_{\delta}},\ -\infty < \mu < x < \infty, \sigma > 0, \delta > 0.$$

Mean = $\mu + \Gamma(1+\frac{1}{\delta})$ and variance = $\sigma^2((\Gamma 1+\frac{2}{\delta}) - (\Gamma(1+\frac{1}{\delta}))^2)$.

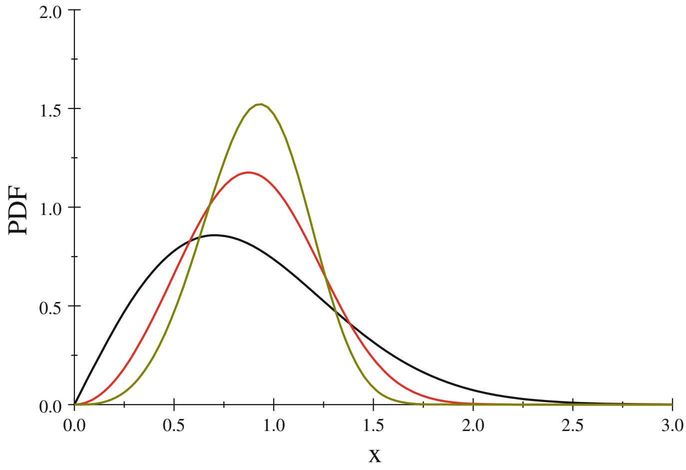

Fig. 2.17 WB(0, 1, 1) *Black*, WB(0, 1, 3) *red* and WB(0, 1, 4) *green*

The Fig. 2.17 gives the pdfs of B(0, 1, 1), WB(0, 1, 3) and B(0, 1, 4).

Chapter 3
Characterizations of Distributions by Independent Copies

In this chapter some characterizations of distributions by the distributional properties based on independent copies of random variables will be presented.

Suppose we have n (≥ 1) independent copies, $X_1, X_2, \ldots, X_n$, of the random variable X. Polya (1920) gave the following characterization theorem of the normal distribution.

3.1 Characterization of Normal Distribution

Theorem 3.1 *If X_1 and X_2 are independent and identically distributed (i.i.d) random variables with finite variance, then $(X_1 + X_2)/\sqrt{2}$ has the same distribution as X_1 if and only if X_1* is *normal* $N(0, \sigma)$.

Proof It is easy to see that $E(X_1) = 0 = E(X_2)$. Let $\phi(t)$ and $\phi_1(t)$ be the characteristic functions of X_1 and $(X_1 + X_2)/\sqrt{2}$ respectively.

If X_1 and X_2 are normal. Then

$$\phi_1(t) = E(e^{\frac{it(X_1+X_2)}{\sqrt{2}}}) = (e^{-\frac{1}{2}(\frac{t}{\sqrt{2}})^2\sigma^2})^2 = e^{-\frac{t^2\sigma^2}{2}}.$$

Thus $(X_1 + X_2)/\sqrt{2}$ is normal.

Suppose that $(X_1 + X_2)/\sqrt{2}$ has the same distribution as X_1, then

$$\phi(t) = \phi_1(t) = E(e^{\frac{it(X_1+X_2)}{\sqrt{2}}}) = (\phi(\frac{t}{\sqrt{2}}))^2.$$

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Atlantis Studies in Probability and Statistics 7,
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Thus

$$\left(\phi(t\sqrt{2}\right) = (\phi(t))^2,$$

and

$$(\phi(2t) = \phi(\sqrt{2}(t\sqrt{2})) = (\phi(t\sqrt{2}))^2 = (\phi(t))^{2^2}$$

By induction it can be shown that

$$\phi\left(t(2^{\frac{k}{2}})\right) = (\phi(t))^{2^k} \text{ for all k} = 0,\ 1,\ 2,\ \ldots$$

Let us find a t_0 such that $\phi(t_0) \neq 0$. We can find such a t_0 since $\phi(t)$ is continuous and $\phi(0) = 1$. Let

$\phi(t_0) = e^{-\sigma^2}$ for $\sigma > 0$, We have

$$\phi\left(t_0 2^{-\frac{k}{2}})\right) = e^{-\sigma^2 2^{-k}},\ \text{k} = 0,\ 1,\ 2,\ \ldots\ldots$$

Thus

$\phi_1(t) = e^{-t^2\sigma^2}$ for all t.

The theorem is proved.

Laha and Lukacs (1960) proved that for X_i, i = 1, 2,…, n independent and identically distributed random variables if the distributions of $\sum_{i=1}^{n} X_i\, and\, X_1$ are identical, then the distribution of X_i, i = 1, 2,…n, is normal.

The following theorem was proved by Cramer (1936).

Theorem 3.2 *Suppose X_1 and X_2 are two independent random variables and Z = X_1 + X_2. If Z is normally distributed, then X_1 and X_2 are normally distributed.*

To prove the theorem, we need the following two lemmas.

Lemma 3.1 (Hadamad Factorial Theorem) *Suppose g(t) is an entire function with zeros $\beta_1, \beta_2, \ldots \beta_p$. and does not vanish at the origin, then we can write $g(t) = m(t)e^{n(t)}$, where $m(t)$ is the canonical product formed with zeros of $\beta_1, \beta_2, \ldots$ and n(t) is a polynomial of degree not exceeding p.*

Lemma 3.2 *If $e^{n(t)}$, where n(t) is a polynomial, is a characteristic function, then the degree of n(t) can not exceed 2.*

Proof of Theorem 3.3 The necessary part is easy to prove. We will proof here the sufficiency part. We will assume that mean of Z is zero and variance is σ^2. Let $\phi(t), \phi_1(t)$ and $\phi_2(t)$ be the characteristic functions of Z, X_1 and X_2 respectively. We can write

$\phi(t) = e^{-\frac{1}{2}\sigma^2 t^2}$ and $\phi(t)$ *is* an entire function without zero.

Since $\phi(t) = \phi_1(t)\ \phi_2(t)$, we can write $\phi_1(\text{t}) = e^{p(t)}$, where p(t) is a polynomial and p(t) must be of degree less than or equal to 2. Let $\text{p(t)} = a_0 + a_1 t + a_2 t^2$, Assume $\text{E}(X_1) = \mu_1$ and variance $= \sigma_1^2$. Since $\phi_1(0) = 1$ *and* $|\phi_1(t)| \leq 1$, we must have a_0 = 0 and a_2 *as negative. Hence* $p(\text{t}) = i\mu_1 t - \sigma_1^2 t^2{}_+$ thus X_1 is distributed as normal Similarly, it can be proved that the distribution of X_2 is normal.

Remark 3.1 If Z is distributed as normal then we can write
$Z = X_1 + X_2 + \cdots + X_n$, where $X_1, X_2, \ldots, X_n$, is normally distributed.

Remark 3.2 Suppose $X_1, X_2, \ldots, X_n$ are n independent and identically distributed random variables with mean = 0 and variance = 1. Let $S_n = \frac{X_1}{\sqrt{n}} + \frac{X_2}{\sqrt{n}} + \cdots + \frac{X_n}{\sqrt{n}}$., By Central Limit Theorem we know that $S_n \to N(0,1)$. But by the Cramer's theorem if S_n is normal, then all the X_i's are normal.

The following characterization theorem of the normal distribution was independently proved by Darmois (1953) and Skitovich (1953).

Theorem 3.3 *Let $X_1, X_2, \ldots, X_n$ be independent random variables. Suppose*
$L_1 = a_1X_1 + a_2X_2 + \cdots + a_nX_n$ *and*
$L_2 = b_1X_1 + b_2X_2 + \cdots + b_nX_n$
If L_1 and L_2 are independent, then for each index i (i = 1, 2,..., n) for which a_i $b_i \neq 0$, X_i is normal.

For an interesting proof of the theorem see Linnik (1964, p. 97).

Heyde (1969) proved that if the conditional distribution of $L_1|L_2$ is symmetric then the X_i's are normally distributed. Kagan et al. (1973) showed that for n $\geq$ 3 if $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d.) with $E(X_i) = 0$, i = 1, 2,..., n, and if $E(\bar{X}|X_1 - \bar{X}, X_2 - \bar{X}, \ldots, X_n - \bar{X}) = 0$, where $\bar{X} = \sum_{i=1}^{x} X_i$, then X_i's (i = 1, 2,..., n) are normally distributed. Rao (1967) showed that if $X_1, X_2, \ldots, X_n$ are independent and identically distributed, $E(X_i) = 0$ and $E\left(X_i^2\right) < \infty$, then if $E(\bar{X}|X_i - \bar{X}) = 0$, for any i = 1, 2,..., n, $\underline{n} \geq 3$, then X_i's are normally distributed. It can be shown that for n = 2, the above result is not true (See Ahsanullah et al. 2014). Kagan and Zinger (1971) proved the normality of the X's under the following conditions.

$$E|X_i|^2 < \infty, i = 1, 2, \ldots, n$$

and

$$E(L_i^{k-1}|L_2) = 0, k = 1, 2, \ldots, n$$

Kagan and Wesolowski (2000) extended the Darmois-Skitovitch theorem for a class of dependent variables.

The following theorem gives a characterization of the normal distribution using the distributional relation of the linear function with chi-squared distribution.

Theorem 3.4 *Suppose $X_1, X_2, \ldots, X_n$ are n independent and identically distributed symmetric around zero random variables. Let $L = a_1X_1 + a_2X_2 + \cdots + a_nX_n$. If L^2 is distributed as CH(0,1,1), then X's are normally distributed.*

Proof If is well known (see Ahsanullah 1987a, b) if Z^2 is distributed as CH(01,1) and g(t) is the characteristic function of Z, then

$$2e^{-\frac{t^2}{2}} = g(t) + g(-t). \tag{3.1.1}$$

Further if Z is symmetric around zero, then $e^{-\frac{t^2}{2}} = g(t)$.

Let $\phi(t)$ be the characteristic function of X_i's, then

$$e^{-\frac{t^2}{2}} = \prod_{i=1}^{n} \phi(a_i t).$$

It is known (see Zinger and Linnik (1964) that if for positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\phi_i(t)$, $i = 1, 2, \ldots, n$ are characteristic function,

$(\phi_1(t))^{\alpha_1}(\phi_2(t))^{\alpha_2} \ldots\ldots(\phi_n(t))^{\alpha_n} = e^{-\frac{t^2}{2}}$, for $|t| < t_0$, $t_0 > 0$ and t_0 is real, then $\phi_i(t)$, $i = 1, 2, \ldots, n$ are characteristic functions of the normal distribution. Thus X_i's, i = 1, 2,…, n are normally distributed.

Remark 3.3 If $a_1 = a_2 = \cdots = a_n = \frac{1}{\sqrt{n}}$, then from Theorem 3.3 it follows that if X_1, X_2,…, X_n are n independent, identically and symmetric around zero random variables, then if $n(\bar{X})^2$ is distributed as CH(0,1,1) where $\bar{X} = (1/n)(X_1 + X_2 + \cdots + X_n)$, then the X_i's, i = 1, 2,…, n are normally distributed.

The following theorem is by Ahsanullah and Hamedani (1988).

Theorem 3.5 *Suppose X_1 and X_{2+} are i.i.d. and symmetric (about zero) random variables and let Z = min(X_1, X_2). If Z^2 is distributed as CH(0,1,1), then X_1 and X_2 are normally distributed.*

Proof Let $\phi(t)$ be the characteristic function of Z, F(x) be the cdf of X_1 and f(x) is the pdf of X.

We have

$$\phi(t) = 2\int_{-\infty}^{\infty} e^{itx}(1 - F(x))f(x)dx$$

$$\begin{aligned}
\phi(t) + \phi(-t) &= 4[\int_{0}^{\infty} \cos(tx)(1 - F(x))f(x)dx + \int_{0}^{\infty} \cos(tx)F(x))f(x)dx \\
&= 4\int_{0}^{\infty} \cos(tx)f(x)dx \\
&= 2[\int_{-\infty}^{\infty} \cos(tx)f(x)dx] \quad \text{by symmetry of X.} \\
&= 2\phi_1(t), \text{ where } \phi_1(t) \text{ is the characteristic function of } X_1.
\end{aligned}$$

Since Z^2 is distributed as ch(0,1,1), we have

$\phi(t) + \phi(-t) = 2e^{-\frac{t^2}{2}}$. Thus $\phi_1(t) = e^{-\frac{t^2}{2}}$ *and* X_1 and X_2 are normally distributed.

Remark 3.4 It is easy to see that Z = min (X_1, X_2) in Theorem 3.4 can be replaced by the max(X_1, X_2).

The following two theorems does not use the symmetry condition on X's.

Theorem 3.6 *Let X_1 and X_2 be independent and identically distributed random variables. Suppose $U = aX_1 + bX_2$ with $0 < a, b < 1$ and $a^2 + b^2 = 1$. If U^2 and X_1^2 are distributed as CH(0,1,1), then X_1 and X_2 are normally distributed.*

Proof Let $\phi_1(t)$ and $\phi(t)$ be the characteristic functions of U and X_1 respectively. We have

$$\begin{aligned} 2e^{-\frac{t^2}{2}} &= \phi_1(t) + \phi_1(-t) \\ &= \phi(at)\phi(bt) + \phi(-at)\phi(-bt) \end{aligned} \tag{3.1.2}$$

Also

$2e^{-\frac{t^2}{2}} = \phi(t) + \phi(-t)$. We can write

$$\phi(at) + \phi(-at) = 2e^{-\frac{(at)^2}{2}}$$

and

$$\phi(bt) + \phi(-bt) = 2e^{-\frac{(bt)^2}{2}}$$

Multiplying the above two equations, we obtain

$$(\phi(at) + \phi(-at))(\phi(bt) + \phi(-bt)) = 4e^{-\frac{t^2}{2}}.$$

$$\begin{aligned} 4e^{-\frac{t^2}{2}} &= (\phi(at) + \phi(-at))(\phi(bt) + \phi(-bt)) \\ &= (\phi(at)\phi(bt)) + (\phi(at)\phi(-bt)) \\ &\quad + (\phi(-at)\phi(bt)) + (\phi(-at)\phi(-bt)) \\ &= 2e^{-\frac{t^2}{2}} + \phi(at)\phi(-bt) + \phi(-at)\phi(bt). \end{aligned}$$

Thus

$$\phi(at)\phi(-bt) + \phi(-at)\phi(bt) = 2e^{-\frac{t^2}{2}}. \tag{3.1.3}$$

We have

$$\begin{aligned} &(\phi(at) - \phi(-at))(\phi(bt) - \phi(-bt)) \\ &= \phi(at)\phi(bt) + \phi(-at)\phi(-bt) \\ &- (\phi(at)\phi(-bt) + \phi(-at)\phi(bt)) = 0 \end{aligned}$$

Thus $\phi(t) = \phi(-t)$ and $\phi(at)\phi(bt) = e^{-\frac{t^2}{2}}$.
Hence the result follows from Cramer's theorem.

Theorem 3.7 *Suppose X_1 and X_2 are independent and identically distributed random variables. Let $Z_1 = aX_1 + a_2X_2$ and $Z_2 = b_1X_1 + b_2X_2$, $-1 < a_1, a_2 < 1$, $-1 < b_1, b_2 < 1$, $1 = a_1^2 + a_2^2 = b_1^2 + b_2^2$ and $a_1b_2 + a_2b_1 = 0$. If Z_1^2 and Z_2^2 are distributed as CH(01,1), then both X_1 and X_2 are normally distributed.*

Proof Let $\phi_1(t)$ and $\phi_2(t)$ be the characteristic functions of Z_1 and Z_2 respectively.
We have

$$2e^{-\frac{(at)^2}{2}} = \phi_1(t) + \phi_1(-t) = \phi_2(t) + \phi_2(-t).$$

Now if $\phi(t)$ be the characteristic function of X_1, then

$$\phi(a_1t)\phi(a_2t) + \phi(-a_1t)\phi(-a_2t) = 2e^{-\frac{t^2}{2}} \tag{3.1.4}$$

and

$$\phi(b_1t)\phi(b_2t) + \phi(-b_1t)\phi(-b_2t) = 2e^{-\frac{t^2}{2}} \tag{3.1.5}$$

Substituting $b_1 = -\frac{a_1b_2}{a_2}$.
In the above equation, we obtain

$$\phi\left(-\frac{a_1b_2}{a_2}t\right)\phi(b_2t) + \phi\left(\frac{a_1b_2}{a_2}t\right)\phi(-b_2t) = 2e^{-\frac{t^2}{2}}.$$

Let $\frac{b_2}{a_2}\mathrm{t} = \mathrm{t}_1$, then we obtain.
In the above equation, we obtain

$$\phi(-a_1t_1)\phi(a_2t_1) + \phi(a_1t_1)\phi(-a_2t_1) = 2e^{-\frac{a_2^2t_1^2}{2b_2^2}} \tag{3.1.6}$$

Now $1 = a_2^2 + a_1^2 = a_2^2(1 + \frac{a_1^2}{a_2^2}) = a_2^2\left(1 + \frac{b_1^2}{b_2^2}\right) = \frac{a_2^2}{b_2^2}$.
From (3.1.6), we obtain

$$\phi(-a_1t)\phi(a_2t) + \phi(a_1t)\phi(-a_2t) = 2e^{-\frac{t^2}{2}} \tag{3.1.7}$$

Now

$$\begin{aligned}
&(\phi(a_1t) - \phi(-a_1t))(\phi(a_2t) - \phi(-a_2t)) \\
&= \phi(a_1t)\phi(a_2t) + \phi(-a_1t)\phi(-a_2t) \\
&= -(\phi(a_1t)\phi(-a_2t) + \phi(-a_1t)\phi(a_2t)) - 0.
\end{aligned}$$

Thus $\phi(t) = \phi(-t)\, for\, all\, t,\ -\infty < t < \infty.$
We have

$$\phi(a_1 t)\phi(a_2 t) = e^{-\frac{t^2}{2}}$$

And by Cramer's theorem it follows that both X_1 and X_2 are normally distributed.

The following theorem has lots of application in statistics.

