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Mohammad Ahsanullah

Extreme Value Distributions

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Mohammad Ahsanullah

Extreme Value Distributions

ATLANTIS PRESS

Mohammad Ahsanullah
Department of Management Sciences
Rider University
Lawrenceville, NJ
USA

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Preface

Extreme values are extremely interesting. The maximum or minimum of large number observations when normalized can only converge to three types of extreme value distributions, Gumbel, Frechet and Weibull. Thus the maximum and minimum order statistics of n observations when normalized converges to the extreme value distributions as n tends to infinity. The local maximum or minimum (records) of a sequence of independent and identically distributed random variables are useful to estimate the parameters of the extreme value distributions. In Chap. 1 of this book some distributional properties of the largest and smallest order statistics from some important distributions are presented. In Chap. 2 some basic properties of record values and inferences based on the distributional properties are given. In Chap. 3 the necessary and sufficient conditions of maximum and minimum order statistics to converge to the extreme value distributions are derived. In Chap. 3 also the normalizing constants of several well-known distributions are derived. In Chap. 4 estimations of parameters of the extreme value distributions are derived. An extensive reference of papers related to ordered random variables is given. This book can be used as a textbook or as a consulting book.

In this book there may be some errors escaped our attention. However, I will be glad to receive any comments from the readers about it. I am grateful to the Atlantis press for publishing this book.

Lawrenceville, NJ, USA Mohammad Ahsanullah

Contents

Chapter 1
Order Statistics

1.1 Distributional Properties

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (I, I, d) absolutely continuous random variables. Suppose that F(x) be their cumulative distribution function (cdf) and f(x) be the corresponding probability density function (pdf). Let $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ be the corresponding order statistics. We denote $F_{k,n}(x)$ and $f_{k,n}(x)$ as the cdf and pdf respectively of $X_{k,n}$, k = 1, 2, …, n. We can write

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!}(F(x))^{k-1}(1-F(x))^{n-k}f(x),$$

The joint probability density function of order statistics $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$ has the form

$$f_{1,2,\ldots,n:n}(x_1,x_2,\ldots,x_n) = n!\prod_{k=1}^{n} f(x_k), -\infty < x_1 < x_2 < \cdots < x_n < \infty$$

and

$$= 0,\ otherwise.$$

There are some simple formulae for distributions of maxima ($X_{n,n}$) and minimum ($X_{1,n}$) of the n random variables.

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The pdfs of the smallest and largest order statistics are given respectively as

$$f_{1,n}(x) = n(1 - F(x))^{n-1} f(x)$$

and

$$f_{n,n}(x) = n(F(x))^{n-1} f(x)$$

The joint pdf $f_{1,n,n}(x,y)$ of $X_{1,n}$ and $X_{n.n}$ is given by

$$f_{1,n,n}(x,y) = n(n-1)(F(y) - F(x))^{n-2} f(x) f(y),$$
$$-\infty < x < y < \infty.$$

Example 1.1 Exponential distribution.

Suppose that X_1, X_2, …, X_n are n i.i.d. random variables with cdf F(x) as $F(x) = 1 - \mathrm{e}^{-x}.x \ge 0$.

The pdfs $f_{1,n}(x)$ of $X_{1,n}$ and $f_{n.n}(x)$ are respectively

$$f_{1,n}(x) = n\mathrm{e}^{-nx}, x \ge 0.$$

and

$$f_{n,n}(x) = n(1 - \mathrm{e}^{-x})^{n-1} \mathrm{e}^{-x}, \quad x \ge 0.$$

It can be seen that $nX_{1,n}$ has the exponential; distribution.

The pdfs of $X_{1.n}$ and $X_{n,n}$ are given respectively in Figs. 1.1 and 1.2 for n = 3, 5 and 10.

The limiting distributions of standardized asymptotic distributions of $X_{1,n}$ and $X_{m,m}$ are given in Chap. 3.

Example 1.2 Uniform distribution.

Suppose that X_1, X_2, …, X_n are n i.i.d. random variables with cdf F(x) as $F(x) = x,\ 0 > x < 1$. We have

The pdfs $f_{1,n}(x)$ of $X_{1,n}$ and $f_{n.n}(x)$ are respectively

$$f_{1,n}(x) = n(1 - x)^{n-1}, \quad 0 < x < 1,$$

and

$$f_{1,n}(x) = nx^{n-1}, \quad 0 < x < 1.$$

The pdfs pf $X_{1.n}$ and $X_{n,n}$ are given respectively in Figs. 1.3 and 1.4 for n = 2, 5 and 10.

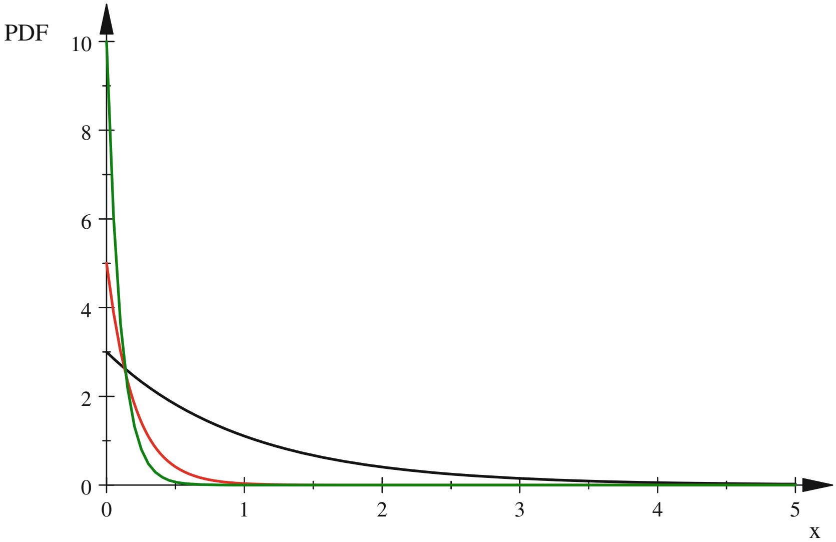

Fig. 1.1 PDFs $f_{1,3}(x)$—*black*, $f_{1,5}$ (x)—*red*, $f_{1.10}(x)$—*green*

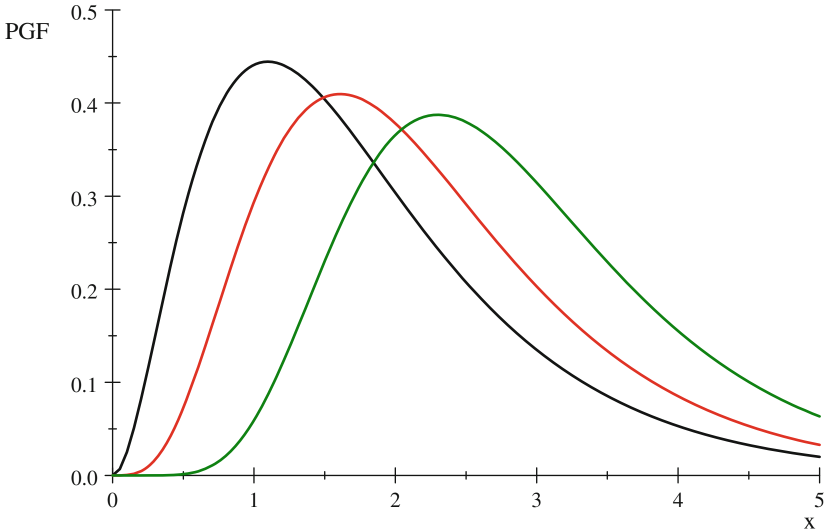

Fig. 1.2 PDFs $f_{3,3}(x)$—*black*, $f_{5,55}$ (x)—*red*, $f_{10.19}(x)$—*green*

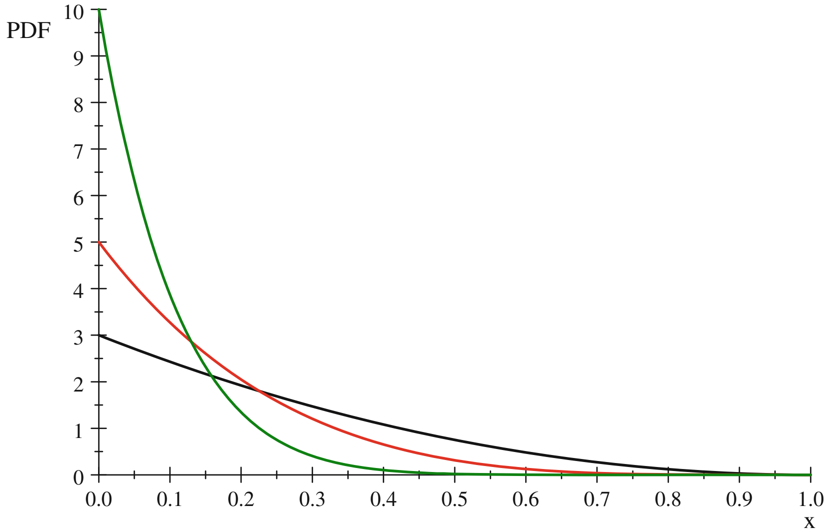

Fig. 1.3 PDFs $f_{1,3}(x)$—*black*, $f_{1,5}(x)$—*red*, $f_{1.10}(x)$—*green*

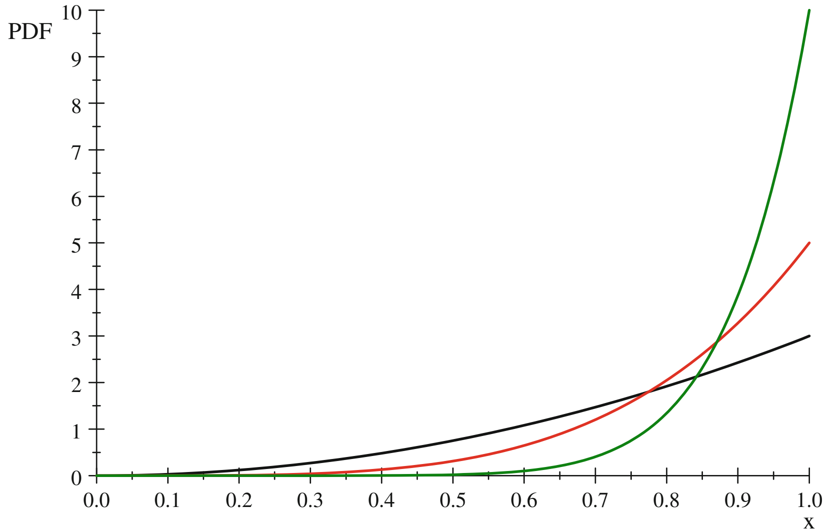

Fig. 1.4 PDFs $f_{3,3}(x)$—*black*, $f_{5,5}(x)$—*red*, $f_{10.19}(x)$—*green*

The limiting distributions of standardized $X_{1,n}$ and $X_{m,m}$ are given in Chap. 3.

Example 1.3 Rayleigh distribution.

Suppose that $X_1, X_2, \ldots, X_n$ are n i.i.d. random variables with cdf

$$F(x) = 1 - e^{-\frac{x^2}{2}}, \quad x \geq 0.$$

We have the pdfs $f_{1,n}(x)$ of $X_{1,n}$ and $f_{n.n}(x)$ are respectively

$$f_{1,n}(x) = nxe^{-\frac{nx^2}{2}}, \quad x \geq 0.,$$

and

$$f_{1,n}(x) = nx(1 - e^{-\frac{x^2}{2}},)^{n-1}e^{-\frac{x^2}{2}}, \quad x \geq 0.$$

The pdfs pf $X_{1.n}$ and $X_{n,n}$ are given respectively in Figs. 1.5 and 1.6 for n = 3, 5 and 10.

Example 1.4 Pareto distribution.

Suppose that $X_1, X_2, \ldots, X_n$ are n i.i.d. random variables with cdf $F(x) = 1 - \frac{1}{x^2}$, $x \geq 1$. We have the pdfs $f_{1,n}(x)$ of $X_{1,n}$ and $f_{n.n}(x)$ are respectively

$$f_{1,n}(x) = \frac{2n}{x^{2n+1}} \quad x \geq 1,$$

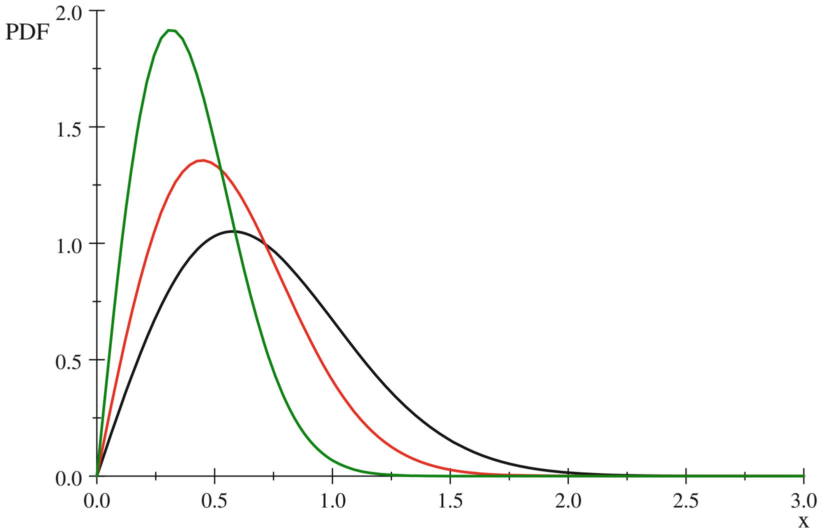

Fig. 1.5 PDFs $f_{3,3}(x)$—*black*, $f_{3,5}(x)_f$—*red*, $f_{1.19}(x)$—*green*

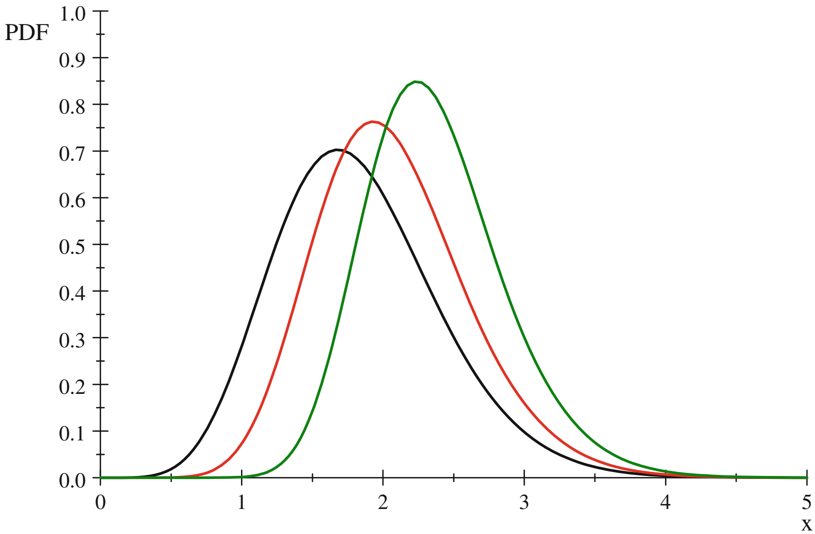

Fig. 1.6 PDFs $f_{3,3}(x)$—*black*, $f_{5,5}(x)$—*red*, $f_{10.19}(x)$—*green*

and

$$f_{n,n}(x) = \frac{2n}{x^3}\left(1 - \frac{1}{x^2},\right)^{n-1}, \quad x \geq 1$$

The pdfs pf $X_{1.n}$ and $X_{n,n}$ are given respectively in Figs. 1.7 and 1.8 for n = 3, 5 and 10.

The joint pdf $f_{r,s,n}$ of two order statistics $X_{1,n}$ and $X_{n,n}$ ($1 \leq r < s < n$) is given by

$$f_{r,s,n}(x,y) = c_{r,s,n}(F(x))^{r-1}(F(y)\_F(x))^{s-r-1}(1 - F(y))^{n-s}f(x)f(y)$$

where $c_{r,s,n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}, 1 \leq r < s \leq n.$

The conditional distribution of $X_{s,n}|X_{r,n}$.

Let $f_{s|r,n}$ be the pdf of $X_{s,n}|X_{r,n}$, then

$$\begin{aligned} f_{s\,|\,r,n}(y\,|\,x) &= \frac{f_{r,s,n(x,y)}}{F_{r,n}(x)} \\ &= \frac{c_{r,s,n}}{c_{r,n}} \frac{(F(y) - F(x))^{s-r-1}(1 - F(y)^{n-s}f(y)}{(1 - F(x))^{n-r}}, \end{aligned}$$

where $1 < r < s < n$, $-\infty < x < y < \infty$ and $c_{r,n} = \frac{n!}{(r-1)!(n-r)!}$.

The distribution of the difference between two order statistics.

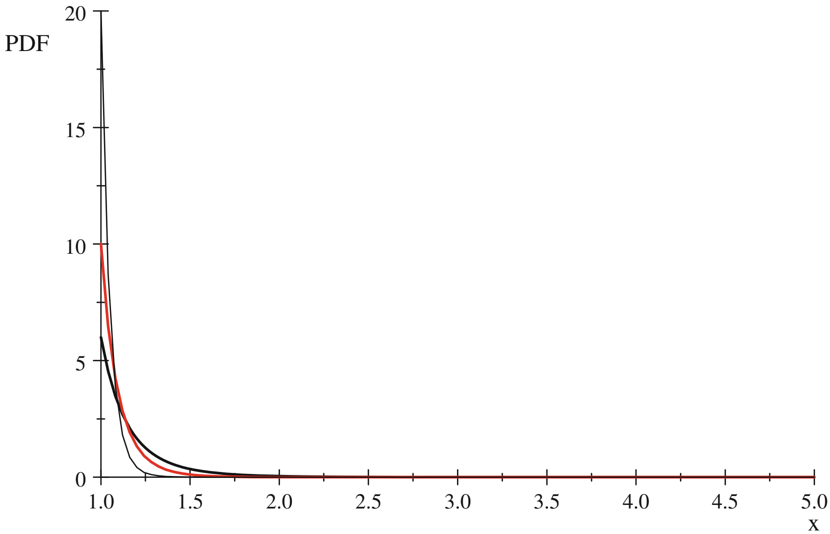

Fig. 1.7 PDFs $f_{3,3}(x)$—*black*, $f_{3,5}(x)_f$—*red*, $f_{1.19}(x)$—*green*

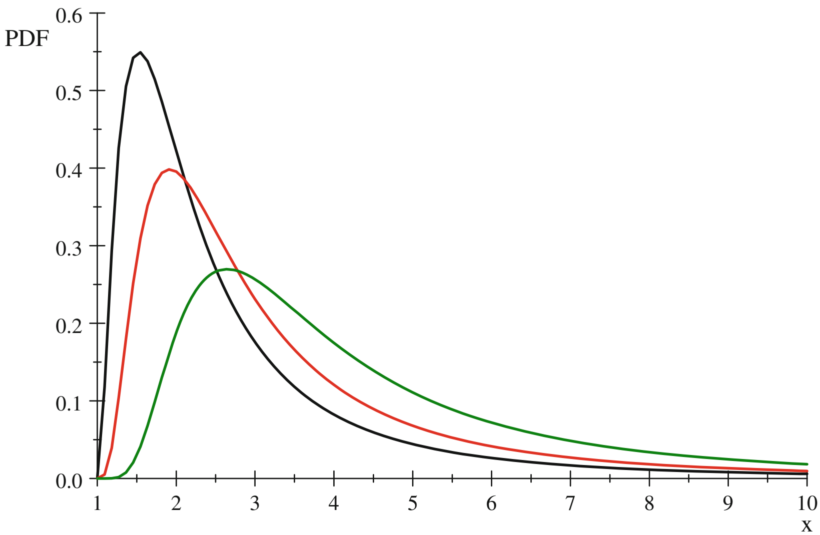

Fig. 1.8 PDFs $f_{3,3}(x)$—*black*, $f_{5,5}(x)$—*red*, $f_{10.19}(x)$—*green*

The using the transformation

$$U = X_{r,n}$$

and

$$V = X_{s,n} - X_{r,n},\ s > r$$

we can write the pdf $f_{U,V}(u, v)$ of U and V as

$$f_{U,V}(u,v) = c_{r,s,n}(F(u))^{r-1}(F(u+v) - F(u))^{s-r-1}(1 - F(u+v))^{n-s}f(u)f(u+v)$$

The pdf $f_V(v)$ov V is given by

$$f_V(v) \int_0^{\infty} c_{r,s,n}(F(u))^{r-1}(F(u+v) - F(u))^{s-r-1}(1 - F(u+v))^{n-s}f(u)f(u+v)du$$

For exponential distribution with $F(x) = 1 - e^{-x}$, $x \geq 0$, the pdf of V is the same as the distribution of $X_{s-r,n-r}$.

1.2 Minimum Variance Linear Unbiased Estimates

Suppose that X has an absolutely continuous distribution function of the form

$$F((x - \mu)/\sigma), \quad -\infty < \mu < \infty,\ \sigma > 0.$$

Further assume

$$\begin{aligned}
E\left(X_{r,n}\right) &= \mu + \alpha_r \sigma, \quad r = 1, 2, \ldots, n, \\
Var\left(X_{r,n}\right) &= V_{rr}\sigma^2, \quad r = 1, 2, \ldots, n, \\
Cov\left(X_{r,n}, X_{s,n}\right) &= Cov\left(X_{s,n}, X_{r,n}\right) = V_{rs}\sigma^2, \quad 1 \leq r < s \leq n.
\end{aligned}$$

Let

$$\mathbf{X}' = \left(X_{1,n}, X_{2,n}, \ldots, X_{n,n}\right).$$

We can write

$$E(\mathbf{X}) = \mu\,\mathbf{1} + \sigma\,\boldsymbol{\alpha}$$

where

$$\mathbf{1} = (1, 1, \ldots, 1)',$$
$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)'$$

and

$$\mathrm{Var}(\mathbf{X}) = \sigma^2\,\boldsymbol{\Sigma},$$

where $\boldsymbol{\Sigma}$ is a matrix with elements V_{rs}, $1 \le r, s \le n$.

Then the MVLUE's of the location and scale parameters μ and σ are

$$\widehat{\mu} = \frac{1}{\Delta}\left\{\alpha'\Sigma^{-1}\alpha 1'\Sigma^{-1} - \alpha'\Sigma^{-1}1\alpha'\Sigma^{-1}\right\}X$$

and

$$\hat{\sigma} = \frac{1}{\Delta}\left\{1'\Sigma^{-1}1\alpha'\Sigma^{-1} - 1'\Sigma^{-1}\alpha 1'\Sigma^{-1}\right\}X,$$

where

$$\Delta = \left(\alpha'\Sigma^{-1}\alpha\right)(1'\Sigma 1) - \left(\alpha'\Sigma^{-1}1\right)^2.$$

The variance and the covariance of these estimators are given as

$$\mathrm{Var}(\hat{\mu}) = \frac{\sigma^2\left(\alpha'\Sigma^{-1}\alpha\right)}{\Delta},$$
$$\mathrm{Var}(\hat{\sigma}) = \frac{\sigma^2\left(1'\Sigma^{-1}1\right)}{\Delta}$$

and

$$\mathrm{Cov}(\widehat{\mu}, \widehat{\sigma}) = -\frac{\sigma^2\left(\alpha'\Sigma^{-1}1\right)}{\Delta}.$$

Note that for any symmetric distribution

$$\alpha_j = -\alpha_{n-j+1},\ 1'\Sigma^{-1}\alpha = \alpha'\Sigma^{-1}1 = 0$$

and

$$\Delta = \left(\alpha'\Sigma^{-1}\alpha\right)\left(1'\Sigma^{-1}1\right).$$

Hence the best linear unbiased estimates of μ and σ for the symmetric case are

$$\widehat{\mu}^* = \frac{1'\Sigma^{-1}X}{1'\Sigma^{-1}1},$$
$$\widehat{\sigma}^* = \frac{\alpha'\Sigma^{-1}X}{\alpha'\Sigma^{-1}\alpha}$$

and the corresponding covariance of the estimators is zero and the their variances are given as

$$\mathrm{Var}(\widehat{\mu}^*) = \frac{\sigma^2}{1'\Sigma^{-1}1}$$

and

$$\mathrm{Var}(\widehat{\sigma}^*) = \frac{\sigma^2}{\alpha'\Sigma^{-1}\alpha}.$$

We can use the above formulas to obtain the MVLUEs of the location and scale parameters for any distribution numerically provided the variances of the order statistics exist. For some distributions the MVLUEs of the location and scale parameters can be expressed in simplified form.

The following Lemma (see Garybill 1983, p. 198) will be useful to find the inverse of the covariance matrix.

Lemma 2.1 Let $\Sigma = (\sigma_{r,s})$ be n × n matrix with elements, which satisfy the relation

$$\sigma_{rs} = \sigma_{sr} = c_r d_s, \quad 1 \le r,\ s \le n,$$

for some positive c_1, …, c_n and d_1 ,…, d_n. Then its inverse

$$\Sigma^{-1} = (\sigma^{r,s})$$

has elements given as follows:

$$\sigma^{1,1} = c_2/c_1(c_2d_1 - c_1d_2),$$
$$\sigma^{n,n} = d_{n-1}/d_n(c_nd_{n-1} - c_{n-1}d_n),$$
$$\sigma^{k+1,k} = \sigma^{k,k+1} = -1/(c_{k+1}d_k - c_kd_{k+1}),$$
$$\sigma^{k,k} = (c_{k+1}d_{k-1} - c_{k-1}d_{k+1})/(\ c_kd_{k-1} - c_{k-1}d_k)\,(c_{k+1}d_k - c_kd_{k+1}),\ k = 2,\ldots,n-1,$$

and

$$\sigma^{i,j} = 0,\ \text{if}|i - j| > 1.$$

Example 2.1 Suppose $X_1, X_2, \ldots, X_n$ are n independent and identically distributed uniform random variables with pdf f(x) given as follows:

$$f(x) = 1/\sigma,$$

if $\mu - \sigma/2 \leq x \leq \mu + \sigma/2$, where $-\infty < \mu < \infty$, $\sigma > 0$, and

$$f(x) = 0, \text{ otherwise}$$

We have moments of the uniform order statistics:

$$E(X_{r,n}) = \mu + \sigma\left(\frac{r}{n+1} - \frac{1}{2}\right),$$
$$\mathrm{Var}(X_{r,n}) = \frac{r(n-r+1)}{(n+1)^2(n+2)}\sigma^2, \quad r = 1, 2, \ldots, n,$$

and

$$Cov(X_{r,n}X_{s,n}) = \frac{r(n-s+1)}{(n+1)^2(n+2)}\sigma^2, \quad 1 \leq r \leq s \leq n.$$

We can write

$$\mathrm{Cov}(X_{r,n}\,X_{s,n}) = \sigma^2 c_r\, d_s, \quad 1 \leq r \leq s \leq n,$$

where

$$c_r = \frac{r}{(n+1)^2}, \quad 1 \leq r \leq n,$$

and

$$d_s = \frac{n-s+1}{n+2}, \quad 1 \leq s \leq n.$$

Using Lemma 2.1, we obtain that

$$\sigma^{j,j} = 2(n+1)(n+2), \quad j = 1, 2, \ldots, n,$$
$$\sigma^{i,j} = -(n+1)(n+2), \quad j = i+1,\ i = 1, 2, \ldots, n-1,$$
$$\sigma^{i,j} = 0,\ |i-j| > 1.$$

It can easily be verified that

$$1'\Sigma^{-1} = ((n+1)(n+2), 0, 0, \ldots, 0, (n+1)(n+2)),$$
$$1'\Sigma^{-1}\underline{1} = 2(n+1)(n+2),$$
$$1'\Sigma^{-1}\alpha = 0,$$
$$\alpha'\Sigma^{-1} = \left(-\frac{(n+1)(n+2)}{2}, 0, 0, \ldots, 0, \frac{(n+1)(n+2)}{2}\right)$$

and

$$\alpha'\Sigma^{-1}\alpha = \frac{(n-1)(n+2)}{2}.$$

Thus, the MVLUEs of the location and scale parameters μ and σ are

$$\widehat{\mu} = \frac{1'\Sigma^{-1}X}{1'\Sigma^{-1}1} = \frac{X_{1,n} + X_{n,n}}{2} \quad \text{and}$$
$$\widehat{\sigma} = \frac{\alpha'\Sigma^{-1}X}{\alpha'\Sigma^{-1}\alpha} = \frac{(n+1)\left(X_{n.n} - X_{1,n}\right)}{n-1}.$$

The corresponding covariance of the estimators is zero and their variances are

$$\mathrm{Var}(\widehat{\mu}) = \frac{\sigma^2}{1'\Sigma^{-1}1} = \frac{\sigma^2}{2(n+1)(n+2)}$$

and

$$\mathrm{Var}(\widehat{\sigma}) = \frac{\sigma^2}{\alpha'\Sigma^{-1}\alpha} = \frac{2\sigma^2}{(n-1)(n+2)}.$$

Example 2.2 Suppose that X_1, X_2, …, X_n are n independent and identically distributed exponential random variables with the probability density function, given as

$$f(x) = (1/\sigma)\exp(-(x-\mu)/\sigma), \quad -\infty < \mu < x < \infty,\ 0 < \sigma < \infty,$$

and

$$f(x) = 0, \text{ otherwise}.$$

From lecture 8 we have means, variances and covariances of the exponential order statistics:

$$\mathrm{E}\left(\mathrm{X}_{\mathrm{r,n}}\right) = \mu + \sigma \sum_{j=1}^{r} \frac{1}{n-j+1},$$
$$\mathrm{Var}\left(X_{r,n}\right) = \sigma^2 \sum_{j=1}^{r} \frac{1}{(n-j+1)^2}, \quad \mathrm{r} = 1, 2, \ldots, \mathrm{n},$$

and

$$\mathrm{Cov}\left(\mathrm{X}_{\mathrm{r,n}}, \mathrm{X}_{\mathrm{s,n}}\right) = \sigma^2 \sum_{j=1}^{r} \frac{1}{(n-j+1)^2}, \quad 1 \le r \le s \le n.$$

One can write that

$$\mathrm{Cov}\left(\mathrm{X}_{\mathrm{r,n}}\mathrm{X}_{\mathrm{s,n}}\right) = \sigma^2 c_r d_s, \quad 1 \le r \le s \le n,$$

where

$$c_r = \sum_{j=1}^{r} \frac{1}{(n-j+1)^2}, \quad 1 \le r \le n,$$

and

$$d_s = 1,\ 1 \le s \le n.$$

Using Lemma 2.1, we obtain (see also Example 13.2) that

$$\sigma^{j,j} = (n-j)^2 + (n-j+1)^2, \quad j = 1, 2, \ldots, n,$$
$$\sigma^{j+1,j} = \sigma^{j,j+1} = (n-j)^2, \quad j = 1,\ i,j = 1, 2, \ldots, n-1,$$

and

$$\sigma^{i,j} = 0,\ \text{if } |i-j| > 1, \quad i,j = 1, 2, \ldots, n.$$

It can easily be shown that

$$1'\Sigma^{-1} = \left(n^2, 0, 0, \ldots, 0\right),\ \alpha'\Sigma^{-1} = (1, 1, \ldots, 1)$$

and

$$\Delta = n^2(n-1).$$

The MVLUEs of the location and scale parameters of μ and σ are respectively

$$\hat{\mu} = \frac{nX_{1,n} - \overline{X}_n}{n-1}$$

and

$$\hat{\sigma} = \frac{n\left(\overline{X}_n - X_{1,n}\right)}{n-1}.$$

The corresponding variances and the covariance of the estimators are

$$Var(\hat{\mu}) = \frac{\sigma^2}{n(n-1)},$$
$$Var(\hat{\sigma}) = \frac{\sigma^2}{n-1}$$

and

$$Cov(\hat{\mu}, \hat{\sigma}) = -\frac{\sigma^2}{n(n-1)}.$$

Exercise 2.1 Suppose that $X_1, X_2, \ldots, X_n$ are n independent random variables having power function distribution with the pdf f(x) given as

$$f(x) = \frac{r}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{\gamma-1}, \; -\infty < \mu < x < \mu + \sigma,$$

where $0 < \sigma < \infty$ and $0 < \gamma < \infty$, and

$$f(x) = 0, \text{ otherwise}.$$

Find MVLUEs of the parameters of μ and σ.

