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Exponentiated Distributions

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Exponentiated Distributions

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To my late father, an educator who taught me how to love and seek knowledge

Essam K. AL-Hussaini

To my wife Masuda

Mohammad Ahsanullah

Preface

Motivated by the fact that adding one or more parameters to a distribution function makes it richer and more flexible to analyzing data, this book is an attempt to collect results, using such distributions, that are useful in theory and practice. Furthermore, the book is devoted to explore some properties of exponentiated distributions (EDs) and their use in statistical inference. New results are obtained that may be added to the existing results.

There are several ways of adding one or more parameters to a distribution function. The simplest way is probably by exponentiating a cumulative distribution function G by a positive real number α.

The ED G^{α} is quite different from the baseline distribution G and needs special investigation. Exponentiating a distribution function by a positive parameter goes back to Gompertz (1825) and Verhulst (1838, 1845, 1847). In many cases, while G accommodates for only monotone hazard rate functions, the ED G^{α} accommodates for both monotone and non-monotone hazard rate functions. Special attention is paid to applications in reliability and life testing.

Chapter 1 is an introductory chapter which includes a historical note and preview, generalized order statistics, the uses of asymmetric loss functions, MCMC, Bayes prediction, and mixtures of EDs.

Inferences (estimations and predictions) and their properties using a general ED G^{α} are discussed in Chap. 2.

In Chap. 3, G is specified to be Weibull, so that the properties and inference of the exponentiated Weibull (EW) distribution are presented. In this chapter, related distributions to the EW distribution and applications are also provided.

In Chap. 4, G is specified to be exponential, so that the properties and inference of the exponentiated exponential (EE) distribution are presented. In this chapter, characterization of the EE distribution is given.

In Chap. 5, G is specified to be Burr type XII, so that the properties and inference of the exponentiated Burr XII (EBXII) distribution are presented. In this chapter, applications, related distributions, and beta-Burr XII distribution are introduced.

Chapter 6 discusses the properties and inference of finite mixtures of exponentiated distributions. Particular emphasis is made when the components are EE distributions.

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Contents

Abbreviations

BHRF	Bathtub hazard rate function
BSEL	Balanced square error loss
CDF	Cumulative distribution function
CHRF	Constant hazard rate function
DHRF	Decreasing hazard rate function
DIDHRF	Decreasing-increasing-decreasing hazard rate function
GOS	Generalized order statistic
HRF	Hazard rate function
IDIHRF	Increasing-decreasing-increasing hazard rate function
IHRF	Increasing hazard rate function
LF	Likelihood function
LINEX	Linear-exponential (loss function)
MCMC	Markov chain Monte Carlo
MGF	Moment generating function
MLE	Maximum likelihood estimate
MSE	Mean square error
OOS	Ordinary order statistic
PDF	Probability density function
PRHRF	Proportional reversed hazard rate function
QUADREX	Quadratic-exponential (loss function)
SBM	Standard Bayes method
SEL	Square error loss
SF	Survival function
UBHRF	Upside down hazard rate function
UORV	Upper ordinary record value

Chapter 1
Class of Exponentiated Distributions Introduction

Contents

1.1 Historical Note and Preview

There are several ways of adding one or more parameters to a distribution function. Such an addition of parameters makes the resulting distribution richer and more flexible for modeling data. A positive parameter was added to a general survival function (SF) by Marshall and Olkin (1997). In their consideration of a countable mixture of Pascal(r, p) mixing proportion and positive integer powers of SF, a SF with two extra parameters was obtained in AL-Hussaini and Ghitany (2005). A new family of distributions as a countable mixture with Poisson added parameter was obtained by AL-Hussaini and Gharib (2009).

Adding a parameter by exponentiation goes back to Gompertz (1825), Verhulst (1838, 1845, 1847). Gompertz suggested the use of a cumulative distribution function (CDF)

$$F_1(t) = \exp(-\alpha \mathrm{e}^{-t/\sigma}), \quad -\infty < t < \infty, \tag{1.1.1}$$

which is an exponentiated extreme value distribution $\exp(-\mathrm{e}^{-t/\sigma})$ by α to graduate mortality tables.

Verhulst (1838) raised his logistic CDF $(1 + \rho \mathrm{e}^{-t/\sigma})^{-1}$ to a positive power α and used

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$$F_2(t) = (1 + \rho e^{-t/\sigma})^{-\alpha}, \quad -\infty < t < \infty, \tag{1.1.2}$$

in Verhulst (1845) and the exponentiated exponential CDF

$$F_3(t) = (1 - \rho e^{-t/\sigma})^{\alpha}, t > \sigma \ln \rho, \tag{1.1.3}$$

in Verhulst (1847) to represent population growth.

If in (1.1.3), $x = t/\sigma, \rho = 1, \beta = 1/\sigma$ and $F = F_3$ then

$$F(x) = (1 - e^{-\beta x})^{\alpha}, x > 0. \tag{1.1.4}$$

This is the form that will be used throughout the book. For details on the exponentiated Gompetrz-Verhulst family, see Ahuja and Nash (1967).

Ahuja and Nash (1967) and several other authors, call such distributions 'generalized' distributions. The word *exponentiated* rather than *generalized* may be more expressive since the latter word could be confused with other generalized concepts. On the other hand, it is not clear why an addition of a parameter or more should 'generalize' a distribution although the resulting distribution belongs to the same family with different parameter space.

The exponentiated distribution (ED) G^{α} is flexible enough to accommodate, in many cases, for both monotone as well as non-monotone hazard rates. In fact, EDs G^{α} are quite different from G and need special investigation. For example, if G is exponential such that $G(x) = 1 - \exp(-\beta x)$, then its corresponding PDF $g(x) = \beta \exp(-\beta x)$ is monotone decreasing on the positive half of the real line. However, $H(x) = [1 - \exp(-\beta x)]^{\alpha}$ has PDF $h(x) = \alpha\beta \exp(-\beta x)[1 - \exp(-\beta x)]^{\alpha - 1}$, which is unimodal on $[0, \infty)$ with mode at $(\ln x)/\beta$. Furthermore, while the exponential distribution G has constant hazard rate β, it can be shown that the exponentiated exponential (EE) H has increasing hazard rate (IHR), if $\alpha > 1$, constant hazard rate (CHR), if $\alpha = 1$, and decreasing hazard rate (DHR) if $\alpha < 1$.

The function

$$H(x) \equiv H(x|\theta) = [G(x|\beta)]^{\alpha} = [G(x)]^{\alpha}, \tag{1.1.5}$$

where $G(x)$ is a CDF and α is a positive real number, $\theta = (\alpha, \beta) \in \Omega$, β is a (vector) of parameters of G and Ω is a parameter space, is known as *exponentiated distribution*, since G is exponentiated by α. It is also known as *proportional reversed hazard rate model* (PRHRM). Notice that the reversed hazard rate function of H is defined by

$$\lambda_H^*(x) = \frac{d}{dx}[\ln H(x)] = \frac{h(x)}{H(x)}.$$

Hence, $\lambda_H^*(x) = \frac{\alpha[G(x)]^{\alpha-1} g(x)}{[G(x)]^{\alpha}} = \alpha \lambda_G^*(x)$.

So that the PRHRF of H is proportional to the PRHRF of G with proportionality parameter α.

If the power α is a positive integer, the ED is also known as *Lehmann alternatives*, due to Lehmann (1953), who defined the model as a non-parametric class of alternatives. See Gupta and Gupta (2007).

Remarks

1. If, in (1.1.5), $\alpha = 1$, then $H(x) = G(x)$, the non-exponentiated distribution, which is also known as *baseline* distribution.
2. If $\alpha = N$, a positive integer, $N = 1, 2, \ldots$, then $H(x) = [G(x)]^N$ is the CDF of the maximum of a random sample of size N drawn from a population whose CDF is G.
3. Any CDF H can be written in terms of its hazard rate function (HRF), denoted by $\lambda(x)$ and its PRHRF $\lambda^*(x)$ as follows

$$H(x) = \frac{\lambda(x)}{\lambda(x) + \lambda^*(x)}.$$

So that the SF and PDF are given by

$$R(x) = \frac{\lambda^*(x)}{\lambda(x) + \lambda^*(x)} \text{ and } h(x) = \lambda(x)R(x) = \frac{\lambda(x)\lambda^*(x)}{\lambda(x) + \lambda^*(x)}.$$

4. AL-Hussaini (2011), constructed a new class of distributions by compounding the PDF corresponding to the exponentiated SF, given by $R_H(x) = [R_G(x)]^\alpha$ with the gamma PDF, given by $\pi(\alpha) = \frac{b_2^{b_1}}{\Gamma(b_1)} \alpha^{b_1 - 1} e^{-b_2 \alpha}$. The resulting PDF takes the form

$$\begin{aligned} h^*(x) &= \int_0^\infty h(x|\alpha)\pi(\alpha) d\alpha \\ &= \int_0^\infty \{\alpha[R_G(x)]^{\alpha-1} g(x)\} \frac{b_2^{b_1}}{\Gamma(b_1)} \{\alpha^{b_1-1} e^{-b_2\alpha}\} d\alpha \\ &= \frac{b_2^{b_1}}{\Gamma(b_1)} \frac{g(x)}{R_G(x)} \int_0^\infty \alpha^{b_1} e^{-[b_2 - \ln R_G(x)]\alpha} d\alpha \\ &= \frac{b_1}{b_2} \lambda_G(x) \left[1 - \frac{\ln R_G(x)}{b_2}\right]^{-b_1-1}, \end{aligned}$$

where $\lambda_G(x) = \frac{g(x)}{R_G(x)}$.

An extension to the multivariate case was given by the following theorem:

Theorem [AL-Hussaini (2011)] *A multivariate PDF of the random vector* $X = (X_1, \ldots, X_n)$ *can be constructed by compounding L with* π *, where*

$$L(\alpha, x) = \prod_{i=1}^{n} h_{X_i|\alpha}(x_i|\alpha),$$

$$h_{X_i|\alpha}(x_i|\alpha) = \theta_i \alpha [R_G(x_i)]^{\theta_i \alpha - 1} g(x_i), \quad i = 1, \ldots, n$$

and $\pi(\alpha)$ *is the gamma PDF as given before, so that*

$$L(\alpha, x) = \alpha^n [\prod_{i=1}^{n} \theta_i \lambda_G(x_i)] \exp[\alpha \sum_{i=1}^{n} \theta_i \ln R_G(x_i)].$$

Therefore

$$\begin{aligned} h^*(x_1, \ldots, x_n) &= \int_0^{\infty} L(\alpha; x) \pi(\alpha) d\alpha \\ &= [\prod_{i=1}^{n} \theta_i \lambda_G(x_i)] \frac{b_2^{b_1}}{\Gamma(b_1)} \int_0^{\infty} \alpha^{n+b_1-1} \exp[-\alpha\{b_2 - \sum_{i=1}^{n} \theta_i \ln R_G(x_i)\}] d\alpha \\ &= \frac{\Gamma(b_1+n)}{\Gamma(b_1)} [\prod_{i=1}^{n} \gamma_i \lambda_G(x_i)][1 - \sum_{i=1}^{n} \gamma_i \ln R_G(x_i)]^{-b_1-n}, \quad x_i > 0, \end{aligned}$$

where $\gamma_i = \theta_i / b_2, \lambda_G(x_i) = g(x_i)/R_G(x_i)$.

1.2 Generalized Order Statistics

Kamps (1995a, b) suggested a unification of several important concepts that were used separately in statistical literature such as ordinary order statistics (OOS), upper ordinary record values, kth records, Pfeifer records, sequential order statistics and progressive type II censored order statistics. He called this unification generalized order statistics (GOS). For details and survey, see Kamps (1995a, b), Cramer and Kamps (2001), Ahsanullah and Nevzorov (2001), Cramer (2002), AL-Hussaini (2004).

The PDF of the first r GOSs $X_1^*, \ldots, X_r^*$ in a random sample of size n drown from a population whose CDF, SF and PDF are $H(.), R_H(.)$ and $h(.)$, is given by

$$f_{X_1^*,\ldots,X_r^*}(x_1,\ldots,x_r) = C_{r-1}\left[\prod_{i=1}^{r-1}[R_H(x_i)]^{m_i}h(x_i)\right][R_H(x_r)]^{\gamma_r-1}h(x_r),$$

$$H^{-1}(0)<x_1<\cdots<x_r<H^{-1}(1).$$

where $X_r^* \equiv X(r,n,k,\tilde{m})$, $C_{r-1}=\prod_{i=1}^{r}\gamma_i, r=1,\ldots,n-1, \gamma_i = k+n-i+M_i$, $M_i=\sum_{j=i}^{n-1}m_j, \tilde{m}=(m_1,\ldots,m_{n-1})$, $n\geq 2$, $1\leq i\leq n-1, k$ and m_j are real numbers such that $k\geq 1$.

An important special case is that in which $m_1,=\ldots=m_{n-1}=m$. Cramer (2002) calls this case m-GOS. In this case, the PDF of $X_r^*\equiv X(r,n,k,m)$, the rth m-GOS, can be shown to be

$$f_{X_r^*}(x) = \frac{C_{r-1}}{(r-1)!}[R(x)]^{\gamma_r-1}h(x)[\xi_m(H(x)]^{r-1}, \quad x\in A, \tag{1.2.1}$$

$\gamma_r = k+(m+1)(n-r)$, A is the set on which $f_{X_r^*}(x)$ is positive and

$$\xi_m(H(x)) = \begin{cases} \{1-[R(x)]^{m+1}\}/(m+1), & m\neq -1,\\ -\ln[R(x)], & m=-1.\end{cases}$$

A special case is that of *OOS*, in which k = 1 and $m=0$. So that $C_{r-1}=\frac{n!}{(n-r)!}$. The PDF of the rth OOS, denoted by $X_{r:n}$, is then given by

$$f_{X_{r:n}}(x) = \frac{1}{B(r,n-r+1)}[R(x)]^{n-r}h(x)(1-[R(x)])^{r-1}. \tag{1.2.2}$$

Another special case is that of the *ordinary upper record value* (OURV), in which k = 1 and $m=-1$. In this case, $\gamma_r=1$ and $C_r=1$, for all r. So that the PDF of X_r^* is then given by

$$f_{X_r^*}(x) = \frac{1}{(r-1)!}h(x)[-\ln R(x)]^{r-1}. \tag{1.2.3}$$

More on GOSs can be found in Kamps (1995a). On ordered random variables, see Ahsanullah and Nevzorov (2001), Ahsanullah and Raqab (2007).

1.3 Why Use Asymmetric Loss Functions?

Before answering this question, it should be remarked that Bayes analysis and statistical decisions are so strongly related that it would be "somewhat unnatural to learn one without the other" Berger (1985).

Several books, and articles, have been published in Bayes (and empirical Bayes) theory using different loss functions. Examples of such books are Savage (1954), Jefferys (1961), Lindley (1965), De Groot (1970), Martz and Waller (1982), Maritz and Lwin (1989), Bernardo and Smith (1994), Press (2003) and many other references (articles and books).

A recent book on Bayes statistics has been written by Savchuk and Tsokos (2013). The book answers general questions on Bayes theory and discusses parametric, quasiparametric and nonparametric Bayes estimation and applied some of the results in reliability.

The square error loss (SEL) function is given by

$$\zeta(\Delta) = d\Delta^2 = d[\hat{u}(\theta) - u(\theta)]^2,$$

where d is a positive constant, usually taken to be 1, $\Delta = \hat{u}(\theta) - u(\theta)$, $u(\theta)$ is the function of θ to be estimated and $\hat{u}(\theta)$ is the SEL estimate of $u(\theta)$. In this case, it is well known that the Bayes estimate of $u(\theta)$, based on SEL function, is the posterior mean, given by

$$\hat{u}_{SEL}(\theta) = E[u(\theta)|\underline{x}] = \int \ldots \int u(\theta)\pi(\theta|\underline{x})d\theta_1 \ldots d\theta_m, \tag{1.3.1}$$

where the integrals are taken over the m-dimensional parameter space, $\pi(\theta|\underline{x})$ is the posterior PDF of the vector of parameters given the vector of observations. For given prior PDF $\pi(\theta)$, the posterior PDF $\pi(\theta|\underline{x})$ is given by

$$\pi(\theta|\underline{x}) \propto L(\theta, \underline{x})\pi(\theta) \tag{1.3.2}$$

where $L(\theta, \underline{x})$ is a likelihood function (LF).

The SEL function has probably been the most popular loss function used in literature. The symmetric nature of the SEL function gives equal weight to over- and under-estimation of the parameters under consideration. However, in life testing, over estimation may be more serious than under estimation or vice versa. The 1986 disaster of the space shuttle Challenger, which was participated by the overestimation of the reliability of key space shuttle components, serves as a dramatic example, Feynman (1988), Dalal et al. (1989).

Research has been directed towards *asymmetric* loss functions. Varian (1975) suggested the use of linear-exponential (LINEX) loss function to be of the form

$$\zeta(\Delta) = b[e^{\kappa\Delta} - \kappa\Delta - 1], \tag{1.3.3}$$

where $|\kappa| \neq 0, b > 0$ and Δ is as before, in which $\hat{u}(\theta)$ is the LINEX estimate of $u(\theta)$.

Notice that for $\Delta = \hat{u}(\theta) - u(\theta) = 0$, $\zeta(\Delta) = 0$. For a > 0, the loss declines almost exponentially for $\Delta = \hat{u}(\theta) - u(\theta) > 0$ and rises approximately linearly when $\Delta = \hat{u}(\theta) - u(\theta) < 0$. For $\kappa < 0$, the reverse is true. By expanding $e^{\kappa\Delta}$, $\zeta(\Delta)$

can be approximated to the SEL function when $\Delta = \hat{u}(\theta) - u(\theta)$ is small. Without loss of generality, b may be taken to be equal to 1.

Using LINEX loss function, the Bayes estimate of u(θ) is given by

$$\hat{u}_{LNX}(\theta) = -\frac{1}{\kappa}\ln E[\mathrm{e}^{-\kappa u(\theta)}|\underline{x}] = -\frac{1}{\kappa}\ln[\int \ldots \int \mathrm{e}^{-\kappa u(\theta)}\pi(\theta|\underline{x})d\theta_1 \ldots d\theta_m]. \quad (1.3.4)$$

Thompson and Basu (1996) generalized the LINEX loss function to the squared-exponential (SQUAREX) loss function of the form

$$\zeta(\Delta) = b[\mathrm{e}^{\kappa\Delta} + c\Delta^2 - \kappa\Delta - 1], \quad (1.3.5)$$

where $c \neq 0, \kappa, b$ and Δ are as before. The SQUAREX loss function reduces to the LINEX loss function if c = 0. If $\kappa = 0$, the SQUAREX loss function reduces to the SEL function, given by (1.3.3).

The Bayes estimate of u(θ), based on SQUAREX loss function is given by

$$\hat{u}_{SQ}(\theta) = \hat{u}_{LIN}(\theta) + \frac{1}{\kappa}\ln[1 + \frac{2c}{\kappa}\{\hat{u}_{SEL}(\theta) - \hat{u}_{SQ}(\theta)\}]. \quad (1.3.6)$$

Other asymmetric loss functions were suggested. Among which is that of Zellner (1994) who introduced the notion of a balanced squared error loss function in the context of a general linear model to reflect both goodness of fit and precision of estimation.

1.4 Markov Chain Monte Carlo (MCMC) Method

To use the MCMC method in computing Bayes estimates of $\alpha, R(x_0), \lambda(x_0)$, at specific value of x_0, we first notice that the general problem is in evaluating the integral $E_\pi[\phi(\theta)] = \int \phi(\theta)\pi(\theta|\underline{x})d\theta$, assuming that $\int|\phi(\theta)|\pi(\theta|\underline{x})d\theta < \infty$, where $\pi(\theta|x)$ is the posterior PDF of θ given data. The vector θ could be of high dimension. This leads to difficulties in the computation of the integral. The high dimensionality of the vector θ remained a problem even with the approximation forms for the computation of the integral, that were suggested, for example, by Lindley (1980), Tierney and Kadane (1986). High dimensionality of the vector is clearly obtained when the distribution considered is finite mixture. On Markov chain for exploring posteriors, see Tierney (1994).

Markov chain Monte Carlo (MCMC) algorithms such as Metropolis-Hastings algorithms (and Gibbs sampler as a special case) have become extremely popular in statistics. The Metropolis-Hastings algorithm was named after Metropolis et al. (1953), Hastings (1970). Its application in statistics started with the early nineties.

MCMC approaches are so named because one uses the previous values to randomly generate the next sample value, generating a Markov chain (as the transition probabilities between sample values are only a function of the most recent sample values).

If we can draw samples $\theta^{(1)}, \ldots, \theta^{(N)}$ from $\pi(\theta|x)$, then Monte Carlo integration allows us to estimate this expectation by the average: $\hat{\phi}_N = \frac{1}{N}\sum_{i=1}^{N} \phi(\theta^{(i)})$. If we generate samples using a Markov chain (aperiodic, irreducible and has a stationary distribution with PDF $\pi(\theta|x)$), then by the ergodic theorem $\hat{\phi}_N \to E_\pi[\phi(\theta)]$, as $N \to \infty$. The estimate $\hat{\phi}_N$ is called an *argodic average*. Also for such chains, if the variance of $\phi(\theta)$ is finite, the central limit theorem holds and convergence occurs geometrically. Early iterations $\theta^{(1)}, \ldots, \theta^{(M)}$, reflect starting value $\theta^{(0)}$. These iterations are called burn-in. After the burn-in, we say that the chain has 'converged'. The burn-in are omitted from the ergodic averages to end up with

$$\hat{\phi} = \frac{1}{N - M} \sum_{i=M+1}^{N} \phi(\theta^{(i)}). \tag{1.4.1}$$

Methods for determining M are called *convergence diagnostics*. For details on the MCMC, see Cowles and Carlin (1996), Gelman and Rubin (1992), Roberts et al. (1997), Gamerman and Lopes (2006).

Associated Bayesian methods based on MCMC tools and novel model diagnostic tools to perform inference based on fully specified models are discussed in Sinha et al. (2008).

The data set is analyzed by applying the provided Gibbs sampler and Metropolis-Hesting algorithm using: WinBUGS1.4.

(http://www.mrc-bsu.cam.ac.uk/bugs/winbugs/shtml), which can be downloaded.

1.5 Bayes Prediction

The general problem of prediction may be described as that of inferring the values of unknown observables (future observations), known as *future sample*, or functions of such variables, from current available observations, known as *informative sample*. According to Geisser (1993), inferring about realizable values not observed based on values that were observed, is the primary purpose of statistical endeavor. The problem of prediction can be solved fully within Bayes framework (Geisser 1993). Bernardo and Smith (1994) stated that: "inference about parameters is thus seen to be a limiting form of predictive inference about observables".

Different sampling schemes are used in prediction. For example, Dunsmore (1974) suggested the *one-sample scheme* to be such that the first r order statistics in a random sample of size n drawn from a population whose CDF H (.), is considered to be the informative sample. The future sample is the set of the remaining order statistics.

In the *two-sample scheme*, two independent samples are assumed to be drawn from the same population, each is ordered and one of the two samples is considered as the informative sample and the other sample as the future sample.

Lingappaiah (1979) suggested an extension to a series of M + 1 independent samples. The aim is to predict a statistic in a future sample based on earlier samples (or stages).

Prediction has its uses in a variety of disciplines such as medicine (medical prognoses, antibiotic assays and preoperative medical diagnosis), engineering (machine tool replacement, quality control and maximization of the yield of an industrial process) and business (determining the difference in future mean performance of competing products and the provision of warranty limits for the future performance of a specified number of systems). For details on the history of statistical prediction, analysis and examples, see Aicheson and Dunsmore (1975), Geisser (1993).

Prediction was reviewed by Patel (1989), Nagaraja (1995), Kaminsky and Nelson (1998), AL-Hussaini (2000).

1.6 Mixtures of Exponentiated Distribution Functions

The study of homogeneous populations with 'single component' distributions was the main concern of statisticians along history. However, Newcomb (1886), Pearson (1894) were two pioneers who approached heterogeneous populations which can be represented by a 'finite mixture' of distributions.

Suppose that $F(x|\theta)$ is a CDF of x given θ and $Q(\theta)$ is a CDF of θ. The function $H(x)$, defined by

$$H(x) = \int_{-\infty}^{\infty} F(x|\theta) dQ(\theta), \tag{1.6.1}$$

was called by Fisher (1936) *compound distribution* of F and Q. Teicher (1960) called H a mixture of F and Q. The function $F(x|\theta)$ is known as *kernel* and $Q(\theta)$ as *mixing distribution*. If the entire mass of the corresponding measure of Q is confined to only a finite number of points $\theta_1, \ldots, \theta_k$, then (1.6.1) becomes a *finite mixture* of k components whose CDF is defined by

$$H(x) = \sum_{j=1}^{k} F(x|\theta_j) G(\theta_j). \tag{1.6.2}$$

To simplify notation, write $F_j(x) \equiv F(x|\theta_j)$ and $p_j \equiv G(\theta_j)$, so that (1.6.2) becomes

$$H(x) = \sum_{j=1}^{k} p_j F_j(x), \tag{1.6.3}$$

where $p_j \geq 0$ and $\sum_{j=1}^{k} p_j = 1$.

In (1.6.3), p_j is known as the jth *mixing proportion* and $F_j(x)$ the jth *component.*

If $F_j(x)$ is absolutely continuous for all j, so that $f_j(x)$ is the corresponding PDF, then the PDF $h(x)$ of a finite mixture of k components, is given by

$$h(x) = \sum_{j=1}^{k} p_j f_j(x). \tag{1.6.4}$$

If $R_j(x)$ is the jth SF, then the SF $R(x)$ of the mixture is given by

$$R(x) = \sum_{j=1}^{k} p_j R_j(x). \tag{1.6.5}$$

It can be shown that the HRF $\lambda_H(x)$ of the mixture H can be written, in terms of the HRFs of the components $\lambda_j(x), j = 1, \ldots, k$ as follows

$$\lambda_H(x) = \sum_{j=1}^{k} \omega_j(x)\lambda_j(x), \tag{1.6.6}$$

where

$$\omega_j(x) = \frac{p_j R_j(x)}{R(x)}, \quad j = 1, \ldots, k. \tag{1.6.7}$$

Notice that $\sum_{j=1}^{k} \omega_j(x) = 1$.

If the components are exponentiated distributions, then (1.6.3) and (1.6.4) become, respectively

$$H(x) = \sum_{j=1}^{k} p_j F_j(x) = \sum_{j=1}^{k} p_j [G_j(x)]^{\alpha},$$
$$h(x) = \sum_{j=1}^{k} p_j f_j(x) = \alpha \sum_{j=1}^{k} p_j [G_j(x)]^{\alpha-1} g_j(x)$$

where $G_j(x)$ is the jth baseline CDF and $g_j(x)$ is its PDF.

The SF and HRF are the same as given by (1.6.5) and (1.6.6) in which the jth SF and HRF are given by $R_j(x) = 1 - [G_j(x)]^{\alpha}$ and $\lambda_j(x) = \frac{\alpha G_j(x)]^{\alpha-1} g_j(x)}{1-[G_j(x)]^{\alpha}}$. Also $\omega_j(x)$ is given by

$$\omega_j(x) = \frac{p_j\{1 - [G_j(x)]^\alpha\}}{\sum_{i=1}^k p_i\{1 - [G_i(x)]^\alpha\}}, \quad j = 1, 2, \ldots, k.$$

Details and applications of finite mixtures can be found in Everitt and Hand (1981), Titterington et al. (1985), McLachlan and Basford (1988), McLachlan and Peel (2000).

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Chapter 2
Basic Properties, Estimation and Prediction Under Exponentiated Distributions

Contents

2.1 Introduction

The class of distributions $\Im = \{H : H(x) = [G(x)]^{\alpha}\}$, where H is defined by (1.1.5), shall be called class of exponentiated distributions. The PDF and SF, corresponding to H, are given, respectively, by

$$h(x) = \alpha[G(x)]^{\alpha-1} g(x), \tag{2.1.1}$$

where $g(x)$ is the PDF corresponding to $G(x)$ and

$$R_H(x) = 1 - [G(x)]^{\alpha}. \tag{2.1.2}$$

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In terms of survival functions (SFs) $R_H(x) = 1 - H(x)$ and $R_G(x) = 1 - G(x)$, corresponding to $H(x)$ and $G(x)$, we could either have

$$R_H(x) = 1 - [1 - R_G(x)]^{\alpha}, \tag{2.1.3}$$

or simply write

$$R_H(x) = [R_G(x)]^{\alpha}. \tag{2.1.4}$$

Notice that SF (2.1.3) corresponds to CDF (1.1.5) whereas SF $R_H(x)$, in (2.1.4), is obtained by exponentiating SF $R_G(x)$ by α. If, in (2.1.3), $\alpha = N$, a positive integer, the SF of the minimum of N independently, identically distributed (iid) random variables from G is obtained.

Cramer and Kamps (1996) were concerned with obtaining and studying the properties of the model parameters in a sequential k-out-of-n structure based on the exponentiated SF (2.1.4) after indexing the parameter α by $i, i = 1, \ldots, n$ and writing (2.1.4) in terms of the CDF's. That is

$$H_i(x) = 1 - [1 - G(x)]^{\alpha_i}, i = 1, \ldots, n. \tag{2.1.5}$$

The specific choice of CDF's, as given by (2.1.5), in the definition of sequential order statistics with CDF G and positive real numbers $\alpha_1, \ldots, \alpha_n$ leads to the following important cases [see Cramer and Kamps (1996)]:

(i) $\alpha_1 = \cdots = \alpha_i \Rightarrow$ ordinary order statistics.
(ii) $\alpha_i = \gamma_i/(n - i + 1) \Rightarrow$ generalized order statistics.
(iii) $\alpha_i = k/(n - i + 1) \Rightarrow$ kth record values.
(iv) $\alpha_i = (N + 1 - i - \sum_{\ell=1}^{i-1} R_\ell)/(n - i + 1) \Rightarrow$ progressive type II censoring, where $\alpha_i = \gamma_i/(n - i + 1)$ and $N = n + \sum_{\ell=1}^{n} R_\ell, (R_1, \ldots, R_n)$ is the censoring scheme at the beginning of the experiment.

Nagaraja and Hoffman (2001) used (1.1.5) as a record model and described the exact as well as the asymptotic distributions of the inter-arrival times of upper record values from the G^{α} record model when $\{X_n, n \geq 1\}$ is a sequence of independent random variables such that $X_n \sim G^{\alpha}$. Hoffman and Nagaraja (2000) studied the model in which $X_i \sim G^{\alpha_i}, i \geq 1$, are independent random variables assuming that the number of observations is random and independent of the observations and that the $\alpha_i' s$ are positive constants. They called this model a *random* G^{α} *model*. Hoffman and Nagaraja (2002) introduced the *random power record model* where for every $n \geq 1$, the joint CDF of $X_1, \ldots, X_n$ of a sequence $\{X_n, n \geq 1\}$ of random variables, not necessarily independent nor identically distributed, is given by

$$H_n(x_1, \ldots, x_n) = E\{G^{\alpha_1}(x_1), \ldots, G^{\alpha_n}(x_n)\}, x_i \in R, i = 1, \ldots, n,$$

where the expectation is taken with respect to the $\alpha' s$ which are assumed to be almost sure finite positive random variables. They established a hierarchical relationship between several previously studied record models and showed that this model incorporates all of them.

2.2 Properties of the Exponentiated Class of Distributions

Motivated by the fact that any (absolutely continuous) SF $R_G(x)$ can be written in the form

$$R_G(x) \equiv R_G(x;\beta) = \exp[-u(x;\beta)] \equiv \exp[-u(x)], 0 \le a < x < b \le \infty, \qquad (2.2.1)$$

where $u(x) = -\ln R_G(x)$, is such that $R_G(x)$ is a SF, so that $u(x)$ is a continuous, monotone increasing, differentiable function of x such that $u(x) \to 0$ as $x \to a^+$ and $u(x) \to \infty$ as $x \to b^-$, in which a and b are real numbers that may assume the values 0 and ∞, respectively. We shall use (2.2.1), sometimes, instead of the direct use of $G(x)$.

Remarks

1. The expression for $R_G(x)$, given by (2.2.1), holds true for any distribution defined over the whole real line if a is allowed to assume the value $-\infty$. However, we shall restrict the domain to the positive half of the real line, or subset of it, as given in (2.2.1), which is more appropriate for x to be used as time variable.
2. Class (2.2.1) includes all SF's with positive support (or subset of it). In particular, it includes among others, the Weibull (exponential and Rayleigh as special cases), compound Weibull (or Burr type XII), (compound exponential (or Lomax) and compound Rayleigh as special cases), beta, Pareto I, Gomperz and compound Gompertz SF's.
3. Although (2.1.3) and (2.1.4) are both exponentiated models, they are not quite the same. Substituting (2.2.1) in (2.1.3) and (2.1.4), we obtain, for $x > 0$,

$$R_H(x) = 1 - \{1 - \exp[-u(x)]\}^{\alpha} \qquad (2.2.2)$$

and

$$R_H(x) = \exp[-\alpha u(x)]. \qquad (2.2.3)$$

Notice that SF (2.2.3) is of the same form as that given by (2.1.4) with, say, $u^*(x) = \alpha u(x)$. So, we shall concentrate on the class of SFs (2.2.2).

4. It is easy to see that if $Z = -\ln G(X)$, where X has CDF $H(x) = [G(x)]^{\alpha}$ then Z has the exponential distribution with HRF α.

5. Suppose that $H(x) = [G(x)]^{\alpha}$. Gupta et al. (1998) showed that: if $\alpha > 1$ and G admits increasing HRF, then F admits increasing HRF and if $\alpha < 1$ and G admits decreasing HRF, then F admits decreasing HRF.

2.2.1 Moments

The CDF and PDF corresponding to SF (2.2.2) are given, for $x > 0$, by

$$H(x) = (1 - \exp[-u(x)])^{\alpha}, \tag{2.2.4}$$

and

$$h(x) = \alpha u'(x) \exp[-u(x)]\{1 - \exp[-u(x)]\}^{\alpha-1}. \tag{2.2.5}$$

It can be shown that the ℓth moment of a random variable X following CDF (2.2.4) is given by

$$E(X^{\ell}) = \ell \sum_{j=1}^{v} c_j I_j(\ell), \tag{2.2.6}$$

where

$$v = \begin{cases} \alpha = 1, 2, 3, \ldots \\ \infty, \quad \alpha \,\text{is a positive non-integral value}, \end{cases} \tag{2.2.7}$$

$$\begin{aligned} c_j &= (-1)^{j-1}\alpha(\alpha-1)\ldots(\alpha-j+1)/j!, \\ I_j(\ell) &= \int_0^{\infty} x^{\ell-1} \exp[-ju(x)]dx. \end{aligned} \tag{2.2.8}$$

In the non-exponentiated case ($\alpha = N = 1$), the ℓth moment of $H(x) = 1 - \exp[-u(x)]$ is

$$E(X^{\ell}) = \ell I_1(\ell) = \ell \int_0^{\infty} x^{\ell-1} \exp[-u(x)]dx.$$

Result (2.2.6) can be shown by observing that integration by parts yields

$$E(X^{\ell}) = \ell \int_0^{\infty} x^{\ell-1} R_H(x)dx,$$

where, from (2.2.2), $R_H(x) = \sum_{j=1}^{N} c_j \exp[-ju(x)]$, c_j is given by (2.2.8) when $\alpha = N$ is an integer ≥1 and $R_H(x) = \sum_{j=1}^{\infty} c_j \exp[-ju(x)]$, c_j is given by (2.2.8) when α takes positive non-integral values. Substituting $R_H(x)$ in the integral of $E(X^\ell)$, we obtain (2.2.6).

Remark

1. If $j - 1 = i$, then (2.2.6) becomes $E(X^\ell) = \ell \sum_{i=0}^{v-1} c_i I_i(\ell)$,

   where

$$c_i = (-1)^i \alpha(\alpha - 1)\ldots(\alpha - i)/(i + 1)! = \frac{\alpha}{i+1} c_i^*,$$
$$c_i^* = (-1)^i (\alpha - 1)\ldots(\alpha - i)/i!,$$
$$I_i(\ell) = \int_0^\infty x^{\ell-1} \mathrm{e}^{-(i+1)u(x)} dx$$

Therefore $E(X^\ell) = \ell\alpha \sum_{i=0}^{v-1} \frac{c_i^*}{i+1} I_i(\ell)$.

For example, if $u(x) = \beta x$ (the exponential baseline distribution) and $v = \alpha$ is a positive integer, then

$$E(X^\ell) = \frac{\ell!\alpha}{\beta^\ell} \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha - 1}{i} \left(\frac{1}{i+1}\right)^{\ell+1}.$$

This result agrees with that given in Gradshteln and Ryshlik (1980), p. 1077.

Table (2.1) gives the ℓth moment $E(X^\ell)$ for some members of class $\Im$, where, for $j = 1, \ldots, v$, c_j is given by (2.2.8) and v by (2.2.7). The letter E preceding the name of the distribution stands for exponentiated. The letter C for compound, W for Weibull, Ray for Rayliegh, Par 1 for Pareto type1 and Gomp for Gompertz.

2.2.2 *Quantiles*

The quantile x_q of the absolutely continuous distribution (2.2.4) is given by

$$x_q = u^{-1}[-\ln(1 - q^{1/\alpha})], \tag{2.2.9}$$

where $u^{-1}(.)$ is the inverse function of $u(.)$. This is true since the quantile is the value of x_q satisfying $q = H(x_q) = \{1 - \exp[-u(x_q)]\}^\alpha$. Table (2.2) displays the medians of some members of the exponentiated class $\Im$. It may be observed that in the non-exponentiated case $(\alpha = 1)$ the median reduces to $median = u^{-1}(\ln 2)$.

Table 2.1 ℓth moments of some members of the exponentiated class $\Im$

Distribution	$u(x)$	ℓth moment
EW$(\alpha, \beta_1, \beta_2)$	$\beta_1\, x^{\beta_2}$	$\frac{\Gamma(1+\ell/\beta_2)}{\beta_1^{\ell/\beta_2}} \sum_{j=1}^{v} \left(\frac{c_j}{j^{\ell/\beta_2}}\right)$
EE(α, β)	βx	$\frac{\Gamma(1+\ell)}{\beta^{\ell}} \sum_{j=1}^{v} \left(\frac{c_j}{j^{\ell}}\right)$
ERay(α, β)	βx^2	$\frac{\Gamma(1+\ell/2)}{\beta^{\ell/2}} \sum_{j=1}^{v} \left(\frac{c_j}{j^{\ell/2}}\right)$
ECW(Burr XII) $(\alpha, \beta_1, \beta_2, \delta)$	$\beta_2 \ln(1+\delta x^{\beta_1})$	$\frac{\Gamma(1+\ell/\beta_1)}{\delta^{\ell/\beta_1}} \sum_{j=1}^{v} \left(\frac{c_j \Gamma(j\beta_2-\ell/\beta_1)}{\Gamma(j\beta_2)}\right)$
ECE(ELomax) (α, β, δ)	$\beta \ln(1+\delta x)$	$\frac{\Gamma(1+\ell)}{\delta^{\ell}} \sum_{j=1}^{v} \left(\frac{c_j \Gamma(j\beta_2-\ell)}{\Gamma(j\beta_2)}\right)$
ECRay(α, β, δ)	$\beta \ln(1+\delta x^2)$	$\frac{\Gamma(1+\ell/2)}{\delta^{\ell/2}} \sum_{j=1}^{v} \left(\frac{c_j\, \Gamma(j\beta_2-\ell/2)}{\Gamma(j\beta_2)}\right)$
EPar I$(\alpha, \beta_1, \beta_2)$	$-\ln(x/\beta_1)^{\beta_2}, (x > \beta_1^{1/\beta_2})$	$\ell \beta_1^{\beta_2+\ell/\beta_2} \sum_{j=1}^{v} \left(\frac{c_j}{(j\beta_2-\ell)\beta_1^{j}}\right)$
EBeta(α, β)	$-\ln(1-x^{\beta}), 0<x<1$	$\Gamma(1+\ell/\beta) \sum_{j=1}^{v} \left(\frac{c_j \Gamma(1+j)}{\Gamma(1+j+\ell/\beta)}\right)$
EGomp$(\alpha, \beta_1, \beta_2)$	$\beta_1 \exp(\beta_2 x)$	$\frac{1}{\beta_2^{\ell}} \sum_{j=1}^{v} c_j I_j(\ell)$[a]
ECGomp$(\alpha, \beta_1, \beta_2, \delta)$	$\delta \ln\left[1+\frac{e^{\beta_2 x}-1}{\beta_1 \beta_2}\right]$	$\ell \sum_{j=1}^{v} c_j I_j^{*}(\ell)$[b]

[a] $I_j(\ell) = \int_0^{\infty} \left[\ln(1+\frac{z}{j\beta_1}\right]^{\ell} e^{-z} dz, \quad z = j\beta_1(e^{\beta_2 x} - 1).$

[b] $I_j^{*}(\ell) = \int_0^{\infty} \frac{x^{\ell-1}}{\left[(\beta_1\delta-1)+e^{-\beta_2 x}\right]^{j\ell}} dx.$

2.2.3 Mode

The logarithm of PDF (2.2.5) is given by

$$\ln h(x) = \ln \alpha + \ln u'(x) - u(x) + (\alpha - 1) \ln\, [1 - e^{-u(x)}].$$

So that

$$\frac{h'(x)}{h(x)} = \frac{u''(x)}{u'(x)} - u'(x) + (\alpha - 1)\left[\frac{u'(x)e^{-u(x)}}{1 - e^{-u(x)}}\right] \Rightarrow$$
$$0 = u''(x)(1 - e^{-u(x)}) - [u'(x)]^2[1 - (\alpha - 1)e^{-u(x)}]$$

The value of x which satisfies this equation is a mode of the PDF, given by (2.2.5).

