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Qualitative and Quantitative Analysis of Nonlinear Systems

Theory and Applications

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"A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories."

Stefan Banach

Preface

Scope

In an abstract form, the evolutional nonlinear system is a mathematical model that describes how physical, chemical, biological, economic, or even mathematical phenomena evolve in time. As a rule, it contains ordinary/partial/stochastic differential equations or inclusions that tell us how the system at hand changes "from one instant to the next." The main goal is to gain information about solutions of this system and then translate this mathematical information into the scientific context. The main challenge addressed by this book is to take this short-term information and obtain information about long-term overall behavior. The study of nonlinear systems has three parts: exact methods, quantitative methods and qualitative methods. But even if we solve the system symbolically, the question of computing values remains.

In this book, we concentrate on the following topics, specific for nonlinear systems:

(a) constructive existence results and regularity theorems for all weak (generalized) solutions;
(b) convergence results for solutions and their approximations in strongest topologies of the natural phase and extended phase spaces;
(c) uniform global behavior of solutions in time;
(d) pointwise behavior of solutions for autonomous problems with possible gaps by the phase variables.

With numerous applications including nonlinear parabolic equations of divergent form, parabolic problems with nonpolynomial growth, nonlinear stochastic equations of parabolic type, unilateral problems with possibly nonmonotone potential, nonlinear problems on manifolds with or without boundary, contact piezoelectric problems with nonmonotone potential, viscoelastic problems with nonlinear "reaction-displacement" and "reaction-velocity" laws as well as particular examples like a model of conduction of electrical impulses in nerve axons, a climate energy balance model, FitzHugh–Nagumo system, Lotka–Volterra system with diffusion, Ginzburg–Landau equations, Belousov–Zhabotinsky equations, and the 3D Navier–Stokes equations. This book is also distinguished with the solutions of a number of applied problems in physics, chemistry, biology, economics, etc.

Contents

This book consists of three parts: Existence and Regularity Results, Quantitative Methods and Their Convergence (Part I), Convergence Results in Strongest Topologies (Part II), and Uniform Global Behavior of Solutions: Uniform Attractors, Flattening and Entropy (Part III). Part I presents several numerical methods for approximate solution of nonlinear systems, their convergence, and regularity results and also discusses recent advances in regularity problem for the 3D Navier–Stokes equations. Part II covers three major topics: (1) strongest convergence results for weak solutions of nonautonomous reaction–diffusion equations with Carathéodory's nonlinearity with applications to FitzHugh–Nagumo systems, Lotka–Volterra systems with diffusion, Ginzburg–Landau equations, Belousov–Zhabotinsky equations, etc; (2) strongest convergence results for weak solutions of feedback control problems with applications to impulse feedback control mechanical problems and mathematical problems of biology and climatology; and (3) strongest convergence results for weak solutions of differential-operator equations and inclusions with applications to nonlinear parabolic equations of divergent form, parabolic problems with nonpolynomial growth, nonlinear stochastic equations of parabolic type, general parabolic and hyperbolic problems, unilateral problems with possibly nonmonotone operators, etc. Part III discusses general methodology for the global qualitative and quantitative investigation of dissipative dynamical systems, first- and second-order operator differential equations and inclusions, and evolutional variational inequalities with possibly nonmonotone potential with several applications. Indirect Lyapunov method for autonomous dynamical systems, exponential attractors, and Kolmogorov entropy are also established. All case studies are closely related to theoretical Parts I and II and are examples of applications to solutions of problems (a) and (b).

Audience

This book is aimed at practitioners working in the areas of nonlinear mechanics, mathematical biology, control theory, differential equations, nonlinear boundary value problems, and decision making. It can serve as a quick introduction into the novel methods of qualitative and quantitative analysis of nonlinear systems for the graduate students, engineers, and mathematicians interested in analysis and control of nonlinear processes and fields, mathematical modeling, and dynamical systems in infinite-dimensional spaces, to mention just a few. It can also be used as a supplementary reading for a number of graduate courses including but not limited to those of nonlinear PDEs, control and optimization, stochastic partial differential equations, advanced numerical methods, systems analysis, and advanced engineering economy.

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Kyiv, Ukraine
February 2017

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Contents

Introduction: Special Classes of Extended Phase Spaces of Distributions

Abstract In this introduction, we briefly establish special classes of extended phase spaces of distributions. We consider sums and intersections of Banach spaces; Gelfand triples; special classes of Bochner integrable functions; generalized derivatives; and basic properties of extended phase spaces.

If it is necessary to describe a nonstationary process that evolve in some domain $\Omega \subset \mathbb{R}^n$ during the time interval $[\tau, T]$, we may deal with functions that correspond to each pair $\{x, t\} \in \Omega \times S$ the real number or vector $u(x, t)$. In this approach, the time and the space variables are equivalent. But there is a more convenient approach to the mathematical description for evolution processes [1, 2]: For each point in time t, it is mapped the state function $u(\,\cdot\,, t)$ (e.g., for each point of time we put the temperature distribution or velocity distribution in the domain Ω)

physical process or field $\Rightarrow$ mathematical model (PDE) $y : \Omega \times [\tau, T] \to \mathbb{R}^m$ $\Rightarrow$ differential - operator equation / inclusion $y : [\tau, T] \to (\Omega \to \mathbb{R}^m)$, $y(t, .) \in (\Omega \to \mathbb{R}^m)$, $t \in [\tau, T]$

Thus, we consider the functions defined on $[\tau, T]$ with values in the state functions space (e.g., in the space $H_0^1(\Omega)$). Therefore, to investigate the evolution problem, it is natural to consider the space of functions acting from the time interval $[\tau, T]$ into some infinite-dimensional space V. In particular, it is natural to consider the spaces of integrable and differentiable functions. In this book, we consider only real vector spaces.

In this chapter, we introduce the classes of function spaces used for qualitative and quantitative analysis of nonlinear distributed systems:

$$Lu + A(u) \ni f, \quad u \in D(L), \tag{1}$$

where $A :\to 2^{X^*}$ is possibly multi-valued mapping with nonempty values, X is a Banach space, X^* is its dual space, $L : D(L) \subset X \to X^*$ is a linear operator defined on $D(L)$, and $f \in X^*$. Moreover, in this chapter, we refer to the basic properties for this spaces (see, e.g., [1, 2] and references therein for details).

For Banach spaces X, Y, the following denotation

$$X \subset Y$$

means the embedding in both the set-theoretic and the topological senses.

The following two theorems are frequently used in the qualitative and quantitative analysis of nonlinear systems in infinite-dimensional spaces. The **main idea** is in the following: the uniform prior estimates for solutions of approximative problems and the following theorems allow ones to obtain at least weak convergence (up to a subsequence in the general situation) of these approximations to the exact solution of the problem in hands.

Theorem 1 (The reflexivity criterion) *A Banach space E is reflexive if and only ifeach bounded in E sequence has a subsequence that weakly converges in E.*

Stefan Banach (March 30, 1892–August 31, 1945) was a Polish, Ukrainian, and Soviet mathematician who is generally considered one of the world's most important and influential twentieth-century mathematicians. He was one of the founders of modern functional analysis and an original member of the Lviv School of Mathematics. His major work was the 1932 book, Théorie des opérations linéaires (Theory of Linear Operations), the first monograph on the general theory of functional analysis (Fig. 1).
Born in Kraków, Banach attended IV Gymnasium, a secondary school, and worked on mathematics problems with his friend Witold Wilkosz. After graduating in 1910, Banach moved to Lviv. However, and during World War I Banach returned to Kraków, where he befriended Hugo Steinhaus. After Banach solved some mathematics problems which Steinhaus considered difficult, they published their first joint work. In 1919, with several other mathematicians, Banach formed a mathematical society. In 1920, he received an assistantship at the Lviv Polytechnic.

He soon became a professor at the Lviv Polytechnic and a member of the Polish Academy of Learning. He organized the "Lviv School of

Fig. 1 Stefan Banach

Mathematics." Around 1929, he began writing his Théorie des opérations linéaires.

After the outbreak of World War II, in September 1939, Lviv was taken over by the Soviet Union. Banach became a member of the Academy of Sciences of Ukraine and was dean of Lviv University's Department of Mathematics and Physics.

In 1941, when the Germans took over Lviv, all institutions of higher education were closed to Poles. As a result, Banach was forced to earn a living as a feeder of lice at Rudolf Weigl's Institute for Study of Typhus and Virology. While the job carried the risk of infection with typhus, it protected him from being sent to slave labor in Germany and from other forms of repression. When the Soviets recaptured Lviv in 1944, Banach re-established the university. However, because the Soviets were removing Poles from Soviet-annexed formerly Polish territories, Banach prepared to return to Kraków. Before he could do so, he died in August 1945, having been diagnosed seven months earlier with lung cancer.

Some of the notable mathematical concepts that bear Banach's name include Banach spaces, Banach algebras, the Banach–Tarski paradox, the Hahn–Banach theorem, the Banach–Steinhaus theorem, the Banach–Mazur game, the Banach–Alaoglu theorem, and the Banach fixed-point theorem.

Theorem 2 (Banach–Alaoglu) *In reflexive Banach space, each bounded sequence has a subsequence that weakly converges*

Leonidas (Leon) Alaoglu (March 19, 1914–August 1981) was a mathematician, known for his result, called Alaoglu's theorem on the weak-star compactness of the closed unit ball in the dual of a normed space, also known as the Banach–Alaoglu theorem; Fig. 2.

Fig. 2 Leonidas (Leon) Alaoglu's grave

Sums and Intersections of Banach Spaces

Let us consider the sums and intersections of Banach spaces. Such objects naturally appear under the investigation of number of anisotropic problems. Let $n \geq 2$ be a natural number and $\{X_i\}_{i=1}^{n}$ be a family of Banach spaces. Let us introduce the definition of the interpolation family of Banach spaces.

Definition 1 If there exists a vector topological space (LTS) Y such that

$$X_i \subset Y$$

for each $i = 1 \ldots n$, then the family of Banach spaces $\{X_i\}_{i=1}^{n}$ is called *an interpolation family*. If $n = 2$, then the interpolation family is called *an interpolation pair*.

In the field of mathematical analysis, an **interpolation space** is a space which lies "in between" two other Banach spaces. The main applications are in **Sobolev spaces**, where spaces of functions that have a noninteger number of derivatives are interpolated from the spaces of functions with integer number of derivatives.

The theory of interpolation of vector spaces began by an observation of **Józef Marcinkiewicz**, later generalized and now known as the **Riesz-Thorin theorem**. In simple terms, if a linear function is continuous on a certain space L_p and also on a certain space L_q, then it is also continuous on the space L_r, for any intermediate r between p and q. In other words, L_r is a space which is intermediate between L_p and L_q.

In the development of **Sobolev spaces**, it became clear that the trace spaces were not any of the usual function spaces (with integer number of derivatives), and **Jacques-Louis Lions** discovered that indeed these trace spaces were constituted of functions that have a noninteger degree of differentiability.

Further, let $\{X_i\}_{i=1}^n$ be an interpolation family of Banach spaces. Similar to [1, p. 23], we endow the vector space $X = \cap_{i=1}^n X_i$ with the following norm:

$$\|x\|_X := \sum_{i=1}^{n} \|x\|_{X_i} \qquad \forall x \in X, \tag{2}$$

where $\|\cdot\|_{X_i}$ is the norm in X_i.

Proposition 1 *Let* $\{X, Y, Z\}$ *be an interpolation family. Then*

$$X \cap (Y \cap Z) = (X \cap Y) \cap Z = X \cap Y \cap Z, \quad X \cap Y = Y \cap X$$

both in the sense of equality of sets and in the sense of equality of norms.

Let us consider also the vector space

$$Z := \sum_{i=1}^{n} X_i = \left\{ \sum_{i=1}^{n} x_i \ : x_i \in X_i,\ i = 1 \ldots n \right\}$$

with the norm

$$\|z\|_Z := \inf\left\{ \max_{i=1\ldots n} \|x_i\|_{X_i} \ : x_i \in X_i,\ \sum_{i=1}^{n} x_i = z \right\} \qquad \forall z \in Z. \tag{3}$$

Proposition 2 *Let* $\{X_i\}_{i=1}^n$ *be an interpolation family. Then* $X = \cap_{i=1}^n X_i$ *and* $Z = \sum_{i=1}^n X_i$ *are Banach spaces. Moreover,*

$$X \subset X_i \subset Z \tag{4}$$

for each $i = 1 \ldots n$.

Remark 1 Let Banach spaces X and Y satisfy the following conditions

$$X \subset Y, \qquad X \quad \text{is dense in} \quad Y,$$
$$\|x\|_Y \leq \gamma \|x\|_X \quad \forall x \in X, \qquad \gamma = const.$$

Then

$$Y^* \subset X^*, \qquad \|f\|_{X^*} \leq \gamma \|f\|_{Y^*} \quad \forall f \in Y^*.$$

Moreover, if X is reflexive, then Y^* is dense in X^*.

Józef Marcinkiewicz (March 30, 1910–1940) was a Polish mathematician. He was a student of Antoni Zygmund and later worked with Juliusz Schauder and Stefan Kaczmarz. He was a Professor of the Stefan Batory University in Wilno.

Marcinkiewicz was taken as a Polish POW to a Soviet camp in Starobielsk. The exact place and date of his death remain unknown, but it is believed that he died in the Katyn massacre on the mass murder site near Smolensk. His parents, to whom he gave his manuscripts before the beginning of World War II, were transported to the Soviet Union in 1940 and later died of hunger in a camp (Fig. 3).

Let $\{X_i\}_{i=1}^n$ be an interpolation family. Assume that the Banach space $X := \cap_{i=1}^n X_i$, endowed with the norm defined in (2) is dense in X_i for each $i = 1\ldots n$. Remark 1 yields that each space X_i^* may be considered as a subspace of X^*. Therefore, the vector space $\sum_{i=1}^n X_i^*$ is well-defined, and the following embedding holds:

Fig. 3 Józef Marcinkiewicz

$$\sum_{i=1}^{n} X_i^* \subset \left(\cap_{i=1}^{n} X_i\right)^*. \tag{5}$$

Since X is dense in $Z := \sum_{i=1}^{n} X_i$ for each $i = 1\ldots n$, then each X_i is dense in Z. According to Remark 1, the space Z^* can be considered both as a subspace of X_i^* for each $i = 1\ldots n$ and as a subspace of $\cap_{i=1}^{n} X_i^*$, that is,

$$\left(\sum_{i=1}^{n} X_i\right)^* \subset \cap_{i=1}^{n} X_i^*. \tag{6}$$

G. Olof Thorin (February 23, 1912–February 14, 2004) was a Swedish mathematician working on analysis and probability, who introduced the Riesz–Thorin theorem (Fig. 4).

Fig. 4 G. Olof Thorin

Proposition 3 *Let $\{X_i\}_{i=1}^n$ be an interpolation family of Banach spaces such that the space $X := \cap_{i=1}^n X_i$ endowed with the norm (2) is dense in X_i for each $i = 1\ldots n$. Then the following equalities hold:*

$$\sum_{i=1}^n X_i^* = \left(\cap_{i=1}^n X_i\right)^* \quad \textit{and} \quad \left(\sum_{i=1}^n X_i\right)^* = \cap_{i=1}^n X_i^*$$

in the sense of both equalities of sets and norms.

Sergei Lvovich Sobolev (October 6, 1908–January 3, 1989) was a Soviet mathematician working in mathematical analysis and partial differential equations. Sobolev introduced the notions that are now fundamental for several areas of mathematics. Sobolev spaces can be defined by some growth conditions on the Fourier transform. They and their embedding theorems are an important subject in functional analysis. Generalized functions (later known as distributions) were first introduced by Sobolev in 1935 for weak solutions and further developed by Laurent Schwartz. Sobolev abstracted the classical notion of differentiation, so expanding the range of application of the technique of Newton and Leibniz. The theory of distributions is considered now as the calculus of the modern epoch (Fig. 5).

Gelfand Triple

Let V be a real reflexive separable Banach space V with the norm $\|\cdot\|_V$ and H be a real Hilbert space with the inner product $(\cdot, \cdot$ and respective norm $\|\cdot\|_H$. Assume that

$$\begin{array}{l} V \subset H, \quad V \text{ is dense in } H, \\ \exists \gamma > 0 \ : \ \|v\|_H \le \gamma \|v\|_V \quad \forall v \in V. \end{array} \tag{7}$$

Remark 1 and conditions (7) yield that the dual space H^* to H is a subspace of the dual space V^* to V. Since the Banach space V is reflexive and the set V is dense in the space H, then the set H^* is dense in the space V^* and the following inequality holds:

$$\|f\|_{V^*} \le \gamma \|f\|_{H^*} \quad \forall f \in H^*,$$

where $\|\cdot\|_{V^*}$ and $\|\cdot\|_{H^*}$ are the norms in spaces V^* and H^*, respectively. By applying the Riesz representation theorem, we can identify H^* with H. Therefore,

Fig. 5 Sergei Lvovich Sobolev

H^* is identified with some subspace of V^*; that is, each element $y \in H$ is identified with some $f_y \in V^*$ such that

$$(y, x) = \langle f_y, x \rangle_V \quad \forall x \in V,$$

where $\langle \cdot, \cdot \rangle_V$ is the canonical pairing between V^* and V. Since the elements y and f_y are identified, then conditions (7) imply that the restriction of the pairing $\langle \cdot, \cdot \rangle_V$ on $H \times V$ coincides with the inner product $(\cdot, \cdot)$ on H restricted on the same set. After this identification of H and H^*, we obtain the following tuple of the continuous and dense embeddings

$$V \subset H \subset V^*.$$

Definition 2 The tuple of spaces $(V; H; V^*)$ satisfying the above conditions is called *the evolution triple* (sometimes Gelfand triple).

Israel Moiseevich Gelfand (September 2 [O.S. 20 August], 1913–October 5, 2009) was a prominent Soviet and American mathematician. He made significant contributions to many branches of mathematics, including group theory, representation theory, and functional analysis. The recipient of many awards, including the Order of Lenin and the Wolf Prize, he was a Fellow of the Royal Society and Professor at Moscow State University and, after immigrating to the USA shortly before his 76th birthday, at Rutgers University (Fig. 6).

His legacy continues through his students, who include Endre Szemerédi, Alexandre Kirillov, Edward Frenkel, and Joseph Bernstein, as well as his own son, Sergei Gelfand.

Fig. 6 Israel Moiseevich Gelfand

Special Classes of Bochner Integrable Functions

Let us consider classes of distributions with values in a Banach space. Let Y be a real Banach space, Y^* be its dual space, and S be a compact time interval. We consider the classes of functions defined on S and taking values in Y (or in Y^*, respectively).

In mathematics, the **Bochner integral**, named for **Salomon Bochner**, extends the definition of Lebesgue integral to functions that take values in a Banach space, as the limit of integrals of simple functions.

Let $1 \le p \le +\infty$. The set $L_p(S;Y)$ of all Bochner measurable functions (see [1]) such that

$$\|y\|_{L_p(S;Y)} = \left(\int_S \|y(t)\|_Y^p dt \right)^{1/p} < \infty$$

is a Banach space. If $p = +\infty$, then the norm on $L_\infty(S;Y)$ is defined as follows

$$\| y \|_{L_\infty(S;Y)} = \operatorname*{ess\,supp}_{t \in S} \| y(t) \|_Y .$$

Salomon Bochner (August 20, 1899–May 2, 1982) was an American mathematician of Austrian–Hungarian origin, known for work in mathematical analysis, probability theory, and differential geometry (Fig. 7).

The following theorem establishes the sufficient conditions for the identification of the dual space $(L_p(S;Y))^*$ to $L_p(S;Y)$, $1 \le p < +\infty$, with $L_q(S;Y^*)$, where q is such that $p^{-1} + q^{-1} = 1$. Sometimes, the following theorem is called the Riesz representation theorem for spaces of Bochner integrable functions. We note that $1/\infty := 0$.

Theorem 3 *Let Y be a reflexive and separable Banach space, $1 \le p < +\infty$, and $q > 1$ be such that $p^{-1} + q^{-1} = 1$. Then for each $f \in (L_p(S;Y))^*$ there exists a unique $\xi \in L_q(S;Y^*)$ such that*

$$f(y) = \int_S \langle \xi(t), y(t) \rangle_Y dt$$

Fig. 7 Salomon Bochner

for each $y \in L_p(S;Y)$. *Moreover, this correspondence* $f \to \xi$ *is linear and*

$$\|f\|_{(L_p(S;Y))^*} = \|\xi\|_{L_q(S;Y^*)},$$

that is, this mapping is isometric isomorphism.

Frigyes Riesz (January 22, 1880–February 28, 1956) was a Hungarian mathematician who made fundamental contributions to functional analysis. He was the Rector and a Professor at the University of Szeged, as well as a member of the Hungarian Academy of Sciences. He was the older brother of the mathematician Marcel Riesz (Fig. 8).

Let us consider the sums and intersections of Banach spaces of Bochner integrable functions. These spaces are important for the investigation of nonlinear

Fig. 8 Frigyes Riesz

anisotropic problems and the respective differential-operator equations and inclusions. Let p_i and r_i, $i = 1, 2$ be real numbers such that $1 < p_i \le r_i \le +\infty$ and $p_i < +\infty$. Define the real numbers $q_i \ge r_{i'} \ge 1$ as follows:

$$p_i^{-1} + q_i^{-1} = r_i^{-1} + r_{i'}^{-1} = 1, \quad i = 1, 2.$$

Let $(V_i; H; V_i^*)$, $i = 1, 2$, be evolution triple such that

$$\text{the set } V_1 \cap V_2 \text{ is dense in the spaces } V_1, V_2 \text{ and } H. \tag{8}$$

Consider the following Banach spaces (see Proposition 2):

$$X_i = X_i(S) = L_{q_i}(S; V_i^*) + L_{r_{i'}}(S; H), \quad i = 1, 2$$
$$X = X(S) = L_{q_1}(S; V_1^*) + L_{q_2}(S; V_2^*) + L_{r_{2'}}(S; H) + L_{r_{1'}}(S; H)$$

with the following respective norms

$$\|y\|_{X_i} = \inf\Big\{\max\{\|y_1\|_{L_{q_i}(S;V_i^*)}; \|y_2\|_{L_{r_{i'}}(S;H)}\} \;:\; : \; y_1 \in L_{q_i}(S; V_i^*), y_2 \in L_{r_{i'}}(S; H), y = y_1 + y_2\Big\},$$

for all $y \in X_i$, and

$$\|y\|_X = \inf\{\max_{i=1,2}\{\|y_{1i}\|_{L_{q_i}(S;V_i^*)}; \|y_{2i}\|_{L_{r_{i'}}(S;H)}\} : y_{1i} \in L_{q_i}(S; V_i^*),$$
$$y_{2i} \in L_{r_{i'}}(S; H), i = 1, 2; y = y_{11} + y_{12} + y_{21} + y_{22}\},$$

for each $y \in X$.

If $r_i < +\infty$, then Proposition 1 and Theorem 3 imply that the space X_i is reflexive. Similarly, if $\max\{r_1, r_2\} < +\infty$, then the space X is reflexive. Moreover, for $i = 1, 2$ the dual space $X_i^* = X_i^*(S)$, we identify with $L_{r_i}(S; H) \cap L_{p_i}(S; V_i)$, where

$$\|y\|_{X_i^*} = \|y\|_{L_{r_i}(S;H)} + \|y\|_{L_{p_i}(S;V_i)}$$

for each $y \in X_i^*$. Similarly, for the dual space $X^* = X^*(S)$ we identify with

$$L_{r_1}(S; H) \cap L_{r_2}(S; H) \cap L_{p_1}(S; V_1) \cap L_{p_2}(S; V_2),$$

where

$$\|y\|_{X^*(S)} = \|y\|_{L_{r_1}(S;H)} + \|y\|_{L_{r_2}(S;H)} + \|y\|_{L_{p_1}(S;V_1)} + \|y\|_{L_{p_2}(S;V_2)}$$

for each $y \in X^*$. The pairing on $X(S) \times X^*(S)$ is defined by

$$\langle f, y\rangle = \langle f, y\rangle_S = \int_S (f_{11}(\tau), y(\tau) d\tau + \int_S (f_{12}(\tau), y(\tau) d\tau$$
$$+ \int_S \langle f_{21}(\tau), y(\tau)\rangle_{V_1} d\tau + \int_S \langle f_{22}(\tau), y(\tau)\rangle_{V_2} d\tau = \int_S (f(\tau), y(\tau)) d\tau$$

for each $f \in X$ and $y \in X^*$, where $f = f_{11} + f_{12} + f_{21} + f_{22}$, $f_{1i} \in L_{r_{i'}}(S; H)$, $f_{2i} \in L_{q_i}(S; V_i^*)$, $i = 1, 2$.

If $\max\{r_1, r_2\} < +\infty$, then we will always use the following “standard” denotations [1, p. 171]: for the spaces X^*, X_1^*, and X_2^*, we will denote as X, X_1, and X_2, respectively, and vice versa; for the spaces X, X_1, and X_2, we will denote as X^*, X_1^*, and X_2^*, respectively. These denotations are correct, because Proposition 3 and Theorem 3 yield that these spaces and their dual spaces are reflexive. The following statement directly follows from Proposition 3 and Theorem 3.

Proposition 4 *If* $\max\{r_1, r_2\} < +\infty$, *then the Banach spaces* X, X_1 *and* X_2 *are reflexive.*

Generalized Derivatives

Let S be a time interval. The *space* $\mathscr{D}(S)$ *of test functions on* S is defined as follows. A function $\varphi : S \to \mathbb{R}$ is said to have *compact support* if there exists a compact subset K of S such that $\varphi(x) = 0$ for all $x \in S \backslash K$. The elements of $\mathscr{D}(S)$ are the infinitely differentiable functions $\varphi : S \to \mathbb{R}$ with compact support—also known as bump functions. This is a real vector space. It can be given a topology by defining the limit of a sequence of elements of $\mathscr{D}(S)$. A sequence $\{\varphi_k\}_{k\geq 1} \subset \mathscr{D}(S)$ is said to converge to $\varphi \in \mathscr{D}(S)$ if the following two conditions hold:

(i) There is a compact set $K \subset S$ containing the supports of all φ_k: $\cup_k \mathrm{supp}\,(\varphi_k) \subset K$;
(ii) For each multi-index α, the sequence of partial derivatives $\partial^\alpha \varphi_k$ tends uniformly to $\partial^\alpha \varphi$.

With this definition, $\mathscr{D}(S)$ becomes a complete locally convex topological vector space satisfying the Heine–Borel property.

Let Y be a real reflexive Banach space. The *distribution* on S with values in Y is a continuous linear mapping acting from $\mathscr{D}(S)$ into Y endowed with the weak topology. The space of all distributions on S with values in Y is denoted by $\mathscr{D}^*(S;Y)$. For each $f \in \mathscr{D}^*(S;Y)$, its *generalized derivative* f' is well defined as follows:

$$f'(\varphi) = -f(\varphi')$$

for each $\varphi \in \mathscr{D}(S)$.

We note that each locally integrable in the Bochner sense function u (i.e., $u \in L_1^{\mathrm{loc}}(S;Y)$ if and only if $u \in L_1(K;Y)$ for each compact interval $K \subset S$), we can identify with the distribution $f_u \in \mathscr{D}^*(S;Y)$ defined as follows:

$$f_u(\varphi) = u(\varphi) = \int_S u(t)\varphi(t)dt, \tag{9}$$

for each $\varphi \in \mathscr{D}(S)$, where the integral is regarded in the Bochner sense. Therefore, we interpret $L_1^{\mathrm{loc}}(S;Y)$ as a subspace of $\mathscr{D}^*(S;Y)$, and *regular distributions* (the distributions that admit the representation (9) via the locally Bochner integrable function) are considered as functions from ($S \to Y$). We also note that the following operation $f \to f'$ is continuous in $\mathscr{D}^*(S;Y)$ [1, p.169].

Laurent-Moïse Schwartz (March 5, 1915–July 4, 2002) was a French mathematician. He pioneered the theory of distributions, which gives a well-defined meaning to objects such as the Dirac delta function. He was awarded the Fields Medal in 1950 for his work on the theory of distributions. For several years, he taught at the École Polytechnique (Fig. 9).

Fig. 9 Laurent-Moïse Schwartz

Definition 3 Let $C^m(S;Y)$, $m \geq 0$, be a family of all functions $y : S \to Y$ such that each strong derivative $y^{(i)}$ of order $i = 1, 2, \ldots, m$ is continuous (we note that $y^{(0)} = y$). If S is a compact interval, then $C^m(S;Y)$ is a Banach space with the norm

$$\|y\|_{C^m(S;Y)} = \sum_{i=0}^{m} \sup_{t \in S} \|y^{(i)}(t)\|_Y = \sum_{i=0}^{m} \max_{t \in S} \|y^{(i)}(t)\|_Y.$$

Extended Phase Spaces

Let $(V_i; H; V_i^*)$, $i = 1, 2$, be evolution triple such that assumption (8) holds. Let S be a finite time interval and $X = X(S)$ and $X^* = X^*(S)$ be the spaces introduced in Sect. 3. The extended phase space $W^* = W^*(S)$, where the real (generalized) solutions of nonlinear evolution systems belongs, is defined as follows:

$$W^*(S) = \{y \in X^*(S) \, : \, y' \in X(S)\},$$

where the derivative y' of $y \in X^*$ is considered in the sense of the distributions space $\mathscr{D}^*(S; V^*)$.

By the analogy with Sobolev spaces, it is necessary to establish basic structure properties, embedding and approximations theorems as well as some "rules of work" with the elements of such spaces.

Theorem 4 *The set W^* with the natural operations and graph norm for y':*

$$\|y\|_{W^*} = \|y\|_{X^*} + \|y'\|_X \quad \forall y \in W^*$$

is Banach space.

Theorem 5 *The set $C^1(S; V) \cap W_0^*$ is dense in W_0^*.*

Theorem 6 *$W_0^* \subset C(S; H)$ with continuous embedding. Moreover, for every $y, \xi \in W_0^*$ and $s, t \in S$, the next formula of integration by parts takes place*

$$(y(t), \xi(t)) - (y(s), \xi(s)) = \int_s^t \{(y'(\tau), \xi(\tau)) + (y(\tau), \xi'(\tau))\} d\tau. \tag{10}$$

In particular, when $y = \xi$, we have:

$$\frac{1}{2}(\|y(t)\|_H^2 - \|y(s)\|_H^2) = \int_s^t (y'(\tau), y(\tau)) d\tau. \tag{11}$$

Corollary 1 *$W^* \subset C(S; H)$ with continuous embedding. Moreover, for every $y, \xi \in W^*$ and $s, t \in S$ formula (10) takes place.*

Remark 2 When $\max\{r_1, r_2\} < +\infty$, due to the standard denotations [1, p. 173], we will denote the space W^* as W; " * " will direct on nonreflexivity of the spaces X and W .

Jacques-Louis Lions (May 3, 1928–May 17, 2001) was a French mathematician who made contributions to the theory of partial differential equations and to stochastic control, among other areas. He received the SIAM's John von Neumann prize in 1986 and numerous other distinctions. Lions is listed as an ISI highly cited researcher.

After being part of the French Résistance in 1943 and 1944, J.-L. Lions entered the École Normale Supérieure in 1947. He was a Professor of mathematics at the Université of Nancy, the Faculty of Sciences of Paris, and the École Polytechnique. He joined the prestigious Collège de France as well as the French Academy of Sciences in 1973. In 1979, he was appointed director of the Institut National de la Recherche en Informatique et Automatique (INRIA), where he taught and promoted the use of numerical simulations using finite elements integration. Throughout his career, Lions insisted on the use of mathematics in industry, with a particular involvement in the French space program, as well as in domains such as energy and the environment. This eventually led him to be appointed director of the Centre National d'Etudes Spatiales (CNES) from 1984 to 1992.

Lions was elected President of the International Mathematical Union in 1991 and also received the Japan Prize and the Harvey Prize that same year. In 1991, Lions became a foreign member of the National Academy of Sciences of Ukraine. In 1992, the University of Houston awarded him an honorary doctoral degree. He was elected President of the French Academy of Sciences in 1996 and was also a Foreign Member of the Royal Society (ForMemRS) and numerous other foreign academies.

He has left a considerable body of work, among this more than 400 scientific articles, 20 volumes of mathematics that were translated into English and Russian, and major contributions to several collective works, including the 4000 pages of the monumental Mathematical Analysis and Numerical Methods for Science and Technology (in collaboration with Robert Dautray), as well as the Handbook of Numerical Analysis in 7 volumes (with Philippe G. Ciarlet).

His son Pierre-Louis Lions is also a well-known mathematician who was awarded a Fields Medal in 1994. In fact, both Father and Son have also both received recognition in the form of Honorary Doctorates from Heriot-Watt University in 1986 and 1995, respectively; Fig. 10.

Fig. 10 Jacques-Louis Lions

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Part I
Existence and Regularity Results, Quantitative Methods and Their Convergence

Chapter 1
Qualitative Methods for Classes of Nonlinear Systems: Constructive Existence Results

Abstract In this chapter we establish the existence results for classes of nonlinear systems. Section 2.1 devoted to the first order differential-operator equations and inclusions. In Sect. 2.2 we consider the second order operator differential equations and inclusions in special classes of infinite-dimensional spaces of distributions. Section 2.3 devoted to the existence of strong solutions for evolutional variational inequalities with nonmonotone potential. The penalty method for strong solutions is justified. A nonlinear parabolic equations of divergent form are considered as examples of applications in Sect. 2.4.

1.1 First Order Differential-Operator Equations and Inclusions

1.1.1 Setting of the Problem

Let $(V_1, \|\cdot\|_{V_1})$ and $(V_2, \|\cdot\|_{V_2})$ be real reflexive separable Banach spaces continuously embedded in a Hilbert space $(H, (\cdot, \cdot))$. Assume that

$$\text{the set } V := V_1 \cap V_2 \text{ is dense in spaces } V_1,\ V_2 \text{ and } H. \tag{1.1}$$

After the identification $H \equiv H^*$ we obtain the following tuples of continuous and dense embeddings:

$$V_1 \subset H \subset V_1^*, \qquad V_2 \subset H \subset V_2^*, \tag{1.2}$$

where $(V_i^*, \|\cdot\|_{V_i^*})$ is the dual space to V_i, $i = 1, 2$, with respect to the pairing

$$\langle \cdot, \cdot \rangle_{V_i} : V_i^* \times V_i \to \mathbb{R}$$

which coincides on $H \times V$ with the inner product $(\cdot, \cdot)$ on H.

Let $S = [0, T]$, $0 < T < +\infty$, $1 < p_i \le r_i < +\infty$, $i = 1, 2$. For $i = 1, 2$ we consider the reflexive Banach space

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$$X_i = L_{r_i}(S; H) \cap L_{p_i}(S; V_i)$$

with the norm $\|y\|_{X_i} = \|y\|_{L_{p_i}(S;V_i)} + \|y\|_{L_{r_i}(S;H)}$, $y \in X_i$; see section "Special Classes of Bochner Integrable Functions". The Banach space $X = X_1 \cap X_2$ with the norm $\|y\|_X = \|y\|_{X_1} + \|y\|_{X_2}$ is also reflexive (see section "Special Classes of Bochner Integrable Functions"). We identify the spaces $L_{q_i}(S; V_i^*) + L_{r_{i'}}(S; H)$ and X_i^*. Similarly,

$$X^* = X_1^* + X_2^* \equiv L_{q_1}(S; V_1^*) + L_{q_2}(S; V_2^*) + L_{r_{1'}}(S; H) + L_{r_{2'}}(S; H),$$

where $r_i^{-1} + r_{i'}^{-1} = p_i^{-1} + q_i^{-1} = 1$. Let us define the duality form on $X^* \times X$

$$\langle f, y\rangle = \int_S (f_{11}(\tau), y(\tau))_H d\tau + \int_S (f_{12}(\tau), y(\tau))_H d\tau + \int_S \langle f_{21}(\tau), y(\tau)\rangle_{V_1} d\tau +$$

$$+ \int_S \langle f_{22}(\tau), y(\tau)\rangle_{V_2} d\tau = \int_S (f(\tau), y(\tau)) d\tau,$$

where $f = f_{11} + f_{12} + f_{21} + f_{22}$, $f_{1i} \in L_{r_{i'}}(S; H)$, $f_{2i} \in L_{q_i}(S; V_i^*)$.

Assume that there is a separable Hilbert space V_σ such that $V_\sigma \subset V_1$, $V_\sigma \subset V_2$ with continuous and dense embedding, $V_\sigma \subset H$ with compact and dense embedding. Then the following tuples of continuous and dense embeddings hold:

$$V_\sigma \subset V_1 \subset H \subset V_1^* \subset V_\sigma^*, \quad V_\sigma \subset V_2 \subset H \subset V_2^* \subset V_\sigma^*.$$

For $i = 1, 2$ we set

$$X_{i,\sigma} = L_{r_i}(S; H) \cap L_{p_i}(S; V_\sigma), \quad X_\sigma = X_{1,\sigma} \cap X_{2,\sigma},$$

$$X_{i,\sigma}^* = L_{r_{i'}}(S; H) + L_{q_i}(S; V_\sigma^*), \quad X_\sigma^* = X_{1,\sigma}^* + X_{2,\sigma}^*,$$

$$W_{i,\sigma} = \{y \in X_i \mid y' \in X_{i,\sigma}^*\}, \qquad W_\sigma = W_{1,\sigma} \cap W_{2,\sigma}.$$

For multi-valued (in the general case) map $A : X \rightrightarrows X^*$ let us consider the following problem:

$$\begin{cases} u' + A(u) \ni f, \\ u(0) = a, \ u \in W \subset C(S; H), \end{cases} \tag{1.3}$$

where $a \in H$ and $f \in X^*$ are arbitrary fixed elements. The main purpose of this section is to establish sufficient conditions for the existence of a solution of Problem (1.3) via the Faedo–Galerkin method.

1.1.2 Main Assumptions

Let $d \in X^*$ and $E \subset S$ be a measurable set. Further we will use the following denotations:

$$(d\chi_E)(\tau) = d(\tau)\chi_E(\tau) \text{ for a.e. } \tau \in S; \quad \chi_E(\tau) = \begin{cases} 1, & \tau \in E; \\ 0, & \text{elsewhere.} \end{cases}$$

We recall that the set B belongs to $\mathscr{H}(X^*)$ if for each measurable set $E \subset S$ and $u, v \in B$ the following inclusion $u + (v - u)\chi_E \in B$ holds.

Lemma 1.1 ([45]) *$B \in \mathscr{H}(X^*)$ if and only if for each $n \geq 1$, $\{d_i\}_{i=1}^n \subset B$ and a family of pairwise disjoint measurable sets $\{E_j\}_{j=1}^n \subset S$ such that $\bigcup_{j=1}^n E_j = S$ we have that $\sum_{j=1}^n d_j \chi_{E_j} \in B$.*

Remark 1.1 We note that $\emptyset, X^* \in \mathscr{H}(X^*)$; $\{f\} \in \mathscr{H}(X^*)$ for each $f \in X^*$; if $K : S \rightrightarrows V^*$ is an arbitrary multi-valued map, then

$$\{f \in X^* \mid f(t) \in K(t) \text{ for a.e. } t \in S\} \in \mathscr{H}(X^*).$$

On the other hand, if $v \in V^* \backslash \bar{0}$, then the closed convex set $B = \{f \in X^* \mid f \equiv \alpha v,\ \alpha \in [0, 1]\}$ does not belong to $\mathscr{H}(X^*)$, because $g(\cdot) = v \cdot \chi_{[0;T/2]}(\cdot) \notin B$.

Let Y be a reflexive Banach space, Y^* be its dual, $\langle \cdot, \cdot \rangle_Y : Y^* \times Y \to \mathbb{R}$ be a pairing, $A : Y \rightrightarrows Y^*$ be a *strict* multi-valued map, that is, $A(y) \neq \emptyset$ for each $y \in Y$. Define the *upper* and *lower support functions*:

$$[A(y), z]_+ := \sup_{d \in A(y)} \langle d, z \rangle_Y, \quad [A(y), z]_- := \inf_{d \in A(y)} \langle d, z \rangle_Y;$$

and the *upper* and *lower norms*:

$$\|A(y)\|_+ := \sup_{d \in A(y)} \|d\|_{X^*}, \quad \|A(y)\|_{:} = \inf_{d \in A(y)} \|d\|_{X^*},$$

$y, z \in Y$. For a nonempty set $B \subset Y^*$ let $\text{co}B$ denotes its convex hull, and $\overline{\text{co}}B$ denotes the closed convex hull of the set B (see Fig. 1.1), that is,

$$\text{co}B = \cap_{B \subset C,\ C \in C_1(Y^*)} C, \quad \overline{\text{co}}B = \cap_{B \subset C,\ C \in C_2(Y^*)} C,$$

where $C_1(Y^*)$ ($C_2(Y^*)$) is the family of all nonempty convex (nonempty closed and convex respectively) subsets of Y^*. Consider the following multi-valued mappings: $\text{co}A : Y \rightrightarrows Y^*$ and $\overline{\text{co}}A : Y \rightrightarrows Y^*$ such that

$$(\text{co}A)(y) = \text{co}(A(y)) \text{ and } (\overline{\text{co}}A(y)) = \overline{\text{co}}(A(y)),$$

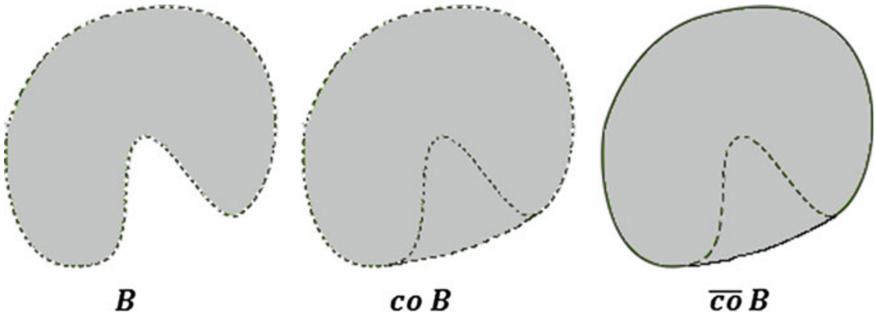

Fig. 1.1 Closed convex hull

for each $y \in Y$. Each strict multi-valued maps $A, B : Y \rightrightarrows Y^*$ satisfy the following properties [20, 31, 50, 53]:

(i) $[A(y), v_1+v_2]_+ \le [A(y), v_1]_+ + [A(y), v_2]_+$, $[A(y), v_1+v_2]_- \ge [A(y), v_1]_- + [A(y), v_2]_-$ $\forall y, v_1, v_2 \in Y$;
(ii) $[A(y), v]_+ = -[A(y), -v]_-$, $[A(y) + B(y), v]_{+(-)} = [A(y), v]_{+(-)} + [B(y), v]_{+(-)}$ $\forall y, v \in Y$;
(iii) $[A(y), v]_{+(-)} = [\overline{\text{co}}A(y), v]_{+(-)}$ $\forall y, v \in Y$;
(iv) $[A(y), v]_{+(-)} \le \|A(y)\|_{+(-)}\|v\|_Y$, $\|A(y) + B(y)\|_+ \le \|A(y)\|_+ + \|B(y)\|_+$ $\forall y \in Y$;
(v) the inclusion $d \in \overline{\text{co}}A(y)$ holds if and only if

$$[A(y), v]_+ \ge \langle d, v\rangle_Y \quad \forall v \in Y;$$

(vi) if $D \subset Y$ and $a(\cdot,\cdot) : D \times Y \to \mathbb{R}$, then for each $y \in D$ the function $w \mapsto a(y, w)$ is positively homogeneous convex and lower semi-continuous if and only if there exists a multi-valued map $A : Y \rightrightarrows Y^*$ such that $D(A) := \{y \in Y : A(y) \ne \emptyset\} = D$ and

$$a(y, w) = [A(y), w]_+ \quad \forall y \in D(A),\ \forall w \in Y.$$

Therefore, the following equalities hold:

$$[A(y), v]_{+(-)} = [\overline{\text{co}}A(y), v]_{+(-)} \quad \text{and} \quad \|A(y)\|_{+(-)} = \|\overline{\text{co}}A(y)\|_{+(-)}$$

for each $y, v \in Y$.

Further, the denotation

$$y_n \rightharpoonup y \text{ in } Y$$

will mean that y_n *converges weakly* to y in a Banach space Y. The family of all nonempty convex closed (weakly star) and bounded subsets of the dual space Y^* (to Y) we denote by $C_v(Y^*)$.

Let W be a normed space such that $W \subset Y$ with the continuous embedding. Consider the basic classes of multi-valued maps acting from Y into Y^* (see also [51] and references therein).

Definition 1.1 A strict multi-valued map $A : Y \rightrightarrows Y^*$ is called:

- *pseudomonotone on* W, if for each sequence $\{y_n, d_n\}_{n\geq 0} \subset W \times Y^*$ satisfying $d_n \in \overline{\text{co}}A(y_n)$ for each $n \geq 1$, $y_n \rightharpoonup y_0$ in W, $d_n \rightharpoonup d_0$ in Y^* as $n \to +\infty$, and

$$\limsup_{n\to\infty} \langle d_n, y_n - y_0\rangle_Y \leq 0, \tag{1.4}$$

  there exists a subsequence $\{y_{n_k}, d_{n_k}\}_{k\geq 1} \subset \{y_n, d_n\}_{n\geq 1}$ such that the following inequality holds:

$$\liminf_{k\to\infty} \langle d_{n_k}, y_{n_k} - w\rangle_Y \geq [A(y_0), y_0 - w]_- \tag{1.5}$$

  for each $w \in Y$;
- *bounded*, if for every $L > 0$ there exists $l > 0$ such that $\|A(y)\|_+ \leq l$ for each $y \in Y$ with $\|y\|_Y \leq L$;
- *coercive*, if there exists the real function $\gamma : \mathbb{R}_+ \to \mathbb{R}$ such that $\gamma(s) \to +\infty$ as $s \to +\infty$ and

$$\inf_{d\in A(y)} \langle d, y\rangle_Y \geq \gamma(\|y\|_Y)\|y\|_Y \quad \forall y \in Y;$$

- *demi-closed*, if for each sequence $\{y_n, d_n\}_{n\geq 0} \subset W \times Y^*$ satisfying $d_n \in \overline{\text{co}}A(y_n)$ for each $n \geq 1$, $y_n \rightharpoonup y_0$ in W, $d_n \rightharpoonup d_0$ in Y^* as $n \to +\infty$, it follows that $d \in \overline{\text{co}}A(y)$.

Definition 1.2 A multi-valued map $A : Y \rightrightarrows Y^*$ satisfies the *property* S_k *on* W, if for each sequence $\{y_n, d_n\}_{n\geq 0} \subset W \times Y^*$ satisfying $d_n \in \overline{\text{co}}A(y_n)$ for each $n \geq 1$, $y_n \rightharpoonup y_0$ in W, $d_n \rightharpoonup d_0$ in Y^* as $n \to +\infty$, and

$$\lim_{n\to\infty} \langle d_n, y_n - y_0\rangle_Y = 0,$$

it follows that $d_0 \in \overline{\text{co}}A(y_0)$.

Definition 1.3 A strict multi-valued map $A : X \rightrightarrows X^*$ is called the *Volterra type operator* (see Fig. 1.2) if for each $u, v \in X$ and $t \in S$ satisfying the equality $u(s) = v(s)$ for a.e. $s \in (0, t)$, it follows that

$$[A(u), \xi_t]_+ = [A(v), \xi_t]_+$$

for each $\xi_t \in X$ such that $\xi_t(s) = 0$ for a.e. $s \in S \setminus [0, t]$.

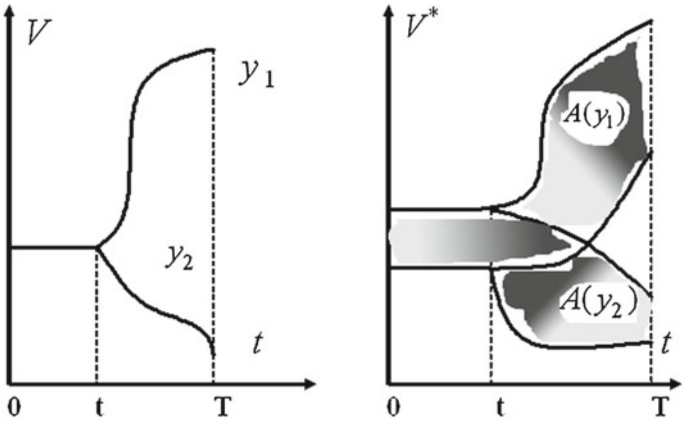

Fig. 1.2 Volterra type operator

1.1.3 Special Basis and Approximations for Multi-valued Mappings

Let us consider the complete vectors system $\{h_i\}_{i\geq 1} \subset V$ such that

(α_1) $\{h_i\}_{i\geq 1}$ orthonormal in H;
(α_2) $\{h_i\}_{i\geq 1}$ orthogonal in V;
(α_3) $(h_i, v)_V = \lambda_i(h_i, v)$ for each $i \geq 1$ and $v \in V$,

where $0 \leq \lambda_1 \leq \lambda_2, \cdots, \lambda_j \to \infty$ as $j \to \infty$, $(\cdot,\cdot)_V$ is the natural inner product in V. This system $\{h_i\}_{i\geq 1}$ is called a *special basis*. Let for each $m \geq 1$ $H_m = \operatorname{span}\{h_i\}_{i=1}^m$, on which we consider the inner product induced from H that we again denote by $(\cdot,\cdot)$. Due to the equivalence of H^* and H it follows that $H_m^* \equiv H_m$; $X_m = L_{p_0}(S; H_m)$, $X_m^* = L_{q_0}(S; H_m)$, $p_0 = \max\{r_1, r_2\}$, $q_0 > 1$: $1/p_0 + 1/q_0 = 1$, $\langle\cdot,\cdot\rangle_{X_m} = \langle\cdot,\cdot\rangle_X|_{X_m^*\times X_m}$, $W_m := \{y \in X_m |\ y' \in X_m^*\}$, where y' is the derivative of an element $y \in X_m$ in the sense of distributions from $\mathscr{D}^*(S, H_m)$.

Let us consider multi-valued maps that act from X_m into X_m^*, $m \geq 1$. Let us remark that embeddings $X_m \subset Y_m \subset X_m^*$ are continuous, and the embedding W_m into X_m is compact.

Definition 1.4 The multi-valued map $\mathscr{A} : X_m \to C_v(X_m^*)$ is called (W_m, X_m^*)-weakly closed if from that fact that $y_n \rightharpoonup y$ in W_m, $d_n \rightharpoonup d$ in X_m^*, $d_n \in \mathscr{A}(y_n)$ $\forall n \geq 1$ it follows that $d \in \mathscr{A}(y)$.

Lemma 1.2 *The multi-valued map $\mathscr{A} : X_m \to C_v(X_m^*)$ satisfies the property S_k on W_m if and only if $\mathscr{A} : X_m \to C_v(X_m^*)$ is (W_m, X_m^*)-weakly closed.*

Proof Let us prove the necessity. Let $y_n \rightharpoonup y$ in W_m, $d_n \rightharpoonup d$ in X_m^*, where $d_n \in \mathscr{A}(y_n)$ $\forall n \geq 1$. Then $y_n \to y$ in X_m and $\langle d_n, y_n - y\rangle_{X_m} \to 0$ as $n \to +\infty$. Therefore, in virtue of $\mathscr{A}$ satisfies the S_k property on W_m, we obtain that $d \in \mathscr{A}(y)$.

Let us prove sufficiency. Let $y_n \rightharpoonup y$ in W_m, $d_n \rightharpoonup d$ in X_m^*, $\langle d_n, y_n - y\rangle_{X_m} \le 0$ as $n \to +\infty$, where $d_n \in \mathscr{A}(y_n)$ $\forall n \ge 1$. Then $y_n \to y$ in X_m and $d \in \mathscr{A}(y)$.

The lemma is proved.

Corollary 1.1 *If the multi-valued map $\mathscr{A} : X_m \to C_v(X_m^*)$ satisfies the property S_k on W_m, then $\mathscr{A}$ is pseudomonotone on W_m.*

Let further $I : X \to X^*$ be the *canonical embedding*. Let us fix $\lambda \in \mathbb{R}$ and set $\varphi_\lambda(t) = e^{-\lambda t}$, $t \in S$. For an arbitrary $y \in X^*$ let us define y_λ (as a map from S into V^*) as follows: $y_\lambda(t) = \varphi_\lambda(t)y(t)$ for a.e. $t \in S$. Let us remark that $(y_\lambda)_{-\lambda} = y$, for all $y \in X^*$. Also we define the element $\varphi_\lambda y$ by $(\varphi_\lambda y)(t) = y(t)\varphi_\lambda(t)$ for a.e. $t \in S$.

Lemma 1.3 *The map $y \mapsto y_\lambda$ is an isomorphism and an homeomorphism as a map acting from X_m into X_m (respectively from X_m^* into X_m^*, from W_m into W_m, from X into X, from X^* into X^*, from Y_m into Y_m, from Y into Y). Moreover, the map $W_m \ni y \longmapsto y_\lambda \in W_m$ is weakly-weakly continuous, i.e. from the fact that $y_n \rightharpoonup y$ in W_m it follows that $y_{n,\lambda} \rightharpoonup y_\lambda$ in W_m. Also, we have $y'_\lambda = \varphi'_\lambda y + \varphi_\lambda y' \in X_m^*$, $\forall y \in W_m$.*

Let us consider the multi-valued map $\mathscr{A} : X \to C_v(X^*)$. Let us define the set $\mathscr{A}_\lambda(y_\lambda) \in C_v(X^*)$ for fixed $y \in X$ by the next relation

$$[\mathscr{A}_\lambda(y_\lambda), \omega]_+ = [\mathscr{A}(y) + \lambda y, \omega_\lambda]_+, \quad \forall \omega \in X.$$

Let us remark that as the functional $\omega \longmapsto [\mathscr{A}(y) + \lambda y, \omega_\lambda]_+$ is semiadditive, positively homogeneous and lower semicontinuous (as the supremum of linear and continuous functionals), $\mathscr{A}_\lambda(y_\lambda)$ is defined correctly.

Lemma 1.4 *If the map $\mathscr{A} : X \to C_v(X^*)$ is bounded, then $\mathscr{A}_\lambda : X \to C_v(X^*)$ is bounded.*

Lemma 1.5 *If $\mathscr{A} : X_m \to C_v(X_m^*)$ is (W_m, X_m^*)-weakly closed, then $\mathscr{A}_\lambda$ is (W_m, X_m^*)-weakly closed.*

Proof Let $y_{n,\lambda} \rightharpoonup y_\lambda$ in W_m, $d_n \rightharpoonup d$ in X_m^*, $d_n \in \mathscr{A}_\lambda(y_{n,\lambda})$. Then, in virtue of Lemma 1.3 we obtain that $y_n := (y_{n,\lambda})_{-\lambda} \rightharpoonup y := (y_\lambda)_{-\lambda}$ in W_m, $y_{n,\lambda} \to y_\lambda$ in X_m and $y_n \to y$ in X_m. Since $[\mathscr{A}(y_n) + \lambda y_n, \omega_\lambda]_+ \ge \langle d_n, \omega\rangle_{X_m}$, for any $\omega \in X_m$, then $d_{n,-\lambda} \in \mathscr{A}(y_n) + \lambda y_n$. Therefore, $g_n := d_{n,-\lambda} - \lambda y_n \in \mathscr{A}(y_n)$. Let us remark that $d_{n,-\lambda} = (d_n)_{-\lambda} \rightharpoonup d_{-\lambda}$ in X_m^*, and since $X_m \subset X_m^*$ continuously, we have $g_n \rightharpoonup g$ in X_m^* for some $g \in X_m^*$. Due to the fact that $\mathscr{A}$ is (W_m, X_m^*)-weakly closed we have that $g \in \mathscr{A}(y)$. Therefore, $d_{n,-\lambda} - \lambda y_n \rightharpoonup g$ in X_m^*, so that $d_{n,-\lambda} \rightharpoonup \lambda y + g$ in X_m^*, and then $d_n = (d_{n,-\lambda})_\lambda \rightharpoonup \lambda y_\lambda + g_\lambda$ in X_m^*. Therefore,

$$\langle d, \omega\rangle_{X_m} = \langle \lambda y_\lambda + g_\lambda, \omega\rangle_{X_m} = \langle \lambda y + g, \omega_\lambda\rangle_{X_m} \le [\mathscr{A}(y) + \lambda y, \omega_\lambda]_+,$$

for all $\omega \in X_m$. Therefore,

$$d \in \mathscr{A}_\lambda(y_\lambda).$$

The lemma is proved.

Since the embedding W_m into X_m is compact, then Lemmas 1.2 and 1.5 yield the following corollary.

Corollary 1.2 *If the multi-valued map* $\mathscr{A} : X_m \to C_v(X_m^*)$ *satisfies the property* S_k *on* W_m, *then* $\mathscr{A}_\lambda$ *is pseudomonotone on* W_m.

1.1.4 Results

The main solvability results for Problem (1.3) are provided in Theorems 1.1 and 1.2, Corollaries 1.3, 1.4, and 1.5, and Proposition 1.1 (see also Fig. 1.3 and [4–11, 14–16, 18, 21, 23, 26, 27, 30, 32, 33, 35–47]).

Theorem 1.1 *Let* $a = \bar{0}$, $A : X \to C_v(X^*) \cap \mathscr{K}(X^*)$ *be coercive bounded map of the Volterra type that satisfies the property* S_k *on* W_σ. *Then for arbitrary* $f \in X^*$ *there exists at least one solution of Problem (1.3) that can be obtained via the Faedo–Galerkin method.*

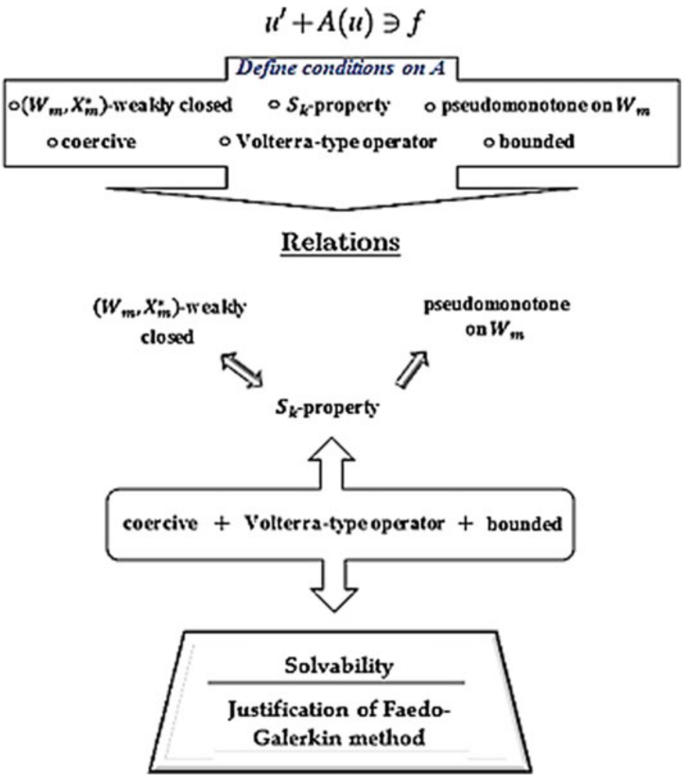

Fig. 1.3 Sufficient conditions of multi-valued mapping for the existence of a weak solution for differential-operator equation/inclusion via the FG method

Proof From coercivity for $A : X \rightrightarrows X^*$ it follows that $\forall\, y \in X$

$$\inf_{d \in A(y)} \langle d, y \rangle_X \geq \gamma(\|y\|_X)\|y\|_X.$$

So, $\exists\, r_0 > 0 : \gamma(r_0) > \|f\|_{X^*} \geq 0$. Therefore,

$$\forall y \in X : \ \|y\|_X = r_0 \quad [A(y) - f, y]_- \geq 0. \tag{1.6}$$

The solvability of approximate problems.
Let us consider the complete vectors system $\{h_i\}_{i \geq 1} \subset V$ such that
(α_1) $\{h_i\}_{i \geq 1}$ orthonormal in H;
(α_2) $\{h_i\}_{i \geq 1}$ orthogonal in V;
(α_3) $\forall\, i \geq 1 \ \ (h_i, v)_V = \lambda_i (h_i, v) \quad \forall v \in V$,
where $0 \leq \lambda_1 \leq \lambda_2, \cdots, \lambda_j \to \infty$ as $j \to \infty$, $(\cdot, \cdot)_V$ is the natural inner product in V, i.e. $\{h_i\}_{i \geq 1}$ is a special basis. Let for each $m \geq 1$ $H_m = \text{span}\,\{h_i\}_{i=1}^m$, on which we consider the inner product induced from H that we again denote by $(\cdot, \cdot)$. Due to the equivalence of H^* and H it follows that $H_m^* \equiv H_m$; $X_m = L_{p_0}(S; H_m)$, $X_m^* = L_{q_0}(S; H_m)$, $p_0 = \max\{r_1, r_2\}$, $q_0 > 1$: $1/p_0 + 1/q_0 = 1$, $\langle \cdot, \cdot \rangle_{X_m} == \langle \cdot, \cdot \rangle_X |_{X_m^* \times X_m}$, $W_m := \{y \in X_m |\ y' \in X_m^*\}$, where y' is the derivative of an element $y \in X_m$ is considered in the sense of $\mathscr{D}^*(S, H_m)$. For any $m \geq 1$ let $I_m \in \mathscr{L}(X_m; X)$ be the canonical embedding of X_m in X, I_m^* be the adjoint operator to I_m. Then

$$\forall\, m \geq 1 \quad \|I_m^*\|_{\mathscr{L}(X_\sigma^*; X_\sigma^*)} = 1. \tag{1.7}$$

Let us consider such maps:

$$A_m := I_m^* \circ A \circ I_m : X_m \to C_v(X^*), \quad f_m := I_m^* f.$$

Therefore, (1.6) and Corollary 1.1 yield that
(j_1) A_m is pseudomonotone on W_m;
(j_2) A_m is bounded;
(j_3) $[A_m(y) - f_m, y]_+ \geq 0\ \forall y \in X_m$: $\|y\|_X = r_0$.
Let us consider the operator $L_m : D(L_m) \subset X_m \to X_m^*$ with the definition domain

$$D(L_m) = \{y \in W_m \mid y(0) = \bar{0}\} = W_m^0,$$

that acts by the rule:

$$\forall y \in W_m^0 \quad L_m y = y',$$

where the derivative y' we consider in the sense of the distributions space $\mathscr{D}^*(S; H_m)$. The operator L_m satisfies the following properties:
(j_4) L_m is linear;
(j_5) $\forall\, y \in W_m^0$ $\langle L_m y, y \rangle \geq 0$;
(j_6) L_m is maximal monotone.

Therefore, conditions (j_1)–(j_6) and [51] guarantees the existence at least one solution $y_m \in D(L_m)$ of the problem:

$$L_m(y_m) + A_m(y_m) \ni f_m, \quad \|y_m\|_X \le r_0,$$

that can be obtained by the method of singular perturbations. This means that y_m is the solution of such problem:

$$\begin{cases} y_m' + A_m(y_m) \ni f_m \\ y_m(0) = \bar{0}, \;\; y_m \in W_m, \;\; \|y_m\|_X \le R, \end{cases} \tag{1.8}$$

where $R = r_0$.

Passing to the limit.

From the inclusion from (1.8) it follows that $\forall m \ge 1 \; \exists d_m \in A(y_m)$:

$$I_m^* d_m = f_m - y_m' \in A_m(y_m) = I_m^* A(y_m). \tag{1.9}$$

1°. The boundedness of $\{d_m\}_{m\ge 1}$ in X^* follows from the boundedness of A and from (1.8). Therefore,

$$\exists c_1 > 0 : \quad \forall m \ge 1 \quad \|d_m\|_{X^*} \le c_1. \tag{1.10}$$

2°. Let us prove the boundedness $\{y_m'\}_{m\ge 1}$ in X_σ^*. From (1.9) it follows that $\forall m \ge 1$ $y_m' = I_m^*(f - d_m)$, and, taking into account (1.7), (1.8) and (1.10) we have:

$$\|y_m'\|_{X_\sigma^*} \le \|y_m\|_{W_\sigma} \le c_2 < +\infty. \tag{1.11}$$

In virtue of (1.8) and the continuous embedding $W_m \subset C(S; H_m)$ we obtain the existence of $c_3 > 0$ such that

$$\forall m \ge 1, \; \forall t \in S \quad \|y_m(t)\|_H \le c_3. \tag{1.12}$$

3°. In virtue of estimations from (1.10)–(1.12), due to the Banach–Alaoglu theorem, taking into account the compact embedding $W \subset Y$, it follows the existence of subsequences

$$\{y_{m_k}\}_{k\ge 1} \subset \{y_m\}_{m\ge 1}, \quad \{d_{m_k}\}_{k\ge 1} \subset \{d_m\}_{m\ge 1}$$

and elements $y \in W$, $d \in X^*$, for which the following converges hold:

$$y_{m_k} \rightharpoonup y \text{ in } W, \quad d_{m_k} \rightharpoonup d \text{ in } X^*$$

$$y_{m_k}(t) \rightharpoonup y(t) \text{ in } H \text{ for each } t \in S \tag{1.13}$$

$$y_{m_k}(t) \to y(t) \text{ in } H \text{ for a.e. } t \in S \text{ as } k \to \infty.$$

Therefore, since $y_{m_k}(0) = \bar{0}$ for each $k \geq 1$, then $y(0) = \bar{0}$.

4°. Let us prove that

$$y' = f - d. \tag{1.14}$$

Let $\varphi \in D(S)$, $n \in \mathbb{N}$ and $h \in H_n$. Then $\forall k \geq 1$: $m_k \geq n$ we have:

$$\left(\int_S \varphi(\tau)(y'_{m_k}(\tau) + d_{m_k}(\tau))d\tau, h\right) = \langle y'_{m_k} + d_{m_k}, \psi\rangle,$$

where $\psi(\tau) = h \cdot \varphi(\tau) \in X_n \subset X$. Let us remark that here we use the property of Bochner integral [12, Theorem IV.1.8, p. 153]. Since for $m_k \geq n$ $H_{m_k} \supset H_n$, then $\langle y'_{m_k} + d_{m_k}, \psi\rangle = \langle f_{m_k}, \psi\rangle$. Therefore, $\forall k \geq 1 : m_k \geq n$

$$\langle f_{m_k}, \psi\rangle = \left(\int_S \varphi(\tau) f(\tau)d\tau, h\right).$$

Hence, for all $k \geq 1$: $m_k \geq n$

$$\left(\int_S \varphi(\tau)y'_{m_k}(\tau)d\tau,\ h\right) = \left\langle f - d_{m_k},\ \psi\right\rangle \to$$

$$\to \left(\int_S \varphi(\tau)(f(\tau) - d(\tau)d\tau,\ h\right) \text{ as } k \to \infty. \tag{1.15}$$

The last follows from the weak convergence d_{m_k} to d in X^*.

From convergence (1.13) we have:

$$\left(\int_S \varphi(\tau)y'_{m_k}(\tau)d\tau,\ h\right) \to \left(y'(\varphi),\ h\right) \text{ as } k \to \infty, \tag{1.16}$$

where

$$\forall \varphi \in \mathscr{D}(S)\ \ y'(\varphi) = -y(\varphi') = -\int_S y(\tau)\varphi'(\tau)d\tau.$$

Therefore, from (1.15) and (1.16) it follows that

$$\forall\varphi \in \mathscr{D}(S)\ \forall h \in \bigcup_{m\geq 1} H_m \quad (y'(\varphi), h) = \left(\int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau, h\right).$$

Since $\bigcup_{m\geq 1} H_m$ is dense in V we have that

$$\forall\varphi \in \mathscr{D}(S) \quad y'(\varphi) = \int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau.$$

Therefore, $y' = f - d \in X^*$.

5°. In order to prove that y is the solution of Problem (1.3) it remains to show that y satisfies the inclusion $y' + A(y) \ni f$. In virtue if identity (1.14), it is sufficient to prove that $d \in A(y)$.

From (1.13) it follows the existence of $\{\tau_l\}_{l\geq 1} \subset S$ such that $\tau_l \nearrow T$ as $l \to +\infty$ and

$$\forall\, l \geq 1 \quad y_{m_k}(\tau_l) \to y(\tau_l) \text{ in } H \text{ as } k \to \infty. \tag{1.17}$$

Let us show that for any $l \geq 1$

$$\langle d, w\rangle \leq [A(y), w]_+ \quad \forall w \in X : \; w(t) = 0 \text{ for a e. } t \in [\tau_l, T]. \tag{1.18}$$

Let us fix an arbitrary $\tau \in \{\tau_l\}_{l\geq 1}$. For $i = 1, 2$ let us set

$$X_{i,\sigma}(\tau) = L_{r_i}(\tau, T; H) \cap L_{p_i}(\tau, T; V_\sigma), \quad X_\sigma(\tau) = X_{1,\sigma}(\tau) \cap X_{2,\sigma}(\tau),$$

$$X^*_{i,\sigma}(\tau) = L_{r_{i'}}(\tau, T; H) + L_{q_i}(\tau, T; V^*_\sigma), \quad X^*_\sigma(\tau) = X^*_{1,\sigma}(\tau) + X^*_{2,\sigma}(\tau),$$

$$W_{i,\sigma}(\tau) = \{y \in X_i(\tau) \mid y' \in X^*_{i,\sigma}(\tau)\}, \quad W_\sigma(\tau) = W_{1,\sigma}(\tau) \cap W_{2,\sigma}(\tau),$$

$$a_0 = y(\tau), \quad a_k = y_{m_k}(\tau), \quad k \geq 1.$$

Similarly we introduce $X(\tau)$, $X^*(\tau)$, $W(\tau)$. From (1.17) it follows that

$$a_k \to a_0 \text{ in } H \text{ as } k \to +\infty. \tag{1.19}$$

For any $k \geq 1$ let $z_k \in W(\tau)$ be such that

$$\begin{cases} z_k' + J(z_k) \ni \bar{0}, \\ z_k(\tau) = a_k, \end{cases} \tag{1.20}$$

where $J : X(\tau) \to C_v(X^*(\tau))$ be the duality (in general multi-valued) mapping, i.e.

$$[J(u), u]_+ = [J(u), u]_- = \|u\|^2_{X(\tau)} = \|J(u)\|^2_+ = \|J(u)\|^2_-, \quad u \in X(\tau).$$

We remark that Problem (1.20) has a solution $z_k \in W(\tau)$ because J is monotone, coercive, bounded and demiclosed (see [1, 3, 12, 26]). Let us also note that for any $k \geq 1$

$$\|z_k(T)\|^2_H - \|a_k\|^2_H = 2\langle z_k', z_k\rangle_{X(\tau)} + 2\|z_k\|^2_{X(\tau)} = 0.$$

Hence,

$$\forall k \geq 1 \quad \|z_k'\|_{X^*(\tau)} = \|z_k\|_{X(\tau)} \leq \frac{1}{\sqrt{2}}\|a_k\|_H \leq c_3.$$

Due to (1.19), similarly to [12, 26], as $k \to +\infty$, z_k weakly converges in W to the unique solution $z_0 \in W$ of Problem (1.20) with initial time value condition $z(0) = a_0$. Moreover,

$$z_k \to z_0 \text{ in } X(\tau) \text{ as } k \to +\infty \tag{1.21}$$

because $\limsup_{k\to+\infty} \|z_k\|^2_{X(\tau)} \le \|z_0\|^2_{X(\tau)}$, $z_k \rightharpoonup z_0$ in $X(\tau)$, and $X(\tau)$ is a Hilbert space.

For any $k \ge 1$ let us set

$$u_k(t) = \begin{cases} y_{m_k}(t), & \text{if } t \in [0,\tau], \\ z_k(t), & \text{elsewhere}, \end{cases} \qquad g_k(t) = \begin{cases} d_{m_k}(t), & \text{if } t \in [0,\tau], \\ \hat{d}_k(t), & \text{elsewhere}, \end{cases}$$

where $\hat{d}_k \in A(u_k)$ is an arbitrary. As $\{u_k\}_{k\ge1}$ is bounded, $A : X \rightrightarrows X^*$ is bounded, then $\{\hat{d}_k\}_{k\ge1}$ is bounded in X^*. In virtue of (1.21), (1.13), (1.17)

$$\lim_{k\to+\infty} \langle g_k, u_k - u\rangle = \lim_{k\to+\infty} \int_0^\tau (d_k(t), y_k(t) - y(t))\,dt =$$

$$= \lim_{k\to+\infty} \int_0^\tau \big(f(t) - y_k'(t), y_k(t) - y(t)\big)\,dt = \lim_{k\to+\infty} \int_0^\tau \big(y_k'(t), y(t) - y_k(t)\big)\,dt =$$

$$= \lim_{k\to+\infty} \frac{1}{2}\left(\|y_k(0)\|_H^2 - \|y_k(\tau)\|_H^2\right) + \lim_{k\to+\infty} \int_0^\tau \big(y_k'(t), y(t)\big)\,dt =$$

$$= \frac{1}{2}\left(\|y(0)\|_H^2 - \|y(\tau)\|_H^2\right) + \int_0^\tau \big(y'(t), y(t)\big)\,dt = 0.$$

So,

$$\lim_{k\to+\infty} \langle g_k, u_k - u\rangle = 0. \tag{1.22}$$

Let us show that $g_k \in A(u_k)$ $\forall k \ge 1$. For any $w \in X$ let us set

$$w_\tau(t) = \begin{cases} w(t), & \text{if } t \in [0,\tau], \\ \bar{0}, & \text{elsewhere}, \end{cases} \qquad w^\tau(t) = \begin{cases} \bar{0}, & \text{if } t \in [0,\tau], \\ w(t), & \text{elsewhere}. \end{cases}$$

In virtue of A is the Volterra type operator we obtain that

$$\langle g_k, w\rangle = \langle d_{m_k}, w_\tau\rangle + \langle \hat{d}_k, w^\tau\rangle \le [A(y_{m_k}), w_\tau]_+ + \langle \hat{d}_k, w^\tau\rangle [A(u_k), w_\tau]_+ + \langle \hat{d}_k, w^\tau\rangle$$

$$\le [A(u_k), w_\tau]_+ + [A(u_k), w^\tau]_+.$$

Due to $A(u_k) \in \mathscr{H}(X^*)$, similarly to [45], we obtain that

$$[A(u_k), w_\tau]_+ + [A(u_k), w^\tau]_+ = [A(u_k), w]_+.$$

Since $w \in X$ is an arbitrary, then $g_k \in A(u_k)$ $\forall k \geq 1$. Due to $\{u_k\}_{k\geq 1}$ is bounded in X, then $\{g_k\}_{k\geq 1}$ is bounded in X^*. Thus, up to a subsequence $\{u_{k_j}, g_{k_j}\}_{j\geq 1} \subset \{u_k, g_k\}_{k\geq 1}$, for some $u \in W$, $g \in X^*$ the following convergence takes place

$$u_{k_j} \rightharpoonup u \text{ in } W_\sigma, \quad g_{k_j} \rightharpoonup g \text{ in } X^* \text{ as } j \to \infty. \tag{1.23}$$

We remark that

$$u(t) = y(t), \quad g(t) = d(t) \text{ for a.e. } t \in [0, \tau]. \tag{1.24}$$

In virtue of (1.22), (1.23), as A satisfies the property S_k on W_σ, we obtain that $g \in A(u)$. Hence, due to (1.24), as A is the Volterra type operator, for any $w \in X$ such that $w(t) = 0$ for a.e. $t \in [\tau, T]$ we have

$$\langle d, w\rangle = \langle g, w\rangle \leq [A(u), w]_+ = [A(y), w]_+.$$

As $\tau \in \{\tau_l\}_{l\geq 1}$ is an arbitrary, we obtain (1.18).

From (1.18), due to the functional $w \to [A(y), w]_+$ is convex and lower semi-continuous on X (hence it is continuous on X) we obtain that for any $w \in X$ $\langle d, w\rangle \leq [A(y), w]_+$. So, $d \in A(y)$.

The theorem is proved.

The following corollary to Theorem 1.1 establishes sufficient conditions for solvability of Problem (1.3) with nonzero initial conditions; see [28].

Corollary 1.3 *Let $A : X \to C_v(X^*) \cap \mathscr{H}(X^*)$ be bounded map of the Volterra type that satisfies the property S_k on W_σ. Moreover, let for some $c > 0$*

$$\frac{\inf_{d\in A(y)}\langle d, y\rangle_X - c\|A(y)\|_+}{\|y\|_X} \to +\infty \tag{1.25}$$

as $\|y\|_X \to +\infty$. Then for each $a \in H$ and $f \in X^*$ there exists at least one solution of Problem (1.3) that can be obtained via the Faedo–Galerkin method.

Proof Let us set $\varepsilon = \frac{\|a\|_H^2}{2c^2}$. We consider $w \in W$ such that

$$\begin{cases} w' + \varepsilon J(w) = \bar{0}, \\ w(0) = a, \end{cases}$$

where $J : X \to C_v(X^*)$ be the duality map. Hence $\|w\|_X \leq c$. We define $\hat{A} : X \to C_v(X^*) \cap \mathscr{H}(X^*)$ by the rule: $\hat{A}(z) = A(z+w)$, $z \in X$. Let us set $\hat{f} = f - w' \in X^*$. If $z \in W$ is the solution of the problem

$$\begin{cases} z' + \hat{A}(z) \ni f, \\ z(0) = \bar{0}, \end{cases}$$

then $y = z + w$ is the solution of Problem (1.3). It is clear that $\hat{A}$ is a bounded map of the Volterra type that satisfies the property S_k on W. Thus, Theorem 1.1 yields that it is sufficient to verify the coercivity for the map $\hat{A}$. This property follows from the following estimates:

$$[\hat{A}(z), z]_- \geq [A(z+w), z+w]_- - [A(z+w), w]_+ \geq$$

$$\geq [A(z+w), z+w]_- - c\|A(z+w)\|_+,$$

$$\|z\|_X \geq \|z+w\|_X - c.$$

The corollary is proved.

Analyzing the proof of Theorem 1.1 the following convergence result holds.

Corollary 1.4 *$A : X \to C_v(X^*) \cap \mathscr{H}(X^*)$ be bounded map of the Volterra type that satisfies the property S_k on W_σ, $\{a_n\}_{n\geq 0} \subset H$: $a_n \to a_0$ in H as $n \to +\infty$, $y_n \in W$, $n \geq 1$ be the corresponding to initial data a_n solution of Problem (1.3). If $y_n \rightharpoonup y_0$ in X, as $n \to +\infty$, then $y \in W$ is the solution of Problem (1.3) with initial data a_0. Moreover, up to a subsequence, $y_n \rightharpoonup y_0$ in $W_\sigma \cap C(S; H)$.*

Now let V and H be real Hilbert spaces, $V_1 = V_2 = V_\sigma := V$; $p_i = r_i = 2$, $i = 1, 2$. Let us set $Y = L_2(S; H)$. Then, according to the identification $H^* \equiv H$, the spaces Y^* and $L_2(S; H)$ are identified.

We note that the vector space $W = \{y \in X \mid y' \in X^*\}$ is a Hilbert space with the norm $\|y\|_W = \|y\|_X + \|y'\|_{X^*}$, where y' is the derivative of $y \in X$ in the sense of the space of distributions $\mathscr{D}^*(S; V^*)$ [12]. For any $v \in X$ and $f \in X^*$ consider the pairing

$$\langle f, v \rangle = \int_S \langle f(\tau), v(\tau) \rangle_V \, d\tau,$$

where $\langle \cdot, \cdot \rangle_V : V^* \times V \to \mathbb{R}$ is the canonical pairing, which coincides with the inner product $(\cdot, \cdot)$ in H on $H \times V$. Hence, $\langle f, v \rangle = \int_S (f(\tau), v(\tau))\, d\tau$ if $f \in Y$. In the sequel, to simplify the conclusions, we shall use the last notation even if $f \in X^*$.

In the following theorem we justify the Faedo–Galerkin method for solutions of Problem (1.3) when the multi-valued mapping A is possibly noncoercive (see also Fig. 1.4).

Theorem 1.2 *Let $a = \bar{0}$, $A : X \to C_v(X^*) \cap \mathscr{H}(X^*)$ be a bounded map of the Volterra type, which satisfies the property S_k on W. Moreover, let for some $\lambda \geq 0$ the map $A + \lambda I$ be coercive. Then for arbitrary $f \in X^*$ there exists at least one solution of Problem (1.3), which can be obtained via the Faedo–Galerkin method.*

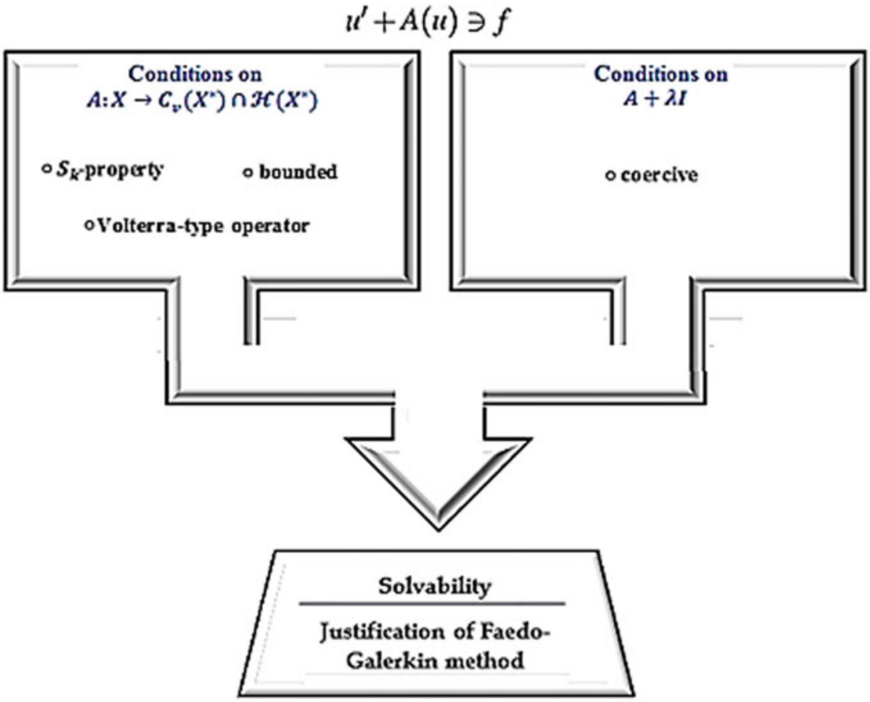

Fig. 1.4 Sufficient conditions for the existence of a solution

Proof We shall provide the proof in several steps.

Step 1: A priory estimate.

At first let us show that there exists a real nondecreasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}$ such that $\gamma(r) \to +\infty$ as $r \to +\infty$, the function in hands is bounded from below on bounded sets and the following inequality holds:

$$\inf_{d \in A(y)} \int_0^T e^{-2\lambda\tau}(d(\tau) + \lambda y(\tau), y(\tau))d\tau \geq \gamma(\|y\|_X)\|y\|_X, \tag{1.26}$$

for each $y \in X$. For an arbitrary $r > 0$ we set

$$\widetilde{\gamma}(r) = \inf_{y \in X,\, \|y\|_X = r} \inf_{d \in A(y)} \frac{\langle d + \lambda y, y \rangle_X}{\|y\|_X}$$

and $\widetilde{\gamma}(0) := 0$. The following properties hold:

(a) As A is bounded and the embedding $X \subset X^*$ is continuous, we have $\widetilde{\gamma}(r) > -\infty$.
(b) From the construction of the function $\widetilde{\gamma}$ we have that for all $y \in X$,

$$[A(y) + \lambda y, y]_- \geq \widetilde{\gamma}(\|y\|_X)\|y\|_X. \tag{1.27}$$

In virtue of the boundedness of A it follows that $\widetilde{\gamma}$ is bounded from below on bounded sets.

(c) From the coercivity of $A + \lambda I$ it follows that $\widetilde{\gamma}(r) \to +\infty$ as $r \to +\infty$.

(d) From (a)–(c) we have $\inf\limits_{r\geq 0} \widetilde{\gamma}(r) =: a > -\infty$.

For an arbitrary $b > a$ let us consider the nonempty bounded set of $\mathbb{R}_+$ given by $A_b = \{c \geq 0 \mid \widetilde{\gamma}(c) \leq b\}$. Let $c_b = \inf_{c\in A_b} c$, $b > a$. Let us remark that $c_{b_2} \leq c_{b_1} < +\infty$, for all $b_1 > b_2 > a$, and $c_b \to +\infty$ as $b \to +\infty$. Let us set

$$\widehat{\gamma}(t) = \begin{cases} a, & t \in [0, c_{a+1}], \\ a + k, & t \in (c_{a+k}, c_{a+k+1}],\ k \geq 1. \end{cases}$$

Then, $\widehat{\gamma} : \mathbb{R}_+ \to \mathbb{R}$ is a bounded from below function on bounded sets of $\mathbb{R}_+$, it is a nondecreasing function such that $\widehat{\gamma}(r) \to +\infty$, as $r \to \infty$, and $\widehat{\gamma}(t) \leq \widetilde{\gamma}(t)$, for any $t \geq 0$.

Let us fix an arbitrary $y \in X$. Since A is the operator of the Volterra type, then

$$\begin{aligned} &\inf_{d\in A(y)} \int_0^t \big(d(\tau) + \lambda y(\tau), y(\tau)\big)d\tau \\ &= \inf_{d\in A(y)} \int_0^T \big(d(\tau) + \lambda y_t(\tau), y_t(\tau)\big)d\tau \\ &\geq \widehat{\gamma}(\|y_t\|_X)\|y_t\|_X = \widehat{\gamma}(\|y\|_{X_t})\|y\|_{X_t}, \end{aligned}$$

for all $t \in S$, where $\|y\|_{X_t} = \|y_t\|_X$, $y_t(\tau) = \begin{cases} y(\tau), & \tau \in [0, t], \\ \bar{0}, & \text{else.} \end{cases}$ Let for an arbitrary $d \in A(y)$

$$\begin{aligned} g_d(\tau) &= \big(d(\tau) + \lambda y(\tau), y(\tau)\big), \quad \text{for a.e.} \quad \tau \in S, \\ h(t) &= \widehat{\gamma}(\|y\|_{X_t})\|y\|_{X_t}, \quad t \in S. \end{aligned}$$

Let us remark that $h(t) \geq \min\{\widehat{\gamma}(0), 0\}\|y\|_X$ and

$$\inf_{d\in A(y)} \int_0^t g_d(\tau)d\tau \geq h(t),\ t \in S.$$

Let us show that

$$\inf_{d\in A(y)} \int_0^T e^{-2\lambda\tau}(d(\tau) + \lambda y(\tau), y(\tau))d\tau \geq$$

$$\geq e^{-2\lambda T} \inf_{d\in A(y)} \int_0^T (d(\tau) + \lambda y(\tau), y(\tau))d\tau + \tag{1.28}$$

$$+\inf_{d\in A(y)}\int_0^T (e^{-2\lambda\tau}-e^{-2\lambda T})(d(\tau)+\lambda y(\tau), y(\tau))d\tau.$$

Let us set $\varphi(\tau) = e^{-2\lambda(T-\tau)}$, $\tau \in [0, T]$ (so $\varphi \in (0, 1]$). For any $n \geq 1$ we put $\varphi_n(\tau) = \sum_{i=0}^{n-1} \varphi\left(\frac{iT}{n}\right)\chi_{\left[\frac{iT}{n},\frac{(i+1)T}{n}\right)}(\tau)$, $\tau \in [0, T]$. Then, $\varphi\left(\frac{iT}{n}\right)d_1 + \left(1-\varphi\left(\frac{iT}{n}\right)\right)d_2 \in A(y), \forall d_1 \in A(y), \forall d_2 \in A(y), \forall i = \overline{0, n-1}$. Let us remark that $|\varphi_n(\tau)-\varphi(\tau)| \leq \frac{2\lambda T}{n}, \forall\tau \in [0, T]$. Lemma 1.1 implies that

$$d = \sum_{i=0}^{n-1}(\varphi\left(\frac{iT}{n}\right)d_1 + \left(1-\varphi\left(\frac{iT}{n}\right)\right)d_2)\chi_{[t_i,t_{i+1})}(\tau) \in A(y),$$

where $t_i = \frac{iT}{n}$. Therefore,

$$\inf_{d\in A(y)}\int_0^T e^{-2\lambda\tau}(d(\tau)+\lambda y(\tau), y(\tau))d\tau \geq$$

$$\geq \int_0^T (d(\tau)+\lambda y(\tau), y(\tau)e^{-2\lambda\tau})d\tau =$$

$$= \int_0^T \varphi_n(\tau)(d_1(\tau)+\lambda y(\tau), y(\tau)e^{-2\lambda\tau})d\tau +$$

$$+\int_0^T (1-\varphi_n(\tau))(d_2(\tau)+\lambda y(\tau), y(\tau)e^{-2\lambda\tau})d\tau \geq$$

$$\geq e^{-2\lambda T}\int_0^T (d_1(\tau)+\lambda y(\tau), y(\tau))d\tau +$$

$$+\int_0^T \left(e^{-2\lambda\tau}-e^{-2\lambda T}\right)(d_2(\tau)+\lambda y(\tau), y(\tau))d\tau -$$

$$-\frac{4\lambda T}{n}\left(\|A(y)\|_+\|y\|_X + \lambda\|y\|_Y^2\right).$$

If $n \to +\infty$, then taking the infimum with respect to $d_1 \in A(y)$ and $d_2 \in A(y)$ in the last inequality we will obtain (1.28). From (1.28) it follows that

$$\inf_{d\in A(y)}\int_0^T e^{-2\lambda\tau}(d(\tau)+\lambda y(\tau), y(\tau))d\tau \geq$$

$$\geq e^{-2\lambda T}h(T) + 2\lambda\inf_{d\in A(y)}\int_0^T e^{-2\lambda s}\int_0^s g_d(\tau)d\tau ds \geq$$

$$\geq e^{-2\lambda T}h(T)+$$

$$+2\lambda T \inf_{d\in A(y)} \inf_{s\in S} e^{-2\lambda s} \int_0^s (d(\tau)+\lambda y(\tau), y(\tau))d\tau.$$

Let us show that

$$\inf_{d\in A(y)} \inf_{s\in S} e^{-2\lambda s} \int_0^s (d(\tau)+\lambda y(\tau), y(\tau))d\tau \geq -c_1\|y\|_X,$$

where $c_1 = \max\{-\widehat{\gamma}(0), 0\} \geq 0$ does not depend on $y \in X$. Let $y \in X$ is fixed. For $s \in S$, $d \in A(y)$ let us set

$$\varphi(s,d) = e^{-2\lambda s} \int_0^s \big(d(\tau)+\lambda y(\tau), y(\tau)\big)d\tau,$$
$$a = \inf_{d\in A(y)} \inf_{s\in S} \varphi(s,d),\ \ S_d = \{s \in S \mid \varphi(s,d) \leq a\}.$$

From the continuity of $\varphi(\cdot, d)$ on S it follows that S_d is a nonempty closed set for an arbitrary $d \in A(y)$. Indeed, for any fixed $d \in A(y)$ there exists $s_d \in S$ such that

$$\varphi(s_d, d) = \min_{\hat{s}\in S} \varphi(\hat{s}, d) \leq a.$$

From the continuity of $\varphi(\cdot, d)$ on S it follows that S_d is closed.