Theorem 3.8 *Suppose $X_1, X_2, \ldots, X_n$ are n independent and identically distributed random variables with $E(X_i) = 0$ and $E(X_i^2) = 1$. Then the mean $\bar{X}(\bar{\bar{X}} = \frac{1}{n}\sum_{i=1}^{n} X_i)$ and the variance $S^2(= \sum_{i=1}^{n}(X_i - \bar{X})^2)$ are independent if and only if the distribution of the X's is N(0,1).*

Proof Suppose the pdf f(x) of X_1 as follows.

$$f(x) = \frac{1}{\sqrt{(2\pi)}} e^{-\frac{1}{2}x^2},\ -\infty < x < \infty.$$

The joint pdf of $X_1, X_2, \ldots, X_n$ can be written as

$$f(x_1, x_2, \ldots, x_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^{n} x_i^2}$$

Let us use the following transformation

$$\begin{aligned} Y_1 &= \bar{X} \\ Y_2 &= X_2 - \bar{X} \\ &\ldots\ldots\ldots\ldots\ldots \\ Y_n &= X_n - \bar{X} \end{aligned}$$

The jacobian of the transformation is n.
Further

$$\begin{aligned} \sum_{i=1}^{n} X_i^2 &= (X_1 - \overline{X})^2 + \sum_{i=2}^{n} (Y_i - \overline{X})^2 + n\overline{X} \\ &= (\textstyle\sum_{i=2}^{n} (Y_2 - \overline{X}))^2 + \sum_{i=2}^{n} (Y_i - \overline{X})^2 + nY_1^2 \end{aligned}$$

We can write the joint pdf of the Y_i' as

$$f(y_1, y_2, \ldots, y_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{\frac{1}{2}((\sum_{i=2}^{n} (y_2 - \bar{\bar{x}}))^2} e^{\sum_{i=2}^{n} (y_i - \bar{\bar{x}})^2} e^{ny_1^2}.$$

Thus $\bar{X} = (\mathrm{Y}_1)$ and $S^2 (= \sum_{i=1}^{n} (X_i - \bar{X})^2)$ are independent.
Suppose $\phi_1(t)$ and $\phi_2(t)$ *be the charateristic* functions of $\bar{X}$ and S^2 respectively.
Let $\phi(t_1, t_2)$ be the joint characteristic function of $\bar{X}$ and S^2.
We can write

$$\phi(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{it_1 \bar{\bar{x}} + it_2 s^2} f(x_1) f(x_2) \ldots f(x_n) dx_1 dx_2 \ldots dx_n,$$

$$\phi_1(t_1) = \phi(t_1, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{it_1 \bar{\bar{x}}} f(x_1) f(x_2) \ldots f(x_n) dx_1 dx_2 \ldots dx_n$$

and

$$\phi_2(t_2) = \phi(0, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{it_2 s^2} f(x_1) f(x_2) \ldots f(x_n) dx_1 dx_2 \ldots dx_n.$$

Since $\bar{X}$ and S^2 are independent, we must have

$$\phi(t_1, t_2) = \phi(t_1, 0)\phi(0, t_2) \tag{3.1.8}$$

Writing $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$, we can write

$$\begin{aligned}\phi_1(t_1) &= \prod_{k=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{it_1 \frac{1}{n} \sum_{i=1}^{n} x_i)} f(x_1) f(x_2) \ldots f(x_n) dx_1 dx_2 \ldots dx_n \\ &= (\phi\left(\frac{t_1}{n}\right))^n,\end{aligned}$$

where $\phi(.)$ is the characteristic function of X_1

$$\begin{aligned}\frac{d}{dt_2}(\phi(t_1, t_2))|_{t_2=0} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} is^2 e^{it_1 \bar{\bar{x}}} f(x_1) f(x_2) \ldots f(x_n) dx_1 dx_2 \ldots dx_n, \\ &= \mathrm{i}(\phi\left(\frac{t_1}{n}\right))^n (E(s^2)), \text{ since } \bar{X} \text{ and } S^2 \text{are independent.} \\ &= (n-1)\mathrm{i}(\phi\left(\frac{t_1}{n}\right))^n\end{aligned}$$

Substituting $S^2 = \frac{n-1}{n} \sum_{i=1}^{n} X_i^2 - \frac{1}{n} \sum_{i \neq j=1}^{x} X_i X_j$ and using

$$\phi'(t) = i \int_{-\infty}^{\infty} e^{itx} f(x) dx, \phi''(t) = -\int_{-\infty}^{\infty} x^2 e^{itx} f(x) dx$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} is^2 e^{it\overline{x}} f(x_1)f(x_2)\ldots f(x_n)dx_a dx_2\ldots dx_n$$
$$= -i(n-1)\Phi''(\frac{t_1}{n})((\Phi\left(\frac{t_1}{n}\right))^{n-1}) - i(n-1)\Phi'(\frac{t_1}{n})((\Phi\left(\frac{t_1}{n}\right))^{n-2}),$$

we obtain

$$\phi''(t)(\phi(t))^{n-1} - (\phi'(t))^2(\phi(t))^{n-2} = -(\phi(t))^n \tag{3.1.9}$$

On simplification, we have

$$\frac{\phi''(t)}{\phi(t)} - \frac{(\phi'(t))^2}{(\phi(t))^2} = -1$$

We can write the above equation as

$$\frac{d^2}{dt^2} ln\phi(t) = -1 \tag{3.1.10}$$

Using the condition E(X_i) = 0 and E(x^2) = 1, we will have

$$\phi(t) = e^{-\frac{t^2}{2}}, -\infty < t < \infty.$$

Thus the distribution of the X_i's is N(0,1).

It is known that if X_1 and X_2 are independently distributed as $N(0,1)$, then X/Y is distributed as CA(0,1). The converse is not true. For the following Theorem that we need some additional condition to characterize the normality of X_1 and X_2.

Theorem 3.9 *Let X_1 and X_2 be independent and identical distributed absolutely continuous random variables with cdff $F(x)$ and pdf $f(x)$. Let $Z = \min(X_1, X_2)$. If Z^2 and $V = \frac{X_1}{X_2}$ are distributed as $CA(0,1)$, then X_1 and X_2 are distributed as $N(0,1)$,*

Proof Since $\frac{X_1}{X_2}$ is distributed as $CA(0,1)$ we have

$$\int_{-\infty}^{\infty} f(uv)f(v)vdv = \frac{1}{\pi(1+u^2)}, -\infty < \mu < \infty. \tag{3.1.11}$$

Or

$$\int_{0}^{\infty} (f(uv)f(v) + f(-uv)f(-v))vdv = \frac{1}{\pi(1+u^2)} \tag{3.1.12}$$

Now letting *u* → *1* <u>*and*</u> *u* → *−1,* we obtain

$$\int_0^\infty \left[(f(v))^2 + (f(-v))^2\right] v dv = \frac{1}{2\pi} \tag{3.1.13}$$

and

$$2\int_0^\infty f(v)f(-v) = \frac{1}{2\pi} \tag{3.1.14}$$

Using (3.1.13) and (3.1.14) we obtain

$$\int_0^\infty [f(v) - f(-v)]^2 v dv - 0 \tag{3.1.15}$$

Thus the distribution of X_1 and X_2 is symmetric and hence their distribution is *N (0,1).*

3.2 Characterization of Levy Distribution

Theorem 3.10 *Let X_1, X_2 and X_3 be independent and identically distributed absolutely continuous random variable with cumulative distribution function F(x) and probability density function f(x). We assume F(0) = 0 and F(x) > 0 for all x > 0 Then X_1 and $(X_2 + X_3)/4$ are identically distributed if and only if F(x) has the Levy distribution with pdf f(x) as*

$$\mathrm{f(x)} = \sqrt{\left(\frac{\sigma}{2\pi}\right)}\frac{e^{-\frac{\sigma}{2x}}}{x^{3/2}},\ x > 0,\ \sigma > 0.$$

Proof Suppose the random variable X_1 has the pdf $\mathrm{f(x)} = \sqrt{\left(\frac{\sigma}{2\pi}\right)}\frac{e^{-\frac{\sigma}{2x}}}{x^{3/2}},\ x > 0,\ \sigma > 0.$ Then the characteristic function $\phi(t)$ is

$$\phi(t) = \int_0^\infty e^{itx}\sqrt{\left(\frac{\sigma}{2\pi}\right)}\frac{e^{-\frac{\sigma}{2x}}}{x^{3/2}}dx = e^{-\sqrt{-2i\sigma t}}.$$

The characteristic function of $(X_2 + X_3)/4$ is

$$e^{-\sqrt{-i\sigma t/2}}.e^{-\sqrt{-i\sigma t/2}} = e^{-\sqrt{-2i\sigma t}}.$$

Thus X_1 and $(X_2 + X_3)/4$ are identically distributed.

Suppose that X_1 and $(X_2 + X_3)/4$ are identically distributed. Let $\varphi(t)$ be their characteristic function, then

$$\varphi(t) = (\varphi(t/2^2))^2 = \cdots = (\varphi(t/2^{2n}))^{2^n},\ n = 1,\ 2,\ \ldots$$

Taking logarithm of both sides of the equation, we obtain

$$(\ln \varphi(t))^2 = 2^{2n}(\ln \varphi(t/2^{2n}))^2$$

Let $\Psi(t) = (\ln \varphi(t))^2$, then

$$\Psi(t) = 2^{2n} \quad \Psi(t/2^{2n})n = 1, 2, \ldots$$

The solution of the above equation is
$\Psi(t) = ct$, where c is a constant.
Hence

$$\varphi(t) = e^{-\sqrt{ct}}$$

Using the condition
$\phi(-t) = \overline{\phi(t)}$, where
$\overline{\phi(t)}$ is the complex conjugate of $\phi(t)$, we can take c = −2i σ, where $i^2 = -1$ and $\sigma > 0$ as a constant.

3.3 Characterization of Wald Distribution

The following theorem (Ahsanullah and Kirmani 1984) gives a characterization of the Wald distribution.

Theorem 3.11 *Let X be an absolutely continuous nom-negative random variable with pdf f(x). Suppose that* $\mathrm{xf(x)} = x^{-2}f(x^{-1})$ *and* X^{-1} *and* $\mathrm{X} + \lambda^{-1}\mathrm{Z}$, *where* $\lambda > 0$ *are identically distributed where Z is distributed as CH(0,1,1). Then X has the Wald distribution.*

Proof Let ϕ_1 *and* ϕ be the characteristic functions of !/X and X respectively. Then we have

$$\begin{aligned}
\phi_1(t) = E\left(e^{itX^{-1}}\right) &= \int_0^\infty e^{itx^{-1}} f(x)dx \\
&= \int_0^\infty e^{ity} y^{-2} f\left(y^{-1}\right)dy \\
&= \int_0^\infty e^{ity} y f(y)dy \\
&= \frac{1}{i}\phi'(t)
\end{aligned}$$

$$\phi_1(t) = \textit{charateristic function of } X + \lambda^{-1}Z = \phi(t)(1 - 2it\lambda^{-1})^{-1/2}.$$

Now

$$\frac{1}{i}\phi^{'}(t)=\phi(t)(1-2it\lambda^{-1})^{-1/2}$$

and hence $\phi(t)=\exp(\lambda(1-\left(1-2it\lambda^{-1}\right)^{\frac{1}{2}}))$ which is the characteristic function of the Wald distribution with pdf f(x) as

$$\mathrm{f(x)}=(\frac{\lambda}{2\pi x^{3}})^{1/2}\exp(-\lambda(x-1)^{2}(2x)^{-1}),\mathrm{x}>0,\lambda>0.$$

3.4 Characterization of Exponential Distribution

Kakosyan et al. (1984) conjectured that the identical distribution of $p\sum_{j=1}^{M}X_{j}$ and $MX_{1,M}$ where $\mathrm{P(M=k)}=p(1-p)^{k-1}, 0<p<1,\ k=1,\ 2,\ \ldots$ characterizes the exponential distribution. The following is a generalization of the conjecture due to Ahsanullah (1988a–c).

Theorem 3.12 *Let X be independent and identically distributed non-negative random variables with cdf F(x) and pdf f(x). We assume M as an integer values random variables with P(M = k) =* $p(1-p)^{k-1}, 0<p<1,\ k=1,\ 2,\ \ldots$ Then *the following two properties are equivalent.*

(a) X's have exponential distribution with $\mathrm{F(x)}=1-e^{-\lambda x}$, $\mathrm{x}\geq 0$,

(b) $\mathrm{p}\sum_{j=1}^{M}X_{j}\ \underline{\underline{d}}\ D_{r,n}$, *where* $D_{r,n}=(n-r+1)(X_{r,n}-X_{r-1,n})$. $1\leq r\leq n$, *n* *2,* $X_{0,n}=0$, *if E(X) is finite,* $X_i\in C_1$ *and* $\lim_{x\to 0}\frac{\bar{F}(x)}{x}=\lambda$,

Proof Let r $\geq$ 2.

Let $\varphi_1(t)$ and $\varphi_2(\mathrm{t})$ be the characteristic functions of $\mathrm{p}\sum_{j=1}^{M}X_j$ and $\mathrm{D_{r,n}}$ respectively

$$\begin{aligned}\varphi_1(t)&=Ee^{itp\sum_{j=1}^{m}X_j}\\&=\textstyle\sum_{k=1}^{m}p(1-p)^{k}(\varphi(pt)^{k},\ \text{where } \varphi(t) \text{ is the characteristic function of the } \mathrm{X}'\mathrm{s}.\\&=\mathrm{p}\varphi(pt)(1-\mathrm{q}\varphi(pt))^{-1},\ \mathrm{q}=1-\mathrm{p}.\end{aligned}\tag{3.3.1}$$

If $\mathrm{F(x)}=1-e^{-\lambda x}$, then $\varphi(t)=\frac{\lambda}{\lambda-it}$ and $\varphi_1(\mathrm{t})=\varphi(t) = \frac{1}{1-\lambda it}$

$$\varphi_2(t)=\int_0^\infty\int_0^\infty \frac{n!e^{itv}}{(r-2)!(n-r)!}(F(u))^{r-2}(1-F\left(u+\frac{v}{n-r+1}\right))^{n-r}f(u)f(u+\frac{v}{n-r+1})\mathrm{dudv}$$
$$=1+\mathrm{it}\frac{n!}{(r-2)!(n-r+1)!}\int_0^\infty e^{itv}(F(u))^{r-2}\left(1-F\left(u+\frac{v}{n-r+1}\right)\right)^{n-r}f(u)f(v)dudv\mathrm{v}$$

Substituting $\mathrm{f(x)}=\lambda e^{-\lambda x}$ and $\mathrm{F(x)}=1-e^{-\lambda x}$, we obtain $\varphi_1(t)=\varphi_2(t)$. Thus (a) $\Rightarrow (b)$.

We now proof (b) $\Rightarrow (a)$.

Since $\varphi_1(t)=\varphi_2(t)$, we get on simplification for $r\geq 2$,

$$\frac{\varphi(pt)-1}{1-q\varphi(pt)}\frac{1}{it}=\frac{n!}{(r-2)!(n-r+1)!}\int_0^\infty\int_0^\infty e^{itv}(F(v))^{r-2}\left(1-F\left(u+\frac{v}{n-r+1}\right)\right)^{n-r+1}f(u)f(v)dudv \tag{3.3.2}$$

Taking limits of both sides of (3.3.2) as t goes to 0, we have

$$\frac{\varphi^{'}(0)}{i}=\frac{n!}{(r-2)!(n-r+1)!}\int_0^\infty\int_0^\infty f(u)(F(v))^{r-2}\left(1-F\left(u+\frac{v}{n-r+1}\right)\right)^{n-r+1}f(v)dudv \tag{3.3.3}$$

Writing $\frac{\varphi^{'}(0)}{i} = \int_0^\infty (1-F(v))dv$, we obtain from (3.3.3)

$$\int_0^\infty\int_0^\infty f(u)(F(v))^{r-2}(1-F(v))^{n-r+1}H(u,v)f(v)dudv \tag{3.3.4}$$

where $\mathrm{H(u,v)}=(\frac{1-F(u+\frac{v}{n-r+1})}{1-F(u)})^{n-r+1}-(1-\mathrm{F(v)})$.

If X belongs to the class c_1, = then it is proved (see Ahsanullah 1988a–c) that H (o,v) = o for all v $\geq$ 0.

$$\text{Thus for all } \mathrm{v}>0, (1-F(\frac{v}{n-r+1})^{n-r+1})=(1-\mathrm{F(v)}) \tag{3.3.5}$$

Since $\lim_{\to 0}\frac{F(x)}{x}=\lambda$, $\lambda>0$ it follows from (3.3.5) that

$\mathrm{F(x)}=1-e^{-\lambda x}$, $\lambda>0$ and x $\geq$ 0.

The proof of the theorem for r = 1 is similar.

3.5 Characterization of Symmetric Distribution

Theorm 3.13 *The following theorem gives a characterization of the symmetric distribution'*

Behboodian (1989) conjectured that if X_1, X_2 and X_3 are independent and if $X_1 + mX_2 - (1 + m)X_3$ for some m. $0 < m \leq 1$ is symmetric about Θ, then X's are symmetric about θ.

The following theorem gives a partial answer to the question.

Suppose X_1, X_2 and X_3 are independent and identically distributed random variable with cdf F(x), pdf f(x) and $\phi(t)$ is the characteristic function of X_1 such that $\phi(t) \neq 0$ for any t, $-\infty < t < \infty$, the random variable $Y = X_1 + m X_2 - (1 + m) X_3$ is symmetric around θ if and only if X's are symmetric around θ

Proof We can write

$$\mathrm{Y} = \mathrm{X}_1 - \theta + \mathrm{m}(\mathrm{X}_2 - \theta) - (1 + \mathrm{m})(\mathrm{X}_3 - \theta).$$

Thus if X's are symmetric about θ, then Y is symmetric about θ.
Let $\varphi_1(t)$ and $\varphi_2(t)$ be the characteristic functions of Y and X's.
We can write

$$\varphi_1(\mathrm{t}) = \varphi_2(t)\varphi_2(mt)\varphi_2(-(1+m)t).$$

Since Y is symmetric about θ, *we must have* $\varphi_1(t) = \varphi_1(-t)$,
i.e.

$$\varphi_2(t)\varphi_2(mt)\varphi_2(-(1+m)t) = \varphi_2(-t)\varphi_2(-mt)\varphi_2((1+m)t).$$

Using $\mathrm{h(t)} = \frac{\varphi_2(t)}{\varphi_2(-t)}$, we obtain

$$\begin{aligned}&\mathrm{h(t)h(mt)} = \mathrm{h((1+m)t)}.\\&\text{Substituting}\,\mathrm{g(t)} = \ln \mathrm{h(t)}, \text{we obtain}\\&\mathrm{g(t)} + \mathrm{g(mt)} = \mathrm{g((1+m)t)}\end{aligned} \tag{3.4.1}$$

The solution of the Eq. (3.4.1) is
g(t) = ct, where c is a constant.
Thus

$$\frac{\varphi_2(t)}{\varphi_2(-t)} = \mathrm{h(t)} = e^{g(t)} = e^{2ct}$$

and

$$\varphi_2(t) = e^{2ct}\varphi_2(-t)$$

Since $|\varphi_2(t)| = |\varphi_2(-t)|$, we must have $c = i\theta$ where $i = \sqrt{-1}$ and θ is any real number. We can write

$$\varphi_2(t)e^{-i\theta t} = \varphi_2(-t)e^{i\theta t}.$$

Thus X's are symmetric about θ.

3.6 Charactetrization of Logistic Disribution

Theorem 3.14 Suppose that the random variable X is continuous and symmetric about 0, the X has the logistic distribution with $F(x) = \frac{1}{1+e^{-\lambda x}}$, $\lambda > 0$ and $x \geq 0$ if and only

$P(=x < X|X < x) = 1 - e^{-\lambda x}$, $\lambda > 0$ and $x \geq 0$.