Example 2.3 Suppose that $X_1, X_2, \ldots, X_n$ are n independent and identically distributed Pareto random variables with pdf f(x), which is given as follows:

$$f(x) = \frac{\gamma}{\sigma}\left(1 + \frac{x-\mu}{\sigma}\right)^{-1-\lambda}, \quad \mu < x < \infty,$$

where $0 < \sigma < \infty$ and $0 < \gamma < \infty$, and

$$f(x) = 0, \text{ otherwise.}$$

Show that the MVLUEs of parameters μ and σ have the form

$$\hat{\mu} = X_{1,n} - (c_1 - 1)\hat{\sigma}$$

and

$$\hat{\sigma} = M_2\left[\sum_{i=1}^{n-1} -P_i X_{i,n} + \sum_{i=1}^{n-1} P_i X_{n.n}\right],$$

where

$$M_2 = \left(c_n \sum_{i=1}^{n-1} P_i - \sum_{i=1}^{n-1} c_i P_i\right)^{-1},$$

with

$$\begin{aligned} P_1 &= D - (\gamma + 1)d_1, \\ Pj &= -(\gamma + 1)d_j, \quad j = 2, \ldots, n - 1, \\ P_n &= (\gamma - 1)d_n \end{aligned}$$

and

$$D = (\gamma + 1)\sum_{i=1}^{n-1} d_i - (\gamma - 1)d_n.$$

The corresponding variances and the covariance of the estimators are

$$\begin{aligned} \mathrm{Var}(\hat{\mu}) &= E\sigma^2, \\ \mathrm{Var}(\widehat{\sigma}) &= \left((n\gamma - 1)^2 E - 1\right)\sigma^2 \end{aligned}$$

and

$$\mathrm{Cov}(\hat{\mu}, \hat{\sigma}) = -\frac{(n\gamma - 1)(n\gamma - 2) - E}{(n\gamma - 2)E}\sigma^2,$$

where

$$E = n\gamma\ (\gamma - 2) - \frac{(n\gamma - 2)^2}{n\gamma - 2 - D}.$$

Suppose that X has an absolutely continuous distribution function of the form F (x/σ), where σ > 0 is an unknown scale parameter. Further assume that

$$E(X_{r,n}) = \alpha_r \sigma, \quad r = 1, 2, \ldots, n,$$
$$Var(X_{r,n}) = V_{rr}\sigma^2, \quad r = 1, 2, \ldots, n,$$
$$Cov(X_{r,n} X_{s,n}) = V_{rs}\sigma^2, \quad 1 \le r < s \le n.$$

Let

$$\mathbf{X}' = (X_{1,n}, X_{2,n}, \ldots, X_{n,n}).$$

Then we can write

$$E(\mathbf{X}) = \sigma \boldsymbol{\alpha}$$

with

$$\boldsymbol{\alpha}' = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

and

$$Var(\mathbf{X}) = \sigma^2 \boldsymbol{\Sigma},$$

where $\boldsymbol{\Sigma}$ is a matrix with elements V_{rs}, $1 \le r \le s \le n$.

Then the MVLUE of the scale parameter σ is given as

$$\hat{\sigma} = \alpha' \Sigma^{-1} X / \alpha' \Sigma^{-1} \alpha$$
$$Var\hat{\sigma} = \sigma^2 / \alpha' \Sigma^{-1} \alpha.$$

Example 2.4 Suppose that $X_1, X_2, \ldots, X_n$ are n independent and identically distributed exponential random variables with pdf given as

$$f(x) = (1/\sigma) \exp(-x/\sigma),\ x > 0,$$

where $0 < \sigma < \infty$, and

$$f(x) = 0,\ \text{otherwise}.$$

We can write that

$$\mathrm{E}\left(\mathrm{X_{r,n}}\right) = \sigma \sum_{j=1}^{r} \frac{1}{n-j+1},$$
$$\mathrm{Var}\left(X_{r,n}\right) = \sigma^2 \sum_{j=1}^{r} \frac{1}{(n-j+1)^2}, \quad \mathrm{r} = 1, 2, \ldots, \mathrm{n},$$

and

$$\mathrm{Cov}\left(\mathrm{X_{r,n}X_{s,n}}\right) = \sigma^2 \sum_{j=1}^{r} \frac{1}{(n-j+1)^2}, \quad 1 \le r \le s \le n.$$

In this situation

$$\mathrm{Cov}\left(\mathrm{X_{r,n}X_{s,n}}\right) = \sigma^2 c_r d_s, \quad 1 \le r \le s \le n,$$

where

$$c_r = \sum_{j=1}^{r} \frac{1}{(n-j+1)^2}, \quad 1 \le r \le n,$$

and

$$d_s = 1, \quad 1 \le s \le n.$$

We have

$$\sigma^{j,j} = (n-j)^2 + (n-j+1)^2, \quad j = 1, 2, \ldots, n,$$
$$\sigma^{j+1,j} = \sigma^{j,j+1} = -(n-j)^2, \quad j = 1,\ i,j = 1, 2, \ldots, n-1,$$

and

$$\sigma^{i,j} = 0 \quad \text{for } |i-j| > 1,\ i,j = 1, 2, \ldots, n.$$

We have in this case

$$\alpha' = (1/n, 1/n + 1/(n-1), \ldots, 1/n + \cdots + 1/2 + 1),$$
$$\alpha'\Sigma^{-1} = (1, 1, \ldots, 1)$$

and

$$\alpha'\Sigma^{-1}\alpha = n.$$

Thus, the MVLUE of σ is given as

$$\hat{\sigma} = \overline{X}$$

and

$$\mathrm{Var}(\widehat{\sigma}) = \sigma/n.$$

Exercise 2.2 Suppose that X_1, X_2, …, X_n are n independent and identically distributed uniform random variables with pdf f(x), which is given as follows:

$$f(x) = 1/\sigma,\ 0 < x < \sigma,$$

where $0 < \sigma < \infty$, and

$$f(x) = 0, \text{ otherwise.}$$

Show that the MVLUE of σ in this case is given as

$$\widehat{\sigma} = \frac{(n+1)X_{n,n}}{n}$$

and

$$\mathrm{Var}\,(\widehat{\sigma}) = \frac{\sigma^2}{n(n+2)}.$$

Suppose **that** smallest r_1 and largest r_2 observations are missing.

We will consider here the minimum variance linear unbiased estimation (MVLUE) of location and scale parameters. Suppose X has an absolutely continuous distribution function of the form $F((x - \mu)/\sigma)$, $-\infty < \mu < \infty$, $\sigma > 0$. Further assume

$$\begin{aligned} E\left(X_{r,n}\right) &= \mu + \alpha_r\sigma, \\ \mathrm{Var}\left(X_{r,n}\right) &= V_{rr}\sigma^2, \quad r_1 < r < r_2 \le n \\ \mathrm{Cov}\left(X_{r,n}X_{s,n}\right) &= V_{rs}\sigma^2, \quad r_1 < r < s < r_2 \le n. \end{aligned}$$

Let $X' = (X_{r_1,n}, X_{r_1+1,n}, \ldots, X_{r_2-1,n})$. Then we can write

$$E(X) = \mu\,1 + \sigma\,\alpha$$

where

$$1' = (1, 1, \ldots, 1)',\ \alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

and

$$\mathrm{Var}(\mathrm{X}) = \sigma^2\Sigma,$$

where Σ is a matrix with elements V_{rs}, $1 \le r_1 \le \underline{r}s \le r_{2<n}$.

Then the MVLUE of the location and scale parameters μ and σ based on the order statistics $X' = X_{r_1,n}, X_{r_1+1,n}, \ldots, X_{r_2-1,n}$ are

$$\hat{\mu} = \frac{1}{\Delta}\{\alpha'\Sigma^{-1}\alpha 1'\Sigma^{-1} - \alpha'\Sigma^{-1}1\alpha'\Sigma^{-1}\}X$$

and

$$\hat{\sigma} = \frac{1}{\Delta}\left\{1'\Sigma^{-1}1\alpha'\Sigma^{-1} - 1'\Sigma^{-1}\alpha 1'\Sigma^{-1}\right\}\mathrm{X}$$

where

$$\Delta = \left(\alpha'\Sigma^{-1}\alpha\right)(1'\Sigma 1) - \left(\alpha'\Sigma^{-1}1\right)^2.$$

The variance and the covariance of these estimators are given as

$$\mathrm{Var}(\hat{\mu}) = \frac{\sigma^2\left(\alpha'\Sigma^{-1}\alpha\right)}{\Delta},$$

$$\mathrm{Var}(\hat{\sigma}) = \frac{\sigma^2\left(1'\Sigma^{-1}1\right)}{\Delta}$$

and

$$\mathrm{Cov}(\hat{\mu}, \hat{\sigma}) = -\frac{\sigma^2\left(\alpha'\Sigma^{=1}1\right)}{\Delta}.$$

Example 2.4 Consider a uniform distribution with cumulative distribution function as

$$F(x) = \frac{2x - 2\mu + \sigma}{2\sigma},\ \mu - \sigma/2 < x < \mu + \sigma/2, -\infty < \mu < \infty \text{ and } \sigma > 0$$

Suppose that the smallest r_1 and the largest r_2 observations are missing, Then considering these $X_{r_1+1,n}, X_{r_2+2,n}, \ldots, X_{n-r_2,n}$ order statistics, it can be shown that the inverse of the corresponding covariance matrix is

$$(n+1)(n+2)\begin{pmatrix} \frac{r_1+2}{r_1+1} & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 2 & -1 & \dots & 0 \\ . & . & . & . & . & . & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{n-r+2}{n-r+1} \end{pmatrix}.$$

The BLUEs of μ and σ are respectively

$$\hat{\mu}^* = \frac{(n-2r_2-1)X_{r_1+1,n} + (n-2r_1-1)X_{n-r_2},n}{2(n-r_i-r_2-1)}$$

and

$$\hat{\sigma}^* = \frac{n+1}{n-r_1-r_2-1}\left(X_{n-r_2,n} - X_{r_1+1,n}\right).$$

The variances and the covariance of the estimators are

$$Var(\hat{\mu}^*) = \frac{(r_1+1)(n-2r_2-1)+(r_2+1)(n-2r_1-1)}{4(n+2)(n+1)(n-r_1-r_2-1)}\,\sigma^2,$$
$$Var(\hat{\sigma}^*) = \frac{r_1+r_2+2}{(n+2)(n-r_1-r_2-1)}\sigma^2,$$

and

$$Cov(\hat{\mu}^*,\hat{\sigma}^*) = \frac{1}{2(n+1)(n+2)}[(n-2r_1-1)(r_2+1)(n-r_2)$$
$$-(n-2r_2-1)(r_1+1)(n-r_1) - 2\left(r_2-r_{1)}\right)(r_1+1)(r_2+1)\big].$$

Note that If $r_1 = r_2 = r$, then $\hat{\mu}^* = \frac{X_r+X_{n-r}}{2}$ and $\mathrm{Cov}(\hat{\mu}^*,\hat{\sigma}^*) = 0$.

Exercise 2.1 Consider an exponential distribution with cumulative distribution

$$F(x) = 1 - \exp(-(x-\mu)/\sigma), -\infty < \mu < x < \infty,\ 0 < \sigma < \infty,$$

We have

$$E\left(X_{r,n}\right) = \mu + \sigma\sum_{j=1}^{r}\frac{1}{n-j+1},\ \mathrm{Var}\left(X_{r,n}\right) = \sigma^2\sum_{j=1}^{r}\frac{1}{(n-k+1)^2},\ r = 1,2,\dots,n,$$

and

$$\mathrm{Cov}\left(\mathrm{X_{r,n}X_{s,n}}\right) = \sigma^2 \sum_{j=1}^{r} \frac{1}{(n-j+1)^2},\ 1 \le r \le s. \le n \le \mathrm{n}.$$

We can write $\mathrm{Cov}\left(\mathrm{X_{r,n}\ X_{s,n}}\right) = \sigma^2 c_r d_s,\ c_r = \sum_{j=1}^{r} \frac{1}{(n-j+1)^2},\ d_s = 1,\ 1 \le r \le s \le n.$

We have $\sigma^{j,j} = (n-j)^2 + (n-j-1)^2,\ j = 1, 2, \ldots, n,$

$$\sigma^{j+1,j} = \sigma^{j,j+1} = (n-j)^2, \quad j = 1,\ i,j = 1, 2, \ldots, n,$$

and

$$\sigma^{j,j} = 0 \quad \text{for } |i-j|\ > 1,\ i,j = 1, 2, \ldots, n.$$

The BLUEs of the location and scale parameter of μ and σ are respectively

$$\hat{\mu} = \frac{\mathrm{nX_{1,n}} - \overline{\mathrm{X}}_{\mathrm{n}}}{\mathrm{n} - 1}$$

and

$$\hat{\sigma} = \frac{n\left(\overline{X}_n - X_{1,n}\right)}{n-1}.$$

The corresponding variances and the covariance of the estimators are

$$Var(\hat{\mu}) = \frac{\sigma^2}{n(n-1)},$$
$$Var(\hat{\sigma}) = \frac{\sigma^2}{n-1}$$

and

$$Cov(\hat{\mu},\ \hat{\sigma}) = -\frac{\sigma^2}{n(n-1)}.$$

Suppose X_1, X_2, …, X_n are n independent and identically distributed as power function distribution with

$$\mathrm{F(x)} = \mathrm{e}^{-(x-\rho)/\sigma},\ -\infty < \mu < \mathrm{x} < \infty,\ 0 < \sigma < \infty.$$

Further assume that the smallest r_1 and the largest r_2 observations are missing, Then considering the order statistics $X_{r_1+1,n}, X_{r_1+2,n}, \ldots, X_{n-r_2,n}$, it can be shown that the corresponding BLUEs of μ and σ are

$$\hat{\mu}^* = X_{r_1+1,n} - \alpha_{r_1+1}\hat{\sigma}^*,\ \alpha_{r_1+1} = \frac{1}{\sigma}E\left(X_{r_1+1,n} - \mu\right) = \sum_{i=1}^{r_1+1} \frac{1}{n-i+1}$$

and

$$\hat{\sigma}^* = \frac{1}{n - r_2 - r_1 - 1}\left\{\sum_{i=r_1+1}^{n-r_2} X_{i,n} - (n - r_1)X_{r_1+1,n} + r_2 X_{n-r_2,n}\right\}.$$

The variances and the covariance of the estimators are

$$Var(\hat{\mu}^*) = \sigma^2\left[\frac{\alpha_{r_1}^2}{n - r_2 - r_1 - 1} + \sum_{i=1}^{r_1+1}\frac{1}{(n-i+1)^2}\right],$$
$$Var(\hat{\sigma}^*) = \frac{\sigma^2}{n - r_1 - r_2 - 1}$$

and

$$Cov(\hat{\mu}^*, \hat{\sigma}^*) = -\frac{\alpha_r \sigma^2}{n - r_2 - r_1 - 1}.$$

Sarhan and Greenberg (1957) have prepared tables of the coefficients of the BLUEs and the variances and covariances of $\hat{\mu}^*$ and $\hat{\sigma}^*$ for $n \leq 10$.

Chapter 2
Record Statistics

In this chapter some of the basic concepts and properties of the record values are presented. For simplicity the descriptions are confined here to the sequence of independent and identically distributed continuous random variables.

2.1 Introduction and Examples of Record Values

Suppose we consider the weighing of objects on a scale missing its spring. An object is placed on the scale and its weight is measured. The 'needle' indicated the correct weight but does not return to zero when the object is removed. If various objects are placed on the scale, only the weights greater than the previous ones can be recorded. These recorded weights are the upper record value sequence. If X_{ij} be the height water level of a river on the jth day of the i-th location. If one is interested to study at each location the local maximum values of X_{ij}, then the local maxima are the upper record values.

Let us consider a sequence of products that may fail under stress. We are interested to determine the minimum failure stress of the products sequentially. We test the first product until it fails with stress less than X_1 then we record its failure stress, otherwise we consider the next product. In general we will record stress X_m of the mth product if $X_m < \min(X_1, \ldots, X_{m-1})$, $m > 2$. The recorded failure stresses are the lower record values. One can go from lower records to upper records by replacing the original sequence of random variables $\{X_j\}$ by $\{-X_j, j \geq 1\}$ or if $P(X_j > 0) = 1$ by $\{1/X_i, i \geq 1\}$.

Chandler (1952) introduced the record values, record times and inter record times. He proved the interesting result that for any given distribution of the random variables the expected value of the inter record time is infinite. Feller (1952) gave some examples of record values with respect to gambling problems.

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2.1.1 Definition of Record Values and Record Times

Suppose that X_1, X_2, is a sequence of independent and identically distributed random variables with cumulative distribution function F(x). Let Y_n = max (min) $\{X_1, X_2, \ldots, X_n\}$ for $n \geq 2$. We say Yj is an upper (lower) record value of $\{X_n, n \geq 1\}$, if $Y_j > (<) Y_{j-1}$, $j > 2$. By definition X_1 is an upper as well as a lower record value. One can transform the upper records to lower records by replacing the original sequence of $\{X_j\}$ by $\{-X_j, j \geq 1\}$ or (if $P(X_i > 0) = 1$ for all i) by $\{1/X_i, i \geq 1\}$; the lower record values of this sequence will correspond to the upper record values of the original sequence.

The indices at which the upper record values occur are given by the record times $\{U(n)\}$, $n > 0$, where $U(n) = \min\{j|j > U(n-1), X_j > X_{U(n-1)}, n > 1\}$ and U(1) = 2. The record times of the sequence $\{X_n\ n \geq 1\}$ are the same as those for the sequence $\{F(X_n), n \geq 1\}$. Since F(X) has a uniform distribution, it follows that the distribution of U(n), $n \geq 1$ does not depend on F. We will denote L(n) as the indices where the lower record values occur. By our assumption U(1) = L(1) = 2. The distribution of L(n) also does not depend on F.

2.2 The Exact Distribution of Record Values

Many properties of the record value sequence can be expressed in terms of the function R(x), where $R(x) = -\ln \bar{F}(x)$,

$0 < \bar{F}(x) < 1$ and $\bar{F}(x) = 1 - F(x)$. Here 'ln' is used for the natural logarithm. If we define $F_n(x)$ as the distribution function of $X_{U(n)}$ for

$n \geq 1$, then we have

$$F_1(x) = P\left[X_{U(1)} \leq x\right] = F(x) \tag{2.2.1}$$

$$\begin{aligned} F_2(x) &= P\left[X_{U(2)} \leq x\right] \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} \sum_{i=1}^{\infty} (F(u))^{i-1} dF(u)dF(y) \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{dF(u)}{1 - F(u)} dF(y) \\ &= \int_{-\infty}^{x} R(y)dF(y) \end{aligned} \tag{2.2.2}$$

If F(x) has a density f(x), then the probability density function (pdf) of $X_{U(2)}$ is

$$f_2(x) = R(x)f(x) \tag{2.2.3}$$

The distribution function

$$\begin{aligned} F_3(x) &= P(X_{U(3)} < x) \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} \sum_{i=0}^{\infty} (F(u))^i R(u)dF(u)dF(y) \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{R(u)}{1 - F(u)} dF(u)dF(y) \\ &= \int_{-\infty}^{x} \frac{(R(u))^2}{2!} dF(u) \end{aligned} \tag{2.2.4}$$

The pdf $f_3(x)$ of $X_{U(3)}$ is

$$f_3(x) = \frac{(R(x))^2}{2!} f(x), \; -\infty < x < \infty \tag{2.2.5}$$

It can similarly be shown that the pdf $F_n(x)$ of $X_{U(n)}$ is

$$\begin{aligned} F_n(x) &= P(X_{U(n)} < x) \\ &= \int_{-\infty}^{x} f(u_n)du_n \int_{-\infty}^{u_n} \frac{f(u_{n-1})}{1 - F(u_{n-1})} du_{n-1} \int_{-\infty}^{u_2} \frac{f(u_1)}{1 - F(u_1)} du_1. \\ &= \int_{-\infty}^{x} \frac{R^{n-1}(u)}{(n-1)!} dF(u), \quad -\infty < x < \infty \end{aligned}$$

This can be expressed as

$$\begin{aligned} F_n(x) &= \int_{-\infty}^{R(x)} \frac{u^{n-1}}{(n-1)!} e^{-u} du, \quad -\infty < x < \infty \\ \bar{F}_n(x) &= 1 - F_n(x) = \bar{F}(x) \sum_{j=0}^{n-1} \frac{(R(x))^j}{j!} \\ &= e^{-R(x)} \sum_{j=0}^{n-1} \frac{(R(x))^j}{j!} \end{aligned} \tag{2.2.6}$$

The pdf $f_n(x)$ of $X_{U(n)}$ is

$$f_n(x) = \frac{R^{n-1}(x)}{(n-1)!} f(x), \quad -\infty < x < \infty. \tag{2.2.7}$$

The joint pdf $f(x_1, x_2, \ldots, x_n)$ of the n record values $(X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)})$ is given by

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= r(x_1) r(x_2) \ldots r(x_{n-1}) f(x_n) \\ &\text{for} \quad -\infty < x_1 < x_2 < \ldots < x_{n-1} < x_n < \infty, \\ \text{where } r(x) &= \frac{d}{dx} R(x) = \frac{f(x)}{1 - F(x)}, \quad 0 < F(x) < 1. \end{aligned} \tag{2.2.8}$$

The function r(x) is known as hazard rate.

The joint pdf of $X_{U(i)}$ and $X_{U(j)}$ is

$$\begin{aligned} f(x_i, x_j) &= \frac{(R(x_i))^{i-1}}{(i-1)!} r(x_i) \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} f(x_j) \\ &\text{for } -\infty < x_i < x_j < \infty. \end{aligned} \tag{2.2.9}$$

In particular for i = 1 and j = n we have

$$\begin{aligned} f(x_1, x_n) &= r(x_1) \frac{(R(x_n) - R(x_1))^{n-2}}{(n-2)!} f(x_n) \\ &\text{for } -\infty < x_1 < x_n < \infty. \end{aligned}$$

Suppose we use the transformation $Y_1 = R(X_{U(i)})$ and $Y_2 = R(X_{U(i)})/R(X_{U(j)})$, $i < j$, then using (2.2.9), it can be shown that the pdf $f_2^*(y)$ of Y_2 is as follows:

$$f_2^*(y) = \frac{\Gamma(j)}{\Gamma(i)} \cdot \frac{1}{\Gamma(j-i)} \cdot y^{i-1} (1-y)^{j-i-1}, 0 < y < \infty. \tag{2.2.10}$$

Thus Y_2 is distributed as Beta distribution with parameters i and j (i.e. B(i, j−i)). The mean and variance of Y_2 are $E(Y_2) = \frac{i}{j}$ and $Var(Y_2) = \frac{ij}{(j+1)j^2}$.

If we use the transformation $V_i = R(X_{U(i)})$, then the joint pdf of V_i, i = 1,2, …, n, is

$$f(v_1, v_2, \ldots, v_n) = e^{-v_n}, \quad 0 < v_1 < v_2 < \ldots < v_n < \infty. \tag{2.2.11}$$

The joint distribution of V_m and V_r, $r > m$, is

$$f(v_m, v_r) = \frac{1}{\Gamma(m)} \cdot \frac{(v_r - v_m)^{r-m-1}}{\Gamma(r-m)} \cdot e^{-v_r} \; 0 < v_m < v_r < \infty$$
$$= 0, otherwise.$$
$$E(V_k^l) = \int_0^\infty t^l \frac{1}{\Gamma(k)} t^{k-1} e^{-t} dt = \frac{\Gamma(k+l)}{\Gamma(k)}.$$

Thus $E(V_k) = k$ and $Var(V_k) = k$. The conditional pdf of

$$X_{U(j)}|X_{U(i)} = x_i \text{ if} \left(x_j|X_{U(i)} = x_i\right) = \frac{f_{ij}(x_i, x_j)}{f_i(x_i)}$$
$$= \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} \frac{f(x_j)}{1 - F(x_i)} \quad (2.2.12)$$
$$\text{for } -\infty < x_i < x_j < \infty.$$

For j = i + 1

$$f\left(x_{i+1}|X_{U(i)} = x_i\right) = \frac{f(x_{i+1})}{1 - F(x_i)} \quad (2.2.13)$$
$$\text{for } -\infty < x_i < x_{i+1} < \infty.$$

For $i > 0$, $1 \le k < m$, the joint conditional pdf of $X_{U(i+k)}$ and $X_{U(i+m)} \mid X_{U(i)}$ is

$$f_{i+k,i+m}\left(x, y|X_{U(i)} = z\right) = \frac{1}{\Gamma(m-k)} \cdot \frac{1}{\Gamma(k)} \cdot [R(y) - R(x)]^{m-k-1} [R(x) - R(z)]^{k-1} \frac{f(y)r(x)}{\bar{F}(z)}$$
$$\text{for } -\infty < z < x < y < \infty.$$

The marginal pdf of the nth lower record value can be derived by using the same procedure as that of the nth upper record value. Let

$H(u) = -\ln F(u)$, $0 < F(u) < 1$ and $h(u) = -\frac{d}{du}H(u)$, then

$$P(X_{L(n)} \le x) = \int_{-\infty}^{x} \frac{\{H(u)\}^{n-1}}{(n-1)!} dF(u) \quad (2.2.14)$$

and the corresponding the pdf $f_{(n)}$ can be written as

$$f_{(n)}(x) = \frac{(H(x))^{n-1}}{(n-1)!} f(x). \quad (2.2.15)$$

The joint pdf of $X_{L(1)}, X_{L(2)}, \ldots, X_{L(m)}$ can be written as

$$f_{(1),(2),\ldots,(m)}(x_1,x_2,\ldots,x_m) = h(x_1)h(x_2)\ldots h(x_{m-1})f(x_m)$$
$$-\infty < x_m < x_{m-1} < \ldots < x_1 < \infty \quad (2.2.16)$$
$$= 0, \text{ otherwise.}$$

The joint pdf of $X_{L(r)}$ and $X_{L(s)}$ is

$$f_{(r),(s)}(x,y) = \frac{(H(x))^{r-1}}{(r-1)!}\frac{[H(y)-H(x)]^{s-r-1}}{(s-r-1)!}h(x)f(y) \quad (2.2.17)$$
$$\text{for } s > r \text{ and } -\infty < y < x < \infty.$$

Using the transformations U = H(y) and W = H(x)/H(y) in (2.2.17), it can be shown easily that W is distributed as B(r, s−r).

Proceeding as in the case of upper record values, we can obtain the conditional pdfs of the lower record values.

Example 2.2.1 Let us consider the exponential distribution with pdf f(x) as

$$f(x) = e^{-x}, 0 \le x < \infty$$

and the cumulative distribution function (cdf) F(x) as

$$F(x) = 1 - e^{-x}, 0 \le x < \infty.$$

Then R(x) = x and

$$f_n(x) = \frac{x^{n-1}}{\Gamma(n)}e^{-x}, x \ge 0$$
$$= 0, \text{otherwise}$$

The joint pdf of $X_{U(m)}$ and $X_{U(n)}$, n > m is

$$f_{m,n}(x,y) = \frac{x^{m-1}}{\Gamma(m)\Gamma(n-m)}(y-x)^{n-m-1}e^{-y}$$
$$0 \le x < y < \infty$$
$$= 0, \text{otherwise.}$$

The conditional pdf of $X_{U(n)} \mid X_{U(m)} = x)$ is

$$f\left(y|X_{U(m)} = x\right) = \frac{(y-x)^{n-m-1}}{\Gamma(n-m)}e^{-(y-x)}$$
$$0 \le x < y < \infty$$
$$= 0, \text{otherwise.}$$

Thus the conditional distribution of $X_{U(n)} - X_{U(m)}$ given $X_{U(m)}$ is the same as the unconditional distribution of $X_{U(n-m)}$ for n > m.

Example 2.2.2 Suppose that the random variable X has the Gumbel distribution with pdf $f(x) = e^{-x}e^{-e^{-x}}, -\infty < x < \infty$. Let $F_{(n)}$ and $f_{(n)}$ be the cdf and pdf of $X_{L(n)}$. It is easy to see that

$$F_{(n)}(x) = \int_{-\infty}^{x} \frac{e^{-nu}}{\Gamma(n)} e^{-e^{-u}} du$$

and

$$f_{(n)}(x) = \frac{e^{-nx}}{\Gamma(n)} e^{-e^{-x}}, \quad -\infty < x < \infty.$$

Let $f_{(m,\ n)}(x, y)$ be the joint pdf of $X_{L(m)}$ and $X_{L(n)}$, m < n. Using (2.2.16), we get for the Gumbel distribution

$$f_{(m,n)}(x,y) = \frac{(e^{-y} - e^{-x})^{n-m-1}}{\Gamma(n-m)} \frac{e^{-mx}}{\Gamma(m)} e^{-y} e^{-e^{-y}},$$
$$-\infty < y < x < \infty$$

Thus the conditional pdf $f_{(n|m)}(y|x)$ of $X_{L(n)} \mid X_{L(m)} = x$ is given by

$$f(n|m)(y|x) = \frac{(e^{-y} - e^{-x})^{n-m-1}}{\Gamma(n-m)} e^{-y} e^{-(e^{-y} - e^{-x})},$$
$$-\infty < y < x < \infty$$

2.3 Moments of Record Values

Let μ_n^r *and* $\mu_{(n)}^r$ be the rth moment of $X_{U(n)}$ and $X_{L(n)}$ respectively, then

$$\mu_n^r = \int_{-\infty}^{\infty} x^r \frac{(R(x))^{n-1}}{\Gamma(n)} f(x) dx \text{ and}$$
$$\mu_{(n)}^r = \int_{-\infty}^{\infty} x^r \frac{(H(x))^{n-1}}{\Gamma(n)} f(x) dx$$

$\mathrm{Var}(X_{U(n)}) = \mu_n^2 - (\mu_n^1)^2$ and $\mathrm{Var}(X_{L(n)}) = \mu_{(n)}^2 - (\mu_{(n)}^1)^2$. We will denote

$$\mu_{m,n}^{r,s} = E\left(X_{U(m)}^r X_{U(n)}^s\right)$$

and

$$\mu^{r,s}_{(m),(n)} = E\left(X^r_{L(m)} X^s_{L(n)}\right)$$

$$\mathrm{Cov}\left(\mathrm{X_{U(m)}}, \mathrm{X_{U(n)}}\right) = \mu^{1,1}_{m,n} - \mu^1_m\, \mu^1_n$$

and

$$\mathrm{Cov}\left(\mathrm{X_{L(m)}}, \mathrm{X_{L(n)}}\right) = \mu^{1,1}_{(m),(n)} - \mu^1_{(m)}\, \mu^1_{(n)}$$

If we take $V_k = R(X_{U(k)})$, k = 1, 2, .., then

$$E(V_m V_r) = \int_0^\infty \int_0^y xy \frac{x^{m-1}}{\Gamma(m)} \frac{(y-x)^{r-m-1}}{\Gamma(r-m)} \mathrm{e}^{-y} dx dy$$

Using the transformation t = yx and w = y, we get on simplification

$$E(V_m V_r) = \frac{\Gamma(m+1)}{\Gamma(m)} \frac{\Gamma(r-m)}{\Gamma(r+1)} \frac{\Gamma(r+2)}{\Gamma(r-m)} = \mathrm{m(r+1), m<r.}$$
$$\mathrm{Cov(V_m V_r) = m(r+1) - m\,r = m = Var(V_m), m<r.}$$

If $\rho_{m,n}$ = the correlation coefficient between V_m and V_n, m < n, is

$$\rho_{m,n} = \sqrt{m/n}$$

Example 2.3.1 For the exponential distribution with $\mathrm{f(x) = e^{-x}, 0 \le x < \infty}$.

$$\mu^r_n = \int_0^\infty x^r \frac{x^{n-1}}{(n-1)!} \mathrm{e}^{-x} dx = \frac{(n+r-1)!}{(n-1)!}$$
$$= n^{(r)},\ where\, x^{(k)} = x(x+1)(x+2)\ldots(x+k-1), k > 0,$$
$$= x^{(k)} = 1\ if\ k = 0$$

Thus $E(X_{U(n)}) = n$, $Var(X_{U(n)}) = n(n + 1) - n^2 = n$.
For m < n,

$$\mu^{r,s}_{m,n} = \int_0^\infty \int_x^\infty x^r y^s \frac{x^{m-1}}{\Gamma(m)\Gamma(n-m)} (y-x)^{n-m-1} \mathrm{e}^{-y} dx\, dy$$
$$= \sum_{k=0}^{s} \frac{\Gamma(m+r+s-k)}{\Gamma(m)} \frac{\Gamma(n-m+k)}{\Gamma(n-m)}$$

and $\mathrm{Cov}(X_{U(m)}, X_{U(n)}) = \mu_{m,n}^{1,1} - \mu_m^1 \mu_n^1 = nm + m - nm = m = \mathrm{Var}(X_{U(m)})$. Let $\rho_{m,n}$ be the correlation between $X_{U(n)}$ and $X_{U(m)}$, then

$$\rho_{m,n} = \sqrt{\frac{m}{n}}.$$

It can easily be shown that $E[X_{U(n)} - X_{U(m)}]^r = (n-m)^{(r)}$.