Let us prove now that the system $\{S_d\}_{d\in A(y)}$ is centered. For fixed $\{d_i\}_{i=1}^n \subset A(y)$, $n \geq 1$, let us set

$$\begin{aligned}
&\psi_i(\cdot) = \big(d_i(\cdot)+\lambda y(\cdot), y(\cdot)\big),\\
&\psi(\cdot) = \max_{i\in\{1,\dots,n\}} \psi_i(\cdot),\\
&E_0 = \emptyset,\\
&E_j = \left\{\tau \in S \setminus \left(\cup_{i=0}^{j-1} E_i\right) \mid \psi_j(\tau) = \psi(\tau)\right\},
\end{aligned}$$

for $j = \overline{1,n}$, and

$$d(\cdot) = \sum_{j=1}^n d_j(\cdot)\chi_{E_j}(\cdot).$$

Let us remark that E_j is measurable for any $j = \overline{1,n}$, $\cup_{j=1}^n E_j = S$, $E_i \cap E_j = \emptyset$, $\forall i \neq j, i, j = \overline{1,n}$. Also, $d \in X^*$. Moreover,

$$\varphi(s, d_i) = e^{-2\lambda s} \int_0^s \psi_i(\tau)d\tau \leq e^{-2\lambda s} \int_0^s \psi(\tau)d\tau =$$

$$= \varphi(s,d), \quad s \in S,\ i = \overline{1,n}.$$

Therefore, in virtue of Lemma 1.1 we have $d \in A(y)$ and for some $s_d \in S$,

$$\varphi(s_d, d_i) \le \varphi(s_d, d) = \min_{\hat{s} \in S} \varphi(\hat{s}, d) \le a, \quad i = \overline{1, n}.$$

So, $s_d \in \cap_{i=1}^n S_{d_i} \neq \emptyset$.

Since S is compact, and the system of closed sets $\{S_d\}_{d \in A(y)}$ is centered, we obtain the existence of $s_0 \in S$ such that $s_0 \in \cap_{d \in A(y)} S_d$. This implies that

$$\begin{aligned}
&\inf_{d \in A(y)} \inf_{s \in S} e^{-2\lambda s} \int_0^s \big(d(\tau) + \lambda y(\tau), y(\tau)\big) d\tau \\
&\ge \inf_{d \in A(y)} e^{-2\lambda s_0} \int_0^{s_0} \big(d(\tau) + \lambda y(\tau), y(\tau)\big) d\tau \\
&= e^{-2\lambda s_0} \inf_{d \in A(y)} \int_0^{s_0} g_d(\tau) d\tau \ge e^{-2\lambda s_0} h(s_0) \\
&\ge e^{-2\lambda s_0} \min\{\widehat{\gamma}(0), 0\} \|y\|_X \\
&\ge -\max\{-\widehat{\gamma}(0), 0\} \|y\|_X = -c_1 \|y\|_X.
\end{aligned}$$

Therefore, for all $y \in X$,

$$\inf_{d \in A(y)} \int_0^T e^{-2\lambda\tau} (d(\tau) + \lambda y(\tau), y(\tau)) d\tau \ge$$

$$\ge \big(e^{-2\lambda T} \widehat{\gamma}(\|y\|_X) - 2\lambda c_1 T\big) \|y\|_X.$$

If we set $\gamma(r) = e^{-2\lambda T} \widehat{\gamma}(r) - 2\lambda c_1 T$, then we will obtain (1.26).

From (1.26), the properties of the real function γ and the conditions of the theorem it follows the existence of $r_0 > 0$ such that $\gamma(r_0) > \|f_\lambda\|_{X^*} \ge 0$ and also that for any $y \in X$,

$$[A_\lambda(y_\lambda), y_\lambda]_- \ge \gamma(\|y\|_X)\|y\|_X \ge \gamma(\|y_\lambda\|_X)\|y_\lambda\|_X.$$

Therefore, for all $y \in X$ satisfying $\|y_\lambda\|_X = r_0$ we have

$$[A_\lambda(y_\lambda) - f_\lambda, y_\lambda]_- \ge (\gamma(r_0) - \|f_\lambda\|_{X^*}) r_0 \ge 0,$$

that is,

$$[A_\lambda(y_\lambda) - f_\lambda, y_\lambda]_- \ge 0. \tag{1.29}$$

Step 2: Finite-dimensional approximations.

We shall consider now a sequence of finite-dimensional approximative problems via the Faedo–Galerkin method.

For any $m \ge 1$ let $I_m \in \mathscr{L}(X_m; X)$ be the canonical embedding of X_m into X, and I_m^* be the adjoint operator to I_m. Then

$$\|I_m^*\|_{\mathscr{L}(X^*;\,X^*)} = 1,\ \forall m \geq 1. \tag{1.30}$$

Let us consider the following maps [25]:

$$\begin{aligned}
&A_m := I_m^* \circ A \circ I_m : X_m \to C_v(X^*),\\
&A_{\lambda,m} := I_m^* \circ A_\lambda \circ I_m : X_m \to C_v(X^*),\\
&A_{m,\lambda} := (A_m)_\lambda : X_m \to C_v(X^*),\\
&f_m := I_m^* f,\ \ f_{\lambda,m} := I_m^* f_\lambda,\ \ f_{m,\lambda} := (f_m)_\lambda.
\end{aligned}$$

Let us remark that

$$A_{\lambda,m} = A_{m,\lambda},\ \ f_{\lambda,m} = f_{m,\lambda}. \tag{1.31}$$

Indeed, in virtue of Lemma 1.3 for any $y, w \in X_m$,

$$[A_{\lambda,m}(y_\lambda), w]_- = [(I_m^* \circ A_\lambda)(y_\lambda), w]_- = [A_\lambda(y_\lambda), w]_-$$

$$= [A(y) + \lambda y, w_\lambda]_- = [I_m^* \circ (A + \lambda I)(y), w_\lambda]_- =$$

$$= [(A_m)_\lambda(y_\lambda), w]_- = [A_{m,\lambda}(y_\lambda), w]_-.$$

So, from (1.29), (1.31), Lemma 1.4, Corollary 1.2, and the conditions of the theorem, applying similar arguments as in [25, pp. 115–117], [19, pp. 197–198], we obtain the following properties:

(j1) $A_{\lambda,m}$ is pseudomonotone on W_m;
(j2) $A_{\lambda,m}$ is bounded;
(j3) $[A_{\lambda,m}(y_\lambda) - f_{\lambda,m}, y_\lambda]_- \geq 0$ for all $y_\lambda \in X_m$ such that $\|y_\lambda\|_X = r_0$.

We note that (j3) is a consequence of (1.29) and the definition of $A_{\lambda,m}$, $f_{\lambda,m}$, whereas (j2) follows from Lemma 1.4 and the boundednesss of I_m, I_m^*. Finally, (j1) is obtained in the following way: since A satisfies the property S_k in W, for A_m the same property holds on W_m; hence, by Corollary 1.2 the operator $A_{m,\lambda} = (A_m)_\lambda$ is pseudomonotone in W_m, and then (1.31) implies (j1).

Let us consider the operator $L_m : D(L_m) \subset X_m \to X_m^*$ with domain

$$D(L_m) = \{y \in W_m \mid y(0) = \bar{0}\} = W_m^0,$$

which is defined by the rule: $L_m y = y'$, $\forall y \in W_m^0$, where the derivative y' we consider in the sense of the space of distributions $\mathscr{D}^*(S; H_m)$. From [25, Lemma 5, p. 117] for the operator L_m the next properties are true:

(j4) L_m is linear;
(j5) $\langle L_m y, y\rangle \geq 0$, $\forall y \in W_m^0$;
(j6) L_m is maximal monotone.

Therefore, conditions (j_1)–(j_6) and Theorem 3.1 from [26] guaranty the existence of at least one solution $z_m \in D(L_m)$ of the problem:

$$L_m(z_m) + A_{\lambda,m}(z_m) \ni f_{\lambda,m}, \quad \|z_m\|_X \leq r_0,$$

which can be obtained by the method of singular perturbations. This means (see (1.31)) that $y_m := (z_m)_{-\lambda} \in W_m$ is the solution of the problem

$$\begin{cases} y_m' + A_m(y_m) \ni f_m, \\ y_m(0) = \bar{0}, \ y_m \in W_m, \ \|y_m\|_X \leq R, \end{cases} \tag{1.32}$$

where $R = r_0 e^{\lambda T}$.

Step 3: Passing to the limit.

From (1.32) it follows that for any $m \geq 1$ there exists $d_m \in A(y_m)$ such that

$$I_m^* d_m = f_m - y_m' \in A_m(y_m) = I_m^* A(y_m). \tag{1.33}$$

Let us prove now that (up to a subsequence) the sequence of solutions of (1.32) converges to a solution of (1.3). Again, we divide this proof in some substeps.

Step 3a.

The boundedness of A and (1.32) imply that $\{d_m\}_{m\geq 1}$ is bounded in X^*. Therefore, there exists $c_1 > 0$ such that

$$\|d_m\|_{X^*} \leq c_1 \quad \forall m \geq 1. \tag{1.34}$$

Step 3b.

Let us prove the boundedness of $\{y_m'\}_{m\geq 1}$ in X^*. From (1.33) it follows that $y_m' = I_m^*(f - d_m)$, $\forall m \geq 1$, and taking into account (1.30), (1.32) and (1.34) we have

$$\|y_m'\|_{X^*} \leq \|y_m\|_W \leq R + \|f\|_{X^*} + c_1 =: c_2. \tag{1.35}$$

In virtue of the continuous embedding $W \subset C(S; H)$ we obtain the existence of $c_3 > 0$ such that

$$\|y_m(t)\|_H \leq c_3 \quad \forall m \geq 1, \ \forall t \in S. \tag{1.36}$$

Step 3c.

In virtue of estimates (1.34)–(1.36), due to the Banach–Alaoglu theorem, and taking into account the continuous embedding $W \subset C(S; H)$ and the compact embedding $W \subset Y$, it follows the existence of subsequences

$$\{y_{m_k}\}_{k\geq 1} \subset \{y_m\}_{m\geq 1}, \quad \{d_{m_k}\}_{k\geq 1} \subset \{d_m\}_{m\geq 1}$$

and elements $y \in W$, $d \in X^*$, for which the next convergences take place:

$$\begin{aligned} &y_{m_k} \rightharpoonup y \text{ in } W, \quad d_{m_k} \rightharpoonup d \text{ in } X^*, \\ &y_{m_k}(t) \rightharpoonup y(t) \text{ in } H \text{ for each } t \in S, \\ &y_{m_k}(t) \to y(t) \text{ in } H \text{ for a.e. } t \in S \text{ as } k \to \infty. \end{aligned} \tag{1.37}$$

From here, as $y_{m_k}(0) = \bar{0}$, $\forall k \geq 1$, we have $y(0) = \bar{0}$.
Step 3d.

Let us prove that
$$y' = f - d. \tag{1.38}$$

Let $\varphi \in D(S),\ n \in \mathbb{N}$, and $h \in H_n$. Then for all $k \geq 1$ such that $m_k \geq n$ we have

$$\left(\int_S \varphi(\tau)(y'_{m_k}(\tau) + d_{m_k}(\tau))d\tau, h\right) = \langle y'_{m_k} + d_{m_k}, \psi\rangle,$$

where $\psi(\tau) = h \cdot \varphi(\tau) \in X_n \subset X$. Let us remark that here we use the properties of Bochner's integral (see [12], Theorem IV.1.8). Since $H_{m_k} \supset H_n$, for $m_k \geq n$ we get $\langle y'_{m_k} + d_{m_k}, \psi\rangle = \langle f_{m_k}, \psi\rangle$. Therefore, for all $k \geq 1$ such that $m_k \geq n$

$$\langle f_{m_k}, \psi\rangle = \left(\int_S \varphi(\tau) f(\tau)d\tau, h\right).$$

Hence, for all $k \geq 1$ such that $m_k \geq n$

$$\left(\int_S \varphi(\tau) y'_{m_k}(\tau)d\tau, h\right) = \langle f - d_{m_k}, \psi\rangle \rightarrow$$

$$\rightarrow \left(\int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau, h\right) \text{ as } k \rightarrow \infty. \tag{1.39}$$

The last convergence follows from the weak convergence d_{m_k} to d in X^*. From (1.37) we have
$$\left(\int_S \varphi(\tau) y'_{m_k}(\tau)d\tau, h\right) \rightarrow \big(y'(\varphi), h\big) \text{ as } k \rightarrow +\infty, \tag{1.40}$$

where
$$y'(\varphi) = -y(\varphi') = -\int_S y(\tau)\varphi'(\tau)d\tau,\ \forall \varphi \in \mathscr{D}(S).$$

Therefore, from (1.39) and (1.40) it follows that for all $\varphi \in \mathscr{D}(S),\ h \in \bigcup_{m\geq 1} H_m$,

$$\big(y'(\varphi), h\big) = \left(\int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau, h\right).$$

Since $\bigcup\limits_{m\geq 1} H_m$ is dense in V we have that

$$y'(\varphi) = \int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau, \forall \varphi \in \mathscr{D}(S).$$

Therefore, $y' = f - d \in X^*$.

Step 3e.

To prove that y is a solution of Problem (1.3) it remains to verify that y satisfies the inclusion $y' + A(y) \ni f$. Thus, according to (1.38), it is sufficient to prove that $d \in A(y)$.

From (1.37) it follows the existence of $\{\tau_l\}_{l\geq 1} \subset S$ such that $\tau_l \nearrow T$ as $l \to +\infty$, and

$$y_{m_k}(\tau_l) \to y(\tau_l) \text{ in } H \ \forall l \geq 1 \ \text{ as } k \to +\infty. \tag{1.41}$$

Let us show that

$$\langle d, w\rangle \leq [A(y), w]_+ \tag{1.42}$$

for each $l \geq 1$ and $w \in X$ such that $w(t) = 0$ for a.e. $t \in [\tau_l, T]$.

Let us fix an arbitrary $\tau \in \{\tau_l\}_{l\geq 1}$. Let us set

$$X(\tau) = L_2(\tau, T; V), \quad X^*(\tau) = L_2(\tau, T; V^*),$$

$$\langle u, v\rangle_{X(\tau)} = \int_\tau^T \langle u(s), v(s)\rangle_V \, ds$$

for $u \in X(\tau),\ v \in X^*(\tau)$, and

$$W(\tau) = \{u \in X(\tau) \mid u' \in X^*(\tau)\},$$

$$a_0 = y(\tau), \quad a_k = y_{m_k}(\tau), \quad k \geq 1.$$

From (1.41) it follows that

$$a_k \to a_0 \text{ in } H \text{ as } k \to +\infty. \tag{1.43}$$

For any $k \geq 1$ let $z_k \in W(\tau)$ be such that

$$\begin{cases} z_k' + J(z_k) = \bar{0}, \\ z_k(\tau) = a_k, \end{cases} \tag{1.44}$$

where $J : X(\tau) \to X^*(\tau)$ is the duality mapping (which is single-valued, as $X(\tau)$ is a Hilbert space), i.e.

$$\langle J(u), u\rangle_{X(\tau)} = \|u\|^2_{X(\tau)} = \|J(u)\|^2_{X^*(\tau)}, \quad u \in X(\tau).$$

We remark that Problem (1.44) has a solution $z_k \in W(\tau)$ because $J : X(\tau) \to X^*(\tau)$ is monotone, coercive, bounded and demicontinuous (see [1, 3, 12, 26]). Let us also note that for any $k \geq 1$,

$$\|z_k(T)\|^2_H - \|a_k\|^2_H = 2\langle z_k', z_k\rangle_{X(\tau)} = -2\|z_k\|^2_{X(\tau)}.$$

Hence,

$$\|z_k'\|_{X^*(\tau)} = \|z_k\|_{X(\tau)} \leq \frac{1}{\sqrt{2}}\|a_k\|_H \leq c_3, \ \forall k \geq 1.$$

Due to (1.43), similarly to [12, 26], z_k converges weakly in W as $k \to +\infty$ to the unique solution $z_0 \in W$ of Problem (1.44) with initial condition $z(\tau) = a_0$. Moreover,

$$z_k \to z_0 \text{ in } X(\tau) \text{ as } k \to +\infty, \tag{1.45}$$

because $\limsup_{k\to+\infty} \|z_k\|^2_{X(\tau)} \leq \|z_0\|^2_{X(\tau)}$, $z_k \rightharpoonup z_0$ in $X(\tau)$, and $X(\tau)$ is a Hilbert space. For any $k \geq 1$ let us set

$$u_k(t) = \begin{cases} y_{m_k}(t), & \text{if } t \in [0, \tau], \\ z_k(t), & \text{elsewhere}, \end{cases}$$

$$g_k(t) = \begin{cases} d_{m_k}(t), & \text{if } t \in [0, \tau], \\ \hat{d}_k(t), & \text{elsewhere}, \end{cases}$$

where $\hat{d}_k \in A(u_k)$ be an arbitrary. As $\{u_k\}_{k\geq1}$ is bounded and $A : X \rightrightarrows X^*$ is bounded, we obtain that $\{\hat{d}_k\}_{k\geq1}$ is bounded in X^*. In virtue of (1.45), (1.37), (1.41) we have

$$\lim_{k\to+\infty} \langle g_k, u_k - u\rangle =$$

$$= \lim_{k\to+\infty} \int_0^\tau \big(d_{m_k}(t), y_{m_k}(t) - y(t)\big)\, dt =$$

$$= \lim_{k\to+\infty} \int_0^\tau \big(f(t) - y'_{m_k}(t), y_{m_k}(t) - y(t)\big)\, dt =$$

$$= \lim_{k\to+\infty} \int_0^\tau \big(y'_{m_k}(t), y(t) - y_{m_k}(t)\big)\, dt =$$

$$= \lim_{k\to+\infty} \frac{1}{2}\big(\|y_{m_k}(0)\|^2_H - \|y_{m_k}(\tau)\|^2_H\big) +$$

$$+ \lim_{k\to+\infty} \int_0^\tau \big(y'_{m_k}(t), y(t)\big)\, dt =$$

$$= \frac{1}{2}\big(\|y(0)\|^2_H - \|y(\tau)\|^2_H\big) + \int_0^\tau \big(y'(t), y(t)\big)\, dt = 0.$$

So,

$$\lim_{k\to+\infty} \langle g_k, u_k - u\rangle = 0. \tag{1.46}$$

Let us show that $g_k \in A(u_k)$, $\forall k \ge 1$. For any $w \in X$ let us set

$$w_\tau(t) = \begin{cases} w(t), & \text{if } t \in [0, \tau], \\ \bar{0}, & \text{elsewhere}, \end{cases}$$

$$w^\tau(t) = \begin{cases} \bar{0}, & \text{if } t \in [0, \tau], \\ w(t), & \text{elsewhere}. \end{cases}$$

Since A is an operator of the Volterra type we obtain that

$$\langle g_k, w\rangle = \langle d_{m_k}, w_\tau\rangle + \langle \hat{d}_k, w^\tau\rangle \le$$

$$\le [A(y_{m_k}), w_\tau]_+ + \langle \hat{d}_k, w^\tau\rangle =$$

$$= [A(u_k), w_\tau]_+ + \langle \hat{d}_k, w^\tau\rangle \le$$

$$\le [A(u_k), w_\tau]_+ + [A(u_k), w^\tau]_+.$$

Since $A(u_k) \in \mathscr{H}(X^*)$, similarly to the proof of (1.28) we obtain that

$$[A(u_k), w_\tau]_+ + [A(u_k), w^\tau]_+ = [A(u_k), w]_+.$$

Since $w \in X$ is an arbitrary, then $g_k \in A(u_k)$ for all $k \ge 1$. Since $\{u_k\}_{k\ge1}$ is bounded in X, $\{g_k\}_{k\ge1}$ is bounded in X^*. Thus, up to a subsequence $\{u_{k_j}, g_{k_j}\}_{j\ge1} \subset \{u_k, g_k\}_{k\ge1}$, for some $u \in W$, $g \in X^*$ the next convergence holds

$$u_{k_j} \rightharpoonup u \text{ in } W, \quad g_{k_j} \rightharpoonup g \text{ in } X^* \text{ as } j \to \infty. \tag{1.47}$$

We remark that

$$u(t) = y(t), \;\; g(t) = d(t) \text{ for a.e. } t \in [0, \tau]. \tag{1.48}$$

In virtue of (1.46), (1.47), as A satisfies the property S_k on W, we obtain that $g \in A(u)$. Hence, due to (1.48), as A is the Volterra type operator, for any $w \in X$ such that $w(t) = 0$ for a.e. $t \in [\tau, T]$ we get

$$\langle d, w\rangle = \langle g, w\rangle \le [A(u), w]_+ = [A(y), w]_+.$$

As $\tau \in \{\tau_l\}_{l\ge1}$ is an arbitrary, we obtain (1.42).

From (1.42), as the functional $w \to [A(y), w]_+$ is convex and lower semicontinuous on X (hence it is continuous on X), we obtain that $\langle d, w\rangle \le [A(y), w]_+$ for each $w \in X$. Therefore, $d \in A(y)$.

The theorem is proved.

Analyzing the proofs of Theorem 1.2 and Corollary 1.3 (see [28]) the following proposition holds.

Proposition 1.1 *Let $A : X \to C_v(X^*) \cap \mathscr{H}(X^*)$ be bounded map of the Volterra type which satisfies the property S_k on W. Moreover, let for some $\lambda_A \geq 0$ and $c > 0$*

$$\frac{[A(y), y]_- - c\|A(y)\|_+ + \lambda_A\|y\|_Y^2}{\|y\|_X} \to +\infty \tag{1.49}$$

as $\|y\|_X \to +\infty$. Then for any $a \in H$, $f \in X^*$ there exists at least one solution of Problem (1.3), which can be obtained via the Faedo–Galerkin method.

Proof Let us set $\varepsilon = \frac{\|a\|_H^2}{2c^2}$. We consider $w \in W$ such that

$$\begin{cases} w' + \varepsilon J(w) = \bar{0}, \\ w(0) = a, \end{cases}$$

where $J : X \to X^*$ is the duality map. Hence $\|w\|_X \leq c$. We define $\hat{A} : X \to C_v(X^*) \cap \mathscr{H}(X^*)$ by the rule: $\hat{A}(z) = A(z + w)$, $z \in X$. Let us set $\hat{f} = f - w'^*$. If $z \in W$ is a solution of the problem

$$\begin{cases} z' + \hat{A}(z) \ni \widehat{f}, \\ z(0) = \bar{0}, \end{cases}$$

then $y = z + w$ is a solution of Problem (1.3). It is clear that $\hat{A}$ is a bounded map of the Volterra type which satisfies the property S_k on W. So, due to Theorem 1.2 it is enough to prove the coercivity for the map $\hat{A} + \lambda_A I$. This property follows from the estimates:

$$[\hat{A}(z), z]_+ \geq [A(z + w), z + w]_+ - [A(z + w), w]_+ \geq$$

$$\geq [A(z + w), z + w]_+ - c\|A(z + w)\|_+,$$

$$\|z\|_Y^2 \geq \|z + w\|_Y^2 - c^2 - 2\|w\|_{X^*}\|z\|_X.$$

$$\|z\|_X \geq \|z + w\|_X - c.$$

The proposition is proved.

Analyzing the proof of Theorem 1.2 we can obtain the following convergence result.

Corollary 1.5 *Let $A : X \to C_v(X^*) \cap \mathscr{H}(X^*)$ be a bounded map of the Volterra type which satisfies the property S_k on W. We consider a sequence $\{a_n\}_{n\geq 0} \subset H$ such that $a_n \to a_0$ in H as $n \to +\infty$. Let $y_n \in W$, $n \geq 1$, be solutions of Problem (1.3) corresponding to the initial data a_n. If $y_n \rightharpoonup y_0$ in X as $n \to +\infty$, then $y_0 \in W$ is solution of Problem (1.3) with initial data a_0. Moreover, up to a subsequence, $y_n \rightharpoonup y_0$ in $W \subset C(S; H)$.*

1.2 Second Order Operator Differential Equations and Inclusions

In this section we provide the existence results for the second order operator differential equations and inclusions. Since the problem in hands naturally considers as the first order integro-differential-operator inclusion, we recall some of them.

Let the following conditions hold:

(H_1) V, Z, H are Hilbert spaces; $H^* \equiv H$ and we have such chain of dense and compact embeddings:

$$V \subset Z \subset H \equiv H^* \subset Z^* \subset V^*;$$

(H_2) $f_0 \in V^*$;
(A_1) $\exists c > 0 : \forall u \in V, \forall d \in A_0(u)\ \|d\|_{V^*} \le c(1 + \|u\|_V)$;
(A_2) $\exists \alpha, \beta > 0 : \forall u \in V, \forall d \in A_0(u)\ \langle d, u\rangle_V \ge \alpha\|u\|_V^2 - \beta$;
(A_3) $A_0 = A_1 + A_2$, where $A_1 : V \to V^*$ is linear, selfconjugated, positive operator, $A_2 : V \rightrightarrows V^*$ satisfies such conditions:

(a) there exists a Hilbert space Z such that the embedding $V \subset Z$ is dense and compact, and the embedding $Z \subset H$ is dense and continuous;

(b) for any $u \in Z$ the set $A_2(u)$ is nonempty, convex, and weakly compact in Z^*;

(c) $A_2 : Z \rightrightarrows Z^*$ is a bounded map, that is, A_2 converts bounded sets from Z into bounded sets in the space Z^*;

(d) $A_2 : Z \rightrightarrows Z^*$ is a demiclosed map, i.e. if $u_n \to u$ in Z, $d_n \to d$ weakly in Z^*, $n \to +\infty$, and $d_n \in A_2(u_n)\ \forall n \ge 1$, then $d \in A_2(u)$;
(B_1) $B_0 : V \to V^*$ is a linear selfconjugated operator;
(B_2) $\exists \gamma > 0 : \langle B_0 u, u\rangle_V \ge \gamma\|u\|_V^2$.
Here $\langle\cdot,\cdot\rangle_V : V^* \times V \to \mathbb{R}$ is the duality in $V^* \times V$, coinciding on $H \times V$ with the inner product $(\cdot,\cdot)$ in Hilbert space H.

Note that from (A_1)–(A_3), [34, 51] it follows that the map A_0 satisfies such condition:
$(A_3)'$ $A_0 : V \rightrightarrows V^*$ is (generalized) pseudomonotone operator, that is,

(a) for any $u \in V$ the set $A_0(u)$ is nonempty, convex, and weakly compact one in V^*;

(b) if $u_n \to u$ weakly in V, $n \to +\infty$, $d_n \in A_0(u_n)\ \forall n \ge 1$, and $\limsup_{n\to\infty}\langle d_n, u_n - u\rangle_V \le 0$, then $\forall \omega \in V\ \exists d(\omega) \in A_0(u)$:

$$\liminf_{n\to+\infty}\langle d_n, u_n - \omega\rangle_V \ge \langle d(\omega), u - \omega\rangle_V;$$

(c) the map A_0 is upper semicontinuous one that acts from an arbitrary finite-dimensional subspace of V into V^* endowed with weak topology.

Thus, we investigate the dynamic of all weak solutions of the second order nonlinear autonomous differential-operator inclusion

$$y''(t) + A_0(y'(t)) + B_0(y(t)) \ni f_0 \quad \text{for a.e. } t > 0 \tag{1.50}$$

as $t \to +\infty$, which are defined as $t \geq 0$, where parameters of the problem satisfy conditions (H_1), (H_2), (A_1)–(A_3), (B_1)–(B_2).

As a *weak solution* of evolution inclusion (1.50) on the interval $[\tau, T]$ we consider a pair of elements $(u(\cdot), u'(\cdot))^T \in L_2(\tau, T; V \times V)$ such that for some $d(\cdot) \in L_2(\tau, T; V^*)$

$$\begin{gathered} d(t) \in A_0(u'(t)) \quad \text{for a.e.} \quad t \in (\tau, T), \\ -\int_\tau^T (\zeta'(t), u'(t))dt + \int_\tau^T \langle d(t), \zeta(t)\rangle_V dt + \\ + \int_\tau^T \langle B_0 u(t), \zeta(t)\rangle_V dt = \int_\tau^T \langle f_0, \zeta(t)\rangle_V \quad \forall \zeta \in C_0^\infty([\tau, T]; V), \end{gathered} \tag{1.51}$$

where u' is the derivative of the element $u(\cdot)$ in the sense of the space of distributions $\mathscr{D}^*([\tau, T]; V^*)$.

Note that the abstract theorems on the existence of solutions for such problems under weaker conditions were considered in [34, 51]. Here we consider Problem 2 from [34], for which we can (as follows from results of the further chapters) have not only the abstract result on existence of weak solution but we can investigate the behavior of all weak solutions as $t \to +\infty$ in the phase space $V \times H$ and study the structure of the global and trajectory attractors. Underline that results concerning multi-valued dynamic of displacements and velocities can be applied to hemivariational inequalities.

Further, without loss the generality we consider the equivalent norm $\|u\|_V = \sqrt{\langle B_0 u, u\rangle_V}$, $u \in V$, in the space V. The given norm is generated by the inner product $(u, v)_V = \langle B_0 u, v\rangle_V$, $u, v \in V$. For fixed $\tau < T$ let us consider

$$X_{\tau,T} = L_2(\tau, T; V), \quad X^*_{\tau,T} = L_2(\tau, T; V^*), \quad W_{\tau,T} = \{u \in X_{\tau,T} | u' \in X^*_{\tau,T}\},$$

$$A_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}, \quad \mathscr{A}_{\tau,T}(y) = \{d \in X^*_{\tau,T} | d(t) \in A_0(y(t)) \text{ for a.e. } t \in (\tau, T)\},$$

$$B_{\tau,T} : X_{\tau,T} \to X^*_{\tau,T}, \quad B_{\tau,T}(y)(t) = B_0(y(t)) \text{ for a.e. } t \in (\tau, T),$$

$$f_{\tau,T} \in X^*_{\tau,T}, \quad f_{\tau,T}(t) = f_0 \text{ for a.e. } t \in (\tau, T).$$

Note that the space $W_{\tau,T}$ is the Hilbert space with the graph norm of the derivative (cf. [50, 51]):

$$\|u\|^2_{W_{\tau,T}} = \|u\|^2_{X_{\tau,T}} + \|u'\|^2_{X^*_{\tau,T}}, \quad u \in W_{\tau,T}. \tag{1.52}$$

From [34, Lemma 7, p. 516], (A_1), (A_2), $(A_3)'$ it follows that $\mathscr{A}_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}$ satisfies the next conditions:

(N_1) $\exists C_1 > 0$: $\|d\|_{X^*_{\tau,T}} \leq C_1(1 + \|y\|_{X_{\tau,T}})$ $\forall y \in X_{\tau,T}$, $\forall d \in \mathscr{A}_{\tau,T}(y)$;

(N_2) $\exists C_2, C_3 > 0$: $\langle d, y\rangle_{X_{\tau,T}} \geq C_2\|y\|^2_{X_{\tau,T}} - C_3$ $\forall y \in X_{\tau,T}$, $\forall d \in \mathscr{A}_{\tau,T}(y)$;

(N_3) $\mathscr{A}_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}$ is (generalized) pseudomonotone on $W_{\tau,T}$, that is,
(a) for any $y \in X_{\tau,T}$ the set $\mathscr{A}_{\tau,T}(y)$ is a nonempty, convex, and weakly compact

one in $X^*_{\tau,T}$;
(b) $\mathscr{A}_{\tau,T}$ is the upper semicontinuous map as the map that acts from an arbitrary finite dimensional subspace from $X_{\tau,T}$ into $X^*_{\tau,T}$ endowed by the weak topology;
(c) if $y_n \to y$ weakly in $W_{\tau,T}$, $d_n \in \mathscr{A}_{\tau,T}(y_n)$ $\forall n \geq 1$, $d_n \to d$ weakly in $X^*_{\tau,T}$, and

$$\limsup_{n\to+\infty} \langle d_n, y_n - y\rangle_{X_{\tau,T}} \leq 0,$$

then $d \in \mathscr{A}_{\tau,T}(y)$ and $\lim\limits_{n\to+\infty} \langle d_n, y_n\rangle_{X_{\tau,T}} = \langle d, y\rangle_{X_{\tau,T}}$.

Here $\langle\cdot,\cdot\rangle_{X_{\tau,T}} : X^*_{\tau,T} \times X_{\tau,T} \to \mathbb{R}$ is the pairing in $X^*_{\tau,T} \times X_{\tau,T}$ coinciding on $L_2(\tau, T; H) \times X_{\tau,T}$ with the inner product in $L_2(\tau, T; H)$, that is,

$$\forall u \in L_2(\tau, T; H),\ \forall v \in X_{\tau,T} \quad \langle u, v\rangle_{X_{\tau,T}} = \int_\tau^T (u(t), v(t))dt.$$

Note also (cf. [12, Theorem IV.1.17, P. 177]) that the embedding $W_{\tau,T} \subset C([\tau, T]; H)$ is continuous and dense. Moreover,

$$(u(T), v(T)) - (u(\tau), v(\tau)) = \int_\tau^T \Big[\langle u'(t), v(t)\rangle_V + \langle v'(t), u(t)\rangle_V\Big]dt, \tag{1.53}$$

for each $u, v \in W_{\tau,T}$.

The definition of derivative in the sense of $\mathscr{D}([\tau, T]; V^*)$ and equality (1.51) yield the following statement.

Lemma 1.6 *Each weak solution* $(y(\cdot), y'(\cdot))^T$ *of Problem (1.50) on the interval* $[\tau, T]$ *belongs to the space* $C([\tau, T]; V) \times W_{\tau,T}$. *Moreover,*

$$y'' + \mathscr{A}_{\tau,T}(y') + B_{\tau,T}(y) \ni f_{\tau,T}. \tag{1.54}$$

Vice versa, *if* $y(\cdot) \in C([\tau, T]; V)$, $y'(\cdot) \in W_{\tau,T}$, *and* $y(\cdot)$ *satisfies (1.54), then* $(y(\cdot), y'(\cdot))^T$ *is a weak solution of (1.50) on* $[\tau, T]$.

A weak solution of Problem (1.50) with initial data

$$y(\tau) = a, \quad y'(\tau) = b \tag{1.55}$$

on the interval $[\tau, T]$ exists for any $a \in V$, $b \in H$. It follows from [34, Theorem 11, p. 523]. Thus, the following lemma holds.

Lemma 1.7 *For any* $\tau < T$, $a \in V$, $b \in H$ *Cauchy Problem (1.50), (1.55) has a weak solution* $(y, y')^T \in X_{\tau,T} \times X_{\tau,T}$. *Moreover, each weak solution* $(y, y')^T$ *of Cauchy Problem (1.50), (1.55) on the interval* $[\tau, T]$ *belongs to the space* $C([\tau, T]; V) \times W_{\tau,T}$ *and* y *satisfies (1.54).*

The similar statement holds also for non-autonomous problems. For this purpose additional measurability assumption for A_2 is claimed.

1.3 Evolutional Variational Inequalities: Penalty Method and Strong Solutions

In investigating unilateral problems, problems on Riemannian manifolds with or without boundary, semi-penetration problems, and in the analysis and control of processes and fields of different natures on the boundary of a domain, a demand arises for the consideration of evolutionary variational inequalities with nonlinear nonmonotone operators in infinite-dimensional spaces. To describe the state functions of such objects, the concepts of strong and weak solutions are naturally introduced. A strong solution does not always adequately describe a system state since, in the majority of cases, such classes of solutions prevents the effects of breaks or unilateral semi-penetration, that is, are too regular to adequately describe states of the processes and fields being investigated. The proof of the existence of strong solutions (especially for equations with nonmonotone reaction laws) is problematic. The concept of a weak solution is too general (this solution not always adequately describes a state function, that is, this class of solutions can formally include not only physical solutions) and, at the same time, is insufficiently regular to adequately implement the numerical analysis of the problems being investigated. Note that a strong solution of an evolutionary variational inequality is, as a rule, a weak solution. A demand arises for the introduction of a new intermediate class of physical solutions to such problems that, on the one hand, must satisfy natural energy equalities and, on the other hand, provide the possibility of substantiation of constructive (and at the same time physical) methods of their existence (for example, the artificial viscosity method for problems of classical hydroaeromechanics in an incompressible continuous medium).

This section introduces the concept of a physical solution on a finite time interval for classes of autonomous evolutionary variational inequalities with nonlinear nonmonotone (in general cases) mappings defined on convex cones. This concept is based on natural energy equalities and continuous dependence of state functions in the phase space on the time variable. For approximate searching for physical solutions, the classical penalty method is used. For the solutions obtained, the possibility of a global description of the behavior of such systems is substantiated on the basis of the results of [13, 17] in their natural phase space with respect to the topology of strong convergence by finite algorithms up to an arbitrary small parameter.

For an evolutionary triple $(V; H; V^*)$ a nonlinear (in the general case) mapping $A : V \to V^*$, and a convex closed cone $K \subseteq V$, the problem of investigating the dynamics of the following autonomous evolutionary variational inequality is considered in the phase space H of all physical solutions $y : \mathbb{R}_+ \to V$, $y(t) \in K$ for a.a. $t > 0$:

$$\langle y'(t) + A(y(t)), v - y(t)\rangle_V \geq 0 \text{ for all } v \in K \text{ and for a.a. } t > 0 \tag{1.56}$$

in which the parameters of the problem satisfy the following conditions.

Assumption 1 $p \geq 2$ and $q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$ and the embedding of V into H is compact.

Assumption 2 $\exists c > 0$: $\|A(u)\|_{V^*} \leq c(1 + \|u\|_V^{p-1})\ \forall u \in V$.

Assumption 3 $\exists \alpha, \beta > 0$: $\langle A(u), u\rangle_V \geq \alpha\|u\|_V^p - \beta\ \forall u \in V$.

Assumption 4 $A : V \to V^*$ is a pseudomonotone operator satisfying the (S)-property, that is, since $u_n \to u$ weakly in V, and $\overline{\lim_{n\to+\infty}} \langle A(u_n), u_n - u\rangle_V \leq 0$, we obtain that $u_n \to u$ in V and $\underline{\lim_{n\to+\infty}} \langle A(u_n), u_n - \omega\rangle_V \geq \langle A(u), u - \omega\rangle_V \forall\omega \in V$.

Assumption 5 $K \subseteq V$ is a convex closed cone such that $int_{V_\sigma} K_\sigma \neq \emptyset$, where $V_\sigma \subseteq V$ is a real reflexive separable Banach space continuously and densely embedded into V, $K_\sigma := K \cap V_\sigma$.

Here, $\langle\cdot,\cdot\rangle_V : V^* \times V \to$ R is a pairing in $V^* \times V$, and this pairing coinsides on $H \times V$ with the scalar product $(\cdot,\cdot)$ in a Hilbert space H. Note that a space V_σ^* is the conjugate of V_σ with respect to the canonical pairing $\langle\cdot,\cdot\rangle_{V_\sigma} : V_\sigma^* \times V_\sigma \to$ R that coincides on $H \times V_\sigma$ with the scalar product $(\cdot,\cdot)$ in H. Then we obtain the following chain of such continuous and dense embeddings: $V_\sigma \subset V \subset H \subset V^* \subset V_\sigma^*$.

Let $0 \leq \tau < T < +\infty$. We set

$$K_{\tau,T} := \{y \in L_p(\tau, T; V) : y(t) \in K \text{ for a.a. } t \in (\tau, T)\}.$$

By a *physical solution* of evolutionary variational inequality (1.56) on an interval $[\tau, T]$ we understand an element y that belongs to the space $K_{\tau,T} \cap C([\tau, T]; H)$ such that

$$-\int_\tau^T (\xi'(t), y(t))dt + \int_\tau^T \langle A(y(t)), \xi(t)\rangle_V dt \geq 0\ \forall\xi \in C_0^\infty([\tau, T]; V) \cap K_{\tau,T}, \tag{1.57}$$

$$||y(t_2)||_H^2 - ||y(t_1)||_H^2 + 2\int_{t_1}^{t_2} \langle A(y(t)), y(t)\rangle_V dt = 0\ \ \forall t_1, t_2 \in [\tau, T]; \tag{1.58}$$

see also Fig. 1.5.

Note that the concept of a physical solution naturally weakens the concept of a strong solution of unilateral Problem (1.56). The concept of a physical solution is a weak solution of Problem (1.56) that is continuous as a mapping from the time interval $[\tau, T]$ into the phase space H and satisfies energy equality (1.58). Of course, each strong solution of Problem (1.56) is a physical solution of this problem. At present, for Assumptions 1, 2, 3, 4 and 5, only the fact of existence of weak solutions is well known.

Now, with the help of the penalty method, we establish the fact of existence of physical solutions to Problem (1.56) for arbitrary initial data from K. The obtained results will be applied in the next sections to the investigation of the dynamics of processes and fields of different nature under unilateral constraints.

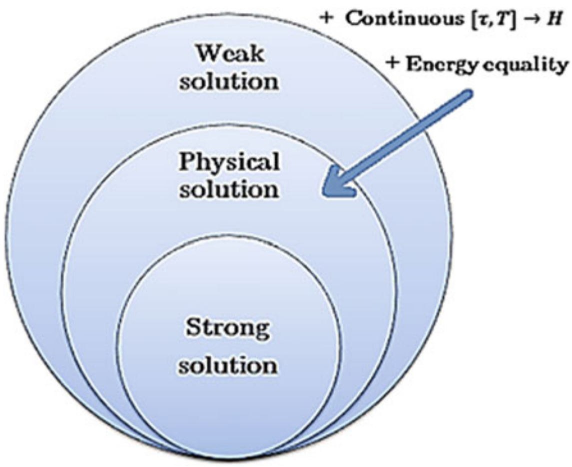

Fig. 1.5 Classes of solutions for evolution variation inequalities

For a fixed $0 \le \tau < T < +\infty$ we consider

$$X_{\tau,T} = L_p(\tau, T; V), \quad X^*_{\tau,T} = L_q(\tau, T; V^*), \quad W_{\tau,T} = \{y \in X_{\tau,T} \ : \ y' \in X^*_{\tau,T}\},$$
$$\mathscr{A}_{\tau,T} : X_{\tau,T} \to X^*_{\tau,T}, \quad (\mathscr{A}_{\tau,T}(y))(t) = A(y(t)) \text{ for a.a. } t \in (\tau, T),$$
$$Y_{\tau,T,\sigma} = L_1(\tau, T; V^*_\sigma), \quad W_{\tau,T,\sigma} = \{y \in X_{\tau,T} \ : \ y' \in Y_{\tau,T,\sigma}\},$$

where y' is the derivative of an element $y \in X_{\tau,T}$ in the sense of the space of distributions $\mathscr{D}([\tau, T]; V^*)$ (see, for example [12, Definition IV.1.10, p. 168]). Note that the space $W_{\tau,T}$ is a reflexive Banach space with the following derivative graph norm (see, for example, [48, Statement 4.2.1, p. 291]): $\|u\|_{W_{\tau,T}} = \|u\|_{X_{\tau,T}} + \|u'\|_{X^*_{\tau,T}}, \quad u \in W_{\tau,T}$.

It follows from [52, Sect. 2.2], [46, pp. 152–157], and Assumptions 1, 2, 3 and 4 that $\mathscr{A}_{\tau,T} : X_{\tau,T} \to X^*_{\tau,T}$ satisfies the following properties.

Property 1. $\exists C_1 > 0$: $\|\mathscr{A}_{\tau,T}(y)\|_{X^*_{\tau,T}} \le C_1(1 + \|y\|^{p-1}_{X_{\tau,T}})$ $\forall y \in X_{\tau,T}$.

Property 2. $\exists C_2, C_3 > 0$: $\langle \mathscr{A}_{\tau,T}(y), y\rangle_{X_{\tau,T}} \ge C_2\|y\|^p_{X_{\tau,T}} - C_3$ $\forall y \in X_{\tau,T}$.

Property 3. $\mathscr{A}_{\tau,T} : X_{\tau,T} \to X^*_{\tau,T}$ is (generalized) pseudomonotone on $W_{\tau,T,\sigma}$ and satisfies the (S)-property, that is, the facts that $y_n \to y$ weakly in $X_{\tau,T}$, $\{y'_n\}_{n=1,2,\ldots}$ is bounded in $Y_{\tau,T,\sigma}$, $\mathscr{A}_{\tau,T}(y_n) \to d$ weakly in $X^*_{\tau,T}$, and $\overline{\lim}_{n\to+\infty} \langle \mathscr{A}_{\tau,T}(y_n), y_n - y\rangle_{X_{\tau,T}} \le 0$, imply that $d = \mathscr{A}_{\tau,T}(y)$ and $y_n \to y$ in $X_{\tau,T}$.

In particular, the following equalities hold:

$$\lim_{n\to+\infty} \langle \mathscr{A}_{\tau,T}(y_n), y_n\rangle_{X_{\tau,T}} = \langle d, y\rangle_{X_{\tau,T}},$$

$$\lim_{n\to+\infty} \int_\tau^T |\langle A(y_n(t)), y_n(t) - y(t)\rangle_V| dt = 0.$$

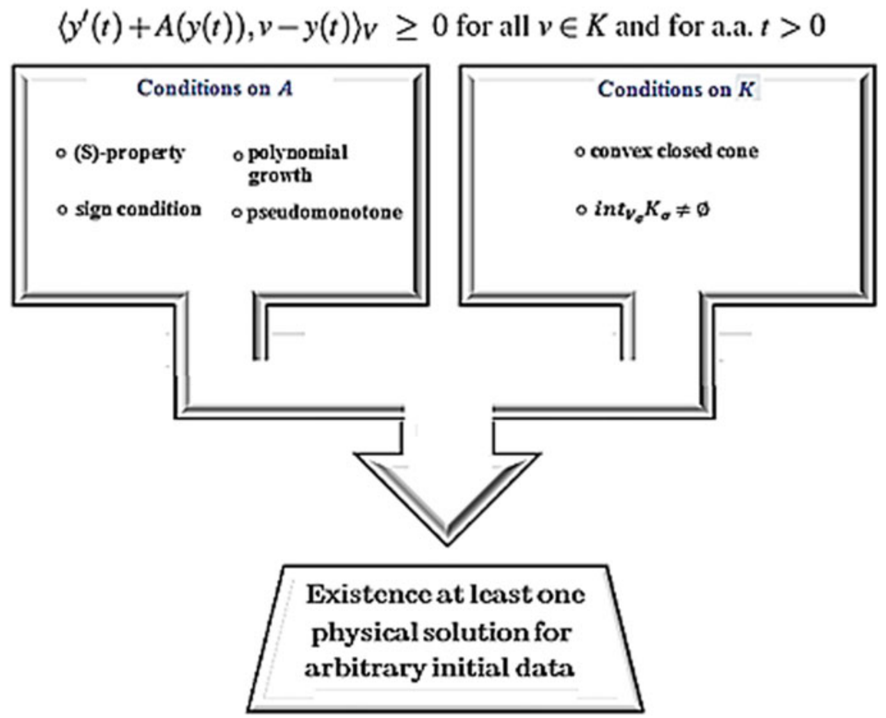

Fig. 1.6 Sufficient conditions for the existence of a solution for evolution variation inequality

Here, $\langle\cdot,\cdot\rangle_{X_{\tau,T}} : X^*_{\tau,T} \times X_{\tau,T} \to \mathrm{R}$ is the pairing in $X^*_{\tau,T} \times X_{\tau,T}$ that coincides with the scalar product in $L_2(\tau, T; H)$ on $L_2(\tau, T; H) \times X_{\tau,T}$, that is,

$$\forall u \in L_2(\tau, T; H),\ \forall v \in X_{\tau,T} \quad \langle u, v\rangle_{X_{\tau,T}} = \int_\tau^T (u(t), v(t))dt.$$

Note also that (see [12, Theorem IV.1.17, p. 177]) the embedding $W_{\tau,T} \subset C([\tau, T]; H)$ is continuous and dense. Moreover,

$$(u(T), v(T)) - (u(\tau), v(\tau)) = \int_\tau^T [\langle u'(t), v(t)\rangle_V + \langle v'(t), u(t)\rangle_V]dt, \qquad (1.59)$$

for each $u, v \in W_{\tau,T}$.

The main result of this section has the following formulation; Fig. 1.6.

Theorem 1.3 *Let Assumptions 1, 2, 3, 4 and 5 be satisfied,* $0 \le \tau < T < +\infty$. *Then, for any* $y_\tau \in K$ *there is at least one physical solution* y *of Problem (1.56) on* $[\tau, T]$, *and this solution is such that* $y(\tau) = y_\tau$.

Proof We use the penalty method. Let P_K be the operator of orthogonal projection of an arbitrary element of the space V onto the convex cone K. Let $J : V \to V^*$ be the dual operator, that is, the mapping satisfying the following two equalities: $\|J(v)\|_{V^*}\|v\|_V = \langle J(v), v\rangle_V$ and $\|J(v)\|_{V^*} = \|v\|_V^{p-1}$ for an arbitrary $v \in V$. By the Asplund theorem, the space V can be renormalized by an equivalent strict norm so that the corresponding norm in the conjugated space V^* is also strict and equivalent to its natural norm. Therefore, the operator J can be considered as single-valued. We will use $\beta(v) = J(v - P_K v)$, $v \in V$, in the capacity of the penalty operator. Note that $\beta(v) = \bar{0}$ if and only if $v \in K$. Moreover, $\beta(\alpha v) = \alpha|\alpha|^{p-2}\beta(v)$ and $\langle \beta(v), v\rangle_V = 0$ for some arbitrary $v \in V$ and $\alpha \in$ R. Hereafter, we consider that $(\beta(y))(t) = \beta(y(t))$ for a.a. $t \in (\tau, T)$ for all $y \in X_{\tau,T}$.

Since $\beta : X_{\tau,T} \to X^*_{\tau,T}$ is a bounded monotone demicontinuous operator, for an arbitrary $\varepsilon > 0$, the mapping $A_\varepsilon(y) := \mathscr{A}_{\tau,T}(y) + \frac{1}{\varepsilon}\beta(y)$, $y \in X_{\tau,T}$, is generalized pseudomonotone on $W_{\tau,T}$ (satisfies property 3). Moreover, the penalty operator definition (defining the properties of the dual mapping J) and properties 1 and 2 for the operator $\mathscr{A}_{\tau,T}$ imply properties 1 and 2 for the new operator A_ε acting from $X_{\tau,T}$ to $X^*_{\tau,T}$. Thus, [51, Theorem 2.4, p. 123] implies the existence of a solution $y_\varepsilon \in W_{\tau,T}$ to the following problem:

$$y'_\varepsilon + \mathscr{A}_{\tau,T}(y_\varepsilon) + \frac{1}{\varepsilon}\beta(y_\varepsilon) = \bar{0}, \quad y_\varepsilon(\tau) = y_\tau. \tag{1.60}$$

Moreover, formula (1.59), the monotonicity of β, and the fact that K is a cone, imply the following relationships:

$$-\langle \xi', y_\varepsilon\rangle_{X_{\tau,T}} + \langle \mathscr{A}_{\tau,T}(y_\varepsilon), \xi\rangle_{X_{\tau,T}} \geq 0 \quad \forall \xi \in C_0^\infty([\tau, T]; V) \cap K_{\tau,T}, \tag{1.61}$$

$$\|y_\varepsilon(t_2)\|_H^2 - \|y_\varepsilon(t_1)\|_H^2 + 2\int_{t_1}^{t_2} \langle A(y_\varepsilon(t)), y_\varepsilon(t)\rangle_V dt = 0 \quad \forall t_1, t_2 \in [\tau, T]. \tag{1.62}$$

Assumptions 2 and 3 imply the existence a constant $C_4 > 0$ such that

$$\|y_\varepsilon\|_{C([\tau,T];H)} \leq C_4, \quad \|y_\varepsilon\|_{X_{\tau,T}} \leq C_4, \quad \|\mathscr{A}_{\tau,T}(y_\varepsilon)\|_{X^*_{\tau,T}} \leq C_4 \quad \forall \varepsilon > 0. \tag{1.63}$$

Let us prove the existence of a constant $C_5 > 0$ such that

$$\|y'_\varepsilon\|_{Y_{\tau,T,\sigma}} \leq C_5 \quad \forall \varepsilon > 0. \tag{1.64}$$

Assumption 5 implies the existence of $v_\sigma \in K_\sigma$ and $r_\sigma > 0$ such that $\{v \in V_\sigma : \|v - v_\sigma\|_{V_\sigma} \leq r_\sigma\} \subset K_\sigma$. Since $K_\sigma = K \cap V_\sigma$ is a cone, without loss of generality, we can consider that $\|v_\sigma\|_{V_\sigma} = 1$ and $r_\sigma \leq 1$. We put $M := \{v \in K_\sigma : \|v - v_\sigma\|_{V_\sigma} \leq 1\}$ and $N := (M - v_\sigma) \cap (v_\sigma - M)$. The set N is convex, closed, absorbing, and balanced. Thus, for the set N the Minkowski functional $\rho_N(\omega) := \inf\{t > 0 : \frac{\omega}{t} \in N\}$, $\omega \in V_\sigma$ is correctly defined. Moreover, ρ_N satisfies the following three properties:

(1) $\|\omega\|_{V_\sigma} \le \rho_N(\omega) \le \frac{1}{r_\sigma}\|\omega\|_{V_\sigma}$ for any $\omega \in V_\sigma$;

(2) $\rho_N(v_\sigma) = 1$;

(3) $\{\omega \in V_\sigma : \rho_N(\omega - v_\sigma) \le 1\} \subset K_\sigma$.

We put $\rho_N^*(g) := \sup\{\langle g, \omega\rangle_{V_\sigma} : \omega \in V_\sigma, \rho_N(\omega) \le 1\}$, $g \in V_\sigma^*$. Property (1) for ρ_N provides the equivalence between the norm ρ_N^* and the natural norm of the space V_σ^*. Consider now $K_\sigma^- := \{g \in V_\sigma^* : \langle g, \omega\rangle_{V_\sigma} \le 0 \forall\omega \in K_\sigma\}$. Properties (2) and (3) of ρ_N imply

$$\rho_N^*(g) = -\langle g, v_\sigma\rangle_{V_\sigma} \quad \forall g \in K_\sigma^-. \tag{1.65}$$

Since $K \subseteq V$ is a cone, the monotonicity of $\beta : V \to V^*$ guarantees that $\beta(\omega) \in K_\sigma^-$ for an arbitrary $\omega \in V$. Therefore, statements (1.59), (1.60), and (1.65) provide the fulfillment of the equalities

$$\begin{aligned} \tfrac{1}{\varepsilon}\|\beta(y_\varepsilon)\|_{Y_{\tau,T,\sigma}} &= -\tfrac{1}{\varepsilon}\int_\tau^T \langle \beta(y_\varepsilon(t)), v_\sigma\rangle_V dt \\ &= \int_\tau^T \langle A(y_\varepsilon(t)), v_\sigma\rangle_V dt + (y_\varepsilon(T) - y_\varepsilon(\tau), v_\sigma)_H \end{aligned}$$

for any $\varepsilon > 0$. Thus, from inequalities (1.63), we obtain

$$\frac{1}{\varepsilon}\|\beta(y_\varepsilon)\|_{Y_{\tau,T,\sigma}} \le C_4\|v_\sigma\|_V(T-\tau)^{\frac{1}{p}} + 2C_4\|v_\sigma\|_H \quad \forall\varepsilon > 0. \tag{1.66}$$

Finally, inequality (1.64) follows from Problem (1.60) and inequalities (1.63) and (1.66) since the embedding $X_{\tau,T}^* \subset Y_{\tau,T,\sigma}$ is continuous and dense.

The following equality holds:

$$\langle \beta(y_\varepsilon), y_\varepsilon\rangle_{X_{\tau,T}} = 0 \quad \forall\varepsilon > 0, \tag{1.67}$$

since $\langle \beta(v), v\rangle_V = 0$ for an arbitrary $v \in V$. Moreover, from the monotonicity of β and the BanachSteinhaus theorem, we obtain

$$\exists C_5 > 0 : \quad \|\beta(y_\varepsilon)\|_{X_{\tau,T}^*} \le C_5 \quad \forall\varepsilon \in (0,1). \tag{1.68}$$

In fact, for an arbitrary $\omega \in X_{\tau,T}$, the monotonicity of β, estimate (1.63) and equality (1.67) imply the following:

$$\begin{aligned} \sup_{\varepsilon\in(0,1)} \langle \beta(y_\varepsilon), \omega\rangle_{X_{\tau,T}} &\le \sup_{\varepsilon\in(0,1)} \langle \beta(y_\varepsilon), \omega - y_\varepsilon\rangle_{X_{\tau,T}} + \sup_{\varepsilon\in(0,1)} \langle \beta(y_\varepsilon), y_\varepsilon\rangle_{X_{\tau,T}} \\ &= \sup_{\varepsilon\in(0,1)} \langle \beta(\omega), \omega - y_\varepsilon\rangle_{X_{\tau,T}} \le \|\beta(\omega)\|_{X_{\tau,T}^*}(\|\omega\|_{X_{\tau,T}} + C_4) < \infty. \end{aligned}$$

Hence, we obtain inequality (1.68) from the Banach–Steinhaus theorem.

From a priori estimates (1.63), (1.64), and (1.66), and the lemma on the compactness of the embedding $W_{\tau,T} \subset L_2(\tau, T; H)$ (by virtue of the compactness of the embedding $V \subset H$), we obtain as a corollary of Banach–Alaoglu theorem that there is a sequence $\varepsilon_n \searrow 0$, $n \to \infty$, and elements $y \in X_{\tau,T}$ and $d \in X_{\tau,T}^*$ such that

$y(\tau) = y_\tau$ and the following convergences take place:

$$\begin{aligned} &y_{\varepsilon_n} \rightharpoonup y \text{ in } X_{\tau,T}, \quad y_{\varepsilon_n}(t) \to y(t) \text{ in } H \text{ for a.a. } t \in (\tau, T), \\ &y_{\varepsilon_n}(T) \rightharpoonup y(T) \text{ in } H, \quad \mathscr{A}_{\tau,T}(y_{\varepsilon_n}) \rightharpoonup d \text{ in } X^*_{\tau,T} \quad n \to \infty. \end{aligned} \tag{1.69}$$

Moreover, from inequalities (1.66) and (1.68), we obtain that

$$\beta(y_{\varepsilon_n}) \to 0 \text{ in } X^*_{\tau,T} \quad n \to \infty. \tag{1.70}$$

Let us show that $y \in K_{\tau,T}$. It follows from convergences (1.69) and (1.70) that $\lim\limits_{n\to\infty} \langle \beta(y_{\varepsilon_n}), y_{\varepsilon_n} - y\rangle_{X_{\tau,T}} = 0$. Since the monotone demicontinuous operator β is pseudomonotone, taking into account convergence (1.70), the following inequality holds:

$$0 = \lim_{n\to\infty} \langle \beta(y_{\varepsilon_n}), y_{\varepsilon_n} - \omega\rangle_{X_{\tau,T}} \geq \langle \beta(y), y - \omega\rangle_{X_{\tau,T}},$$

$\forall \omega \in X_{\tau,T}$. Thus, $\beta(y(t)) \in K$ for a.a. $t \in (\tau, T)$. Therefore, $y \in K_{\tau,T}$ since $y \in X_{\tau,T}$.

Let us show that

$$\overline{\lim_{n\to\infty}} \langle \mathscr{A}_{\tau,T}(y_{\varepsilon_n}), y_{\varepsilon_n} - y\rangle_{X_{\tau,T}} \leq 0. \tag{1.71}$$

In fact, Problem (1.60), the monotonicity of β and formula (1.59) imply

$$\begin{aligned} \langle \mathscr{A}_{\tau,T}(y_{\varepsilon_n}), y_{\varepsilon_n} - v\rangle_{X_{\tau,T}} &= \tfrac{1}{\varepsilon}\langle \beta(y_{\varepsilon_n}), v - y_{\varepsilon_n}\rangle_{X_{\tau,T}} + \langle y'_{\varepsilon_n}, v - y_{\varepsilon_n}\rangle_{X_{\tau,T}} \\ &\leq \tfrac{1}{\varepsilon}\langle \beta(y_{\varepsilon_n}), v - y_{\varepsilon_n}\rangle_{X_{\tau,T}} + \langle v', v - y_{\varepsilon_n}\rangle_{X_{\tau,T}} \\ \leq \tfrac{1}{\varepsilon}\langle \beta(v), v - y_{\varepsilon_n}\rangle_{X_{\tau,T}} &+ \langle v', v - y_{\varepsilon_n}\rangle_{X_{\tau,T}} \leq \langle v', v - y_{\varepsilon_n}\rangle_{X_{\tau,T}}, \end{aligned} \tag{1.72}$$

for arbitrary $n = 1, 2, \ldots$ and $v \in W_{\tau,T} \cap K_{\tau,T}$ since $\beta(v) = \bar{0}$. Thus, convergence (1.69) implies the inequality

$$\overline{\lim_{n\to\infty}} \langle \mathscr{A}_{\tau,T}(y_{\varepsilon_n}), y_{\varepsilon_n}\rangle_{X_{\tau,T}} \leq \langle d, v\rangle_{X_{\tau,T}} + \langle v', v - y\rangle_{X_{\tau,T}}, \tag{1.73}$$

for all $v \in W_{\tau,T} \cap K_{\tau,T}$. Since $\bar{0} \in K_{\tau,T} - \omega_\tau$ for $\omega_\tau \equiv y_\tau \in K_{\tau,T}$, [29, p. 284] implies the existence of a sequence $\{v_j\}_{j=1,2,\ldots} \subset (K_{\tau,T} - \omega_\tau) \cap W_{\tau,T}$ such that

(a) $v_j(\tau) = \bar{0}$ for all $j = 1, 2, \ldots$;
(b) $v_j \to y - \omega_\tau$ in $X_{\tau,T}$ as $j \to \infty$;
(c) $\overline{\lim\limits_{j\to\infty}} \langle v'_j, v_j + \omega_\tau - y\rangle_{X_{\tau,T}} \leq 0$.

Putting $v = v_j + \omega_\tau \in K_{\tau,T} \cap W_{\tau,T}$, $j = 1, 2, \ldots$ in inequality (1.73), we obtain that $\overline{\lim\limits_{n\to\infty}} \langle \mathscr{A}_{\tau,T}(y_{\varepsilon_n}), y_{\varepsilon_n}\rangle_{X_{\tau,T}} \leq \langle d, y\rangle_{X_{\tau,T}}$. The last inequality together with convergences (1.69) and inequalities (1.72) implies inequality (1.71). We will use the pseudomonotonicity of $\mathscr{A}_{\tau,T}$ on $W_{\tau,T,\sigma}$. It follows from inequality (1.64), convergences (1.69), and inequality (1.71) that $d = \mathscr{A}_{\tau,T}(y)$, $\lim\limits_{n\to\infty} \langle \mathscr{A}_{\tau,T}(y_{\varepsilon_n}), y_{\varepsilon_n}\rangle_{X_{\tau,T}} = \langle d, y\rangle_{X_{\tau,T}}$,

and
$$\int_\tau^T \left|\langle A(y_{\varepsilon_n}(t)), y_{\varepsilon_n}(t) - y(t)\rangle_V\right| dt \to 0 \quad n \to \infty. \tag{1.74}$$

Moreover, inequalities (1.72) additionally imply inequality (1.57).

To complete the verification of the fact that y is a physical solution to Problem (1.56) on $[\tau, T]$, it remains to verify that $y \in C([\tau, T]; H)$ and that y satisfies energy equality (1.58).

It follows from formulas (1.59) and (1.60) (see also formula (1.62)) that

$$\frac{d}{dt}\|y_{\varepsilon_n}(t)\|_H^2 = -\langle A(y_{\varepsilon_n}(t)), y_{\varepsilon_n}(t)\rangle_V$$

for a.a. $t \in (\tau, T)$. We obtain from formula (1.74) and the last convergence in formula (1.69) that the sequence $\{t \to \frac{d}{dt}\|y_{\varepsilon_n}(t)\|_H^2\}_{n=1,2,\ldots}$ of measurable real-valued functions on (τ, T) is uniformly integrable, that is, there is a subsequence $\{y_{\varepsilon_m}\}_m \subseteq \{y_{\varepsilon_n}\}_n$ such that the sequence $\{t \to \frac{d}{dt}\|y_{\varepsilon_m}(t)\|_H^2\}_m$ weakly converges in $L_1(\tau, T)$ to an element $-\langle A(y(\cdot)), y(\cdot)\rangle_V \in L_1(\tau, T)$. Hence, the sequence $\{t \to \frac{d}{dt}\|y_{\varepsilon_m}(t)\|_H^2\}_m$, on the one hand, converges in the space $D^*(\tau, T)$ (of generalized functions on $[\tau, T]$) to a regular generalized function $-\langle A(y(\cdot)), y(\cdot)\rangle_V \in L_1(\tau, T)$. On the other hand, the sequence $\{t \to \|y_{\varepsilon_m}(t)\|_H^2\}_m$ converges in the space $D^*(\tau, T)$ to the measurable function $\|y(\cdot)\|_H^2$ essentially bounded on (τ, T). Thus, the sequence $\{t \to \frac{d}{dt}\|y_{\varepsilon_m}(t)\|_H^2\}_m$ converges in the space $D^*(\tau, T)$ to the generalized function $\frac{d}{dt}\|y(\cdot)\|_H^2$. Thus, by virtue of the uniqueness of the limit in the space $D^*(\tau, T)$, $\frac{d}{dt}\|y(\cdot)\|_H^2 = -\langle A(y(\cdot)), y(\cdot)\rangle_V \in L_1(\tau, T)$, which, in view of formula (1.59), implies a priori estimate (1.58).

Let us show that
$$\|y(t) - y_\tau\|_H \to 0 \quad t \searrow \tau_+. \tag{1.75}$$

Since $y_\tau \in K$ and $\langle \beta(v), v\rangle_V = 0$ for arbitrary $v \in V$, formula (1.60) implies

$$\begin{gathered}(y_{\varepsilon_n}(t), y_\tau) - \|y_\tau\|_H^2 = \int_\tau^t (y'_{\varepsilon_n}(s), y_\tau)ds \\ = -\int_\tau^t \langle A(y_{\varepsilon_n}(s)), y_\tau\rangle_V dt - \frac{1}{\varepsilon}\int_\tau^t \langle \beta(y_{\varepsilon_n}(s)), y_\tau\rangle_V dt \\ = -\int_\tau^t \langle A(y_{\varepsilon_n}(s)), y_\tau\rangle_V dt + \frac{1}{\varepsilon}\int_\tau^t \langle \beta(y_{\varepsilon_n}(s)), y_{\varepsilon_n}(s) - y_\tau\rangle_V dt \\ \geq -\int_\tau^t \langle A(y_{\varepsilon_n}(s)), y_\tau\rangle_V dt\end{gathered}$$

for arbitrary $n = 1, 2, \ldots$ and $t \in (\tau, T)$. Otherwise,

$$(y_{\varepsilon_n}(t), y_\tau) - \|y_\tau\|_H^2 \leq \|y_\tau\|_H(\|y_{\varepsilon_n}(t)\|_H - \|y_\tau\|_H)$$

for arbitrary $n = 1, 2, \ldots$.

Thus,

$$\begin{gathered}-\int_\tau^t \langle A(y_{\varepsilon_n}(s)), y_\tau\rangle_V dt \leq (y_{\varepsilon_n}(t), y_\tau) - \|y_\tau\|_H^2 \\ \leq \|y_\tau\|_H(\|y_{\varepsilon_n}(t)\|_H - \|y_\tau\|_H), \quad n = 1, 2, \ldots, \quad t \in (\tau, T),\end{gathered}$$

and, taking into account convergences (1.69), we obtain

$$-\int_\tau^t \langle A(y(s)), y_\tau\rangle_V dt \le (y(t), y_\tau) - \|y_\tau\|_H^2$$
$$\le \|y_\tau\|_H(\|y(t)\|_H - \|y_\tau\|_H) \text{ for a.a. } t \in (\tau, T).$$

Since $\langle A(y(\,\cdot\,)), y_\tau\rangle_V \in L_1(\tau, T)$, energy equality (1.58) provides the last two inequalities for all $t \in [\tau, T]$. Hence

$$(y(t), y_\tau) - \|y_\tau\|_H^2 \to 0 \text{ as } t \searrow \tau_+. \tag{1.76}$$

To complete the proof of property (1.75) note that

$$||y(t) - y_\tau||_H^2 = ||y(t)||_H^2 + ||y_\tau||_H^2 - 2(y(t), y_\tau), t \in [\tau, T].$$

Hence, energy equality (1.58) and property (1.76) imply property (1.75).

From the monotonicity of β and formulas (1.59) and (1.60) we obtain the inequality

$$\|y_{\varepsilon_n}(t+h) - y_{\varepsilon_n}(t)\|_H^2$$
$$\le \|y_{\varepsilon_n}(\tau+h) - y_\tau\|_H^2 - 2\int_\tau^t \langle A(y_{\varepsilon_n}(s+h)) - A(y_{\varepsilon_n}(s)), y_{\varepsilon_n}(s+h) - y_{\varepsilon_n}(s)\rangle_V dt$$

for arbitrary $t \in (\tau, T-h)$ and $h \in (0, T-\tau)$. From property 3, convergences (1.69) and (1.74), and the last inequality we obtain

$$\|y(t+h) - y(t)\|_H^2$$
$$\le \|y(\tau+h) - y_\tau\|_H^2 - 2\int_\tau^t \langle A(y(s+h)) - A(y(s)), y(s+h) - y(s)\rangle_V dt \tag{1.77}$$

for a.a. $t \in (\tau, T-h)$ and arbitrary $h \in (0, T-\tau)$. With allowance for energy equality (1.58), inequality (1.77) takes place for all $t \in (\tau, T-h)$ and $h \in (0, T-\tau)$. Thus, formulas (1.75) and (1.77) imply the property of continuity of y as a mapping from a time interval $[\tau, T]$ into the phase space H.

The theorem is proved.

For fixed $\tau < T$, we introduce the notation:

$$\mathscr{D}_{\tau,T}(y_\tau) = \{y(\cdot) \mid y \text{ is a physical solution to inequality (1.56) on } [\tau, T], \quad y(\tau) = y_\tau\},$$
$$y_\tau \in K.$$

It follows from Theorem 1.3 that $\mathscr{D}_{\tau,T}(y_\tau) \neq \emptyset$ and $\mathscr{D}_{\tau,T}(y_\tau) \subset C([\tau, T]; H)$ $\forall \tau < T$, $y_\tau \in K$. Moreover, the conditions imposed on the parameters of Problem (1.56) and the generalized Gronwall–Bellman lemma [2] imply the existence of C_4, C_5, C_6, $C_7 > 0$ such that, for any finite time interval $[\tau, T]$ each physical solution y to Problem (1.56) on $[\tau, T]$ satisfies the following estimate $\forall t \ge s$, $t, s \in [\tau, T]$:

$$\|y(t)\|_H^2 + C_4 \int_s^t \|y(\xi)\|_V^p d\xi \le \|y(s)\|_H^2 + C_5(t-s), \tag{1.78}$$

$$\|y(t)\|_H^2 \leq \|y(s)\|_H^2 e^{-C_6(t-s)} + C_7. \tag{1.79}$$

Moreover, the translation and concatenation of physical solutions to Problem (1.56) on finite time intervals are physical solutions of this problem on the corresponding intervals. It follows from the proof of Theorem 1.3 that the penalty method guarantees the existence of physical solutions to Problem (1.56) on a finite time interval that are equicontinuous as mappings from a time interval $[\tau, T]$ into the phase space H if they start from a bounded subset of the natural phase space H (that is, the statements of the theorems on the strong convergence of solutions from [24]). Thus (see [24]), physical solutions (a) can be extended to global solutions defined on the positive time semiaxis; (b) tends uniformly to a small (compact) subset of the natural phase space H (as time $t \to +\infty$), and this subset is independent of the bounded set from which they have started. Proceeding from the results of [24], such an attracting set consists of complete trajectories of Problem (1.56) that are defined on the entire real line. Thus, the results of (1.56) allow one to globally describe the dynamics of solutions to such problems by finite algorithms up to an arbitrary small parameter.

1.4 Nonlinear Parabolic Equations of Divergent Form

Consider now an example of the class of nonlinear boundary problems for which the dynamics of solutions can be investigated as $t \to +\infty$. Note that our consideration does not pretend to generality.

Assume that $n \geq 2$, $m \geq 1$, $p \geq 2$, $1 < q \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\Omega \subset \mathrm{R}^n$ is a bounded domain with a sufficiently smooth boundary $\Gamma = \partial\Omega$, N_1 (N_2 accordingly) is the number of differentiations with respect to x of order of $\leq m-1$ (of order of $= m$ accordingly). Let $A_\alpha(x, \eta; \xi)$ be the family of real functions ($|\alpha| \leq m$) that are defined in $\Omega \times \mathrm{R}^{N_1} \times \mathrm{R}^{N_2}$ and satisfy the following conditions:

(i) for a.a. $x \in \Omega$ a function $(\eta, \xi) \to A_\alpha(x, \eta, \xi)$ is continuous in $\mathrm{R}^{N_1} \times \mathrm{R}^{N_2}$;
(ii) $\forall(\eta, \xi) \in \mathrm{R}^{N_1} \times \mathrm{R}^{N_2}$ a function $x \to A_\alpha(x, \eta, \xi)$ is measurable in Ω;
(iii) there are $c_1 \geq 0$ and $k_1 \in L_q(\Omega)$ such that, for a.a. $x \in \Omega$ and $\forall(\eta, \xi) \in \mathrm{R}^{N_1} \times \mathrm{R}^{N_2}$

$$|A_\alpha(x, \eta, \xi)| \leq c_1[|\eta|^{p-1} + |\xi|^{p-1} + k_1(x)];$$

(iv) there are $c_2 > 0$ and $k_2 \in L_1(\Omega)$ such that, for a.a $x \in \Omega$ and $\forall(\eta, \xi) \in \mathrm{R}^{N_1} \times \mathrm{R}^{N_2}$

$$\sum_{|\alpha|=m} A_\alpha(x, \eta, \xi)\xi_\alpha \geq c_2|\xi|^p - k_2(x);$$

(v) there is an increasing real-valued function υ such that, for a.a. $x \in \Omega$, $\forall\eta \in \mathrm{R}^{N_1}$, and $\forall\xi, \xi^* \in \mathrm{R}^{N_2}$, $\xi \neq \xi^*$ the following inequality holds:

$$\sum_{|\alpha|=m} (A_\alpha(x,\eta,\xi) - A_\alpha(x,\eta,\xi^*))(\xi_\alpha - \xi^*_\alpha) \geq (\upsilon(\xi_\alpha) - \upsilon(\xi^*_\alpha))(\xi_\alpha - \xi^*_\alpha).$$

We introduce the denotations $D^k u = \{D^\beta u, |\beta| = k\}$ and $\delta u = \{u, Du, \dots, D^{m-1}u\}$ [48, p. 194].

For an arbitrary fixed external force $f \in L_2(\Omega)$, we will investigate the dynamics of the following problem for all weak (generalized) solutions defined on $[0, +\infty)$ as $t \to +\infty$:

$$\frac{\partial y(x,t)}{\partial t} + \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha \left(A_\alpha(x, \delta y(x,t), D^m y(x,t))\right) = f(x) \text{ in } \Omega \times (0,+\infty), \tag{1.80}$$

$$\begin{aligned} D^\alpha y(x,t) = 0 \text{ on } \Gamma \times (0,+\infty), \quad |\alpha| \leq m-1, \\ y(x,t) \geq 0 \text{ for a.a. } (x,t) \in \Omega \times (0,+\infty). \end{aligned} \tag{1.81}$$

We introduce the following denotations [48, p. 195]: $H = L_2(\Omega)$, $V = W_0^{m,p}(\Omega)$, $V_\sigma = W_0^{m+\sigma,p}(\Omega)$, $\sigma \gg 1$, is the Sobolev real space, $K = \{y \in W_0^{m,p}(\Omega) : \ y(x) \geq 0 \text{ for a.a. } x \in \Omega\}$, and

$$a(u,\omega) = \sum_{|\alpha|\leq m} \int_\Omega A_\alpha(x, \delta u(x), D^m u(x)) D^\alpha \omega(x) dx, \quad u, \omega \in V.$$

Taking into account conditions (i)–(v) and [29, pp. 192–199], the operator $A : V \to V^*$ defined by the formula $\langle A(u), \omega\rangle_V = a(u,\omega)$ $\forall u, \omega \in V$ satisfies the basic assumptions. Therefore, we can pass from Problems (1.80), (1.81) to corresponding Problem (1.56). Note that

$$A(u) = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha \left(A_\alpha(x, \delta u, D^m u)\right) \quad \forall u \in C_0^\infty(\Omega).$$

Thus, for physical solutions to Problems (1.80), (1.81), all the statements from the previous sections hold.

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Chapter 2
Regularity of Solutions for Nonlinear Systems

Abstract In this chapter we establish sufficient conditions for regularity of all weak solutions for nonlinear systems. We note that the respective Cauchy problems may have nonunique weak solution. In Sect. 2.1 we establish regularity of all weak solutions for parabolic feedback control problems. Section 2.2 devoted to artificial control method for nonlinear partial differential equations and inclusions. The regularity of all weak solutions is obtained. In Sect. 2.3 we consider regularity results of all weak solutions for nonlinear reaction-diffusion systems with nonlinear growth. In Sect. 2.4 we consider the following examples of applications: a parabolic feedback control problem; a model of conduction of electrical impulses in nerve axons; a climate energy balance model; FitzHugh–Nagumo System; a model of combustion in porous media.

2.1 Regularity of All Weak Solutions for a Parabolic Feedback Control Problem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be bounded and open subset with a smooth boundary $\partial\Omega$, $\underline{f}$, $\overline{f} : \mathbb{R} \to \mathbb{R}$ are some real functions. We consider the semilinear reaction-diffusion inclusion:

$$u_t - \triangle u + [\underline{f}(u), \overline{f}(u)] \ni 0 \text{ in } \Omega \times (\tau, T), \quad (-\infty < \tau < T < +\infty), \tag{2.1}$$

with boundary condition

$$u\big|_{\partial\Omega} = 0, \tag{2.2}$$

where $[a, b] = \{\alpha a + (1 - \alpha)b \mid \alpha \in [0, 1]\}, a, b \in \mathbb{R}$. We suppose that $f = [\underline{f}, \overline{f}] : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ satisfies the growth condition

$$\exists c_0 > 0 : \quad -c_0(1 + |u|) \leq \underline{f}(u) \leq \overline{f}(u) \leq c_0(1 + |u|) \quad \forall u \in \mathbb{R}. \tag{2.3}$$

Suppose also that $\underline{f}$ is lower semi-continuous, and $\overline{f}$ is upper semi-continuous.

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We shall use the following standard notations: $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, V' is the dual space of V. The function $u(\cdot) \in L^2(\tau, T; V)$ is a *weak solution* of Problem (2.1) and (2.2) on $[\tau, T]$, if there exists a measurable function $d : \Omega \times (\tau, T) \to \mathbb{R}$ such that

$$d(x,t) \in [\underline{f}(u(x,t)), \overline{f}(u(x,t))] \quad \text{for a.e. } (x,t) \in \Omega \times (\tau, T); \tag{2.4}$$

$$-\int_\tau^T \left\langle u, \frac{d\xi}{dt} \right\rangle dt + \int_\tau^T \int_\Omega (\nabla u, \nabla \xi)\, dxdt + \int_\tau^T \int_\Omega (d, \xi)\, dxdt = 0 \tag{2.5}$$

for all $\xi \in C_0^\infty(\Omega \times (\tau, T))$, where $\langle \cdot, \cdot \rangle$ denotes the pairing in the space V.

We note that Problem (2.1) and (2.2) arises in many important models for distributed parameter control problems and that large class of identification problems enter this formulation. Let us indicate a problem which is one of motivations for the study of the autonomous evolution inclusion (2.1) (cf. [37, 56] and references therein). In a subset Ω of $\mathbb{R}^3$, we consider the nonstationary heat conduction equation (Figs. 2.1 and 2.2):

$$\frac{\partial y}{\partial t} - \triangle y = f \text{ in } \Omega \times (0, +\infty)$$

with initial conditions and suitable boundary ones. Here $y = y(x,t)$ represents the temperature at the point $x \in \Omega$ and time $t > 0$. It is supposed that $f = f_1 + f_2$, where f_2 is given and f_1 is a known function of the temperature of the form

$$-f_1(x,t) \in \partial j(x, y(x,t)) \text{ a.e. } (x,t) \in \Omega \times (0, +\infty);$$

Figure 2.3 Here $\partial j(x, \xi)$ denotes generalized gradient of Clarke (see [12]) with respect to the last variable of a function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ which is assumed to be locally Lipschitz in ξ (cf. [37] and references therein). The multi-valued function $\partial j(x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is generally nonmonotone and it includes the vertical jumps. In a physicist's language it means that the law is characterized by the generalized gradient of a nonsmooth potential j (cf. [39]).

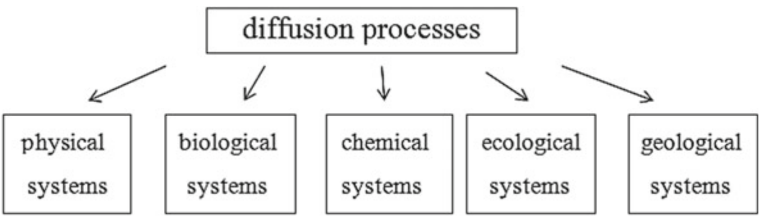

Fig. 2.1 Diffusion processes

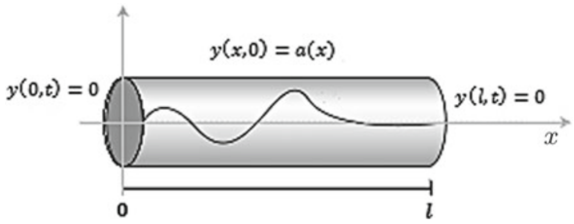

Fig. 2.2 Idealized physical setting for heat conduction in a rod with homogeneous boundary conditions

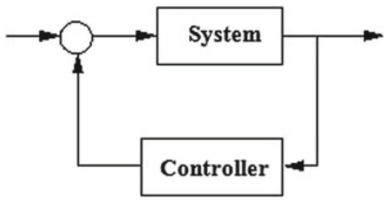

Fig. 2.3 Feedback control diagram

Another motivations connected with parabolic equations with a discontinuous nonlinearity. In [43] it is considered the case, when f is the difference of maximal monotone maps. Global attractor in phase space H for such type equations is considered there. Obtained inclusion is a particular case of an abstract differential inclusion generated by a difference of subdifferential maps of proper convex lower semicontinuous functionals [38]. Models of physical interest includes also the next (cf. [3] and references therein):

- a model of combustion in porous media;
- a model of conduction of electrical impulses in nerve axons;
- a climate energy balance model;

etc. The main purpose of this subsection is to investigate regularity properties of all globally defined weak solutions for Problem (2.1) and (2.2) with initial data $u_\tau \in H$ under listed above assumptions.

Further we need to consider the restriction of $v : [\tau, T] \to V^*$ on $[s, T]$, $s \in (\tau, T)$, $\tau < T$. To simplify conclusions denote it by the same symbol $v(\cdot)$.

Theorem 2.1 *Let $u(\cdot)$ be an arbitrary weak solution of Problem (2.1) and (2.2) on $[\tau, T]$. Then for any $\varepsilon \in (0, T - \tau)$ $u(\cdot) \in C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; H^2(\Omega) \cap V)$ and $u_t(\cdot) \in L^2(\tau + \varepsilon, T; H)$.*

Proof Let $u(\cdot)$ be an arbitrary weak solution of Problem (2.1) and (2.2) on $[\tau, T]$. Then there exists a measurable function $d : \Omega \times (\tau, T) \to \mathbb{R}$ such that $u(\cdot)$ and $d(\cdot)$ satisfy (2.4) and (2.5). As $u(\cdot) \in L^2(\Omega \times (\tau, T))$ and the growth condition (2.3) holds, then $d(\cdot) \in L^2(\Omega \times (\tau, T))$. The set

$$\mathscr{D} := \{s \in (\tau, T) \mid u(s) \in V\}$$

is dense in $[\tau, T]$. For any arbitrary fixed $s \in \mathscr{D}$ we note that $u(\cdot)$ is the unique weak solution on $[s, T]$ of the problem

$$\begin{cases} z_t - \triangle z = -d(x,t) \text{ in } \Omega \times (s,T), \\ z\big|_{\partial\Omega} = 0, \\ z(x,s) = u(x,s) \text{ in } \Omega. \end{cases} \tag{2.6}$$

Moreover, $u(\cdot) \in L^2(s,T;H^2(\Omega)\cap V)\cap C([s,T];V)$ and $u_t(\cdot) \in L^2(s,T;H)$, $s \in \mathscr{D}$ (cf. [40, Chap. 4.I], [42, Chap. III] and references therein). Thus for any $\varepsilon \in (0, T-\tau)$ $u(\cdot) \in C([\tau+\varepsilon,T];V)\cap L^2(\tau+\varepsilon,T;H^2(\Omega)\cap V)$ and $u_t(\cdot) \in L^2(\tau+\varepsilon,T;H)$.

The theorem is proved.

2.2 Artificial Control Method for Nonlinear Partial Differential Equations and Inclusions: Regularity of All Weak Solutions

Let $(V;H;V^*)$ be evolution triple , where V be a real Hilbert space, such that $V \subset H$ with compact imbedding. Let $A : V \to V^*$ be a linear symmetric operator such that $\exists c > 0 : \langle Av, v\rangle_V \geq c\|v\|_V^2$, for each $v \in V$ and let $D(A) = \{u \in V : Au \in H\}$. We note that the mapping $v \to \|Av\|_H$ defines the equivalent norm on $D(A)$; Temam [42, Chap. III]. Let $B : \mathbb{R} \times V \to 2^H \setminus \{\emptyset\}$ be set-valued (in the general case) mapping such that the following assumption holds: there exists $c_1 > 0$ such that $\|y\|_H \leq c_1(1+\|u\|_V)$, for a.e. t and each $u \in V$ and $y \in B(t,u)$.

For a set $D \subset H$ let $\overline{\text{co}}D$ be a closed convex hull of a set D. We consider the differential-operator inclusion:

$$\frac{du}{dt} + Au(t) + B(t,u(t)) \ni \bar{0} \quad (-\infty < \tau < T < +\infty). \tag{2.7}$$

The function $u(\cdot) \in L^2(\tau,T;V)$ is called a *weak solution* of Problem (2.7) on $[\tau,T]$, if there exists a Bochner-measurable function $d : (\tau,T) \to H$ such that

$$d(t) \in \overline{\text{co}}B(t,u(t)) \text{ for a.e. } t \in (\tau,T); \text{ and} \tag{2.8}$$

$$\int_\tau^T \left[-\langle u,v\rangle\, \xi'(t) + \langle Au,v\rangle\, \xi(t) + \langle d,v\rangle\, \xi(t)\right] dt = 0, \tag{2.9}$$

for all $\xi \in C_0^\infty(\tau,T)$ and for all $v \in V$, where $\langle\cdot,\cdot\rangle$ denotes the pairing in the space V.

The main regularity result of this section has the following formulation.

Theorem 2.2 *Let $-\infty < \tau < T < +\infty$ and $u_\tau \in H$. If $u(\cdot)$ is a weak solution of Problem (2.7) on $[\tau,T]$, then $u(\cdot) \in C([\tau+\varepsilon,T];V)\cap L^2(\tau+\varepsilon,T;D(A))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau+\varepsilon,T;H)$ for each $\varepsilon \in (0,T-\tau)$.*

Proof Let $u(\cdot)$ be an arbitrary weak solution of Problem (2.7) on $[\tau, T]$. According to the definition of a weak solution of Problem (2.7) on $[\tau, T]$, there exist $d \in L^2(\tau, T; H)$ such that $u(\cdot) \in L^2(\tau, T; V)$ and $d(\cdot)$ satisfy (2.8) and (2.9). Note that the set

$$\mathscr{D} := \{s \in (\tau, T) \mid u(s) \in V\}$$

is dense in $[\tau, T]$. For an arbitrary fixed $s \in \mathscr{D}$ we remark that $u(\cdot)$ is the unique weak solution on $[\tau, T]$ of the problem

$$\begin{cases} \frac{dz}{dt} + Az(t) = -d(t) \text{ on } (s, T), \\ z(s) = u(s). \end{cases} \tag{2.10}$$

Therefore, $u(\cdot) \in L^2(s, T; D(A)) \cap C([s, T]; V)$ and $\frac{du}{dt}(\cdot) \in L^2(s, T; H)$, $s \in \mathscr{D}$ (cf. [40, Chap. 4.I], [42, Chap. III] and references therein). Thus $u(\cdot) \in C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; D(A))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau + \varepsilon, T; H)$ for any $\varepsilon \in (0, T - \tau)$.

The theorem is proved.

Remark 2.1 Theorem 2.2 implies that each weak solution of Problem (2.7) on $[\tau, T]$ is *regular*, that is, $u(\cdot) \in L^2(\varepsilon, T; D(A)) \cap C([\varepsilon, T]; V)$ and $\frac{du}{dt}(\cdot) \in L^2(\varepsilon, T; H)$, for each $\varepsilon \in (0, T - \tau)$.

Let $B(t, u) := \partial J_1(u) - \partial J_2(u)$ for each $u \in V$ and $t \in \mathbb{R}$, where $J_i : H \to \mathbb{R}$ be a convex, lower semi-continuous function such that the following assumptions hold: (i) (growth condition) there exists $c_1 > 0$ such that $\|y\|_H \le c_1(1 + \|u\|_H)$, for each $u \in H$ and $y \in \partial J_i(u)$ and $i = 1, 2$; (ii) (sign condition) there exist $c_2 > 0$, $\lambda \in (0, c)$ such that $(y_1 - y_2, u)_H \ge -\lambda \|u\|_H^2 - c_2$, for each $y_i \in \partial J_i(u)$, $u \in H$, where $\partial J_i(u)$ the subdifferential of $J_i(\cdot)$ at a point u. Note that $u^* \in \partial J_i(u)$ if and only if $u^*(v - u) \le J_i(v) - J_i(u)\ \forall v \in H$; $i = 1, 2$. For such B Problem (2.7) has the following formulation:

$$\frac{du}{dt} + Au(t) + \partial J_1(u(t)) - \partial J_2(u(t)) \ni \bar{0} \quad (-\infty < \tau < T < +\infty). \tag{2.11}$$

We recall that the function $u(\cdot) \in L^2(\tau, T; V)$ is called a *weak solution* of Problem (2.11) on $[\tau, T]$, if there exist Bochner measurable functions $d_i : (\tau, T) \to H$; i = 1,2, such that

$$d_i(t) \in \partial J_i(u(t)) \text{ for a.e. } t \in (\tau, T), \text{ i=1,2; and} \tag{2.12}$$

$$\int_\tau^T \left[- \langle u, v\rangle \, \xi'(t) + \langle Au, v\rangle \, \xi(t) + \langle d_1, v\rangle \, \xi(t) - \langle d_2, v\rangle \, \xi(t)\right] dt = 0, \tag{2.13}$$

for all $\xi \in C_0^\infty(\tau, T)$ and for all $v \in V$.