Proof We have $P(=x < X|X < x) = \frac{2F(x)-1}{F(x)}$, if $F(x) = \frac{1}{1+e^{-\lambda x}}$ Then

$$P(=x < X|\frac{2F(x)-1}{F(x)}X < x) = \frac{2F(x)-1}{F(x)} = 1 - e^{-\lambda x}.$$

$$\text{Suppose}\frac{2F(x)-1}{F(x)} = 1 - e^{-\lambda x}. Then\, F(x) = 1 - e^{-\lambda x}. \qquad (3.5.1)$$

3.7 Characterization of Distributions by Truncated Statistics

We will use the following two lemmas to characterize some distributions by truncated distributions.

Lemma 3.1 *Suppose the random variable X is absolutely continuous with cdf F(x) and pdf f(x). Let*

$a = \inf\{x|F(x) > 0\}$, $\beta = \sup\{x|F(x) < 1\}$, $h(x)$ *is a continuous of x for* $\alpha < x < \beta$.

We assume E(h(x)) exists. If $E(h(X)|X \leq x) = g(x)\frac{f(x)}{F(x)}$, *where g(x) is a differential function for all x,* $\alpha < x < \beta$ *and* $\int_\alpha^x \frac{h(u)-g'(u)}{g(u)}du$ *is finite for all* $a < x < \beta$, *then* $f(x) = ce\int_\alpha^x \frac{h(u)-g'(u)}{g(u)}du$, *where c is determined by the condition* $\int_\alpha^\beta f(x)dx = 1$.

Proof

$$\text{We have } g(x) = \frac{\int_{\alpha}^{x} h(u)f(u)du}{f(x)} \text{ and } \int_{\alpha}^{x} h(u)f(u)du = g(x)f(x). \tag{3.6.1}$$

Differentiating both sides of the above equation with respect to x, we obtain

$$\frac{f^{'}(x)}{f(x)} = \frac{h(x) - g^{'}(x)}{g(x)} \tag{3.6.2}$$

On integrating the above, we obtain

$$f(x) = ce \int_{\alpha}^{x} uf(u)du \tag{3.6.3}$$

where c is determined by the condition $\int_{\alpha}^{\beta} f(x)dx = 1$.

Lemma 3.2 *Suppose the random variable X is absolutely continuous with cdf F(x) and pdf f(x). Let*

$\alpha = \inf\{x|F(x) > 0\}, \beta = \sup\{x|F(x) < 1\}, m(x)$ *is a continuous function of x for* $\alpha < x < \beta$.

We assume E(m(x)) exists. If $E(m(X)|X \geq x) = n(x)\frac{f(x)}{1 - F(x)}$, *where g(x) is a differential function for all* $x, \alpha < \beta$ *and* $\int_{\alpha}^{\beta} \frac{m(u) + n'(u)}{g(u)} du$ *is finite for all* $\alpha < x < \beta$, *then* $f(x) = ce^{-\int_{\alpha}^{x} \frac{m(u) - n^{'}(u)}{n(u)} du}$, *where c is determined by the condition* $\int_{\alpha}^{\beta} f(x)dx = 1$.

Proof We have $n(x) = \frac{\int_{z}^{\beta} m(u)f(u)du}{f(x)}$ and

$$\int_{x}^{\beta} m(u)f(u)du = n(x)f(x). \tag{3.6.4}$$

Differentiating both sides of the above equation with respect to x, we obtain $-m(x)f(x) = n(x)f^{'}(x) + n^{'}(x)f(x)$. On simplification

$$\frac{f^{'}(x)}{f(x)} = -\frac{m(x) + n^{'}(x)}{n(x)} \tag{3.6.5}$$

On integrating the above, we obtain

$$f(x) = ce^{-\int_{\alpha}^{x} \frac{m(u) + n^{'}(u)}{n(u)} du} \tag{3.6.6}$$

where c is determined by the condition $\int_{\alpha}^{\beta} f(x)dx = 1$.

3.7.1 Characterization of Semi Circular Distribution

The following theorem characterized semi-circular distribution using the right truncation of the random variable *X*. A random variable *X* has the standard semi-circular distribution if the *pdf F(x) of X* is as follows:

$$f(x) = \frac{2}{\pi}\sqrt{1-x^2,}\ -1 < x < 1. \tag{3.5.7}$$

Theorem 3.15 *Suppose that IXI is an absolutely continuous random variable with* $cdf(F)x$ *and* $pdff(x)$.

We assume $F(-1) = 0, F(x) > 0$ *for* $x > -1$ *and* $F(1) = 1$. *Then* $E(X|X \geq x) = g(x)\frac{f(x)}{F(x)}.\ x > -1$, *where* $g(x) = \frac{x^2-1}{3}$ *if and only if* $f(x) = \frac{2}{n}\sqrt{1-x^2},\ -1 < x < 1$.

Proof If *f(x)* =

We have

$f(x) = \frac{2}{\pi}\sqrt{1-x^2}$ then $g(x) = \frac{\int_{-1}^{x} u\sqrt{1-u^2}}{\sqrt{1=x^2}} du = \frac{x^2-1}{3}$

Suppose $g(x) = \frac{x^2-1}{3}$, then $g'(x) = \frac{2x}{3}$

By Lemma 3.1,

$$\frac{f(x)}{fx(x)} = \frac{x - g'(x)}{g(x)} = \frac{-x}{1-x^2} \tag{3.6.8}$$

On integrating the above equations, we obtain $f(x) - c\sqrt{1-x^2}$, where c is a constant.

Using the condition $\int_{=1}^{1} f(x) = 1$,

we obtain

$$f(x) = \frac{2}{\pi}\sqrt{1-x^2},\ -1 < x < 1.$$

If *h(x)* and *g(x)* satisfy the conditions given in the Lemma 3.1 then knowing *h(x)* and *g(x),* we can use Lemma 3.1 to characterize various distributions.

3.7.2 Characterization of Lindley Distribution

We use Lemma 3.2 to characterize Lindley distribution.

A random variable X is said to have Lindley distribution if the pdf f(x) is of the following form:

$$f(x) = \frac{\beta^2}{1+\beta}(1+x)e^{-\beta x}, \; x \geq 0, \; \beta > 0. \tag{3.6.9}$$

Theorem 3.16 *Suppose that the random variable X has an absolutely continuous with cdf F(x) and pdf f(x). We assume that* $F(0) = 0, F(x) > 0$ *for all x > 0 and E* (X^n) *exists for some fixed n > 0. Then* $E(X^n|X \geq x) = g(x)\frac{f(x)}{1-F(x)}$, *where* $g(x) = \frac{\sum_{k=0}^{n+1} c_k x^k}{1+x}$, $c_0 = \frac{n!(n+1+\beta)}{\beta^{n+1}}$, $c_{k+1} = \frac{\beta}{k+1}$, $k = 0, 1, 2, \ldots, n-1$, $c_{n+1} = \frac{1}{\beta}$, *if and only if* $f(x) = \frac{\beta^2}{1+\beta}(1+x)e^{-\beta x}$, $x \geq 0$, $\beta > 0$.

Proof If $f(x) = \frac{\beta^2}{1+\beta}(1+x)e^{-\beta x}$, then

$$g(x) = \frac{\int_x^\infty u^n f(u)du}{1-F(x)} \frac{\int_x^\infty u^n(1+u)e^{\beta u}du}{(1+x)e^{-\beta x}} = \frac{\sum_{k=0}^{n+1} c_k x^k}{1+x}.$$

Suppose $h(x) = x^n$ and $g(x) = \frac{\sum_{k=0}^{n+1} c_k x^k}{1+x}$.
We have

$$\begin{aligned}\beta \sum_{k=0}^{n+1} c_k x^k - \sum_{k=0}^{n+1} k c_k x^{k-1} &= \delta c_{n+1} x^{n+1} + [\beta c_n - (n+1)x^n] \\ &\quad + \sum_{k=0}^{n-1} (\beta c_k - (k+1)c_{k+1})x^k \\ &= x^n(1+x)\end{aligned}$$

Thus

$$\frac{h(x)}{g(x)} = \frac{x^n(1+x)}{\sum_{k=0}^{n+1} c_k x^k} = \beta - \frac{\sum_{k=0}^{n+1} k c_k x^{k-1}}{\sum_{k=0}^{n+1} c_k x^k}.$$

Since $1ng(x) = -1n(1+x) + 1n(\sum_{k=0}^{n+1} c_k x^k)$.
Now

$$\frac{g'(x)}{g(x)} = \frac{1}{1+x} + \frac{\sum_{k=0}^{n+1} k c_k x^{k-1}}{\sum_{k=0}^{n+1} c_k x^k} = -\frac{1}{1+x} + \beta - \frac{h(x)}{g(x)}.$$

Thus

$$\frac{h(x) + g(x)}{g(x)} = -\frac{1}{1+x} + \beta$$

By Lemma 3.2

$$\frac{f'(x)}{f(x)} = -\frac{h(x)+g'(x)}{g(x)} - (\beta - \frac{1}{1+x}) \tag{3.6.10}$$

On integrating the above equation with respect to x, we obtain $f(x) = c(1+x)e^{-\beta x}$, where c is a constant. Using the condition $\int_0^\infty f(x)dx = 1$, we obtain

$$f(x) = \frac{\beta^2}{1+\delta}(1+x)e^{-\beta x}, x \geq 0, \beta > 0.$$

3.7.3 Characterization of Rayleigh Distribution

Theorem 3.17 *Suppose the random variable X has an absolutely continuous cdf F(x) and pdf f(x). We assume F(0) = o and F(x) > 0 for all x >. If* $(E(X^{2n}))$ *is finite for any n > 0, then X has a Rayleigh distribution with* $F(x) = 1 - e^{-cx^2}, c > 0, x \geq 0$ *if and only if*

$\mathrm{E}(X^{2n}|X > t) = \sum_{k=0}^{n} \frac{n^{(k)}}{c^k} t^{2(n-k)}$, *where* $n^{(i)} = \mathrm{n}(\mathrm{n}-1)\ldots(\mathrm{n}-\mathrm{i}+1)$

Proof of this theorem can be established following the proof of Theorem 3.16.

There is a similar characterization using truncated odd moments. For details of this and some other characterization of Rayleigh distribution, see Ahsanullah and Shakil (2011a, b).

Chapter 4
Characterizations of Univariate Distributions by Order Statistics

In this chapter several characterizations of univariate continuous distributions based on order statistics will be presented.

4.1 Characterizations of Student's t Distribution

We will consider the random variable X has an absolutely continuous distribution with cdf as F(x) and pdf f(x). Suppose $\alpha(F) = \inf\{x|F(x) > 0\}$ and $e(F) = \sup\{x|F(x) < 1\}$.

Let $f_{ST}(x, n)$ be the pdf of Student's t distribution with n degrees of freedom (ST (n)). We have

$$f_{ST}(x,n) = \frac{1}{\sqrt{n}}\frac{1}{B\left(\frac{n}{2},\frac{1}{2}\right)}\left(1+\frac{x^2}{n}\right)^{-(n+1)/2}, \ -\infty < x < \infty, \ n \geq 1. \tag{4.1.1}$$

The Student's t distribution with 2 degrees of freedom has the pdf $f_{ST}(x, 2)$ where

$$f_{ST}(x,2) = \frac{1}{2\sqrt{2}}\left(1+\frac{x^2}{2}\right)^{-\frac{3}{2}}, \ -\infty < x < \infty. \tag{4.1.2}$$

The corresponding cdf $F_{ST}(x, 2)$ is

$$F_{ST}(x,2) = \frac{1}{2}\left\{1+\frac{x}{\sqrt{(2+x^2)}}\right\}, \ -\infty < x < \infty. \tag{4.1.3}$$

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It can be seen that

$$F(x)(1-F(x))^{3/2} = cf(x), \text{ where } c = 2^{-3/2} \tag{4.1.4}$$

Let $W_n = (X_{1,n} + X_{n,n})/2$ and $M_n = X_{(n+1)/2,n}$ for odd n.
It can be shown that

$$\begin{aligned} E(X_{1,3}|M_3 = x) &= \frac{\int_{-\infty}^{x} \frac{1}{2\sqrt{2}}\left(1+\frac{u^2}{2}\right)^{-3/2} du}{\frac{1}{2}\left\{1+\frac{x}{\sqrt{2+x^2}}\right\}} \\ &= \frac{-2}{x+\sqrt{(2+x^2)}} \end{aligned} \tag{4.1.5}$$

and

$$\begin{aligned} E(X_{2,3}|M_3 = x) &= \frac{\int_{x}^{\infty} \frac{1}{2\sqrt{2}}\left(1+\frac{u^2}{2}\right)^{-3/2} du}{\frac{1}{2}\left\{1-\frac{x}{\sqrt{2+x^2}}\right\}} \\ &= \frac{2}{\sqrt{2+x^2}-x} \end{aligned} \tag{4.1.6}$$

Thus

$$E(W_3|M_3 = x) = x \tag{4.1.7}$$

The relation shown in (4.1.7) is a characterizing property of the Student's t distribution of 2 degrees of freedom.

We have the following theorem due to Nevzorov et al. (2003).

Theorem 4.1 L*et X_1, X_2, X_3 be independent and identically distributed random variables with cdf F(x) and pdf f(x). We assume $E(X_1)$ exists*

The regression function $\varphi(x) = E(W_3|M_3 = x) = x$, $\alpha(F) < x < e(F)$,

If and only cdf of X_1 is of the following form

$$F_{ST}(x, 2) = \frac{1}{2}\left\{1+\frac{x}{\sqrt{2+x^2}}\right\}, \ -\infty < x < \infty. \tag{4.1.8}$$

Proof The proof of "if" condition is given in (4.1.7). We will give here the proof that $\varphi(x) = x$ implies that the cdf of X is as given in (4.1.8). We know (see Nagaraja and Nevzorov 1997) that

$$E(X_1|X_{k,n} = x) = \frac{x}{n} + \frac{k-1}{n}E(X|X \le x) + \frac{n-k}{n}E(X \ge x), \ 1 \le k \le n.$$

Thus we have

$$\begin{aligned}\mathrm{E}(\mathrm{X}_1|\mathrm{M}_3=\mathrm{x}) &= \frac{x}{3}+\frac{1}{3}E(X|X\leq x)+\frac{1}{3}E(X\geq x).\\ &=\frac{1}{3}\{x+\frac{1}{F(x)}\int_{-\infty}^{x}uf(u)du+\frac{1}{1-F(x)}\int_{x}^{\infty}uf(u)du\end{aligned}$$

We have $\lim_{x\to\infty}xF(x)=\lim_{x\to\infty}x(1-F(x))=0.$
We have

$$\frac{1}{F(x)}\int_{-\infty}^{x}uf(u)dou=x-\frac{1}{F(x)}\int_{-\infty}^{x}F(u)du$$

and

$$\frac{1}{1-F(x)}\int_{x}^{\infty}uf(u)du=\mathrm{x}+\frac{1}{1-F(x)}\int_{x}^{\infty}(1-F(u)du$$

Thus $\mathrm{E}(\mathrm{X}_1|\mathrm{X}_{1,3}=\mathrm{x}\ )=\mathrm{x}-\frac{1}{F(x)}\int_{-\infty}^{x}F(u)du+\frac{1}{1-F(x)}\int_{x}^{\infty}(1-F(u))du$
and

$$\begin{aligned}\mathrm{x}=\phi(x)=E(W_3|M_3=x)&=E\left(\frac{1}{2}(X_{1,3}+X_{2.3})|M_3=x\right)\\ 2\mathrm{x}=\mathrm{E}(3\bar{x}-x|M_3=x).&\\ \mathrm{x}=\ \mathrm{E}(\bar{x}|M_3=x)\mathrm{E}(\mathrm{X}_1|\mathrm{M}_3=\mathrm{x})=\ \mathrm{E}(\mathrm{X}_1|\mathrm{X}_{1,3}=\mathrm{x}).&\end{aligned}$$

We have

$$\mathrm{F(x)}\int_{x}^{\infty}(1-F(u))du-(1-F(x))\int_{-\infty}^{x}F(u)du=0 \tag{4.1.9}$$

We can write (4.9) as

$$\frac{d}{dx}\left[\int_{-\infty}^{x}F(u)du\int_{x}^{\infty}(1-F(u))du\right]=0$$

i.e.

$$\int_{-\infty}^{x}F(u)du\int_{x}^{\infty}(1-F(u))du=\mathrm{c} \tag{4.1.10}$$

where c is a constant.

We rewrite (4.1.10) as

$$\int_x^\infty (1 - F(u))du \quad \frac{c}{\int_{-\infty}^x F(u)du}$$

Differentiating the above equation with respect to x, we obtain

$$\mathrm{I} - \mathrm{F(x)} = \frac{cF(x)}{\{\int_{-\infty}^x F(u)du\}^2},$$

which is equivalent to

$$\int_{-\infty}^x F(u)du = \left\{\frac{cF(x)}{1 - F(x)}\right\}^{1/2}$$

Differentiating the above equation with respect to x, we obtain

$$\{F(x)(1 - F(x)\}^{3/2} = cf(x), c > 0 \tag{4.1.11}$$

This is the equation we have seen in (4.4).

Nevzorov et al. (2003) showed that the unique solution of the above equation is the Student's t-distribution with 2 degrees of freedom. The cdf F(x) with location parameter μ and σ is

$$\mathrm{F(x)} = \frac{1}{2}\left[1 + \frac{x - \mu}{\sqrt{\left\{\sigma^2 + (x - \mu)\right]^2\right\}}}\right], \quad -\infty < \mu < x < \infty, \sigma > 0.$$

Let Q(x) be the quantile function of a random variable X with cdf F(x) i.e. F(Q (x)) = x for 0 < x < 1. Akhundov et al. (2004) proved that for $0 < \lambda < 1$, the relation

$$E(\lambda X_{1.3} + (1 - \lambda)X_{3,3}|X_{2.3} = x) = x$$

characterizes a family of probability distributions with quantile function

$$Q_\lambda(x) = \frac{c(x - \lambda)}{\lambda * (1 - \lambda)(1 - x)^\lambda x^{1-\lambda}} + d$$

where $0 < \mathrm{c} < \infty$ and $-\infty < d < \infty$. We will call this family as Q family.

The Student's t distribution with 2 degrees of freedom belongs to the Q family with the quantile function $Q_{1/2}(x) = \frac{2^{1/2}(x - 1/2)}{x^{1/2}(1 - x)^{1/2}}$

Yanev and Ahsanullah (2012) characterized Student's t distribution with more than 2 degrees of freedom They proved that a random variable X belongs to the Q family if $E(|X|) < \infty$, *and* for some k, $2 \le k \le n-1$, $0 < \lambda < 1$ *and*

$$\lambda E\left(\frac{1}{k-1}\sum_{j=1}^{k-1} X_{j,n}|X_{k,n}\right) + (1-\lambda)E\left(\frac{1}{n-k}\sum_{j=k+1}^{x} X_{j,n}|X_{k,n}=x\right) = x.$$

For k = 2 this is the result of Akhundov et al. (2004).