Example 2.3.2 For the Gumbel distribution with $f(x) = e^{-x}e^{-e^{-x}}, -\infty < x < \infty$, $E(X_{L(r)}) = \int_{-\infty}^{\infty} x \frac{e^{-rx}}{\Gamma(r)} e^{-e^{-x}} dx = -\frac{d}{dr} \ln \Gamma(r) = -\psi(r)$, where $\psi(r)$ is the Psi (Digamma) function. Thus

$$\begin{aligned} E(X_{L(r)}) &= \upsilon_r^* \\ \upsilon_1^* &= \upsilon \\ \upsilon_j^* &= \upsilon_{j-1}^* - (j-1)^{-1}, j \geq 2. \end{aligned}$$

Here υ is the Euler's constant. Let $f_{(m),(n)}(x, y)$ be the joint pdf of $X_{L(m)}$ and $X_{L(n)}$, m < n. Using (2.2.17), we get for the Gumbel distribution

$$f_{(m),(n)}(x,y) = \frac{(e^{-y} - e^{-x})^{n-m-1}}{\Gamma(n-m)} \frac{e^{-x}}{\Gamma(m)} e^{-my} e^{-e^{-y}}$$
$$-\infty < y < x < \infty.$$

Thus the conditional pdf $f_{(n|m)}(y|x)$ of $X_{L(n)} | X_{L(m)} = x$ is given by

$$\begin{aligned} f_{(n|m)}(y|x) &= \frac{(e^{-y} - e^{-x})^{n-m-1}}{\Gamma(n-m)} e^{-y} e^{(-e^{-y} - e^{-x})}, -\infty < y < x < \infty \\ E(X_{L(m)} X_{L(n)}) &= \int_{-\infty}^{\infty} \int_{y}^{\infty} xy f_{(m),(n)}(x,y) dx\, dy \\ &= \int_{-\infty}^{\infty} \int_{y}^{\infty} xy \frac{(e^{-y} - e^{-x})^{n-m-1}}{\Gamma(n-m)} \frac{e^{-mx}}{\Gamma(m)} e^{-y} e^{-e^{-y}} dx\, dy. \end{aligned}$$

Substituting y−x = t, we get on simplification

$$\int_{-\infty}^{\infty} \int_{y}^{\infty} xy \frac{(e^{-y} - e^{-x})^{n-m-1}}{\Gamma(n-m)} \frac{e^{-mx}}{\Gamma(m)} e^{-y} e^{-e^{-y}} dx\, dy$$
$$= E\left(X_{L(n)}^2\right) + E(T)E(X_{L(n)})$$

where $E(T) = \int_0^\infty \Gamma(n)(\Gamma(m)\Gamma(n-m-1))^{-1}(1-e^{-t})^{n-m-1}e^{-mt}\,dt$

Similarly it can be shown that

$$E(X_{L(m)}) = E(X_{L(n)}) + E(T)$$

Thus $Cov\left(X_{L(m)}X_{L(n)}\right) = Var(X_{L(n)})$ and

$$\begin{aligned} Var(X_{L(r)}) &= \int_{-\infty}^{\infty} x^2 f_{(r)}(x)dx\_\left(\int_0^\infty x f_{(r)}(x)dx\right)^2 \\ &= \frac{d}{dr}\psi(r) \\ &= \frac{\pi^2}{6} - \sum_{k=1}^{r-1}\frac{1}{k^2}, k > 1 \end{aligned}$$

and

$$= \frac{\pi^2}{6} \text{ for } k = 2.$$

Let $Var(X_{L(r)}) = V^*_{r,r}$, r = 1, 2, …, then

$$\begin{aligned} V^*_{1,1} &= \frac{\pi^2}{6} \\ V^*_{j,j} &= V^*_{j-1,j-1} - (j-1)^{-2}, \quad j \geq 2 \end{aligned}$$

Further

$$\begin{aligned} E(X_{L(m)}) &= E(X_{L(n)}) + \sum_{p=m}^{n-1}\frac{1}{p} \\ Var(X_{L(n-1)}) - Var(X_{L(n)}) &= (n-1)^{-2} \end{aligned}$$

Let ρ(m,n) be the correlation coefficient between $X_{L(m)}$ and $X_{L(n)}$, then

$$\rho(m,n) = \sqrt{\frac{Var(X_{(n)})}{Var(X_{(m)})}}.$$

Example 2.3.3 A random variable is said to have generalized Pareto distribution if its probability density function is of the following form:

$$f_0(x,\mu,\sigma,\beta) = \frac{1}{\sigma}\left(1+\beta\left(\frac{x-\mu}{\sigma}\right)\right)^{-(1+\beta^{-1})}$$
$$x \geq \mu,\ for\ \beta > 0,$$
$$\mu < x \leq \mu - \sigma\beta^{-1},\ for\ \beta < 0$$
$$= \frac{1}{\sigma}e^{-(x-\mu)\sigma^{-1}}, x \geq \mu,\ for\ \beta = 0$$
$$= 0, \quad \text{otherwise.}$$

It can be shown that for $\beta \neq 0$

$$X_{U(n)} \stackrel{d}{=} \mu - \frac{\sigma}{\beta} + \frac{\sigma}{\beta}\prod_{i=1}^{n} U_i$$

where $U_1, U_2, \ldots, U_n$ are independently and identically distributed with

$$P(U_i \leq x) = 1 - (x)^{-\beta^{-1}}, \quad x \geq 1, \beta > 0,$$
$$= (x)^{-\beta^{-1}}, \quad \beta < 0, 0 < x < 1.$$

For $\beta = 0$, we have

$$X_{U(n)} \stackrel{d}{=} \mu + \sigma\sum_{i=1}^{n} Z_i$$

where Z_1, Z_2, ….., Z_n are independently and identically distributed with P ($Z_i \leq z$) = 1 − e^{-z}, z > 0, here $\stackrel{d}{=}$ denotes the equality in distribution.

For $\beta \neq 0$, we have

$$E(X_{U(n)}) = \mu + \frac{\sigma}{\beta}\{(1-\beta)^{-n} - 1\}, \beta < 1$$
$$Var(X_{U(n)}) = \sigma^2\beta^{-2}\left\{(1-2\beta)^{-n} - (1-\beta)^{-2n}\right\}, \beta < \frac{1}{2}$$

$Cov(X_{U(m)}, X_{U(n)}) = \sigma^2\beta^{-2}(1-\beta)^{m-n}\left\{(1-2\beta)^{-m} - (1-\beta)^{-2m}\right\}$ Let $\rho_{m,n}$ be the correlation coefficient between $X_{U(m)}$ and $X_{U(n)}$, then

$$\rho_{m,n} = (1-\beta)^{m-n}\left[\frac{(1-2\beta)^{-m} - (1-\beta)^{-2m}}{(1-2\beta)^{-n} - (1-\beta)^{-2n}}\right]^{\frac{1}{2}}, \beta < 1/2.$$
$$= \left\{(t^m - 1)/(t^n - 1)\right\}^{1/2}, \quad \text{where } t = \frac{(1-\beta)^2}{1-2\beta} \text{ and } \beta < 1/2.$$

As $\beta \to 0$, $\rho_{m,n} \to \sqrt{(m/n)}$ which is the correlation coefficient between $X_{U(m)}$ and $X_{U(n)}$ when $\beta = 0$ i.e. for the exponential distribution.

Example 2.3.4 A random variable is said to have Type II extreme value distribution if its cumulative distribution function is of the following form:

$$F(x) = e^{-\left(\frac{x-\mu}{\sigma}\right)^{-\delta}}, x > \mu, \sigma > 0, \delta > 0.$$

Suppose $X_{L(1)}$, $X_{L(2)}$ .... be the sequence of lower record values and $f_{(n)}(x)$ is the pdf of $X_{L(n)}$, n = 1, 2, ...We can write

$$\begin{aligned} f_{(n)}(x) &= \frac{(H(x))^{n-1}}{\Gamma(n)} f(x) \\ &= \frac{\delta^n \left(\frac{x-\mu}{\sigma}\right)^{-(n\delta+1)}}{\sigma \Gamma(n)} e^{-\left(\frac{x-\mu}{\sigma}\right)^{-\delta}} \end{aligned}$$

Here $H(x) = -\ln F(x) = e^{-x}$. We can write $\frac{X_{L(n)}-\mu}{\sigma} \stackrel{d}{=} (W_1 + W_2 + \ldots\ldots + W_n)^{-\frac{1}{\delta}}$, where $W_1, W_2, \ldots, W_n$ are independent and identically distributed as exponential with unit mean.

Let $Y_{L(n)} = \frac{X_{L(n)}-\mu}{\sigma}$, and $U_n = W_1 + W_2 + \cdots + W_n$, then

$$\begin{aligned} E(Y_{L(n)}) &= E((U_n) - 1/\delta) \\ &= \int_0^\infty \frac{u^{-\frac{1}{\delta}} u^{n-1} e^{-u}}{\Gamma(n)} du = \frac{\Gamma\left(n - \frac{1}{\delta}\right)}{\Gamma(n)} \\ E(Y_{L(n)})^2 &= E((U_n) - 2/\delta). \\ &= \int_0^\infty \frac{u^{-\frac{2}{\delta}} u^{n-1} e^{-x}}{\Gamma(n)} du = \frac{\Gamma\left(n - \frac{2}{\delta}\right)}{\Gamma(n)} \end{aligned}$$

Thus

$$\begin{aligned} E(X_{L(n)}) &= \mu + \sigma \frac{\Gamma\left(n - \frac{1}{\delta}\right)}{\Gamma(n)}, \\ \mathrm{Var}(X_{L(n)}) &= \sigma^2 \left[\frac{\Gamma\left(n - \frac{2}{\delta}\right)}{\Gamma(n)} - \left(\frac{\Gamma\left(n - \frac{1}{\delta}\right)}{\Gamma(n)} \right)^2 \right] \end{aligned}$$

For m < n,

$$E(Y_{L(m)}.Y_{L(n)}) = \int_0^\infty \int_0^\infty \frac{u^{-\frac{1}{\delta}}(u+v)^{-\frac{1}{\delta}}}{\Gamma(m)\Gamma(n-m)} e^{-u} u^{m-1} e^{-v} v^{n-m-1} du dv$$

Substituting

$$y_1 = u$$
$$y_2 = \frac{u}{u+v}$$

we get on simplification,

$$E(Y_{L(m)}.Y_{L(n)}) = \int_0^\infty \int_0^1 \frac{(y_1)^{n-1-\frac{2}{\delta}} e^{-y_1} (1-y_2)^{n-m-1}}{\Gamma(m)\Gamma(n-m)} (y_2)^{m-1-\frac{1}{\delta}} dy_1 dy_2$$
$$= \frac{\Gamma\left(n-\frac{2}{\delta}\right)\Gamma\left(m-\frac{1}{\delta}\right)}{\Gamma(m)\Gamma\left(n-\frac{1}{\delta}\right)}$$

Thus

$$\mathrm{Cov}\left(X_{L(m)}X_{L(n)}\right) = \sigma^2 \left\{ \frac{\Gamma\left(m-\frac{1}{\delta}\right)}{\Gamma(m)} \left[\frac{\Gamma\left(n-\frac{2}{\delta}\right)}{\Gamma\left(n-\frac{1}{\delta}\right)} - \frac{\Gamma\left(n-\frac{1}{\delta}\right)}{\Gamma(n)} \right] \right\}$$

We rewrite the covariance expression as $\mathrm{Cov}\left(X_{L(m)}X_{L(n)}\right) = \sigma^2 a_m b_n$, where

$$a_m = \frac{\Gamma\left(m-\frac{1}{\delta}\right)}{\Gamma(m)} \text{ and } b_n = \frac{\Gamma\left(n-\frac{2}{\delta}\right)}{\Gamma\left(n-\frac{1}{\delta}\right)} - \frac{\Gamma\left(n-\frac{1}{\delta}\right)}{\Gamma(n)}, 1 \le m \le n.$$

$$\mathrm{Corr}(X_{L(m)}X_{L(n)}) = \sqrt{\frac{a_m}{a_n} \cdot \frac{b_n}{b_n}}$$

The following theorem gives the condition for the existence of the moments of the nth record value.

Theorem 2.3.1 *If* $\int_{-\infty}^{\infty} |x|^{r+\delta} dF(x) < \infty$, *for some* $\delta > 0$, *then* $E(X_{U(n)})^r$ *is finite for all* $n \geq 2$.

Proof We define the inverse function $R^{-1}(y) = \inf\{x : R(x) \geq y\}$

$$\begin{aligned}
\mathrm{E}\left(\left|\mathrm{X}_{\mathrm{U(n)}}\right|^{\mathrm{r}}\right) &= \int_{-\infty}^{\infty} \frac{1}{\Gamma(\mathrm{n})}|x|^{r+\delta}(R(x))^{n-1}dF(x) < \infty, \text{for } \delta > 0, \\
&= \frac{1}{\Gamma(\mathrm{n})}\int_0^{\infty} \left|R^{-1}y\right|^{r} y^{n-1}\mathrm{e}^{-y}dy \\
&= \frac{1}{\Gamma(\mathrm{n})}\left(\int_0^{\infty} \left|R^{-1}y\right|^{rp}\mathrm{e}^{-y}dy\right)^{1/p}\left(\int_0^{\infty} y^{nq}\mathrm{e}^{-y}dy\right)^{1/q}
\end{aligned}$$

by Holder's inequality, where $\frac{1}{p}+\frac{1}{q}=1, p>1, q>1,$

$$= \frac{1}{\Gamma(n)}\left(\int_0^{\infty} \left|R^{-1}(y)\right|^{r+\delta}\mathrm{e}^{-y}dy\right)^{1/p}\left(\int_0^{\infty} y^{nq}\mathrm{e}^{-y}dy\right)^{1/q},$$

$$\text{where} \quad p = \frac{r+\delta}{r};$$

$$= \frac{1}{\Gamma(n)}\left(E(|x|)^{r+\delta}\right)^{1/p}\left(\int_0^{\infty} y^{nq}\mathrm{e}^{-y}dy\right)^{1/q} < \infty.$$

Theorem 2.3.2 *If E(X) = 0 and Var(X) = 1, then* $\left|\mathrm{E}(\mathrm{X}_{\mathrm{U(n+1)}})\right| \le \sqrt{\binom{2n}{n}-1}.$

Proof Let

$$\begin{aligned}
\mathrm{F}^{-1}(u) &= \mathrm{Sup}\{x{:}\,\mathrm{F}(x)\le u\}, 0<u<1, \\
\mathrm{F}^{-1}(1) &= \mathrm{Sup}\left\{\mathrm{F}^{-1}(u), u<1\right\}
\end{aligned}$$

$$0 = \mathrm{E(X)} = \int_0^{\infty} x\,\mathrm{f}(x)dx = \int_0^{\infty} \bar{\mathrm{F}}^{-1}t dt.$$

$$1 = \mathrm{E}(\mathrm{X}^2) = \int_0^{\infty} x^2\mathrm{f}(x)dx = \int_0^{\infty} \left\{\bar{\mathrm{F}}^{-1}(t)\right\}^2 dt.$$

$$\begin{aligned}
\mathrm{E}(\mathrm{X}_{\mathrm{U(n+1)}}) &= \int_{-\infty}^{\infty} x\frac{\{-\ln\bar{\mathrm{F}}(x)\}^n}{\Gamma(n+1)}\mathrm{f}(x)dx \\
&= \int_0^1 \bar{\mathrm{F}}^{-1}(t)\frac{\{-\ln(1-t)\}^n}{\Gamma(n+1)}dt \\
&= \int_0^1 \bar{\mathrm{F}}^{-1}(t)\left[\frac{\{-\ln(1-t)\}^n}{\Gamma(n+1)}-\lambda\right]dt.
\end{aligned}$$

Using Cauchy and Schwarz inequality, we get

$$\left|E(X_{U(n+1)})\right| \leq \left\{\int_0^1 \left[\bar{F}^{-1}(t)\right]^2 dt\right\}^{\frac{1}{2}} \left\{\int_0^1 \left(\frac{(-\ln(1-t))^n}{\Gamma(n+1)} - \lambda\right)^2 dt\right\}^{\frac{1}{2}}.$$

Now

$$\int_0^1 \left\{\bar{F}^{-1}(t)\right\}^2 dt = 1,$$

and

$$\int_0^1 \left(\frac{(-\ln(1-t))^n}{\Gamma(n+1)} - \lambda\right)^2 dt = \binom{2n}{n} + \lambda^2 - 2\lambda$$

Since the minimum value of $\lambda^2 - 2\lambda$ is -1, we get

$$E(X_{U(n)})| \leq \sqrt{\binom{2n}{n} - 1} \approx \frac{2^{2n}}{\sqrt{n\pi}}, \text{for large n.} \tag{2.3.1}$$

For symmetric distribution the upper bound of $|E(X_{U(n)})|$ is smaller. The bound of the symmetric distribution is given in the following theorem.

Theorem 2.3.3 *Suppose the random variable X is symmetric about zero and has variance 1, then* $E(X_{U(n+1)}) < \frac{1}{\sqrt{2}}\left\{\binom{2n}{n} - \frac{1}{\left[\Gamma(n+1)\right]^2}\int_0^\infty [\ln(1-u)\ln u]^n du\right\}^{\frac{1}{2}}$.

Proof

$$\begin{aligned} E(X_{U(n+1)}) &= \int_{-\infty}^{\infty} x\frac{\left\{-\ln \vec{F}(x)\right\}^n}{\Gamma(n+1)} f(x)dx \\ &= \int_0^\infty x\frac{\left\{-\ln \vec{F}(x)\right\}^n}{\Gamma(n+1)} f(x)dx - \int_0^\infty x\frac{\{-\ln F(x)\}^n}{\Gamma(n+1)} f(x)dx \\ &= \frac{1}{2\Gamma(n+1)}\int_0^1 F^{-1}(u)[\{-\ln(1-u)\}^n - \{-\ln u\}^n]du. \end{aligned}$$

Now

$$\int_0^1 \left\{F^{-1}(u)\right\}^2 du = 1$$

and

$$\int_0^1 [\{-\ln(1-u)\}^n - \{-\ln u\}^n]^2 du$$
$$= 2\Gamma(2n+1) - 2\int_0^1 [\ln(1-u)\ln u]^n du$$

Hence using the Cauchy and Schwarz inequality, we get

$$\mathrm{E}\left| \mathrm{X}_{\mathrm{U(n+1)}} \right| \le \frac{1}{\sqrt{2}}\left\{\binom{2n}{n} - \frac{1}{[\Gamma(n+1)]^2}\int_0^\infty [\ln(1-u)\ln u]^n du\right\}^{\frac{1}{2}}. \tag{2.3.2}$$

The following table gives the upper bounds of the inequalities given by (2.3.1) and (2.3.2). For large n, the ratio of the bounds as given by (2.3.2) and (2.3.1) is approximately $\sqrt{2}$.

Let

$$\mathrm{h(n)} = \frac{1}{\sqrt{2}}\left\{\binom{2n}{n} - \frac{1}{[\Gamma(n+1)]^2}\int_0^\infty [\ln(1-u)\ln u]^n du\right\}^{\frac{1}{2}},$$
$$\mathrm{g(n)} = \sqrt{\binom{2n}{n} - 1} \text{ and } \mathrm{b(n)} = \frac{g(n)}{h(n)}.$$

Thus g(n) is the upper bound of $|\mathrm{E}(\mathrm{X}_{\mathrm{U(n)}}|$ and h(n) is the upper bound of $\mathrm{E}(\mathrm{X}_{\mathrm{U(n)}})$, when the distribution of $\mathrm{X_i}$, i = 1, 2, … is symmetric (Table 2.1).

Nevzerov (1992) gave an interesting upper bounds of the correlation coefficient between any two upper record values. The result is given in the following theorem.

Theorem 2.3.4 *Let { X_i, i = 1, 2, …} be a sequence of independent and identically distributed random variables and suppose that for* $1 \le m < n$, $E(X_1^2(\ln(1-F(X_1)))^{j-1} < \infty$, *for j = n. Then*

Table 2.1 Values of h(n), g(n) and b(n)

N	h(n)	g(n)	b(n)
1	0.906896	1	2.102662
2	2.726929	2.236068	2.294824
3	3.162147	4.358899	2.378462
4	5.916078	8.306624	2.404076
5	12.224972	15.84298	2.411405
6	22.494185	30.380915	2.413448
7	42.424630	58.574739	2.414008
8	80.218452	113.441615	2.414159
9	155.916644	220.497166	2.41419
10	303.937494	429.831362	2.414210

$$\rho(X_{U(m)}, X_{U(n)}) \leq \sqrt{\frac{m}{n}},$$

where ρ(X, Y) is the correlation between X and Y. The equality holds if and only if X1 has an

Theorem 2.3.5 *Let* $\{X_n,\ n \geq 1\}$ *be a sequence of independent and identically distributed random variables with distribution function F(x) and the corresponding density function f(x). If* $E(X_n)$*,* $n \geq 1$ *is finite and F belongs to the class C1, then E* $\{X_{U(m+1)} - X_{U(m)}\} \leq (\geq)\ E(X_n)$*, for any fixed m and n according as F is NBU (NWU).*

Proof From Eq. (2.2.4), we can write the $E\{X_{U(m+1)} - X_{U(m)}\}$ as

$$E\{X_{U(m+1)} - X_{U(m)}\} = \int_0^\infty \int_0^\infty \frac{1}{\Gamma(n)} (R(u))^{n-1} f(u) \frac{\bar{F}(u+z)}{\bar{F}(u)} dudz$$

$$\leq (\geq) \int_0^\infty \int_0^\infty \frac{1}{\Gamma(n)} (R(u))^{n-1} f(u) \bar{F}(z) dudz,$$

according as $\bar{F}(x+y) \leq (\geq) \bar{F}(x)\bar{F}(y)$. Hence $E\{X_{U(m+1)} - X_{U(m)}\} \leq (\geq) E(X_n)$ according as F is NBU(NWU).

If F(x) has the density f(x), the ratio $r(x) = \frac{f(x)}{\bar{F}(x)}$, for $\bar{F}(x) > 0$ is called the failure (hazard) rate hazard rate, we will say F belongs to the class C_2 if the failure rate, r(x), is either monotone increasing (IFR) or monotone decreasing (DFR).

Theorem 2.3.6 *Let* $\{X_i,\ i = 1, 2, \ldots\}$ *be ac sequence of i.i.d. continuous non-negative rv's with common cdf F(x) and pdf f(x). Suppose that* $X_{U(1)}, X_{U(2)}, \ldots$ *are the upper record values of this sequence and* $Z_{n+1,\ n} = X_{U(n+1)} - X_{U(n)}$*, n = 1, 2, ...*

with $X_{U(0)} = 0$. If $E(D_{n+1})$ exists and F belongs to class C_2, then $E(Z_{n+1}) > (<) E(Z_n)$ according as F is IFR or DFR.

Proof For n = 1, 2, …, the joint pdf of $X_{U(n)}$ and $X_{U(n+1)}$ is given by

$$f_{n,n+1}(x,y) = \frac{(R(x))^{n-1}}{(n-1)!} r(x) f(y)$$
$$\text{for } -\infty < x < y < \infty.$$

The joint pdf of $X_{U(n)}$ and $Z_{n+1,n}$ is

$$f_{n,z}(x,z) = \frac{(R(x))^{n-1}}{(n-1)!} r(x) f(z+x)$$
$$\text{for } 0 < x, z < \infty.$$

Now

$$E(Z_{n+1,n}) = \int_0^\infty \int_0^\infty z \frac{(R(x))^{n-1)}}{(n-1)!} r(x) f(z + x) dx dz.$$

Since $\int_0^\infty z f(z+x) dz = \int_0^\infty \vec{F}(z+x) dz$, we obtain

$$E(Z_{n+1,n}) = \int_0^\infty \int_0^\infty \frac{(R(x))^{n-1)}}{(n-1)!} r(x) \vec{F}(z+x) dx dz.$$

On integrating by parts and using the relation R'(x) = r(x), we get

$$E(Z_{n+1,n}) = \int_0^\infty \int_0^\infty \frac{(R(x))^n}{n!} f(z+x) dx dz$$
$$= \int_0^\infty \int_0^\infty \frac{(R(x))^n}{n!} r(z+x) \bar{F}(z+x) dx dz$$
$$\geq (\leq) \int_0^\infty \int_0^\infty \frac{(R(x))^n}{n!} r(z) \bar{F}(z+x) dx dz$$
$$\text{according as r(x) is IFR or DFR}$$
$$= E(Z_{n+2,n+1})..$$

2.4 Entropies of Record Values

Let X be a continuous random variable with the pdf f(x), then the entropy H(x) of X is defined as

$$H(x) = -\int_{-\infty}^{\infty} f(x) \ln f(x) dx$$

where f(x) ln f(x) is integrable.

For a discrete random variable X taking values on $x_1, x_2, \ldots$, with h probabilities $p_1, p_2, \ldots$ the entropy H(x) is defined as

$$H(X) = \sum_{i=1}^{\infty} p_i x_i$$

provided the summation is finite.

In the case of discrete distribution the transformation

$$Y = a + bX, -\infty < a < \infty, b > 0,$$

Does not change the probabilities $p_1, p_2, \ldots$ and we have

$$H(Y) = H(X)$$

In the case of continuous random variable the Y = a+bX will change the entropy of Y as

$$\begin{aligned} H(Y) &= -\int_{-\infty}^{\infty} \frac{1}{b} f\left(\frac{x-a}{b}\right) \ln f\left(\frac{x-a}{b}\right) dx \\ &= -\int_{-\infty}^{\infty} f(x) \ln\left(\frac{1}{b} f(x)\right) dx \\ &= \ln b + H(x) \end{aligned}$$

The concept of entropy has recently been used in statistical inference. Shannon was the first to compute the entropies of the normal, exponential and uniform distribution. We will discuss here the entropies of upper record values. The entropies of lower record values are similar.

Let $H_n(x)$ be the entropy of $X_{U(n)}$ for a continuous random variable, then

$$
\begin{aligned}
-\mathrm{H_n}(x) &= \int_{-\infty}^{\infty} \mathrm{f}_n(x)\ln \mathrm{f}_n(x)dx \\
&= -\ln\Gamma(n) + (n-1)\int_{-\infty}^{\infty} \ln \mathrm{R}(x)\frac{(\mathrm{R}(x))^{n-1}}{\Gamma(n)}\mathrm{f}(x)dx \\
&\quad + \int_{-\infty}^{\infty} \ln \mathrm{f}(x)\frac{(\mathrm{R}(x))^{n-1}}{\Gamma(n)}\mathrm{f}(x)dx \\
&= -\ln\Gamma(n) + (n-1)\int_{0}^{\infty} \frac{t^{n-1}}{\Gamma(n)}\mathrm{e}^{-t}\ln t\, dt + I, \\
&= -\ln\Gamma(n) + (n-1)\psi(n) + I
\end{aligned}
$$

where

$$
\mathrm{I} = \int_{-\infty}^{\infty} \ln \mathrm{f}(x)\frac{(R(x))^{n-1}}{\Gamma(n)}\mathrm{f}(x)dx = \int_{-\infty}^{\infty} \mathrm{f}_n(x)\ln \mathrm{f}(x)dx,
$$

and $\psi(n)$ is the digamma function i.e. $\psi(\mathrm{n}) = \frac{d}{dn} = \ln\Gamma(n) = \frac{\Gamma'(n)}{\Gamma(n)}$.