Table 2.2 Medians of some members of the exponentiated class $\Im$

Distribution	u(x)	$u^{-1}(y)$	Median
$\mathrm{EW}(\alpha, \beta_1, \beta_2)$	$\beta_1 x^{\beta_2}$	$(y/\beta_1)^{1/\beta_2}$	$\left[\ln\left(1-2^{-1/\alpha}\right)^{-1/\beta_1}\right]^{1/\beta_2}$
$\mathrm{EE}(\alpha, \beta)$	βx	y/β	$\ln\left(1-2^{-1/\alpha}\right)^{-1/\beta}$
$\mathrm{ERay}(\alpha, \beta)$	βx^2	$(y/\beta)^{1/2}$	$\left[\ln\left(1-2^{-1/\alpha}\right)^{-1/\beta}\right]^{1/2}$
$\mathrm{ECW(Burr\ XII)}(\alpha, \beta_1, \beta_2, \delta)$	$\beta_2 \ln(1+\delta x^{\beta_1})$	$[\frac{1}{\delta}(e^{y/\beta_2}-1)]^{1/\beta_1}$	$\left[\left\{\left(1-2^{-1/\alpha}\right)^{-1/\beta_2}-1\right\}/\delta\right]^{1/\beta_1}$
$\mathrm{ECE(ELomax)}(\alpha, \beta, \delta)$	$\beta \ln(1+\delta x)$	$\frac{1}{\delta}(e^{y/\beta}-1)$	$\left\{\left(1-2^{-1/\alpha}\right)^{-1/\beta}-1\right\}/\delta$
$\mathrm{ECRay}(\alpha, \beta, \delta)$	$\beta \ln(1+\delta x^2)$	$[(e^{y/\beta}-1)]^{1/2}$	$\left[\left\{\left(1-2^{-1/\alpha}\right)^{-1/\beta}-1\right\}/\delta\right]^{1/2}$
$\mathrm{EPar\,I}(\alpha, \beta_1, \beta_2)$	$-\ln(x/\beta_1)^{\beta_2}, x > \beta_1^{1/\beta_2}$	$\beta_1^{1/\beta_2} e^{y/\beta_2}$	$\beta_1^{1/\beta_2}\left[1-2^{-1/\alpha}\right]^{-1/\beta_2}$
$\mathrm{EBeta}(\alpha, \beta)$	$-\ln(1-x^{\beta}), (0<x<1)$	$(1-e^{-y})^{1/\beta}$	$2^{-1/(\alpha\beta)}$
$\mathrm{EGomp}(\alpha, \beta_1, \beta_2)$	$\beta_1 e^{\beta_2 x}$	$[\ln(1+y/\beta_1)]/\beta_2$	$\ln\left[1-\ln\left(1-2^{-1/\alpha}\right)^{1/\beta_1}\right]^{1/\beta_2}$
$\mathrm{ECGomp}(\alpha, \beta_1, \beta_2, \delta)$	$\delta \ln\left[1+\frac{e^{\beta_2 x}-1}{\beta_1\beta_2}\right]$	$[\ln\{1+\beta_1\beta_2(e^{x/\beta_2}-1)\}]/\beta_2$	$\ln\left[1+\beta_1\beta_2\left\{\left(1-2^{-1/\alpha}\right)^{-1/\delta}-1\right\}\right]^{1/\beta_2}$

2.2.4 Hazard Rate Function

The hazard rate function (HRF) corresponding to the exponentiated CDF (1.1.5) is given, for $x > 0$, by

$$\lambda_H(x) = \frac{h(x)}{R_H(x)} = \frac{\alpha[G(x)]^{\alpha-1} g(x)}{1-[G(x)]^{\alpha}} = \alpha[1- \in_{\alpha}(x)]\lambda_G(x), \tag{2.2.10}$$

where $\lambda_G(x) = g(x)/R_G(x)$ and $\in_{\alpha}(x) = \frac{1-G^{\alpha-1}(x)}{1-G^{\alpha}(x)}$.

Notice that, since $G(x)$ is a CDF on $[0, \infty)$, then

If $0<\alpha<1$, then $-\infty < \in_{\alpha}(x) \leq 1 \Rightarrow 1- \in_{\alpha}(x) \geq \frac{1}{\alpha} \Rightarrow \lambda_H(x) \geq \lambda_G(x)$.

If $\alpha \geq 1$, then $\frac{\alpha-1}{\alpha} \leq \in_{\alpha}(x) \leq 1 \Rightarrow 0 \leq \alpha[1- \in_{\alpha}(x)] \leq 1 \Rightarrow 0 \leq \lambda_H(x) \leq \lambda_G(x)$. $\in_{\alpha}(0) = 1$ and $\in_{\alpha}(\infty) = \lim_{x\to\infty}\left[\frac{1-G^{\alpha-1}(x)}{1-G^{\alpha}(x)}\right] = \frac{\alpha-1}{\alpha}$. So that, $\frac{\alpha-1}{\alpha} \leq \in_{\alpha}(x) \leq 1$, for all $x \in [0, \infty)$. Hence, $0 \leq \alpha[1- \in_{\alpha}(x)] \leq 1$.

By differentiating $\lambda_H(x)$, given by (2.2.10) with respect to x and simplifying, it can be shown that, provided that $G(x)g'(x) < g^2(x)$,

H has an increasing hazard rate (IHR), if:

$$G^{\alpha}(x) > 1 - \frac{\alpha}{1-\{G(x)g'(x)/g^2(x)\}}. \tag{2.2.11}$$

H has a decreasing hazard rate (DHR), if:

$$G^{\alpha}(x) < 1 - \frac{\alpha}{1-\{G(x)g'(x)/g^2(x)\}}. \tag{2.2.12}$$

If equality holds, then critical points at which extrema for $H(x)$ may be obtained and so other shapes for the HRF of $H(x)$ are expected to take place.

2.2.5 Proportional Reversed Hazard Rate Function

The proportional reversed hazard rate function (PRHRF) of H, denoted by $\lambda_H^*(x)$ is defined by

$$\lambda_H^*(x) = \frac{d}{dx}[\ln H(x)] = \frac{h(x)}{H(x)}.$$

It may be noticed, from (2.2.10), that the HRF $\lambda_H(x)$ of H(x) is not proportional to the HRF $\lambda_G(x)$ of G(x). However, the PRHRF $\lambda_H^*(x)$ of H(x) can be seen to be proportional to the PRHRF $\lambda_G^*(x)$ of G(x). In fact,

$$\lambda_H^*(x) = \frac{h(x)}{H(x)} = \frac{\alpha[G(x)]^{\alpha-1}g(x)}{[G(x)]^{\alpha}} = \alpha\lambda_G^*(x). \tag{2.2.13}$$

This is why the exponentiated model is equivalently called PRHRM.

It may also be noted that $\lambda_H^*(x)$ dx provides the probability of failing in (x − dx, x), when a unit is found failed at time x. In general, the PRHRF has been found to be useful in estimating the SF for left censored data.

It can be seen that the CDF $H(x)$ can be written, in terms of the HRF $\lambda_H(x)$ and PRHRM $\lambda_H^*(x)$ of H as follows

$$H(x) = \frac{\lambda_H(x)}{\lambda_H(x) + \lambda_H^*(x)} \tag{2.2.14}$$

From which, the SF and PDF are given, respectively, by

$$R(x) = \frac{\lambda_H^*(x)}{\lambda_H(x) + \lambda_H^*(x)} \quad \text{and} \quad h(x) = \lambda_H(x)R(x) = \frac{\lambda_H^*(x)\lambda_H(x)}{\lambda_H(x) + \lambda_H^*(x)}.$$

2.2.6 Density Function of the rth m-Generalized Order Statistic

The PDF of the rth m-GOS based on an absolutely continuous CDF $H(x)$, whose SF is $R_H(x)$ and PDF is $h(x)$, and positive numbers $\gamma_1, \ldots, \gamma_r$ is given by (1.2.1).

The following theorem gives an expression for the PDF of the rth m-GOS based on an exponentiated distribution.

Theorem 2.1 *The PDF of the* rth *m-GOS based on an exponentiated distribution, whose CDF* $H(x) = [1 - e^{-u(x)}]^\alpha$ *and SF* $R_H(x) = 1 - [1 - e^{-u(x)}]^\alpha$, *is given from* (1.2.1) *in case* $(m \neq -1)$, *by*

$$f_{X_r^*}(x) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \left(1 - [1 - e^{-u(x)}]^\alpha\right)^{\gamma_r - 1} \alpha u'(x) e^{-u(x)} [1 - e^{-u(x)}]^{\alpha - 1} \left(1 - [1 - e^{-u(x)}]^{(m+1)\alpha}\right)^{r-1}. \tag{2.2.15}$$

where C_{r-1}, γ_r *are as given in Chap.* 1.

In the case $m = -1, \gamma_i = k, C_{r-1} = k^r$, *the PDF of the rth OURV is given by*

$$f_{X_r^*}(x) = \frac{k^r}{(r-1)!} [1 - (1 - e^{-u(x)})^\alpha]^{k-1} \alpha u'(x) e^{-u(x)} (1 - e^{-u(x)})^{\alpha - 1} [-\ln\{1 - (1 - e^{-u(x)})^\alpha\}]^{r-1}. \tag{2.2.16}$$

The PDF of the rth ordinary order statistic (OOS) is obtained by setting $k = 1$ *and* $m = 0$ in (2.2.15), or equivalently by the direct use of (1.2.2), to get

$$\begin{aligned} f_{X_{r:n}}(x) &= \frac{1}{B(r, n-r+1)} \left(1 - [1 - e^{-u(x)}]^\alpha\right)^{n-r} \alpha u'(x) e^{-u(x)} [1 - e^{-u(x)}]^{\alpha-1} (1 - e^{-u(x)})^{\alpha(r-1)}. \\ &= \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} \frac{1}{B(r, n-r+1)} (1 - e^{-u(x)})^{\alpha(j+r)-1} \alpha u'(x) e^{-u(x)}, \\ &= \sum_{j=0}^{n-r} \omega_j h_j^*(x), \end{aligned} \tag{2.2.17}$$

where

$$\omega_j = \frac{(-1)^j n!}{(n-r-j)!(r-1)! j! (r+j)} \tag{2.2.18}$$

and

$$h_j^*(x) = \alpha(r+j) u'(x) e^{-u(x)} (1 - e^{-u(x)})^{\alpha(r+j)-1}. \tag{2.2.19}$$

Also, from (2.2.16), the PDF of OURV is obtained by setting $k = 1$ and $m = -1$ to get

$$f_{X_r^*}(x) = \frac{1}{(r-1)!} \alpha u'(x) e^{-u(x)} (1 - e^{-u(x)})^{\alpha-1} [-\ln\{1 - (1 - e^{-u(x)})^\alpha\}]^{r-1}. \tag{2.2.20}$$

(i) An s-out-of-n structure functions if at least s of its components function, or equivalently, that the life of the s-out-of-n structure is the (n − s + 1) largest of the component lifetimes. So, if r is replaced by n − s + 1 in the PDF of the rth order statistic (2.2.18), we obtain the PDF of life of an s-out-of-n structure.

$$f_{n-s+1:n}(x) = \binom{n}{n-s+1}(n-s+1)\alpha u'(x)\mathrm{e}^{-u(x)} \left[1-\mathrm{e}^{-u(x)}\right]^{\alpha(n-s+1)-1}\left[1-\left(1-\mathrm{e}^{-u(x)}\right)^{\alpha}\right]^{s-1}. \tag{2.2.21}$$

(ii) The PDFs of a series (n-out-of-n) and parallel (1-out of n) structures are obtained, for $x > 0$, from (2.2.21), respectively, as follows:

$$f_{1:n}(x) = n\alpha u'(x)\mathrm{e}^{-u(x)}(1-\mathrm{e}^{-u(x)})^{\alpha-1}\left[1-(1-\mathrm{e}^{-u(x)})^{\alpha}\right]^{n-1}. \tag{2.2.22}$$

$$f_{n:n}(x) = n\alpha u'(x)\mathrm{e}^{-u(x)}(1-\mathrm{e}^{-u(x)})^{n\alpha-1}. \tag{2.2.23}$$

Notice that in the non-exponentiated case $(\alpha = 1), f_{1:n}(x) = nu'(x)\mathrm{e}^{-nu(x)}$ and $f_{n:n}(x) = nu'(x)\mathrm{e}^{-u(x)}(1-\mathrm{e}^{-u(x)})^{n-1}$, which agree with the PDFs of the minimum and maximum order statistics based on a population with CDF $1-\mathrm{e}^{-u(x)}$.

(iii) Expression (2.2.17) agrees with the expression obtained by Sarabia and Castillo (2005), for the PDF of the rth order statistic, with the appropriate parameters. This expression makes it easy to obtain the corresponding CDF, SF and moments.

(iv) Mudholkar and Hutson (1996) obtained asymptotic distributions of the extreme order statistics $X_{1:n}$ and $X_{n:n}$ and the extreme spacings $X_{2:n} - X_{1:n}$ and $X_{n::n} - X_{n-1:n}$.

2.3 Estimation of $\alpha, R(x_0), \lambda(x_0)$ (All Parameters of G are Known)

2.3.1 Maximum Likelihood Estimation of $\alpha, R(x_0), \lambda(x_0)$

In this section, the parameter α, SF $R(x_0)$ and HRF $\lambda(x_0)$, at x_0, are estimated using the maximum likelihood (ML) and Bayes methods.

Suppose that α is the only unknown parameter (that is all of the parameters of G are known). We are going to show that an unbiased estimator $\hat{\alpha}$ of α which is also consistent and asymptotically efficient, is given by

$$\hat{\alpha} = (n-1)/\sum_{i=1}^{n} Z_i. \tag{2.3.1}$$

It may be noticed that the transformation $Z = -\ln G(X)$, where X is distributed as $H(X) = [G(X)]^{\alpha}$, transforms X to an exponential random variable $Z$with HRF α, denoted by Exp (α). In fact

$$\begin{aligned} F_Z(z) &= P[Z \le z] = P[-\ln G(X) \le z] = P[X > G^{-1}(\mathrm{e}^{-z})] = 1 - H_X[G^{-1}(\mathrm{e}^{-z})] \\ &= 1 - \mathrm{e}^{-\alpha z}, z > 0 \end{aligned}$$

Suppose that $X_1, < \cdots < X_r$ are the first r order statistics in a random sample of size n drawn from a population whose CDF is given by $H(X) = [G(X)]^{\alpha}$ (type II censoring). Let $Z_i = -\ln G(X_i)$, then $Z_{1:n} > \cdots > Z_{r:n}$ where $Z_{j:n}$ is the jth order statistic of a random sample $Z_1, \ldots, Z_n$ of size n from Exp (α). It then follows that the LF is given by

$$\begin{aligned} L(\alpha; \underline{z}) &\propto [\prod_{i=1}^{r} h(z_i)][H(z_r)]^{n-r} \\ &\propto [\prod_{i=1}^{r} \alpha \mathrm{e}^{-\alpha z_i}][1 - \mathrm{e}^{-\alpha z_r}]^{n-r} \\ &\propto \alpha^r \mathrm{e}^{-\alpha T}[1 - \mathrm{e}^{-\alpha z_r}]^{n-r}, \end{aligned} \tag{2.3.2}$$

where $\underline{z} = (z_1, \ldots, z_r)$ and

$$T = \sum_{i=1}^{r} Z_i = -\sum_{i=1}^{r} \ln G(X_i). \tag{2.3.3}$$

The log-likelihood function is given, from (2.3.2), by

$$\ell(\alpha, \underline{x}) \equiv \ln L(\alpha, \underline{x}) \propto r \ln \alpha - \alpha T + (n - r) \ln(1 - \mathrm{e}^{-\alpha z_r}).$$

Differentiating both sides with respect to α and then equating to zero, we obtain

$$\frac{r}{\alpha} - T - \frac{(n-r) z_r \mathrm{e}^{-\alpha z_r}}{1 - \mathrm{e}^{-\alpha z_r}} = 0. \tag{2.3.4}$$

The solution of (2.3.4) is the MLE $\tilde{\alpha}_{ML}$ of α. Such solution could not be obtained analytically and numerical solution may be necessary.

In the complete sample case ($r = n$), it follows, from (2.3.4), that

$$\tilde{\alpha}_{ML} = n/T. \tag{2.3.5}$$

where $Z_i = -\ln[G(X_i)]$ are independently, identically distributed random variables from the exponential distribution with parameter α. It then follows that $T = \sum_{i=1}^{n} Z_i$ has a gamma (n, α) distribution. Therefore,

$$E(\tilde{\alpha}_{ML}) = E(n/T) = \int_0^\infty \frac{n}{t} \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha t} dt = \frac{n}{n-1} \alpha$$

So that, from (2.3.5),

$$\hat{\alpha}_{ML} = \frac{n-1}{n} \tilde{\alpha} = \frac{n-1}{T}, \tag{2.3.6}$$

is unbiased for α. Furthermore, it can be shown that the distribution (1.1.5) belongs to the exponential class, so that $\sum_{i=1}^{n} \ln[G(x_i)]$ is sufficient and complete for α. The efficiency of the estimator [see, for example, Hogg et al. (2005), p. 324] is given by $e = \frac{RCLB}{V(\hat{\alpha})} = 1 - \frac{2}{n} \to 1, as\ n \to \infty$. Notice that Rao-Cramer lower bound ($RCLB$) is the reciprocal of n times Fisher information $I(\alpha)$, given by the variance of the score function. That is, $RCLB = \frac{1}{nI(\alpha)} = \frac{1}{nV(\partial \ln h/\partial \alpha)} = \frac{\alpha^2}{n}$ and it can be shown that the variance of $\hat{\alpha}$ is given by $V(\hat{\alpha}) = \frac{\alpha^2}{n-2}$. The estimator $\hat{\alpha}$ is unbiased, consistent estimator for α. It then follows that $\sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow{D} N(0, \alpha^2)$.

Remarks

1. If all of the parameters are unknown, the MLE's of the unknown parameters of G are obtained (by solving the likelihood equations involved) and then substituted in G to get $Z_i = -\ln[G(X_i)]$ and hence $\hat{\alpha}_{ML}$.
2. The invariance principle of MLEs can be used in estimating the SF and HRF by replacing the parameters by their estimates.
3. The above estimators of $\alpha, R(x_0), \lambda(x_0)$, may be of use when $G(x)$ is in 'standard form' or can be transformed to standard form, where all of its parameters are known and interest is in estimating α.
4. In the complete sample case, the MLE $\tilde{\alpha}_{ML} = -n/\sum_{i=1}^{n} \ln G(X_i)$ agrees with the result obtained by Gupta and Gupta (2007).

2.3.2 Bayes Estimation of $\alpha, R(x_0), \lambda(x_0)$

Assuming that the prior belief of the experimenter about α is gamma (b_1, b_2) with PDF

$$\pi(\alpha) \propto \alpha^{b_1-1}\mathrm{e}^{-b_2\alpha}, \alpha > 0, (b_1, b_2 > 0). \tag{2.3.7}$$

The posterior PDF is given, from (2.3.2) and (2.3.7) by

$$\begin{aligned} &\pi(\alpha|\underline{z}) \propto L(\alpha;\underline{z})\pi(\alpha) = A\alpha^{r+b_1-1}\mathrm{e}^{-(b_2+T)\alpha}[1-\mathrm{e}^{-\alpha z_r}]^{n-r} \\ &\Rightarrow \pi(\alpha|\underline{z}) = A\sum\nolimits_{j_1=0}^{n-r} C_{j_1}\alpha^{r+b_1-1}\exp[-T_{0j_1}\alpha], \end{aligned} \tag{2.3.8}$$

where

$$T_{0j_1} = b_2 + j_1 z_r + \sum_{i=1}^{r} z_i \tag{2.3.9}$$

and A is a normalizing constant, which can be shown to be given by

$$A = \frac{1}{\Gamma(r+b_1)S_0}, \tag{2.3.10}$$

$$S_0 = \sum_{j_1=0}^{n-r}\left(\frac{C_{j_1}}{T_{0j_1}^{r+b_1}}\right), C_{j_1} = (-1)^{j_1}\binom{n-r}{j_1} \tag{2.3.11}$$

and T_{0j_1} is given by (2.3.9).

Based on squared error loss function, the Bayes estimators of $\alpha, R(x_0), \lambda(x_0)$ were obtained in AL-Hussaini (2010a), using (1.3.1), as follows

$$\left.\begin{aligned} \hat{\alpha}_{SEL} &= E(\alpha|\underline{z}) = \frac{(r+b_1)S_1}{S_0}, \\ \hat{R}_{SEL}(z_0) &= E[R(z_0)|\underline{z}] = 1 - \frac{S_2}{S_0}, \\ \hat{\lambda}_{SEL}(z_0) &= E[\lambda(z_0)|\underline{z}] = \frac{(r+b_1)\lambda_G^*(z_0)S_3}{S_0} \end{aligned}\right\} \tag{2.3.12}$$

where S_0 and C_{j_1} are given by (2.3.11) and $\lambda_G^*(z_0) = \frac{g(z_0)}{G(z_0)}$,

$$S_1 = \sum_{j_1=0}^{n-r}\left(\frac{C_{j_1}}{T_{0j_1}^{r+b_1+1}}\right), S_2 = \sum_{j_1=0}^{n-r}\left(\frac{C_{j_1}}{T_{1j_1}^{r+b_1}}\right), S_3 = \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{\infty}\left(\frac{C_{j_1}}{T_{j_1,j_2}^{r+b_1+1}}\right), \tag{2.3.13}$$

$$T_{1j_1} = T_{0j_1} - \ln G(z_0), \tag{2.3.14}$$

$$T_{j_1,j_2} = T_{0j_1} - (j_2+1)\ln G(z_0). \tag{2.3.15}$$

For proof, see AL-Hussaini (2010a). This development shall be called ‘standard Bayes method’ (SBM).

Remarks

1. In the complete sample case, the Bayes estimator $\hat{\alpha}_{SEL} = \frac{n+b_1}{b_2 - \sum_{i=1}^{n} \ln G(X_i)}$, based on the SEL function, agrees with the result obtained by AL-Hussaini (2010b).
2. In the complete sample case, $\hat{\alpha}_{ML}$ and $\hat{\alpha}_{SEL}$ coincide for non-informative prior of α (the case in which $b_1 = b_2 = 0$).
3. It may be observed that $\tilde{\alpha}_{SEL} \to \tilde{\alpha}_{ML}$ as $n \to \infty$, indicating that $\tilde{\alpha}_{SEL}$ has the same properties as $\tilde{\alpha}_{ML}$ for large values of n.

The following theorem gives the Bayes estimates under the LINEX loss function, using standard Bayes method (SBM).

Theorem 2.2 *Based on LINEX loss function, the Bayes estimators of* $\alpha, R(x_0), \lambda(x_0)$ *are given, using* (1.3.4), *by the following*:

$$\hat{\alpha}_{LNX} = -\frac{1}{\kappa}\ln\int_0^\infty e^{-\alpha\kappa}\pi(\alpha|\underline{z})d\alpha = -\frac{1}{\kappa}\ln\left(\frac{S_1^*}{S_0}\right), \tag{2.3.16}$$

$$\hat{R}_{LNX}(x_0) = -\frac{1}{\kappa}\ln\int_0^\infty e^{-R(x_0)\kappa}\pi(\alpha|\underline{z})d\alpha = -\frac{1}{\kappa}\ln\left(\frac{S_2^*}{S_0}\right), \tag{2.3.17}$$

$$\hat{\lambda}_{LNX}(x_0) = -\frac{1}{\kappa}\ln\int_0^\infty e^{-\lambda(x_0)\kappa}\pi(\alpha|\underline{z})d\alpha = -\frac{1}{\kappa}\ln\left(\frac{S_3^*}{S_0}\right), \tag{2.3.18}$$

where S_0 *is given by* (2.3.11),

$$S_1^* = \sum_{j_1=0}^{n-r}\frac{C_{j_1}}{(\kappa + T_{0j_1})^{r+b_1}}, \tag{2.3.19}$$

$$S_2^* = \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{\infty} e^{-\kappa}\frac{\kappa^{j_2}}{j_2!}\frac{C_{j_1}}{[T_{0j_1} - j_2\ln G(z_0)]^{r+b_1}}, \tag{2.3.20}$$

$$S_3^* = \sum C\left(\frac{[\lambda_G^*(z_0)]^{j_2}\Gamma(r+b_1+j_2)}{\Gamma(r+b_1)[T_{0j_1} - (j_2+j_3)\ln G(z_0)]^{r+b_1+j_2}}\right), \tag{2.3.21}$$

where T_{0j_1} *is given by* (2.3.9),

$$\sum = \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} \quad \text{and} \quad C = C_{j_1}C_{j_2}C_{j_3} \tag{2.3.22}$$

C_{j_1} is given by (2.3.11),

$$C_{j_2} = \frac{(-1)^{j_2}\kappa^{j_2}}{j_2!}, \quad C_{j_3} = \binom{j_2 + j_3 - 1}{j_3}. \tag{2.3.23}$$

Proof Such estimators can be seen to be obtained, by using (1.3.4) and the posterior PDF (2.3.8), as follows:

$$\begin{aligned}
\hat{\alpha}_{LNX} &= -\frac{1}{\kappa}\ln\int_0^\infty \mathrm{e}^{-\alpha\kappa}\pi(\alpha|\underline{z})d\alpha \\
&= -\frac{1}{\kappa}\ln A\sum_{j_1=0}^{n-r} C_{j_1}\int_0^\infty \alpha^{r+b_1-1}\mathrm{e}^{-\alpha[T_{0j_1}+\kappa]}d\alpha \\
&= -\frac{1}{\kappa}\ln\left(\frac{S_1^*}{S_0}\right),
\end{aligned}$$

where A is given by (2.3.10), S_0 by (2.3.11) and S_1^* by (2.3.19).

$$\begin{aligned}
\hat{R}_{LNX}(z_0) &= -\frac{1}{\kappa}\ln\int_0^\infty \mathrm{e}^{-R_H(z_0)\kappa}\pi(\alpha|\underline{z})d\alpha, \\
&= -\frac{1}{\kappa}\ln\int_0^\infty \mathrm{e}^{-[1-\{G(z_0)\}^\alpha]\kappa}\pi(\alpha|\underline{z})d\alpha, \\
&= -\frac{1}{\kappa}\ln\sum_{j_2=0}^\infty \mathrm{e}^{-\kappa}\frac{\kappa^{j_2}}{j_2!}\int_0^\infty \mathrm{e}^{j_2\alpha\ln G(z_0)}\pi(\alpha|\underline{z})d\alpha, \\
&= -\frac{1}{\kappa}\ln\sum_{j_2=0}^\infty \mathrm{e}^{-\kappa}\frac{\kappa^{j_2}}{j_2!}\int_0^\infty \mathrm{e}^{j_2\alpha\ln G(z_0)}A\sum_{j_1=0}^{n-r}C_{j_1}\alpha^{r+b_1-1}\mathrm{e}^{-T_{0j_1}\alpha}d\alpha, \\
&= -\frac{1}{\kappa}\ln A\sum_{j_1=0}^{n-r}C_{j_1}\sum_{j_2=0}^\infty \mathrm{e}^{-\kappa}\frac{\kappa^{j_2}}{j_2!}\int_0^\infty \alpha^{r+b_1-1}\mathrm{e}^{-(T_{0j_1}-j_2\ln G(z_0))\alpha}d\alpha, \\
&= -\frac{1}{\kappa}\ln\left(\frac{S_2^*}{S_0}\right),
\end{aligned}$$

where S_0 is given by (2.3.11) and S_2^* by (2.3.20).

Finally, the LINEX estimator of $\lambda_H(x_0)$ is given by

$$\begin{aligned}
\hat{\lambda}_{LNX}(z_0) &= -\frac{1}{\kappa}\ln\int_0^\infty \mathrm{e}^{-\lambda(z_0)\kappa}\pi(\alpha|\underline{z})d\alpha \\
&= -\tfrac{1}{\kappa}\ln\int_0^\infty \sum_{j_2=0}^\infty \frac{[-\lambda(z_0)\kappa]^{j_2}}{j_2!}\pi(\alpha|\underline{z})d\alpha \\
&= -\tfrac{1}{\kappa}\ln\int_0^\infty \sum_{j_2=0}^\infty \frac{(-\kappa)^{j_2}}{j_2!}\left[\frac{\alpha[G(z_0)]^{\alpha-1}g(z_0)}{1-[G(z_0)]^\alpha}\right]^{j_2}\pi(\alpha|\underline{z})d\alpha \\
&= -\frac{1}{\kappa}\ln\sum_{j_2=0}^\infty [\lambda_G^*(z_0)]^{j_2}\frac{(-\kappa)^{j_2}}{j_2!}\int_0^\infty \alpha^{j_2}[G(z_0)]^{j_2\alpha}\{1-[G(z_0)]^\alpha\}^{-j_2}\pi(\alpha|\underline{z})d\alpha \\
&= -\frac{1}{\kappa}\ln\sum_{j_2=0}^\infty [\lambda_G^*(z_0)]^{j_2}\frac{(-\kappa)^{j_2}}{j_2!}\sum_{j_3=0}^\infty C_{j_3}\int_0^\infty \alpha^{j_2}[G(z_0)]^{(j_2+j_3)\alpha}\pi(\alpha|\underline{z})d\alpha
\end{aligned}$$

where $\lambda_G^*(z_0) = g(z_0)/[G(z_0)]$ and C_{j_3} is given by (2.3.23).

So that

$$\begin{aligned}
\hat{\lambda}_{LNX}(z_0) &= -\frac{1}{\kappa}\ln\sum_{j_2=0}^\infty\sum_{j_3=0}^\infty C_{j_3}[\lambda_G^*(z_0)]^{j_2}\frac{(-\kappa)^{j_2}}{j_2!}A\sum_{j_1=0}^{n-r}C_{j_1} \\
&\qquad \int_0^\infty \alpha^{r+b_1+j_2-1}\mathrm{e}^{-[T_{0j_1}-(j_2+j_3)\ln G(z_0)]\alpha}d\alpha \\
\hat{\lambda}_{LNX}(x_0) &= -\frac{1}{\kappa}\ln A\sum C[\lambda_G^*(x_0)]^{j_2}\cdot\frac{\Gamma(r+b_1+j_2)}{[T_{0j_1}-(j_2+j_3)\ln G(x_0)]^{r+b_1+j_2}}.
\end{aligned}$$

Therefore

$$\hat{\lambda}_{LIN}(x_0) = -\frac{1}{\kappa}\ln\left(\frac{S_3^*}{S_0}\right),$$

where S_3^* is given by (2.3.21), $\sum$ and C by (2.3.22). □

2.4 Estimation of $(\boldsymbol{\alpha}, \boldsymbol{\beta_1}, \ldots, \boldsymbol{\beta_k})$, $\boldsymbol{R_H(x_0)}$ and $\boldsymbol{\lambda_H(x_0)}$ (All Parameters of H are Unknown)

This section is devoted to the estimation of the vector of parameters (α, β), $R_H(x_0)$ and $\lambda_H(x_0)$, where $\beta = (\beta_1, \ldots, \beta_k)$ using the ML and Bayes methods.

2.4.1 Maximum Likelihood Estimation of $(\alpha, \beta_1, \ldots, \beta_k), R_H(x_0), \lambda_H(x_0)$

In this section, G is assumed to depend on k-dimensional vector $\beta = (\beta_1, \ldots, \beta_k)$ of unknown parameters. So that H will depend on the $(k+1)$-dimensional vector of unknown parameters (α, β). All parameters are assumed to be positive. In this case, the LF is given by

$$\begin{aligned} L(\theta; \underline{x}) &\propto [\prod_{i=1}^{r} h(x_i|\theta)][R_H(x_r|\theta)]^{n-r} \\ &\propto [\prod_{i=1}^{r} \alpha\{G(x_i|\beta)\}^{\alpha-1} g(x_i|\beta)][1 - \{G(x_r|\beta)\}^{\alpha}]^{n-r}, \end{aligned} \tag{2.4.1}$$

where $\underline{x} = (x_1, \ldots, x_r)$ are the first r order statistics, $\theta = (\alpha, \beta)$, $\beta = (\beta_1, \ldots, \beta_k)$,

So that the LLF, denoted by $\ell(\theta; \underline{x})$ is given by

$$\begin{aligned} \ell(\theta; \underline{x}) \equiv \ln L(\theta; \underline{x}) &= r \ln \alpha + (\alpha - 1) \sum_{i=1}^{r} \ln G(x_i|\beta) + \sum_{i=1}^{r} \ln g(x_i|\beta) \\ &+ (n-r) \ln[1 - \{G(x_r|\beta)\}^{\alpha}] \end{aligned} \tag{2.4.2}$$

The likelihood equations (LEs) are then given by

$$\frac{\partial \ell}{\partial \alpha} : 0 = \frac{r}{\alpha} + \sum_{i=1}^{r} \ln G(x_i|\beta) - \frac{(n-r)\{G(x_r|\beta)\}^{\alpha} \ln G(x_r|\beta)}{1 - \{G(x_r|\beta)\}^{\alpha}} \tag{2.4.3}$$

and for $j = 1, \ldots, k$,

$$\begin{aligned} \frac{\partial \ell}{\partial \beta_j} : 0 = (\alpha - 1) \sum_{i=1}^{r} \frac{1}{G(x_i|\beta)} \frac{\partial G(x_i|\beta)}{\partial \beta_j} + \sum_{i=1}^{r} \frac{1}{g(x_i|\beta)} \frac{\partial g(x_i|\beta)}{\partial \beta_j} \\ - \frac{(n-r)\alpha\{G(x_r|\beta)\}^{\alpha-1}}{1 - \{G(x_r|\beta)\}^{\alpha}} \frac{\partial G(x_r|\beta)}{\partial \beta_j} \end{aligned} \tag{2.4.4}$$

By solving this system of equations, we obtain the MLEs of $\alpha, \beta_1, \ldots, \beta_k$. The invariance property of MLEs can then be applied to obtain the MLEs of $R_H(x_0)$ and $\lambda_H(x_0)$, for some x_0, by replacing the parameters by their MLEs.

2.4.2 Bayes Estimation of $(\alpha, \beta_1, \ldots, \beta_k), R_H(x_0), \lambda_H(x_0)$

AL-Hussaini (2010a) gave expressions for the Bayes estimators of the parameters, SF and HRF of H, under the SEL function. In what follows, such estimators are obtained under LINEX loss function.

Theorem 2.3 *Suppose that the CDF G depends on an unknown k-dimensional vector of parameters* $\beta = (\beta_1, \ldots, \beta_k)$, *so that H depends on the* $(k+1)$ *unknown parameters* (α, β).

Given by

$$\pi(\theta) \equiv \pi(\alpha, \beta) = \pi_1(\alpha)\pi_2(\beta), \tag{2.4.5}$$

where

$$\pi_1(\alpha) \propto \alpha^{b_1 - 1} \exp(-b_2\alpha), \tag{2.4.6}$$

and $\pi_2(\beta)$ *is a k-variate PDF. Then*

1. *The LINEX estimators of the parameters are*

$$\hat{\alpha}_{LNX} = -\frac{1}{\kappa}\ln\left(S_1^{**}/S_0^{**}\right), \tag{2.4.7}$$

$$\hat{\beta}_{v,LNX} = -\frac{1}{\kappa}\ln\left(S_v^{**}/S_0^{**}\right), \quad v = 2, \ldots, k. \tag{2.4.8}$$

2. The LINEX estimators of the SF $R_H(x_0)$ and HRF $\lambda_H(x_0)$ are given, for some x_0, by

$$\hat{R}_{LNX}(x_0) = -\frac{1}{\kappa}\ln\left(S_{k+2}^{**}/S_0^{**}\right), \tag{2.4.9}$$

$$\hat{\lambda}_{LNX}(x_0) = -\frac{1}{\kappa}\ln\left(S_{k+3}^{**}/S_0^{**}\right), \tag{2.4.10}$$

where

$$S_0^{**} = \sum_{j_1=0}^{n-r} C_{j_1} I_0, \quad I_0 = \int_\beta \frac{w(\beta, \underline{x})\pi_2(\beta)}{[T_{0j_1}(x_r, \beta)]^{r+b_1}} d\beta, \tag{2.4.11}$$

$$S_1^{**} = \sum_{j_1=0}^{n-r} C_{j_1} I_1, \quad I_1, = \int_\beta \frac{e^{-T_0(\beta)}\pi_2(\beta)}{[\kappa + T_{0j_1}(\beta)]^{r+b_1}} d\beta, \tag{2.4.12}$$

$$S_v^{**} = \sum_{j_1=0}^{n-r} C_{j_1} I_v, \quad I_v = \int_\beta \frac{e^{-T_0(\beta)-\kappa\beta_v}\pi_2(\beta)}{[T_{0j_1}(\beta)]^{r+b_1}} d\beta, \quad v = 2, \ldots, k+1, \tag{2.4.13}$$

$$S_{k+2}^{**} = e^{-\kappa} \sum_{j_1=0}^{n-r} \sum_{j_2=0}^{\infty} \frac{C_{j_1}}{j_2!} I_{k+2}, I_{k+2} = \int_\beta \frac{e^{-T_0(\beta)}\pi_2(\beta)}{[T_{j_1,j_2}(\beta)]^{r+b_1}} d\beta, \tag{2.4.14}$$

$$\begin{aligned} S_{k+3}^{**} &= \sum_{j_1=0}^{n-r} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \frac{C_{j_1}(-\kappa)^{j_2}\Gamma(r+b_1+j_2)}{j_2!\Gamma(r+b_1)} I_{k+3}, \\ I_{k+3} &= \int_\beta \frac{e^{-T_0(\beta)}\pi_2(\beta)[\lambda^*(x_0)]^{j_2}}{[T_{j_1,j_2,j_3}(\beta)]^{r+b_1+j_2}} d\beta. \end{aligned} \tag{2.4.15}$$

$$T_0(\beta) = \sum_{i=1}^{r} \ln G(x_i|\beta) + \sum_{i=1}^{r} \ln g(x_i|\beta), \tag{2.4.16}$$

$$T_{0j_1}(\beta) = b_2 + T_{j_1}(\beta), \tag{2.4.17}$$

$$T_{j_1}(\beta) = -\left[\sum_{i=1}^{r} \ln G(x_i|\beta) + j_1 \ln G(x_r|\beta)\right], \tag{2.4.18}$$

$$T_{j_1,j_2}(\beta) = T_{0j_1} - (j_2+1) \ln G(x_0|\beta), \tag{2.4.19}$$

$$T_{j_1,j_2,j_3}(\beta) = T_{0j_1}(\beta) - (j_2+j_3) \ln G(x_0|\beta). \tag{2.4.20}$$

Proof The LF (2.4.1) can be written as

$$L(\theta; \underline{x}) \propto w(\beta) \sum_{j_1=0}^{n-r} C_{j_1} \alpha^r \exp[-\alpha T_{j_1}(\beta)], \tag{2.4.21}$$

where $\theta = (\alpha, \beta_1, \ldots, \beta_k), w(\beta) = \prod_{i=1}^r \lambda_G^*(x_i, \beta) = \prod_{i=1}^r \frac{g(x_i,\beta)}{G(x_i,\beta)}$ and $T_{j_1}(\beta)$ is given by (2.4.18).

The posterior PDF is given, from (2.4.5) and (2.4.21), by

$$\pi(\theta|\underline{x}) \propto L(\theta;\underline{x})\pi(\theta) = Aw(\beta)\pi_2(\beta)\sum_{j_1=0}^{n-r} C_{j_1}\alpha^{r+b_1-1}\exp[-\alpha T_{0j_1}(\beta)], \tag{2.4.22}$$

where A is a normalizing constant, given by

$$\begin{aligned} A &= \frac{1}{\Gamma(r+b_1)S_0^{**}}, \\ S_0^{**} &= \sum_{j_1=0}^{n-r} C_{j_1} I_0, \quad I_0 = \int_0^\infty \frac{w(\beta)\pi_2(\beta)}{[T_{0j_1}(\beta)]^{r+b_1}} d\beta, \end{aligned} \tag{2.4.23}$$

$T_{0j_1}(\beta)$ is given by (2.4.17) and $T_{j_1}(\beta)$ by (2.4.18).

It then follows, from (1.3.2), that

$$\begin{aligned} \hat{\alpha}_{LNX} &= -\frac{1}{\kappa}\ln\int_\beta\int_0^\infty \mathrm{e}^{-\kappa\alpha}\pi(\alpha,\beta|\underline{x})d\alpha d\beta \\ &= -\frac{1}{\kappa}\ln\left[\int_\beta\int_0^\infty \mathrm{e}^{-\kappa\alpha}\left(Aw(\beta)\pi_2(\beta)\sum_{j_1=0}^{n-r} C_{j_1}\alpha^{r+b_1-1}\exp[-\alpha T_{0j_1}(\beta)]\right)\right]d\alpha d\beta \\ &= -\frac{1}{\kappa}\ln\left[A\sum_{j_1=0}^{n-r} C_{j_1}\int_\beta w(\beta)\pi_2(\beta)\int_0^\infty \alpha^{r+b_1-1}\exp[-\alpha\{\kappa+T_{0j_1}(\beta)\}]\,d\alpha d\beta\right] \\ &= -\frac{1}{\kappa}\ln\left[A\Gamma(r+b_1)S_1^{**}\right] \\ &= -\frac{1}{\kappa}\ln\left[S_1^{**}/S_0^{**}\right] \end{aligned}$$

where S_0^{**} is given by (2.4.23),

$$S_1^{**} = \sum_{j_1=0}^{n-r} C_{j_1} I_1, \quad I_1 = \int_\beta \frac{w(\beta)\pi_2(\beta)}{[\kappa+T_{0j_1}(\beta)]^{r+b_1}} d\beta$$

For $v = 1, \ldots, k$, the LINEX Bayes estimator of β_v is given by

$$\begin{aligned}
\hat{\beta}_{v,LNX} &= -\frac{1}{\kappa}\ln \int_{\beta}\int_{0}^{\infty} \mathrm{e}^{-\kappa\beta_v}\pi(\alpha,\beta|\underline{x})d\alpha d\beta \\
&= -\frac{1}{\kappa}\ln\left[\int_{\beta}\int_{0}^{\infty} \mathrm{e}^{-\kappa\beta_v}\left(Aw(\beta)\pi_2(\beta)\sum_{j_1=0}^{n-r} C_{j_1}\alpha^{r+b_1-1}\exp[-\alpha T_{0j_1}(\beta)]\right)\right]d\alpha d\beta \\
&= -\frac{1}{\kappa}\ln\left[A\sum_{j_1=0}^{n-r} C_{j_1}\int_{\beta} \mathrm{e}^{-\kappa\beta_v}w(\beta)\pi_2(\beta)\int_{0}^{\infty}\alpha^{r+b_1-1}\exp[-\alpha T_{0j_1}(\beta)]d\alpha d\beta\right] \\
&= -\frac{1}{\kappa}\ln\left[A\Gamma(r+b_1)S_i^{**}\right] \\
&= -\frac{1}{\kappa}\ln\left[S_i^{**}/S_0^{**}\right]
\end{aligned}$$

where, for $i = 2, \ldots, k+1$,

$$S_i^{**} = \sum_{j_1=0}^{n-r} C_{j_1}I_i, \quad I_i = \int_{\beta}\frac{\mathrm{e}^{-\kappa\beta_i}w(\beta)\pi_2(\beta)}{[T_{0j_1}(\beta)]^{r+b_1}}d\beta$$

The Bayes estimator of $R_H(x_0)$, at some x_0, is given by

$$\hat{R}_{LNX}(x_0) = -\frac{1}{\kappa}\ln\int_{\beta}\int_{0}^{\infty} \mathrm{e}^{-R(x_0)\kappa}\pi(\alpha,\beta|\underline{x})\,d\beta d\alpha$$

Since $R(x_0) = 1 - [G(x_0)]^{\alpha}$, then

$$\begin{aligned}
\mathrm{e}^{-\kappa R(x_0)} = \mathrm{e}^{-\kappa\{1-[G(x_0)]^{\alpha}\}} &= \mathrm{e}^{-\kappa}\sum_{j_2=0}^{\infty}\frac{\kappa^{j_2}[G(x_0)]^{\alpha j_2}}{j_2!} \\
&= \mathrm{e}^{-\kappa}\sum_{j_2=0}^{\infty}\frac{\kappa^{j_2}\mathrm{e}^{\alpha j_2\ln[G(x_0)]}}{j_2!}.
\end{aligned}$$

Therefore,

$$\hat{R}_{LNX}(x_0) = -\frac{1}{\kappa}\ln\int\limits_{\beta}\int\limits_{0}^{\infty}\left(Aw(\beta)\pi_2(\beta)\sum_{j_1=0}^{n-r}C_{j_1}\alpha^{r+b_1-1}\exp[-\alpha T_{0j_1}(\beta)]\right)$$

$$e^{-\kappa}\sum_{j_2=0}^{\infty}\frac{\kappa^{j_2}e^{\alpha j_2\ln[G(x_0)]}}{j_2!}d\alpha d\beta$$

$$= -\frac{1}{\kappa}\ln Ae^{-\kappa}\sum_{j_1=0}^{n-r}\sum_{j_2=0}^{\infty}\frac{\kappa^{j_2}C_{j_1}}{j_2!}\int\limits_{\beta}w(\beta)\pi_2(\beta)$$

$$\int\limits_{0}^{\infty}\alpha^{r+b_1-1}e^{-\alpha[T_{0j_1}(\beta)-j_2\ln G(x_0)]}d\alpha d\beta$$

$$= -\frac{1}{\kappa}\ln Ae^{-\kappa}\sum_{j_1=0}^{n-r}\sum_{j_2=0}^{\infty}\frac{\kappa^{j_2}C_{j_1}}{j_2!}\int\limits_{\beta}w(\beta)\pi_2(\beta)\frac{\Gamma(r+b_1)}{[T_{j_1,j_2}(\beta)]^{r+b_1}}d\alpha d\beta$$

$$= -\frac{1}{\kappa}\ln\left(\frac{S_{k+2}^{**}}{S_0^{**}}\right)$$

where

$T_{j_1,j_2}(\beta)$ is given by (2.4.19) and S_0^{**} by (2.4.23).