4.2 Characterizations of Distributions by Conditional Expectations (Finite Sample)

We assume that E(X) exists. We consider that $E(X_{j,n}|X_{i,.n} = x) = ax + b$, $j > i$. Fisz (1958) considered the characterization of exponential distribution by considering j = 2, i = 1 and a = 1. Roger (1963) characterized the exponential distribution by considering j = i +1 and a = 1. Ferguson (1963) characterized the following distributions with j = i + 1.

(i) Exponential distribution with a = 1
(ii) Pareto distribution with a > 1
(iii) Power function distribution with a < 1.

Gupta and Ahsanullah (2004a, b) proved the following theorem.

Theorem 4.2 *Under some mild conditions on* $\psi(x)$ *and g(x) the relation*

$$E(\psi(X_{i+s,n})|X_{i,n} = x) = g(x) \tag{4.2.1}$$

uniquely determines the distribution F(x).

The relation (4.2.1) for s = 1 will lead to the equation

$$r(x) = \frac{g'(x)}{(n-i)(g(x)-\psi(x))} \tag{4.2.2}$$

Here r(x) = f(x)/(1 − F(x)), the hazard rate of X. If

$\psi(x) = x$ and g(x) = ax + b, then we obtain from (4.2.2)

$$r(x) = \frac{a}{(n-i)((a-1)x+b)} \tag{4.2.3}$$

From (4.2.3) we have

(i) $a = 1$, then r(x) = constant and X has the exponential distribution with $F(x) = 1 - e^{-\lambda(x-\mu)}, x \geq \mu$,
$\lambda = \frac{1}{b(n-i)}$ and $x \geq \mu$.

(ii) $a > 1$, then X will have the Pareto distribution
with $F(x) = 1 - \left(x + \frac{b}{a-1}\right)^{-\frac{a}{(a-1)(n-i)}}, x \geq 1 - \frac{b}{a-1}$

(iii) $a < 1$, then X will have power function distribution with $F(x) = 1 - \left(\frac{b}{1-a} - x\right)^{\frac{a}{(1-a)(n-i)}}, \frac{b}{1-a} - 1 \leq x \leq \frac{b}{1-a}$.

Wesolowski and Ahsanullah (2001) gave the following generalization Ferguson's (1963) result.

Theorem 4.3 *Suppose that X is an absolutely continuous random variables with cumulative distribution function F(x) and probability distribution function f(x). If* $E(X_{k+2.n}) < \infty$, *1 ≤ k ≤ n − 2, then* $E(X_{k+2,n}|X_{k,n} = x) = ax + b$ *if and only if*

(i) $a > 1$, $F(x) = 1 - \left(\frac{\mu+\delta}{x+\delta}\right)^{\theta}$, $x \geq \mu, \theta > 1$
where μ *is a real number,* $\delta = b/(a-1)$ *and*

$$\theta = \frac{a(2n-2k-1) + \sqrt{a^2 + 4a(n-k)(n-k-1)}}{2(a-1)(n-k)(n-k-1)}$$

(ii) $a = 1$, $F(x) = 1 - e^{-\lambda(x-\mu)}, x \geq \mu$,

$$b = \frac{(2n-3k-k)!}{\lambda(n-k)(n-k-1)!}$$

(iii) $a < 1$, $F(x) = 1 - \left(\frac{\nu-x}{\nu-\mu}\right)^{\theta}, \mu \leq x \leq \nu, \nu = \frac{b}{1-a}$ *and*

$$\theta = \frac{a(2n-2k-1) + \sqrt{a^2 + 4a(n-k)(n-k-1)}}{2(1-a)(n-k)(n-k-1)}.$$

Dembińska and Wesolowski (1998) gave the following general result.

Theorem 4.4 *Suppose that X is an absolutely continuous random variables with cumulative distribution function F(x) and probability distribution function f(x). If* $E(X_{k+r,n}) < \infty$, *1 ≤ k ≤ n − r, r ≥ 1, then* $E(X_{k+r,n}|X_{k,n} = x) = ax + b$ *iff*

(i) $a > 1$, $F(x) = 1 - \left(\frac{\mu+\delta}{x+\delta}\right)^{\theta}$, $x \geq \mu, \theta > 1$

where μ is a real number,

$$a = \frac{\theta(n-k)!}{(n-k-r)!}\sum_{m=0}^{r-1}\frac{1}{m!(r-1-m)!}\frac{(-1)^m}{\theta(n-k--r+1+m)[\theta(n-k--r+1+m)+1)]}$$
$$b = \delta\frac{\theta(n-k)!}{(n-k-r)!}\sum_{m=0}^{r-1}\frac{1}{m!(r-1-m)!}\frac{(-1)^m}{\theta(n-k--r+1+m)[\theta(n-k--r+1+m)+1]}$$

(ii) a < 1, $F(x) = 1 - (\frac{\nu-x}{\nu-\mu})^{\theta}$, $\mu \le x \le \nu$,

$$b = \nu\frac{\theta(n-k)!}{(n-k-r)!}\sum_{m=0}^{r-1}\frac{1}{m!(r-1-m)!}\frac{(-1)^m}{\theta(n-k--r+1+m)[\theta(n-k--r+1+m)-1]}$$

(iii) $a = 1, F(x) = 1 - e^{-\lambda(x-\mu)}, x \ge \mu,$

$$b = \frac{(n-k)!}{\lambda(n-k-r)!}$$
$$\sum_{m=0}^{r-1}\frac{1}{m!(r-1-m)!}\frac{(-1)^m}{(n-k--r+1+m)[(n-k-r+1+m)^2}$$

4.3 Characterizations of Distributions by Conditional Expectations (Extended Sample)

Consider the extended sample case. Suppose in addition to n sample, we take another m observations from the same distribution. We order the m + n observations. The combined order statistics is, $X_{1,m+n} \le X_{2,m+n} < \ldots < X_{m+n,m+n}$. We assume F(x) is the cdf of the observations.

Ahsanullah and Nevzorov (1999) proved the following theorem

Theorem 4.5 *If $E(X_{1,n}|X_{1,+n} = x) = x + m(x)$, then*

(i) *then F(x) is exponential with $F(x) = 1 - \exp(-x)$, $x > 0$ and* $m(x) = \frac{m}{n(m+n)}$

(ii) *then F(x) is Pareto with* $tF(x) = 1 - (x-1)^{-\partial}$, *$x > 1$,* $\partial > 0$ *and* $m(x) = \frac{m(x-1)}{(m+n)(m\partial+1)}$

(iii) *then F(x) is Power function with* $F(x) = 1 - (1-x)^{\partial}, 0 < x < 1, \partial > 0$ *and* $m(x) = \frac{m(1-x)}{(m+n)(m\partial+1)}$.

4.4 Characterizations Using Spacings

Ahsanullah (1977) gave the following characterization of the exponential distribution based on the equality of the distribution of X and standardized spacings of the order statistics.

Theorem 4.6 *Let be a non-negative random variable having an absolutely continuous cdf F(x)* that *is strictly increasing on* $(0, \infty)$. *Then the following statements are equivalent.*

(a) *X has an exponential distribution with* $\mathrm{F(x)} = 1 - e^{-\lambda x}, \lambda > 0$ *and,* $x \geq 0$.
(b) *For some i and n,* $1 \leq i < n$, *the statistics* $(n-i)(X_{i+1,n} - X_{i,n})$ *and X are identically distributed and X belongs to class* C_2.

Proof It is known (see Galambos 1975a, b) that $(\mathrm{a}) \Rightarrow (b)$. We will prove here $(\mathrm{b}) \Rightarrow (a)$.

We can write the pdf $f_Z(z)$ of $Z = (n-i)(X_{i+1,n} - X_{i,n})$ as

$$f_Z^{(z)} = \frac{n!}{(i-1)!(n-i)!}\int_0^\infty (F(u))^{i-1}(1-F(u+\frac{d}{n-i}))^{n-i-1}f(u)f(u+\frac{z}{n-i})du.$$

Using the assumption $f_Z(z) = f(z)$, where f(z) is the pdf of x, and writing $\int_0^\infty *F(u)^{i-1}(1-F(u))^{n-i}f(u)du = B(i,(n-i+1)) = \frac{(i-1)!(n-i)!}{n!}$, we obtain

$$0 = \int_0^\infty (F(u))^{i-1}g(u,z)f(u)du, \quad \text{for all } z \geq 0 \tag{4.4.1}$$

where $\mathrm{g(u,z)} = f(z)(1-F(u))^{n-i} - (1-F(u+z(n-i)^{-1})^{n-i-1}f(u+z(n-i)^{-1})$.

Integrating (4.15) with respect z from 0 to z_1, we obtain

$$0 = \int_0^\infty (F(u))^{i-1}(1-F(u))^{n-i}G(u,z)f(u)du, \text{ for all } z_1 \geq 0, \tag{4.4.2}$$

where

$$\mathrm{G(u, z_1)} = (1-F(u+z_1(n-i)^{-1}))/(1-F(u)))^{n-i} - (1-F(z_1))$$

If F is NBU, then for any integer $k > 0$, $1-F(x/k)) \geq (1-F(u))^{1/k}$, so $G(0, z_1) \geq 0$. Thus if (4.16) holds, then $G(0, z_1) = 0$. Similarly if F is NWU, then $G(0, z_1) \leq 0$ and hence for (4.16) to be true, we must have $G(0, z_1) = 0$. Writing $G(0, z_1)$ in terms of F, we obtain

$$(1-F(z_1(n-i)^{-1})))^{n-i} = (1-F(z_1)) \tag{4.4.3}$$

The solution of the above equation (see Aczel 1966) with the boundary conditions F(0) = 0, F(x) > 0 for all x and $F(\infty) = 1$ *is* $F(x) = 1 - e^{-\lambda x}, \lambda > 0$ and $x \geq 0$.

The following theorem (Ahsanullah 1976) gives a characterization of exponential distribution based on the equality of two standardized spacings.

Theorem 4.7 *Let X be a non-negative random variable with an absolutely continuous cumulative distribution function F(x) that is strictly increasing in* $[0, \infty)$ *and having probability density function f(x). Then the following two conditions are identical.*

(a) *F(x) has an exponential distribution with* $F(x) = 1 - e^{-\lambda x}, x \geq 0$
(b) *for some i, j and* $0 \leq i < j < n$ *the statistics* $D_{j,n}$ *and* $D_{i.n}$ *are identically distributed and F belongs to the class* C_2.

Proof We have already seen (a) ⇒ (*b*). We will give here the proof of (b) ⇒ (a)

The conditional pdf of $D_{j,n}$ given $X_{i,n}$ = x is given by

$$f_{D_{i,n}}(d|X_{i,n}) = k\int_0^\infty (\overline{F}(x) - \overline{F}(x+s))(\overline{F}(x))^{-(n-i-1)} \\ (\overline{F}(x+s+\tfrac{d}{n-j})/(\overline{F}(x))^{-1})^{n-j-1} \\ \frac{f(x+s)}{\overline{F}(x)}\frac{f(x+s+\frac{d}{n-j})}{\overline{F}(x)}ds \tag{4.4.4}$$

where k = $\frac{(n-i)!}{(j-i-1)!((n-j)!}$.

Integrating the above equation with respect to d from d to ∞, we obtain

$$\overline{F}_{D_{j,n}}(d|X_{i,n} = x) = k\int_0^\infty (\overline{F}(x) - \overline{F}(x+s))(\overline{F}(x))^{-(i-i-1)} \\ (\overline{F}(x+s+\tfrac{d}{n-j})/(\overline{F}(x))^{-1})^{n-j-1} \\ \frac{f(x+s)}{\overline{F}(x)}ds$$

The conditional probability density function $f_{i.n}$ of $D_{i,n}$ given $X_{i,n}$ = x is given by

$$f_{D_{i+1,n}}(d|X_{i,n} = x) = (n-i)\frac{\left(\bar{F}\left(d + \frac{x}{n-r}\right)\right)^{n-i-1}}{(\bar{F}(x))^{n-i}}\frac{f\left(u + \frac{x}{n-i}\right)}{\bar{F}(x)}$$

The corresponding cdf $F_{D_{i+1,n}}$ is giving by

$$1 - F_{D_{i+1,n}} = \frac{(\overline{F}(d + \frac{x}{n-i}))^{n-i}}{(\overline{F}(x))^{n-i}}$$

Using the relations

$\frac{1}{k} = \int_0^\infty (\frac{\overline{F}(x+s)}{\overline{F}(x)})^{n-j}(\frac{\overline{F}(x)-\overline{F}(x+s)}{\overline{F}(x)})^{j-i-1}\frac{f(x+s)}{\overline{F}(x)}ds$ and the equality of the distribution of $D_{i,n}$ and $D_{j,n}$ given $X_{i,n,}$ we obtain

$$\int_0^\infty (\frac{\overline{F}(x+s)}{\overline{F}(x)})^{n-j}(\frac{\overline{F}(x)-\overline{F}(x+s)}{\overline{F}(x)})^{j-i-1}G(x,d,s)\frac{f(x+s)}{\overline{F}(x)}ds=0 \tag{4.4.5}$$

where

$$G(x,\ d,\ s)=(\frac{\overline{F}(x+\frac{d}{n-i})}{\overline{F}(x)})^{n-i}-(\frac{\overline{F}(x+s+\frac{d}{n-j})}{\overline{F}(x+s)})^{n-j}. \tag{4.4.6}$$

Differentiating (4.4.5) with respect to s, we obtain

$$\frac{\partial}{\partial s}G(x,\ s,\ d)=(\frac{\overline{F}(x+s+\frac{d}{n-j})}{\overline{F}(x+s)})^{n-j}(r(x+s+\frac{d}{n-i})-r(x+s)) \tag{4.4.7}$$

(i) If F has IHR, then G(x, s, d) is increasing with s. Thus (4.19) to be true, we must have G(x, 0, d) = 0

If F has IFR, then $\ln\overline{F}$ is concave and

$$\ln(\overline{F}(x+\frac{d}{nn-i}) \geq \frac{j-i}{n-i}\ln(\overline{F}(x))+\frac{n-j}{n-i}\ln(\overline{F}(x+\frac{d}{n-j}))$$

i.e.

$$(\overline{F}(x+\frac{d}{nn-i}))^{n-i}\geq(\overline{F}(x))^{j-i}(\overline{F}(x+\frac{d}{n-j}))^{n-j}.$$

Thus G(x, 0, d) $\geq$ 0. Thus (4.19) to be true we must have G(x, 0, d) = 0 for all d and any given x.

(ii) If F has DHR, then similarly we get G(x, 0, d) = 0. Taking x = o, we obtain from G(x, 0, d) as

$$(\overline{F}(\frac{d}{n-i})^{n-i}=(\overline{F}(\frac{d}{n-j}))^{n-j} \tag{4.4.8}$$

for all d $\geq$ 0 and some i, j, n with 1 $\leq i<j<n.$

Using $\varphi(d)=\ln(\overline{F}(d))$ we obtain

(n − i) $\varphi\left(\frac{d}{n-i}\right)=(n-j)\varphi(\frac{d}{n-j})$

Putting $\frac{d}{n-i}=t$, we obtain

$$\varphi(t) = \frac{n-j}{n-i}\varphi(\frac{n-i}{n-j}t) \tag{4.4.9}$$

The non zero solution of (4.4.9) is

$$\varphi(x) = x. \text{ for all x} > 0 \tag{4.4.10}$$

for all x $\geq$ 0.

Using the boundary conditions F(x) = 0 and $F(\infty) = 1$, we obtain

$$F(x) = 1 - e^{-\lambda x}, x \geq 0, \lambda > 0. \tag{4.4.11}$$

for all x $\geq$ 0 and $\lambda > 0$.

4.5 Characterizations of Symmetric Distribution Using Order Statistics

The following theorem is due to Ahsanullah (1992a, b).

Theorem 4.8 *Suppose $X_1, X_2, \ldots, X_n$ ($n \geq 2$) are independent and identically distributed continuous random variable with cdf F(x) and pdf f(x). If $X_{1,n}^2$ and $X_{n,n}^2$ are identically distributed for some fixed n, the X's are distributed symmetrically about zero.*

Proof A random variable X has a symmetric about zero if F(−x) = 1 − F(x) for all x or equivalently if the pdf f(x) exists, then f(−x) = f(x) for all x.

The pdf $f_{n.n}$ of $X_{n,n}$ is

$$f_{n,n}(x) = n(F(x))^{n-1}f(x)$$

and the pdf $f_{1,n}$ (x) is

$$f_{1,n}(x) = n(1 - F(x))^{n-1}f(x).$$

$$P(X_{n,n}^2 \leq u^2) = P(-u < X_{n.n} < u) = (F(u))^n - (F(-u))^n$$

$$\begin{aligned} P\left(X_{1,n} \leq u^2\right) &= P(X_{1,n} \geq -u) - P(X_{1,n} \geq u) \\ &= (1 - F(-u))^n - (1 - F(u))^n \end{aligned}$$

Since $P(X_{n,n}^2 \leq u^2) = P(X_{1,n}^2 \leq u^2)$ for all u, we must have

$$(F(u))^n - (F(-u))^n = = (1 - F(-u))^n - (1 - F(u))^n \text{ for all u.}$$

We can write

$$(F(u))^n - (1 - F(-u))^n = = (F(-u))^n - (1 - F(u))^n \text{ for all u. and some n } \geq 2.$$

Thus
F(u) = 1 − F(-u) for all u.
Hence the result.

4.6 Characterization of Exponential Distribution Using Conditional Expectation of Mean

The following theorem gives a characterization of the exponential distribution by the condition expectation of $\bar{X}|X_{1.n}$.

Theorem 4.9 *Suppose X_1, X_2,…, X_n are independent and identically distributed random variables with cdf F(x) and pdf f(x). we assume $E(X_1)$ exists. Then $E(\bar{X}|X_{1,n} = y) = y + c$, where c is a constant if and only if*

$$\mathrm{F(x)} = 1 - e^{-\lambda(x-\mu)}, \lambda > 0, -\infty < \mu < x < \infty., \lambda = \frac{n-1}{nc}.$$

Proof

$$\mathrm{E(X_i|X_{1,n} = y)} = \frac{y}{n} + \frac{n-1}{n} \frac{\int_y^\infty x f(x) dx}{1 - F(y)}.$$

$$\text{If } \mathrm{F(x)} = = 1 - e^{-\lambda(x-\mu)}, \text{ then}$$

$$\begin{aligned}\mathrm{E(X_i|X_{1,n} = y)} &= \frac{y}{n} + \frac{n-1}{n} \frac{\int_y^\infty \lambda x e^{-\lambda(x-\mu)} dx}{e^{-\lambda(y-\mu)}}.\\ &= \frac{y}{n} + \frac{n-1}{n}(y + \frac{1}{\lambda})\\ &= \mathrm{y + c,\ c} = \frac{n-1}{n\lambda}.\end{aligned}$$

Thus

$$E(\bar{X}|X_{1,n} = y) = E(X_i|X_{1.n} = y) = y + c.$$

Suppose

$$E(\bar{X}|X_{1,n}=y)=y+c.$$

Since $E(\bar{X}|X_{1,n}=y)=E(X_i|X_{1.n}=y)$, we must have

$$\mathrm{E}(\mathrm{X_i}|\mathrm{X}_{1,\mathrm{n}}=\mathrm{y}) = \frac{y}{n}+\frac{n-1}{n}\frac{\int_y^\infty xf(x)dx}{1-F(y)} = \mathrm{y}+\mathrm{c}.$$

Thus

$$\frac{\int_y^\infty xf(x)dx}{1-F(y)}=y+c_1.c_1=\frac{n}{n-1}c \tag{4.6.1}$$

From (4.6.1), we obtain

$$\int_y^\infty xf(x)dx=y(1-F(y))+c_1(1-F(y)) \tag{4.6.2}$$

Differentiating both sides of (4.6.2) with respect to y, we obtain $-yf(y) = 1 - F(y) - yf(y) - c_1f(y)$, i.e.

$$\frac{f(y)}{1-F(y)}=\frac{1}{c_1}$$

Thus X has the exponential distribution.