Example 2.4.1 Suppose the sequence of independent and identically distributed random variables X_n, $n \geq 1$, has the Rayleigh distribution with pdf f(x), where

$$
\begin{aligned}
\mathrm{f(x)} &= \frac{x}{b^2}\mathrm{e}^{-x^2/\left(2b^2\right)}, 0 < \mathrm{x} < \infty, \text{then} \\
\mathrm{I} &= \int_{0}^{\infty} \mathrm{f}_n(x)\ln \mathrm{f}(x)dx \\
&= -2\ln \mathrm{b} + \int_{0}^{\infty} \ln x\, \mathrm{f}_n(x)dx - \int_{0}^{\infty} \frac{x^2}{2b^2}\mathrm{f}_n(x)dx \\
&= -2\ln \mathrm{b} + \frac{1}{2}\psi(n) + \frac{1}{2}\ln 2 + \ln \mathrm{b} \\
&= \frac{1}{2}\ln 2 - \ln b + \frac{1}{2}\ln \psi(\mathrm{n}) - \mathrm{n}.
\end{aligned}
$$

Hence

$$H_n(x) = \ln\Gamma(n) - (n - 1/2)\psi(n) + \ln b - 1/2\ln 2 + n$$

Example 2.4.2 Suppose that the sequence of i.i.d. random variables X_n has the Weibull pdf, f(x) where

$$f(x) = \frac{c}{a} x^{c-1} x^{-x^c/a}, 0 < x, a, c < \infty.$$

In this case, we have

$$\begin{aligned} I &= \int_0^\infty f_n(x) \ln f(x) dx \\ &= \ln\frac{c}{a} + (c-1)\int_0^\infty \ln x f_n(x) dx - \int_0^\infty \frac{x^c}{a} f_n(x) dx \\ &= \ln\frac{c}{a} + (c-1)(\ln a + \psi(n)) - n \\ &= \ln c - \frac{1}{c}\ln a + \frac{c-1}{c}\psi(n) - n. \end{aligned}$$

Hence

$$H_n(x) = \ln\Gamma(n) - \left(n - \frac{1}{c}\right)\psi(n) - \ln c + \frac{1}{c}\ln a + n.$$

2.5 Estimation of Parameters and Predictions of Records

2.5.1 Exponential Distribution

We will consider here the two parameter exponential distribution with pdf f(x) as given by

$$\begin{aligned} f(x) &= \sigma^{-1}\exp\left(-\sigma^{-1}(x-\mu)\right), x \geq \mu \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{2.5.1}$$

2.5.1.1 Minimum Variance Linear Unbiased Estimates (MVLUE) of μ and σ

Suppose that $X(1), X(2), \ldots, X(m)$ are the m (upper) record values from $E(\mu, \sigma)$ with pdf as given in (2.5.1).
Let

$$Y_i = \sigma^{-1}(X(i) - \mu), \mathrm{i} = 1, 2, \ldots, \mathrm{m}, \text{then}$$
$$E(Y_i) = i = Var(Y_i), \quad i, = 1, 2, \ldots, m,$$

and $Cov(Y_i, Y_j) = \min(i, j)$.
Let

$$X = (X(1), X(2), \ldots, X(m)), \text{then}$$
$$E(X) = \mu L + \sigma\delta$$
$$Var(X) = \sigma^2 V,$$

where

$$L' = (1, 1, \ldots, 1)', \delta' = (1, 2, \ldots m)'$$
$$V = (V_{ij}), V_{ij} = \min(i, j), \quad i, j = 1, 2, \ldots, m.$$

The inverse $V^{-1}(= V^{ij})$ can be expressed as

$$V^{ij} = \begin{array}{lll} 2 & if & i = j = 12, \ldots m - 1 \\ 1 & if & i = j = m \\ -1 & if & |i - j| = 1, i, j = 1, 2 \ldots, m \\ 0 & & otherwise. \end{array}$$

The minimum variance linear unbiased estimates (MVLUE) $\hat{\mu}, \hat{\sigma}$ of μ and σ respectively are

$$\hat{\mu} = -\delta' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta$$
$$\hat{\sigma} = L' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta,$$

where

$$\Delta = (L' V^{-1} L)(\delta' V^{-1}\delta) - (L' V^{-1}\delta)^2$$

and

$$Var(\hat{\mu}) = \sigma^2 L' V^{-1}\delta/\Delta$$
$$Var(\hat{\sigma}) = \sigma^2 L' V^{-1} L/\Delta$$
$$Cov(\hat{\mu}, \hat{\sigma}) = -\sigma^2 L' V^{-1}\delta/\Delta.$$

It can be shown that

$$L'V^{-1} = (1,0,0,\ldots,0), \delta' V^{-1} = (0,0,0,\ldots,1), \quad \delta' V^{-1}\delta = m \quad \text{and}$$
$$\Delta = m - 1.$$

On simplification, we get

$$\hat{\mu} = (m(X(1)) - (X(m)))/(m-1)$$
$$\hat{\sigma} = (X(m) - X(1))/(m-1)$$

with

$$Var(\hat{\mu}) = m\,\sigma^2/(m-1), Var(\hat{\sigma}) = \sigma^2/(m-1) \text{ and}$$
$$Cov(\hat{\mu},\hat{\sigma}) = -\sigma^2/(m-1).$$

Exercise 2.5.1.1 **If** $\mu = 0$, then the MVLUE $\hat{\sigma}_0$ of σ_0 is

$$\hat{\sigma}_0 = \frac{X(m)}{m}$$

2.5.1.2 Best Linear Invariant Estimators (BLIE) of μ and σ Are

The best linear invariant (in the sense of minimum mean squared error and invariance with respect to the location parameter μ) estimators (BLIE) $\tilde{\mu}$ $\tilde{\sigma}$ of μ and σ are

$$\tilde{\mu} = \hat{\mu} - \hat{\sigma}\left(\frac{E_{12}}{1+E_{22}}\right)$$

and

$$\tilde{\sigma} = \hat{\sigma}/(1+E_{22}),$$

where $\hat{\mu}$ and $\hat{\sigma}$ are MVLUE of μ and σ and

$$\begin{pmatrix} Var(\hat{\mu}) & Cov(\hat{\mu},\hat{\sigma}) \\ Cov(\hat{\mu},\hat{\sigma}) & Var(\hat{\sigma}) \end{pmatrix} = \sigma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}$$

The mean squared errors of these estimators are

$$MSE(\tilde{\mu}) = \sigma^2\left(E_{11} - E_{12}^2(! \ + \ E_{22})^{-1}\right) \text{ and}$$
$$MSE(\tilde{\sigma}) = \sigma^2 E_{22}(1+E_{22})^{-1}$$

We have

$$E(\tilde{\mu}-\mu)(\tilde{\sigma}-\sigma) = \sigma^2 E_{12}(1+E_{22})^{-1}.$$

Using the values of E_{11}, E_{12} and E_{22} from (2.3.2), we obtain

$$\hat{\mu} = ((m+1)X(1) - X(m))/m,$$
$$\hat{\sigma} = (X(m) - X(1))/m$$
$$Var(\tilde{\mu}) = \frac{m+1}{m}\sigma^2 \text{ and } \quad Var(\hat{\sigma}) = \frac{m-1}{m^2}\sigma^2$$

2.5.1.3 Maximum Likelihood Estimate. of μ and σ Are

The log likelihood equation based on the m upper records X(1), X(2), …, X(m) can be written as

$$\ln \mathrm{L} = -\mathrm{m}\ln\sigma - \frac{1}{\sigma}(X(m)-\mu), \mu < X(1) < X(2)\ldots < X(m) < \infty$$

The maximum likelihood estimate $\hat{\mu}_{ml}$ and $\hat{\sigma}_{ml}$ of μ and σ are respectively

$$\hat{\mu}_{ml} = \mathrm{X}(1)$$

and

$$\hat{\sigma}_{ml} = \frac{1}{m}(X(m) - X(1))$$
$$E(\hat{\mu}_{ml}) = \mu + \sigma, Var(\hat{\mu}_{ml}) = \sigma^2,$$
$$E(\hat{\sigma}_{ml}) = \frac{(m-1)\sigma}{m}, Var(\hat{\sigma}_{ml}) = \frac{(m-1)\sigma^2}{m^2}$$

and $Cov(\hat{\mu}_{ml}\hat{\sigma}_{mj}) = 0$

Exercise 2.5.1.3 Show that in the case of one parameter exponential with $F(x) = 1 - e^{=x/\sigma}$, x $\geq$ 0, $\sigma > 0$. The maximum likelihood estimate σ_{ml}^* of σ based on m upper records X(1), X(3), …, X(m) is

$$\sigma^*_{ml} = \frac{\mathrm{x(m)}}{\mathrm{m}} \quad with \quad E\left(\sigma^*_{ml}\right) = \sigma + \frac{\mu}{m} \text{and}\, Var\left(\sigma^*_{ml}\right) = \frac{\sigma^2}{m}.$$

2.5.1.4 Prediction of Record Values

We will predict the sth upper record value based on the first m record values for s > m. Let $W' = (W_1, W_2, \ldots, W_m)$, where

$$\sigma^2 W_{ij} Cov(X(i), X(j)), i = 1, \ldots, m \text{ and } \alpha^* = \sigma^{-1} E(X(i) - \mu).$$

The best linear unbiased predictor of X(s) is $\hat{X}(s)$ where

$$\hat{X}(s) = \hat{\mu} + \hat{\sigma}\alpha^* + W' V^{-1}(X - \hat{\mu} L - \hat{\sigma}\delta), \hat{X}_{U(s)},$$

$\hat{\mu}, \hat{\sigma}$ are the MVLUE of μ, σ respectively. It can be shown that $W' V^{-1}(X - \hat{\mu} L - \hat{\sigma}\delta) = 0$.

$$\begin{aligned} \hat{X}(s) &= ((s-1)X(m) + (m-s)X(1))/(m-1) \\ E(\hat{X}(s)) &= \mu + s\sigma \\ Var(\hat{X}(s)) &= \sigma^2\left(m + s^2 - 2s\right)/(m-1). \end{aligned}$$

Let $\tilde{X}(s)$ be the best linear invariant predictor of $X(s)$. Then it can be shown that

$$\tilde{X}(s) = \hat{X}(s) - C_{12}(1 + E_{22})^{-1}\hat{\sigma},$$

where

$$C_{12}\sigma^2 = Cov\left(\hat{\sigma}, (L - W' V^{-1} L)\hat{\mu} + (\alpha^* - W' V^{-1}\delta)\hat{\sigma}\right)$$

and $\sigma^2 E_{22} = Var(\hat{\sigma})$. On simplification we get

$$\begin{aligned} \tilde{X}(s) &= \frac{m-s}{m} X(1) + \frac{s}{m} X(m) \\ E(\tilde{X}(s)) &= \mu + \left(\frac{ms + m - s}{m}\right)\sigma \\ Var(\tilde{X}(s)) &= \sigma^2\left(m^2 + ms^2 - s^2\right)/m^2. \end{aligned}$$

It is well known that the best (unrestricted) least squares predictor $\tilde{\tilde{X}}$ of X(s) is

$$\hat{\hat{X}}(s) = E(X(s)|X(1), \ldots, X(m)) \\ = X(m) + (s - m)\sigma\ldots$$

But $\hat{\hat{X}}_{U(s)}$ depends on the unknown parameter σ. If we substitute the minimum variance linear unbiased estimate $\hat{\sigma}$ for σ, then $\hat{\hat{X}}(s)$
becomes equal to $\hat{X}(s)$. Now

$$E\left(\hat{\hat{X}}(s)\right) = \mu + s\,\sigma = E(X(s))$$
$$Var\left(\hat{\hat{X}}(s)\right) = m\sigma^2$$

2.5.2 *Generalized Pareto Distribution*

We will consider the generalized Pareto distribution with the following pdf f(x)

$$\begin{aligned} f(x) &= \frac{1}{\sigma}\left(1 + \beta\left(\frac{x - \mu}{\sigma}\right)\right)^{-1(1+\beta)^{-1}} \\ & x \geq \mu, \text{ for } \beta > 0, \\ & \mu < x < \mu - \sigma/\beta, \text{ for } \beta < 0, \\ &= \frac{1}{\sigma} e^{-1(x-\mu)\sigma^{-1}}, x \geq \mu \text{ for } \beta = 0, \quad \text{for } \sigma > 0. \\ &= 0, \text{ otherwise,} \end{aligned} \tag{2.5.2}$$

2.5.2.1 Minimum Variance Linear Unbiased Estimator of μ and σ When β Is Known

Theorem 2.5.2.1 *The minimum variance linear unbiased estimators* $\hat{\mu}$ *and* $\hat{\sigma}$ *of* μ *and* σ *based on the observed upper record values X(1), X(2), ..., X(m)*

$$\hat{\mu} = X(1)_1 - (1 - \beta)^{-1}\hat{\sigma}.$$

$$\hat{\sigma} = (1 - \beta)(\beta - D)^{-1}(1 - 2\beta)^3 X(1) + D^{-1}\beta(1 - \beta)\sum_{i=2}^{m-1}(1 - 2\beta)^{i+1}X(i) \\ + D^{-1}(1 - \beta)^2(! - 2\beta)^{m+1}X(m)$$

where

$$D = \sum_{I=2}^{m} (1-2\beta)^{i+1} \text{and } \beta < 1/2$$

Proof We assume GP(μ, σ, β) with β ≠ 0 and with finite variance. Let R be the m × 1 vector corresponding to X(i), i = 1,2, …, m, then we can write

$$E(R) = \mu L + \sigma\delta$$

where

$$R' = (X(1),X(2),\ldots,X(m))$$
$$L' = (1,1,\ldots,1), \delta' = (\alpha_1, \alpha_1, \alpha_1, \ldots, \alpha_m)$$
$$\alpha_i = \beta^{-1}(1-\beta)^{-i},$$

and

$$\alpha_i = \beta^{-1}(1-\beta)^{-i}, i = 1,2,\ldots,m.$$

We can write $V(R) = \sigma^2 V, V = (V_{i,j}), V_{i,j} = \beta^{-2} a_i b_j, 1 < i < j < m$ and $V_{i,j} = V_{j,i}$. The inverse $V^{-1}(= V^{i,j})$ can be expressed as

$$V^{i+1,i} = V^{i,i+1} = -\frac{1}{a_{i+1}b_i - a_i b_{i+1}} = -(1-2\beta)^{i+1}(1-\beta), i = 1,2,\ldots,m-1,$$
$$V^{i,i} = \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})}, i = 1,2,\ldots,n, V^{i,j} = 0, \text{for } |i-j| > 1,$$

where $a_o = 0 = b_{n+1}$ *and* $b_o = 1 = a_{n+1}$.
On simplification, we obtain

$$V^{i,i} = (1-2\beta)^i (2 - 4\beta + \beta^2), i = 1,2,\ldots,m-1$$

and

$$V^{m,m} = (1-2\beta)^m (1-\beta).$$

The minimum variance linear unbiased estimators (MVLUE) $\hat{\mu}, \hat{\sigma}$ of μ and σ are respectively based on the upper record values are

$$\hat{\mu} = -\delta' V^{-1}(L\delta' - \delta L')V^{-1}R/\Delta,$$

and

$$\hat{\sigma} = L'V^{-1}(L\delta' - \delta L\prime)V^{-1}R/\Delta,$$

where

$$\Delta = (\mathrm{L}'\mathrm{V}^{-1}\mathrm{L})(\mathrm{d}'\mathrm{V}^{-1}\delta) - (\mathrm{L}'\mathrm{V}^{-1}\delta)^2.$$

On substituting the values for δ and V^{-1} and subsequent simplification, it can be shown that

$$\hat{\mu} = X(1) - \hat{\sigma}(1-\beta)^{-1} \, and$$
$$\hat{\sigma} = (1-\beta)\left(\beta - D^{-1}(1-2\beta)^3 X(1) r_1\right) + D^{-1}\beta(1-\beta)\sum_{i=2}^{m}(1-2\beta)X(i)_i$$

where

$$\mathrm{D} = \sum_{i=2}^{m}(1-2\beta)^{i+1}.$$

The corresponding variances and the covariance of the estimates are

$$Var(\hat{\mu}) = \sigma^2 \frac{T}{D}$$
$$Var(\hat{\sigma}) = \sigma^2 \frac{\beta T - (1-2\beta)}{D}$$
$$Cov(\hat{\mu}, \hat{\sigma}) = \sigma^2 \frac{\left\{(1-2\beta)^2 + \beta^2 T\right\}}{D}$$

and

$$\mathrm{T} = \sum_{i=2}^{m}(1-2\beta)^i.$$

Exercise 2.5.2.1 Find the MVLUE $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ based on n upper record values X(1), X(2), …, X(n) of the Pareto Type II (Lomax) distribution with pdf f(x) as $f(x) = \frac{v}{\sigma}\left(1 + \frac{x-\mu}{\sigma}\right)^{-(v+1)}, x > \mu.\sigma > 0$ and $v > 0$,

2.5.2.2 Best Linear Invariant Estimators (BLIE)

Theorem 2.5.2.2 *The best linear invariant (in the sense of minimum mean squared error and invariance with respect to the location parameter* μ*) estimators* $\tilde{\mu}, \tilde{\sigma}$ *of* μ *and* σ *are respectively*

$$\tilde{\mu} = \hat{\mu} - \frac{\beta T - (1-2\beta)}{T(1-\beta)^2}\hat{\sigma} \text{ and}$$
$$\tilde{\sigma} = \frac{D}{T(1-\beta)^2}\hat{\sigma}, \text{where}$$
$$D = \sum_{i=2}^{m}(1-2\beta)^{i+1}, T = \sum_{i=1}^{m}(1-2\beta)^{i}.$$

and $\hat{\mu}$ *and* $\hat{\sigma}$ *are MVLUE of* μ *and* σ.

Proof The BLIE $\tilde{\mu}$ *and* $\tilde{\sigma}$ can be written as

$$\hat{\mu} = \hat{\mu} - \frac{E_{12}}{1+E_{22}}\hat{\sigma}.$$

and

$$\tilde{\sigma} = \frac{1}{1+E_{22}}\tilde{\sigma},$$

where

$$\begin{pmatrix} Var(\hat{\mu}) & Cov(\hat{\mu},\hat{\sigma}) \\ Cov(\hat{\mu},\hat{\sigma}) & Var(\hat{\sigma}) \end{pmatrix} = \sigma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}.$$

The mean squared errors of $\tilde{\mu}$ *and* $\tilde{\sigma}$ are

$$MSE(\tilde{\mu}) = \sigma^2\left(E_{11} - \frac{E_{12}^2}{1+E_{22}}\right),$$
$$MSE(\tilde{\sigma}) = \sigma^2\left(\frac{E_{22}}{1+E_{22}}\right).$$

Substituting the values of E_{11}, E_{12} and E_{22} in terms of β, T and D, we get the result.

2.5.2.3 Estimator of β for Known μ and σ

A Moment Estimator of β. We have seen that for μ = 0 and σ = 1. $E(X_{U(m)}) = \beta - 1\{(1-\beta)^{-m} - 1\}$. Thus

$$E(\bar{X}) = E\{(X(1) + X(2) + \ldots\ldots + X(m)\}/m\} = \frac{1}{m\beta^2}\{(1-\beta)^{-m} - 1\} - \frac{1}{\beta}$$
$$= \frac{X(m) - m}{m\beta}$$

Thus we can take $\tilde{\beta}$ as an estimator of β where

$$\tilde{\beta} = \frac{X(m) - m}{X(1) + X(2) + \ldots + X(m)}, for\ rx(1) + X(2) + \ldots + X(m) \neq 0$$

2.5.3 *Power Function Distribution*

We will consider the following pdf f(x) of power function distribution

$$f(x, \alpha, \beta, \gamma) = \gamma\, \beta^{-\gamma}(\alpha + \beta - x)^{\gamma-1}, \quad \text{for}\ \alpha < x < \alpha + \beta, \beta > 0, \gamma > 0,$$
$$= 0, \text{ otherwise.} \tag{2.5.3}$$

We will say a rv X ∈ PF(α, β, γ) if its pdf is given by (5.0.1). This is a Pearson's Type I distribution. If γ = 1, then f(x, α, β, γ) as given by (5.3.3) coincides with the uniform distribution in the interval (α, α + β). If we take $Y = (\alpha+\beta)^{\gamma}$, the Y has the uniform distribution in (0, 1). If γ is an integer, then the pdf of X as given in (5.0.1) can be consider as the pdf of ξ, where $\xi = \max(X_1, X_2, \ldots, X_\gamma)$.

2.5.3.1 The Minimum Variance Linear Unbiased Estimate of α and β When γ Is Known and γ ≠ 0

We will consider the following pdf f(x) for X.

$$f(x, \alpha, \beta, \gamma) = \gamma\, \beta^{-\gamma}(\alpha + \beta - x)^{\gamma-1}, \quad \text{for}\ \alpha < x < \alpha + \beta, \beta > 0, \gamma > 0,$$
$$= 0, \text{otherwise.}$$

We will say a rv X ∈ PF(α, β, γ) if its pdf is given by (5.0.1). This is a Pearson's Type I distribution. If γ = 1, then f(x, α, β, γ) as given by (5.0.1) coincides with the uniform distribution in the interval (α, α + β). If we take $Y = (\alpha+\beta)^{\gamma}$, the Y has the uniform distribution in (0, 1). If γ is an integer, then the pdf of X as given in (5.0.1) can be consider as the pdf of ξ, where $\xi = \max(X_1, X_2, \ldots, X_\gamma)$. Let X(1), X(2),…, X(m) be the first m upper records from this distribution. Let

$$W_k = c_k(X(k) - \frac{\gamma}{\gamma+1} X(k-1)), k = 1, 2, .., m$$

with X(0) = 0, and $c_k = (\gamma+1)\left(\frac{\gamma+2}{\gamma}\right)^{k/2}, k = 1, 2, \ldots, m.$
Now

$$E(W_1) = \left(\frac{\gamma+2}{\gamma}\right)^{1/2} \{(\gamma+1)\alpha + \beta\},$$
$$E(W_k) = \left(\frac{\gamma+2}{\gamma}\right)^{k/2} (\alpha + k), k = 1, 2, \ldots, m.$$
$$\mathrm{Var}(W_k) = \beta^2, k = 1, 2, \ldots, m$$
$$\mathrm{Cov}(W_i W_j) = 0, i \neq j, \ 1 \leq i, \ j \leq m.$$

Let $W' = (W_1, W_2, \cdots, W_n)$, then $E(W) = X\theta$, where

$$X = \begin{bmatrix} (\gamma+2/\gamma)^{1/2}(\gamma+1) & \left((\gamma+2/\gamma)^{1/2}\right) \\ (\gamma+2)/\gamma & (\gamma+2)/\gamma \\ . & . \\ \left((\gamma+2/\gamma)^{n/2}\right) & \left((\gamma+2/\gamma)^{n/2}\right) \end{bmatrix}, \theta = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

We can write X'X as

$$XX' = \begin{pmatrix} (\gamma\ +\ 2)^2 + T & \gamma\ +\ 2 + T \\ \gamma\ +\ 2 + T & T \end{pmatrix}$$
$$T = \sum_{k=1}^{m} \left(\frac{\gamma+2}{\gamma}\right)^k$$
$$(X'X)^{-1} = D_0^{-1} \begin{pmatrix} T & -(\gamma+2+T) \\ -(\gamma+2+T) & (\gamma+2)^2 + T \end{pmatrix}$$
$$D_0 = (\gamma+2)(\gamma T - \gamma - 2)$$
$$X'W = \begin{pmatrix} \frac{V_1}{V_2} \end{pmatrix}$$
$$V_1 = (\gamma(\gamma+2))^{1/2} W_1 + V_2$$
$$V_2 = \sum_{k=1}^{m} \left(\frac{\gamma+2}{\gamma}\right)^{k/2} W_k$$

Theorem 2.5.3.1 *The minimum variance unbiased estimates of α and β respectively based on Y_1, …, Y_n(assuming γ as known) are*

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = (X'X)^{-1}X'W$$

On simplification, we get

$$\hat{\alpha} = \frac{1}{D_0}\left[(\gamma(\gamma+2)^{1/2})W_1 - \sum_{k=2}^{n}((\gamma+2)/\gamma)^{k/2}W_k\right]$$
$$\hat{\beta} = \frac{1}{D_0}\left[-(\mathrm{T}+\gamma+2)(\gamma(\gamma+2))^{1/2}W_1 + (\gamma+2)(\gamma+1)\sum_{k=1}^{n}((\gamma+2)/\gamma)^{k/2}W_k\right]$$

The variances and covariance of are given by

$$\mathrm{Var}(\hat{\alpha}) = \beta^2 T D_0^{-1},$$
$$\mathrm{Var}(\hat{\beta}) = \beta^2((\gamma+2)^2+T)D_0^{-1}$$

and

$$\mathrm{Cov}(\hat{\alpha},\hat{\beta}) = -\beta^2(\gamma+2+T)D_0^{-1}$$

2.5.3.2 Minimum Variance Invariance Estimators

Theorem 2.5.3.2 *The best linear invariant (in the sense of minimum mean squared error and invariance with respect to the location parameter α) estimators $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β are respectively*

$$\tilde{\alpha} = \hat{\alpha} - \frac{\gamma+2+T}{(\gamma+1)\{(\gamma+1)T-(\gamma+2)\}}\hat{\beta}$$
$$\text{and } \tilde{\beta} = \frac{D_0}{(\gamma+1)\{(\gamma+1)T-(\gamma+2)\}}\hat{\beta}$$

where

$$D_0 = (\gamma+2)\{\gamma T-(\gamma+2)\}, T = \sum_{i=1}^{m}\left(\frac{\gamma+2}{\gamma}\right)^i.$$

and $\hat{\alpha}$ and $\hat{\beta}$ are MVLUEs of α and β.

Proof The BLIE $\tilde{\alpha}$ *and* $\tilde{\beta}$ *of* α *and* β can be written as

$$\hat{\alpha} = \hat{\alpha} - \frac{E_{12}}{1+E_{22}}\hat{\beta}.$$

and

$$\tilde{\beta} = \frac{1}{1+E_{22}}\hat{\beta},$$

where

$$\begin{pmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\beta}) \\ Cov(\hat{\alpha},\hat{\beta}) & Var(\hat{\beta}) \end{pmatrix} = \gamma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}.$$

The mean squared errors of $\tilde{\alpha}$ *and* $\tilde{\beta}$ *of* α *and* β are

$$MSE(\tilde{\alpha}) = \gamma^2\left(E_{11} - \frac{E_{12}^2}{1+E_{22}}\right),$$
$$MSE(\hat{\beta}) = \gamma^2\left(\frac{E_{22}}{1+E_{22}}\right).$$

Substituting the values of E_{11}, E_{12} and E_{22} in terms of γ, we get the results.

2.5.3.3 Maximum Estimator of β for Known μ and σ

Without any loss of generality we will assume $\mu = 0$ and $\sigma = 1$. The log likelihood function can be written as

$$\ln L = m \ln \gamma - \sum_{i=1}^{m} \frac{1}{1 - x(i)} + \gamma \ln(1 - x(m))$$

Differentiating with respect γ and equating to zero, we get $\breve{\gamma}$ as the maximum likelihood estimator of γ as

$$\breve{\gamma} = \frac{m}{\ln(1 - x(m))}$$

A moment Estimator of γ. Taking $\alpha = 0$ and $\beta = 1$, we get $E(X(i)) = \left(\frac{\gamma}{\lambda+1}\right)^i - 1$ and

$$E(X(1)+X(2)+\ldots+X(m)) = \gamma\left\{\left(\frac{\gamma}{\gamma+2}\right)^m - 1\right\} - m.$$

Thus we can have a moment estimator based on the m record values X(1), X(2), …, X(m) is

$$\ddot{\lambda} = \frac{X(1)+\ldots+X(m)+m}{x(m)}.$$

2.5.4 Rayleigh Distribution

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d random variables from standard Rayleigh distribution with pdf

$$f(x) = x e^{-x^2/2}, x > 0 \tag{2.5.4}$$

and cdf

$$F(x) = 1 - e^{-x^2/2}, x > 0$$

We say X€ RH(0,1) if the pdf of X is given by (2.5.6.1)

Theorem 2.5.4.1 Let

$$\mu_n = E(X_{U(n)}), V_{n,n} = Var(X_{U(n)}) \text{ and } V_{m,n} = Cov(X_{U(m)}X_{U(n)}),$$

then

$$\mu_n = \sqrt{2}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n)}, V_{n,n} = 2\left[n - \left(\frac{\Gamma(n+1/2)}{\Gamma(n)}\right)^2\right] \text{ and}$$

$$V_{m,n} = 2\left[\frac{\Gamma(m+1/2)}{\Gamma(m)}\right]\left[\frac{\Gamma(n+1)}{\Gamma(n+1/2)} - \left[\frac{\Gamma(n+1/2)}{\Gamma(n)}\right]\right], \text{for } 1 \leq m \leq n.$$

Proof

$$\mu_n = \frac{1}{\Gamma(n)}\int_0^\infty x\{-\ln(1-F(x))\}^{n-1} f(x)dx$$

$$= \frac{1}{\Gamma(n)}\int_0^\infty x\left(\frac{x^2}{2}\right)^{n-1} e^{-x^2/2} x dx$$

$$= \frac{1}{\Gamma(n)}\sqrt{2}\int_0^\infty u^{1/2}u^{n-1}e^{-u}du$$

$$= \sqrt{2}\frac{\Gamma(n+1/2)}{\Gamma(n)}.$$

Similarly it can be shown that

$$\mu_n^2 = E\left(X_{U(n)}^2\right) = 2\frac{\Gamma(n+1)}{\Gamma(n)} = 2n$$

$$\mu_{\mathrm{m,n}} = \frac{1}{\Gamma(m)\Gamma(n-m)}\int_0^\infty \int_0^y xy\left(\frac{x^2}{2}\right)^{m-1} x\left(\frac{y^2}{2}+\frac{x^2}{2}\right)^{n-m-1} ye^{-y^2/2}dxdy$$

$$= \frac{1}{\Gamma(m)\Gamma(n-m)}\frac{2}{2^{m-1}}\int_0^\infty y\left(\frac{y^2}{2}\right)^{n-m-1} ye^{-y^2/2}I_y\,dy,$$

where

$$\mathrm{I_Y} = \int_0^y (x^2)^m\left(1-\frac{x^2}{y^2}\right)^{n-m-1} dx$$

$$= \frac{1}{2}y^{2m+!}B(m+1/2, n-m),$$

with

$$\mathrm{B(a,b)} = \frac{\Gamma(\mathrm{a})\Gamma(\mathrm{b})}{\Gamma(\mathrm{a+b})}.$$

On simplification we get

$$\mathrm{V_{n,n}} = 2\left[n-\left(\frac{\Gamma(n+1/2)}{\Gamma(n)}\right)^2\right] \text{and}$$

$$\mathrm{V_{m,n}} = \left[\frac{\Gamma(m+1/2)}{\Gamma(m)}\right]\left[\frac{\Gamma(n+1)}{\Gamma(n+1/2)}-\frac{\Gamma(n+1/2)}{\Gamma(n)}\right], \text{for } 1 \le \mathrm{m} \le \mathrm{n}.$$

$$= \left[\frac{\Gamma(m+1/2)}{\Gamma(m)}\right]\left[\frac{\Gamma(n)}{\Gamma(n+1/2)}\right]\mathrm{V_{n,n}}$$

We will consider the estimation of μ and σ based on the observed record values X(1), X(2), …, X(m) of the two parameter Rayleigh distribution with the pdf

$$\mathrm{f(x,\mu,\sigma)} = \frac{x-\mu}{\sigma}\mathrm{e}^{\frac{(x-\mu)^2}{2\sigma^2}}, \mu < x < \infty, \sigma > 0$$

2.5.4.1 Minimum Variance Linear Unbiased Estimators of μ and σ

Theorem 2.5.4.2 *The minimum variance linear unbiased estimators* $\hat{\mu}$ and $\hat{\sigma}$ *of* μ *and* σ *based on the X(1), X(2),..., X(m) are*

$$\hat{\mu} = \sum_{k=1}^{m} c_k X(k), \text{and } \hat{\sigma} \sum_{k=1}^{m} d_k X(k),$$

where

$$c_1 = \frac{3}{2}\frac{\alpha_m b_m}{D}, c_i = \frac{2}{2i}\frac{\alpha_m b_m}{D}, i = 2, 3, \ldots, m-1,$$
$$c_m = 1 - \frac{\alpha_m b_m}{2D}\left[3 + \sum_{i=2}^{m-1}\frac{1}{i}\right], d_1 = \frac{3}{2}\frac{b_m}{D}, d_i = \frac{2}{2i}\frac{b_m}{D}, i = 2, 3, \ldots, m-1,$$
$$d_m = \frac{1}{2}\frac{b_m}{D}\left\{3 + \sum_{i=2}^{m-1}\frac{1}{i}\right\},$$

where

$$D = \alpha_m b_m T - 1, T = \left[\frac{3}{2} + \sum_{i=2}^{m-1}\frac{1}{2i} + (2m-1)\left(\frac{b_{m-1}}{b_m} - 1\right)\right]$$
$$\alpha_k = \sqrt{2}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k)} = a_k \text{ and } b_k = \sqrt{2}\left\{\frac{\Gamma(k+1)}{\Gamma\left(k+\frac{1}{2}\right)} - \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k)}\right\},$$
$$k = 1, 2, \ldots, m.$$

Proof Let R be the m × 1 vector corresponding to X(k), ki = 1, 2, …, m, then we have

$$E(R) = \mu L + \sigma\delta$$

where

$$R' = (X(1), X(2), \ldots, X(m))$$
$$L' = (1, 1, \ldots, 1), \delta' = (\alpha_1, \alpha_1, \ldots, \alpha_m,)$$
$$\alpha_i = \sqrt{2}\frac{\Gamma(i+1/2)}{\Gamma(i)}, i = 1, 2, \ldots, m.$$

We can write

$$V(R) = \sigma^2 V, V = (V_{i,j}), V_{i,j} = a_i b_j, 1 < i < j < m \text{ and } V_{i,j} = V_{j,i}.$$

The inverse V^{-1} (= $V^{i,j}$) can be expressed as

$$\begin{aligned}
V^{i+1,i} &= V^{i,i+1} = -\frac{1}{a_{i+1}b_i - a_i b_{i+1}} = -(2i+1), i = 1,2,\ldots,m-1,\\
V^{i,i} &= \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})}, i = 1,2,\ldots,n,\\
V^{i,j} &= 0, \text{for} |i-j| > 1,
\end{aligned}$$

where $a_0 = 0 = b_{n+1}$ *and* $b_0 = 1 = a_{n+1}$.