The Bayes estimator of $\lambda_H(x_0)$, at some x_0, is given by $\hat{\lambda}_{LNX}(x_0) = -\frac{1}{\kappa}\ln\int\limits_{\beta}\int\limits_{0}^{\infty}e^{-\kappa\lambda(x_0)}\pi(\alpha,\beta|\underline{x})d\beta d\alpha$.

Since

$$\lambda_H(x_0) = \frac{h(x_0)}{R_H(x_0)} = \frac{\alpha[G(x_0)]^{\alpha-1}g(x_0)}{1-[G(x_0)]^{\alpha}} = \alpha\,\lambda^*(x_0)[G(x_0)]^{\alpha}\{1-[G(x_0)]^{\alpha}\}^{-1},$$

where $\lambda^*(x_0) = \frac{g(x_0)}{G(x_0)}$, then

$$e^{-\kappa\lambda_H(x_0)} = \sum_{j_2=0}^{\infty}\frac{[-\kappa\lambda_H(x_0)]^{j_2}}{j_2!}$$

$$= \sum_{j_2=0}^{\infty}\frac{(-\kappa)^{j_2}\alpha^{j_2}[\lambda^*(x_0)]^{j_2}[G(x_0)]^{\alpha j_2}}{j_2!}\sum_{j_3=0}^{\infty}C_{j_3}[G(x_0)]^{\alpha j_3}$$

$$= \sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}C_{j_2}C_{j_3}[\lambda^*(x_0)]^{j_2}\alpha^{j_2}e^{(j_2+j_3)\alpha\ln G(x_0)}$$

$$C_{j_2} = \frac{(-\kappa)^{j_2}}{j_2!},\; C_{j_3} = (-1)^{j_3}\binom{j_2+j_3-1}{j_3},\; \lambda^*(x_0) = \frac{g(x_0)}{G(x_0)}$$

Therefore,

$$\hat{\lambda}_{LNX}(x_0) = -\frac{1}{\kappa}\ln \int_{\beta}\int_{0}^{\infty} \left(Aw(\beta)\pi_2(\beta)\sum_{j_1=0}^{n-r} C_{j_1}\alpha^{r+b_1-1}\exp[-\alpha T_{0j_1}(\beta)]\right)$$

$$\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} C_{j_2}C_{j_3}[\lambda^*(x_0)]^{j_2}\,\alpha^{j_2}\mathrm{e}^{(j_2+j_3)\alpha\,\ln\,G(x_0)}d\alpha d\beta,$$

$$= -\frac{1}{\kappa}\,\ln\,A\sum_{j_1=0}^{n-r}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} C_{j_1}C_{j_2}C_{j_3}\int_{\beta} w(\beta)\pi_2(\beta)[\lambda^*(x_0)]^{j_2}I(\beta)d\beta$$

$$I(\beta) = \int_{0}^{\infty}\alpha^{r+b_1+j_2-1}\,\mathrm{e}^{-\alpha[T_{0j_1}(\beta)-(j_2+j_3)\,\ln\,G(x_0)]}d\alpha$$

$$= \frac{\Gamma(r+b_1+j_2)}{[T_{j_1,j_2,j_3}(\beta)]^{r+b_1+j_2}}.$$

So that

$$\hat{\lambda}_{LNX}(x_0) = -\frac{1}{\kappa}\ln A\sum C\int_{\beta} w(x_0)\pi_2(\beta)[\lambda^*(x_0)]^{j_2}$$

$$\frac{\Gamma(r+b_1+j_2)}{[T_{j_1,j_2,j_3}(\beta)]^{r+b_1+j_2}}d\beta$$

$$= -\frac{1}{\kappa}\ln\left(\frac{S^{**}_{k+3}}{S^{**}_0}\right).$$

where, for $k = 0, 1, 2, \ldots,$

$$S^{**}_{k+3} = \sum CI_{k+3}, I_{k+3} = \frac{\Gamma(r+b_1+j_2)}{\Gamma(r+b_1)}\int_{\beta}\frac{w(x_0)\,\pi_2(\beta)[\lambda^*(x_0)]^{j_2}}{[T_{j_1,j_2,j_3}(\beta)]^{r+b_1+j_2}}d\beta,$$

and $T_{j_1,j_2,j_3}(\beta)$ is given by (2.4.20) and S^{**}_0 by (2.4.23). □

2.5 Bayes One-Sample Prediction of Future Observables (All Parameters of H are Unknown)

2.5.1 One-Sample Scheme

AL-Hussaini (2010a) obtained 100 $(1-\tau)$ % predictive intervals based on the two-sample scheme. In the one-sample scheme, the informative sample consists of the first r order statistics $X_1 < \cdots < X_r$ of a random sample of size n drawn from a population whose CDF is $H(x)$. The future sample consists of the remaining order statistics $X_{r+1} < \cdots < X_n$. Let $Y_s = X_{r+s}, s = 1, \ldots, n-r$. Write $f_r(y_s|\theta)$ to denote the PDF of the sth unit to fail, given that the rth unit had already failed. Then

$$f_r(y_s|\theta) \propto [H(y_s|\theta) - H(x_r|\theta)]^{s-1}[1 - H(y_s|\theta)]^{n-r-s} [R(x_r|\theta)]^{-(n-r)} h(y_s|\theta)$$

The binomial expansion of each of the first two terms on the right-hand side then yields

$$f_r(y_s|\theta) \propto \sum_{j_1=0}^{s-1} \sum_{j_2=0}^{n-r-s} D_{j_1} D_{j_2} [H(y_s|\theta)]^{s-1-(j_1-j_2)} [H(x_r|\theta)]^{j_1} [R_H(x_r|\theta)]^{-(n-r)} h(y_s|\theta),$$

where
$\theta = (\alpha, \beta), \beta = (\beta_1, \ldots, \beta_k), D_{j_1}$ and D_{j_2} are given by

$$D_{j_1} = (-1)^{j_1} \binom{s-1}{j_1}, \quad D_{j_2} = (-1)^{j_2} \binom{n-r-s}{j_2}. \tag{2.5.1}$$

Substitution of $H(.|\theta) = [G(.|\beta)]^{\alpha})$ and $h(.|\theta) = \alpha G(.|\beta)]^{\alpha-1} g(.|\beta)$ then yields

$$\begin{aligned} f_r(y_s|\theta) &\propto \sum_{j_1=0}^{s-1} \sum_{j_2=0}^{n-r-s} D_{j_1} D_{j_2} [G(y_s|\beta)]^{\alpha[s-1-j_1+j_2)]} \\ &[G(x_r|\beta)]^{\alpha j_1} \{\alpha G(y_s|\beta)]^{\alpha-1} g(y_s|\beta)\} [R_H(x_r|\theta)]^{-(n-r)} \\ &\propto \lambda^*(y_s|\beta) \sum_{j_1=0}^{s-1} \sum_{j_2=0}^{n-r-s} D_{j_1} D_{j_2} \alpha e^{\alpha[s-j_1+j_2)] \ln G(y_s|\beta) + j_1 \ln G(x_r|\beta)]} \\ &[R_H(x_r|\theta)]^{-(n-r)}, \end{aligned} \tag{2.5.2}$$

where $\lambda^*(y_s|\beta) = \frac{g(y_s|\beta)}{G(y_s|\beta)]}$.

The posterior PDF is given, from (2.4.1) and (2.4.3), by

$$\begin{aligned}
\pi(\theta|\underline{x}) &\propto L(\theta;\underline{x})\pi(\theta) \propto [\prod_{i=1}^{r} h(x_i|\theta)][R_H(x_r|\theta)]^{n-r}\alpha^{b_1-1}\exp(-b_2\alpha)\pi_2(\beta) \\
&= \{\prod_{i=1}^{r} \alpha[G(x_i|\beta)]^{\alpha-1}g(x_i|\beta)\}\alpha^{b_1-1}\exp(-b_2\alpha)\pi_2(\beta)[R_H(x_r|\theta)]^{n-r} \\
&= [\prod_{i=1}^{r} \lambda^*(x_i|\beta)][\prod_{i=1}^{r} [G(x_i|\beta)]^{\alpha}]\alpha^{r+b_1-1}\exp(-b_2\alpha)\pi_2(\beta)[R_H(x_r|\theta)]^{n-r} \\
&= w(\beta,\underline{x})\pi_2(\beta)\alpha^{r+b_1-1}\exp\{-[b_2 - \sum_{i=1}^{r}\ln G(x_i|\beta)]\alpha\}[R_H(x_r|\theta)]^{n-r},
\end{aligned} \tag{2.5.3}$$

where $w(\beta,\underline{x}) = \prod_{i=1}^{r}\lambda^*(x_i|\beta)]$.

It follows, from (2.5.2) and (2.5.3), that the predictive density function of Y_s is given by

$$\begin{aligned}
f^*(y_s|\underline{x}) &= \int_{\Theta} f(y_s|\theta)\pi(\theta|\underline{x})d\theta \\
&= A\sum_{j_1=0}^{s-1}\sum_{j_2=0}^{n-r-s} D_{j_1}D_{j_2}\int_{\beta} w(\beta,\underline{x})\lambda^*(y_s|\beta)\pi_2(\beta)\int_0^{\infty}\alpha^{r+b_1}\mathrm{e}^{-\alpha T^*_{j_1j_2}(\beta)}d\alpha d\beta \\
&= A\Gamma(r+b_1+1)\sum_{j_1=0}^{s-1}\sum_{j_2=0}^{n-r-s} D_{j_1}D_{j_2}\int_{\beta}\frac{w(\beta,\underline{x})\lambda^*(y_s|\beta)\pi_2(\beta)}{T^*_{j_1j_2}(\beta)]^{r+b_1+1}}d\beta,
\end{aligned} \tag{2.5.4}$$

where $\lambda^*(z|\beta) = \frac{g(z|\beta)}{G(z|\beta)}$ and $T_{0j_1}(\beta)$ is given by (2.4.12) and

$$T^*_{j_1j_2}(\beta) = T_{0j_1} - [s-(j_1-j_2)]\ln G(y_s|\beta), \tag{2.5.5}$$

The predictive SF is then given by

$$\begin{aligned}
P[Y_s > v|\underline{x}] &= \int_v^{\infty} f^*(y_s|\underline{x})dy_s \\
&= A\Gamma(r+b_1+1)\sum_{j_1=0}^{s-1}\sum_{j_2=0}^{n-r-s} D_{j_1}D_{j_2}I^*(v), \\
&= A\Gamma(r+b_1+1)S^{***}(v), \quad v > x_r,
\end{aligned} \tag{2.5.6}$$

where

$$S^{***}(v) = \sum_{j_1=0}^{s-1} \sum_{j_2=0}^{n-r-s} D_{j_1} D_{j_2} I^*(v), \tag{2.5.7}$$

$$I^*(v) = \int_\beta w(\beta, \underline{x}) \pi_2(\beta) \int_v^\infty \frac{\lambda^*(y_s|\beta)}{[T^*_{j_1j_2}(\beta)]^{r+b_1+1}} dy_s d\beta.$$

The inner integral of $I^*(v)$ is given by

$$\int_v^\infty \frac{\lambda^*(y_s|\beta)}{[T^*_{j_1j_2}(\beta)]^{r+b_1+1}} dy_s = \int_v^\infty \frac{\lambda^*(y_s|\beta)}{[T_{0j_1}(\beta) - \{s - (j_1 - j_2)\} \ln G(y_s|\beta)]^{r+b_1+1}} dy_s$$

By applying the transformation $z = T_{0j_1}(\beta) - \{s - (j_1 - j_2)\} \ln G(y_s|\beta)$,
$dz = -\{s - j_1 + j_2\} \lambda^*(y_s|\beta) dy_s$ and $(v, \infty) \to (z_1, z_2)$, where $z_1 = T_{0j_1}(\beta) - (s - j_1 + j_2) \ln G(v|\beta)$ and $z_2 = T_{0j_1}(\beta)$
Therefore

$$\begin{aligned}
\int_v^\infty \frac{\lambda^*(y_s|\beta)}{[T^*_{j_1j_2}(\beta)]^{r+b_1+1}} dy_s &= \frac{-1}{s - j_1 + j_2} \int_{z_1}^{z_2} z^{-r-b_1-1} dz \\
&= \frac{1}{(r + b_1)(s - j_1 + j_2)} z^{-r-b_1} \Big|_{z_1}^{z_2} = \frac{1}{(r + b_1)(s - j_1 + j_2)} [z_2^{-r-b_1} - z_1^{-r-b_1}] \\
&= \frac{[T_{0j_1}(\beta)]^{-r-b_1} - [T_{0j_1}(\beta) - \{s - j_1 + j_2\} \ln G(v|\beta)]^{-r-b_1}}{(r + b_1)(s - j_1 + j_2)}.
\end{aligned}$$

So that

$$I^*(v) = \int_\beta w(\beta, \underline{x}) \pi_2(\beta) \left(\frac{[T_{0j_1}(\beta)]^{-r-b_1} - [T_{0j_1}(\beta) - \{s - j_1 + j_2\} \ln G(v|\beta)]^{-r-b_1}}{(r + b_1)(s - j_1 + j_2)} \right) d\beta \tag{2.5.8}$$

Since, from (2.5.6),
$1 = P[Y_s > x_r|\underline{x}] = A\Gamma(r + b_1 + 1) S^{***}(x_r)$, then $A = \frac{1}{\Gamma(r+b_1+1) S^{***}(x_r)}$.
where $S^{***}(v)$ is given by (2.5.7).
It then follows, from (2.5.6), that

$$P[Y_s > v|\underline{x}] = \frac{S^{***}(v)}{S^{***}(x_r)}, \quad v > x_r. \tag{2.5.9}$$

A 100 $(1-\tau)$ % two-sided predictive interval for the sth future order statistic Y_s has lower and upper bounds L and U, given by

$$1-(\tau/2) = P[Y_s > L|\underline{x}] = \frac{S^{***}(L)}{S^{***}(x_r)} \quad \text{and} \quad \tau/2 = P[Y_s > U|\underline{x}] = \frac{S^{***}(U)}{S^{***}(x_r)}.$$

Equivalently, L and U are given by the solution of the following equations

$$\left.\begin{array}{l} S^{***}(L) - [1-(\tau/2)]S^{***}(x_r) = 0, \\ S^{***}(U) - (\tau/2)S^{***}(x_r) = 0. \end{array}\right\} \tag{2.5.10}$$

where $S^{***}(v)$ is given by (2.5.7).

Remarks

1. The one-sided predictive interval of the form $Y_s < L$ is such that

$$0 = S^{***}(L) - (1-\tau)S^{***}(x_r)$$

and of the form $Y_s > U$ is such that

$$0 = S^{***}(U) - \tau S^{***}(x_r).$$

2. Two-sample scheme

In the case of two-sample scheme, we have two independent samples of sizes n and m. The informative sample consists of the first r order statistics $X_1 < \cdots < X_r$ of a random sample of size n. The future sample is assumed to consist of the order statistics $Y_\ell, \ell = 1, \ldots, m$. It is also assumed that all observations are drawn from the same population whose CDF is $H(x) = [G(x)]^\alpha$. Derivations of the estimators and predictive interval of the future observable $Y_\ell, \ell = 1, \ldots, m$ are the same as in the one-sample case, by replacing $f_r(y_s|\theta)$, given by (2.5.2) by

$$f(y_\ell|\theta) \propto [H(y_\ell|\theta)]^{\ell-1}[1-H(y_\ell|\theta)]^{m-\ell}h(y_\ell|\theta).$$

Proceeding as in the one-sample case, we finally obtain the estimators of $\alpha, \beta_i, (i = 1, \ldots, k)$, $R(x_0)$ and $\lambda(x_0)$ are given by

$$\hat{\alpha} = \frac{(r+b_1)S_1^*}{S_0^*}, \beta_i = \frac{S_{i+1}^*}{S_0^*}, (i = 1, \ldots, k),$$
$$R(x_0) = 1 - \frac{S_{k+2}^*}{S_0^*}, \lambda(x_0) = \frac{(r+b_1)S_{k+3}^*}{S_0^*} \tag{2.5.11}$$

The predictive SF of the future observable $Y_\ell, \ell = 1, \ldots, k$, is given by

$$P[Y_\ell > v|\underline{x}] = \frac{S_{k+5}^*(v)}{S_{02}}, \tag{2.5.12}$$

where, for $i = 0, 1, \ldots, k+2$,

$$S_i^* = \sum_{j_1=0}^{n-r} C_{j_1} I_{ij_1} \quad \text{and} \quad S_{k+3}^* = \sum_{j_1=0}^{n-r} \sum_{j_2=0}^{\infty} C_{j_1} I_{k+3,j_1,j_2}, \tag{2.5.13}$$

$$\begin{aligned}
I_{0j_1} &= \int_\beta \mathrm{e}^{-T_0(\beta)} \pi_2(\beta) / T_{0j_1}^{r+b_1}(\beta) d\beta, \\
I_{1j_1} &= \int_\beta \mathrm{e}^{-T_0(\beta)} \pi_2(\beta) / T_{0j_1}^{r+b_1+1}(\beta) d\beta, \\
I_{i+1,j_1} &= \int_\beta \beta_i \mathrm{e}^{-T_0(\beta)} \pi_2(\beta) / T_{0j_1}^{r+b_1+1}(\beta) d\beta, \quad i = 1, \ldots, k, \\
I_{k+2,j_1} &= \int_\beta \mathrm{e}^{-T_0(\beta)} \pi_2(\beta) / T_{0j_1}^{r+b_1}(\beta) d\beta, \\
I_{k+3,j_1,j_2} &= \int_\beta \frac{g(x_0|\beta) \mathrm{e}^{-T_0(\beta)} \pi_2(\beta)}{G(x_0|\beta) T_{j_1,j_2}^{r+b_1+1}(\beta)} d\beta, \\
I_{k+4,j_1,j_2} &= \int_\beta \frac{g(y_\ell|\beta) \mathrm{e}^{-T_0(\beta)} \pi_2(\beta)}{G(y_\ell|\beta) T_{j_1,j_2}^{r+b_1+1}(\beta)} d\beta,
\end{aligned} \tag{2.5.14}$$

$$I_{k+5,j_1,j_3}(v) = \int_\beta \left[\{T_{0j_1}(\beta)\}^{-(r+b_1)} - \{T_{0j_1}(\beta) - (\ell + j_3) \ln G(v|\beta)\}^{-(r+b_1)} \right]$$
$$\mathrm{e}^{-T_0(\beta)} \pi_2(\beta) d\beta. \tag{2.5.15}$$

$T_0(\beta)$ and $T_{0j_1}(\beta)$ are given by (2.4.11) and (2.4.12) and $T_{j_1,j_2}(\beta)$ by (2.4.14).

So that the lower and upper bounds L and U of the $(1-\tau)$ % predictive interval of $Y_\ell, \ell = 1, \ldots, m$ are given by the solution of the equations

$$\left.\begin{array}{l} S^*_{k+5}(L) - (1-(\tau/2))S_{02} = 0, \\ S^*_{k+5}(U) - (\tau/2)S_{02} = 0, \end{array}\right\} \tag{2.5.16}$$

where

$$S_{02} = \sum_{j_1=0}^{n-r}\sum_{j_3=0}^{m-\ell}\left[\frac{C_{j_1}C_{j_3}}{\ell + j_3}\right] I_{0j_1} \quad \text{and} \quad S^*_{k+5}(v) = \sum_{j_1=0}^{n-r}\sum_{j_3=0}^{m-\ell}\left[\frac{C_{j_1}C_{j_3}}{\ell + j_3}\right] I_{k+5,j_1,j_3} \tag{2.5.17}$$

$$C_{j_1} = (-1)^{j_1}\binom{n-r}{j_1}, C_{j_3} = (-1)^{j_3}\binom{m-\ell}{j_3}, \tag{2.5.18}$$

I_{0j_1} and I_{k+5,j_1,j_3} are given by (2.5.14) and (2.5.15).

For details, See AL-Hussaini (2010a).

2.6 Numerical Computations Applied to Three Examples

Three examples are given: in one of which the base distribution G is in standard form (with no parameters involved), the second depends on one parameter β and, in the third, G depends on two parameters (β_1, β_2). The computations using Bayes method in the three examples are based on the square error loss function.

1. $G(x) = 1 - e^{-x}$: the base distribution G does not depend on any unknown parameters.
2. $G(x) = 1 - e^{-\beta x}$: the base distribution G depends on one parameter. In this case, the exponentiated distribution depends on the two parameters (α, β).
3. $G(x) = 1 - e^{-\beta_1 x^{\beta_2}}$: the base distribution depends on two parameters. In this case, the exponentiated distribution depends on the three parameters $(\alpha, \beta_1, \beta_2)$.

Example 2.1 $G(x) = 1 - e^{-x}, x > 0,$ so that $H(x) = (1 - e^{-x})^{\alpha}$, where α is unknown.

- **Maximum likelihood estimation**

  To compute the MLEs of $\alpha, R(x_0), H(x_0)$ at some x_0,

  (i) Generate $n = 20$ uniform (0, 1) random numbers $u_1, \ldots, u_{20}$.
  (ii) Compute the corresponding $x_1, \ldots, x_{20}$, where $x_i = -\ln(1 - u_i^{1/\alpha})$, where U_i is uniform on the interval (0, 1). Choose $\alpha = 2.5$.
  (iii) Order the $x's$ and censor at r (r = 20, 18, 15).
  (iv) Use (2.3.4), with $G(x) = 1 - e^{-x}$, to compute $\hat{\alpha}_{ML}$. The MLEs $\hat{\alpha}, \hat{R}(x_0), \hat{H}(x_0)$ and their MSEs are displayed in Table 2.3a–c.

Table 2.3 $n = 20, b_1 = 3, b_2 = 0.6, x_0 = 0.2$ and 1,000 repetitions

Parameters	SBM		MCMC		MLE		Actual values
	Bayes	MSE	Bayes	MSE	ML	MSE	
(**a**) r = 20 (*complete sample case*)							
α	2.5890	0.0091	2.5780	0.0091	2.5920	0.0092	2.5
$R(x_0)$	0.9855	8.17×10^{-5}	0.9855	8.17×10^{-5}	0.9855	8.17×10^{-5}	0.986
$\lambda(x_0)$	0.165	0.006	0.165	0.006	0.166	0.006	0.16
(**b**) r = 18							
α	2.678	0.0943	2.678	0.0943	2.6820	0.0950	2.5
$R(x_0)$	0.988	0.0001	0.9850	0.0001	0.9850	0.0001	0.986
$\lambda(x_0)$	0.165	0.0008	0.165	0.0008	0.166	0.0008	0.16
(**c**) r = 15							
α	2.7095	0.165	2.7096	0.1652	2.7125	0.169	2.5
$R(x_0)$	0.9980	0.0001	0.9833	0.0001	0.9840	0.0001	0.986
$\lambda(x_0)$	0.1663	0.0042	0.1662	0.0042	0.1680	0.0050	0.16

The following Bayes methods are based on SEL.

- **Standard Bayes Method (SBM)**

Given b_1, b_2 and x_0, the Bayes estimates $\hat{\alpha}, \hat{R}(x_0), \hat{H}(x_0)$ are computed by using the expressions in (2.3.11). Based on 1,000 samples, each of size $n = 20$, censored at r = 20, 18, 15, when $b_1 = 3, b_2 = 0.6, x_0 = 0.2$, the average values of the estimates and their mean square errors (MSEs) over the 1,000 samples are given in Table 2.3a–c. We mean by the MSE, in the Bayes case, the overall risk function.

- **MCMC**

The data set is analyzed by applying Gibbs sampler and Metropolis-Hastings algorithm using WinBUGS 1.4 (http://www.nrc-bsu.cam.ac.uk/bugs/winbugs/contents.smtm1) which can be downloaded and used.

Step 0: Take some initial guess of $\alpha^{(0)}$.

Step 1: From $i = 1$ to N, generate $\alpha^{(i)}$ from the posterior PDF $\pi(\alpha|\underline{z})$, given by (2.3.8).

Step 2: Calculate the Bayes estimator of α by: $\hat{\alpha} = \frac{1}{N-M}\sum_{i=M+1}^{N} \alpha^{(i)}$, where M is the burn-in period.

Step 3: For a given time x_0, the Bayes estimators of the SF and HRF are given, respectively, by

$$\hat{R}(x_0) = \frac{1}{N-M}\sum_{i=M+1}^{N}\left[1-\left(1-e^{-x_0}\right)^{\alpha^{(i)}}\right]$$

and

$$\hat{\lambda}(x_0) = \frac{1}{N-M}\sum_{i=M+1}^{N}\left(\frac{a^{(i)}(1-\mathrm{e}^{-x_0})^{\alpha^{(i)}-1}\mathrm{e}^{-x_0}}{1-(1-\mathrm{e}^{-x_0})^{\alpha^{(i)}}}\right),$$

where M is the burn-in period.

The estimates and their MSEs, obtained by the MCMC method, are reported in Table 2.3a–c. The same parameter, and hyper-parameter values, used in SBM, are used here and computations are based on 1,000 samples.

- **Bayes prediction (two-sample scheme)**

The 95 % predictive intervals $(\tau = 0.05), n = 20, r = 20, 18, 15$ when $b_1 = 3, b_2 = 0.6$ for the first future observable Y_1 in a sample of size $m = 10$ future observables, are obtained by solving the two equations, given by (2.5.11). The intervals are found to be:

$$\begin{array}{lll} 0.0821 < Y_1 < 1.00845, & \text{length} = 1.0024, & (r = 20) \\ 0.0854 < Y_1 < 1.0921, & \text{length} = 1.0067, & (r = 18) \\ 0.0862 < Y_1 < 1.0949, & \text{length} = 1.0087, & (r = 15) \end{array}$$

where the lower bound of each interval is the average of the lower bounds L computed to satisfy the first equation of (2.5.16) for each one of the 1,000 samples, respectively and similarly for the upper bounds.

Example 2.2 $G(x) = 1 - \mathrm{e}^{-\beta x}$, $x > 0$, $(\beta > 0)$, the base distribution G depends on a single parameter β.

- **Maximum likelihood estimation**

With $k = 1$, Eqs. (2.4.3) and (2.4.4) reduce to only two equations and we write β for β_1. The solution of the two equations, using some iteration scheme, such as Newton-Raphson, yields MLEs $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$ of α and β. The MLEs of $\hat{R}_{ML}(x_0)$ and $\hat{\lambda}_{ML}(x_0)$ are obtained by applying the invariance property of MLEs. The average values of the estimates and their mean square errors (MSEs) over the 1,000 samples are given in Table 2.4a–c.

- **Standard Bayes method**

Suppose that α and β are independent and that α is distributed as gamma (b_1, b_2) whose PDF is given by (2.3.7) and β is distributed as gamma (b_3, b_4) whose PDF is given by

$$\pi_2(\beta) \propto \beta^{b_3-1}\exp(-b_4\beta), \quad \beta > 0, (b_3, b_4 > 0). \tag{2.6.1}$$

Table 2.4 $n = 20, b_1 = 3, b_2 = 0.6, b_3 = 2, b_4 = 3, x_0 = 0.2$ and 1,000 repetitions

Parameters	SBM		MCMC		MLE		Actual values
	Bayes	MSE	Bayes	MSE	ML	MSE	
(**a**) r = *20 (complete sample case)*							
α	2.5480	0.0254	2.5261	0.0071	2.5920	0.0092	2.5
β	1.523	0.0006	1.519	0.0004	1.574	0.0013	1.5
$R(x_0)$	0.9855	8.17×10^{-5}	0.9855	8.17×10^{-5}	0.9855	8.17×10^{-5}	0.9658
$\lambda(x_0)$	0.3776	0.0006	0.3771	0.0006	0.3702	0.0022	0.3795
(**b**) r = 18							
α	2.6078	0.1325	2.5628	0.0442	2.6640	0.5821	2.5
β	1.4523	0.0154	1.4569	0.0152	1.4491	0.0283	1.5
$R(x_0)$	0.9618	1.2×10^{-4}	0.9618	1.2×10^{-4}	0.9608	1.5×10^{-4}	0.9658
$\lambda(x_0)$	0.3761	0.0092	0.3755	0.0092	0.3681	0.0191	0.3795
(**c**) r = 15							
α	2.6875	0.5960	2.6940	0.5573	2.7930	0.6956	2.5
β	1.4375	0.0687	1.4436	0.0638	1.4017	0.0854	1.5
$R(x_0)$	0.9580	0.0004	0.9583	0.0004	0.9574	5.88×10^{-4}	0.9658
$\lambda(x_0)$	0.3758	0.0130	0.3750	0.0130	0.3651	0.0282	0.3795

Accordingly, the Bayes estimators of $\alpha, \beta, R(x_0), \lambda(x_0)$ are given by

$$\hat{\alpha} = \frac{(r+b_1)S_1^*}{S_0^*},\ \hat{\beta} = \frac{S_2^*}{S_0^*},\ \hat{R}(x_0) = 1 - \frac{S_3^*}{S_0^*},\ \hat{\lambda}(x_0) = \frac{(r+b_1)S_4^*}{S_0^*}. \tag{2.6.2}$$

The predictive PDF and SF of the future ℓ th observable $Y_\ell, \ell = 1, \ldots, k$, are given by

$$f^*(y_\ell|\underline{x}) = \frac{(r+b_1)S_5^*}{S_{02}} \text{ and } P[Y_\ell > v|\underline{x}] = \frac{S_6^*(v)}{S_{02}},$$

where, for $v = 0, 1, 2, 3$, S_v^* and $S_4^* - S_6^*(.)$ are given by (2.5.17) with $k = 1$.

The integrals involved are given as follows, from (2.5.17), when $k = 1$ and $\pi_2(\beta)$ is given by (2.6.1). So that

$$I_{0j_1} = \int_0^\infty [\beta^{b_3-1} e^{-T^*(\beta)} / T_{0j_1}^{r+b_1}(\beta)] d\beta,$$

$T^*(\beta) = b_4\beta + T_0(\beta), T_0(\beta)$
is given by (2.4.11)

$$I_{1j_1} = \int_0^\infty [\beta^{b_3-1} e^{-T^*(\beta)} / T_{0j_1}^{r+b_1+1}(\beta)] d\beta,$$

$$I_{2j_1} = \int_0^\infty [\beta^{b_3} e^{-T^*(\beta)} / T_{0j_1}^{r+b_1}(\beta)] d\beta,$$

$$I_{3j_1} = \int_0^\infty [\beta^{b_3-1} e^{-T^*(\beta)} / T_{1j_1}^{r+b_1}(\beta)] d\beta,$$

$$I_{4j_1j_2} = \int_0^\infty \frac{g(x_0|\beta)\beta^{b_3-1} e^{-T^*(\beta)}}{G(x_0|\beta) T_{j_1j_2}^{r+b_1+1}(\beta)} d\beta,$$

$$I_{5j_1j_2} = \int_0^\infty \frac{g(y_\ell|\beta)\beta^{b_3-1} e^{-T^*(\beta)}}{G(y_\ell|\beta) T_{j_1j_2}^{r+b_1+1}(\beta)} d\beta,$$

$$I_{6j_1j_2} = \int_0^\infty \left[\frac{1}{\{T_{0j_1}(\beta)\}^{r+b_1}} - \frac{1}{\{T_{0j_1}(\beta) - (\ell + j_3)\ln G(v|\beta)\}^{r+b_1}} \right] \beta^{b_3-1} e^{-T^*(\beta)} d\beta$$

Computations are carried out in the same manner as in Example 2.1, with the obvious changes in which $H(x) = [G(x)]^\alpha, G(x) = 1 - e^{-\beta x}$, with $n = 20$, censored at r = 20, 18, 15, when $b_1 = 3, b_2 = 0.6$. So that $x_i = -\frac{1}{\beta}\ln(1 - u_i^{1/\alpha}),\ i = 1, \ldots, 20$, where α and β are chosen to be $\alpha = 2.5$ and $\beta = 1.5$. Computations are also based on chosen values of $x_0 = 0.2, b_1 = 3, b_2 = 0.6, b_3 = 2, b_4 = 3$.

- **MCMC**

In this case, samples are generated from the posterior distributions. Bayes estimates of α and β and their functions are computed according to the following steps:

Step 0: Take some initial guess of α and β, say $\alpha^{(0)}$ and $\beta^{(0)}$.
Step 1: Generate $\alpha^{(1)}$ and $\beta^{(1)}$ from the posterior PDFs, given by

$$\pi(\alpha|\beta, \underline{x}) = \frac{\pi(\alpha, \beta|\underline{x})}{\int_0^\infty \pi(\alpha, \beta|\underline{x}) d\alpha} = \frac{\alpha^{r+b_1-1} \sum_{j_1=0}^{n-r} C_{j_1} e^{-\alpha T_{0j_1}(\beta)}}{\Gamma(r + b_1) \sum_{j_1=0}^{n-r} C_{j_1} / [T_{0j_1}(\beta)]^{r+b_1}},$$

$$\pi(\beta|\alpha, \underline{x}) = \frac{\pi(\alpha, \beta|\underline{x})}{\int_0^\infty \pi(\alpha, \beta|\underline{x}) d\beta} = \frac{\sum_{j_1=0}^{n-r} C_{j_1} \beta^{b_3-1} e^{-\alpha T_{0j_1}(\beta) - T^*(\beta)}}{\sum_{j_1=0}^{n-r} C_{j_1} \int_0^\infty \beta^{b_3-1} e^{-\alpha T_{0j_1}(\beta) - T^*(\beta)} d\beta},$$

where C_{j_1} and $T_{0j_1}(\beta)$ are given by (2.4.12), $T^*(\beta) = b_4\beta + T_0(\beta)$, $T_0(\beta)$ is given by (2.4.11).

Step 2: From $i = 1$ to $N - 1$, generate $\alpha^{(i+1)}$ and $\beta^{(i+1)}$ from $\pi\,(\alpha|\,\beta^{(i+1)},\, \underline{x})$ and $\pi\,(\beta|\,\alpha^{(i+1)},\, \underline{x})$, respectively.

Step 3: Calculate the Bayes estimates of α and β from $\hat{\alpha} = \frac{1}{N-M}\sum_{i=M+1}^{N} \alpha^{(i)}$ and $\hat{\beta} = \frac{1}{N-M}\sum_{i=M+1}^{N} \beta^{(i)}$, respectively.

For a given x_0, calculate the Bayes estimates of the SF and HRF from:

$$\hat{R}\,(x_0) = \frac{1}{N-M}\sum_{i=M+1}^{N} \left[1 - (1 - \mathrm{e}^{-\beta^{(i)}\,x_0})^{\alpha^{(i)}}\right]$$

and

$$\hat{\lambda}(x_0) = \frac{1}{N-M}\sum_{i=M+1}^{N} \left(\frac{a^{(i)}\beta^{(i)}(1 - \mathrm{e}^{-\beta^{(i)}x_0})^{\alpha^{(i)}-1}\mathrm{e}^{-\beta^{(i)}x_0}}{1 - (1 - \mathrm{e}^{-\beta^{(i)}x_0})^{\alpha^{(i)}}} \right).$$

It may be noted that in Kundu and Gupta (Kundu and Gupta 2008), the Bayes estimates of the parameters α and β (α and λ in their notation) were developed in the complete sample case, assuming that α and β are independent and each has a gamma prior.

The estimates using the ML method, SBM, MCMC are displayed in Table 2.4a–c.

- **Bayes prediction (two-sample scheme)**

The 95 % predictive interval $(\tau = 0.05), n = 20, r = 20, 18, 15$ when $b_1 = 3, b_2 = 0.6$, $b_3 = 2, b_4 = 3$, for the first future observable Y_1, in a sample of size $m = 10$ future observables are found to be

$$\begin{aligned} &0.0321 < Y_1 < 0.6542, \quad \text{length} = 0.6221, \quad (r = 20)\\ &0.0412 < Y_1 < 0.7320, \quad \text{length} = 0.6908, \quad (r = 18)\\ &0.0447 < Y_1 < 0.7397, \quad \text{length} = 0.6950, \quad (r = 15) \end{aligned}$$

where the lower bound of each interval is the average of the lower bounds L computed to satisfy the first equation of (2.5.16) for each one of the 1,000 samples, respectively. Similarly for the upper bounds.

Example 2.3 $G(x) = 1 - \mathrm{e}^{-\beta_1 x^{\beta_2}}, x > 0$, $(\beta_1, \beta_2 > 0)$, the base (Weibull) distribution depends on two unknown parameters β_1, β_2.

- **Maximum likelihood estimation**

With $k = 2$, the system of LEs (2.4.3) and (2.4.4) reduce to three equations in the three unknowns α, β_1, β_2. By solving such equations, using some iteration method we obtain the MLEs of these parameters. The MLEs of $R(x_0)$ and $\lambda(x_0)$ are computed by applying the invariance property of MLEs.

- Standard Bayes method

Suppose that α and $\beta = (\beta_1, \beta_2)$ are independent and that α is distributed as gamma (b_1, b_2) whose PDF is given by (2.3.7) and β is such that

$$\begin{aligned}\pi_2(\beta) &= \pi_3(\beta_2|\beta_1)\pi(\beta_1)\\ &\propto [\beta_2^{b_4-1}\mathrm{e}^{-\beta_1\beta_2}][\beta_1^{b_3-1}\mathrm{e}^{-b_5\beta_1}],\\ &\quad \beta_1, \beta_2 > 0, (b_3, b_4, b_5 > 0)\\ &\propto \beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-\beta_1(b_5+\beta_2)}.\end{aligned} \tag{2.6.3}$$

So that the prior PDF of θ, is given, from (2.3.7) and (2.5.17), by

$$\pi(\theta) = \pi_1(\alpha)\pi_2(\beta) \propto \alpha^{b_1-1}\,\beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-b_2\,\alpha-\beta_1(b_5+\beta_2)}. \tag{2.6.4}$$

According to Theorem 2.3, the Bayes estimators of $\alpha, \beta_1, \beta_2, R(x_0), \lambda\,(x_0)$ are given by

$$\hat{\alpha} = \frac{(r+b_1)\,S_1^*}{S_0^*},\ \hat{\beta}_1 = \frac{S_2^*}{S_0^*},\ \hat{\beta}_2 = \frac{S_3^*}{S_0^*},\ \hat{R}(x_0) = 1 - \frac{S_4^*}{S_0^*},\ \hat{\lambda}(x_0) = \frac{(r+b_1)S_5^*}{S_0^*} \tag{2.6.5}$$

where, for $\ell = 0, 1, \ldots, 5$, S_ℓ^* is given by (2.5.13), in which

$$\begin{aligned}
I_{0j_1} &= \int_0^\infty\int_0^\infty \frac{\beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-T^*(\beta_1,\beta_2)}}{T_{0j_1}^{r+b_1}}d\beta_1 d\beta_2,\\
T^*(\beta_1,\beta_2) &= T_0(\beta_1,\beta_2) + \beta_1(b_5+\beta_2),\\
I_{1j_1} &= \int_0^\infty\int_0^\infty \frac{\beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-T^*(\beta_1,\beta_2)}}{T_{0j_1}^{r+b_1+1}}d\beta_1 d\beta_2,\\
I_{2j_1} &= \int_0^\infty\int_0^\infty \frac{\beta_1^{b_3}\beta_2^{b_4-1}\mathrm{e}^{-T^*(\beta_1,\beta_2)}}{T_{0j_1}^{r+b_1}}d\beta_1 d\beta_2,\\
I_{3j_1} &= \int_0^\infty\int_0^\infty \frac{\beta_1^{b_3-1}\beta_2^{b_4}\mathrm{e}^{-T^*(\beta_1,\beta_2)}}{T_{0j_1}^{r+b_1}}d\beta_1 d\beta_2,\\
I_{4j_1} &= \int_0^\infty\int_0^\infty \frac{\beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-T^*(\beta_1,\beta_2)}}{T_{1j_1}^{r+b_1}}d\beta_1 d\beta_2,\\
I_{5j_1j_2} &= \int_0^\infty\int_0^\infty \frac{g(x_0|\beta_1,\beta_2)\beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-T^*(\beta_1,\beta_2)}}{G(x_0|\,\beta_1,\beta_2)T_{j_1j_2}^{r+b_1+1}(\beta_1,\beta_2)}d\beta_1 d\beta_2,
\end{aligned}$$

In this example, $G(x) = 1 - \mathrm{e}^{-\beta_1 x^{\beta_2}}$, so that $X_i = [-\frac{1}{\beta_1}\ln(1 - U_i^{1/\alpha})]^{1/\beta_2}$.

- **MCMC**

Bayes estimates of α, β_1, β_2 and their functions are computed according to the following steps:

Step 0: Take some initial guess of α, β_1, β_2, say $\alpha^{(0)}, \beta_1^{(0)}, \beta_2^{(0)}$.