4.7 Characterizations of Power Function Distribution by Ratios of Order Statistics

Ahsanullah (1989) gave some characterizations of the power function and uniform distributions based on the spacings of the order statistics. For proving the results the following restriction on the cdf F(x) were used.

We say that cdf F(x) is "super additive" if $F(X + y) \le F(x) + F(y)$, $x, y \ge 0$. and F(x) is sub additive if $F(x + y) \le F(x) + F(y)$. We will say that F(x) belongs to the class C_0 if F(x) is either Supper additive or sub additive.

Is it a characteristic property of the uniform distribution on [0, 1] that X and $X_{1,n}/X_{2,n}$ are identically distributed. The answer is no. In fact identical distribution of X and $X_{1,n}/X_{2,n}$ characterizes a family of distributions of which the uniform distribution is a member. We have the following theorem.

Theorem 4.10 *Let X be a positive and bounded random variable having an absolutely continuous distribution function F(x). We assume without any loss of generality* $\inf\{x|F(x) > 0\} = 0$, $F(x) > 0$ *for* $0 < x < 1$ *and* $F(1) = 1$. *Then the following two statements are equivalent.*

(i) *If the cdf F(x) of X is* $\mathrm{F(x)} = x^{\alpha}$ $0 \le x \le 1, \alpha \ge 1$, *then* $X_{1,n}/X_{2,n}$ *and X are identically distributed.*

(ii) *If for some fixed* $n \ge 2$, $X_{1.n}|X_{2,n}$ *and X are identically distributed and F belongs to class* C_0, *then the cdf F(x) of X is* $\mathrm{F(x)} = x^{\alpha}$ $0 \le x \le 1, \alpha \ge 1.$

(iii) *Proof. The statement (i) can easily be verified. We proof here the statement (ii). Let* $U_1 = X_{1,n}/X_{2,n}$. *The pdf* $f_{U_1}(u)$ *of* U_1 *is given by*

$$f_{U_1}(u) = \int_0^1 n(n-1)(1-F(v))^{n-2} f(uv)vf(v)dv, 0 \le u \le 1.$$

The corresponding cdf $F_{U_1}(u)$ is

$$F_{U_1}(u) = \int_0^1 n(n-1)(1-F(v))^{n-2} F(uv)f(v)dv, 0 \le u \le 1.$$

Substituting $\mathrm{F(x)} = x^{\alpha}$ it follows that

$$F_{U_1}(u) = x^{\alpha},\ 0 \le \mathrm{x} \le 1,\ \alpha \ge 1.$$

Suppose that $X_{1.n}|X_{2,n}$ and X are identically distributed. Then we have

$$\int_0^1 n(n-1)(1-F(v))^{n-2} f(uv)vf(v)dv = \mathrm{f(u)}$$

Integrating both sides of the equation, with respect u from 0 to u_0. We obtain

$$\int_0^1 n(n-1)(1-F(v))^{n-2} F(u_0 v)f(v)dv. = F(u_0) \tag{4.7.1}$$

Writing $\frac{1}{n(n-1)} = \int_0^1 (1-F(v))^{n-2} F(v)f(u)du$ and substituting in (4.7.1), we obtain

$$\int_0^1 n(n-1)(1-F(v))^{n-2} F(u_0 v)G(u_0, v)f(v)dv. = 0 \tag{4.7.2}$$

where $G(u_0 v) = F(u_0 v) - F(u_0)F(v)$.

If F(x) is supper additive, then $G(u_0 v) \leq 0$ for all u_0 and V. Thus (4.7.2) to be true, we must have

$$F(u_0 v) = F(u_0)F(v) \tag{4.7.3}$$

The solution of the Eq. (4.7.3) with boundary condition F(0) = 0 and F(1) = 1 is $F(x) = x^\alpha$, $0 \leq x \leq 1$, $\alpha \geq 1$.

Similarly if F(x) is sub additive, then we will obtain the Eq. (4.7.3). Hence $F(x) = x^\alpha$, $0 \leq x \leq 1$, $\alpha \geq 1$.

If X is distributed as U(0, 10), then for all k, $1 \leq k \leq n$, $U_{k,n} = X_{k+1,n} - X_{k,n}$ with $U_{n,n} = 1 - X_{n,n}$ and $U_{0,n} = X_{1,n}$ are identically distributed as U(0, 1). Huang et al. proved that if F(x) belongs to the class C_3, then identical distribution of $U_{k,n}$ and $U_{0,n}$ characterize the uniform, U(0, 1) distribution. Is the uniform distribution by (U(0, 1)) the only distribution having the property $U_{i,n}$ and $U_{j,n}$ $1 \leq \, < j \leq n$ are identically distributed?

As an answer to this question we have the following theorem with some restriction on the pdf f(x). We say F (x) belongs to the class C_4 if the corresponding pdf f(x) satisfies the either $f(x_1) \geq f(x_2)$ or $f(x_1) \leq f(x_2)$ for all x_1 and x_2 with $x_1 > x_2$.

Theorem 4.11 *Let X be a positive and bounded random variable having an absolutely continuous distribution function F(x). We assume without any loss of generality sup{x|F(x) > 0} = 0, F(x) > 0 for 0 < x < 1 and F(1) = 1. Then if $U_{i,n}$ and $U_{i+1,n}$, $0 \leq i < n$, $i \neq (n-1)/2$ for odd are identically distributed and F belongs to the class C_4, then F(x) = x, $0 \leq x \leq 1$.*

For proof see Ahsanullah (1989).

4.8 Characterization of Uniform Distribution Using Range

The following theorem give a characterization of the uniform (U(0, 1)) distribution using identical distribution of the $(X_{n,n} - X_{1,n})$ and $X_{n-1,n}$.

Theorem 4.12 *Suppose the random variable X is a bounded absolutely continuous random with cdf F(x) and pdf f(x). Let Inf{x|F(x) > 0) = 0, F(x) > 0 for x 0 < x 1 and F(x) = 1, Then the following two statements are equivalent.*

(i) *X is distributed as U[0, 1].*
(ii) *$X_{n,n} - X_{1,n}$ and $X_{n-1,n}$ are identically distributed and F belongs to the class C_4.*

Proof The pdf $f_{1,n}(v)$ of $V = X_{n,n} - X_{1,n}$ is given as

$$f_V(v) = \int_0^{1-v} n(n-1)((F(u+v) - F(u))^{n-2} f(u+v) f(u) du, \; 0 < v < 1, \; n \geq 2.$$

Substituting F(x) = x and f(x) = 1, we obtain

$$\mathrm{f_V(v)} = \int_0^{1-v} n(n-1)v^{n-2}du$$
$$= \mathrm{n(n-1)(1-v)}v^{n-2},\ 0 \le \mathrm{v} \le 1.$$

We will prove now (ii) implies (i).

The cdf $F_V(v)$ of V is

$$\mathrm{F_V(v)} = n\int_0^{1-v} (F(u+v) - F(u))^{n-1} f(u)du + 1 - F(1-v)$$

The cdf $f_{n-1,n}$ (x) is

$$\mathrm{F_{n-1,n}(x)} = n(F(x))^{n-1} - (n-1)(F(x))^n$$

Using F(x)s symmetric, we have F(x) = F(1 − x). Using the symmetric property and the equality of $F_V(v)$ and

$F_{n-1,n}$ (v), we obtain on simplification

$$\int_0^{\infty} f(u)g(u,v)du = 0., \tag{4.8.1}$$

where $\mathrm{g(u,v)} = (F(u+v) - F(u))^{n-1} - (F(u))^{n-1}$

If F(x) I supper additive, then g(u, v) ≤ 0 and (4.8.1) to be true, we must have g (u, v) = 0 for all v, 0 ≤ v ≤ 1 and almost all u, 0 ≤ u ≤ 1. Now g(u, v) = 0 implies

$$\mathrm{F(u+v) = F(u) + F(v)} \tag{4.8.2}$$

The only continuous solution of (4.8.2) with the boundary conditions F(0) = 0 and F(1) =1 is

$$\mathrm{F(x) =\ x,\ 0 \le x \le 1.} \tag{4.8.3}$$

If F(x) is sub additive, then similarly we get the equation (4.8.2) and hence we obtain the solution (4.8.3).

4.9 Characterization by Truncated Order Statistics

It can be shown easily that if F(0) = 0, F(x) > 0 for all x, 0 < x ≤ b, F(b) = 1, then for 2 ≤ i ≤ n.

$$E(X_{i,n}^{\alpha}|X_{i-1}=t)=t^{\alpha}+\frac{\int_t^b \alpha x^{\alpha-1}(1-F(x))^{n-i+1}dx}{(1-F(t))^{n-i+1}}$$

Using above conditional expectation we have the following theorem.

Theorem 4.13 *Let X:* $\Omega\to(a,b)$, *a ≥ 0 be an absolutely continuous random variable with cdf F(x) and* $\lim_{x\to b}x^{\alpha}(1-F(x))=0$ *for α > 0. We assume g(x, i, n) is a differentiable function with* $\int_a^{\infty}\frac{\alpha x^{\alpha-1}}{g(x,i,n)}dx=\infty$. *Then*

$$E(X_{i,n}^{\alpha}|X_{i-1.n}=t)=t^{\alpha}+g(t,i,n),\ a\le \mathrm{t}\le \mathrm{b}$$

Implies

$$\mathrm{F(x)}=1-\left(\frac{g(a,i,n)}{g(x,i,n)}\right)^{\frac{1}{n-i+1}}e^{-\int_a^x\frac{\alpha t^{\alpha-1}}{(n-i+1)g(t,i,n)}dt}$$

Suppose $\alpha=1, a=0, b=\infty$ *and* g(t.i.n) $=\frac{1}{n-i+1}$, *then*

$$\mathrm{F(x)}=1-e^{-x},\mathrm{x}\ge 0.$$

Recently Ahsanullah and Anis (2016) proved the following theorem.

Theorem 4.14 *Let* X_n, *n = 1, 2,…, n, be n independent and identically distributed random variables with absolutely continuous cdf F(x) with F(0) = 0 and F(x) > 0 for all x > 0. Let* $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$ *be the corresponding* order statistics. *If F belongs to class* C_1, *then the following two statements are equivalent:*

(a) $\mathrm{F(x)}=1-e^{-\lambda x}$, x > 0, λ > 0,
(b) $X_{n,n}-X_{1,n}$ *and* $X_{n-1,n-1}$ *are identically distributed and F(x) belongs to class* C_1.

Chapter 5
Characterizations of Distributions by Record Values

In this chapter, we will discuss the characterizations of univariate continuous distributions by record values.

5.1 Characterizations Using Conditional Expectations

Suppose $\{X_i, i = 1, 2, \ldots\}$ be a sequence of independent and identically distributed random variables with cdf F(x) and pdf f(x). We assume $E(X_i)$ exists. Let X(n), $n \geq 1$ be the corresponding upper records. We have the following theorem for the determination of F(x) based on the conditional expectation.

Theorem 5.1.1 *The condition*

$$E(\psi(X(k+s)|X(k)=z)=\mathrm{g(z)}$$

where k, s $\geq$ *1 and* $\psi(x)$ *is a continuous function, determines the distribution F(x) uniquely.*

Proof

$$E(\psi(X(k+s)|X(k)=z)=\int_z^\infty \frac{\psi(x)(R(x)-R(z))^{s-1}}{\bar{F}(z)}f(x)dx \tag{5.1.1}$$

where R(x) = -ln $\bar{F}(x)$.

Case s = 1

Using the Eq. (5.1.1), we obtain

$$\int_z^\infty \psi(x)f(x)dx=g(z)\bar{F}(z) \tag{5.1.2}$$

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Differentiating both sides of (5.1.2) with respect to z and simplifying, we obtain

$$r(z) = \frac{f(z)}{\bar{F}(z)} = \frac{g^{'}(z)}{g(z) - \psi(z)} \tag{5.1.3}$$

where r(z) is the failure rate of the function. Hence the result.If $\psi(x) = x$ and g (x) = ax + b, a, b $\geq$ 0, then

$$r(x) = \frac{a}{(a-1)x+b} \tag{5.1.4}$$

If $a \neq 1$, then $F(x) - 1 - ((a-1)x+b)^{-\frac{a}{a-1}}$, which is the power function distribution for $a < 1$ and the Pareto distribution with $a > 1$. For $a = 1$, (5.1.4) will give exponential distribution. Nagaraja (1977) gave the following characterization theorem.

Theorem 5.1.2 *Let F be a continuous cumulative distribution function. If for some constants, a and b,*

E(X(n) | X(n − 1) = x) = ax + b, *then except for a change of location and scale,*

(i) $F(x) = 1 - (-x)^{\theta}$, $-1 < x < 0$, if $0 < a < 1$
(ii) $F(x) = 1 - e^{-x}$, $x \geq 0$, if $a = 1$
(iii) $F(x) = 1 - x^{\theta}$, $x > 1$ if $a > 1$,

where $\theta = a/(1 - a)$. Here $a > 0$.

Proof of Theorem 5.1.1 for s = 2

In this case, we obtain

$$\int_z^{\infty} \psi(x)(R(x) - R(z)f(x)dx = g(z)\bar{F}(z) \tag{5.1.5}$$

Differentiating both sides of the above equation with respect to z, we obtain

$$-\int_z^{\infty} \psi(x)f(z)dx = g^{'}(z)\frac{(\bar{F}(z))^2}{f(z)} - g(z)\bar{F}(z) \tag{5.1.6}$$

Differentiating both sides of (5.1.6) with respect to z and using the relation $\frac{f^{'}(z)}{f(z)} = \frac{r^{'}(z)}{r(z)} - r(z)$ we obtain on simplification

$$g^{'}(z)\frac{r^{'}(z)}{r(z)} + 2g^{'}(z)r(z) = g^{''}(z) + (r(z))^2(g(z) - \psi(z)) \tag{5.1.7}$$

Thus $r^{'}(z)$ is expressed in terms of r(z) and known functions. The solution of r(x) is unique (for details see Gupta and Ahsanullah 2004a, b).

Putting $\psi(x) = x$ and g(x) = ax + b, we obtain from (5.1.7)

$$a\frac{r'(z)}{r(z)} + 2ar(z) = (r(z))^2((a-1)a+b) \quad (5.1.8)$$

The solution of (5.1.8) is

$$\mathrm{r(x)} = \frac{\mathrm{a} + \sqrt{\mathrm{a}}}{(\mathrm{a}-1)\ \mathrm{x}\ + \mathrm{b}}.$$

Thus X will have (i) exponentially distributed if a = 1, (ii) power function distribution if a < 1 and (iii) Pareto distribution if a > 1.

Ahsanullah and Wesolowski (1998) extended the result Theorem 5.1.2 for non-adjacent record values. Their result is given in the following theorem.

Theorem 5.1.3 *If* E(X(n + 2) | X(n)) = a X(n) + b. n ≥ 1.
where a and b are constants, then if:

(a) *$a = 1$ then X_i has the exponential distribution,*
(b) *$a < 1$, then X_I has the power function distribution*
(c) *$a > 1$ X_I has the Pareto distribution*

Proof of Theorem 5.1.1 for s > 2

In this case, the problem becomes more complicated because of the nature of the resulting differential equation.

Lopez-Blazquez and Moreno-Rebollo (1997) also gave characterizations of distributions by using the following linear property

$$E(X(k)|X(k+s) = z) = az + b.\ \mathrm{s.k} > 1.$$

Raqab (2002) considered this problem for non-adjacent record values under some stringent smoothness assumptions on the distribution function $F(.)$. Dembinska and Wesolowski (2000) characterized the distribution by means of the relation

$$E(X(s+k)||X(k) = z) = a\,z + b, \quad \mathrm{for\,k.s} \geq 1.$$

They used a result of Rao and Shanbhag (1994) which deals with the solution of extended version of integrated Cauchy functional equation. It can be pointed out earlier that Rao and Shanbhag's result is applicable only when the conditional expectation is a linear function.

Bairamov et al. (2005) gave the following characterization,

Theorem 5.1.4 *Let X be an absolutely continuous random variable with cdf F(x)* with F(0) = 0 and F(x) > 0 for all x > 0 *and pdf f(x), then*

(a) *For* 1 ⩽ *k* ⩽ *n* – *1*,

$$E((X(n)|X(n-k)=x),X(n+1)=y)=\frac{u+kv}{k+1},0<u<v<\infty$$

If and only if

$$F(x)=1-e^{\lambda x},x\geq 0,\lambda>0,$$

(b) for 2 ⩽ *k* ⩽ *n-1,*

$$E((X(n)|X(n-k+1)=x),X(n+2)=y)_{-}=\frac{2u+(k-1)v}{k+1},0<u<v<\infty$$

If and only if

$F(x)=1-e^{=\lambda x},x\geq 0,\lambda>0,$ Yanev et al. (2007) extended these results for general cases of nonadjacent record values. Under the conditions of Theorem 5.1.4., Akhundov and Nevzorov (200&) proved that

$$E\left(\frac{X(2)\_X(3)+\ \dots\ +X(n)}{n-1}|X(1)=u,X(n+1)=v\right)=\frac{u+v}{2}$$

characterizes the exponential distribution under mild condition on F(x).

5.2 Characterization by Independence Property

Tata (1969) presented a characterization of the exponential distribution by the independence of the random variables X(1) and X(2) − X(1), The result is given in the following theorem.