The following theorem provides sufficient conditions for the existence and regularity of all weak solutions for Problem (2.11).

Theorem 2.3 *Let* $-\infty < \tau < T < +\infty$ *and* $u_\tau \in H$. *If* $u(\cdot)$ *is a weak solution of Problem (2.11) on* $[\tau, T]$*, then* $u(\cdot) \in C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; D(A))$ *and* $\frac{du}{dt}(\cdot) \in L^2(\tau + \varepsilon, T; H)$ *for any* $\varepsilon \in (0, T - \tau)$.

Proof The regularity of each weak solution follows from Theorem 2.2.
The theorem is proved.

2.3 Regularity of All Weak Solutions for Nonlinear Reaction-Diffusion Systems with Nonlinear Growth

In this section we establish sufficient conditions for regularity of weak solutions for both reaction-diffusion equations (Sect. 2.3.1) as well as systems of reaction-diffusion equations (Sect. 2.3.2) separately.

2.3.1 Reaction-Diffusion Equations

In a bounded domain $\Omega \subset \mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$ we consider the problem

$$\begin{cases} u_t - \Delta u + f(u) = h, \quad x \in \Omega, \ t > 0, \\ u|_{\partial\Omega} = 0, \\ u(0) = u_0, \end{cases} \tag{2.14}$$

where

$$\begin{gathered} h \in L^2(\Omega), \\ f \in C(\mathbb{R}), \\ |f(u)| \le C_1(1 + |u|^{p-1}), \quad \forall u \in \mathbb{R}, \end{gathered} \tag{2.15}$$

with $2 \le p \le 3$, $C_1, C_2, \alpha > 0$.

We denote by A the operator $-\Delta$ with Dirichlet boundary conditions, so that $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. As usual, denote the eigenvalues and the eigenfunctions of A by $\lambda_i, e_i, i = 1, 2 \ldots$

Denote $F(u) = \int_0^u f(s)ds$. From (2.15) we have that $\liminf\limits_{|u|\to\infty} \frac{f(u)}{u} = \infty$, and for some $D_1 0$,

$$|F(u)| \le D_1(1 + |u|^p), \qquad \forall u \in \mathbb{R}. \tag{2.16}$$

In what follows we denote $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, and $\|\cdot\|$, $(\cdot, \cdot)$ will be the norm and the scalar product in $L^2(\Omega)$. We denote by $\|\cdot\|_X$ the norm in the abstract Banach space X, whereas $(\cdot, \cdot)_Y$ will be the scalar product in the abstract Hilbert space Y. Also, $P(X)$ will be the set of all non-empty subsets of X.

On the other hand, we define the usual sequence of spaces

$$V^{2\alpha} = D\left(A^{\alpha}\right) = \{u \in H : \sum_{i=1}^{\infty} \lambda_i^{2\alpha} \left|(u, e_i)\right|^2 < \infty\},$$

where $\alpha \geq 0$. We recall the following well known result, which is a particular case of [40, Lemma 37.8] for our operator $A = -\Delta$ in a three-dimensional domain.

Lemma 2.1 *$D\left(A^{\alpha}\right) \subset W^{k,q'}\left(\Omega\right)$ whenever $q' \geq 2$ and k is an integer such that*

$$k - \frac{3}{q'} < 2\alpha - \frac{3}{2}.$$

Also, it is well known that $V^s \subset H^s\left(\Omega\right)$ for all $s \geq 0$ (see [49, Chap. IV] or [34]).

A function $u \in L^2_{loc}(0, +\infty; V) \bigcap L^p_{loc}(0, +\infty; L^p(\Omega))$ is called a *weak solution* of (2.14) on $(0, +\infty)$ if for all $T > 0$, $v \in V$, $\eta \in C_0^{\infty}(0, T)$

$$-\int_0^T (u, v)\eta_t dt + \int_0^T \left((u, v)_V + (f(u), v) - (h, v)\right)\eta dt = 0.$$

It is well known [1, Theorem 2] or [9, p. 284] that for any $u_0 \in$ there exists at least one weak solution of (2.14) with $u(0) = u_0$ (and it may be non unique) and that any weak solution of (2.14) belongs to $C\left([0, +\infty); H\right)$. Moreover, the function $t \mapsto \|u(t)\|^2$ is absolutely continuous and

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|u(t)\|_V^2 + (f(u(t)), u(t)) - (h, u(t)) = 0 \text{ a.e.} \tag{2.17}$$

The function $u \in L^2_{loc}(0, +\infty; V) \bigcap L^p_{loc}(0, +\infty; L^p(\Omega))$ is called a *regular solution* of (2.14) on $(0, +\infty)$ if for all $T > 0$, $v \in V$, and $\eta \in C_0^{\infty}(0, T)$ we have

$$-\int_0^T (u, v)\eta_t dt + \int_0^T \left((u, v)_V + (f(u), v) - (h, v)\right)\eta dt = 0, \tag{2.18}$$

and

$$u \in L^{\infty}\left(\varepsilon, T; V\right), \tag{2.19}$$

$$u_t \in L^2\left(\varepsilon, T; H\right), \ \forall 0 < \varepsilon < T. \tag{2.20}$$

Any regular solution u satisfies

$$u \in L^2\left(\varepsilon, T; D\left(A\right)\right). \tag{2.21}$$

In this section we will prove that every weak solution is in fact a regular solution.

Theorem 2.4 *Assume that $2 \leq p \leq 3$ in condition (2.15). Then any weak solution $u(\cdot)$ satisfies $u \in C([\varepsilon, T]; V) \cap L^2(\varepsilon, T; D(A))$, $u_t \in L^2(\varepsilon, T; H)$ for all $\varepsilon > 0$, that is, it is a regular solution.*

Proof From

$$\int_{\Omega} |f(u(x,t))|^{\frac{p}{p-1}} dx \leq C_1 + C_2 \int_{\Omega} |u(x,t)|^p dx$$

we obtain that

$$\|f(u(t))\|^2_{L^{\frac{p}{p-1}}(\Omega)} \leq C_3 + C_4 \|u(t)\|^{2p-2}_{L^p(\Omega)}.$$

Using the Sobolev embedding $H^r(\Omega) \subset L^p(\Omega)$ if $r = \left(\frac{3}{2} - \frac{3}{p}\right) \leq \frac{1}{2}$ (as $p \leq 3$) and the Gagliardo–Nirenberg inequality

$$\|v\|_{H^r(\Omega)} \leq C_5 \|v\|_{H^{\frac{1}{2}}(\Omega)} \leq C_6 \|v\|^{\frac{1}{2}} \|v\|^{\frac{1}{2}}_{H^1(\Omega)},$$

we have

$$\begin{aligned} \|f(u(t))\|^2_{L^{\frac{p}{p-1}}(\Omega)} &\leq C_3 + C_7 \|u(t)\|^{p-1} \|u(t)\|^{p-1}_{H^1(\Omega)} \\ &\leq C_8 + C_9 \|u(t)\|^2 \|u(t)\|^2_{H^1(\Omega)}. \end{aligned}$$

Thus,

$$\|f(u)\|_{L^2\left(0,T;L^{\frac{p}{p-1}}(\Omega)\right)} \leq C_{10}\left(1 + \|u\|_{C([0,T];H)} \|u\|_{L^2(0,T;H^1(\Omega))}\right).$$

Set $d(x,t) = f(u(x,t))$ for $(x,t) \in (0,T) \times \Omega$. Then $d \in L^2\left(0, T; L^{\frac{p}{p-1}}(\Omega)\right) \subset L^2\left(0, T; H^{-r}(\Omega)\right) \subset L^2\left(0, T; V^{-r}\right)$.

We consider the problem

$$\begin{cases} v_t - \Delta v = -d(x,t) + h(x), \quad x \in \Omega,\ t > 0, \\ v|_{\partial\Omega} = 0, \\ v(\tau) = u(\tau). \end{cases}$$

We note that $u(\tau) \in V \subset V^r$ for a.a. $\tau > 0$. For such τ in view of [40, p. 163, Theorem 42.12] there exists a unique weak solution $v(\cdot)$ such that $v \in C([\tau, T]; V^r) \cap L^2\left(\tau, T; V^{r+1}\right)$. Hence, $u \in C([\varepsilon, T]; V^r) \cap L^2\left(\varepsilon, T; V^{r+1}\right)$ for all $\varepsilon > 0$.

We shall prove that $f(u(\cdot)) \in L^2(\varepsilon, T; H)$. As this is obvious if $p = 2$, we consider that $2 < p \leq 3$. We note that $V^r \subset H^r(\Omega) \subset L^p(\Omega)$. Also, by Lemma 2.1 with $\alpha = \frac{r+1}{2}$, $r = 3\left(\frac{1}{2} - \frac{1}{p}\right)$, $k = 1$ we obtain that $V^{r+1} \subset W^{1,q'}(\Omega)$ for any $q' < p$. On the other hand, by the Sobolev embedding theorems we have

$W^{1,q'}(\Omega) \subset L^q(\Omega)$, for $q < \frac{3p}{3-p}$. Thus, the inequality $p(p-1) < \frac{3p}{3-p}$, for all $2 \le p \le 3$, implies that $u \in C([\varepsilon, T]; L^p(\Omega)) \cap L^2\left(\varepsilon, T; L^{p(p-1)}(\Omega)\right)$. By (2.15) we have

$$\|f(u(t))\|^2 = \int_\Omega |f(u(x,t))|^2\,dx \le C_{11} + C_{12}\int_\Omega |u(x,t)|^{2(p-1)}\,dx$$
$$\le C_{13} + C_{14}\,\|u(t)\|^{p-1}_{L^p(\Omega)}\,\|u(t)\|^{p-1}_{L^{p(p-1)}(\Omega)}.$$

Therefore, $f(u(\cdot)) \in L^2(\varepsilon, T; H)$. Then standard results imply that $u \in C([\varepsilon, T]; V) \cap L^2(\varepsilon, T; D(A))$ and $u_t \in L^2(\varepsilon, T; H)$.

The lemma is proved.

Remark 2.2 Theorem 2.4 was proved in [23].

2.3.2 Systems of Reaction-Diffusion Equations

Let us consider the following reaction-diffusion system (RD-system for short)

$$\begin{cases} u_t = a\Delta u - f(u) + h(x), \quad x \in \Omega,\ t > 0, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.22}$$

where $u = u(x,t) = (u^1(x,t), \ldots, u^N(x,t))$ is unknown vector-function, a is a real $N \times N$ matrix with positive symmetric part $\frac{1}{2}(a + a^*) \ge \beta I,\ \beta > 0,\ h = (h_1, \ldots, h_N),\ f = (f_1, \ldots, f_N)$ are given functions,

$$h \in (L^2(\Omega))^N, \quad f \in C(\mathbb{R}^N; \mathbb{R}^N),$$

and for given numbers $C_1, C_2 \ge 0,\ \gamma > 0,\ p_i \ge 2,\ i = \overline{1, N}$ the following conditions hold:

$$\forall\, v \in \mathbb{R}^N \quad \sum_{i=1}^N |f_i(v)|^{q_i} \le C_1(1 + \sum_{i=1}^N |v^i|^{p_i}), \tag{2.23}$$

$$\forall\, v \in \mathbb{R}^N \quad \sum_{i=1}^N f_i(v)v^i \ge \gamma \sum_{i=1}^N |v^i|^{p_i} - C_2, \tag{2.24}$$

where $\frac{1}{p_i} + \frac{1}{q_i} = 1,\ i = \overline{1, N}$. In further arguments we will use the standard functional spaces

$$H = (L^2(\Omega))^N \text{ with the norm } |v|^2 = \int_\Omega \sum_{i=1}^N |v^i(x)|^2 dx,$$

$$V = (H_0^1(\Omega))^N \text{ with the norm } \|v\|^2 = \int_\Omega \sum_{i=1}^N |\nabla v^i(x)|^2 dx.$$

Let us denote $V' = H^{-1}(\Omega))^N$, $\mathbf{p} = (p_1, \ldots, p_N)$, $L^{\mathbf{p}}(\Omega) = L^{p_1}(\Omega) \times \cdots \times L^{p_N}(\Omega)$,

$$W = L^{\mathbf{p}}_{loc}(0, +\infty; L^{\mathbf{p}}(\Omega)) \cap L^2_{loc}(0, +\infty; V).$$

Definition 2.1 The function $u = u(x, t) \in W$ is called a (global) weak solution of Problem (2.22) on $(0, +\infty)$ if for all $T > 0$, $v \in V \cap L^{\mathbf{p}}(\Omega)$,

$$\frac{d}{dt}\int_\Omega u(x,t)v(x)dx + \int_\Omega \Big(a\nabla u(x,t)\nabla v(x) + f(u(x,t))v(x) - h(x)v(x)\Big)dx = 0 \tag{2.25}$$

in the sense of scalar distributions on $(0, T)$.

From (2.23) and Sobolev embedding theorem we see that every solution of (2.22) satisfies $u_t \in L^{\mathbf{q}}_{loc}(0, +\infty; H^{-\mathbf{r}}(\Omega))$, where $\mathbf{r} = (r_1, \ldots, r_N)$, $r_k = max\{1, n(\frac{1}{2} - \frac{1}{p_k})\}$. The well-known result on global resolvability of (2.22) for initial conditions from the phase space H established in [9]. Under conditions (2.23), (2.24) for every $u_0 \in H$ there exists at least one weak solution of (2.22) on $(0, +\infty)$ with $u(0) = u_0$. Every weak solution of (2.22) belongs to $C([0, +\infty); H)$, the function $t \mapsto |u(t)|^2$ is absolutely continuous and for a.a. $t \geq 0$ the following energy equality holds

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + (a\nabla u(t), \nabla u(t)) + (f(u(t)), u(t)) = (h, u(t)). \tag{2.26}$$

The function $u = u(x, t) \in W$ is called a *regular solution* of Problem (2.22) on $(0, +\infty)$ if it is weak solution on $(0, +\infty)$ and, additionally,

$$u \in L^\infty\left(\varepsilon, T; V \cap L^{\mathbf{p}}(\Omega)\right), \tag{2.27}$$

$$u_t \in L^2(\varepsilon, T; H) \ \forall\, 0 < \varepsilon < T. \tag{2.28}$$

Let us consider the following additional condition on vector-function f [51]:

$$\forall\, v \in \mathbb{R}^N \quad f(v) = \nabla F(v) + g(v), \tag{2.29}$$

where ∇F satisfies (2.23), (2.24), and $g \in C(\mathbb{R}^N; \mathbb{R}^N)$ is such that for some constants $C_3 \geq 0$, $C_4 \geq 0$,

$$|g(v)|^2 \leq C_3 F(v) + C_4(|v|^2 + 1), \ \forall\, v \in \mathbb{R}^N. \tag{2.30}$$

If $N = 1$ (scalar case), then (2.29), (2.30) hold with $F(v) = \int_0^v f(s)ds$, $g \equiv 0$.

Conditions (2.29), (2.30) also take place if

$$f_i(v) = \alpha_i v^i |v^i|^{p_i - 2} + g_i(v),\ i = \overline{1, N},$$

where $\alpha_i > 0$, $g \in C(\mathbb{R}^N; \mathbb{R}^N)$, and $|g(v)| \leq C_4(1 + |v|)$. Another example is the FitzHugh–Nagumo system (see the example in Sect. 3.4.4 below).

Let us briefly analyze conditions (2.29), (2.30).

Using the equality

$$F(v) - F(0) = \int_0^1 \nabla F(sv) \cdot v ds = \int_0^{\frac{1}{(|v|+1)^2}} \nabla F(sv) \cdot v ds + \int_{\frac{1}{(|v|+1)^2}}^1 \nabla F(sv) \cdot v ds$$

and condition (2.24), we deduce that for some $\alpha > 0$

$$\forall\, v \in \mathbb{R}^N \quad F(v) \geq \alpha \sum_{i=1}^N |v^i|^{p_i} - C_5. \tag{2.31}$$

Again using the equality $F(v) - F(0) = \int_0^1 \nabla F(sv) v ds$, Young's inequality and condition (2.23), we obtain

$$|F(v)| \leq C_6 (\sum_{i=1}^N |v^i|^{p_i} + 1). \tag{2.32}$$

Theorem 2.5 *Under conditions (2.23), (2.24), (2.29), (2.30) for every $u_0 \in H$ there exists at least one regular solution $u(\cdot)$ of (2.22) such that $u(0) = u_0$, and for some positive constants $C(g)$, $D(g)$, which depend on the function g but not on $u(\cdot)$, the following energy inequality holds for a.e. $s > 0$ and each $t \geq s$*

$$E(u(t)) + \int_s^t |u_r|^2 dr \leq E(u(s)) + C(g) \int_s^t E(u(p)) dp + D(g)(t - s), \tag{2.33}$$

where $E(u(t)) = \|u(t)\|^2 + 2(F(u(t)), 1) - 2(h, u(t))$. *Moreover,* $C(g) = D(g) = 0$ *if in condition (2.29) we have* $g \equiv 0$.

Proof We take as in [9, p.281] the Galerkin approximations using the basis of eigenfunctions $\{w_j(x),\ j \in \mathbb{N}\}$, of the Laplace operator with Dirichlet boundary conditions. Let $X_n = \{w_1, \ldots, w_n\}$ and let P_n be the orthogonal projector from H onto X_n. Then $u^n(x, t) = \sum_{j=i}^n a_{j,m}(t) w_j(x)$ will be a solution of the system of ordinary differential equations

$$\frac{du^n}{dt} = P_n \Delta u^n - P_n f\left(u^n\right) + P_n h,\ u^n(0) = P_n u_0. \tag{2.34}$$

It is proved in [9, p.281] that (2.34) is globally resolved, and for every $T > 0$ passing to a subsequence u^n converges to a weak solution u of (2.22) in $C([0,T]; H)$, weakly in $L^{\mathbf{p}}(0,T; L^{\mathbf{p}}(\Omega))$ and weakly in $L^2(0,T;V)$. Also, $u_t^n \to u_t$ weakly in $L^{\mathbf{q}}\left(0,T; H^{-\mathbf{r}}(\Omega)\right)$.

Multiplying the equation in (2.34) by u_t^n we get

$$\frac{d}{dt}\left(\|u^n\|^2 + 2(F(u^n),1) - 2(h,u^n)\right) + 2|u_t^n|^2 = -2(g(u^n),u_t^n). \tag{2.35}$$

Using (2.30), we deduce from (2.35) that

$$\begin{aligned}&\frac{d}{dt}\left(\|u^n\|^2 + 2(F(u^n),1) - 2(h,u^n)\right) + |u_t^n|^2\\ &\le C_7(g)\left(\|u^n\|^2 + 2(F(u^n),1) - 2(h,u^n)\right) + C_8(g).\end{aligned} \tag{2.36}$$

In particular, u^n satisfies (2.33) $\forall\ t \ge s \ge 0$. We note that if $g \equiv 0$, then $C_7(g) = C_8(g) = 0$, so that $C(g) = D(g) = 0$ holds.

On the other hand, multiplying (2.22) by u^n and using (2.23) in a standard way we obtain

$$\frac{d}{dt}|u^n|^2 + \lambda_1|u^n|^2 + \left\|u^n\right\|^2 + \gamma\sum_{i=1}^{N}\left\|u_i^n\right\|_{L^{p_i}}^{p_i} \le K + |h|^2. \tag{2.37}$$

By Gronwall's lemma we obtain

$$|u^n(t)|^2 \le e^{-\lambda_1 t}|u_0^n|^2 + \frac{1}{\lambda_1}\left(K + |h|^2\right). \tag{2.38}$$

Thus integrating (2.37) over $(t, t+r)$ with $r > 0$ we have

$$\begin{aligned}&|u^n(t+r)|^2 + \int_t^{t+r}\left\|u^n\right\|^2 ds + \gamma\int_t^{t+r}\sum_{i=1}^{N}\left\|u_i^n(s)\right\|_{L^{p_i}}^{p_i} ds\\ &\le |u^n(t)|^2 + r\left(K + |h|^2\right)\\ &\le e^{-\lambda_1 t}|u_0^n|^2 + \left(\frac{1}{\lambda_1} + r\right)\left(K + |h|^2\right).\end{aligned} \tag{2.39}$$

Then from (2.32),

$$\int_t^{t+r}\left(\left\|u^n\right\|^2+2\left(F\left(u^n(s)\right),1\right)-2\left(h,u^n\right)\right)ds$$
$$\leq \int_t^{t+r}\left\|u^n\right\|^2 ds+2C_6\int_t^{t+r}\sum_{i=1}^N\left\|u_i^n(s)\right\|_{L^{p_i}}^{p_i}ds+r|h|^2+\int_t^{t+r}|u^n|^2ds+2C_6|\Omega|r$$
$$\leq C_9(e^{-\lambda_1 t}|u_0^n|^2+r+1). \tag{2.40}$$

Now we can apply uniform Gronwall Lemma [46] to inequality (2.36) and obtain

$$\left\|u^n(t+r)\right\|^2+2\left(F\left(u^n(t+r)\right),1\right)-2\left(h,u^n(t+r)\right) \tag{2.41}$$
$$\leq C_{10}\Big(\frac{e^{-\lambda_1 t}|u_0^n|^2+1}{r}+1\Big)e^r \text{ for all } 0\leq t\leq t+r.$$

From the last inequality and (2.31) we have

$$\left\|u^n(t+r)\right\|^2+\sum_{i=1}^N\left\|u_i^n(t+r)\right\|_{L^{p_i}}^{p_i} \tag{2.42}$$
$$\leq C_{11}\left(\Big(\frac{e^{-\lambda_1 t}|u_0^n|^2+1}{r}+1\Big)e^r+1\right) \text{ for all } 0\leq t\leq t+r.$$

Therefore, the sequence $u^n(\cdot)$ is bounded in $L^\infty(r,T;V\cap L^{\mathbf{p}}(\Omega))$ for all $0<r<T$.

Integrating (2.36) over (r,T), we have

$$\int_r^T|u_t^n|^2dt\leq C_7\int_r^T\left(\left\|u^n(s)\right\|^2+2\left(F\left(u^n(s)\right),1\right)-2\left(h,u^n(s)\right)\right)ds \tag{2.43}$$
$$+\left\|u^n(r)\right\|^2+2\left(F\left(u^n(r)\right),1\right)-2\left(h,u^n(r)\right)+C_8(T-r)+2C_5|\Omega|+|h|^2+|u^n(T)|^2.$$

So from (2.38), (2.40), (2.41) and the last inequality we deduce that u_t^n is bounded in $L^2(r,T;H)$ for all $0<r<T$.

Thus for the limit function u we can claim that it is regular solution of (2.22) and $u(0)=u_0$.

Let us prove that u satisfies the energy inequality (2.33). As u^n is bounded in $L^\infty(r,T;L^{\mathbf{p}}(\Omega))$, so $f(u^n)$ is bounded in $L^\infty(r,T;L^{\mathbf{q}}(\Omega))$. Therefore from [45] up to subsequence

$$u^n\to u \text{ in } L^2(r,T;V)\cap L^{\mathbf{p}}(r,T;L^{\mathbf{p}}(\Omega)). \tag{2.44}$$

and, in particular,

$$u^n(t)\to u(t) \text{ in } V \text{ for a.a. } t\in(r,T).$$

Also, it is standard to check that $u^n \to u$ in $C([r, T], H)$, for all $0 < r < T$, and that $u^n(t) \to u(t)$ weakly in V for all $0 < t \leq T$.

Then by the dominated convergence theorem $F(u^n(t)) \to F(u(t))$ in $L^1(\Omega)$ for a.a. $t \in [r, T]$. Also, for any $0 < t \leq T$ we have $F(u^n(x,t)) \to F(u(x,t))$ for a.a. x. Then $F(u^n(x,t)) \geq -C_5$ and Fatou's lemma imply

$$\int_\Omega F(u(x,t))dx \leq \liminf \int_\Omega F(u^n(x,t))dx$$

and

$$E(u(t)) \leq \liminf\ E(u^n(t)).$$

Hence, we can pass to the limit in (2.33) and obtain the required result.

The theorem is proved.

Remark 2.3 Theorem 2.5 yields only existence but not regularity of each weak solution of Problem (2.22). This theorem was proved in [24].

2.4 Examples of Applications

In this section we provide examples of applications to theorems established in Sects. 2.1–2.3. We consider a parabolic feedback control problem (Sect. 2.4.1), a model of conduction of electrical impulses in nerve axons (Sect. 2.4.2), a climate energy balance model (Sect. 2.4.3); FitzHugh–Nagumo system (Sect. 2.4.4); and a model of combustion in porous media (Sect. 2.4.5).

2.4.1 A Parabolic Feedback Control Problem

Let Ω be an open and bounded subset of $\mathbb{R}^3$. Let us consider the following nonstationary heat conduction equation

$$\frac{\partial y}{\partial t} - \triangle y = f \text{ in } \Omega \times \mathbb{R} \tag{2.45}$$

with initial condition and Dirichlet homogeneous boundary condition. Here $y = y(x,t)$ represents the temperature at the point $x \in \Omega$ and time $t > 0$.

Let $j : \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function in ξ (cf. [37] and references therein) and $\partial j(x, \xi)$ denotes generalized gradient of Clarke (see [12]) with respect to the last variable. Note that the multi-valued function $\partial j(x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is generally nonmonotone and it includes the vertical jumps.

We assume that $f = f_1 + f_2$, where $f_2 = f_2(x)$ is given and f_1 is a known function of the temperature of the form

$$-f_1(x,t) \in \partial j(x, y(x,t)) \text{ a.e. } (x,t) \in \Omega \times \mathbb{R}. \tag{2.46}$$

In a physicist's language it means that the law is characterized by the generalized gradient of a nonsmooth potential j (cf. [39]).

Assume also that ∂j satisfies the growth condition

$$\exists c_0 > 0: \ |p| \leq c_0(1+|u|) \text{ for a.e. } x \in \Omega, \text{ and each } u \in \mathbb{R}, \text{ and } d \in \partial j(x,u);$$

and the sign condition

$$\underline{\lim}_{u \to +\infty} \frac{\inf_{d \in \partial j(x,u)} d}{u} > -\lambda_1; \quad \underline{\lim}_{u \to -\infty} \frac{\sup_{d \in \partial j(x,u)} d}{u} > -\lambda_1,$$

where λ_1 is the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$. According to Theorem 2.2, for any $-\infty < \tau < T < +\infty$ each weak solution $u_\tau \in L^2(\Omega)$ of Problem (2.45) and (2.46) on $[\tau, T]$ belongs to $C([\tau+\varepsilon, T]; H_0^1(\Omega)) \cap L^2(\tau+\varepsilon, T; H^2(\Omega) \cap H_0^1(\Omega))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau+\varepsilon, T; L^2(\Omega))$ for each $\varepsilon \in (0, T-\tau)$.

2.4.2 A Model of Conduction of Electrical Impulses in Nerve Axons

Consider the problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u \in \lambda H(u-a), & (x,t) \in (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) = 0, & t \in \mathbb{R}, \end{cases} \tag{2.47}$$

where $a \in \left(0, \frac{1}{2}\right)$; Terman [47, 48]. Since Problem (2.47) is a particular case of Problem (2.1) and (2.2), then for each $-\infty < \tau < T < +\infty$ and a weak solution $u_\tau \in L^2((0,\pi))$ of Problem (2.47) on $[\tau, T]$ belongs to $C([\tau+\varepsilon, T]; H_0^1((0,\pi))) \cap L^2(\tau+\varepsilon, T; H^2((0,\pi)) \cap H_0^1((0,\pi)))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau+\varepsilon, T; L^2((0,\pi)))$ for each $\varepsilon \in (0, T-\tau)$; Figs. 2.4, 2.5, 2.6, and 2.7.

2.4.3 Climate Energy Balance Model

Let $(\mathscr{M}, \mathbf{g})$ be a C^∞ compact connected oriented two-dimensional Riemannian manifold without boundary (as, e.g. $\mathscr{M} = S^2$ the unit sphere of $\mathbb{R}^3$). Consider the problem:

$$\frac{\partial u}{\partial t} - \Delta u + R_e(x,u) \in QS(x)\beta(u), \quad (x,t) \in \mathscr{M} \times \mathbb{R}, \tag{2.48}$$

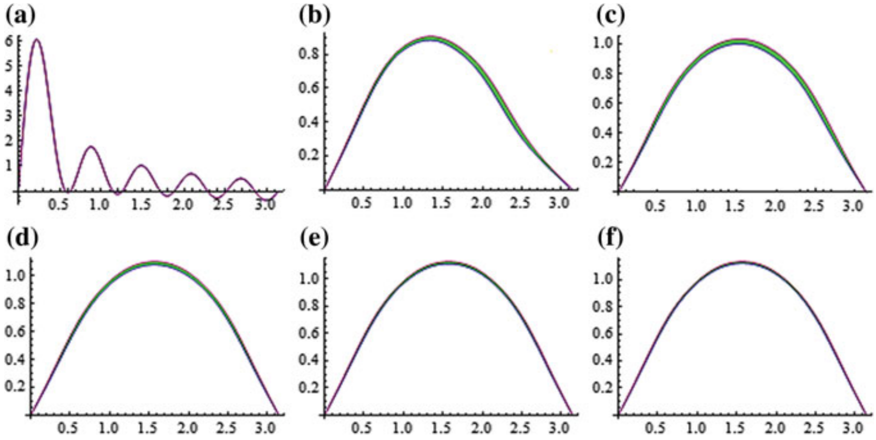

Fig. 2.4 Graphics of solutions of problem (2.47) with $a = 0.49$, $\lambda = 2$, $n = 10$, $h = 0, 001$, $N = 100$ in a moment **a** $t = 0$; **b** $t = 0.8$; **c** $t = 1.6$; **d** $t = 2.4$; **e** $t = 3.2$; **f** $t = 4$

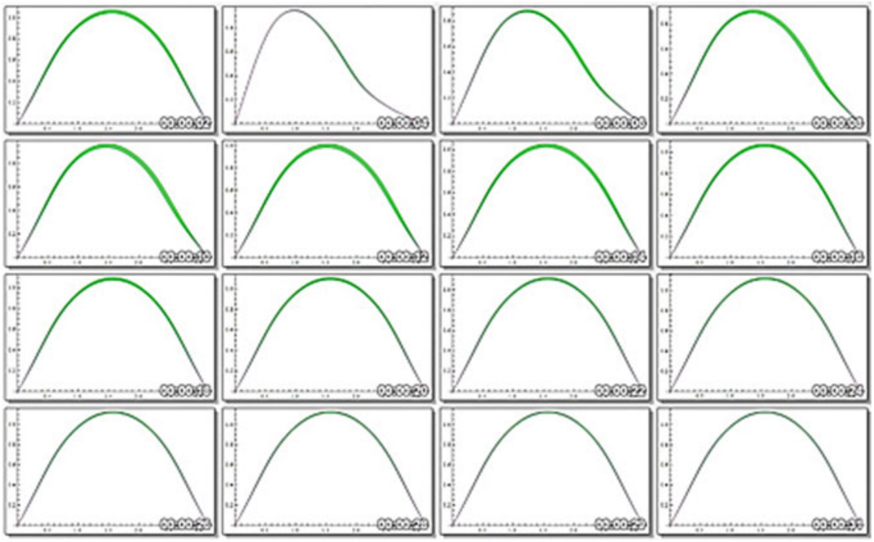

Fig. 2.5 Screenlist of animation for dynamics of solutions of problem (2.47) in 2D

where $\Delta u = \mathrm{div}_{\mathscr{M}} (\nabla_{\mathscr{M}} u)$; $\nabla_{\mathscr{M}}$ is understood in the sense of the Riemannian metric **g**. Note that (2.48) is the so-called climate energy balance model. It was proposed in Budyko [8] and Sellers [41] and examined also in Díaz et al. [13–15]. The unknown $u(x, t)$ represents the average temperature of the Earth's surface. In Budyko [8] the energy balance is expressed as

$$\text{heat variation} \;=\; R_a - R_e + D.$$

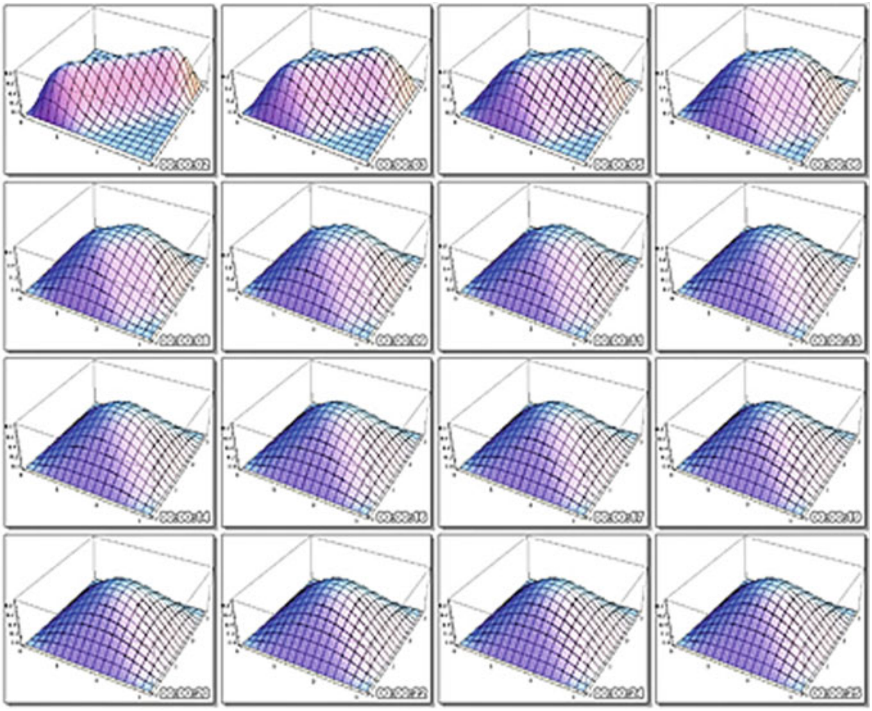

Fig. 2.6 Screenlist of animation for dynamics of solutions of problem (2.47) in 3D

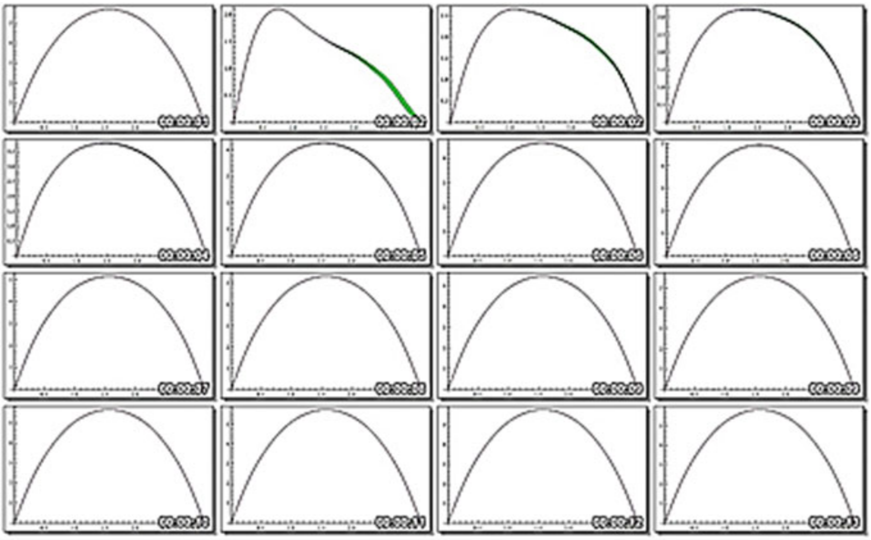

Fig. 2.7 Screenlist of animation for dynamics of solutions of problem (2.47) in section

Here $R_a = QS(x)\beta(u)$. It represents the solar energy absorbed by the Earth, $Q > 0$ is a solar constant, $S(x)$ is an insolation function, given the distribution of solar radiation falling on upper atmosphere, β represents the ratio between absorbed and incident solar energy at the point x of the Earth's surface (so-called co-albedo function). The term R_e represents the energy emitted by the Earth into space, as usual, it is assumed to be an increasing function on u. The term D is heat diffusion, we assume (for simplicity) that it is constant.

As usual, the term R_e may be chosen according to the Newton cooling law as linear function on u, $R_e = Bu + C$ (here B, C are some positive constants) [8], or according to the Stefan-Boltzman law, $R_e = \sigma u^4$ [41]. In this subsection we consider $R_e = Bu$ as in Budyko [8].

Let $S : \mathscr{M} \to \mathbb{R}$ be a function such that $S \in L^\infty(\mathscr{M})$ and there exist $S_0, S_1 > 0$ such that

$$0 < S_0 \le S(x) \le S_1.$$

Suppose also that β is a bounded maximal monotone graph of $\mathbb{R}^2$, that is there exist $m, M \in \mathbb{R}$ such that for all $s \in \mathbb{R}$ and $z \in \beta(s)$

$$m \le z \le M.$$

Let us consider real Hilbert spaces

$$H := L^2(\mathscr{M}), \quad V := \{u \in L^2(\mathscr{M}) \, : \, \nabla_{\mathscr{M}} u \in L^2(T\mathscr{M})\}$$

with respective standard norms $\|\cdot\|_H$, $\|\cdot\|_V$, and inner products $(\cdot,\cdot)_H$, $(\cdot,\cdot)_V$, where $T\mathscr{M}$ represents the tangent bundle and the functional spaces $L^2(\mathscr{M})$ and $L^2(T\mathscr{M})$ are defined in a standard way; see, for example, Aubin [2]. According to Theorem 2.2, for any $-\infty < \tau < T < +\infty$ each weak solution $u_\tau \in L^2(\Omega)$ of Problem (2.48) on $[\tau, T]$ belongs to $C([\tau+\varepsilon, T]; H_0^1(\Omega)) \cap L^2(\tau+\varepsilon, T; H^2(\Omega) \cap H_0^1((0,\pi)))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau+\varepsilon, T; L^2(\Omega))$ for each $\varepsilon \in (0, T-\tau)$.

2.4.4 FitzHugh–Nagumo System

Let us consider generalized FitzHugh–Nagumo system [46]:

$$u_t = d_1 \Delta u - f_1(u) - v, \tag{2.49}$$

$$v_t = d_2 \Delta v + \delta u - \gamma v, \tag{2.50}$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \tag{2.51}$$

where $\Omega = (0, L)$, d_1, d_2, δ, γ are positive constants, $f_1 \in C(\mathbb{R})$,

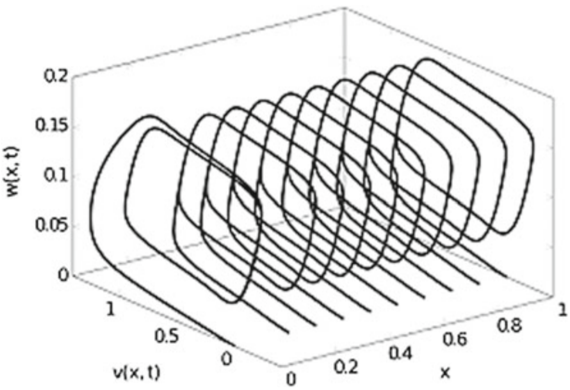

Fig. 2.8 Trajectories of FitzHugh–Nagumo system

$$|f_1(u)| \leq C_1(1+|u|^3); \quad f^1(u)u \geq \alpha|u|^4 - C_2. \tag{2.52}$$

For the vector-function

$$f(u,v) = \begin{pmatrix} f_1(u)+v \\ -\delta u + \gamma v \end{pmatrix}$$

conditions (2.23), (2.24) hold with $p_1 = 4$, $p_2 = 2$. Moreover, $f = \nabla F + g$, where $F = F(u,v) = \int\limits_0^u f_1(s)ds + \frac{\gamma}{2}v^2$, $g = g(u,v) = \begin{pmatrix} v \\ -\delta u \end{pmatrix}$ and conditions (2.29), (2.30) also hold. Then all statements of Theorem 2.5 hold; Fig. 2.8.

2.4.5 A Model of Combustion in Porous Media

Let us consider the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - f(u) \in \lambda H(u-1), & (x,t) \in (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) = 0, \quad t \in \mathbb{R}, \end{cases} \tag{2.53}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous and nondecreasing function satisfying growth and sign assumptions, $\lambda > 0$, and $H(0) = [0,1]$, $H(s) = \mathbf{I}\{s > 0\}$, $s \neq 0$; Feireisl and Norbury [17]. Since Problem (2.53) is a particular case of Problem (2.1) and (2.2), then for any $-\infty < \tau < T < +\infty$ each weak solution $u_\tau \in L^2((0,\pi))$ of Problem (2.53) on $[\tau, T]$ belongs to $C([\tau+\varepsilon, T]; H_0^1((0,\pi))) \cap L^2(\tau+\varepsilon, T; H^2((0,\pi)) \cap H_0^1((0,\pi)))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau+\varepsilon, T; L^2((0,\pi)))$ for each $\varepsilon \in (0, T-\tau)$; Fig. 2.9.

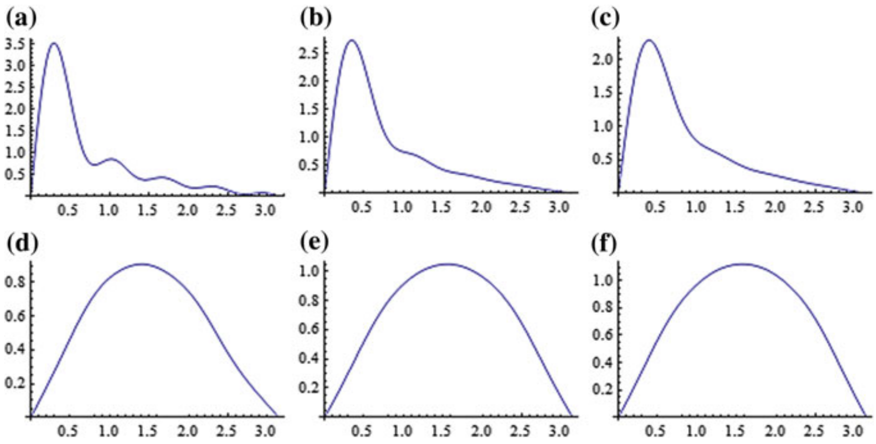

Fig. 2.9 Graphics of solutions with $f(u) = u$, $\lambda = 2$, $\varepsilon = 0.1$, $M = 100$ in moment **a** $t = 0$; **b** $t = 0.8$; **c** $t = 1.6$; **d** $t = 2.4$; **e** $t = 3.2$; **f** $t = 4$

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Chapter 3
Advances in the 3D Navier-Stokes Equations

Abstract In this chapter we provide a criterion for the existence of global strong solutions for the 3D Navier-Stokes system for any regular initial data. Moreover, we establish sufficient conditions for Leray-Hopf property of a weak solution for the 3D Navier-Stokes system. Under such conditions this weak solution is rightly continuous in the standard phase space H endowed with the strong convergence topology.

3.1 Weak, Leray-Hopf and Strong Solutions

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with rather smooth boundary $\Gamma = \partial\Omega$, and $[\tau, T]$ be a fixed time interval with $-\infty < \tau < T < +\infty$. We consider 3D Navier-Stokes system in $\Omega \times [\tau, T]$

$$\begin{cases} \dfrac{\partial y}{\partial t} - \nu \triangle y + (y \cdot \nabla) y = -\nabla p + f, \ \operatorname{div} y = 0, \\ y\big|_{\Gamma} = 0, \qquad\qquad\qquad\qquad\qquad y\big|_{t=\tau} = y_\tau, \end{cases} \tag{3.1}$$

where $y(x,t)$ means the unknown velocity, $p(x,t)$ is the unknown pressure, $f(x,t)$ is the given exterior force, and $y_\tau(x)$ is the given initial velocity with $t \in [\tau, T]$, $x \in \Omega$, $\nu > 0$ means the viscosity constant; see also Figs. 3.1 and 3.2.

Throughout this note we consider generalized setting of Problem (3.1). For this purpose define the usual function spaces

$$\mathscr{V} = \{u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u = 0\}, \ V_\sigma = \operatorname{cl}_{(H_0^\sigma(\Omega))^3} \mathscr{V}, \ \sigma \ge 0,$$

where cl_X denotes the closure in the space X. Set $H := V_0$, $V := V_1$. It is well known that each V_σ, $\sigma > 0$, is a separable Hilbert space and identifying H and its dual H^* we have $V_\sigma \subset H \subset V_\sigma^*$ with dense and compact embedding for each $\sigma > 0$. We denote by $(\cdot,\cdot)$, $\|\cdot\|$ and $((\cdot,\cdot))$, $\|\cdot\|_V$ the inner product and norm in H and V,

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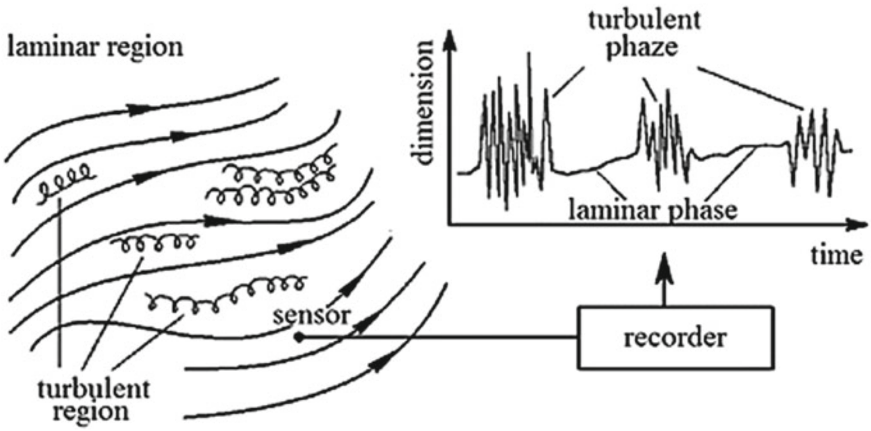
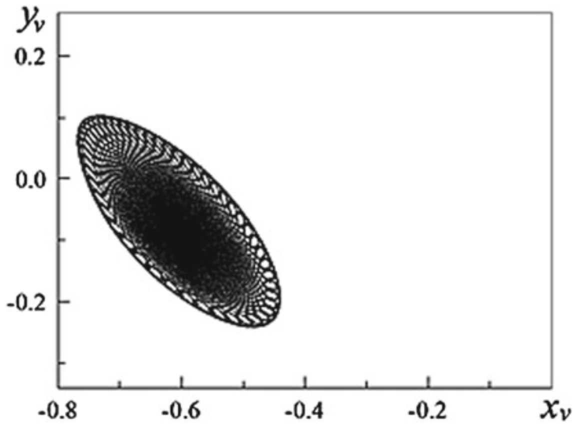


Fig. 3.1 Alternating turbulence

Fig. 3.2 Poincare intersections of vortices in groove in perturbed flow



respectively; $\langle \cdot, \cdot \rangle$ will denote pairing between V and V^* that coincides on $H \times V$ with the inner product $(\cdot, \cdot)$. Let H_w be the space H endowed with the weak topology. For $u, v, w \in V$ we put

$$b(u, v, w) = \int_\Omega \sum_{i,j=1}^{3} u_i \frac{\partial v_j}{\partial x_i} w_j dx.$$

It is known that b is a trilinear continuous form on V and $b(u, v, v) = 0$, if $u, v \in V$. Furthermore, there exists a positive constant C such that

$$|b(u, v, w)| \leq C \|u\|_V \|v\|_V \|w\|_V, \tag{3.2}$$

for each $u, v, w \in V$; see, for example, Sohr [18, Lemma V.1.2.1] and references therein.

Let $f \in L^2(\tau, T; V^*) + L^1(\tau, T; H)$ and $y_\tau \in H$. Recall that the function $y \in L^2(\tau, T; V)$ with $\frac{dy}{dt} \in L^1(\tau, T; V^*)$ is a *weak solution* of Problem (3.1) on $[\tau, T]$, if for all $v \in V$

$$\frac{d}{dt}(y, v) + \nu((y, v)) + b(y, y, v) = \langle f, v\rangle \tag{3.3}$$

in the sense of distributions, and

$$y(\tau) = y_\tau. \tag{3.4}$$

The weak solution y of Problem (3.1) on $[\tau, T]$ is called a *Leray-Hopf* solution of Problem (3.1) on $[\tau, T]$, if y satisfies the energy inequality:

$$V_\tau(y(t)) \le V_\tau(y(s)) \quad \text{for all } t \in [s, T],\ \text{a.e. } s > \tau \text{ and } s = \tau, \tag{3.5}$$

where

$$V_\tau(y(\varsigma)) := \frac{1}{2}\|y(\varsigma)\|^2 + \nu\int_\tau^\varsigma \|y(\xi)\|_V^2 d\xi - \int_\tau^\varsigma \langle f(\xi), y(\xi)\rangle d\xi, \quad \varsigma \in [\tau, T]. \tag{3.6}$$

For each $f \in L^2(\tau, T; V^*) + L^1(\tau, T; H)$ and $y_\tau \in H$ there exists at least one Leray-Hopf solution of Problem (3.1); see, for example, Temam [19, Chapter III] and references therein. Moreover, $y \in C([\tau, T], H_w)$ and $\frac{dy}{dt} \in L^{\frac{4}{3}}(\tau, T; V^*) + L^1(\tau, T; H)$. If $f \in L^2(\tau, T; V^*)$, then, additionally, $\frac{dy}{dt} \in L^{\frac{4}{3}}(\tau, T; V^*)$. In particular, initial condition (3.4) makes sense.

Let $A : V \to V^*$ be the linear operator associated to the bilinear form $((u, v)) = \langle Au, v\rangle$. Then A is an isomorphism from $D(A)$ onto H with $D(A) = (H^2(\Omega))^3 \cap V$. We recall that the embedding $D(A) \subset V$ is dense and continuous. Moreover, we assume $\|Au\|_H$ as the norm on $D(A)$, which is equivalent to the one induced by $(H^2(\Omega))^3$. Problem (3.3) can be rewritten as

$$\begin{cases} \frac{dy}{dt} + \nu Ay + B(y, y) = f \text{ in } V^*, \\ y(\tau) = y_\tau, \end{cases} \tag{3.7}$$

where the first equation we understand in the sense of distributions on (τ, T). Now we write

$$\mathscr{D}(y_\tau, f) = \{y : y \text{ is a weak solution of Problem (3.3) on } [\tau, T]\}.$$

It is well known (cf. [19]) that if $f \in L^2(\tau, T; V^*)$, and if $y_\tau \in H$, then $\mathscr{D}(y_\tau, f)$ is not empty.

A weak solution y of Problem (3.3) on $[\tau, T]$ is called a *strong* one, if it additionally belongs to Serrin's class $L^8(\tau, T; (L^4(\Omega))^3)$. We note that any strong solution y of Problem (3.3) on $[\tau, T]$ belongs to $C([\tau, T]; V) \cap L^2(\tau, T; D(A))$ and $\frac{dy}{dt} \in L^2(\tau, T; H)$ (cf. [18, Theorem 1.8.1, p. 296] and references therein).

For any $f \in L^\infty(\tau, T; H)$ and $y_\tau \in V$ it is well known the only local existence of strong solutions for the 3D Navier-Stokes equations (cf. [18–20] and references therein).

3.2 Leray-Hopf Property for a Weak Solution of the 3D Navier-Stokes System: Method of Artificial Control

Let $-\infty < \tau < T < +\infty$. We consider the following space of parameters:

$$\mathbb{U}_{\tau,T} := (L^2(\tau, T; V)) \times \left(L^2\left(\tau, T; V^*\right) + L^1(\tau, T; H)\right) \times H.$$

Each triple $(u, g, z_\tau) \in \mathbb{U}_{\tau,T}$ is called *admissible* for the following auxiliary control problem.

Problem **(C)** on $[\tau, T]$ with $(u, g, z_\tau) \in \mathbb{U}_{\tau,T}$: find $z \in L^2(\tau, T; V)$ with $\dfrac{dz}{dt} \in L^1(\tau, T; V^*)$ such that $z(\tau) = z_\tau$ and for all $v \in V$

$$\frac{d}{dt}(z, v) + \nu((z, v)) + b(u, z, v) = \langle g, v\rangle \tag{3.8}$$

in the sense of distributions; cf. Kapustyan et al. [10, 11]; Kasyanov et al. [12, 13]; Melnik and Toscano [15]; Zgurovsky et al. [20, Chap. 6].

As usual, let $A : V \to V^*$ be the linear operator associated with the bilinear form $((u, v)) = \langle Au, v\rangle$, $u, v \in V$. For $u, v \in V$ we denote by $B(u, v)$ the element of V^* defined by $\langle B(u, v), w\rangle = b(u, v, w)$, for all $w \in V$. Then Problem (C) on $[\tau, T]$ with $(u, g, z_\tau) \in \mathbb{U}_{\tau,T}$ can be rewritten as: find $z \in L^2(\tau, T; V)$ with $\dfrac{dz}{dt} \in L^1(\tau, T; V^*)$ such that

$$\frac{dz}{dt} + \nu Az + B(u, z) = g, \text{ in } V^*, \text{ and } z(\tau) = z_\tau. \tag{3.9}$$

We recall, that $\{w_1, w_2, \cdots\} \subset \mathscr{V}$ is the *special basis* if $((w_j, v)) = \lambda_j(w_j, v)$ for each $v \in V$ and $j = 1, 2, \cdots$, where $0 < \lambda_1 \le \lambda_2 \le \cdots$ is the sequence of eigenvalues. Let P_m be the projection operator of H onto $H_m := \text{span}\{w_1, \cdots, w_m\}$, that is $P_m v = \sum_{i=1}^m (v, w_i)w_i$ for each $v \in H$ and $m = 1, 2, \cdots$. Of course we may consider P_m as a projection operator that acts from V_σ onto H_m for each $\sigma > 0$ and, since $P_m^* = P_m$, we deduce that $\|P_m\|_{\mathscr{L}(V_\sigma^*;V_\sigma^*)} \le 1$. Note that $(w_j, v)_{V_\sigma} = \lambda_j^\sigma (w_j, v)$ for each $v \in V_\sigma$ and $j = 1, 2, \cdots$.

The following theorem establishes sufficient conditions for the existence of an unique solution for Problem (C). This is the main result of this section.

Theorem 3.1 *Let* $-\infty < \tau < T < +\infty$, $y_\tau \in H$, $f \in L^2(\tau, T; V^*) + L^1(\tau, T; H)$, *and* y *be a weak solution of Problem (3.1) on* $[\tau, T]$. *If Problem (C) on* $[\tau, T]$ *with* $(u, \bar{0}, \bar{0}) \in \mathbb{U}_{\tau,T}$ *has the unique solution* $z \equiv \bar{0}$, *then* $(y, f, y_\tau) \in \mathbb{U}_{\tau,T}$ *and Problem (C) on* $[\tau, T]$ *with* $(y, f, y_\tau) \in \mathbb{U}_{\tau,T}$ *has the unique solution* $z = y$. *Moreover,* y *satisfies inequality (3.5).*

Before the proof of Theorem 3.1 we remark that $AC([\tau, T]; H_m)$, $m = 1, 2, \cdots$, will denote the family of absolutely continuous functions acting from $[\tau, T]$ into H_m, $m = 1, 2, \cdots$.

Proof of Theorem 3.1. Prove that $z = y$ is the unique solution of Problem (C) on $[\tau, T]$ with $(y, f, y_\tau) \in \mathbb{U}_{\tau,T}$. Indeed, y is the solution of Problem (C) on $[\tau, T]$ with $(y, f, y_\tau) \in \mathbb{U}_{\tau,T}$, because y is a weak solution of Problem (3.1) on $[\tau, T]$. Uniqueness holds, because if z is a solution of Problem (C) on $[\tau, T]$ with $(y, f, y_\tau) \in \mathbb{U}_{\tau,T}$, then $z - y \equiv \bar{0}$ is the unique solution of Problem (C) on $[\tau, T]$ with $(y, \bar{0}, \bar{0}) \in \mathbb{U}_{\tau,T}$.

The rest of the proof establishes that y satisfies inequality (3.5). We note that y can be obtained via standard Galerkin arguments, that is, if $y_m \in AC([\tau, T]; H_m)$ with $\frac{d}{dt} y_m \in L^1(\tau, T; H_m)$, $m = 1, 2, \cdots$, is the approximate solution such that

$$\frac{dy_m}{dt} + \nu A y_m + P_m B(y, y_m) = P_m f, \text{ in } H_m, \quad y_m(\tau) = P_m y(\tau), \tag{3.10}$$

then the following statements hold:

(i) y_m satisfy the following energy equality:

$$\frac{1}{2}\|y_m(t_1)\|^2 + \nu \int_s^{t_1} \|y_m(\xi)\|_V^2 d\xi - \int_s^{t_1} \langle f(\xi), y_m(\xi)\rangle d\xi = \frac{1}{2}\|y_m(t_2)\|^2 + \nu \int_s^{t_2} \|y_m(\xi)\|_V^2 d\xi - \int_s^{t_2} \langle f(\xi), y_m(\xi)\rangle d\xi, \tag{3.11}$$

for each $t_1, t_2 \in [\tau, T]$, for each $m = 1, 2, \cdots$;

(ii) there exists a subsequence $\{y_{m_k}\}_{k=1,2,\dots} \subseteq \{y_m\}_{m=1,2,\dots}$ such that the following convergence (as $k \to \infty$) hold:

(ii)$_1$ $y_{m_k} \to y$ weakly in $L^2(\tau, T; V)$;
(ii)$_2$ $y_{m_k} \to y$ weakly star in $L^\infty(\tau, T; H)$;
(ii)$_3$ $P_{m_k} B(y, y_{m_k}) \to B(y, y)$ weakly in $L^2(\tau, T; V_{\frac{3}{2}}^*)$;
(ii)$_4$ $P_{m_k} f \to f$ strongly in $L^2(\tau, T; V^*) + L^1(\tau, T; H)$;
(ii)$_5$ $\frac{dy_{m_k}}{dt} \to \frac{dy}{dt}$ weakly in $L^2(\tau, T; V_{\frac{3}{2}}^*) + L^1(\tau, T; H)$.

Indeed, convergences $(\mathrm{ii})_1$ and $(\mathrm{ii})_2$ follow from (3.11) (see also Temam [19, Remark III.3.1, pp. 264, 282]) and Banach-Alaoglu theorem. Since there exists $C_1 > 0$ such that $|b(u,v,w)| \le C\|u\|_V\|w\|_V\|v\|_V^{\frac{1}{2}}\|v\|^{\frac{1}{2}}$, for each $u, v, w \in V$ (see, for example, Sohr [18, Lemma V.1.2.1]), then $(\mathrm{ii})_1$, $(\mathrm{ii})_2$ and Banach-Alaoglu theorem imply $(\mathrm{ii})_3$. Convergence $(\mathrm{ii})_4$ holds, because of the basic properties of the projection operators $\{P_m\}_{m=1,2,\ldots}$. Convergence $(\mathrm{ii})_5$ directly follows from $(\mathrm{ii})_3$, $(\mathrm{ii})_4$ and (3.10). We note that we may not pass to a subsequence in $(\mathrm{ii})_1$–$(\mathrm{ii})_5$, because $z = y$ is the unique solution of Problem (C) on $[\tau, T]$ with $(y, f, y_\tau) \in \mathbb{U}_{\tau,T}$.

Moreover, there exists a subsequence $\{y_{k_j}\}_{j=1,2,\cdots} \subseteq \{y_{m_k}\}_{k=1,2,\cdots}$ such that

$$y_{k_j}(t) \to y(t) \text{ strongly in } H \text{ for a.e. } t \in (\tau, T) \text{ and } t = \tau, \quad j \to \infty. \tag{3.12}$$

Indeed, according to (3.10), (3.11) and $(\mathrm{ii})_3$, the sequence $\{y_{m_k}\}_{k=1,2,\cdots}$ is bounded in a reflexive Banach space $W_{\tau,T} := \{w \in L^2(\tau, T; V) : \frac{d}{dt}w \in L^1(\tau, T; V^*_{\frac{3}{2}})\}$. Compactness lemma yields that $W_{\tau,T} \subset L^2(\tau, T; H)$ with compact embedding. Therefore, $(\mathrm{ii})_1$–$(\mathrm{ii})_5$ imply that $y_{m_k} \to y$ strongly in $L^2(\tau, T; H)$ as $k \to \infty$. Thus, there exists a subsequence $\{y_{k_j}\}_{j=1,2,\cdots} \subseteq \{y_{m_k}\}_{k=1,2,\cdots}$ such that (3.12) holds.

Due to convergences $(\mathrm{ii})_1$–$(\mathrm{ii})_5$ and (3.12), if we pass to the limit in (3.11) as $m_{k_j} \to \infty$, then we obtain that y satisfies the inequality

$$\frac{1}{2}\|y(t)\|^2 + \nu\int_s^t \|y(\xi)\|_V^2 d\xi - \int_s^t \langle f(\xi), y(\xi)\rangle d\xi \le \frac{1}{2}\|y(\tau)\|^2, \tag{3.13}$$

for a.e. $t \in (s, T)$, a.e. $s \in (\tau, T)$ and $s = \tau$.

Since $y \in L^\infty(\tau, T; H) \cap C([\tau, T]; V^*)$ and $H \subset V^*$ with continuous embedding, then $y \in C([\tau, T]; H_w)$; Temam [19, Chap. III]. Thus, equality (3.13) yields

$$\frac{1}{2}\|y(t)\|^2 + \nu\int_s^t \|y(\xi)\|_V^2 d\xi - \int_s^t \langle f(\xi), y(\xi)\rangle d\xi \le \frac{1}{2}\|y(\tau)\|^2,$$

for each $t \in [\tau, T]$, a.e. $s \in (\tau, T)$ and $s = \tau$. Therefore, y satisfies inequality (3.5).

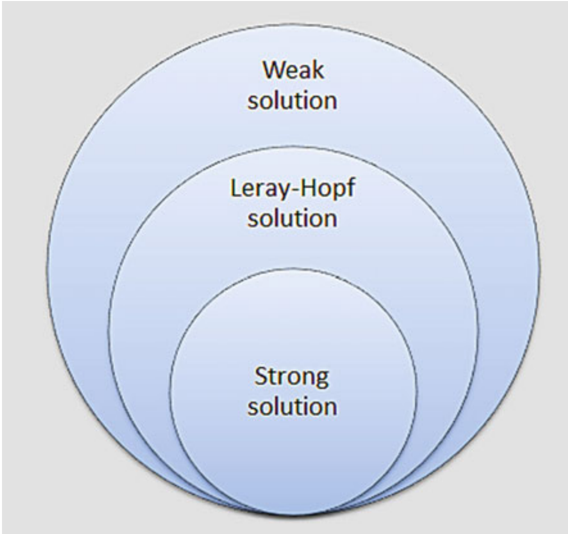
The theorem is proved.

3.3 The Existence of Strong Solutions and 1-Dimensional Dynamical Systems

Let $T > 0$. The main result of this section has the following formulation (see also Figs. 3.3 and 3.4).

Theorem 3.2 *Let $f \in L^2(0, T; H)$ and $y_0 \in V$. Then either for any $\lambda \in [0, 1]$ there is an $y_\lambda \in C([0, T]; V) \cap L^2(0, T; D(A))$ such that $y_\lambda \in \mathscr{D}(\lambda y_0, \lambda f)$, or the set*

Fig. 3.3 Relations between the types of solutions for the 3D Navier-Stokes system



$$\{y \in C([0,T]; V) \cap L^2(0,T; D(A)) \ : \ y \in \mathscr{D}(\lambda y_0, \lambda f), \lambda \in (0,1)\} \tag{3.14}$$

is unbounded in $L^8(0,T; (L^4(\Omega))^3)$.

In the proof of Theorem 3.2 we use an auxiliary statement connected with continuity property of strong solutions on parameters of problem (3.3) in Serrin's class $L^8(0,T; (L^4(\Omega))^3)$.

Theorem 3.3 *Let* $f \in L^2(0,T;H)$ *and* $y_0 \in V$. *If* y *is a strong solution for Problem (3.3) on* $[0,T]$, *then there exist* $L, \delta > 0$ *such that for any* $z_0 \in V$ *and* $g \in L^2(0,T;H)$, *satisfying the inequality*

$$\|z_0 - y_0\|_V^2 + \|g - f\|_{L^2(0,T;H)}^2 < \delta, \tag{3.15}$$

the set $\mathscr{D}(z_0, g)$ *is one-point set* $\{z\}$ *which belongs to* $C([0,T]; V) \cap L^2(0,T; D(A))$, *and*

$$\|z - y\|_{C([0,T];V)}^2 + \frac{\nu}{4}\|z - y\|_{D(A)}^2 \le L\left(\|z_0 - y_0\|_V^2 + \|g - f\|_{L^2(0,T;H)}^2\right). \tag{3.16}$$

Remark 3.1 We note that from Theorem 3.3 with $z_0 \in V$ and $g \in L^2(0,T;H)$ with sufficiently small $\|z_0\|_V^2 + \|g\|_{L^2(0,T;H)}^2$, Problem (3.3) has only one global strong solution.

Remark 3.2 Theorem 3.3 provides that, if for any $\lambda \in [0,1]$ there is an $y_\lambda \in L^8(0,T; (L^4(\Omega))^3)$ such that $y_\lambda \in \mathscr{D}(\lambda y_0, \lambda f)$, then the set

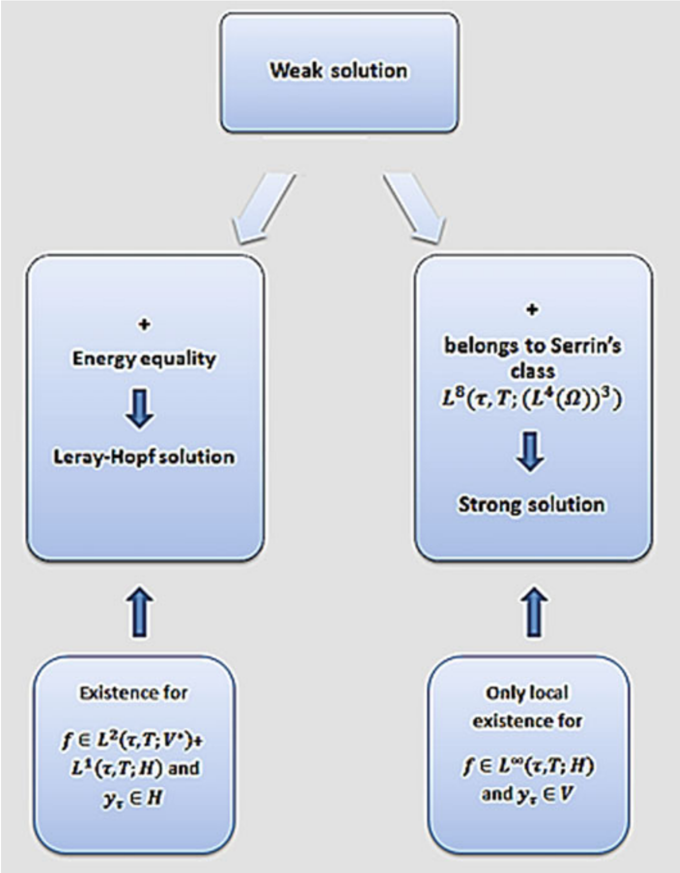


Fig. 3.4 Existence of solutions for the 3D Navier-Stokes System

$$\{y \in C([0,T];V) \cap L^2(0,T;D(A)) \;:\; y \in \mathscr{D}(\lambda y_0, \lambda f), \lambda \in (0,1)\}$$

is bounded in $L^8(0,T;(L^4(\Omega))^3)$.

If Ω is a C^∞-domain and if $f \in C_0^\infty(\overline{(0,T)\times\Omega})^3$, then any strong solution y of Problem (3.3) on $[0,T]$ belongs to $C^\infty((0,T]\times\Omega)^3$ and $p \in C^\infty((0,T]\times\Omega)$ (cf. [18, Theorem 1.8.2, p. 300] and references therein). This fact directly provides the next corollary of Theorems 3.2 and 3.3.

Corollary 3.1 *Let Ω be a C^∞-domain, $f \in C_0^\infty(\overline{(0,T)\times\Omega})^3$. Then either for any $y_0 \in V$ there is a strong solution of Problem (3.3) on $[0,T]$, or the set*

$$\{y \in C^\infty((0,T]\times\Omega)^3 \;:\; y \in \mathscr{D}(\lambda y_0, \lambda f), \lambda \in (0,1)\}$$

is unbounded in $L^8(0,T;(L^4(\Omega))^3)$ for some $y_0 \in C_0^\infty(\Omega)^3$.

Proof of Theorem 3.3. Let $f \in L^2(0, T; H)$, $y_0 \in V$, and $y \in C([0, T]; V) \cap L^2(0, T; D(A))$ be a strong solution of Problem (3.3) on $[0, T]$. Due to [17], [19, Chap. 3] the set $\mathscr{D}(y_0, f) = \{y\}$. Let us now fix $z_0 \in V$ and $g \in L^2(0, T; H)$ satisfying (3.15) with

$$\delta = \min\left\{1; \frac{\nu}{4}\right\} e^{-2TC}, \ C = \max\left\{\frac{27c^4}{2\nu^3}; \frac{7^7c^8}{2^9\nu^7}\right\}\left(\|y\|^4_{C([0,T];V)} + 1\right)^2, \quad (3.17)$$

$c > 0$ is a constant from the inequalities (cf. [18, 19])

$$|b(u, v, w)| \le c\|u\|_V \|v\|_V^{\frac{1}{2}} \|v\|_{D(A)}^{\frac{1}{2}} \|w\|_H \quad \forall u \in V, \ v \in D(A), \ w \in H; \quad (3.18)$$

$$|b(u, v, w)| \le c\|u\|_{D(A)}^{\frac{3}{4}} \|u\|_V^{\frac{1}{4}} \|v\|_V \|w\|_H \quad \forall u \in D(A), \ v \in V, \ w \in H. \quad (3.19)$$

The auxiliary problem

$$\begin{cases} \frac{d\eta}{dt} + \nu A\eta + B(\eta, \eta) + B(y, \eta) + B(\eta, y) = g - f \text{ in } V^*, \\ \eta(0) = z_0 - y_0, \end{cases} \quad (3.20)$$

has a strong solution $\eta \in C([0, T]; V) \cap L^2(0, T; D(A))$ with $\frac{d\eta}{dt} \in L^2(0, T; H)$, i.e.

$$\frac{d}{dt}(\eta, v) + \nu((\eta, v)) + b(\eta, \eta, v) + b(y, \eta, v) + b(\eta, y, v) = \langle g - f, v\rangle \quad \text{for all } v \in V,$$

in the sense of distributions on $(0, T)$. In fact, let $\{w_j\}_{j\ge1} \subset D(A)$ be a special basis (cf. [19]), i.e. $Aw_j = \lambda_j w_j$, $j = 1, 2, ..., 0 < \lambda_1 \le \lambda_2 \le ...,\ \lambda_j \to +\infty, j \to +\infty$. We consider Galerkin approximations $\eta_m : [0, T] \to \text{span}\{w_j\}_{j=1}^m$ for solutions of Problem (3.20) satisfying

$$\frac{d}{dt}(\eta_m, w_j) + \nu((\eta_m, w_j)) + b(\eta_m, \eta_m, w_j) + b(y, \eta_m, w_j) + b(\eta_m, y, w_j) = \langle g - f, w_j\rangle,$$

with $(\eta_m(0), w_j) = (z_0 - y_0, w_j)$, $j = \overline{1, m}$. Due to (3.18), (3.19) and Young's inequality we get

$$2\langle g - f, A\eta_m\rangle \le 2\|g - f\|_H \|\eta_m\|_{D(A)} \le \frac{\nu}{4}\|\eta_m\|^2_{D(A)} + \frac{4}{\nu}\|f - g\|^2_H;$$

$$-2b(\eta_m, \eta_m, A\eta_m) \le 2c\|\eta_m\|_V^{\frac{3}{2}} \|\eta_m\|_{D(A)}^{\frac{3}{2}} \le \frac{\nu}{2}\|\eta_m\|^2_{D(A)} + \frac{27c^4}{2\nu^3}\|\eta_m\|^6_V;$$

$$-2b(y,\eta_m,A\eta_m)\le 2c\|y\|_V\|\eta_m\|_V^{\frac12}\|\eta_m\|_{D(A)}^{\frac32}\le\frac{\nu}{2}\|\eta_m\|_{D(A)}^2+\frac{27c^4}{2\nu^3}\|y\|_{C([0,T];V)}^4\|\eta_m\|_V^2;$$

$$-2b(\eta_m,y,A\eta_m)\le 2c\|\eta_m\|_{D(A)}^{\frac74}\|\eta_m\|_V^{\frac14}\|y\|_V\le\frac{\nu}{2}\|\eta_m\|_{D(A)}^2+\frac{7^7c^8}{2^9\nu^7}\|y\|_{C([0,T];V)}^8\|\eta_m\|_V^2.$$

Thus,

$$\frac{d}{dt}\|\eta_m\|_V^2+\frac{\nu}{4}\|\eta_m\|_{D(A)}^2\le C(\|\eta_m\|_V^2+\|\eta_m\|_V^6)+\frac{4}{\nu}\|g-f\|_H^2,$$

where $C>0$ is a constant from (3.17). Hence, the absolutely continuous function $\varphi=\min\{\|\eta_m\|_V^2,1\}$ satisfies the inequality $\frac{d}{dt}\varphi\le 2C\varphi+\frac{4}{\nu}\|g-f\|_H^2$, and therefore $\varphi\le L(\|z_0-y_0\|_V^2+\|g-f\|_{L^2(0,T;H)}^2)<1$ on $[0,T]$, where $L=\delta^{-1}$. Thus, $\{\eta_n\}_{n\ge1}$ is bounded in $L^\infty(0,T;V)\cap L^2(0,T;D(A))$ and $\{\frac{d}{dt}\eta_n\}_{n\ge1}$ is bounded in $L^2(0,T;H)$. In a standard way we get that the limit function η of η_n, $n\to+\infty$, is a strong solution of Problem (3.20) on $[0,T]$. Due to [17], [19, Chapter 3] the set $\mathscr{D}(z_0,g)$ is one-point $z=y+\eta\in L^8(0,T;(L^4(\Omega))^3)$. So, z is strong solution of Problem (3.3) on $[0,T]$ satisfying (3.16).

The theorem is proved.

Proof of Theorem 3.2. Let $f\in L^2(0,T;H)$ and $y_0\in V$. We consider the 3D controlled Navier-Stokes system (cf. [10, 15])

$$\begin{cases}\frac{dy}{dt}+\nu Ay+B(z,y)=f,\\ y(0)=y_0,\end{cases}\tag{3.21}$$

where $z\in L^8(0,T;(L^4(\Omega))^3)$.

By using standard Galerkin approximations (see [19]) it is easy to show that for any $z\in L^8(0,T;(L^4(\Omega))^3)$ there exists an unique weak solution $y\in L^\infty(0,T;H)\cap L^2(0,T;V)$ of Problem (3.21) on $[0,T]$, that is,

$$\frac{d}{dt}(y,v)+\nu((y,v))+b(z,y,v)=\langle f,v\rangle\text{, for all }v\in V,\tag{3.22}$$

in the sense of distributions on $(0,T)$. Moreover, by the inequality

$$|b(u,v,Av)|\le c_1\|u\|_{(L^4(\Omega))^3}\|v\|_V^{\frac14}\|v\|_{D(A)}^{\frac74}\le\frac{\nu}{2}\|v\|_{D(A)}^2+c_2\|u\|_{(L^4(\Omega))^3}^8\|v\|_V^2,\tag{3.23}$$

for all $u\in(L^4(\Omega))^3$ and $v\in D(A)$, where $c_1,c_2>0$ are some constants that do not depend on u, v (cf. [19]), we find that $y\in C([0,T];V)\cap L^2(0,T;D(A))$ and $B(z,y)\in L^2(0,T;H)$, so $\frac{dy}{dt}\in L^2(0,T;H)$ as well. We add that, for any $z\in L^8(0,T;(L^4(\Omega))^3)$ and corresponding weak solution $y\in C([0,T];V)\cap L^2(0,T;D(A))$ of (3.21) on $[0,T]$, by using Gronwall inequality, we obtain

$$\begin{aligned} &\|y(t)\|_V^2 \le \|y_0\|_V^2 e^{2c_2 \int_0^t \|z(t)\|^8_{(L^4(\Omega))^3} dt}, \qquad \forall t \in [0,T]; \\ &\nu \int_0^T \|y(t)\|^2_{D(A)} dt \le \|y_0\|_V^2 \left[1 + 2c_2 e^{2c_2 \int_0^T \|z(t)\|^8_{(L^4(\Omega))^3} dt} \|z\|^8_{L^8(0,T;(L^4(\Omega))^3)} \right]. \end{aligned} \tag{3.24}$$

Let us consider the operator $F : L^8(0, T; (L^4(\Omega))^3) \to L^8(0, T; (L^4(\Omega))^3)$, where $F(z) \in C([0, T]; V) \cap L^2(0, T; D(A))$ is the unique weak solution of (3.21) on $[0, T]$ corresponded to $z \in L^8(0, T; (L^4(\Omega))^3)$.

Let us check that F is a compact transformation of Banach space $L^8(0, T; (L^4(\Omega))^3)$ into itself (cf. [7]). In fact, if $\{z_n\}_{n\ge1}$ is a bounded sequence in $L^8(0, T; (L^4(\Omega))^3)$, then, due to (3.23) and (3.24), the respective weak solutions y_n, $n = 1, 2, ...,$ of Problem (3.21) on $[0, T]$ are uniformly bounded in $C([0, T]; V) \cap L^2(0, T; D(A))$ and their time derivatives $\frac{dy_n}{dt}$, $n = 1, 2, ...,$ are uniformly bounded in $L^2(0, T; H)$. So, $\{F(z_n)\}_{n\ge1}$ is a precompact set in $L^8(0, T; (L^4(\Omega))^3)$. In a standard way we deduce that $F : L^8(0, T; (L^4(\Omega))^3) \to L^8(0, T; (L^4(\Omega))^3)$ is continuous mapping. Since F is a compact transformation of $L^8(0, T; (L^4(\Omega))^3)$ into itself, Schaefer's Theorem (cf. [7, p. 133] and references therein) and Theorem 3.3 provide the statement of Theorem 3.2. We note that Theorem 3.3 implies that the set $\{z \in L^8(0, T; (L^4(\Omega))^3) : z = \lambda F(z),\ \lambda \in (0, 1)\}$ is bounded in $L^8(0, T; (L^4(\Omega))^3)$ iff the set defined in (3.14) is bounded in $L^8(0, T; (L^4(\Omega))^3)$.

The theorem is proved.

3.4 Extremal Solutions: Existence and Continuity Results in Strongest Topologies

We consider the 3D controlled Navier-Stokes system

$$\begin{cases} \frac{dy}{dt} + Ay + B(u, y) = f, \\ y(\tau) = y_\tau \in H, \end{cases} \tag{3.25}$$

where $f \in H$ and

$$u(\cdot) \in \mathbf{U}_\tau = \begin{cases} u \in L_\infty(\tau, +\infty; H) \cap L_2^{loc}(\tau, +\infty; V) \cap L_\infty^{loc}(\tau, +\infty; \mathbf{L}_4(\Omega)), \\ \int_\tau^{+\infty} \|u(p)\|^2 e^{-\delta p} dp < \infty,\ |u(p)| \le R_0 \text{ for a.a. } p \ge \tau, \\ \|u(t)\|_{\mathbf{L}_4} \le \alpha \text{ for a.a. } t > \tau, \end{cases} \tag{3.26}$$

$$J_\tau(u, y) = \int_\tau^{+\infty} \|y(p) - u(p)\|^2 e^{-\delta p} dp \to \inf, \tag{3.27}$$

with $\delta = \lambda_1 \nu,\ R_0 = \frac{|f|}{\nu\lambda_1}$, and where λ_1 is the first eigenvalue of the Stokes operator A and $\alpha > 0$ is some constant.

By using standard Galerkin approximations it is easy to show that for any $y_\tau \in H$ and $u(\cdot) \in U_\tau$ there exists a unique weak solution $y(\cdot) \in L_\infty(\tau, +\infty; H) \cap L_2^{loc}(\tau, +\infty; V)$ of (3.25), that is,

$$\frac{d}{dt}(y, v) + \nu((y, v)) + b(u, y, v) = \langle f, v \rangle \text{, for all } v \in V. \tag{3.28}$$

Indeed, let us prove existence of a weak solution of (3.25). Let $\{w_i\} \subset D(A)$ be the sequence of eigenfunctions of A, which are an orthonormal basis of H. Let $y^m(t) = \sum_{i=1}^m g_{im}(t)\, w_i$ be the Galerkin approximations of (3.25), i.e.

$$\begin{cases} \frac{dy^m}{dt} + \nu A y^m + P_m B(u, y^m) = P_m f, \\ y^m(\tau) = y_\tau^m, \end{cases} \tag{3.29}$$

where P_m is the projection onto the finite dimensional subspace generated by the set $\{w_1, ..., w_m\}$. Also, y_τ^m belongs to this subspace and $y_\tau^m \to y_\tau$ in H.

We need to obtain some a priori estimates for the approximative functions $\{y^m\}$. Multiplying (3.29) by y^m we obtain

$$\frac{1}{2}\frac{d}{dt}|y^m|^2 + \nu \left\| y^m \right\|^2 = \left(f, y^m\right), \tag{3.30}$$

where we have used the equalities

$$\left(P_m B\left(u, y^m\right), y^m\right) = \left(B\left(u, y^m\right), y^m\right) = b\left(u, y^m, y^m\right) = 0.$$

Also from (3.30) we obtain for all $p \in [s, T]$, $s \in [\tau, T]$ that

$$\frac{1}{2}|y^m(p)|^2 + \nu \int_s^p \left\| y^m(\tau) \right\|^2 d\tau \le \int_s^p \left(f(\tau), y^m(\tau)\right) d\tau + \frac{1}{2}|y^m(s)|^2. \tag{3.31}$$

In view of (3.31) we conclude that $\{y^m\}$ is bounded in $L_2(\tau, T; V) \cap L_\infty(\tau, T; H)$.

Therefore, passing to a subsequence we obtain $y^m \to y$ weakly in $L_2(\tau, T; V)$ and weakly star in $L_\infty(\tau, T; H)$. From the inequalities

$$|b(u, y^m, w)| \le d\|u\|_{L_4}\|y^m\|\|w\|,\ \ \forall w \in V,$$

and

$$\left\| P_m B\left(u, y^m\right)\right\|_{V^*} \le \left\| B\left(u, y^m\right)\right\|_{V^*},$$

due to the choice of the spacial basis, we immediately obtain that $P_m B(u, y^m)$ is bounded in $L_2(\tau, T; V^*)$. Then

$$\frac{d}{dt}y^m \to \frac{d}{dt}y \text{ weakly in } L_2\left(\tau, T; V^*\right),\ m \to \infty,$$

so that $y \in C([\tau, T]; H)$ and by the Compactness Lemma we have

$$y^m \to y \text{ strongly in } L_2(\tau, T; H),\ m \to \infty.$$

Hence, $y^m(t) \to y(t)$ strongly in H for a.e. $t \in (\tau, T)$, $m \to \infty$. Since one can easily prove using the Ascoli-Arzelà theorem that $y^m \to y$, $m \to \infty$, in $C([\tau, T]; V^*)$, a standard argument implies that $y^m(t) \to y(t)$ weakly in H for all $t \in [\tau, T]$, $m \to \infty$. In particular, $y(\tau) = y_\tau$.

On the other hand, from

$$\left\|u_i y_j^m\right\|^2_{L_2(\tau,T;L_2(\Omega))} \le \int_\tau^T \|u_i\|^2_{L_4(\Omega)} \left\|y_j^m\right\|^2_{L_4(\Omega)} dt \le C$$

we obtain $u_i y_j^m \to u_i y_j$ weakly in $L_2(\tau, T; L_2(\Omega))$, $m \to \infty$, so that

$$\int_\tau^T b(u, y^m - y, w)dt = -\sum_{i,j=1}^3 \int_\tau^T \int_\Omega u_i \left(y_j^m - y_j\right) \frac{\partial w_j}{\partial x_i} dxdt \to 0,\ m \to \infty,$$

for any $w \in L_2(\tau, T; V)$. This implies

$$B(u, y^m) \to B(u, y) \text{ weakly in } L_2\left(\tau, T; V^*\right),\ m \to \infty.$$

So we can pass to the limit in (3.29) and deduce that y is solution of (3.25). To prove uniqueness we should note that if y_1, y_2 are solutions of (3.25), corresponding the same control function u, then

$$\frac{d}{dt}|y_1 - y_2|^2 = 2(\frac{d(y_1 - y_2)}{dt}, y_1 - y_2),$$

$$b(u, y_1 - y_2, y_1 - y_2) = 0.$$

So after simple calculations we have

$$\frac{d}{dt}|y_1 - y_2|^2 \le C|y_1 - y_2|^2,$$

and therefore $y_1 \equiv y_2$.

Moreover, by the inequality

$$|b(u, y, v)| = |b(u, v, y)| \leq c_1 \|u\|_{\mathbf{L}_4} \|v\| \|y\|_{\mathbf{L}_4} \leq c_2 c_1 \|u\|_{\mathbf{L}_4} \|v\| \|y\|, \forall u, y, v \in V,$$

and (3.26) we have $B(u(\cdot), y(\cdot)) \in L_2^{loc}(\tau, +\infty; V^*)$, so $\frac{dy}{dt} \in L_2^{loc}(\tau, +\infty; V^*)$ as well. Hence, it follows that $y(\cdot) \in C([\tau, +\infty); H)$ (so the initial condition $y(\tau) = y_\tau$ makes sense for any $y_\tau \in H$) and standard arguments imply that for all $t \geq s \geq \tau$,

$$F(y(t)) := \left(|y(t)|^2 - R_0^2\right)e^{\delta t} \leq F(y(s)), \tag{3.32}$$

$$V_\tau(y(t)) := \frac{1}{2}|y(t)|^2 + \nu \int_\tau^t \|y(p)\|^2 dp - \int_\tau^t (f, y(p))dp \leq V_\tau(y(s)), \tag{3.33}$$

$$|y(t)|^2 + \nu \int_\tau^t \|y(p)\|^2 dp \leq |y_\tau|^2 + \frac{|f|^2}{\nu\lambda_1}(t - \tau). \tag{3.34}$$

Indeed, multiplying the equation by $y(t)$ and using the property $b(u, y, y) = 0$ we obtain

$$\frac{1}{2}\frac{d}{dt}|y|^2 + \nu \|y\|^2 = (f, y). \tag{3.35}$$

After integration over (s, t) we obtain

$$\frac{1}{2}|y(t)|^2 + \nu \int_s^t \|y(p)\|^2 dp = \int_s^t (f, y(p))\, dp + \frac{1}{2}|y(s)|^2, \tag{3.36}$$

and then (3.33) follows. Taking $s = \tau$ in (3.36) and using the inequality

$$(f, y(p)) \leq |f|\,|y(p)| \leq \frac{1}{\sqrt{\lambda_1}}|f|\,\|y(p)\| \leq \frac{|f|^2}{2\lambda_1\nu} + \frac{\nu}{2}\|y(p)\|^2$$

we have

$$|y(t)|^2 + \nu \int_s^t \|y(p)\|^2 dp \leq |y(\tau)|^2 + \frac{|f|^2}{\lambda_1\nu}(t - \tau).$$

Finally, from (3.35) we obtain

$$\frac{d}{dt}|y|^2 + \lambda_1\nu |y|^2 \leq \frac{|f|^2}{\lambda_1\nu}.$$

Multiplying the last inequality by $e^{\nu\lambda_1 t}$ and integrating we get

$$|y(t)|^2 e^{\nu\lambda_1 t} \leq |y(s)|^2 e^{\nu\lambda_1 s} + \frac{|f|^2}{\lambda_1^2\nu^2}(e^{\lambda_1\nu t} - e^{\nu\lambda_1 s}),$$

and then (3.32) holds.

So, for all $n \geq 0$,

$$\int_{\tau+n}^{\tau+(n+1)} \|y(p) - u(p)\|^2 e^{-\delta p} dp \leq$$

$$\leq 2e^{-\delta(n+\tau)} \int_{\tau+n}^{\tau+(n+1)} \|y(p)\|^2 dp + 2 \int_{\tau+n}^{\tau+(n+1)} \|u(p)\|^2 e^{-\delta p} dp$$

$$\leq \frac{2}{\nu} e^{-\delta(n+\tau)} (|y_\tau|^2 + \frac{|f|^2}{\nu\lambda_1}) + 2 \int_{\tau+n}^{\tau+(n+1)} \|u(p)\|^2 e^{-\delta p} dp.$$

From this

$$J_\tau(u, y) = \sum_{n=0}^{\infty} \int_{\tau+n}^{\tau+(n+1)} \|y(p) - u(p)\|^2 e^{-\delta p} dp$$

$$\leq \frac{2e^{-\delta\tau}}{\nu} (|y_\tau|^2 + \frac{|f|^2}{\nu\lambda_1}) \sum_{n=0}^{\infty} e^{-\delta n} + 2 \sum_{n=0}^{\infty} \int_{\tau+n}^{\tau+(n+1)} \|u(p)\|^2 e^{-\delta p} dp < \infty.$$

Therefore, the functional J_τ and the optimal control problem (3.25), (3.26) and (3.27) is correctly defined.