Step 1: Generate $\alpha^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}$ from the posterior PDFs, given, respectively, by

$$\begin{aligned}
\pi(\alpha|\beta_1, \beta_2, \underline{x}) &= \frac{\pi(\alpha, \beta_1, \beta_2|\underline{x})}{\int_0^\infty \pi(\alpha, \beta_1, \beta_2|\underline{x})d\alpha} \\
&= \frac{\alpha^{r+b_1-1}\sum_{j_1=0}^{n-r} C_{j_1} \mathrm{e}^{-\alpha T_{0j_1}(\beta_1,\beta_2)}}{\Gamma(r+b_1)\sum_{j_1=0}^{n-r} C_{j_1}/[T_{0j_1}(\beta_1,\beta_2)]^{r+b_1}} \\
\pi(\beta_1|\alpha, \beta_2, \underline{x}) &= \frac{\pi(\alpha, \beta_1, \beta_2|\underline{x})}{\int_0^\infty \pi(\alpha, \beta_1, \beta_2|\underline{x})d\beta_1} \\
&= \frac{\sum_{j_1=0}^{n-r} C_{j_1} \beta_1^{b_3-1} \mathrm{e}^{-\alpha T_{0j_1}(\beta_1,\beta_2)-T^*(\beta_1,\beta_2)}}{\sum_{j_1=0}^{n-r} C_{j_1} \int_0^\infty \beta_1^{b_3-1} \mathrm{e}^{-\alpha T_{0j_1}(\beta_1,\beta_2)-T^*(\beta_1,\beta_2)} d\beta_1}, \\
\pi(\beta_2|\alpha, \beta_1, \underline{x}) &= \frac{\pi(\alpha, \beta_1, \beta_2|\underline{x})}{\int_0^\infty \pi(\alpha, \beta_1, \beta_2|\underline{x})d\beta_2} \\
&= \frac{\sum_{j_1=0}^{n-r} C_{j_1} \beta_2^{b_4-1} \mathrm{e}^{-\alpha T_{0j_1}(\beta_1,\beta_2)-T^*(\beta_1,\beta_2)}}{\sum_{j_1=0}^{n-r} C_{j_1} \int_0^\infty \beta_2^{b_4-1} \mathrm{e}^{-\alpha T_{0j_1}(\beta_1,\beta_2)-T^*(\beta_1,\beta_2)} d\beta_2}.
\end{aligned}$$

where C_{j_1} is given by (2.3.11), $T_{0j_1}(\beta)$ by (2.4.17) and $T^*(\beta_1, \beta_2) = b_4\beta + T_0(\beta)$, $T_0(\beta)$ is given by (2.4.16).

Step 2: From $i = 1$ to $N-1$, generate $\alpha^{(i+1)}, \beta_1^{(i+1)}$ and $\beta_2^{(i+1)}$ from $\pi(\alpha|\beta_1^{(i)}, \beta_2^{(i)}, \underline{x})$, $\pi(\beta_1|\alpha^{(i)}, \beta_2^{(i)}, \underline{x})$ and $\pi(\beta_2|\alpha^{(i)}, \beta_1^{(i)}, \underline{x})$, respectively.

Step 3: Calculate the Bayes estimates of α,β_1 and β_2 from $\hat{\alpha} = \frac{1}{N-M}\sum_{i=M+1}^{N} \alpha^{(i)}$, $\hat{\beta}_1 = \frac{1}{N-M}\sum_{i=M+1}^{N} \beta_1^{(i)}$ and $\hat{\beta}_2 = \frac{1}{N-M}\sum_{i=M+1}^{N} \beta_2^{(i)}$, respectively.

For a given x_0, calculate the Bayes estimates of the SF and HRF from:

$$\hat{R}(x_0) = \frac{1}{N-M}\sum_{i=M+1}^{N} [1 - (1 - \mathrm{e}^{-\beta_1^{(i)} x_0^{\beta_2^{(i)}}})^{\alpha^{(i)}}]$$

and

$$\hat{\lambda}(x_0) = \frac{1}{N-M}\sum_{i=M+1}^{N} \left(\frac{a^{(i)}\beta_1^{(i)}\beta_2^{(i)} x_0^{\beta_2^{(i)}-1} \left[1 - \mathrm{e}^{-\beta_1^{(i)} x_0^{\beta_2^{(i)}}}\right]^{\alpha^{(i)}-1} \mathrm{e}^{-\beta_1^{(i)} x_0^{\beta_2^{(i)}}}}{1 - \left[1 - \mathrm{e}^{-\beta_1^{(i)} x_0^{\beta_2^{(i)}}}\right]^{\alpha^{(i)}}} \right)$$

Table 2.5 $n = 20, b_1 = 3, b_2 = 0.6, b_3 = 2, b_4 = 3, b_5 = 0.8, x_0 = 0.2$ and 1,000 repetitions

Parameters	SBM		MCMC		MLE		Actual values
	Bayes	MSE	Bayes	MSE	ML	MSE	
(**a**) r = 20 (*complete sample case*)							
α	2.4012	0.0054	2.4865	0.0005	2.3122	0.0731	2.5
β_1	1.3942	0.0121	1.4572	0.0013	1.2647	0.0596	1.5
β_2	0.6012	0.0143	0.5231	0.0018	0.6412	0.0596	0.5
$R(x_0)$	0.8341	0.0002	0.8333	0.0001	0.8631	0.0012	0.833
$\lambda(x_0)$	0.8371	0.0082	0.8568	0.0023	0.8134	0.0121	0.8792
(**b**) r = 18							
α	2.2143	0.1251	2.4431	0.0851	2.0122	0.2845	2.5
β_1	1.2491	0.0649	1.4025	0.0056	1.1831	0.1314	1.5
β_2	0.6582	0.0423	0.6021	0.0214	0.7342	0.0921	0.5
$R(x_0)$	0.8352	0.0009	0.8.339	0.0002	0.8921	0.0116	0.833
$\lambda(x_0)$	0.8021	0.0142	0.8352	0.0092	0.7985	0.0163	0.8792
(**c**) r = 15							
α	2.0051	0.3325	2.4258	0.1369	1.9683	0.3452	2.5
β_1	1.1638	0.1272	1.3253	0.0509	1.1276	0.2134	1.5
β_2	0.7420	0.0883	0.6807	0.0655	0.8535	0.1049	0.5
$R(x_0)$	0.8374	0.0015	0.8344	0.0016	0.9840	0.0145	0.833
$\lambda(x_0)$	0.7786	0.0295	0.8081	0.0270	0.7621	0.0352	0.8792

where M is the burn-in period.

Computations are carried out as before for $n = 20, r = 20, 18, 15, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 0.5, b_1 = 3, b_2 = 0.6, b_3 = 2, b_4 = 3, b_5 = 0.8, x_0 = 0.2$. The estimates obtained by the above methods and their MSEs are displayed in Table 2.5.

- Bayes prediction (two-sample scheme)

The 95 % predctive intervals, for the first future observable Y_1 in a sample of size $m = 10$ future observables, when $n = 20, r = 20, 18, 15, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 0.5, b_1 = 3, b_2 = 0.6, b_3 = 2, b_4 = 3, b_5 = 0.8, x_0 = 0.2$. are obtained by solving the two equations, given by (2.5.16). The intervals are found to be:

$$\begin{array}{lll} 0.00193 < Y_1 < 0.54554, & \text{length} = 0.54361, & (r = 20) \\ 0.00215 < Y_1 < 0.57045, & \text{length} = 0.56830, & (r = 18) \\ 0.00309 < Y_1 < 0.58464, & \text{length} = 0.58155, & (r = 15) \end{array}$$

where the lower bound of each interval is the average of the lower bounds L computed to satisfy the first equation of (2.5.16) for each one of the 1,000 samples, respectively and similarly for the upper bounds, where

$$I_{6j_1j_2} = \int_0^\infty \int_0^\infty \frac{g(x_\ell|\beta_1,\beta_2)\beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-T^*(\beta_1,\beta_2)}}{G(x_\ell|\beta_1,\beta_2)T_{j_1j_2}^{r+b_1+1}(\beta_1,\beta_2)} d\beta_1 d\beta_2$$

$$I_{7j_1j_3'} = \int_0^\infty \int_0^\infty \left[\{T_{0j_1}(\beta_1,\beta_2)\}^{-(r+b_1)} - \{T_{0j_1}(\beta_1,\beta_2) - (\ell+j_3)\ln G(v|\beta_1,\beta_2)\}^{-(r+b_1)}\right] \times \beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-T^*(\beta_1,\beta_2)} d\beta_1 d\beta_2.$$

Remarks

1. It may be noticed, in the three examples, that the Bayes estimates, using the MCMC, performs best in most cases, in the sense of having smallest MSEs then comes the estimates using SBM and finally those based on MLEs.
2. Even with censored samples (r = 15), the estimates are still reasonable.
3. Indeed, predictive intervals for $Y_\ell, \ell = 2, \ldots, k$, can be obtained as that computed for Y_1, by following the same steps.

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Chapter 3
Family of Exponentiated Weibull Distributions

Contents

3.1 Introduction

A handbook, by Rinne (2009), covers the Weibull distribution in many of its aspects. The study of the family of exponentiated Weibull (EW) distributions and their applications attracted the interest of researchers in the nineties. Such interest is growing since then.

If the baseline distribution is Weibull (β_1, β_2), with CDF

$$G(x) = 1 - \exp[-\beta_1 x^{\beta_2}], \quad x > 0 \ , \ (\beta_1, \beta_2 > 0), \tag{3.1.1}$$

then the family of exponentiated Weibull (EW) distributions has CDF

$$H(x) = \left(1 - \exp[-\beta_1 x^{\beta_2}]\right)^{\alpha}. \tag{3.1.2}$$

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We shall write $X \sim EW(\alpha, \beta_1, \beta_2)$ to mean that the random variable X follows the exponentiated Weibull distribution with parameters α, β_1, β_2. The family of distributions (3.1.2) includes the following important distributions, as special cases,

(i) $\beta_2 = 1 \Rightarrow H(x) = (1 - \exp[-\beta_1 x])^{\alpha} \Rightarrow X \sim EE(\alpha, \beta_1)$,
(ii) $\beta_2 = 2 \Rightarrow H(x) = (1 - \exp[-\beta_1 x^2])^{\alpha} \Rightarrow X \sim ERay(\alpha, \beta_1)$,
(iii) $\alpha = 1 \Rightarrow H(x) = 1 - \exp[-\beta_1 x^{\beta_2}] \Rightarrow X \sim W(\beta_1, \beta_2)$,
(iv) $\alpha = 1, \beta_2 = 1 \Rightarrow H(x) = 1 - \exp[-\beta_1 x] \Rightarrow X \sim Exp(\beta_1)$,
(v) $\alpha = 1, \beta_2 = 2 \Rightarrow H(x) = 1 - \exp[-\beta_1 x^2] \Rightarrow X \sim Ray(\beta_1)$

where, W, Exp, Ray, EE and ERay stand for Weibull, Exponential, Rayleigh, Exponentiated Exponential and Exponentiated Rayleigh, respectively.

The SF and PDF corresponding to (3.1.2) are given, respectively by

$$R_H(x) = 1 - \left(1 - \exp[-\beta_1 x^{\beta_2}]\right)^{\alpha}, \tag{3.1.3}$$

$$h(x) = \alpha\beta_1\beta_2 x^{\beta_2 - 1} \exp[-\beta_1 x^{\beta_2}]\left(1 - \exp[-\beta_1 x^{\beta_2}]\right)^{\alpha - 1}. \tag{3.1.4}$$

It then follows, from (3.1.3) and (3.1.4), that the corresponding HRF is given by

$$\lambda_H(x) = \frac{h(x)}{R_H(x)} = \frac{\alpha\left(1 - \exp[-\beta_1 x^{\beta_2}]\right)^{\alpha-1} \beta_1\beta_2 x^{\beta_2 - 1} \exp[-\beta_1 x^{\beta_2}]}{1 - (1 - \exp[-\beta_1 x^{\beta_2}])^{\alpha}}. \tag{3.1.5}$$

As mentioned before, an ED differs substantially from its baseline distribution.

For example, while the Weibull (β_1, β_2) distribution allows three shapes for the HRF which are: DHR, if $\beta_2 < 1$, CHR, if $\beta_2 = 1$, IHR, if $\beta_2 > 1$, the $EW(\alpha, \beta_1, \beta_2)$ allows two additional shapes for the HRF: BTHR and UBTHR. In fact, Modholkar et al. (1995) showed that the $EW(\alpha, \beta_1, \beta_2)$ distribution with HRF, given by (3.1.5) (when $\beta_1 = 1$) has the following shapes

(i) CHR $= \beta_1$, if and only if $\alpha = \beta_2 = 1$.
(ii) DHR, if $\beta_2 \leq 1$ and $\alpha\beta_2 \leq 1$.
(iii) IHR, if $\beta_2 \geq 1$ and $\alpha\beta_2 \geq 1$.
(iv) Bathtub hazard rate (BTHR), if $\beta_2 > 1$ and $\alpha\beta_2 < 1$.
(v) Upside down bathtub hazard rate (UBTHR), if $\beta_2 < 1$ and $\alpha\beta_2 > 1$.

So, while the Weibull (β_1, β_2) distribution allows for three shapes for the hazard rate function: DHR if $\beta_2 < 1$, CHR if $\beta_2 = 1$ and IHR if $\beta_2 > 1$, the $EW(\alpha, \beta_1, \beta_2)$ allows two additional shapes for the HRF: the BTHR and UBTHR. The figures on p. 63 show cases (ii)–(v), for different parameter values.

Remarks

1. The EW distribution has a natural physical interpretation, that if the lifetimes of n units in a system, connected in parallel, are iid as EW, then the system's lifetime is also EW.

3.2 Properties of the Exponentiated Weibull Family

3.2.1 Moments

The CDF, SF and PDF of $EW(\alpha, \beta_1, \beta_2)$ distribution are given by (3.1.2), (3.1.3) and (3.1.4), respectively. The ℓth moment of an ED is given, for $\ell = 1, 2, \ldots$ by (2.2.6), where

$$I_j(\ell) = \int_0^\infty x^{\ell-1} \exp[-ju(x)]dx.$$

In the $EW(\alpha, \beta_1, \beta_2)$ case, $u(x) = \beta_1 x^{\beta_2}$, so that

$$I_j(\ell) = \int_0^\infty x^{\ell-1} \exp[-j\beta_1 x^{\beta_2}]dx.$$

By applying the transformation $z = x^{\beta_2}$, the integral then becomes

$$I_j(\ell) = \frac{1}{\beta_2} \int_0^\infty z^{(\ell/\beta_2)-1} \exp[-j\beta_1 z]dz.$$

It then follows, from (2.2.6), that

$$E(X^\ell) = \ell \sum_{j=1}^{v} c_j I_j(\ell) = \frac{\Gamma[1 + (\ell/\beta_2)]}{\beta_1^{\ell/\beta_2}} \sum_{j=1}^{v} \frac{c_j}{j^{\ell/\beta_2}}, \tag{3.2.1}$$

where c_j and v are given by (2.2.7).

This is the result given in Table 2.1, for the $EW(\alpha, \beta_1, \beta_2)$ distribution.

It may be remarked that if $j - 1 = i$, then

$$E(X^\ell) = \frac{\alpha\Gamma[1 + (\ell/\beta_2)]}{\beta_1^{\ell/\beta_2}} \sum_{i=0}^{v} \frac{c_i}{(i+1)^{\ell/\beta_2}}$$

and if α is a positive integer, then

$$E(X^{\ell}) = \frac{\alpha\Gamma[1 + (\ell/\beta_2)]}{\beta_1^{\ell/\beta_2}} \sum_{i=0}^{\alpha-1} (-1) \binom{\alpha - 1}{i} \frac{1}{(i+1)^{\ell/\beta_2}}.$$

This expression agrees with that given in Eq. 5 by Nadarajah et al. (2013) after renaming the parameters $(\lambda = \beta_1^{1/c}, c = \beta_2, k = \ell)$.

The mean, variance, skewness and kurtosis can be computed by using the appropriate values of ℓ. For details, see Mudholkar and Hutson (1996), Eissa (2005). Other forms, rather than (3.2.1) were considered by some authors. For example, Gupta and Kundu (2009) noticed that one can express the ℓth moment as follows

$$E(X^{\ell}) = \alpha \int_0^1 \left[H_0^{-1}(z)\right]^{\ell} z^{\alpha-1} dz,$$

where $H_0^{-1}(z)$ is the quantile function of a two-parameter Weibull distribution with parameters (β_1, β_2).

More forms for $E(X^{\ell})$ can be found in Nadarajah et al. (2013).

3.2.2 Mean Residual Life (MRL) Function

In addition to the important concept of HRF, used in life-testing and reliability, another important concept is the MRL function $m(x)$, which is defined by

$$m(x) = E[T - x | T > x], \tag{3.2.2}$$

where T is a positive r.v. with PDF $h(t)$ and CDF $H(t)$. If $R(t) = 1 - H(t)$ is the corresponding SF, then $m(x)$ is given by

$$m(x) = \int_x^{\infty} (t - x) h(t) dt / R(x).$$

Integrating by parts, it can be seen that an alternative definition of $m(x)$ is given by

$$m(x) = \frac{\int_x^{\infty} R(t) dt}{R(x)}. \tag{3.2.3}$$

If $H(x) = [G(x)]^{\alpha}$, then $R(x) = 1 - [G(x)]^{\alpha}$, so that (3.2.3) becomes

$$m(x) = \frac{\int_x^{\infty} \{1 - [G(t)]^{\alpha}\}dt}{1 - [G(x)]^{\alpha}}. \tag{3.2.4}$$

The MRL of the EW$(\alpha, \beta_1, \beta_2)$ is obtained by substituting the baseline distribution $G(t) = 1 - \exp(-\beta_1 t^{\beta_2})$ in (3.2.4). In this case

$$m(x) = \frac{\int_x^{\infty} \{1 - [1 - \exp(-\beta_1 t^{\beta_2})]^{\alpha}\}dt}{R(x)}. \tag{3.2.5}$$

It can be shown that the MRL of EW$(\alpha, \beta_1, \beta_2)$ distribution is given by

$$m(x) = \frac{\sum_{j=1}^{v} \frac{c_j}{\beta_2 (j\beta_1)^{1/\beta_2}} \{\Gamma(1/\beta_2) - \Gamma(1/\beta_2, jz_0)\}}{1 - [1 - \exp(-\beta_1 x^{\beta_2})]^{\alpha}}, \tag{3.2.6}$$

where c_j is given by (2.2.8), v by (2.2.7), $z_0 = \beta_1 x^{\beta_2}$, $\Gamma(1/\beta_2)$ and $\Gamma(1/\beta_2, jz_0)$ are the gamma and incomplete gamma functions, as defined by

$$\left.\begin{aligned} \Gamma(\alpha) &= \int_0^{\infty} z^{\alpha-1} \exp(-z)dz, \\ \Gamma(\alpha, w_0) &= \int_0^{w_0} w^{\alpha-1} \exp(-w)dw. \end{aligned}\right\} \tag{3.2.7}$$

In fact, the numerator of $m(x)$, in (3.2.5), is given by

$$\int_x^{\infty} \{1 - [1 - \exp(-\beta_1 t^{\beta_2})]^{\alpha}\}dt = \sum_{j=1}^{v} c_j I_j, \tag{3.2.8}$$

where c_j is given by (2.2.8), v by (2.2.7) and

$$I_j = \int_x^{\infty} \exp[-j\beta_1 t^{\beta_2}]dt.$$

Let $z = \beta_1 t^{\beta_2}$, then $t = \left(\frac{z}{\beta_1}\right)^{1/\beta_2}$, $(x, \infty) \to (z_0, \infty)$, $z_0 = \beta_1 x^{\beta_2}$ and $dt = \frac{z^{1/\beta_2 - 1}}{\beta_1^{1/\beta_2} \beta_2} dz.$

So that

$$\begin{aligned}
I_j &= \int_{z_0}^{\infty} \exp[-jz] \frac{z^{1/\beta_2 - 1}}{\beta_1^{1/\beta_2} \beta_2} dz \\
&= \frac{1}{\beta_2 \beta_1^{1/\beta_2}} \int_{z_0}^{\infty} z^{1/\beta_2 - 1} \exp[-jz] dz \\
&= \frac{1}{\beta_2 \beta_1^{1/\beta_2}} \left\{ \int_0^{\infty} z^{1/\beta_2 - 1} \exp[-jz] dz - \int_0^{z_0} z^{1/\beta_2 - 1} \exp[-jz] \mathrm{dz} \right\} \\
&= \frac{1}{\beta_2 \beta_1^{1/\beta_2}} \left\{ \frac{\Gamma(1/\beta_2)}{j^{1/\beta_2}} - \frac{\Gamma(1/\beta_2, jz_0)}{j^{1/\beta_2}} \right\} \\
&= \frac{1}{\beta_2 (j\beta_1)^{1/\beta_2}} \{ \Gamma(1/\beta_2) - \Gamma(1/\beta_2, jz_0) \}
\end{aligned}$$

where $\Gamma(\alpha)$ and $\Gamma(\alpha, jw_0)$ are given by (3.2.7) .

Therefore, from (3.2.8) and (3.2.5),

$$m(x) = \frac{1}{R(x)} \sum_{j=1}^{v} \frac{c_j}{\beta_2 (j\beta_1)^{1/\beta_2}} \{ \Gamma(1/\beta_2) - \Gamma(1/\beta_2, jz_0) \}, \quad z_0 = \beta_1 x^{\beta_2}. \tag{3.2.9}$$

If α is a positive integer, then

$$m(x) = \frac{1}{R(x)} \sum_{j=1}^{\alpha} (-1)^{j-1} \binom{\alpha}{j} \frac{\{ \Gamma(1/\beta_2) - \Gamma(1/\beta_2, jz_0) \}}{\beta_2 (j\beta_1)^{1/\beta_2}}, \quad z_0 = \beta_1 x^{\beta_2}.$$

It may be observed that, if $\alpha = 1$, then (3.2.9) reduces to the form

$$m(x) = \frac{\{ \Gamma(1/\beta_2) - \Gamma(1/\beta_2, z_0) \}}{\beta_2 \beta_1^{1/\beta_2} \exp(-\beta_1 x^{\beta_2})}, \quad z_0 = \beta_1 x^{\beta_2},$$

which is the MRL of the $W(\beta_1, \beta_2)$ distribution.

For the case α is a positive integer, Nassar and Eissa (2003) obtained an expression for the MRL function using the form $H(x) = [1 - \exp\{-(\frac{x}{\lambda})^c\}]^{\alpha}$.

By following similar steps, it can be shown that

$$E[X^n | X > x] = x^n + \frac{n \sum_{j=1}^{v} \frac{c_j}{j^{(n+1)/\beta_2}} [\Gamma\{(n+1)/\beta_2\} - \Gamma\{(n+1)/\beta_2, jz_0\}]}{\beta_1^{(n+1)/\beta_2} \beta_2 [1 - \{1 - \exp(-\beta_1 x^{\beta_2})\}^{\alpha}]}, \tag{3.2.10}$$

where c_j is given by (2.2.8), v by (2.2.7), $z_0 = \beta_1 x^{\beta_2}$ and $\Gamma(\alpha)$, $\Gamma(\alpha, jz_0)$ are given by (3.2.6). To see this,

$$E[X^n|X > x] = \frac{1}{R(x)} \int_x^{\infty} y^n h(y) dy.$$

Integrating by parts, we then have

$$E[X^n|X > x] = \frac{1}{R(x)}[x^n R(x) + nI_n(x)] = x^n + \frac{nI_n(x)}{R(x)}, \tag{3.2.11}$$

where

$$\begin{aligned} I_n(x) &= \int_x^{\infty} y^{n-1} R(y) dy \\ &= \int_x^{\infty} y^{n-1}[1 - \{1 - \exp(-\beta_1 y^{\beta_2})\}^{\alpha}] dy \\ &= \sum_{j=1}^{v} c_j \int_x^{\infty} y^{n-1} \exp(-j\beta_1 y^{\beta_2}) dy \end{aligned}$$

Let $z = \beta_1 y^{\beta_2}$, then $y = \left(\frac{z}{\beta_1}\right)^{1/\beta_2}$, $(x, \infty) \to (z_0, \infty)$, $z_0 = \beta_1 x^{\beta_2}$ and $dy = \frac{z^{1/\beta_2 - 1}}{\beta_1^{1/\beta_2} \beta_2} - dz$. So that

$$\begin{aligned} I_n(x) &= \sum_{j=1}^{v} c_j \int_{z_0}^{\infty} \left(\frac{z}{\beta_1}\right)^{n/\beta_2} \exp(-jz) \frac{z^{1/\beta_2 - 1}}{\beta_1^{1/\beta_2} \beta_2} dz \\ &= \frac{1}{\beta_1^{(n+1)/\beta_2} \beta_2} \sum_{j=1}^{v} c_j \int_{z_0}^{\infty} z^{(n+1)/\beta_2 - 1} \exp(-jz) dz \\ &= \frac{1}{\beta_1^{(n+1)/\beta_2} \beta_2} \sum_{j=1}^{v} c_j \left[\frac{\Gamma\{(n+1)/\beta_2\}}{j^{(n+1)/\beta_2}} - \frac{\Gamma\{(n+1)/\beta_2, jz_0\}}{j^{(n+1)/\beta_2}} \right] \\ &= \frac{1}{\beta_1^{(n+1)/\beta_2} \beta_2} \sum_{j=1}^{v} \frac{c_j}{j^{(n+1)/\beta_2}} [\Gamma\{(n+1)/\beta_2\} - \Gamma\{(n+1)/\beta_2, jz_0\}], \end{aligned}$$

where v is given by (2.2.7), c_j by (2.2.8) and, $\Gamma(\alpha)$,$\Gamma(\alpha, \beta)$ are as defined by (3.2.6). So that, if $I_n(x)$ is substituted in (2.2.11), we then have

$$E[X^n|X > x] = x^n + \frac{nI_n(x)}{R(x)},$$
$$= x^n + \frac{n\sum_{j=1}^{v} \frac{c_j}{j^{(n+1)/\beta_2}} [\Gamma\{(n+1)/\beta_2\} - \Gamma\{(n+1)/\beta_2, jz_0\}]}{\beta_1^{(n+1)/\beta_2}\beta_2[1 - \{1 - \exp(-\beta_1 x^{\beta_2})\}^{\alpha}]}.$$

Remarks

1. The MRL function, as well as the HRF are important since each of them could be used to determine a unique corresponding lifetime distribution.
2. As in HRFs, lifetime distributions could exhibit increasing MRL (IMRL), decreasing MRL (DMRL), decreasing-increasing MRL(bathtub BMRL), increasing-decreasing MRL (upside down MRL (UMRL) or other shapes such as decreasing-increasing-decreasing (DIDMRL) or increasing-decreasing-increasing (IDIMRL) .
3. The MRL and HRF are connected by a relation, given by

$$\lambda(x) = \frac{1 + m'(x)}{m(x)}, \tag{3.2.12}$$

where $\lambda(x)$ is the HRF, $m(x)$ is the MRLF and $m'(x)$ is the first derivative of $m(x)$ with respect to x . This follows by observing that, since

$$m(x) = \frac{\int_x^{\infty} R(t)dt}{R(x)}, \quad \text{then } m'(x) = -1 - \frac{\int_x^{\infty} R(t)dtR'(x)}{[R(x)]^2}$$

$\Rightarrow m'(x) + 1 = m(x)\lambda(x)$ and (3.2.18) follows.
4. The relationship between the shapes of the MRL and HRF of a distribution were studied by Shanbhag (1970), Park (1985), Mi (1995), Ghitany (1998), and Tang et al. (1999), among others. Some of their results are summarized in the following theorem.

Theorem [Eissa (2005)] *For a non-negative continuous r.v. X, with PDF $h(x)$, finite mean μ and differentiable HRF, the MRL is*

(i) *Constant = μ, if X is the exponential distribution.*
(ii) *DMRL (IMRL), if $\lambda(x)$ is increasing (decreasing).*
(iii) *UMRL (BMRL) with a unique change point x_m if $\lambda(x)$ is bathtub (upside down bathtub) shape with a unique change point x_r, $0 < x_m < x_r < \infty$ and $f(0)\mu > 1$ (<1).*

It can be shown that if $X \sim EW(\alpha, \beta_1, \beta_2)$*, then the MRL is given by*

1. $m(x) = \beta_1$*, if* $\alpha = \beta_2 = 1$,
2. $m(x)$ *is DMRL, if* $\beta_2 \geq 1, \alpha\beta_2 \geq 1$,
3. $m(x)$ *is IMRL, if* $\beta_2 < 1, \alpha\beta_2 < 1$,
4. $m(x)$ *is BMRL with a unique change point* x_m *if* $\beta_2 < 1$ *and* $\alpha\beta_2 > 1$,
5. $m(x)$ *is UBMRL with a unique change point* x_m *if* $\beta_2 > 1$ *and* $\alpha\beta_2 < 1$.

3.2.3 Quantiles

The quantile x_q of the absolutely continuous distribution (2.2.4) is given, from (2.2.9) by

$$x_q = u^{-1}[-\ln(1 - q^{1/\alpha})],$$

where $u^{-1}(.)$ is the inverse function of $u(.)$.

This is true since the quantile is the value of x_q satisfying

$$q = H(x_q) = \{1 - \exp[-u(x_q)]\}^{\alpha}.$$

In particular, the median m of a distribution with CDF H(.) is given by

$$m = x_{1/2} = u^{-1}[-\ln(1 - 2^{-1/\alpha})].$$

It may be observed that in the non-exponentiated case $(\alpha = 1)$, the median reduces to

$$m = u^{-1}(\ln 2).$$

For the $EW(\alpha, \beta_1, \beta_2)$ distribution, $u(x) = \beta_1 x^{\beta_2} \Rightarrow u^{-1}(y) = \left(\frac{y}{\beta_1}\right)^{1/\beta_2}$. Substitution in (3.2.6) then yields the qth quantile to be given by

$$x_q = \left[\ln\left(1 - q^{1/\alpha}\right)^{-1/\beta_1}\right]^{1/\beta_2}. \qquad (3.2.13)$$

The median of the $EW(\alpha, \beta_1, \beta_2)$ distribution is given by

$$x_{1/2} = \left[\ln\left(1 - 2^{-1/\alpha}\right)^{-1/\beta_1}\right]^{1/\beta_2}, \qquad (3.2.14)$$

which is the corresponding value obtained in Table 2.2.

The EW quantile can be useful for fitting the distribution to frequency data using the method of probability plotting.

3.2.4 Modes

It follows, from (3.1.4), that

$$\ln h(x) = \ln(\alpha\beta_1\beta_2) + (\alpha - 1)\ln\left(1 - \exp[-\beta_1 x^{\beta_2}]\right) + (\beta_2 - 1)\ln x - \beta_1 x^{\beta_2}.$$

Differentiating both sides with respect to x, we obtain

$$\begin{aligned}\frac{h'(x)}{h(x)} &= \frac{(\alpha-1)\beta_1\beta_2 x^{\beta_2-1}\exp[-\beta_1 x^{\beta_2}]}{(1-\exp[-\beta_1 x^{\beta_2}])} + (\beta_2-1)x^{-1} - \beta_1\beta_2 x^{\beta_2-1}\\ \Rightarrow 0 &= (\alpha-1)\beta_1\beta_2 x^{\beta_2-1}\left(\exp[\beta_1 x^{\beta_2}]-1\right)^{-1} + (\beta_2-1)x^{-1} - \beta_1\beta_2 x^{\beta_2-1}.\end{aligned} \tag{3.2.15}$$

The value of x which satisfies this equation is the mode of the $\mathrm{EW}(\alpha, \beta_1, \beta_2)$ distribution. The following cases are particularly important

1. $\alpha = 1$ [the case of $\mathrm{W}(\beta_1, \beta_2)$]
   In this case, (3.2.15) reduces to

$$\begin{aligned}0 &= (\beta_2 - 1)x^{-1} - \beta_1\beta_2 x^{\beta_2-1}\\ \Rightarrow \text{mode } &= \left(\frac{\beta_2 - 1}{\beta_1\beta_2}\right)^{1/\beta_2}, \quad \beta_2 > 1,\end{aligned} \tag{3.2.16}$$

   which agrees with the mode of the $\mathrm{W}(\beta_1, \beta_2)$.
   It may be noticed that for $\beta_2 < 1$ and $\alpha\beta_2 > 1$, the mode of the $\mathrm{EW}(\alpha, \beta_1, \beta_2)$ is the same as in (3.2.16). The case $\beta_2 < 1, \alpha\beta_2 \le 1$ corresponds to a monotone decreasing PDF and that of $\beta_2 > 1, \alpha\beta_2 \le 1$ corresponds to a bathtub shape. See Fig. 3.1.
2. $\beta_2 = 1, \alpha > 1$ (the case of $\mathrm{EE}(\alpha, \beta_1)$).
   In this case, $\alpha\beta_2 > 1$ and (3.2.15) reduces to

$$(\alpha - 1)(\exp[\beta_1 x] - 1)^{-1} = 1 \Rightarrow x = \frac{1}{\beta_1}\ln\alpha.$$

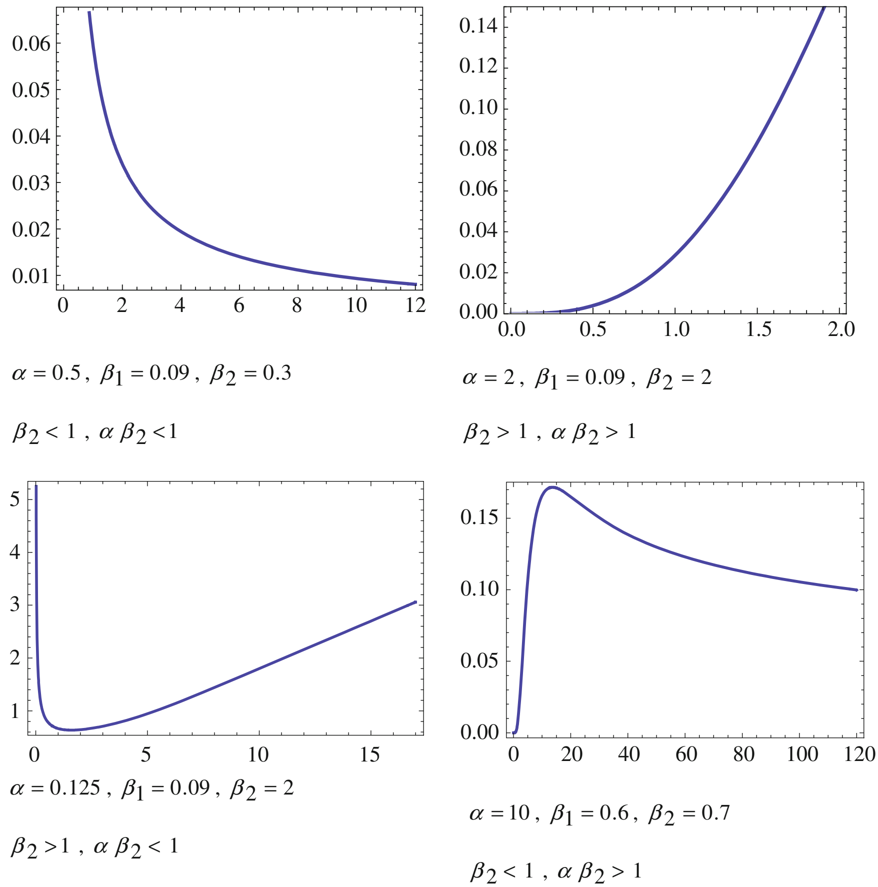

Fig. 3.1 HRFs of EW($(\alpha, \beta_1, \beta_2)$, for different parameter values

Since

$$\ln \alpha = \frac{2}{\beta_1}\left[\frac{\alpha-1}{\alpha+1} + \frac{1}{3}\left(\frac{\alpha-1}{\alpha+1}\right)^3 + \frac{1}{5}\left(\frac{\alpha-1}{\alpha+1}\right)^5 + \cdots\right],$$

see Abramowitz and Stegun (1970), p. 67, then

$$x = \frac{1}{\beta_1}\ln \alpha = \frac{2}{\beta_1}\left[\frac{\alpha-1}{\alpha+1} + \frac{1}{3}\left(\frac{\alpha-1}{\alpha+1}\right)^3 + \frac{1}{5}\left(\frac{\alpha-1}{\alpha+1}\right)^5 + \cdots\right]$$

A first approximation for the mode is obtained by neglecting the powers greater than one to obtain

$$x = \frac{2}{\beta_1}\left[\frac{\alpha - 1}{\alpha + 1}\right]. \tag{3.2.17}$$

This is the mode of the EE(α, β_1), when $\alpha > 1$. If $\alpha \leq 1$, so that $\alpha\beta_2 \leq 1$, the density function is monotone decreasing on the positive half of the real line.

3. $0 < \beta_2 < 1, \alpha > 1$, such that $\alpha\beta_2 > 1$

In this case, the density function is unimodal with mode given by

$$M_1 = \begin{cases} \left[\frac{2(\alpha\beta_2 - 1)}{\beta_1\beta_2(\alpha+1)}\right]^{1/\beta_2}, & \alpha\beta_2 > 1, \\ 0, & \alpha\beta_2 \leq 1. \end{cases} \tag{3.2.18}$$

This mode is obtained by observing that

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} \frac{\gamma_{2k}}{(2k)!} z^{2k}, \quad z < 2\pi \tag{3.2.19}$$

where, $\gamma_2 = \frac{1}{6}, \quad \gamma_4 = -\frac{1}{30}, \quad \gamma_6 = \frac{1}{42}, \quad \gamma_8 = -\frac{1}{30}, \ldots$

The γ's are Bernoulli's numbers.

From (3.2.15),

$$0 = (\beta_2 - 1) - \beta_1\beta_2 x^{\beta_2} + \frac{(\alpha - 1)\beta_2\beta_1 x^{\beta_2}}{(\exp[\beta_1 x^{\beta_2}] - 1)}.$$

By writing $z = \beta_1 x^{\beta_2}$ and using the first approximation in expansion (3.2.15), then

$$0 = (\beta_2 - 1) - \beta_2 z + (\alpha - 1)\beta_2[1 - (z/2)] \Rightarrow z = \frac{2(\alpha\beta_2 - 1)}{(\alpha + 1)\beta_2}$$

$$\Rightarrow M_1 = \left[\frac{2(\alpha\beta_2 - 1)}{(\alpha + 1)\beta_1\beta_2}\right]^{1/\beta_2}, \quad \alpha\beta_2 > 1.$$

It may be noticed that the mode M_1 and median are equal when there exist values of $\beta_2 > 1$ and $\alpha > 1$ such that

$$\beta_2[2\alpha + (\alpha + 1)\ln(1 - 2^{-1/\alpha})] - 2 = 0.$$

For $\alpha\beta_2 \leq 1$, the PDF of the EW(α, β_1, β_2) is monotone decreasing. The case $0 < \beta_2 < 1, \alpha < 1$ also leads to a monotone decreasing PDF.

Mudholkar and Hutson (1996) have derived another approximate formula for the mode, say M_2, in the case $\alpha\beta_2 > 1$.

The mode M_2, for $\beta_1 = 1$, is given by

$$M_2 = \left[-\ln\left(\frac{3}{2} + \frac{1}{2\alpha} - \frac{\sqrt{\beta_2(\beta_2 - 8\alpha + 2\alpha\beta_2 + 9\alpha^2\beta_2)}}{2\alpha\beta_2} \right) \right]^{1/\beta_2}.$$

Eissa (2005) made an extensive computational comparison between M_1 with M_2. Their differences were computed and their decrease (or increase) was observed, by fixing some of the parameters and changing others. For details, see Eissa (2005) or Nassar and Eissa (2003).

Using expression (3.2.15), the mode of the EW and EE distributions are given by (3.2.16) and (2.3.17), respectively. The mode of ERay (α, β_2) distribution, can be obtained by taking $\beta_2 = 2$, in (3.2.18), to get the mode = $\left[\frac{2\alpha-1}{\beta_1(\alpha+1)}\right]^{1/2}$, $2\alpha > 1$.

3.2.5 Hazard Rate Function

The hazard rate function (HRF) corresponding to the exponentiated CDF (1.1.5) is given, for x > 0, by

$$\lambda_H(x) = \frac{h(x)}{R_H(x)} = \frac{\alpha[G(x)]^{\alpha-1}g(x)}{1 - [G(x)]^{\alpha}} = \alpha[1 - \in_\alpha(x)]\lambda_G(x), \tag{3.2.20}$$

where $\lambda_G(x) = g(x)/R_G(x)$ and $\in_\alpha(x) = \frac{1-G^{\alpha-1}(x)}{1-G^{\alpha}(x)}$.

If $0 < \alpha < 1$, then $-\infty < \in_\alpha(x) \le 1 \Rightarrow 1 - \in_\alpha(x) \ge \frac{1}{\alpha} \Rightarrow \lambda_H(x) \ge \lambda_G(x)$.
If $\alpha \ge 1$, then $\frac{\alpha-1}{\alpha} \le \in_\alpha(x) \le 1 \Rightarrow 0 \le \alpha[1 - \in_\alpha(x)] \le 1 \Rightarrow 0 \le \lambda_H(x) \le \lambda_G(x)$.

Notice that, since G(x) is a CDF on $[0, \infty)$, then $\in_\alpha(0) = 1$ and $\in_\alpha(\infty) = \lim_{x\to\infty}\left[\frac{1-G^{\alpha-1}(x)}{1-G^{\alpha}(x)}\right] = \frac{\alpha-1}{\alpha}$. So that, $\frac{\alpha-1}{\alpha} \le \in_\alpha(x) \le 1$, for all $x \in [0, \infty)$. Hence, $0 \le \alpha[1 - \in_\alpha(x)] \le 1$.

By differentiating $\lambda_H(x)$, given by (3.2.16) with respect to **x** and simplifying, it can be shown that, provided that $G(x)g'(x) < g^2(x)$,

H has an increasing hazard rate (IHR), if:

$$G^{\alpha}(x) > 1 - \frac{\alpha}{1 - \{G(x)g'(x)/g^2(x)\}}. \tag{3.2.21}$$

H has a decreasing hazard rate (DHR), if:

$$G^{\alpha}(x) < 1 - \frac{\alpha}{1 - \{G(x)g'(x)/g^2(x)\}}. \tag{3.2.22}$$

If equality holds, then critical points at which extrema for $H(x)$ may be obtained and so other shapes for the HRF of $H(x)$ are expected to take place.

It can be shown that if $X \sim EW(\alpha, \beta_1, \beta_2)$, then the HRF $\lambda_H(x)$ is:

(i) CHR $= \beta_1$, if and only if $\alpha = \beta_2 = 1$.
(ii) DHR, if $\beta_2 \le 1$ and $\alpha\beta_2 \le 1$.
(iii) IHR, if $\beta_2 \ge 1$ and $\alpha\beta_2 \ge 1$.
(iv) Bathtub hazard rate (BTHR), if $\beta_2 > 1$ and $\alpha\beta_2 < 1$.
(v) Upside down bathtub hazard rate (UBTHR), if $\beta_2 < 1$ and $\alpha\beta_2 > 1$.