On simplification, we obtain

$$V^{i,i} = \frac{8i^2+1}{2i}, i = 1, 2, \ldots, m-1,$$

and

$$V^{m,m} = (2m-1)\frac{b_{m-1}}{b_m}.$$

The minimum variance linear unbiased estimates (MVLUE) $\hat{\mu}, \hat{\sigma}$ of μ and σ respectively are

$$\begin{aligned}
\hat{\mu} &= -\delta' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta\\
\hat{\sigma} &= L' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta,
\end{aligned}$$

where

$$\Delta = (L' V^{-1} L)(\delta' V^{-1}\delta) - (L' V^{-1}\delta)^2$$

and

$$\begin{aligned}
Var(\hat{\mu}) &= \sigma^2 L' V^{-1}\delta/\Delta,\\
Var(\hat{\sigma}) &= \sigma^2 L' V^{-1} L/\Delta\\
Cov(\hat{\mu}, \hat{\sigma}) &= -\sigma^2 L' V^{-1}\delta/\Delta.
\end{aligned}$$

On simplification, we obtain the MVLUE $\hat{\mu}, \hat{\sigma}$ of μ and σ.

The corresponding variances and the covariance of the estimates are

$$Var(\hat{\mu}) = \sigma^2 \frac{\alpha_n b_n}{D}$$
$$Var(\hat{\sigma}) = \sigma^2 \frac{b_n^2 T}{D}$$
$$Cov(\hat{\mu}, \hat{\sigma}) = -\sigma^2 \frac{b_n}{D}.$$

2.5.4.2 Best Linear Invariant Estimators (BLIEs) of μ and σ

Theorem 2.5.4.3 *The best linear invariant (in the sense of minimum mean squared error and invariance with respect to the location parameter* μ*) estimators (BLIEs)* $\tilde{\mu}$ $\tilde{\sigma}$ *of* μ *and* σ *are*

$$\tilde{\mu} = \hat{\mu} - \hat{\sigma}\left(\frac{E_{12}}{1 + E_{22}}\right)$$

and

$$\tilde{\sigma} = \hat{\sigma}/(1 + E_{22}),$$

$\hat{\mu}$ *and* $\hat{\sigma}$ *are* MVLUE *of* μ *and* σ *and*

$$\begin{pmatrix} Var(\hat{\mu}) & Cov(\hat{\mu}, \hat{\sigma}) \\ Cov(\hat{\mu}, \hat{\sigma}) & Var(\hat{\sigma}) \end{pmatrix} = \sigma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}.$$

The mean squared errors of these estimators are

$$\mathrm{MSE}(\tilde{\mu}) = \sigma^2\left(E_{11} - E_{12}^2(!1 + E_{22})^{-1}\right)$$

and

$$\mathrm{MSE}(\tilde{\sigma}) = \sigma^2 E_{22}(1 + E_{22})^{-1}$$

Using the values of E_{11}, E_{12} and E_{22} from (2.3.4), we obtain

$$\tilde{\mu} = \hat{\mu} + \hat{\sigma}\left(\frac{b_m}{D + b_m^2 T}\right)$$

and

$$\tilde{\sigma} = \hat{\sigma}\frac{D}{D + b_m^2 T}.$$

Exercise 2.5.4.1 Show that if μ = 0, then MVLUE of σ based on X(1), X(2), …, X(m) is

$$\hat{\sigma} = cX(m)$$

where

$$c = \frac{\sigma}{E(X(m))} = \frac{1}{\sqrt{2}}\frac{\Gamma(m)}{\Gamma\left(m + \frac{1}{\gamma}\right)}$$

Exercise 2.5.4.2 Show that the minimum varaiance linear unbiased predictor $\hat{X}(s)$ of X(s) based on X(1), X(2), …, X(m), s > m is $\hat{X}(s) = \hat{\mu} + \alpha_{s\hat{\sigma}}$. Where $\hat{\mu}$ and σ are the MVLUEs of μ and σ, Respectively.

2.5.5 Two Parameter Uniform Distribution

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables from a uniform distribution with the following pdf

$$\mathrm{f(x)} = \frac{1}{\theta_1 - \theta_2}, \theta_1 < x < \theta_2 \tag{2.5.5}$$

and cdf

$$\mathrm{F(x)} = \frac{x - \theta_1}{\theta_2 - \theta_1}, \theta_1 < x < \theta_2.$$

We will say $X \in U(\theta_1, \theta_2)$ if the pdf of X is as given in Eq (2.5.5).The pdf $f_n(x)$ of X(n) can be written as

$$\mathrm{f_n(x)} = \frac{1}{\Gamma(n)}\frac{1}{\theta_2 - \theta_1}\left\{\ln\frac{\theta_2 - \theta_1}{\theta_2 - x}\right\}^{n-1}, \theta_1 < x < \theta_2$$
$$\mathrm{E(X(m))} = 2^{-n}\theta_1 + (1 - 2^{-n})\theta_2$$
$$\mathrm{Var(X(m))} = (3^{-n} - 4^{-n})(\theta_2 - \theta_1)^2.$$

The joint pdf of X(m) and X(n) is

$$\mathrm{f_{m,n}(x,y)} = \frac{1}{\Gamma(m)}\frac{1}{\Gamma(n-m)}\frac{1}{\theta_2-\theta_1}\frac{1}{\theta_2-x}\left\{\ln\frac{\theta_2-\theta_1}{\theta_2-x}\right\}^{m-1}\left\{\ln\frac{\theta_2-\theta_1}{\theta_2-y}\right\}^{n-m-1},$$
$$\text{for } \theta_1 < x < y < \theta_2$$

We have

$$\mathrm{E(X(n)||X(m) = y_m) = 2^{m-n}y_m + (1-2^{m-n})}\theta_2.$$

and

$$\mathrm{Cov(X(m)X(n)) = 2^{m-n}Var\left(X_{U(m)}\right)}.$$

2.5.6 Minimum Variance Linear Unbiased Estimate of θ_1 and θ_2

We will consider here the estimation of θ_1 and θ_2 based on the observed m upper record values X(1), X(2), …, X(m).Consider the following transformation

$$\begin{aligned} W_1 &= X_{U(1)} \\ W_i &= (3)^{(i-1)/2}\left(X_{U(i)} - \frac{1}{2}X_{U(i-1)}\right), i = 2,3,\ldots\ldots,m \end{aligned} \tag{2.5.6}$$

It can easily be verified that

$$\begin{aligned} \mathrm{E(W_1)} &= \frac{\theta_1+\theta_2}{2}, \\ E(W_k) &= \frac{3^{\frac{i-1}{2}}}{2}\theta_2, k = 2,3,\ldots,m. \\ E(W_i) &= \frac{3^{i-1}}{2}\theta_2, i = 2,3,\ldots,m \\ \mathrm{Var(W_i)} &= \frac{\sigma^2}{12}, i = 2,3,\ldots,m \end{aligned}$$

and

$$\mathrm{Cov(W_i,W_j)} = 0, \mathrm{i} \neq \mathrm{j}.$$

Let W′ = (W_1, W_2, …., W_m), then E(W) = H θ, where

$$H = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2}\left(3^{\frac{i-1}{2}}\right) \\ & \cdots \\ 0 & \frac{1}{2}(3)^{(m-1)} \end{bmatrix}, \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}.$$

We have

$$(H'H)^{-1} = \frac{32}{3(3^{m-1} - !)} \begin{bmatrix} \frac{3^m-1}{8} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Thus, expressing W's in terms of the X(1), X(2), …, X(m) we obtain

$$\hat{\theta}_1 = 2X(1) - \hat{\theta}_2$$

and

$$\hat{\theta}_1 = \frac{4}{3(3^{m-1} - 1)} \left(3^{m-1}X(m) - \frac{3^{m-2}}{2}x(m-1) - \ldots - \frac{3}{2}X(2) - \frac{3}{2}X(1) \right)$$

The variances covariance of these estimates are

$$Var(\hat{\theta}) = \frac{1}{9}\frac{3^m - 1}{3^{m-1} - 1}(\theta_2 - \theta_1)^2,$$
$$Var(\hat{\theta}_2) = \frac{2}{9}\frac{1}{3^{n-1} - 1}(\theta_2 - \theta_1)^2$$

and

$$Cov(\hat{\theta}_1, \hat{\theta}_2) = \frac{2}{9}\frac{1}{3^{m-1} - 1}(\theta_2 - \theta_1)^2.$$

The generalized variance $\hat{\sum}\left(\hat{\sum} = \text{var}\theta_1.\text{var}\theta_2 - (\text{cov}(\theta_1\theta_2))^2\right)$ is

$$\frac{2}{27} \cdot \frac{1}{3^{n-1} - 1}(\theta_1 - \theta_2)^2.$$

Exercise 2.5.6.1 Suppose X(1), X(2), …, X(m) are m upper record values from a one parameter uniform distribution with pdf $f_U(u)$ as $f_U(u) = \frac{1}{\theta}$, $0 < x < \theta, \theta > 0$. Then the MVLUE $\hat{\theta}$ of θ is

$$\hat{\theta} = \frac{2}{3^n - 1}\left(2.3^{n-1}X(n) - 3^{n-2}X(n-1) - 3^{n-3}X(n-2) - \ldots - X(1)\right)$$

Proof Let X″ = (X(1), X(2), …, X(m)); We have E(X′) = $\delta\theta$
and Var(X) $= \theta^2 V$. V + (V_{ij})
where $\delta' = (\delta_1, \delta_2, \ldots, \delta_m)$, $\delta_i = 1 - \frac{1}{2^i}, i = 1, 2, \ldots m$
Vii = $\frac{1}{3^i} - \frac{1}{4^i}, i = 1, 2, \ldots, m$ and
Let V = (V_{ij}), then), then

$$V_{ii} = \frac{1}{3^i} - \frac{1}{4^i}, \mathrm{i} = 1, 2, \ldots, \mathrm{m}$$
$$V_{ij} = 2^{i-j}\left(\frac{1}{3^i} - \frac{1}{4^i}\right), i < j < m.$$

Let $V^{-1} - (V^{ij})$, **t**hen $V^{ii} = 73^i$, **i = 1,2, .., m–1.**
$V^{mm} = 4.3^m$, $V^{i\,i+1} = -2, 3^{i+1}. = V^{i+1i}$ and $V^{ij} = 0$ for |i−j|.
The MVLUE $\hat{\sigma}$ of σ is

$$\hat{\sigma} = \frac{\delta^{/} V^{-1} X}{\delta^{/} V^{-1} \delta}$$
$$= \frac{2}{3^m - 1}\left(2.3^{m-1}X(m) - 3^{m-2}X(m-1).,,,, - -X(1)\right)$$
$$Var(\hat{\sigma}) = \frac{2\sigma^2}{3(3^n - 1)}.$$

2.5.7 *One Parameter Uniform Distribution*

Suppose γ = 1 and α = 0, i.e. when X is distributed uniformly in the interval (0, β), We have in this case the pdf $f_n(x)$ of X(n) as

$$f_n(x) = \frac{1}{\Gamma(n)}\left[\ln\frac{\beta}{x}\right]^{n-1}, 0 < x < \beta. \quad (2.5.7)$$

Using (2.5.2.1), we obtain

$$\mathrm{E(X(n))} = (1 - 2^{-n})\beta.$$
$$\mathrm{Var(X(n))} = (3^{-n} - 4^{-n})\beta^2$$

The joint pdf of X(m) and X(n), n > m is

$$f_{m,n}(x,y) = \frac{1}{\Gamma(m)}\frac{1}{\Gamma(n-m)}\frac{1}{\beta}\frac{1}{\beta-x}\left[\ln\frac{\beta}{\beta-x}\right]^{m-1}\left[\ln\frac{\beta}{\beta-y}\right]^{n-m-1},$$

$$\mathrm{n} > \mathrm{m} > 0, 0 < \mathrm{x} < \mathrm{y} < \beta.$$

It follows from (2.2.6) that

$$\mathrm{E}(\mathrm{X}(\mathrm{n})|\mathrm{X}(\mathrm{m}) = \mathrm{x_m}) = 2^{\mathrm{m-n}}\mathrm{x_m} + (1 - 2^{\mathrm{m-n}})\beta.$$

and

$$\mathrm{Cov}(\mathrm{X}(\mathrm{n})\mathrm{X}(\mathrm{m})) = 2^{\mathrm{m-n}}\mathrm{Var}(\mathrm{X}(\mathrm{m})), \mathrm{m} < \mathrm{n},\ 1 < \mathrm{m} < \mathrm{n}$$

The correlation coefficient $\rho_{\mathrm{m,n}}$ of X(m) and X(n)–s

$$\rho_{m,n} = \left(\left(\frac{4}{3}\right)^m - 1\right)^{\frac{1}{2}}\left(\left(\frac{4}{3}\right)^n - 1\right)^{\frac{1}{2}}, m < n$$

2.5.7.1 Minimum Variance Unbiased Estimator of β

Using the following transformation

$$\begin{aligned}
\mathrm{W_1} &= \mathrm{X}(1)\\
W_i &= 3^{\frac{i-1}{2}}\left(X(i) - \frac{1}{2}(X-i)\right), i = 2,\ldots,n\\
\mathrm{E}(\mathrm{W_i}) &= (1/2)(3)^{(\mathrm{i}-1)/2}\beta\\
\mathrm{Var}(\mathrm{W_i}) &= \frac{\beta^2}{12},\\
\mathrm{Cov}(\mathrm{W_i},\mathrm{W_j}) &= 0,\ \mathrm{i} \neq \mathrm{j},\ \mathrm{i},\mathrm{j} = 1,2,\ldots,\mathrm{n}.
\end{aligned}$$

Let

$$\mathrm{X}' = \left(\frac{1}{2}, \frac{1}{2}(3)^{1/2}, \frac{1}{2}(3), \ldots, \frac{1}{2}(3)^{n-1}\right)$$

and

$$\mathrm{W}' = (\mathrm{W}_1, \mathrm{W}_2, \ldots, \mathrm{Wn}),$$

then minimum variance linear unbiased estimator $\hat{\beta}$ of β based on the first n record values is

$$\begin{aligned}
\hat{\beta} &= (\mathrm{X}'\mathrm{X}) - 1\mathrm{X}'\mathrm{W} \\
&= \frac{4}{3^n - 1}\left(\sum_{i=1}^{n} (3)^{(i-1)/2} W_{\mathrm{i}}\right) \\
&= \frac{4}{3^n - 1}\left(3^{n-1}X(n) - \frac{3^{n-2}}{2}X(n-1) - \frac{3^{n-3}}{2}X(n-2) - \ldots - \frac{1}{2}X(1)\right)
\end{aligned}$$

Since $\mathrm{X}'\mathrm{X} = \frac{3^n - 1}{8}$ and $\mathrm{Var}(\mathrm{W}_{\mathrm{i}}) = \frac{\beta^2}{12}$, we have

$$\begin{aligned}
\mathrm{var}(\hat{\beta}) &= (X'X)^{-1}\frac{\beta^2}{12} \\
&= \frac{2\beta^2}{3(3^n - 1)}
\end{aligned}$$

2.5.7.2 Minimum Mean Square Estimate of β

If we drop the condition of unbiasedness, then the estimator $\tilde{\beta}$, where

$$\tilde{\beta} = \frac{3(3^n - 1)}{3^{n+1} - 1}\hat{\beta}$$

has minimum mean squared error.

$$\text{Bias of } \tilde{\beta} = \mathrm{E}(\tilde{\beta}) - \beta = \frac{2}{3^{n+1} - 1}\beta$$

and

$$\mathrm{MSE}(\hat{\beta}) = \frac{2\beta^2}{3^{n+1} - 1}$$

Exercise 2.5.7.1 Find the maximum likelihood estimate of β.

2.5.8 Prediction of Record Values

Writing

$$\mathrm{Y}_{\mathrm{n+s}} = Y_{n+s} - \frac{1}{2}Y_{n+s-1} + \frac{1}{2}(Y_{n+s-2}) + \cdots + \frac{1}{2^{n+s-2}}\left(Y_2 - \frac{1}{2}Y_1\right) + \frac{1}{2^{n+s-1}}Y_1,$$

it can be shown that

$$\mathrm{Cov}(\ \mathrm{Y}_{\mathrm{n+s}}, \mathrm{W}_{\mathrm{i}}) = ,\ \mathrm{i} = 1{,}2{,}\ldots{,}\mathrm{n}.$$

It can be shown that the best linear unbiased predictor (BLUP) of $\mathrm{Y}_{\mathrm{n+s}}$ is $\hat{Y}_{n+s}$, where

$$\hat{Y}_{n+s} = \left(1 - \frac{1}{2^{n+s}}\right)\beta + c'V^{-1}\left(W - X\hat{\beta}\right)$$

where

$$\mathrm{c}' = (c_1{,}c_2{,}\ldots{,}c_n), \mathrm{V}^{-1} = (\mathrm{X}'\mathrm{X})^{-1} \text{ and } \mathrm{c_i}\mathrm{Var}(\mathrm{W_i}) = \mathrm{Cov}(\ \mathrm{Y}_{\mathrm{n+s}}, \mathrm{W}_{\mathrm{i}}), \mathrm{s} > 1.$$

Thus

$$\hat{Y}_{n+s} = \left(1 - \frac{1}{2^{n+s}}\right)\hat{\beta} + \frac{8}{3^n - 1}\left[\sum_{i=1}^{n} - \frac{1}{2^{n+s-i}} \cdot \frac{W}{3^{(i-1)/2}} - \frac{\hat{\beta}}{2^s}\left(1 - \frac{1}{2^n}\right)\right]$$

The best linear (unrestricted) least square predictor of $\mathrm{Y}_{\mathrm{n+s}}$ is $\widetilde{Y}_{r+s}$, where

$$\overleftrightarrow{Y}_{r+s} = E(Y_{n+s}|Y_1,Y_2,\cdots Y_n) = \frac{y_n}{2^s} + \left(1 - \frac{1}{2^s}\right)\beta,$$

Substituting $\hat{\beta}$ for β, we get the best linear least squares predictor as

$$\frac{y_n}{2^s} + \left(1 - \frac{1}{2^s}\right) \cdot \frac{4}{3^n - 1}\left(3^{n-1}y_n - \frac{1}{3}(3)^{n-2}y_{n-1} - \ldots - \frac{1}{2}y_1\right).$$

2.6 Weibull Distribution

Let {X_n, n $\geq$ 1} be a sequence of i.i.d random variables from standard Weibull distribution with pdf

$$\mathrm{f(x)} = x^{\gamma-1}\mathrm{e}^{-x^{\gamma}}, \mathrm{x} > 0, \gamma > 0, \tag{2.6.1}$$

and cdf

$$\mathrm{F(x)} = 1 - \mathrm{e}^{-\frac{1}{\gamma}x^{\gamma}}, \mathrm{x} > 0, \mathrm{g} > 0,$$

Let μn = E(X(n), $V_{n,n}$ = Var(X(n)) and V_{mn} = Cov(X(m)X(n)), m < n, then

$$\mu_n = \gamma^{1/\gamma}\frac{\Gamma\left(n+\frac{1}{\gamma}\right)}{\Gamma(n)}, V_{n,n} = \gamma^{2/\gamma}\left\{\frac{\Gamma\left(n+\frac{2}{\gamma}\right)}{\Gamma\left(n+\frac{1}{\gamma}\right)} - \left(\frac{\sqrt{\left(n+\frac{1}{2}\right)}}{\sqrt{(n)}}\right)^2\right\}.$$

and

$$V_{m,n} = \frac{\Gamma\left(m+\frac{1}{\gamma}\right)}{\Gamma(m)}\gamma^{2/\gamma}\left\{\frac{\Gamma\left(n+\frac{2}{\gamma}\right)}{\Gamma\left(n+\frac{1}{\gamma}\right)} - \frac{\Gamma\left(n+\frac{1}{\gamma}\right)}{\Gamma(n)}\right\}, \text{for } 1 < m < n.$$

We will consider the following pdf $f(x, \mu, \sigma)$, for Weibull distribution,

$$f(x, \mu, \sigma) = \frac{(x-\mu)^{\gamma-1}}{\sigma^{\gamma}} e^{-\frac{1}{\gamma}\left(\frac{x-\mu}{\sigma}\right)^{\gamma}} \quad -\infty < \mu < x < \infty, \sigma > 0.$$

2.6.1 Minimum Variance Linear Unbiased Estimators of μ and σ

Theorem 2.6.1 *The minimum variance linear unbiased estimators* $\hat{\mu}$ and $\hat{\sigma}$ *of* μ *and* σ *based on the record values X(1),X(2,,..,X(n)are*

$$\hat{\mu} = \sum_{k=1}^{m} c_k X(k), \text{ and } \hat{\sigma} = \sum_{k=1}^{m} d_k X(k),$$

where

$$c_1 = \frac{\alpha_m b_m}{D}\frac{(\gamma+1)\gamma^{-2/\gamma}}{\Gamma\left(1+\frac{1}{\gamma}\right)}, c_i = \frac{\alpha_m b_m}{D}\gamma^{-2/\gamma}(\gamma - !)\frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)}, i = 2, 3, \ldots, m-1,$$

$$c_m = 1 - \frac{\alpha_m b_m}{D}\gamma^{-2/\gamma}\left[\frac{\gamma+1}{\Gamma\left(1+\frac{2}{\gamma}\right)} + (\gamma-1)\sum_{i=2}^{m-1}\frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)}\right],$$

$$d_1 = -\frac{b_m(\gamma+1)\gamma^{-2/\gamma}}{D},$$

$$d_i = -\frac{b_m}{D}(\gamma-1)\gamma^{-2/\gamma}\frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)}, i = 2, 3, \ldots, m-1,$$

$$d_m = \frac{b_m}{D}\gamma^{-\frac{2}{\gamma}}\left[\frac{\gamma+1}{\Gamma\left(1+\frac{2}{\gamma}\right)} + (\gamma-1)\sum_{i=2}^{m-1}\frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)}\right],$$

where

$$D = \alpha_m b_m T - 1,$$

$$T=\gamma^{-2/\gamma}\left[\frac{\gamma+1}{\Gamma\left(1+\frac{2}{\gamma}\right)} + (\gamma-1)\sum_{i=2}^{m-1}\frac{\Gamma(i)}{\Gamma\left(i+\frac{2}{\gamma}\right)} + \frac{\Gamma(m)}{\Gamma\left(m+\frac{2}{\gamma}\right)}(m\gamma-\gamma-1)(m\gamma-\gamma+2)\left(\frac{b_{m-1}}{b_m}-1\right)\right]$$

$$\alpha_m = \gamma^{1/\gamma}\frac{\left(m+\frac{1}{\gamma}\right)}{(m)} \text{ and } b_m = \gamma^{1/\gamma}\left\{\frac{\left(n+\frac{2}{\gamma}\right)}{\left(n+\frac{1}{\gamma}\right)} - \frac{\left(n+\frac{1}{\gamma}\right)}{\Gamma(n)}\right\}$$

We can write

$$V(R) = \sigma^2 V, V = (V_{i,j}), V_{i,j} = a_i b_j, 1 < i < j < m \text{ and } V_{i,j} = V_{j,i}.$$

The inverse V^{-1} (= $V^{i,j}$) can be expressed as

$$V^{i+1,i} = V^{i,i+1} = -\frac{1}{a_{i+1}b_i - a_i b_{i+1}} = -(2i+1), i = 1,2,\ldots,m-1,$$

$$V^{i,i} = \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})}, i = 1,2,\ldots,n,$$

$$V^{i,j} = 0, \text{for} |i-j| > 1,$$

where $a_0 = 0 = b_{n+1}$ *and* $b_0 = 1 = a_{n+1}$. On simplification, we obtain

$$V^{i,i} = \gamma^{-2/\gamma} \frac{\Gamma(i)}{\Gamma\left(i+\frac{1}{\gamma}\right)} \left[\gamma^2(2i^2 - 2i + 1) + \gamma(4i+2) + 1\right], i = 1,2,\ldots,m-1,$$

$$V^{m,m} = \gamma^{-2/\gamma} \frac{\Gamma(n)}{\Gamma\left(n+\frac{2}{\gamma}\right)} \frac{b_{n-1}}{b_n} [(n\gamma - \gamma + 1)(n\gamma - \gamma + 2)].$$

The minimum variance linear unbiased estimates (MVLUE) $\hat{\mu}, \hat{\sigma}$ of μ and σ respectively are

$$\hat{\mu} = -\delta' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta$$
$$\hat{\sigma} = L' V^{-1}(L\delta' - \delta L')V^{-1}X/\Delta,$$

where

$$\Delta = (L'V^{-1}L)(\delta' V^{-1}\delta) - (L'V^{-1}\delta)^2,$$
$$X' = (\ X(1), X(2), \ldots .X(n))$$

and

$$Var(\hat{\mu}) = \sigma^2 L' V^{-1}\delta/\Delta,$$
$$Var(\hat{\sigma}) = \sigma^2 L' V^{-1} L/\Delta$$
$$Cov(\hat{\mu}, \hat{\sigma}) = -\sigma^2 L' V^{-1}\delta/\Delta.$$

On simplification, we obtain the MVLUEs $\hat{\mu}, \hat{\sigma}$ of μ and σ. The corresponding variances and the covariance of the estimates are

$$Var(\hat{\mu}) = \sigma^2 \frac{\alpha_n b_n}{D}$$
$$Var(\hat{\sigma}) = \sigma^2 \frac{b_n^2 T}{D}$$
$$Cov(\hat{\mu}, \hat{\sigma}) = -\sigma^2 \frac{b_n}{D}.$$

Best Linear Invariant Estimators (BLIEs) of μ and σ.

Theorem 2.6.2 *The best linear invariant (in the sense of minimum mean squared error and invariance with respect to the location parameter μ) estimators (BLIEs) $\tilde{\mu}, \tilde{\sigma}$ of μ and σ are*

$$\tilde{\mu} = \hat{\mu} - \hat{\sigma}\left(\frac{E_{12}}{1 + E_{22}}\right)$$

and

$$\tilde{\sigma} = \hat{\sigma}/(1 + E_{22}),$$

where $\hat{\mu}$ *and* $\hat{\sigma}$ *are* MVLUE of μ *and* σ *and*

$$\begin{pmatrix} Var(\hat{\mu}) & Cov(\hat{\mu}, \hat{\sigma}) \\ Cov(\hat{\mu}, \hat{\sigma}) & Var(\hat{\sigma}) \end{pmatrix} = \sigma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}$$

The mean squared errors of these estimators are

$$\mathrm{MSE}(\tilde{\mu}) = \sigma^2 \left(E_{11} - E_{12}^2 (1 + E_{22})^{-1} \right)$$

and

$$\mathrm{MSE}(\tilde{\sigma}) = \sigma^2 E_{22} (1 + E_{22})^{-1}.$$

Using the values of E_{11}, E_{12} and E_{22} from (2.62.4), we obtain

$$\tilde{\mu} = \hat{\mu} + \hat{\sigma} \left(\frac{b_m}{D + b_m^2 T} \right)$$

and

$$\tilde{\sigma} = \hat{\sigma} \frac{D}{D + b_m^2 T}.$$

Exercise 2.6.2 Show that if μ = 0, then MVLUE estimator of σ based on the record values X(1),X(2),…,X(m) for known v is

$$\bar{\sigma} = c_0 X(m),$$

Chapter 3
Extreme Value Distributions

3.1 Introduction

In this chapter some distributional properties of extreme value distributions will be presented.

Extreme value distributions arise in probability theory as limit distributions of maximum or minimum of n independent and identically distributed random variables with r some normalizing constants. For example if X_1, X_2, …, X_n are n independent and identically distributed random variables. Then the largest order statistic $X_{n,n}$, if it has a non degenerate limiting distribution, then with some normalizing constants its distribution will tend to one of the following three types of limiting extreme value distributions as $n \to \infty$.

(1) Type 1: (Gumbel) $F(x) = \exp(-e^{-x})$, for all x,
(2) Type 2: (Frechet) $F(x) = \exp(-x^{-\delta})$, $x > 0$, $\delta > 0$
(3) Type 3: (Weibull) $F(x) = \exp(-(-x)^{\delta})$, $x < 0$, $\delta > 0$,

Since the smallest order statistic $X_{1,n} = Y_{n,n}$, where $Y = -X$, $X_{1,n}$ with some appropriate normalizing constants will also converse to one of the above three limiting distributions if we change to $-X$ in (1), (2) and (3). Gumbel (1958) has given various applications of these distributions.

Suppose X_1, X_2, … X_n be i.i.d random variable having the distribution function F(X) with $F(x) = 1 - e^{-x}$. Then with normalizing constant $a_n = \ln n$ and $b_n = 1$, P $(X_{n.n} < a_n + b_n x) = P(X_{n,n} \le \ln n + x) = (1 - e^{-(\ln n + x)})^n = (1 - \frac{e^{-x}}{n})^n \to e^{-e^{-x}}$ as $n \to \infty$. Thus the limiting distribution of $X_{n,n}$ when X's are distributed as exponential with unit mean is Type 1 extreme value distribution as given above. It can be shown that Type 1 (Gumbel distribution) is the limiting distribution of $X_{n,n}$ when F(x) is normal, log normal, logistic, gamma etc. The type 2 and type 3 distributions can be transformed to Type 1 distribution by the transformations ln X and—ln X respectively. We will denote the Type 1 distribution as T_{10} and Type 2 and Type 3 distribution as $T_{2\delta}$ and $T_{3\delta}$ respectively. If the $X_{n,n}$ of n independent random

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variables from a distribution F has the limiting distribution T, then we will say that F belongs to the domain of attraction of T and write $F \in D(T)$.

The extreme value distributions were originally introduced by Fisher and Tippet (1928). These distributions have been used in the analysis of data concerning floods, extreme sea levels and air pollution problems; for details see Gumbel (1958), Horwitz (1980), Jenkinson (1955) and Roberts (1979).

3.2 The Pdfs of the Extreme Values Distributions

3.2.1 Type 1 Extreme Value for $X_{n,n}$

The cumulative distribution function of type 1 extreme value distribution (T_{10}) is given in Fig. 3.1.