Lemma 3.1 *For any $\tau \in \mathbb{R}$ and $y_\tau \in H$ the optimal control problem (3.25), (3.26) and (3.27) has at least one solution $\{y(\cdot), u(\cdot)\}$, and, moreover, $\frac{dy}{dt} \in L_2^{loc}(\tau, +\infty; V^*)$, $y(\cdot) \in C([\tau, +\infty); H)$ and (3.32), (3.33) and (3.34) hold.*

Proof Let $\{y_n, u_n\}$ be a minimizing sequence such that

$$\int_{\tau}^{+\infty} \|y_n(p) - u_n(p)\|^2 e^{-\delta p} dp \leq d + \frac{1}{n},\ \forall n \geq 1\ ,$$

where $d = \inf J_\tau(u, y)$. Thus, for all $T > \tau$ and $n \geq 1$

$$\begin{aligned} &\int_\tau^T \|y_n(p) - u_n(p)\|^2 e^{-\delta p} dp \leq d + \tfrac{1}{n}, \\ &\int_\tau^T \|y_n(p) - u_n(p)\|^2 dp \leq (d + \tfrac{1}{n}) e^{\delta T}. \end{aligned} \tag{3.37}$$

From (3.32), (3.33) and (3.34) we obtain that $\{y_n\}$ is bounded in $L_\infty(\tau, T; H) \cap L_2(\tau, T; V)$. Hence, (3.37) implies that $\{u_n\}$ is bounded in $L_2(\tau, T; V)$ and from the definition of $\mathbf{U}_\tau$ it follows that

$$\begin{aligned} |u_n(p)| &\leq R_0, \ \forall p \geq \tau, \\ \|u_n(p)\|_{\mathbf{L}^4} &\leq \alpha \text{ for a.e. } p > \tau \text{ and for all } n \geq 1. \end{aligned}$$

Therefore, there exist $u \in L_\infty(\tau, T; H) \cap L_2(\tau, T; V) \cap L_\infty(\tau, T; \mathbf{L}_4(\Omega))$ and $y \in L_\infty(\tau, T; H) \cap L_2(\tau, T; V)$ such that

$$\begin{aligned} u_n &\to u \text{ weakly in } L_2(\tau, T; V), \\ u_n &\to u \text{ weakly star in } L_\infty(\tau, T; H), \\ u_n &\to u \text{ weakly star in } L_\infty(\tau, T; \mathbf{L}^4(\Omega)), \\ y_n &\to y \text{ weakly in } L_2(\tau, T; V), \\ y_n &\to y \text{ weakly star in } L_\infty(\tau, T; H), \ n \to \infty. \end{aligned}$$

Moreover, $\|B(u_n, y_n)\|_{V^*} \leq c_1 \|y_n\| \|u_n\|_{\mathbf{L}_4}$. Hence, $\frac{dy_n}{dt}$ is bounded in $L_2(\tau, T; V^*)$. From this using standard arguments, we obtain that $y(\cdot) \in C([\tau, T]; H)$ is the solution of (3.25) with control $u(\cdot)$, $y(\cdot)$ satisfies (3.32), (3.33) and (3.34), and for this control the following relations hold:

$$|u(p)| \leq R_0, \quad \text{for a.a. } p \geq \tau,$$

$$\|u(p)\|_{\mathbf{L}_4} \leq \alpha \quad \text{for a.a.} \quad p > \tau,$$

$$u \in L_2(\tau, T; V),$$

$$\int_\tau^T \|y(p) - u(p)\|^2 e^{-\delta p} dp \leq d.$$

The fact that $y(\cdot)$ is a solution with control $u(\cdot)$ is proved in a standard way. Indeed, as $\frac{dy_n}{dt}$ is bounded in $L_2(\tau, T; V^*)$, up to subsequence

$$\frac{d}{dt} y_n \to \frac{d}{dt} y \text{ weakly in } L_2\left(\tau, T; V^*\right), \ n \to \infty.$$

Thus, $y \in C([\tau, T]; H)$ and arguing as in the proof of the existence of solution for (3.25) we obtain

$$\begin{aligned} y_n &\to y \text{ strongly in } L_2(\tau, T; H), \\ y_n(t) &\to y(t) \text{ strongly in } H \text{ for a.a } t \in (\tau, T), \\ y_n(t) &\to y(t) \text{ weakly in } H \text{ for all } t \in [\tau, T],\ n \to \infty. \end{aligned}$$

From

$$\left\| u_i^n y_j^n \right\|^2_{L_2(\tau,T;L_2(\Omega))} \le \int_\tau^T \left\| u_i^n \right\|^2_{L_4(\Omega)} \left\| y_j^n \right\|^2_{L_4(\Omega)} dt \le C$$

we obtain $u_i y_j^m \to u_i y_j$ weakly in $L_2(\tau, T; L_2(\Omega))$, $n \to \infty$, so that

$$\int_\tau^T b(u^n, y^n, w)dt = -\sum_{i,j=1}^{3} \int_\tau^T \int_\Omega u_i^n y_j^n \frac{\partial w_j}{\partial x_i} dx dt \to \int_\tau^T b(u, w, y)dt,\ n \to \infty,$$

for any $w \in L_2(\tau, T; V)$. This implies

$$B(u, y^m) \to B(u, y) \text{ weakly in } L_2\left(\tau, T; V^*\right),\ n \to \infty.$$

Hence we can pass to the limit in (3.25) and obtain that $\{u, y\}$ is a solution. Also, $y(\tau) = y_\tau$.

By using a standard diagonal procedure we can claim that $y(\cdot)$ and $u(\cdot)$ are defined on $[\tau, +\infty)$, $y_n \to y$, $u_n \to u$ in the previous sense on every $[\tau, T]$, $n \to \infty$, and

$$\int_\tau^{+\infty} \|y(p) - u(p)\|^2 e^{-\delta p} dp \le d. \tag{3.38}$$

By (3.34), arguing as before,

$$\begin{aligned} \int_\tau^\infty \|y(p)\|^2 e^{-\delta p} dp &= \sum_{n=0}^\infty \int_{\tau+n}^{\tau+n+1} \|y(p)\|^2 e^{-\delta p} dp \\ &\le \frac{e^{-\delta\tau}}{\nu} \left(|y_\tau|^2 + \frac{|f|^2}{\nu\lambda_1} \right) \sum_{n=0}^\infty e^{-\delta n} < \infty. \end{aligned}$$

and from (3.38) we have

$$\int_\tau^{+\infty} \|u(p)\|^2 e^{-\delta p} dp < \infty.$$

It follows that $u(\cdot) \in \mathbf{U}_\tau$ and from (3.38) we obtain that $\{y(\cdot), u(\cdot)\}$ is an optimal pair of problem (3.25), (3.26) and (3.27).

The lemma is proved.

Remark 3.3 Lemma 3.1 was proved in [10].

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Part II
Convergence Results in Strongest Topologies

Chapter 4
Strongest Convergence Results for Weak Solutions of Non-autonomous Reaction-Diffusion Equations with Carathéodory's Nonlinearity

Abstract In this chapter we consider the problem of uniform convergence results for all globally defined weak solutions of non-autonomous reaction-diffusion system with Carathéodory's nonlinearity satisfying standard sign and polynomial growth assumptions. The main contributions of this chapter are: the uniform convergence results for all globally defined weak solutions of non-autonomous reaction-diffusion equations with Carathéodory's nonlinearity and sufficient conditions for the convergence of weak solutions in strongest topologies.

Let $N, M = 1, 2, \ldots$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. We consider a problem of long-time behavior of all globally defined weak solutions for the non-autonomous parabolic problem (named RD-system)

$$\begin{cases} y_t = a\Delta y - f(x, t, y), & x \in \Omega,\ t > 0, \\ y|_{\partial\Omega} = 0, \end{cases} \tag{4.1}$$

as $t \to +\infty$, where $y = y(x, t) = (y^{(1)}(x, t), \ldots, y^{(M)}(x, t))$ is unknown vector-function, $f = f(x, t, y) = (f^{(1)}(x, t, y), \ldots, f^{(M)}(x, t, y))$ is given function, a is real $M \times M$ matrix with positive symmetric part.

4.1 Translation-Compact, Translation-Bounded and Translation Uniform Integrable Functions

To introduce the assumptions on parameters of Problem (4.1) we need to present some additional constructions. Let $\gamma \geq 1$ and $\mathscr{E}$ be a real separable Banach space. As $L_\gamma^{\mathrm{loc}}(\mathbb{R}_+; \mathscr{E})$ we consider the Fréchet space of all locally integrable functions with values in $\mathscr{E}$, i.e. $\varphi \in L_\gamma^{\mathrm{loc}}(\mathbb{R}_+; \mathscr{E})$ if and only if for any finite interval $[\tau, T] \subset \mathbb{R}_+$ the restriction of φ on $[\tau, T]$ belongs to the space $L_\gamma(\tau, T; \mathscr{E})$. If $\mathscr{E} \subseteq L_1(\Omega)$, then any function φ from $L_\gamma^{\mathrm{loc}}(\mathbb{R}_+; \mathscr{E})$ can be considered as a measurable mapping that acts from $\Omega \times \mathbb{R}_+$ into $\mathbb{R}$. Further, we write $\varphi(x, t)$, when we consider this mapping as a function from $\Omega \times \mathbb{R}_+$ into $\mathbb{R}$, and $\varphi(t)$, if this mapping is considered as an element from $L_\gamma^{\mathrm{loc}}(\mathbb{R}_+; \mathscr{E})$; cf. Gajewski et al. [3, Chap. III]; Temam [23]; Babin and Vishik [1]; Chepyzhov and Vishik [5]; Zgurovsky et al. [28] and references therein.

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A function $\varphi \in L_\gamma^{\rm loc}(\mathbb{R}_+; \mathscr{E})$ is called *translation bounded* in $L_\gamma^{\rm loc}(\mathbb{R}_+; \mathscr{E})$, if

$$\sup_{t\geq 0} \int_t^{t+1} \|\varphi(s)\|_{\mathscr{E}}^\gamma ds < +\infty; \tag{4.2}$$

Chepyzhov and Vishik [7, p. 105]. A function $\varphi \in L_1^{\rm loc}(\mathbb{R}_+; L_1(\Omega)$ is called *translation uniform integrable (t.u.i.)* in $L_1^{\rm loc}(\mathbb{R}_+; L_1(\Omega))$, if

$$\lim_{K\to+\infty} \sup_{t\geq 0} \int_t^{t+1} \int_\Omega |\varphi(x,s)| \chi_{\{|\varphi(x,s)|\geq K\}} dx ds = 0. \tag{4.3}$$

Dunford–Pettis compactness criterion provides that a function $\varphi \in L_1^{\rm loc}(\mathbb{R}_+; L_1(\Omega))$ is t.u.i. in $L_1^{\rm loc}(\mathbb{R}_+; L_1(\Omega))$ if and only if for every sequence of elements $\{\tau_n\}_{n\geq 1} \subset \mathbb{R}_+$ the sequence $\{\varphi(\cdot + \tau_n)\}_{n\geq 1}$ contains a subsequence which converges weakly in $L_1^{\rm loc}(\mathbb{R}_+; L_1(\Omega))$. We note that for any $\gamma > 1$ Hölder's and Chebyshev's inequalities imply that every translation bounded in $L_\gamma^{\rm loc}(\mathbb{R}_+; L_\gamma(\Omega))$ function is t.u.i. in $L_1^{\rm loc}(\mathbb{R}_+; L_1(\Omega))$, because

$$\int_t^{t+1} \int_\Omega |\varphi(x,s)| \chi_{\{|\varphi(x,s)|\geq K\}} dx ds \leq \frac{1}{K^{\gamma-1}} \sup_{t\geq 0} \int_t^{t+1} \int_\Omega |\varphi(x,s)|^\gamma dx ds \to 0 \text{ as } K \to +\infty.$$

4.2 Setting of the Problem

Throughout this chapter we suppose that the listed below assumptions hold.

Assumption I. Let $p_i \geq 2$ and $q_i > 1$ are such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$, for any $i = 1, 2, \ldots, M$. Moreover, there exists a positive constant d such that $\frac{1}{2}(a+a^*) \geq dI$, where I is unit $M \times M$ matrix, a^* is a transposed matrix for a.

Assumption II. The interaction function $f = (f^{(1)}, \ldots, f^{(M)}) : \Omega \times \mathbb{R}_+ \times \mathbb{R}^M \to \mathbb{R}^M$ satisfies the standard Carathéodory's conditions, i.e. the mapping $(x,t,u) \to f(x,t,u)$ is continuous in $u \in \mathbb{R}^M$ for a.e. $(x,t) \in \Omega \times \mathbb{R}_+$, and it is measurable in $(x,t) \in \Omega \times \mathbb{R}_+$ for any $u \in \mathbb{R}^M$.

Assumption III. (Growth Condition). There exist a t.u.i. in $L_1^{\rm loc}(\mathbb{R}_+; L_1(\Omega))$ function $c_1 : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $c_2 > 0$ such that

$$\sum_{i=1}^M \left|f^{(i)}(x,t,u)\right|^{q_i} \leq c_1(x,t) + c_2 \sum_{i=1}^M \left|u^{(i)}\right|^{p_i}$$

for any $u = (u^{(1)}, \dots, u^{(M)}) \in \mathbb{R}^M$, and a.e. $(x, t) \in \Omega \times \mathbb{R}_+$.

Assumption IV. (Sign Condition). There exists a constant $\alpha > 0$ and a t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ function $\beta : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sum_{i=1}^{M} f^{(i)}(x, t, u)u^{(i)} \geq \alpha \sum_{i=1}^{M} \left|u^{(i)}\right|^{p_i} - \beta(x, t)$$

for any $u = (u^{(1)}, \dots, u^{(M)}) \in \mathbb{R}^M$, and a.e. $(x, t) \in \Omega \times \mathbb{R}_+$.

In further arguments we will use standard functional Hilbert spaces $H = (L_2(\Omega))^M$, $V = (H_0^1(\Omega))^M$, and $V^* = (H^{-1}(\Omega))^M$ with standard respective inner products and norms $(\cdot, \cdot)_H$ and $\|\cdot\|_H$, $(\cdot, \cdot)_V$ and $\|\cdot\|_V$, and $(\cdot, \cdot)_{V^*}$ and $\|\cdot\|_{V^*}$, vector notations $\mathbf{p} = (p_1, p_2, \dots, p_M)$ and $\mathbf{q} = (q_1, q_2, \dots, q_M)$, and the spaces

$$\begin{aligned}
&\mathbf{L_p}(\Omega) := L_{p_1}(\Omega) \times \cdots \times L_{p_M}(\Omega), \quad \mathbf{L_q}(\Omega) := L_{q_1}(\Omega) \times \cdots \times L_{q_M}(\Omega),\\
&\mathbf{L_p}(\tau, T; \mathbf{L_p}(\Omega)) := L_{p_1}(\tau, T; L_{p_1}(\Omega)) \times \cdots \times L_{p_M}(\tau, T; L_{p_M}(\Omega)),\\
&\mathbf{L_q}(\tau, T; \mathbf{L_q}(\Omega)) := L_{q_1}(\tau, T; L_{q_1}(\Omega)) \times \cdots \times L_{q_M}(\tau, T; L_{q_M}(\Omega)), \ 0 \leq \tau < T < +\infty.
\end{aligned}$$

Let $0 \leq \tau < T < +\infty$. A function $y = y(x, t) \in \mathbf{L}_2(\tau, T; V) \cap \mathbf{L_p}(\tau, T; \mathbf{L_p}(\Omega))$ is called a *weak solution* of Problem (4.1) on $[\tau, T]$, if for any function $\varphi = \varphi(x) \in (C_0^\infty(\Omega))^M$, the following identity holds

$$\frac{d}{dt}\int_\Omega y(x, t) \cdot \varphi(x)dx + \int_\Omega \{a\nabla y(x, t) \cdot \nabla\varphi(x) + f(x, t, y(x, t)) \cdot \varphi(x)\}dx = 0 \tag{4.4}$$

in the sense of scalar distributions on (τ, T).

In the general case Problem (4.1) on $[\tau, T]$ with initial condition $y(x, \tau) = y_\tau(x)$ in Ω has more than one weak solution with $y_\tau \in H$ (cf. Balibrea et al. [2] and references therein).

4.3 Preliminary Properties of Weak Solutions

Let $\langle\cdot, \cdot\rangle : (V^* + \mathbf{L_q}(\Omega)) \times (V \cap \mathbf{L_p}(\Omega)) \to \mathbb{R}$ be the pairing in $(V^* + \mathbf{L_q}(\Omega)) \times (V \cap \mathbf{L_p}(\Omega))$, that coincides on $H \times (V \cap \mathbf{L_p}(\Omega))$ with the inner product $(\cdot, \cdot)_H$ on the Hilbert space H, i.e. $\langle u, v\rangle = (u, v)_H$ for any $u \in H$ and $v \in V \cap \mathbf{L_p}(\Omega)$.

For fixed nonnegative τ and T, $\tau < T$, let us consider the spaces

$$\begin{aligned}
X_{\tau,T}^{(i)} &= L_2(\tau, T; H_0^1(\Omega)) \cap L_{p_i}(\tau, T; L_{p_i}(\Omega)),\\
X_{\tau,T}^{(i)*} &= L_2(\tau, T; H^{-1}(\Omega)) + L_{q_i}(\tau, T; L_{q_i}(\Omega)),\\
X_{\tau,T} &= X_{\tau,T}^{(1)} \times \cdots \times X_{\tau,T}^{(M)}, \quad X_{\tau,T}^* = X_{\tau,T}^{(1)*} \times \cdots \times X_{\tau,T}^{(M)*},\\
W_{\tau,T}^{(i)} &= \{y \in X_{\tau,T}^{(i)} \mid y' \in X_{\tau,T}^{(i)*}\}, \quad W_{\tau,T} = W_{\tau,T}^{(1)} \times \cdots \times W_{\tau,T}^{(M)},
\end{aligned}$$

where y' is a derivative of an element $y \in X^{(i)}_{\tau,T}$ ($y \in X_{\tau,T}$) in the sense of $\mathscr{D}([\tau, T]; H^{-1}(\Omega) + L_q(\Omega))$ ($\mathscr{D}([\tau, T]; V^* + \mathbf{L_q}(\Omega))$ respectively); Gajewski et al. [3, Definition IV.1.10]. Note that the space $W_{\tau,T}$ is a reflexive Banach space with the graph norm of a derivative $\|u\|_{W_{\tau,T}} = \|u\|_{X_{\tau,T}} + \|u'\|_{X^*_{\tau,T}}$, $u \in W_{\tau,T}$. Let $\langle\cdot,\cdot\rangle_{X_{\tau,T}} : X^*_{\tau,T} \times X_{\tau,T} \to \mathbb{R}$ be the pairing in $X^*_{\tau,T} \times X_{\tau,T}$, that coincides on $L_2(\tau, T; H) \times X_{\tau,T}$ with the inner product in $L_2(\tau, T; H)$, i.e. $\langle u, v\rangle_{X_{\tau,T}} = \int_\tau^T (u(t), v(t))_H dt$ for any $u \in L_2(\tau, T; H)$ and $v \in X_{\tau,T}$. Gajewski et al. [3, Theorem IV.1.17] provide that the embedding $W^{(i)}_{\tau,T} \subset C([\tau, T]; L_2(\Omega))$ is continuous and dense, $i = 1, 2, \dots, M$. Thus, the embedding $W_{\tau,T} \subset C([\tau, T]; H)$ is continuous and dense. Moreover,

$$(u(T), v(T))_H - (u(\tau), v(\tau))_H = \int_\tau^T \Big[\langle u'(t), v(t)\rangle + \langle v'(t), u(t)\rangle\Big]dt, \tag{4.5}$$

for any $u, v \in W_{\tau,T}$.

If $y(x, t) \in \mathbf{L}_\mathrm{p}(\tau, T; \mathbf{L}_\mathrm{p}(\Omega))$, then Assumptions I–III yield

$$f(x, t, y(x, t)) \in \mathbf{L_q}(\tau, T; \mathbf{L_q}(\Omega)),$$

and

$$\begin{aligned} &\sum_{i=1}^M \|f^{(i)}(y(\cdot))\|^{q_i}_{L_{q_i}(\tau,T;L_{q_i}(\Omega))} \\ \le\, & c_2 \sum_{i=1}^M \|y^{(i)}(\cdot)\|^{p_i}_{L_{p_i}(\tau,T;L_{p_i}(\Omega))} + \int_{\Omega\times(\tau,T)} c_1(x, t)dxdt. \end{aligned} \tag{4.6}$$

Moreover, if $y(x, t) \in \mathbf{L}_2(\tau, T; V)$, then $a\triangle y(x, t) \in \mathbf{L}_2(\tau, T; V^*)$.

Assumptions I–IV and Chepyzhov and Vishik [7, pp. 283–284] (see also Zgurovsky et al. [27, Chap. 2] and references therein) provide the existence of a weak solution of Cauchy problem (4.1) with initial data $y(\tau) = y^{(\tau)}$ on the interval $[\tau, T]$, for any $y^{(\tau)} \in H$. The proof is provided by standard Faedo–Galerkin approximations and using local existence Carathéodory's theorem instead of classical Peano results. A priori estimates are similar. Formula (4.4) and definition of the derivative for an element from $\mathscr{D}([\tau, T]; V^* + \mathbf{L_q}(\Omega))$ yield that each weak solution $y \in X_{\tau,T}$ of Problem (4.1) on $[\tau, T]$ belongs to the space $W_{\tau,T}$. Moreover, each weak solution of Problem (4.1) on $[\tau, T]$ satisfies the equality:

$$\int_\tau^T \int_\Omega \left[\frac{\partial y(x,t)}{\partial t} \cdot \psi(x, t) + a\nabla y(x, t) \cdot \nabla\psi(x, t) + f(x, t, y(x, t)) \cdot \psi(x, t)\right] dxdt = 0, \tag{4.7}$$

for any $\psi \in X_{\tau,T}$. For fixed τ and T, such that $0 \le \tau < T < +\infty$, we denote

$$\mathscr{D}_{\tau,T}(y^{(\tau)}) = \{y(\cdot) \mid y \text{ is a weak solution of (4.1) on } [\tau, T],\ y(\tau) = y^{(\tau)}\}, \quad y^{(\tau)} \in H.$$

We remark that $\mathscr{D}_{\tau,T}(y^{(\tau)}) \neq \emptyset$ and $\mathscr{D}_{\tau,T}(y^{(\tau)}) \subset W_{\tau,T}$, if $0 \leq \tau < T < +\infty$ and $y^{(\tau)} \in H$. Moreover, the concatenation of Problem (4.1) weak solutions is a weak solutions too, i.e. if $0 \leq \tau < t < T$, $y^{(\tau)} \in H$, $y(\cdot) \in \mathscr{D}_{\tau,t}(y^{(\tau)})$, and $v(\cdot) \in \mathscr{D}_{t,T}(y(t))$, then

$$z(s) = \begin{cases} y(s), & s \in [\tau, t], \\ v(s), & s \in [t, T], \end{cases}$$

belongs to $\mathscr{D}_{\tau,T}(y^{(\tau)})$; cf. Zgurovsky et al. [28, pp. 55–56].

Listed above properties of solutions and Gronwall lemma provide that for any finite time interval $[\tau, T] \subset \mathbb{R}_+$ each weak solution y of Problem (4.1) on $[\tau, T]$ satisfies estimates

$$\|y(t)\|_H^2 - 2\int_\tau^t \int_\Omega \beta(x,\xi)dxd\xi + 2\alpha \sum_{i=1}^M \int_s^t \|y^{(i)}(\xi)\|_{L_{p_i}(\Omega)}^{p_i} d\xi + 2d\int_s^t \|y(\xi)\|_V^2 d\xi \leq \|y(s)\|_H^2 - 2\int_\tau^s \int_\Omega \beta(x,\xi)dxd\xi, \tag{4.8}$$

$$\|y(t)\|_H^2 \leq \|y(s)\|_H^2 e^{-2d\lambda_1(t-s)} + 2\int_s^t \int_\Omega \beta(x,\xi)e^{-2d\lambda_1(t-\xi)}dxd\xi, \tag{4.9}$$

for any $t, s \in [\tau, T]$, $t \geq s$, where λ_1 is the first eigenvalue of the scalar operator $-\Delta$ with Dirichlet boundary conditions; cf. Chepyzhov and Vishik [7, p. 285]; Vishik et al. [28, p. 56]; Valero and Kapustyan [24] and references therein. We note that the same term with β appears both on the left and right hand side of inequality (4.9). This was done on purpose to comply the inequality with the definition (4.18) of J and J_k below.

Therefore, any weak solution y of Problem (4.1) on a finite time interval $[\tau, T] \subset \mathbb{R}_+$ can be extended to a global one, defined on $[\tau, +\infty)$. For arbitrary $\tau \geq 0$ and $y^{(\tau)} \in H$ let $\mathscr{D}_\tau(y^{(\tau)})$ be the set of all weak solutions (defined on $[\tau, +\infty)$) of Problem (4.1) with initial data $y(\tau) = y^{(\tau)}$. Let us consider the family $\mathscr{K}_\tau^+ = \cup_{y^{(\tau)} \in H} \mathscr{D}_\tau(y^{(\tau)})$ of all weak solutions of Problem (4.1) defined on the semi-infinite time interval $[\tau, +\infty)$.

In further arguments as a Banach space $\mathscr{F}_{t_1,t_2}$ we consider either $C([t_1, t_2]; H)$ or W_{t_1,t_2} with respective topologies of strong convergence, where $0 \leq t_1 < t_2 < +\infty$. Consider the Fréchet space

$$\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+) := \{y : \mathbb{R}_+ \to H \ : \ \Pi_{t_1,t_2} y \in \mathscr{F}_{t_1,t_2} \text{ for any } [t_1, t_2] \subset \mathbb{R}_+\},$$

where Π_{t_1,t_2} is the restriction operator to the interval $[t_1, t_2]$; Chepyzhov and Vishik [5, p. 918]. We remark that the sequence $\{f_n\}_{n\geq 1}$ converges (converges weakly respectively) in $\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)$ towards $f \in \mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)$ as $n \to +\infty$ if and only if the sequence $\{\Pi_{t_1,t_2} f_n\}_{n\geq 1}$ converges (converges weakly respectively) in $\mathscr{F}_{t_1,t_2}$ towards $\Pi_{t_1,t_2} f$ as $n \to +\infty$ for any finite interval $[t_1, t_2] \subset \mathbb{R}_+$.

We denote $T(h)y(\cdot) = y_h(\cdot)$, where $y_h(t) = y(t+h)$ for any $y \in \mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)$ and $t, h \geq 0$.

In the autonomous case, when $f(x, t, y)$ does not depend on t, the long-time behavior of all globally defined weak solutions for Problem (4.1) is described by using trajectory and global attractors theory; Chepyzhov and Vishik [7, Chap. XIII]; Vishik et al. [25]; Melnik and Valero [19]; Kasyanov [11, 12], Zgurovsky et al. [28, Chap. 2] and references therein; see also Balibrea et al. [2]. In this situation the set $\mathscr{K}^+ := \mathscr{K}_0^+$ is *translation semi-invariant*, i.e. $T(h)\mathscr{K}^+ \subseteq \mathscr{K}^+$ for any $h \geq 0$. As trajectory attractor it is considered a classical global attractor for translation semigroup $\{T(h)\}_{h\geq 0}$, that acts on $\mathscr{K}^+$.

In the non-autonomous case we notice that $T(h)\mathscr{K}_0^+ \not\subseteq \mathscr{K}_0^+$. Therefore, we need to consider *united trajectory space* that includes all globally defined on any $[\tau, +\infty) \subseteq \mathbb{R}_+$ weak solutions of Problem (4.1) shifted to $\tau = 0$:

$$\mathscr{K}_\cup^+ := \bigcup_{\tau \geq 0} \left\{ y(\cdot + \tau) \in W^{\mathrm{loc}}(\mathbb{R}_+) \,:\, y(\cdot) \in \mathscr{K}_\tau^+ \right\}, \tag{4.10}$$

Note that $T(h)\{y(\cdot + \tau) \,:\, y \in \mathscr{K}_\tau^+\} \subseteq \{y(\cdot + \tau + h) \,:\, y \in \mathscr{K}_{\tau+h}^+\}$ for any $\tau, h \geq 0$. Therefore,

$$T(h)\mathscr{K}_\cup^+ \subseteq \mathscr{K}_\cup^+$$

for any $h \geq 0$. Further we consider *extended united trajectory space* for Problem (4.1) (see Fig. 4.1):

$$\mathscr{K}^+_{\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)} = \mathrm{cl}_{\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)} \left[\mathscr{K}_\cup^+ \right], \tag{4.11}$$

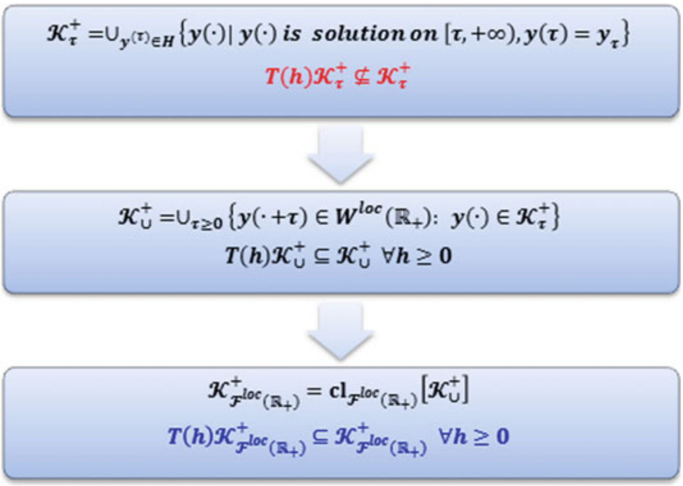

Fig. 4.1 The extended united trajectory space construction scheme

where $\mathrm{cl}_{\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)}[\,\cdot\,]$ is the closure in $\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)$. We note that

$$T(h)\mathscr{K}^+_{\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)} \subseteq \mathscr{K}^+_{\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)}$$

for each $h \geq 0$, because

$$\rho_{\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)}(T(h)u, T(h)v) \leq \rho_{\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)}(u, v) \text{ for any } u, v \in \mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+),$$

where $\rho_{\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)}$ is a standard metric on Fréchet space $\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)$; cf. Vishik, Zelik, and Chepyzhov [25]; Chepyzhov and Vishik [5]; Vishik et al. [25].

4.4 Strongest Convergence Results in $C^{\mathrm{Loc}}(\mathbb{R}_+; H)$

Let us provide the result characterizing the compactness properties of shifted solutions of Problem (4.1) in the induced topology from $C^{\mathrm{loc}}(\mathbb{R}_+; H)$.

Theorem 4.1 *Let Assumptions* I–IV *hold. If* $\{y_n\}_{n\geq 1} \subset \mathscr{K}^+_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ *be an arbitrary sequence, which is bounded in* $L_\infty(\mathbb{R}_+; H)$*, then there exist a subsequence* $\{y_{n_k}\}_{k\geq 1} \subseteq \{y_n\}_{n\geq 1}$ *and an element* $y \in \mathscr{K}^+_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ *such that*

$$\|\Pi_{\tau,T}\, y_{n_k} - \Pi_{\tau,T}\, y\|_{C([\tau,T];H)} \to 0, \quad k \to +\infty, \tag{4.12}$$

for any finite time interval $[\tau, T] \subset (0, +\infty)$*. Moreover, for any* $y \in \mathscr{K}^+_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ *the estimate holds*

$$\|y(t)\|_H^2 \leq \|y(0)\|_H^2 e^{-c_3 t} + c_4, \tag{4.13}$$

for any $t \geq 0$*, where positive constants* c_3 *and* c_4 *do not depend on* $y \in \mathscr{K}^+_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ *and* $t \geq 0$*.*

Proof Assume that $\{y_n\}_{n\geq 1} \subset \mathscr{K}^+_{\cup}$ be an arbitrary sequence, which is bounded in $L_\infty(\mathbb{R}_+; H)$. Let us fix $n \geq 1$. Formula (4.10) provides the existence of $\tau_n \geq 0$ and $z_n(\,\cdot\,) \in \mathscr{K}^+_{\tau_n}$ such that $y_n(\,\cdot\,) = z_n(\,\cdot + \tau_n)$. Then, formulas (4.8) and (4.9) yield that

$$\|y_n(t)\|_H^2 - 2\int_0^t \int_\Omega \beta_n(x,\xi)dxd\xi + 2\alpha \sum_{i=1}^M \int_s^t \|y_n^{(i)}(\xi)\|^{p_i}_{L_{p_i}(\Omega)} d\xi + 2\int_s^t \|y_n(\xi)\|_V^2 d\xi \leq \|y_n(s)\|_H^2 - 2\int_0^s \int_\Omega \beta_n(x,\xi)dxd\xi, \tag{4.14}$$

$$\|y_n(t)\|_H^2 \leq \|y_n(s)\|_H^2 e^{-2d\lambda_1(t-s)} + 2\int_s^t \int_\Omega \beta_n(x,\xi)e^{-2d\lambda_1(t-\xi)}dxd\xi, \tag{4.15}$$

for any $t \geq s \geq 0$, where $\beta_n(x,t) := \beta(x, t+\tau_n)$ for a.e. $(x,t) \in \Omega \times \mathbb{R}_+$.

Note that formula (4.15) and t.u.i. of β in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ provide formula (4.13) for any $y \in \mathscr{K}_\cup^+$, where positive constants c_3 and c_4 do not depend on respective y and t; cf. Chepyzhov and Vishik [7, p. 35]. Formula (4.13) holds for any $y \in \mathscr{K}^+_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$, because the set $\mathscr{K}_\cup^+$ is dense in $\mathscr{K}^+_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ endowed with strong local convergence topology of $C^{\mathrm{loc}}(\mathbb{R}_+; H)$. Therefore, the second statement of the theorem (estimate (4.13)) is proved.

Let us continue the proof of the first statement of the theorem (formula (4.12)). Further, to simplify arguments we set

$$d_n(x,t) := f(x, t+\tau_n, y_n(x,t)) \text{ for a.e. } (x,t) \in \Omega \times \mathbb{R}_+ \text{ and } n \geq 1.$$

Estimates (4.14) and (4.15), formula (4.7), t.u.i. of β and c_1 in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$, and Assumptions III and IV, provide that the sequence $\{y_n, d_n\}_{n\geq 1}$ is bounded in $W^{\mathrm{loc}}(\mathbb{R}_+) \times \mathbf{L_q}(\tau, T; \mathbf{L_q}(\Omega))$. Banach–Alaoglu theorem (cf. Zgurovsky et al. [27, Chap. 1]; Kasyanov [11] and references therein) yields that there exist a subsequence $\{y_{n_k}, d_{n_k}\}_{k\geq 1} \subseteq \{y_n, d_n\}_{n\geq 1}$ and elements $(y, d) \in W^{\mathrm{loc}}(\mathbb{R}_+) \times \mathbf{L_q}(\tau, T; \mathbf{L_q}(\Omega))$, and $\bar{\beta} \in L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ such that

$$\begin{array}{lll} (y_{n_k}, d_{n_k}) \to (y, d) & \text{weakly in } W^{\mathrm{loc}}(\mathbb{R}_+) \times \mathbf{L_q^{\mathrm{loc}}}(\mathbb{R}_+; \mathbf{L_q}(\Omega)), & \\ y_{n_k} \to y & \text{weakly in } C^{\mathrm{loc}}(\mathbb{R}_+; H), & \\ y_{n_k} \to y & \text{in } L_2^{\mathrm{loc}}(\mathbb{R}_+; H), & \\ y_{n_k}(t) \to y(t) & \text{in } H \text{ for a.e. } t > 0, & \\ \beta_{n_k} \to \bar{\beta} & \text{weakly in } L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega)), \quad k \to +\infty. & \end{array} \tag{4.16}$$

Note that the second convergence holds, because the embedding $W^{\mathrm{loc}}(\mathbb{R}_+) \subset C^{\mathrm{loc}}(\mathbb{R}_+; H)$ is continuous, the third one follows from the compact embedding of $W^{\mathrm{loc}}(\mathbb{R}_+)$ into $L_2^{\mathrm{loc}}(\mathbb{R}_+; H)$ (cf. Zgurovsky et al. [27, Chap. 1]), the fourth convergence follows from the third one, and the last statement in (4.16) follows from Dunford–Pettis compactness criterion.

Let us prove that

$$y_{n_k}(t) \to y(t) \text{ in } H \text{ for any } t > 0, \text{ as } k \to +\infty. \tag{4.17}$$

We consider continuous and nonincreasing (by formula (4.14)) functions on $\mathbb{R}_+$:

$$J_k(t) = \|y_n(t)\|_H^2 - 2\int_0^t \int_\Omega \beta_n(x,\xi)dxd\xi, \; J(t) = \|y(t)\|_H^2 - 2\int_0^t \int_\Omega \bar{\beta}(x,\xi)dxd\xi, \; k \geq 1; \tag{4.18}$$

cf. Kapustyan et al. [14]. The fourth and the last statements in (4.16) imply

$$J_k(t) \to J(t), \text{ as } k \to +\infty, \text{ for a.e. } t > 0. \tag{4.19}$$

Similarly to Zgurovsky et al. [28, p. 57] (see book and references therein) we show that

$$\limsup_{k\to+\infty} J_k(t) \leq J(t) \quad \forall t > 0. \tag{4.20}$$

Indeed, formula (4.19) and continuity of J imply that for any $t > 0$ and $\varepsilon > 0$ there exists $\bar{t} \in (0, t)$ such that $|J(\bar{t}) - J(t)| < \varepsilon$ and $\lim\limits_{k\to+\infty} J_k(\bar{t}) = J(\bar{t})$. Hence,

$$J_k(t) - J(t) \leq J_k(\bar{t}) - J(t) \leq |J_k(\bar{t}) - J(\bar{t})| + |J(\bar{t}) - J(t)| < \varepsilon + |J_k(\bar{t}) - J(\bar{t})|,$$

for any $k \geq 1$. Therefore, $\limsup\limits_{k\to+\infty} J_k(t) \leq J(t) + \varepsilon$, for each $t > 0$ and $\varepsilon > 0$. Thus, inequality (4.20) holds.

Formula (4.20) and last statement of (4.16) yield the inequality

$$\limsup_{k\to+\infty} \|y_{n_k}(t)\|_H^2 \leq \|y(t)\|_H^2 \quad \forall t > 0.$$

Convergence (4.17) holds, because of the last inequality and the pointwise weak convergence in H of the sequence $\{y_{n_k}\}_{k\geq 1}$ towards y, as $k \to +\infty$ (see the second statement in (4.16)).

Let us prove (4.12). By contradiction suppose the existence of a positive constant $L > 0$, a finite interval $[\tau, T] \subset (0, +\infty)$, and a subsequence $\{y_{k_j}\}_{j\geq 1} \subseteq \{y_{n_k}\}_{k\geq 1}$ such that

$$\forall j \geq 1 \quad \max_{t\in[\tau,T]} \|y_{k_j}(t) - y(t)\|_H = \|y_{k_j}(t_j) - y(t_j)\|_H \geq L.$$

Suppose also that $t_j \to t_0 \in [\tau, T]$, as $j \to +\infty$. Continuity of $\Pi_{\tau,T}\, y : [\tau, T] \to H$ implies

$$\liminf_{j\to+\infty} \|y_{k_j}(t_j) - y(t_0)\|_H \geq L. \tag{4.21}$$

On the other hand we prove that

$$y_{k_j}(t_j) \to y(t_0) \text{ in } H, \;\; j \to +\infty. \tag{4.22}$$

For this purpose we firstly note that

$$y_{k_j}(t_j) \to y(t_0) \text{ weakly in } H, \quad j \to +\infty. \tag{4.23}$$

Indeed, for a fixed $h \in (C_0^\infty(\Omega))^M$ from (4.16) it follows that the sequence of real functions $\{(\Pi_{\tau,T}\, y_{n_k}(\cdot), h)_H : [\tau, T] \to \mathbb{R}\}_{k\geq 1}$ is uniformly bounded and equicontinuous. Taking into account the boundedness of $\{\Pi_{\tau,T}\, y_{n_k}\}_{k\geq 1}$ in $W_{\tau,T}$ and the density of the set $(C_0^\infty(\Omega))^M$ in H we obtain that $y_{n_k}(t) \to y(t)$ weakly in H uniformly on $[\tau, T]$, as $k \to +\infty$. So, we obtain (4.23).

Secondly we prove that

$$\limsup_{j\to+\infty} \|y_{k_j}(t_j)\|_H \le \|y(t_0)\|_H. \tag{4.24}$$

We consider continuous nonincreasing functions J and J_{k_j}, $j \ge 1$, defined in (4.18). Let us fix an arbitrary $\varepsilon > 0$. Continuity of J and (4.19) provide the existence of $\bar{t} \in (\tau, t_0)$ such that $\lim_{j\to+\infty} J_{k_j}(\bar{t}) = J(\bar{t})$ and $|J(\bar{t}) - J(t_0)| < \varepsilon$. Then,

$$J_{k_j}(t_j) - J(t_0) \le |J_{k_j}(\bar{t}) - J(\bar{t})| + |J(\bar{t}) - J(t_0)| \le |J_{k_j}(\bar{t}) - J(\bar{t})| + \varepsilon,$$

for rather large $j \ge 1$. Thus, $\limsup_{j\to+\infty} J_{k_j}(t_j) \le J(t_0)$ and inequality (4.24) holds.

Thirdly note that the convergence (4.22) holds, because of (4.23) and (4.24); cf. Gajewski et al. [3, Chap. I]. Finally we remark that statement (4.22) contradicts to assumption (4.21). Therefore, the first statement of the theorem holds for any sequence $\{y_n\}_{n\ge1} \subset \mathscr{K}_{\cup}^{+}$.

To finish the proof of the theorem we consider the first statement in the general case. Let $\{y_n\}_{n\ge1} \subset \mathscr{K}^{+}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ be an arbitrary sequence, which is bounded in $L_\infty(\mathbb{R}_+; H)$. Since the set $\mathscr{K}_{\cup}^{+}$ is dense in a Polish space $\mathscr{K}^{+}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ we have that for any $n \ge 1$ there exists $\bar{y}_n \in \mathscr{K}_{\cup}^{+}$ such that $\rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}(y_n, \bar{y}_n) \le \frac{1}{n}$. A priori estimate (4.13) provides that the sequence $\{\bar{y}_n\}_{n\ge1}$ is bounded in $L_\infty(\mathbb{R}_+; H)$. The first statement of the theorem, applied for the sequence $\{\bar{y}_n\}_{n\ge1} \subset \mathscr{K}_{\cup}^{+}$, yields that there exist a subsequence $\{\bar{y}_{n_k}\}_{k\ge1} \subset \{\bar{y}_n\}_{n\ge1}$ and an element $y \in \mathscr{K}^{+}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ such that $\|\Pi_{\tau,T}\bar{y}_{n_k} - \Pi_{\tau,T}y\|_{C^{\mathrm{loc}}([\tau,T];H)} \to 0$, as $k \to +\infty$, for any finite time interval $[\tau, T] \subset (0, +\infty)$. Therefore, formula (4.12) holds for any $[\tau, T] \subset (0, +\infty)$.

4.5 Strongest Convergence Results for Solutions in the Natural Extended Phase Space

For convergence results in the strong topology of the natural extended phase space $W^{\mathrm{loc}}(\mathbb{R}_+)$ it is necessary to claim that additional assumption holds (see Example 8.1). To formulate this additional assumption we provide some auxiliary constructions. A function $\varphi \in L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ is called *translation-compact (tr.-c.)* in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$, if the set $\{\varphi(\cdot + h) : h \ge 0\}$ is precompact in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$; cf. Chepyzhov and Vishik [5, p. 917]. Note that a function $\varphi \in L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ is tr.-c. in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ if and only if two conditions hold: a) the set $\left\{\int_t^{t+h} \varphi(s)ds : t \ge 0\right\}$ is precompact in $L_1(\Omega)$ for any $h > 0$; b) there exists a function $\psi(s)$, $\psi(s) \to 0+$ as $s \to 0+$ such that

$$\int_t^{t+1}\int_\Omega |\varphi(x,s) - \varphi(x,s+h)|dxds \le \psi(|h|) \text{ for any } t \ge 0 \text{ and } h \ge -t;$$

Chepyzhov and Vishik [5, Proposition 6.5].

Assumption V. Let the conditions hold:

(i) the functions c_1 and β from Assumptions (III) and (IV) respectively are tr.-c. in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$;

(ii) the set $\left\{\frac{1}{h}\int_t^{t+h} f(\cdot, s, u)ds \,:\, t \geq 0,\ h \in (0, h_0),\ \|u\|_{\mathbb{R}^M} \leq R\right\}$ is precompact in $(L_1(\Omega))^M$ for any $R > 0$ and some $h_0 = h_0(R) > 0$;

(iii) for any $r > 0$ there exist a nondecreasing function $\psi(s, r) : \mathbb{R}_+^2 \to \mathbb{R}_+$, $\psi(s, r) \to 0+$ as $s \to 0+$, and $h_0 = h_0(r) > 0$ such that

$$\frac{1}{h_1}\sum_{i=1}^{M}\int_t^{t+h_1}\int_\Omega \left|f^{(i)}(x, s, u) - f^{(i)}(x, s + h_2, v)\right| dxds \leq \psi(|h_2| + \|u - v\|_{\mathbb{R}^M}, r)$$

for each $t \geq 0$, $h_1 \in (0, h_0)$, $h_2 \geq -t$, and $u, v \in \mathbb{R}^M$ such that $\|u\|_{\mathbb{R}^M}, \|v\|_{\mathbb{R}^M} \leq r$.

Remark 4.1 Let us discuss sufficient conditions for Assumption V.

(i) The autonomous case. Let f does not depend on the time variable t and it satisfies Assumptions I–IV with $c_1, \beta \in L_1(\Omega)$ (in particular, assumptions from Vishik et al. [25] hold). Then Assumption V hold. Indeed, Assumptions V(i) holds, because c_1 and β do not depend on t; Assumptions II, III and the dominated convergence theorem imply Assumption V(ii). Assumption V(iii) follows from Heine–Cantor theorem and continuity of the mapping $u \to \int_\Omega f(x, u)dx$. The last follows from the dominated convergence theorem and Assumptions I–III.

(ii) The non-autonomous case. Let $f = f(t, u)$ is jointly continuous mapping, it satisfies Assumptions I–IV with positive constants c_1 and β, and f being tr.-c. in $C^{\mathrm{loc}}(\mathbb{R}_+; C(\mathbb{R}^M))$, that is

$$\|f(t, u) - f(s, v)\|_{\mathbb{R}^M} \leq \omega(|t - s| + \|u - v\|_{\mathbb{R}^M}, K),$$

for all $t, s \in \mathbb{R}_+$, $\|u\|_{\mathbb{R}^M}, \|v\|_{\mathbb{R}^M} \leq K$, $K > 0$, where $\omega(l, K) \to 0$, as $l \to 0+$; see, for example, Chepyzhov and Vishik [7, p.105], Kapustyan and Valero [16, 24], where uniform global in H and uniform trajectory in $C^{\mathrm{loc}}(\mathbb{R}_+; H)$ attractors were investigated. Then Assumption V holds.

(iii) The sufficient condition for Assumption V(iii) is: for any $r > 0$ there exist a nondecreasing function $\psi(s, r) : \mathbb{R}_+^2 \to \mathbb{R}_+$, $\psi(s, r) \to 0+$ as $s \to 0+$, such that

$$\sum_{i=1}^{M}\int_\Omega \left|f^{(i)}(x, t, u) - f^{(i)}(x, t + h, v)\right| dxds \leq \psi(|h| + \|u - v\|_{\mathbb{R}^M}, r)$$

for each $t \geq 0$, $h \geq -t$, and $u, v \in \mathbb{R}^M$ such that $\|u\|_{\mathbb{R}^M}, \|v\|_{\mathbb{R}^M} \leq r$.

Note that Assumption V is a generalization of the above assumptions to the case when f depends on the space, time and state variables simultaneously and it is not necessarily continuous by t. Meanwhile, Example 8.1 below provide piecewise continuous function f that satisfies Assumptions I–IV, but it does not satisfy

Assumption V. Moreover, the statement of Theorem 4.2 below does not hold for Problem (4.1) with such interaction function.

Now let us provide the result characterizing the compactness properties of shifted solutions of Problem (4.1) in the induced topology from $W^{\text{loc}}(\mathbb{R}_+)$.

Theorem 4.2 *Let Assumptions* I–V *hold. If* $\{y_n\}_{n\ge 1} \subset \mathscr{K}^+_{W^{\text{loc}}(\mathbb{R}_+)}$ *be an arbitrary sequence, which is bounded in* $L_\infty(\mathbb{R}_+; H)$*, then there exist a subsequence* $\{y_{n_k}\}_{k\ge 1} \subseteq \{y_n\}_{n\ge 1}$ *and an element* $y \in \mathscr{K}^+_{W^{\text{loc}}(\mathbb{R}_+)}$ *such that*

$$\|\Pi_{\tau,T}\, y_{n_k} - \Pi_{\tau,T}\, y\|_{W_{\tau,T}} \to 0, \quad \text{as } k \to +\infty, \tag{4.25}$$

for any finite time interval $[\tau, T] \subset (0, +\infty)$*. Moreover, for any* $y \in \mathscr{K}^+_{W^{\text{loc}}(\mathbb{R}_+)}$ *the estimate (4.13) holds for any* $t \ge 0$*, where positive constants* c_3 *and* c_4 *do not depend on* $y \in \mathscr{K}^+_{W^{\text{loc}}(\mathbb{R}_+)}$ *and* $t \ge 0$.

Proof The embedding $\mathscr{K}^+_{W^{\text{loc}}(\mathbb{R}_+)} \subseteq \mathscr{K}^+_{C^{\text{loc}}(\mathbb{R}_+;H)}$ and Theorem 4.1 yield the second statement of the theorem. Let us provide the first one.

We consider an arbitrary sequence $\{y_n\}_{n\ge 1} \subset \mathscr{K}^+_{W^{\text{loc}}(\mathbb{R}_+)}$, which is bounded in $L_\infty(\mathbb{R}_+; H)$. Since the set $\mathscr{K}^+_{\cup}$ is dense in a Polish space $\mathscr{K}^+_{W^{\text{loc}}(\mathbb{R}_+)}$ and the estimate (4.13) holds for any $y \in \mathscr{K}^+_{W^{\text{loc}}(\mathbb{R}_+)}$, there exists a sequence $\{\bar{y}_n\}_{n\ge 1} \subset \mathscr{K}^+_{\cup}$, which is bounded in $L_\infty(\mathbb{R}_+; H)$ and $\rho_{W^{\text{loc}}(\mathbb{R}_+)}(y_n, \bar{y}_n) \le \frac{1}{n}$ for any $n \ge 1$. Therefore, to provide the first statement of the theorem, we may additionally suppose that the sequence $\{y_n\}_{n\ge 1}$ belongs to $\mathscr{K}^+_{\cup}$.

Note that Assumptions III, IV, and V, and Young's inequality yield that there exist positive constants $\alpha', \beta' > 0$ and a tr.-c. in $L_1^{\text{loc}}(\mathbb{R}_+; L_1(\Omega))$ function $c' : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sum_{i=1}^{M} f^{(i)}(x,t,u)(u^{(i)} - v^{(i)}) \ge \alpha' \sum_{i=1}^{M} \left|u^{(i)}\right|^{p_i} - \beta' \sum_{i=1}^{M} \left|v^{(i)}\right|^{p_i} - c'(x,t), \tag{4.26}$$

for any $u, v \in \mathbb{R}^M$ and a.e. $(x,t) \in \Omega \times (0,+\infty)$. Let $\mathscr{H}(c') := \text{cl}_{L_1^{\text{loc}}(\mathbb{R}_+;L_1(\Omega))} \{c'(\cdot + h) \,:\, h \ge 0\}$ be the hull of tr.-c. function c' in $L_1^{\text{loc}}(\mathbb{R}_+; L_1(\Omega))$. This is a compact set in $L_1^{\text{loc}}(\mathbb{R}_+; L_1(\Omega))$; Chepyzhov and Vishik [5].

Let us fix an arbitrary $n \ge 1$. Formula (4.10) provides the existence of $\tau_n \ge 0$ and $z_n(\cdot) \in \mathscr{K}^+_{\tau_n}$ such that $y_n(\cdot) = z_n(\cdot + \tau_n)$. Following to the statement of Theorem 4.1 and its proof (see formula (4.16) and conclusions above it), there exist a subsequence $\{y_{n_k}, d_{n_k}\}_{k\ge 1} \subseteq \{y_n, d_n\}_{n\ge 1}$ and elements $(y, d) \in W^{\text{loc}}(\mathbb{R}_+) \times \mathbf{L_q}(\tau, T; \mathbf{L_q}(\Omega))$, and $\bar{\beta} \in L_1^{\text{loc}}(\mathbb{R}_+; L_1(\Omega))$ such that convergences (4.16) and (4.12) hold. Here we again use the notation:

$$d_n(x,t) := f(x, t+\tau_n, y_n(x,t)) \text{ for a.e. } (x,t) \in \Omega \times \mathbb{R}_+ \text{ and } n \ge 1.$$

Since the sets $\mathscr{H}(c_1)$ and $\mathscr{H}(c')$ are compact in $L_1^{\text{loc}}(\mathbb{R}_+; L_1(\Omega))$, taking into account the third statement of (4.16), we may additionally claim (passing to a subsequence if necessary) the existence of elements $\bar{c}_1$ and $\bar{c}'$ from $L_1^{\text{loc}}(\mathbb{R}_+; L_1(\Omega))$ such that

$$\begin{aligned} y_{n_k}(x,t) &\to y(x,t) \\ c_{1,n_k}(x,t) &\to \bar{c}_1(x,t), \\ c'_{n_k}(x,t) &\to \bar{c}'(x,t), \text{ as } k \to +\infty, \qquad \text{for a.e. } (x,t) \in \Omega \times \mathbb{R}_+; \\ c'_{n_k} &\to \bar{c}', \\ c_{1,n_k} &\to \bar{c}_1, \qquad \text{in } L_1^{\text{loc}}(\mathbb{R}_+; L_1(\Omega)), \text{ as } k \to +\infty, \end{aligned} \tag{4.27}$$

where $c_{1,n_k} := c_1(x, t + \tau_{n_k})$ and $c'_{n_k}(x,t) := c'(x, t + \tau_{n_k})$ for a.e. $(x,t) \in \Omega \times \mathbb{R}_+$ and any $k \geq 1$.

Assumption III yields that

$$\sum_{i=1}^{M} \left| d_{n_k}^{(i)}(x,t) \right|^{q_i} \leq c_{1,n_k}(x,t) + c_2 \sum_{i=1}^{M} \left| y_{n_k}^{(i)}(x,t) \right|^{p_i} \tag{4.28}$$

for a.e. $(x,t) \in \Omega \times \mathbb{R}_+$ and any $k \geq 1$. Therefore, the first two statements of (4.27) provide

$$d_{n_k}(x,t) \cdot (y_{n_k} - y)(x,t) \to 0 \text{ as } k \to +\infty, \text{ for a.e. } (x,t) \in \Omega \times \mathbb{R}_+. \tag{4.29}$$

Now let fix an arbitrary finite time interval $[\tau, T] \subset (0, +\infty)$. Prove that

$$\begin{aligned} \|\Pi_{\tau,T} y_{n_k} - \Pi_{\tau,T} y\|^2_{X_{\tau,T}} &= \|\Pi_{\tau,T} y_{n_k} - \Pi_{\tau,T} y\|^2_{L_2(\tau,T;V)} \\ &+ \|\Pi_{\tau,T} y_{n_k} - \Pi_{\tau,T} y\|^2_{\mathbf{L_p}(\tau,T;\mathbf{L_p}(\Omega))} \to 0, \text{ as } k \to +\infty. \end{aligned} \tag{4.30}$$

Formulas (4.7) and (4.5) yield

$$\begin{aligned} &(y_{n_k}(\tau), y_{n_k}(\tau) - y(\tau))_H - (y_{n_k}(T), y_{n_k}(T) - y(T))_H \\ &= \int_\tau^T \int_\Omega a \nabla y_{n_k}(x,t) \cdot \nabla (y_{n_k} - y)(x,t) dx dt + \int_\tau^T \int_\Omega d_{n_k}(x,t) \cdot (y_{n_k} - y)(x,t) dx dt, \end{aligned} \tag{4.31}$$

for any $k \geq 1$. Formula (4.12) provides

$$(y_{n_k}(\tau), y_{n_k}(\tau) - y(\tau))_H - (y_{n_k}(T), y_{n_k}(T) - y(T))_H \to 0, \text{ as } k \to +\infty. \tag{4.32}$$

The first statement of (4.16) implies

$$\begin{aligned} &\liminf_{k \to +\infty} \int_\tau^T \int_\Omega a \nabla y_{n_k}(x,t) \cdot \nabla (y_{n_k} - y)(x,t) dx dt \\ &= \liminf_{k \to +\infty} \int_\tau^T \int_\Omega a \nabla y_{n_k}(x,t) \cdot \nabla y_{n_k}(x,t) dx dt - \int_\tau^T \int_\Omega a \nabla y(x,t) \cdot \nabla y(x,t) dx dt \geq 0, \end{aligned} \tag{4.33}$$

and

$$\liminf_{k\to+\infty}\int_\tau^T\int_\Omega\sum_{i=1}^M\left|y_{n_k}^{(i)}(x,t)\right|^{p_i}dxdt \ge \int_\tau^T\int_\Omega\sum_{i=1}^M\left|y^{(i)}(x,t)\right|^{p_i}dxdt, \qquad (4.34)$$

because the sequence $\{\Pi_{\tau,T}y_{n_k}\}_{k\ge1}$ converges weakly to $\Pi_{\tau,T}y$ in $X_{\tau,T}$ as $k\to+\infty$. Therefore, formulas (4.31)–(4.33) yield

$$\limsup_{k\to\infty}\int_\tau^T\int_\Omega d_{n_k}(x,t)\cdot(y_{n_k}-y)(x,t)dxdt \le 0. \qquad (4.35)$$

Let us apply Fatou's lemma to the sequence $\{\psi_k\}_{k\ge1}$ of Lebesgue integrable non-negative (see formula (4.26)) functions

$$\psi_k(x,t) := d_{n_k}(x,t)\cdot(y_{n_k}-y)(x,t) - \alpha'\sum_{i=1}^M\left|y_{n_k}^{(i)}(x,t)\right|^{p_i}$$
$$+\beta'\sum_{i=1}^M\left|y^{(i)}(x,t)\right|^{p_i} + c'_{n,k}(x,t), \quad (x,t)\in\Omega\times\mathbb{R}_+,\ k\ge1.$$

We obtain

$$-\alpha'\int_\tau^T\int_\Omega\sum_{i=1}^M\left|y^{(i)}(x,t)\right|^{p_i}dxdt + \beta'\int_\tau^T\int_\Omega\sum_{i=1}^M\left|y^{(i)}(x,t)\right|^{p_i}dxdt + \int_\tau^T\int_\Omega\bar{c}'(x,t)dxdt$$
$$=\int_\tau^T\int_\Omega\liminf_{k\to\infty}\left\{d_{n_k}(x,t)\cdot(y_{n_k}-y)(x,t)-\alpha'\sum_{i=1}^M\left|y_{n_k}^{(i)}(x,t)\right|^{p_i}+\beta'\sum_{i=1}^M\left|y^{(i)}(x,t)\right|^{p_i}+\bar{c}'_{n_k}(x,t)\right\}dxdt$$
$$\le\liminf_{k\to\infty}\left\{\int_\tau^T\int_\Omega d_{n_k}(x,t)\cdot(y_{n_k}-y)(x,t)dxdt-\alpha'\int_\tau^T\int_\Omega\sum_{i=1}^M\left|y_{n_k}^{(i)}(x,t)\right|^{p_i}dxdt\right.$$
$$\left.+\beta'\int_\tau^T\int_\Omega\sum_{i=1}^M\left|y^{(i)}(x,t)\right|^{p_i}dxdt+\int_\tau^T\int_\Omega c'_{n_k}(x,t)dxdt\right\}$$
$$\le-\alpha'\limsup_{k\to\infty}\int_\tau^T\int_\Omega\sum_{i=1}^M\left|y_{n_k}^{(i)}(x,t)\right|^{p_i}dxdt+\beta'\int_\tau^T\int_\Omega\sum_{i=1}^M\left|y^{(i)}(x,t)\right|^{p_i}dxdt+\int_\tau^T\int_\Omega\bar{c}'(x,t)dxdt,$$

where the equality follows from the first and third statements of (4.27) and formula (4.29); the first inequality follows from Fatou's lemma applied to the sequence $\{\psi_k\}_{k\ge1}$; the second inequality follows from (4.35) and the last statement of (4.27). Therefore, due to inequality (4.34), we have

$$\int_\tau^T\int_\Omega\sum_{i=1}^M\left|y_{n_k}^{(i)}(x,t)\right|^{p_i}dxdt\to\int_\tau^T\int_\Omega\sum_{i=1}^M\left|y^{(i)}(x,t)\right|^{p_i}dxdt,\ \text{as } k\to\infty. \qquad (4.36)$$

Moreover,

$$\liminf_{k\to\infty}\int_\tau^T\int_\Omega d_{n_k}(x,t)\cdot(y_{n_k}-y)(x,t)dxdt\geq 0. \tag{4.37}$$

Inequalities (4.35) and (4.37) provide

$$\int_\tau^T\int_\Omega d_{n_k}(x,t)\cdot(y_{n_k}-y)(x,t)dxdt\to 0,\ \text{as } k\to\infty. \tag{4.38}$$

Passing to the limit as $k\to+\infty$ in formula (4.31), taking into account (4.32) and (4.38), we obtain

$$\int_\tau^T\int_\Omega a\nabla y_{n_k}(x,t)\cdot\nabla y_{n_k}(x,t)dxdt\to\int_\tau^T\int_\Omega a\nabla y(x,t)\cdot\nabla y(x,t)dxdt,\ \text{as } k\to\infty. \tag{4.39}$$

Statement (4.30) holds, because the sequence $\{\Pi_{\tau,T}y_{n_k}\}_{k\geq 1}$ converges weakly to $\Pi_{\tau,T}y$ in the uniformly convex (superreflexive) Banach space $X_{\tau,T}=L_2(\tau,T;V)\cap\mathbf{L_p}(\tau,T;\mathbf{L_p}(\Omega))$ as $k\to+\infty$ (see the first statement of (4.16)), and statements (4.36) and (4.39) hold. We note that the mapping

$$z\to\sqrt{\int_\tau^T\int_\Omega a\nabla z(x,t)\cdot\nabla z(x,t)dxdt}$$

defines a norm, that is equivalent to the natural one, defined on Hilbert space $L_2(\tau,T;V)$. Statement (4.30) is proved.

To finish the proof of the theorem we provide that there exist a subsequence $\{y_{k_m}\}_{m\geq 1}\subseteq\{y_{n_k}\}_{k\geq 1}$ such that $\|\Pi_{\tau,T}\frac{\partial}{\partial t}y_{k_m}-\Pi_{\tau,T}\frac{\partial}{\partial t}y\|_{L_2(\tau,T;V^*)+\mathbf{L_q}(\tau,T;\mathbf{L_q}(\Omega))}\to 0$, as $m\to+\infty$, for any finite time interval $[\tau,T]\subset(0,+\infty)$. Since $\triangle y_{n_k}\to\triangle y$ in $L_2^{\mathrm{loc}}(\mathbb{R}_+;V^*)$, as $k\to+\infty$ (see formula (4.30)), it is sufficient to prove that for any finite time interval $[\tau,T]\subset(0,+\infty)$ the sequence $\{\Pi_{\tau,T}d_{n_k}\}_{k\geq 1}$ is precompact in $\mathbf{L_q}(\tau,T;\mathbf{L_q}(\Omega))$ (see Problem (4.1)).

On the contrary assume that $\{\Pi_{\tau,T}d_{n_k}\}_{k\geq 1}$ is not precompact in $\mathbf{L_q}(\tau,T;\mathbf{L_q}(\Omega))$ for some finite time interval $[\tau,T]\subset(0,+\infty)$. Therefore, there exist a subsequence of $\{\Pi_{\tau,T}d_{n_k}\}_{k\geq 1}$ (we denote it again by $\{\Pi_{\tau,T}d_{n_k}\}_{k\geq 1}$), a finite time interval $[\tau,T]\subset(0,+\infty)$, and $\varepsilon^*>0$ such that

$$\|\Pi_{\tau,T}d_{n_k}-\Pi_{\tau,T}d_{n_m}\|_{\mathbf{L_q}(\tau,T;\mathbf{L_q}(\Omega))}\geq\varepsilon^*,\ \text{for any } k,m\geq 1. \tag{4.40}$$

The last statement of (4.27), statements (4.28) and (4.30), and dominated convergence theorem yield

$$\lim_{K\to+\infty}\sup_{k\geq 1}\int_\tau^T\int_\Omega\sum_{i=1}^M\left|d_{n_k}^{(i)}(x,t)\right|^{q_i}\chi_{A_{k,K}}(x,t)dxdt$$
$$\leq \lim_{K\to+\infty}\sup_{k\geq 1}\int_\tau^T\int_\Omega\left[c_{1,n_k}(x,t)+c_2\sum_{i=1}^M\left|y_{n_k}^{(i)}(x,t)\right|^{p_i}\right]\chi_{A_{k,K}}(x,t)dxdt=0,$$

where $A_{k,K}:=\{(x,t)\in\Omega\times(\tau,T):\|y_{n_k}(x,t)\|_{\mathbb{R}^M}\geq K\}$, $K>0$, $k\geq 1$. Therefore, without loss of generality, we may additionally assume that there exists a rather large $K>0$ such that

$$\|y_{n_k}(x,t)\|_{R^M}\leq K \text{ for a.e. } (x,t)\in\Omega\times(\tau,T), \text{ and any } k\geq 1. \tag{4.41}$$

Krasnosel'skii [17, Chap. 1] (see book and references therein) implies that for any $k=1,2,\ldots,$ there exists a simple function $z_k:\Omega\times(\tau,T)\to\mathbb{R}^M$,

$$z_k(x,t)=\sum_{j=1}^{N_k}b_{j,k}\chi_{B_{j,k}}(x,t) \text{ for a.e. } (x,t)\in\Omega\times(\tau,T),$$

where $N_k\geq 1$, $\{b_{j,k}\}_{j=1}^{N_k}\subset\mathbb{R}^M$, $\{B_{j,k}\}_{k=1}^{N_k}\subset\Omega\times(\tau,T)$ be a family of disjoint measurable sets, such that $\|z_k(x,t)\|_{\mathbb{R}^M}\leq K$, for a.e. $(x,t)\in\Omega\times(\tau,T)$, and

$$\|\Pi_{\tau,T}y_{n_k}-z_k\|_{\mathbf{L_p}(\tau,T;\mathbf{L_p}(\Omega))}+\int_\tau^T\int_\Omega\sum_{i=1}^M\left|d_{n_k}^{(i)}(x,t)-f^{(i)}(x,t+\tau_{n_k},z_k(x,t))\right|^{q_i}dxdt\leq\frac{1}{k}. \tag{4.42}$$

For any $h\in(0,h_0)$, where $h_0=h_0(K)$ be a positive constant from Assumption V, let us define the mapping $F_h:\Omega\times\mathbb{R}_+\times\mathbb{R}^M\to\mathbb{R}^M$,

$$F_h(x,t,u):=\frac{1}{h}\int_t^{t+h}f(x,s,u)ds,\quad (x,t,u)\in\Omega\times\mathbb{R}_+\times\mathbb{R}^M.$$

Assumption V(iii) yields that

$$\int_\tau^T\int_\Omega\sum_{i=1}^M\left|f^{(i)}(x,t+\tau_{n_k},z_k(x,t))-F_h^{(i)}(x,t+\tau_{n_k},z_k(x,t))\right|dxdt\leq\psi(|h|,K)(T-\tau)N_k,$$

for any $k\geq 1$ and $h\in(0,h_0)$. Since, $\psi(s,K)\to 0+$ as $s\to 0+$, for each $k\geq 1$ there exists $h_k\in(0,h_0)$ such that

$$\psi(|h_k|,K)(T-\tau)N_k\leq\frac{1}{k}.$$

Thus,

$$\int_{\tau}^{T}\int_{\Omega}\sum_{i=1}^{M}\left|f^{(i)}(x,t+\tau_{n_k},z_k(x,t))-F_{h_k}^{(i)}(x,t+\tau_{n_k},z_k(x,t))\right|dxdt\le\frac{1}{k},\quad(4.43)$$

for any $k\ge 1$.

Assumptions V(ii) and V(iii) and Arzelà–Ascoli theorem (see Warga [26, Chap. I] and references therein) provide the existence of a mapping $G\in C([\tau,T]\times\bar{B}_K;(L_1(\Omega))^M)$, where $\bar{B}_K:=\{u\in\mathbb{R}^M : \|u\|_{\mathbb{R}^M}\le K\}$, and a subsequence $\{(t,u)\to F_{h_{k_m}}(\cdot,t+\tau_{k_m},u)\}_{m\ge1}\subseteq\{(t,u)\to F_{h_k}(\cdot,t+\tau_{n_k},u)\}_{k\ge1}$ such that

$$\sup_{t\in[\tau,T],\ \|u\|_{\mathbb{R}^M}\le K}\int_{\Omega}\sum_{i=1}^{M}\left|F_{h_{k_m}}^{(i)}(x,t+\tau_{k_m},u)-G^{(i)}(x,t,u)\right|dx\le\frac{1}{mN_{k_m}},\ \text{for any } m\ge1.$$

Therefore,

$$\int_{\tau}^{T}\int_{\Omega}\sum_{i=1}^{M}\left|F_{h_{k_m}}^{(i)}(x,t+\tau_{k_m},z_{k_m}(x,t))-G^{(i)}(x,t,z_{k_m}(x,t))\right|dxdt\le\frac{T-\tau}{m},\ \text{for any } m\ge1.\quad(4.44)$$

Since $G\in C([\tau,T]\times\bar{B}_K;(L_1(\Omega))^M)$, Heine–Cantor theorem provides the existence of nondecreasing function $\Psi:\mathbb{R}_+\to\mathbb{R}_+$, $\Psi(s)\to0+$, as $s\to0+$, such that

$$\int_{\Omega}\sum_{i=1}^{M}\left|G^{(i)}(x,t_1,u_1)-G^{(i)}(x,t_2,u_2)\right|dx\le\Psi(|t_1-t_2|+\|u_1-u_2\|_{\mathbb{R}^M}),$$

for any $t_1,t_2\in[\tau,T]$ and $u_1,u_2\in\bar{B}_K$. Therefore,

$$\begin{aligned}&C\cdot\text{meas}\left\{(x,t)\in\Omega\times(\tau,T) : \sum_{i=1}^{M}\left|G^{(i)}(x,t,u_1)-G^{(i)}(x,t,u_2)\right|\ge C\right\}\\&\le\int_{\tau}^{T}\int_{\Omega}\sum_{i=1}^{M}\left|G^{(i)}(x,t,u_1)-G^{(i)}(x,t,u_2)\right|dxdt\le(T-\tau)\Psi(\|u_1-u_2\|_{\mathbb{R}^M})\end{aligned}\quad(4.45)$$

for any $C>0$ and $u_1,u_2\in\bar{B}_K$, where meas$(\cdot)$ is a standard Lebeasgue measure on $\Omega\times(\tau,T)$.

We note that the sequence of mappings $\{(x,t)\to G(x,t,z_{k_m}(x,t))\}_{m\ge1}$, defined on $\Omega\times(\tau,T)$, converges in measure towards the mapping $(x,t)\to G(x,t,y(x,t))$ as $m\to+\infty$, i.e. for any $C>0$ and $\varepsilon>0$ there exists $\bar{M}\ge1$ such that meas $(\mathscr{A}_{m,C}) \le \varepsilon$, for each $m \ge \bar{M}$, where

$$\mathscr{A}_{m,C} := \left\{ (x,t) \in \Omega \times (\tau, T) \ : \ \sum_{i=1}^{M} \left| G^{(i)}(x,t,z_{k_m}(x,t)) - G^{(i)}(x,t,y(x,t)) \right| \ge C \right\}.$$

Indeed, there exists $\delta > 0$ such that $\Psi(\delta)(T-\tau) \le \frac{C\varepsilon}{2}$. Therefore, meas$(\mathscr{A}_{m,C} \setminus \{(x,t) \in \Omega \times (\tau,T) \ : \ \|z_{k_m}(x,t) - y(x,t)\|_{\mathbb{R}^M} \ge \delta\}) \le \frac{\varepsilon}{2}$ for any $m \ge 1$; see formula (4.45). Since $z_{k_m}(x,t) \to y(x,t)$ as $m \to +\infty$ for a.e. $(x,t) \in \Omega \times (\tau,T)$ (see (4.42) and (4.27)), we obtain the necessary statement.

Since the sequence of mappings $\{(x,t) \to G(x,t,z_{k_m}(x,t))\}_{m\ge 1}$, defined on $\Omega \times (\tau,T)$, converges in measure towards the mapping $(x,t) \to G(x,t,y(x,t))$ as $m \to +\infty$, inequalities (4.42)–(4.44) yield that the sequence $\{\Pi_{\tau,T} d_{k_m}\}_{m\ge 1}$ converges in measure towards the mapping $(x,t) \to G(x,t,y(x,t))$ as $m \to +\infty$. Thus, formulas (4.27), (4.28), (4.41) and dominated convergence theorem yield that

$$\lim_{m\to+\infty} \int_{\tau}^{T} \int_{\Omega} \sum_{i=1}^{M} \left| d_{k_m}^{(i)}(x,t) - G^{(i)}(x,t,y(x,t)) \right|^{q_i} dxdt = 0.$$

This is a contradiction with (4.40).

4.6 Examples of Applications

As applications we may consider the following examples: FitzHugh-Nagumo system (signal transmission across axons), complex Ginzburg–Landau equation (theory of superconductivity), Lotka–Volterra system with diffusion (ecology models), Belousov-Zhabotinsky system (chemical dynamics) and many other reaction-diffusion type systems (see Smoller [22]), whose dynamics are well studied in autonomous case (see Temam [23], Chepyzhov and Vishik [7]) and in non-autonomous case, when all coefficients are uniformly continuous on time variable (see Chepyzhov and Vishik [7], Zgurovsky et al. [28] and references therein). Now results of Theorems 4.1 and 4.2 allow us to study these systems with Carathéodory's nonlinearities.

4.6.1 Non-autonomous Complex Ginzburg–Landau Equation

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with smooth boundary $\partial\Omega$. Consider the non-autonomous complex Ginzburg–Landau equation:

$$\begin{cases} \frac{\partial u}{\partial t} = (1+\eta i)\,\Delta u + R(t)\,u - (1+i\beta(t))\,|u|^2 u + g(x,t), \\ u\,|_{\partial\Omega} = 0,\ u(x,\tau) = u_\tau(x), \end{cases} \tag{4.46}$$

where $u = u(x,t) = u^1(x,t) + iu^2(x,t)$, $(x,t) \in \Omega \times \mathbb{R}_+$; $g(t) = g^1(t) + g^2(t)\,i \in L_2(\Omega;\mathbb{C})$ for a.e. $t > 0$; $\eta, \beta(t) \in \mathbb{R}$; and $R(t) > 0$ for a.e. $t > 0$. We assume that $g^i \in L_2^{\text{loc}}(\mathbb{R}_+; L_2(\Omega))$ with $\sup_{t>0} \int_t^{t+1} \|g^i(\cdot,s)\|^2_{L_2(\Omega)} ds < +\infty$ and also that the functions $R(t)$ and $\beta(t)$ are measurable and essentially bounded.

For $v = \left(u^1, u^2\right)$, $u = u^1 + iu^2$, Eq. (4.46) can be writen as the system

$$\frac{\partial v}{\partial t} = \begin{pmatrix} 1 & -\eta \\ \eta & 1 \end{pmatrix} \Delta v + \begin{pmatrix} R(t)\,u^1 - \left(\left|u^1\right|^2 + \left|u^2\right|^2\right)\left(u^1 - \beta(t)\,u^2\right) \\ R(t)\,u_2 - \left(\left|u^1\right|^2 + \left|u^2\right|^2\right)\left(\beta(t)\,u^1 + u^2\right) \end{pmatrix} + \begin{pmatrix} g^1(t,x) \\ g^2(t,x) \end{pmatrix}$$

and Assumptions I–IV hold with $\mathbf{p} = (4,4)$. Indeed, since

$$f(t,v) = \left(-R(t)\,u^1 + |v|^2\left(u^1 - \beta(t)\,u_2\right), -R(t)\,u_2 + |v|^2\left(\beta(t)\,u^1 + u_2\right)\right),$$

then the Young's inequality yields that

$$\left|f^1(t,v)\right|^{\frac{4}{3}} + \left|f^2(t,v)\right|^{\frac{4}{3}} \le K_1 \left(|R(t)|^{\frac{4}{3}} \left(\left|u^1\right|^{\frac{4}{3}} + \left|u^2\right|^{\frac{4}{3}}\right)\right.$$

$$\left. + |v|^{\frac{8}{3}} \left(1 + |\beta(t)|^{\frac{4}{3}}\right) \left(\left|u^1\right|^{\frac{4}{3}} + \left|u^2\right|^{\frac{4}{3}}\right)\right) \le K_2 \left(\left|u^1\right|^4 + \left|u^2\right|^4\right) + K_3,$$

because $R(t)$, $\beta(t)$ are essentially bounded in $\mathbb{R}$. Moreover,

$$(f(t,v), v) = -R(t)\,|v|^2 + |v|^4 \ge \frac{|v|^4}{2} - K_4 \ge \frac{\left|u^1\right|^4 + \left|u^2\right|^4}{2} - K_4.$$

Hence, all statements of Theorem 4.1 hold. Furthermore, if the functions $R(t)$ and $\beta(t)$ satisfy

$$|\beta(t) - \beta(s)| \le a(|t-s|),\ |R(t) - R(s)| \le b(|t-s|), \tag{4.47}$$

for all $t, s \in \mathbb{R}$, where $a(l) \to 0$, $b(l) \to 0$, as $l \to 0^+$, then, additionally, Assumption V holds and, thus, all statements of Theorem 4.2 hold.

4.6.2 Non-autonomous Lotka–Volterra System with Diffusion

Let D_i be positive constants, $\Omega \subset \mathbb{R}^3$ be an open bounded subset with sufficiently smooth boundary $\partial\Omega$, and $a_i(t)$, $a_{ij}(t)$ be positive measurable and bounded functions on $\mathbb{R}_+$. Consider the Lotka–Volterra system with diffusion:

$$\begin{cases} \frac{\partial u^1}{\partial t} = D_1 \Delta u^1 + u^1 \left(a_1(t) - u^1 - a_{12}(t) u^2 - a_{13}(t) u^3\right), \\ \frac{\partial u^2}{\partial t} = D_2 \Delta u^2 + u^2 \left(a_2(t) - u^2 - a_{21}(t) u^1 - a_{23}(t) u^3\right), \\ \frac{\partial u^3}{\partial t} = D_3 \Delta u^3 + u^3 \left(a_3(t) - u^3 - a_{31}(t) u^1 - a_{32}(t) u^2\right), \end{cases} \tag{4.48}$$

with Neumann boundary conditions $\frac{\partial u^1}{\partial \nu} |_{\partial\Omega} = \frac{\partial u^2}{\partial \nu} |_{\partial\Omega} = \frac{\partial u^3}{\partial \nu} |_{\partial\Omega} = 0$, where $u^i = u^i(x,t) \geq 0$. In this case the function f is given by

$$f(t,u) = \begin{pmatrix} -u^1 \left(a_1(t) - u^1 - a_{12}(t) u^2 - a_{13}(t) u^3\right) \\ -u^2 \left(a_2(t) - u^2 - a_{21}(t) u^1 - a_{23}(t) u^3\right) \\ -u^3 \left(a_3(t) - u^3 - a_{31}(t) u^1 - a_{32}(t) u^2\right) \end{pmatrix}.$$

Then Assumptions I–IV hold for $u \in \mathbb{R}^3_+$ with $\mathbf{p} := (3,3,3)$. Indeed, since $u^i \geq 0$, then the Young's inequality implies that

$$\begin{aligned} (f(t,u), u) &\geq \left(u^1\right)^3 + \left(u^2\right)^3 + \left(u^3\right)^3 - a_1(t)\left(u^1\right)^2 - a_2(t)\left(u^2\right)^2 - a_3(t)\left(u^3\right)^2 \\ &\geq \tfrac{1}{2}\left(\left(u^1\right)^3 + \left(u^2\right)^3 + \left(u^3\right)^3\right) - K_1, \end{aligned}$$

and

$$\sum_{i=1}^{3} |f_i(t,u)|^{\frac{3}{2}} \leq K_2 \left(\left(u^1\right)^3 + \left(u^2\right)^3 + \left(u^3\right)^3 + \left(u^1\right)^{\frac{3}{2}} + \left(u^2\right)^{\frac{3}{2}} + \left(u^3\right)^{\frac{3}{2}}\right.$$

$$\left. + \left(u^1 u^2\right)^{\frac{3}{2}} + \left(u^2 u^3\right)^{\frac{3}{2}} + \left(u^1 u^3\right)^{\frac{3}{2}}\right) \leq K_3 \left(\left(u^1\right)^3 + \left(u^2\right)^3 + \left(u^3\right)^3\right) + K_4.$$

Hence, all statements of Theorem 4.1 hold. Furthermore, if the functions $a_i(t)$ and $a_{ij}(t)$ satisfy (4.47), then, additionally, Assumption V holds and, thus, all statements of Theorem 4.2 hold.

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Chapter 5
Strongest Convergence Results for Weak Solutions of Feedback Control Problems

Abstract In this chapter we establish strongest convergence results for weak solutions of feedback control problems. In Sect. 5.1 we set the problem. Section 5.2 devoted to the regularity of all weak solutions and their additional properties. In Sect. 5.3 we consider convergence of weak solutions results in the strongest topologies. As examples of applications we consider a model of combustion in porous media; a model of conduction of electrical impulses in nerve axons; and a climate energy balance model.

5.1 Setting of the Problem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be bounded and open subset with a smooth boundary $\partial\Omega$, $\underline{f}$, $\overline{f} : \mathbb{R} \to \mathbb{R}$ are some real functions. We consider the semilinear reaction-diffusion inclusion

$$u_t - \triangle u + [\underline{f}(u), \overline{f}(u)] \ni 0 \text{ in } \Omega \times (\tau, T), \quad (-\infty < \tau < T < +\infty), \tag{5.1}$$

with boundary condition

$$u\big|_{\partial\Omega} = 0, \tag{5.2}$$

where $[a, b] = \{\alpha a + (1 - \alpha)b \mid \alpha \in [0, 1]\}, a, b \in \mathbb{R}$. We suppose that $f = [\underline{f}, \overline{f}] : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ satisfies the growth condition

$$\exists c_0 > 0 : \quad -c_0(1 + |u|) \leq \underline{f}(u) \leq \overline{f}(u) \leq c_0(1 + |u|) \quad \forall u \in \mathbb{R}, \tag{5.3}$$

and the sign condition

$$\underline{\lim}_{u \to +\infty} \frac{\underline{f}(u)}{u} > -\lambda_1; \quad \underline{\lim}_{u \to -\infty} \frac{\overline{f}(u)}{u} > -\lambda_1, \tag{5.4}$$

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where λ_1 is the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$. Suppose also that $\underline{f}$ is lower semi-continuous, and $\overline{f}$ is upper semi-continuous (see Sect. 2.1).

We shall use the following standard notations: $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, V' is the dual space of V and $\langle \cdot, \cdot \rangle$ denotes the pairing in the space V.

5.2 Regularity of All Weak Solutions and Their Additional Properties

Further by $\|\cdot\|_E$ we denote the norm in a real Banach space E. Assumption (5.4) is equivalent to the next one

$$\exists \lambda \in (0, \lambda_1),\ \exists c_1 > 0 :\ f(u) \cdot u \geq -\lambda u^2 - c_1\ \forall u \in \mathbb{R}. \tag{5.5}$$

Sign condition (5.5), the variational characterization of λ_1, and Gronwall-Bellman inequality imply that for any $\tau < T$ and for any weak solution $u(\cdot)$ of Problems (5.1) and (5.2) on $[\tau, T]$ we have

$$\|u(t)\|_H^2 \leq \|u(s)\|_H^2 e^{-2\varepsilon^*(t-s)} + \frac{c_2}{\varepsilon^*} \quad \forall \tau \leq s \leq t \leq T, \tag{5.6}$$

where $\varepsilon^* = \lambda_1 - \lambda$ and $c_2 = c_1 \cdot \text{meas}(\Omega)$ (cf. [56, p. 56]).

We note that the mapping $v \to \|\triangle v\|_H$ defines an equivalent norm on $V \cap H^2(\Omega)$ (cf. [42, Chapter III]). The next theorem provides additional a priory estimates for all weak solutions of Problems (5.1) and (5.2).

Theorem 5.1 *There exists $C > 0$ such that for any $\tau < T$ each weak solution $u(\cdot)$ of Problems (5.1) and (5.2) on $[\tau, T]$ belongs to $C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; D(A))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau + \varepsilon, T; H)$ for each $\varepsilon \in (0, T - \tau)$. Moreover, the following inequality holds*

$$(t - \tau)\|u(t)\|_V^2 + \int_\tau^t (s - \tau)\|u(s)\|_{H^2(\Omega)\cap V}^2 ds \leq C(1 + \|u(\tau)\|_H^2 + (t - \tau)^2) \quad \forall t \in (\tau, T].$$

Proof Let $\tau < T$ and $u(\cdot)$ be an arbitrary weak solution of Problems (5.1) and (5.2) on $[\tau, T]$. We fix $\varepsilon \in (0, T - \tau)$. Theorem 2.1 implies that $u(\cdot) \in C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; H^2(\Omega) \cap V)$ and $u_t \in L^2(\tau + \varepsilon, T; H)$. Then $\|u(\cdot)\|_V^2$ and $\|u(\cdot)\|_H^2$ are absolutely continuous on $[\tau + \varepsilon, T]$ and for a.e. $s \in (\tau + \varepsilon, T)$ we have $\frac{d}{ds}\left[\frac{1}{2}\|u(s)\|_V^2\right] = (u'(s), -\triangle u(s))$ and $\frac{d}{ds}\left[\frac{1}{2}\|u(s)\|_H^2\right] = (u'(s), u(s))$ (cf. [18, Chapter IV]). Thus due to grows and sign assumptions (5.3) and (5.5) for a.e. $s \in (\tau + \varepsilon, T)$ in a standard way we obtain

$$
\begin{aligned}
&\frac{d}{ds}\left[(s-\tau-\varepsilon)\|u(s)\|_V^2+\frac{1}{2}\|u(s)\|_H^2\right]+(s-\tau-\varepsilon)\|u(s)\|^2_{H^2(\Omega)\cap V}\\
&\le \|u(s)\|_H^2\left(c_0^2+\frac{1}{2}+2c_0^2(s-\tau-\varepsilon)\right)+\left(c_0^2+2c_0^2(s-\tau-\varepsilon)\right)\mathrm{meas}(\Omega)\\
&\le \left(c_0^2+\frac{1}{2}+2c_0^2(s-\tau-\varepsilon)\right)\left(\|u(s)\|_H^2+\mathrm{meas}(\Omega)\right) \qquad (5.7)\\
&\le \left(c_0^2+\frac{1}{2}+2c_0^2(s-\tau-\varepsilon)\right)\left(\|u(\tau)\|_H^2 e^{-2\varepsilon^*(s-\tau)}+\frac{c_2}{\varepsilon^*}+\mathrm{meas}(\Omega)\right),
\end{aligned}
$$

where the last inequality follows from (5.6). We fix an arbitrary $t\in[\tau+\varepsilon,T]$. Integrating the inequality (5.7) from $\tau+\varepsilon$ to t, we have

$$
(t-\tau-\varepsilon)\|u(t)\|_V^2+\frac{1}{2}\|u(t)\|_H^2-\frac{1}{2}\|u(\tau+\varepsilon)\|_H^2+\int_{\tau+\varepsilon}^{t}(s-\tau-\varepsilon)\|u(s)\|^2_{H^2(\Omega)\cap V}ds
$$

$$
\le\left(\frac{1}{2}c_0^2+\frac{1}{4}+\left(\frac{3}{2}c_0^2+\frac{1}{4}\right)(t-\tau)^2\right)\times\left(\|u(\tau)\|_H^2+\frac{c_2}{\varepsilon^*}+\mathrm{meas}(\Omega)\right).
$$

Let $\varepsilon\searrow 0+$. Then $\forall t\in(\tau,T]$

$$
\|u(t)\|_V^2(t-\tau)+\int_{\tau}^{t}(s-\tau)\|u(s)\|^2_{H^2(\Omega)\cap V}ds\le C((t-\tau)^2+\|u(\tau)\|_H^2+1),
$$

where $C>0$ is a constant that does not depend on $\tau,\ T,\ \varepsilon$, and $u(\cdot)$.

The theorem is proved.

5.3 Convergence of Weak Solutions in the Strongest Topologies

For each $u_\tau\in H$ we set $\mathscr{D}_{\tau,T}(u_\tau)=\{u(\cdot)\in L^2(\tau,T;V)\big|\ u(\cdot)$ is a weak solution of Problems (5.1) and (5.2) and $u(\tau)=u_\tau\}$. We note that the existence of a weak solution for this problem was considered in [56] (see also Sect. 1.1).

The compactness in V of global attractor and compactness in $L^2_{loc}(\mathbb{R}_+;H^2(\Omega)\cap V)\cap C(\mathbb{R}_+;V)$ of trajectory attractor for Problems (5.1) and (5.2) with initial data from H is based on properties of the family of weak solutions of Problems (5.1) and (5.2), related to the asymptotic compactness of the generated m-semiflow of solutions and its absorbing (cf. [10, 30–32, 35, 44] and references therein). Theorem 5.2 below on dependence of weak solutions in V on initial data from H and Theorem 5.1 allow us to investigate the dynamics of all weak solutions of Problems (5.1) and (5.2) in V as $t\to+\infty$.

Theorem 5.2 *Let $\tau < T$ and $u_{\tau,n} \to u_\tau$ weakly in H, $u_n(\cdot) \in \mathscr{D}_{\tau,T}(u_{\tau,n})$ for any $n \geq 1$. Then there exist a subsequence $\{u_{n_k}(\cdot)\}_{k\geq 1} \subset \{u_n(\cdot)\}_{n\geq 1}$ and $u(\cdot) \in \mathscr{D}_{\tau,T}(u_\tau)$ such that*

$$\forall \varepsilon \in (0, T-\tau) \quad \sup_{t\in[\tau+\varepsilon,T]} \|u_{n_k}(t) - u(t)\|_V \to 0, \ k \to +\infty. \tag{5.8}$$

Proof Let $\tau < T$, $u_{\tau,n} \to u_\tau$ weakly in H, $u_n(\cdot) \in \mathscr{D}_{\tau,T}(u_\tau)$ $\forall n \geq 1$. Theorem 1 from [26] (cf. also [56, Theorem 2.1, p. 56]) implies the existence of a subsequence $\{u_{n_k}(\cdot)\}_{k\geq 1} \subseteq \{u_n(\cdot)\}_{n\geq 1}$ and $u(\cdot) \in \mathscr{D}_{\tau,T}(u_\tau)$ such that

$$\forall \varepsilon \in (0, T-\tau) \quad \sup_{t\in[\tau+\varepsilon,T]} \|u_{n_k}(t) - u(t)\|_H \to 0, \ k \to +\infty. \tag{5.9}$$

We fix an arbitrary $\varepsilon \in (0, T-\tau)$. Theorem 2.1 implies that the restrictions of $u_{n_k}(\cdot)$ and $u(\cdot)$ on $[\tau+\varepsilon, T]$ belong to $L^2(\tau+\varepsilon, T; H^2(\Omega)\cap V) \cap C([\tau+\varepsilon, T]; V)$. Moreover, $u_{n_k,t}(\cdot)$ and $u_t(\cdot)$ belong to $L^2(\tau+\varepsilon, T; H)$. Theorem 5.1 imply that $\{u_{n_k}(\cdot)\}_{k\geq 1}$ is bounded in $C([\tau+\varepsilon, T]; V) \cap L^2(\tau+\varepsilon, T; H^2(\Omega)\cap V)$. Moreover, $\{u_{n_k,t}(\cdot)\}_{k\geq 1}$ is bounded in $L^2(\tau+\varepsilon, T; H)$. Thus in virtue of (5.9) and of the compact and dense embedding $H^2(\Omega)\cap V \subset V \subset H \subset V^*$, we have

$$\begin{aligned} &u_{n_k}(\cdot) \to u(\cdot) \text{ weakly in } L^2(\tau+\varepsilon, T; H^2(\Omega)\cap V),\\ &u_{n_k,t}(\cdot) \to u_t(\cdot) \text{ weakly in } L^2(\tau+\varepsilon, T; H), \ k \to +\infty. \end{aligned} \tag{5.10}$$

Moreover,

$$u_{n_k}(\cdot) \to u(\cdot) \text{ in } C([\tau+\varepsilon, T]; V_w), \ k \to +\infty. \tag{5.11}$$

Without loss of generality, in virtue of the compact embedding theorem (cf. [33, Section 5.1]), the next convergences hold

$$\begin{aligned} &u_{n_k}(t) \to u(t) \text{ in } V \text{ for a.e. } t \in (\tau+\varepsilon, T),\\ &u_{n_k}(\cdot) \to u(\cdot) \text{ in } L^2(\tau+\varepsilon, T; V), \ k \to +\infty. \end{aligned} \tag{5.12}$$

We consider the dense subset of $[\tau, T]$:

$$\mathscr{D} := \{t \in [\tau, T] \,\big|\, u_{n_k}(t) \to u(t) \text{ in } V, k \to +\infty\}.$$

Let us fix an arbitrary $\varepsilon > 0$ such that $\tau + \varepsilon \in \mathscr{D}$. Then

$$\sup_{t\in[\tau+\varepsilon,T]} \|u_{n_k}(t) - u(t)\|_V = \|u_{n_k}(t_{n_k}) - u(t_{n_k})\|_V, \tag{5.13}$$

where $t_{n_k} \in [\tau+\varepsilon, T]$ for any $k \geq 1$.