3.2.6 Proportional Reversed Hazard Rate Function

The proportional reversed hazard rate function (PRHRF) of H, denoted by $\lambda^*_H(x)$ is defined by

$$\lambda^*_H(x) = \frac{d}{dx}[\ln H(x)] = \frac{h(x)}{H(x)}.$$

It may be noticed, from (3.2.20), that the HRF $\lambda_H(x)$ of H(x) is not proportional to the $\lambda_G(x)$ of G(x). However, the PRHRF $\lambda^*_H(x)$ of H(x) can be seen to be proportional to the PRHRF $\lambda^*_G(x)$ of G(x). In fact,

$$\lambda^*_H(x) = \frac{h(x)}{H(x)} = \frac{\alpha[G(x)]^{\alpha-1}g(x)}{[G(x)]^{\alpha}} = \alpha\lambda^*_G(x) \tag{3.2.23}$$

This is why the exponentiated model is equivalently called PRHRM.

It may also be noted that $\lambda^*_H(x)$ dx provides the probability of failing in (x-d x, x), when a unit is found failed at time x. In general, the PRHRF has been found to be useful in estimating the SF for left censored data.

It can be seen that the CDF H(x) can be written, in terms of the HRF $\lambda_H(x)$ and PRHRM $\lambda^*_H(x)$ of H as follows

$$H(x) = \frac{\lambda_H(x)}{\lambda_H(x) + \lambda^*_H(x)}. \tag{3.2.24}$$

So that the SF and PDF are given, respective by

$$R_H(x) = \frac{\lambda^*_H(x)}{\lambda_H(x) + \lambda^*_H(x)} \text{ and } h(x) = R_H(x)\,\lambda_H(x) = \frac{\lambda^*_H(x)\,\lambda_H(x)}{\lambda_H(x) + \lambda^*_H(x)}.$$

3.2.7 Density Function of the rth m-Generalized Order Statistic

If $u(x) = \beta_1 x^{\beta_2}$, then (2.2.17) becomes, when $m \neq -1$,

$$
\begin{aligned}
f_{X_r^*}(x) = {} & \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \left(1 - [1 - \exp(-\beta_1 x^{\beta_2})]^{\alpha}\right)^{\gamma_r - 1} \\
& \alpha\beta_1\beta_2 x^{\beta_2 - 1} \exp(-\beta_1 x^{\beta_2})[1 - \exp(-\beta_1 x^{\beta_2})]^{\alpha - 1} \\
& \left[1 - \left(1 - \exp(-\beta_1 x^{\beta_2})\right)^{(m+1)\alpha}\right]^{r-1}.
\end{aligned} \tag{3.2.25}
$$

From (2.2.17), the PDF of the rth OOS is given by $f_{X_{r:n}}(x) = \sum_{j=1}^{n-r} \omega_j h_j^*(x)$, where ω_j is given by (2.2.18) and

$$
h^*(x) = \alpha(r+j)\beta_1\beta_2 x^{\beta_2 - 1} \exp(-\beta_1 x^{\beta_2})[1 - \exp(-\beta_1 x^{\beta_2})]^{\alpha(r+j)-1}. \tag{3.2.26}
$$

Also, if $m = -1$, it follows from (2.2.20), that the PDF of OURV is given by

$$
\begin{aligned}
f_{X_r^*}(x) = {} & \frac{1}{(r-1)!} \alpha\beta_1\beta_2 x^{\beta_2 - 1} \mathrm{e}^{-\beta_1 x^{\beta_2}} (1 - \mathrm{e}^{-\beta_1 x^{\beta_2}})^{\alpha - 1} \\
& [-\ln\{1 - (1 - e^{-\beta_1 x^{\beta_2}})^{\alpha}\}]^{r-1}.
\end{aligned} \tag{3.2.27}
$$

- The PDF of the s-out-of-n structure is given, from (2.2.6), by

$$
\begin{aligned}
f_{n-s+1:n}(x) = {} & \binom{n}{n-s+1} (n-s+1)\alpha\beta_1\beta_2 x^{\beta_2 - 1} \mathrm{e}^{-\beta_1 x^{\beta_2}} \\
& \left[1 - \mathrm{e}^{-\beta_1 x^{\beta_2}}\right]^{\alpha(n-s+1)-1} \left[1 - \left(1 - \mathrm{e}^{-\beta_1 x^{\beta_2}}\right)^{\alpha}\right]^{s-1}.
\end{aligned} \tag{3.2.28}
$$

- The PDFs of a series (n-out-of-n) and parallel (1-out-of-n) structures are obtained, for x > 0, from (3.2.4), respectively, as follows:

$$
f_{1:n}(x) = n\alpha\beta_1\beta_2 x^{\beta_2 - 1} \mathrm{e}^{-\beta_1 x^{\beta_2}} (1 - \mathrm{e}^{-\beta_1 x^{\beta_2}})^{\alpha - 1} \left[1 - (1 - \mathrm{e}^{-\beta_1 x^{\beta_2}})^{\alpha}\right]^{n-1}. \tag{3.2.29}
$$

$$
f_{n:n}(x) = n\alpha\beta_1\beta_2 x^{\beta_2 - 1} \mathrm{e}^{-\beta_1 x^{\beta_2}} (1 - \mathrm{e}^{-\beta_1 x^{\beta_2}})^{n\alpha - 1}. \tag{3.2.30}
$$

Notice that in the non-exponentiated case ($\alpha = 1$),

$$f_{1:n}(x) = n\beta_1\beta_2 x^{\beta_2-1}\mathrm{e}^{-\beta_1 x^{\beta_2}},$$
$$f_{n:n}(x) = n\beta_1\beta_2 x^{\beta_2-1}\mathrm{e}^{-\beta_1 x^{\beta_2}}(1-\mathrm{e}^{-\beta_1 x^{\beta_2}})^{n-1},$$

which agree with the PDFs of the minimum and maximum order statistics based on a population with CDF $1-\mathrm{e}^{-\beta_1 x^{\beta_2}}$.

1. Expression (3.2.1) agrees with the expression obtained by Sarabia and Castillo (2005), for the PDF of the rth ordinary order statistic, from $\mathrm{EW}(\alpha, \beta_1, \beta_2)$. This expression makes it easy to obtain the corresponding CDF, **SF,** moments and product moments.
2. Mudholkar and Hutson (1996) obtained asymptotic distributions of the extreme order statistics $X_{1:n}$ *and* $X_{n:n}$ and the extreme spacings $X_{2:n} - X_{1:n}$ and $X_{n::n} - X_{n-1:n}$. They showed that $n^{1/(\alpha\beta_2)}X_{1:n}$ approaches in distribution $Z^{1/(\alpha\beta_2)}$, where Z is a standard exponential random variable. Also,

$$n^{1/(\alpha\beta_2)}[X_{2:n} - X_{1:n}] \to Z^{1/(\alpha\beta_2)} - X^{1/(\alpha\beta_2)}$$

and

$$(\ln n)^{1-1/\beta_2}[X_{n:n} - X_{n-1:n}] \to \frac{1}{\beta_2}[\ln Z - \ln X],$$

where (Z, X) has the joint PDF

$$f(z,x) = \begin{cases} \exp(-z), & 0<x<z \\ 0, & otherwise \end{cases}.$$

Furthermore,

$$X_{2:n} - X_{1:n} = O_p\left(n^{1/(\alpha\beta_2)}\right) \text{ and } X_{n:n} - X_{n-1:n} = O_p\left((\ln n)^{1/\beta_2-1}\right).$$

Explicit expressions for single and product moments of order statistics from the ERay (or, Burr type X) which is $[\mathrm{EW}(\alpha, \beta_1 = 1,\ \beta_2 = 2)]$ distribution were obtained by Raqab (1998). He also obtained expressions for percentiles and suggested an estimate for α that is based on order statistics. Ahmad (2001) gave recurrence relations for single and product moments of order statistics from doubly truncated EW distribution. Ahmad and Al-Matrafi (2006) derived recurrence relations for moments and conditional moments of generalized order statistics from the EW distribution. Khan et al. (2008) established recurrence relations for single and product moments of dual generalized order statistics from the EW distribution.

3.3 Estimation of $\alpha, \beta_1, \beta_2,\ R_H(x_0)$ and $\lambda_H(x_0)$, (Allparameters are Unknown)

The case in which α is the only unknown parameter of H shall not be discussed here and we shall be satisfied with the discussion given in Chap. 2. In this section, all of the parameters α, β_1, β_2, of H are assumed to be unknown.

3.3.1 Maximum Likelihood Estimation

The three LEs, are given, from (2.4.3) and (2.4.4) (when $k = 2$) by

$$\frac{\partial \ell}{\partial \alpha} : 0 = \frac{r}{\alpha} + \sum_{i=1}^{r} \ln G(x_i|\beta) - \frac{(n-r)\{G(x_r|\beta)\}^{\alpha} \ln G(x_r|\beta)}{1 - \{G(x_r|\beta)\}^{\alpha}} \tag{3.3.1}$$

and for $j = 1, 2$,

$$\begin{aligned} \frac{\partial \ell}{\partial \beta_j} : 0 = (\alpha - 1) \sum_{i=1}^{r} \frac{1}{G(x_i|\beta)} \frac{\partial G(x_i|\beta)}{\partial \beta_j} + \sum_{i=1}^{r} \frac{1}{g(x_i|\beta)} \frac{\partial g(x_i|\beta)}{\partial \beta_j} \\ - \frac{(n-r)\alpha\{G(x_r|\beta)\}^{\alpha-1}}{1 - \{G(x_r|\beta)\}^{\alpha}} \frac{\partial G(x_r|\beta)}{\partial \beta_j}, \end{aligned} \tag{3.3.2}$$

where ℓ is the log-likelihood function, given by (2.4.2). The base distribution G is given by $G(x|\beta) = 1 - \exp(-\beta_1 x^{\beta_2}), \beta = (\beta_1, \beta_2)$ and $g(x|\beta)$ is the corresponding PDF of G, given by

$$g(x|\beta) = \beta_1 \beta_2 x^{\beta_2 - 1} \exp(-\beta_1 x^{\beta_2}).$$

Substitution of G and g and their derivatives with respect to β_1 and β_2, in the above three LEs, and solving such system, we obtain the MLEs $\hat{\alpha}_{ML}, \hat{\beta}_{1,ML}, \hat{\beta}_{2,ML}$ The MLEs $\hat{R}_{ML}(x_0)$ and $\hat{\lambda}_{ML}(x_0)$ can be computed by applying the invariance property of MLEs. This system of equations can be solved by using Matlab, Mathematica or IMSL routine.

In the complete sample case, the three equations reduce to

$$0 = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln G(x_i|\beta),$$

and, for $j = 1, 2,$

$$\frac{\partial \ell}{\partial \beta_j} : 0 = (\alpha - 1) \sum_{i=1}^{r} \frac{1}{G(x_i|\beta)} \frac{\partial G(x_i|\beta)}{\partial \beta_j} + \sum_{i=1}^{r} \frac{1}{g(x_i|\beta)} \frac{\partial g(x_i|\beta)}{\partial \beta_j},$$

where $G(x_i|\beta) = 1 - \exp(-\beta_1 x_i^{\beta_2})$ and $g(x_i|\beta) = \beta_1 \beta_2 x_i^{\beta_2 - 1} \exp(-\beta_1 x_i^{\beta_2})$. For interval estimation and testing hypotheses, it is useful to have explicit expressions for the elements of Fisher information matrix. Asymptotic normality of MLEs implies that the distribution of the vector: $\sqrt{n}(\hat{\alpha} - \alpha, \hat{\beta}_1 - \beta_1, \hat{\beta}_2 - \beta_2)$ tends to the distribution of a trivariate random vector with mean zero and variance-covariance matrix I^{-1}, where I is Fisher information matrix. So that, 100(1−τ)% confidence intervals for α, β_1, β_2 are given, respectively, by

$$\hat{\alpha} - z_{\tau/2}\hat{\sigma}(\hat{\alpha})/\sqrt{n} < \alpha < \hat{\alpha} + z_{\tau/2}\hat{\sigma}(\hat{\alpha})/\sqrt{\text{n}},$$

where $\hat{\sigma}(\hat{\alpha})$ is the estimated standard deviation of $\hat{\alpha}$. Variance $\hat{\alpha}$, is the diagonal element in the variance-covariance matrix I^{-1}, that corresponds to α and $z_{\tau/2}$ is the $\tau/2$ quantile under the standard normal curve.

3.3.2 Fisher Information Matrix

Qian (2011) obtained the 3 × 3 Fisher information matrix I for 3-parameter EW distribution under type II censoring. He used F instead of H and the base distribution G is given by

$$G(x) = 1 - \exp(x/\sigma)^{\beta}, \tag{3.3.3}$$

So that the parameters β, σ are related to β_1 and β_2 (used in this book) by the relations $\beta_1 = (1/\sigma)^{\beta}$, $\beta_2 = \beta$ or $\sigma = \beta_1^{-1/\beta_2}$, $\beta = \beta_2$.

Using $F(x) = [1 - \exp(x/\sigma)^{\beta}]^{\alpha}$, Qian (2011) obtained the elements of Fisher information matrix in the following theorem.

Theorem *Qian* (2011) *Let* $\frac{r}{n} \to p = F(a_p, \theta) \in (0, 1)$, *where* a_p *is the 100 p percentile of* $F(x; \theta)$.

For EW family with parameters $\theta = (\alpha, \beta, \sigma)^T$ *under type II censoring, we have*

$$I_p^{11}(\theta) = \frac{1}{\alpha^2} \int_0^p \left(1 + \frac{\ln x}{1-x}\right) dx,$$

$$I_p^{22}(\theta) = \frac{\alpha}{\beta^2} \int_0^{p^{1/\alpha}} \{1 + \ln[-\ln(1-x)]\psi(x;\ \alpha)\}^2 dx,$$

$$I_p^{33}(\theta) = \alpha \left(\frac{\beta}{\alpha}\right)^2 \int_0^{p^{1/\alpha}} \psi^2(x;\ \alpha) x^{\alpha-1} dx,$$

$$I_p^{12}(\theta) = I_p^{21}(\theta) = \frac{\alpha}{\beta} \int_0^{p^{1/\alpha}} \left(\frac{1}{\alpha} + \frac{\ln x}{1-x^\alpha}\right) \{1 + \ln[-\ln(1-x)]\psi(x;\alpha)\} x^{\alpha-1} dx$$

$$I_p^{13}(\theta) = I_p^{31}(\theta) = \frac{\alpha\beta}{\sigma} \int_0^{p^{1/\alpha}} \left(\frac{1}{\alpha} + \frac{\ln x}{1-x^\alpha}\right) \psi(x;\alpha) x^{\alpha-1} dx,$$

$$I_p^{23}(\theta) = I_p^{23}(\theta) = -\frac{\alpha}{\sigma} \int_0^{p^{1/\alpha}} \{1 + \ln[-\ln(1-x)]\psi(x;\alpha)\} \psi(x;\alpha) x^{\alpha-1} dx,$$

where

$$\psi(x;\alpha) = 1 - \left(\frac{x}{\sigma}\right)^\beta + \frac{(\alpha-1)xf(x;\theta)}{\alpha\beta F(x;\theta)} + \frac{xf(x;\theta)}{\beta(1-F(x;\theta)}.$$

For proof, see Qian (2011).

In the complete sample case $(r = n)$, $p = 1$. So that the upper limits of all of the above integrals will be 1.

3.3.3 Bayes Estimation of $\boldsymbol{\alpha}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{R}_{\boldsymbol{H}}(\boldsymbol{x}_0), \boldsymbol{\lambda}_{\boldsymbol{H}}(\boldsymbol{x}_0)$

- SBM

By using the prior, given by (2.5.18), the Bayes estimators, based on the LINEX loss function, are given by (2.4.7)–(2.4.10).

Nassar and Eissa (2004) considered Bayes estimation of the two-parameter EW distribution whose CDF is given by

$$F(x) = [1 - \exp(-x^{\alpha})]^{\theta}. \tag{3.3.4}$$

This is a special case of the three-parameter EW$(\alpha, \beta_1, \beta_2)$ distribution in which $\beta_1 = 1, \beta_2 = \alpha, \alpha = \theta$. They estimated the parameters, SF and HRF, based on SEL and LINEX loss functions for type II censoring and used a subjective prior of the form:

$$\pi(\alpha, \theta) = \pi_1(\alpha)\pi_2(\theta|\alpha),$$

where $\pi_2(\theta|\alpha)$ is gamma and $\pi_1(\alpha)$ is exponential. In their computations, they used an approximation form due to Lindely(1980).

Singh et al. (2005) considered Bayes estimation of the three-parameter EW distribution whose CDF takes the form

$$F(x) = [1 - \exp(x/\sigma)^{\beta}]^{\alpha}, \tag{3.3.5}$$

where the parameters α, β, σ are all positive. They estimated the three parameters based on SEL and LINEX loss functions for type II censoring and used an objective, non-informative prior, assuming independence of α, β, σ. In their computations, they suggested the use of 16-points Gauss quadrature formula.

- MCMC

Example 3 in Chap. 2 describes the steps to be followed to obtain Bayes estimators, based on the LINEX loss function, using the MCMC algorithm.

3.4 Bayes Prediction of Future Observables

A $100(1 - \tau)\%$ Bayes prediction interval, for the sth future observable (based on the one-sample scheme), has bounds L and U, given by the solution of (2.5.10). In the two-sample case, the bounds can be obtained by the solution of (2.5.16). In both cases, the EW distribution is assumed to be the underlying base line distribution.

In the two-sample case the sample size m of the future sample was assumed to be fixed.

3.5 Related Distributions to the EW Family

Before discussing relations between the EW and other distributions, we shall first show how to get the so called beta-G distribution.

Beta-G Distributions

Given two absolutely continuous CDFs F and G, so that f and g are their corresponding PDFs, one can obtain a new distribution H by composing

(i) F with G, so that $H(x) = F[G(x)$ is a CDF, (3.5.1)

(ii) F with $\bar{\eta}(x) = -\ln R_G(x)$, so that $H(x) = F[\bar{\eta}(x)]$ is a CDF, or

$$F \textit{ with } \eta(x) = -\ln G(x), \textit{ so that } R_H(x) = F[\eta(x)] \textit{ is a SF}$$

Eugene et al. (2002), suggested the use of a PDF, given by

$$h(x) = \frac{1}{B(a,b)}[G(x)]^{a-1}(1 - G(x)]^{b-1}g(x), \tag{3.5.2}$$

where a and b are positive real numbers, $B(a, b)$ is the beta function and $G(x)$ is a normal CDF.

With $G(x)$ being a baseline distribution, Eugene et al. (2002) defined a generalized class of distributions as follows:

$$F(x) = \frac{1}{B(a,b)}\int_0^{G(x)} y^{a-1}(1-y)^{b-1}dy \tag{3.5.3}$$

in which $G(x)$ was chosen to be normal.

Differentiating both sides with respect to x, we obtain the PDF $h(x)$, given by (3.5.2). The CDF $F(x)$ is known as the *beta-G distribution*. For example, if G is Weibull, then $H(x)$ is *beta-Weibull distribution* and so on.

Distributions obtained by composition of this kind, such as the beta-normal, beta-Fréchet, beta-Gumble, beta-exponential, beta-exponentiated exponential, beta-Burr type XII, beta-power function, were studied by Eugene et al. (2002), Nadarajah and Gupta (2005), Nadarajah and Kotz (2006), Barreto-Souza et al. (2010), Paranoíba et al. (2011) and Cordeiro and Brito (2012).

Two important special cases result from the beta-G distribution:

1. When $b = 1$ in (3.5.4), then
   $F(x) = [(G(x)]^a$, the CDF of the exponentiated-G distribution.
2. When $a = 1$ in (3.5.2), then

$$H(x) = 1 - [1 - G(x)]^b \Leftrightarrow R_H(x) = [R_G(x)]^b.$$

That is, the SF of H is the exponentiated SF of G.

Relations of the EW distribution to other distributions are given by the following:

It was indicated, in Sect. 3.1, that the $\mathrm{EW}(\alpha,\beta_1,\beta_2)$ distribution includes as special cases, $\mathrm{EE}(\alpha,\beta_1)$, when $\beta_2=1$, ERay (α,β_1), when $\beta_2=2$, Weibull(β_1,β_2),when $\alpha=1$, Exp (β_1),when $\alpha=1,\beta_2=1$ and Ray (β_1) when $\alpha=1,\beta_2=2$.

Chapter 4 is devoted to cover the EE distribution.

The ERay (or, equivalently, Burr Type X) distribution, was considered by several researchers, among whom are Sartawy and Abu-Salih (1991), Raqab (1998), Ahmad (2001), Ahmad and Al-Matrafi (2006), Alshunnar et al. (2010) and Montazer and Shayib (2010).

The following distributions are related to the EW distribution:

1. Carrasco et al. (2008) introduced a *generalized modified Weibull* distribution

$$H(x)=[1-\exp\{-\alpha x^{\gamma}\mathrm{e}^{\lambda x}\}]^{\beta},\quad x>0, \tag{3.5.4}$$

where all of the four parameters $\alpha,\beta,\gamma,\lambda$ are positive.

Important distributions result as special cases, for example:

   (i) If $\lambda=0$, we get $H(x)=[1-\exp\{-\alpha x^{\gamma}\}]^{\beta}$, which is the CDF of the $EW(\alpha,\beta,\gamma)$ distribution.
   (ii) If $\beta=1,\gamma=0$, we get $H(x)=1-\exp[-\alpha\mathrm{e}^{\lambda x}]$, which is the CDF of *extreme value* distribution with parameters (α,λ).
   (iii) If $\beta=1$, we get $H(x)=1-\exp\{-\alpha x^{\gamma}\mathrm{e}^{\lambda x}\}$, which is the CDF of the *modified Weibull* MW (α,γ,λ) distribution. See Lai et al. (2003).
   (iv) The *beta integrated* distribution, see Lai et al. (2003), has a CDF of the form

$$H(x)=\exp\left[-ax^{b}(1-dx)^{c}\right],\quad 0<x<1\,l\,d.$$

If $d=1/n, c=n\lambda$, then $\left(1-\frac{x}{n}\right)^{-n\lambda}\to\exp(\lambda x)$, as $n\to\infty$. This yields

$$H(x)=\exp[-ax^{b}\mathrm{e}^{\lambda x}],$$

which is the CDF of the $\mathrm{MW}(a,b,\lambda)$.

2. Famoye et al. (2005) introduced the *beta-Weibull* distribution, whose PDF is given, from (3.5.4), by

$$h(x)=\frac{\beta_1\beta_2}{B(a,b)}x^{\beta_2-1}\exp(-b\beta_1x^{\beta_2})[1-\exp(-\beta_1x^{\beta_2})]^{a-1},\quad x>0, \tag{3.5.5}$$

If $b=1$, PDF (3.5.5) reduces to the $\mathrm{EW}(a,\beta_1,\beta_2)$.

Famoye et al. (2005), Lee et al. (2007), Wahed et al. (2009) and Cordeiro et al. (2008, 2011a, b) studied the distribution which allows for DHR, IHR, BTHR and UBTHR functions.

3. Cordeiro et al. (2008) introduced the 4-parameter Kumarswamy-Weibull distribution whose PDF is given by

$$h(x) = \frac{\alpha\gamma\beta_1\beta_2 x^{\beta_2-1}\exp(-\beta_1 x^{\beta_2})[1-\exp(-\beta_1 x^{\beta_2})]^{\alpha-1}}{[1-\{1-\exp(-\beta_1 x^{\beta_2})\}^{\alpha}]^{1-\gamma}}, \quad x > 0, \qquad (3.5.6)$$

where all 4 parameters are positive. Such PDF includes, among others, the $\mathrm{EW}(a, \beta_1, \beta_2)$, when $\gamma = 1$. This distribution allows for DHR, IHR, BTHR and UBTHR functions.

4. Silva et al. (2010), introduced the five-parameter beta-modified Weibull distribution, whose PDF is given, from (3.5.4), by

$$h(x) = \frac{ab\delta x^{\gamma-1}(\gamma+\lambda x)\exp(\lambda x)\exp\{-b\delta x^{\gamma}\exp(\lambda x)\}}{B(a,b)[1-\exp\{-b\delta x^{\gamma}\exp(\lambda x)\}]^{1-a}}, \quad x > 0, \qquad (3.5.7)$$

where all of the five parameters are positive.
It includes some distributions, among which is the $\mathrm{EW}(a, \gamma, \delta)$, that is obtained by taking $b = 1, \lambda = 0$.

5. Alexander et al. (2012) introduced a class of 'generalized beta-generated distributions' whose PDF is obtained, from (3.5.4), as

$$\begin{aligned} h(x) &= \frac{1}{B(a,b)}[G^c(x)]^{a-1}[1-G^c(x)]^{b-1}[cG^{c-1}(x)g(x)], \\ &= \frac{c}{B(a,b)}[G(x)]^{ca-1}[1-G^c(x)]^{b-1}g(x), \end{aligned} \qquad (3.5.8)$$

by taking the base distribution to be $[G(x)]^c$, where all of the parameters are positive, $G(x)$ is an absolutely continuous CDF and $g(x)$ is the corresponding PDF.

If c = 1, the PDF reduces to the beta generated PDF.
If a = 1, the PDF reduces to the Kumuraswamy generated PDF.
If b = 1, the PDF reduces to a family of PDFs, among which it includes the EW PDF (when $G(x)$ is Weibull).

6. Cordeiro and de Castro (2011) introduced a family of Kumuraswamy generated distribution whose CDF is given by

$$H(x) = 1 - [1 - G^{\alpha}(x)]^{\beta}, \quad x > 0, \qquad (3.5.9)$$

where $\alpha, \beta > 0$ and $G(.)$ is a CDF.

The EW CDF is obtained from (3.5.10) by taking $\beta = 1$ and $G(.)$ to be Weibull. Lemonte et al. (2011) introduced the family of *exponentiated Kumuraswamy distributions*, generalizing (3.5.10), whose CDF is given by

$$H(x) = \left(1 - [1 - G^{\alpha}(x)]^{\beta}\right)^{\gamma}, \quad x > 0, \tag{3.5.10}$$

If $\gamma = \beta = 1$ and $G(.)$ is Weibull, the distribution reduces to the EW CDF.

7. Cardeiro et al. (2011c) introduced the family of beta extended Weibull distributions with PDF

$$h(x) = \frac{ck(x)}{B(a,b)}[1 - \exp(-cK(x))]^{a-1}\exp(-bcK(x)), \quad x > 0, \tag{3.5.11}$$

where all parameters are positive, $K(x) \geq 0$ and $k(x) = \frac{dK(x)}{dx}$.
This PDF can be obtained by applying (3.5.4), in which $G(x) = 1 - \exp(-cK(x))$.
If $K(x) = x^{\beta}$ and $b = 1$, then

$$h(x) = ac\beta x^{\beta-1}\exp(-cx^{\beta})[1 - \exp(-cx^{\beta})]^{a-1}, \quad x > 0,$$

which is the PDF of $\text{EW}(a, c, \beta)$.
Other families of distributions can also be obtained from (3.5.12). It allows for DHR, IHR, BTHR and UBTHR functions.

8. Zaindin and Sarhan (2011) introduced a generalized Weibull distribution whose CDF is given by

$$H(x) = [1 - \exp(-\alpha x - \beta x^{\gamma})]^{\lambda}, \quad x > 0, \tag{3.5.12}$$

This specializes to the EW distribution when $\alpha = 0$. It allows for DHR, IHR, BTHR functions.
Some other relations and generalizations are given in Nadarajah et al. (2013).

3.6 Applications

Applications of the EW model have been widespread. For example: modeling of extreme value data using floods, statistically optimal accelerated life test plans, modeling for carbon fibrous composites, modeling tree diameters, modeling firmware system failure, modeling the SF pattern of test subjects after a treatment is administered to them, modeling of distributions for excess-of-loss insurance data, software reliability modeling, models for reliability prediction, models for future toughness, modeling Markovian migration in finance and medicine, estimating the number of ozone peaks, modeling bus-motor failure data and mean residual life computation of (n − k + 1)-out-of-n systems.

In particular the EW family was applied in analyzing bathtub failure data by Modholkar and Srivastava (1993). This family was also applied to the bus-motor-failure data in Davis (1952) and to head-and-neck clinical data in Efron (1988), see modeling extreme value data to analyze the flood of the Floyd River at James, Iowa.

Bokhari et al. (2000) suggested statistically optimal life test plans for items whose lifetime follows the EW distribution under periodic inspection and type I censoring.

As a failure model, the EW distribution was used in accelerated life tests applications, see Ahmad et al. (2006).

Ahmad et al. (2008) studied the EW software reliability growth model with various testing efforts and optimal released efforts and optimal released policy: a performance analysis.

Jiang (2010) compared between the fitted EW bus-motor-failure data with competing risk model with parameters being functions of the number of successive failures.

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Chapter 4
Family of Exponentiated Exponential Distribution

Contents

4.1 Introduction

As was mentioned, in Chap. 1, that Gompertz (1825) raised the extreme value distribution to a positive parameter. Verhulst (1847) introduced the following CDF of a random variable X

$$F(x) = (1 - \rho e^{-\beta x})^{\alpha}, \ldots x > \ln \rho,$$

for ρ, β *and* α are positive real parameters. Verhulst (1838, 1845, 1847) used this and the exponentiated logistic distributions in mortality tables to represent population growth. Gupta and Kundu (1999) used this distribution with $\rho = 1$ and called it "generalized exponential" distribution. Unfortunately Gupta and Kundu did not refer to the works of Gompertz and Verhulst. In this chapter we will discuss some inferences of Verhulst distribution and will call this distribution exponentiated exponential distribution. An absolutely continuous (with respect to Lebesgue measure) random variable is said to have the exponentiated exponential distribution EED if the PDF and the corresponding CDF are given, for x $\geq$ 0, respectively, by

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$$f(x) = \alpha\beta e^{-\beta x}\left(1 - e^{-\beta x}\right)^{\alpha-1}, \tag{4.1.1}$$

$$F(x) = (1 - e^{-\beta x})^{\alpha}, \tag{4.1.2}$$

where β and α are positive parameters.

Many authors studied various properties of the *EED*. See, for example, Abdel-Hamid and AL-Hussaini (2009), Ahsanullah et al. (2013), AL-Hussaini (2010, 2011), AL-Hussaini and Hussein (2011), Ellah (2009), Escalante-Sandolva (2007), Kundu and Pradham (2009), Gupta and Kundu (1999, 2001, 2007), Madi and Raqab (2007, 2009), Nadarajah and Kotz (2006), Raqab (2002), Raqab and Ahsanullah (2001), Raqab et al. (2008), Sarhan (2007), Tripathi (2007), and Zheng (2002), among others. Nadarajah (2011) surveyed the *EE* distribution.

It was Verhulst (1847), who introduced a distribution with CDF $F(x)$ as

$$F(x) = (1 - \rho e^{-\beta x})^{\alpha}, \quad x > \frac{1}{\beta}\ln\rho, \tag{4.1.3}$$

where ρ, β and α are real, positive parameters.

Verhulst (1838, 1845, 1847) also presented some distributional results of the above exponentiated distribution. Gupta and Kundu (1999) gave several distributional properties of (4.1.3) when $\rho = 1$ they called this distribution generalized exponential distribution. Figure 4.1 gives the PDF of EED(α, β) for α = 0.5, β = 1 (Blue), α = 2, β = 3 (red) and α = 3 and β = 5 (Blue).

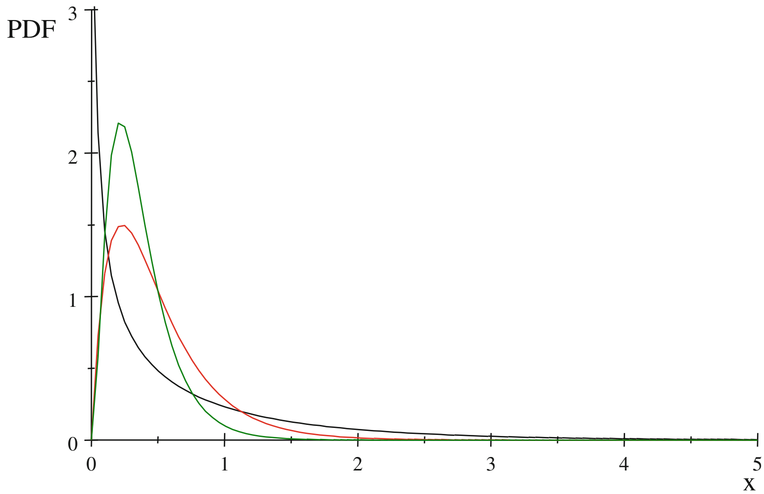

Fig. 4.1 PDF of EED

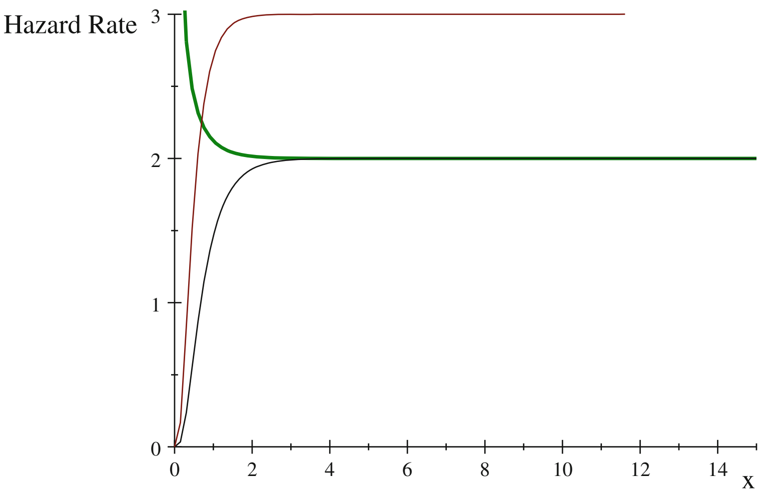

Fig. 4.2 Hazard Rate

Figure 4.2 shows the HRF for different values of α when β = 1.

$$\lambda_{5,3}(x) - \text{Red}, \lambda_{5,2}(x) - \text{Black}, \lambda_{0.2,2}(x) - \text{Blue}$$

The PRHRF $\lambda^*(x)$ is given as

$$\lambda^*(x) = \frac{f(x)}{F(x)} = \frac{\alpha\beta e^{-\beta x}}{1 - e^{-\beta x}}.$$

The HRF $\lambda_{\alpha,\beta}(x)$ is given by

$$\lambda_{\alpha,\beta}(x) = \frac{f(x)}{1 - F(x)} = \frac{\alpha\beta e^{-\beta x}(1 - e^{-\beta x})^{\alpha-1}}{1 - (1 - e^{-\beta x})^{\alpha}}$$

If $\alpha = 1$, then $\lambda_{\alpha,\beta}(x) = \beta$ (the hazard rate of exponential distribution) and if $\alpha < 1$, then $\lambda_{\alpha,\beta}(x)$ decreases from ∞ to β as x goes from 0 to ∞. If $\alpha > 1$, then $\lambda_{\alpha,\beta}(x)$ increases from zero to β as x goes from 0 to ∞.

4.2 Stress-Strength Reliability

Let the strength (or demand) X_1 and stress (or supply) X_2 be independent and distributed as $EED(\beta_1, \alpha_1)$ and $EED(\beta_2, \alpha_2)$ respectively. The reliability R = P $(X_2 < X_1)$ is known as stress-strength reliability. So that [see Raqab et al. (2008)]

$$\begin{aligned}
R = P(X_2 < X_1) &= \iint\limits_{y<x} f_{X_1,X_2}(x,y)dxdy \\
&= \int_0^\infty \int_0^x f_{X_1}(x) f_{X_2}(z) dz dx \\
&= \int_0^\infty F_{X_2}(x) f_{X_1}(x) dx \\
&= \int_0^\infty \alpha_1 \beta_1 \mathrm{e}^{-\beta_1 x} \left(1 - \mathrm{e}^{-\beta_1 x}\right)^{\alpha_1 - 1} \left(1 - \mathrm{e}^{-\beta_2 x}\right)^{\alpha_2} dx, u = \mathrm{e}^{-\beta_1 x} \\
&= \alpha_1 \int_0^1 (1-u)^{\alpha_1 - 1} (1 - u^{\beta_2/\beta_1})^{\alpha_2} du.
\end{aligned}$$

If $\beta_2/\beta_1 = \delta$, then

$$\begin{aligned}
R &= \alpha_1 \int_0^1 \sum_{j=0}^{v} C_j u^{j^\delta} (1-u)^{\alpha_1 - 1} du \\
&= \alpha_1 \sum_{j=0}^{v} C_j B(\alpha_1, 1 + j\delta),
\end{aligned}$$

where

$$v = \begin{cases} \alpha_2 = 0, 1, 2, \ldots \\ \infty, \alpha_2 \text{ is a positive fraction} \end{cases}, \tag{4.2.1}$$

$$C_j = (-1)^j \alpha_2 (\alpha_2 - 1) \ldots (\alpha_2 - j + 1). \tag{4.2.2}$$

If $\delta = 1$, then

$$\mathrm{R} = \alpha_1 \int_0^1 (1-u)^{\alpha_1 + \alpha_2 - 1} du = \frac{\alpha_1}{\alpha_1 + \alpha_2}$$

If $\delta = 2$, then

$$\begin{aligned}
\mathrm{R} &= \alpha_1 \int_0^1 (1-u)^{\alpha_1 + \alpha_2 - 1} (1+u)^{\alpha_2} du \\
&= \frac{1}{\alpha_1 + \alpha_2} {}_2F_1(1 - \alpha_2; 2 + \alpha_1 + \alpha_2 : -1),
\end{aligned}$$

where ${}_2F_1(a_1, a_2;\ b_1 : x)$ is a hypergeometric function given by

$${}_2F_1(a_1, a_2; b_1 : x) = \sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j}{(b_1)_j} x_j \quad \text{and} \quad (c)_j = c(c+1)\ldots(c+j-1).$$

4.3 Entropy

In information theory, entropy is a measure of uncertainty in a random variable. We will consider here Renyi's and Shannon entropies.

Renyi's entropy (Renyi 1961), denoted by $E_R(\gamma)$, is defined by

$$E_R(\gamma) = \frac{1}{1-\gamma} \ln \left\{ \int_0^{\infty} f^{\gamma}(x) dx \right\},$$

where $\gamma > 0$ and $\gamma \neq 1$.

Using the PDF, given in (4.1.1), we have

$$\int_0^{\infty} f^{\gamma}(x) dx = (\alpha\beta)^{\gamma} \int_0^{\infty} \mathrm{e}^{-\gamma\beta x} (1 - \mathrm{e}^{-\beta x})^{\gamma(\alpha-1)} dx.$$

By using the transformation $u = \mathrm{e}^{-\beta x}$, we obtain

$$\begin{aligned} \int_0^{\infty} f^{\gamma}(x) dx &= \alpha^{\gamma} \beta^{\gamma-1} \int_0^{1} u^{\gamma-1} (1-u)^{\gamma(\alpha-1)} du \\ &= \alpha^{\gamma} \beta^{\gamma-1} B(\gamma, \gamma\alpha - \gamma + 1). \end{aligned}$$

So that

$$E_R(\gamma) = -\ln(\alpha\beta) + \frac{\ln \alpha + \ln B(\gamma, \gamma\alpha - \gamma + 1)}{1-\gamma}. \tag{4.3.1}$$

If $\alpha = 1$, then

$$E_R(\gamma) = \frac{\gamma \ln \beta - \ln(\gamma\beta)}{1-\gamma},$$

which is the Renyi's entropy of the exponential distribution.

Shannon's entropy E_S (1951) is defined by

$$\begin{aligned} E_S &= E(-\ln f(x)) \\ &= -\ln(\alpha\beta) + \beta E(X) - (\alpha - 1)E[\ln(1 = \mathrm{e}^{-\beta X})] \\ E(X) &= \frac{1}{\beta}\{\psi(a+1) - \psi(1)\}, \end{aligned}$$

where $\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, is the digamma function.

$$E[\ln(1 - \mathrm{e}^{-\beta x})] = \alpha\beta\int_0^\infty \ln(1 - \mathrm{e}^{-\beta x})\mathrm{e}^{-\beta x}(1 - \mathrm{e}^{-\beta x})^{\alpha-1}dx$$

Let $u = \mathrm{e}^{-\beta x}$, then

$$\begin{aligned} E[\ln(1 - \mathrm{e}^{-\beta x})] &= \alpha\int_0^1 \ln(1-u)(1-u)^{\alpha-1}du \\ &= \frac{-1}{\alpha}. \end{aligned}$$

Thus

$$E_S = -\ln(\alpha\beta) + \psi(\alpha+1) - \psi(1) + \frac{\alpha - 1}{\alpha}. \tag{4.3.2}$$

Shannon's entropy (4.3.2) is a special case of Reni's entropy (4.3.1) as $\gamma \uparrow 1$.

4.4 Moments and Cumulants

Suppose that X has $EED(\alpha, \beta)$ distribution. Let $M(t)$ be its moment generating function (MGF). Then

$$M(t) = \alpha\beta\int_0^\infty \mathrm{e}^{(1-\beta)x}[1 - \mathrm{e}^{-\beta x}]^{\alpha-1}dx.$$

Using $u = \mathrm{e}^{-\beta x}$, then

$$\begin{aligned} M(t) &= \alpha\int_0^1 u^{-t/\beta}(1-u)^{\alpha-1}du \\ &= \alpha B\left(1 - \frac{t}{\beta}, a\right), \quad 0 < t < \beta, \end{aligned} \tag{4.4.1}$$

where

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

Gupta and Kundu (2001, 2007) gave expressions for the mean, variance, third and fourth central moments of X when X has EED (α, β). Using (4.4.1), we obtain

$$\begin{aligned} E(X) &= \frac{1}{\beta}\{\psi(a+1) - \psi(1)\} \\ Var(X) &= \frac{1}{\beta^2}(\psi'(1) - \psi'(\alpha+1)) \\ E(X^n) &= \frac{\alpha\Gamma(n+1)}{\beta^n}\sum_{i=0}^{\infty}(-1)^i \frac{p(\alpha,i)}{(i+1)^n} \end{aligned}$$

where

$$\begin{aligned} \psi(x) &= \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma(x)}{\Gamma(x)}, \psi'(x) = \frac{d}{dx}\psi(x), \\ P(\alpha,i) &= \frac{(\alpha-1)\ldots(\alpha-i)}{i!}, P(\alpha,0) = 1. \end{aligned}$$

The variance, third central moment, skewness, fourth central moment and kurtosis are, respectively

$$\text{Var(X)} = \{\pi^2 - 6\psi'(\alpha+1)\}/\{6\beta^2\}.$$

This expression is the same as that given before, since $\psi'(1) = \pi^2/6$ implies that $Var(X) = \frac{1}{\beta^2}(\psi'(1) - \psi'(\alpha+1))$. Also

$$\begin{aligned} \mathrm{E}\left[\{X - \mathrm{E}(X)\}^3\right] &= \{2\eta(3) + \psi''(\alpha+1)\}/\beta^3, \\ \text{Skewness} &= [6\sqrt{6}\{2\eta(3) + \psi''(\alpha+1)\}]/\left\{\pi^2 - 6\psi'(\alpha+1)\right\}^{3/2}, \\ \mathrm{E}\left[\{X - \mathrm{E}(X)\}^4\right] &= [3\pi^4 + 60\{\psi'(\alpha+1)\}^2 - 20\pi^2\psi'(\alpha+1)' \\ &\quad - 20\psi'''(\alpha+1)]/\left(20\beta^4\right), \end{aligned}$$

where $\eta(x) = \sum_{i=1}^{\infty}\frac{1}{i^x}$ and

$$\text{Kurtosis} = \frac{9[3\pi^4 + 60\{\psi'(\alpha+1)\}^2 - 20\pi^2\psi'(\alpha+1) - 20\psi'''(\alpha+1)]}{[5\{\pi^2 - 6\psi'(\alpha+1)\}^2]}.$$

Note that the skewness and kurtosis measures depend only on α. Both $E(X)$ and $\mathrm{Var}(X)$ increase monotonically with α.