The type I extreme value is unimodal with mode at 0 and the points of inflection are at $\pm\ln\left((3+\sqrt{5})/2\right)$. The pth percentile η_p, (0 < p < 1) of the curve can be calculated by the relation $\eta_p = -\ln(-\ln p)$. The median of X is $-\ln\ln 2$. The moment generating function $M_{10}(t)$, of the distribution for some t, $0 < |t| < \delta$, is

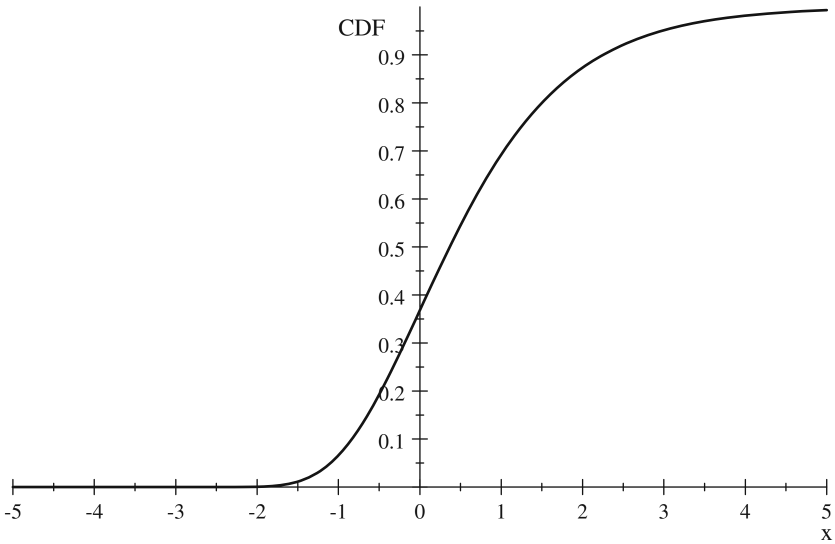

Fig. 3.1 CDF of T_{10}

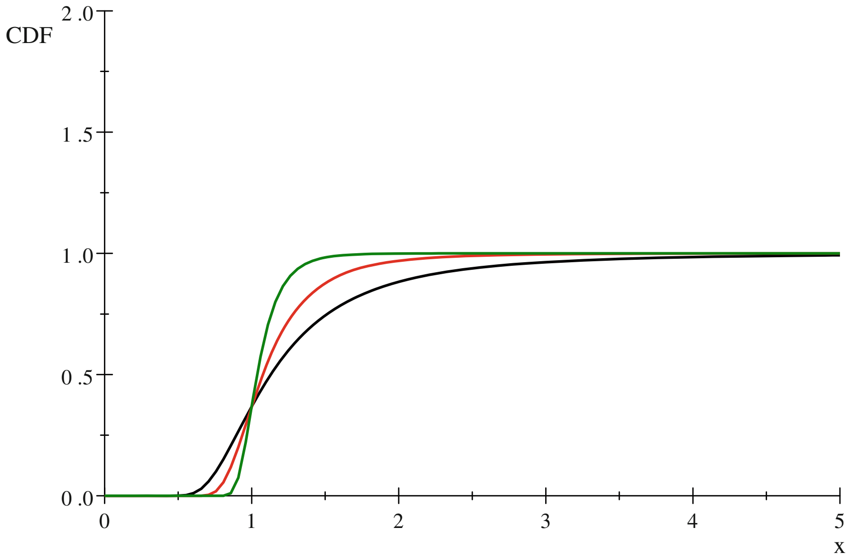

Fig. 3.2 CDFs $T_{2,3}$—*Black*, $T_{2,5}$—*Red*, $T_{2,10}$—*Green*

$M_{10}(t) = \int_{-\infty}^{\infty} e^{tx} e^{-x} e^{-e^{-x}} dx = e^{t}\Gamma(1-t)$. The mean = γ, the Euler's constant and the variance = $\pi^2/6$.

3.2.2 *Type 2 Extreme Value Distribution for $X_{n,n}$*

The cumulative distribution function of $T_{2,3}$, $T_{2,5}$ and $T_{2.10}$ are given in Fig. 3.2.

The mode of T_{21} is at x = 1/2. For $T_{2\delta}$ the mode is at 1/(δ + 1), for δ > 1, $E(X) = \Gamma(1-\frac{1}{\delta})$ and for $\delta > 2, Var(X) = \Gamma(1-\frac{2}{\delta}) - (\Gamma(1-\frac{1}{\delta}))^2$.

3.2.3 *Type 3 Extreme Value Distribution for $X_{n,n}$*

The cumulative distribution function of r type 3 for δ = 3, 5 and 10 are given in Fig. 3.3. Note for δ = 1 T_{31} is the reverse exponential distribution.

The mode of the type 3 distribution is at $\left(\frac{\delta-1}{\delta}\right)^{\frac{1}{\delta}}$. For type 3 distribution, $E(X) = \Gamma(1+\frac{1}{\delta})$ and $Var(X) = \Gamma(1+\frac{2}{\delta}) - (\Gamma(1+\frac{1}{\delta}))^2$.

Table 3.1 gives the percentile points of T_{10}, T_{21}, T_{31} and T_{32} for some selected values of p.

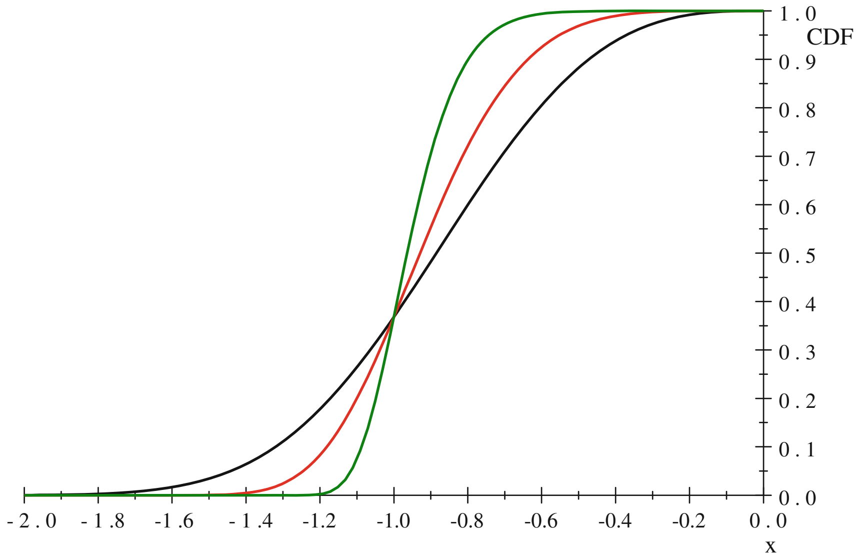

Fig. 3.3 CDFs $T_{3,3}$—*Black*, $T_{3,5}$—*Red*, $T_{3,10}$—*Green*

Table 3.1 Percentile points of T_{10}, T_{21}, T_{31} and T_{32}

P	T_{10}	T_{21}	T_{31}	T_{32}
0.1	−0.83403	0.43429	−2.30259	−1.51743
0.2	−0.47589	0.62133	−1.60844	−1.26864
−0.3	−0.18563	0.83058	−1.20397	−1.09726
0.4	0.08742	1.09136	−0.91629	−0.95723
−0.5	0.36651	1.44270	−0.69315	−0.83255
−0.6	0.67173	1.95762	−0.51083	−0.71472
0.7	1.03093	2.80367	−0.35667	−9.59722
0.8	1.49994	4.48142	−0.22314	−0.47239
0.9	2.2504	9.49122	−0.10536	−0.324598

3.3 Domain of Attraction

In this section we will study the domain of attraction of various distributions. The maximum order statistics $X_{n,n}$ of n independent and identically distributed random variable will considered first. We will say that $X_{n,n}$ will belong to the domain of attraction of T(x) if the $\lim_{n\to\infty} P(X_{nn} \le a_n + b_n x) = \mathrm{T(x)}$ for some sequence of normalizing constants a_n and b_n

For example consider the uniform distribution with pdf f(x) = 1, 0 < x < 1. Then for t < 0, $P(X_{nn} \le 1 + t/n) = (1 + t/n)^n \to e^t$. Thus X_{nn} from the uniform distribution belong to the domain of attraction of T(x), $T(x) = e^x$, $-\infty < x < 0$.

The following lemma will be helpful in proving the theorems of the domain of attraction.

Lemma 3.3.1 Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with distribution function F. Consider a sequence $\{e_n, n \geq 1\}$ of real numbers. Then for any ξ, $0 \leq \xi < \infty$, the following two statements are equivalent

(i) $\lim_{n\to\infty} n(\overline{F}(e_n)) = \xi$

(ii) $\lim_{n\to\infty} P\left(X_{n,n} \leq e_n\right) = \mathrm{e}^{-\xi}$.

Proof Suppose (i) is true, then

$$\lim_{n\to\infty} P(X_{n,n} \leq e_n) = \lim_{n\to\infty} F^n(e_n) = \lim_{n\to\infty} (1 - \overline{F}(e_n))^n$$
$$= \lim_{n\to\infty} (1 - \xi/\mathrm{n} + \mathrm{o}(1))^{\mathrm{n}} = \mathrm{e}^{-\xi}.$$

Suppose (ii) is true, then

$$\mathrm{e}^{-\xi} = \lim_{n\to\infty} P(X_{n,n} \leq e_n) = \lim_{n\to\infty} F^n(e_n) = \lim_{n\to\infty} (1 - \overline{F}(e_n))^n$$

Taking the logarithm of the above expression, we get

$$\lim_{n\to\infty} n \ln(1 - \overline{F}(e_n)) = -\xi.n\overline{F}(e_n)(1 + o(1)) \to \xi$$

Note: The above theorem is true if $\xi = \infty$.

3.3.1 Domain of Attraction of Type I Extreme Value Distribution for $X_{n,n}$

The following theorem is due to Gnedenko (1943).

Theorem 3.3.1 Let X_1, X_2, … be a sequence of i.i.d random variables with distribution function F and $\beta(F) = \sup\{x: F(x) < 1\}$. Then $F \in T_{10}$ if there exists a positive function g(t) such that

$$\lim_{t\to e(F)} \frac{\hat{F}(t + xg(t))}{\hat{F}(t)} = \mathrm{e}^{-x}, \overleftarrow{F} = 1 - F \text{ for all real x.}$$

Proof We choose the normalarizing constants a_n and b_n of $X_{n,n}$ such that $a_n = \inf\{x: \overline{F}(x) \le \frac{1}{n}\}$ $b_n = g(a_n)$. $a_n \to \beta(F)$ as $n \to \infty$. Suppose $\lim_{t \to e(F)} \frac{\hat{F}(t + xg(t))}{\hat{F}(t)} = e^{-x}, \overleftarrow{F} = 1 - F$, then $\lim_{n\to\infty} n\overline{F}(a_n + b_n x) = \lim_{n\to\infty} n\bar{F}(a_n)\left(\frac{\overline{F}(a_n + b_n x)}{\overline{F}(a_n)}\right) = e^{-x} \lim_{n\to\infty} n\overline{F}(a_n) = e^{-x}$. By Lemma 3.3.1. we have $P(X_{n,n} \le a_n + b_n x) = e^{-e^{-x}}$.

Suppose $P(X_{n,n} \le a_n + b_n x) = e^{-e^{-x}}$ we have by Lemma 2.1.1, $\lim_{n\to\infty} n\overline{F}(a_n + b_n x) = e^{-x}$.

$$
\begin{aligned}
e^{-x} &= \lim_{n\to\infty} n\bar{F}(a_n + b_n x) = \lim_{n\to\infty} n\bar{F}(a_n)\left(\frac{\overline{F}(a_n + b_n x)}{\overline{F}(a_n)}\right) = \lim_{n\to\infty}\left(\frac{\overline{F}(a_n + b_n x)}{\overline{F}(a_n)}\right) \\
&= \lim_{t\to e(F)} \frac{\hat{F}(t + xg(t))}{\hat{F}(t)}
\end{aligned}
$$

The following Lemma (see Von Mises 1936) gives a sufficient condition for the domain of attraction of Type 1 extreme value distribution for $X_{n,n}$.

Lemma 3.3.2 Suppose the distribution function F has a derivative on $[c_0, \beta(F)]$ for some c_0, $0 < c_0 < \beta(F)$, then if $\lim_{x \uparrow \beta(F)} \frac{f(x)}{\overline{F}(x)} = c, c > 0$, then $F \in D(T_{10})$.

Example 3.3.1 The exponential distribution $F(x) = 1 - e^{-x}$ satisfies the sufficient condition, since $\lim_{x\to\infty} \frac{f(x)}{\overline{F}(x)} = 1$. For the logistic distribution $F(x) = \frac{1}{1+e^{-x}}$, $\lim_{x\to\infty} \frac{f(x)}{\overline{F}(x)} = \lim_{x\to\infty} \frac{1}{1+e^{-x}} = 1$. Thus the logistic distribution satisfies the sufficient condition.

Example 3.3.2 For the standard normal distribution with $x > 0$, (see Abramowitz and Stegun 1968, p. 932)

$\overline{F}(x) = \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}} h(x)$, where $h(x) = 1 - \frac{1}{x^2} + \frac{1.3}{x^4} + \cdots + \frac{(-1)^n 1.3\ldots(2n-1)}{x^{2n}} + R_n$ and $R_n = (-1)^{n+1} 1.3\ldots(2n+1) \int_x^\infty \frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}u^{2n+2}} du$ which is less in absolute value than the first neglected term.

It can be shown that $g(t) = 1/t + 0(t^3)$. Thus $\lim_{t\to\infty} \frac{\overline{F}(t + xg(t))}{\overline{F}(t)} = \lim_{t\to\infty} \frac{te^{\frac{t^2}{2}}}{(t + xg(t))e^{\frac{1}{2}(t+xg(t))^2}} \frac{h(t+xg(t))}{h(t)} = \lim_{t\to\infty} \frac{e^{-xm(t,x)}}{t+xg(t)}$, where $m(t, x) = g(t)\,(t + \frac{1}{2}xg(t)$. Since as $t \to \infty$, $m(t, x) \to 1$, we $\lim_{t\to\infty} \frac{\overline{F}(t + xg(t))}{\overline{F}(t)} = e^{-x}$. Thus normal distribution belong to the domain of attraction of Type I distribution.

Since $\lim_{x\to\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x\overline{F}(x)} = \lim_{x\to\infty} h(x) = 1$, the standard normal distribution does not satisfy the Von Mises sufficient condition for the domain of attraction of the type I distribution.

We can take $a_n = \frac{1}{b_n} - \frac{b_n}{2}(\ln \ln n + 4\pi)$ and $b_n = (2 \ln \ln n)^{-1/2}$. However this choice of a_n and b_n is not unique. The rate of convergence of $P(X_{n,n} \le a_n + b_n x)$ to $T_{10}(x)$ depends on the choices of a_n and b_n.

3.3.2 Domain of Attraction of Type 2 Extreme Value Distribution for $X_{n,n}$

Theorem 3.3.2 Let $X_1, X_2, \ldots$ be a sequence of i.i.d random variables with distribution function F and e(F) = sup{x: F(x) < 1}. If β(F) = ∞, then $F \in T_{2\delta}$ if $\lim_{t\to\infty} \frac{\bar{F}(tx)}{\bar{F}(t)} = x^{-\delta}$ for x > 0 and some constant δ > 0.

Proof Let $a_n = \inf\left\{x: \bar{F}(x) \le \frac{1}{n}\right\}$, then $a_n \to \infty$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} n(\bar{F}(a_n x)) = \lim_{n\to\infty} n(\bar{F}(a_n)) \frac{\bar{F}(a_n x)}{\bar{F}(a_n)} = x^{-\delta} \lim_{n\to\infty} n\bar{F}(a_n).$$

It is easy to show that $\lim_{n\to\infty} n\bar{F}(a_n) = 1$. Thus $\lim_{n\to\infty} n(\bar{F}(a_n x)) = x^{-\delta}$ and the proof of the Theorem follows from Lemma 2.1.1.

Example 3.3.3 For the Pareto distribution with $\bar{F}(x) = \frac{1}{x^\delta}, \delta > 0, 0 < x < \infty$ $\lim_{t\to\infty} \frac{\bar{F}(tx)}{\bar{F}(t)} = \frac{1}{x^\delta}$. Thus the Pareto distribution belongs to $T_{2\delta}$.

The following Theorem gives a necessary and sufficient condition for the domain of attraction of Type 2 distribution for $X_{n,n}$ when e(F) < ∞.

Theorem 3.3.3 Let $X_1, X_2, \ldots$ be a sequence of i.i.d random variables with distribution function F and β(F) = sup{x: F(x) < 1}. If β(F) < ∞, then $F \in T_{2\delta}$ if $\lim_{t\to\infty} \frac{\bar{F}(e(F) - \frac{1}{tx})}{\bar{F}(e(F) - \frac{1}{t})} = x^{-\delta}$ for x > 0 and some constant δ > 0.

Proof Similar to Theorem 3.3.2.

Example 3.3.4 The truncated Pareto distribution $f(x) = \frac{\delta}{x^{\delta+1}} \cdot \frac{1}{1-b^\delta}$, $1 \le x < b$, $b > 1$, $\lim_{t\to\infty} \frac{\bar{F}(e(F) - \frac{1}{tx})}{\bar{F}(e(F) - \frac{1}{t})} = \lim_{t\to\infty} \frac{\bar{F}(b - \frac{1}{tx})}{\bar{F}(b - \frac{1}{t})} = \lim_{t\to\infty} \frac{\left(b - \frac{1}{tx}\right)^{-\delta} - b^{-\delta}}{\left(b - \frac{1}{t}\right)^{-\delta} - b^{-\delta}} = x^{-1}$. Thus the truncated Pareto distribution belongs to the domain of attraction of Type T_{21} distribution.

The following Lemma (see Von Mises 1936) gives a sufficient condition for the domain of attraction of Type 2 extreme value distribution for $X_{n,n}$.

Lemma 3.3.3 Suppose the distribution function F is absolutely continuous in $[c_0$, e(F)] for some c_0, $0 < c_0 < e(F)$, then if $\lim_{x \uparrow e(F)} \frac{xf(x)}{\bar{F}(x)} = \delta,\ \delta > 0$, then $F \in D(T_{2\delta})$.

Proof Let $q(x) = \frac{xf(x)}{\overline{F}(x)}$, then $q(x) = -x\frac{d}{dx}\left(\ln \overline{F}(x)\right)$. Thus $\overline{F}(x) = ke^{-\int_{c\alpha\alpha}^{x}\frac{q(u)}{u}du}$, where k is a positive constant and $c_0 < \alpha < e(F)$.

Now $\lim_{t\to\infty}\frac{\overline{F}(tx)}{\overline{F}(t)} = \lim_{t\to\infty} e^{-\int^{ttx}\frac{q(u)}{u}du} = \lim_{t\to\infty} e^{-\int_1^{xx}\frac{q(tu)}{u}du} = e^{-\delta\ln x} = x^{-\delta}$

Example 3.3.5 The truncated Pareto distribution $f(x) = \frac{\delta}{x^{\delta+1}}\cdot\frac{1}{1-b^{-\delta}}$, $1 \le x < b$, $b > 1$, $\lim_{x\to\infty}\frac{xf(x)}{\overline{F}(x)} = \lim_{x\to b}\frac{\delta x^{-\delta}}{x^{-\delta}-b^{-\delta}} = \infty$. Thus the truncated Pareto distribution does not satisfy the Von Mises sufficient condition. However it belongs to the domain of attraction of the type 2 extreme value distribution, because $\lim_{t\to\infty}\frac{\overline{F}(e(F)-\frac{1}{tx})}{\overline{F}(e(F)-\frac{1}{t})} = x^{-\delta}$ for $x > 0$ and some constant δ>.

3.3.3 Domain of Attraction of Type 3 Extreme Value Distribution for $X_{n,n}$

The following theorem gives a necessary and sufficient condition for the domain of attraction of type 3 distribution for $X_{n,n}$.

Theorem 3.3.4 Let $X_1, X_2, \ldots$ be a sequence of i.i.d random variables with distribution function F and $e(F) = \sup\{x\colon F(x) < 1\}$. If $e(F) < \infty$, then $F \in T_{3\delta}$ if $\lim_{t\to 0^+}\frac{\overline{F}(e(F)+tx)}{\overline{F}(e(F)-t)} = (-x)^{\delta}$ for $x < 0$ and some constant $\delta > 0$.

Proof Similar to Theorem 3.3.3.

Suppose X is a negative exponential distribution truncated at $x = b > 0$. The pdf of X is $f(x) = \frac{e^{-x}}{F(b)}$, then for $x < 0$, $P(X_{nn} \le b + \frac{x(e^{b}-1)}{n}) = \left(\frac{1-e^{-(b+\frac{x(e^{b}-1)}{n})}}{1-e^{-b}}\right)^n \to e^x$ as $n \to \infty$.

Thus the truncated exponential distribution belongs to T_{31}.

Since $\lim_{t\to 0^+}\frac{\overline{F}(e(F)+tx)}{\overline{F}(e(F)-t)} = \lim_{t\to 0^+}\frac{e^{-(b+tx)}-e^{-b}}{e^{-(b-t)}-e^{-b}} = -x$, the truncated exponential distribution satisfies the necessary and sufficient condition for the domain of attraction of type 3 distribution for maximum.

The following Lemma gives Von Mises sufficient condition for the domain of attraction of type 3 distribution for $X_{n,n}$.

Lemma 3.3.4 Suppose the distribution function F is absolutely continuous in $[c_0, e(F)]$ for some c_0, $0 < c_0 < e(F) < \infty$, then if $\lim_{x\uparrow e(F)}\frac{(e(F)-x)f(x)}{\overline{F}(x)} = \delta, \delta > 0.$, then $F \in D(T_{3\delta})$.

Proof Similar to Lemma 2.1.3.

Example 3.3.6 Suppose X is a negative exponential distribution truncated at x = b > 0, then the pdf of X is $f(x) = \frac{e^{-x}}{F(b)}$. Now $\lim_{x \uparrow e(F)} \frac{(e(F)-x)f(x)}{\bar{F}(x)} = \lim_{x \uparrow b} \frac{(b-x)e^{-x}}{e^{-x}-e^{-b}} = 1$. Thus the truncated exponential distribution satisfies the Von Mises sufficient condition for the domain of attraction to type 3 distribution.

A distribution that belongs to the domain of attraction of Type 2 distribution cannot have finite e(F). A distribution that belongs to the domain of attraction of Type 3 distribution must have finite e(F). The normalizing constants of $X_{n,n}$ are not unique for any distribution. From the table it is evident that two different distributions (exponential and logistic) belong to he domain of attraction of the same distribution and have the same normalizing constants. The normalizing constants depends on F and the limiting distribution. It may happen that $X_{n,n}$ with any normalizing constants may not converge in distribution to a non degenerate limiting distribution but W_{nn} where W = u(X), a function of X, may with some normalizing constants may converge in distribution to one of the three limiting distribution. We can easily verify that the rv X whose pdf, $f(x) = \frac{1}{x(\ln x)^2}, x \geq e$ does not satisfy the necessary and sufficient conditions for the convergence in distribution of $X_{n,n}$ to any of the extreme value distributions. Suppose $W = \ln X$, then $F_W(x) = 1 - 1/x$ for y > 1. Thus with as $a_n = 0$ and $b_n = 1/n$, $P(W_{n,n} \leq x) \to T_{31}$ as $n \to \infty$.

Following Pickands (1975), the following theorem gives a necessary and sufficient condition for the domain of attraction of $X_{n,n}$ from a continuous distribution.

Theorem 3.3.5 For a continuous random variable the necessary and sufficient condition for $X_{n,n}$ to belong to the domain of attraction of the extreme value distribution of the maximum is

$$\lim_{c \to 0} \frac{F^{-1}(1-c) - F^{-1}(1-2c)}{F^{-1}(1-2c) - F^{-1}(1-4c)} = 1 \quad \text{if } F \in T_{10},$$

$$\lim_{c \to 0} \frac{F^{-1}(1-c) - F^{-1}(1-2c)}{F^{-1}(1-2c) - F^{-1}(1-4c)} = 2^{1/\delta} \quad \text{if } F \in T_{2\delta}$$

and

$$\lim_{c \to 0} \frac{F^{-1}(1-c) - F^{-1}(1-2c)}{F^{-1}(1-2c) - F^{-1}(1-4c)} = 2^{-1/\delta} \quad \text{if } F \in T_{3\delta}$$

Example 3.3.7 For the exponential distribution, E(0, σ), with pdf $f(x) = \sigma^{-1} e^{-\sigma^{-1} x}, x > 0$, $F^{-1}(x) = -\sigma^{-1} \ln(1-x)$ and $\lim_{c \to 0} \frac{F^{-1}(1-c) - F^{-1}(1-2c)}{F^{-1}(1-2c) - F^{-1}(1-4c)} = \lim_{c \to 0} \frac{-\ln\{1-(1-c)\} + \ln\{1-(1-2c)\}}{-\ln\{1-(1-2c)\} + \ln\{1-(1-4c)\}} = 1$. Thus the domain of attraction of X_{nn} from the exponential distribution, E(0, σ), is T_{10}.

For the Pareto distribution, P(0, 0, α) with pdf $f(x) = \alpha x^{-(\alpha+1)}, x > 1, \alpha > 0$, $F^{-1}(x) = (1-x)^{-1/\alpha}$ and $\lim_{c\to 0} \frac{F^{-1}(1-c)-F^{-1}(1-2c)}{F^{-1}(1-2c)-F^{-1}(1-4c)} = \lim_{c\to 0} \frac{c^{-1/\alpha}-(2c)^{-1/\alpha}}{(2c)^{-1/\alpha}-(4c)^{=1/\alpha}} = 2^{1/\alpha}$. Hence the domain of attraction of X_{nn} from the Pareto distribution, P(0, 0, α) is $T_{2\alpha}$.

For the uniform distribution, U(−1/2, 1/2), with pdf $f(x) = \frac{1}{2}$, $-\frac{1}{2} < x < \frac{1}{2}$, $F^{-1}(x) = 2x - 1$. We have $\lim_{c\to 0} \frac{F^{-1}(1-c)-F^{-1}(1-2c)}{F^{-1}(1-2c)-F^{-1}(1-4c)} = \lim_{c\to 0} \frac{2(1-c)-1-2(1-2c)+1}{2(1-2c)-1-2(1-4c)+1} = 2^{-1}$. Consequently the domain of attraction of X_{nn} from the uniform distribution, U (−1/2, 1/2) is T_{31}.

It may happen that X_{nn} from a continuous distribution does not belong to the domain of attraction of any one of the distribution. In that case $X_{n,n}$ has a degenerate limiting distribution. Suppose the rv X has the pdf f(x), where $f(x) = \frac{1}{x(\ln x)^2}$, $x \geq e$. $F^{-1}(x) = e^{\frac{1}{1-x}}, 0 < x < 1$.

Then

$$\lim_{c\to 0} \frac{F^{-1}(1-c) - F^{-1}(1-2c)}{F^{-1}(1-2c) - F^{-1}(1-4c)} = \lim_{c\to 0} \frac{e^{\frac{1}{c}} - e^{\frac{1}{2c}}}{e^{\frac{1}{2c}} - e^{\frac{1}{4c}}} = \lim_{c\to 0} \frac{e^{\frac{1}{c}} - 1}{1 - e^{\frac{1}{2c}}}$$
$$= \lim_{c\to 0} \frac{2e^{\frac{1}{c}}}{e^{\frac{1}{2c}}} = \lim_{c\to 0} 2e^{\frac{1}{2c}} = \infty$$

Thus the limit does not exit. Hence the rv X does not satisfy the necessary and sufficient condition given in Theorem 2.1.4.

Theorems 3.3.1–3.3.5 are also true for discrete distributions. If $X_{n,n}$ is from discrete random variable with finite number of points of support, then $X_{n,n}$ can not converge to one of the extreme value distributions. Thus $X_{n,n}$ from binomial and discrete uniform distribution will converge to degenerate distributions. The following Lemma (Galambos 1987, p. 85) is useful to determine whether $X_{n,n}$ from a discrete distribution will have degenerate distribution.

Lemma 3.3.5 Suppose X is a discrete random variable with infinite number points in its support and taking values on non negative integers with $P(X = k) = p_k$. Then a necessary condition for the convergence of $P(X_{nn} \leq a_n + b_n x)$ for a suitable sequence of a_n and b_n to one of the three extreme value distributions is $\lim_{k\to\infty} \frac{p_k}{P(X \geq k)} = 0$.

For the geometric distribution, $P(X = k) = p(1 - p)^{k-1}$, $k \geq 1$, $0 < p < 1$, $\frac{p_k}{P(X \geq k)} = p$. Thus X_{nn} from the geometric distribution will have degenerate distribution as limiting distribution of X_{nn}.

Consider the distribution: $P(X = k) = \frac{1}{k(k+1)}$, k = 1, 2, …, then $P(X > k) = \frac{1}{k}$ and $\lim_{k\to\infty} \frac{p_k}{P(X \geq k)} = \lim_{k\to\infty} \frac{1}{k+1} = 0$. But $\lim_{t\to\infty} \frac{\bar{F}(tx)}{\bar{F}(t)} = x^{-1}$. Thus X belongs to the domain of attraction of T_{21}. The normalizing constants are $a_n = 0$ and $b_n = n$.

However the condition $\lim_{k\to\infty} \frac{p_k}{P(X \geq k)} = 0$ is necessary but not sufficient.

Consider the discrete probability distribution whose $P(X = k) = \frac{c}{k(\ln(k+1))^6}$, k = 1, 2, … where $1/c = \sum_{k=1}^{\infty} \frac{1}{k(\ln(k+1))^6} \cong 9.3781$.

Since $1 - \sum_{k=1}^{n} \frac{1}{k(\ln(k+1))^6} \propto \frac{1}{(\ln n)^5}, \frac{P(X=n)}{1-\sum_{k=1}^{n-1} P(X=k)} \to 0$ as $n \to \infty$. But this probability distribution does not satisfy the necessary and sufficient conditions for the convergence of $X_{n,n}$ the extreme value distributions.

We can use the following lemma to calculate the normalizing constants for various distributions belonging to the domain of attractions of T(x).

Lemma 3.3.6 Suppose $P(X_{n,n} < a_n + b_n x) \to T(x)$ as $n \to \infty$, then

(i) $a_n = F^{-1}(1 - \frac{1}{n}), b_n = F^{-1}(1 - \frac{1}{ne}) - F^{-1}(1 - \frac{1}{n})$ if $T(x) = T_{10}(x)$,
(ii) $a_n = 0, b_n = F^{-1}(1 - \frac{1}{n})$ if $T(x) = T_{2\delta}(x)$,
(iii) $a_n = F^{-1}(1), b_n = F^{-1}(1) - F^{-1}(1 - \frac{1}{n})$ if $T(x) = T_{3\delta}(x)$.

We have seen that the normalizing constants are not unique. However we can use the following Lemma to select simpler normalizing constants.

Lemma 3.3.7 Suppose a_n and b_n is a sequence of normalizing constants for X_{nn} for the convergence to the domain of attraction of any one of the extreme value distributions. If a_n^* and b_n^* is another sequence such that $\lim_{n\to\infty} \frac{a_n - b_n^*}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n^*}{b_n} = 0$, then a_n^* and b_n^* can be substituted for as the normalizing constants a_n and b_n for $X_{n,n}$.

Example 3.3.8 We have seen that for the Cauchy distribution with pdf $f(x) = \frac{1}{\pi(1+x^2)}, -\infty < x < \infty$ the normalizing constants as $a_n = 0$ and $b_n = \cot(\pi/n)$. However we can take $a_n^* = 0$ and $b_n^* = \frac{n}{\pi}$.