Let us show that

$$\|u_{n_k}(t_{n_k}) - u(t_{n_k})\|_V \to 0, \ k \to +\infty. \tag{5.14}$$

We prove this statement by contradiction. If (5.14) does not hold, then without loss of generality we assume that for some $t_0 \in [\tau + \varepsilon, T]$

$$t_{n_k} \to t_0, \ \ k \to +\infty, \tag{5.15}$$

and there exists $\delta^* > 0$ such that

$$\|u_{n_k}(t_{n_k}) - u(t_{n_k})\|_V \geq \delta^* \ \forall k \geq 1. \tag{5.16}$$

As $u(\cdot) \in C([\tau + \varepsilon, T]; V)$ then (5.15) and (5.16) imply

$$\|u_{n_k}(t_{n_k}) - u(t_0)\|_V \geq \delta^* \text{ for } k \text{ rather large.} \tag{5.17}$$

On the other hand from (5.12) we get

$$\mathscr{V}(u_{n_k})(s) \to \mathscr{V}(u)(s) \text{ for each } s \in (\tau + \varepsilon, T) \cap \mathscr{D}, \ \ k \to +\infty, \tag{5.18}$$

where for any weak solution $z(\cdot)$ of Problems (5.1) and (5.2) on $[\tau, T]$ and any $s \in [\tau + \varepsilon, T]$

$$\mathscr{V}(z)(s) = \|z(s)\|_V^2 - 2c_0^2 \text{meas}(\Omega)s - 2c_0^2 \int_\tau^s \|u(\xi)\|_H^2 d\xi.$$

We note that

$$\mathscr{V}(u_{n_k})(t) \leq \mathscr{V}(u_{n_k})(s) \quad \forall \tau \leq s \leq t \leq T, \ \forall k \geq 1. \tag{5.19}$$

Let us prove the inequality

$$\overline{\lim_{k\to+\infty}} \mathscr{V}(u_{n_k})(t_{n_k}) \leq \mathscr{V}(u)(t_0). \tag{5.20}$$

We need to consider two cases.

Case 1: $t_0 > \tau + \varepsilon$. Let us fix an arbitrary $\delta > 0$. As $u(\cdot) \in C([\tau + \varepsilon, T]; V)$, then the density of $\mathscr{D}$ in $[\tau, T]$ implies the existence of $\bar{s} \in [\tau + \varepsilon, t_0) \cap \mathscr{D}$ such that

$$\mathscr{V}(u)(\bar{s}) - \mathscr{V}(u)(t_0) < \delta.$$

In virtue of (5.15)–(5.19) for any $\delta > 0$ we obtain

$$\overline{\lim_{k\to+\infty}} \mathscr{V}(u_{n_k})(t_{n_k}) - \mathscr{V}(u)(t_0) \leq \lim_{k\to+\infty} \mathscr{V}(u_{n_k})(\bar{s}) - \mathscr{V}(u)(t_0) = \mathscr{V}(u)(\bar{s}) - \mathscr{V}(u)(t_0) < \delta.$$

Thus inequality (5.20) holds.

Case 2: $t_0 = \tau + \varepsilon$. As $\tau + \varepsilon \in \mathscr{D}$, then in virtue of (5.15)–(5.19) we obtain

$$\overline{\lim_{k\to+\infty}} \mathscr{V}(u_{n_k})(t_{n_k}) - \mathscr{V}(u)(t_0) \le \lim_{k\to+\infty} \mathscr{V}(u_{n_k})(t_0) - \mathscr{V}(u)(t_0) = 0.$$

Thus inequality (5.20) is true. Therefore, (5.9), (5.11) and (5.15) imply

$$\overline{\lim_{k\to+\infty}} \|u_{n_k}(t_{n_k})\|_V \le \|u(t_0)\|_V$$

that together with (5.11) provides

$$u_{n_k}(t_{n_k}) \to u(t_0) \text{ in } V,\ \ k \to +\infty,$$

which contradicts (5.17). Therefore (5.14) holds.

The theorem is proved.

5.4 Examples of Applications

In this section we provide examples of applications to theorems established in Sects. 5.1–5.3. We consider a model of combustion in porous media (sect. 5.4.1), a model of conduction of electrical impulses in nerve axons (sect. 5.4.2), a climate energy balance model (sect. 5.4.2); and a model of combustion in porous media (sect. 5.4.3).

5.4.1 A Model of Combustion in Porous Media

Let us consider the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - f(u) \in \lambda H(u-1), & (x,t) \in (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) = 0, \quad t \in \mathbb{R}, \end{cases} \tag{5.21}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous and nondecreasing function satisfying growth and sign assumptions, $\lambda > 0$, and $H(0) = [0,1]$, $H(s) = \mathbf{I}\{s > 0\}$, $s \ne 0$; Feireisl and Norbury [17] (see also sect. 2.4.5 and Fig. 5.1). For each $u_\tau \in L^2((0,\pi))$ we set $\mathscr{D}_{\tau,T}(u_\tau) = \{u(\cdot) \in L^2(\tau,T; H_0^1((0,\pi)))\big|\ u(\cdot)$ is a weak solution of Problem (5.21) and $u(\tau) = u_\tau\}$. Since Problem (5.21) is a particular case of Problems (5.1) and (5.2), then the following statement holds: if $\tau < T$ and $u_{\tau,n} \to u_\tau$ weakly in $L^2((0,\pi))$, $u_n(\cdot) \in \mathscr{D}_{\tau,T}(u_{\tau,n})$ for any $n \ge 1$, then there exist a subsequence $\{u_{n_k}(\cdot)\}_{k\ge1} \subset \{u_n(\cdot)\}_{n\ge1}$ and $u(\cdot) \in \mathscr{D}_{\tau,T}(u_\tau)$ such that

$$\forall \varepsilon \in (0, T-\tau) \ \sup_{t\in[\tau+\varepsilon,T]} \|u_{n_k}(t) - u(t)\|_{H_0^1((0,\pi))} \to 0,\ \ k \to +\infty.$$

Fig. 5.1 Porous media

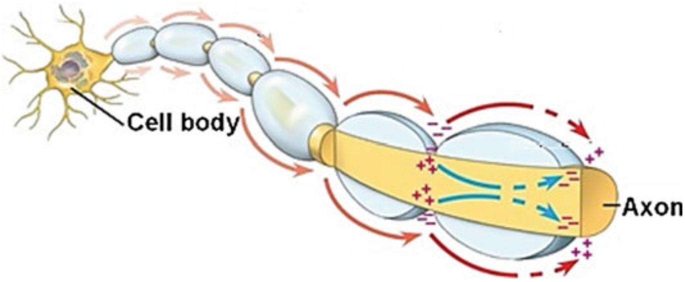

Fig. 5.2 Structure of the peripheral nerve

5.4.2 A Model of Conduction of Electrical Impulses in Nerve Axons

Consider the problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u \in \lambda H(u-a), & (x,t) \in (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) = 0, & t \in \mathbb{R}, \end{cases} \tag{5.22}$$

where $a \in \left(0, \frac{1}{2}\right)$; Terman [47, 48] (see also sect. 2.4.2 and Fig. 5.2). Since Problems (5.22) is a particular case of Problems (5.1) and (5.2), then then the following statement holds: if $\tau < T$ and $u_{\tau,n} \to u_\tau$ weakly in $L^2((0,\pi))$, $u_n(\cdot) \in \mathscr{D}_{\tau,T}(u_{\tau,n})$ for any $n \geq 1$, then there exist a subsequence $\{u_{n_k}(\cdot)\}_{k\geq 1} \subset \{u_n(\cdot)\}_{n\geq 1}$ and $u(\cdot) \in \mathscr{D}_{\tau,T}(u_\tau)$ such that

$$\forall \varepsilon \in (0, T-\tau) \sup_{t\in[\tau+\varepsilon,T]} \|u_{n_k}(t) - u(t)\|_{H_0^1((0,\pi))} \to 0,\ k \to +\infty.$$

5.4.3 *Climate Energy Balance Model*

Let $(\mathcal{M}, \mathbf{g})$ be a C^∞ compact connected oriented two-dimensional Riemannian manifold without boundary (as, e.g. $\mathcal{M} = S^2$ the unit sphere of $\mathbb{R}^3$). Consider the Budyko model (see also sect. 2.4.3 and Figs. 5.3 and 5.4):

$$\frac{\partial u}{\partial t} - \Delta u + Bu \in QS(x)\beta(u), \quad (x, t) \in \mathcal{M} \times \mathbb{R}, \tag{5.23}$$

where $\Delta u = \operatorname{div}_{\mathcal{M}} (\nabla_{\mathcal{M}} u)$; $\nabla_{\mathcal{M}}$ is understood in the sense of the Riemannian metric $\mathbf{g}$ (see sect. 2.4.3, Budyko [8] and Sellers [41]).

Let $S : \mathcal{M} \to \mathbb{R}$ be a function such that $S \in L^\infty(\mathcal{M})$ and there exist $S_0, S_1 > 0$ such that

$$0 < S_0 \leq S(x) \leq S_1.$$

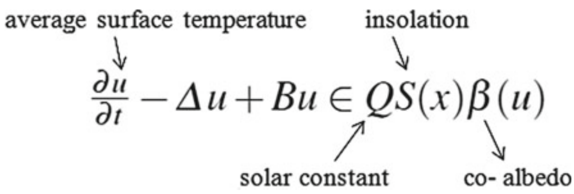

Fig. 5.3 Budyko model

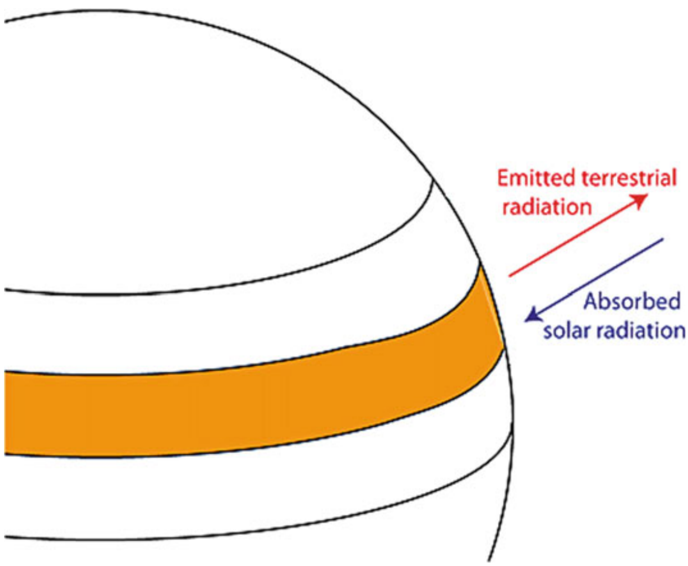

Fig. 5.4 Climate energy

Suppose also that β is a bounded maximal monotone graph of $\mathbb{R}^2$, that is there exist $m, M \in \mathbb{R}$ such that for all $s \in \mathbb{R}$ and $z \in \beta(s)$

$$m \leq z \leq M.$$

Let us consider real Hilbert spaces

$$H := L^2(\mathscr{M}), \quad V := \{u \in L^2(\mathscr{M}) : \nabla_{\mathscr{M}} u \in L^2(T\mathscr{M})\}$$

with respective standard norms $\|\cdot\|_H$, $\|\cdot\|_V$, and inner products $(\cdot,\cdot)_H$, $(\cdot,\cdot)_V$, where $T\mathscr{M}$ represents the tangent bundle and the functional spaces $L^2(\mathscr{M})$ and $L^2(T\mathscr{M})$ are defined in a standard way; see, for example, Aubin [2]. According to Theorem 2.2, for any $-\infty < \tau < T < +\infty$ each weak solution $u_\tau \in L^2(\Omega)$ of Problem (5.23) on $[\tau, T]$ belongs to $C([\tau+\varepsilon, T]; H_0^1(\Omega)) \cap L^2(\tau+\varepsilon, T; H^2(\Omega) \cap H_0^1((0,\pi)))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau+\varepsilon, T; L^2(\Omega))$ for each $\varepsilon \in (0, T-\tau)$. Consider the generalized setting of Problem (5.23):

$$\frac{du}{dt} + Au(t) + \partial J_1(u(t)) - \partial J_2(u(t)) \ni \bar{0} \text{ on } (-\infty < \tau < T < +\infty), \tag{5.24}$$

where $A : V \to V^*$ be a linear symmetric operator such that $\exists c > 0 : \langle Av, v\rangle_V \geq c\|v\|_V^2$, for each $v \in V$ and $J_i : H \to \mathbb{R}$ be a convex, lower semi-continuous function such, that the following assumptions hold: (i) (growth condition) there exists $c_1 > 0$ such that $\|y\|_H \leq c_1(1 + \|u\|_H)$, for each $u \in H$ and $y \in \partial J_i(u)$ and $i = 1, 2$; (ii) (sign condition) there exist $c_2 > 0$, $\lambda \in (0, c)$ such that $(y_1 - y_2, u)_H \geq -\lambda\|u\|_H^2 - c_2$, for each $y_i \in \partial J_i(u)$, $u \in H$, where $\partial J_i(u)$ the subdifferential of $J_i(\cdot)$ at a point u. Note that $u^* \in \partial J_i(u)$ if and only if $u^*(v - u) \leq J_i(v) - J_i(u)$ $\forall v \in H$; $i = 1, 2$. Let $D(A) = \{u \in V : Au \in H\}$. We note that the mapping $v \to \|Av\|_H$ defines the equivalent norm on $D(A)$; Temam [42, Chapter III].

We recall that the function $u(\cdot) \in L^2(\tau, T; V)$ is called a *weak solution* of Problem (5.24) on $[\tau, T]$, if there exist Bochner measurable functions $d_i : (\tau, T) \to H$; $i = 1, 2$, such that

$$d_i(t) \in \partial J_i(u(t)) \text{ for a.e. } t \in (\tau, T),\ \ i = 1, 2; \text{ and} \tag{5.25}$$

$$\int_\tau^T \big[-\langle u, v\rangle\, \xi'(t) + \langle Au, v\rangle\, \xi(t) + \langle d_1, v\rangle\, \xi(t) - \langle d_2, v\rangle\, \xi(t)\big]\, dt = 0, \tag{5.26}$$

for all $\xi \in C_0^\infty(\tau, T)$ and for all $v \in V$.

The following theorem provides sufficient conditions for the existence and regularity of all weak solutions for Problem (5.24).

Theorem 5.3 *Let $-\infty < \tau < T < +\infty$ and $u_\tau \in H$. Problem (5.24) has at least one weak solution $u(\cdot) \in L^2(\tau, T; V)$ on $[\tau, T]$ such that $u(\tau) = u_\tau$. Moreover, if $u(\cdot)$ is a weak solution of Problem (5.24) on $[\tau, T]$, then $u(\cdot) \in C([\tau+\varepsilon, T]; V) \cap L^2(\tau+\varepsilon, T; D(A))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau+\varepsilon, T; H)$ for any $\varepsilon \in (0, T-\tau)$.*

Proof We note that for any $u_\tau \in H$ there exists at least one weak solution of Problem (5.24) on $[\tau, T]$ with initial condition $u(\tau) = u_\tau$; see Kasyanov [26] and references therein. The regularity of each weak solution follows from Theorem 2.3.

The theorem is proved.

Denote by $\mathscr{K}_+$ the family of all, globally defined on $[0, +\infty)$, weak solutions of Problem (5.24). Let us set

$$E(u) = \frac{1}{2}\langle Au, u\rangle + J_1(u) - J_2(u), \quad u \in V. \tag{5.27}$$

Theorem 5.4 *For each $u \in \mathscr{K}_+$ and all τ and T, $0 < \tau < T < \infty$, the energy equality holds*

$$E(u(T)) - E(u(\tau)) = -\int_\tau^T \|\frac{du}{ds}(s)\|_H^2 ds. \tag{5.28}$$

Proof Suppose $u(\cdot) \in \mathscr{K}_+$ be arbitrary fixed and let $0 < \tau < T < +\infty$. To simplify conclusions, let the symbol $u(\cdot)$ denotes the restriction of $u(\cdot)$ on $[\tau, T]$. Theorem 5.3 implies that $u(\cdot) \in C([\tau, T]; V) \cap L^2(\tau, T; D(A))$ and $\frac{du}{dt}(\cdot) \in L^2$ $(\tau, T; H)$, because $\tau > 0$. Barbu [7, Lemma 2.1, p. 189] yields that the functions $J_i(u(\cdot))$, $i = 1, 2$, are absolutely continuous on $[\tau, T]$ and the equality holds:

$$\frac{d}{dt} J_i(u(t)) = \langle h_i(t), \frac{du}{dt}(t)\rangle_H, \text{ for a.e. } t \in (\tau, T), \tag{5.29}$$

for all $h_i(\cdot) \in L^2(\tau, T; H)$ such that $h_i(t) \in \partial J_i(s)|_{s=u(t)}$ for a.e. $t \in (\tau, T)$, i=1,2.

We remark that the mapping $t \to \langle Au(t), u(t)\rangle_V$ is absolutely continuous on $[\tau, T]$ and the equality holds:

$$\frac{d}{dt}\langle Au(t), u(t)\rangle = 2\langle Au(t), \frac{du}{dt}(t)\rangle_H, \text{ for a.e. } t \in (\tau, T) \tag{5.30}$$

Thus, the function $E(u(\cdot))$ is absolutely continuous on $[\tau, T]$ as the linear combination of absolutely continuous on $[\tau, T]$ functions. According to formulae (5.29) and (5.30), $\frac{d}{dt} E(u(t)) = -\|\frac{du}{dt}(t)\|_H^2$ for a.e. $t \in (\tau, T)$.

The theorem is proved.

Repeating several lines from the proof of Theorem 5.1 we obtain that there exists $C > 0$ such that for any $\tau < T$ and for each weak solution $u(\cdot)$ of Problem (5.24) on $[\tau, T]$ the inequality holds

$$(t-\tau)\|u(t)\|_V^2 + \int_\tau^t (s-\tau)\|u(s)\|_{D(A)}^2 ds \le C(1 + \|u(\tau)\|_H^2 + (t-\tau)^2), \tag{5.31}$$

for each $t \in (\tau, T]$.

Let

$$\mathscr{D}_{\tau,T}(u_\tau) = \{u(\cdot) \in L^2(\tau, T; V) \big| \; u(\cdot) \text{ is a weak solution of Problem (5.24) and } u(\tau) = u_\tau\},$$

for any $u_\tau \in H$. Let us provide the main convergence result for all weak solutions of Problem (5.24) in the strongest topologies.

Theorem 5.5 *Let $\tau < T$, $u_{\tau,n} \to u_\tau$ weakly in H, $u_n(\cdot) \in D_{\tau,T}(u_{\tau,n})$ for any $n \geq 1$. Then there exists a subsequence $\{u_{n_k}(\cdot)\}_{k\geq 1} \subseteq \{u_n(\cdot)\}_{n\geq 1}$ and $u(\cdot) \in D_{\tau,T}(u_\tau)$ such that*

$$\sup_{t\in[\tau+\varepsilon,T]} \|u_{n_k}(t) - u(t)\|_V \to 0, \tag{5.32}$$

$$\int_{\tau+\varepsilon}^{T} \|\frac{du_{n_k}}{dt}(t) - \frac{du}{dt}(t)\|_H^2 dt \to 0, \tag{5.33}$$

as $k \to +\infty$, for all $\varepsilon \in (0, T - \tau)$.

Proof The inequality (5.31), Kasyanov et al. [29, Theorem 3], Banach-Alaoglu theorem, and Cantor diagonal arguments (alternatively we may repeat several lines from the proof of Theorem 5.2) yield that there exist a subsequence $\{u_{n_k}(\cdot)\}_{k\geq 1} \subseteq \{u_n(\cdot)\}_{n\geq 1}$ and $u(\cdot) \in D_{\tau,T}(u_\tau)$ such that the following statements hold:

(a) the restrictions of $u_{n_k}(\cdot)$ and $u(\cdot)$ on $[\tau + \varepsilon, T]$ belong to $C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; D(A))$ and $\frac{du_{n_k}}{dt}(\cdot)$, $\frac{du}{dt}(\cdot) \in L^2(\tau + \varepsilon, T; H)$;

(b) the following convergence hold:

$$\begin{aligned} &u_{n_k}(\cdot) \to u(\cdot) \text{ weakly in } L^2(\tau + \varepsilon, T; D(A)),\\ &u_{n_k}(\cdot) \to u(\cdot) \text{ strongly in } C([\tau + \varepsilon, T]; V),\\ &\tfrac{du_{n_k}}{dt}(\cdot) \to \tfrac{du}{dt}(\cdot) \text{ weakly in } L^2(\tau + \varepsilon, T; H), \end{aligned} \tag{5.34}$$

as $k \to \infty$, for each $\varepsilon \in (0, T - \tau)$, that imply statement (5.32). Let us prove (5.33). Theorem 5.4 yields the following energy equalities

$$\int_{\tau+\varepsilon}^{T} \|\frac{du}{dt}(t)\|_H^2 dt = E(u(\tau + \varepsilon)) - E(u(T)), \tag{5.35}$$

$$\int_{\tau+\varepsilon}^{T} \|\frac{du_{n_k}}{dt}(t)\|_H^2 dt = E(u_{n_k}(\tau + \varepsilon)) - E(u_{n_k}(T)), \tag{5.36}$$

$k \geq 1$, $\varepsilon \in (0, T - \tau)$. Continuity of E on V and (5.32) imply

$$E(u_{n_k}(\tau + \varepsilon)) - E(u_{n_k}(T)) \to E(u(\tau + \varepsilon)) - E(u(T)), \; m \to \infty. \tag{5.37}$$

Therefore, formulae (5.35)–(5.37) yield

$$\int_{\tau+\varepsilon}^{T} \|\frac{du_{n_k}}{dt}(t)\|_H^2 dt \to \int_{\tau+\varepsilon}^{T} \|\frac{du}{dt}(t)\|_H^2 dt, \tag{5.38}$$

as $k \to \infty$, for each $\varepsilon \in (0, T-\tau)$. Since, $L^2(\tau+\varepsilon; T)$ is a Hilbert space, (5.34) and (5.38) imply (5.33).

The theorem is proved.

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Chapter 6
Strongest Convergence Results for Weak Solutions of Differential-Operator Equations and Inclusions

Abstract In this chapter we establish strongest convergence results for weak solutions of differential-operator equations and inclusions. In Sect. 6.1 we consider first order differential-operator equations and inclusions. Section 6.2 devoted to convergence results for weak solutions of second order operator differential equations and inclusions. In Sect. 6.3 we consider the following examples of applications: nonlinear parabolic equations of divergent form; nonlinear problems on manifolds with and without boundary: a climate energy balance model; a model of conduction of electrical impulses in nerve axons; viscoelastic problems with nonlinear "reaction-displacement" law.

6.1 First Order Differential-Operator Equations and Inclusions

In this section we consider strongest convergence results for both the autonomous first order differential-operator equations as well as nonautonomous evolution inclusions.

6.1.1 Convergence Results for Autonomous Evolution Equations

Let us consider the first-order general nonlinear evolution equations of the form

$$u'(t) + A(u(t)) = \bar{0}, \tag{6.1}$$

It is assumed that the nonlinear operator $A : V \to V^*$, acts in a Banach space V, which is reflexive and separable and, for some Hilbert space H, the embeddings $V \Subset H \equiv H \subset V^*$ are valid. Suppose that the nonlinear operator A is *pseudomonotone* and satisfies dissipation conditions of the form

$$\langle A(u), u\rangle_V \geq \alpha\|u\|_V^p - \beta \quad \forall u \in V, \tag{6.2}$$

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where $p \geq 2$, and $\alpha, \beta > 0$, and also power growth conditions of the form

$$\|A(u)\|_{V^*} \leq c(1 + \|u\|_V^{p-1}) \quad \forall u \in V, \tag{6.3}$$

for some $c > 0$. Here $\langle \cdot, \cdot \rangle_V : V^* \times V \to \mathbb{R}$ is the pairing in $V^* \times V$ coinciding on $H \times V$ with the inner product$(\cdot, \cdot)$ in the Hilbert space H.

By a *weak solution* of operator differential equation (6.1) on a closed interval $[\tau, T]$ we mean an element u of the space $L_p(\tau, T; V)$ such that

$$\forall \xi \in C_0^\infty([\tau, T]; V) \quad -\int_\tau^T (\xi'(t), u(t))dt + \int_\tau^T \langle A(u(t)), \xi(t) \rangle_V dt = 0. \tag{6.4}$$

Many evolution partial differential equations in a domain Ω whose leading part is a pth power nonlinear monotone differential operator and which may contain lower (now nonmonotone) summands with subordinate nonlinearity growth can be reduced to the form (6.1). In this case, the space V is a Sobolev space of the corresponding order, while the space H is $H = L_2(\Omega)$. Such equations are very often used to describe complicated evolution processes in various models in physics and mechanics. For equations of the form (6.1), there is a well-developed technique for constructing global (i.e., for all $t \geq 0$) weak solutions $u(t)$, $t \geq 0$, from the space $L_p^{loc}(\mathbb{R}_+; V)$ such that $u'(\cdot) \in L_q^{loc}(\mathbb{R}_+; V^*)$ (here $1/p + 1/q = 1$). It is well known that such weak solutions $u(t)$ are continuous functions with values in H, i.e., $u(\cdot) \in C(\mathbb{R}_+; H)$.

The problem is to study the asymptotic behavior as $t \to +\infty$ of the families of weak solutions $\{u(t)\}$ of Problem (6.1) in the norm of H under the assumption that the initial data $\{u(0)\}$ constitute a bounded set in H (see also [1, 2, 4, 7, 9, 11, 19–21, 28]).

Note that, under certain additional conditions on the nonlinear operator $A(u)$ ensuring, for Problem (6.1), the unique solvability of the Cauchy problem $u|_{t=0} = u_0$ for any $u_0 \in H$, the study of the class of weak solutions under consideration involves the highly fruitful theory of dynamical semigroups and their global attractors in infinite-dimensional phase spaces. This theory has been successfully developed over a period of more than thirty years; its foundations were created by Ladyzhenskaya, Babin, Vishik, Hale, Temam and other well-known mathematicians [14, 15, 17, 18].

The problem becomes significantly more complicated if the corresponding Cauchy problem is not uniquely solvable or the proof of the relevant theorem is not known. Such a situation often occurs in complicated mathematical models. In this case, the "classical" method based on unique semigroups and global attractors cannot be applied directly. However, two approaches to the study of the dynamics of the corresponding weak solutions are well known. The first method is based on the theory of multi-valued semigroups; it was developed in ground-breaking papers of Babin and Vishik (see, for example, [3]). The second approach uses the method of trajectory attractors; it was proposed in the papers [5, 6] of Chepyzhov and Vishik as well as in the independent work [25] of Sell.

The new results contained in the present section consist in the application of these two approaches to the study of the strongest convergence results for weak solutions of equations of the form (6.1) with general nonlinear pseudomonotone operator $A(u)$ satisfying (S)-property without any conditions guaranteeing the unique solvability of the Cauchy problem.

For fixed $\tau < T$ let us set

$$X_{\tau,T} = L_p(\tau, T; V), \quad X^*_{\tau,T} = L_q(\tau, T; V^*), \quad W_{\tau,T} = \{u \in X_{\tau,T} \mid u' \in X^*_{\tau,T}\},$$

where u' is a derivative of an element $u \in X_{\tau,T}$ in the sense of the space of distributions $\mathscr{D}([\tau, T]; V^*)$ (see, for example, [12, Definition IV.1.10, p. 168]). We note that

$$A(u)(t) = A(u(t)), \quad \text{for any } u \in X_{\tau,T} \text{ and a.e. } t \in (\tau, T).$$

The space $W_{\tau,T}$ is a reflexive Banach space with the graph norm of a derivative (see, for, example [15, Proposition 4.2.1, p. 291]):

$$\|u\|_{W_{\tau,T}} = \|u\|_{X_{\tau,T}} + \|u'\|_{X^*_{\tau,T}}, \quad u \in W_{\tau,T}. \tag{6.5}$$

Properties of A and (V, H, V^*) provide the existence of a weak solution of Cauchy problem (6.1) with initial data

$$u(\tau) = u_\tau \tag{6.6}$$

on the interval $[\tau, T]$ for an arbitrary $y_\tau \in H$. Therefore, the next result takes place:

According to Proposition 1.1, for any $\tau < T$, $y_\tau \in H$ Cauchy problem (6.1), (6.6) has a weak solution on the interval $[\tau, T]$. Moreover, each weak solution $u \in X_{\tau,T}$ of Cauchy problem (6.1), (6.6) on the interval $[\tau, T]$ belongs to $W_{\tau,T} \subset C([\tau, T]; H)$.

For fixed $\tau < T$ we denote

$$\mathscr{D}_{\tau,T}(u_\tau) = \{u(\cdot) \mid u \text{ is a weak solution of (6.1) on } [\tau, T],\ u(\tau) = u_\tau\}, \quad u_\tau \in H.$$

From Proposition 1.1 it follows that $\mathscr{D}_{\tau,T}(u_\tau) \neq \emptyset$ and $\mathscr{D}_{\tau,T}(u_\tau) \subset W_{\tau,T}$ $\forall \tau < T,\ u_\tau \in H$.

We note that the translation and concatenation of weak solutions is a weak solution too.

Lemma 6.1 (Zgurovsky et al. [15]) *If $\tau < T$, $u_\tau \in H$, $u(\cdot) \in \mathscr{D}_{\tau,T}(u_\tau)$, then $v(\cdot) = u(\cdot + s) \in \mathscr{D}_{\tau-s,T-s}(u_\tau)$ $\forall s$. If $\tau < t < T$, $u_\tau \in H$, $u(\cdot) \in \mathscr{D}_{\tau,t}(u_\tau)$ and $v(\cdot) \in \mathscr{D}_{t,T}(u(t))$, then*

$$z(s) = \begin{cases} u(s), & s \in [\tau, t], \\ v(s), & s \in [t, T] \end{cases}$$

belongs to $\mathscr{D}_{\tau,T}(u_\tau)$.

As a rule, the proof of the existence of compact global and trajectory attractors for equations of type (6.1) is based on the properties of the set of weak solutions of problem (6.1) related to the absorption of the generated m-semiflow of solutions and its asymptotic compactness (see, for example, [24, 27] and the references therein). The following lemma on a priori estimates of solutions and Theorem 6.1 on the dependence of solutions on initial data will play a key role in the study of the dynamics of the solutions of Problem (6.1) as $t \to +\infty$.

Lemma 6.2 (Zgurovsky et al. [15]) *There exist $c_4, c_5, c_6, c_7 > 0$ such that for any finite interval of time $[\tau, T]$ every weak solution u of problem (6.1) on $[\tau, T]$ satisfies estimates: $\forall t \geq s$, $t, s \in [\tau, T]$*

$$\|u(t)\|_H^2 + c_4 \int_s^t \|u(\xi)\|_V^p d\xi \leq \|u(s)\|_H^2 + c_5(t - s), \tag{6.7}$$

$$\|u(t)\|_H^2 \leq \|u(s)\|_H^2 e^{-c_6(t-s)} + c_7. \tag{6.8}$$

We recall that $A : V \to V^*$ satisfies (S)-*property*, if from $u_n \to u$ weakly in V and $\langle A(u_n), u_n - u\rangle_V \to 0$, as $n \to \infty$, it follows that $u_n \to u$ strongly in V, as $n \to +\infty$.

Further we assume that A satisfies (S)-property.

Theorem 6.1 *Let $\tau < T$, $\{u_n\}_{n\geq 1}$ be an arbitrary sequence of weak solutions of (6.1) on $[\tau, T]$ such that $u_n(\tau) \to \eta$ weakly in H. Then there exist $\{u_{n_k}\}_{k\geq 1} \subset \{u_n\}_{n\geq 1}$ and $u(\cdot) \in \mathscr{D}_{\tau,T}(\eta)$ such that*

$$\forall \varepsilon \in (0, T - \tau) \quad \max_{t\in[\tau+\varepsilon,T]} \|u_{n_k}(t) - u(t)\|_H + \int_{\tau+\varepsilon}^T \|u_{n_k}(t) - u(t)\|_V^p dt \to 0, \quad k \to +\infty. \tag{6.9}$$

Before the proof of Theorem 6.1 let us provide some auxiliary statements.

Lemma 6.3 *Let $\tau < T$, $y_n \to y$ weakly in $W_{\tau,T}$, and*

$$\limsup_{n\to+\infty} \langle A(y_n), y_n - y\rangle_{X_{\tau,T}} \leq 0. \tag{6.10}$$

Then

$$\lim_{n\to+\infty} \int_\tau^T |\langle A(y_n(t)), y_n(t) - y(t)\rangle_V|\, dt = 0. \tag{6.11}$$

Proof There exists a set of measure zero, $\Sigma_1 \subset (\tau, T)$ such that for $t \notin \Sigma_1$, we have that

$$y_n(t) \in V \text{ for all } n \geq 1.$$

Similarly to [17, p. 7] we verify the following claim.

Claim: Let $y_n \to y$ weakly in $W_{\tau,T}$ and let $t \notin \Sigma_1$. Then

$$\varliminf_{n\to+\infty} \langle A(y_n(t)), y_n(t) - y(t)\rangle_V \geq 0.$$

Proof of the claim. Fix $t \notin \Sigma_1$ and suppose to the contrary that

$$\varliminf_{n\to+\infty} \langle A(y_n(t)), y_n(t) - y(t)\rangle_V < 0. \tag{6.12}$$

Then up to a subsequence $\{y_{n_k}\}_{k\geq1} \subset \{y_n\}_{n\geq1}$ we have

$$\lim_{k\to+\infty} \langle A(y_{n_k}(t)), y_{n_k}(t) - y(t)\rangle_V = \varliminf_{n\to+\infty} \langle A(y_n(t)), y_n(t) - y(t)\rangle_V < 0. \tag{6.13}$$

Therefore, for all rather large k, growth and dissipation conditions imply

$$\alpha\|y_{n_k}(t)\|_V^p - \beta \leq \|A(y_{n_k}(t))\|_{V^*}\|y(t)\|_V \leq c(1 + \|y_{n_k}(t)\|_V^{p-1})\|y(t)\|_V.$$

which implies that the sequences $\{\|y_{n_k}(t)\|_V\}_{k\geq1}$ and consequently $\{\|A(y_{n_k}(t))\|_{V^*}\}_{k\geq1}$ are bounded sequences. In virtue of the continuous embedding $W_{\tau,T} \subset C([\tau, T]; H)$ we obtain that $y_{n_k}(t) \to y(t)$ weakly in H. Due to boundedness of $\{y_{n_k}(t)\}_{k\geq1}$ in V we finally have

$$\forall t \in [\tau, T]\backslash\Sigma_1 \;\; y_{n_k}(t) \to y(t) \text{ weakly in } V,\;\; k \to +\infty. \tag{6.14}$$

The pseudomonotony of A, (6.12)–(6.14) imply that

$$\begin{aligned}\varliminf_{n\to+\infty} \langle A(y_n(t)), y_n(t) - y(t)\rangle_V &\geq \langle A(y(t)),\\ y(t) - y(t)\rangle_V = 0 &> \varliminf_{n\to+\infty} \langle A(y_n(t)), y_n(t) - y(t)\rangle_V.\end{aligned}$$

We obtain a contradiction.

The claim is proved.

Now let us continue the proof of Lemma 6.3. The claim provides that for a.e. $t \in [\tau, T]$, in fact for any $t \notin \Sigma_1$, we have

$$\varliminf_{n\to+\infty} \langle A(y_n(t)), y_n(t) - y(t)\rangle_V \geq 0. \tag{6.15}$$

Dissipation and growth conditions imply that, if $\omega \in X_{\tau,T}$, then

$$\langle A(y_n(t)), y_n(t) - \omega(t)\rangle_V \geq \alpha\|y_n(t)\|_V^p - \beta - c(1 + \|y_n(t)\|_V^{p-1})\|\omega(t)\|_V$$
$$\text{for a.e. } t \in [\tau, T]\backslash\Sigma_1.$$

Using $p - 1 = \frac{p}{q}$, the right side of the above inequality equals to

$$\alpha\|y_n(t)\|_V^p - \beta - c\|y_n(t)\|_V^{\frac{p}{q}}\|\omega(t)\|_V - c\|\omega(t)\|_V.$$

Now using Young's inequality, we can obtain a constant $c(c,\alpha)$ depending on c, α such that

$$c\|y_n(t)\|_V^{\frac{p}{q}}\|\omega(t)\|_V \le \frac{\alpha}{2}\|y_n(t)\|_V^p + \|\omega(t)\|_V^p \cdot c(c,\alpha).$$

Letting $\bar{c} = \max\{\beta + \frac{c}{q}; c(c,\alpha) + \frac{c}{p}\}$ it follows that

$$\langle A(y_n(t)), y_n(t) - \omega(t)\rangle_V \ge -\bar{c}(1 + \|\omega(t)\|_V^p) \text{ for a.e. } t \in [\tau, T]. \tag{6.16}$$

Letting $\omega = y$, we can use Fatou's lemma and we obtain

$$\varliminf_{n\to+\infty} \int_0^T [\langle A(y_n(t)), y_n(t) - y(t)\rangle_V + \bar{c}(1 + \|y(t)\|_V^p)]dt \ge$$

$$\ge \int_0^T \varliminf_{n\to+\infty} [\langle A(y_n(t)), y_n(t) - y(t)\rangle_V + \bar{c}(1 + \|y(t)\|_V^p)]dt \ge \bar{c}\int_0^T (1 + \|y(t)\|_V^p)dt.$$

Therefore,

$$0 \ge \limsup_{n\to+\infty}\langle A(y_n), y_n - y\rangle_{X_{\tau,T}} \ge \varliminf_{n\to+\infty}\int_\tau^T \langle A(y_n(t)), y_n(t) - y(t)\rangle_V dt =$$

$$= \varliminf_{n\to+\infty} \langle A(y_n), y_n - y\rangle_{X_{\tau,T}} \ge \int_\tau^T \varliminf_{n\to+\infty} \langle A(y_n(t)), y_n(t) - y(t)\rangle_V dt = 0,$$

showing that

$$\lim_{n\to+\infty}\langle A(y_n), y_n - y\rangle_{X_{\tau,T}} = 0. \tag{6.17}$$

From (6.16),

$$\forall n \ge 1 \;\forall t \notin \Sigma_1 \;\; 0 \le \langle A(y_n(t)), y_n(t) - y(t)\rangle_V^- \le \bar{c}(1 + \|y(t)\|_V^p),$$

where $a^- = \max\{0, -a\}$, for $a \in \mathbb{R}$. Due to (6.15) we know that for a.e. t, $\langle A(y_n(t)), y_n(t) - y(t)\rangle_V \ge -\varepsilon$ for all rather large n. Therefore, for such n, $\langle A(y_n(t)), y_n(t) - y(t)\rangle_V^- \le \varepsilon$, if $\langle A(y_n(t)), y_n(t) - y(t)\rangle_V < 0$ and $\langle A(y_n(t)), y_n(t) - y(t)\rangle_V^- = 0$, if $\langle A(y_n(t)), y_n(t) - y(t)\rangle_V \ge 0$. Therefore, $\lim_{n\to+\infty}\langle A(y_n(t)), y_n(t) - y(t)\rangle_V^- = 0$ and we can apply the dominated convergence theorem and from (6.15) we conclude that

$$\lim_{n\to+\infty}\int_{\tau}^{T}\langle A(y_n(t)), y_n(t)-y(t)\rangle_V^- = \int_{\tau}^{T}\lim_{n\to+\infty}\langle A(y_n(t)), y_n(t)-y(t)\rangle_V^- dt = 0.$$

Now by (6.17) and the above equation we have

$$\lim_{n\to+\infty}\int_{\tau}^{T}\langle A(y_n(t)), y_n(t)-y(t)\rangle_V^+ dt =$$

$$= \lim_{n\to+\infty}\int_{0}^{T}[\langle A(y_n(t)), y_n(t)-y(t)\rangle_V + \langle A(y_n(t)), y_n(t)-y(t)\rangle_V^-]dt =$$

$$= \lim_{n\to+\infty}\langle A(y_n), y_n - y\rangle_{X_{\tau,T}} = 0.$$

Therefore,

$$\lim_{n\to+\infty}\int_{\tau}^{T}|\langle A(y_n(t)), y_n(t)-y(t)\rangle_V| dt = 0.$$

The lemma is proved.

Lemma 6.4 *Let $\tau < T$, $y_n \to y$ weakly in $W_{\tau,T}$, and (6.10) holds. Then there exists a subsequence $\{y_{n_k}\}_{k\geq 1} \subset \{y_n\}_{n\geq 1}$ such that for a.e. $t \in (\tau, T)$ we have that $y_{n_k}(t) \to y(t)$ weakly in V, and $\langle A(y_{n_k}(t)), y_{n_k}(t) - y(t)\rangle_V \to 0$, $k \to +\infty$.*

Proof Let $y_n \to y$ weakly in $W_{\tau,T}$ and

$$\limsup_{n\to+\infty}\langle A(y_n), y_n - y\rangle_{X_{\tau,T}} \leq 0.$$

In virtue of Lemma 6.3 we obtain

$$\lim_{n\to+\infty}\int_{\tau}^{T}|\langle A(y_n(t)), y_n(t)-y(t)\rangle_V| dt = 0. \tag{6.18}$$

Due to the continuous embedding $W_{\tau,T} \subset C([\tau, T]; H)$ we have

$$\forall t \in [\tau, T]\ \ y_n(t) \to y(t) \text{ weakly in } H,\ \ n \to +\infty. \tag{6.19}$$

From (6.18) it follows that there exists a subsequence $\{y_{n_k}\}_{k\geq 1} \subset \{y_n\}_{n\geq 1}$ such that

$$\langle A(y_{n_k}(t)), y_{n_k}(t) - y(t)\rangle_V \to 0,\ \ k \to +\infty, \quad \text{for a.e. } t \in [\tau, T].$$

Let $\Sigma_1 \subset [\tau, T]$ be a set of measure zero such that for $t \notin \Sigma_1$ $y_{n_k}(t),\ y(t)$ are well-defined $\forall k \geq 1$, and

$$\langle A(y_{n_k}(t)), y_{n_k}(t) - y(t)\rangle_V \to 0,\ \ k \to +\infty.$$

In virtue of growth and dissipation conditions we obtain

$$\forall t \notin \Sigma_1\ \forall k \geq 1\ \limsup_{k\to+\infty}\left(\alpha\|y_{n_k}(t)\|_V^p - \beta - c(1 + \|y_{n_k}(t)\|_V^{p-1})\|y(t)\|_V\right) \leq 0.$$

Thus $\forall t \notin \Sigma_1$

$$\limsup_{k\to+\infty}\|y_{n_k}(t)\|_V^p \leq c(c, \alpha, \beta, p)(1 + \|y(t)\|_V^p).$$

Therefore, due to (6.19) we obtain that for a.e. $t \in (\tau, T)$ $y_{n_k}(t) \to y(t)$ weakly in V, $k \to +\infty$.

The lemma is proved.

Proof of Theorem 6.1. Let $\tau < T$, $\{u_n\}_{n\geq 1}$ be an arbitrary sequence of weak solutions of (6.1) on $[\tau, T]$ such that $u_n(\tau) \to \eta$ weakly in H. Theorem 1 from [15] implies the existence of a subsequence $\{u_{n_k}\}_{k\geq 1} \subset \{u_n\}_{n\geq 1}$ and $u(\cdot) \in \mathscr{D}_{\tau,T}(\eta)$ such that

$$\forall \varepsilon \in (0, T - \tau)\quad \max_{t\in[\tau+\varepsilon,T]}\|u_{n_k}(t) - u(t)\|_H \to 0,\quad k \to +\infty. \tag{6.20}$$

Let us prove that

$$\forall \varepsilon \in (0, T - \tau)\quad \int_{\tau+\varepsilon}^{T}\|u_{n_k}(t) - u(t)\|_V^p dt \to 0,\quad k \to +\infty. \tag{6.21}$$

On the contrary, without loss of generality we assume that for some $\varepsilon \in (0, T - \tau)$ and $\delta > 0$ it is fulfilled

$$\int_{\tau+\varepsilon}^{T}\|u_{n_k}(t) - u(t)\|_V^p dt \geq \delta,\quad \forall k \geq 1. \tag{6.22}$$

In virtue of (6.7), without loss of generality we claim that

$$u_{n_k} \to u \text{ weakly in } W_{\tau+\varepsilon,T},\quad k \to +\infty. \tag{6.23}$$

Moreover, due to (6.20), we have

$$\limsup_{k\to\infty} \int_{\tau+\varepsilon}^{T} \langle A(u_{n_k}(t)), u_{n_k}(t) - u(t)\rangle_V dt \leq 0. \tag{6.24}$$

Thus, Lemma 6.4 and (S)-property for A imply that up to a subsequence which we denote again as $\{u_{n_k}\}_{k\geq 1}$ for a.e. $t \in (\tau+\varepsilon, T)$ we have that $u_{n_k}(t) \to u(t)$ strongly in V, $k \to +\infty$. Moreover, Lemma 6.3 provides that

$$\lim_{k\to+\infty} \int_{\tau+\varepsilon}^{T} \left|\langle A(u_{n_k}(t)), u_{n_k}(t) - u(t)\rangle_V\right| dt = 0.$$

Dissipation and growth conditions follow the existence a constant $C > 0$ such that

$$\|u_{n_k}(t) - u(t)\|_V^p \leq C(1 + \|u(t)\|_V^p + \left|\langle A(u_{n_k}(t)), u_{n_k}(t) - u(t)\rangle_V\right|)$$

for a.e. $t \in (\tau+\varepsilon, T)$ and any $k \geq 1$. Therefore,

$$\lim_{k\to+\infty} \int_{\tau+\varepsilon}^{T} \|u_{n_k}(t) - u(t)\|_V^p dt = 0.$$

We obtain a contradiction.

The theorem is proved.

6.1.2 *Convergence Results for Nonautonomous Evolution Inclusions*

For evolution triple $(V_i; H; V_i^*)$[1] and multi-valued map $A_i : \mathbb{R}_+ \times V \rightrightarrows V^*$, $i = 1, 2, \ldots, N$, $N = 1, 2, \ldots$ we consider a problem of long-time behavior of all globally defined weak solutions for nonautonomous evolution inclusion

$$y'(t) + \sum_{i=1}^{N} A_i(t, y(t)) \ni \bar{0}, \tag{6.25}$$

as $t \to +\infty$. Let $\langle \cdot, \cdot \rangle_{V_i} : V_i^* \times V_i \to \mathbb{R}$ be the pairing in $V_i^* \times V_i$, that coincides on $H \times V_i$ with the inner product $(\cdot, \cdot)$ in the Hilbert space H.

[1] I.e., V_i is a real reflexive separable Banach space continuously and densely embedded into a real Hilbert space H, H is identified with its topologically conjugated space H^*, V_i^* is a dual space to V_i. So, there is a chain of continuous and dense embeddings: $V_i \subset H \equiv H^* \subset V_i^*$ (see, for example, Gajewski, Gröger, and Zacharias [12, Chap. I]).

Note that Problem (6.25) arises in many important models for distributed parameter control problems and that large class of identification problems enter this formulation.

Throughout this subsection we suppose that the listed below assumptions hold:

Assumption I. Let $p_i \geq 2,\ q_i > 1$ are such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$, for each for $i = 1, 2, \ldots, N$, and the embedding $V_i \subset H$ is compact one, for some for $i = 1, 2, \ldots, N$.

Assumption II (Grows Condition). There exist a translation uniform integrable (t.u.i.) function in $L_1^{\text{loc}}(\mathbb{R}_+)$ function $c_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $c_2 > 0$ such that

$$\max_{i=1}^{N} \|d_i\|_{V_i^*}^{q} \leq c_1(t) + c_2 \sum_{i=1}^{N} \|u\|_{V_i}^{p}$$

for any $u \in V_i$, $d_i \in A_i(t, u)$, $i = 1, 2, \ldots, N$, and a.e. $t > 0$.

Assumption III (Sign Assumption). There exist a constant $\alpha > 0$ and a t.u.i. in $L_1^{\text{loc}}(\mathbb{R}_+)$ function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sum_{i=1}^{N} \langle d_i, u\rangle_{V_i} \geq \alpha \sum_{i=1}^{N} \|u\|_{V_i}^{p} - \beta(t)$$

for any $u \in V_i$, $d_i \in A_i(t, u)$, $i = 1, 2, \ldots, N$, and a.e. $t > 0$.

Assumption IV (Strong Measurability). If $C \subseteq V_i{}^*$ is a closed set, then the set $\{(t, u) \in (0, +\infty) \times V_i\ :\ A_i(t, u) \cap C \neq \emptyset\}$ is a Borel subset in $(0, +\infty) \times V_i$.

Assumption V (Pointwise Pseudomonotonicity). Let for each $i = 1, 2, \ldots, N$ and a.e. $t > 0$ two assumptions hold:

(a) for every $u \in V_i$ the set $A_i(t, u)$ is nonempty, convex, and weakly compact one in $V_i{}^*$;
(b) if a sequence $\{u_n\}_{n\geq 1}$ converges weakly in V_i towards $u \in V_i$ as $n \to +\infty$, $d_n \in A_i(t, u_n)$ for any $n \geq 1$, and $\limsup_{n\to+\infty}\langle d_n, u_n - u\rangle_{V_i} \leq 0$, then for any $\omega \in V_i$ there exists $d(\omega) \in A_i(t, u)$ such that

$$\liminf_{n\to+\infty}\langle d_n, u_n - \omega\rangle_{V_i} \geq \langle d(\omega), u - \omega\rangle_{V_i}.$$

Let $0 \leq \tau < T < +\infty$. As a *weak solution* of evolution inclusion (6.25) on the interval $[\tau, T]$ we consider an element $u(\cdot)$ of the space $\cap_{i=1}^{N} L_{p_i}(\tau, T; V_i)$ such that for some $d_i(\cdot) \in L_{q_i}(\tau, T; V_i{}^*)$, $i = 1, 2, \ldots, N$, it is fulfilled:

$$-\int_{\tau}^{T} (\xi'(t), y(t))dt + \sum_{i=1}^{N} \int_{\tau}^{T} \langle d_i(t), \xi(t)\rangle_{V_i} dt = 0 \quad \forall \xi \in C_0^{\infty}([\tau, T]; V_i), \quad (6.26)$$

and $d_i(t) \in A_i(t, y(t))$ for each $i = 1, 2, \ldots, N$ and a.e. $t \in (\tau, T)$.

For fixed nonnegative τ and T, $\tau < T$, let us consider

$$X_{\tau,T} = \cap_{i=1}^{N} L_{p_i}(\tau, T; V_i), \quad X^*_{\tau,T} = \sum_{i=1}^{N} L_{q_i}(\tau, T; V_i{}^*), \quad W_{\tau,T} = \{y \in X_{\tau,T} \mid y' \in X^*_{\tau,T}\},$$

$$\mathscr{A}_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}, \quad \mathscr{A}_{\tau,T}(y) = \{d \in X^*_{\tau,T} \mid d(t) \in A(t, y(t)) \text{ for a.e. } t \in (\tau, T)\},$$

where y' is a derivative of an element $u \in X_{\tau,T}$ in the sense of $\mathscr{D}([\tau, T]; \sum_{i=1}^{N} V_i{}^*)$ (see, for example, Gajewski, Gröger, and Zacharias [12, Definition IV.1.10]). Note that the space $W_{\tau,T}$ is a reflexive Banach space with the graph norm of a derivative $\|u\|_{W_{\tau,T}} = \|u\|_{X_{\tau,T}} + \|u'\|_{X^*_{\tau,T}}$, $u \in W_{\tau,T}$. Let $\langle \cdot, \cdot \rangle_{X_{\tau,T}} : X^*_{\tau,T} \times X_{\tau,T} \to \mathbb{R}$ be the pairing in $X^*_{\tau,T} \times X_{\tau,T}$, that coincides on $L_2(\tau, T; H) \times X_{\tau,T}$ with the inner product in $L_2(\tau, T; H)$, i.e., $\langle u, v\rangle_{X_{\tau,T}} = \int_\tau^T (u(t), v(t))dt$ for any $u \in L_2(\tau, T; H)$ and $v \in X_{\tau,T}$. Gajewski, Gröger, and Zacharias [12, Theorem IV.1.17] provide that the embedding $W_{\tau,T} \subset C([\tau, T]; H)$ is continuous and dense one. Moreover,

$$(u(T), v(T)) - (u(\tau), v(\tau)) = \int_\tau^T \Big[\langle u'(t), v(t)\rangle_{V_i} + \langle v'(t), u(t)\rangle_{V_i}\Big]dt, \tag{6.27}$$

for any $u, v \in W_{\tau,T}$.

Migórski [22, Lemma 7, p. 516] (see paper and references therein) and Assumptions I–V provide the existence of multi-valued Nemitsky operator $\mathscr{A}_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}$ for $\sum_{i=1}^{N} A_i$ that satisfies the following properties:

Property I. The mapping $\mathscr{A}_{\tau,T}$ transforms an each bounded set in $X_{\tau,T}$ onto bounded subset of $X^*_{\tau,T}$;

Property II. There exist positive constants $C_1 = C_1(\tau, T)$ and $C_2 = C_2(\tau, T)$ such that $\langle d, y\rangle_{X_{\tau,T}} \geq C_1\|y\|^p_{X_{\tau,T}} - C_2$ for any $y \in X_{\tau,T}$ and $d \in \mathscr{A}_{\tau,T}(y)$.

Property III. The multi-valued mapping $\mathscr{A}_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}$ is (generalized) pseudomonotone on $W_{\tau,T}$, i.e., (a) for every $y \in X_{\tau,T}$ the set $\mathscr{A}_{\tau,T}(y)$ is a nonempty, convex and weakly compact one in $X^*_{\tau,T}$; (b) $\mathscr{A}_{\tau,T}$ is upper semicontinuous from every finite dimensional subspace $X_{\tau,T}$ into $X^*_{\tau,T}$ endowed with the weak topology; (c) if a sequence $\{y_n, d_n\}_{n\geq 1} \subset W_{\tau,T} \times X^*_{\tau,T}$ converges weakly in $W_{\tau,T} \times X^*_{\tau,T}$ towards $(y, d) \in W_{\tau,T} \times X^*_{\tau,T}$, $d_n \in \mathscr{A}_{\tau,T}(y_n)$ for any $n \geq 1$, and $\limsup_{n\to+\infty}\langle d_n, y_n - y\rangle_{X_{\tau,T}} \leq 0$, then $d \in \mathscr{A}_{\tau,T}(y)$ and $\lim_{n\to+\infty}\langle d_n, y_n\rangle_{X_{\tau,T}} = \langle d, y\rangle_{X_{\tau,T}}$.

Formula (6.26) and definition of the derivative for an element from $\mathscr{D}([\tau, T]; \sum_{i=1}^{N} V_i^*)$ yield that each weak solution $y \in X_{\tau,T}$ of Problem (6.25) on $[\tau, T]$ belongs to the space $W_{\tau,T}$ and $y' + \mathscr{A}_{\tau,T}(y) \ni \bar{0}$. Vice versa, if $y \in W_{\tau,T}$ satisfies the last inclusion, then y is a weak solution of Problem (6.25) on $[\tau, T]$.

Assumption I, Properties I–III, and Denkowski, Migórski, and Papageorgiou [10, Theorem 1.3.73] (see also Zgurovsky, Mel'nik, and Kasyanov [30, Chap. 2] and references therein) provide the existence of a weak solution of Cauchy problem (6.25) with initial data $y(\tau) = y^{(\tau)}$ on the interval $[\tau, T]$, for any $y^{(\tau)} \in H$.

For fixed τ and T, such that $0 \le \tau < T < +\infty$, we denote

$$\mathscr{D}_{\tau,T}(y^{(\tau)}) = \{y(\cdot) \mid y \text{ is a weak solution of (6.25) on } [\tau, T],\ y(\tau) = y^{(\tau)}\}, \quad y^{(\tau)} \in H.$$

We remark that $\mathscr{D}_{\tau,T}(y^{(\tau)}) \neq \emptyset$ and $\mathscr{D}_{\tau,T}(y^{(\tau)}) \subset W_{\tau,T}$, if $0 \le \tau < T < +\infty$ and $y^{(\tau)} \in H$. Moreover, the concatenation of Problem (6.25) weak solutions is a weak solutions too, i.e., if $0 \le \tau < t < T$, $y^{(\tau)} \in H$, $y(\cdot) \in \mathscr{D}_{\tau,t}(y^{(\tau)})$, and $v(\cdot) \in \mathscr{D}_{t,T}(y(t))$, then

$$z(s) = \begin{cases} y(s), & s \in [\tau, t], \\ v(s), & s \in [t, T], \end{cases}$$

belongs to $\mathscr{D}_{\tau,T}(y^{(\tau)})$; cf. Zgurovsky et al. [31, pp. 55–56].

Gronwall lemma provides that for any finite time interval $[\tau, T] \subset \mathbb{R}_+$ each weak solution y of Problem (6.25) on $[\tau, T]$ satisfies estimates

$$\|y(t)\|_H^2 - 2\int_0^t \beta(\xi)d\xi + 2\alpha \sum_{i=1}^N \int_s^t \|y(\xi)\|_{V_i}^p d\xi \le \|y(s)\|_H^2 - 2\int_0^s \beta(\xi)d\xi, \tag{6.28}$$

$$\|y(t)\|_H^2 \le \|y(s)\|_H^2 e^{-2\alpha\gamma(t-s)} + 2\int_s^t (\beta(\xi) + \alpha\gamma) e^{-2\alpha\gamma(t-\xi)} d\xi, \tag{6.29}$$

where $t, s \in [\tau, T]$, $t \ge s$; γ is a constant that does not depend on y, s, and t; cf. Zgurovsky et al. [31, p. 56]. In the proof of (6.29) we used the inequality $\|u\|_H^2 - 1 \le \|u\|_H^p$ for any $u \in H$.

Therefore, any weak solution y of Problem (6.25) on a finite time interval $[\tau, T] \subset \mathbb{R}_+$ can be extended to a global one, defined on $[\tau, +\infty)$. For arbitrary $\tau \ge 0$ and $y^{(\tau)} \in H$ let $\mathscr{D}_\tau(y^{(\tau)})$ be the set of all weak solutions (defined on $[\tau, +\infty)$) of Problem (6.25) with initial data $y(\tau) = y^{(\tau)}$. Let us consider the family $\mathscr{K}_\tau^+ = \cup_{y^{(\tau)} \in H} \mathscr{D}_\tau(y^{(\tau)})$ of all weak solutions of Problem (6.25) defined on the semi-infinite time interval $[\tau, +\infty)$.

Assumptions (II) and (III) yield that there exist a positive constant $\alpha' > 0$ and a t.u.i. function c' in $L_1^{\text{loc}}(\mathbb{R}_+)$ such that $A(t, u) \subseteq \mathscr{A}_{c'(t)}(u)$ for each $u \in \cap_{i=1}^N V_i$ and a.e. $t > 0$, where

$$\mathscr{A}_{c'(t)}(u) := \left\{ \sum_{i=1}^N p_i \ :\ p_i \in V_i^*,\ \sum_{i=1}^N \langle p_i, u \rangle_{V_i} \ge \alpha' \max_{i=1}^N \left\{ \|u\|_{V_i}^p; \|p\|_{V_i^*}^q \right\} - c'(t) \right\}.$$

Let $\mathscr{H}(c')$ be the hull of t.u.i. function c' in $L_{1,w}^{\text{loc}}(\mathbb{R}_+)$, i.e., $\mathscr{H}(c') = \text{cl}_{L_1^{\text{loc}}(\mathbb{R}_+)}\{c'(\cdot + h) : h \ge 0\}$. This is a weakly compact set in $L_1^{\text{loc}}(\mathbb{R}_+)$; Gorban et al. [13].

Let us consider the family of problems

$$y' = \mathscr{A}_\sigma(y), \quad \sigma \in \Sigma := \mathscr{H}(c'). \tag{6.30}$$

To each $\sigma \in \Sigma$ there corresponds a space of all globally defined on $[0, +\infty)$ weak solutions $\mathscr{K}_\sigma^+ \subset C^{\mathrm{loc}}(\mathbb{R}_+; H)$ of Problem (6.30). We set $\mathscr{K}_\Sigma^+ = \cup_{\sigma\in\Sigma}\mathscr{K}_\sigma^+$.

We remark that any element from $\mathscr{K}_\Sigma^+$ satisfies prior estimates.

Lemma 6.5 *There exist positive constants c_3 and c_4 such that for any $\sigma \in \Sigma$ and $y \in \mathscr{K}_\sigma^+$ the inequalities hold:*

$$\|y(t)\|_H^2 - 2\int_0^t \sigma(\xi)d\xi + 2\alpha' \sum_{i=1}^N \int_s^t \|y(\xi)\|_{V_i}^p d\xi \le \|y(s)\|_H^2 - 2\int_0^s \sigma(\xi)d\xi, \tag{6.31}$$

$$\|y(t)\|_H^2 \le \|y(s)\|_H^2 e^{-c_3(t-s)} + c_4 \int_s^t \sigma(\xi)e^{-c_3(t-\xi)}d\xi, \tag{6.32}$$

for any $t \ge s \ge 0$.

Proof The proof naturally follows from conditions for the parameters of Problem (6.30) and Gronwall lemma.

Let us provide the result characterizing the compactness properties of solutions for the family of Problems (6.30).

Theorem 6.2 *Let $\{y_n\}_{n\ge1} \subset \mathscr{K}_\Sigma^+$ be an arbitrary sequence, that is bounded in $L_\infty(\mathbb{R}_+; H)$. Then there exist a subsequence $\{y_{n_k}\}_{k\ge1} \subset \{y_n\}_{n\ge1}$ and an element $y \in \mathscr{K}_\Sigma^+$ such that*

$$\max_{t\in[\tau,T]} \|y_{n_k}(t) - y(t)\|_H \to 0, \quad k \to +\infty, \tag{6.33}$$

for any finite time interval $[\tau, T] \subset (0, +\infty)$.

Proof For any $n \ge 1$ there exists $\sigma_n \in \Sigma$ such that $y_n \in \mathscr{K}_\sigma^+$. Furthermore, the definition of weak solution of evolution inclusion yields that for any $n \ge 1$ and $i = 1, 2, \ldots, N$, there exists $d_{n,i} \in L_{q_i}^{\mathrm{loc}}(\mathbb{R}_+; V_i{}^*)$ such that $y_n'(t) + \sum_{i=1}^N d_{n,i}(t) = \bar{0}$ for a.e. $t > 0$. The definition of $\mathscr{A}_\sigma$ and estimates (6.31) and (6.32) provide that the sequence $\{y_n, y_n', d_{n,i}\}_{n\ge1}$ is bounded in $\cap_{i=1}^N L_{p_i}^{\mathrm{loc}}(\mathbb{R}_+; V_i) \times \sum_{i=1}^N L_{q_i}^{\mathrm{loc}}(\mathbb{R}_+; V_i{}^*) \times L_{q_i}^{\mathrm{loc}}(\mathbb{R}_+; V_i{}^*)$, $i = 1, 2, \ldots, N$. Since Σ is a weakly compact set in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$, Banach–Alaoglu theorem (cf. Zgurovsky, Mel'nik, and Kasyanov [30, Chap. 1]; Kasyanov [15]) yields that there exist a subsequence $\{y_{n_k}, d_{n_k,i}\}_{k\ge1} \subset \{y_n, d_n\}_{n\ge1}$ and elements $d_i \in L_{q_i}^{\mathrm{loc}}(\mathbb{R}_+; V_i{}^*)$, $y \in \cap_{i=1}^N L_{p_i}^{\mathrm{loc}}(\mathbb{R}_+; V_i)$, and $\sigma \in \Sigma$, such that $y' \in \sum_{i=1}^N L_q^{\mathrm{loc}}(\mathbb{R}_+; V_i{}^*)$ and for each $i = 1, 2, \ldots, N$ the following convergence hold:

$$\begin{array}{ll} y_{n_k} \to y & \text{weakly in } \cap_{i=1}^N L_{p_i}^{\text{loc}}(\mathbb{R}_+; V_i), \\ y'_{n_k} \to y' & \text{weakly in } \sum_{i=1}^N L_{q_i}^{\text{loc}}(\mathbb{R}_+; V_i^*), \\ d_{n_k,i} \to d_i & \text{weakly in } L_{q_i}^{\text{loc}}(\mathbb{R}_+; V_i^*), \\ y_{n_k} \to y & \text{weakly in } C^{\text{loc}}(\mathbb{R}_+; H), \\ y_{n_k} \to y & \text{in } L_2^{\text{loc}}(\mathbb{R}_+; H), \\ y_{n_k}(t) \to y(t) & \text{in } H \text{ for a.e. } t > 0, \\ \sigma_{n_k} \to \sigma & \text{weakly in } L_1^{\text{loc}}(\mathbb{R}_+), \quad k \to +\infty. \end{array} \tag{6.34}$$

Formula (6.33) follows from Zgurovsky et al. [31, Steps 1 and 5, p. 58]. We remark that in the proof we need to consider continuous and nonincreasing (by Lemma 6.5) functions on $\mathbb{R}_+$:

$$J_k(t) = \|y_{n_k}(t)\|_H^2 - 2\int_0^t \sigma_{n_k}(\xi)d\xi, \;\; J(t) = \|y(t)\|_H^2 - 2\int_0^t \sigma(\xi)d\xi, \;\; k \ge 1. \tag{6.35}$$

The two last statements in (6.34) imply $J_k(t) \to J(t)$, as $k \to +\infty$, for a.e. $t > 0$.

The definition of a weak solution of evolution inclusion (cf. Zgurovsky et al. [31, p. 58]) and (6.34) yield $y'(t) = -\sum_{i=1}^N d_i(t)$ for a.e. $t > 0$. To finish the proof it is necessary to provide that

$$\sum_{i=1}^N d_i(t) \in \mathscr{A}_{\sigma(t)}(y(t)) \text{ for a.e. } t > 0. \tag{6.36}$$

Let $\varphi \in C_0^\infty((0, +\infty))$, $\varphi \ge 0$. Then

$$\int_{\mathbb{R}_+} \varphi(t)\big(\alpha' \max_{i=1}^N \left\{\|y(t)\|_{V_i}^p; \|d_i(t)\|_{V_i^*}^q\right\} - \sigma(t)\big)dt \le$$
$$\liminf_{k\to+\infty} \int_{\mathbb{R}_+} \varphi(t)\big(\alpha' \max_{i=1}^N \left\{\|y_{n_k}(t)\|_{V_i}^p; \|d_{n_k,i}(t)\|_{V_i^*}^q\right\} - \sigma_{n_k}(t)\big)dt \le$$
$$\lim_{k\to+\infty} \int_{\mathbb{R}_+} \varphi(t) \sum_{i=1}^N \langle d_{n_k,i}(t), y_{n_k}(t)\rangle_V dt = \lim_{k\to+\infty} \frac{1}{2}\int_{\mathbb{R}_+} \|y_{n_k}(t)\|_H^2 \frac{d}{dt}\varphi(t)dt =$$
$$\frac{1}{2}\int_{\mathbb{R}_+} \|y(t)\|_H^2 \frac{d}{dt}\varphi(t)dt = \sum_{i=1}^N \int_{\mathbb{R}_+} \varphi(t)\langle d_i(t), y(t)\rangle_{V_i} dt,$$

where the first inequality holds, because the convex functional

$$(y, d) \to \int_{\mathbb{R}_+} \varphi(t)\big(\alpha' \max_{i=1}^N \left\{\|y(t)\|_{V_i}^p; \|d_i(t)\|_{V_i^*}^q\right\}\big)dt$$

is weakly lower semi-continuous on $\cap_{i=1}^N L_{p_i}^{\text{loc}}(\mathbb{R}_+; V_i) \times L_{q_1}^{\text{loc}}(\mathbb{R}_+; V_1^*) \times L_{q_2}^{\text{loc}}(\mathbb{R}_+; V_2^*) \times \ldots \times L_{q_N}^{\text{loc}}(\mathbb{R}_+; V_N^*)$; the second inequality follows from the definition of $\mathscr{A}_\sigma$; the first and the third equalities follow from formula (6.27), because $y'_{n_k}(t) + \sum_{i=1}^{N} d_{n_k,i}(t) = y'(t) + \sum_{i=1}^{N} d_i(t) = \bar{0}$ for any $k \geq 1$ and a.e. $t > 0$; the second equality holds, because $y_{n_k} \to y$ in $L_2^{\mathrm{loc}}(\mathbb{R}_+; H)$, as $k \to +\infty$. As a nonnegative function $\varphi \in C_0^\infty((0, +\infty))$ is an arbitrary, then, by definition of $\mathscr{A}_\sigma$, formula (6.36) holds.

The theorem is proved.

6.2 Second Order Operator Differential Equations and Inclusions

In this section we consider damped wave equations with possibly nonmonotone discontinuous nonlinearities. Then, we generalize the results to the autonomous second order differential-operator inclusions with possibly nonmonotone potential.

Let $\beta > 0$ be a constant, $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Consider the problem

$$\begin{cases} u_{tt} + \beta u_t - \triangle u + f(u) = 0, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{6.37}$$

where $u(x, t)$ is unknown state function defined on $\Omega \times \mathbb{R}_+$; $f : \mathbb{R} \to \mathbb{R}$ is an interaction function such that

$$\lim_{|u|\to\infty} \frac{f(u)}{u} > -\lambda_1, \tag{6.38}$$

where λ_1 is the first eigenvalue for $-\triangle$ in $H_0^1(\Omega)$;

$$\exists\ D \geq 0 : \quad |f(u)| \leq D(1 + |u|), \quad \forall u \in \mathbb{R}. \tag{6.39}$$

Further, we use such denotation

$$\overline{f}(s) := \limsup_{t\to s} f(t), \quad \underline{f}(s) := \varliminf_{t\to s} f(t), \quad G(s) := [\underline{f}(s), \overline{f}(s)], \quad s \in \mathbb{R}.$$

Let us set $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. The space $X = V \times H$ is a phase space of Problem (6.37). For the Hilbert space X as $(\cdot, \cdot)_X$ and $\|\cdot\|_X$ denote the inner product and the norm in X respectively.

Definition 6.1 Let $T > 0$, $\tau < T$. The function $\varphi(\cdot) = (u(\cdot), u_t(\cdot))^T \in L^\infty(\tau, T; X)$ is called *a weak solution* of Problem (6.37) on (τ, T) if for a.e. $(x, t) \in \Omega \times (\tau, T)$, there exists $l = l(x, t) \in L^2(\tau, T; L^2(\Omega))$ $l(x, t) \in G(u(x, t))$, such that $\forall \psi \in H_0^1(\Omega)$, $\forall \eta \in C_0^\infty(\tau, T)$,

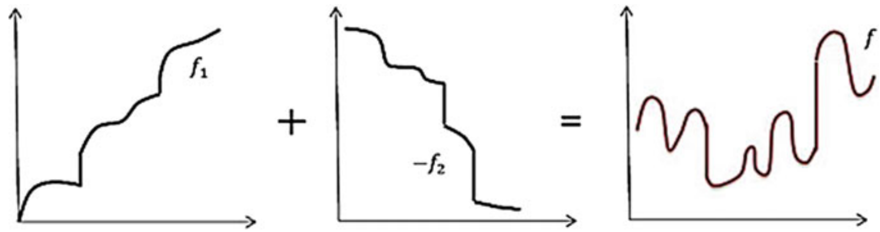

Fig. 6.1 Difference of nondecreasing functions

$$-\int_{\tau}^{T}(u_t,\psi)_H\eta_t dt+\int_{\tau}^{T}(\beta(u_t,\psi)_H+(u,\psi)_V+(l,\psi)_H)\eta dt=0. \tag{6.40}$$

The main goal of the manuscript is to obtain the existence of the global attractor generated by the weak solutions of Problem (6.37) in the phase space X.

Lemma 6.6 (Zgurovsky et al. [31]) *For any $\varphi_\tau=(u_0,u_1)^T\in X$ and $\tau<T$ there exists a weak solution $\varphi(\cdot)$ of Problem (6.37) on (τ,T) such that $\varphi(\tau)=\varphi_\tau$.*

Further, we assume that

$$f(s)=f_1(s)-f_2(s),\quad s\in\mathbb{R},$$

where $f_i:\mathbb{R}\to\mathbb{R}$, $i=1,2$, are nondecreasing functions (see Fig. 6.1).

We remark that

$$[\underline{f}(s),\overline{f}(s)]\subseteq[\underline{f_1}(s),\overline{f_1}(s)]-[\underline{f_2}(s),\overline{f_2}(s)],\quad s\in\mathbb{R}.$$

Thus we consider more general evolution inclusion

$$\begin{cases} u_{tt}+\beta u_t-\Delta u+[\underline{f_1}(u),\overline{f_1}(u)]-[\underline{f_2}(u),\overline{f_2}(u)]\ni 0,\\ u|_{\partial\Omega}=0. \end{cases} \tag{6.41}$$

Let us set

$$G_i(s):=\int_0^s f_i(\xi)d\xi,\ \ J_i(u):=\int_\Omega G_i(u(x))dx,\ \ J(u)=J_1(u)-J_2(u),\ \ u\in H,\ i=1,2.$$

The functionals G_i and J_i are locally Lipschitz and regular; Clarke [8, Chap. I]. Thus the next result holds.

Lemma 6.7 (Kasyanov et al. [31]) *Let $u\in C^1([\tau,T];H)$. Then for a.e. $t\in(\tau,T)$, the functions $J_i\circ u$ are classically differentiable at the point t. Moreover,*

$$\frac{d}{dt}(J_i \circ u)(t) = (p, u_t(t)) \ \forall p \in \partial J_i(u(t)), \ \ i = 1, 2,$$

and $\frac{d}{dt}(J_i \circ u)(\cdot) \in L_1(\tau, T)$.

Consider $W_\tau^T = C([\tau, T]; X)$. Lebourgue's mean value theorem (see Clarke [8, Chap. 2]) provides the existence of constants $c_1, c_2 > 0$ and $\mu \in (0, \lambda_1)$ such that

$$|J(u)| \leq c_1(1 + \|u\|_H^2), \quad J(u) \geq -\frac{\mu}{2}\|u\|_H^2 - c_2 \quad \forall u \in H. \tag{6.42}$$

The weak solution of Problem (6.37) with initial data

$$u(\tau) = a, \quad u'(\tau) = b \tag{6.43}$$

on the interval $[\tau, T]$ exists for any $a \in V$, $b \in H$. It follows from Zadoianchuk and Kasyanov [29, Theorem 1.4]. Thus the next lemma holds true (see Kasyanov et al. [16, Lemma 3.2]).

Lemma 6.8 (Kasyanov et al. [16, Lemma 3.2]) *For any* $\tau < T$, $a \in V$, $b \in H$, *Cauchy Problem (6.37), (6.43) has the weak solution* $(u, u_t)^T \in L_\infty(\tau, T; X)$. *Moreover, each weak solution* $(u, u_t)^T$ *of Cauchy Problem (6.37), (6.43) on the interval* $[\tau, T]$ *belongs to the space* $C([\tau, T]; X)$ *and* $u_{tt} \in L_2(\tau, T; V^*)$.

For any $\varphi_\tau = (a, b)^T \in X$, denote

$$\mathscr{D}_{\tau,T}(\varphi_\tau) = \left\{ (u(\cdot), u_t(\cdot))^T \ \middle| \begin{array}{l} (u, u_t)^T \text{is a weak solution of Problem (6.37) on } [\tau, T], \\ u(\tau) = a, u_t(\tau) = b \end{array} \right\}.$$

From Lemma 6.8 it follows that $\mathscr{D}_{\tau,T}(\varphi_\tau) \subset C([\tau, T]; X) = W_\tau^T$. Let us check that translation and concatenation of weak solutions are weak solutions too.

Lemma 6.9 *If* $\tau < T$, $\varphi_\tau \in X$, $\varphi(\cdot) \in \mathscr{D}_{\tau,T}(\varphi_\tau)$, *then* $\forall s$ $\psi(\cdot) = \varphi(\cdot + s) \in \mathscr{D}_{\tau-s,T-s}(\varphi_\tau)$. *If* $\tau < t < T$, $\varphi_\tau \in X$, $\varphi(\cdot) \in \mathscr{D}_{\tau,t}(\varphi_\tau)$ *and* $\psi(\cdot) \in \mathscr{D}_{t,T}(\varphi_\tau)$, *then*

$$\theta(s) = \begin{cases} \varphi(s), & s \in [\tau, t], \\ \psi(s), & s \in [t, T] \end{cases} \in \mathscr{D}_{\tau,T}(\varphi_\tau).$$

Proof The proof is trivial (see Kasyanov et al. [16, Lemma 4.1]).

Let $\varphi = (a, b)^T \in X$ and

$$\mathscr{V}(\varphi) = \frac{1}{2}\|\varphi\|_X^2 + J_1(a) - J_2(a). \tag{6.44}$$

Lemma 6.10 *Let* $\tau < T$, $\varphi_\tau \in X$, $\varphi(\cdot) = (u(\cdot), u_t(\cdot))^T \in \mathscr{D}_{\tau,T}(\varphi_\tau)$. *Then* $\mathscr{V} \circ \varphi : [\tau, T] \to \mathbb{R}$ *is absolutely continuous and for a.e.* $t \in (\tau, T)$, $\frac{d}{dt}\mathscr{V}(\varphi(t)) = -\beta\|u_t(t)\|_H^2$.

Proof Let $-\infty < \tau < T < +\infty$, $\varphi(\cdot) = (u(\cdot), u_t(\cdot))^T \in W_\tau^T$ be an arbitrary weak solution of Problem (6.37) on (τ, T). Since $\partial J(u(\cdot)) \subset L_2(\tau, T; H)$, from Temam [26] and Zgurovsky et al. [31, Chap. 2] we obtain that the function $t \to \|u_t(t)\|_H^2 + \|u(t)\|_V^2$ is absolutely continuous and for a.e. $t \in (\tau, T)$,

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\left[\|u_t(t)\|_{\mathrm{H}}^2 + \|u(t)\|_{\mathrm{V}}^2\right] = (u_{tt}(t) - \triangle u(t), u_t(t))_{\mathrm{H}} = \\ = -\beta\|u_t(t)\|_{\mathrm{H}}^2 - (d_1(t), u_t(t))_{\mathrm{H}} + (d_2(t), u_t(t))_{\mathrm{H}}, \end{aligned} \tag{6.45}$$

where $d_i(t) \in \partial J_i(u(t))$ for a.e. $t \in (\tau, T)$ and $d_i(\cdot) \in L_2(\tau, T; H)$, $i = 1, 2$. As $u(\cdot) \in C^1([\tau, T]; H)$ and $J_i : H \to \mathbb{R}$, $i = 1, 2$ is regular and locally Lipschitz, due to Lemma 6.7 we obtain that for a.e. $t \in (\tau, T)$, $\exists \frac{d}{dt}(J_i \circ u)(t)$, $i = 1, 2$. Moreover, $\frac{d}{dt}(J_i \circ u)(\cdot) \in L_1(\tau, T)$, $i = 1, 2$ and for a.e. $t \in (\tau, T)$, $\forall p \in \partial J_i(u(t))$,

$$\frac{d}{dt}(J_i \circ u)(t) = (p, u_t(t))_H, \quad i = 1, 2.$$

In particular for a.e. $t \in (\tau, T)$, $\frac{d}{dt}(J_i \circ u)(t) = (d_i(t), u_t(t))_H$. Taking into account (6.45) we finally obtain the necessary statement.

This completes the proof.

Lemma 6.11 *Let* $T > 0$. *Then any weak solution of Problem (6.37) on* $[0, T]$ *can be extended to a global one defined on* $[0, +\infty)$.

Proof The statement of this lemma follows from Lemmas 6.8–6.10, (6.42) and from the next estimates

$$\forall \tau < T, \ \ \forall t \in [\tau, T], \ \ \forall \varphi_\tau \in X, \ \ \forall \varphi(\cdot) = (u(\cdot), u_t(\cdot))^T \in \mathscr{D}_{\tau,T}(\varphi_\tau),$$

$$\begin{aligned} 2c_1 + \left(1 + \tfrac{2c_1}{\lambda_1}\right)\|u(\tau)\|_V^2 + \|u_t(\tau)\|_H^2 \ \ \geq 2\mathscr{V}(\varphi(\tau)) \geq 2\mathscr{V}(\varphi(t)) \ \ = \\ = \|u(t)\|_V^2 + \|u_t(t)\|_H^2 + 2J(u(t)) \geq \left(1 - \tfrac{\mu}{\lambda_1}\right)\|u(t)\|_V^2 + \|u_t(t)\|_H^2 - 2c_2. \end{aligned}$$

The lemma is proved.

For an arbitrary $\varphi_0 \in X$ let $\mathscr{D}(\varphi_0)$ be the set of all weak solutions (defined on $[0, +\infty)$) of Problem (6.37) with initial data $\varphi(0) = \varphi_0$. We remark that from the proof of Lemma 6.11 we obtain the next corollary.

Corollary 6.1 *For any* $\varphi_0 \in X$ *and* $\varphi \in \mathscr{D}(\varphi_0)$, *the next inequality is fulfilled*

$$\|\varphi(t)\|_X^2 \leq \frac{\lambda_1 + 2c_1}{\lambda_1 - \mu}\|\varphi(0)\|_X^2 + \frac{2(c_1 + c_2)\lambda_1}{\lambda_1 - \mu} \quad \forall t > 0. \tag{6.46}$$

From Corollary 6.1 in a standard way we obtain such statement.

Theorem 6.3 *Let* $\tau < T$, $\{\varphi_n(\cdot)\}_{n\geq 1} \subset W_\tau^T$ *be an arbitrary sequence of weak solutions of Problem (6.37) on* $[\tau, T]$ *such that* $\varphi_n(\tau) \to \varphi_\tau$ *weakly in* X, $n \to +\infty$, *and let* $\{t_n\}_{n\geq 1} \subset [\tau, T]$ *be a sequence such that* $t_n \to t_0$, $n \to +\infty$. *Then there exists* $\varphi \in \mathscr{D}_{\tau,T}(\varphi_\tau)$ *such that up to a subsequence* $\varphi_n(t_n) \to \varphi(t_0)$ *weakly in* X, $n \to +\infty$.

Proof We prove this theorem in several steps.

Step 1. Let $\tau < T$, $\{\varphi_n(\cdot) = (u_n(\cdot), u_n'(\cdot))\}_{n\geq 1} \subset W_\tau^T$ be an arbitrary sequence of weak solutions of Problem (6.37) on $[\tau, T]$ and $\{t_n\}_{n\geq 1} \subset [\tau, T]$ such that

$$\varphi_n(\tau) \to \varphi_\tau \text{ weakly in } X, \quad t_n \to t_0, \quad n \to +\infty. \tag{6.47}$$

In virtue of Corollary 6.1 we have that $\{\varphi_n(\cdot)\}_{n\geq 1}$ is bounded on $W_\tau^T \subset L_\infty(\tau, T; X)$. Therefore up to a subsequence $\{\varphi_{n_k}(\cdot)\}_{k\geq 1} \subset \{\varphi_n(\cdot)\}_{n\geq 1}$ we have

$$\begin{gathered}
u_{n_k} \to u \text{ weakly star in } L_\infty(\tau, T; V),\ k \to +\infty,\\
u_{n_k}' \to u' \text{ weakly star in } L_\infty(\tau, T; H),\ k \to +\infty,\\
u_{n_k}'' \to u'' \text{ weakly star in } L_\infty(\tau, T; V^*),\ k \to +\infty,\\
d_{n_k,i} \to d_i \text{ weakly star in } L_\infty(\tau, T; H),\ i = \overline{1, 2},\ k \to +\infty,\\
u_{n_k} \to u \text{ in } L_2(\tau, T; H),\ k \to +\infty,\\
u_{n_k}(t) \to u(t) \text{ in } H \text{ for a.e. } t \in [\tau, T],\ k \to +\infty,\\
u_{n_k}'(t) \to u'(t) \text{ in } V^* \text{ for a.e. } t \in (\tau, T),\ k \to +\infty,\\
\triangle u_{n_k} \to \triangle u \text{ weakly in } L_2(\tau, T; V^*),\ k \to +\infty,
\end{gathered} \tag{6.48}$$

where $\forall n \geq 1$ $d_{n,i} \in L_2(\tau, T; H)$ and

$$\begin{gathered}
u_n''(t) + \beta u_n'(t) + d_{n,1}(t) - d_{n,2}(t) - \triangle u_n(t) = \bar{0},\\
d_{n,i}(t) \in \partial J_i(u_n(t)), \quad i = 1, 2, \quad \text{for a.e. } t \in (\tau, T).
\end{gathered} \tag{6.49}$$

Step 2. ∂J_i, $i = 1, 2$ are demiclosed. So, by a standard way we get that $d_i(\cdot) \in \partial J_i(u(\cdot))$, $i = 1, 2$, $\varphi := (u, u') \in \mathscr{D}_{\tau,T}(\varphi_\tau) \subset W_\tau^T$.

Step 3. From (6.48) it follows that for arbitrary fixed $h \in V$ the sequences of real functions $(u_{n_k}(\cdot), h)_H$, $(u_{n_k}'(\cdot), h)_H : [\tau, T] \to \mathbb{R}$ are uniformly bounded and equipotentionally continuous. Taking into account (6.48), (6.46) and density of the embedding $V \subset H$ we obtain that $u_{n_k}'(t_{n_k}) \to u'(t_0)$ weakly in H and $u_{n_k}(t_{n_k}) \to u(t_0)$ weakly in V as $k \to +\infty$.

The theorem is proved.

Theorem 6.4 *Let* $\tau < T$, $\{\varphi_n(\cdot)\}_{n\geq 1} \subset W_\tau^T$ *be an arbitrary sequence of weak solutions of Problem (6.37) on* $[\tau, T]$ *such that* $\varphi_n(\tau) \to \varphi_\tau$ *strongly in* X, $n \to +\infty$, *then up to a subsequence* $\varphi_n(\cdot) \to \varphi(\cdot)$ *in* $C([\tau, T]; X)$, $n \to +\infty$.

Proof Let $\tau < T$, $\{\varphi_n(\cdot) = (u_n(\cdot), u_n'(\cdot))^T\}_{n\geq 1} \subset W_\tau^T$ be an arbitrary sequence of weak solutions of Problem (6.37) on $[\tau, T]$ and $\{t_n\}_{n\geq 1} \subset [\tau, T]$:

$$\varphi_n(\tau) \to \varphi_\tau \text{ strongly in } X,\ n \to +\infty. \tag{6.50}$$

From Theorem 6.3 we have that there exists $\varphi \in \mathscr{D}_{\tau,T}(\varphi_\tau)$ such that up to the subsequence $\{\varphi_{n_k}(\cdot)\}_{k\geq 1} \subset \{\varphi_n(\cdot)\}_{n\geq 1}$ $\varphi_n(\cdot) \to \varphi(\cdot)$ weakly in X, uniformly on $[\tau, T]$, $k \to +\infty$. Let us prove that

$$\varphi_{n_k} \to \varphi \text{ in } W_\tau^T, \quad k \to +\infty. \tag{6.51}$$

By contradiction, suppose the existence of $L > 0$ and the subsequence $\{\varphi_{k_j}\}_{j\geq 1} \subset \{\varphi_{n_k}\}_{k\geq 1}$ such that $\forall j \geq 1$,

$$\max_{t\in[\tau,T]} \|\varphi_{k_j}(t) - \varphi(t)\|_X = \|\varphi_{k_j}(t_j) - \varphi(t_j)\|_X \geq L.$$

Without loss of generality we suggest that $t_j \to t_0 \in [\tau, T]$, $j \to +\infty$. Therefore by virtue of a continuity of $\varphi : [\tau, T] \to X$ we have

$$\varliminf_{j\to+\infty} \|\varphi_{k_j}(t_j) - \varphi(t_0)\|_X \geq L. \tag{6.52}$$

On the other hand, we prove that

$$\varphi_{k_j}(t_j) \to \varphi(t_0) \text{ in } X,\ \ j \to +\infty. \tag{6.53}$$

First we remark that

$$\varphi_{k_j}(t_j) \to \varphi(t_0) \text{ weakly in } X, \quad j \to +\infty \tag{6.54}$$

(see Theorem 6.3 for details). Secondly let us prove that

$$\limsup_{j\to+\infty} \|\varphi_{k_j}(t_j)\|_X \leq \|\varphi(t_0)\|_X. \tag{6.55}$$

Since J is sequentially weakly continuous, $\mathscr{V}$ is sequentially weakly lower semi-continuous on X. Hence we obtain

$$\begin{aligned} &\mathscr{V}(\varphi(t_0)) \leq \varliminf_{j\to+\infty} \mathscr{V}(\varphi_{k_j}(t_j)), \\ &\int_\tau^{t_0} \|u'(s)\|_H^2 ds \leq \varliminf_{j\to+\infty} \int_\tau^{t_j} \|u'_{k_j}(s)\|_H^2 ds \end{aligned} \tag{6.56}$$

and

$$\mathscr{V}(\varphi(t_0)) + \beta \int_\tau^{t_0} \|u'(s)\|_H^2 ds \leq \varliminf_{j\to+\infty} \left(\mathscr{V}(\varphi_{k_j}(t_j)) + \beta \int_\tau^{t_j} \|u'_{k_j}(s)\|_{\mathrm{H}}^2 ds \right). \tag{6.57}$$

Since by the energy equation both sides of (6.57) equal $\mathscr{V}(\varphi(\tau))$ (see Lemma 6.10), it follows from (6.56) that $\mathscr{V}(\varphi_{k_j}(t_j)) \to \mathscr{V}(\varphi(t_0))$, $j \to +\infty$ and (6.55). Convergence (6.53) directly follows from (6.54), (6.55) and Gajewski et al. [12, Chap. I]. To finish the proof of the theorem we remark that (6.53) contradicts (6.52). Therefore (6.51) holds.