We have $\mathrm{Var}(X \geq E(X)$ for all $\alpha \leq 1$ and $\mathrm{Var}(X) \leq E(X)$ for all $\alpha \geq 1$. Both skewness and kurtosis measure decrease monotonically with α. The skewness is always less than the kurtosis measure for all α.

The cumulant generating function $\mathrm{K}(t)$ is defined as

$$\mathrm{K}(t) = \ln M(t)$$

It follows, from Eq. (4.4.1), that

$$\mathrm{K}(t) = \ln \Gamma(\alpha + 1) + \ln \Gamma(1 - \frac{t}{\beta}) - \ln \Gamma(1 + \alpha - \frac{t}{\beta}). \tag{4.4.2}$$

The mth cumulant can be easily obtained from (4.4.2).

For example

$$\begin{aligned}
\kappa_1 &= \frac{1}{\beta}[\psi(1+\alpha) - \psi(1)] \\
\kappa_2 &= \frac{1}{\beta^2}[\psi'(1+\alpha) - \psi'(1)], \\
\kappa_3 &= \frac{1}{\beta^3}[\psi^{(2)}(1+\alpha) - \psi^{(2)}(1)], \\
\kappa_4 &= \frac{1}{\beta^4}[\psi^{(3)}(1+\alpha) - \psi^{(3)}(1)]
\end{aligned}$$

where $\psi(z) = \frac{d}{dz}[\ln\Gamma(z), \psi^r(z) = \frac{d^r}{dx^r}\psi(z)] = \frac{d^{r+1}}{dx^{r+1}}[\ln\Gamma(z)], r = 1, 2, 3, \ldots$ is the poly-gamma function. For example, $\psi'(z)$ is known as the tri-gamma function, $\psi^{(2)}(z)$ the tetra- gamma function and so on. For details, on the poly-gamma function, see Abramowitz and Stegun (1970).

4.5 Generalized Order Statistics

The PDF of the rth m-generalized order statistics (m-gos) [see Kamps (1995)] based on EED is given by (4.5.1) in case $m \neq -1$ and by (4.5.2), in case $m = -1$, in which $u(x) = \beta x$. So that,

$$\text{For}\, m \neq -1,$$

$$\begin{aligned}
f_{X^*}(x) = {} & \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}[1-(1-\mathrm{e}^{-\beta x})^{\alpha}]^{\gamma_r - 1}\alpha\beta\mathrm{e}^{-\beta x}(1-\mathrm{e}^{-\beta x})^{\alpha-1} \\
& \times [1 - \{1-(1-\mathrm{e}^{-\beta x})^{(m+1)\alpha}\}]^{r-1}.
\end{aligned} \tag{4.5.1}$$

where $C_{r-1} = \gamma_1\gamma_2 \ldots \gamma_r$ and $\gamma_j = k + (n-r)(m+1)$ $J = 1, 2, \ldots, n$

For $m = -1$,

$$f_{X_r^*}(x) = \frac{k^r}{(r-1)!}[1-(1-\mathrm{e}^{-\beta x})^{\alpha}]^{k-1}\alpha\beta\mathrm{e}^{-\beta x}(1-\mathrm{e}^{-\beta x})^{\alpha-1} \times [-\ln\{1-(1-\mathrm{e}^{-\beta x})^{\alpha}\}]^{r-1}. \tag{4.5.2}$$

Several important special cases can be obtained from the PDFs of (4.5.1) and (4.5.2). Among others, the PDFs of ordinary order statistics (ORS) and upper record value (URV) can be seen to be of the respective forms:

(i) The PDF $f_{r:n}(x)$ of the rth OOS $X_{r:n}$ is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}\alpha\beta\mathrm{e}^{-\beta x}(1-\mathrm{e}^{-\beta x})^{\alpha r-1}\left(1-(1-\mathrm{e}^{-\beta x})^{\alpha}\right)^{n-r}. \tag{4.5.3}$$

(ii) The PDF $f_{X_r^*}(x)$ of the rth OURV X_r^* is given by

$$f_{X_r^*}(x) = \frac{\alpha\beta}{(r-1)!}\mathrm{e}^{-\beta x}(1-\mathrm{e}^{-\beta x})^{\alpha-1}[-\ln\{1-(1-\mathrm{e}^{-\beta x})^{\alpha}\}]^{r-1}. \tag{4.5.4}$$

By expanding the last term in (4.5.3), the PDF of the rth OOS can be written in the form

$$f_{r:n}(x) = \sum_{j=0}^{n-r}\omega_j h_j^*(x), \tag{4.5.5}$$

where $\omega_j = \binom{n}{r}\binom{n-r}{n-r-jj}\frac{r(-1)^j}{r+j}$ and

$$h_j^*(\omega) = \alpha(r+j)\beta\mathrm{e}^{-\beta x}(1-\mathrm{e}^{-\beta x})^{\alpha(r+j)-1}, \tag{4.5.6}$$

which is the PDF of $EED[\alpha(r+j),\beta]$. So that the PDF of the rth order statistic is a linear combination of EE densities.

An s-out-of-n structure functions if at least s of its components function. Equivalently the life of an s-out-of-n structure is the $(n-s+1)$ largest of the component lifetime.

It follows, from (4.5.3), that the PDF of an s-out-of-n structure is given by

$$f_{n-s+1:n}(x) = \frac{n!}{(n-s)!(s-1)!}\alpha\beta\mathrm{e}^{-\beta x}\left(1-\mathrm{e}^{-\beta x}\right)^{\alpha(n-s+1)}\left(1-(1-\mathrm{e}^{-\beta x})^{\alpha}\right)^{s-1}. \tag{4.5.7}$$

From (4.5.7), the PDF of n-out-of-n structure (series system) is given by

$$f_{1:n}(x) = n\alpha\beta \mathrm{e}^{-\beta x}\left(1 - \mathrm{e}^{-\beta x}\right)^{\alpha-1)}\left(1 - (1 - \mathrm{e}^{-\beta x})^{\alpha}\right)^{n-1}. \tag{4.5.8}$$

The corresponding CDF is given by

$$F_{1:n}(x) = 1 - [1 - (1 - \mathrm{e}^{-\beta x})^{\alpha}]^{n}. \tag{4.5.9}$$

Notice that this is the CDF of the minimum of a random sample of size n from EED with CDF $G(x) = (1 - \mathrm{e}^{-\beta x})^{\alpha}$. Equivalently, (4.5.9) can be written as $R_{1:n}(x) = [R(x)]^{\alpha}$. That is, the SF of the minimum is the SF of $EED(\alpha, \beta)$.

Also, we have the PDF of 1-out-of-n structure (parallel system), given by

$$f_{n:n}(x) = n\alpha\beta \mathrm{e}^{-\beta x}(1 - \mathrm{e}^{-\beta x})^{\alpha-1}. \tag{4.5.10}$$

The corresponding CDF is given by

$$F_{n:n}(x) = (1 - \mathrm{e}^{-\beta x})^{n\alpha}, \tag{4.5.11}$$

which is $EED(n\alpha, \beta)$. This is the CDF of the maximum of a random sample of size n from EED with CDF $G(x) = (1 - \mathrm{e}^{-\beta x})^{\alpha}$.

Without loss of generality, we assume β = 1. The moment generating function (MGF) of the order statistic $X_{r:n}$ is given by

$$M_{r:n}(t) = \int_0^{\infty} \mathrm{e}^{tx} \mathrm{f}_{r:n}(x)\, dx,$$

where

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1 - F(x)]^{n-r} f(x).$$

Substituting $F(x) = (1 - \mathrm{e}^{-\beta x})^{\alpha}$ and $f(x) = \alpha\beta \mathrm{e}^{-\beta x}(1 - \mathrm{e}^{-\beta x})^{\alpha-1}$ and then integrating we obtain

$$\mathrm{M}_{r:n}(t) = \frac{n!\alpha}{(r-n)!(n-r)!}\sum_{j=0}^{n-r} D_j \frac{\Gamma[\alpha(r+j)]\Gamma(1-t)}{\Gamma[\alpha(r+j) - t + 1]}, \tag{4.5.12}$$

where t < 1 and

$$D_j = (-1)^j \binom{n-r}{j}. \tag{4.5.13}$$

Differentiating (4.5.12) with respect to t and then putting t = 0, we obtain

$$E(X_{r:n}) = \frac{n!}{(r-n)!(n-r-1)!}\sum_{j=0}^{n-r}\frac{D_j}{r+j}[\psi\{(r+j)\alpha\} - \psi(1)]$$

$$E(X_{r:n}^2) = \frac{n!}{(r-n)!(n-r-1)!}\sum_{j=0}^{n-r}\frac{D_j}{r+j}[\psi\{(r+j)\alpha\} - \psi(1)]^2$$
$$- [\psi^{'}(r+j)\alpha - \psi^{'}(1)].$$

Similarly we can obtain higher moments of the rth order statistic from (4.5.12). As a special case of (4.5.3),

$$f_{n:n}(x) = n\,\alpha\,\beta\,\mathrm{e}^{-\beta x}(1-\mathrm{e}^{-\beta x})^{n\,\alpha-1}$$

$\Rightarrow X_{n:n} \sim EED(n\alpha, \beta)$. The corresponding CDF is given by $F_{n:n}(x) = (1-\mathrm{e}^{-\beta x})^{n\alpha}$. The inverse function of the exponentiated CDF F, given by (4.1.2), is

$$F^{-1}(x) = -\frac{1}{\beta}\ln\left(1-u^{1/\alpha}\right).$$

So that

$$\lim_{n\to\infty}\left\{\frac{F^{-1}\left(1-\frac{1}{n}\right) - F^{-1}\left(1-\frac{2}{n}\right)}{F^{-1}\left(1-\frac{2}{n}\right) - F^{-1}\left(1-\frac{4}{n}\right)}\right\} = 1$$

Thus by Theorem 2.1.5 of Ahsanullah and Nevzorov (2001), see also Leadbetter (1987), it follows that with suitable normalizing constants a_n, b_n

$$P(X_{n:n} \le a_n + b_n x) \to \mathrm{e}^{-\mathrm{e}^{-x}}, \quad -\infty < x < \infty,$$

where

$$a_n = F^{-1}\left(1-\frac{1}{n}\right) = -\frac{1}{\beta}\ln\left(1-|1-\frac{1}{n}\right)^{1/\alpha} \to \ln n\alpha$$

$$b_n = -\frac{1}{\beta}\ln\left(1-\left(1-\frac{1}{n}\right)^{1/\alpha}\right) + \frac{1}{\beta}\ln\left(1-\left(1-\frac{1}{ne}\right)^{1/\alpha}\right) \to 1$$

Furthermore,

$$\lim_{n\to\infty}\left\{\frac{F^{-1}\left(\frac{1}{n}\right) - F^{-1}\left(\frac{2}{n}\right)}{F^{-1}\left(\frac{2}{n}\right) - F^{-1}\left(\frac{4}{n}\right)}\right\} = 2^{-1/\alpha}$$

Thus by Theorem 2.1.5 of Ahsanullah and Nevzorov (2001), it follows that with proper normalizing constants

$$P[X_{n:1} \le C_n + d_n x] = 1 - \mathrm{e}^{-x^{\alpha}}, \quad 0 < x < \infty,$$

where

$$C_n = 0 \text{ and } d_n = F^{-1}\left(\frac{1}{n}\right) \cong n^{-1/\alpha}$$

Assuming that $\beta = 1$, the joint PDF of any two order statistics $X_{r:n}$ and $X_{s:n}(1 \le \mathrm{r} < \mathrm{s} \le \mathrm{n})$ can be written, for $0 \le x < y < \infty$, as

$$f_{r,s:n}(x,y) = c_{r,s:n}\alpha^2 \sum_{i=0}^{n-s} \sum_{j=0}^{s-r-1} D_{ij}(1-\mathrm{e}^{-x})^{(r+i)\alpha-1}(1-\mathrm{e}^{-y})^{(s-r-j+1)\alpha-1}\mathrm{e}^{-(x+y)}$$

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \text{ and } D_{ij} = (-1)^{i+j}\binom{s-r-1}{j}\binom{n-s}{i}.$$

For details, see Raqab and Ahsanullah (2001).

The joint moment generating function $M_{r,s,:\,n}(t_1,t_2)$ is given by

$$\begin{aligned} M_{r,s:n}(t_1 t_2) = c_{r,s:n}\alpha \sum_{i=0}^{n-s} \sum_{j=0}^{s-r-j} \frac{D_{ij}}{r+j} B((s+r)\alpha, 1-t_2) \\ \times\ {}_3F_2((s+i)\alpha, (r+j)\alpha, t_1; (r+j)\alpha+1, (s+i)\alpha - t_2 + 1; 1), \\ t_2 < t_1 - 2, \end{aligned} \tag{4.5.14}$$

where

$${}_pF_q(\varepsilon_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q; u) = \sum_{j=0}^{\infty} \frac{(\alpha_1)_j \ldots (\alpha_p)_j u^j}{(\beta_1)_j \ldots (\beta_q)_j j!}$$

and $(\alpha) = \alpha(\alpha+1)\ldots(\alpha+j1) = \Gamma(\alpha+j)/\mathrm{I}(\alpha), j = 1, 2, \ldots.$

Using (4.5.14) it can be shown that, for $1 \le r < s < n$,

$$E(X_{r:n}X_{s:n}) = c_{r,s:n}\alpha^2 \sum_{k=1}^{\infty} \sum_{i=0}^{n-s} \sum_{j=0}^{s-r-1} D_{ij} \frac{\psi((s+i)\alpha + k + 1 - \psi(1)}{k[(s+i)\alpha + k][(r+j)\alpha + k]}.$$

(For details see Raqab and Ahsanullah 2001).

4.6 Distributions of Sums S_2

Suppose X_1 and X_2 are independent and distributed as $EED(\alpha_1, \beta_1)$ and $EED(\alpha_2, \beta_2)$. Let $S_2 = X_1 + X_2$, then the PDF of the function of S_2 is given by

$$fs_2(x) = \int_0^x fx_1(u) fx_2(x-u)\, du$$

$$= \int_0^x \alpha_2\beta_2 e^{-\beta_2(x-u)}(1 - e^{-\beta_2(x-u)})^{-\alpha_2-1}\alpha_1\beta_2 e^{-\beta_1 u}(1 - e^{-\beta_1 u})^{-\alpha_1-1} du$$

$$= \alpha_1\alpha_2\beta_1\beta_2 e^{-\beta_2 x}\int_0^x \sum_{j=0}^{v} c_j e^{-j\beta_2(x-u)} e^{\beta_2 u} e^{-\beta_1 u}(1 - e^{-\beta_1 u})^{\alpha_1-1} du,$$

where $v = \begin{cases} \alpha_2 - 1 = 0, 1, 2, \ldots \\ \infty, \textit{otherwise} \end{cases}$ and

$$c_j = (-1)^j(\alpha_2 - 1)\ldots(\alpha_2 - j).$$

$$fs_2(x) = \alpha_1\alpha_2\beta_2 e^{-\beta_2 x}\sum_{j=0}^{v} c_j e^{-\beta_1 jx} B_\delta\left(\alpha_1, 1 - \frac{\beta_2(j+1)}{\beta_1},\right)$$

where $\delta = 1 - e^{-\beta_1 x}$ and $B_\delta(p,q) = \int_0^\delta u^{p-1}(1-u)du$, is the incomplete beta function.

The corresponding CDF is

$$F_{S_2}(x) = \alpha_1 \sum_{j=0}^{v} c_j e^{-j\beta_1 x} B_\delta\left(\alpha_1, 1 - \frac{j\beta_2}{\beta_1}\right)$$

If $\beta_1 = \beta_2 = \beta$ and $\alpha_1 = \alpha_2 = \alpha$, then

$$fs_2(x) = \alpha^2\beta e^{-\beta x}\sum_{j=0}^{v} c_j e^{-j\beta x} B_\delta(\alpha, -j),$$

The corresponding CDF $Fs_2(x)$ is given by

$$Fs_2(x) = \alpha\beta e^{-\beta x}\sum_{j=0}^{v} e^{-j\beta x} B_\delta(\alpha_1 - j),$$

If $\alpha = 1$, then

$$fs_2(x) = \beta^2 x e^{-\beta x}$$

and the corresponding CDF is

$$Fs_2(x) = 1 - e^{-\beta x} - \beta x e^{-\beta x}, \quad x \geq 0, \beta > 0.$$

This is the distribution of two identical[y distributed exponential distribution.

4.6.1 Distribution of the Sum S_n

Let $X = X_1 + X_2 + \cdots + X_n$

We assume without loss of generality β = 1. Let $U_i = e^{-X_1}$, where X_1 is EED(α, 1). Then U_i is distributed as $B(1, \alpha)$ and the PDF $f_{U_1}(u)$ is given by

$$U_i e^{-X_i} \text{ and } U = \prod_{i=1}^{n} U_i = \exp[-\sum_{i=1}^{n} X_i] = e^{-X}$$

We can write $U_i e^{-X_i}$ and $U = \prod_{i=1}^{n} U = \exp[-\sum_{i=1}^{n} X_i] = e^{-X}$

The Mellin transformation of Y_i is

$$M_{Y_i}(t) = \int_0^1 \alpha x^t (1-x)^{\alpha-1} dx = \frac{\Gamma(1+t)\Gamma(1+\alpha)}{\Gamma(1+t+\alpha)}$$

The Mellin transformation of $M_Y(t)$ for $\mathrm{Y} = \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{Y}_i$ is

$$M(t) = \left[\frac{\Gamma(1+t)\Gamma(1+\alpha)}{\Gamma(1+t+\alpha)}\right]^n$$

Using the inverse of the Mellin transformation we can obtain the PDF $f_U(u)$ of Y [see for details Gupta and Kundu (1999)] as

$$f_U(u) = \sum_{j=0}^{\infty} C_j f_B(u : \alpha^* + j),$$

where $\alpha^* = \alpha_1 + \cdots + \alpha_n, C_0 = \frac{\prod_{j=1}^{n} \Gamma(\alpha_j+1)}{\Gamma(\alpha^*+1)}, C_j = \frac{C_0 \alpha^*}{(\alpha^*+j)} C_j^n, C_j^2 = \frac{(\alpha_1)_j (\alpha_2)_j}{j!(\alpha_1+\alpha_2)_j},$

$C_j^k = \frac{(\alpha_1 + \cdots + \alpha_{k-1})_j}{(\alpha_1 + \cdots + \alpha_k)_j} \sum_{i=0}^{j} \frac{(\alpha_k)_j}{j!} C_{j-1}^{(k-1)}, k = 3, \ldots \quad j = 1, 2, \ldots n = 2, 3, \ldots$

$$(\alpha)_j = \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)} \text{ and} f_B(u : 1, \alpha) = \alpha(1 - u)^{\alpha - 1}.$$

Using the transformation $X = -\ln u$, we obtain the PDF $f_X(x)$ of X as

$$f_X(x) = \sum_{j=0}^{\infty} C_j(\alpha^* + j)\mathrm{e}^{-x}(1 - \mathrm{e}^{-x})^{\alpha^* + j - 1}, \; x > 0.$$

4.7 Distribution of the Product and the Ratio

Let $P_2 = X_1X_2$ where X_1 and X_2 are independent and distributed as $EED(\alpha_1, \beta_1)$ and $EED(\alpha_1, \beta_1)$ respectively. Then the PDF of S_2 is given by

$$\begin{aligned} f_{P_2}(x) &= \alpha_1\alpha_2\beta_1\beta_2 \int_0^{\infty} \frac{1}{u}\mathrm{e}^{-\beta_1 x/u}(1 - \mathrm{e}^{-\beta_2 x/u})^{\alpha_2 - 1}\mathrm{e}^{-\beta_1 u}(1 - \mathrm{e}^{-\beta_1 u})^{\alpha_1 - 1} du \\ &= \alpha_1\alpha_2\beta_1\beta_2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\alpha_1 - 1}{i}\binom{\alpha_2 - 1}{j}(-1)^{i+j} \int_0^{\infty} \frac{1}{u}\exp[-\frac{\beta_2(i+1)x}{u} - \beta_1(j+1)u]du \\ &= \alpha_1\alpha_2\beta_1\beta_2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\alpha_1 - 1}{i}\binom{\alpha_2 - 1}{j}(-1)^{i+j} B_2(2\sqrt{(i+1)j\beta_1\beta_2 x}) \end{aligned}$$

where $B_2(x)$ is the modified Bessel function of the second kind and $B_2(x)$ is given by

$$B_0(x) = \int_0^{\infty} \frac{\cos tx}{\sqrt{1 + t^2}} dt$$

If $\beta_1 = \beta_2 = \beta$ and $\alpha_1 = \alpha_2 = \alpha$ then the pdf of S_2 reduces to

$$f_{P_2}(x) = \alpha^2\beta^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\alpha_1 - 1}{i}\binom{\alpha_2 - 1}{j}(-1)^{i+j} B_2(2\alpha\sqrt{(i+1)(j+1)x}),$$

$$\mathrm{Var}(X) = \{\pi^2 - 6\psi'(\alpha + 1)\}/\{6\beta^2\},$$

Let $R_2 = \frac{X_1}{X_2}$ where X_1 and X_2 are independent and are distributed as $EED(\alpha_1, \beta_1)$ and $EED(\alpha_1, \beta_1)$ respectively. Then the pdf of R_2 is given by

$$f_{R_2}(x) = \alpha_1\alpha_2\beta_1\beta_2 \int_0^\infty u e^{-\beta_1 xu}(1 - e^{-\beta_1 xu})^{\alpha_2 - 1} e^{-\beta_1 u}(1 - e^{-\beta_1 xu})^{\alpha_1 - 1} du$$

$$= \alpha_1\alpha_2\beta_1\beta_2 \sum_{j=0}^\infty \binom{\alpha_2 - 1}{j} (-1)^j \int_0^\infty u e^{-[\beta_1(j+1)+\beta_1 x]u}(1 - e^{\beta_1 xu})^{\alpha_2 - 1} du$$

$$\frac{\alpha_1\alpha_2\beta_2}{\beta_1 x^2}{}_2 \sum_{j=0}^\infty \binom{\alpha_2 - 1}{j} (-1)^j B(\alpha_1, \frac{\beta_2(j+1)}{\beta_1 x} + 1)[\psi \frac{\beta_2(j+1)}{\beta_1 x} + 1 = \psi(\frac{\beta_2(j+1)}{\beta_1 x} + \alpha_1)]$$

The corresponding CDF is

$$F_{R_2}(x) = \frac{\alpha_1\beta_3}{\beta_1 x}{}_2 \sum_{j=0}^\infty \binom{\alpha_2 - 1}{j} (-1)^j B(\alpha_1 + 1, \frac{\beta_2(j+1)}{\beta_1 x}).$$

If $\beta_1 = \beta_2 = \beta$ and $\alpha_1 = \alpha_2 = \alpha$, then the pdf of R_2 reduces to

$$f_{R_2}(x) = \frac{\alpha^2}{x^2} \sum_{j=0}^\infty \binom{\alpha - 1}{j} (-1)^j B(\alpha + 1, \frac{j+1}{x})$$

and the corresponding CDF

$$F_{R_2}(x) = \frac{\alpha^2}{x^2} \sum_{j=0}^\infty \binom{\alpha - 1}{j} (-1)^j B(\alpha + 1, \frac{j+1}{x}).$$

4.8 Maximum Likelihood Estimation

Let $X_1, \ldots, X_n$ be a random sample of EED(α, β). The log likelihood function can be written as

$$l(\alpha, \beta) = n\ln\alpha + n\ln\beta - \beta \sum_{i=1}^n x_i + (\alpha + 1) \sum_{i=1}^n \ln(1 - e^{-\beta x_i}) \tag{4.8.1}$$

Differentiating the above equation with respect to α and β and equating to zero,

$$\frac{d}{\alpha} l(\alpha, \beta) = \frac{n}{\alpha} + \sum_{i=1}^n \ln(1 - e^{-\beta x}) = 0 \tag{4.8.2}$$

$$\frac{d}{\beta}l(\alpha,\beta) = \frac{n}{\beta} + (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i e^{-\beta x_i}}{1-e^{-\beta x_i}}\right) - \sum_{i=1}^{n} x_i = 0. \tag{4.8.3}$$

From (4.8.2) we get maximum likelihood estimate (MLE) $\hat{\alpha}$ of α as a function of β as

$$\hat{\alpha}(\beta) = n/\sum_{i=1}^{n}\ln(1-e^{-\beta x_i}). \tag{4.8.4}$$

If β is known, then MLE of α can be obtained from (4.8.2). If both parameters are unknown, then we can obtain first the estimate of β by maximizing the equation

$$\xi(\beta) = l(\hat{\alpha}(\beta),\beta) = C - n\ln\beta - n\ln(-\sum_{i=1}^{n}\ln(1-e^{-\beta x_i})$$
$$-\sum_{i=1}^{n}\ln(1-e^{-\beta x_i}) - \beta\sum_{i=1}^{n}x_i,$$

where C is a constant, independent of β. Once the estimate of β is obtained, we can use Eq. (4.8.4) to obtain the MLE of α.

Raqab and Ahsanullah (2001) studied the estimates of location and scale parameters of the EED distribution with the following PDF

$$f(x) = \alpha e^{-(x-\mu)/\sigma}(1-e^{-(x-\mu)/\sigma})^{\alpha-1}, \quad -\infty<\mu<x_{,a,}\ \sigma>0. \tag{4.8.5}$$

Using (4.5.3) we can calculate the variances and covariance's of $X_{r:n}$ and $X_{s:n}$. They gave the best linear estimates (BLUEs) of μ and σ based on order statistics of n independent observations having the PDF as given in (4.8.5). They provided tables for BLUEs of μ and σ for various values of α.

4.9 Characterization

We will give here a characterization of EED(α,β). The PDF of EED(α,β) is $f(x) = \alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}$, $x>0$, α and β are positive real numbers.

Theorem 4.9.1 *Suppose that a random variable X is absolutely continuous (with respect to Lebesgue measure) with CDF F(x) and PDF f(x). We assume that* $E(X)$ *and* $f'(x)$ *exist, for* $x\in(\gamma,\delta)$, $\gamma=\inf\{x: F(x)>0\}$ *and* $\delta=\sup\{x: F(x)<1\}$. *Then* $E(X|X\le x) = g(x)\eta(x)$, *where*

$$\lambda^*(x) = \frac{f(x)}{F(x)}, \quad x>0,$$

and

$$g(x) = \frac{x(e^{\beta x} - 1)}{\alpha\beta} + \frac{(1 - e^{-\beta x})^1}{\alpha\beta} = \sum_{j=1}^{\infty} \frac{(1 - e^{-\beta x})^j}{\alpha + j},$$

α *and* β *if and only if*

$$f(x) = \alpha\beta e^{-\beta x}(1 - e^{-\beta x})^{\alpha - 1}.$$

To prove the theorem, we need the following Lemma 4.9.1.

Lemma 4.9.1 *Suppose that X is an absolutely continuous (with respect to Lebesgue measure) random variable with CDF F(x) and PDF f(x). We assume* $\gamma = \inf\{x|F(x) > 0\}$, $\delta = \sup\{x|F(x) < 1\}$, $E(X)$ *and* $f(x)$ *exist for all* $x \in (\gamma, \delta)$. *If* $E(X|X \le x) = g(x)\lambda^*(x)$, *where g(x) is a differentiable function, and* $\lambda^*(x) = \frac{f(x)}{F(x)}$ *for all* $x > 0$, *then we have*

$$f(x) = A \exp[\int_{\gamma}^{x} \frac{u - g'(u)}{g(u)} du]$$

where A is determined by taking $\int_{\gamma}^{\delta} f(x)dx = 1$.

Proof of Lemma 4.9.1 We have

$$\int_{\gamma}^{x} uf(u)du/F(x) = g(x)f(x)/F(x).$$

Thus

$$\int_{\gamma}^{x} uf(u)du = g(x)f(x).$$

□

Differentiating both sides of the above equation with respect to x, we obtain

$$xf(x) = g'(x)f(x) + g(x)f'(x).$$

Simplification, we get

$$f'(x)/f(x) = [x - g'(x)]/g(x).$$

Integrating the above equation, we obtain

$$f(x) = c \exp\left[\int_{\gamma}^{x} \left(\frac{u - g'(u)}{g(u)}\right) du\right]$$

where c is determined such that $\int_{\gamma}^{\delta} f(x)dx = 1$.

Proof of Theorem 4.9.1 Suppose

$$f(x) = \alpha\beta e^{-\beta x}(1 - e^{-\beta x})^{\alpha-1}, \quad x > 0.$$

Then

$$\begin{aligned} g(x) &= \int_{0}^{x} uf(u)du/f(x) \\ &= \frac{\int_0^x \alpha\beta u e^{-\beta u}(1 - e^{-\beta u})^{\alpha-1} du}{\alpha\beta e^{-\beta x}(1 - e^{-\beta x})^{\alpha-1}} \\ &= \frac{x(e^{\beta x} - 1)}{\alpha\beta} - \frac{e^{\beta x}}{\alpha\beta(1 - e^{\beta x})} \sum_{j=1}^{v} c_j \frac{(1 - e^{j-\beta jx})^{j+2}}{\beta j}, \end{aligned}$$

where $v = \begin{cases} \alpha = 1, 2, 3, \ldots \\ \infty, \quad otherwise \end{cases}$ and $c_j = (-1)^j \alpha(\alpha - 1) \ldots (\alpha - j + 1)$.

Suppose that

$$g(x) = \frac{x(e^{\beta x} - 1)}{\alpha\beta} - \frac{e^{\beta x}}{\alpha\beta(1 - e^{\beta x})} \sum_{j=1}^{v} c_j \frac{(1 - e^{j-\beta jx})^{j+2}}{\beta j}.$$

then

$$\begin{aligned} g'(x) = x &+ \frac{e^{\beta x}(1 - e^{-\beta x})}{\alpha\beta} - x\frac{e^{\beta x}(1 - e^{-\beta x})}{\alpha\beta} H(x) - \frac{e^{\beta x}(1 - e^{-\beta x})}{\alpha\beta} \\ &+ \frac{1}{\alpha\beta e^{-\beta x}(1 - e^{-\beta x})^{\alpha-1}} \sum_{j=1}^{v} c_j \frac{(1 - e^{-\beta jx})^{j+2}}{\beta j} H(x), \end{aligned}$$

where $H(x) = -\beta + \frac{(\alpha-1)\beta e^{-\beta x}}{1 - e^{-\beta x}}$.

Thus

$$g'(x) = x - g(x)H(x),$$

and

$$f'(x)/f(x) = [x - g(x)]/g(x) = H(x) = -\beta + \frac{(\alpha - 1)\beta e^{-\beta x}}{1 - e^{-\beta x}}.$$

Integrating the above equation, we have

$$f(x) = A \exp\left[\int_0^x H(u)du\right].$$

Now

$$\int_0^x H(u)du = \int_0^x \left[-\beta + \frac{(\alpha - 1)\beta e^{-\beta x}}{1 - e^{-\beta x}}\right]du$$
$$= -\beta x + (\alpha - 1)\ln(1 - e^{-\beta u}).$$

Thus

$$f(x) = Ae^{-\beta x}(1 - e^{-\beta x})^{\alpha - 1},$$

where $\frac{1}{A} = \int_0^x e^{-\beta u}(1 - e^{-\beta u})^{\alpha-1}du = \frac{1}{\alpha\beta}$. This completes the proof of Theorem 4.9.1. □

For $\alpha = 1$, EED$(1, \beta)$ is an exponential distribution with, x $\geq$ 0 and $\beta > 0$. There are many characterizations of the exponential distribution, see for example Ahsanullah and Hamedani (2010, Chaps. 5 and 6).

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Chapter 5
Family of Exponentiated Burr Type XII Distributions

Contents

5.1 Introduction

Analogous to the Pearson system of distributions, Burr (1942) introduced a system that includes twelve types of CDFs which yield a variety of density shapes. This system is obtained by considering CDFs satisfying a differential equation which has a solution, given by

$$G(x) = [1 + \exp\{-\int \eta(x)dx\}]^{-1},$$

where $\eta(x)$ is chosen such that $G(x)$ is a CDF on the real line. Twelve choices for $\eta(x)$ made by Burr, resulted in twelve distributions from which types III, X and XII have been frequently used. The flexibilities of Burr XII distribution were

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investigated by Hatke (1949), Burr and Cislak (1968), Rodrigues (1977), Tadikamalla (1980). In a different direction, it was Takahasi (1965) who first noticed that the 3-parameter Burr XII PDF can be obtained by compounding a Weibull PDF with a gamma PDF. That is, if $X|\theta \sim$ Weibull (θ, β_1) and $\theta \sim$ gamma (β_2, δ), then the compound PDF, is given by

$$\begin{aligned} g(x;\beta_1,\beta_2,\delta) &= \int_0^\infty [\theta\beta_1 x^{\beta_1-1}\mathrm{e}^{-\theta x^{\beta_1}}]\left[\frac{1}{\Gamma(\beta_2)\delta^{\beta_2}}\theta^{\beta_2-1}\mathrm{e}^{-\theta/\delta}\right]d\theta \\ &= \delta\beta_1\beta_2 x^{\beta_1-1}(1+\delta x^{\beta_1})^{-\beta_2-1}, \quad x>0, \end{aligned} \tag{5.1.1}$$

The CDF, SF and HRF of the 3-parameter Burr XII $(\beta_1, \beta_2, \delta)$ are given, for $x > 0$, respectively, by

$$G(x) \equiv G(x;\beta_1,\beta_2,\delta) = 1-(1+\delta x^{\beta_1})^{-\beta_2}, \tag{5.1.2}$$

$$R_G(x) \equiv R_G(x;\beta_1,\beta_2) = 1-G(x;\beta_1,\beta_2,\delta) = (1+\delta x^{\beta_1})^{-\beta_2}, \tag{5.1.3}$$

$$\lambda_G(x) \equiv \lambda_G(x;\beta_1,\beta_2,\delta) = \frac{g(x;\beta_1,\beta_2,\delta)}{R_G(x;\beta_1,\beta_2,\delta)} = \frac{\delta\beta_1\beta_2 x^{\beta_1-1}}{1+x^{\beta_1}}. \tag{5.1.4}$$

The PDF (5.1.1) of the Burr XII $(\beta_1, \beta_2, \delta)$ distribution is monotone decreasing if $\beta_1 \leq 1$ and unimodal with mode $x^* = \left(\frac{\beta_1-1}{\delta(\beta_1\beta_2+1)}\right)^{1/\beta_1}$ if $\beta_1 > 1$.

It can be seen that the HRF (5.1.4) of the Burr XII $(\beta_1, \beta_2, \delta)$ distribution is decreasing if $\beta_1 \leq 1$ and has an UBT shape if $\beta_1 > 1$. It attains its maximum at $\left[\frac{\beta_1-1}{\delta}\right]^{1/\beta_1}$.

The Burr XII and its reciprocal Burr III distributions have been used in many applications such as actuarial science (Embrechts et al. 1977; Klugman 1986), quantal bio-assay (Drane et al. 1978), economics (McDonald and Richards 1978; Morrison and Schmittlin 1980; McDonald 1984) forestry (Lindsay et al. 1996), exotoxicology (Shao 2000), life testing and reliability (Dubey 1972, 1973; Papadopoulos 1978; Lewis 1981; Evans and Ragab 1983; Lingappaiah 1983; Jaheen 1990; AL-Hussaini et al. 1997; Shah and Gokhale 1993; AL-Hussaini and Jaheen 1992, 1994; Moore and Papadopoulos 2000), among others. Khan and Khan (1987), AL-Hussaini (1991) characterized the Burr XII distribution. Lewis (1981) proposed the use of the Burr XII distribution as a model in accelerated life test data representing times to break down of an insulated fluid. Constant partially accelerated life tests for Burr XII distribution with progressive type two censoring was investigated by Abdel-Hamid (2009). Prediction of future observables from the Burr XII distribution was studied by Nigm (1988), AL-Hussaini and Jaheen (1995, 1996), AL-Hussaini (2003), AL-Hussaini and Ahmad (2003), among others. The extended 3-parameter Burr XII distribution was applied in flood frequency analysis by Shao et al. (2004).

5.2 Properties of the Exponentiated Burr XII Distributions

5.2.1 Moments

The ℓth moment, $\ell = 1, 2, \ldots$, is given by

$$E(X^{\ell}) = \frac{\Gamma(1 + \ell/\beta_1)}{\delta^{\ell/\beta_1}} \sum_{j=1}^{v} \frac{c_j \Gamma[j\beta_2 - (\ell/\beta_1)]}{\Gamma(j\beta_2)}, \quad j\beta_1\beta_2 > \ell, \tag{5.2.1}$$

where v and c_j are given by (2.2.7) and (2.2.8).

This can be shown as follows:

$$E(X^{\ell}) = \ell \int_0^{\infty} x^{\ell-1} R_H(x) dx,$$

where

$$R_H(x) = 1 - H(x) = 1 - [1 - R_G(x)]^{\alpha} = \sum_{j=1}^{v} c_j [R_G(x)]^j = \sum_{j=1}^{v} c_j (1 + \delta x^{\beta_1})^{-\beta_2}$$

$$\Rightarrow E(x^{\ell}) = \ell \sum_{j=1}^{v} c_j I_j(\ell),$$

where v and c_j are given by (2.2.7) and (2.2.8) and

$$I_j(\ell) = \int_0^{\infty} x^{\ell-1} (1 + \delta x^{\beta_1})^{-j\beta_2} dx.$$

By using the transformation $z = (1 + \delta x^{\beta_1})^{-1}$, we obtain

$$\begin{aligned} I_j(\ell) &= \frac{1}{\beta_1 \delta^{\ell/\beta_1}} \int_0^{1} z^{j\beta_2 - (\ell/\beta_1) - 1} (1 - z)^{(\ell/\beta_1) - 1} dz, \\ &= \frac{1}{\beta_1 \delta^{\ell/\beta_1}} \frac{\Gamma[j\beta_2 - (\ell/\beta_1)]\Gamma(\ell/\beta_1)}{(\Gamma j\beta_2)}, \end{aligned}$$

provided that $j\beta_1\beta_2 > \ell$. So that

$$E(x^{\ell}) = \frac{\Gamma(1 + \ell/\beta_1)}{\delta^{\ell/\beta_1}} \sum_{j=1}^{v} \frac{c_j \Gamma[j\beta_2 - (\ell/\beta_1)]}{\Gamma(j\beta_2)}, \tag{5.2.2}$$

which is the same as that obtained in Table 2.1 for the ℓth moment of the EBurr XII $(\alpha, \beta_1, \beta_2, \delta)$ distribution.

Special Cases

1. If $\beta_1 = 1$, then

$$E(x^\ell) = \frac{\Gamma(1+\ell)}{\delta^\ell} \sum_{j=1}^{v} \frac{c_j \Gamma[j\beta_2 - \ell]}{\Gamma(j\beta_2)}, \quad j\beta_2 > \ell, \tag{5.2.3}$$

   which is the ℓth moment of the EED (α, β_2). See Table 2.1.
2. If $\beta_1 = 2$, then

$$E(x^\ell) = \frac{\Gamma(1+\ell/2)}{\beta^{\ell/2}} \sum_{j=1}^{v} \frac{c_j \Gamma[j\beta_2 - \ell/2]}{\Gamma(j\beta_2)}, \ 2j\beta_2 > \ell, \tag{5.2.4}$$

   which is the ℓth moment of the ERay (α, β_2). See Table 2.1.
3. If $\delta = 1$, then

$$E(x^\ell) = \Gamma(1+\ell/\beta_1) \sum_{j=1}^{v} \frac{c_j \Gamma[j\beta_2 - \ell/\beta_2]}{\Gamma(j\beta_2)}, \ j\beta_1\beta_2 > \ell \tag{5.2.5}$$

   which is the ℓth moment of the EBurrXII $(\alpha, \delta, \beta_2)$ in which the Burr XII distribution has only two parameters (α, β_2). See Table 2.1.
4. If $\alpha = 1, \ \delta = 1$, then

$$E(X^\ell) = \frac{\ell}{\beta_1} B(\beta_2 - \ell/\beta_1, \ell/\beta_1), \ \beta_1\beta_2 > \ell, \tag{5.2.6}$$

   which is the ℓth moment of the two-parameter EBurr XII (β_1, β_2) distribution, see Table 2.1.

5.2.2 Mean Residual Life Function

The MRL $m(x)$ is given by

$$m(x) = \int_x^\infty R_H(t)dt / R_H(x),$$

where the SF $R_H(x)$ is given by

$$R_H(x) = 1 - H(x) = 1 - [1 - R_G(x)]^\alpha = \sum_{j=1}^{v} c_j [R_G(x)]^j = \sum_{j=1}^{v} c_j [1 + \delta x^{\beta_1}]^{-j\beta_2}$$

Therefore

$$\int_x^\infty R_H(t)dt = \sum_{j=1}^{v} c_j I_j(t)dt,$$

where

$$I_j(x) = \int_x^\infty [1 + \delta t^{\beta_1}]^{-j\beta_2} dt.$$

Applying the transformation $z = [1 + \delta t^{\beta_1}]^{-1}$, then $(x, \infty) \to (z_0, 0)$, where $z_0 = [1 + \delta x^{\beta_1}]^{-1}, t = \left[\frac{1}{\delta}\left(\frac{1}{z} - 1\right)\right]^{1/\beta_1}$,

$$dt = \left(\frac{1}{\delta}\right)^{1/\beta_1} \left(\frac{1}{\beta_1}\right) \left(\frac{1}{z} - 1\right)^{1/\beta_1 - 1} \left(-\frac{dz}{z^2}\right) = -\frac{1}{\beta_1 \delta^{1/\beta_1}} \frac{(1-z)^{1/\beta_1 - 1}}{z^{1/\beta_1 + 1}} dz.$$

$$\begin{aligned} I_j(x) &= \frac{1}{\beta_1 \delta^{1/\beta_1}} \int_0^{z_0} z^{j\beta_2 - 1/\beta_1 - 1} (1-z)^{1/\beta_1 - 1} dz \\ &= \frac{B_{z_0}(j\beta_2 - 1/\beta_1, 1/\beta_1)}{\beta_1 \delta^{1/\beta_1}}, \quad j\beta_1\beta_2 > 1, \end{aligned}$$

where $B_z(\alpha, \beta) = \int_0^z t^{\alpha-1}(1-t)^{\beta-1} du$, is the incomplete beta function.