Theorem 5.2.1 *Let* $\{X_n,\ n\ \geq\ 1\}$ *be an i.i.d. sequence of non-negative continuous random variables with cdf* $F(x)$ *and pdf* $f(x)$. *We assume* $F(0) = 0$ *and* $F(x) > 0$ *for all* $x > 0$. *Then for* X_n *to have the cdf,* $F(x)=1-e^{-x/\sigma},x\geq 0,\sigma>0,$ *it is necessary and sufficient that* $X(2) - X(1)$ *and* $X(1)$ *are independent.*

Proof The necessary condition is easy to establish, we give here the proof of the sufficiency condition. The property of the independence of X(2) -X(1) and X(1) will lead to the functional equation

$$\bar{F}(0)\bar{F}(x+y)=\bar{F}(x)\bar{F}(y),0<x,y<\infty \tag{5.2.1}$$

The continuous solution of this functional equation with the boundary conditions F(0) = 0 and F(∞) = 1, is

$$\bar{F}(x) = e^{-x\sigma^{-1}}, \mathrm{x} > 0, \sigma > 0.$$

The following generalization theorem was given by Ahsanullah (1979)

Theorem 5.2.2 *Let* $\{X_n, n \geq 1\}$ *be a sequence of i.i.d. random variables with common distribution function F which is absolutely continuous with pdf f. Assume that* $F(0) = 0$ *and* $F(x) > 0$ *for all* $x > 0$. *Then* X_n *to have the cdf,* $F(x) = 1 - e^{-x/\sigma}, x \geq 0, \sigma > 0,$ *it is necessary and sufficient that* X(n) – X(n − 1) and X(n − 1) are independent.

Proof It is easy to establish that if X_n has the cdf, $F(x) = 1 - e^{-x/\sigma}, x \geq 0, \sigma > 0,$ then X(n) –X(n-1) and X(n − 1) are independent. Suppose that X(n + 1) − X(n) and X(n), n ≥ 1, are independent. Now the joint pdf f(z, u) of Z = X(n − 1) − X (n) and U = $X(n)_1$ can be written as

$$\begin{aligned} \mathrm{f}(\mathrm{z}, \mathrm{u}) &= \frac{[R(u)]^{n-1}}{\Gamma(n)} r(u) f(u+z), \quad 0 < \mathrm{u}, \mathrm{z} < \infty. \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{5.2.2}$$

But the pdf fn (u) of X(n) can be written as

$$\begin{aligned} \mathrm{F}_{\mathrm{n}-1}(\mathrm{u}) &= \frac{[R(u)]^{n-1}}{\Gamma(n)} f(u), 0 < u < \infty, \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{5.2.3}$$

Since Z and U are independent, we get from (5.2.2) and (5.2.3)

$$\frac{f(u+z)}{\bar{F}(u)} = g(z), \tag{5.2.4}$$

where g(z) is the pdf of u. Integrating (5.2.4) with respect z from 0 to z_1, we obtain on simplification

$$\bar{F}(u) - \bar{F}(u + z_1) = \bar{F}(u) G(z_1). \tag{5.2.5}$$

Since $G(z_1) = \int_0^{z_1} g(z)\, dz$. Now u → 0^+ and using the boundary condition $\bar{F}(0) = 1,$ we see that $G(z_1) = F(z_1)$. Hence, we get from (8.25)

$$\bar{F}(u + z_1) = \bar{F}(u) \bar{F}(z_1). \tag{5.2.6}$$

The only continuous solution of (5.2.6) with the boundary condition F(0) = 0, is

$$\bar{F}(x) = e^{-\sigma^{-1} x}, \mathrm{x} \geq 0 \tag{5.2.7}$$

where σ is an arbitrary positive real number.

The following theorem (Theorem 5.2.3) is a generalization of the Theorem 5.2.2.

Theorem 5.2.3 *Let* $\{X_n, n \geq 1\}$ *be independent and identically distributed with common distribution function F which is absolutely continuous and* $F(0) = 0$ and F(x) < 1 for all x > 0. Then X_n has the cdf $F(x) = 1 - e^{-\sigma x}, x \geq 0, \sigma > 0,$ *it is necessary and sufficient that X(n) –X(m) and X(m) are independent.*

Proof The necessary condition is easy to establish. To proof the sufficient condition, we need the following lemma.

Lemma 5.2.1 Let F(x) be a continuous distribution function and $\bar{F}(x) > 0$, for all x > 0. Suppose that $\bar{F}(u+v)(\bar{F}(v))^{-1}$ = exp{-q(u, v)} and h(u, v) = $\{q(u, v)\}^r$ exp{-q(u, v)} $\frac{\partial}{\partial u}$ q(u, v), for r ≥ 0. Further if h(u,v) ≠ 0, and $\frac{\partial}{\partial u} q(u, v) \neq 0$ for any positive u and v. If h(u,v) is independent of v, then q(u,v) is a function of u only.

Proof: of the sufficiency of Theorem 5.2.4.

The conditional pdf of Z = X(n) − X(m) given V(m) = x is

$$f(z|X(m) = x) = \frac{1}{\Gamma(n-m)}[R(z+x) - R(x)]^{n-m-1}\frac{f(z+x)}{\bar{F}(x)}, 0 < z < \infty, 0 < x < \infty.$$

Since Z and X(m) are independent, we will have for all z > 0,

$$(R(z+x) - R(x))^{n-m-1}\frac{f(z+x)}{\bar{F}(x)} \tag{5.2.8}$$

as independent of x. Now let

$$R(z+x) - R(x) = -\ln\frac{\bar{F}(z+x)}{\bar{F}(x)} = q(z,x), \quad \text{say.}$$

Writing (8.1.9) in terms of q(z, x), we get

$$[q(z,x)]^{n-m-1}\exp\{-q(z,x)\}\frac{\partial}{\partial z}q(z,x), \tag{5.2.9}$$

as independent of x. Hence by the lemma 5.1.1, we have

$$-\ln\left\{\bar{F}(z+x)(\bar{F}(x))^{-1}\right\} = q(z+x) = c(z), \tag{5.2.10}$$

where c(z) is a function of z only. Thus

$$\bar{F}(z+x)(\bar{F}(x))^{-1}=c_1(z), \qquad (5.2.11)$$

and $c_1(z)$ is a function of z only.

The relation (5.2.11) is true for all z ≥ 0 and any arbitrary fixed positive number x. The continuous solution of (5.2.11) with the boundary conditions, $\bar{F}(0)=1$ *and* $\bar{F}(\infty)=0$ is

$$\bar{F}(x)=\exp(-x\sigma^{-1}), \qquad (5.2.12)$$

for x ≥ 0 and any arbitrary positive real number σ. The assumption of absolute continuity of F(x) in the Theorem can be replaced by the continuity of F(x).

Chang (2007) gave an interesting characterization of the Pareto distribution. Unfortunately, the statement and the proof of the theorem were wrong. Here we will give a correct statement and proof of his theorem.

Theorem 5.2.4 *Let* $\{X_n,\ n \ge 1\}$ *be independent and identically distributed with common distribution function F which is continuous and* $F(1) = 0$ *and* $F(x) < 1$ *for all* $x > 1$. *Then* X_n *has the cdf* $F(x)=1-x^{-\theta}, x\ge 1, \theta>0$, *it is necessary and sufficient that* $\frac{X(n)}{X(n+1)-X(n)}$ *and* $X(m)$, $n \ge 1$ *are independent.*

Proof If $F(x)=1-x^{-\theta}, x\ge 1, \theta>0$, then the joint pdf $f_{n,n+1}(x, y)$ of X(n) and X (n + 1) is

$$f_{n,n+1}(x,y)=\frac{1}{\Gamma(n)}\frac{\theta^{n+1}(\ln x)^{n-1}}{xy^{\theta+1}}, 1<x<y<\infty, \theta>0.$$

Using the transformation, U = X(n) and $V=\frac{X(n)}{X(n+1)-X(n)}$. The joint pdf f_{UV} (u, v) can be written as

$$f_{U,V}(w,v)=\frac{1}{\Gamma(n)}\frac{\theta^{n+1}(\ln u)^{n-1}}{u^{\theta+3}}\left(\frac{v}{1+v}\right)^{\theta+1}, 1<u,v<\infty, \theta>0. \qquad (5.2.13)$$

Thus, U and V are independent.

The proof of sufficiency.

The joint pdf of W and V can be written as

$$f_{W,V}(u,v)=\frac{(R(u))^{n-1}}{\Gamma(n)}r(u)f\left(\frac{1+v}{v}u\right)\frac{u}{V^2}, 1<u,v<\infty, \qquad (5.2.14)$$

where R(x) = −ln(1 − F(x)), $r(x)=\frac{d}{dx}R(x)$.

We have the pdf $f_U(u)$ of U as

$f_U(u) = \frac{(R(u))^{n-1}}{\Gamma(n)} f(u)$. Since U and V are independent, we must have the pdf $f_V(v)$ of V as

$$f_V(v) = f\left(\frac{1+v}{v}u\right)\frac{w}{V^2}\frac{1}{1-F(u)}, 0 < v < \infty,. \tag{5.2.15}$$

Integrating he above pdf from v_0 to ∞, we obtain

$$1 - F(v_0) = \frac{1 - F\left(\frac{1+v_0}{v_0}u\right)}{1-F(u)} \tag{5.2.16}$$

Since $F(v_0)$ is independent of U, we must have

$$\frac{1 - F\left(\frac{1+v_0}{v_0}u\right)}{1-F(u)} = G(v_0) \tag{5.2.17}$$

where $G(v_0)$ is independent of u

Letting u → 1, we obtain $G(v_0) = 1 - F\left(\frac{1+v_0}{v_0}\right)$.

We can rewrite (5.2.17) as

$$1 - F\left(\frac{1+v_0}{v_0}u\right) = \left(1 - F\left(\frac{1+v_0}{v_0}\right)(1 - F(u)\right) \tag{5.2.18}$$

$F(x) = 1 - x^{\beta}$. Since F(1) = 0 and $F(F(\infty) = 0$, we must have

$$F(x) = 1 - x^{-\theta}, \quad x \geq 1 \text{ and } \theta > 0. \tag{5.2.19}$$

The following theorem is a generalization of Theorem 5.2.4.

Theorem 5.2.4 *Let* $\{X_n, n \geq 1\}$ *be independent and identically distributed with common distribution function F which is continuous and* $F(1) = 0$ *and* $F(x) < 1$ *for all* $x > 0$. *Then* X_n *has the cdf,* $F(x) = 1 - x^{-\theta}, x \geq 1, \theta > 0$, *it is necessary and sufficient that* $\frac{X(m)}{X(n) - X(m)}$, $1 \leq m < n$ *and* $X(m)$ *are independent.*

Proof The joint pdf $f_{m,n}$ (x,y) of X(m) and X(n),n > m, is

$$f_{m,n}(x,y) = \frac{(R(x))^{m-1}}{\Gamma(m)}\frac{(R(y) - R(x))^{n-m-1}}{\Gamma(n-m)} r(x) f(y), \tag{5.2.20}$$

We have $F(x) = 1 - x^{-\theta}$, $R(x) = \theta \ln x, r(x) = \frac{\theta}{x}$, thus we obtain

$$f_{m,n}(x,y) = \frac{(\theta \ln x)^{m-1}}{\Gamma(m)} \frac{(\ln y - \ln x)^{n-m-1}}{\Gamma(n-m)} \frac{1}{xy^{\theta+1}}. \tag{5.2.21}$$

where $1 \leq x < y < \infty, \theta > 0$.

Using the transformation U = X(m) and $V = \frac{X(m)}{X(n) - X(m)}$, we obtain the pdf $f_{U,V}$ (u,v) of U and V as

$$f_{U,V}(u,v) = \frac{\theta^n (\ln u)^{n-1}}{\Gamma(n)} \frac{\left(\ln\left(\frac{1+v}{v}\right)\right)^{n-m-1}}{\Gamma(n-m)} \frac{v^{\theta-1}}{u^{\theta+1}(1+v)^{\theta+1}}$$

Thus X(m) and $\frac{X(m)}{X(n) - X(m)_-}$ are independent.

Proof of sufficiency

Using U = X(m) and $V = \frac{X(m)}{X(n) - X(m)_-}$, we can obtain the pdf $f_{U,V}$ of U and V from (5.2.20) as

$$f_{U,V}(u,v) = \frac{(Ru)^{m-1}}{\Gamma(m)} \frac{\left(R\left(\frac{u(1+v)}{v}\right) - R(u)\right)^{n-m-1}}{\Gamma(n-m)} r(u) f\left(\frac{u(1+v)}{v}\right), \tag{5.2.22}$$

We can write the conditional pdf $f_{V|U}(v|u)$ of V|U as

$$f_{V|UV}(v|u) = \frac{\left(R\left(\frac{u(1+v)}{v}\right) - R(u)\right)^{n-m-1}}{\Gamma(n-m)} \frac{uf\left(\frac{u(1+v)}{v}\right),}{v^2 \bar{F}(u)}, 1 < u < \infty, o) < v < \infty. \tag{5.2.23}$$

Using the relation R(x) = -ln $\bar{F}(x)$, we obtain from (5.2.23) that

$$f_{V|U} v|u) = \frac{\left(-\ln\left(\frac{\bar{F}\left(\frac{u(1+v)}{v}\right)}{\bar{F}(u)}\right)\right)^{n-m-1}}{\Gamma(n-m)} \frac{d}{dv}\left(\frac{\bar{F}\left(\frac{u(1+v)}{v}\right)}{\bar{F}(u)}\right), 1 < u < \infty, 0 < v < \infty. \tag{5.2.24}$$

Since V and U are independent, we must have $\frac{\bar{F}\left(\frac{u(1+v)}{v}\right)}{\bar{F}(u)}$ independent of U.

Let

$$\frac{\bar{F}\left(\frac{u(1+v)}{v}\right)}{\bar{F}(u)} = G(v),$$

Letting $u \to 1$, we obtain

$$\bar{F}\left(\frac{u(1+v)}{v}\right) = \bar{F}(u)\bar{F}\left(\frac{1+v}{v}\right), \tag{5.2.25}$$

For all u, 1 < u < ∞ and all v, 0 < v < ∞.

The continuous solution of (5.2.25) with the boundary condition F(1) = 0 and F (∞) = 1 is

$$\mathrm{F(x)} = 1 - x^{-\theta}, x \geq 1 \text{ and } \theta > 0.$$

5.3 Characterizations Based on Identical Distribution

Theorem 5.3.1 *Let X_n, $n \geq 1$ be a sequence of i.i.d. random variables which has absolutely continuous distribution function F with pdf f and F(0) = 0. Assume that F(x) < 1 for all x > 0. If X_n belongs to the class C_1 and $I_{n\text{-}1,\mathrm{n}} = X(n) - X(n-1)$, $\underline{n} > 1$, has an identical distribution with X_k, $k \geq 1$, then X_k has the cdf $F(x) = 1 - e^{-\sigma x}$, $x \geq 0, \sigma > 0$.*

Proof The if condition is easy to establish. We will proof here the only if condition.

By the assumption of the identical distribution of $\mathrm{I}_{\text{n-1,n}}$ and $\mathrm{X_k}$, we must have

$$\int_0^\infty [R(u)]^{n-1}\frac{r(u)}{\Gamma(n)}f(u+z)du = f(z) \quad , \text{for all z} > \text{o}. \tag{5.3.1}$$

Substituting

$$\int_0^\infty [R(u)]^{n-1} f(u)du = \Gamma(n), \tag{5.3.2}$$

we have

$$\int_0^\infty [R(u)]^{n-1} r(u)f(u+z)du = f(z)\int_0^\infty [R(u)]^{n-1} f(u)\,du, \ \mathrm{z} > 0. \tag{5.3.3}$$

Thus

$$\int_0^\infty [R(u)]^{n-1} f(u)\left[f(u+z)(\bar{F}(u))^{-1} - f(z)\right]du = 0, \ \mathrm{z} > 0. \tag{5.3.4}$$

Integrating the above expression with respect to z from z_1 to ∞, we get from (5.3.5)

$$\int_0^{\infty} [R(u)]^{n-1} f(u) \left[\bar{F}(u+z_1)(\bar{F}(u))^{-1} - \bar{F}(z_1)\right] du = 0, \ \mathrm{z}_1 > 0. \tag{5.3.5}$$

If F(x) is NBU, then (5.3.5) is true if

$$\bar{F}(u+z_1)(\bar{F}(u))^{-1} = \bar{F}(z_1), \ \mathrm{z}_1 > 0. \tag{5.3.6}$$

The only continuous solution of (5.3.6) with the boundary conditions $\bar{F}(0)=1$ and $\bar{F}(\infty)=0$ is $\bar{F}(x)=e^{-\sigma x}$, where σ is an arbitrary real positive number. Similarly, if F is NWU then (5.3.6) is true if (5.3.5) is satisfied and X_k has the cdf F(x) = $1-e^{-\sigma x}$ $x \geq 0, \sigma > 0$. k ≥ 1.

Theorem 5.3.2 *Let* X_n, $n \geq 1$ *be a sequence of independent and identically distributed non-negative random variables with absolutely continuous distribution function F(x) with f(x) as the corresponding density function. If* $F \in C_2$ *and for some fixed* n, m, $1 \leq m < n < \infty$, $I_{m,n} \stackrel{d}{=} X(n-m)$, *then* X_k *has the cdf* $F(x) = 1 - e^{-\sigma x}$, $x \geq 0, \sigma > 0.$, $k \geq 1$.

Proof The pdfs $f_1(x)$ of $X_{(n\text{-}m)}$ and $f_2(x)$ of $I_{m,n}$ (= R_n - R_m) can be written as

$$f_1(x) = \frac{1}{\Gamma(n-m)} [R(x)]^{n-m-1} f(x), \quad \text{for } 0 < \mathrm{x} < \infty, \tag{5.3.7}$$

and

$$f_2(x) = \int_0^{\infty} \frac{[R(u)]^{m-1}}{\Gamma(m)} \frac{[R(x+u) - R(x)]^{n-m-1}}{\Gamma(n-m)} r(u) f(u+x) du, \quad 0 < \mathrm{x} < \infty. \tag{5.3.8}$$

Integrating (5.3.7) and (5.3.8) with respect to x from 0 to x_0, we get

$$\mathrm{F}_1(\mathrm{x}_0) = 1 - \ \mathrm{g}_1(\mathrm{x}_0), \tag{5.3.9}$$

where

$$g_1(x_0) = \sum_{j=1}^{n-m} \frac{[R(x_0)]^{j-1}}{\Gamma(j)} e^{-R(x_o)},$$

and

$$\mathrm{F}_2(\mathrm{x}_o) = 1 - \mathrm{g}_2(\mathrm{x}_o, \mathrm{u}), \tag{5.3.10}$$

where

$$g_2(x_0,u)=\sum_{j=1}^{n-m}\frac{[R(u+x_0)-R(u)]^{j-1}}{\Gamma(j)}\exp\{-(R(u+x_0)-R(u))\}.$$

Now equating (5.3.9) and (5.3.10), we get

$$\int_0^\infty \frac{[R(y)]^{m-1}}{\Gamma(m)}f(u)[g_2(u,x_0)-g_1(x_0)]du=0,\ \ \mathrm{x}_0>0. \tag{5.3.11}$$

Now $g_2(x_0, 0) = g_1(0)$ and

$$0=\frac{[R(u)-R(u)]^{n-m-1}}{\Gamma(n-m)}\exp\{-(R(u+x_0)-R(u)\}[r(x_0)-r(u+x_0)].$$

Thus if F $\in$ C_2, then (5.3.15) is true if

$$\mathrm{r(u+x_0)=r(u)} \tag{5.3.12}$$

for almost all u and any fixed $x_0 \geq 0$. Hence X_k has the cdf F(x) = $1-e^{-\sigma x}$, $x\geq 0,\sigma>0$. k $\geq$ 1. Here σ is an arbitrary positive real number. Substituting m = n-1, we get $I_{n\text{-}1,n} \stackrel{d}{=} X_1$ as a characteristic property of the exponential distribution.