The theorem is proved.

Now let us consider autonomous second order differential-operator inclusions with possibly nonmonotone potential. Let V and H be real separable Hilbert spaces such that $V \subset H$ with compact and dense embedding. Let V^* be the dual space of V. We identify H with H^* (dual space of H). For the linear operators $A : V \to V^*$, $B : V \to V^*$ and locally Lipschitz functional $J : H \to \mathbb{R}$ we consider a problem of investigation of dynamics for all weak solutions defined for $t \geq 0$ of non-linear second order autonomous differential-operator inclusion:

$$u''(t) + Au'(t) + Bu(t) + \partial J_1(u(t)) - \partial J_2(u(t)) \ni \bar{0}. \tag{6.58}$$

We need the following hypotheses:.

$\underline{H(A)}$ $A : V \to V^*$ is a linear symmetric such that $\exists c_A > 0 : \langle Av, v\rangle_V \geq c_A\|v\|_V^2$ $\forall v \in V$;

$\underline{H(B)}$ $B : V \to V^*$ is linear, symmetric and $\exists c_B > 0 : \langle Bv, v\rangle_V \geq c_B\|v\|_V^2$ $\forall v \in V$[2];

$\underline{H(J)}$ $J_i : H \to \mathbb{R}$, $i = 1, 2$, is a function such that

(i) $J_i(\cdot)$ is locally Lipschitz and regular [8, Chap. II], i.e.,

• for any $x, v \in H$, the usual one-sided directional derivative $J'(x; v) = \lim\limits_{t\searrow 0} \frac{J(x+tv)-J(x)}{t}$ exists,

• for all $x, v \in H$, $J'(x; v) = J^\circ(x; v)$, where $J^\circ(x; v) = \overline{\lim\limits_{y\to x,\, t\searrow 0}} \frac{J(y+tv)-J(y)}{t}$;

(ii) $\exists c_1 > 0$: $\sup\limits_{d\in\partial J_1(v)-\partial J_2(v)} \|d\|_H \leq c_1(1 + \|v\|_H)$ $\forall v \in H$;

(iii) $\exists c_2 > 0$:

$$\inf_{d\in\partial J_1(v)-\partial J_2(v)} (d, v)_H \geq -\lambda\|v\|_H^2 - c_2 \quad \forall v \in H,$$

where $\partial J_i(v) = \{p \in H \mid (p, w)_H \leq J_i^\circ(v; w)\ \forall w \in H\}$ denotes the Clarke subdifferential of $J_i(\cdot)$ at a point $v \in H$ (see [8] for details), $\lambda \in (0, \lambda_1)$, $\lambda_1 > 0$: $\|v\|_V^2 \geq \lambda_1\|v\|_H^2$ $\forall v \in V$, $i = 1, 2$.

We note that in (6.89) we can consider $g = \bar{0}$. Indeed, let $g \in V^*$, then $u^* = B^{-1}g \in V \subset H$. If $u(\cdot)$ is a weak solution of (6.89), then $\bar{u}(\cdot) = u(\cdot) - u^*$ is a weak

[2] We remark that operators A and B are continuous on V [12, Chap. III].

solution of

$$\bar{u}''(t) + A\bar{u}'(t) + B\bar{u}(t) + \partial J_3(\bar{u}(t)) - \partial J_4(\bar{u}(t)) \ni \bar{0} \quad \text{a.e.} \quad t > 0,$$

where $J_{i+2}(\cdot) = J_i(\cdot + u^*)$, $i = 1, 2$, satisfies $\underline{H(J)}$ with respective parameters. Thus, to simplify our conclusions, without loss of generality, further we will consider problem (6.58).

The phase space for Problem (6.58) is the Hilbert space:

$$E = \{(a, b)^T \mid a \in V, \ b \in H\},$$

where $(a, b)^T = \begin{pmatrix} a \\ b \end{pmatrix}$ with $\|(a, b)^T\|_E = (\|a\|_V^2 + \|b\|_H^2)^{1/2}$. Let $-\infty < \tau < T < +\infty$.

Definition 6.2 The function $(u(\cdot), u'(\cdot))^T \in L_\infty(\tau, T; E)$ with $u'(\cdot) \in L_2(\tau, T; V)$ is called a *weak solution* for (6.58) on $[\tau, T]$, if there exists $d(\cdot) \in L_2(\tau, T; H)$, $d(t) \in \partial J_1(u(t)) - \partial J_2(u(t))$ for a.e. $t \in (\tau, T)$, such that $\forall \psi \in V, \forall \eta \in C_0^\infty(\tau, T)$

$$-\int_\tau^T (u'(t), \psi)_H \eta'(t) dt + \int_\tau^T \left[\langle Au'(t), \psi\rangle_V + \langle Bu(t), \psi\rangle_V + (d(t), \psi)_H\right] \eta(t) dt = 0.$$

Evidently if $(u(\cdot), u'(\cdot))^T$ is a weak solution of (6.58) on $[\tau, T]$, then $u''(\cdot) \in L_2(\tau, T; V^*)$, $(u(\cdot), u'(\cdot))^T \in C([\tau, T]; E)$ and $d(\cdot) \in L_\infty(\tau, T; H)$.

We consider the class of functions $W_\tau^T = C([\tau, T]; E)$. Further c_1, c_2, λ, λ_1, c_A, c_B we recall parameters of Problem (6.58). The main purpose of this work is to investigate the long-time behavior (as $t \to +\infty$) of all weak solutions for the problem (6.58) on $[0, +\infty)$.

To simplify our conclusions, since condition $\underline{H(B)}$, we suppose that

$$(u, v)_V = \langle Bu, v\rangle_V, \ \|v\|_V^2 = \langle Bv, v\rangle_V, \ c_B = 1, \ \forall u, v \in V. \tag{6.59}$$

Lebourg mean value theorem [8, Chap. 2] provides the existence of constants $c_3, c_4 > 0$ and $\mu \in (0, \lambda_1)$:

$$|J(u)| \le c_3(1 + \|u\|_H^2), \quad J(u) \ge -\frac{\mu}{2}\|u\|_H^2 - c_4 \quad \forall u \in H, \tag{6.60}$$

where $J(v) := J_1(v) - J_2(v)$, $v \in H$.

Lemma 6.12 *Let $J : H \to \mathbb{R}$ be a locally Lipschitz and regular functional, $y \in C^1([\tau, T]; H)$. Then for a.e. $t \in (\tau, T)$ there exists $\frac{d}{dt}(J \circ y)(t) = (p, y'(t))$ for all $p \in \partial J(y(t))$. Moreover, $\frac{d}{dt}(J \circ y)(\cdot) \in L_1(\tau, T)$.*

Proof Since $y \in C^1([\tau, T]; H)$ then y is strictly differentiable at the point t_0 for any $t_0 \in (\tau, T)$. Hence, taking into account the regularity of J and [8, Theorem 2.3.10], we obtain that the functional $J \circ y$ is regular one at $t_0 \in (\tau, T)$ and $\partial(J \circ y)(t_0) = \{(p, y'(t_0))|\ p \in \partial J(y(t_0))\}$. On the other hand, since $y \in C^1([\tau, T]; H)$, $J : H \to \mathbb{R}$ is locally Lipschitz then $J \circ y : [\tau, T] \to \mathbb{R}$ is globally Lipschitz and therefore it is absolutely continuous. Hence for a.e. $t \in (\tau, T)$ $\exists \frac{d(J\circ y)(t)}{dt}$, $\frac{d(J\circ y)(\cdot)}{dt} \in L_1(\tau, T)$ and $\int\limits_s^t \frac{d}{d\xi}(J \circ y)(\xi)d\xi = (J \circ y)(t) - (J \circ y)(s)$ $\forall \tau \le s < t \le T$. At that taking into account the regularity of $J \circ y$, note that $(J \circ y)^\circ(t_0, v) = (J \circ y)'(t_0, v) = \frac{d(J\circ y)(t_0)}{dt} \cdot v$ for a.e. $t_0 \in (\tau, T)$ and all $v \in \mathbb{R}$. This implies that for a.e. $t_0 \in (\tau, T)$ $\partial(J \circ y)(t_0) = \{\frac{d(J\circ y)(t_0)}{dt}\}$.

At the inclusion (6.58) on $[\tau, T]$ we associate the conditions

$$u(\tau) = a, \quad u'(\tau) = b \tag{6.61}$$

where $a \in V$ and $b \in H$. From [29] we get the following lemma.

Lemma 6.13 *For any $\tau < T$, $a \in V$, $b \in H$ the Cauchy problem (6.58), (6.61) has a weak solution $(y, y')^T \in L_\infty(\tau, T; E)$. Moreover, each weak solution $(y, y')^T$ of the Cauchy problem (6.58), (6.61) on the interval $[\tau, T]$ belongs to the space $C([\tau, T]; E)$ and $y' \in L_2(\tau, T; V)$, $y'' \in L_2(\tau, T; V^*)$.*

Let us consider the next denotations: $\forall \varphi_\tau = (a, b)^T \in E$ we set $\mathscr{D}_{\tau,T}(\varphi_\tau) = \{ (u(\cdot), u'(\cdot))^T \mid (u, u')^T$ is a weak solution of (6.58) on $[\tau, T]$, $u(\tau) = a$, $u'(\tau) = b \}$. From Lemma 6.13 it follows that $\mathscr{D}_{\tau,T}(\varphi_\tau) \subset C([\tau, T]; E) = W_\tau^T$.

Let us check that translation and concatenation of weak solutions are weak solutions too.

Lemma 6.14 *If $\tau < T$, $\varphi_\tau \in E$, $\varphi(\cdot) \in \mathscr{D}_{\tau,T}(\varphi_\tau)$, then $\psi(\cdot) = \varphi(\cdot + s) \in \mathscr{D}_{\tau-s,T-s}(\varphi_\tau)$ $\forall s$. If $\tau < t < T$, $\varphi_\tau \in E$, $\varphi(\cdot) \in \mathscr{D}_{\tau,t}(\varphi_\tau)$ and $\psi(\cdot) \in \mathscr{D}_{t,T}(\varphi(t))$, then $\theta(s) = \begin{cases} \varphi(s), & s \in [\tau, t], \\ \psi(s), & s \in [t, T] \end{cases}$ belongs to $\mathscr{D}_{\tau,T}(\varphi_\tau)$.*

Proof The first part of the statement of this lemma follows from the autonomy of the inclusion (6.58). The proof of the second part follows from the definition of the solution of (6.58) and from that fact that $z \in W_{\tau,T}$ as soon as $v \in W_{\tau,t}$, $u \in W_{t,T}$ and $v(t) = u(t)$, where

$$z(s) = \begin{cases} v(s), & s \in [\tau, t], \\ u(s), & s \in [t, T] \end{cases}$$

Let $\varphi = (a, b)^T \in E$ and

$$\mathscr{V}(\varphi) = \frac{1}{2}\|\varphi\|_E^2 + J(a). \tag{6.62}$$

Lemma 6.15 *Let $\tau < T$, $\varphi_\tau \in E$, $\varphi(\cdot) = (y(\cdot), y'(\cdot))^T \in \mathscr{D}_{\tau,T}(\varphi_\tau)$. Then $\mathscr{V} \circ \varphi : [\tau, T] \to \mathbb{R}$ is absolutely continuous and for a.e. $t \in (\tau, T)$ $\frac{d}{dt}\mathscr{V}(\varphi(t)) = -\langle Ay'(t), y'(t)\rangle_V \le -\gamma \|y'(t)\|_V^2$, where $\gamma > 0$ depends only on parameters of Problem (6.58).*[3]

Proof Let $-\infty < \tau < T < +\infty$, $\varphi(\cdot) = (y(\cdot), y'(\cdot))^T \in W_\tau^T$ be an arbitrary weak solution of (6.58) on (τ, T). From [12, Chap. IV] we get that the function $t \to \|y'(t)\|_H^2 + \|y(t)\|_V^2$ is absolutely continuous and for a.e. $t \in (\tau, T)$

$$\frac{1}{2}\frac{d}{dt}\left[\|y'(t)\|_H^2 + \|y(t)\|_V^2\right] = \langle y''(t) + By(t), y'(t)\rangle_V =$$

$$= -\langle Ay'(t), y'(t)\rangle_V - (d(t), y'(t))_H \le -\gamma\|y'(t)\|_V^2 - (d(t), y'(t))_H, \quad (6.63)$$

where $d(t) \in \partial J_1(y(t)) - \partial J_2(y(t))$ for a.e. $t \in (\tau, T)$, $d(\cdot) \in L_2(\tau, T; H)$ and $\gamma > 0$ depends only on parameters of Problem (6.58), in virtue of $u \to \sqrt{\langle Au, u\rangle_V}$ is equivalent norm on V. Since $y(\cdot) \in C^1([\tau, T]; H)$ and $J_i : H \to \mathbb{R}$, $i = 1, 2$, is regular and locally Lipschitz, then Lemma 6.12 yields that for a.e. $t \in (\tau, T)$ there exists $\frac{d}{dt}(J_i \circ y)(t)$. Moreover, $\frac{d}{dt}(J_i \circ y)(\cdot) \in L_1(\tau, T)$ and for a.e. $t \in (\tau, T)$ and all $p \in \partial J_1(y(t)) - \partial J_2(y(t))$ we have $\frac{d}{dt}(J \circ y)(t) = (p, y'(t))_H$. In particular, for a.e. $t \in (\tau, T)$ $\frac{d}{dt}(J \circ y)(t) = (d(t), y'(t))_H$. Taking into account (6.63) we finally obtain the necessary statement.

The lemma is proved.

Lemma 6.16 *Let $T_0 > 0$. If $(u(\cdot), u'(\cdot))^T$ is a weak solution of (6.58) on $[0, T_0]$, then there exists an its extension on $[0, +\infty)$ $(\bar{u}(\cdot), \bar{u}'(\cdot))^T$ which is weak solution of (6.58) on $[0, +\infty)$, i.e., $(\bar{u}(\cdot), \bar{u}'(\cdot))^T \in C(\mathbb{R}_+; E) \cap L_\infty(\mathbb{R}_+; E)$ with $\bar{u}'(\cdot) \in L_2(0, T; V)$ $\forall T > 0$ and there exists $d(\cdot) \in L_\infty(0, +\infty; H)$, $d(t) \in \partial J_1(\bar{u}(t)) - \partial J_2(\bar{u}(t))$ for a.e. $t \in (0, +\infty)$, such that $\forall \psi \in V$, $\forall \eta \in C_0^\infty(0, +\infty)$*

$$-\int_0^{+\infty} (\bar{u}'(t), \psi)_H \eta'(t)dt + \int_0^{+\infty} [\langle A\bar{u}'(t), \psi\rangle_V + \langle B\bar{u}(t), \psi\rangle_V + (d(t), \psi)_H]\eta(t)dt = 0.$$

Proof The statement of this lemma follows from Lemmas 6.13–6.15, Conditions (6.59), (6.60) and from the next estimates: $\forall \tau < T, \forall \varphi_\tau \in E, \forall \varphi(\cdot) = (y(\cdot), y'(\cdot))^T \in \mathscr{D}_{\tau,T}(\varphi_\tau)$, $\forall t \in [\tau, T]$ $2c_3 + \left(1 + \frac{2c_3}{\lambda_1}\right)\|y(\tau)\|_V^2 + \|y'(\tau)\|_H^2 \ \ge 2\mathscr{V}(\varphi(\tau)) \ge 2\mathscr{V}(\varphi(t)) = \|y(t)\|_V^2 + \|y'(t)\|_H^2 + 2J(y(t)) \ge \left(1 - \frac{\mu}{\lambda_1}\right)\|y(t)\|_V^2 + \|y'(t)\|_H^2 - 2c_4$.

The lemma is proved.

For an arbitrary $\varphi_0 \in E$ let $\mathscr{D}(\varphi_0)$ be the set of all weak solutions (defined on $[0, +\infty)$) of problem (6.58) with initial data $\varphi(0) = \varphi_0$. We remark that from the proof of Lemma 6.16 we obtain the next corollary.

Corollary 6.2 *For any $\varphi_0 \in E$ and $\varphi \in \mathscr{D}(\varphi_0)$ the next inequality is fulfilled:*

[3] We remark that $\sqrt{\langle Au, u\rangle_V}$ is equivalent norm on V, generated by inner product $\langle Au, v\rangle_V$.

$$\|\varphi(t)\|_E^2 \le \frac{\lambda_1 + 2c_3}{\lambda_1 - \mu}\|\varphi(0)\|_E^2 + \frac{2(c_3+c_4)\lambda_1}{\lambda_1 - \mu} \quad \forall t > 0. \tag{6.64}$$

From Corollary 6.2 and Conditions $\underline{H(A)}$, $\underline{H(B)}$, $\underline{H(J)}$ in a standard way we obtain such proposition.

Theorem 6.5 *Let $\tau < T$, $\{\varphi_n(\cdot)\}_{n\ge 1} \subset W_\tau^T$ be an arbitrary sequence of weak solutions of (6.58) on $[\tau, T]$ such that $\varphi_n(\tau) \to \varphi_\tau$ weakly in E, $n \to +\infty$. Then there exist $\varphi \in \mathscr{D}_{\tau,T}(\varphi_\tau)$ and $\{\varphi_{n_k}(\cdot)\}_{k\ge 1} \subset \{\varphi_n(\cdot)\}_{n\ge 1}$ such that $\varphi_{n_k}(\cdot) \to \varphi(\cdot)$ weakly in E uniformly on $[\tau, T]$, $k \to +\infty$, i.e., $\varphi_{n_k}(t_k) \to \varphi(t_0)$ weakly in E, $k \to +\infty$, for any $\{t_k\}_{k\ge 1} \subset [\tau, T]$ with $t_k \to t_0$, $k \to +\infty$.*

Theorem 6.6 *Let $\tau < T$, $\{\varphi_n(\cdot)\}_{n\ge 1} \subset W_\tau^T$ be an arbitrary sequence of weak solutions of (6.58) on $[\tau, T]$ such that $\varphi_n(\tau) \to \varphi_\tau$ strongly in E, $n \to +\infty$. Then there exist $\varphi \in \mathscr{D}_{\tau,T}(\varphi_\tau)$ such that up to a subsequence $\varphi_n(\cdot) \to \varphi(\cdot)$ in $C([\tau, T]; E)$, $n \to +\infty$.*

Proof Let $\{\varphi_n(\cdot) = (u_n(\cdot), u_n'(\cdot))^T\}_{n\ge 1} \subset W_\tau^T$ be an arbitrary sequence of weak solutions of (6.58) on $[\tau, T]$ such that

$$\varphi_n(\tau) \to \varphi_\tau \text{ strongly in } E,\ n \to +\infty. \tag{6.65}$$

Let $\varphi = (u(\cdot), u'(\cdot))^T \in \mathscr{D}_{\tau,T}(\varphi_\tau)$ and $\{\varphi_{n_k}(\cdot)\}_{k\ge 1} \subseteq \{\varphi_n(\cdot)\}_{n\ge 1}$ as in Theorem 6.5. It is important to remark that in the proof of Theorem 6.5, by using the inequality (Lemma 6.15, Corollary 6.2, (6.60))

$$\gamma \|u_n'(\cdot)\|_{L_2(\tau,T;V)} \le \mathscr{V}(\varphi_n(\tau)) - \mathscr{V}(\varphi_n(T)) \le \sup_{n\ge 1}\left[\mathscr{V}(\varphi_n(\tau)) + \frac{\mu}{2}\|u_n(T)\|_H^2\right] + c_4 < +\infty,$$

we establish that

$$u_{n_k}'(\cdot) \to u'(\cdot) \text{ weakly in } L_2(\tau, T; V),\ k \to +\infty.$$

Let us prove that

$$\varphi_{n_k} \to \varphi \text{ in } W_\tau^T, \quad k \to +\infty. \tag{6.66}$$

By contradiction suppose the existence of $L > 0$ and subsequence $\{\varphi_{k_j}\}_{j\ge 1} \subset \{\varphi_{n_k}\}_{k\ge 1}$ such that $\forall j \ge 1$ $\max_{t\in[\tau,T]} \|\varphi_{k_j}(t) - \varphi(t)\|_E = \|\varphi_{k_j}(t_j) - \varphi(t_j)\|_E \ge L$. Without loss of generality we suppose that $t_j \to t_0 \in [\tau, T]$, $j \to +\infty$. Therefore, by virtue of the continuity of $\varphi : [\tau, T] \to E$, we have

$$\varliminf_{j\to+\infty} \|\varphi_{k_j}(t_j) - \varphi(t_0)\|_E \ge L. \tag{6.67}$$

On the other hand we prove that

$$\varphi_{k_j}(t_j) \to \varphi(t_0) \text{ in } E,\ j \to +\infty. \tag{6.68}$$

Firstly we remark that (Theorem 6.5)

$$\varphi_{k_j}(t_j) \to \varphi(t_0) \text{ weakly in } E, \quad j \to +\infty. \tag{6.69}$$

Secondly let us prove that

$$\lim_{j\to+\infty} \|\varphi_{k_j}(t_j)\|_E = \|\varphi(t_0)\|_E. \tag{6.70}$$

Since J is sequentially weakly continuous on V, $\mathscr{V}$ is sequentially weakly lower semicontinuous on E. Hence, we obtain

$$\mathscr{V}(\varphi(t_0)) \le \underline{\lim}_{j\to+\infty} \mathscr{V}(\varphi_{k_j}(t_j)), \quad \int_\tau^{t_0} \langle Au'(s), u'(s)\rangle_V ds \le \underline{\lim}_{j\to+\infty} \int_\tau^{t_j} \langle Au'_{k_j}(s), u'_{k_j}(s)\rangle_V ds, \tag{6.71}$$

and hence

$$\begin{aligned} \mathscr{V}(\varphi(t_0)) + \int_\tau^{t_0} \langle Au'(s), u'(s)\rangle_V ds &\le \underline{\lim}_{j\to+\infty} \mathscr{V}(\varphi_{k_j}(t_j)) + \int_\tau^{t_0} \langle Au'(s), u'(s)\rangle_V ds \le \\ &\le \overline{\lim}_{j\to+\infty} \mathscr{V}(\varphi_{k_j}(t_j)) + \int_\tau^{t_0} \langle Au'(s), u'(s)\rangle_V ds \\ &\le \overline{\lim}_{j\to+\infty} \mathscr{V}(\varphi_{k_j}(t_j)) + \underline{\lim}_{j\to+\infty} \int_\tau^{t_j} \langle Au'_{k_j}(s), u'_{k_j}(s)\rangle_V ds \le \\ &\le \overline{\lim}_{j\to+\infty} \left(\mathscr{V}(\varphi_{k_j}(t_j)) + \int_\tau^{t_j} \langle Au'_{k_j}(s), u'_{k_j}(s)\rangle_V ds \right). \end{aligned} \tag{6.72}$$

We remark that the last inequality in (6.71) follows from weak convergence of $u'_{n_k}(\cdot)$ to $u'(\cdot)$ in $L_2(\tau, T; V)$ and because of the functional $v \to \int_\tau^T \langle Av(s), v(s)\rangle_V ds$ is sequentially weakly lower semi-continuous on $L_2(\tau, T; V)$.

Since the energy equation and (6.65) both sides of (6.72) are equal to $\mathscr{V}(\varphi(\tau))$ (see Lemma 6.15), it follows that $\mathscr{V}(\varphi_{k_j}(t_j)) \to \mathscr{V}(\varphi(t_0))$, $j \to +\infty$ and then (6.70). Convergence (6.68) directly follows from (6.69), (6.70). Finally we remark that (6.68) contradicts (6.67). Therefore, (6.66) is true.

The theorem is proved.

6.3 Examples of Applications

In this section we consider the following examples of applications: nonlinear parabolic equations of divergent form, nonlinear problems on manifolds with and without boundary: a climate energy balance model; a model of conduction of electrical impulses in nerve axons; and viscoelastic problems with nonlinear "reaction-displacement" law.

6.3.1 Nonlinear Parabolic Equations of Divergent Form

Consider an example of the class of nonlinear boundary value problems for which we can investigate the dynamics of solutions as $t \to +\infty$. Note that in discussion we do not claim generality.

Let $n \geq 2, m \geq 1, p \geq 2, 1 < q \leq 2, \frac{1}{p} + \frac{1}{q} = 1, \Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\Gamma = \partial \Omega$. We denote a number of differentiations by x of order $\leq m-1$ (correspondingly of order $= m$) by N_1 (correspondingly by N_2). Let $A_\alpha(x, \eta; \xi)$ be a family of real functions ($|\alpha| \leq m$), defined in $\Omega \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and satisfying the next properties:

(C_1) for a.e. $(x, t) \in \Omega \times (0, \infty)$ the function $(\eta, \xi) \to A_\alpha(x, t, \eta, \xi)$ is continuous one in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$;

(C_2) for each $(\eta, \xi) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ the function $x \to A_\alpha(x, t, \eta, \xi)$ is measurable on $\Omega \times (0, \infty)$;

(C_3) there exist $c_1 \geq 0$ and $k_1 \in L_q(\Omega)$ such that

$$|A_\alpha(x, t, \eta, \xi)| \leq c_1[|\eta|^{p-1} + |\xi|^{p-1} + k_1(x)]$$

for a.e. $(x, t) \in \Omega \times (0, \infty)$ and for each $(\eta, \xi) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$;

(C_4) there exist $c_2 > 0$ and $k_2 \in L_1(\Omega)$ such that

$$\sum_{|\alpha|=m} A_\alpha(x, t, \eta, \xi)\xi_\alpha \geq c_2|\xi|^p - k_2(x)$$

for a.e. $(x, t) \in \Omega \times (0, \infty)$ and for each $(\eta, \xi) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$;

(C_5) there exists an increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ such that the following inequality holds:

$$\sum_{|\alpha|=m} (A_\alpha(x, t, \eta, \xi) - A_\alpha(x, t, \eta, \xi^*))(\xi_\alpha - \xi_\alpha^*) \geq (\varphi(|\xi_\alpha|) - \varphi(|\xi_\alpha^*|))(|\xi_\alpha| - |\xi_\alpha^*|)$$

for a.e. $(x, t) \in \Omega \times (0, \infty)$ and each $\eta \in \mathbb{R}^{N_1}$ and $\xi, \xi^* \in \mathbb{R}^{N_2}, \xi \neq \xi^*$.

Consider the following notations:

$$D^k u = \{D^\beta u, |\beta| = k\}, \quad \delta u = \{u, Du, \ldots, D^{m-1}u\}.$$

Let us examine the dynamics of all weak (generalized) solutions defined on $[0, +\infty)$ for the following problem:

$$\frac{\partial y(x,t)}{\partial t} + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^{\alpha}\left(A_{\alpha}(x, \delta y(x,t), D^m y(x,t))\right) = 0 \text{on} \Omega \times (0,+\infty), \tag{6.73}$$

$$D^{\alpha} y(x,t) = 0 \text{on } \Gamma \times (0,+\infty), \quad |\alpha| \le m-1. \tag{6.74}$$

as $t \to +\infty$.

Consider such denotations: $H = L_2(\Omega)$, $V = W_0^{m,p}(\Omega)$ is a real Sobolev space,

$$a(t,u,\omega) = \sum_{|\alpha|\le m} \int_{\Omega} A_{\alpha}(x,t,\delta u(x), D^m u(x)) D^{\alpha}\omega(x)dx, \quad u,\omega \in V.$$

Note that the operator $A(t) : V \to V^*, t \ge 0$, defined by the formula $\langle A(t,u),\omega\rangle_V = a(t,u,\omega)$, $t \ge 0$, $u,\omega \in V$, satisfies Assumptions (C1)–(C5). Therefore, we pass from Problem (6.73) and (6.74) to the respective problem in the "generalized" setting (6.1). Here we note that

$$A(t,u) = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^{\alpha}\left(A_{\alpha}(x,t,\delta u, D^m u)\right), \quad u \in C_0^{\infty}(\Omega),\ t \ge 0.$$

Therefore, all statements from Sect. 6.1 hold for all weak (generalized) solutions of Problem (6.73) and (6.74).

6.3.2 Nonlinear Non-autonomous Problems on Manifolds with and Without Boundary: A Climate Energy Balance Model

Let $(\mathscr{M}, \mathbf{g})$ be a C^{∞} compact connected oriented two-dimensional Riemannian manifold without boundary (as, e.g. $\mathscr{M} = S^2$ the unit sphere of $\mathbb{R}^3$). Consider the problem (see Sect. 2.4.3 for autonomous setting):

$$\tfrac{\partial u}{\partial t} - \Delta u + R_e(x,t,u) \in QS(x,t)\beta(u), \ \ (x,t) \in \mathscr{M} \times \mathbb{R}, \tag{6.75}$$

where $\Delta u = \text{div}_{\mathscr{M}}(\nabla_{\mathscr{M}} u)$; $\nabla_{\mathscr{M}}$ is understood in the sense of the Riemannian metric $\mathbf{g}$. Note that (6.75) is the so-called climate energy balance model (see Sect. 2.4.3). The unknown $u(x,t)$ represents the average temperature of the Earth's surface. The energy balance is expressed as

$$\text{heat variation} \ = \ R_a - R_e + D.$$

Here $R_a = QS(x,t)\beta(u)$. It represents the solar energy absorbed by the Earth, $Q > 0$ is a solar constant, $S(x,t)$ is an insolation function, given the distribution of solar radiation falling on upper atmosphere, β represents the ratio between absorbed and incident solar energy at the point x of the Earth's surface (so-called co-albedo function). The term R_e represents the energy emitted by the Earth into space, as usual, it is assumed to be an increasing function on u. The term D is heat diffusion, we assume (for simplicity) that it is constant. We consider $R_e = Bu$ as in Budyko; see [31] and references therein.

Let $S : \mathscr{M} \to \mathbb{R}$ be a function such that $S \in L^\infty(\mathscr{M})$ and there exist $S_0, S_1 > 0$ such that

$$0 < S_0 \le S(x,t) \le S_1.$$

Suppose also that β is a bounded maximal monotone graph of $\mathbb{R}^2$, that is there exist $m, M \in \mathbb{R}$ such that for all $s \in \mathbb{R}$ and $z \in \beta(s)$

$$m \le z \le M.$$

Let us consider real Hilbert spaces

$$H := L^2(\mathscr{M}), \quad V := \{u \in L^2(\mathscr{M}) \, : \, \nabla_{\mathscr{M}} u \in L^2(T\mathscr{M})\}$$

with respective standard norms $\| \cdot \|_H, \| \cdot \|_V$, and inner products $(\cdot, \cdot)_H, (\cdot, \cdot)_V$, where $T\mathscr{M}$ represents the tangent bundle and the functional spaces $L^2(\mathscr{M})$ and $L^2(T\mathscr{M})$ are defined in a standard way. Therefore, all statements from Sect. 6.1 hold for weak solutions of Problem (6.75).

6.3.3 A Model of Conduction of Electrical Impulses in Nerve Axons

Consider the problem (see Sect. 2.4.2 and Fig. 6.2):

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u \in \lambda H(u-a), & (x,t) \in (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) = 0, & t \in \mathbb{R}, \end{cases} \tag{6.76}$$

where $a \in \left(0, \frac{1}{2}\right)$ (see Sect. 2.4.2) Therefore, all statements from Sect. 6.1 hold for weak solutions of Problem (6.76).

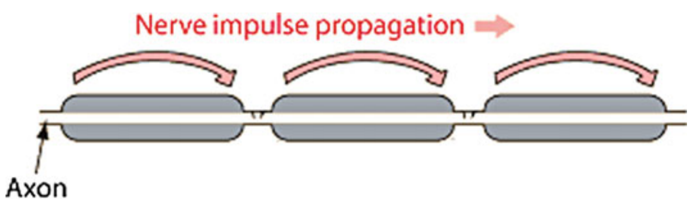

Fig. 6.2 Nerve impulse propagation

6.3.4 Viscoelastic Problems with Nonlinear "Reaction-Displacement" Law

Let a viscoelastic body occupy a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ in applications, and it is acted upon by volume forces and surface tractions.[4] The boundary Γ of Ω is supposed to be Lipschitz continuous and it is partitioned into two disjoint measurable parts Γ_D and Γ_N such that $meas(\Gamma_D) > 0$. We consider the process of evolution of the mechanical state on the interval $(0, +\infty)$. The body is clamped on Γ_D and thus the displacement vanishes there. The forces field of density f_0 acts in Ω, the surface tractions of density g_0 are applied on Γ_N. We denote by $u = (u_1, \ldots, u_d)$ the displacement vector, by $\sigma = (\sigma_{ij})$ the stress tensor and by $\varepsilon(u) = (\varepsilon_{ij}(u))$ the linearized (small) strain tensor $(\varepsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j))$, where $i, j = 1, \ldots, d$.

The mechanical problem consists in finding the displacement field $u : \Omega \times (0, +\infty) \to \mathbb{R}^d$ such that

$$u''(t) - \mathrm{div}\sigma(t) = f_0 \quad \text{in } \Omega \times (0, +\infty), \tag{6.77}$$

$$\sigma(t) = \mathscr{C}\varepsilon(u'(t)) + \mathscr{E}\varepsilon(u(t)) \quad \text{in } \Omega \times (0, +\infty), \tag{6.78}$$

$$u(t) = 0 \quad \text{on } \Gamma_D \times (0, +\infty), \tag{6.79}$$

$$\sigma n(t) = g_0 \quad \text{on } \Gamma_N \times (0, +\infty), \tag{6.80}$$

$$u(0) = u_0, \; u'(0) = u_1 \quad \text{in } \Omega, \tag{6.81}$$

where $\mathscr{C}$ and $\mathscr{E}$ are given linear constitutive functions, n being the outward unit normal vector to Γ.

In the above model dynamic equation (6.77) is considered with the viscoelastic constitutive relationship of the Kelvin–Voigt type (6.78) while (6.79) and (6.80) represent the displacement and traction boundary conditions, respectively. The functions u_0 and u_1 are the initial displacement and the initial velocity, respectively. In order to formulate the skin effects, we suppose that the body forces of density f_0 consists of two parts: f_1 which is prescribed external loading and f_2 which is the reaction of constrains introducing the skin effects, i.e., $f_0 = f_1 + f_2$. Here f_2 is a possibly multi-valued function of the displacement u. We consider the reaction-displacement law of the form

$$-f_2(x, t) \in \partial j(x, u(x, t)) \text{ in } \Omega \times (0, +\infty), \tag{6.82}$$

where $j : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz function in u and ∂j represents the Clarke subdifferential with respect to u. Let $\mathscr{Y}_d$ be the space of second-order symmetric tensors on $\mathbb{R}^d$.

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[4] This section is based on results of [23] and references therein.

We consider the following problem:

examine the long-time (as $t \to +\infty$) behavior of all (weak, generalized) solutions for (6.77)–(6.81) and (6.82).

In [23] for finite time interval it was presented the hemivariational formulation of problems similar to (6.77)–(6.82) and an existence theorem for evolution inclusions with pseudomonotone operators. We give now variational formulation of the above problem. To this aim let $H = L_2(\Omega, \mathbb{R}^d)$, $H_1 = H^1(\Omega, \mathbb{R}^d)$, $\mathscr{H} = L_2(\Omega, \mathscr{Y}_d)$ and V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_D\}.$$

On V we consider the inner product and the corresponding norm given by

$$(u, v)_V = \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathscr{H}}, \quad \|v\|_V = \|\varepsilon(v)\|_{\mathscr{H}} \text{ for } u, v \in V.$$

From the Korn inequality $\|v\|_{H_1} \le C_1 \|\varepsilon(v)\|_{\mathscr{H}}$ for $v \in V$ with $C_1 > 0$, it follows that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are the equivalent norms on V. Identifying H with its dual, we have an evolution triple $V \subset H \subset V^*$ (see e.g. [12]) with dense and compact embeddings. We denote by $\langle \cdot, \cdot \rangle_V$ the duality of V and its dual V^*, by $\|\cdot\|_{V^*}$ the norm in V^*. We have $\langle u, v \rangle_V = (u, v)_H$ for all $u \in H$ and $v \in V$.

We admit the following hypotheses:

H($\mathscr{C}$). The linear symmetric viscosity operator $\mathscr{C} : \Omega \times \mathscr{Y}_d \to \mathscr{Y}_d$ satisfies the Carathéodory condition (i.e., $\mathscr{C}(\cdot, \varepsilon)$ is measurable on Ω for all $\varepsilon \in \mathscr{Y}_d$ and $\mathscr{C}(x, \cdot)$ is continuous on $\mathscr{Y}_d$ for a.e. $x \in \Omega$) and

$$\mathscr{C}(x, \varepsilon) : \varepsilon \ge C_2 \|\varepsilon\|^2_{\mathscr{Y}_d} \text{ for all } \varepsilon \in \mathscr{Y}_d \text{ and a.e. } x \in \Omega \text{ with } C_2 > 0. \tag{6.83}$$

H($\mathscr{E}$). The elasticity operator $\mathscr{E} : \Omega \times \mathscr{Y}_d \to \mathscr{Y}_d$ is of the form $\mathscr{E}(x, \varepsilon) = \mathbb{E}(x)\varepsilon$ (Hooke's law) with a symmetric elasticity tensor $\mathbb{E} \in L_\infty(\Omega)$, i.e., $\mathbb{E} = (g_{ijkl})$, $i, j, k, l = 1, \ldots, d$ with $g_{ijkl} = g_{jikl} = g_{lkij} \in L_\infty(\Omega)$. Moreover,

$$\mathscr{E}(x, \varepsilon) : \varepsilon \ge C_3 \|\varepsilon\|^2_{\mathscr{Y}_d} \text{ for all } \varepsilon \in \mathscr{Y}_d \text{ and a.e. } x \in \Omega \text{ with } C_3 > 0.$$

H(j). $j : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a function such that

(i) $j(\cdot, \xi)$ is measurable for all $\xi \in \mathbb{R}^d$ and $j(\cdot, 0) \in L_1(\Omega)$;

(ii) $j(x, \cdot)$ is locally Lipschitz and it admits the representation via the difference of regular functions [8] for all $x \in \Omega$;

(iii) $\|\eta\| \le C_4(1 + \|\xi\|)$ for all $\eta \in \partial j(x, \xi)$, $x \in \Omega$ with $C_4 > 0$;

(iv) $j^0(x, \xi; -\xi) \le C_5(1 + \|\xi\|)$ for all $\xi \in \mathbb{R}^d$, $x \in \Omega$, with $C_5 \ge 0$, where $j^0(x, \xi; \eta)$ is the directional derivative of $j(x, \cdot)$ at the point $\xi \in \mathbb{R}^d$ in the direction $\eta \in \mathbb{R}^d$.

H(f). $f_1 \in V^*$, $g_0 \in L_2(\Gamma_N; \mathbb{R}^d)$, $u_0 \in V$ and $u_1 \in H$.

Next we need the spaces $\mathscr{V} = L_2(\tau, T; V)$, $\hat{\mathscr{H}} = L_2(\tau, T; H)$ and $\mathscr{W} = \{w \in \mathscr{V} : w' \in \mathscr{V}^*\}$, where the time derivative involved in the definition of $\mathscr{W}$ is understood in the sense of vector-valued distributions, $-\infty < \tau < T < +\infty$. Endowed with the norm $\|v\|_{\mathscr{W}} = \|v\|_{\mathscr{V}} + \|v'\|_{\mathscr{V}^*}$, the space $\mathscr{W}$ becomes a separable reflexive Banach space. We also have $\mathscr{W} \subset \mathscr{V} \subset \hat{\mathscr{H}} \subset \mathscr{V}^*$. The duality for the pair $(\mathscr{V}, \mathscr{V}^*)$ is denoted by $\langle z, w\rangle_{\mathscr{V}} = \int_\tau^T \langle z(s), w(s)\rangle_V ds$. It is well known (cf. [12]) that the embedding $\mathscr{W} \subset C([\tau, T]; H)$ and $\{w \in \mathscr{V} : \ w' \in \mathscr{W}\} \subset C([\tau, T]; V)$ are continuous. Next we define $g \in V^*$ by

$$\langle g, v\rangle_V = \langle f_1, v\rangle_V + \langle g_0, v\rangle_{L^2(\Gamma_N;\mathbb{R}^d)} \text{ for } v \in V. \tag{6.84}$$

According to condition (6.82), we obtain the following variational formulation of our problem:

$$\begin{cases} \langle u''(t), v\rangle_V + (\sigma(t), \varepsilon(v))_{\mathscr{H}} + \int_\Omega j^0(x, u(t); v)dx \geq \\ \geq \langle g, v\rangle_V \text{ for all } v \in V \text{ and a.e. } t \in (0, +\infty), \\ \sigma(t) = \mathscr{C}(\varepsilon(u'(t))) + \mathscr{E}(\varepsilon(u(t))) \text{ for a.e. } t \in (0, +\infty), \\ u(0) = u_0, \ u'(0) = u_1. \end{cases} \tag{6.85}$$

We define the operators $A : V \to V^*$ and $B : V \to V^*$ by

$$\langle A(u), v\rangle_V = (\mathscr{C}(x, \varepsilon(u)), \varepsilon(v))_{\mathscr{H}} \text{ for } u, v \in V, \tag{6.86}$$

$$\langle Bu, v\rangle_V = (\mathscr{E}(x, \varepsilon(u)), \varepsilon(v))_{\mathscr{H}} \text{ for } u, v \in V. \tag{6.87}$$

Note that bilinear forms (6.86) and (6.87) are symmetric, continuous, and coercive.

Let us introduce the functional $J : L_2(\Omega; \mathbb{R}^d) \to \mathbb{R}$ defined by

$$J(v) = \int_\Omega j(x, v(x))dx \text{ for } v \in L_2(\Omega; \mathbb{R}^d). \tag{6.88}$$

From [8, Chap. II] under Assumptions **H(j)**, the functional J defined by (6.88) satisfies

H(J). $J : L_2(\Omega; \mathbb{R}^d) \to \mathbb{R}$ is a functional such that:

(i) $J(\cdot)$ is well defined, locally Lipschitz (in fact, Lipschitz on bounded subsets of $L_2(\Omega; \mathbb{R}^d)$) and it admits the representation via the difference of regular functions J_1 and J_2 on H;

(ii) $\zeta \in \partial J_1(v) - \partial J_2(v)$ implies $\|\zeta\|_{L_2(\Omega;\mathbb{R}^d)} \leq C_6(1 + \|v\|_{L_2(\Omega;\mathbb{R}^d)})$ for $v \in L_2(\Omega; \mathbb{R}^d)$ with $C_6 > 0$;

(iii) $J^0(v; -v) \leq C_7(1 + \|v\|_{L_2(\Omega;\mathbb{R}^d)})$ for $v \in L_2(\Omega; \mathbb{R}^d)$ with $C_7 \geq 0$, where $J^0(u; v)$ denotes the directional derivative of $J(\cdot)$ at a point $u \in L_2(\Omega; \mathbb{R}^d)$ in the direction $v \in L_2(\Omega; \mathbb{R}^d)$.

We can now formulate the second-order evolution inclusions associated with the variational form of our problem

$$\begin{cases} \text{Find } u \in C([0,+\infty);V) \text{ with } u' \in C([0,+\infty);H) \cap L_2^{loc}(0,+\infty;V) \\ \text{ and } u'' \in L_2^{loc}(0,+\infty;V^*) \text{ such that} \\ u''(t) + Au'(t) + Bu(t) + \partial J_1(u(t)) - \partial J_2(u(t)) \ni g \text{ a.e. } t \in (0,+\infty), \\ u(0) = u_0, \;\; u'(0) = u_1. \end{cases} \tag{6.89}$$

Theorem 6.6 yields that, if $\tau < T$, $\{\varphi_n(\cdot)\}_{n\geq 1} \subset W_\tau^T$ is an arbitrary sequence of weak solutions of (6.89) on $[\tau, T]$ such that $\varphi_n(\tau) \to \varphi_\tau$ strongly in E, $n \to +\infty$, then there exists $\varphi \in \mathscr{D}_{\tau,T}(\varphi_\tau)$ such that up to a subsequence $\varphi_n(\cdot) \to \varphi(\cdot)$ in $C([\tau, T]; E)$, $n \to +\infty$ (see Sect. 6.2 for details).

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Part III
Uniform Global Behavior of Solutions: Uniform Attractors, Flattening and Entropy

Chapter 7
Uniform Global Attractors for Non-autonomous Dissipative Dynamical Systems

Abstract In this chapter we consider sufficient conditions for the existence of uniform compact global attractor for non-autonomous dynamical systems in special classes of infinite-dimensional phase spaces. The obtained generalizations allow us to avoid the restrictive compactness assumptions on the space of shifts of non-autonomous terms in particular evolution problems. The results are applied to several evolution inclusions.

7.1 General Methodology

The standard scheme of investigation of uniform the long-time behavior for all solutions of non-autonomous problems covers non-autonomous problems of the form

$$\partial_t u(t) = A_{\sigma(t)}(u(t)), \tag{7.1}$$

where $\sigma(s)$, $s \geq 0$, is a functional parameter called the time symbol of Eq. (7.1) (t is replaced by s). In applications to mathematical physics equations, a function $\sigma(s)$ consists of all time-dependent terms of the equation under consideration: external forces, parameters of mediums, interaction functions, control functions, etc.; Chepyzhov and Vishik [4, 5, 8]; Sell [36]; Zgurovsky et al. [48] and references therein; see also Hale [16]; Ladyzhenskaya [30]; Mel'nik and Valero [32]; Iovane, Kapustyan and Valero [17]. In the mentioned above papers and books it is assumed that the symbol σ of Eq. (7.1) belongs to a Hausdorff topological space Ξ_+ of functions defined on $\mathbb{R}_+$ with values in some complete metric space. Usually, in applications, the topology in the space Ξ_+ is a local convergence topology on any segment $[t_1, t_2] \subset \mathbb{R}_+$. Further, they consider the family of Eq. (7.1) with various symbols $\sigma(s)$ belonging to a set $\Sigma \subseteq \Xi_+$. The set Σ is called the symbol space of the family of Eq. (7.1). It is assumed that the set Σ, together with any symbol $\sigma(s) \in \Sigma$, contains all positive translations of $\sigma(s)$: $\sigma(t + s) = T(t)\sigma(s) \in \Sigma$ for any $t, s \geq 0$. The symbol space Σ is invariant with respect to the translation semigroup $\{T(t)\}_{t\geq 0}$: $T(t)\Sigma \subseteq \Sigma$ for any $t \geq 0$. To prove the existence of uniform trajectory attractors they suppose that the symbol space Σ with the topology induced from Ξ_+ is a

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compact metric space. Mostly in applications, as a symbol space Σ it is naturally to consider the hull of translation-compact function $\sigma_0(s)$ in an appropriate Hausdorff topological space Ξ_+. The direct realization of this approach to differential-operator inclusions, PDEs with Caratheodory's nonlinearities, optimization problems, etc., is problematic without any additional assumptions for parameters of Problem (7.1) and requires the translation-compactness of the symbol $\sigma(s)$ in some compact Hausdorff topological space of measurable multi-valued mappings acts from $\mathbb{R}_+$ to some metric space of operators from $(V \to 2^{V^*})$, where V is a Banach space and V^* is its dual space, satisfying (possibly) only growth and sign assumptions. To avoid this technical difficulties we present an alternative approach for the existence and construction of the uniform global attractor for classes of non-autonomous dynamical systems in special classes of infinite-dimensional phase spaces; see also [1, 6, 12–15, 18, 21, 22, 24–29, 37–44, 46].

7.2 Main Constructions and Results

Let $p \geq 2$ and $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, $(V; H; V^*)$ to be evolution triple such that $V \subset H$ with compact embedding. For each $t_1, t_2 \in \mathbb{R}$, $0 \leq t_1 < t_2 < +\infty$, consider the space

$$W_{t_1,t_2} := \{y(\cdot) \in L_p(t_1, t_2; V) \; : \; y'(\cdot) \in L_q(t_1, t_2; V^*)\},$$

where $y'(\cdot)$ is a derivative of an element $y(\cdot) \in L_p(t_1, t_2; V)$ in the sense of distributions $\mathscr{D}^*([t_1, t_2]; V^*)$. The space W_{t_1,t_2} endowed with the norm

$$\|y\|_{W_{t_1,t_2}} := \|y\|_{L_p(t_1,t_2;V)} + \|y'\|_{L_q(t_1,t_2;V^*)}, \quad y \in W_{t_1,t_2},$$

is a reflexive Banach space. Note that $W_{t_1,t_2} \subset C([t_1, t_2]; H)$ with continuous and dense embedding; Gajewsky et al. [11, Chap. IV]. For each $\tau \geq 0$, consider the Fréchet space

$$W^{\text{loc}}([\tau, +\infty)) := \{y : [\tau, +\infty) \to H \; : \; \Pi_{t_1,t_2} y \in W_{t_1,t_2} \text{ for each } [t_1, t_2] \subset [\tau, +\infty)\},$$

where Π_{t_1,t_2} is the restriction operator to the finite time interval $[t_1, t_2]$. We recall that the sequence $\{f_n\}_{n\geq 1}$ converges in $W^{\text{loc}}([\tau, +\infty))$ (in $C^{\text{loc}}([\tau, +\infty); H)$ respectively) to $f \in W^{\text{loc}}([\tau, +\infty))$ (to $f \in C^{\text{loc}}([\tau, +\infty); H)$ respectively) as $n \to +\infty$ if and only if the sequence $\{\Pi_{t_1,t_2} f_n\}_{n\geq 1}$ converges in W_{t_1,t_2} (in $C([t_1, t_2]; H)$ respectively) to $\Pi_{t_1,t_2} f$ as $n \to +\infty$ for each finite time interval $[t_1, t_2] \subset [\tau, +\infty)$. Further we denote that

$$T(h)y(\cdot) = \Pi_{0,+\infty}\, y(\cdot + h), \quad y \in W^{\text{loc}}(\mathbb{R}_+), \; h \geq 0,$$

where $\mathbb{R}_+ = [0, +\infty)$ and $\Pi_{0,+\infty}$ is the restriction operator to the time interval $[0, +\infty)$.

Throughout the chapter we consider the *family of solution sets* $\{\mathcal{K}_\tau^+\}_{\tau\geq 0}$ such that $\mathcal{K}_\tau^+ \subset W^{\mathrm{loc}}([\tau,+\infty))$ for each $\tau \geq 0$ and $\mathcal{K}_{\tau_0}^+ \neq \emptyset$ for some $\tau_0 \geq 0$. In the most of applications as $\mathcal{K}_\tau^+$ can be considered the family of globally defined on $[\tau,+\infty)$ weak solutions for particular non-autonomous evolution problem (see Sect. 7.4).

To state the main assumptions on the family of solution sets $\{\mathcal{K}_\tau^+\}_{\tau\geq 0}$ it is necessary to formulate two auxiliary definitions.

A function $\varphi \in L_\gamma^{\mathrm{loc}}(\mathbb{R}_+)$, $\gamma > 1$, is called *translation bounded* function in $L_\gamma^{\mathrm{loc}}(\mathbb{R}_+)$ if

$$\sup_{t\geq 0}\int_t^{t+1} |\varphi(s)|^\gamma ds < +\infty;$$

Chepyzhov and Vishik [7, p. 105]. A function $\varphi \in L_1^{\mathrm{loc}}(\mathbb{R}_+)$ is called *a translation uniform integrable (t.u.i.)* function in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$ if

$$\lim_{K\to+\infty}\sup_{t\geq 0}\int_t^{t+1} |\varphi(s)|\mathbf{I}\{|\varphi(s)| \geq K\}ds = 0;$$

Gorban et al. [14]. Note that Dunford–Pettis compactness criterion provides that $\varphi \in L_1^{\mathrm{loc}}(\mathbb{R}_+)$ is a t.u.i. function in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$ if and only if for every sequence of elements $\{\tau_n\}_{n\geq 1} \subset \mathbb{R}_+$, the sequence $\{\varphi(\cdot + \tau_n)\}_{n\geq 1}$ contains a subsequence converging weakly in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$. Note that for each $\gamma > 1$, every translation bounded in $L_\gamma^{\mathrm{loc}}(\mathbb{R}_+)$ function is t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$; Gorban et al. [14].

Main assumptions. Let the following two assumptions hold:

(A1) there exist a t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$ function $c_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $\alpha_1 > 0$ such that for each $\tau \geq 0$, $y \in \mathcal{K}_\tau^+$, and $t_2 \geq t_1 \geq \tau$, the following inequality holds:

$$\|y(t_2)\|_H^2 - \|y(t_1)\|_H^2 + \alpha_1 \int_{t_1}^{t_2} \|y(t)\|_V^p dt \leq \int_{t_1}^{t_2} c_1(t)dt; \tag{7.2}$$

(A2) there exist a t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$ function $c_2 : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $\alpha_2 > 0$ such that for each $\tau \geq 0$, $y \in \mathcal{K}_\tau^+$, and $t_2 \geq t_1 \geq \tau$, the following inequality holds:

$$\int_{t_1}^{t_2} \|y'(t)\|_{V^*}^q dt \leq \alpha_2 \int_{t_1}^{t_2} \|y(t)\|_V^p dt + \int_{t_1}^{t_2} c_2(t)dt. \tag{7.3}$$

To characterize the uniform long-time behavior of solutions for non-autonomous dissipative dynamical system consider the *united trajectory space* $\mathcal{K}_\cup^+$ for the family of solutions $\{\mathcal{K}_\tau^+\}_{\tau\geq 0}$ shifted to zero:

$$\mathcal{K}_\cup^+ := \bigcup_{\tau\geq 0} \left\{T(h)y(\cdot + \tau) \,:\, y(\cdot) \in \mathcal{K}_\tau^+,\ h \geq 0\right\} \subset W^{\mathrm{loc}}(\mathbb{R}_+), \tag{7.4}$$

and the *extended united trajectory space* for the family $\{\mathcal{K}_\tau^+\}_{\tau\geq 0}$:

$$\mathcal{K}^+ := \mathrm{cl}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}\left[\mathcal{K}_\cup^+\right], \tag{7.5}$$

where $\mathrm{cl}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}[\,\cdot\,]$ is the closure in $C^{\mathrm{loc}}(\mathbb{R}_+;H)$. Since $T(h)\mathcal{K}_\cup^+ \subseteq \mathcal{K}_\cup^+$ for each $h \geq 0$, then

$$T(h)\mathcal{K}^+ \subseteq \mathcal{K}^+ \text{ for each } h \geq 0, \tag{7.6}$$

due to

$$\rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}(T(h)u, T(h)v) \leq \rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}(u,v) \text{ for each } u, v \in C^{\mathrm{loc}}(\mathbb{R}_+;H),$$

where $\rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ is the standard metric on Fréchet space $C^{\mathrm{loc}}(\mathbb{R}_+;H)$. Therefore the set

$$\mathbb{X} := \{y(0) \ : \ y \in \mathcal{K}^+\} \tag{7.7}$$

is closed in H (it follows from Theorem 7.2). We endow this set $\mathbb{X}$ with metric

$$\rho_{\mathbb{X}}(x_1, x_2) = \|x_1 - x_2\|_H, \quad x_1, x_2 \in \mathbb{X}.$$

Then we obtain that $(\mathbb{X}, \rho)$ is a Polish space (complete separable metric space).

Let us define the multi-valued semiflow (*m-semiflow*) $G : \mathbb{R}_+ \times \mathbb{X} \to 2^{\mathbb{X}}$:

$$G(t, y_0) := \{y(t) \ : \ y(\cdot) \in \mathcal{K}^+ \text{ and } y(0) = y_0\}, \quad t \geq 0, \ y_0 \in \mathbb{X}. \tag{7.8}$$

According to (7.6), (7.7), and (7.8) for each $t \geq 0$ and $y_0 \in \mathbb{X}$ the set $G(t, y_0)$ is nonempty. Moreover, the following two conditions hold:

(i) $G(0,\cdot) = I$ is the identity map;
(ii) $G(t_1 + t_2, y_0) \subseteq G(t_1, G(t_2, y_0))$, $\forall t_1, t_2 \in \mathbb{R}_+$, $\forall y_0 \in \mathbb{X}$,

where $G(t, D) = \underset{y\in D}{\cup} G(t, y)$, $D \subseteq \mathbb{X}$.

We denote by $\mathrm{dist}_{\mathbb{X}}(C, D) = \sup_{c\in C} \inf_{d\in D} \rho(c, d)$ the *Hausdorff semidistance* between nonempty subsets C and D of the Polish space $\mathbb{X}$. Recall that the set $\mathfrak{R} \subset \mathbb{X}$ is a *global attractor* of the m-semiflow G if it satisfies the following conditions:

(i) $\mathfrak{R}$ attracts each bounded subset $B \subset \mathbb{X}$, i.e.

$$\mathrm{dist}_{\mathbb{X}}(G(t, B), \mathfrak{R}) \to 0, \quad t \to +\infty; \tag{7.9}$$

(ii) $\mathfrak{R}$ is negatively semi-invariant set, i.e. $\mathfrak{R} \subseteq G(t, \mathfrak{R})$ for each $t \geq 0$;
(iii) $\mathfrak{R}$ is the minimal set among all nonempty closed subsets $C \subseteq \mathbb{X}$ that satisfy (7.9).

In this chapter we examine the uniform long-time behavior of solution sets $\{\mathcal{K}_\tau^+\}_{\tau\geq 0}$ in the strong topology of the natural phase space H (as time $t \to +\infty$) in the sense of the existence of a compact global attractor for m-semiflow G generated by the family of solution sets $\{\mathscr{K}_\tau^+\}_{\tau\geq 0}$ and their shifts. The following theorem is the main result of the chapter.

Theorem 7.1 *Let assumptions (A1)–(A2) hold. Then the m-semiflow G, defined in (7.8), has a compact global attractor* $\Re$ *in the phase space* $\mathbb{X}$.

7.3 Proof of Theorem 7.1

Before the proof of Theorem 7.1 we provide the following statement characterizing the compactness properties of the family $\mathscr{K}^+$ in the topology induced from $C^{\mathrm{loc}}(\mathbb{R}_+; H)$.

Theorem 7.2 *Let assumptions (A1)–(A2) hold. Then the following two statements hold:*

(a) for each $y \in \mathscr{K}^+$*, the following estimate holds*

$$\|y(t)\|_H^2 \leq \|y(0)\|_H^2 e^{-c_3 t} + c_4, \quad t \geq 0, \tag{7.10}$$

where the positive constants c_3 *and* c_4 *do not depend on* $y \in \mathscr{K}^+$ *and* $t \geq 0$*;*

(b) for any bounded in $L_\infty(\mathbb{R}_+; H)$ *sequence* $\{y_n\}_{n\geq 1} \subset \mathscr{K}^+$*, there exist an increasing sequence* $\{n_k\}_{k\geq 1} \subseteq \mathbb{N}$ *and an element* $y \in \mathscr{K}^+$ *such that*

$$\|\Pi_{\tau,T}\, y_{n_k} - \Pi_{\tau,T}\, y\|_{C([\tau,T];H)} \to 0, \quad k \to +\infty, \tag{7.11}$$

for each finite time interval $[\tau, T] \subset (0, +\infty)$*. If, additionally, there exists* $y_0 \in H$ *such that* $y_{n_k}(0) \to y_0$ *in* H*, then* $y(0) = y_0$.

Proof Let us prove statement (a). If statement (a) holds for each $y \in \mathscr{K}_\cup^+$, then inequality (7.10) holds for each $y \in \mathscr{K}^+$, due to (7.5). The rest of the proof of statement (a) establishes inequality (7.10) for each $y \in \mathscr{K}_\cup^+$.

For an arbitrary $y \in \mathscr{K}_\cup^+$, there exist $\tau, h \geq 0$ and $z(\cdot) \in \mathscr{K}_\tau^+$ such that $y(\cdot) = T(\tau + h)z(\cdot)$. Assumption (A1) implies the following inequality:

$$\|y(t_2)\|_H^2 - \|y(t_1)\|_H^2 + \alpha_1 \int_{t_1}^{t_2} \|y(t)\|_V^p dt \leq \int_{t_1}^{t_2} c_1(t + \tau + h)dt, \tag{7.12}$$

for each $t_2 \geq t_1 \geq 0$, where $c_1(\cdot)$ is t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$. Since the embedding $V \subset H$ is compact, then this embedding is continuous. So, there exists a constant $\beta > 0$ such that $\|b\|_H \leq \beta \|b\|_V$ for each $b \in V$. According to (7.12), since the inequality $a^2 \leq 1 + a^p$ holds for each $a \geq 0$, then the following inequality holds:

$$\|y(t_2)\|_H^2 - \|y(t_1)\|_H^2 + \alpha_3 \int_{t_1}^{t_2} \|y(t)\|_H^2 dt \leq \int_{t_1}^{t_2} [c_1(t + \tau + h) + \alpha_3]\, dt, \tag{7.13}$$

for each $t_2 \geq t_1 \geq 0$, where $\alpha_3 = \frac{\alpha_1}{\beta^p}$. Let us set

$$\rho(t) := \|y(t)\|_H^2 + \alpha_3 \int_0^t \|y(s)\|_H^2 ds - \int_0^t [c_1(s+\tau+h)+\alpha_3]\,ds, \quad t \geq 0.$$

Inequality (7.13) and Ball [3, Lemma 7.1] yield that $\frac{d}{dt}\rho \leq 0$ in $D^*((0,+\infty))$, where $\frac{d}{dt}$ is the derivative operation in the sense of $D^*((0,+\infty))$. Thus,

$$\frac{d}{dt}\|y(t)\|_H^2 + \alpha_3\|y(t)\|_H^2 - [c_1(t+\tau+h)+\alpha_3] \leq 0 \text{ in } D^*((0,+\infty)).$$

Therefore,

$$\frac{d}{dt}\left[\|y(t)\|_H^2 e^{\alpha_3 t}\right] - e^{\alpha_3 t}\,[c_1(t+\tau+h)+\alpha_3] \leq 0 \text{ in } D^*((0,+\infty)). \tag{7.14}$$

Ball [3, Lemma 7.1] and inequality (7.14) imply

$$\|y(t_2)\|_H^2 \leq \|y(t_1)\|_H^2 e^{-\alpha_3(t_2-t_1)} + \int_{t_1}^{t_2} e^{-\alpha_3(t_2-t)}\,[c_1(t+\tau+h)+\alpha_3]\,dt, \tag{7.15}$$

for each $t_2 \geq t_1 \geq 0$. Therefore,

$$\begin{aligned}\|y(t_2)\|_H^2 \leq & \|y(t_1)\|_H^2 e^{-\alpha_3(t_2-t_1)} + \int_{t_1}^{t_2} e^{-\alpha_3(t_2-t)}\,[c_1(t+\tau+h)+\alpha_3]\,dt \leq \\ & \|y(t_1)\|_H^2 e^{-\alpha_3(t_2-t_1)} + 1 + \int_{t_1+\tau+h}^{t_2+\tau+h} e^{-\alpha_3(t_2-t+\tau+h)} c_1(t)dt \leq \\ & \|y(t_1)\|_H^2 e^{-\alpha_3(t_2-t_1)} + 1 + \frac{K}{\alpha_3} + \\ & \int_{t_1+\tau+h}^{t_2+\tau+h} e^{-\alpha_3(t_2-t+\tau+h)} |c_1(t)| \mathbf{I}\{|c_1(t)| \geq K\} dt,\end{aligned}$$

for each $K > 0$, $t_2 \geq t_1 \geq 0$. Since the function $c_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$ (see assumption (A1)), then there exists $K_0 > 0$ such that

$$\sup_{t\geq 0} \int_t^{t+1} |c_1(s)| \mathbf{I}\{|c_1(s)| \geq K_0\} ds \leq 1.$$

Thus,

$$\|y(t_2)\|_H^2 \leq \|y(t_1)\|_H^2 e^{-\alpha_3(t_2-t_1)} + 1 + \frac{K_0}{\alpha_3} + e^{\alpha_3} + 1,$$

that yields estimate (7.10) with $c_3 := \alpha_3$ and $c_4 := 1 + \frac{K_0}{\alpha_3} + e^{\alpha_3} + 1$, where the positive constants c_3 and c_4 do not depend on $y \in \mathscr{K}^+$ and $t \geq 0$.

Let us prove statement (b). Let $\{y_n\}_{n\ge1} \subset \mathscr{K}^+$ be an arbitrary sequence that is bounded in $L_\infty(\mathbb{R}_+; H)$. Since $\mathscr{K}_\cup^+$ is the dense set in a Polish space $\mathscr{K}^+$ endowed with the topology induced from $C^{\mathrm{loc}}(\mathbb{R}_+; H)$, then for each $n \ge 1$ there exists $u_n \in \mathscr{K}_\cup^+$ such that

$$\rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}(y_n, u_n) \le \frac{1}{n}, \text{ for each } n \ge 1. \tag{7.16}$$

Note that a priori estimate (7.10) provides that the sequence $\{u_n\}_{n\ge1}$ is bounded in $L_\infty(\mathbb{R}_+; H)$. Therefore, the rest of the proof establishes statement (b) for the sequence $\{u_n\}_{n\ge1}$.

Let us fix $n \ge 1$. Formula (7.4) provides the existence of $\tau_n, h_n \ge 0$ and $z_n(\,\cdot\,) \in \mathscr{K}_{\tau_n}^+$ such that $u_n(\,\cdot\,) = z_n(\,\cdot\, + \tau_n + h_n)$. Then, assumptions (A1) and (A2) yield

$$\begin{aligned} \|u_n(t_2)\|_H^2 - \|u_n(t_1)\|_H^2 + \alpha_1 \int_{t_1}^{t_2} \|u_n(t)\|_V^p dt &\le \int_{t_1}^{t_2} c_1(t + \tau_n + h_n)dt, \\ \int_{t_1}^{t_2} \|u_n{}'(t)\|_{V^*}^q dt \le \alpha_2 \int_{t_1}^{t_2} \|u_n(t)\|_V^p dt &+ \int_{t_1}^{t_2} c_2(t + \tau_n + h_n)dt, \end{aligned} \tag{7.17}$$

for each $t_2 \ge t_1 \ge 0$ and $n \ge 1$.

We remark that

$$\sup_{n\ge1} \int_{t_1}^{t_2} |c_1(t + \tau_n + h_n)|dt < \infty \text{ and } \sup_{n\ge1} \int_{t_1}^{t_2} |c_2(t + \tau_n + h_n)|dt < \infty, \tag{7.18}$$

for each $t_2 \ge t_1 \ge 0$, since the functions $c_1, c_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$.

Formulae (7.17) and (7.18) imply that the sequence $\{u_n\}_{n\ge1}$ is bounded in $W^{\mathrm{loc}}(\mathbb{R}_+)$. Thus, Banach–Alaoglu theorem and Zgurovsky et al. [47, Theorems 1.16 and 1.21] yield that there exist an increasing sequence $\{n_k\}_{k\ge1} \subseteq \mathbb{N}$ and elements $y \in W^{loc}(\mathbb{R}_+) \subset C^{\mathrm{loc}}(\mathbb{R}_+; H)$ and $\bar{c}_1 \in L_1^{\mathrm{loc}}(\mathbb{R}_+)$ such that

$$\begin{array}{ll} u_{n_k} \to y & \text{weakly in } L_p^{\mathrm{loc}}(\mathbb{R}_+; V), \\ u'_{n_k} \to y' & \text{weakly in } L_q^{\mathrm{loc}}(\mathbb{R}_+; V^*), \\ u_{n_k} \to y & \text{weakly in } C^{\mathrm{loc}}(\mathbb{R}_+; H), \\ u_{n_k}(t) \to y(t) & \text{in } H \text{ for a.e. } t > 0, \\ c_1(\,\cdot\, + \tau_{n_k} + h_{n_k}) \to \bar{c}_1 & \text{weakly in } L_1^{\mathrm{loc}}(\mathbb{R}_+), \quad k \to \infty, \end{array} \tag{7.19}$$

where the last convergence holds due to the fact that $c_1 \in L_1^{\mathrm{loc}}(\mathbb{R}_+)$ is t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$. According to (7.19), we can pass to the limit in (7.2). So, we obtain that y satisfies (7.2).

We consider the continuous and nonincreasing (by assumption (A1)) functions on $\mathbb{R}_+$:

$$J_k(t) = \|u_{n_k}(t)\|_H^2 - \int_0^t c_1(s + \tau_{n_k} + h_{n_k})ds,$$
$$J(t) = \|y(t)\|_H^2 - \int_0^t \bar{c}_1(s)ds, \quad k \geq 1; \tag{7.20}$$

cf. Kapustyan and Valero [19]. The last two statements in (7.19) imply

$$J_k(t) \to J(t), \text{ as } k \to +\infty, \text{ for a.e. } t > 0. \tag{7.21}$$

Similarly to Zgurovsky et al. [48, p. 57] (see the book and references therein) we show that (7.11) holds. By contradiction suppose the existence of a positive constant $L > 0$, a finite interval $[\tau, T] \subset (0, +\infty)$, and a subsequence $\{u_{k_j}\}_{j\geq 1} \subseteq \{u_{n_k}\}_{k\geq 1}$ such that

$$\max_{t\in[\tau,T]} \|u_{k_j}(t) - y(t)\|_H = \|u_{k_j}(t_j) - y(t_j)\|_H \geq L,$$

for each $j \geq 1$. Suppose also that $t_j \to t_0 \in [\tau, T]$, as $j \to +\infty$. Continuity of $\Pi_{\tau,T}y : [\tau, T] \to H$ implies

$$\liminf_{j\to+\infty} \|u_{k_j}(t_j) - y(t_0)\|_H \geq L. \tag{7.22}$$

On the other hand, we prove that

$$u_{k_j}(t_j) \to y(t_0) \text{ in } H, \ \ j \to +\infty. \tag{7.23}$$

For this purpose we firstly note that from (7.19) we have

$$u_{k_j}(t_j) \to y(t_0) \text{ weakly in } H, \quad j \to +\infty. \tag{7.24}$$

Secondly we prove that

$$\limsup_{j\to+\infty} \|u_{k_j}(t_j)\|_H \leq \|y(t_0)\|_H. \tag{7.25}$$

We consider the continuous nonincreasing functions J and J_{k_j}, $j \geq 1$, defined in (7.20). Let us fix an arbitrary $\varepsilon > 0$. The continuity of J and (7.21) provide the existence of $\bar{t} \in (\tau, t_0)$ such that $\lim_{j\to\infty} J_{k_j}(\bar{t}) = J(\bar{t})$ and $|J(\bar{t}) - J(t_0)| < \varepsilon$. Then,

$$J_{k_j}(t_j) - J(t_0) \leq |J_{k_j}(\bar{t}) - J(\bar{t})| + |J(\bar{t}) - J(t_0)| \leq |J_{k_j}(\bar{t}) - J(\bar{t})| + \varepsilon,$$

for rather large $j \geq 1$. Thus, $\limsup_{j\to+\infty} J_{k_j}(t_j) \leq J(t_0)$ and inequality (7.25) holds.

Thirdly note that the convergence (7.23) holds due to (7.24) and (7.25); cf. Gajewski et al. [11, Chap. I]. Finally, we remark that statement (7.23) contradicts assumption

(7.22). Therefore, according to (7.16), the first statement of the theorem holds for each sequence $\{y_n\}_{n\geq 1} \subset \mathscr{K}^+$.

To finish the proof of statement (b) we note that if, additionally, there exists $y_0 \in H$ such that $y_{n_k}(0) \to y_0$ in H, then, according to the third convergence in (7.19), $y(0) = y_0$.

Let us provide the proof of the main result.

Proof (Proof of Theorem 7.1) Theorem 7.2 implies the following properties for the m-semiflow G, defined in (7.8):

(a) for each $t \geq 0$ the mapping $G(t, \cdot) : \mathbb{X} \to 2^{\mathbb{X}} \setminus \{\emptyset\}$ has a closed graph;
(b) for each $t \geq 0$ and $y_0 \in \mathbb{X}$ the set $G(t, y_0)$ is compact in $\mathbb{X}$;
(c) the set $G(1, \tilde{C})$, where $\tilde{C} := \{z \in \mathbb{X} : \|z\|_H^2 < c_4 + 1\}$, is precompact and attracts each bounded subset $C \subset \mathbb{X}$.

Indeed, property (a) follows from Theorem 7.2 (see formulae (7.5) and (7.8)); property (b) directly follows from (a) and Theorem 7.2(b); property (c) holds, since $G(1, \tilde{C})$ is precompact in $\mathbb{X}$ (Theorem 7.2(b) and formula (7.8)) and the following inequalities and equality hold:

$$\text{dist}_{\mathbb{X}}(G(t, C), G(1, \tilde{C})) \leq \text{dist}_{\mathbb{X}}(G(1, G(t-1, C)), G(1, \tilde{C})) \leq$$

$$\text{dist}_{\mathbb{X}}(G(1, \tilde{C}), G(1, \tilde{C})) = 0,$$

for sufficiently large t.

According to properties (a)–(c), Mel'nik and Valero [31, Theorems 1, 2, Remark 2, Proposition 1] yields that the m-semiflow G has a compact global attractor $\mathfrak{R}$ in the phase space $\mathbb{X}$.

7.4 Example of Applications

In the following three examples we examine the uniform global attractor for the family of solution sets $\{\mathscr{K}_\tau^+\}$ generated by particular evolution problems. In all the cases we assume that

$$\forall z \in H \ \forall \tau \geq 0 \ \exists y \in \mathscr{K}_\tau^+ \text{ such that } y(\tau) = z.$$

This assumption guarantees the equality $\mathbb{X} = H$.

7.4.1 Autonomous Evolution Problem

Let $\{\mathcal{K}_\tau^+\}$ be a family of solutions for an autonomous problem on $[\tau, +\infty)$, $\tau \geq 0$. Then we have:

$$\forall h \geq 0 \;\; T(h)\mathcal{K}_0^+ \subset \mathcal{K}_0^+; \tag{7.26}$$

$$\forall \tau \geq 0 \;\forall y \in \mathcal{K}_\tau^+ \;\; y(\cdot + \tau) \in \mathcal{K}_0^+. \tag{7.27}$$

So, $\mathcal{K}_\cup^+ = \mathcal{K}_0^+$. If additionally we have that

$$\mathcal{K}_0^+ \text{ is closed in } C^{\text{loc}}(\mathbb{R}_+; H), \tag{7.28}$$

then

$$\mathcal{K}^+ = \mathcal{K}_0^+.$$

It implies that the m-semiflow G (defined by (7.8)) is a classical multi-valued semigroup generated by an autonomous evolution problem.

7.4.2 Non-autonomous Evolution Problem

Let $\{\mathcal{K}_\tau^+\}$ be a family of solutions for non-autonomous problem on $[\tau, +\infty)$, $\tau \geq 0$, and the following condition holds:

$$\forall s \geq \tau \geq 0 \;\forall y \in \mathcal{K}_\tau^+ \quad \Pi_{s,+\infty} y(\cdot) \in \mathcal{K}_s^+. \tag{7.29}$$

Then, according to Kapustyan et al. [23], formula

$$U(t, \tau, z) = \{y(t) \;:\; y(\cdot) \in \mathcal{K}_\tau^+, \;\; y(\tau) = z\} \tag{7.30}$$

defines a m-semiprocess, that is

$$\forall t \geq s \geq \tau \quad U(t, \tau, z) \subset U(t, s, U(s, \tau, z)).$$

One of the most important objects for m-semiprocess (7.30) is uniform global attractor; Chepyzhov and Vishik [7], Kapustyan et al. [20], Zgurovsky et al. [48]. It is a set Θ such that for every bounded subset $C \subset H$

$$\sup_{\tau \geq 0} \text{dist}_H(U(t + \tau, \tau, C), \Theta) \to 0, \quad t \to \infty, \tag{7.31}$$

and Θ is minimal among all closed sets satisfying this property. Then under assumptions (A1), (A2) and from (7.29) it follows that the m-semiprocess (7.30) has the compact uniform global attractor $\Theta \subseteq \Re$, where $\Re$ is the global attractor for the m-semiflow (7.8).

Indeed,

$$\forall t \geq \tau \geq 0 \ \forall z \in H \quad U(t+\tau, \tau, z) \subset G(t, z). \tag{7.32}$$

So, if $\Re$ is a compact global attractor for the m-semiflow G then, according to Kapustyan et al. [20], there exists a compact uniform global attractor Θ for m-semiprocess U and, moreover, $\Theta \subset \Re$.

In the following example we examine the existence of uniform global attractor for non-autonomous differential-operator inclusion. The uniform trajectory attractors for classes of non-autonomous inclusions and equations were proved to exist in Zgurovsky and Kasyanov [45] (see also Gorban et al. [14]).

7.4.3 Non-autonomous Differential-Operator Inclusion

For the multi-valued map $A : \mathbb{R}_+ \times V \to 2^{V^* \setminus \{\emptyset\}}$ we consider the problem of long-time behavior of all globally defined weak solutions for non-autonomous evolution inclusion

$$y'(t) + A(t, y(t)) \ni \bar{0}, \tag{7.33}$$

as $t \to +\infty$. Let $\langle \cdot, \cdot \rangle_V : V^* \times V \to \mathbb{R}$ be the pairing in $V^* \times V$, that coincides on $H \times V$ with the inner product $(\cdot, \cdot)$ in the Hilbert space H.

We note that Problem (7.33) arises in many important models for distributed parameter control problems and that large class of identification problems enter this formulation. Let us indicate a problem which is one of the motivations for the study of the non-autonomous evolution inclusion (7.33) (see, for example, Migórski and Ochal [34]; Zgurovsky et al. [48] and references therein). In a subset Ω of $\mathbb{R}^3$, we consider the nonstationary heat conduction equation

$$\frac{\partial y}{\partial t} - \triangle y = f \text{ in } \Omega \times (0, +\infty)$$

with initial conditions and suitable boundary ones. Here $y = y(x, t)$ represents the temperature at the point $x \in \Omega$ and time $t > 0$. It is supposed that $f = f_1 + f_2$, where f_2 is given and f_1 is a known function of the temperature of the form (see Fig. 7.1)

$$-f_1(x, t) \in \partial j(x, t, y(x, t)) \text{ a.e. } (x, t) \in \Omega \times (0, +\infty).$$

Here $\partial j(x, t, \xi)$ denotes generalized gradient of Clarke (see Clarke [9]) with respect to the last variable of a function $j : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ which is assumed to be locally Lipschitz in ξ (cf. Migórski and Ochal [34] and references therein). The multi-valued function $\partial j(x, t, \cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is generally nonmonotone and it includes the vertical jumps. In a physicist's language it means that the law is characterized

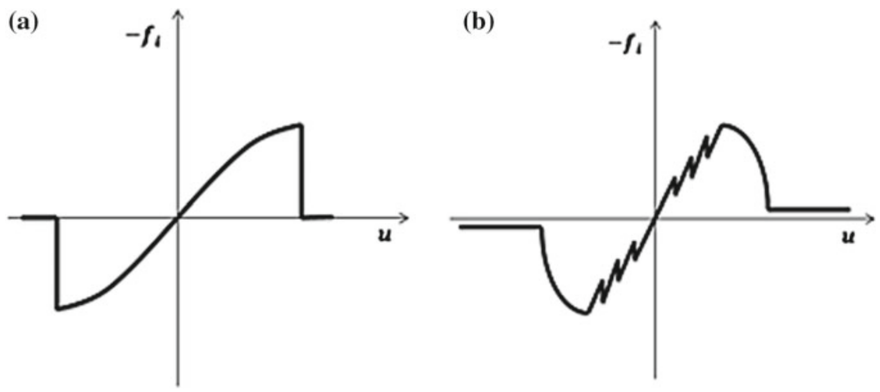

Fig. 7.1 Control laws

by the generalized gradient of a nonsmooth potential j (cf. Panagiotopoulos [35]). Models of physical interest includes also the next (see, for example, Balibrea et al. [2] and references therein): a model of combustion in porous media; a model of conduction of electrical impulses in nerve axons; a climate energy balance model; etc.

Let the following assumptions hold:

(H1) (*Growth condition*) There exist a t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$ function $c_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $c_2 > 0$ such that $\|d\|_{V^*}^q \le c_1(t) + c_2\|u\|_V^p$ for any $u \in V$, $d \in A(t,u)$, and a.e. $t > 0$;

(H2) (*Sign condition*) There exist a constant $\alpha > 0$ and a t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$ function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\langle d, u\rangle_V \ge \alpha\|u\|_V^p - \beta(t)$ for any $u \in V$, $d \in A(t,u)$, and a.e. $t > 0$;

(H3) (*Strong measurability*) If $C \subseteq V^*$ is a closed set, then the set $\{(t,u) \in (0,+\infty) \times V \;:\; A(t,u) \cap C \ne \emptyset\}$ is a Borel subset in $(0,+\infty) \times V$;

(H4) (*Pointwise pseudomonotonicity*) Let for a.e. $t > 0$ the following two assumptions hold:

a) for every $u \in V$ the set $A(t,u)$ is nonempty, convex, and weakly compact one in V^*;

b) if a sequence $\{u_n\}_{n\ge1}$ converges weakly in V towards $u \in V$ as $n \to +\infty$, $d_n \in A(t,u_n)$ for any $n \ge 1$, and $\limsup_{n\to+\infty}\langle d_n, u_n - u\rangle_V \le 0$, then for any $\omega \in V$ there exists $d(\omega) \in A(t,u)$ such that

$$\liminf_{n\to+\infty}\langle d_n, u_n - \omega\rangle_V \ge \langle d(\omega), u - \omega\rangle_V.$$

Let $0 \le \tau < T < +\infty$. As a *weak solution* of evolution inclusion (7.33) on the interval $[\tau, T]$ we consider an element $u(\cdot)$ of the space $L_p(\tau, T; V)$ such that for some $d(\cdot) \in L_q(\tau, T; V^*)$ it is fulfilled:

$$-\int_{\tau}^{T}(\xi'(t),y(t))dt+\int_{\tau}^{T}\langle d(t),\xi(t)\rangle_V dt=0\quad\forall\xi\in C_0^{\infty}([\tau,T];V),\qquad(7.34)$$

and $d(t)\in A(t,y(t))$ for a.e. $t\in(\tau,T)$. For fixed nonnegative τ and T, $\tau<T$, let us consider

$$X_{\tau,T}=L_p(\tau,T;V),\quad X^*_{\tau,T}=L_q(\tau,T;V^*),$$

$$W_{\tau,T}=\{y\in X_{\tau,T}\mid y'\in X^*_{\tau,T}\},\quad \mathscr{A}_{\tau,T}:X_{\tau,T}\to 2^{X^*_{\tau,T}}\setminus\{\emptyset\},$$

$$\mathscr{A}_{\tau,T}(y)=\{d\in X^*_{\tau,T}\mid d(t)\in A(t,y(t))\text{ for a.e. }t\in(\tau,T)\},$$

where y' is a derivative of an element $u\in X_{\tau,T}$ in the sense of $\mathscr{D}([\tau,T];V^*)$ (see, for example, Gajewski, Gröger, and Zacharias [11, Definition IV.1.10]). Gajewski, Gröger, and Zacharias [11, Theorem IV.1.17] provide that the embedding $W_{\tau,T}\subset C([\tau,T];H)$ is continuous and dense. Moreover,

$$(u(T),v(T))-(u(\tau),v(\tau))=\int_{\tau}^{T}\Big[\langle u'(t),v(t)\rangle_V+\langle v'(t),u(t)\rangle_V\Big]dt,\qquad(7.35)$$

for any $u,v\in W_{\tau,T}$.

Migórski [33, Lemma 7, p. 516] (see the paper and references therein) and the assumptions above provide that the multi-valued mapping $\mathscr{A}_{\tau,T}:X_{\tau,T}\to 2^{X^*_{\tau,T}}\setminus\{\emptyset\}$ satisfies the listed below properties:

(P1) There exists a positive constant $C_1=C_1(\tau,T)$ such that $\|d\|_{X^*_{\tau,T}}\le C_1(1+\|y\|^{p-1}_{X_{\tau,T}})$ for any $y\in X_{\tau,T}$ and $d\in\mathscr{A}_{\tau,T}(y)$;

(P2) There exist positive constants $C_2=C_2(\tau,T)$ and $C_3=C_3(\tau,T)$ such that $\langle d,y\rangle_{X_{\tau,T}}\ge C_2\|y\|^p_{X_{\tau,T}}-C_3$ for any $y\in X_{\tau,T}$ and $d\in\mathscr{A}_{\tau,T}(y)$;

(P3) The multi-valued mapping $\mathscr{A}_{\tau,T}:X_{\tau,T}\to 2^{X^*_{\tau,T}}\setminus\{\emptyset\}$ is (generalized) pseudo-monotone on $W_{\tau,T}$, i.e.

(a) for every $y\in X_{\tau,T}$ the set $\mathscr{A}_{\tau,T}(y)$ is a nonempty, convex and weakly compact one in $X^*_{\tau,T}$;

(b) $\mathscr{A}_{\tau,T}$ is upper semi-continuous from every finite dimensional subspace $X_{\tau,T}$ into $X^*_{\tau,T}$ endowed with the weak topology;

(c) if a sequence $\{y_n,d_n\}_{n\ge1}\subset W_{\tau,T}\times X^*_{\tau,T}$ converges weakly in $W_{\tau,T}\times X^*_{\tau,T}$ towards $(y,d)\in W_{\tau,T}\times X^*_{\tau,T}$, $d_n\in\mathscr{A}_{\tau,T}(y_n)$ for any $n\ge1$, and $\limsup_{n\to+\infty}\langle d_n,y_n-y\rangle_{X_{\tau,T}}\le0$, then $d\in\mathscr{A}_{\tau,T}(y)$ and $\lim_{n\to+\infty}\langle d_n,y_n\rangle_{X_{\tau,T}}=\langle d,y\rangle_{X_{\tau,T}}$.

Formula (7.34) and the definition of the derivative for an element from $\mathscr{D}([\tau,T];V^*)$ yield that each weak solution $y\in X_{\tau,T}$ of Problem (7.33) on $[\tau,T]$ belongs to the space $W_{\tau,T}$ and $y'+\mathscr{A}_{\tau,T}(y)\ni\bar{0}$. On the contrary, suppose that $y\in W_{\tau,T}$ satisfies the last inclusion, then y is a weak solution of Problem (7.33) on $[\tau,T]$.

Assumption (H1), properties (P1)–(P3), and Denkowski, Migórski, and Papageorgiou [10, Theorem 1.3.73] (see also Zgurovsky, Mel'nik, and Kasyanov

[47, Chap. 2] and references therein) provide the existence of a weak solution of Cauchy problem (7.33) with initial data $y(\tau) = y^{(\tau)}$ on the interval $[\tau, T]$, for any $y^{(\tau)} \in H$.

For fixed τ and T, such that $0 \le \tau < T < +\infty$, we denote

$$\mathscr{D}_{\tau,T}(y^{(\tau)}) = \{y(\cdot) \mid y \text{ is a weak solution of (7.33) on } [\tau, T],\ y(\tau) = y^{(\tau)}\}, \quad y^{(\tau)} \in H.$$

We remark that $\mathscr{D}_{\tau,T}(y^{(\tau)}) \neq \emptyset$ and $\mathscr{D}_{\tau,T}(y^{(\tau)}) \subset W_{\tau,T}$, if $0 \le \tau < T < +\infty$ and $y^{(\tau)} \in H$. Moreover, the concatenation of weak solutions of Problem (7.33) is a weak solutions too, i.e. if $0 \le \tau < t < T$, $y^{(\tau)} \in H$, $y(\cdot) \in \mathscr{D}_{\tau,t}(y^{(\tau)})$, and $v(\cdot) \in \mathscr{D}_{t,T}(y(t))$, then

$$z(s) = \begin{cases} y(s), & s \in [\tau, t], \\ v(s), & s \in [t, T], \end{cases}$$

belongs to $\mathscr{D}_{\tau,T}(y^{(\tau)})$; cf. Zgurovsky et al. [48, pp. 55–56].

Gronwall's lemma provides that for any finite time interval $[\tau, T] \subset \mathbb{R}_+$ each weak solution y of Problem (7.33) on $[\tau, T]$ satisfies the estimates

$$\|y(t)\|_H^2 - 2\int_0^t \beta(\xi)d\xi + 2\alpha \int_s^t \|y(\xi)\|_V^p d\xi \le \|y(s)\|_H^2 - 2\int_0^s \beta(\xi)d\xi, \quad (7.36)$$

$$\|y(t)\|_H^2 \le \|y(s)\|_H^2 e^{-2\alpha\gamma(t-s)} + 2\int_s^t (\beta(\xi) + \alpha\gamma)e^{-2\alpha\gamma(t-\xi)}d\xi, \qquad (7.37)$$

where $t, s \in [\tau, T]$, $t \ge s$; $\gamma > 0$ is a constant such that $\gamma\|u\|_H^p \le \|u\|_V^p$ for any $u \in V$; cf. Zgurovsky et al. [48, p. 56]. In the proof of (7.37) we used the inequality $\|u\|_H^2 - 1 \le \|u\|_H^p$ for any $u \in H$.

Therefore, any weak solution y of Problem (7.33) on a finite time interval $[\tau, T] \subset \mathbb{R}_+$ can be extended to a global one, defined on $[\tau, +\infty)$. For arbitrary $\tau \ge 0$ and $y^{(\tau)} \in H$ let $\mathscr{D}_\tau(y^{(\tau)})$ be the set of all weak solutions (defined on $[\tau, +\infty)$) of Problem (7.33) with initial data $y(\tau) = y^{(\tau)}$. Let us consider the family $\mathscr{K}_\tau^+ = \cup_{y^{(\tau)} \in H} \mathscr{D}_\tau(y^{(\tau)})$ of all weak solutions of Problem (7.33) defined on the semi-infinite time interval $[\tau, +\infty)$.

Properties (P1)–(P2) imply assumptions (A1) and (A2). Therefore, Theorem 7.1 yields that the m-semiflow G, defined in (7.8), has a compact global attractor $\mathfrak{R}$ in the phase space H.

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Chapter 8
Uniform Trajectory Attractors for Non-autonomous Nonlinear Systems

Abstract In this chapter we study uniform trajectory attractors for non-autonomous nonlinear systems. In Sect. 8.1 we establish the existence of uniform trajectory attractor for non-autonomous reaction-diffusion equations with Carathéodory's nonlinearity. Section 8.2 devoted to structural properties of the uniform global attractor for non-autonomous reaction-diffusion system in which uniqueness of Cauchy problem is not guarantied. In the case of translation compact time-depended coefficients it is established that the uniform global attractor consists of bounded complete trajectories of corresponding multi-valued processes. Under additional sign conditions on nonlinear term we also prove (and essentially use previous result) that the uniform global attractor is, in fact, bounded set in $L^\infty(\Omega) \cap H_0^1(\Omega)$. Section 8.3 devoted to uniform trajectory attractors for nonautonomous dissipative dynamical systems. As applications we may consider FitzHugh–Nagumo system (signal transmission across axons), complex Ginzburg–Landau equation (theory of superconductivity), Lotka–Volterra system with diffusion (ecology models), Belousov–Zhabotinsky system (chemical dynamics) and many other reaction-diffusion type systems from Sect. 2.4.