Therefore, the MRLF is given by

$$m(x) = \frac{1}{\beta_1 \delta^{1/\beta_1}} \sum_{j=0}^{v} c_j B_{z_0}(j\beta_2 - 1/\beta_1, 1/\beta_1), \quad j\beta_1\beta_2 > 1,$$

where v, c_j are given by (2.2.7) and (2.2.8), $B_{z_0}(j\beta_2 - 1/\beta_1, 1/\beta)$ is the incomplete beta function and $z_0 = (1 + x^{\beta_1})^{-1}$.

5.2.3 Quantiles

From (2.2.9), where $u(x) = \beta_2 \ln(1 + \delta x^{\beta_1})$,

$$u^{-1}(y) = [(1/\delta)\exp(y/\beta_2) - 1]^{1/\beta_1},$$

so that the qth quartile is given by

$$x_q = [(1/\delta)\{(1 - q^{1/\alpha})^{-1/\beta_2} - 1\}]^{1/\beta_1}. \quad (5.2.7)$$

$$\Rightarrow \text{Median} = x_{1/2} = [(1/\delta)\{(1 - 2^{-1/\alpha})^{-1/\beta_2} - 1\}]^{1/\beta_1}, \quad (5.2.8)$$

which is the same median value obtained for EBurr XII $(\alpha, \beta_1, \beta_2, \delta)$ in Table 2.2 with $\delta = 1$.

5.2.4 Mode

The mode M of the PDF $h(x) = \alpha[G(x)]^{\alpha-1}g(x)$ is the value of x which maximizes $h(x)$. This is equivalent to maximizing $\ln\ h(x) = \ln\alpha + (\alpha+1)\ln G(x) + \ln g(x)$.

Differentiating both sides with respect to x and then equating to zero, we get

$$0 = \frac{h'(x)}{h(x)} = (\alpha - 1)\frac{g(x)}{G(x)} + \frac{g'(x)}{g(x)}. \quad (5.2.9)$$

$$G(x) = 1 - (1 + x^{\beta_1})^{-\beta_2} \Rightarrow g(x) = \beta_2\beta_1 x^{\beta_1 - 1}(1 + x^{\beta_1})^{-\beta_2 - 1}$$
$$\Rightarrow \frac{g'(x)}{g(x)} = \frac{\beta_1 - 1 - (\beta_1\beta_2 + 1)x^{\beta_1}}{x(1 + x^{\beta_1})}.$$

So that, from (5.2.9), the mode is the value of x which satisfies

$$0 = \frac{\beta_1 - 1 - (\beta_1\beta_2 + 1)x^{\beta_1}}{x(1 + x^{\beta_1})} + (\alpha - 1)\frac{\beta_2\beta_1 x^{\beta_1 - 1}(1 + x^{\beta_1})^{-\beta_2 - 1}}{1 - (1 + x^{\beta_1})^{-\beta_2}} \quad (5.2.10)$$

$$\Rightarrow (1 + x^{\beta_1})^{-\beta_2} = \frac{\beta_1 - 1 - (\beta_1\beta_2 + 1)x^{\beta_1}}{\beta_1 - 1 - (\alpha\beta_1\beta_2 + 1)x^{\beta_1}}. \quad (5.2.11)$$

A numerical iterative procedure is required to solve this non-linear equation.

Remarks

1. If $\alpha = 1$ Eq. (5.2.10) reduces to the equation

$$0 = \beta_1 - 1 - (\beta_1\beta_2 + 1)x^{\beta_1} \Rightarrow$$
$$M = \left[\frac{\beta_1 - 1}{\beta_1\beta_2 + 1}\right]^{1/\beta_1},$$

   which is the mode of the two-parameter Burr XII (β_1, β_2) distribution when $\beta_1 > 1$.
2. If $\beta_1 = 1,\ \alpha > 1$, then

$$M = \left[\frac{\alpha\beta_2 + 1}{\beta_2 + 1}\right]^{1/\beta_2} - 1,$$

   which is the mode of the $ELomax(\alpha, \beta_2)$ or $ECompE(\alpha, \beta_2)$ distribution.
3. If $\beta_2 = 2,\ \alpha > 1$, then

$$M = \left[\frac{\beta_1 - 1}{\beta_1 + 1}\right]^{1/\beta_1},$$

   which is the mode of the $ECompRay(\alpha, \beta_2)$ distribution.
4. If $\beta_2 = 1,\ \alpha\beta_1 > 1$, then

$$M = \left[\frac{\alpha\beta_1 - 1}{\beta_1 + 1}\right]^{1/\beta_2}, \alpha\beta_1 > 1,$$

5. *If* $\alpha\beta_1 \le 1$, the PDF is monotone decreasing on $(0, \infty)$.

5.2.5 HRF

The hazard rate function (HRF) corresponding to the exponentiated CDF (1.1.5) is given, for x > 0 by

$$\begin{aligned} \lambda_H(x) &= \frac{h(x)}{R_H(x)} \\ &= \frac{\alpha\beta_2\beta_1 x^{\beta_1 - 1}(1 + x^{\beta_1})^{-\beta_2 - 1}[1 - (1 + x^{\beta_1})^{-\beta_2}]^{\alpha - 1}}{1 - [1 - (1 + x^{\beta_1})^{-\beta_2}]^{\alpha}}. \end{aligned} \tag{5.2.12}$$

The PDFs and their corresponding HRFs are plotted for different values of the parameters. See Figs. 5.1 and 5.2.

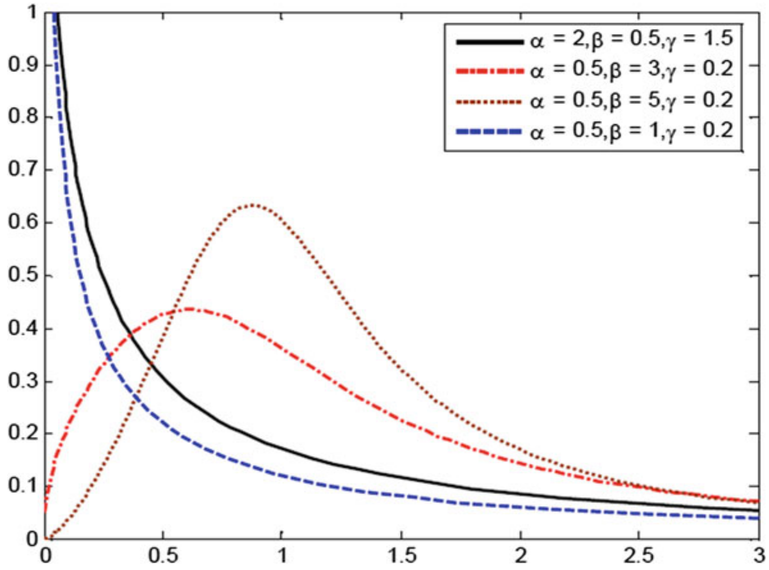

Fig. 5.1 The PDF of EBurr(α, β, γ) distributions

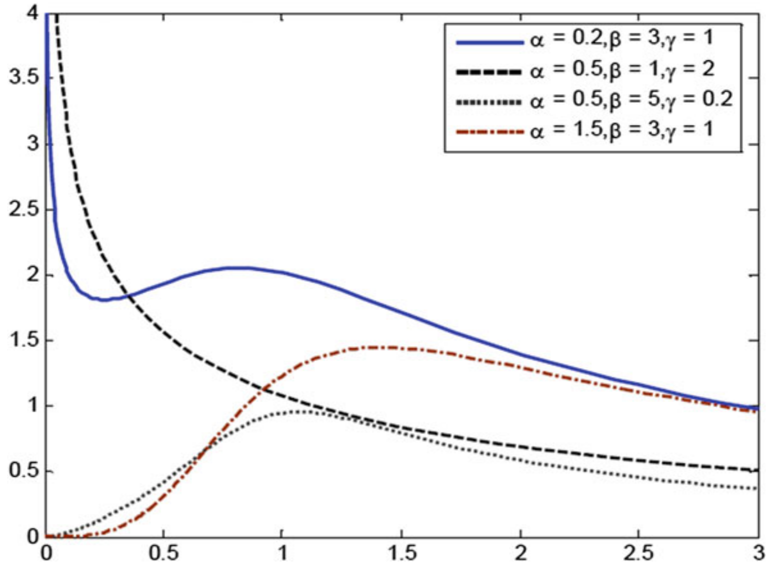

Fig. 5.2 The HRF of EBurr(α, β, γ) distribution

It may be noticed that corresponding to decreasing and unimodal lEBurr XII PDFs, one can obtain DHRF, UBTHRF, DIDHRF.

5.2.6 Proportional Reversed Hazard Rate Function

The PRHRF corresponding to Burr XII distribution is given, from (2.2.13), by

$$\lambda_H^*(x) = \alpha\lambda_G(x) = \frac{\alpha g(x)}{G(x)} = \frac{\alpha\beta_2\beta_1 x^{\beta_1-1}(1+x^{\beta_1})^{-\beta_2-1}}{1-(1+x^{\beta_1})^{-\beta_2}}. \quad (5.2.13)$$

5.2.7 Density Function of the rth m-Generalized Order Statistic

Substitution of $u(x) = \beta_2 \ln(1+\delta x^{\beta_1}),\ u'(x) = \frac{\delta\beta_1\beta_2 x^{\beta_1-1}}{1+\delta x^{\beta_1}}$ in (2.2.15) and (2.2.16) yields the PDFs of the rth m-GOS when $m \neq -1$ and $m = -1$, respectively, given by

$$m \neq -1 \Rightarrow$$

$$f_{X_r^*}(x) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\left(1-[1-(1+\delta x^{\beta_1})^{-\beta_2}]^{\alpha}\right)^{\gamma_r-1}\frac{\alpha\delta\beta_1\beta_2 x^{\beta_1-1}}{1+\delta x^{\beta_1}}$$
$$(1+\delta x^{\beta_1})^{-\beta_2}[1-(1+\delta x^{\beta_1})^{-\beta_2}]^{\alpha-1}\left(1-[1-(1+\delta x^{\beta_1})^{-\beta_2}]^{(m+1)\alpha}\right)^{r-1}. \quad (5.2.14)$$

$$m = -1 \Rightarrow$$

$$f_{X_r^*}(x) = \frac{k^r}{(r-1)!}\left(1-[1-(1+\delta x^{\beta_1})^{-\beta_2}]^{\alpha}\right)^{k-1}\frac{\alpha\delta\beta_1\beta_2 x^{\beta_1-1}}{1+\delta x^{\beta_1}}$$
$$\left(1-(1+\delta x^{\beta_1})^{-\beta_2}\right)^{\alpha-1}\left[-\ln\{1-(1-(1+\delta x^{\beta_1})^{-\beta_2})^{\alpha}\}\right]^{r-1}. \quad (5.2.15)$$

where C_{r-1}, γ_r, k are as defined in Chap. 2.

The PDF of the rth OOS [$k = 1, m = 0$ *in* (5.2.14)] is given by $f_{X_{r:n}}(x) = \sum_{j=1}^{n-r}\omega_j h_j^*(x)$, where ω_j is given by (2.2.18) and

$$h_j^*(x) = \alpha(r+j)\frac{\delta\beta_1\beta_2 x^{\beta_1-1}}{1+\delta x^{\beta_1}}(1+\delta x^{\beta_1})^{-\beta_2}$$
$$[1-(1+\delta x^{\beta_1})^{-\beta_2}]^{\alpha(r+j)-1}. \quad (5.2.16)$$

Also, from (2.2.20), the PDF of OURV is given by

$$f_{X_r^*}(x) = \frac{1}{(r-1)!}\frac{\alpha\delta\beta_1\beta_2 x^{\beta_1-1}}{1+\delta x^{\beta_1}}(1+\delta x^{\beta_1})^{-\beta_2} \left[1-(1+\delta x^{\beta_1})^{-\beta_2}\right]^{\alpha-1}\left[-\ln\{1-(1-(1+\delta x^{\beta_1})^{-\beta_2})^{\alpha}\}\right]^{r-1}. \tag{5.2.17}$$

(i) The PDF of life of an s-out-of-n structure is given by

$$f_{n-s+1:n}(x) = \binom{n}{n-s+1}(n-s+1)\frac{\alpha\delta\beta_1\beta_2 x^{\beta_1-1}}{1+\delta x^{\beta_1}}(1+\delta x^{\beta_1})^{-\beta_2} \left[1-(1+\delta x^{\beta_1})^{-\beta_2}\right]^{\alpha(n-s+1)-1}\left[1-(1-(1+\delta x^{\beta_1})^{-\beta_2})^{\alpha}\right]^{s-1}. \tag{5.2.18}$$

(ii) The PDFs of a series (n-out-of-n) and parallel (1-out-of-n) structures are obtained, for $x > 0$ from (2.2.21), respectively, as follows:

$$f_{1:n}(x) = \frac{n\alpha\delta\beta_1\beta_2 x^{\beta_1-1}}{1+\delta x^{\beta_1}}(1+\delta x^{\beta_1})^{-\beta_2}\{1-(1+\delta x^{\beta_1})^{-\beta_2}\}^{\alpha-1}, \left[1-\left\{1-(1+\delta x^{\beta_1})^{-\beta_2}\right\}^{\alpha}\right]^{s-1} \tag{5.2.19}$$

$$f_{n:n}(x) = \frac{n\alpha\delta\beta_1\beta_2 x^{\beta_1-1}}{1+\delta x^{\beta_1}}(1+\delta x^{\beta_1})^{-\beta_2}\{1-(1+\delta x^{\beta_1})^{-\beta_2}\}^{n\alpha-1}. \tag{5.2.20}$$

5.3 Estimation: All Parameters of H are Unknown

5.3.1 Maximum Likelihood Estimation of $(\alpha, \beta_1, \beta_2), R_H(x_0), \lambda_H(x_0)$

Assuming that $(k = 2, \delta = 1), \theta = (\alpha, \beta), \beta = (\beta_1, \beta_2)$ so that the LEs, are given from (2.4.3) and (2.4.4) by

$$0 = \frac{r}{\alpha} + \sum_{i=1}^{r} \ln G(x_i|\beta) - \frac{(n-r)\{G(x_r|\beta)\}^{\alpha}\ln G(x_r|\beta)}{1-\{G(x_r|\beta)\}^{\alpha}}, \tag{5.3.1}$$

$$0 = (\alpha - 1)\sum_{i=1}^{r} A_{i\beta_1}(\beta) + \sum_{i=1}^{r} B_{i\beta_1}(\beta) - K_r(\theta) \tag{5.3.2}$$

$$0 = (\alpha - 1)\sum_{i=1}^{r} A_{i\beta_2}(\beta) + \sum_{i=1}^{r} B_{i\beta_2}(\beta) - K_r(\theta), \tag{5.3.3}$$

where

$$A_{i\beta_1}(\beta) = \frac{1}{G(x_i,\beta)}\frac{\partial G(x_i,\beta)}{\partial \beta_1} \Big/ \frac{\partial G(x_r,\beta)}{\partial \beta_1}$$

$$A_{i\beta_2}(\beta) = \frac{1}{G(x_i,\beta)}\frac{\partial G(x_i,\beta)}{\partial \beta_2} \Big/ \frac{\partial G(x_r,\beta)}{\partial \beta_2}$$

$$B_{i\beta_1}(\beta) = \frac{1}{g(x_i,\beta)}\frac{\partial g(x_i,\beta)}{\partial \beta_1} \Big/ \frac{\partial G(x_r,\beta)}{\partial \beta_1}$$

$$B_{i\beta_2}(\beta) = \frac{1}{g(x_i,\beta)}\frac{\partial g(x_i,\beta)}{\partial \beta_1} \Big/ \frac{\partial G(x_r,\beta)}{\partial \beta_1}$$

$$K_r(\theta) = \frac{(n-r)\alpha[G(x_r,\beta)]^{\alpha-1}}{1-[G(x_r,\beta)]^{\alpha}}.$$

The baseline distribution G is given by

$$G(x|\beta) = 1 - (1 + x^{\beta_1})^{-\beta_2}, \beta = (\beta_1, \beta_2)$$

and $g(x|\beta)$ is the corresponding PDF of G, given by

$$g(x|\beta) = \beta_1\beta_2 x^{\beta_1 - 1}(1 + x^{\beta_1})^{-\beta_2 - 1}.$$

Substitution of G and g and their derivatives with respect to β_1 and β_2, in the above three LEs, and solving such system, we obtain the MLEs $\hat{\alpha}_{ML}, \hat{\beta}_{1,ML}, \hat{\beta}_{2,ML}$.

The MLE $\hat{\alpha}$ of α can be written in the from

$$\hat{\alpha}_{ML} = 1 - \frac{\sum_{i=1}^{r}[B_{i,\beta_1}(\beta) - B_{i,\beta_2}(\beta)]}{\sum_{i=1}^{r}[A_{i,\beta_1}(\beta) - A_{i,\beta_2}(\beta)]}. \tag{5.3.4}$$

Once the MLEs $\hat{\beta}_{1,ML}, \hat{\beta}_{2,ML}$ are obtained (by maximizing the log-LF with respect to β_1, β_2), the MLEs $\hat{R}_H(x_0), \hat{\lambda}_H(x_0)$ of $R_H(x_0), \lambda_H(x_0)$ at time x_0 are computed by applying the invariance property of MLEs.

5.3.2 Bayes Estimation of $(\boldsymbol{\alpha}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2), \boldsymbol{R_H}(\boldsymbol{x}_0), \boldsymbol{\lambda_H}(\boldsymbol{x}_0)$

Following the same steps of Example 3 in Chap. 2, the Bayes estimates of the parameters, SF and HRF for the EBurr XII (α, β, γ) distribution, based on SEL or LINEX loss functions were developed by AL-Hussaini and Hussein (2011). This was done using SBM and MCMC methods, as explained in Example 3 of Chap. 2.

Simulation comparisons of various estimation methods are made when $n = 20$ and censored data ($r = 15, 18, 20$).

Example 5.1 (Simulation) A random sample of size $n = 20$ is drawn from EBurr XII with parameters $(\alpha = 2.5, \beta_1 = 1.5, \beta_2 = 2)$ according to the expression $X = \left[\left(1 - U^{1/\alpha}\right)^{-1/\beta_2} - 1\right]^{1/\beta_1}, U \sim uniform(0, 1)$

An ordered set of data ($n = 20$) is given by:

0.3163, 0.3703, 0.4688, 0.5366, 0.5440, 0.6469, 0.6626, 0.8013, 0.8207, 0.8495 0.9015, 1.0728, 1.2344, 1.2932, 1.3510, **1.7918**, **1.8123**, **2.6583**, **2.7362**, **5.0043**

This is a typical sample used. Generating 1,000 of such samples are used in estimating the parameters, SF and HRF. Suppose that the prior belief of the experimenter is measured by the PDF given by (2.5.18) with hyper-parameters: $b_1 = 0.6$, $b_2 = 0.6$, $b_3 = 2$, $b_4 = 2$, $b_5 = 3$. The averages of the estimates over the 1,000 samples are reported in Table 5.1a, b, c.

The Bayes estimates are, generally, better than the MLEs against the prior used, in the sense of having smaller MSEs, whether using SE or LINEX loss function. Naturally, by increasing r, the estimates should improve, till the complete sample case is reached, where the estimates are better than any censored case.

Example 5.2 (Real Life Data) The breaking strength of 64 (=n) single carbon fibers of length 10 (Lawless 1983, p. 573) are:

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2,616, 2,618, 2.624, 2.650, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2 937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3,501, 3.537, 3.554, 3.562, 3.628, **3.852**, **3.871**, **3.886**, **3.971**, **4.024**, **4.027**, **4.225**, **4.395**, **5.020**.

In the complete sample case ($n = r$), the estimates of the parameters, SF, HRF at $x_0 = 3$ and the corresponding p-values of KS goodness of fit tests are presented in Table 5.2a. The Bayes estimates (SBM and MCMC) are calculated for the hyper-parameters $b_1 = 180$, $b_2 = 0.6$, $b_3 = 2$, $b_4 = 3$, $b_5 = 2$. We have used the same values for b_2, b_3, b_4, b_5 as in the simulation study. To give a value for b_1, we have noticed that the MLE of α is quite large. In the Bayes case, the mean of the gamma (b_1, b_2)

Table 5.1 **a** Complete sample (r = 20), **b** censored sample (r = 18), and **c** censored sample (r = 15)

Estimate	MLE	SEL		LINEX					
				κ = −0.5		κ = 0.01		κ = 0.1	
		SBM	MCMC	SBM	MCMC	SBM	MCMC	SBM	MCMC
a									
$\hat{\alpha}$	2.9072	2.5833	2.5860	3.0262	3.2393	2.5738	2.5737	2.4968	2.4723
MSE	(0.4738)	(0.0619)	(0.0763)	(0.3901)	(0.4160)	(0.0595)	(0.0733)	(0.0485)	(0.0601)
$\hat{\beta}_1$	1.5050	1.4428	1.4632	1.5055	1.5268	1.4418	1.4620	1.4313	1.4519
MSE	(0.0329)	(0.0268)	(0.0163)	(0.0191)	(0.0181)	(0.0268)	(0.0164)	(0.0210)	(0.0169)
$\hat{\beta}_2$	2.1921	1.9392	1.9229	2.0572	2.0615	1.9366	1.9202	1.9163	1.8908
MSE	(0.1281)	(0.0119)	(0.0202)	(0.0103)	(0.0421)	(0.0122)	(0.0206)	(0.0015)	(0.0244)
$\hat{R}(x_0)$	0.7250	0.6795	0.6793	0.6816	0.6810	0.6795	0.6801	0.6798	0.6789
MSE	(0.0067)	(0.0035)	(0.0036)	(0.0033)	(0.0034)	(0.0035)	(0.0036)	(0.0035)	(0.0036)
$\hat{\lambda}(x_0)$	1.0643	0.9626	0.9732	0.9720	0.9826	0.9624	0.9731	0.9608	0.9709
MSE	(0.0841)	(0.0082)	(0.0081)	(0.0073)	(0.0072)	(0.0081)	(0.0081)	(0.0085)	(0.0083)
b									
$\hat{\alpha}$	3.2976	2.5507	2.5658	3.0674	3.2586	2.5416	2.5536	2.4659	2.4528
MSE	(1.1163)	(0.0674)	(0.0828)	(0.4306)	(0.4412)	(0.0657)	(0.0803)	(0.0593)	(0.0708)
$\hat{\beta}_1$	1.4280	1.4632	1.4780	1.5277	1.5438	1.4620	1.4768	1.4514	1.4663
MSE	(0.0307)	(0.0246)	(0.0213)	(0.0262)	(0.0252)	(0.0246)	(0.0164)	(0.0210)	(0.0216)
$\hat{\beta}_2$	2.3623	1.9266	1.9141	2.0468	2.0550	1.9240	1.9114	1.9033	1.8875
MSE	(0.2651)	(0.0171)	(0.0273)	(0.0119)	(0.0296)	(0.0175)	(0.0277)	(0.0213)	(0.0320)
$\hat{R}(x_0)$	0.7237	0.6802	0.6799	0.6823	0.6816	0.6802	0.6801	0.6799	0.6789
MSE	(0.0070)	(0.0035)	(0.0036)	(0.0033)	(0.0035)	(0.0035)	(0.0036)	(0.0036)	(0.0036)
$\hat{\lambda}(x_0)$	1.0808	0.9685	0.9767	0.9795	0.9909	0.9683	0.9764	0.9664	0.9739

(continued)

Table 5.1 (continued)

Estimate	MLE	SEL		LINEX					
				κ = −0.5		κ = 0.01		κ = 0.1	
		SBM	MCMC	SBM	MCMC	SBM	MCMC	SBM	MCMC
MSE	(0.0993)	(0.0113)	(0.0177)	(0.0105)	(0.0113)	(0.0113)	(0.0118)	(0.0115)	(0.0119)
c									
$\hat{\alpha}$	3.7674	2.5148	2.5536	3.1082	3.2168	2.5059	2.5415	2.4319	2.4411
MSE	(2.5013)	(0.0684)	(0.0848)	(0.4689)	(0.6130)	(0.0674)	(0.0826)	(0.0661)	(0.0754)
$\hat{\beta}_1$	1.3723	1.4797	1.4880	1.5470	1.5570	1.4785	1.4867	1.4674	1.4757
MSE	(0.0416)	(0.0268)	(0.0241)	(0.0307)	(0.0263)	(0.0268)	(0.0214)	(0.0270)	(0.0240)
$\hat{\beta}_2$	2.5580	1.9171	1.9167	2.0386	2.0613	1.9141	1.9114	1.8937	1.8894
MSE	(0.5274)	(0.0263)	(0.0410)	(0.0177)	(0.0421)	(0.0414)	(0.0277)	(0.0311)	(0.0455)
$\hat{R}(x_0)$	0.7214	0.6802	0.6794	0.6823	0.6810	0.6794	0.6801	0.6798	0.6790
MSE	(0.0074)	(0.0035)	(0.0036)	(0.0033)	(0.0035)	(0.0037)	(0.0036)	(0.0035)	(0.0037)
$\hat{\lambda}(x_0)$	1.1170	0.0178	0.9847	0.9923	1.0047	0.9870	0.9764	0.9757	0.9840
MSE	(0.1336)	(0.0178)	(0.0201)	(0.0175)	(0.0206)	(0.0202)	(0.0118)	(0.0179)	(0.0201)

Table 5.2 **a** Complete sample (n = 64), and **b** censored sample (n = 64, r = 55)

Estimate	MLE	SEL		LINEX					
				κ = −0.5		κ = 0.01		κ = 0.1	
		SBM	MCMC	SBM	MCMC	SBM	MCMC	SBM	MCMC
a									
$\hat{\alpha}$	414.57	302.12	302.20	324.882	331.37	299.99	300.28	281.91	280.34
$\hat{\beta}_1$	2.5019	2.1284	2.1000	2.1321	2.1833	2.1280	2.1734	2.1247	2.1654
$\hat{\beta}_2$	2.3322	2.5581	2.5610	2.5617	2.0613	2.5577	2.5523	2.5543	2.5425
$\hat{R}(x_0)$	0.4458	0.4758	0.4719	0.4752	0.6810	0.4751	0.4711	0.4751	0.4720
$\hat{\lambda}(x_0)$	1.3422	1.1342	1.1640	1.1315	1.1689	1.1345	1.1668	1.1368	1.1665
p-value	0.6190	0.7906	0.6545	0.7208	0.5318	0.7706	0.3605	0.5812	0.2792
b									
$\hat{\alpha}$	403.08	300.90	302.65	325.25	330.99	300.52	299.11	282.50	279.45
$\hat{\beta}_1$	2.3935	2.0980	2.1170	2.1205	2.1800	2.1167	2.1600	2.1136	2.1600
$\hat{\beta}_2$	2.4090	1.5590	2.5694	2.5728	2.5600	2.5691	2.5500	2.5659	2.5505
$\hat{R}(x_0)$	0.4539	0.4781	0.4747	0.4774	0.4800	0.4747	0.48000	0.4747	0.4800
$\hat{\lambda}(x_0)$	1.3057	1.1520	1.1336	1.1309	1.1500	1.1339	1.1600	1.1363	1.1500
p-value	0.7359	0.6398	0.7881	0.7086	0.5813	0.7849	0.4970	0.5908	0.2565

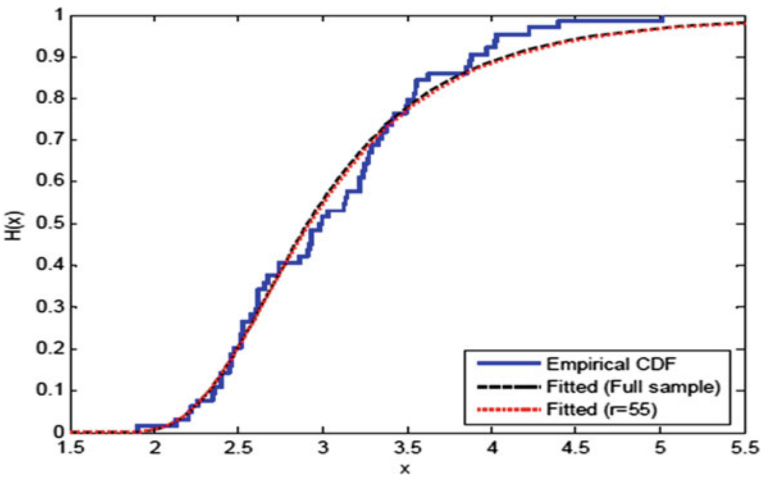

Fig. 5.3 Empirical CDF, fitted to the complete and censored data of Example 2

prior depends on b_1, b_2. For fixed b_2 at 0.6, this mean is large if b_1 is large. After some fitting trials, we found that b_1 = 180 gives a good fit. See Fig. 5.3.

Suppose that the test is terminated after the first 55 (=r) observations. The estimates of the parameters, SF and HRF at x_0 = 3 and the corresponding p-values of KS goodness of fit tests are presented in Table 5.2.

5.4 Prediction of Future Observables

A 100 $(1 - \tau)$% Bayes prediction interval, for the ℓth future observable (based on the one sample scheme), has bounds L and U, given by the solution of (2.5.10).

In the two-sample case, the bounds can be obtained by the solution of (2.5.16).

In both cases the EBurr XII distribution is used.

5.4.1 Random Sample Size

In the two-sample case, the size m of the future sample was assumed to be fixed. If, however, m is random, Gupta and Gupta (1984) suggested the use of the predictive PDF of Y_ℓ to be given in the form

$$f^{**}(y_\ell|\underline{x}) = \frac{1}{P(m \geq \ell)} \sum_{m=\ell}^{\infty} p(m) f^*(y_\ell|\underline{x}), \tag{5.4.1}$$

where $p(m)$ is the probability mass function (PMF) of the random variable m and $f^*(y_\ell|\underline{x})$ is the predictive PDF of Y_ℓ when m is fixed.

AL-Hussaini and Hussein (2011) considered the case when m has truncated Poisson mass function with parameter μ, given by

$$p(m) = \frac{e^{-\mu}\mu^m}{m!(1-e^{-\mu})},\ m = 1, 2, 3\ldots \tag{5.4.2}$$

They showed that, in this case, the lower and upper bounds L and U of a 100 $(1-\tau)\%$ predictive interval, for the future order statistic Y_ℓ are given by the solution of the two equations

$$\left.\begin{aligned} 0 &= \sum_{m=\ell}^{\infty} p(m)[S(L)/S_{02}] - [1-(\tau/2)]\sum_{m=\ell}^{\infty} p(m) \\ 0 &= \sum_{m=\ell}^{\infty} p(m)[S(U)/S_{02}] - (\tau/2)\sum_{m=\ell}^{\infty} p(m) \end{aligned}\right\}, \tag{5.4.3}$$

where $p(m)$ is given by (5.4.2), S_{02} and $S(v)$ (which are functions of m) are given by

$$S_{02} = \int_{j_1=0}^{n-r}\int_{j_2=0}^{m-\ell}\left[\frac{C_{j_1}C_{j_2}}{\ell+j_2}\right]I_{0j_1},\ I_{0j_1} = \int_0^\infty\int_0^\infty \frac{\beta_1^{b_3+b_4-1}\beta_2^{b_3-1}e^{-T_0}}{[T_{0j_1}]^{r+b_1}}d\beta_1 d\beta_2$$

$$S(v) = \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{m-\ell}\left[\frac{C_{j_1}C_{j_2}}{\ell+j_2}\right]I_{j_1,j_2}(v),$$

$$\begin{aligned} I_{j_1,j_2}(v) = \int_0^\infty\int_0^\infty &\left[\frac{1}{[T_{0j_1}]^{r+b_1}} - \frac{1}{[T_{0j_1}-(\ell+j_2)\ln G(v)]^{r+b_1}}\right] \\ &\times \beta_1^{b_3+b_4-1}\beta_2^{b_3-1}e^{-T_0}d\beta_1 d\beta_2, \end{aligned}$$

where

$$C_{j_1} = (-1)^{j_1}\binom{n-r}{j_1} \text{ and } C_{j_2} = (-1)^{j_2}\binom{m-\ell}{j_2},$$

$$T_{j_1j_2} \equiv T_{j_1j_2}(\beta_1,\beta_2) = T_{0j_1}(\beta_1,\beta_2) - (\ell+j_2)\ln G(x_r|\beta_1,\beta_2),$$

$$T_{0j_1} \equiv T_{0j_1}(\beta_1,\beta_2) = b_2 - \left[\sum_{i=1}^{r}\ln G(x_i|\beta_1,\beta_2) + j_1\ln G(x_r|\beta_1,\beta_2)\right],$$

$$T_0 \equiv T_0(\beta_1,\beta_2) = \left[\sum_{i=1}^{r}\ln G(x_i|\beta_1,\beta_2) - \sum_{i=1}^{r}\ln g(x_i|\beta_1,\beta_2)\right]$$

For details, see AL-Hussaini (2010).

Table 5.3 One- and two-sample predictive intervals based on simulated data and 1,000 repetitions: $(n = 20, r = 15, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 2)$ for some future observable

	One-sample			Two-sample					
				m-fixed			m-random		
	Y_1	Y_2	Y_{10}	Y_1	Y_2	Y_{10}	Y_1	Y_2	Y_{10}
L	1.56	1.62	1.88	0.13	0.25	0.37	0.14	0.29	0.30
U	2.59	3.18	6.99	0.82	0.94	1.64	0.89	1.24	1.87
Length	1.03	1.56	5.11	0.69	0.69	1.27	0.75	0.95	1.57

Table 5.4 One-sample predictive intervals ($n = 64$, $r = 55$) based on real life data and two-sample predictive intervals ($m = 10$ is fixed and m $\sim$ truncated Poisson) for some future observables

	One-sample			Two-sample					
				m-fixed			m-random		
	Y_1	Y_2	Y_{10}	Y_1	Y_2	Y_{10}	Y_1	Y_2	Y_{10}
L	3.63	3.66	4.41	0.69	1.38	1.72	0.57	1.06	1.20
U	4.14	4.20	5.03	2.32	3.12	3.52	2.41	2.98	3.18
Length	0.51	0.54	0.62	1.63	1.74	1.80	1.84	1.92	1.98

Using the same generated sets of data obtained in estimation, the lower and upper bounds of 95 % predictive intervals of $X_{r+1} = Y_1, X_{r+2} = Y_2, X_n = Y_{n-r}$ based on one- and two-sample schemes are reported in Table 5.3. In the two-sample case, the bounds are computed for three of the future observables: Y_1, Y_2, Y_{10}, when m is fixed ($m = 10$) and when m has truncated Poisson distribution with parameter $\mu = 10$.

The real life data affirm the simulation results, in that whether one- or two-sample scheme is used and whether m is fixed or random, the lengths of intervals increase as the index of future observables increase. Comparing the lengths of predictive intervals when m is fixed and when m is random Table 5.4 shows that the predictive intervals, using fixed m have shorter lengths than when m is random.

5.5 On Beta—Burr XII Distribution

Adding one or more parameters to a distribution makes it more flexible to analyzing data. This might have been a motivation for Paraníaba et al. (2011) to study the beta-Burr XII distribution which has two more parameters than the Burr XII distribution.

The beta-G distribution has been briefly discussed in Sect. 3.5. The CDF of this distribution is given by (3.5.5). If $G(x)$ is the CDF of the Burr XII, then the CDF of the beta-Burr XII distribution is given, for $x > 0$, by

$$H(x) = \frac{1}{B(a,b)} \int_0^{1-[1+(x/s)^c]} y^{a-1}(1-y)^{b-1}dy.$$

The corresponding PDF is given by

$$h(x) = \frac{ckx^{c-1}}{s^c B(a,b)} \left[1 + \left(\frac{x}{s}\right)^c\right]^{-(kb+1)} \left\{1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-k}\right\}^{a-1}.$$

The five-parameter distribution was investigated and its properties studied. The distribution was compared with other distributions such as the Burr XII, Weibull, EW, beta-W and log-logistic distributions. One application of the beta-Burr XII on life-time data shows that it could provide a better fit than the above mentioned models used in lifetime data analysis. Paranaíba et al. (2011) derived the moment generating function (MGF) of the beta-Burr XII distribution and, as a special case, the MGF of the Burr XII distribution. They provided expressions for the moments, mean deviations, two representations for the moments of order statistics, Bonferroni and Lorenz curves and the stress-strength reliability and estimated the model parameters by using the ML method. For interval estimation of the model parameters, they computed the 5×5 observed information matrix, I and used the fact that the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$ is $N_5(0, I^{-1})$.

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Chapter 6
Finite Mixture of Exponentiated Distributions

Contents

6.1 Introduction

The study of homogeneous populations was the main concern of statisticians along history. However, Newcomb (1886) and Pearson (1894) were two pioneers who approached heterogeneous populations with 'finite mixture' distributions.

With the advent of computing facilities, the study of heterogeneous populations which is the case with many real world populations (see Titterington et al. 1985), attracted the interest of several researchers in the last sixty years. Monographs and books by Everitt and Hand (1981), Titterington et al. (1985), MacLachlan and Basford (1988), Lindsay (1995) and MacLachlan and Peel (2000), collected and organized the research done up to the year 2000, analyzed data and gave examples of possible practical applications in different domains. Reliability and hazard based on finite mixtures were surveyed by AL-Hussaini and Sultan (2001).

The CDF of a finite mixture of k *components* is defined by

$$H(x) = \sum_{j=1}^{k} p_j H_j(x), \tag{6.1.1}$$

E.K. AL-Hussaini and M. Ahsanullah, *Exponentiated Distributions*,
Atlantis Studies in Probability and Statistics 5,
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where, for $j = 1, \ldots, k$, the *mixing proportions* p_j are non-negative and their sum adds up to 1. That is, $p_j \geq 0$ and $\sum_{j=1}^{k} p_j = 1$. The CDF $H_j(x)$ is known as the jth component.

In this chapter, concentration will be on the study of a finite mixture of two exponentiated exponential components. Due to the exponentiation of each component by a positive integer, the model is so flexible that it shows different shapes of HRFs. The CDF, PDF and SF of a mixture of two *EE*components are given, respectively, for $q = 1 - p$ by

$$\begin{aligned} H(x) &= pH_1(x) + qH_2(x), \quad \text{where for} \quad j = 1, 2, \\ H_j(x) &= (1 - \mathrm{e}^{-\beta_j x})^{\alpha_j}, \end{aligned} \tag{6.1.2}$$

$$\begin{aligned} h(x) &= ph_1(x) + qh_2(x), \quad \text{where for } j = 1, 2, \\ h_j(x) &= \alpha_j \beta_j \mathrm{e}^{-\beta_j x}(1 - \mathrm{e}^{-\beta_j x})^{\alpha_j}, \end{aligned} \tag{6.1.3}$$

$$R(x) = pR_1(x) + qR_2(x), \text{where for} j = 1, 2,\ R_j(x) = 1 - (1 - \mathrm{e}^{-\beta_j x})^{\alpha_j}, \tag{6.1.4}$$

so that the HRF of the mixture is given by

$$\lambda(x) = B(x)\,\lambda_1(x) + [1 - B(x)]\lambda_2(x), \tag{6.1.5}$$

where

$$B(x) = \frac{p\,R_1(x)}{R(x)} \; \textit{and for} \, j = 1, 2$$

$$\lambda_j(x) = \frac{h_j(x)}{R_j(x)} = \frac{\alpha_j \beta_j \mathrm{e}^{-\beta_j x}(1 - \mathrm{e}^{-\beta_j x})^{\alpha_j}}{1 - (1 - \mathrm{e}^{-\beta_j x})^{\alpha_j}}. \tag{6.1.6}$$

6.2 Properties of Finite Mixtures

6.2.1 Moments

The ℓth moment of a finite mixture of k components is given by

$$\mu_\ell = E\left(X^\ell\right) = \int x^\ell h(x)dx = \sum_{j=1}^{k} p_j \int x^\ell h_j(x)dx = \sum_{j=1}^{k} p_j\, \mu_{j\ell}, \tag{6.2.1}$$

where $\mu_{j\ell} = \int x^\ell h_j(x)\,dx$ is the ℓth moment of the jth component. So that the ℓth moment of a finite mixture of k components is given by the finite mixture of the ℓth moments of the k components.

For a finite mixture of $k\, EE(\alpha_j, \beta_j)$, it follows, from (6.2.1) and Table 2.1, that

$$\mu_\ell = \sum_{j=1}^{k} p_j \mu_{j\ell} = \Gamma(1+\ell) \sum_{j=1}^{k} \frac{p_j}{\beta_j^\ell} \sum_{i=1}^{v_j} \left(\frac{cij}{i^\ell} \right), \tag{6.2.2}$$

where

$$v_j = \begin{cases} \alpha_j = 1, 2, \ldots \\ otherwise \end{cases}, c_{ij} = (-1)^{i-1} \alpha_j (\alpha_j - 1) \cdots (\alpha_j - i + 1)/i!. \tag{6.2.3}$$

The ℓth moment of a finite mixture of two $EE(\alpha_j, \beta_j)$ components, j = 1, 2 is given by

$$\mu_\ell = \Gamma(1+\ell) \left[\frac{p}{\beta_1^\ell} \sum_{i=1}^{v_1} \left(\frac{c_{i1}}{i^\ell} \right) + \frac{q}{\beta_2^\ell} \sum_{i=1}^{v_2} \left(\frac{c_{i2}}{i^\ell} \right) \right], \tag{6.2.4}$$

where, for $i = 1, \ldots, v_j$ and $j = 1, 2, c_{ij}$, and v_j are given by (6.2.3).

6.2.2 MRLF

The MRLF $m(x)$ of a finite mixture with k components, in terms of the MRLFs of the components, is given by

$$m(x) = \sum_{j=1}^{k} B_j(x) m_j(x), \tag{6.2.5}$$

where $B_j(x)$ is given by

$$B_j(x) = \frac{p_j R_j(x)}{R(x)}, \quad j = 1, \ldots, k, \tag{6.2.6}$$

and $m_j(x)$ is the MRLF of the jth component.

This follows by observing that $\sum_{j=1}^{k} B_j(x) = \sum_{j=1}^{k} \frac{p_j R_j(x)}{R(x)} = 1$, so that the MRLF of a finite mixture is a finite mixture of the MRLFs of the components, with mixing proportions $B_j(x)$.