Table 3.2 gives the normalizing constants for some well known distributions belonging to the domain of attraction of the extreme value distributions.

Pdfs of Extreme Value distributions for $X_{1,n}$.

Let us consider $X_{1,n}$ of n i.i.d random variables. Suppose $P(X_{1n} \le \underline{c_n} + d_n x) \to H(x)$ as $n \to \infty$, then the following three types of distributions are possible for H(x).

Type 1 distribution $H_{10}(x) = 1 - e^{-e^x}, -\infty < x < \infty$.

Type 2 distribution $H_{2\delta}(x) = 1 - e^{-(-x)^{-\delta}}, x < 0, \delta > 0$.

Type 3 distribution $H_{3\delta}(x) = 1 - e^{-x^{\delta}}, x > 0, \delta > 0$.

It may happen that $X_{n,n}$ and X_{1n} may belong to different types of extreme value distributions. For example consider the exponential distribution, $f(x) = e^{-x}$, x > 0. The X_{nn} belongs to the domain of attraction of the type 1 distribution of the maximum, T_{10}. Since $P(X_{1n} > n^{-1}x) = e^{-x}$, X_{1n} belongs to the domain of attraction of Type 2 distribution of the minimum, H_{21}. It may happen that $X_{n,n}$ does not belong to any one of the three limiting distributions of the maximum but X_{1n} belong to the domain of attraction of one of the limiting distribution of the minimum. Consider the rv X whose pdf, $f(x) = \frac{1}{x(\ln x)^2}, x \ge e$. We have seen that F does not

Table 3.2 Normalizing constants for $X_{n,n}$

Distribution	f(x)	α_n	β_n	Domain
Beta	$c\,x^{\alpha-1}(1-x)^{\beta-1}$ $c=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ $\alpha>0, \beta>0$ $0<x<1$	1	$\left(\frac{\beta}{nc}\right)^{1/\beta}$	$T_{3\beta}$
Cauchy	$\frac{1}{\pi(1+x^2)}, -\infty<x<\infty$	0	$\cot(\frac{\pi}{n})$	T_{21}
Discrete Pareto	$P(X=k)=[k]^{\theta}-[k+1]^{\theta}$, $\theta>0$, $k\geq 1$, [] represents the greatest integer contained in.	0	$n^{1/\theta}$	$T_{2\theta}$
Exponential	$\sigma e^{-\sigma x}$, $0<x<\infty$, $\sigma>0$	$\frac{1}{\sigma}\ln n$	$\frac{1}{\sigma}$	T_{10}
Gamma	$\frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}, 0<x<\infty$	$\ln n+\ln\Gamma(\alpha)-(\alpha-1)\ln\ln n$	1	T_{10}
Laplace	$\frac{1}{2}e^{-\lvert x\rvert}, -\infty<x<\infty$	$\ln(\frac{n}{2})$	1	T_{10}
Logistic	$\frac{e^{-x}}{(1+e^{-x})^2}$	$\ln n$	1	T_{10}
Lognormal	$\frac{1}{x\sqrt{2\pi}}e^{-\frac{1}{2}(\ln x)^2}$ $0<x<\infty$	e^{α_n}, $\alpha_n=\frac{1}{\beta_n}-\frac{\beta_n D_n}{2}$, $D_n=\ln\ln n+\ln 4\pi$ $\beta_n=(2\ln n)^{-1/2}$	$(2\ln n)^{-1/2}e^{\alpha_n}$	T_{10}
Normal	$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}, -\infty<x<\infty$	$\frac{1}{\beta_n}-\frac{\beta_n D_n}{2}$, $D_n=\ln\ln n+\ln 4\pi$ $\beta_n=(2\ln n)^{-1/2}$	$(2\ln n)^{-1/2}$	T_{10}
Pareto	$\alpha x^{-(\alpha+1)}\ x>1, \alpha>0$	0	$n^{1/\alpha}$	$T_{2\alpha}$
Power function	$\alpha x^{\alpha-1}, 0<x<1, \alpha>0$	1	$\frac{1}{n\alpha}$	T_{31}
Rayleigh	$\frac{2x}{\sigma^2}e^{-\frac{x^2}{\sigma^2}}, x>0$	$\sigma(\ln n)^{\frac{1}{2}}$	$\frac{\sigma}{2}(\ln n)^{-\frac{1}{2}}$	T_{10}
t distribution	$\frac{k}{\left(1+\frac{x^2}{\upsilon}\right)^{(\upsilon+1)/2}}$	0	$\left(\frac{kn}{\upsilon}\right)^{1/\upsilon}$	$T_{2\upsilon}$

(continued)

Table 3.2 (continued)

Distribution	f(x)	α_n	β_n	Domain
	$k = \frac{\Gamma((\upsilon+1)/2)}{(\pi\upsilon)^{1/2}\Gamma(\upsilon/2)}$			
Truncated exponential	$Ce^{-x}, C = 1/(1 - e^{-e(F)})0 < x < e(F) < \infty$	E(F)	$\frac{e^{e(F)}-1}{n}$	T_{31}
Type 1	$e^{-x}e^{-e^{-x}}$	ln n	1	T_{10}
Type 2	$\alpha x^{-(\alpha+1)}e^{-x^{-\alpha}}\ x > 0, \alpha > 0$	0	$n^{1/\alpha}$	$T_{2\alpha}$
Type 3	$\alpha(-x)^{\alpha-1}.e^{-(-x)^{\alpha}},\ x < 0, \alpha > 0$	0	$n^{-1/\alpha}$	$T_{3\alpha}$
Uniform	$1/\theta,\ 0 < x < \theta$	θ	θ/n	T_{31}
Weibull	$\alpha x^{\alpha-1}e^{-x^{\alpha}},\ x > 0, \alpha > 0$	$(\ln n)^{1/\alpha}$	$\frac{(\ln n)^{\frac{1-\alpha}{\alpha}}}{\alpha}$	$T_{1\alpha}$

satisfy the necessary and sufficient conditions for the convergence in distribution of $X_{n,n}$ to any of the extreme value distributions. However it can be shown that $P(X_{1,n} > \alpha_n + \beta_n x) \to e^{-x}$ as $n \to \infty$ for $\alpha_n = e$ and $b_n = e^{-\frac{n-1}{n}} - e$. Thus the X_{1n} belongs to the domain of attraction of H_{21}.

If X is a symmetric random variable and $X_{n,n}$ belongs to the domain of attraction of $T_i(x)$, then X_{1n} will belong to the domain of attraction of the corresponding $H_i(x)$, i = 1, 2, 3.

3.4 Domain of Attraction for $X_{1,n}$

The following Lemma is needed to prove the necessary and sufficient conditions for the convergence of X_{1n} to one of the limiting distributions H(x).

Lemma 3.4.1 Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with distribution function F. Consider a sequence $(e_n, n \geq 1\}$ of real numbers. Then for any ξ, $0 \leq \xi < \infty$, the following two statements are equivalent

(iii) $\lim_{n\to\infty} n(F(e_n)) = \xi$

(iv) $\lim_{n\to\infty} P(X_{n,n} > e_n) = e^{-\xi}$.

Proof The proof of the Lemma follows from Lemma 2.1.1 by considering the fact $P(X_{1n} > e_n) = (1 - F(e_n))^n$.

3.4.1 Domain of Attraction for Type 1 Extreme Value Distribution for $X_{1,n}$

The following theorem gives a necessary and sufficient condition for the convergence of X_{1n} to $H_{10}(x)$.

Theorem 3.4.1 Let $X_1, X_2, \ldots$ be a sequence of i.i.d random variables with distribution function F. Assume further that $E(X|X \leq t)$ is finite for some $t > \alpha(F)$ and $h(t) = E(t - X|X \leq t)$. Then $F \in H_{10}$ if $\lim_{t\to\alpha(F)} \frac{F(t+xh(t))}{F(t)} = e^x$ for all real x.

Proof Similar to Theorem 3.3.1.

Example 3.4.1 Suppose the logistic distribution with $F(x) = \frac{1}{1+e^{-x}}$, $-\infty < x < \infty$. Now $h(t) = E(t - x|X \leq t) = t - (1+e^{-t}) \int_{-\aleph}^{t} \frac{xe^{-x}}{(1+e^{-x})^2} dx = (1+e^{-t}) \ln(1+e^t)$.

It can easily be shown that h(t) → 1 as t → −∞. We have $\lim_{t\to\alpha(F)} \frac{F(t+xh(t))}{F(t)} = \lim_{t\to-\infty} \frac{1+e^{-t}}{1+e^{-(t+xh(t))}} = \lim_{t\to-\infty} \frac{e^{t+xh(t)}+e^{xh(t)}}{1+e^{t+xh(t)}} = e^x$. Thus X_{1n} from logistic distribution belongs to the domain of H_{10}.

3.4.2 Domain of Attraction of Type 2 Distribution for $X_{1,n}$

Theorem 3.4.2 Let X_1, X_2, … be a sequence of i.i.d random variables with distribution function F then $F \in H_{2\delta}$ if $\alpha(F) = -\infty$ and $\lim_{t\to\alpha(F)} \frac{F(tx)}{F(t)} = x^{\delta}$ for all x > 0.

Proof Suppose $H_{2\delta}(x) = 1 - e^{-(-x)^{-\delta}}$, $x<0, \delta > 0$, then we have

$$\lim_{t\to\alpha(F)} \frac{F(tx)}{F(t)} = \lim_{t\to-\infty} \frac{1 - e^{-(-tx)^{-\delta}}}{1 - e^{-(-t)^{=\delta}}} = \lim_{t\to-\infty} \frac{\delta x(-tx)^{-(\delta+1)} e^{-(-tx)^{-\delta}}}{\delta(-t)^{-(\delta+1)} e^{-(-t)^{=\delta}}} = x^{-\delta}, \ \delta > 0.$$

Let $\lim_{t\to\alpha(F)} \frac{F(tx)}{F(t)} = x^{-\delta}$, $\delta > 0$. We can write Let $a_n = \inf\{x: \overline{F}(x) \le \frac{1}{n}\}$, then $a_n \to -\infty$ as $n \to \infty$.

Thus $\lim_{n\to-\infty} n(F(a_n x)) = \lim_{n\to-\infty} n(F(a_n)) \frac{F(a_n x)}{F(a_n)} = x^{-\delta} \lim_{n\to-\infty} nF(a_n)$.

It is easy to show that $\lim_{n\to-\infty} nF(a_n) = 1$. Thus $\lim_{n\to-\infty} n(F(a_n x)) = x^{-\delta}$ and the proof of the follows.

Example 3.4.2 For the Cauchy distribution $F(x) = \frac{1}{2} + \tan^{-1}(x)$. Thus

$$\lim_{t\to\alpha(F)} \frac{F(tx)}{F(t)} = \lim_{t\to-\infty} \frac{\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(tx)}{\frac{1}{2} + -\frac{1}{\pi}\tan^{-1}(i)} = \lim_{t\to-\infty} \frac{x(1+t^2)}{1+(tx)^2} = x^{-1}.$$

Thus F belongs to the domain of attraction of H_{21}.

3.4.3 Domain of Attraction of Type 3 Extreme Value Distribution

Theorem 3.4.3 Let X_1, X_2, … be a sequence of i.i.d random variables with distribution function F then $F \in H_{3\delta}$ if $\alpha(F)$ is finite and $\lim_{t\to 0} \frac{F(\alpha(F)+tx)}{F(\alpha(F)+t)} = x^{\delta}, \delta > 0$ and for all x > 0.

Proof The proof is similar to Theorem 3.4.2.

Example 3.4.3 Suppose X has the uniform distribution with F(x) = x, 0 < x < 1. Then $\lim_{t\Rightarrow 0}\frac{F(tx)}{F(t)} = x$. Thus then F $\in$ H_{31}.

Following Pickands (1975), the following theorem gives a necessary and sufficient condition for the domain of attraction of X_{1n} from a continuous distribution.

Theorem 3.4.4 For a continuous random variable the necessary and sufficient condition for X_{1n} to belong to the domain of attraction of the extreme value distribution of the minimums

$$\lim_{c\to 0}\frac{F^{-1}(c)-F^{-1}(2c)}{F^{-1}(2c)-F^{-1}(4c)} = 1 \quad \text{if } F \in H_{10},$$

$$\lim_{c\to 0}\frac{F^{-1}(c)-F^{-1}(2c)}{F^{-1}(2c)-F^{-1}(4c)} = 2^{1/\delta} \quad \text{if } F \in H_{2\delta}$$

and

$$\lim_{c\to 0}\frac{F^{-1}(c)-F^{-1}(2c)}{F^{-1}(2c)-F^{-1}(4c)} = 2^{-1/\delta} \quad \text{if } F \in H_{3\delta}$$

Example 3.4.4 For the logistic distribution with F(x) $=\frac{1}{1+e^{-x}}$, $F^{-1}(x) = \ln x - \ln(1-x)$ $\lim_{c\to 0}\frac{F^{-1}(c)-F^{-1}(2c)}{F^{-1}(2c)-F^{-1}(4c)} = \lim_{c\to 0}\frac{\ln c-\ln(1-c)-\ln 2c+\ln(1-2c)}{\ln 2c-\ln(1-2c)-\ln 4c+\ln(1-4c)} = 1$. Thus the domain of attraction of X_{1n} from the logistic distribution is T_{10}.

For the Cauchy distribution with $F(x) = \frac{1}{2} + \tan^{-1}(x)$. We have $F^{-1}(x) = \tan\pi(x-\frac{1}{2}) = -\frac{1}{\pi x}$ = for small x. Thus $\lim_{c\to 0}\frac{F^{-1}(c)-F^{-1}(2c)}{F^{-1}(2c)-F^{-1}(4c)} = \frac{\frac{1}{2\pi c}-\frac{1}{\pi c}}{\frac{1}{4\pi c}-\frac{1}{2\pi c}} = 2$. Thus the domain of attraction of X_{1n} from the Cauchy distribution is T_{21}.

For the exponential distribution, E(0, σ), with pdf $f(x) = \sigma^{-1}e^{-\sigma^{-1}x}$, x > 0, $F^{-1}(x) = -\sigma^{-1}\ln(1-x)$ and $\lim_{c\to 0}\frac{F^{-1}(c)-F^{-1}(2c)}{F^{-1}(2c)-F^{-1}(4c)} = \lim_{c\to 0}\frac{-\ln\{1-c)+\ln\{1-2c)}{-\ln\{1-2c)+\ln\{1-4c)} = 2^{-1}$. Thus the domain of attraction of X_{1n} from the exponential distribution, E(0, σ), is T_{31}.

We can use the following lemma to calculate the normalizing constants for various distributions belonging to the domain of attractions of H(x).

Lemma 3.4.2 Suppose $P(X_{1n} < c_n + d_n x) \to H(x)$ as n → ∞, then

(i) $c_n = F^{-1}(\frac{1}{n}), d_n = F^{-1}(\frac{1}{n}) - F^{-1}(\frac{1}{ne})$ if $H(x) = H_{10}(x)$,
(ii) $c_n = 0, b_n = |F^{-1}(\frac{1}{n})|$ if $H(x) = H_{2\delta}(x)$,
(iii) $c_n = \alpha(F), b_n = F^{-1}(\frac{1}{n}) - \alpha(F)$ if $H(x) = H_{3\delta}(x)$.

We have seen (Lemma 2.1.6) that the normalizing constants are not unique for $X_{n,n}$. The same is also true for the X_{1n}.

Table 3.3 Normalizing Constants for $X_{1,n}$

Distribution	f(x)	C_n	D_n	Domain
Beta	$c\,x^{\alpha-1}(1-x)^{\beta-1} c=\frac{1}{B(\alpha,\beta)}\ \alpha>0,\beta>0$ $0<x<1$	0	$\left(\frac{c\alpha}{n}\right)^{1/\alpha}$ $c=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$	$H_{3\alpha}$
Cauchy	$\frac{1}{\pi(1+x^2)},\ -\infty<x<\infty$	0	$\cot\left(\frac{\pi}{n}\right)$	H_{21}
Exponential	$\sigma e^{-\sigma x},\ 0<x<\infty,\ \sigma>0$	0	$\frac{1}{n\sigma}$	H_{31}
Gamma	$\frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)},\ 0<x<\infty$	0	$\frac{\Gamma(\alpha)}{n}$	H_{31}
Laplace	$\frac{1}{2}e^{-\lvert x\rvert},\ -\infty<x<\infty$	$\ln\left(\frac{n}{2}\right)$	1	H_{10}
Logistic	$\frac{e^{-x}}{(1+e^{-x})^2}$	$-\ln n$	1	H_{10}
Lognormal	$\frac{1}{x\sqrt{2\pi}}e^{-\frac{1}{2}(\ln x)^2}$ $0<x<\infty$	$e^{n\alpha_n},\ \alpha_n=\frac{1}{b_n}-\frac{b_nD_n}{2},$ $D_n=\ln\ln n+\ln 4\pi$ $b_n=(2\ln n)^{-1/2}$	$(2\ln n)^{-1/2}e^{\alpha_n}$	H_{10}
Normal	$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2},\ -\infty<x<\infty$	$\frac{1}{c_n}-\frac{c_nD_n}{2},$ $D_n=\ln\ln n+\ln 4\pi$ $b_n=(2\ln n)^{-1/2}$	$(2\ln n)^{-1/2}$	H_{10}
Pareto	$\alpha x^{-(\alpha+1)}\ x>1,\alpha>0$	0	$\left(\frac{n}{n-1}\right)^{1/\alpha}$	H_{21}
Power function	$\alpha x^{\alpha-1},\ 0<x<1,\ \alpha>0$	0	$\frac{1}{n^{1/\alpha}}$	H_{31}
Rayleigh	$\frac{2x}{\sigma^2}e^{-\frac{x^2}{\sigma^2}},\ x>0$	0	$\sigma\sqrt{\frac{2}{n}}$	H_{32}
T distribution	$\frac{k}{\left(1+\frac{x^2}{\upsilon}\right)^{(\upsilon+1)/2}}$ $k=\frac{\Gamma((\upsilon+1)/2)}{(\pi\upsilon)^{1/2}\Gamma(\upsilon/2)}$	0	$\left(\frac{kn}{\upsilon}\right)^{1/\upsilon}$	$H_{2\upsilon}$
Type 1 (for minimum)	$e^{x}e^{-e^{x}}$	$-\ln n$	1	H_{10}
Type 2 (for minimum)	$\alpha(-x)^{-(\alpha+1)}e^{-x^{-\alpha}}\ x<0,\alpha>0$	0	$n^{1/\alpha}$	$H_{2\alpha}$

(continued)

Table 3.3 (continued)

Distribution	f(x)	C_n	D_n	Domain
Type 3 (for minimum)	$\alpha x^{\alpha-1}.e^{x^{\alpha}}\ x > 0, \alpha > 0$	0	$n^{-1/\alpha}$	$H_{3\alpha}$
Uniform	$1/\theta,\ 0 < x < \theta$	0	θ/n	H_{31}
Weibul	$\alpha x^{\alpha-1}e^{-x^{\alpha}},\ x > 0, \alpha > 0$	0	$\frac{1}{n^{1/\alpha}}$	$H_{3\alpha}$

Example 3.4.3 For the logistic distribution with $F(x) = \frac{1}{1+e^{-x}}, X_{1,n}$ when normalized converge in distribution to Type 1 (H_{10}) distribution. The normalizing constants are $c_n = F^{-1}(\frac{1}{n}) = \ln\left(\frac{1/n}{1-(1/n)}\right) \cong -\ln n$ and $d_n = F^{-1}(\frac{1}{n}) - F^{-1}(\frac{1}{ne}) = 1.$

For Cauchy distribution with $F(x)\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x), X_{1n}$ when normalized converge in distribution to Type 2 (H_{21}) distribution. The normalizing constants are $c_n = 0$ and $d_n = |F^{-1}(\frac{1}{n})| = \tan\pi(\frac{1}{2} - \frac{1}{n}), n > 2$ (Table 3.3).

For the uniform distribution with F(x) = x, X_{1n} when normalized converge in distribution to Type 3 (H_{31}) distribution. The normalizing constants are $c_n = 0, b_n = F^{-1}(\frac{1}{n}) = \frac{1}{n}$.

Chapter 4
Inferences of Extreme Value Distributions

In this chapter some inferences of extreme value distributions will be given. Some characterizations of the extreme value distributions are presented.

4.1 Type 1 Extreme Value (Gumbel) Distribution

Estimation of Parameters.

We will consider the following cdf f(x) of the type I extreme value distribution.

$$F(x) - \exp(-\exp(\frac{x-\mu}{\sigma})), \quad \sigma > 0,$$
$$-\infty < x - \mu < \infty, \quad \mu \text{ is any real number.}$$

The Type 1 (Gumbel distribution) is the limiting distribution of $X_{n,n}$ when F(x) is normal, log normal, logistic, gamma etc.

Estimation of parameters of Type I extreme values based on lower record values are in closed form. We will consider the estimation of the parameters based on the lower record values.

For a given set of n observations, let $X_{1,n} < \ldots. < X_{n,n}$ be the associated order statistics. Suppose that $P\{a_n (X_{n,n} - b_n) < x\} \to G(x)$ as $n \to \infty$ for some suitable constants a_n and b_n. Then it is known (see Leadbetter et al. 1983, p. 33) that

$$P\{a_n(X_{n-m,n} - b_n) \le x\} \xrightarrow{d} G(x)\sum_{s=0}^{m-1}\frac{[-\ln G(x)]^s}{s!} \qquad (4.1.1)$$

We have already seen that the right hand side of the above expression is the cdf of the mth lower record value from the distribution function G(x).

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Thus the limiting distribution of the (n − m + 1)th order statistic (m finite) as $n \to \infty$ from the generalized extreme value distribution is the same as the mth lower record value from the generalized extreme value distribution. In this chapter we will study the lower record values of GEV (μ, σ, γ).

The pdf F(x) satisfies the following condition.

It is known (see Ahsanullah 1995) that if $X_{L(m)}$ is mth lower record from the cdf as given in (4.1.1), then

$$X_{L(m)} \underline{\underline{d}} X - \sigma\left(W_1 + \frac{W_2}{2} + \cdots + \frac{W_{m-1}}{m-1}\right), \quad \gamma = 0, \tag{4.1.2}$$

where X is the cdf as given in (4.1.2) and W_1, W_2, W_m are independent and identically distributed with cdf F(x) = 1 − exp(x), x ≥ 0.

Thus

$$\begin{aligned} &\mathrm{Var}\left(X_{L(m)}\right) = \sigma^2 V^*_{r,r}, \quad r = 1, 2, \ldots \\ &\mathrm{Cov}\left(X_{L(r)}, X_{L(m)}\right) = \mathrm{Var}\left(X_{L(m)}\right), \quad r < m, \end{aligned}$$

with

$$\begin{aligned} &\upsilon^*_1 = \upsilon \\ &\upsilon^*_j = \upsilon^*_{j-1} - (j-1)^{-1}, \quad j \geq 2, \\ &V^*_{1,1} = \frac{\pi^2}{6}, \\ &\ldots \\ &V^*_{j,j} = V^*_{j-1,j-1} - (j-1)^{-2}, \quad j \geq 2 \\ &V_{m,n} = \frac{(n-1)^{(n-m)}}{(n-1+\gamma)^{(n-m)}} V_{n,n}. \end{aligned}$$

Here

$$\begin{aligned} r^{(-i)} &= r(r-1)\ldots(r-i+1) \quad \text{for } i = 1, 2, \ldots \\ &= 0 \quad \text{for } i = 0. \end{aligned}$$

4.1.1 Minimum Variance Linear Unbiased Estimates (MVLUE)

The Theorem follows by putting these relations in $\hat{\mu}$ *and* $\hat{\sigma}_o$ and in their variances and covariance.

Theorem 4.1.1 *The MVLUE* $\hat{\mu}$ and $\hat{\sigma}$ *of* μ *and* σ *respectively based on the observed record values* $r_1, r_2, \ldots, r_m$ *are*:

$$\hat{\mu} = r_m - v_m^* \hat{\sigma}$$
$$\hat{\sigma} = (m-1)^{-1} \sum_{i=1}^{m-1} r_i - r_m$$

Their corresponding variances and covariance are

$$Var(\hat{\mu}) = \sigma^2 \{(\upsilon_m^*)^2 (m-1)^{-1} + V_{mm}^*\}$$
$$Var(\hat{\sigma}) = \sigma^2/(m-1), \text{ and}$$
$$Cov(\hat{\mu}, \hat{\sigma}) = \sigma^2\, \upsilon_m^*/(m-1),$$

where

$$v_m^* = E(X_{L(m)}) \text{ and } v_{mm}^* = \mathrm{Var}(\mathrm{X_L(m)}).$$

Proof For $\gamma = 0$, we have

$$\mathrm{E}(\mathrm{X}_{\mathrm{L(r)}}) = \mu + \upsilon_r^* \sigma$$
$$\mathrm{Var}(\mathrm{X}_{\mathrm{L(m)}}) = \sigma^2 V_{r,r}^*, \quad r = 1, 2, \ldots$$
$$\mathrm{Cov}(\mathrm{X}_{\mathrm{L(r)}}, \mathrm{X}_{\mathrm{L(m)}}) = \mathrm{Var}(\mathrm{X}_{\mathrm{L(m)}}), \quad r < m,$$

with

$$\upsilon_1^* = \upsilon$$
$$\upsilon_j^* = \upsilon_{j-1}^* - (j-1)^{-1}, \quad j \geq 2,$$
$$V_{1,1}^* = \frac{\pi^2}{6},$$
$$\ldots$$
$$V_{j,j}^* = V_{j-1,j-1}^* - (j-1)^{-2}, \quad j \geq 2$$

Here υ is the Euler's constant.

Let $\Omega = V^{-1} = (V^{ij})$, then

$$
\begin{aligned}
V^{ii} &= i^2 + (i-1)^2, \quad i = 1, 2, \ldots, m-1 \\
V^{ij} &= \min(i^2, j^2), \quad i \neq j, \quad |i-j| = 1 \\
&= 0, \quad \text{if} |i-j| > 1 \\
V^{mm} &= (m-1)^2 + 1/V^*_{mm}. \\
1'V^{-1} &= (0, 0, \ldots, 1/V^*_{mm}) \\
\alpha'V^{-1} &= (1, 1, \ldots, \alpha_m/V^*_{mm} - (m-1) \\
\alpha'V^{-1}1 &= \alpha_m/V^*_{mm}, \quad \alpha'V^{-1}\alpha = (\alpha_m)^2/V^*_{mm} + m - 1
\end{aligned}
$$

and

$$\Delta = (m-1)/V^*_{mm}.$$

Substituting these values in the expression of $\hat{\mu}$ *and* $\hat{\sigma}$, where

$$
\begin{aligned}
\hat{\mu} &= -\alpha'V^{-1}\{1\alpha' - \alpha 1'\}V^{-1}r/\Delta \\
\hat{\sigma} &= 1'V^{-1}\{1\alpha' - \alpha 1'\}V^{-1}r/\Delta
\end{aligned}
$$

the results follow. For example for m = 6, the MVLUE of μ and σ for the type I extreme value distribution is given by

$$\hat{\sigma} = 0.2(r_1 + r_2 + r_3 + r_4 + r_5) - r_6$$

and

$$
\begin{aligned}
&\hat{\mu} = 0.3412(r_1 + r_2 + r_3 + r_4 + r_5) - 0.7061 r_6 \\
&Var(\hat{\mu}) = 0.7635\sigma^2, \\
&Var(\hat{\sigma}) = 0.2000\sigma^2, \textit{and} \\
&Cov(\hat{\mu}, \hat{\sigma}) = -0.3412\sigma^2.
\end{aligned}
$$

4.1.2 Best Invariant Estimates (BLIE)

Theorem 4.1.2 *The BLIE* $\tilde{\mu}$ and $\tilde{\sigma}$ *of* μ *and* σ *are*:

$$
\begin{aligned}
\tilde{\mu} &= \hat{\mu} - \upsilon^*_m \hat{\sigma}/m \\
\tilde{\sigma} &= \hat{\sigma}(m-1)/m \\
MSE(\tilde{\mu}) &= \sigma^2[V^*_{mm} + (\upsilon^*_m)^2/m]
\end{aligned}
$$

and

$$MSE(\tilde{\sigma}) = \sigma^2/m.$$

where $\tilde{\mu}$ *and* $\tilde{\sigma}$ are the MVLUE of μ and σ when $\gamma = 0$.

Proof We know

$$Var(\hat{\mu}) = \sigma^2\{(\upsilon_m^*)^2(m-1)^{-1} + V_{mm}^*\}$$
$$Var(\hat{\sigma}) = \sigma^2/(m-1), \quad \text{and}$$
$$Cov(\hat{\mu}, \hat{\sigma}) = \sigma^2\upsilon_m^*/(m-1),$$

Since $1 + E_{22} = \frac{m}{m-1}$, on simplification, we get the results. For m = 6,

$$\hat{\sigma} = \frac{1}{6}(r_1 + r_2 + r_3 + r_4 + r_5) - \frac{5}{6}r_6,$$
$$\hat{\mu} = 0.3981(r_1 + r_2 + r_3 + r_4 + r_5) - 0.9904r_6$$
$$MSE(\tilde{\mu}) = 0.6665\sigma^2$$

and

$$MSE(\tilde{\sigma}) = 0.1667\sigma^2.$$

4.2 Maximum Likelihood Estimates (MLE)

The solutions of the equations as given in (4.2.1) will give the MLE of μ and σ as

$$\begin{aligned} \overset{\circ}{\widehat{\sigma}}_\ell &= \bar{r} - r_m \\ \overset{\circ}{\widehat{\mu}}_\ell &= r_m + \overset{\circ}{\widehat{\sigma}}_\ell \ln m \end{aligned} \quad (4.2.1)$$

where

$$\bar{r} = (r_1 + \cdots + r_m)/m.$$

It can easily be shown that

$$E(\overset{\circ}{\widehat{\sigma}}_\ell) = \frac{m}{m-1}\sigma.$$

The bias in $\overset{\circ}{\widehat{\sigma}}_\ell$ is $-\frac{\sigma^2}{m-1}$. The variance of $\overset{\circ}{\widehat{\sigma}}_\ell$ is

$$Var(\hat{\sigma}^{\circ}_{\ell}) = \left(\frac{m}{m-1}\right)^2 \frac{\sigma^2}{m}.$$

$$\mathrm{E}(\hat{\mu}^{\circ}_{\ell}) = \mu + \sigma(v^*_m + \frac{m-1}{m}(\ln m)).$$

Since $v^*_m = \upsilon - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{m-1}$, the bias in $\mathrm{E}(\hat{\mu}^{\circ}_{\ell})$ is slight. We obtain on simplification,

$$\mathrm{Var}(\hat{\mu}^{\circ}_{\ell}) = \sigma^2[v_{r,r} + (\frac{m-1}{m})^2 \frac{(\ln m)^2}{m}].$$

4.2.1 Characterization

We have that for the type I extreme value distribution, $S_{(m)} = m(X_{L(m)} - X_{L(m+1)})$, m = 1, 2, … as identically distributed negative exponential. Random variables. Arnold and Villasenor (1997) raised the question whether the identical distribution of S1 and $2S_2$ are i.i.d. negative exponential with unit mean can characterize the Gumbel distribution. Al-Zaid and Ahsanullah (2003) proved the following theorem.