8.1 Uniform Trajectory Attractor for Non-autonomous Reaction-Diffusion Equations with Carathéodory's Nonlinearity

Let $N, M = 1, 2, \ldots$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. We consider a problem of long-time behavior of all globally defined weak solutions for the non-autonomous parabolic problem (named reaction-diffusion or RD-system; see Chap. 4 and [1–23]).

$$\begin{cases} y_t = a\Delta y - f(x, t, y), & x \in \Omega,\ t > 0, \\ y|_{\partial\Omega} = 0, \end{cases} \tag{8.1}$$

as $t \to +\infty$, where $y = y(x,t) = (y^{(1)}(x,t), ..., y^{(M)}(x,t))$ is unknown vector-function, $f = f(x,t,y) = (f^{(1)}(x,t,y), ..., f^{(M)}(x,t,y))$ is given function, a is real $M \times M$ matrix with positive symmetric part.

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Throughout this section we suppose that the listed below assumptions hold (see Chap. 5).

Assumption I Let $p_i \geq 2$ and $q_i > 1$ are such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$, for any $i = 1, 2, \ldots, M$. Moreover, there exists a positive constant d such that $\frac{1}{2}(a+a^*) \geq dI$, where I is unit $M \times M$ matrix, a^* is a transposed matrix for a.
Assumption II The interaction function $f = (f^{(1)}, ..., f^{(M)}) : \Omega \times \mathbb{R}_+ \times \mathbb{R}^M \to \mathbb{R}^M$ satisfies the standard Carathéodory's conditions, i.e. the mapping $(x, t, u) \to f(x, t, u)$ is continuous in $u \in \mathbb{R}^M$ for a.e. $(x, t) \in \Omega \times \mathbb{R}_+$, and it is measurable in $(x, t) \in \Omega \times \mathbb{R}_+$ for any $u \in \mathbb{R}^M$.
Assumption III (Growth Condition) There exist a t.u.i. in $L_1^{\text{loc}}(\mathbb{R}_+; L_1(\Omega))$ function $c_1 : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $c_2 > 0$ such that

$$\sum_{i=1}^{M} \left| f^{(i)}(x, t, u) \right|^{q_i} \leq c_1(x, t) + c_2 \sum_{i=1}^{M} \left| u^{(i)} \right|^{p_i}$$

for any $u = (u^{(1)}, ..., u^{(M)}) \in \mathbb{R}^M$, and a.e. $(x, t) \in \Omega \times \mathbb{R}_+$.
Assumption IV (Sign Condition). There exists a constant $\alpha > 0$ and a t.u.i. in $L_1^{\text{loc}}(\mathbb{R}_+; L_1(\Omega))$ function $\beta : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sum_{i=1}^{M} f^{(i)}(x, t, u) u^{(i)} \geq \alpha \sum_{i=1}^{M} \left| u^{(i)} \right|^{p_i} - \beta(x, t)$$

for any $u = (u^{(1)}, ..., u^{(M)}) \in \mathbb{R}^M$, and a.e. $(x, t) \in \Omega \times \mathbb{R}_+$.

In further arguments we will use standard functional Hilbert spaces $H = (L_2(\Omega))^M$, $V = (H_0^1(\Omega))^M$, and $V^* = (H^{-1}(\Omega))^M$ with standard respective inner products and norms $(\cdot, \cdot)_H$ and $\| \cdot \|_H$, $(\cdot, \cdot)_V$ and $\| \cdot \|_V$, and $(\cdot, \cdot)_{V^*}$ and $\| \cdot \|_{V^*}$, vector notations $\mathbf{p} = (p_1, p_2, ..., p_M)$ and $\mathbf{q} = (q_1, q_2, ..., q_M)$, and the spaces

$$\begin{aligned}
&\mathbf{L}_{\mathbf{p}}(\Omega) := L_{p_1}(\Omega) \times ... \times L_{p_M}(\Omega), \quad \mathbf{L}_{\mathbf{q}}(\Omega) := L_{q_1}(\Omega) \times ... \times L_{q_M}(\Omega),\\
&\mathbf{L}_{\mathbf{p}}(\tau, T; \mathbf{L}_{\mathbf{p}}(\Omega)) := L_{p_1}(\tau, T; L_{p_1}(\Omega)) \times ... \times L_{p_M}(\tau, T; L_{p_M}(\Omega)),\\
&\mathbf{L}_{\mathbf{q}}(\tau, T; \mathbf{L}_{\mathbf{q}}(\Omega)) := L_{q_1}(\tau, T; L_{q_1}(\Omega)) \times ... \times L_{q_M}(\tau, T; L_{q_M}(\Omega)), \ 0 \leq \tau < T < +\infty.
\end{aligned}$$

Let $0 \leq \tau < T < +\infty$. We recall that a function $y = y(x, t) \in \mathbf{L}_2(\tau, T; V) \cap \mathbf{L}_{\mathbf{p}}(\tau, T; \mathbf{L}_{\mathbf{p}}(\Omega))$ is a *weak solution* of Problem (8.1) on $[\tau, T]$, if for any function $\varphi = \varphi(x) \in (C_0^\infty(\Omega))^M$, the following identity holds

$$\frac{d}{dt} \int_\Omega y(x, t) \cdot \varphi(x) dx + \int_\Omega \{a \nabla y(x, t) \cdot \nabla \varphi(x) + f(x, t, y(x, t)) \cdot \varphi(x)\} dx = 0$$

in the sense of scalar distributions on (τ, T).

In the general case Problem (8.1) on $[\tau, T]$ with initial condition $y(x, \tau) = y_\tau(x)$ in Ω has more than one weak solution with $y_\tau \in H$ (cf. Zgurovsky et al. [23] and references therein). Thus, for investigation of the long-time behavior as $t \to +\infty$ of all weak solutions of Problem (8.1) with initial data from H, the results for uniform global and trajectory attractors of multi-valued semi-processes in infinite-dimensional spaces were applied; Babin and Vishik [2], Chepyzhov and Vishik [6], Mel'nik and Valero [14, 15] and references therein. Theses approaches were applied to various non-autonomous problems of the form

$$\partial_t y(t) = A_{\sigma(t)}(y(t)), \tag{8.2}$$

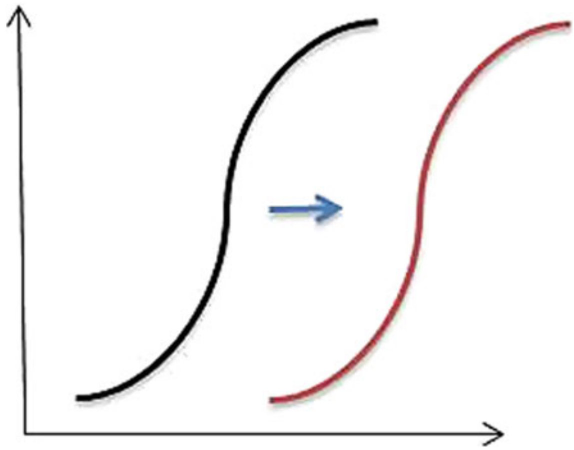
where $\sigma(s)$, $s \geq 0$, is a functional parameter called the time symbol of Eq. (8.2) (t is replaced by s). In applications to mathematical physics equations, a function $\sigma(s)$ consists of all time-dependent terms of the equation under consideration: external forces, parameters of mediums, interaction functions, control functions, etc. It is assumed that the symbol σ of Eq. (8.2) belongs to a Hausdorff topological space Ξ_+ of functions defined on $\mathbb{R}_+$ with values in some complete metric space. Usually, in applications, the topology in the space Ξ_+ is a local convergence topology on any segment $[t_1, t_2] \subset \mathbb{R}_+$. Further, they consider the family of Eq. (8.2) with various symbols $\sigma(s)$ belonging to a set $\Sigma \subseteq \Xi_+$. The set Σ is called the symbol space of the family of Eq. (8.2). It is assumed that the set Σ, together with any symbol $\sigma(s) \in \Sigma$, contains all positive translations of $\sigma(s)$: $\sigma(t+s) = T(t)\sigma(s) \in \Sigma$ for any $t, s \geq 0$. The symbol space Σ is invariant with respect to the translation semigroup $\{T(t)\}_{t\geq 0}$: $T(t)\Sigma \subseteq \Sigma$ for any $t \geq 0$. To prove the existence of uniform trajectory attractor they supposed that the symbol space Σ with the topology induced from Ξ_+ is a compact metric space. Mostly in applications, as a symbol space Σ it is natural to consider the hull of translation-compact function $\sigma_0(s)$ in an appropriate Hausdorff topological space Ξ_+. The direct realization of this approach for Problem (8.1) is problematic without any additional assumptions for parameters of Problem (8.1) and requires the translation-compactness of the symbol $\sigma(s) = f(\cdot, s, \cdot)$ in some compact Hausdorff topological space of mappings act from $\mathbb{R}_+$ to some metric space of Carathéodory's vector-functions satisfying growth and signed assumptions. To avoid this technical difficulties we present the alternative direct approach for the existence and construction of the uniform trajectory attractor for all weak solutions for Problem (8.1).

The main purpose of this section is to investigate uniform long-time behavior of all globally defined weak solutions for Problem (8.1) with initial data $u_\tau \in H$ under listed above assumptions. The main results of this paper are: (i) the existence of uniform trajectory attractor for all globally defined weak solutions of non-autonomous reaction-diffusion equations with Carathéodory's nonlinearity (Theorem 8.1), and (ii) sufficient conditions for the existence of uniform trajectory attractor in strongest topologies (Theorem 8.2).

In further arguments as a Banach space $\mathscr{F}_{t_1,t_2}$ we consider either $C([t_1, t_2]; H)$ or W_{t_1,t_2} with respective topologies of strong convergence, where $0 \leq t_1 < t_2 < +\infty$. Consider the Fréchet space

$$\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+) := \{y : \mathbb{R}_+ \to H \ : \ \Pi_{t_1,t_2} y \in \mathscr{F}_{t_1,t_2} \text{ for any } [t_1, t_2] \subset \mathbb{R}_+\},$$

Fig. 8.1 Translation operation



where Π_{t_1,t_2} is the restriction operator to the interval $[t_1, t_2]$; Chepyzhov and Vishik [6, p. 918]. We remark that the sequence $\{f_n\}_{n\geq 1}$ converges (converges weakly respectively) in $\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)$ towards $f \in \mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)$ as $n \to +\infty$ if and only if the sequence $\{\Pi_{t_1,t_2} f_n\}_{n\geq 1}$ converges (converges weakly respectively) in $\mathscr{F}_{t_1,t_2}$ towards $\Pi_{t_1,t_2} f$ as $n \to +\infty$ for any finite interval $[t_1, t_2] \subset \mathbb{R}_+$.

We denote $T(h)y(\cdot) = y_h(\cdot)$, where $y_h(t) = y(t+h)$ for any $y \in \mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)$ and $t, h \geq 0$ (see Fig. 8.1).

In the autonomous case, when $f(x, t, y)$ does not depend on t, the long-time behavior of all globally defined weak solutions for Problem (8.1) is described by using trajectory and global attractors theory. In this situation the set $\mathscr{K}^+ := \mathscr{K}_0^+$ is *translation semi-invariant*, i.e. $T(h)\mathscr{K}^+ \subseteq \mathscr{K}^+$ for any $h \geq 0$. As trajectory attractor it is considered a classical global attractor for translation semigroup $\{T(h)\}_{h\geq 0}$, that acts on $\mathscr{K}^+$.

In the non-autonomous case we notice that $T(h)\mathscr{K}_0^+ \nsubseteq \mathscr{K}_0^+$. Therefore, we need to consider *united trajectory space* that includes all globally defined on any $[\tau, +\infty) \subseteq \mathbb{R}_+$ weak solutions of Problem (8.1) shifted to $\tau = 0$:

$$\mathscr{K}_{\cup}^+ := \bigcup_{\tau\geq 0} \left\{ y(\cdot + \tau) \in W^{\mathrm{loc}}(\mathbb{R}_+) \,:\, y(\cdot) \in \mathscr{K}_\tau^+ \right\}, \tag{8.3}$$

Note that $T(h)\{y(\cdot + \tau) \,:\, y \in \mathscr{K}_\tau^+\} \subseteq \{y(\cdot + \tau + h) \,:\, y \in \mathscr{K}_{\tau+h}^+\}$ for any $\tau, h \geq 0$. Therefore, $T(h)\mathscr{K}_{\cup}^+ \subseteq \mathscr{K}_{\cup}^+$ for any $h \geq 0$.

To define an uniform trajectory attractor, the united trajectory space need to be a closed subset of a Polish space. Further we consider *extended united trajectory space* for Problem (8.1):

$$\mathcal{K}^{+}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})} = \mathrm{cl}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})}\left[\mathcal{K}^{+}_{\cup}\right], \tag{8.4}$$

where $\mathrm{cl}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})}[\,\cdot\,]$ is the closure in $\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})$. We note that

$$T(h)\mathcal{K}^{+}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})} \subseteq \mathcal{K}^{+}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})} \text{ for any } h \geq 0,$$

because

$$\rho_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})}(T(h)u, T(h)v) \leq \rho_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})}(u, v) \text{ for any } u, v \in \mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+}),$$

where $\rho_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})}$ is a standard metric on Fréchet space $\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})$; cf. Chepyzhov and Vishik [6]; Vishik et al. [21].

A set $\mathcal{P} \subset \mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+}) \cap L_{\infty}(\mathbb{R}_{+}; H)$ is said to be a *uniformly attracting set* (cf. Chepyzhov and Vishik [6, p. 921]) for the extended united trajectory space $\mathcal{K}^{+}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})}$ of Problem (8.1) in the topology of $\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})$, if for any bounded in $L_{\infty}(\mathbb{R}_{+}; H)$ set $\mathcal{B} \subseteq \mathcal{K}^{+}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})}$ and any segment $[t_1, t_2] \subset \mathbb{R}_{+}$ the following relation holds:

$$\mathrm{dist}_{\mathcal{F}_{t_1,t_2}}(\Pi_{t_1,t_2}T(t)\mathcal{B}, \Pi_{t_1,t_2}\mathcal{P}) \to 0, \quad t \to +\infty, \tag{8.5}$$

where $\mathrm{dist}_{\mathcal{F}_{t_1,t_2}}$ is the Hausdorff semi-metric.

A set $\mathcal{U} \subset \mathcal{K}^{+}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})}$ is said to be a *uniform trajectory attractor* (cf. Chepyzhov and Vishik [6, p. 921] and references therein) of the translation semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{K}^{+}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})}$ in the induced topology from $\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})$, if

(i) $\mathcal{U}$ is a compact set in $\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})$ and bounded in $L_{\infty}(\mathbb{R}_{+}; H)$;
(ii) $\mathcal{U}$ is strictly invariant with respect to $\{T(h)\}_{h\geq 0}$, i.e. $T(h)\mathcal{U} = \mathcal{U}$ $\forall h \geq 0$;
(iii) $\mathcal{U}$ is a minimal uniformly attracting set for $\mathcal{K}^{+}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})}$ in the topology of $\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})$, i.e. $\mathcal{U}$ belongs to any compact uniformly attracting set $\mathcal{P}$ of $\mathcal{K}^{+}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})}$: $\mathcal{U} \subseteq \mathcal{P}$.

Note that uniform trajectory attractor of the translation semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{K}^{+}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})}$ in the induced topology from $\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})$ coincides with the classical global attractor for the continuous semi-group $\{T(t)\}_{t\geq 0}$ defined on $\mathcal{K}^{+}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_{+})}$.

Assumptions I–IV are sufficient conditions for the existence of uniform trajectory attractor for weak solutions of Problem (8.1) in the topology of $C^{\mathrm{loc}}(\mathbb{R}_{+}; H)$.

Theorem 8.1 *Let Assumptions* I–IV *hold. Then there exists an uniform trajectory attractor* $\mathcal{U} \subset \mathcal{K}^{+}_{C^{\mathrm{loc}}(\mathbb{R}_{+};H)}$ *of the translation semigroup* $\{T(t)\}_{t\geq 0}$ *on* $\mathcal{K}^{+}_{C^{\mathrm{loc}}(\mathbb{R}_{+};H)}$ *in the induced topology from* $C^{\mathrm{loc}}(\mathbb{R}_{+}; H)$. *Moreover, there exists a compact in* $C^{\mathrm{loc}}(\mathbb{R}_{+}; H)$ *uniformly attracting set* $\mathcal{P} \subset C^{\mathrm{loc}}(\mathbb{R}_{+}; H) \cap L_{\infty}(\mathbb{R}_{+}; H)$ *for the* *extended united trajectory space* $\mathcal{K}^+_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ *of Problem (8.1) in the topology of* $C^{\mathrm{loc}}(\mathbb{R}_+; H)$ *such that* $\mathcal{U}$ *coincides with* ω*-limit set of* $\mathcal{P}$*:*

$$\mathcal{U} = \bigcap_{t \geq 0} \mathrm{cl}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)} \left[\bigcup_{h \geq t} T(h)\mathcal{P} \right]. \tag{8.6}$$

For the existence of uniform trajectory attractor in the strong topology of the natural extended phase space $W^{\mathrm{loc}}(\mathbb{R}_+)$ it is necessary to claim that additional assumption holds (see Example 8.1). To formulate this additional assumption we provide some auxiliary constructions. A function $\varphi \in L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ is called *translation-compact (tr.-c.)* in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$, if the set $\{\varphi(\cdot + h) : h \geq 0\}$ is precompact in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$; cf. Chepyzhov and Vishik [6, p. 917]. Note that a function $\varphi \in L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ is tr.-c. in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ if and only if two conditions hold: (a) the set $\left\{\int_t^{t+h} \varphi(s)ds : t \geq 0\right\}$ is precompact in $L_1(\Omega)$ for any $h > 0$; (b) there exists a function $\psi(s)$, $\psi(s) \to 0+$ as $s \to 0+$ such that

$$\int_t^{t+1} \int_\Omega |\varphi(x,s) - \varphi(x, s+h)| dx ds \leq \psi(|h|) \text{ for any } t \geq 0 \text{ and } h \geq -t;$$

Chepyzhov and Vishik [6, Proposition 6.5].

Assumption V Let the conditions hold:

(i) the functions c_1 and β from Assumptions (III) and (IV) respectively are tr.-c. in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$;

(ii) the set $\left\{\frac{1}{h}\int_t^{t+h} f(\cdot, s, u)ds : t \geq 0,\ h \in (0, h_0),\ \|u\|_{\mathbb{R}^M} \leq R\right\}$ is precompact in $(L_1(\Omega))^M$ for any $R > 0$ and some $h_0 = h_0(R) > 0$;

(iii) for any $r > 0$ there exist a nondecreasing function $\psi(s, r) : \mathbb{R}_+^2 \to \mathbb{R}_+$, $\psi(s, r) \to 0+$ as $s \to 0+$, and $h_0 = h_0(r) > 0$ such that

$$\frac{1}{h_1} \sum_{i=1}^{M} \int_t^{t+h_1} \int_\Omega \left| f^{(i)}(x,s,u) - f^{(i)}(x, s+h_2, v) \right| dx ds \leq \psi(|h_2| + \|u - v\|_{\mathbb{R}^M}, r)$$

for each $t \geq 0$, $h_1 \in (0, h_0)$, $h_2 \geq -t$, and $u, v \in \mathbb{R}^M$ such that $\|u\|_{\mathbb{R}^M}, \|v\|_{\mathbb{R}^M} \leq r$.

Remark 8.1 Let us discuss sufficient conditions for Assumption V.

(i) The autonomous case. Let f does not depend on the time variable t and it satisfies Assumptions I–IV with $c_1, \beta \in L_1(\Omega)$ (in particular, assumptions from Vishik et al. [21] hold). Then Assumption V hold; see Remark 4.1.

(ii) The non-autonomous case. Let $f = f(t, u)$ is jointly continuous mapping, it satisfies Assumptions I–IV with positive constants c_1 and β, and f being tr.-c. in $C^{\mathrm{loc}}(\mathbb{R}_+; C(\mathbb{R}^M))$, that is,

$$\|f(t,u) - f(s,v)\|_{\mathbb{R}^M} \leq \omega(|t-s| + \|u-v\|_{\mathbb{R}^M}, K)$$

for all $t, s \in \mathbb{R}_+$, $\|u\|_{\mathbb{R}^M}, \|v\|_{\mathbb{R}^M} \le K$, $K > 0$, where $\omega(l, K) \to 0$, as $l \to 0+$; see, for example, Chepyzhov and Vishik [6, p. 105], Kapustyan and Valero [8–10], where uniform global in H and uniform trajectory in $C^{\mathrm{loc}}(\mathbb{R}_+; H)$ attractors were investigated. Then Assumption V holds.

(iii) The sufficient condition for Assumption V(iii) is: for any $r > 0$ there exist a nondecreasing function $\psi(s, r) : \mathbb{R}_+^2 \to \mathbb{R}_+$, $\psi(s, r) \to 0+$ as $s \to 0+$, such that

$$\sum_{i=1}^{M} \int_{\Omega} \left| f^{(i)}(x, t, u) - f^{(i)}(x, t+h, v) \right| dxds \le \psi(|h| + \|u - v\|_{\mathbb{R}^M}, r)$$

for each $t \ge 0$, $h \ge -t$, and $u, v \in \mathbb{R}^M$ such that $\|u\|_{\mathbb{R}^M}, \|v\|_{\mathbb{R}^M} \le r$.

Note that Assumption V is a generalization of the above assumptions to the case when f depends on the space, time and state variables simultaneously and it is not necessarily continuous by t. Meanwhile, Example 8.1 below provide piecewise continuous function f that satisfies Assumptions I-IV, but it does not satisfy Assumption V. Moreover, the statement of Theorem 8.2 below does not hold for Problem (8.1) with such interaction function.

The main result on the existence of uniform trajectory attractor for weak solutions of Problem (8.1) in the topology of $W^{\mathrm{loc}}(\mathbb{R}_+)$ has the following form:

Theorem 8.2 *Let Assumptions* I–V *hold. Then there exists an uniform trajectory attractor $\mathscr{U} \subset \mathscr{K}^+_{W^{\mathrm{loc}}(\mathbb{R}_+)}$ of the translation semigroup $\{T(t)\}_{t\ge 0}$ on $\mathscr{K}^+_{W^{\mathrm{loc}}(\mathbb{R}_+)}$ in the induced topology from $W^{\mathrm{loc}}(\mathbb{R}_+)$. Moreover, there exists a compact in $W^{\mathrm{loc}}(\mathbb{R}_+)$ uniformly attracting set $\mathscr{P} \subset W^{\mathrm{loc}}(\mathbb{R}_+) \cap L_\infty(\mathbb{R}_+; H)$ for the extended united trajectory space $\mathscr{K}^+_{W^{\mathrm{loc}}(\mathbb{R}_+)}$ of Problem (8.1) in the topology of $W^{\mathrm{loc}}(\mathbb{R}_+)$ such that $\mathscr{U}$ coincides with ω-limit set of $\mathscr{P}$:*

$$\mathscr{U} = \bigcap_{t\ge 0} \mathrm{cl}_{W^{\mathrm{loc}}(\mathbb{R}_+)} \left[\bigcup_{h \ge t} T(h)\mathscr{P} \right]. \tag{8.7}$$

Remark 8.2 All statements of Theorems 8.1 and 8.2 hold for the function $f(x, t, y)$ equals to the sum of an interaction function $f_1(x, t, y)$, satisfying Assumptions I–IV (Assumptions I–V respectively), and an external force $g \in L_2^{\mathrm{loc}}(\mathbb{R}_+; V^*)$. In Theorem 8.1 g is need to be translation bounded in $L_2^{\mathrm{loc}}(\mathbb{R}_+; V^*)$ and g is translation compact in $L_2^{\mathrm{loc}}(\mathbb{R}_+; V^*)$ in Theorem 8.2 respectively. The proofs are similar with some standard technical modifications. To simplify the conclusions, further we consider the case $g \equiv 0$.

Proof of Theorems 8.1 *and* 8.2 The proofs of both two theorems are similar and based on the respective statements of Theorems 4.1 and 4.2. To avoid reduplication we set $\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+) := C^{\mathrm{loc}}(\mathbb{R}_+; H)$ for the proof of Theorem 8.1 and $\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+) := W^{\mathrm{loc}}(\mathbb{R}_+)$ for the proof of Theorem 8.2 respectively.

We provide the proof in several steps. First, let us show that there exists a uniform trajectory attractor $\mathscr{U} \subset \mathscr{K}^+_{\mathscr{F}^{\mathrm{loc}}(\mathbb{R}_+)}$ of the translation semigroup $\{T(t)\}_{t\ge 0}$ on

$\mathcal{K}^+_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_+)}$ in the induced topology from $\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_+)$. Theorem 4.1, if $\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_+) = C^{\mathrm{loc}}(\mathbb{R}_+; H)$, and Theorem 4.2, if $\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_+) = W^{\mathrm{loc}}(\mathbb{R}_+)$, yields that the translation semigroup $\{T(t)\}_{t\geq 0}$ has a compact absorbing (and, therefore, an uniformly attracting) set in the space of trajectories $\mathcal{K}^+_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_+)}$; Zgurovsky et al. [23] and references therein. This set can be constructed as follows: 1) consider $\mathcal{P}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_+)}$, the intersection of $\mathcal{K}^+_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_+)}$ with a ball in the space of bounded continuous functions on $\mathbb{R}_+$ with values in H, $C_b(\mathbb{R}_+; H)$, of sufficiently large radius; 2) shift the resulting set by any fixed distance $h > 0$. Thus, we obtain $T(h)\mathcal{P}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_+)}$, a set with the required properties. Recall that the semigroup $\{T(t)\}_{t\geq 0}$ is continuous. Therefore, the set $\mathcal{P}_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_+)}$ is a compact absorbing (and, therefore, an uniformly attracting) for $\mathcal{K}^+_{\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_+)}$ with the induced topology of $\mathcal{F}^{\mathrm{loc}}(\mathbb{R}_+)$. Then we can apply, for example, Theorem 2.2 from Chepyzhov and Vishik [6, Chap. XI] and finish the proof. In particular, formula (8.6) holds; cf. Babin and Vishik [2]; Melnik and Valero [14], Temam [20] etc.

The example provided below implies that additional Assumption V in Theorem 4.2 is essential for the existence of uniform trajectory attractor in strongest topology of an extended united phase space of weak solutions for Problem (4.1).

Example 8.1 Interaction function $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^M \to \mathbb{R}^M$ satisfies Assumptions I–IV, Assumption V does not hold, and the statement of Theorem 4.2 does not hold. Let $N, M = 1$, $\Omega = (0, \pi)$, $a = 1$, $f(x, t, u) := u - \sin(x) \cdot \sin(\pi[t]t)$, $(x, t, u) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}$, where $[t]$ is a largest integer, that does not exceed t. The verifying of Assumptions I–IV and V(i,ii) is trivial. Assumption V(iii) does not hold, because $|\sin(\pi k^2) - \sin(\pi(k + \frac{1}{2k})k)| = 1 \not\to 0,$ as $k \to +\infty$.

The statement of Theorem 4.2 does not hold. On the contrary assume that there exists an uniform trajectory attractor $\mathcal{U} \subset \mathcal{K}^+_{W^{\mathrm{loc}}(\mathbb{R}_+)}$ of the translation semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{K}^+_{W^{\mathrm{loc}}(\mathbb{R}_+)}$ in the induced topology from $W^{\mathrm{loc}}(\mathbb{R}_+)$. By definition of an uniform trajectory attractor, since $\mathcal{U}$ is a compact set in $W^{\mathrm{loc}}(\mathbb{R}_+)$, for each $y(\cdot) \in \mathcal{K}^+_{W^{\mathrm{loc}}(\mathbb{R}_+)}$ (we note that the set $\{T(h)y(\cdot) : h \geq 0\}$ is bounded in $L_\infty(\mathbb{R}_+; H)$) any monotone increasing unbounded sequence $\{h_n\}_{n\geq 1} \subset \mathbb{R}_+$ has a subsequence $\{h_{n_k}\}_{k\geq 1} \subseteq \{h_n\}_{n\geq 1}$ such that $\{T(h_{n_k})y(\cdot)\}_{k\geq 1}$ is precompact in $W^{\mathrm{loc}}(\mathbb{R}_+)$. On the other hand, let

$$y(x, t) := \sin(x) \int_0^t \sin(\pi[s]s)e^{-2(t-s)}ds, \quad (x, t) \in \Omega \times \mathbb{R}_+,$$

and $h_n := n$, $n = 1, 2, \ldots$. Note that $y \in \mathcal{K}^+_0 \subseteq \mathcal{K}^+_{W^{\mathrm{loc}}(\mathbb{R}_+)}$, and

$$\frac{\partial}{\partial t} y(x, t+n) = -2\sin(x) \int_0^{t+n} \sin(\pi[s](s))e^{-2(t+n-s)}ds + (-1)^n \sin(x) \cdot \sin(\pi nt),$$

$(x, t) \in \Omega \times (0, 1)$, $n = 1, 2, \ldots$. Therefore,

$$\liminf_{n,m\to+\infty} \|\Pi_{0,1}T(n)y(\cdot) - \Pi_{0,1}T(m)y(\cdot)\|_{W_{0,1}} > 0,$$

because the sequence of functions $\{(x,t) \to \sin(x) \int_0^{t+n} \sin(\pi[s]s)e^{-2(t+n-s)}ds\}_{n\geq 1}$, restricted on $\Omega \times (0,1)$, converges strongly in $L_2(0,1;V) \subset X^*_{0,1}$ as $n \to +\infty$, and the sequence of functions $\{(x,t) \to \frac{2}{\sqrt{\pi}} \sin(x) \cdot \sin(\pi n t)\}_{n\geq 1}$, restricted on $\Omega \times (0,1)$, is orthonormal in $X^*_{0,1}$. This is a contradiction with the existence of a subsequence $\{h_{n_k}\}_{k\geq 1} \subseteq \{h_n\}_{n\geq 1}$ such that $\{T(h_{n_k})y(\cdot)\}_{k\geq 1}$ is precompact in $W^{\mathrm{loc}}(\mathbb{R}_+)$. Therefore, the statement of Theorem 4.2 does not hold.

8.2 Structure of Uniform Global Attractor for Non-autonomous Reaction-Diffusion Equations

In this section we study the structural properties of the uniform global attractor of non-autonomous reaction-diffusion system in which the nonlinear term satisfy suitable growth and dissipative conditions on the phase variable, suitable translation compact conditions on time variable, but there is no condition ensuring uniqueness of Cauchy problem. In autonomous case such system generates in the general case a multi-valued semiflow having a global compact attractor (see [8, 12, 23]). Also, it is known [12], that the attractor is the union of all bounded complete trajectories of the semiflow. Here we prove the same result for non-autonomous system. More precisely, we prove that the family of multi-valued processes, generated by weak solutions of reaction-diffusion system, has uniform global attractor which is union of bounded complete trajectories of corresponding processes. Using this result, we can prove that under additional restrictions on nonlinear term obtained uniform global attractor is bounded set in the space $L^\infty(\Omega) \cap H^1_0(\Omega)$.

In a bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$ we consider the following non-autonomous parabolic problem (named RD-system)

$$\begin{cases} u_t = a\Delta u - f(t,u) + h(t,x), & x \in \Omega,\ t > \tau, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{8.8}$$

where $\tau \in \mathbb{R}$ is initial moment of time, $u = u(t,x) = (u^1(t,x), ..., u^N(t,x))$ is unknown vector-function, $f = (f^1, ..., f^N)$, $h = (h^1, ..., h^N)$ are given functions, a is real $N \times N$ matrix with positive symmetric part $\frac{1}{2}(a + a^*) \geq \beta I$, $\beta > 0$,

$$h \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^N), \quad f \in C(\mathbb{R} \times \mathbb{R}^N; \mathbb{R}^N), \tag{8.9}$$

$\exists\, C_1, C_2 > 0,\ \gamma_i > 0,\ p_i \geq 2,\ i = \overline{1,N}$ such that $\forall\, t \in \mathbb{R},\ \forall\, v \in \mathbb{R}^N$

$$\sum_{i=1}^N |f^i(t,v)|^{\frac{p_i}{p_i-1}} \leq C_1(1 + \sum_{i=1}^N |v^i|^{p_i}), \tag{8.10}$$

$$\sum_{i=1}^{N} f^i(t,v)v^i \geq \sum_{i=1}^{N} \gamma_i |v^i|^{p_i} - C_2. \tag{8.11}$$

In further arguments we will use standard functional spaces

$$H = (L^2(\Omega))^N \text{ with the norm } |v|^2 = \int_\Omega \sum_{i=1}^{N} |v^i(x)|^2 dx,$$

$$V = (H_0^1(\Omega))^N \text{ with the norm } \|v\|^2 = \int_\Omega \sum_{i=1}^{N} |\nabla v^i(x)|^2 dx.$$

Let us denote $V' = (H^{-1}(\Omega))^N$, $q_i = \frac{p_i}{p_i - 1}$, $p = (p_1, ..., p_N)$, $q = (q_1, ..., q_N)$, $L^p(\Omega) = L^{p_1}(\Omega) \times ... \times L^{p_N}(\Omega)$.

Definition 8.1 The function $u = u(t,x) \in L^2_{loc}(\tau, +\infty; V) \bigcap L^p_{loc}(\tau, +\infty; L^p(\Omega))$ is called a (weak) solution of the problem (8.8) on $(\tau, +\infty)$ if for all $T > \tau$, $v \in V \cap L^p(\Omega)$

$$\frac{d}{dt}\int_\Omega u(t,x)v(x)dx + \int_\Omega \Big(a\nabla u(t,x)\nabla v(x) + f(t,u(t,x))v(x) - h(t,x)v(x)\Big)dx = 0 \tag{8.12}$$

in the sense of scalar distributions on (τ, T).

From (8.10) and Sobolev embedding theorem we see that every solution of (8.8) satisfies inclusion $u_t \in L^q_{loc}(\tau, +\infty; H^{-r}(\Omega))$, where $r = (r_1, ..., r_N)$, $r_k = max\{1, n(\frac{1}{2} - \frac{1}{p_k})\}$. The following theorem is well-known result about global resolvability of (8.8) for initial conditions from the phase space H.

Theorem 8.3 *([1, Theorem 2] or [6, p.284]) Under conditions (8.10), (8.11) for every $\tau \in \mathbb{R}$, $u_\tau \in H$ there exists at least one weak solution of (8.8) on $(\tau, +\infty)$ with $u(\tau) = u_\tau$ (and it may be non unique) and any weak solution of (8.8) belongs to $C([\tau, +\infty); H)$. Moreover, the function $t \mapsto |u(t)|^2$ is absolutely continuous and for a.a. $t \geq \tau$ the following energy equality holds*

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + (a\nabla u(t), \nabla u(t)) + (f(t,u(t)), u(t)) = (h(t), u(t)). \tag{8.13}$$

Under additional not-restrictive conditions on function f and h it is known that solution of (8.8) generate non-autonomous dynamical system (two-parametric family of m-processes), which has uniform global attractor. The aim of this paper is to give description of the attractor in terms of bounded complete trajectories and show some regularity property of this set.

Let (X, ρ) be a complete metric space. The Hausdorff semidistance from A to B is given by

$$dist(A, B) = \sup_{x\in A} \inf_{y\in B} \rho(x, y),$$

By $\overline{A}$ and $O_\varepsilon(A) = \{x \in X \mid \inf_{y\in A} \rho(x, y) < \varepsilon\}$ we denote closure and ε-neighborhood of the set A. Denote by $P(X)$ ($\beta(X)$, $C(X)$, $K(X)$) the set of all non-empty (not-empty bounded, not-empty closed, not-empty compact) subsets of X,

$$\mathbb{R}_d = \{(t, \tau) \in \mathbb{R}^2 | t \geq \tau\}.$$

Let Σ be some complete metric space, $\{T(h) : \Sigma \mapsto \Sigma\}_{h\geq 0}$ be a continuous semigroup, acting on Σ. Note, that in most applications $T(h)$ is shift semigroup.

Definition 8.2 Two-parameter family of multi-valued mappings $\{U_\sigma : \mathbb{R}_d \times X \mapsto P(X)\}_{\sigma\in\Sigma}$ is said to be the family of m-processes (family of MP), if $\forall\, \sigma \in \Sigma, \tau \in \mathbb{R}$:

(1) $U_\sigma(\tau, \tau, x) = x \;\; \forall x \in X$,
(2) $U_\sigma(t, \tau, x) \subseteq U_\sigma(t, s, U_\sigma(s, \tau, x)), \;\; \forall t \geq s \geq \tau \;\; \forall x \in X$,
(3) $U_\sigma(t + h, \tau + h, x) \subseteq U_{T(h)\sigma}(t, \tau, x) \;\; \forall t \geq \tau \;\; \forall h \geq 0, \; \forall\, x \in X$,

where for $A \subset X, B \subset \Sigma$ $U_B(t, s, A) = \bigcup_{\sigma\in B} \bigcup_{x\in A} U_\sigma(t, s, x)$, in particular

$$U_\Sigma(t, \tau, x) = \bigcup_{\sigma\in\Sigma} U_\sigma(t, \tau, x);$$

see also Fig. 8.2.

Family of MP $\{U_\sigma | \sigma \in \Sigma\}$ is called strict, if in conditions (2), (3) equality take place.

Definition 8.3 A set $A \subset X$ is called uniformly attracting for the family of MP $\{U_\sigma | \sigma \in \Sigma\}$, if for arbitrary $\tau \in \mathbb{R}$, $B \in \beta(X)$

$$dist(U_\Sigma(t, \tau, B), A) \to 0, \; t \to +\infty, \tag{8.14}$$

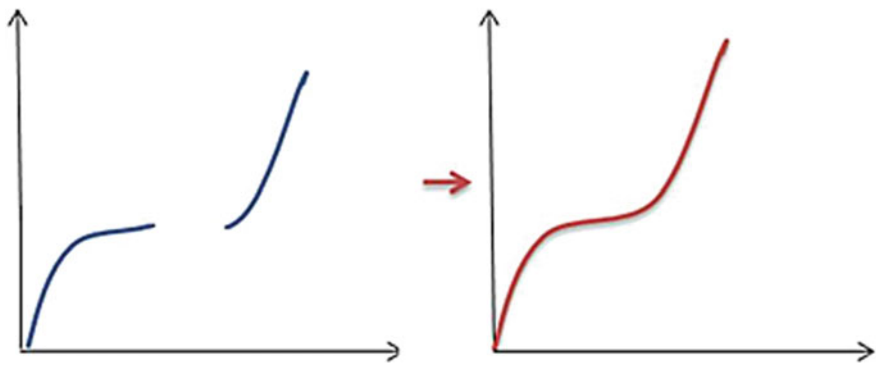


Fig. 8.2 Concatenation

that is $\forall\, \varepsilon > 0$, $\tau \in \mathbb{R}$ and $B \in \beta(X)$ there exists $T = T(\tau, \varepsilon, B)$ such that

$$U_\Sigma(t, \tau, B) \subset O_\varepsilon(A) \;\; \forall\, t \geq T.$$

For fixed $B \subset X$ and $(s, \tau) \in \mathbb{R}_d$ let us define the following sets

$$\gamma^\tau_{s,\sigma}(B) = \bigcup_{t \geq s} U_\sigma(t, \tau, B), \;\; \gamma^\tau_{s,\Sigma}(B) = \bigcup_{t \geq s} U_\Sigma(t, \tau, B),$$

$$\omega_\Sigma(\tau, B) = \bigcap_{s \geq \tau} cl_X(\gamma^\tau_{s,\Sigma}(B)).$$

It is clear that $\omega_\Sigma(\tau, B) = \bigcap\limits_{s \geq p} cl_X(\gamma^\tau_{s,\Sigma}(B)) \;\; \forall p \geq \tau$.

Definition 8.4 The family of MP $\{U_\sigma | \sigma \in \Sigma\}$ is called uniformly asymptotically compact, if for arbitrary $\tau \in \mathbb{R}$ and $B \in \beta(X)$ there exists $A(\tau, B) \in K(X)$ such that

$$U_\Sigma(t, \tau, B) \to A(\tau, B), \; t \to +\infty \;\text{ in } X.$$

It is known [9] that if $\forall \tau \in \mathbb{R}$, $\forall B \in \beta(X)\, \exists T = T(\tau, B)\; \gamma^\tau_{T,\Sigma}(B) \in \beta(X)$, then the condition of uniformly asymptotically compactness is equivalent to the following one:

$$\forall \tau \in \mathbb{R}\, \forall B \in \beta(X)\, \forall t_n \nearrow \infty$$

$$\text{every sequence } \; \xi_n \in U_\Sigma(t_n, \tau, B) \text{ is precompact.}$$

Definition 8.5 A set $\Theta_\Sigma \subset X$ is called uniform global attractor of the family of MP $\{U_\sigma | \sigma \in \Sigma\}$, if:

(1) Θ_Σ is uniformly attracting set;
(2) for every uniformly attracting set Y we have $\Theta_\Sigma \subset cl_X Y$.

Uniform global attractor $\Theta_\Sigma \subset X$ is called invariant (semiinvariant), if $\forall\, (t, \tau) \in \mathbb{R}_d$

$$\Theta_\Sigma = U_\Sigma(t, \tau, \Theta_\Sigma) \;\; (\Theta_\Sigma \subset U_\Sigma(t, \tau, \Theta_\Sigma)).$$

If Θ_Σ is compact, invariant uniform global attractor, then it is called stable if $\forall\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, (t, \tau) \in \mathbb{R}_d$

$$U_\Sigma(t, \tau, O_\delta(\Theta_\Sigma)) \subset O_\varepsilon(\Theta_\Sigma).$$

The following sufficient conditions we can obtain with slight modifications from [9].

Theorem 8.4 *(I) Let us assume that the family of MP $\{U_\sigma | \sigma \in \Sigma\}$ satisfies the following conditions:*

(1) $\exists\ B_0 \in \beta(X)\ \forall\ B \in \beta(X)\ \forall \tau \in \mathbb{R}\ \exists\ T = T(\tau, B)$

$$\forall\ t \geq T\ U_\Sigma(t, \tau, B) \subset B_0;$$

(2) $\{U_\sigma | \sigma \in \Sigma\}$ *is uniformly asymptotically compact.*
Then $\{U_\sigma\}_{\sigma\in\Sigma}$ *has compact uniform global attractor*

$$\Theta_\Sigma = \bigcup_{\tau\in\mathbb{R}} \bigcup_{B\in\beta(X)} \omega_\Sigma(\tau, B) = \omega_\Sigma(0, B_0) = \omega_\Sigma(\tau, B_0)\ \ \forall \tau \in \mathbb{R}. \tag{8.15}$$

(II) If $\{U_\sigma\}_{\sigma\in\Sigma}$ *satisfy 1),2),* Σ *is compact and* $\forall t \geq \tau$ *the mapping*

$$(x, \sigma) \mapsto U_\sigma(t, \tau, x) \tag{8.16}$$

has closed graph, then Θ_Σ *is semiinvariant.*

If, moreover, $\forall h \geq 0\ T(h)\Sigma = \Sigma$ *and the family MP* $\{U_\sigma | \sigma \in \Sigma\}$ *is strict, then* Θ_Σ *is invariant.*

(III) If $\{U_\sigma\}_{\sigma\in\Sigma}$ *satisfy (1), (2),* Σ *is connected and compact,* $\forall t \geq \tau$ *the mapping (8.16) is upper semicontinuous and has connected values,* B_0 *is connected set, then* Θ_Σ *is connected set.*

(IV) If $\{U_\sigma | \sigma \in \Sigma\}$ *has compact, invariant uniform global attractor* Θ_Σ *and the following condition hold:*

$$if\ \ y_n \in U_\Sigma(t_n, \tau, x_n),\ \ t_n \to t_0,\ \ x_n \to x_0,$$

$$then\ up\ to\ subsequence\ y_n \to\ \ y_0 \in U_\Sigma(t_0, \tau, x_0), \tag{8.17}$$

then Θ_Σ *is stable.*

Proof (I) From conditions (1), (2) due to [9] we have that $\forall \tau \in \mathbb{R}\ \forall B \in \beta(X)$ $\omega_\Sigma(\tau, B) \neq \emptyset$, is compact, $\omega_\Sigma(\tau, B) \subset B_0$ and the set

$$\Theta_\Sigma = \bigcup_{\tau\in\mathbb{R}} \bigcup_{B\in\beta(X)} \omega_\Sigma(\tau, B)$$

is uniform global attractor. Let us prove that $\omega_\Sigma(\tau, B) \subset \omega_\Sigma(\tau_0, B_0)\ \forall \tau, \tau_0 \in \mathbb{R}$.

$$U_\sigma(t, \tau, B) \subset U_\sigma(t, \frac{t}{2}, U_\sigma(\frac{t}{2}, \tau, B)) \subset U_{T(\frac{t}{2}-\tau_0)\sigma}(\frac{t}{2} + \tau_0, \tau_0, U_\sigma(\frac{t}{2}, \tau, B)) \subset$$

$$\subset U_\Sigma(\frac{t}{2} + \tau_0, \tau_0, B_0),\ \text{if}\ \ \frac{t}{2} \geq T(\tau, B) + |\tau_0| + |\tau| := T.$$

So, for $t \geq 2T$

$$U_\Sigma(t, \tau, B) \subset U_\Sigma(\frac{t}{2} + \tau_0, \tau_0, B_0).$$

Then for $s \geq 2T$

$$\bigcup_{t\geq s} U_\Sigma(t,\tau,B) \subset \bigcup_{t\geq s} U_\Sigma(\frac{t}{2}+\tau_0,\tau_0,B_0) = \bigcup_{p\geq \frac{s}{2}+\tau_0} U_\Sigma(p,\tau_0,B_0),$$

$$\bigcap_{s\geq 2T} \overline{\bigcup_{t\geq s} U_\Sigma(t,\tau,B)} = \omega_\Sigma(\tau,B) \subset \bigcap_{s\geq 2T} \overline{\bigcup_{p\geq \frac{s}{2}+\tau_0} U_\Sigma(p,\tau_0,B_0)} =$$

$$\bigcap_{s'\geq T+\tau_0} \overline{\bigcup_{p\geq s'} U_\Sigma(p,\tau_0,B_0)} = \omega_\Sigma(\tau_0,B_0).$$

So we deduce equality (8.15).

(II) Due to (8.15) $\forall \xi \in \Theta_\Sigma = \omega_\Sigma(\tau,B_0)$ $\exists t_n \nearrow +\infty$, $\exists \sigma_n \in \Sigma$ $\exists \xi_n \in U_{\Sigma_n}(t_n,\tau,B_0)$ such that $\xi = \lim_{n\to\infty} \xi_n$. Then

$$\xi_n \in U_{\sigma_n}(t_n - t - \tau + t + \tau, \tau, B_0) \subset$$

$$\subset U_{\sigma_n}(t_n - t - \tau + t + \tau, t_n - t + \tau, U_{\sigma_n}(t_n - t + \tau, \tau, B_0)) \subset U_{T(t_n-t)\sigma_n}(t,\tau,\eta_n),$$

where $\eta_n \in U_{\sigma_n}(t_n - t + \tau, \tau, B_0)$, $t \geq \tau$ and for sufficiently large $n \geq 1$.

From uniform asymptotically compactness we have that on some subsequence $\eta_n \to \eta \in \omega_\Sigma(\tau,B_0) = \Theta_\Sigma$,

$$T(t_n - t)\sigma_n \to \sigma \in \Sigma.$$

Then from (8.16) we deduce:

$$\xi \in U_\Sigma(t,\tau,\Theta_\Sigma),$$

and therefore $\Theta_\Sigma \subset U_\Sigma(t,\tau,\Theta_\Sigma)$.

Other statements of the theorem are proved similarly to [9].

Theorem is proved.

Corollary 8.1 *If for the family of MP $\{U_\sigma\}_{\sigma\in\Sigma}$ we have:*

(1) $\forall h \geq 0$ $T(h)\Sigma = \Sigma$;
(2) $\forall (t,\tau) \in \mathbb{R}_d$ $\forall h \geq 0$ $\forall \sigma \in \Sigma$ $\forall x \in X$

$$U_\sigma(t+h,\tau+h,x) = U_{T(h)\sigma}(t,\tau,x),$$

then all conditions of previous theorem can be verified only for $\tau = 0$.

Proof Under conditions (1), (2) $\forall t \geq \tau$ if $\tau \geq 0$ then

$$U_\sigma(t,\tau,x) = U_{T(\tau)\sigma}(t-\tau,0,x),$$

and if $\tau \le 0$ then $\exists \sigma' \in \Sigma : \sigma = T(-\tau)\sigma'$, so

$$U_\sigma(t,\tau,x) = U_{T(-\tau)\sigma'}(t,\tau,x) = U_{\sigma'}(t-\tau,0,x).$$

In the single-valued case it is known [6], that the uniform global attractor consists of bounded complete trajectories of processes $\{U_\sigma\}_{\sigma\in\Sigma}$.

Definition 8.6 The mapping $\varphi : [\tau,+\infty) \mapsto X$ is called trajectory of MP U_σ, if $\forall t \ge s \ge \tau$

$$\varphi(t) \in U_\sigma(t,s,\varphi(s)). \tag{8.18}$$

If for $\varphi : \mathbb{R} \mapsto X$ the equality (8.18) takes place $\forall t \ge s$, then φ is called complete trajectory.

Now we assume that for arbitrary $\sigma \in \Sigma$ and $\tau \in \mathbb{R}$ we have the set K_σ^τ of mappings $\varphi : [\tau,+\infty) \mapsto X$ such that:

(a) $\forall x \in X\ \exists \varphi(\cdot) \in K_\sigma^\tau$ such, that $\varphi(\tau) = x$;
(b) $\forall \varphi(\cdot) \in K_\sigma^\tau\ \ \forall s \ge \tau\ \ \varphi(\cdot)|_{[s,+\infty)} \in K_\sigma^s$;
(c) $\forall h \ge 0\ \forall \varphi(\cdot) \in K_\sigma^{\tau+h}\ \ \varphi(\cdot + h) \in K_{T(h)\sigma}^\tau$.

Let us put

$$U_\sigma(t,\tau,x) = \{\varphi(t) | \varphi(\cdot) \in K_\sigma^\tau,\ \varphi(\tau) = x\}. \tag{8.19}$$

Lemma 8.1 *Formula (8.19) defines the family of MP* $\{U_\sigma\}_{\sigma\in\Sigma}$, *and* $\forall \varphi(\cdot) \in K_\sigma^\tau$

$$\forall t \ge s \ge \tau \quad \varphi(t) \in U_\sigma(t,s,\varphi(s)). \tag{8.20}$$

Proof Let us check conditions of the Definition 8.2.

(1) $U_\sigma(\tau,\tau,x) = \varphi(\tau) = x$;

(2) $\forall \xi \in U_\sigma(t,\tau,x)\ \ \xi = \varphi(t)$, where $\varphi \in K_\sigma^\tau$, $\varphi(\tau) = x$. Then for $s \in [\tau,t]$ $\varphi(s) \in U_\sigma(s,\tau,x)$ and from $\varphi|_{[s,+\infty)} \in K_\sigma^s$ we have $\varphi(t) \in U_\sigma(t,s,\varphi(s))$. So

$$\xi \in U_\sigma(t,s,U_\sigma(s,\tau,x)).$$

(3) $\forall \xi \in U_\sigma(t+h,\tau+h,x)\ \xi = \varphi(t+h)$, where $\varphi \in K_\sigma^{\tau+h}$, $\varphi(\tau+h) = x$. Then $\psi(\cdot) = \varphi(\cdot+h) \in K_{T(h)\sigma}^\tau$, $\psi(\tau) = x$, so $\xi = \psi(t) \in U_{T(h)\sigma}(t,\tau,x)$. Lemma is proved.

It is easy to show that under conditions a)-c), if $\forall s \ge \tau\ \forall \psi \in K_\sigma^\tau, \forall \varphi \in K_\sigma^s$ such that $\psi(s) = \varphi(s)$, we have

$$\theta(p) = \begin{cases} \psi(p), & p \in [\tau,s] \\ \varphi(p), & p > s, \end{cases} \in K_\sigma^\tau, \tag{8.21}$$

then in the condition (2) of Definition 8.2 equality takes place.

If $\forall h \ge 0\ \forall \varphi \in K_{T(h)\sigma}^\tau\ \varphi(\cdot - h) \in K_\sigma^{\tau+h}$, then in the condition 3) of Definition 8.2 equality takes place.

From (8.20) we immediately obtain that if for mapping $\varphi(\cdot):\mathbb{R}\mapsto X$ for arbitrary $\tau\in\mathbb{R}$ we have $\varphi(\cdot)|_{[\tau,+\infty)}\in K_\sigma^\tau$, then $\varphi(\cdot)$ is complete trajectory of U_σ.

The next result is generalization on non-autonomous case results from [5].

Lemma 8.2 *Let the family of MP $\{U_\sigma\}_{\sigma\in\Sigma}$ be constructed by the formula (8.19), $\forall\varphi(\cdot)\in K_\sigma^\tau$ is continuous on $[\tau,+\infty)$, the condition (8.21) takes place and the following one:*
if $\varphi_n(\cdot)\in K_\sigma^\tau$, $\varphi_n(\tau)=x$, then $\exists\varphi(\cdot)\in K_\sigma^\tau$, $\varphi(\tau)=x$ such that on some subsequence

$$\varphi_n(t)\to\varphi(t)\ \ \forall t\ge\tau.$$

Then every continuous on $[\tau,+\infty)$ trajectory of MP U_σ belongs to K_σ^τ.

Proof Let $\psi:[\tau,+\infty)\mapsto X$ be continuous trajectory. Let us construct sequence $\{\varphi_n(\cdot)\}_{n=1}^\infty\subset K_\sigma^\tau$ such that

$$\varphi_n(\tau+j2^{-n})=\psi(\tau+j2^{-n}),\quad j=0,1,...,n2^n.$$

For $\varphi_1(\cdot)$ we have

$$\psi(\tau+\frac{1}{2})\in U_\sigma(\tau+\frac{1}{2},\tau,\psi(\tau)),$$

$$\psi(\tau+1)\in U_\sigma(\tau+1,\tau+\frac{1}{2},\psi(\tau+\frac{1}{2})).$$

So there exists $\widetilde{\varphi}(\cdot)\in K_\sigma^\tau$, there exists $\widetilde{\widetilde{\varphi}}(\cdot)\in K_\sigma^{\tau+\frac{1}{2}}$ such that

$$\psi(\tau+\frac{1}{2})=\widetilde{\varphi}(\tau+\frac{1}{2}),\ \ \widetilde{\varphi}(\tau)=\psi(\tau),$$

$$\psi(\tau+1)=\widetilde{\widetilde{\varphi}}(\tau+1),\ \ \widetilde{\widetilde{\varphi}}(\tau+\frac{1}{2})=\psi(\tau+\frac{1}{2}).$$

Therefore due to (8.21) for function

$$\varphi_1(p)=\begin{cases}\widetilde{\varphi}(p), & \tau\le p\le\tau+\frac{1}{2},\\ \widetilde{\widetilde{\varphi}}(p), & p>\tau+\frac{1}{2}\end{cases}\quad \text{we have:}$$

$$\varphi_1(\cdot)\in K_\sigma^\tau,\ \ \varphi_1(\tau)=\psi(\tau),\ \ \varphi_1(\tau+\frac{1}{2})=\psi(\tau+\frac{1}{2}),\ \ \varphi_1(\tau+1)=\psi(\tau+1).$$

Further, using (8.21), we obtain required property for every $n\ge1$. As $\varphi_n(\tau)=\psi(\tau)$, so $\exists\varphi(\cdot)\in K_\sigma^\tau$, $\varphi(\tau)=\psi(\tau)$ such that on subsequence $\forall t\ge\tau\ \ \varphi_n(t)\to\varphi(t)$. As $\forall t=\tau+j2^{-n}\ \ \varphi(t)=\psi(t)$, so from continuity $\varphi(t)=\psi(t)\ \ \forall t\ge\tau$. Lemma is proved.

The following theorem declare structure of uniform global attractor in terms of bounded complete trajectories of corresponding m-processes. It should be noted that this result is known for single-valued case [6] and in multi-valued case for very special class of strict processes, generated by strict compact semiprocesses, which act in Banach spaces [22].

Theorem 8.5 *Let Σ is compact, $T(h)\Sigma = \Sigma$ $\forall\, h \geq 0$, the family of MP $\{U_\sigma\}_{\sigma\in\Sigma}$ satisfies (8.19), in condition (3) of Definition 8.2 equality takes place, the mapping $(x, \sigma) \mapsto U_\sigma(t, 0, x)$ has closed graph. Let us assume that there exists Θ_Σ - compact uniform global attractor of the family $\{U_\sigma\}_{\sigma\in\Sigma}$, and one of two conditions hold: either the family of MP $\{U_\sigma\}_{\sigma\in\Sigma}$ is strict, or*

$$for\ every\ \sigma_n \to \sigma_0,\ \ x_n \to x_0\ if\ \varphi_n(\cdot) \in K^0_{\sigma_n},\ \varphi_n(0) = x_n,$$

$$so\ \exists\, \varphi(\cdot) \in K^0_{\sigma_0},\ \varphi(0) = x_0\ such\ that\ on\ subsequnce\ \forall t \geq 0\ \ \varphi_n(t) \to \varphi(t). \tag{8.22}$$

Then the following structural formula holds

$$\Theta_\Sigma = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(0), \tag{8.23}$$

where $\mathcal{K}_\sigma$ is the set of all bounded complete trajectories of MP U_σ.

Proof First let us consider situation when the family of MP $\{U_\sigma\}_{\sigma\in\Sigma}$ is strict. In this case one can consider multi-valued semigroup (m-semiflow) on the extended phase space $X \times \Sigma$ by the rule

$$G(t, \{x, \sigma\}) = \{U_\sigma(t, 0, x), T(t)\sigma\}. \tag{8.24}$$

Then G is strict, has closed graph and compact attracting set $\Theta_\Sigma \times \Sigma$. So G has compact invariant global attractor

$$\mathcal{A} = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} G(t, \Theta_\Sigma \times \Sigma)} = \{\gamma(0) | \gamma \text{ is bounded complete trajectories of } G\}.$$

Here under complete trajectory of m-semiflow G we mean the mapping $\mathbb{R} \ni t \mapsto \gamma(t)$ such that

$$\forall\, t \in \mathbb{R}\ \forall\, s \geq 0\ \gamma(t+s) \in G(s, \gamma(t)).$$

Let us consider two projectors Π_1 and Π_2, $\Pi_1(u, \sigma) = u$, $\Pi_2(u, \sigma) = \sigma$. As $T(t)\Sigma = \Sigma$, so $\Pi_2\mathcal{A} = \Sigma$. Let us prove that $\Pi_1\mathcal{A} = \Theta_\Sigma$.

As $\forall\, B \in \beta(X)$ $G(t, B \times \Sigma) \to \mathcal{A}$, $t \to +\infty$, so

$$U_\Sigma(t, \tau, B) \to \Pi_1\mathcal{A},$$

so $\Theta_\Sigma \subset \Pi_1\mathscr{A}$. Let us prove that $\Pi_1\mathscr{A} = \bigcup_{\sigma\in\Sigma} \mathscr{K}_\sigma(0)$. For this purpose we take $(u_0, \sigma_0) \in \mathscr{A}$. Then there exists $\gamma(\cdot) = \{u(\cdot), \sigma(\cdot)\}$, which is bounded complete trajectory of G and such that $\gamma(0) = (u_0, \sigma_0)$. Then $\forall t \geq \tau$

$$u(t) \in U_{\sigma(\tau)}(t-\tau, 0, u(\tau)), \;\; \sigma(t) = T(t-\tau)\sigma(\tau).$$

If $\tau \geq 0$, then $\sigma(\tau) = T(\tau)\sigma_0$, that is

$$u(t) \in U_{T(\tau)\sigma_0}(t-\tau, 0, u(\tau)) = U_{\sigma_0}(t, \tau, u(\tau)).$$

If $\tau < 0$, then $\sigma_0 = T(-\tau)\sigma(\tau)$, so

$$u(t) \in U_{\sigma(\tau)}(t-\tau, \tau-\tau, u(\tau)) = U_{T(-\tau)\sigma(\tau)}(t, \tau, u(\tau)) = U_{\sigma_0}(t, \tau, u(\tau)).$$

Therefore $u_0 = u(0) \in \mathscr{K}_{\sigma_0}(0) \subset \bigcup_{\sigma\in\Sigma} \mathscr{K}_\sigma(0)$.

Now let $u_0 = u(0) \in K_{\sigma_0}(0)$, $\;u(t) \in U_{\sigma_0}(t, \tau, u(\tau)) \;\; \forall t \geq \tau$. As $T(t)\Sigma = \Sigma$, so there exists $\sigma(s)$, $s \in \mathbb{R}$, such that $\sigma(t) = T(t-\tau)\sigma(\tau)$, $\;\forall t \geq \tau$, $\;\sigma(0) = \sigma_0$. Then for $s \geq 0$ we have

$$G(t, \{u(s), \sigma(s)\}) = (U_{\sigma(s)}(t, 0, u(s)), T(t)\sigma(s)) =$$

$$= (U_{T(s)\sigma_0}(t, 0, u(s)), \sigma(t+s)) = (U_{\sigma_0}(t+s, s, u(s)), \sigma(t+s)),$$

$$\{u(t+s), \sigma(t+s)\} \in (U_{\sigma_0}(t+s, s, u(s)), \sigma(t+s)).$$

If $s < 0$, then $\sigma_0 = T(-s)\sigma(s)$, and

$$u(t+s) \in U_{\sigma_0}(t+s, s, u(s)) = U_{T(-s)\sigma(s)}(t+s, s, u(s)) = U_{\sigma(s)}(t, 0, u(s)).$$

Then $u_0 \in \Pi_1\mathscr{A}$ and $\Pi_1\mathscr{A} = \bigcup_{\sigma\in\Sigma} \mathscr{K}_\sigma(0)$.

Since for arbitrary attracting set P and for arbitrary bounded complete trajectory $\Gamma = \{u(s)\}_{s\in\mathbb{R}}$ of the process U_σ we have

$$u(0) \in U_\sigma(0, -n, u(-n)) = U_{T(n)\sigma(-n)}(0, -n, u(-n)) \subset$$

$$\subset U_\Sigma(n, 0, \Gamma) \to P, \;\; n \to +\infty,$$

so $u(0) \in P$, and we obtain (8.23).

Now let us consider another case, when family of m-processes is not strict, but the condition (8.22) holds. Let us show that $\mathscr{K}_\sigma(0) \subset \Theta_\Sigma$. If $z \in \mathscr{K}_\sigma(0)$, then there exists bounded complete trajectory $\varphi(\cdot)$ of m-process U_σ, such that $\varphi(0) = z$. Let us denote $\Gamma = \bigcup_{t\in\mathbb{R}} \varphi(t) \in \beta(X)$. Then for $z = \varphi(0)$ we have

$$\varphi(0) \in U_\sigma(0, -n, \varphi(-n)) = U_{T(n)\sigma_n}(0, -n, \varphi(-n)) \subset U_\Sigma(n, 0, \Gamma).$$

Since $\forall \varepsilon > 0 \ \exists n_0 \ \forall n \geq n_0 \ U_\Sigma(n, 0, \Gamma) \subset O_\varepsilon(\Theta_\Sigma)$, then $z \in \Theta_\Sigma$ and we obtain required embedding.

Now let $z \in \Theta_\Sigma = \omega_\Sigma(0, B_0)$. Then $z = \lim\limits_{n\to+\infty} \xi_n$, $\xi_n \in U_\Sigma(t_n, 0, B_0)$. Therefore on some subsequence

$$z = \lim_{n\to+\infty} \varphi_n(t_n), \quad \varphi_n(\cdot) \in K^0_{\sigma_n}, \quad \varphi_n(0) \in B_0, \quad \sigma_n \to \sigma.$$

For $\forall n \geq 1$ let us consider

$$\psi_n(\cdot) := \varphi_n(\cdot + t_n) \in K^{-t_n}_{T(t_n)\sigma_n},$$

that is $\psi_n(\cdot) \in K^{-t_n}_{\tilde{\sigma_n}}$, where $\tilde{\sigma_n} = T(t_n)\sigma_n$. Then $\psi_n(\cdot) \in K^0_{\tilde{\sigma_n}}, \quad \tilde{\sigma_n} \to \tilde{\sigma}, \quad \psi_n(0) = \varphi_n(t_n) \to z$, so there exists $\psi^{(0)}(\cdot) \in K^0_{\tilde{\sigma}}, \quad \psi^{(0)}(0) = z$, such that

$$\forall t \geq 0 \ \ \psi_n(t) = \varphi_n(t + t_n) \to \psi^{(0)}(t).$$

For $\tau = -1 \ \forall n \geq n_1 \ \ -t_n < -1$, therefore $\psi_n(\cdot) \in K^{-1}_{\tilde{\sigma_n}}$ and on some subsequence

$$\psi_n(-1) = \varphi_n(t_n - 1) \to z_1.$$

Herewith there exists $\psi^{(-1)}(\cdot) \in K^{-1}_{\tilde{\sigma}}$ such that on subsequence

$$\psi_n(t) = \varphi_n(t + t_n) \to \psi^{(-1)}(t) \ \ \forall t \geq -1,$$

and $\forall t \geq 0 \ \ \psi^{(0)}(t) = \psi^{(-1)}(t)$. By standard diagonal procedure we construct sequence of functions

$$\psi^{(-k)}(\cdot) \in K^{-k}_{\tilde{\sigma}}, \quad k \geq 0,$$

with $\psi^{(-k+1)}(t) = \psi^{(-k)}(t) \ \ \forall t \geq -k + 1$. Let us put

$$\psi(t) := \psi^{(-k)}(t), \ if \ t \geq -k.$$

Then the function $\psi(\cdot)$ is correctly defined, $\psi : \mathbb{R} \mapsto X$.

Moreover $\forall \tau < 0 \ \exists k$ such that $[\tau, +\infty) \subset [-k, +\infty)$, on $[-k, +\infty) \ \ \psi(\cdot) \equiv \psi^{(-k)}$, so $\psi(\cdot) \in K^{-k}_{\tilde{\sigma}}$, and from this

$$\psi(\cdot) \in K^\tau_{\tilde{\sigma}}, \quad \psi(0) = \psi^{(0)}(0) = z.$$

Since on subsequence

$$\forall t \in \mathbb{R} \ \ \psi(t) = \lim_{n\to+\infty} \varphi_n(t + t_n) \in \omega_\Sigma(0, B_0) \in \beta(X),$$

then $z = \psi(0) \in \mathcal{K}_{\tilde{\sigma}}$ and theorem is proved.

Definition 8.7 Let Θ be some topological space of functions from $\mathbb{R}$ to topological space E. The function $\xi \in \Theta$ is called translation compact in Θ, if the set

$$H(\xi) = cl_{\Theta}\{\xi(\cdot + s) \mid s \in \mathbb{R}\}$$

is compact in Θ.

To construct family of m-processes for the problem (8.8) we suppose that time-depended functions f and h are translation compact in natural spaces [6]. More precisely, we will assume that

$$h \text{ is translation compact in } L^{2,w}_{loc}(\mathbb{R}; H), \tag{8.25}$$

where $L^{2,w}_{loc}(\mathbb{R}; H)$ is the space $L^2_{loc}(\mathbb{R}; H)$ with the local weak convergence topology, and

$$f \text{ is translation compact in } C(\mathbb{R}; C(\mathbb{R}^N, \mathbb{R}^N)), \tag{8.26}$$

where $C(\mathbb{R}; C(\mathbb{R}^N, \mathbb{R}^N))$ equipped with local uniform convergence topology.

It is known that condition (8.25) is equivalent to

$$|h|_+^2 := \sup_{t\in\mathbb{R}} \int_t^{t+1} |h(s)|^2 ds < \infty \tag{8.27}$$

It is also known that condition (8.26) is equivalent to

$$\begin{aligned} &\forall\, R > 0 \; f \text{ is bounded and uniformly continuous on} \\ &Q(R) = \{(t, v) \in \mathbb{R} \times \mathbb{R}^N \mid |v|_{\mathbb{R}^N} \le R\}. \end{aligned} \tag{8.28}$$

If conditions (8.25), (8.26) take place, then the symbol space

$$\Sigma = cl_{C(\mathbb{R}; C(\mathbb{R}^N, \mathbb{R}^N)) \times L^{2,w}_{loc}(\mathbb{R}; H)}\{(f(\cdot + s), h(\cdot + s)) \mid s \in \mathbb{R}\} \tag{8.29}$$

is compact, and $\forall\, s \ge 0$ $T(s)\Sigma = \Sigma$, where $T(s)$ is translation semigroup, which is continuous on Σ.

For every $\sigma = (f_\sigma, h_\sigma) \in \Sigma$ we consider the problem

$$\begin{cases} u_t = a\Delta u - f_\sigma(t, u) + h_\sigma(t, x), \quad x \in \Omega,\; t > \tau, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{8.30}$$

It is proved in [9] that $\forall\, \sigma \in \Sigma$ f_σ satisfies (8.10), (8.11) with the same constants C_1, C_2, γ_i, $|h_\sigma|_+ \le |h|_+$. So we can apply Theorem 2 and obtain that $\forall\, \tau \in \mathbb{R}$, $u_\tau \in H$ the problem (8.30) has at least one solution on $(\tau, +\infty)$, each solution of (8.30) belongs to $C([\tau, +\infty); H)$ and satisfies energy equality (8.13). For every $\sigma \in \Sigma$, $\tau \in \mathbb{R}$ we define

$$K_\sigma^\tau = \{u(\cdot) \mid u(\cdot) \; is \; solution \; of \; (8.30) \; on \; (\tau, +\infty)\} \tag{8.31}$$

and according to (8.19) we put $\forall\, \sigma \in \Sigma, \forall\, t \geq \tau, \forall\, u_\tau \in H$

$$U_\sigma(t, \tau, u_\tau) = \{u(t)|u(\cdot) \in K_\sigma^\tau,\; u(\tau) = u_\tau\}. \tag{8.32}$$

From [9] and Theorem 8.5 we obtain the following result. The following theorem was proved in [11].

Theorem 8.6 *Under conditions (8.10), (8.11), (8.25), (8.26) formula (8.32) defines a strict family of MP $\{U_\sigma\}_{\sigma\in\Sigma}$ which has compact, invariant, stable and connected uniform global attractor Θ_Σ, which consists of bounded complete trajectories, that is*

$$\Theta_\Sigma = \bigcup_{\sigma\in\Sigma} \mathscr{K}_\sigma(0), \tag{8.33}$$

where $\mathscr{K}_\sigma$ is the set of all bounded complete trajectories of MP U_σ.

Now we want to use formula (8.33) for proving that the uniform global attractor of RD-system is bounded set in the space $(L^\infty(\Omega))^N \cap V$.

First let us consider the following conditions:

$$\exists\, M_i > 0,\; i = \overline{1, N} \; such \; that \; for \; all \; v = (v^1, ..., v^N) \in \mathbb{R}^N \; for \; a.a. \; x \in \Omega \; \forall\, t \in \mathbb{R}$$

$$\sum_{i=1}^{N} (f^i(t, v) - h^i(t, x))(v^i - M_i)^+ \geq 0 \tag{8.34}$$

$$\sum_{i=1}^{N} (f^i(t, v) - h^i(t, x))(v^i + M_i)^- \leq 0 \tag{8.35}$$

where $\varphi^+ = max\{0, \varphi\}, \varphi^- = max\{0, -\varphi\}, \varphi = \varphi^+ - \varphi^-$.

Let us consider some example, which allow to verify conditions (8.34), (8.35).

Lemma 8.3 *If $N = 1$ (scalar equation), then from (8.10), (8.11) and $h \in L^\infty(\mathbb{R} \times \Omega)$ we have (8.34), (8.35).*

Proof From (8.10) and $h \in L^\infty(\Omega)$ for a.a. $x \in \Omega$ and $u \in \mathbb{R}$,

$$\tilde{\gamma}|u|^p - \tilde{C}_2 \leq g(t, x, u)u \leq \tilde{C}_1|u|^p + \tilde{C}_1,$$

where $g(t, x, u) = f(t, u) - h(t, x)$, $\tilde{\gamma}$ does not depend on t, u, x.

If $u \le M$, then $g(t, x, u)(u - M)^+ = 0$.
If $u > M$, then

$$g(t,x,u)(u-M)^+ = g(t,x,u)u\frac{(u-M)^+}{u} = g(t,x,u)u(1-\frac{M}{u})$$
$$\ge (\tilde{\gamma}u^p - \tilde{C}_2)(1-\frac{M}{u}) \ge (\tilde{\gamma}M^p - \tilde{C}_2)(1-\frac{M}{u})$$

and if we choose $M = \left(\frac{\tilde{C}_2}{\tilde{\gamma}}\right)^{\frac{1}{p}}$, then $g(t, x, u)(u - M)^+ \ge 0$ a.e.

Lemma 8.4 *If for arbitrary* $N \ge 1$ $h \equiv 0$, $f(t,u) = (f^1(t,u), \dots f^N(t,u))$, *where* $f^i(t,u) = (\sum_{i=1}^{N} |u^i|^2 - R^2)u^i$, $R > 0$ *is positive constant, then conditions (8.34), (8.35) hold for* $M_i = R$.

Proof If $\sum_{i=1}^{N} |u^i|^2 < R^2$, so $\forall\, i = \overline{1, N}$ $|u^i| < R$ and

$$\sum_{i=1}^{N} f^i(t,u)(u^i - R)^+ = 0,$$

$$\sum_{i=1}^{N} f^i(t,u)(u^i + R)^- = 0.$$

If $\sum_{i=1}^{N} |u^i|^2 \ge R^2$, then

$$\sum_{i=1}^{N} f^i(t,u)(u^i - R)^+ = \left(\sum_{i=1}^{N} |u^i|^2 - R^2\right)\sum_{i=1}^{N} u^i(u^i - R)^+ \ge 0,$$

$$\sum_{i=1}^{N} f^i(t,u)(u^i + R)^- = \left(\sum_{i=1}^{N} |u^i|^2 - R^2\right)\sum_{i=1}^{N} u^i(u^i + R)^- \le 0.$$

Theorem 8.7 *If conditions (8.10), (8.11), (8.25), (8.26), (8.34), (8.35) hold and matrix a is diagonal, then the uniform global attractor* Θ_Σ *is bounded set in the space* $(L^\infty(\Omega))^N \cap V$.

Proof First let us prove that $\forall\, \sigma \in \Sigma$ functions f_σ, h_σ satisfy (8.34), (8.35). Indeed, there exists sequence $t_n \nearrow \infty$ such that $\forall\, T > 0,\ R > 0,\ \eta \in L^2((-T,T)\times\Omega)$

$$\sup_{|t|\le T}\sup_{|v|\le R}\sum_{i=1}^{N} |f^i(t+t_n, v) - f^i_\sigma(t,v)|^2 \to 0,\ n \to \infty,$$

$$\sum_{i=1}^{N}\int_{-T}^{T}\int_{\Omega}(h^i(t+t_n,x)-h^i_\sigma(t,x))\eta(t,x)dxdt \to 0,\ n\to\infty.$$

From (8.34)

$$\sum_{i=1}^{N}(f^i(t+t_n,v)-h^i(t+t_n,x))(v^i-M_i)^+ \geq 0. \tag{8.36}$$

Therefore for fixed v and for arbitrary $\varepsilon > 0$ there exists $N \geq 1$ such that $\forall\, n \geq N$

$$\sum_{i=1}^{N}h^i(t+t_n,x)(v^i-M_i)^+ \leq \sum_{i=1}^{N}f^i(t+t_n,v)(v^i-M_i)^+ < \sum_{i=1}^{N}f^i_\sigma(t,v)(v^i-M_i)^+ + \varepsilon.$$

Because

$$\sum_{i=1}^{N}h^i(t+t_n,x)(v^i-M_i)^+ \to \sum_{i=1}^{N}h^i_\sigma(t,x)(v^i-M_i)^+\ weakly\ in\ L^2((-T,T)\times\Omega),$$

from Mazur Theorem we deduce that

$$\sum_{i=1}^{N}h^i_\sigma(t,x)(v^i-M_i)^+ \leq \sum_{i=1}^{N}f^i_\sigma(t,v)(v^i-M_i)^+ + \varepsilon\ for\ a.a.\ x\in\Omega.$$

From arbitrary choice of ε we can obtain required result.

It is easy to obtain that for arbitrary weak solution of (8.8) and for every $\eta \in C_0^\infty(\tau, T)$

$$\int_{\tau}^{T}(u_t,u^+)\eta dt = -\frac{1}{2}\int_{\tau}^{T}|u^+|^2\eta_t dt. \tag{8.37}$$

Then putting $g_\sigma = f_\sigma - h_\sigma$ and for numbers M_1, ..., M_N from condition (8.34) we have

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N}|(u^i-M_i)^+|^2+\beta\sum_{i=1}^{N}\|(u^i-M_i)^+\|^2+\int_{\Omega}\sum_{i=1}^{N}g^i_\sigma(t,x,u)(u^i-M_i)^+dx=0.$$

Then from (8.34)

$$\frac{d}{dt}\sum_{i=1}^{N}|(u^i-M_i)^+|^2+2\beta\sum_{i=1}^{N}|(u^i-M_i)^+|^2 \leq 0$$

and for all $t > \tau$

$$\sum_{i=1}^{N} |(u^i - M_i)^+(t)|^2 \leq \sum_{i=1}^{N} |(u^i - M_i)^+(\tau)|^2 e^{-2\lambda_1 \beta(t-\tau)}. \tag{8.38}$$

If $u(\cdot) \in \mathscr{K}_\sigma$ then from (8.38) taking $\tau \to -\infty$ we obtain $u^i(x,t) \leq M_i$, $i = \overline{1, N}$, $\forall t \in \mathbb{R}$, for a.a. $x \in \Omega$.

In the same way we will have $u^i(x,t) \geq M_i$ (using $(u^i + M_i)^-$).

Then

$$ess \sup_{x\in\Omega} |z^i(x)| \leq M_i \ \ \forall z = (z^1, ..., z^N) \in \Theta_\Sigma.$$

So we obtain that Θ_Σ is bounded set in the space $(L^\infty(\Omega))^N$. From the equality $\Theta_\Sigma = U_\Sigma(t, \tau, \Theta_\Sigma)$ $\forall\, t \geq \tau$ we deduce that $\forall\, \sigma \in \Sigma$ $U_\sigma(t, \tau, \Theta_\Sigma) \subset \Theta_\Sigma$. Now let us consider arbitrary complete trajectory $u(\cdot) \in \mathscr{K}_\sigma$. Due to definition of weak solution for a.a. $t \in \mathbb{R}$ $u(t) \in V$. We take such $\tau \in \mathbb{R}$ that $u(\tau) \in V$ and consider the following Cauchy problem

$$\begin{cases} v_t = a\Delta v - f_\sigma(t, u) + h_\sigma(t, x), \quad x \in \Omega,\ t > \tau, \\ v|_{\partial\Omega} = 0, \\ v|_{t=\tau} = u(\tau). \end{cases} \tag{8.39}$$

Because $\forall\, t \geq \tau$ $u(t) \in \Theta_\Sigma$, which is bounded in $(L^\infty(\Omega))^N$, we have that $f_\sigma(t, u(t,x)) \in (L^\infty(\Omega))^N$. Thus for linear problem (8.39) from well-known results one can deduce that $\forall\, T > \tau$ $v \in C([\tau, T]; V)$. So from uniqueness of the solution of Cauchy problem (8.39) $v \equiv u$ on $[\tau, +\infty)$ and, therefore, $\forall\, t \geq \tau$ $u(t) \in V$. It means that $\forall\, t \in \mathbb{R}$ $u(t) \in V$ and from the formula (8.33) $\Theta_\Sigma \subset V$.

From the energy equality, applying to function u, and boundness of Θ_Σ in the space H we deduce, that $\exists\, C > 0$, which does not depend on σ, such that $\forall\, t \in \mathbb{R}$

$$\int_t^{t+1} \|u(s)\|^2 ds \leq C(1 + \int_t^{t+1} |h_\sigma(s)|^2 ds).$$

From translation compactness of h we have

$$\int_t^{t+1} \|u(s)\|^2 ds \leq C(1 + |h|_+^2).$$

So for arbitrary $t \in \mathbb{R}$ we find $\tau \in [t, t+1]$ such that $\|u(\tau)\|^2 \leq C(1 + |h|_+^2)$. Then for the problem (8.39) we obtain inequality

$$\forall\, t \geq \tau \ \ \|v(t)\|^2 \leq e^{-\delta(t-\tau)} \|u(\tau)\|^2 + D,$$

where positive constants $\delta,\ D$ do not depend on σ. Thus

$$\forall\, t \in \mathbb{R} \ \ \|u(t)\|^2 \le C(1 + |h|_+^2) + D$$

and theorem is proved.

8.3 Uniform Trajectory Attractors for Nonautonomous Dissipative Dynamical Systems

For evolution triple $(V_i; H; V_i^*)$[1] and multi-valued map $A_i : \mathbb{R}_+ \times V \rightrightarrows V^*$, $i = 1, 2, \ldots, N$, $N = 1, 2, \ldots$ we consider a problem of long-time behavior of all globally defined weak solutions for nonautonomous evolution inclusion

$$y'(t) + \sum_{i=1}^{N} A_i(t, y(t)) \ni \bar{0}, \tag{8.40}$$

as $t \to +\infty$. Let $\langle \cdot, \cdot \rangle_{V_i} : V_i{}^* \times V_i \to \mathbb{R}$ be the pairing in $V_i{}^* \times V_i$, that coincides on $H \times V_i$ with the inner product $(\cdot, \cdot)$ in the Hilbert space H.

Note that Problem (8.40) arises in many important models for distributed parameter control problems and that large class of identification problems enter this formulation.

Throughout this subsection we suppose that the listed below assumptions hold:

Assumption I Let $p_i \ge 2$, $q_i > 1$ are such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$, for each for $i = 1, 2, \ldots, N$, and the embedding $V_i \subset H$ is compact one, for some for $i = 1, 2, \ldots, N$.

Assumption II (Grows Condition) There exist a t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$ function $c_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $c_2 > 0$ such that

$$\max_{i=1}^{N} \|d_i\|_{V_i^*}^{q} \le c_1(t) + c_2 \sum_{i=1}^{N} \|u\|_{V_i}^{p}$$

for any $u \in V_i$, $d_i \in A_i(t, u)$, $i = 1, 2, \ldots, N$, and a.e. $t > 0$.

Assumption III (Signed Assumption) There exist a constant $\alpha > 0$ and a t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$ function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sum_{i=1}^{N} \langle d_i, u \rangle_{V_i} \ge \alpha \sum_{i=1}^{N} \|u\|_{V_i}^{p} - \beta(t)$$

[1] i.e. V_i is a real reflexive separable Banach space continuously and densely embedded into a real Hilbert space H, H is identified with its topologically conjugated space H^*, V_i^* is a dual space to V_i. So, there is a chain of continuous and dense embeddings: $V_i \subset H \equiv H^* \subset V_i^*$.

for any $u \in V_i, d_i \in A_i(t,u), i = 1,2,\dots,N$, and a.e. $t > 0$.

Assumption IV (Strong Measurability) If $C \subseteq V_i{}^*$ is a closed set, then the set $\{(t,u) \in (0,+\infty) \times V_i \;:\; A_i(t,u) \cap C \neq \emptyset\}$ is a Borel subset in $(0,+\infty) \times V_i$.

Assumption V (Pointwise Pseudomonotonicity) Let for each $i = 1,2,\dots,N$ and a.e. $t > 0$ two assumptions hold:

(a) for every $u \in V_i$ the set $A_i(t,u)$ is nonempty, convex, and weakly compact one in $V_i{}^*$;
(b) if a sequence $\{u_n\}_{n\geq 1}$ converges weakly in V_i towards $u \in V_i$ as $n \to +\infty$, $d_n \in A_i(t,u_n)$ for any $n \geq 1$, and $\limsup\limits_{n\to+\infty} \langle d_n, u_n - u\rangle_{V_i} \leq 0$, then for any $\omega \in V_i$ there exists $d(\omega) \in A_i(t,u)$ such that

$$\liminf_{n\to+\infty} \langle d_n, u_n - \omega\rangle_{V_i} \geq \langle d(\omega), u - \omega\rangle_{V_i}.$$

Let $0 \leq \tau < T < +\infty$. As a *weak solution* of evolution inclusion (8.40) on the interval $[\tau, T]$ we consider an element $u(\cdot)$ of the space $\cap_{i=1}^N L_{p_i}(\tau,T;V_i)$ such that for some $d_i(\cdot) \in L_{q_i}(\tau,T;V_i{}^*)$, $i = 1,2,\dots,N$, it is fulfilled:

$$-\int_\tau^T (\xi'(t), y(t))dt + \sum_{i=1}^N \int_\tau^T \langle d_i(t), \xi(t)\rangle_{V_i} dt = 0 \quad \forall \xi \in C_0^\infty([\tau,T];V_i), \tag{8.41}$$

and $d_i(t) \in A_i(t, y(t))$ for each $i = 1,2,\dots,N$ and a.e. $t \in (\tau,T)$.

For fixed nonnegative τ and T, $\tau < T$, let us consider

$$X_{\tau,T} = \cap_{i=1}^N L_{p_i}(\tau,T;V_i), \quad X^*_{\tau,T} = \sum_{i=1}^N L_{q_i}(\tau,T;V_i{}^*), \quad W_{\tau,T} = \{y \in X_{\tau,T} \mid y' \in X^*_{\tau,T}\},$$
$$\mathscr{A}_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}, \quad \mathscr{A}_{\tau,T}(y) = \{d \in X^*_{\tau,T} \mid d(t) \in A(t,y(t)) \text{ for a.e. } t \in (\tau,T)\},$$

where y' is a derivative of an element $u \in X_{\tau,T}$ in the sense of $\mathscr{D}([\tau,T]; \sum_{i=1}^N V_i{}^*)$ (see, for example, Sect. 6.1). Note that the space $W_{\tau,T}$ is a reflexive Banach space with the graph norm of a derivative $\|u\|_{W_{\tau,T}} = \|u\|_{X_{\tau,T}} + \|u'\|_{X^*_{\tau,T}}$, $u \in W_{\tau,T}$. Let $\langle\cdot,\cdot\rangle_{X_{\tau,T}} : X^*_{\tau,T} \times X_{\tau,T} \to \mathbb{R}$ be the pairing in $X^*_{\tau,T} \times X_{\tau,T}$, that coincides on $L_2(\tau,T;H) \times X_{\tau,T}$ with the inner product in $L_2(\tau,T;H)$, i.e. $\langle u, v\rangle_{X_{\tau,T}} = \int_\tau^T (u(t),v(t))dt$ for any $u \in L_2(\tau,T;H)$ and $v \in X_{\tau,T}$. The embedding $W_{\tau,T} \subset C([\tau,T];H)$ is continuous and dense one. Moreover,

$$(u(T),v(T)) - (u(\tau),v(\tau)) = \int_\tau^T \Big[\langle u'(t), v(t)\rangle_{V_i} + \langle v'(t), u(t)\rangle_{V_i}\Big]dt, \tag{8.42}$$

for any $u, v \in W_{\tau,T}$.

For fixed τ and T, such that $0 \le \tau < T < +\infty$, we denote

$$\mathscr{D}_{\tau,T}(y^{(\tau)}) = \{y(\cdot) \mid y \text{ is a weak solution of (8.40) on } [\tau, T],\ y(\tau) = y^{(\tau)}\}, \quad y^{(\tau)} \in H.$$

We remark that $\mathscr{D}_{\tau,T}(y^{(\tau)}) \ne \emptyset$ and $\mathscr{D}_{\tau,T}(y^{(\tau)}) \subset W_{\tau,T}$, if $0 \le \tau < T < +\infty$ and $y^{(\tau)} \in H$. Moreover, the concatenation of Problem (8.40) weak solutions is a weak solutions too, i.e. if $0 \le \tau < t < T$, $y^{(\tau)} \in H$, $y(\cdot) \in \mathscr{D}_{\tau,t}(y^{(\tau)})$, and $v(\cdot) \in \mathscr{D}_{t,T}(y(t))$, then

$$z(s) = \begin{cases} y(s), & s \in [\tau, t], \\ v(s), & s \in [t, T], \end{cases}$$

belongs to $\mathscr{D}_{\tau,T}(y^{(\tau)})$; see Sect. 6.1.

Gronwall lemma provides that for any finite time interval $[\tau, T] \subset \mathbb{R}_+$ each weak solution y of Problem (8.40) on $[\tau, T]$ satisfies estimates

$$\|y(t)\|_H^2 - 2\int_0^t \beta(\xi)d\xi + 2\alpha \sum_{i=1}^{N} \int_s^t \|y(\xi)\|_{V_i}^p d\xi \le \|y(s)\|_H^2 - 2\int_0^s \beta(\xi)d\xi, \tag{8.43}$$

$$\|y(t)\|_H^2 \le \|y(s)\|_H^2 e^{-2\alpha\gamma(t-s)} + 2\int_s^t (\beta(\xi) + \alpha\gamma)e^{-2\alpha\gamma(t-\xi)}d\xi, \tag{8.44}$$

where $t, s \in [\tau, T]$, $t \ge s$; γ is a constant that does not depend on y, s, and t; cf. Sect. 6.1. In the proof of (8.44) we used the inequality $\|u\|_H^2 - 1 \le \|u\|_H^p$ for any $u \in H$.

Therefore, any weak solution y of Problem (8.40) on a finite time interval $[\tau, T] \subset \mathbb{R}_+$ can be extended to a global one, defined on $[\tau, +\infty)$. For arbitrary $\tau \ge 0$ and $y^{(\tau)} \in H$ let $\mathscr{D}_\tau(y^{(\tau)})$ be the set of all weak solutions (defined on $[\tau, +\infty)$) of Problem (8.40) with initial data $y(\tau) = y^{(\tau)}$. Let us consider the family $\mathscr{K}_\tau^+ = \cup_{y^{(\tau)} \in H} \mathscr{D}_\tau(y^{(\tau)})$ of all weak solutions of Problem (8.40) defined on the semi-infinite time interval $[\tau, +\infty)$. Consider the Fréchet space $C^{\text{loc}}(\mathbb{R}_+; H)$. We remark that the sequence $\{f_n\}_{n\ge1}$ converges in $C^{\text{loc}}(\mathbb{R}_+; H)$ towards $f \in C^{\text{loc}}(\mathbb{R}_+; H)$ as $n \to +\infty$ iff the sequence $\{\Pi_{t_1,t_2} f_n\}_{n\ge1}$ converges in $C([t_1, t_2]; H)$ towards $\Pi_{t_1,t_2} f$ as $n \to +\infty$ for any finite interval $[t_1, t_2] \subset \mathbb{R}_+$, where Π_{t_1,t_2} is the restriction operator to the interval $[t_1, t_2]$; Chepyzhov and Vishik [6, p. 918]. We denote $T(h)y(\cdot) = y_h(\cdot)$, where $y_h(t) = y(t + h)$ for any $y \in C^{\text{loc}}(\mathbb{R}_+; H)$ and $t, h \ge 0$.