6.2.3 HRF

The HRF of a finite mixture of k components, is a finite mixture of the HRFs of the components with mixing proportions $B_j(x),\ j = 1, \ldots, k$. This is so true, since

$$\lambda(x) = \frac{h(x)}{R(x)} = \frac{\sum_{j=1}^{k} p_j h_j(x)}{R(x)} = \sum_{j=1}^{k} B_j(x)\lambda_j(x), \tag{6.2.7}$$

where for $j = 1, \ldots, k$, $\lambda_j(x) = \frac{h_j(x)}{R_j(x)}$, $h_j(x)$ and $R_j(x)$ are the HRE, PDF and SF of the jth component and $B_j(x)$ is given by $B_j(x) = \frac{p_j R_j(x)}{R(x)}$, so that $\sum_{j=1}^{k} B_j(x) = 1$.

In the particular case, $k = 2$, the HRF of a mixture of two components, in terms of the HRFs of the components, is given by

$$\begin{aligned} \lambda(x) &= \frac{h(x)}{R(x)} = B_1(x)\,\lambda_1(x) + B_2(x)\,\lambda_2(x), \\ B_1(x) &= \frac{pR_1(x)}{R(x)}, \quad B_2(x) = 1 - B_1(x) = \frac{qR_2(x)}{R(x)}, \quad q = 1 - p. \end{aligned} \tag{6.2.8}$$

If

$$\lambda_j(x) = \frac{h_j(x)}{R_j(x)} = \frac{\alpha_j \beta_j e^{-\beta_j x}(1 - e^{-\beta_j x})^{\alpha_j}}{1 - (1 - e^{-\beta_j x})^{\alpha_j}}, \tag{6.2.9}$$

$R_1(x)$ and $R(x)$ are given by (6.1.4), then (6.2.8) represents the HRF of a mixture of two EE components.

It is well known that the exponential distribution has a constant HRF on the positive half of the real line and that a finite mixture of two exponential distributions has a decreasing HRF on the same domain. If each of the exponential components is exponentiated by a positive parameter, more flexible model is obtained in that several shapes of the HRF of the mixture are obtained. Figure 6.1 shows six different shapes of CDFs and their corresponding HRFs of the given vector of parameters of a mixture of two EEs. Examples of such shapes are:

$$\begin{aligned} \text{DHR}&: (p = 0.9, \alpha_1 = 0.5, \alpha_2 = 1.5, \beta_1 = 2, \beta_2 = 3) \\ \text{IHR}&: (p = 0.1, \alpha_1 = 1.5, \alpha_2 = 3, \beta_1 = 2, \beta_2 = 2) \\ \text{BHR}&: (p = 0.1, \alpha_1 = 0.5, \alpha_2 = 3, \beta_1 = 3, \beta_2 = 0.5) \\ \text{UBHR}&: (P = 0.1, \alpha_1 = 0.5, \alpha_2 = 1.5, \beta_1 = 2, \beta_2 = 3) \\ \text{DIDHR}&: (p = 0.25, \alpha_1 = 0.5, \alpha_2 = 3, \beta_1 = 0.5, \beta_2 = 1.5) \\ \text{IDIHR}&: (p = 0.1, \alpha_1 = 2, \alpha_2 = 3, \beta_1 = 3, \beta_2 = 0.5) \end{aligned}$$

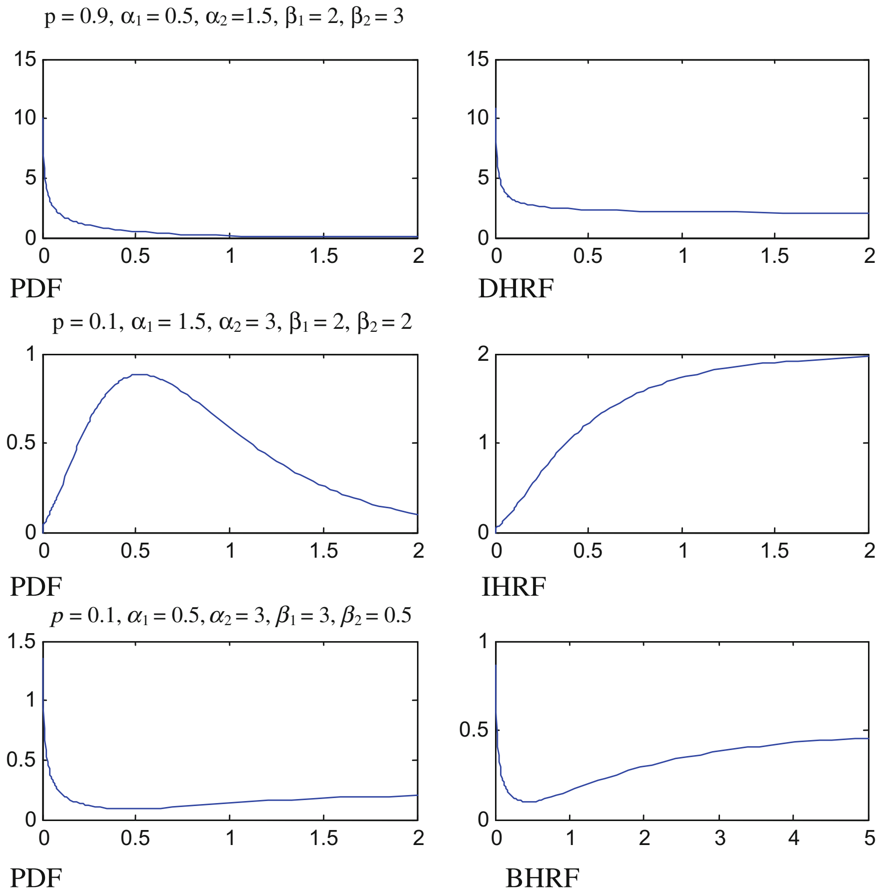

Fig. 6.1 Different shapes of PDFs and their corresponding HRFs *D* decreasing, *I* increasing, *B* bathtub, *UB* upside down bathtub, *DID* decreasing-increasing-decreasing, *IDI* increasing-decreasing-increasing

So, a finite mixture of two exponentiated components allows for monotone as well as non-monotone HRFs.

6.2.4 PRHRF

An expression, similar to that given by (6.2.7) can be obtained for PRHRF as follows

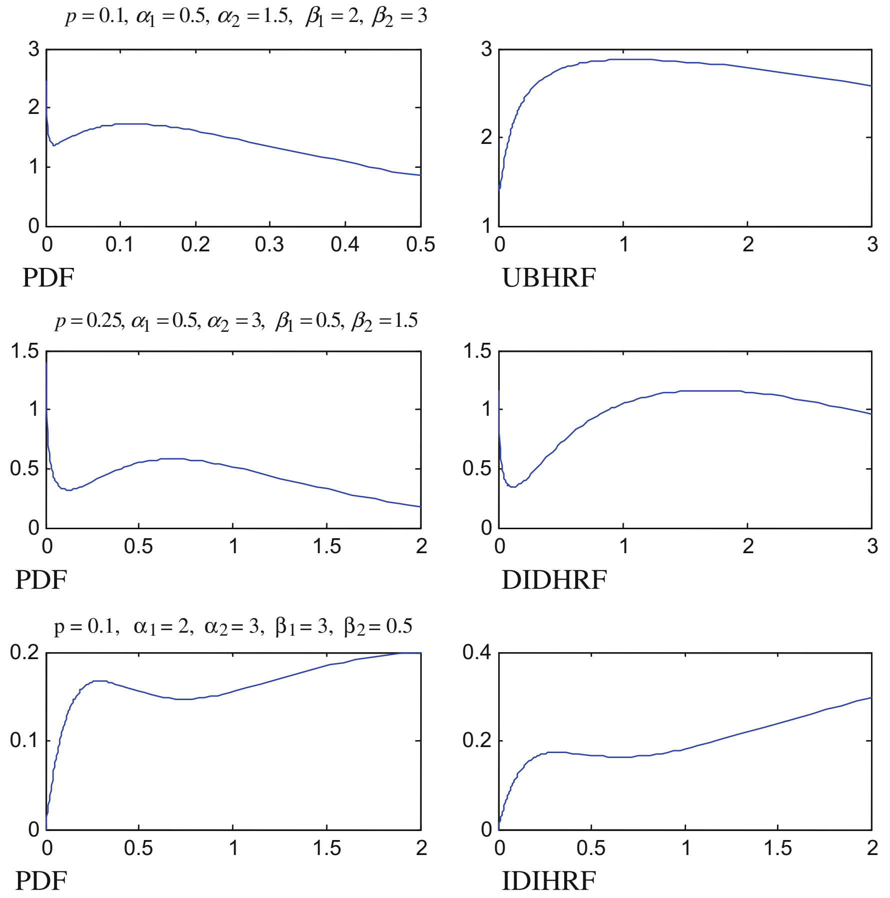

Fig. 6.1 (continued)

$$\lambda^*(x) = \sum_{j=1}^{k} B_j^*(x)\, \lambda_j^*(x), \tag{6.2.8}$$

where for $j = 1, \ldots, k$, $\lambda_j^*(x) = \frac{h_j(x)}{H_j(x)}$, $\lambda_j^*(x), h_j(x)$ and $H_j(x)$ are the PRHRF, PDF and CDF of the jth component and $B_j^*(x)$ is given by $B_\ell^*(x) = \frac{p_j H_j(x)}{H(x)}$, so that $\sum_{j=1}^{k} B_j^*(x) = 1$. This is because

$$\lambda^*(x) = \frac{h(x)}{H(x)} = \frac{\sum_{j=1}^{k} p_j h_j(x)}{H(x)} = \sum_{\ell=1}^{k} \frac{p_j H_j(x)}{H(x)}\, \lambda_j^*(x) = \sum_{j=1}^{k} B_j^*(x)\, \lambda_j^*(x).$$

In the particular case, $k = 2$, the PRHRF of a mixture of two components, in terms of the PRHRFs of the components, is given by

$$\begin{aligned} \lambda^*(x) &= \frac{h(x)}{H(x)} = B_1^*(x)\lambda_1^*(x) + B_2^*(x)\lambda_2^*(x), \\ B_1^*(x) &= \frac{pH_1(x)}{H(x)},\ B_2^*(x) = 1 - B_1^*(x). \end{aligned} \tag{6.2.9}$$

If $R_1(x)$ and $R(x)$ are given by (6.1.4), then (6.2.9) represents the PRHRF of two EE components.

6.3 Point Estimation Based on Balanced Square Error Loss Function

MLE and Bayes estimates using SEL function are obtained and used in finding Bayes estimates of the parameters, SF and HRF based on an asymmetric loss function, known as *balanced square error loss* function (BSEL).

Ahmadi et al. (2009), suggested the use of the so called *balanced square error loss* (BSEL) function, which was originated by Zellner (1994), to be of the form

$$L^*(\theta, \delta) = \omega\, \rho\,(\delta_0, \delta) + (1 - \omega)\, \rho(\theta, \delta), \tag{6.3.1}$$

where $\rho(\theta, \delta)$ is an arbitrary loss function, δ_0 is a chosen "target" estimator of δ and the weight $\omega \in [0, 1]$. The balanced square error loss function (6.3.1) specializes to various choices of loss functions, such as the absolute error loss, LINEX and generalizes SEL functions.

If $\rho(\theta, \delta) = (\delta - \theta)^2$ is substituted in (6.3.1), we obtain the BSEL function, given by

$$L^*(\theta, \delta) = \omega\,(\delta - \delta_0)^2 + (1 - \omega)(\delta - \theta)^2,$$

The estimator $\hat{u}_{BSEL}$ of a function $u(\theta)$, using BSEL, is given by

$$\hat{u}_{BSEL} = \omega\hat{u}_{ML} + (1 - \omega)\hat{u}_{SEL}, \tag{6.3.2}$$

where $\hat{u}_{ML}$ and $\hat{u}_{SEL}$ are the ML and Bayes estimates of u, based on SEL.

Notice that if $\omega = 0$ then $\hat{u}_{BSEL} = \hat{u}_{ML}$ and if $\omega = 1$ then $\hat{u}_{BSEL} = \hat{u}_{SEL}$.

The estimator of a function, using BSEL is actually a mixture of the MLE of the function and the BE, using SEL. Other estimators, such as the least squares estimator may replace the MLE. Also, a LINEX or QUADREX loss function could be used for $\rho(\theta, \delta)$. Having obtained the MLE and BE based on SEL, the estimators based on BSEL function are given, from (6.3.2), by

$$\begin{aligned}
\hat{p}_{BSEL} &= \omega \hat{p}_{ML} + (1-\omega)\hat{p}_{SEL},\\
\hat{\alpha}_{1BSEL} &= \omega \hat{\alpha}_{1ML} + (1-\omega)\hat{\alpha}_{1SEL},\\
\hat{\alpha}_{2BSEL} &= \omega \hat{\alpha}_{2ML} + (1-\omega)\hat{\alpha}_{2SEL},\\
\hat{\beta}_{1BSEL} &= \omega \hat{\beta}_{1ML} + (1-\omega)\hat{\beta}_{1SEL}\\
\hat{\beta}_{2BSEL} &= \omega \hat{\beta}_{2ML} + (1-\omega)\hat{\beta}_{2SEL},\\
\hat{R}_{BSEL}(x_0) &= \omega \hat{R}_{ML}(x_0) + (1-\omega)\hat{R}_{SEL}(x_0),\\
\hat{\lambda}_{BSEL}(x_0) &= \omega \hat{\lambda}_{ML}(x_0) + (1-\omega)\hat{\lambda}_{SEL}(x_0).
\end{aligned}$$

6.3.1 Maximum Likelihood Estimation

Suppose that r units have failed during the interval (0, x_r): r_1 units from the first sub-population and r_2 units from the second such that $r_1 + r_2 = r$ and $n-r$ units, which cannot be identified as to sub-population, are still functioning. Let, for $\ell = 1, 2$ and $i = 1, \ldots, r_\ell$, $x_{\ell i}$ denote the failure time of the ith unit belonging to the ℓth sub-population and that $x_{\ell i} \le x_0$. The likelihood function (LF) is given by Mendenhall and Hader (1958), as

$$L(\theta, \underline{x}) \;\propto \prod_{i=1}^{r_1} p h_1(x_{1i}) \prod_{i=1}^{r_2} p h_2(x_{2i}) [R(x_r)]^{n-r}, \tag{6.3.3}$$

where θ is the vector of parameters involved and $\underline{x} = (x_{11}, \ldots, x_{1r_1}, x_{21}, \ldots, x_{2r_2})$, $x_{\ell i} \le x_0$.

By substituting (6.1.3) and (6.1.4) in (6.3.3), we obtain

$$L(\theta, \underline{x}) \propto p^{r_1} q^{r_2} \alpha_1^{r_1} \beta_1^{r_1} \alpha_2^{r_2} \beta_2^{r_2} \ \exp\left[-\beta_1 \sum_{i=1}^{r_1} x_{1i} - \beta_2 \sum_{i=1}^{r_2} x_{2i}\right]$$
$$\prod_{i=1}^{r_1} (1 - \mathrm{e}^{-\beta_1 x_{1i}})^{\alpha_1 - 1} \prod_{i=1}^{r_2} (1 - \mathrm{e}^{-\beta_2 x_{2i}})^{\alpha_2 - 1} [R(x_r)]^{n-r}. \tag{6.3.4}$$

The log-LF is given by

$$\begin{aligned}
\ell^*(\theta, x) = \mathrm{In}\, L(\theta, x) \propto\ & r_1 \,\mathrm{In}\, p + r_2 \,\mathrm{In}\ \mathrm{q} + \mathrm{r}_1 \,\mathrm{In}\alpha_1 + \mathrm{r}_1 \,\mathrm{In}\beta_1 + r_2 \,\mathrm{In}\alpha_2 + \mathrm{r}_2 \,\mathrm{In}\, \beta_2\\
& - \beta_1 \sum_{i=1}^{r_1} x_{1i} - \beta_2 \sum_{i=1}^{r_2} x_{2i} + (\alpha_1 - 1) \sum_{i=1}^{r_1} \mathrm{In}(1 - \mathrm{e}^{-\beta_1 x_{1i}})\\
& + (\alpha_2 - 1) \sum_{i=1}^{r_2} \mathrm{In}(1 - \mathrm{e}^{-\beta_2 x_{2i}}) + (n - r)\mathrm{In}[\mathrm{R}(x_r)].
\end{aligned} \tag{6.3.5}$$

The MLEs $\hat{p}_{ML}, \hat{\alpha}_{1ML}, \hat{\alpha}_{2ML}, \hat{\beta}_{1ML}, \hat{\beta}_{2ML}$, of the five parameters are obtained by solving the following system of likelihood equations

$$0 = \frac{\partial \ell^*}{\partial p} = \frac{r_1}{p} - \frac{r_2}{q} - \frac{n-r}{R(x_r)}\left[(1-\mathrm{e}^{-\beta_1 x_r})^{\alpha_1} - (1-\mathrm{e}^{-\beta_2 x_r})^{\alpha_2}\right],$$

$$0 = \frac{\partial \ell^*}{\partial \alpha_1} = \frac{r_1}{\alpha_1} + \sum_{i=1}^{r_1} \mathrm{In}(1-\mathrm{e}^{-\beta_1 x_{1i}}) - \frac{(n-r)p}{R(x_r)}(1-\mathrm{e}^{-\beta_1 x_r})^{\alpha_1}\,\mathrm{In}(1-\mathrm{e}^{-\beta_1 x_r}),$$

$$0 = \frac{\partial \ell^*}{\partial \alpha_2} = \frac{r_2}{\alpha_2} + \sum_{i=1}^{r_2} \mathrm{In}(1-\mathrm{e}^{-\beta_2 x_{2i}}) - \frac{(n-r)q}{R(x_r)}(1-\mathrm{e}^{-\beta_2 x_r})^{\alpha_2}\mathrm{In}(1-\mathrm{e}^{-\beta_2 x_r}),$$

$$0 = \frac{\partial \ell^*}{\partial \beta_1} = \frac{r_1}{\beta_1} + \sum_{i=1}^{r_1} x_{1i} + (\alpha_1 - 1)\sum_{i=1}^{r_1} \frac{x_{1i}\mathrm{e}^{-\beta_1 x_{1i}}}{1-\mathrm{e}^{-\beta_1 x_{1i}}} - \frac{(n-r)p\alpha_1 x_r}{R(x_r)}\mathrm{e}^{-\beta_1 x_r}(1-\mathrm{e}^{-\beta_1 x_r})^{\alpha_1 - 1},$$

$$0 = \frac{\partial \ell^*}{\partial \beta_2} = \frac{r_2}{\beta_2} + \sum_{i=1}^{r_2} x_{2i} + (\alpha_2 - 1)\sum_{i=1}^{r_2} \frac{x_{2i}\mathrm{e}^{-\beta_2 x_{2i}}}{1-\mathrm{e}^{-\beta_2 x_{2i}}} - \frac{(n-r)p\alpha_2 x_r}{R(x_r)}\mathrm{e}^{-\beta_2 x_r}(1-\mathrm{e}^{-\beta_2 x_r})^{\alpha_2 - 1},$$

where

$$R(x_r) = 1 - \left[p(1-\mathrm{e}^{-\beta_1 x_r})^{\alpha_1} + q(1-\mathrm{e}^{-\beta_2 x_r})^{\alpha_2}\right].$$

The invariance property of MLEs enables us to obtain the MLEs $\hat{R}_{ML}(x_0)$ and $\hat{\lambda}_{ML}(x_0)$ by replacing the parameters by their MLEs in $R(x_0)$ and $\lambda(x_0)$.

Remarks

1. If $n = r$ (complete sample case), then

$$\hat{p}_{ML} = \frac{r_1}{n},$$

$$\hat{\alpha}_{1ML} = \frac{r_1}{\sum_{i=1}^{r_2} \mathrm{In}\left(1-\mathrm{e}^{-\hat{\beta}_1 x_{1i}}\right)^{-1}},$$

$$\hat{\alpha}_{2ML} = \frac{r_2}{\sum_{i=1}^{r_1} \mathrm{In}\left(1-\mathrm{e}^{-\hat{\beta}_2 x_{2i}}\right)^{-1}},$$

$$\hat{\beta}_{1ML} = \frac{r_1}{\sum_{i=1}^{r_1} x_{1i}\left\{1-(\hat{\alpha}_{1ML}-1)\left[\mathrm{e}^{-\hat{\beta}_1 x_{1i}}/(1-\mathrm{e}^{-\hat{\beta}_1 x_{1i}})\right]\right\}},$$

$$\hat{\beta}_{2ML} = \frac{r_2}{\sum_{i=1}^{r_2} x_{2i}\left\{1-(\hat{\alpha}_{2ML}-1)\left[\mathrm{e}^{-\hat{\beta}_2 x_{2i}}/(1-\mathrm{e}^{-\hat{\beta}_2 x_{2i}})\right]\right\}},$$

2. It can be numerically shown that the vector of parameters $\theta = (p, \alpha_1, \alpha_2, \beta_1, \beta_2)$ actually maximizes LF (6.3.3). This is done by applying Theorem (7–9) on p. 153 of Apostol (1957).
3. The parameters of the components are assumed to be distinct, so that the mixture is *identifiable*. For the concept of identifiability and examples, see Titterington et al. (1985), Yokowitz and Spragins (1968), AL-Hussaini and Ahmad (1981) and Ahmad and AL-Hussaini (1982).

6.3.1.1 Approximate Confidence Intervals

Let $\theta = (\theta_1 = p, \theta_2 = \alpha_1, \theta_3 = \alpha_2, \theta_5 = \beta_1, \theta_5 = \beta_2)$. The observed Fisher information matrix (see Nelson 1990), for the MLEs of the parameters is the 5 × 5 symmetric matrix I of the negative second partial derivatives of log-LF (6.3.4) with respect to the parameters. That is

$$I = -\left(\frac{\partial^2 \ell^*}{\partial\theta_i\,\partial\theta_j}\right), \tag{6.3.6}$$

evaluated at the vector of MLEs $\hat{\theta}$. The inverse of I is the local estimate of the asymptotic variance-covariance matrix V of the vector

$$(\hat{\theta}_1 = \hat{p}_{ML}, \hat{\theta}_2 = \hat{\alpha}_{1ML}, \hat{\theta}_3 = \hat{\alpha}_{2ML}, \hat{\theta}_5 = \hat{\beta}_{1ML}, \hat{\theta}_5 = \hat{\beta}_{2ML}).$$

That is

$$V = I^{-1} = (\hat{\sigma}_{ij}), \tag{6.3.7}$$

where $\hat{\sigma}_{ij}$ is the estimated $Cov(\hat{\theta}_i, \hat{\theta}_j)$, $i, j = 1, \ldots, 5$.

The observed Fisher information matrix enables us to construct approximate confidence intervals for the parameters based on the limiting normal distribution. Following the general asymptotic theory of MLEs, the sampling distribution of $(\hat{\theta}_i - \theta_i)/\sqrt{\hat{\sigma}_{ii}}$, $i = 1, \ldots, 5$, can be approximated by a standard normal distribution.

An approximate two-sided 100 $(1 - \tau)$ % confidence interval for θ_i, is given by

$$\hat{\theta}_i - z_1 - (\tau/2)\sqrt{\hat{\sigma}_{ii}} < \theta_i < \hat{\theta}_i + z_{1-(\tau/2)}\sqrt{\hat{\sigma}_{ii}}, i = 1, \ldots, 5, \tag{6.3.8}$$

where z_{1-c} is the percentile of the standard normal distribution with right-tale probability of c.

6.3.2 Bayes Estimation

Suppose that an objective, non-informative prior is used, in which $p, \alpha_1, \alpha_2, \beta_1, \beta_2$ are independent and that $p \sim$ *uniform* on the interval (0,1), so that the prior PDF is given by

$$\pi(\theta) \propto \frac{1}{\alpha_1\beta_1\alpha_2\beta_2}. \tag{6.3.9}$$

The following theorem gives expressions for the Bayes estimators, using SEL.

Theorem 6.1 *The Bayes estimators of the parameters, SF and HRF, assuming the prior given by* (6.3.9) *and using SEL function are given by:*

$$\hat{p}_{SEL} = \frac{S_1}{S_2},\ \hat{\alpha}_{1SEL} = \frac{r_1 S_2}{S_0},\ \hat{\alpha}_{2SEL} = \frac{r_2 S_3}{S_0},\ \beta_{1SEL} = \frac{S_4}{S_0},\ \beta_{2SEL}\frac{S_5}{S_0}, \tag{6.3.10}$$

$$\hat{R}_{SEL} = 1 - \frac{S_6}{S_0},\ \hat{\lambda}_{SEL} = \frac{S_7}{S_0}. \tag{6.3.11}$$

where

$$\begin{aligned}
S_0 &= \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{j_1} C_{j_1 j_2} B(\kappa_{j_2}, \kappa_{j_1 j_2}) I_0\\
S_1 &= \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{j_1} C_{j_1 j_2} B(\kappa_{j_2} + 1, \kappa_{j_1 j_2}) I_0\\
S_2 &= \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{j_1} C_{j_1 j_2} B(k_{j_2}, k_{j_1 j_2}) I_1\\
S_3 &= \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{j_1} C_{j_1 j_2} B(k_{j_2}, k_{j_1 j_2}) I_2\\
S_4 &= \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{j_1} C_{j_1 j_2} B(k_{j_2}, k_{j_1 j_2}) I_3\\
S_5 &= \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{j_1} C_{j_1 j_2} B(k_{j_2}, k_{j_1 j_2}) I_4\\
S_6 &= \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{j_1} C_{j_1 j_2} [B(k_{j_2} + 1, k_{j_1 j_2}) I_5 + B(k_{j_2}, k_{j_1 j_2} + 1)] I_6,\\
S_7 &= \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{\infty}\sum_{j_4=0}^{j_3} C_{j_1 j_2 j_3 j_4} [r_1 B(k_{j_2 j_4} + 1, k_{j_1 j_2 j_3 j_4}) I_7 + r_2 B(k_{j_2 j_4}, k_{j_1 j_2 j_3 j_4} + 1)] I_8,
\end{aligned} \tag{6.3.12}$$

$$C_{j_1 j_2} = (-1)^{j_1}\binom{n-r}{j_1}\binom{j_1}{j_2},\ C_{j_1 j_2 j_3 j_4} = C_{j_1 j_2} C_{j_3 j_4},\ C_{j_3 j_4} = \binom{j_3}{j_4}, \tag{6.3.13}$$

$$\begin{aligned}
k_{j_2} &= r_1 + j_2 + 1, k_{j_1 j_2} = r_2 + j_1 - j_2 + 1, k_{j_2 j_4} = r_1 + j_2 + j_4,\\
k_{j_1 j_2 j_3 j_4} &= r_2 + j_1 - j_2 + j_3 - j_4.
\end{aligned} \tag{6.3.14}$$

and $B(a, b)$ *is the beta function. The integrals involved are given by*

$$
\begin{aligned}
I_0 &= \int_0^\infty \int_0^\infty \frac{\beta_1^{r_1-1}\beta_2^{r_2-1}\mathrm{e}^{-T_0}}{T_{j_2}^{r_1} T_{j_1 j_2}^{r_2}}\, d\beta_1 d\beta_2 \\
I_1 &= \int_0^\infty \int_0^\infty \frac{\beta_1^{r_1-1}\beta_2^{r_2-1}\mathrm{e}^{-T_0}}{T_{j_2}^{r_1+1} T_{j_1 j_2}^{r_2}}\, d\beta_1 d\beta_2 \\
I_2 &= \int_0^\infty \int_0^\infty \frac{\beta_1^{r_1-1}\beta_2^{r_2-1}\mathrm{e}^{-T_0}}{T_{j_2}^{r_1} T_{j_1 j_2}^{r_2+1}}\, d\beta_1 d\beta_2 \\
I_3 &= \int_0^\infty \int_0^\infty \frac{\beta_1^{r_1}\beta_2^{r_2-1}\mathrm{e}^{-T_0}}{T_{j_2}^{r_1} T_{j_1 j_2}^{r_2}}\, d\beta_1 d\beta_2 \\
I_4 &= \int_0^\infty \int_0^\infty \frac{\beta_1^{r_1-1}\beta_2^{r_2}\mathrm{e}^{-T_0}}{T_{j_2}^{r_1} T_{j_1 j_2}^{r_2}}\, d\beta_1 d\beta_2 \qquad (6.3.15)\\
I_5 &= \int_0^\infty \int_0^\infty \frac{\beta_1^{r_1-1}\beta_2^{r_2-1}\mathrm{e}^{-T_0}}{[T_{j_2} - \ln(1-\mathrm{e}^{-\beta_1 x_0})]^{r_1} T_{j_1 j_2}^{r_2}}\, d\beta_1 d\beta_2 \\
I_6 &= \int_0^\infty \int_0^\infty \frac{\beta_1^{r_1-1}\beta_2^{r_2-1}\mathrm{e}^{-T_0}}{T_{j_2}^{r_1} [T_{j_1 j_2} - \ln(1-\mathrm{e}^{-\beta_2 x_0})]^{r_2}}\, d\beta_1 d\beta_2 \\
I_7 &= \int_0^\infty \int_0^\infty \frac{\beta_1^{r_1}\beta_2^{r_2-1}\mathrm{e}^{-T_3}}{T_1^{r_1+1} T_2^{r_2}}\, d\beta_1 d\beta_2 \\
I_8 &= \int_0^\infty \int_0^\infty \frac{\beta_1^{r_1-1}\beta_2^{r_2}\mathrm{e}^{-T_6}}{T_4^{r_1} T_5^{r_2+1}}\, d\beta_1 d\beta_2,
\end{aligned}
$$

$$
\begin{aligned}
T_0 &= \sum_{i=1}^{r_1} [\beta_1 x_{1i} + \ln(1-\mathrm{e}^{-\beta_1 x_{1i}})] + \sum_{i=1}^{r_2} [\beta_2 x_{2i} + \ln(1-\mathrm{e}^{-\beta_2 x_{1i}})], \\
T_{j_2} &= \beta_1 - \left[j_2 \ln(1-\mathrm{e}^{-\beta_1 x_r}) + \sum_{i=1}^{r_1} \ln(1-\mathrm{e}^{-\beta_1 x_{1i}}) \right], \\
T_{j_1 j_2} &= \beta_2 - \left[(j_1 - j_2) \ln(1-\mathrm{e}^{-\beta_1 x_r}) + \sum_{i=1}^{r_1} \ln(1-\mathrm{e}^{-\beta_1 x_{1i}}) \right], \qquad (6.3.16)\\
T_1 &= T_{j_2} - (j_4+1)\,\mathrm{In}\,(1-\mathrm{e}^{-\beta_1 x_0}), \\
T_2 &= T_{j_1 j_2} - (j_3 - j_4)\,\mathrm{In}\,(1-\mathrm{e}^{-\beta_2 x_0}) \\
T_3 &= T_0 + \beta_1 x_0 + \mathrm{In}\,(1-\mathrm{e}^{-\beta_1 x_0}), \\
T_4 &= T_{j_2} - j_4\,\mathrm{In}\,(1-\mathrm{e}^{-\beta_1 x_0}), \\
T_5 &= T_{j_1 j_2} - (j_3 - j_4 + 1)\,\mathrm{In}\,(1-\mathrm{e}^{-\beta_2 x_0}).
\end{aligned}
$$

Proof By expanding the last term in LF (6.3.4), using the binomial expansion, it follows that

$$[R(x_r)]^{n-r} = \left[1 - \left\{p\left(1 - \mathrm{e}^{-\beta_1 x_r}\right)^{\alpha_1} + q\left(1 - \mathrm{e}^{-\beta_1 x_r}\right)^{\alpha_2}\right\}\right]^{n-r}$$
$$= \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{j_1} C_{j_1 j_2} p^{j_2} q^{j_1 - j_2} \exp[\alpha_1 j_2 \,\mathrm{In}\,(1 - \mathrm{e}^{-\beta_1 x_r}) + \alpha_2 (j_1 - j_2)\mathrm{In}(1 - \mathrm{e}^{-\beta_2 x_r})]$$

where $C_{j_1 j_2}$ is given by (6.3.13). So that

$$L(\theta, \underline{x}) \propto \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{j_1} C_{j_1 j_2} p^{r_1 + j_2} q^{r_2 + j_1 - j_2} \alpha_1^{r_1} \beta_1^{r_1} \alpha_2^{r_2} \beta_2^{r_2}$$
$$\times \exp[-\alpha_1 T_{j_2} - \alpha_2 T_{j_1 j_2} - T_0]$$

where T_0, $T_{j_2}, T_{j_1 j_2}$ are given by (6.3.16).

Suppose that the prior PDF is as given by (6.3.9). The posterior is then given by

$$\pi(\theta|\underline{x}) \propto L(\theta, \underline{x})\pi(\theta)$$
$$\Rightarrow \pi(\theta|\underline{x})$$
$$= A \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{j_1} C_{j_1 j_2} p^{\kappa_{j_2} - 1} q^{\kappa_{j_1 j_2} - 1} \alpha_1^{r_1 - 1} \alpha_2^{r_2 - 1} \beta_1^{r_1 - 1} \beta_2^{r_2 - 1} \mathrm{e}^{-\alpha_1 T_{j_2} - \alpha_2 T_{j_1 j_2} - T_0}, \tag{6.3.17}$$

where $\kappa_{j_2}, \kappa_{j_1 j_2}$ are given by (6.3.14) and A is a normalizing constant, given by

$$A^{-1} = \int \pi(\theta|\underline{x}) d\theta = \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{j_1} C_{j_1 j_2} B(\kappa_{j_2}, \kappa_{j_1 j_2}) \int_0^\infty \int_0^\infty \frac{\Gamma(r_1)\Gamma(r_2)}{T_{j_2}^{*r_1} T_{j_1 j_2}^{*r_2}}$$
$$\times \beta_1^{\mathrm{r}_1 - 1} \beta_2^{\mathrm{r}_2 - 1} \mathrm{e}^{-\mathrm{T}_0} \mathrm{d}\beta_1 \mathrm{d}\beta_2$$
$$= \Gamma(\mathrm{r}_1)\Gamma(\mathrm{r}_2)\mathrm{S}_0,$$

where S_0 is given by (6.3.12) and I_0 by (6.3.15). It then follows that the Bayes estimates using SEL function are given by

$$\hat{p}_{SEL} = E(p|\underline{x}) = \frac{S_1}{S_0}, \hat{\alpha}_{1SEL} = E(\alpha_1|\underline{x}) = r_1 \frac{S_2}{S_0}, \hat{\alpha}_{2SEL} = E(\alpha_2|\underline{x}) = r_2 \frac{S_3}{S_0},$$
$$\hat{\beta}_{1SEL} = E(\beta_1|\underline{x}) = \frac{S_4}{S_0}, \hat{\beta}_{2SEL} = E(\beta_2|\underline{x}) = \frac{S_5}{S_0},$$
$$\hat{R}_{SEL}(x_0) = E[R(x_0)] = 1 - \int [p\mathrm{e}^{\alpha_1 \mathrm{In}\,(1 - \mathrm{e}^{-\beta_1 x_0})} + q\mathrm{e}^{\alpha_1 \mathrm{In}\,(1 - \mathrm{e}^{-\beta_1 x_0})}]\, \pi(\theta|\underline{x}) d\theta$$
$$= 1 - \frac{S_6}{S_0}.$$

$$\hat{\lambda}_{SEL}(x_0) = E[\lambda(x_0)|\underline{x}] = \frac{h(x_0)}{R(x_0)} = [ph(x_0) + qh(x_0)][1 - \{pH_1(x_0) + qH_2(x_0)\}]^{-1}$$

Since

$$[1 - \{pH_1(x_0) + qH_2(x_0)\}]^{-1} = \sum_{j_3=0}^{\infty} \{pH_1(x_0) + qH_2(x_0)\}^{j_3}$$
$$= \sum_{j_3=3}^{\infty}\sum_{j_4=0}^{j_3} C_{j_3 j_4} p^{j_3} q^{j_3-j_4} [H_1(x_0)]^{j_3} [H_2(x_0)]^{j_3-j_4},$$

then

$$\lambda(x_0) = \sum_{j_3=0}^{\infty}\sum_{j_4=0}^{j_3} C_{j_3 j_4} p^{j_3} q^{j_3-j_4} [H_1(x_0)]^{j_3} [H_2(x_0)]^{j_3-j_4} \{pH_1(x_0) + qH_2(x_0)\} \quad (6.3.18)$$

It then follows that

$$\hat{\lambda}_{SEL}(x_0) = E[\lambda(x_0)|\underline{x}] = \int \lambda(x_0)\pi(\theta|\underline{x})d\theta. \quad (6.3.19)$$

Substitution of (6.3.17) and (6.3.18) in (6.3.19), then finally yields $\hat{\lambda}_{SEL}(x_0)$, where $S_0, \ldots, S_7$ are given by (6.3.11).

6.4 Numerical Example

6.4.1 Point Estimation of the Parameters, SF and HRF

A sample is generated from the mixture in such a way that $x_{\ell i} < x_0$, $\ell = 1, 2, i = 1, \ldots, r_\ell$. We generate 100 samples of size $n = 50$ each, from a finite mixture of two exponentiated exponential components, whose PDF is given by (6.1.3), as follows:

1. Generate u_1 and u_2 from Uniform (0,1) distribution.
2. For given values of p, $\alpha_1, \alpha_2, \beta_1, \beta_2$, generate x according to the expression:

$$x = \begin{cases} -\frac{1}{\beta_1}\ln\left(1-u_1^{1/\alpha_1}\right), & u_1 \le p, \\ -\frac{1}{\beta_2}\ln\left(1-u_1^{1/\alpha_2}\right), & u_1 \le p. \end{cases}$$

An observation x_{ji} belongs to sub-population 1, if $u_{ji} \le p$ and to sub-population 2, if $u_{ji} \le p$, where the sample is generated from the mixture in such a way that $x_{ji} < x_0$, $j = 1, 2, i = 1, \ldots, r_j$.

3. Repeat until you get a sample of size n. The observations are ordered and only the first $r = 45$ (90 % of n) out of the $n = 50$ observations are assumed to be known. Now we have r_1 observations from the first component of the mixture and r_2 observations from the second component ($r = r_1 + r_2 = 45$).

The value x_0 is chosen to be equal to 1.

The estimates of $p, \alpha_1, \alpha_2, \beta_1, \beta_2, R_H(x_0), \lambda_H(x_0)$ and absolute biases are computed by using the ML and Bayes methods. The Bayes estimates are obtained under SEL function. An estimator of a function, using BSEL, is actually a mixture (ω = 0.2, 0.4, 0.6, 0.8) of the MLE of the function and the BE, using SEL.

The MLEs are computed using the built-in MATLAB® function "*ga*" to find the maximum of the log-LF (6.3.5) using the genetic algorithm. This is better than solving the system of five likelihood equations in the five unknowns, by using some iteration scheme (Table 6.1).

Nevertheless, the system of equations is needed for the computation of the asymptotic variance-covariance matrix.

The (arbitrarily) chosen actual population values are $p = 0.4, \alpha_1 = 2, \alpha_2 = 3, \beta_1 = 2$ and $\beta_2 = 3$. For $x_0 = 1$ the actual values for $R(x_0)$ and $\lambda(x_0)$ are given, respectively, by 0.1862 and 2.3096.

The estimates of the parameters, SF and HRF under the BSEL function are given in Table 5.1, for different weights ω. It may be noticed that when $\omega = 1$, we obtain the MLEs while the case $\omega = 0$, yields the Bayes estimates under SEL function.

Table 6.1 Estimates of the parameters, SF and BSEL function and absolute biases

Estimate	$\omega = 0$ "SEL"	$\omega = 0.2$	$\omega = 0.4$	$\omega = 0.6$	$\omega = 0.8$	$\omega = 1$ "MLE"
$\hat{p}$	0.416229	0.416139	0.416049	0.415959	0.415869	0.415779
	0.016229	0.016139	0.016049	0.015959	0.015869	0.015779
$\hat{\alpha}_1$	2.380074	2.432918	2.485761	2.538598	2.591441	2.644284
	0.380074	0.432918	0.485761	0.538598	0.591441	0.644284
$\hat{\alpha}_2$	2.712658	2.840706	2.968753	3.096802	3.224850	3.352898
	0.287342	0.159294	0.031247	0.096802	0.224850	0.352898
$\hat{\beta}_1$	2.371177	2.407887	2.444589	2.481294	2.517997	2.554707
	0.371177	0.407887	0.444589	0.481294	0.517997	0.554707
$\hat{\beta}_2$	2.586853	2.705871	2.824898	2.943908	3.062934	3.181952
	0.413147	0.294129	0.175102	0.056092	0.062934	0.181952
$R(x_0)$	0.253611	0.236527	0.219444	0.202361	0.185277	0.168194
	0.067411	0.050327	0.033244	0.016161	0.000923	0.018006
$\hat{\lambda}(x_0)$	2.106139	2.190786	2.27544	2.360092	2.444747	2.529393
	0.203461	0.118814	0.03416	0.050492	0.135147	0.219793

6.4.2 Interval Estimation of the Parameters

By inverting Fisher information matrix (by computing the second partial derivatives of the log-likelihood function), given by (6.3.5), evaluated at the vector of MLEs $\hat{p}_{ML} = 0.4158$, $\hat{\alpha}_{1ML} = 2.6443$, $\hat{\alpha}_{2ML} = 3.3588$, $\hat{\beta}_{1ML} = 2.5547$, $\hat{\beta}_{2ML} = 3.1819$ ($\omega = 1$ in Table 5.2). The variance-covariance matrix (6.3.6) is found to be

$$V = \begin{pmatrix} 0.00546 & -0.0106 & 0.00941 & 0.01345 & 0.01097 \\ & 1.1651 & -0.3266 & 0.66019 & -0.23199 \\ & & 1.0099 & -0.23629 & 0.54097 \\ & & & 0.68764 & -0.21231 \\ & & & & 0.49433 \end{pmatrix}$$

So that the asymptotic variances of the estimators of the parameters are given by:

$$V(\hat{p}) = 0.00546,\ V(\hat{\alpha}_1) = 1.1651,\ V(\hat{\alpha}_2) = 1.0099,\ V(\hat{\beta}_1) = 0.68764,\ V(\hat{\beta}_2) = 0.49433$$

It then follows that approximate 95 % confidence intervals of the parameters are given, based on 1,000 samples, by

$$0.2710 < p < 0.5610, \quad 0.5287 < \alpha_1 < 4.7599, \quad 0.9294 < \beta_1 < 4.1800$$
$$1.3832 < \alpha_2 < 5.3226, \quad 1.8039 < \beta_2 < 4.5600.$$

In this chapter, point estimation of the five parameters, SF and HRF are obtained when the underlying population is a finite mixture of two exponentiated exponential components based on the balanced squared error loss function which is a weighted average of two loss functions: one of which reflects precision of estimation and the other reflects goodness-of-fit. This asymmetric loss function may be considered as a compromise between Bayes and non-Bayes estimates.

We have also estimated the parameters of the mixture by obtaining the asymptotic variance-covariance matrix and hence the approximate confidence intervals.

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