Theorem 4.2.1 *Let* $\{X_j, j = 1,\ldots.\}$ *be a sequence of independent and identically distributed random variables with absolutely continuous (with respect to Lebesgue measure) distribution function* F(x). *Then the following two statements are identical.*

(a) $\mathrm{F(x)} = e^{-e^{-x}}, -\infty < x < \infty$,
(b) *for a fixed* m > 1, *the condition* $X_{L(m)} \underline{\underline{d}} X_{L(m+1)} + \frac{W}{m}$, *where* W *is independent of* $X_{L(n)}$ *and* $X_{L(n+1)}$ *and* $X_{L(n+1)}$ *and is distributed as exponential mean unity.*

Proof It is easy to show that (a) ⇒ (b),

We will prove here that (b) ⇒ (a).

Suppose that for a fixed m > 1, $X_{L(m)} \underline{\underline{d}} X_{L(m+1)} + \frac{W}{m}$, then

$$\begin{aligned} F_{(m)}(x) &= \int_{-\infty}^{x} P(W \le m(x-y) f_{(m+1)}(y)dy \\ &= \int_{-\infty}^{x} [1 - e^{-m(x-y)} f_{(m+1)}(y)dy \\ &= \mathrm{F}_{(m+1)}(\mathrm{x}) - \int_{-\infty}^{x} e^{-m(x-y)} f_{(m+1)}(y)dy. \end{aligned}$$

Thus

$$e^{mx}[F_{(m+1)}(x) - F_{(m)}(x)] = \int_{-\infty}^{x} e^{my} f_{(m+1)}(y)dy$$

Using the relation (1.1.7), we obtain

$$e^{mx} \frac{F(x)H^m(x)}{\Gamma(m+1)} = \int_{-\infty}^{x} e^{my} f_{(m+1)}(y)dy$$

Taking the derivatives of both sides we obtain

$$\frac{d}{dx}\left[e^{mx} \frac{H^m(x)}{\Gamma(m+1)} F(x)\right] = e^{mx} f_{(n+1)}(x)$$

This implies that

$$\frac{d}{dx}\left[e^{mx} \frac{H^m(x)}{\Gamma(m+1)}\right] F(x) = 0.$$

Thus

$$\frac{d}{dx}\left[e^{mx} \frac{H^m(x)}{\Gamma(m+1)}\right] = 0.$$

Hence

$$\mathrm{H(x) = ce^{-x}}, -\infty < \mathrm{x} < \infty$$

Thus

$$\mathrm{F(x)} = e^{-ce^{-x}}, -\infty < \mathrm{x} < \infty.$$

Since F(x) is a distribution, assuming F(0) = e^{-1}, we obtain

$$\mathrm{F(x)} = e^{-e^{-x}}, \quad -\infty < x < \infty.$$

Corollary 4.2.4.1 *If for some fixed* m > 1, $X_{U(m+1)} \underline{\underline{d}} X_{U(m)} + \frac{W}{m}$, *then we get a characterization of the Gumbel distribution with* $F(x) = 1 - e^{-e^x}$, $-\infty < \mathrm{x} < \infty$.

Corollary 4.2.4.2 *If* m = 1, *then relation* $X_{U(2)} \underline{\underline{d}} X_{U(1)} + W$, *will give a characterization of the negative exponential distribution.*

Remark 4.2.1 The condition that any one of the statistics $m(X_{L(m)} - X_{L(m+1)})$, $m(X_{U(m+1)} - X_{U(m)})$, $(X_{L(m)} - X_{L(m+1)})$ or $(X_{U(m+1)} - X_{U(m)})$ is distributed as negative exponential do not characterize any distribution.

4.2.2 Applications

Example 4.2.1 Table 4.1 shows the one hour mean concentration of $S0_2$ from Long Beach, California (taken from Roberts 1979) from 1979 to 1974. Roberts (1979) fitted the Gumbel distribution $F(x) = \exp(-e^{-a(x-b)})$ to the annual maxima of the hourly concentration of $S0_2$. He obtained by using complete data with a variant of the least squared method, the estimates $\hat{a}, \hat{b}$ of a and b as $\hat{a} = 0.081$ and $\hat{b} = 31.5$. In terms of our notation $\sigma = 1/a$ and $\mu = b$. From the annual maxima of the hourly concentration of $S0_2$, we obtain 47, 41, 32, 27, 20 and 18 as lower records values.

For the estimations of the location and scale parameters, we will use the first six lower records i.e. m = 6. The minimum variance linear unbiased estimators of μ and σ are

$$\hat{\sigma} = 0.2(47 + 41 + 32 + 27 + 20) - 18 = 15.4$$
$$\hat{\mu} = 0.3412(47 + 41 + 32 + 27 + 20) - 0.7061(18) = 44.3$$

Table 4.1 Sulfur dioxide, 1-h average concentration (p phm). Monthly and annual maxima

Year	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec	Max
1956	47	31	44	12	13	3	14	21	33	33	40	32	47
1957	22	19	20	32	20	23	18	16	13	14	41	25	41
1958	15	13	30	12	24	13	37	20	32	27	27	68	68
1959	20	32	20	15	3	6	8	15	17	15	29	20	32
1960	22	18	23	20	8	13	14	9	13	16	27	20	27
1981	25	20	20	16	10	10	8	10	12	16	14	43	43
1962	20	13	15	18	10	1	10	10	11	11	14	7	20
1963	12	16	27	21	2	7	4	4	15	19	18	18	27
1964	16	10	3	3	19	9	16	25	4	14	18	21	25
1965	16	18	9	14	8	10	18	18	14	12	12	14	18
1966	27	33	25	10	17	30	13	18	22	15	25	23	32
1967	30	40	32	10	8	7	8	26	10	40	18	17	40
1968	51	30	18	22	10	19	22	25	26	29	50	40	51
1969	37	13	55	14	9	10	13	17	33	13	15	44	55
1970	23	18	19	11	15	12	25	40	25	20	12	8	40
1971	22	26	20	28	10	15	20	55	38	41	26	25	55
1972	30	32	18	27	37	13	23	19	21	31	25	13	37
1973	10	8	8	12	11	16	25	16	11	28	10	23	28
1974	8	9	9	13	8	34	9	9	25	11	19	15	34

The best linear invariance estimators of μ and σ are

$$\tilde{\mu} = \hat{\mu} - \upsilon_m^* \hat{\sigma}/m = 48.7$$
$$\tilde{\sigma} = \hat{\sigma}(m-1)/m = 12.8$$

The maximum likelihood estimators of μ and σ are

$$\overset{\circ}{\widehat{\sigma}}_\ell = \bar{r} - r_m = 12.8$$
$$\overset{\circ}{\widehat{\mu}}_\ell = r_m + \overset{\circ}{\widehat{\sigma}}_\ell \ln m = 51.7$$

4.3 Type II and Type II Distributions

The three types of extreme value distributions can be combined in one distribution known as generalized order statistics.

A random variable X is said to have the generalized extreme value distribution if its cumulative distribution function is of the following form:

$$\mathrm{F(X)} = \exp[-\{1 - \gamma\sigma^{-1}(\mathrm{x} - \mu)\}^{1/\gamma}]$$

where σ > 0, γ ≠ 0 and

$$\mathrm{x} < \mu + \sigma\gamma^{-1}, \quad \text{for}\, \gamma > 0$$
$$\mathrm{x} > \mu + \sigma\gamma^{-1}, \quad \text{for}\, \gamma > 0.$$

If γ = 0 then

$$\mathrm{F(x)} = \exp[-\exp\{-(\mathrm{x} - \mu)/\sigma\}], \quad \sigma > 0, -\infty < \mathrm{x} < \infty.$$

We will write X ∈ GEV(μ, σ, γ) if X has the cdf as given in (4.0.1).
Since

$$\lim_{\gamma \to 0} \{1 - \gamma\sigma^{-1}(x - \mu)\}^{1/\gamma} = \exp\{-\sigma^{-1}(x - \mu)\}, \quad \text{we can take}$$
$$\lim_{\gamma \to 0} GEV(\mu, \sigma, \gamma) = GEV(\mu, \sigma, 0).$$

The density function of GEV(μ, σ, γ) is

$$\mathrm{f(x)} = \sigma^{-1}\{1 - \gamma\sigma^{-1}(x - \mu)\}^{\frac{1-\gamma}{\gamma}} \exp[-\{1 - \gamma\sigma^{-1}(\mathrm{x} - \mu)\}1/\gamma], \quad \gamma \neq 0$$
$$\mathrm{x} < 1/\gamma, \quad \text{for}\ \gamma > 0,$$
$$\mathrm{x} > 1/\gamma, \quad \text{for}\ \gamma < 0,$$

and

$$\mathrm{f(x)} = e^{-x}\exp(-e^{-x}), \quad \text{for } \gamma = 0, \text{ for all x.}$$

Since we have already discussed about type I distribution $\gamma = 0$, we will consider here the generalized extreme value $\gamma \neq 0$, Further the estimation of location and scale parameters can be estimated in closed form using record values, we will restrict ourselves to record values of the generalized order statistics for $\gamma \neq 0$.

4.3.1 Distributional Properties

It is known (see Leadbetter et al. 1980) that normalized $X_{n-k+1,n}$ converges to kth lower records. We will consider here the distributional properties of lower records of the type 2 and type 3 distribution as part of generalized extreme value distribution. If X ∈ GEV(μ, σ, γ), then we can write for $\gamma \neq 0$, the pdf f(m) of the mth lower record value as

$$\mathrm{f_{(m)}(x)} = \{1 - \gamma\sigma^{-1}(x-\mu)\}^{(m-1)/\gamma} f_m^*(x)$$

where

$$f_m^*(x) = \frac{\{1-\gamma\sigma^{-1}(x-\mu)\}^{(1-\gamma)/\gamma}}{\sigma(m-1)!}\exp\{-(1-\gamma\sigma^{-1}(x-\mu))\}^{1/\gamma}$$

From (4.1.1) and (4.1.2) it can be shown that

$$X_{L(m)} \underline{d} \mu + \sigma\gamma^{-1}\{1-(W_1+\cdots+W_m)^{\gamma}\}, \quad \gamma \neq 0,$$

where W_1, W_2, W_m are independently distributed as exponential random variables with mean unity.

$$\begin{aligned} E\left(X_{L(m)}\right) &= \mu + \sigma\gamma^{-1}.\{1-\Gamma(m+\gamma)/\Gamma(m)\}. \\ \mathrm{Var}\left(X_{L(m)}\right) &= \sigma^2\gamma^{-2}\left[E(W_1+\cdots+W_m)^{2m} - \{E(W_1+\cdots+W_m)\}^2\right] \\ &= \sigma^2\gamma^{-2}\left[\frac{\Gamma(m+2\gamma)}{\Gamma(m)} - \{\frac{\Gamma(m+\gamma)}{\Gamma(m)}\}^2\right]. \end{aligned}$$

For r < m

$$\gamma^2\sigma^{-2}\mathrm{Cov}\left(X_{L(r)}, X_{L(m)}\right) = \{ (\sum_{j=1}^{r} W_j)^{\gamma}(\sum_{j=1}^{m} W_j)^{\gamma} - E(\sum_{j=1}^{r} W_j)^{\gamma}E(\sum_{j=1}^{m} W_j)^{\gamma}$$
$$= \int_0^{\infty}\int_0^{\infty} u^{\gamma}(u+v)^{\gamma}\frac{e^{-u}u^{r-1}}{\Gamma(r)}\frac{e^{-v}v^{m-r-1}}{\Gamma(m-r)}dudv$$
$$= \frac{\Gamma(r+\gamma)\Gamma(r+2\gamma)}{\Gamma(r)\Gamma(r+\gamma)} - \frac{\Gamma(r+\gamma)\Gamma(m+\gamma)}{\Gamma(r)\Gamma(m)},$$

since u and v are independent. Thus

$$\mathrm{Cov}\left\{X_{L(r)}, X_{L(m)}\right\} = \sigma_0^2 a_r b_m, \quad r<m$$

where

$$a_r = \frac{\Gamma(r+\gamma)}{\Gamma(r)}, \quad b_m = \frac{\Gamma(m+2\gamma)}{\Gamma(m+\gamma)} - \frac{\Gamma(m+\gamma)}{\Gamma(m)} \quad \mathit{and} \quad \sigma_o^2 = \frac{\sigma^2}{\gamma^2}.$$

Tables 4.2 and 4.3 give the values of $E(X_{L(n)})$ and $Var(X_{L(n)})$ for some selected values of n and γ.

4.3.2 Estimation of Parameters

Estimation of μ and σ for known γ.

Table 4.2 Expected values of $X_{L(n)}$

n\γ	0.5	1.0	1.5
5	−2.3619	−4.0000	−7.3301
10	−4.2460	−9.0000	−21.1944
15	5.6817	−14.0000	−39.0221
20	−6.8886	−19.0000	−60.0718
25	−7.9501	−24.0000	−84.9094
30	−8.9089	−29.0000	−110.2405

Table 4.3 Variances of $X_{L(n)}$

n\γ	γ = 0.5	γ = 1.0	γ = 1.5
5	0.9738	5.0000	29.3843
10	0.9872	10.0000	108.7898
15	0.9915	15.0000	238.1350
20	0.9937	20.0000	417.5101
25	0.9950	25.0000	646.8852
30	0.9958	30.0000	926.2602

4.3.2.1 Minimum Variance Linear Unbiased Estimates (MVLUE)

Theorem 4.3.2.1 *Then the MVLUE* $\hat{\mu}$ *and* $\hat{\sigma}_o$ *of* μ *and* σ_o *respectively, based on the observed* m *record values* $r_1,\ r_2,\ldots,r_m$ *are*:

$$\hat{\mu} = D^{-1}\{r_m(1'V^{-1}\alpha) - \alpha_m 1'V^{-1}r\}$$
$$\hat{\sigma}_o = -D^{-1}\{r_m\{1'V^{-1}1) - 1'V^{-1}r\}$$

where

$$\mathrm{D} = \Gamma(m+k)\{\frac{1'V^{-1}1}{\Gamma(m)} - \frac{1}{b_m}\},\quad V = \{V^{ij}\},$$
$$V^{11} = \frac{(1+\gamma)^2}{\gamma^2}\cdot\frac{1}{\Gamma(1+2\gamma)},\quad V^{mm} = \frac{b_{m-1}}{b_m}\frac{m-1+\gamma}{\gamma^2}\cdot\frac{\Gamma(m)}{\Gamma(m-1+\gamma)}$$
$$V^{ii} = \frac{\Gamma(i)}{\Gamma(i+2\gamma)}\cdot\frac{1}{\gamma^2}\{\{i+\gamma)^2 + (i-1)(i-1+2\gamma)\},\quad i = 2,\ldots,m-1,$$
$$V^{ij} = V^{ji} = -\frac{i+\gamma}{\gamma^2}\frac{\Gamma(i+1)}{\Gamma(i+2\gamma)},\quad j = i+1,\ \ i = 1,\ldots,m-1$$
$$V^{ij} = 0,\quad \text{if } \ |\mathrm{i}-\mathrm{j}| > 1,$$
$$1' = (1,\ldots,1),\quad \mathrm{r}' = (r_1,\ldots,r_m),\ \ \alpha' = (\alpha_1,\ldots,\alpha_m),$$
$$\alpha_i = 1 - \frac{\Gamma(i+\gamma)}{\Gamma(i)},\quad \mathrm{i} = 1,2,\ldots,\mathrm{m},$$
$$Var(\hat{\mu}) = \sigma_o^2\{b_m(1'V^{-1}1) - 2 + \frac{\Gamma(m+\gamma)}{\Gamma(m)}\}/D$$
$$Var(\hat{\sigma}_o) = \sigma_o^2 b_m\{1'V^{-1}1\}/D$$

and

$$Cov(\hat{\mu},\hat{\sigma}_o) = -\sigma_o^2\quad\{b_m(1'V^{-1}1-1\}/D.$$

Proof Let R = (XL(1), …, XL(m)). Then we can write

$$\mathrm{E(R)} = \mu 1 + \sigma_0\alpha$$
$$\mathrm{Var(R)} = \sigma_0^2\mathrm{V},$$

where

$$\alpha' = (\alpha_1,\ldots,\alpha_m),\quad \alpha_i = 1 - \frac{\Gamma(i+\gamma)}{\Gamma(i)},\ \ 1' = (1,\ldots,1),\ \ \mathrm{V} = \{\mathrm{V_{ij}}\},\ \ \mathrm{V_{ij}} = \mathrm{a_i b_j},$$
$$1 < \mathrm{i},\ \ \mathrm{j} < \mathrm{m},\ \ \mathrm{a_i} = \frac{\Gamma(i+\gamma)}{\Gamma(i)},\ \ b_i = \frac{\Gamma(i+2\gamma)}{\Gamma(i+\gamma)} - \frac{\Gamma(i+\gamma)}{\Gamma(i)}\ \text{ and }\ \sigma_0^2 = \frac{\sigma^2}{\gamma^2}.$$

Let $V^{-1} = \{V^{ij}\}$. Then

$$\begin{aligned}
V^{11} &= \frac{a_2}{a_1(a_2b_1 - a_1b_2)} = \frac{1}{\gamma^2}\frac{(1+\gamma)^2}{\Gamma(1+2\gamma)}\\
V^{ii} &= \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_ib_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_ib_{i+1})}\\
&= \frac{\Gamma(i)}{\Gamma(i+2\gamma)}\frac{1}{\gamma^2}\{(i+\gamma)^2 + (i-1)(i-1+2\gamma)\}, \quad \mathrm{i} = 2,\ldots,\mathrm{m}-1\\
V^{mm} &= \frac{b_{m-1}}{b_m}\frac{1}{a_mb_{m-1} - a_{m-1}b_m} = \frac{b_{m-1}}{b_m}\frac{m+1-\gamma}{\gamma^2}\frac{\Gamma(m)}{\Gamma(m-1+\gamma)},\\
V^{ij} &= V^{ji} = -\frac{1}{a_{i+1}b_i - a_ib_{i+1}} = -\frac{i+\gamma}{\gamma^2}\frac{\Gamma(i+1)}{\Gamma(i+2\gamma)}\\
&\mathrm{j} = \mathrm{i}+1, \mathrm{i} = 1,2,\ldots,\mathrm{m}-1,\\
\mathrm{Vij} &= 0, \quad \mathrm{if}|\mathrm{i}-\mathrm{j}| > 1.
\end{aligned}$$

It follows from the method of Lloyd (1952) that the MVLUE of μ and σ_o based on the observed value r of R are, respectively,

$$\begin{aligned}
\hat{\mu} &= -\alpha' V^{-1}\{1\alpha' - \alpha 1'\}V^{-1}r/\Delta\\
\hat{\sigma}_o &= 1'V^{-1}\{1\alpha' - \alpha 1'\}V^{-1}r/\Delta
\end{aligned}$$

where

$$\Delta = \{1'\mathrm{V}^{-1}1\}\{\alpha'\mathrm{V}^{-1}\alpha\} - \{1'\mathrm{V}^{-1}\alpha\}^2$$

and

$$\begin{aligned}
Var(\hat{\mu}) &= \sigma_o^2(\alpha' V^{-1}\alpha)/\Delta\\
Var(\hat{\sigma}_o) &= \sigma_o^2(1'V^{-1}1)/\Delta\\
Cov(\hat{\mu}, \hat{\sigma}_o) &= \sigma_o^2(1'V^{-1}\alpha)/\Delta.
\end{aligned}$$

It can be shown that, upon simplification,

$$\begin{aligned}
1'\mathrm{V} - 1\alpha &= 1'\mathrm{V}^{-1}1 - 1/\mathrm{b_m}\\
\alpha'\mathrm{V}^{-1}\alpha &= 1'\mathrm{V}^{-1}1 - 1/\mathrm{b_m} + \mathrm{a_m}/\mathrm{b_m}\\
\alpha'\mathrm{V}^{-1}\mathrm{r} &= 1'\mathrm{V}^{-1}\mathrm{r} - \mathrm{r_m}/\mathrm{b_m},
\end{aligned}$$

and

$$\Delta = \Gamma(m+\gamma)\{\frac{1'V^{-1}1}{b_m\Gamma(m)}\} - \frac{1}{b_m^2}.$$

4.3.2.2 Minimum Mean Squared Invariance Estimator (MMSIE)x

Theorem 4.3.2.2 *The best linear invariant (best in the sense of minimum mean squared error and invariant with respect to the location parameter* μ) *estimators* $\tilde{\mu}$ *and* $\tilde{\sigma}_o$ *of* μ *and* σ_o *are respectively*

$$\tilde{\mu} = \hat{\mu} - c_1\hat{\sigma}_o,$$
$$\tilde{\sigma} = c_2\hat{\sigma}_o$$

where

$$c_1 = \frac{b_m\{(1'V^{-1})b_m - 1\}}{\{\Gamma(m+2\gamma)/\Gamma(m+\gamma)\{b_m(1'V^{-1}1) - 1\}}$$

and

$$c_2 = \frac{D}{D + b_m(1'V^{-1}1)}.$$

Proof The BLIE $\tilde{\mu}$ *and* $\tilde{\sigma}_o$ of μ and σ_0 are:

$$\tilde{\mu} = \hat{\mu} - \hat{\sigma}_o\{E_{12}(1+E_{22})^{-1}\},$$

and

$$\tilde{\sigma}_o = \hat{\sigma}_o(1+E_{22})^{-1},$$

where

$$\sigma_o^2\begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}$$

defines the covariance matrix of the MVLUEs of $\tilde{\mu}$ *and* $\tilde{\sigma}_o$. The mean squares errors (MSE) of $\tilde{\mu}$ *and* $\tilde{\sigma}_o$ are:

$$MSE(\tilde{\mu}) = \sigma_o^2\{E_{11} - E_{12}^2(1+E_{22})^{-1}\},$$
$$MSE(\tilde{\sigma}_o) = \sigma_o^2 E_{22}(1+E_{22})^{-1},$$
$$E(\tilde{\mu}-\mu)(\tilde{\sigma}_o-\sigma) = \sigma_o^2 E_{12}(1+E_{22})^{-1}$$

Substituting the values of E_{11}, E_{12} and E_{22}, the results follow on simplification.

4.3.2.3 Maximum Likelihood Estimates (MLE)

We can write the log likelihood function L based on the observed record values $r_1, r_2, \ldots, r_m$ are:

$$\ln L = \sum_{i=1}^{m-1} \ln\{\frac{f(r_i)}{F(r_i)}\} + \ln f(r_m). \tag{4.3.1}$$

Differentiating (4.3.1) with respect to μ and equating to zero, we get

$$(-1+\gamma^{-1})\sum_{i=1}^{m}(\gamma;\sigma^{-1})\{1+\gamma\sigma^{-1}(r_i-\mu)\}^{-1} - \sigma^{-1}\{1+\gamma\sigma^{-1}(r_m-\mu)\}^{-1+\gamma^{-1}} = 0 \tag{4.3.2}$$

Differentiating (4.3.1) with respect to σ and equating to zero, we get

$$-m\sigma^{-1} - (1+\gamma^{-1})\sum_{i=1}^{m}\gamma(r_i-\mu)\sigma^{-2}\{1+\gamma\sigma^{-1}(r_i-\mu)\}$$
$$-\gamma\sigma^{-2}(r_m-\mu)\{1+\gamma\sigma^{-1}(r_m-\mu)\}^{-1+\gamma^{-1}} = 0.$$

From the above equations we obtain the maximum likelihood estimators $\widehat{\mu}_\ell$ *and* $\widehat{\sigma}_\ell$ of μ and σ assuming γ as known are the solutions of the following equations.

$$\widehat{\mu}_\ell = r_m - \frac{\sigma}{\gamma}(1-m^\gamma)$$
$$\sum_{i=1}^{m-1}\frac{1}{m^\gamma - \gamma(r_i-r_m)/\sigma} = \frac{m^{1-\gamma}}{1-\gamma} - m^{-\gamma}$$

A closed form solution can be found if m = 2:

$$\widehat{\sigma}_\ell = \frac{(1+\gamma)(r_1 - r_2)}{2^{\gamma+1}}$$

$$\widehat{\mu}_\ell = r_2 - \frac{\widehat{\sigma}_\ell}{\gamma}(1 - 2^\gamma)$$

A closed form solution can also be found if γ = 0. In this case:

The solutions of the equations as given in (4.4.4.6) will give the MLE of μ and σ as

$$\begin{aligned} \overset{\circ}{\sigma}_\ell &= \bar{r} - r_m \\ \overset{\circ}{\mu}_\ell &= r_m + \overset{\circ}{\sigma}_\ell \ln m \end{aligned} \tag{4.3.3}$$

where

$$\bar{r} = (r_1 + \cdots + r_m)/m.$$

It can easily be shown that

$$E(\widehat{\overset{\circ}{\sigma}}_\ell) = \frac{m}{m-1}\sigma.$$

The bias in $\widehat{\overset{\circ}{\sigma}}_\ell$ is $-\frac{\sigma^2}{m-1}$. The variance of $\widehat{\overset{\circ}{\sigma}}_\ell$ is

$$Var(\widehat{\overset{\circ}{\sigma}}_\ell) = \left(\frac{m}{m-1}\right)^2 \frac{\sigma^2}{m}.$$

$$\mathrm{E}(\widehat{\overset{\circ}{\mu}}_\ell) = \mu + \sigma(v_m^* + \frac{m-1}{m}(\ln m)).$$

Since $v_m^* = \upsilon - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{m-1}$, the bias in $\mathrm{E}(\widehat{\overset{\circ}{\mu}}_\ell)$ is slight. We obtain on simplification, $Var(\widehat{\overset{\circ}{\mu}}_\ell) = \sigma^2[v_{r,r} + (\frac{m-1}{m})^2 \frac{(\ln m)^2}{m}]$.

4.3.2.4 Estimation of γ for Known μ and σ

We will assume without any loss of generality that μ = 0 and σ = 1.

Using the following two identities:

$$E\left(\sum_{i=1}^{m-1} Y_i\right) = m - 1 - \gamma E\left(\sum_{i=1}^{m-1} Y_i\right)$$

where

$$Y_i = i\left(X_{L(i)} - X_{L(i+1)}\right)$$

and

$$(i + \gamma)E\left(X_{L(i)}\right) = 1 + iE\left(X_{L(i+1)}\right), \quad i = 1, 2, \ldots, m - 1,$$

we can consider two moment estimators of γ as:

$$\overset{\circ}{\gamma}_1 = \frac{1 - \bar{y}}{\bar{r}},$$

where $\bar{y} = \sum_{i=1}^{m-1} y_i/(m-1)$, $y_i = i(r_i - r_{i+1})$, y_i is the corresponding observed value of Y_i, i = 1, 2, …, m and

$$\overset{\circ}{\gamma}_2 = \frac{(m-1)(r_m + 1)}{\sum_{i=1}^{m-1} r_i} - 1$$

Picands (1975) proposed the following estimate $\hat{\gamma}_p$ of γ_e as

$$\hat{\gamma}_p = \frac{1}{\ln 2} \ln \frac{X_{n-k+1,n} - X_{n=2k+1,n}}{X_{n-2k+1,n} - X_{n=4k+1,n}}$$

And showed that this estimate is consistent if $k \to \infty$, $n \to \infty$ and $\frac{k}{n} \to 0$. Dekkers and de Hann (1980) showed that this estimator is strongly consistent if $k \ln \ln n \to \infty$. Hill (1975) gave the following estimator $\hat{\gamma}_h$ for γ

$$\hat{\gamma}_h = \frac{1}{k} \sum_{i=1}^{k} (X_{n-i+1} - \ln X_{n-k+1,n}).$$

4.4 Characterizations

Here we give a characterization of the type II distribution based on records. The characterization of type III distribution is similar.

Theorem 4.4.1 *Let* $(X_i$, i = 1, 2, …} *be a sequence of i.i.d. absolutely continuous random variables with* cdf F(x), pdf f(x) *and* $\text{var}(X_i^{-\delta}) = 1, \delta > 0$.

Then the following two statements are equivalent.

(a) $F(x) - e^{-x^{-\delta}},\ x \geq 0, \delta > 0,$
(b) $Var(X_{L(n)}^{-\delta} - X_{L(n-1)}^{-\delta} | X_{L(n-1)}^{-\delta} = x) = b,\ n \geq 2,$
where b *is a constant independent of* X.

Proof It is easy to prove that (a) ⇒ (b).

We will that (b) ⇒ (a).

Let $Y - X^{-\delta}$, then $Var(X_{L(n)}^{-\delta} - X_{L(n-1)}^{-\delta} | X_{L(n-1)}^{-\delta} = x) = b$ is equivalent to $Var(Y_{U(n)} - Y_{U(n-1)|} | Y_{U(n-1)} = x) = b.$

Using the transformation
$Z_n = Y_{U(n)} - Y_{U(n-1)}$, we have

$$b = E(Z_n^2 | Y_{U(n-1)} = y) - (E(Z_n | Y_{U(n-1)} = y))^2$$

$$E(Z_n^2 | Y_{U(n-1)} = y) - \int_0^\infty z^2 \frac{g(z+y)}{1 - G(y)} dz$$

$$= - \int_0^\infty 2z \frac{1 - G(z+y)}{1 - G(y)} dz$$

$$E(Z_n | Y_{U(n-1)} = y) - \int_0^\infty z \frac{g(z+y)}{1 - G(y)} dz$$

$$= \int_0^\infty \frac{1 - G(z+y)}{1 - G(y)} dz$$

where G(y) and g(y) are the cdf and pdf of Y respectively. We have

$$\int_0^\infty 2z \frac{G(z+y)}{1 - G(y)} dz - (\int_0^\infty \frac{1 - G(z+y)}{1 - G(y)} dz)^2 = b.$$

Let H(y) $= \int_0^\infty z(1 - G(z+y))dz$, then

$$\mathrm{H}'(\mathrm{y}) = \frac{dH(y)}{dy} = \int_0^\infty (1 - G(z+y)dz\ , H^{(2)}(y) = \frac{d^2H(y)}{dy^2} = 1 - G(y)$$

and

$$H^{(3)}(y) - \frac{d^3 H(y)}{dy^3} = -g(y)$$

Thus we have

$$\frac{2H(y)}{H^{(2)}(y)} - (\frac{H^{(1)}(y)}{H^{(2)}(y)})^2 = b$$

Differentiating both sides of the above equation with respect to y, we obtain

$$\frac{2H^{(1)}(y)}{H^{(2)}(y)} - \frac{2H(y)H^{(3)}(y)}{(H^{(2)}(y))^2} - \frac{2H^{(1)}(y)}{H^{(2)}(y)} + \frac{2(H^{(1)}(y))^2 H^{(3)}(y)}{(H^{(2)}(y))^3} = 0$$

Since $H^{(3)}(y) \neq 0$ for any y. we must have

$$(H^{(1)}(y))^2 - H(y)H^{(2)}(y) = 0 = 0.$$

We can write the above equation as

$$\frac{d}{dy}(H(y)(H^{(1)}(y)) = 0$$

The solution of the above equation is $H(y) - ae^{cy}$, where a and c are constant.
Since $1 - G(x) = H^{(2)}(x) = ac^2 e^{cx}$
Since G(x) is a cdf with F(0) = 0, $F(\infty) = 1$ and E(Y) = 1, we must have

$$\mathrm{G(x)} = 1 - e^{-x}, \quad x \geq 0.$$

Now

$$\begin{aligned} P(X \leq x) &= P(X^{\delta} \geq x^{-\delta}), \quad \delta > 0 \\ &= P(Y \geq x^{-\delta}) \\ &= e^{-x^{-\delta}}. \end{aligned}$$

Note: The characterization of type III distribution follows similarly.

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