In the autonomous case, when $A(t, y)$ does not depend on t, the long-time behavior of all globally defined weak solutions for Problem (8.40) is described by using trajectory and global attractors theory. In this situation the set $\mathscr{K}^+ := \mathscr{K}_0^+$ is *translation invariant*, i.e. $T(h)\mathscr{K}^+ \subseteq \mathscr{K}^+$ for any $h \ge 0$. As trajectory attractor it is considered a classical global attractor for translation semigroup $\{T(h)\}_{h\ge0}$, that acts on $\mathscr{K}^+$.

In the nonautonomous case we notice that $T(h)\mathscr{K}_0^+ \nsubseteq \mathscr{K}_0^+$. Therefore, we need to consider *united trajectory space* that includes all globally defined on any $[\tau, +\infty) \subseteq \mathbb{R}_+$ weak solutions of Problem (8.40) shifted to $\tau = 0$:

$$\mathscr{K}^+ = \mathrm{cl}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}\left[\bigcup_{\tau\geq 0}\{y(\cdot+\tau) : y \in \mathscr{K}_\tau^+\}\right],$$

where $\mathrm{cl}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}[\,\cdot\,]$ is the closure in $C^{\mathrm{loc}}(\mathbb{R}_+; H)$. Note that $T(h)\{y(\cdot+\tau) : y \in \mathscr{K}_\tau^+\} \subseteq \{y(\cdot+\tau+h) : y \in \mathscr{K}_{\tau+h}^+\}$ for any $\tau, h \geq 0$. Moreover,

$$T(h)\mathscr{K}^+ \subseteq \mathscr{K}^+ \text{ for any } h \geq 0,$$

because

$$\rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}(T(h)u, T(h)v) \leq \rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}(u, v) \text{ for any } u, v \in C^{\mathrm{loc}}(\mathbb{R}_+; H),$$

where $\rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ is a standard metric on Fréchet space $C^{\mathrm{loc}}(\mathbb{R}_+; H)$.

A set $\mathscr{P} \subset C^{\mathrm{loc}}(\mathbb{R}_+; H) \cap L_\infty(\mathbb{R}_+; H)$ is said to be a *uniformly attracting set* (cf. Chepyzhov and Vishik [6, p. 921]) for the united trajectory space $\mathscr{K}^+$ of Problem (8.40) in the topology of $C^{\mathrm{loc}}(\mathbb{R}_+; H)$, if for any bounded in $L_\infty(\mathbb{R}_+; H)$ set $\mathscr{B} \subseteq \mathscr{K}^+$ and any segment $[t_1, t_2] \subset \mathbb{R}_+$ the following relation holds:

$$\mathrm{dist}_{C([t_1,t_2];H)}(\Pi_{t_1,t_2}T(t)\mathscr{B}, \Pi_{t_1,t_2}\mathscr{P}) \to 0, \quad t \to +\infty, \tag{8.45}$$

where $\mathrm{dist}_{C([t_1,t_2];H)}$ is the Hausdorff semi-metric.

A set $\mathscr{U} \subset \mathscr{K}^+$ is said to be a *uniform trajectory attractor* (cf. Chepyzhov and Vishik [6, p. 921]) of the translation semigroup $\{T(t)\}_{t\geq 0}$ on $\mathscr{K}^+$ in the induced topology from $C^{\mathrm{loc}}(\mathbb{R}_+; H)$, if

(i) $\mathscr{U}$ is a compact set in $C^{\mathrm{loc}}(\mathbb{R}_+; H)$ and bounded in $L_\infty(\mathbb{R}_+; H)$;
(ii) $\mathscr{U}$ is strictly invariant with respect to $\{T(h)\}_{h\geq 0}$, i.e. $T(h)\mathscr{U} = \mathscr{U}$ $\forall h \geq 0$;
(iii) $\mathscr{U}$ is a minimal uniformly attracting set for $\mathscr{K}^+$ in the topology of $C^{\mathrm{loc}}(\mathbb{R}_+; H)$, i.e. $\mathscr{U}$ belongs to any compact uniformly attracting set $\mathscr{P}$ of $\mathscr{K}^+$: $\mathscr{U} \subseteq \mathscr{P}$.

Note that uniform trajectory attractor of the translation semigroup $\{T(t)\}_{t\geq 0}$ on $\mathscr{K}^+$ in the induced topology from $C^{\mathrm{loc}}(\mathbb{R}_+; H)$ coincides with the classical trajectory attractor for the continuous semi-group $\{T(t)\}_{t\geq 0}$ defined on $\mathscr{K}^+$ (see, for example, Chepyzhov and Vishik [6, Definition 1.1]).

Presented construction is coordinated with the theory of uniform trajectory attractors for nonautonomous problems of the form

$$\partial_t u(t) = A_{\sigma(t)}(u(t)), \tag{8.46}$$

where $\sigma(s)$, $s \geq 0$, is a functional parameter called the time symbol of Eq. (8.46) (t is replaced by s). In applications to mathematical physics equations, a function $\sigma(s)$ consists of all time-dependent terms of the equation under consideration: external forces, parameters of mediums, interaction functions, control functions, etc. In mentioned above papers and books it is assumed that the symbol σ of Eq. (8.46) belongs to a Hausdorff topological space Ξ_+ of functions defined on $\mathbb{R}_+$ with values in some complete metric space. Usually, in applications, the topology in the space Ξ_+ is a local convergence topology on any segment $[t_1, t_2] \subset \mathbb{R}_+$. Further, they consider the family of Eq. (8.46) with various symbols $\sigma(s)$ belonging to a set $\Sigma \subseteq \Xi_+$. The set Σ is called the symbol space of the family of Eq. (8.46). It is assumed that the set Σ, together with any symbol $\sigma(s) \in \Sigma$, contains all positive translations of $\sigma(s)$: $\sigma(t+s) = T(t)\sigma(s) \in \Sigma$ for any $t, s \geq 0$. The symbol space Σ is invariant with respect to the translation semigroup $\{T(t)\}_{t\geq 0}$: $T(t)\Sigma \subseteq \Sigma$ for any $t \geq 0$. To prove the existence of uniform trajectory attractor they suppose that the symbol space Σ with the topology induced from Ξ_+ is a compact metric space. Mostly in applications, as a symbol space Σ it is naturally to consider the hull of translation-compact function $\sigma_0(s)$ in an appropriate Hausdorff topological space Ξ_+. The direct realization of this approach for Problem (8.40) is problematic without any additional assumptions for parameters of Problem (8.40) and requires the translation-compactness of the symbol $\sigma(s) = A(s, \cdot)$ in some compact Hausdorff topological space of measurable multi-valued mappings acts from $\mathbb{R}_+$ to some metric space of pseudomonotone operators from ($V_i \to 2^{V_i{}^*}$) satisfying grows and signed assumptions. To avoid this technical difficulties we present the alternative approach for the existence and construction of the uniform trajectory attractor for all weak solutions for Problem (8.40). Note that Assumptions (I)–(V) are natural and guaranty, in the general case, only existence of weak solution for Cauchy problem on any finite time interval $[\tau, T] \subset \mathbb{R}_+$ and for any initial data form H.

The main result of this section has the following form.

Theorem 8.8 *Let Assumptions* (I)–(V) *hold. Then there exists an uniform trajectory attractor $\mathscr{U} \subset \mathscr{K}^+$ of the translation semigroup $\{T(t)\}_{t\geq 0}$ on $\mathscr{K}^+$ in the induced topology from $C^{\text{loc}}(\mathbb{R}_+; H)$. Moreover, there exists a compact in $C^{\text{loc}}(\mathbb{R}_+; H)$ uniformly attracting set $\mathscr{P} \subset C^{\text{loc}}(\mathbb{R}_+; H) \cap L_\infty(\mathbb{R}_+; H)$ for the united trajectory space $\mathscr{K}^+$ of Problem (8.40) in the topology of $C^{\text{loc}}(\mathbb{R}_+; H)$ such that $\mathscr{U}$ coincides with ω-limit set of $\mathscr{P}$:*

$$\mathscr{U} = \bigcap_{t\geq 0} \text{cl}_{C^{\text{loc}}(\mathbb{R}_+;H)} \left[\bigcup_{h\geq t} T(h)\mathscr{P} \right]. \tag{8.47}$$

Before the proof of Theorem 8.8 we provide some auxiliary constructions (see Sect. 6.1).

Assumptions (II) and (III) yield that there exist a positive constant $\alpha' > 0$ and a t.u.i. function c' in $L_1^{\text{loc}}(\mathbb{R}_+)$ such that $A(t, u) \subseteq \mathscr{A}_{c'(t)}(u)$ for each $u \in \cap_{i=1}^N V_i$ and a.e. $t > 0$, where

$$\mathscr{A}_{c'(t)}(u) := \left\{ \sum_{i=1}^N p_i \ : \ p_i \in V_i{}^*, \ \sum_{i=1}^N \langle p_i, u\rangle_{V_i} \geq \alpha' \max_{i=1}^N \left\{ \|u\|_{V_i}^p; \|p\|_{V_i{}^*}^q \right\} - c'(t) \right\}.$$

Let $\mathscr{H}(c')$ be the hull of t.u.i. function c' in $L_{1,w}^{\text{loc}}(\mathbb{R}_+)$, i.e. $\mathscr{H}(c') = \text{cl}_{L_1^{\text{loc}}(\mathbb{R}_+)}\{c'(\cdot + h) \ : \ h \geq 0\}$. This is a weakly compact set in $L_1^{\text{loc}}(\mathbb{R}_+)$.

Let us consider the family of problems

$$y' = \mathscr{A}_\sigma(y), \quad \sigma \in \Sigma := \mathscr{H}(c'). \tag{8.48}$$

To each $\sigma \in \Sigma$ there corresponds a space of all globally defined on $[0, +\infty)$ weak solutions $\mathscr{K}_\sigma^+ \subset C^{\mathrm{loc}}(\mathbb{R}_+; H)$ of Problem (8.48). We set $\mathscr{K}_\Sigma^+ = \cup_{\sigma\in\Sigma}\mathscr{K}_\sigma^+$.

We remark that (see Sect. 6.1) any element from $\mathscr{K}_\Sigma^+$ satisfies prior estimates.

Lemma 8.5 *There exist positive constants c_3 and c_4 such that for any $\sigma \in \Sigma$ and $y \in \mathscr{K}_\sigma^+$ the inequalities hold:*

$$\|y(t)\|_H^2 - 2\int_0^t \sigma(\xi)d\xi + 2\alpha' \sum_{i=1}^N \int_s^t \|y(\xi)\|_{V_i}^p d\xi \le \|y(s)\|_H^2 - 2\int_0^s \sigma(\xi)d\xi, \tag{8.49}$$

$$\|y(t)\|_H^2 \le \|y(s)\|_H^2 e^{-c_3(t-s)} + c_4 \int_s^t \sigma(\xi) e^{-c_3(t-\xi)} d\xi, \tag{8.50}$$

for any $t \ge s \ge 0$.

Moreover, the following result characterizing the compactness properties of solutions for the family of Problems (8.48) holds:

Theorem 8.9 *Let $\{y_n\}_{n\ge1} \subset \mathscr{K}_\Sigma^+$ be an arbitrary sequence, that is bounded in $L_\infty(\mathbb{R}_+; H)$. Then there exist a subsequence $\{y_{n_k}\}_{k\ge1} \subset \{y_n\}_{n\ge1}$ and an element $y \in \mathscr{K}_\Sigma^+$ such that*

$$\max_{t\in[\tau,T]} \|y_{n_k}(t) - y(t)\|_H \to 0, \quad k \to +\infty, \tag{8.51}$$

for any finite time interval $[\tau, T] \subset (0, +\infty)$.

Proof of Theorem 8.8 First, let us show that there exists a uniform trajectory attractor $\mathscr{U} \subset \mathscr{K}^+$ of the translation semigroup $\{T(t)\}_{t\ge0}$ on $\mathscr{K}^+$ in the induced topology from $C^{\mathrm{loc}}(\mathbb{R}_+; H)$. Lemma 8.5 and Theorem 8.9 yields that the translation semigroup $\{T(t)\}_{t\ge0}$ has a compact absorbing (and, therefore, an uniformly attracting) set in the space of trajectories $\mathscr{K}_\Sigma^+$. This set can be constructed as follows: 1) consider $\mathscr{P}$, the intersection of $\mathscr{K}_\Sigma^+$ with a ball in the space of bounded continuous functions on $\mathbb{R}_+$ with values in H, $C_b(\mathbb{R}_+; H)$, of sufficiently large radius; 2) shift the resulting set by any fixed distance $h > 0$. Thus, we obtain $T(h)\mathscr{P}$, a set with the required properties. Recall that the semigroup $\{T(t)\}_{t\ge0}$ is continuous. Therefore, the set $\mathscr{P}_1 := \mathscr{P} \cap \mathscr{K}^+$ is a compact absorbing (and, therefore, an uniformly attracting) in the space $\mathscr{K}^+$ with the induced topology of $C^{\mathrm{loc}}(\mathbb{R}_+; H)$. In fact, here one can apply the classical theorem on the global attractor of a (unique) continuous semigroup in a complete metric space, the semigroup in question having a compact attracting.

8.4 Notes on Applications

As applications we may consider FitzHugh–Nagumo system (signal transmission across axons), complex Ginzburg–Landau equation (theory of superconductivity), Lotka–Volterra system with diffusion (ecology models), Belousov–Zhabotinsky system (chemical dynamics) and many other nonlinear systems (see Sects. 2.4 and 4.6).

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Chapter 9
Indirect Lyapunov Method for Autonomous Dynamical Systems

Abstract In this chapter we establish indirect Lyapunov method for autonomous dynamical systems. Section 9.1 devoted to the first order autonomous differential-operator equations and inclusions. In Sect. 9.2 we consider the second order autonomous operator differential equations and inclusions. In Sect. 9.3 we examine examples of applications. In particular, a model of combustion in porous media; a model of conduction of electrical impulses in nerve axons; viscoelastic problems with nonlinear "reaction-displacement" law etc.

9.1 First Order Autonomous Differential-Operator Equations and Inclusions

Let $(\mathscr{M}, \mathbf{g})$ be a C^∞ compact connected oriented two-dimensional Riemannian manifold without boundary (as, e.g. $\mathscr{M} = S^2$ the unit sphere of $\mathbb{R}^3$). Consider the Budyko model:

$$\tfrac{\partial u}{\partial t} - \Delta u + Bu \in QS(x)\beta(u), \quad (x, t) \in \mathscr{M} \times \mathbb{R}, \tag{9.1}$$

where $\Delta u = \operatorname{div}_{\mathscr{M}} (\nabla_{\mathscr{M}} u)$; $\nabla_{\mathscr{M}}$ is understood in the sense of the Riemannian metric $\mathbf{g}$ (see Sect. 2.4.3).

Let $S : \mathscr{M} \to \mathbb{R}$ be a function such that $S \in L^\infty(\mathscr{M})$ and there exist $S_0, S_1 > 0$ such that

$$0 < S_0 \le S(x) \le S_1.$$

Suppose also that β is a bounded maximal monotone graph of $\mathbb{R}^2$, that is there exist $m, M \in \mathbb{R}$ such that for all $s \in \mathbb{R}$ and $z \in \beta(s)$

$$m \le z \le M.$$

Let us consider real Hilbert spaces

$$H := L^2(\mathscr{M}), \quad V := \{u \in L^2(\mathscr{M}) : \nabla_{\mathscr{M}} u \in L^2(T\mathscr{M})\}$$

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with respective standard norms $\|\cdot\|_H$, $\|\cdot\|_V$, and inner products $(\cdot,\cdot)_H$, $(\cdot,\cdot)_V$, where $T\mathscr{M}$ represents the tangent bundle and the functional spaces $L^2(\mathscr{M})$ and $L^2(T\mathscr{M})$ are defined in a standard way. According to Theorem 2.2, for any $-\infty < \tau < T < +\infty$ each weak solution $u_\tau \in L^2(\Omega)$ of Problem (9.1) on $[\tau, T]$ belongs to $C([\tau+\varepsilon, T]; H_0^1(\Omega)) \cap L^2(\tau+\varepsilon, T; H^2(\Omega)\cap H_0^1((0,\pi)))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau+\varepsilon, T; L^2(\Omega))$ for each $\varepsilon \in (0, T-\tau)$.

Consider the generalized setting of Problem (9.1):

$$\frac{du}{dt} + Au(t) + \partial J_1(u(t)) - \partial J_2(u(t)) \ni \bar{0} \text{ on } (-\infty < \tau < T < +\infty), \tag{9.2}$$

where $A : V \to V'$ be a linear symmetric operator such that $\exists c > 0 : \langle Av, v\rangle_V \geq c\|v\|_V^2$, for each $v \in V$ and $J_i : H \to \mathbb{R}$ be a convex, lower semi-continuous function such, that the following assumptions hold: (i) (growth condition) there exists $c_1 > 0$ such that $\|y\|_H \leq c_1(1 + \|u\|_H)$, for each $u \in H$ and $y \in \partial J_i(u)$ and $i = 1, 2$; (ii) (sign condition) there exist $c_2 > 0$, $\lambda \in (0, c)$ such that $(y_1 - y_2, u)_H \geq -\lambda\|u\|_H^2 - c_2$, for each $y_i \in \partial J_i(u)$, $u \in H$, where $\partial J_i(u)$ the subdifferential of $J_i(\cdot)$ at a point u. Note that $u^* \in \partial J_i(u)$ if and only if $u^*(v-u) \leq J_i(v) - J_i(u)$ $\forall v \in H$; $i = 1, 2$. Let $D(A) = \{u \in V : Au \in H\}$. We note that the mapping $v \to \|Av\|_H$ defines the equivalent norm on $D(A)$.

We recall that the function $u(\cdot) \in L^2(\tau, T; V)$ is called a *weak solution* of Problem (9.2) on $[\tau, T]$, if there exist Bochner measurable functions $d_i : (\tau, T) \to H$; $i = 1, 2$, such that

$$d_i(t) \in \partial J_i(u(t)) \text{ for a.e. } t \in (\tau, T),\ i = 1, 2; \text{ and} \tag{9.3}$$

$$\int_\tau^T \left[-\langle u, v\rangle\, \xi'(t) + \langle Au, v\rangle\, \xi(t) + \langle d_1, v\rangle\, \xi(t) - \langle d_2, v\rangle\, \xi(t)\right] dt = 0, \tag{9.4}$$

for all $\xi \in C_0^\infty(\tau, T)$ and for all $v \in V$.

We note that for any $u_\tau \in H$ there exists at least one weak solution of Problem (9.2) on $[\tau, T]$ with initial condition $u(\tau) = u_\tau$. The regularity of each weak solution follows from Theorem 2.3.

Denote by $\mathscr{K}_+$ the family of all, globally defined on $[0, +\infty)$, weak solutions of Problem (9.2). Let us set

$$E(u) = \frac{1}{2}\langle Au, u\rangle + J_1(u) - J_2(u), \quad u \in V. \tag{9.5}$$

For each $u \in \mathscr{K}_+$ and all τ and T, $0 < \tau < T < \infty$, the energy equality holds

$$E(u(T)) - E(u(\tau)) = -\int_\tau^T \|\frac{du}{ds}(s)\|_H^2 ds. \tag{9.6}$$

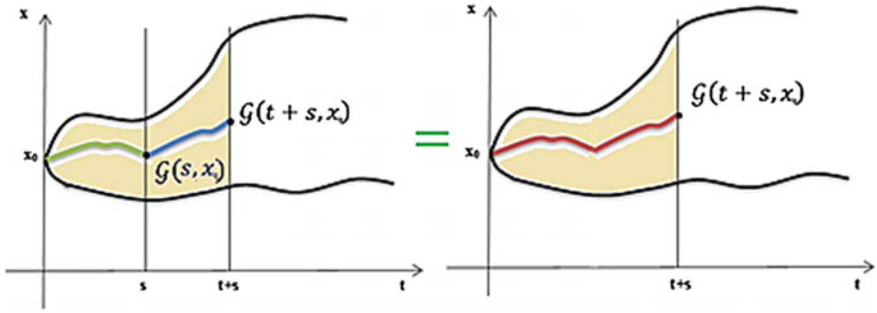


Fig. 9.1 Multi-valued semiflow

Thus, the function $E(u(\cdot))$ is absolutely continuous on $[\tau, T]$ as the linear combination of absolutely continuous on $[\tau, T]$ functions.

Let

$$\mathscr{D}_{\tau,T}(u_\tau) = \{u(\cdot) \in L^2(\tau, T; V) \big| \ u(\cdot) \text{ is a weak solution of Problem (9.2) and } u(\tau) = u_\tau\},$$

for any $u_\tau \in H$.

Define real Banach space

$$W(M_1, M_2) = \{u(\cdot) \in C([M_1, M_2]; V) \ : \ \frac{du}{dt}(\cdot) \in L^2(M_1, M_2; H)\}$$

with the norm $\|u\|_{W(M_1,M_2)} = \|u\|_{C([M_1,M_2];V)} + \|\frac{du}{dt}\|_{L^2(M_1,M_2;H)}$, $u \in W(M_1, M_2)$, $-\infty < M_1 < M_2 < +\infty$.

We denote the set of all nonempty (nonempty bounded) subsets of H by $P(H)$ ($\mathscr{B}(H)$). Let us define the strict m-semiflow $G : \mathbb{R}_+ \times H \to P(H)$ in the following way: $G(t, u_0) = \{u(t) : u(\cdot) \in \mathscr{K}_+, \ u(0) = u_0\}$. We recall that the multi-valued map $G : \mathbb{R}_+ \times H \to P(H)$ is said to be a *strict multi-valued semiflow (strict m-semiflow)* if (see also Fig. 9.1):

(a) $G(0, \cdot) = \text{Id}$ (the identity map);

(b) $G(t + s, x) = G(t, G(s, x)) \ \forall x \in H, t, s \in \mathbb{R}_+$.

We recall that the set $\mathscr{A} \subseteq H$ is said to be *an invariant global attractor* of G if:

(1) $\mathscr{A}$ is invariant (that is $\mathscr{A} = G(t, \mathscr{A}) \ \forall t \geq 0$);

(2) $\mathscr{A}$ is attracting set, that is,

$$\text{dist}_H(G(t, B), \mathscr{A}) \to 0, \quad t \to +\infty \quad \forall B \in \mathscr{B}(H), \tag{9.7}$$

where $\text{dist}_H(C, D) = \sup_{c \in C} \inf_{d \in D} \|c - d\|_H$ is the Hausdorff semidistance;

(3) for any closed set $Y \subseteq H$ satisfying (9.7), we have $\mathscr{A} \subseteq Y$ (minimality).

Let $\{T(h)\}_{h \geq 0}$ be the translation semigroup acting on $\mathscr{K}_+$, that is $T(h)u(\cdot) = u(\cdot + h)$, $h \geq 0$, $u(\cdot) \in \mathscr{K}_+$. On $\mathscr{K}_+$ we consider the topology induced from the

Fréchet space $C^{loc}(\mathbb{R}_+; H)$. Note that $f_n(\cdot) \to f(\cdot)$ in $C^{loc}(\mathbb{R}_+; H)$ if and only if $\forall M > 0\, \Pi_M f_n(\cdot) \to \Pi_M f(\cdot)$ in $C([0, M]; H)$, where Π_M is the restriction operator to the interval $[0, M]$.

A set $\mathscr{U} \subset \mathscr{K}_+$ is said to be *trajectory attractor* in the trajectory space $\mathscr{K}_+$ with respect to the topology of $C^{loc}(\mathbb{R}_+; H)$, if $\mathscr{U} \subset \mathscr{K}_+$ is a global attractor for the translation semigroup $\{T(h)\}_{h\geq 0}$ acting on $\mathscr{K}_+$.

The following theorem completely describes the long-time behavior of all weak solutions, as time $t \to +\infty$, for the problem in hands. The structure properties of global and trajectory attractors and the strongest convergence results of solutions are provided.

Theorem 9.1 *The following statements hold:*

(i) *the strict m-semiflow $G : \mathbb{R}_+ \times H \to P(H)$ has the invariant global attractor $\mathscr{A}$;*

(ii) *there exists the trajectory attractor $\mathscr{U} \subset \mathscr{K}_+$ in the space $\mathscr{K}_+$;*

(iii) *the following equalities hold: $\mathscr{U} = \Pi_+ \mathscr{K} = \{y \in \mathscr{K}_+ \mid y(t) \in \mathscr{A} \ \forall t \in \mathbb{R}_+\}$;*

(iv) *$\mathscr{A}$ is a compact subset of V;*

(v) *for each $B \in \mathscr{B}(H)$ $\mathrm{dist}_V(G(t, B), \mathscr{A}) \to 0$, as $t \to \infty$;*

(vi) *$\mathscr{U}$ is a bounded subset of $L^\infty(\mathbb{R}_+; V)$ and compact subset of $W^{\mathrm{loc}}(\mathbb{R}_+)$, that is $\Pi_M \mathscr{U}$ is compact in $W(0, M)$ for each $M > 0$;*

(vii) *for any bounded in $L^\infty(\mathbb{R}_+; H)$ set $\mathbf{B} \subset \mathscr{K}_+$ and any $M \geq 0$ the following relation holds: $\mathrm{dist}_{W(0,M)}(\Pi_M T(t)\mathbf{B}, \Pi_M \mathscr{U}) \to 0, \quad t \to +\infty$;*

(viii) *$\mathscr{K}$ is a bounded subset of $L^\infty(\mathbb{R}; V)$ and compact subset of $W^{\mathrm{loc}}(\mathbb{R})$, that is $\Pi_{M_1,M_2} \mathscr{U}$ is compact in $W(M_1, M_2)$ for each M_1, M_2, $-\infty < M_1 < M_2 < +\infty$;*

(ix) *for each $u \in \mathscr{K}$ the limit sets*

$$\alpha(u) = \{z \in V \mid u(t_j) \to z \text{ in } V \text{ for some sequence } t_j \to -\infty\},$$

$$\omega(u) = \{z \in V \mid u(t_j) \to z \text{ in } V \text{ for some sequence } t_j \to +\infty\}$$

are connected subsets of Z on which E is constant. If Z is totally disconnected (in particular, if Z is countable) the limits in V

$$z_- = \lim_{t\to-\infty} u(t), \quad z_+ = \lim_{t\to+\infty} u(t) \tag{9.8}$$

exist and z_-, z_+ are rest points; furthermore, $u(t)$ tends in V to a rest point as $t \to +\infty$ for every $u \in \mathscr{K}_+$.

Proof Statements (i)–(v) of Theorem 9.1 follow from Kasyanov et al. [2, Theorems 4–6]. Statements (vi)–(viii) of Theorem 9.1 follow from Theorem 5.5 and Kasyanov et al. [2, Theorem 6]. Statement (ix) of Theorem 9.1 follows from Theorem 5.4 and Ball [1, Theorem 2.7].

9.2 Second Order Autonomous Operator Differential Equations and Inclusions

Let $\beta > 0$ be a constant, $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Consider the problem

$$\begin{cases} u_{tt} + \beta u_t - \triangle u + f(u) = 0, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{9.9}$$

where $u(x,t)$ is unknown state function defined on $\Omega \times \mathbb{R}_+$; $f : \mathbb{R} \to \mathbb{R}$ is an interaction function such that

$$\lim_{|u|\to\infty} \frac{f(u)}{u} > -\lambda_1, \tag{9.10}$$

where λ_1 is the first eigenvalue for $-\triangle$ in $H_0^1(\Omega)$;

$$\exists\ D \geq 0 : \quad |f(u)| \leq D(1+|u|), \quad \forall u \in \mathbb{R}. \tag{9.11}$$

Further, we use such denotation

$$\overline{f}(s) := \limsup_{t\to s} f(t), \quad \underline{f}(s) := \underline{\lim}_{t\to s} f(t), \quad G(s) := [\underline{f}(s), \overline{f}(s)], \quad s \in \mathbb{R}.$$

Let us set $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. The space $X = V \times H$ is a phase space of Problem (9.9). For the Hilbert space X as $(\cdot,\cdot)_X$ and $\|\cdot\|_X$ denote the inner product and the norm in X respectively.

Definition 9.1 Let $T > 0, \tau < T$. The function $\varphi(\cdot) = (u(\cdot), u_t(\cdot))^T \in L^\infty(\tau, T; X)$ is called *a weak solution* of Problem (9.9) on (τ, T) if for a.e. $(x,t) \in \Omega \times (\tau, T)$, there exists $l = l(x,t) \in L^2(\tau, T; L^2(\Omega))$ $l(x,t) \in G(u(x,t))$, such that $\forall \psi \in H_0^1(\Omega), \forall \eta \in C_0^\infty(\tau, T)$,

$$-\int_\tau^T (u_t, \psi)_H \eta_t dt + \int_\tau^T (\beta(u_t,\psi)_H + (u,\psi)_V + (l,\psi)_H)\eta dt = 0. \tag{9.12}$$

The main goal of the manuscript is to obtain the existence of the global attractor generated by the weak solutions of Problem (9.9) in the phase space X.

Thus we consider more general evolution inclusion

$$\begin{cases} u_{tt} + \beta u_t - \triangle u + [\underline{f_1}(u), \overline{f_1}(u)] - [\underline{f_2}(u), \overline{f_2}(u)] \ni 0, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{9.13}$$

Let us set

$$G_i(s) := \int_0^s f_i(\xi)d\xi, \quad J_i(u) := \int_\Omega G_i(u(x))dx, \quad J(u) = J_1(u) - J_2(u), \quad u \in H,\ i = 1, 2.$$

The functionals G_i and J_i are locally Lipschitz and regular. Consider $W_\tau^T = C([\tau, T]; X)$. For any $\varphi_\tau = (a, b)^T \in X$, denote

$$\mathscr{D}_{\tau,T}(\varphi_\tau) = \left\{ (u(\cdot), u_t(\cdot))^T \ \middle|\ \begin{array}{l} (u, u_t)^T \text{is a weak solution of Problem (9.9) on } [\tau, T], \\ u(\tau) = a, u_t(\tau) = b \end{array} \right\};$$

see Sect. 7.2.

Define the m-semiflow $\mathscr{G}$ as

$$\mathscr{G}(t, \xi_0) = \{\xi(t) \mid \xi(\cdot) \in \mathscr{D}(\xi_0)\}, \quad t \geq 0.$$

Denote the set of all nonempty (nonempty bounded) subsets of X by $P(X)(\beta(X))$. Note that the multi-valued map $\mathscr{G} : \mathbb{R}_+ \times X \to P(X)$ is *a strict m-semiflow*, i.e., (see Lemma 6.9)
1. $\mathscr{G}(t, \cdot) = \mathrm{Id}$ (the identity map);
2. $\mathscr{G}(t + s, x) = \mathscr{G}(t, \mathscr{G}(s, x))\ \forall x \in X, t, s \in \mathbb{R}_+$.
Further, $\varphi \in \mathscr{G}$ means that $\varphi \in \mathscr{D}(\xi_0)$ for some $\xi_0 \in X$.

Definition 9.2 $\mathscr{G}$ is called *an asymptotically compact* m-semiflow if for any sequence $\{\varphi_n\}_{n\geq 1} \subset \mathscr{G}$ with $\{\varphi_n(0)\}_{n\geq 1}$ bounded, and for any sequence $\{t_n\}_{n\geq 1}$: $t_n \to +\infty$, $n \to \infty$, the sequence $\{\varphi_n(t_n)\}_{n\geq 1}$ has a convergent subsequence Ball [1, p. 35].

Theorem 9.2 *$\mathscr{G}$ is an asymptotically compact m-semiflow.*

Proof Let $\xi_n \in \mathscr{G}(t_n, v_n)$, $v_n \in B$, $B \in \beta(X)$, $n \geq 1$, $t_n \to +\infty$, $n \to +\infty$. Let us check a precompactness of $\{\xi_n\}_{n\geq 1}$ in X. Without loss of the generality, we extract a convergent in X subsequence from $\{\xi_n\}_{n\geq 1}$. From Corollary 6.1 we obtain that there exists $\{\xi_{n_k}\}_{k\geq 1}$ and $\xi \in X$ such that $\xi_{n_k} \to \xi$ weakly in X, $\|\xi_{n_k}\|_X \to a \geq \|\xi\|_X$, $k \to +\infty$. Show that $a \leq \|\xi\|_X$.

Let us fix an arbitrary $T_0 > 0$. Then for rather big $k \geq 1$, $\mathscr{G}(t_{n_k}, v_{n_k}) \subset \mathscr{G}(T_0, \mathscr{G}(t_{n_k} - T_0, v_{n_k}))$. Hence $\xi_{n_k} \in \mathscr{G}(T_0, \beta_{n_k})$, where $\beta_{n_k} \in \mathscr{G}(t_{n_k} - T_0, v_{n_k})$ and $\sup_{k\geq 1} \|\beta_{n_k}\|_X < +\infty$ (see Corollary 6.1). From Theorem 6.3 for some $\{\xi_{k_j}, \beta_{k_j}\}_{j\geq 1} \subset \{\xi_{n_k}, \beta_{n_k}\}_{k\geq 1}$, $\beta_{T_0} \in X$, we obtain

$$\xi \in \mathscr{G}(T_0, \beta_{T_0}), \quad \beta_{k_j} \to \beta_{T_0} \text{ weakly in } X, \quad j \to +\infty. \tag{9.14}$$

From the definition of $\mathscr{G}$ we set $\forall j \geq 1$, $\xi_{k_j} = (u_j(T_0), u_j'(T_0))^T$, $\beta_{k_j} = (u_j(0), u_j'(0))^T$, $\xi = (u_0(T_0), u_0'(T_0))^T$, $\beta_{T_0} = (u_0(0), u_0'(0))^T$, where $\varphi_j = (u_j, u_j')^T \in C([0, T_0]; X)$, $u_j'' \in L_2(0, T_0; V^*)$, $d_j \in L_\infty(0, T_0; H)$,

$$u_j''(t) + \beta u_j'(t) - \triangle u_j(t) + d_{j,1}(t) - d_{j,2}(t) = \bar{0},$$

$$d_{j,i}(t) \in \partial J_i(u_j(t)), \ \ i = 1, 2 \quad \text{for a.e.} \quad t \in (0, T_0).$$

Let for every $t \in [0, T_0]$,

$$I(\varphi_j(t)) := \frac{1}{2}\|\varphi_j(t)\|_X^2 + J_1(u_j(t)) - J_2(u_j(t)) + \frac{\beta}{2}(u_j'(t), u_j(t))_H.$$

Then in virtue of Lemma 6.7

$$\frac{dI(\varphi_j(t))}{dt} = -\beta I(\varphi_j(t)) + \beta \mathscr{H}(\varphi_j(t)), \text{ for a.e. } t \in (0, T_0),$$

where

$$\mathscr{H}(\varphi_j(t)) = J_1(u_j(t)) - \frac{1}{2}(d_{j,1}(t), u_j(t)) - J_2(u_j(t)) + \frac{1}{2}(d_{j,2}(t), u_j(t))_H.$$

From (9.14) we have $\exists \bar{R} > 0 : \forall j \geq 0, \forall t \in [0, T_0]$,

$$\|u_j'(t)\|_{\mathrm{H}}^2 + \|u_j(t)\|_{\mathrm{V}}^2 \leq \bar{R}^2.$$

Moreover,

$$\begin{gathered} u_j \to u_0 \text{ weakly in } L_2(0, T_0; V), \ \ j \to +\infty, \\ u_j' \to u_0' \text{ weakly in } L_2(0, T_0; H), \ \ j \to +\infty, \\ u_j \to u_0 \text{ in } L_2(0, T_0; H), \ \ j \to +\infty, \\ d_{j,i} \to d_i \text{ weakly in } L_2(0, T_0; H), \ \ i = 1, 2, \ \ j \to +\infty, \\ u_j'' \to u_0'' \text{ weakly in } L_2(0, T_0; V^*), \ \ j \to +\infty, \\ \forall t \in [0, T_0] \ \ u_j(t) \to u_0(t) \text{ in } H, \ \ j \to +\infty. \end{gathered} \tag{9.15}$$

For every $j \geq 0$ and $t \in [0, T_0]$,

$$I(\varphi_j(t)) = I(\varphi_j(0))e^{-\beta t} + \int_0^t \mathscr{H}(\varphi_j(s))e^{-\beta(t-s)}ds.$$

In particular $I(\varphi_j(T_0)) = I(\varphi_j(0))e^{-\beta T_0} + \int_0^{T_0} \mathscr{H}(\varphi_j(s))e^{-\beta(T_0-s)}ds.$

From (9.15) and Lemma 6.7 we have

$$\int_0^{T_0} \mathscr{H}(\varphi_j(s))e^{-\beta(T_0-s)}ds \to \int_0^{T_0} \mathscr{H}(\varphi_0(s))e^{-\beta(T_0-s)}ds, \ \ j \to +\infty.$$

Therefore

$$\limsup_{j\to+\infty} I(\varphi_j(T_0)) \le \limsup_{j\to+\infty} I(\varphi_j(0))e^{-\beta T_0} + \int_0^{T_0} \mathscr{H}(\varphi_0(s))e^{-\beta(T_0-s)}ds =$$
$$= I(\varphi_0(T_0)) + \left[\limsup_{j\to+\infty} I(\varphi_j(0)) - I(\varphi_0(0))\right] e^{-\beta T_0} \le I(\varphi_0(T_0)) + c_3 e^{-\beta T_0},$$

where c_3 does not depend on $T_0 > 0$.

On the other hand, from (9.15) we have

$$\limsup_{j\to+\infty} I(\varphi_j(T_0)) \ge \frac{1}{2}\lim_{j\to+\infty} \|\varphi_j(T_0)\|_X^2 + J(u_0(T_0)) + \frac{\beta}{2}(u_0'(T_0), u_0(T_0)).$$

Therefore we obtain $\frac{1}{2}a^2 \le \frac{1}{2}\|\xi\|_X^2 + c_3 e^{-\beta T_0}\ \forall T_0 > 0$.

Thus, $a \le \|\xi\|_X$.

The Theorem is proved.

Let us consider the family $\mathscr{K}_+ = \cup_{u_0\in X}\mathscr{D}(u_0)$ of all weak solutions of Problem (9.9) defined on $[0, +\infty)$. Note that $\mathscr{K}_+$ is *translation invariant one*, i.e., $\forall u(\cdot) \in \mathscr{K}_+$, $\forall h \ge 0$, $u_h(\cdot) \in \mathscr{K}_+$, where $u_h(s) = u(h+s)$, $s \ge 0$. On $\mathscr{K}_+$ we set the *translation semigroup* $\{T(h)\}_{h\ge 0}$, $T(h)u(\cdot) = u_h(\cdot)$, $h \ge 0$, $u \in \mathscr{K}_+$. In view of the translation invariance of $\mathscr{K}_+$ we conclude that $T(h)\mathscr{K}_+ \subset \mathscr{K}_+$ as $h \ge 0$.

On $\mathscr{K}_+$ we consider a topology induced from the Fréchet space $C^{loc}(\mathbb{R}_+; X)$. Note that

$$f_n(\cdot) \to f(\cdot) \text{ in } C^{loc}(\mathbb{R}_+; X) \iff \forall M > 0,\ \Pi_M f_n(\cdot) \to \Pi_M f(\cdot) \text{ in } C([0, M]; X),$$

where Π_M is the restriction operator to the interval $[0, M]$. We denote the restriction operator to $[0, +\infty)$ by Π_+.

Let us consider Problem (9.9) on the entire time axis. Similarly to the space $C^{loc}(\mathbb{R}_+; X)$ the space $C^{loc}(\mathbb{R}; X)$ is endowed with the topology of a local uniform convergence on each interval $[-M, M] \subset \mathbb{R}$. A function $u \in C^{loc}(\mathbb{R}; X) \cap L_\infty(\mathbb{R}; X)$ is said to be *a complete trajectory* of Problem (9.9) if $\forall h \in \mathbb{R}$, $\Pi_+ u_h(\cdot) \in \mathscr{K}_+$.

Let $\mathscr{K}$ be a family of *all complete trajectories* of Problem (9.9). Note that $\forall h \in \mathbb{R}$, $\forall u(\cdot) \in \mathscr{K}$ $u_h(\cdot) \in \mathscr{K}$. We say that the complete trajectory $\varphi \in \mathscr{K}$ is *stationary* if $\varphi(t) = z$ for all $t \in \mathbb{R}$ for some $z \in X$. Following Ball [1, p. 486] we denote by $Z(\mathscr{G})$ the set of all rest points of $\mathscr{G}$. Note that

$$Z(\mathscr{G}) = \{(\bar{0}, u) \mid u \in V,\ -\triangle(u) + \partial J(u) \ni \bar{0}\}.$$

Lemma 9.1 *$Z(\mathscr{G})$ is an bounded set in X.*

The existence of a Lyapunov function for $\mathscr{G}$ follows from Lemma 6.10 (see Ball [1, p. 486]).

Lemma 9.2 *A functional* $\mathscr{V} : X \to \mathbb{R}$ *defined by (6.44) is a Lyapunov function for* $\mathscr{G}$.

We recall that the set $\mathscr{A}$ is said to be a *global attractor* $\mathscr{G}$ if
(1) $\mathscr{A}$ is negatively semiinvariant (i.e., $\mathscr{A} \subset \mathscr{G}(t, \mathscr{A})$ $\forall t \geq 0$);
(2) $\mathscr{A}$ is attracting set, i.e.,

$$\operatorname{dist}(\mathscr{G}(t, B), \mathscr{A}) \to 0, \quad t \to +\infty, \quad \forall B \in \beta(X), \tag{9.16}$$

where $\operatorname{dist}(C, D) = \sup\limits_{c \in C} \inf\limits_{d \in D} \|c - d\|_X$ is the Hausdorff semidistance;
(3) for any closed set $Y \subset H$ satisfying (9.16), we have $\mathscr{A} \subset Y$ (minimality).

The global attractor is said to be *invariant* if $\mathscr{A} = \mathscr{G}(t, \mathscr{A})$, $\forall t \geq 0$.

Note that by definition a global attractor is unique.

We prove the existence of an invariant compact global attractor.

Theorem 9.3 *The m-semiflow* $\mathscr{G}$ *has an invariant compact in the phase space* X *global attractor* $\mathscr{A}$. *For each* $\psi \in \mathscr{K}$ *the limit sets*

$$\alpha(\psi) = \{z \in \mathrm{X} | \ \psi(t_j) \to z \ \textit{ for some sequence } \ t_j \to -\infty\},$$

$$\omega(\psi) = \{z \in \mathrm{X} | \ \psi(t_j) \to z \ \textit{ for some sequence } \ t_j \to +\infty\}$$

are connected subsets of $Z(\mathscr{G})$ *on which* $\mathscr{V}$ *is constant. If* $Z(\mathscr{G})$ *is totally disconnected (in particular if* $Z(\mathscr{G})$ *is countable) the limits*

$$z_- = \lim_{t \to -\infty} \psi(t), \quad z_+ = \lim_{t \to +\infty} \psi(t)$$

exist and z_-, z_+ *are rest points; furthermore,* $\varphi(t)$ *tends to a rest point as* $t \to +\infty$ *for every solution* $\varphi \in \mathscr{K}_+$.

Proof The existence of a global attractor for Second Order Evolution Inclusions directly follows from Lemmas 6.8, 6.9, 9.1 and 9.2, Theorems 6.3, 6.4, 9.2 and Ball [1, Theorem 2.7].

9.3 Examples of Applications

In this section we provide examples of applications to theorems established in previous sections. We consider a model of combustion in porous media, a model of conduction of electrical impulses in nerve axons, a climate energy balance model etc. (see also [4–21, 23–46, 49–96]).

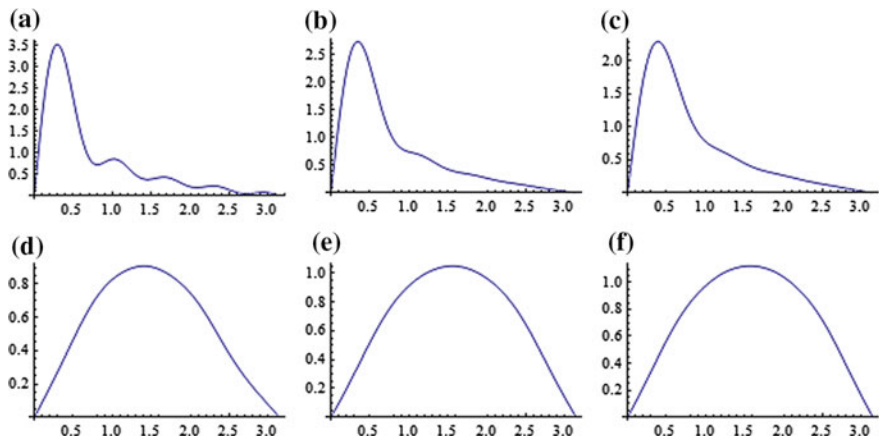


Fig. 9.2 Approximations for trajectories to model of combustion in porous media in a moment **a** $t = 0$; **b** $t = 0.8$; **c** $t = 1.6$; **d** $t = 2.4$; **e** $t = 3.2$; **f** $t = 4$

9.3.1 A Model of Combustion in Porous Media

Let us consider the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - f(u) \in \lambda H(u-1), & (x,t) \in (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) = 0, \quad t \in \mathbb{R}, \end{cases} \tag{9.17}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous and nondecreasing function satisfying growth and sign assumptions, $\lambda > 0$, and $H(0) = [0,1]$, $H(s) = \mathbf{I}\{s > 0\}$, $s \neq 0$. For each $u_\tau \in L^2((0,\pi))$ we set $\mathscr{D}_{\tau,T}(u_\tau) = \{u(\cdot) \in L^2(\tau, T; H_0^1((0,\pi)))\big|\ u(\cdot)$ is a weak solution of Problem (9.17) and $u(\tau) = u_\tau\}$. Since Problem (9.17) is a particular case of Problem (9.2), then all statements from Sect. 10.1 hold; Fig. 9.2.

9.3.2 A Model of Conduction of Electrical Impulses in Nerve Axons

Consider the problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u \in \lambda H(u-a), & (x,t) \in (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) = 0, \quad t \in \mathbb{R}, \end{cases} \tag{9.18}$$

where $a \in \left(0, \frac{1}{2}\right)$; Terman [47, 48]. Since Problem (9.18) is a particular case of Problem (9.2), then all statements from Sect. 9.1 hold; Fig. 9.3.

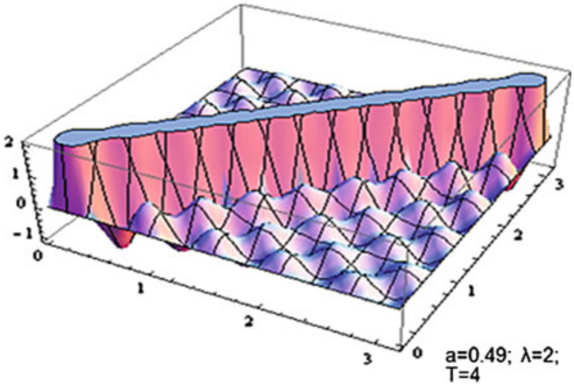


Fig. 9.3 Modeling of solutions for a model of conduction of electrical impulses in nerve axons

9.3.3 Viscoelastic Problems with Nonlinear "Reaction-Displacement" Law

Let a viscoelastic body occupy a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ in applications, and it is acted upon by volume forces and surface tractions.[1] The boundary Γ of Ω is supposed to be Lipschitz continuous and it is partitioned into two disjoint measurable parts Γ_D and Γ_N such that $meas(\Gamma_D) > 0$. We consider the process of evolution of the mechanical state on the interval $(0, +\infty)$. The body is clamped on Γ_D and thus the displacement vanishes there. The forces field of density f_0 act in Ω, the surface tractions of density g_0 are applied on Γ_N. We denote by $u = (u_1, \ldots, u_d)$ the displacement vector, by $\sigma = (\sigma_{ij})$ the stress tensor and by $\varepsilon(u) = (\varepsilon_{ij}(u))$ the linearized (small) strain tensor ($\varepsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$), where $i, j = 1, \ldots, d$.

The mechanical problem consists in finding the displacement field $u : \Omega \times (0, +\infty) \to \mathbb{R}^d$ such that

$$u''(t) - \mathrm{div}\sigma(t) = f_0 \quad \text{in } \Omega \times (0, +\infty), \tag{9.19}$$

$$\sigma(t) = \mathscr{C}\varepsilon(u'(t)) + \mathscr{E}\varepsilon(u(t)) \quad \text{in } \Omega \times (0, +\infty), \tag{9.20}$$

$$u(t) = 0 \quad \text{on } \Gamma_D \times (0, +\infty), \tag{9.21}$$

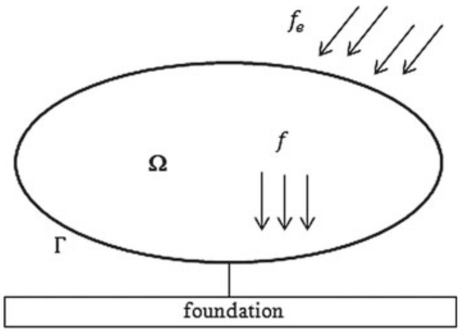
$$\sigma n(t) = g_0 \quad \text{on } \Gamma_N \times (0, +\infty), \tag{9.22}$$

$$u(0) = u_0, \; u'(0) = u_1 \quad \text{in } \Omega, \tag{9.23}$$

where $\mathscr{C}$ and $\mathscr{E}$ are given linear constitutive functions, n being the outward unit normal vector to Γ.

[1]This section is based on results of [22] and references therein.

Fig. 9.4 Foundation, body, and main forces



In the above model the dynamic equation (9.19) is considered with the viscoelastic constitutive relationship of the Kelvin-Voigt type (9.20) while (9.21) and (9.22) represent the displacement and traction boundary conditions (Fig. 9.4), respectively. The functions u_0 and u_1 are the initial displacement and the initial velocity, respectively. In order to formulate the skin effects, we suppose that the body forces of density f_0 consists of two parts: f_1 which is prescribed external loading and f_2 which is the reaction of constrains introducing the skin effects, i.e. $f_0 = f_1 + f_2$. Here f_2 is a possibly multi-valued function of the displacement u. We consider the reaction-displacement law of the form

$$-f_2(x,t) \in \partial j(x, u(x,t)) \text{ in } \Omega \times (0, +\infty), \tag{9.24}$$

where $j : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz function in u and ∂j represents the Clarke subdifferential with respect to u. Let $\mathscr{Y}_d$ be the space of second-order symmetric tensors on $\mathbb{R}^d$.

We consider the following problem:

examine the long-time (as $t \to +\infty$) behavior of all (weak, generalized) solutions for (9.19)–(9.23) and (9.24).

In [22] for finite time interval it was presented the hemivariational formulation of problems similar to (9.19)–(9.24) and an existence theorem for evolution inclusions with pseudomonotone operators. We give now variational formulation of the above problem. To this aim let $H = L_2(\Omega, \mathbb{R}^d)$, $H_1 = H^1(\Omega, \mathbb{R}^d)$, $\mathscr{H} = L_2(\Omega, \mathscr{Y}_d)$ and V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_D\}.$$

On V we consider the inner product and the corresponding norm given by

$$(u, v)_V = \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathscr{H}}, \quad \|v\|_V = \|\varepsilon(v)\|_{\mathscr{H}} \text{ for } u, v \in V.$$

From the Korn inequality $\|v\|_{H_1} \le C_1 \|\varepsilon(v)\|_{\mathscr{H}}$ for $v \in V$ with $C_1 > 0$, it follows that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are the equivalent norms on V. Identifying H with its dual, we have an evolution triple $V \subset H \subset V^*$ (see e.g. [3]) with dense and compact embeddings. We denote by $\langle \cdot, \cdot \rangle_V$ the duality of V and its dual V^*, by $\| \cdot \|_{V^*}$ the norm in V^*. We have $\langle u, v \rangle_V = (u, v)_H$ for all $u \in H$ and $v \in V$.

We admit the following hypotheses:

H($\mathscr{C}$). The linear symmetric viscosity operator $\mathscr{C} : \Omega \times \mathscr{Y}_d \to \mathscr{Y}_d$ satisfies the Carathéodory condition (i.e. $\mathscr{C}(\cdot, \varepsilon)$ is measurable on Ω for all $\varepsilon \in \mathscr{Y}_d$ and $\mathscr{C}(x, \cdot)$ is continuous on $\mathscr{Y}_d$ for a.e. $x \in \Omega$) and

$$\mathscr{C}(x, \varepsilon) : \varepsilon \geq C_2 \|\varepsilon\|^2_{\mathscr{Y}_d} \text{ for all } \varepsilon \in \mathscr{Y}_d \text{ and a.e. } x \in \Omega \text{ with } C_2 > 0. \tag{9.25}$$

H($\mathscr{E}$). The elasticity operator $\mathscr{E} : \Omega \times \mathscr{Y}_d \to \mathscr{Y}_d$ is of the form $\mathscr{E}(x, \varepsilon) = \mathbb{E}(x)\varepsilon$ (Hooke's law) with a symmetric elasticity tensor $\mathbb{E} \in L_\infty(\Omega)$, i.e. $\mathbb{E} = (g_{ijkl})$, $i, j, k, l = 1, \ldots, d$ with $g_{ijkl} = g_{jikl} = g_{lkij} \in L_\infty(\Omega)$. Moreover,

$$\mathscr{E}(x, \varepsilon) : \varepsilon \geq C_3 \|\varepsilon\|^2_{\mathscr{Y}_d} \text{ for all } \varepsilon \in \mathscr{Y}_d \text{ and a.e. } x \in \Omega \text{ with } C_3 > 0.$$

H(**j**). $j : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a function such that

(i) $j(\cdot, \xi)$ is measurable for all $\xi \in \mathbb{R}^d$ and $j(\cdot, 0) \in L_1(\Omega)$;

(ii) $j(x, \cdot)$ is locally Lipschitz and regular [8] for all $x \in \Omega$;

(iii) $\|\eta\| \leq C_4(1 + \|\xi\|)$ for all $\eta \in \partial j(x, \xi)$, $x \in \Omega$ with $C_4 > 0$;

(iv) $j^0(x, \xi; -\xi) \leq C_5(1 + \|\xi\|)$ for all $\xi \in \mathbb{R}^d$, $x \in \Omega$, with $C_5 \geq 0$, where $j^0(x, \xi; \eta)$ is the directional derivative of $j(x, \cdot)$ at the point $\xi \in \mathbb{R}^d$ in the direction $\eta \in \mathbb{R}^d$.

H(**f**). $f_1 \in V^*$, $g_0 \in L_2(\Gamma_N; \mathbb{R}^d)$, $u_0 \in V$ and $u_1 \in H$.

Next we need the spaces $\mathscr{V} = L_2(\tau, T; V)$, $\hat{\mathscr{H}} = L_2(\tau, T; H)$ and $\mathscr{W} = \{w \in \mathscr{V} : w' \in \mathscr{V}^*\}$, where the time derivative involved in the definition of $\mathscr{W}$ is understood in the sense of vector-valued distributions, $-\infty < \tau < T < +\infty$. Endowed with the norm $\|v\|_{\mathscr{W}} = \|v\|_{\mathscr{V}} + \|v'\|_{\mathscr{V}^*}$, the space $\mathscr{W}$ becomes a separable reflexive Banach space. We also have $\mathscr{W} \subset \mathscr{V} \subset \hat{\mathscr{H}} \subset \mathscr{V}^*$. The duality for the pair $(\mathscr{V}, \mathscr{V}^*)$ is denoted by $\langle z, w \rangle_{\mathscr{V}} = \int_\tau^T \langle z(s), w(s) \rangle_V ds$. It is well known (cf. [13]) that the embedding $\mathscr{W} \subset C([\tau, T]; H)$ and $\{w \in \mathscr{V} : w' \in \mathscr{W}\} \subset C([\tau, T]; V)$ are continuous. Next we define $g \in V^*$ by

$$\langle g, v \rangle_V = \langle f_1, v \rangle_V + \langle g_0, v \rangle_{L^2(\Gamma_N; \mathbb{R}^d)} \text{ for } v \in V. \tag{9.26}$$

Taking into account the condition (9.24), we obtain the following variational formulation of our problem:

$$\begin{cases} \langle u''(t), v \rangle_V + (\sigma(t), \varepsilon(v))_{\mathscr{H}} + \int_\Omega j^0(x, u(t); v) dx \geq \\ \geq \langle g, v \rangle_V \text{ for all } v \in V \text{ and a.e. } t \in (0, +\infty), \\ \sigma(t) = \mathscr{C}(\varepsilon(u'(t))) + \mathscr{E}(\varepsilon(u(t))) \text{ for a.e. } t \in (0, +\infty), \\ u(0) = u_0, \ u'(0) = u_1. \end{cases} \tag{9.27}$$

We define the operators $A : V \to V^*$ and $B : V \to V^*$ by

$$\langle A(u), v\rangle_V = (\mathscr{C}(x, \varepsilon(u)), \varepsilon(v))_{\mathscr{H}} \text{ for } u, v \in V, \tag{9.28}$$

$$\langle Bu, v\rangle_V = (\mathscr{E}(x, \varepsilon(u)), \varepsilon(v))_{\mathscr{H}} \text{ for } u, v \in V. \tag{9.29}$$

Obviously the bilinear forms (9.28) and (9.29) are symmetric, continuous and coercive.

Let us introduce the functional $J : L_2(\Omega; \mathbb{R}^d) \to \mathbb{R}$ defined by

$$J(v) = \int_{\Omega} j(x, v(x))dx \text{ for } v \in L_2(\Omega; \mathbb{R}^d). \tag{9.30}$$

From Sect. 7.3, under Assumptions **H(j)**, the functional J defined by (9.30) satisfies

H(J). $J : L_2(\Omega; \mathbb{R}^d) \to \mathbb{R}$ is a functional such that:

(i) $J(\cdot)$ is well defined, locally Lipschitz (in fact, Lipschitz on bounded subsets of $L_2(\Omega; \mathbb{R}^d)$) and admits the representation via the difference of convex functions;

(ii) $\zeta \in \partial J(v)$ implies $\|\zeta\|_{L_2(\Omega;\mathbb{R}^d)} \le C_6(1 + \|v\|_{L_2(\Omega;\mathbb{R}^d)})$ for $v \in L_2(\Omega; \mathbb{R}^d)$ with $C_6 > 0$;

(iii) $J^0(v; -v) \le C_7(1 + \|v\|_{L_2(\Omega;\mathbb{R}^d)})$ for $v \in L_2(\Omega; \mathbb{R}^d)$ with $C_7 \ge 0$, where $J^0(u; v)$ denotes the directional derivative of $J(\cdot)$ at a point $u \in L_2(\Omega; \mathbb{R}^d)$ in the direction $v \in L_2(\Omega; \mathbb{R}^d)$.

We can now formulate the second-order evolution inclusions associated with the variational form of our problem

$$\begin{cases} \text{Find } u \in C([0, +\infty); V) \text{ with } u' \in C([0, +\infty); H) \cap L_2^{loc}(0, +\infty; V) \\ \text{ and } u'' \in L_2^{loc}(0, +\infty; V^*) \text{ such that} \\ u''(t) + Au'(t) + Bu(t) + \partial J(u(t)) \ni g \text{ a.e. } t \in (0, +\infty), \\ u(0) = u_0, \;\; u'(0) = u_1. \end{cases} \tag{9.31}$$

Theorem 6.6 yields that, if $\tau < T$, $\{\varphi_n(\cdot)\}_{n\ge1} \subset W_\tau^T$ is an arbitrary sequence of weak solutions of (9.31) on $[\tau, T]$ such that $\varphi_n(\tau) \to \varphi_\tau$ strongly in E, $n \to +\infty$, then there exist $\varphi \in \mathscr{D}_{\tau,T}(\varphi_\tau)$ such that up to a subsequence $\varphi_n(\cdot) \to \varphi(\cdot)$ in $C([\tau, T]; E)$, $n \to +\infty$ (see Sect. 7.3 for details).

We define the m-semiflow $\mathscr{G}$ as $\mathscr{G}(t, \xi_0) = \{\xi(t) \mid \xi(\cdot) \in \mathscr{D}(\xi_0)\}, t \ge 0$. Denote the set of all nonempty (nonempty bounded) subsets of E by $P(E)$ ($\beta(E)$). We remark that the multi-valued map $\mathscr{G} : \mathbb{R}_+ \times E \to P(E)$ is *strict m-semiflow*, i.e. $\mathscr{G}(0, \cdot) =$ Id (the identity map), $\mathscr{G}(t + s, x) = \mathscr{G}(t, \mathscr{G}(s, x))$ $\forall x \in E, t, s \in \mathbb{R}_+$. Further $\varphi \in \mathscr{G}$ will mean that $\varphi \in \mathscr{D}(\xi_0)$ for some $\xi_0 \in E$.

Definition 9.3 ([1, p. 35]) The m-semiflow $\mathscr{G}$ is called *asymptotically compact*, if for any sequence $\varphi_j \in \mathscr{G}$ with $\varphi_j(0)$ bounded, and for any sequence $t_j \to +\infty$, the sequence $\varphi_j(t_j)$ has a convergent subsequence.

Theorem 9.4 *The m-semiflow $\mathscr{G}$ is asymptotically compact.*

Proof Let $\xi_n \in \mathscr{G}(t_n, v_n)$, $v_n \in B \in \beta(E)$, $n \geq 1$, $t_n \to +\infty$, $n \to +\infty$. Let us check the precompactness of $\{\xi_n\}_{n\geq 1}$ in E. In order to do that without loss of the generality it is sufficiently to extract a convergent in E subsequence from $\{\xi_n\}_{n\geq 1}$. From Corollary 6.2 we obtain that there exist $\{\xi_{n_k}\}_{k\geq 1}$ and $\xi \in E$ such that $\xi_{n_k} \to \xi$ weakly in E, $\|\xi_{n_k}\|_E \to a \geq \|\xi\|_E$, $k \to +\infty$. We show that $a \leq \|\xi\|_E$. Let us fix an arbitrary $T_0 > 0$. Then for rather big $k \geq 1$ $\mathscr{G}(t_{n_k}, v_{n_k}) = \mathscr{G}(T_0, \mathscr{G}(t_{n_k} - T_0, v_{n_k}))$. Hence $\xi_{n_k} \in \mathscr{G}(T_0, \beta_{n_k})$, where $\beta_{n_k} \in \mathscr{G}(t_{n_k} - T_0, v_{n_k})$ and

$$\delta := \sup_{k\geq 1} \|\beta_{n_k}\|_E < +\infty$$

(see Corollary 6.2). From Theorem 6.5 for some $\{\xi_{k_j}, \beta_{k_j}\}_{j\geq 1} \subset \{\xi_{n_k}, \beta_{n_k}\}_{k\geq 1}$, $\beta_{T_0} \in E$ we obtain:

$$\xi \in \mathscr{G}(T_0, \beta_{T_0}), \quad \beta_{k_j} \to \beta_{T_0} \text{ weakly in } E, \quad j \to +\infty. \tag{9.32}$$

From the definition of $\mathscr{G}$ we set: $\forall j \geq 1$ $\xi_{k_j} = (y_j(T_0), y_j'(T_0))^T$, $\beta_{k_j} = (y_j(0), y_j'(0))^T$, $\xi = (y_0(T_0), y_0'(T_0))^T$, $\beta_{T_0} = (y_0(0), y_0'(0))^T$, where $\varphi_j = (y_j, y_j')^T \in C([0, T_0]; E)$, $y_j' \in L_2(0, T_0; V)$, $y_j'' \in L_2(0, T_0; V^*)$, $d_j \in L_\infty(0, T_0; H)$,

$$y_j''(t) + Ay_j'(t) + By_j(t) + d_j(t) = \bar{0}, \quad d_j(t) \in \partial J(y_j(t)) \quad \text{for a.e.} \quad t \in (0, T_0), \ \forall j \geq 0.$$

Now we fix an arbitrary $\varepsilon > 0$. Let for each $t \in [0, T_0]$, $j \geq 0$

$$I_\varepsilon(\varphi_j(t)) := \frac{1}{2}\|\varphi_j(t)\|_E^2 + J(y_j(t)) + \varepsilon(y_j'(t), y_j(t))_H.$$

Then,

$$\frac{dI_\varepsilon(\varphi_j(t))}{dt} = -2\varepsilon I_\varepsilon(\varphi_j(t)) + 2\varepsilon \mathscr{H}_\varepsilon(\varphi_j(t)) - \varepsilon\langle Ay_j'(t), y_j(t)\rangle_V - \langle Ay_j'(t), y_j'(t)\rangle_V,$$

for a.e. $t \in (0, T_0)$, where

$$\mathscr{H}_\varepsilon(\varphi_j(t)) = J(y_j(t)) - \frac{1}{2}(d_j(t), y_j(t))_H + \|y_j'(t)\|_H^2 + \varepsilon(y_j'(t), y_j(t))_H, \text{ for a.e. } t \in (0, T_0).$$

Thus, for any $j \geq 0$ and $t \in [0, T_0]$

$$I_\varepsilon(\varphi_j(T_0)) = I_\varepsilon(\varphi_j(0))e^{-2\varepsilon T_0} + 2\varepsilon \int_0^{T_0} \mathscr{H}_\varepsilon(\varphi_j(t))e^{-2\varepsilon(T_0-t)}dt-$$

$$-\varepsilon\int_0^{T_0}\langle Ay_j(t), y_j'(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt - \int_0^{T_0}\langle Ay_j'(t), y_j'(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt.$$

From (6.64), (9.32) and Lemma 6.15 we have

$$\|y_j'(t)\|_H^2 + \|y_j(t)\|_V^2 + \gamma\int_0^t \|y_j'(t)\|_V^2 dt \le \bar{R} \quad \forall j\ge 0,\ \forall t\in[0,T_0],$$

where $\bar{R} > 0$ is a constant. Moreover,

$$\begin{aligned}
&y_j \to y_0 \text{ weakly in } L_2(0,T_0;V), \quad y_j' \to y_0' \text{ weakly in } L_2(0,T_0;V),\\
&y_j \to y_0 \text{ strongly in } L_2(0,T_0;H),\ y_j' \to y_0' \text{ strongly in } L_2(0,T_0;H),\\
&y_j'' \to y_0'' \text{ weakly in } L_2(0,T_0;V^*),\ d_j \to d_0 \text{ weakly in } L_2(0,T_0;H),\\
&\forall t\in[0,T_0]\quad y_j(t)\to y_0(t) \text{ in } H,\ y_j'(t)\to y_0'(t) \text{ weakly in } H,\ j\to+\infty.
\end{aligned} \tag{9.33}$$

Therefore,

$$\int_0^{T_0}\mathscr{H}_\varepsilon(\varphi_j(t))e^{-2\varepsilon(T_0-t)}dt \to \int_0^{T_0}\mathscr{H}_\varepsilon(\varphi_0(t))e^{-2\varepsilon(T_0-t)}dt, \quad j\to+\infty,$$

$$\varliminf_{j\to+\infty}\int_0^{T_0}\langle Ay_j'(t), y_j'(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt \ge \int_0^{T_0}\langle Ay_0'(t), y_0'(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt.$$

The last inequality holds, because of the functional $v(\cdot)\to \int_0^{T_0}\langle Av(t), v(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt$ is sequentially weakly lower semi-continuous on $L_2(\tau, T_0; V)$. Furthermore,

$$\varepsilon\int_0^{T_0}\langle Ay_j(t), y_j'(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt = \frac{\varepsilon}{2}\langle Ay_j(T_0), y_j(T_0)\rangle_V - \frac{\varepsilon}{2}\langle Ay_j(0), y_j(0)\rangle_V e^{-2\varepsilon T_0} -$$

$$-\varepsilon^2\int_0^{T_0}\langle Ay_j(t), y_j(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt \quad \forall j\ge 0,$$

from which, by Corollary 6.2, we have

$$\left|\varepsilon\int_0^{T_0}\langle Ay_j(t), y_j'(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt\right| \le 2\varepsilon\gamma\left[\frac{\lambda_1+2c_3}{\lambda_1-\mu}\delta^2 + \frac{2(c_3+c_4)\lambda_1}{\lambda_1-\mu}\right], \quad \forall j\ge 0.$$

Thus,

$$\overline{\lim_{j\to+\infty}} I_\varepsilon(\varphi_j(T_0)) \le I_\varepsilon(\varphi_0(0))e^{-2\varepsilon T_0} + \left[\overline{\lim_{j\to+\infty}} I_\varepsilon(\varphi_j(0)) - I_\varepsilon(\varphi_0(0))\right] e^{-2\varepsilon T_0} +$$

$$+2\varepsilon \int_0^{T_0} \mathscr{H}_\varepsilon(\varphi_0(t))e^{-2\varepsilon(T_0-t)}dt - \varepsilon \int_0^{T_0} \langle Ay_0(t), y_0'(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt -$$

$$-\int_0^{T_0} \langle Ay_0'(t), y_0'(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt + 4\varepsilon\gamma \left[\frac{\lambda_1+2c_3}{\lambda_1-\mu}\delta^2 + \frac{2(c_3+c_4)\lambda_1}{\lambda_1-\mu}\right] \le$$

$$\le I_\varepsilon(\varphi_0(T_0)) + \delta^2 e^{-2\varepsilon T_0} + 4\varepsilon\gamma \left[\frac{\lambda_1+2c_3}{\lambda_1-\mu}\delta^2 + \frac{2(c_3+c_4)\lambda_1}{\lambda_1-\mu}\right]$$

and, due to (9.33), for any $T_0 > 0$ and $\varepsilon > 0$

$$\frac{1}{2}\|\xi\|_E^2 \le \frac{1}{2}a^2 = \frac{1}{2}\lim_{j\to+\infty} \|\xi_{k_j}\|_E^2 \le \frac{1}{2}\|\xi\|_E^2 + \delta^2 e^{-2\varepsilon T_0} + 4\varepsilon\gamma \left[\frac{\lambda_1+2c_3}{\lambda_1-\mu}\delta^2 + \frac{2(c_3+c_4)\lambda_1}{\lambda_1-\mu}\right].$$

Hence, for all $\varepsilon > 0$ we have

$$\frac{1}{2}\|\xi\|_E^2 \le \frac{1}{2}a^2 \le \frac{1}{2}\|\xi\|_E^2 + 4\varepsilon\gamma \left[\frac{\lambda_1+2c_3}{\lambda_1-\mu}\delta^2 + \frac{2(c_3+c_4)\lambda_1}{\lambda_1-\mu}\right].$$

Thus, $a = \|\xi\|_E$.

The theorem is proved.

Let us consider the family $\mathscr{K}_+ = \cup_{y_0\in E}\mathscr{D}(y_0)$ of all weak solutions of the inclusion (9.31), defined on $[0, +\infty)$. Note that $\mathscr{K}_+$ is *translation invariant one*, i.e. for all $u(\cdot) \in \mathscr{K}_+$ and all $h \ge 0$ we have $u_h(\cdot) \in \mathscr{K}_+$, where $u_h(s) = u(h+s)$, $s \ge 0$. On $\mathscr{K}_+$ we set the *translation semigroup* $\{T(h)\}_{h\ge 0}$, $T(h)u(\cdot) = u_h(\cdot)$, $h \ge 0$, $u \in \mathscr{K}_+$. In view of the translation invariance of $\mathscr{K}_+$ we conclude that $T(h)\mathscr{K}_+ \subset \mathscr{K}_+$ as $h \ge 0$.

On $\mathscr{K}_+$ we consider the topology induced from the Fréchet space $C^{loc}(\mathbb{R}_+; E)$. Note that

$$f_n(\cdot) \to f(\cdot) \text{ in } C^{loc}(\mathbb{R}_+; E) \iff \forall M > 0\, \Pi_M f_n(\cdot) \to \Pi_M f(\cdot) \text{ in } C([0, M]; E),$$

where Π_M is the restriction operator to the interval $[0, M]$. We denote the restriction operator to $[0, +\infty)$ by Π_+.

Let us consider the autonomous inclusion (9.31) on the entire time axis. Similarly to the space $C^{loc}(\mathbb{R}_+; E)$ the space $C^{loc}(\mathbb{R}; E)$ is endowed with the topology of local uniform convergence on each interval $[-M, M] \subset \mathbb{R}$. A function $u \in C^{loc}(\mathbb{R}; E) \cap$

$L_\infty(\mathbb{R}; E)$ is said to be *a complete trajectory* of the inclusion (9.31), if $\forall h \in \mathbb{R}$ $\Pi_+ u_h(\cdot) \in \mathcal{K}_+$. Let $\mathcal{K}$ be a family of *all complete trajectories* of the inclusion (9.31). Note that $\forall h \in \mathbb{R}, \forall u(\cdot) \in \mathcal{K}$ $u_h(\cdot) \in \mathcal{K}$. We say that the complete trajectory $\varphi \in \mathcal{K}$ is *stationary* if $\varphi(t) = z$ for all $t \in \mathbb{R}$ for some $z = (u, \bar{0})^T \in E$ (rest point). We denote the set of rest points of $\mathcal{G}$ by $Z(\mathcal{G})$. We remark that $Z(\mathcal{G}) = \{(u, \bar{0})^T \mid u \in V,\ B(u) + \partial J(u) \ni \bar{0}\}$.

From Conditions $\underline{H(B)}$ and $\underline{H(J)}$ it follows that

Lemma 9.3 *The set $Z(\mathcal{G})$ is nonempty and bounded in E.*

From Lemma 6.10 the existence of Lyapunov function for $\mathcal{G}$ follows.

Lemma 9.4 *The functional $\mathcal{V} : E \to \mathbb{R}$, defined by*

$$\mathcal{V}(\varphi) = \frac{1}{2}\|\varphi\|_E^2 + J(a). \tag{9.34}$$

, is a Lyapunov function for $\mathcal{G}$.

We recall that the set $\mathcal{A} \subset E$ is said to be a *global attractor* for $\mathcal{G}$, if
(1) $\mathcal{A}$ is negatively semiinvariant (i.e. $\mathcal{A} \subset \mathcal{G}(t, \mathcal{A})$ $\forall t \geq 0$);
(2) $\mathcal{A}$ is attracting set i.e.

$$\operatorname{dist}(\mathcal{G}(t, B), \mathcal{A}) \to 0, \quad t \to +\infty \quad \forall B \in \beta(E), \tag{9.35}$$

where $\operatorname{dist}(C, D) = \sup_{c \in C} \inf_{d \in D} \|c - d\|_E$ is the Hausdorff semidistance;
(3) for any closed set $Y \subset E$, satisfying (9.35), we have $\mathcal{A} \subset Y$ (minimality).

The global attractor is said to be *invariant*, if $\mathcal{A} = \mathcal{G}(t, \mathcal{A})$ $\forall t \geq 0$.

Note that from the definition of the global attractor it follows that it is unique.

We prove the existence of the invariant compact global attractor.

Theorem 9.5 *The m-semiflow $\mathcal{G}$ has the invariant compact in the phase space E global attractor $\mathcal{A}$. For each $\psi \in \mathcal{K}$ the limit sets*

$$\alpha(\psi) = \{z \in E |\ \psi(t_j) \to z \ for\ some\ sequence\ t_j \to -\infty\},$$

$$\omega(\psi) = \{z \in E |\ \psi(t_j) \to z \ for\ some\ sequence\ t_j \to +\infty\}$$

are connected subsets of $Z(\mathcal{G})$ on which $\mathcal{V}$ is constant. If $Z(\mathcal{G})$ is totally disconnected (in particular, if $Z(\mathcal{G})$ is countable) the limits

$$z_- = \lim_{t \to -\infty} \psi(t), \quad z_+ = \lim_{t \to +\infty} \psi(t)$$

exists and z_-, z_+ are rest points; furthermore, $\varphi(t)$ tends to a rest point as $t \to +\infty$ for every $\varphi \in \mathcal{K}_+$.

Proof The existence of the global attractor with required properties directly follows from previous theorems and [1, Theorem 2.7].

We remark in advance

$$\forall h \in \mathbb{R},\ \forall u(\cdot) \in \mathscr{K} \quad u_h(\cdot) \in \mathscr{K}. \tag{9.36}$$

Lemma 9.5 *The set* $\mathscr{K}$ *is nonempty and*

$$\forall \xi(\cdot) \in \mathscr{K},\ \forall t \in \mathbb{R} \quad \xi(t) \in \mathscr{A}, \tag{9.37}$$

where $\mathscr{A}$ *is the global attractor from Theorem 9.5.*

Proof Let us show that $\mathscr{K} \neq \emptyset$. Note that in virtue of Lemma 9.3, the set $Z(\mathscr{G})$ is nonempty and bounded in E. Let $(v, \bar{0})^T \in Z(\mathscr{G})$. We set $u(t) = v\ \forall t \in \mathbb{R}$. Then $(u, u')^T \in \mathscr{K} \neq \emptyset$.

Let us prove (9.37). For any $y \in \mathscr{K}$ $\exists d > 0$: $\|y(t)\|_E \le d$ $\forall t \in \mathbb{R}$. We set $B = \cup_{t\in\mathbb{R}}\{y(t)\} \in \beta(E)$. Note that $\forall \tau \in \mathbb{R}$, $\forall t \in \mathbb{R}_+$ $y(\tau) = y_{\tau-t}(t) \in \mathscr{G}(t, y_{\tau-t}(0)) \subset \mathscr{G}(t, B)$. From Theorem 9.5 and from (9.35) it follows that $\forall \varepsilon > 0$ $\exists T > 0$: $\forall \tau \in \mathbb{R}$ $\mathrm{dist}(y(\tau), \mathscr{A}) \le \mathrm{dist}(\mathscr{G}(T, B), \mathscr{A}) < \varepsilon$. Hence taking into account the compactness of $\mathscr{A}$ in E, it follows that $y(\tau) \in \mathscr{A}$ for any $\tau \in \mathbb{R}$.

Lemma 9.6 *The set* $\mathscr{K}$ *is compact in* $C^{loc}(\mathbb{R}; E)$ *and bounded in* $L_\infty(\mathbb{R}; E)$.

Proof The boundedness of $\mathscr{K}$ in $L_\infty(\mathbb{R}_+; E)$ follows from (9.37) and from the boundedness of $\mathscr{A}$ in E.

Let us check the compactness of $\mathscr{K}$ in $C^{loc}(\mathbb{R}; E)$. In order to do that it is sufficient to check the precompactness and completeness.

Step 1. Let us check the precompactness of $\mathscr{K}$ in $C^{loc}(\mathbb{R}; E)$. If it is not true then in view of (9.36), $\exists M > 0$: $\Pi_M \mathscr{K}$ is not precompact set in $C([0, M]; E)$. Hence there exists a sequence $\{v_n\}_{n\ge1} \subset \Pi_M \mathscr{K}$, that has not a convergent subsequence in $C([0, M]; E)$. On the other hand $v_n = \Pi_M u_n$, where $u_n \in \mathscr{K}$, $v_n(0) = u_n(0) \in \mathscr{A}$, $n \ge 1$. Since $\mathscr{A}$ is compact set in E (see Theorem 9.5), then in view of Theorem 6.4, $\exists\{v_{n_k}\}_{k\ge1} \subset \{v_n\}_{n\ge1}$, $\exists \eta \in E$, $\exists v(\cdot) \in \mathscr{D}_{0,M}(\eta)$: $v_{n_k}(0) \to \eta$ in E, $v_{n_k} \to v$ in $C([0, T]; E)$, $k \to +\infty$. We obtained a contradiction.

Step 2. Let us check the completeness of $\mathscr{K}$ in $C^{loc}(\mathbb{R}; E)$. Let $\{v_n\}_{n\ge1} \subset \mathscr{K}$, $v \in C^{loc}(\mathbb{R}; E)$: $v_n \to v$ in $C^{loc}(\mathbb{R}; E)$, $n \to +\infty$. From the boundedness of $\mathscr{K}$ in $L_\infty(\mathbb{R}; E)$ it follows that $v \in L_\infty(\mathbb{R}; E)$. From Theorem 6.4 we have that $\forall M > 0$ the restriction $v(\cdot)$ to the interval $[-M, M]$ belongs to $\mathscr{D}_{-M,M}(v(-M))$. Therefore $v(\cdot)$ is a complete trajectory of the inclusion (9.31). Thus, $v \in \mathscr{K}$.

Lemma 9.7 *Let* $\mathscr{A}$ *be the global attractor from Theorem 9.5. Then*

$$\forall y_0 \in \mathscr{A} \quad \exists y(\cdot) \in \mathscr{K}: \quad y(0) = y_0. \tag{9.38}$$

Proof Let $y_0 \in \mathscr{A}, u(\cdot) \in \mathscr{D}(y_0)$. From Theorem 9.5 $\mathscr{G}(t, \mathscr{A}) = \mathscr{A}\ \forall t \in \mathbb{R}_+$. Therefore,

$$u(t) \in \mathscr{A}\ \forall t \in \mathbb{R}_+,$$

$$\forall \eta \in \mathscr{A} \quad \exists \xi \in \mathscr{A},\ \exists \varphi_\eta(\cdot) \in \mathscr{D}_{0,1}(\xi): \quad \varphi_\eta(1) = \eta.$$

For any $t \in \mathbb{R}$ we set

$$y(t) = \begin{cases} u(t), & t \in \mathbb{R}_+, \\ \varphi_{y(-k+1)}(t+k), & t \in [-k, -k+1),\ k \in \mathbb{N}. \end{cases}$$

Note that $y \in C^{loc}(\mathbb{R}; E)$, $y(t) \in \mathscr{A}\ \forall t \in \mathbb{R}$ (hence $y \in L_\infty(\mathbb{R}; E)$) $y \in \mathscr{K}$. Moreover $y(0) = y_0$.

Now we shall construct the attractor of the translation semigroup $\{T(h)\}_{h \geq 0}$, acting on $\mathscr{K}_+$. We recall that the set $\mathscr{P} \subset C^{loc}(\mathbb{R}_+; E) \cap L_\infty(\mathbb{R}_+; E)$ is said to be *an attracting one* for the trajectory space $\mathscr{K}_+$ of the inclusion (9.31) in the topology of $C^{loc}(\mathbb{R}_+; E)$, if for any bounded (in $L_\infty(\mathbb{R}_+; E)$) set $\mathscr{B} \subset \mathscr{K}_+$ and an arbitrary number $M \geq 0$ the next relation

$$\text{dist}_{C([0,M];E)}(\Pi_M T(t)\mathscr{B}, \Pi_M \mathscr{P}) \to 0, \quad t \to +\infty \tag{9.39}$$

holds.

A set $\mathscr{U} \subset \mathscr{K}_+$ is said to be *trajectory attractor* in the trajectory space $\mathscr{K}_+$ with respect to the topology of $C^{loc}(\mathbb{R}_+; E)$ (cf. [5, Definition 1.2, p. 179]), if
(i) $\mathscr{U}$ is a compact set in $C^{loc}(\mathbb{R}_+; E)$ and bounded in $L_\infty(\mathbb{R}_+; E)$;
(ii) $\mathscr{U}$ is strictly invariant with respect to $\{T(h)\}_{h \geq 0}$, i.e. $T(h)\mathscr{U} = \mathscr{U}\ \forall h \geq 0$;
(iii) $\mathscr{U}$ is an attracting set in the trajectory space $\mathscr{K}_+$ in the topology $C^{loc}(\mathbb{R}_+; E)$.

Note that from the definition of the trajectory attractor it follows that it is unique.

The existence of the trajectory attractor and its structure properties follow from such theorem:

Theorem 9.6 *Let $\mathscr{A}$ be the global attractor from Theorem 9.5. Then there exists the trajectory attractor $\mathscr{P} \subset \mathscr{K}_+$ in the space $\mathscr{K}_+$ and we have*

$$\mathscr{P} = \Pi_+ \mathscr{K} = \{y \in \mathscr{K}_+ \mid y(t) \in \mathscr{A}\ \forall t \in \mathbb{R}_+\}. \tag{9.40}$$

Proof The proof intersects with proofs of previous sections results.

From Lemmas 9.5, 9.6 and the continuity of the operator $\Pi_+ : C^{loc}(\mathbb{R}; E) \to C^{loc}(\mathbb{R}_+; E)$ it follows that the set $\Pi_+\mathscr{K}$ is nonempty, compact in $C^{loc}(\mathbb{R}_+; E)$ and bounded in $L_\infty(\mathbb{R}_+; E)$. Moreover, the second equality in (9.40) holds (Lemma 9.5 and the proof of Lemma 9.7). The strict invariance of $\Pi_+\mathscr{K}$ follows from the autonomy of the inclusion (9.31).

Let us prove that $\Pi_+\mathscr{K}$ is the attracting set for the trajectory space $\mathscr{K}_+$ in the topology of $C^{loc}(\mathbb{R}_+; E)$. Let $B \subset \mathscr{K}_+$ be a bounded set in $L_\infty(\mathbb{R}_+; E)$, $M \geq 0$.

Let us suppose $M > 0$. Let us check (9.39). If it is not true, then there exist $\varepsilon > 0$, the sequences $t_n \to +\infty$, $v_n(\cdot) \in B$ such that

$$\forall n \geq 1 \quad \mathrm{dist}_{C([0,T];E)}(\Pi_M v_n(t_n + \cdot), \Pi_M \mathscr{K}) \geq \varepsilon. \tag{9.41}$$

On the other hand, from the boundedness of B in $L_\infty(\mathbb{R}_+; E)$ it follows that $\exists R > 0$: $\forall v(\cdot) \in B, \forall t \in \mathbb{R}_+ \ \|v(t)\|_E \leq R$. Hence, taking into account (9.35) and the asymptotic compactness of m-semiflow $\mathscr{G}$ (Theorem 9.4) we obtain that $\exists \{v_{n_k}(t_{n_k})\}_{k\geq 1} \subset \{v_n(t_n)\}_{n\geq 1}$, $\exists z \in \mathscr{A}$: $v_{n_k}(t_{n_k}) \to z$ in E, $k \to +\infty$. Further, $\forall k \geq 1$ we set $\varphi_k(t) = v_{n_k}(t_{n_k} + t)$, $t \in [0, M]$. Note that $\forall k \geq 1$ $\varphi_k(\cdot) \in \mathscr{D}_{0,M}(v_{n_k}(t_{n_k}))$. Then from Theorem 6.4 there exists a subsequence $\{\varphi_{k_j}\}_{j\geq 1} \subset \{\varphi_k\}_{k\geq 1}$ and an element $\varphi(\cdot) \in \mathscr{D}_{0,M}(z)$:

$$\varphi_{k_j} \to \varphi \text{ in } C([0, M]; E), \quad j \to +\infty. \tag{9.42}$$

Moreover, taking into account the invariance of $\mathscr{A}$ (see Theorem 9.5), for all $t \in [0, M]$ $\varphi(t) \in \mathscr{A}$. From Lemma 9.7 there exist $y(\cdot), v(\cdot) \in \mathscr{K}$: $y(0) = z$, $v(0) = \varphi(M)$. For any $t \in \mathbb{R}$ we set

$$\psi(t) = \begin{cases} y(t), & t \leq 0, \\ \varphi(t), & t \in [0, M], \\ v(t - M), & t \geq M. \end{cases}$$

Therefore, from (9.41) we obtain:

$$\forall k \geq 1 \quad \|\Pi_M v_{n_k}(t_{n_k} + \cdot) - \Pi_M \psi(\cdot)\|_{C([0,M];E)} = \|\varphi_k - \varphi\|_{C([0,M];E)} \geq \varepsilon,$$

that contradicts with (9.42). We reason in the same way when $M = 0$.

Thus, the set $\mathscr{P}$ constructed in (9.40) is the trajectory attractor in the trajectory space $\mathscr{K}_+$ with respect to the topology of $C^{loc}(\mathbb{R}_+; E)$.

Let $\mathscr{A}$ be the global attractor from Theorem 9.5, $\mathscr{P}$ be the trajectory attractor from Theorem 9.6. From previous sections results we have:

$$\mathscr{A} \text{ is a compact set in the space } E; \tag{9.43}$$

$$\mathscr{P} \text{ is a compact set in the space } C^{loc}(\mathbb{R}_+; E); \tag{9.44}$$

$$\mathscr{P} = \Pi_+ \mathscr{K} = \{y \in \mathscr{K}_+ \mid y(t) \in \mathscr{A} \ \forall t \in \mathbb{R}_+\} = \{y \in \mathscr{K}_+, \mid y(0) \in \mathscr{A}\}, \tag{9.45}$$

where $\mathscr{K}$ is the family of all complete trajectories of the inclusion (9.31), Π_+ is the restriction operator on $\mathbb{R}_+$. Moreover,

$$\mathscr{K} \text{ is a compact in the space } C^{loc}(\mathbb{R}; E); \tag{9.46}$$

$$\forall \xi(\cdot) \in \mathscr{K} \text{ and } \forall t \in \mathbb{R}\ \xi(t) \in \mathscr{A}; \tag{9.47}$$

$$\forall y_0 \in \mathscr{A} \text{ and } \forall t_0 \in \mathbb{R}\ \exists y(\cdot) \in \mathscr{K} :\ \ y(t_0) = y_0. \tag{9.48}$$

For any $y \in \mathscr{K}$ let us set

$$\mathscr{H}(y) = \mathrm{cl}_{C^{loc}(\mathbb{R};E)}\{y(\cdot + s) |\, s \in \mathbb{R}\} \subset C^{loc}(\mathbb{R}; E) \cap L^{\infty}(\mathbb{R}; E).$$

Such family is said to be the hull of function $y(\cdot)$ in $\Xi = C^{loc}(\mathbb{R}; E)$.

Definition 9.4 The function $y(\cdot) \in \Xi$ is said to be translation-compact (tr.-c.) in Ξ if the hull $\mathscr{H}(y)$ is compact in Ξ.

Definition 9.5 The family $\mathscr{U} \subset \Xi$ is said to be translation-compact, if $\mathscr{H}(\mathscr{U}) = \mathrm{cl}_{\Xi}\{y(\cdot + s)|\ y(\cdot) \in \mathscr{U},\ \ s \in \mathbb{R}\}$ is a compact in Ξ.

From the autonomy of problem (9.31) and (9.46) it follows that

Corollary 9.1 *$\mathscr{K}$ is translation-compact set in Ξ.*

From autonomy of system (9.31), applying the Arzelá-Ascoli compactness criterion, we obtain the translation compactness criterion for the family $\mathscr{K}$:

a) the set $\{y(t)|\, y \in \mathscr{K}\}$ is a compact in E $\forall t \in \mathbb{R}$;

b) there exists a positive function $\alpha(s) \to 0+$ $(s \to 0+)$ such that

$$\|y(t_1) - y(t_2)\|_E \leq \alpha(|t_1 - t_2|)\ \forall y \in \mathscr{K} \text{ and } \forall t_1, t_2 \in \mathbb{R}.$$

Similarly if we set $\Xi_+ = C^{loc}(\mathbb{R}_+; E)$ we obtain:

Corollary 9.2 *$\mathscr{P}$ is translation-compact set in Ξ_+.*

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