Theorem 5.3.3 *Let* $\{X_n,\ n \geq 1\}$ *be a sequence of independent and identically distributed non-negative random variables with absolutely continuous distribution function F(x) and the corresponding density function f(x). If F belongs to* C_2 *and for some m, m > 1, X(n) and X(n − 1) + U are identically distributed, where U is independent of X(n) and X(n − 1) is distributed as* X_n*'s, then* X_k has the cdf $F(x)=1-e^{-\sigma x},\ x\geq 0,\sigma>0,\ k\ \geq\ 1.$

Proof The pdf f_m (x) of R_m, m $\geq$ 1, can be written as

$$\begin{aligned}f_m(y)&=\frac{[R(y)]^m}{\Gamma(m+1)}f(y),0<y<\infty,\\&=\frac{d}{dy}\left(-\bar{F}(y)\int_0^y\frac{[R(x)]^{m-1}}{\Gamma(m)}r(x)dx+\int_0^y\frac{[R(x)]^m}{\Gamma(m)}f(x)dx\right),\end{aligned}$$

The pdf f_2 (y) of X(n − 1) + U can be written as

$$\begin{aligned}f_2(y)&=\int_0^y\frac{[R(x)]^{m-1}}{\Gamma(m)}f(y-x)f(x)dy\\&=\frac{d}{dy}\left(-\frac{[R(x)]^{m-1}}{\Gamma(m)}\bar{F}(y-x)f(x)dx+\int_0^y\frac{[R(x)]^{m-1}}{\Gamma(m)}f(x)dx\right).\end{aligned}$$

Equating (8.3.9) and (8.3.12), we get on simplification

$$\int_0^y \frac{[R(x)]^{m-1}}{\Gamma(m-1)} f(x) H_1(x,y) dx = 0, \tag{5.3.13}$$

where $H_1(x,y) = \bar{F}(y-x) - \bar{F}(y)(\bar{F}(x))^{-1}, 0 < x < y < \infty$. Since F $\in$ C_1, therefore for (8.2.13) to be true, we must have

$$\mathrm{H}_1(\mathrm{x}, \mathrm{y}) = 0, \tag{5.3.14}$$

for almost all x, 0 < x < y < ∞.

This implies that

$$\bar{F}(y-x)\bar{F}(x) = \bar{F}(y), \tag{5.3.15}$$

for almost all x, 0 < x < y < ∞. The only continuous solution of (5.3.15) with the boundary conditions $\overline{F}(0) = 1,$ *and* $\overline{F}(\infty) = 0,$ *is*

$$\overline{F}(x) = e^{-x\sigma^{-1}}, \tag{5.3.16}$$

where σ is an arbitrary positive number.

Theorem 5.3.4 *Let* $\{X_n,\ n \geq 1\}$ *be a sequence of independent and identically distributed non-negative random variables with absolutely continuous distribution function* $F(x)$ *and the corresponding probability We assume* $F(0) = 0$ *and* $F(x) > 0$ *for all* $x > 0$ *density function* $f(x)$. *Then the following two conditions are equivalent.*

(a) *X's has an exponential distribution with* $F(x) = 1 - e^{-\theta x}$, x ≥ 0, $\theta > 0$.

(b) $X(n) \stackrel{d}{=} X(n-2) + \mathrm{W}$

where w has the pdf $f_W(w)$ as $f_W(w) = \frac{\theta^2 w e^{-\theta w}}{\Gamma(2)}$, w ≥ 0, $\theta > 0$.

For proof, see Ahsanullah and Aliev (2008).

Theorem 5.3.5 *Let* $X_1, X_2, \ldots, X_m, \ldots$ *be independent and identically distributed random variables with probability density function* $f(x)$, $x \geq 0$ *and m is an integer valued random variable independent of X's and* $\mathrm{P}(\mathrm{m} = \mathrm{k}) = \mathrm{p}(1 - \mathrm{p})^{\mathrm{k}-1}$, $\mathrm{k} = 1, 2, \ldots,$ *and* $0 < p < 1$. *Then the following two properties are equivalent:*

(a) *X's are distributed as* $E(0, \sigma)$, *where* σ *is a positive real number*

(b) $\mathrm{p} \sum_{j=1}^{m} X_j \stackrel{d}{=} I_{n-1,n}$, *for some fixed n,* $n \geq 2$, $X_j \in C_2$ *and* $E(X_j) < \infty$.

Proof It is easy to verify (a) ⇒ (b). We will prove here that (b) ⇒ (a). Let ϕ_1 (t) be the characteristic function of $I_{n-1,n}$ then

$$\begin{aligned}\phi_1(t) &= \int_0^\infty \int_0^\infty \frac{1}{\Gamma(n)} e^{itx}[R(u)]^{n-1} r(u) f(u+x) du\, dx \\ &= 1 + it \int_0^\infty \int_0^\infty \frac{1}{\Gamma(n)} e^{itx}[R(u)]^{n-1} r(u) \bar{F}(u+x) du\, dx\end{aligned} \tag{5.3.17}$$

The characteristic function $\phi_2(t)$ of p $\sum_{j=1}^{m} X_j$ can be written as

$$\begin{aligned}\Phi_2(t) &= E\left(e^{itp\sum_{j=1}^{m} X_j}\right) \\ &= \sum_{k=1}^{\infty} [\Phi(tp)]^k p(1-p)^{k-1}, \\ &= \mathrm{p}(\Phi(\mathrm{tp}))\,(1-\mathrm{q}\,\Phi(\mathrm{pt}))^{-1}, \mathrm{q} \ = 1-\mathrm{p},\end{aligned} \tag{5.3.18}$$

where Φ(t) is the characteristic function of X's.

Equating (5.3.17) and (5.3.18), we get on simplification

$$\frac{\Phi(pt)-1}{1-q\Phi(pt)} \frac{1}{it} = \int_0^\infty \int_0^\infty \frac{1}{\Gamma(n)} e^{itx}[R(u)]^{n-1} r(u) \bar{F}(u+x) du\, dx \tag{5.3.19}$$

Now taking limit of both sides of (5.3.19) as t goes to zero, we have

$$\frac{\Phi^{'}(0)}{i} = \int_0^\infty \int_0^\infty \frac{1}{\Gamma(n)} [R(u)]^{n-1} r(u) \bar{F}(u+x) dudx. \tag{5.3.20}$$

$$\Gamma(n)\frac{\Phi^{'}(0)}{i} = E(x) = \int_0^\infty (R(u))^{n-1} f(u)du \int_o^\infty (F(x))dx$$

Thus

$$\int_0^\infty (R(u))^{n-1} f(u)[(\overline{F}(x) - \frac{\overline{F}(u+x)}{\overline{F}(u)})dudx = 0$$

Since X's belong to C_1, we must have

$$\bar{F}(u+x) = \bar{F}(x)\bar{F}(u), \tag{5.3.21}$$

for almost all x, u, 0 < u, x < ∞. The only continuous solution of (5.2.21) with the boundary condition $\bar{F}(0) = 1 \; and \; \bar{F}(\infty) = 0,$ is

$$\bar{F}(x) = \exp(-x\sigma^{-1}), x \geq 0, \tag{5.3.22}$$

where σ is an arbitrary positive real number.

It is known that (see Ahsanullah and Holland (1994), p. 475) for Gumbel distribution

$$X(m)^* \stackrel{d}{=} X - (W_1 + \frac{W_2}{2} + \ldots + \frac{W_{m-1}}{m-1} +, m > 1$$

where $X(n)^*$ is the nth lower record from the Gumbel distribution, $W_0 = 0$ and W_1, $W_2, \ldots, W_{m-1}$ are independently distributed as exponential with $F(w) = 1 - e^{-w}$, $w > 0$. $X^*(1) = X$. Thus $S_{(m)} = m\,(X^*(m-1) - X^*(m)$, m = 2, …, are identically distributed as exponential. Similarly if we consider the upper records from the distribution, $F(x) = e^{-e^x}, -\infty < x < \infty$, then for any $m \geq 1$, $S_m = m(X(m) - X(m+1))$, $m = 2, \ldots.$, where X(m) is the upper record, are identically distributed as exponential distribution. It can be shown that for one fixed m, $S_{(m)}$ or S_m distributed as exponential does not characterize the Gumbel distribution.

Arnold and Villasenor (1997) raised the question suppose that S_1 and $2S_2$ are i.i.d. exponential with unit mean, can we consider that X_j's are (possibly translated) Gumbel variables? Here, we will prove that for a fixed m > 1, the condition $X^*(n-1) = X^*(n) + \frac{W}{n-1}$ where W is distributed as exponential distribution with mean unity characterizes the Gumbel distribution.

Theorem 5.3.4 *Let* $\{X_j, j = 1, \ldots,\}$ *be a sequence of independent and identically distributed random variables with absolutely continuous (with respect to Lebesgue measure) distribution function* F(x). *Then the following two statements are identical.*

(a) $F(x) = e^{-e^{-x}}, -\infty < x < \infty$,
(b) *For a fixed* $m \geq 1$, *the condition* $X^*(m) = X^*(m+1) + \frac{W}{m}$ *where W is distributed as exponential with mean unity.*

Proof It is enough to show that (b) ⇒ (a). Suppose that for a fixed m > 1, $X(m) \stackrel{d}{=} X(m-1) + \frac{W}{m}$, then

$$\begin{aligned} F_{(m)}(x) &= \int_{-\infty}^{x} P(W \leq m(x-y) f_{(m+1)}(y)dy \\ &= \int_{-\infty}^{x} [1 - e^{-m(x-y)}] f_{(m+1)}(y)dy \\ &= F_{(m+1)(x)} - \int_{-\infty}^{x} e^{-m(x-y)} f_{(m+1)}(y)dy. \end{aligned} \tag{5.3.23}$$

Thus

$$e^{mx}[F_{(m+1)}(x) - F_{(m)}(x)] = \int_{-\infty}^{x} e^{my} f_{(m+1)}(y)dy \tag{5.3.24}$$

Using the relation

$$e^{mx}\frac{F(x)[H(x)]^m}{\Gamma(m+1)} = e^{H(x)}\sum_{j=0}^{m}\frac{[H(x)]^j}{m!},$$

we obtain

$$e^{mx}\frac{F(x)(H(x))^m}{\Gamma(m+1)} = \int_{-\infty}^{x} e^{my} f_{(m+1)}(y)dy \tag{5.3.25}$$

Taking the derivatives of both sides of (5.3.25), we obtain

$$\frac{d}{dx}\left[e^{mx}\frac{(H(x))^m}{\Gamma(m+1)}F(x)\right] = e^{mx} f_{(m+1)}(x) \tag{5.3.26}$$

This implies that

$$\frac{d}{dx}\left[e^{mx}\frac{H^m(x)}{\Gamma(m+1)}\right]F(x) = 0. \tag{5.3.27}$$

Thus

$$\frac{d}{dx}\left[e^{mx}\frac{(H(x))^m}{\Gamma(m+1)}\right] = 0. \tag{5.3.28}$$

Hence

$$\mathrm{H(x) = c\, e^{-x}}, \quad -\infty < \mathrm{x} < \infty$$

Thus

$$\mathrm{F(x)} = e^{-c e^{-x}}, \quad \infty < \mathrm{x} < \infty. \tag{5.3.29}$$

Since F(x) is a distribution function we must have c as positive. Assuming F (0) = e^{-1}, we obtain

$$\mathrm{F(x)} = e^{-e^{-x}}, \quad -\infty < x < \infty. \tag{5.3.30}$$

Ahsanullah and Malov (2004) proved the following characterization theorem.

Theorem 5.3.5 *Let X_1, X_2, …, be a sequence of independent and identically distributed r.v.'s with distribution function $F(x)$. If $X(m) \overset{d}{=} X(m-2) + \frac{W_1}{m-2} + \frac{W_2}{m-1}$, $m > 2$, for twice differrentiable $F(x)$, where W_1 and W_2 are independent as exponential distribution with unit mean then $F(x) = 1 - e^{-e^{x}}$, $-\infty < x < \infty$.*

Chapter 6
Characterizations of Distributions by Generalized Order Statistics

In this chapter characterizations of distributions based on generalized order statistics will be considered.

6.1 Characterizations by Conditional Expectations

We have seen that for a continuous random variable X if

$$\mathrm{E}(\mathrm{X}_{r+s,n}|X_{r,n} = x) = ax + b, 1 \le r < n, 1 \le s \le n - s$$

or

$$\mathrm{E}(\mathrm{R}(\mathrm{r}+\mathrm{s})|\mathrm{R}(\mathrm{s}) = \mathrm{x}) = ax + b, r, s \ge 1. \text{ Then if}$$

(1) $a = 1$, then X has the exponential distribution.
(2) $a > 1$, then X has the Pareto distribution,
(3) $a < 1$, then X has the power function distribution.

These results are special cases of the following theorem.

Theorem 6.1 *Suppose that X(r, n, m, k), r = 1, 2,…, n are n generalized order statistics from an absolutely continuous cdf F(x) and pdf f(x). We assume F(0) = 0, F(x) > 0 for all x > 0 and E(X(r, n, m, k)), 1 ≤ r < n is finite.*
For

$$1 \le r < s \le n, \text{if } \mathrm{E}(\mathrm{X}(\mathrm{s},\mathrm{n},\mathrm{m},\mathrm{k})|\mathrm{X}(\mathrm{r},\mathrm{n},\mathrm{m},\mathrm{k}), = \mathrm{x}) = a\mathrm{x} + \mathrm{b}, \text{then}$$

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(1) a = 1, F(x) is exponential
(2) $a > 1$, *F(x) is Pareto*
(3) $a < 1$, *F(x) is power function.*

To prove the theorem, we need the following Lemma (for details, see Rao and Shanbag 1994).

Lemma 6.1 *Consider the integral equation* $\int_{R_+} H(x+y)\mu(dy) = H(x) + c, x \in R_+,$ *where c is a real constant,* μ *is a non-arithmetic* σ *finite measure on* R_+ *such that* $\mu(\{0\}) < 1$ *and H:* $R_+ \longleftrightarrow R_+$ *is a Borel measureable either non-decreasing or non-increasing function that is not identically equal to a constant. Then there is a* γ *in* R_+ *such that* $\int_{R_+} \exp(\gamma x)\mu(dx) = 1$ *and H has the form*

$$\begin{aligned} \mathrm{H(x)} &= \lambda + \alpha(1 - \exp(\gamma x)), \text{if } \gamma \neq 0 \\ &= \lambda + \beta x, \text{if } \gamma = 0 \end{aligned}$$

where λ, α, β *are constants. If c = 0, then* $\lambda - -\alpha$ *and* $\beta = 0$.

Proof of Theorem 6.1 We can write (see Kamps and Cramer 2001)

$$\mathrm{E(X(s,n,m,k)|X(r,n,m,k),=x)} = \int_x^\infty \frac{c_{s-1}}{c_{r-1}} y \sum_{i=r+1}^{s} a_{(i)}^{(r)}(s) \left(\frac{\bar F(y)}{\bar F(x)}\right)^{\gamma_i} \frac{f(y)}{1-F(x)} dy, \tag{6.1.1}$$

where $a_{(i)}^{(r)}(s) = \Pi_{j=r+1, j\neq i}^{s} \frac{1}{\gamma_j - \gamma_i}, r+1 < i \leq s.\gamma_j \neq \gamma_i$.

Now we have

$$\int_x^\infty \frac{c_{s-1}}{c_{r-1}} y \sum_{i=r+1}^{s} a_{(i)}^{(r)}(s) \left(\frac{\bar F(y)}{\bar F(x)}\right)^{\gamma_i} \frac{f(y)}{\bar F(x)} dy = ax + b \tag{6.1.2}$$

Substituting $\mathrm{t} = \frac{\bar F(y)}{\bar F(x)} i.e. y = (\bar F^{-1}(t(\bar F(x))$ and $\bar F(x) = w$, we obtain from (6.1.2)

$$\int_x^\infty \frac{c_{s-1}}{c_{r-1}} y \sum_{i=r+1}^{s} a_{(i)}^{(r)}(s) t^{\gamma_i - 1} (\bar F)^{-1}(tw) dt = a(\bar F)^{-1}(w) + b$$

i.e.

$$\int_x^\infty \frac{c_{s-1}}{a\, c_{r-1}} \sum_{i=r+1}^{s} a_{(i)}^{(r)}(s) t^{\gamma_i - 1} (\bar F)^{-1}(tw) dt = (\bar F)^{-1}(w) + \frac{b}{a} \tag{6.1.3}$$

Putting $t = e^{-u}$ and $w = e^{-w}$, we have from (6.1.3)

$$\int_x^\infty \frac{c_{s-1}}{a\, c_{r-1}} \sum_{i=r+1}^{s} a_{(i)}^{(r)}(s) t^{\gamma_i - 1} (\bar F)^{-1}(e^{-(u+v)}) e^{-u\gamma_i} du = (\bar F)^{-1}(e^{-v}) + \frac{b}{a}$$

We can write

$$\frac{c_{s-1}}{a\,c_{r-1}} \sum_{i=r+1}^{s} a_{(i)}^{(r)}(s) e^{-u\gamma_i} du = \mu(d\mu)$$

Here μ is a finite measure and we can find a η such that

$$\frac{c_{s-1}}{a\,c_{r-1}} \sum_{i=r+1}^{s} a_{(i)}^{(r)}(s) \int_0^\infty e^{-x(\gamma_r-\eta)} dx = 1$$

i,e.

$$\frac{c_{s-1}}{a\,c_{r-1}} \sum_{i=r+1}^{s} a_{(i)}^{(r)}(s) \frac{1}{\gamma_r - \eta} = 1 \tag{6.1.4}$$

Further $\int_x^\infty \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_{(i)}^{(r)}(s) \left(\frac{\bar{F}(x)}{\bar{F}(y)}\right)^{\gamma_i} \frac{f(x)}{1-F(y)} dy = 1$
i.e.

$$\frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_{(i)}^{(r)}(s) \frac{1}{\gamma_r} = 1 \tag{6.1.5}$$

It is obvious that

1. $a = 1$, iff $\eta = 0$,
2. $a > 1$ iff $\eta > 1$,
3. $a < 1$ iff $\eta < 1$.

Consider the three cases:
If $a = 1$, then $\bar{F}^{-1}(e^{-x}) = \gamma + \beta x$
Hence

$$\bar{F}(x) = e^{-(\frac{x-\gamma}{\beta})} = e^{-\lambda(x-\gamma)},\ x > \gamma \text{ and } \lambda = \frac{1}{\beta}.$$

Hence the random variable X has the exponential distribution.
If $a > 1$, then $\eta > 0$, and

$$\bar{F}(x) = \left(\frac{-\alpha}{x-\alpha-\gamma}\right)^{1/\eta} = \left(\frac{\gamma-(\alpha+\gamma)}{x-(\alpha+\gamma)}\right) = \left(\frac{\mu+\delta}{x+\delta}\right) \cdot\ x > \delta = \alpha+\gamma, \mu = \gamma.$$

The random variable X has the Pareto distribution.
If $a < 1$, then $\eta < 0$, and

$$\bar{F}(x) = \left(\frac{\alpha+\gamma-x}{\alpha}\right)^{-1/\eta} = \left(\frac{\alpha+\gamma-x}{\alpha+\gamma-\gamma}\right)^{-1/\eta} = \left(\frac{\nu-x}{\upsilon-\mu}\right)^{\theta} \cdot\ \mu < x < \nu, \mu = \gamma, \nu = \alpha+\gamma \text{ and } \theta - -\frac{1}{\eta}.$$

Thus X has the power function distribution.

Theorem 6.2 *Suppose that X(r, n, m, k), r = 1, 2,..., n are n generalized order statistics from an absolutely continuous cdf F(x) and pdf f(x). We assume F(0) = 0, F(x) > 0 for all x > 0 and E(X(r, n, m, k)), 1 ≤ r < n is finite. Let g(x) be a continuous such that* $\lim_{x\to 0} g(x) = 1$ *and* $\lim_{x\to\infty} g(x) = 0$*, then and if 1 ≤ r < n, if*

$$\mathrm{E(X(r+1,n,m,k))} \ge x|\mathrm{X(r,n,m,k)=t)} = (\mathrm{g(x-t)})^{\gamma_{r+1}}, \mathrm{then\,F(x)} = 1 - \mathrm{e}^{-\lambda x}, x \ge 0, \lambda > 0.$$

Proof We have

$$\mathrm{E(X(r+1,n,m,k))} \ge x|\mathrm{X(r,n,m,k)=t)} = \left(\frac{1-F(x)}{1-F(t)}\right)^{\gamma_{r+1}}.$$

Thus

$$\left(\frac{1-F(x)}{1-F(t)}\right)^{\gamma_{r+1}} = (g(x-t))^{\gamma_{r+1}}$$

The proofs of the following two theorems are similar.

Theorem 6.3 *Suppose that X(r, n, m, k), r = 1, 2,..., n are n generalized order statistics from an absolutely continuous cdf F(x) and pdf f(x). We assume F(1) = 0, F(x) > 0 for all x > 1 and E(X(r, n, m, k)), 1 ≤ r < n is finite. Let g(x) be a continuous function such that* $\lim_{x\to 1} g(x) = 1$ *and* $\lim_{x\to\infty} g(x) = 0$*, then if 1 ≤ r < n, and*

$$\mathrm{E(X(r+1,n,m,k))} \ge x|\mathrm{X(r,n,m,k)=t)} = \left(\mathrm{g}\left(\frac{x}{t}\right)\right)^{\gamma_{r+1}}, \mathrm{then\,F(x)} = 1 - \mathrm{x}^{\lambda}, x \ge 0, \lambda < 0.$$

Theorem 6.4 *Suppose that X(r, n, m, k), r = 1, 2,..., n are n generalized order statistics from an absolutely continuous cdf F(x) and pdf f(x). We assume F(1) = 0, F(x) > 0 for all x > 0 and E(X(r, n, m, k)), 1≤ r < n is finite. Let g(x) be a continuous such that* $\lim_{x\to 0} g(x) = 1$ *and* $\lim_{x\to 1} g(x) = 0$*, then if 1 ≤ r < n, if*

$$\mathrm{E(X(r+1,n,m,k))} \ge x|\mathrm{X(r,n,m,k)=t)} = \left(\mathrm{g}\left(\frac{1-x}{1-t}\right)\right)^{\gamma_{r+1}}, \mathrm{then\,F(x)} = 1 - (1-\mathrm{x})^{\lambda}, 0 \le x \le 1, \lambda > 0.$$

Theorem 6.5 *Let* $X_j, j = 1, 2, \ldots, n$ *be i.i.d random variables on (a, b) with an absolutely continuous cfd F, pdf f and* $\lim_{x\to b} s(x)(1-F(x))^{\gamma_{r+1}} = 0$ *where s(x) is a differentiable function on (a, b). Let X(r, n, k), r = 1, 2,..., n be the first n gos from F and let h(x) be a positive differentiable function on (a, b) such that* $\lim_{x\to b} h(x) e^{\int_a^x \frac{g'(t)}{h(t)} dt} = \infty$. *Then for* $m \ge -1, E(s(X(r+1,n,m.k)|X(r,n,m,k) = t) = s(t) + h(t), a < t < b$ *if and only if* $F(x) = 1 - \left(\left(\frac{h(x)}{h(a)}\right) e^{\int_a^x \frac{g'(t)}{h(t)} dt}\right)^{-\frac{1}{\gamma_{r+1}}}.$

Proof It is easy to proof the if condition. We will prove here the only if condition.

We have

$$E(s(X(r+1,n,m.k))|X(,n,m,k)=t)=\int_t^b s(x)\gamma_{r+1}\left(\frac{1-F(x)}{1-F(t)}\right)^{\gamma_{r+1}-1}\left(\frac{f(x)}{1-F(t)}\right)$$

Thus

$$\int_t^b s(x)\gamma_{r+1}\left(\frac{1-F(x)}{1-F(t)}\right)^{\gamma_{r+1}-1}\left(\frac{f(x)}{1-F(t)}\right)=s(t)+h(t) \tag{6.1.6}$$

Differentiating both sides of (6.1.6) and simplifying, we obtain

$$\frac{f(t)}{1-F(t)}=\frac{1}{\gamma_{r+1}}\left[\frac{s'(t)}{h(t)}+\frac{h'(t)}{h(t)}\right] \tag{6.1.7}$$

Integrating both sides of (6.1.8) from a to x, we obtain

$$-\ln(1-F(x))=\frac{1}{\gamma_{r+1}}\left(\int_a^x\frac{s'(t)}{h(t)}dt+\ln\frac{h(x)}{h(a)}\right)$$

Thus

$$\mathrm{F(x)}=1-\left(\left(\frac{h(x)}{h(a)}\right)e^{\int_a^x\frac{g'(t)}{h(t)}dt}\right)^{-\frac{1}{\gamma_{r+1}}}.$$

The next theorem gives a characterization based on the conditional expectation of X(r, n, m, k) given X(r+1, n, m, k).

Theorem 6.6 *Let $X_j, j=1,2,\ldots,n$ be i.i.d random variables on (a, b) with an absolutely continuous cdf F, pdf f and $\lim_{x\to b}s(x)(1-(1-F(x))^{m+1})^r=0$ where s(x) is a differentiable function on (a, b). Let X(r, n, k), r = 1, 2,…, n be the first n gos from F and let h(x) be a positive differentiable function on (a, b) such that $\lim_{x\to b}h(x)e^{\int_a^x\frac{g'(t)}{h(t)}dt}=\infty$. Then for $m>-1, E(s(X(r,n,m.k)|X(r+,n,m,k)=t)=s(t)-h(t), a<t<b$ if and only if $F(x)=1-\left(1-\left(\left(\frac{h(x)}{h(b)}\right)e^{\int_x^b\frac{g'(t)}{h(t)}dt}\right)^{-1}\right)^{\frac{1}{m+1.}}$.*

Proof The if condition is easy to prove. We will give here a proof of the only if condition.

We have

$$E(s(r,n,m,k)|X(r+1,n,m,k)=t)=\int_a^t s(x)r(1-F(x))^r(\frac{1-(1-F(x))^{m+1}}{m+1})^{r-1}(\frac{1-(1-F(t))^{m+1}}{m+1})^{-r}$$

$$xf(x)dx.$$

Thus

$$\int_a^t s(x) r(1-F(x))^r \left(\frac{1-(1-F(x))^{m+1}}{m+1}\right)^{r-1} \left(\frac{1-(1-F(t))^{m+1}}{m+1}\right)^{-r} x f(x) dx - = s(t) - h(t), a < t < b \tag{6.1.8}$$

Differentiating both sides of (6.1.8) and solemnifying, we obtain

$$\frac{(m+1)(1-F(t))^m}{1-(1-F(t))^{m+1}} = \frac{1}{r}\left[\frac{s'(t)}{h(t)} - \frac{h'(t)}{h(t)}\right] \tag{6.1.9}$$

Integrating both sides of (6.1.9) with respect to t from x to b, we obtain

$$\mathrm{F(x)} = 1 - \left(1 - \left(\left(\frac{h(x)}{h(b)}\right) e^{\int_x^b \frac{g'(t)}{h(t)} dt}\right)^{-1}\right)^{\frac{1}{m+1}}.$$

Theorem 6.7 *Let $X_j, j = 1, 2, \ldots, n$ be i.i.d random variables on (a, b) with an absolutely continuous cfd F, pdf f and $\lim_{x \to b} s(x)(1-F(x))^{\gamma_{r+1}} = 0$ where s(x) is a differentiable function on (a, b). Let X(r, n, k), r = 1, 2,..., n be the first n gos from F and let h(x) be a positive differentiable function on (a, b) such that $\lim_{x \to b} h(x) e^{\int_a^x \frac{g'(t)}{h(t)} dt} = \infty$. Then for $m \geq -1, E(s(X(r+1, n, m.k) | X(r, n, m, k) = t) = s(t) + h(t), a < t < b$ if and only if $F(x) = 1 - \left(\left(\frac{h(x)}{h(a)}\right) e^{\int_a^x \frac{g'(t)}{h(t)} dt}\right)^{-\frac{1}{\gamma_{r+1}}}$.*

Proof It is easy to proof the if condition. We will prove here the only if condition.
We have

$$E(s(X(r+1, n, m.k) | X(, n, m, k) = t) = \int_t^b s(x) \gamma_{r+1} \left(\frac{1-F(x)}{1-F(t)}\right)^{\gamma_{r+1}-1} \left(\frac{f(x)}{1-F(t)}\right)$$

Thus

$$\int_t^b s(x) \gamma_{r+1} \left(\frac{1-F(x)}{1-F(t)}\right)^{\gamma_{r+1}-1} \left(\frac{f(x)}{1-F(t)}\right) = s(t) + h(t) \tag{6.1.10}$$

Differentiating both sides of (6.1.10) and simplifying, we obtain

$$\frac{f(t)}{1-F(t)} = \frac{1}{\gamma_{r+1}}\left[\frac{s'(t)}{h(t)} + \frac{h'(t)}{h(t)}\right] \tag{6.1.11}$$

Integrating both sides of (6.1.11) from *a* to x, we obtain

$$-\ln(1-F(x)) = \frac{1}{\gamma_{r+1}}\left(\int_a^x \frac{s'(t)}{h(t)} dt + \ln \frac{h(x)}{h(a)}\right)$$

Thus

$$F(x) = 1 - \left(\left(\frac{h(x)}{h(a)} \right) e^{\int_a^x \frac{g'(t)}{h(t)} dt} \right)^{-\frac{1}{\gamma_{r+1}}}.$$

6.2 Characterizations by Equality of Expectations of Normalized Spacings

We define the normalized spacings as $D(1, n, m, k) = \gamma_1 X(1, n, m, k)$ $D(r, n, m, k) = \gamma_r(X(r, n, m, k) - X(r-1, n, m, k)), 2 \leq r \leq n$. Kamp and Gather (1997) gave the following characterization theorems.

Theorem 6.8 *Let F(x) be absolutely continuous cdf with pdf f(x) with* $F(0) = 0$ *and suppose that F(x) is strictly increasing in* $(0, \infty)$*, and F belongs to* C_1*, Then* $F(x)1 - e^{-\lambda x}, \lambda > 0$ *and* $x \geq 0$*, if and only if there exist integers r, s and n,* $1 \leq r < s \leq n$*, such* $E(D(r, n, m, k)) = E(D(s, n, m, k))$.

Theorem 6.9 *Let F(x) be absolutely continuous cdf with pdf f(x) with* $F(0) = 0$ *and suppose that F(x) is strictly increasing in* $(0, \infty)$*, and F belongs to* C_2*, Then* $F(x)$ $1 - e^{-\lambda x}, \lambda > 0$ *and* $x \geq 0$*, if and only if there exist integers r, s and n,* $1 \leq r < s \leq n$*, such* $E(X(r+1, n, m, k)) - E(X(r, n, m, k)) = E(X(1, n-r, m, k))$

Remark 6.3 Without further assumption, the equation

$$E(X(r+1, n, m, k)) - E(X(r, n, m, k)) = E(X(1, n-r, m, k)) \text{ for just one pair (r,n), } 1 \leq r \leq n-1$$

does not characterize the exponential distribution. For example the distribution

$$\begin{aligned} F(x) = 1 - (1 + cx^d)^{-1/(m+1)}, & c > .x \in (0, \infty), m > -1 \\ & c < 0, x \in \left(0, (-1/c))^{1,d}\right) \;\; m < -1 \\ & \text{and } d = \frac{\gamma_1}{\gamma_2}. \end{aligned}$$

6.3 Characterizations by Equality of Distributions

Here a characterization using lower generalized order statistics will be presented.

Theorem 6.10 *Let X be an absolutely continuous bounded random variable with cdf F(x) and pdf f(x). We assume without any loss of geberality F(0) = 0 · F(x)>* 0, 0 *<x < 1 and F(x) = 1 for all x ≥* 1. *Then the following two statements are equivalent:*

(a) X is uniformly distributed as uniform on [0, 1],

(b) $X^*(r+1, n, m, k)) \stackrel{d}{=} X^*(r,n,m,k) W_{r+1}$,

where W_{r+1} *is independent of X*(r+1, n, m, k) and the pdf of* $f_{r+1}(w)$ *of* W_{r+1} *is as follows:* $f_{r+1}(w) = \gamma_{r+1} w^{\gamma_{r+1}-1}, 0 \le w \le 1.$

Proof (a)⟶ (*b*). Let Y = X*(r, n, m, k)W_{r+1}. The cdf $F_Y(y)$ of Y ias as follows:

$$F_Y(x) = F^*_{r,n,m,k}(x) + \int_x^1 \left(\frac{x}{u}\right)^{\gamma_{r+1}} f^*_{r,n,m,k}(u)du$$

Differentiating both sides of the above equation with respect to x, we obtain

$$f_Y(x) = f^*_{r,n,m,k}(x) - f^*_{r,n,m,k}(x) + \int_x^1 \frac{\gamma_{r+1}}{x}\left(\frac{x}{u}\right)^{\gamma_{r+1}} f^*_{r,n,m,k}(u)du$$

i.e.

$$\frac{f_Y(x)}{x^{\gamma_{r+1}-1}} = \int_x^1 \gamma_{r+1}\left(\frac{1}{u}\right)^{\gamma_{r+1}} f^*_{r,n,m,k}(u)du \qquad (6.3.1)$$

Differentiating both sides of (6.3.1) with respect to x, we obtain

$$\frac{f'_Y(x)}{x^{\gamma_{r+1}-1}} - \frac{f_Y(x)}{x^{\gamma_{r+1}}}(\gamma_{r+1} - 1) = -\gamma_{r+1}\left(\frac{1}{x}\right)^{\gamma_{r+1}} f^*_{r,n,m,k}(x)$$

On simplification we obtain from above,

$$f'_Y(x) - \frac{\gamma_{r+1} - 1}{x} f_Y(x) = -c_r \frac{x^{\gamma_r - 1}}{\Gamma(r)x}\left[\frac{1 - x^{m+1}}{m+1}\right]^{r-1}$$

Multiplying by $x^{-(\gamma_{r+1}-1)}$ and we obtain

$$\frac{d}{dx}(f_Y(x)x^{-(\gamma_{r+1}-1)}) = -c_r \frac{x^m}{\Gamma(r)x}\left[\frac{1 - x^{m+1}}{m+1}\right]^{r-1}.$$

Thus

$$f_Y(x)x^{-(\gamma_{r+1}-1)} = c - \int c_r \frac{x^m}{\Gamma(r)x}\left[\frac{1 - x^{m+1}}{m+1}\right]^{r-1} dx$$

$$= c + \frac{c_r}{\Gamma(r+1)}\left[\frac{1 - x^{m+1}}{m+1}\right]^r$$

Thus for uniform distribution.

$$f_Y(x) = cx^{\gamma_{r+1}-1} + \frac{c_r x^{\gamma_{r+1}-1}}{\Gamma(r+1)}\left[\frac{1 - x^{m+1}}{m+1}\right]^r$$

Since $F_Y(0) = 0$ and $F_y(1) = 1$, we must have c = 0 and

$$f_Y(x) = \frac{c_r x^{\gamma_{r+1}-1}}{\Gamma(r+1)}\left[\frac{1-x^{m+1}}{m+1}\right]^r$$

Thus $\mathrm{Y} = \mathrm{X}^*(r+1, n, m, k)$.
Prove of (b) → (a).

$$\begin{aligned}F^*{}_{r+1,n,m,k}(x) &= P(X^*(r,n,m,k)W_{r+1} \le x)\\ &= \int_0^1 F^*_{r,n,m,k}(\frac{x}{u})u^{\gamma_{r+1}-1}du\\ &= \mathrm{x}^{\gamma_{r+1}} + \gamma_{r+1}\int_x^1 F^*_{r,n,m,k}\left(\frac{x}{u}\right)u^{\gamma_{r+1}-1}du\end{aligned}$$

Substituting $\mathrm{t} = \frac{x}{u}$ in the above integral, we obtain

$$\mathrm{F}^*{}_{r+1,n,m,k}(x) = \mathrm{x}^{\gamma_{r+1}} + \gamma_{r+1}x^{\gamma_{r+1}}\int_x^1 F^*_{r,n,m,k}(t)\left(\frac{1}{t}\right)^{\gamma_{r+1}+1}dt \tag{6.3.2}$$

Differentiating both sides of (6.3.2) with respect to x, we obtain

$$\mathrm{f}^*{}_{r+1,n,m,k}(x) = \gamma_{r+1}\mathrm{x}^{\gamma_{r+1}-1} - \gamma_{r+1}x^{\gamma_{r+1}}f^*_{r,n,m,k}(x)\left(\frac{1}{x}\right)^{\gamma_{r+1}+1}$$

$$+(\gamma_{r+1})^2x^{\gamma_{r+1}-1}\int_x^1 F^*_{r,n,m,k}(t)\left(\frac{1}{t}\right)^{\gamma_{r+1}+1}dt \tag{6.3.3}$$

Using (6.3.2), we will have from (6.3.3)

$$\mathrm{f}^*{}_{r+1,n,m,k}(x) = \frac{f(x)}{F(x)}\mathrm{f}^*{}_{r+1,n,m,k}(x)x^{-1}$$

Thus

$$\frac{f(x)}{F(x)} = \frac{1}{x} \tag{6.3.4}$$

Integrating both sides of (6.3.4) and using the boundary conditions $\mathrm{F}(0) = 0$ and $\mathrm{F}(1) = 1$, we get (a).

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