Advances in Mathematics Education

Stephen J. Hegedus
Jeremy Roschelle *Editors*

The SimCalc Vision and Contributions

Democratizing Access to Important Mathematics

Springer

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Series Preface

The fourth volume in the series *Advances in Mathematics Education* deals with contributions from the SimCalc research programme, a highly prestigious and long-term research programme going back to the visions of Jim Kaput. Jim Kaput started to develop his visions in the eighties of the last century; the SimCalc-Project is implementing the Kaputian research ideas into practice. The papers are contributing to an eminent important and topical theme, namely a democratic access to important mathematics can be offered based on dynamic representations.

The book starts with a section focusing on more general aspects such as the philosophical foundation of the programme dealing with eminent important aspects such as the transfer from static to dynamic perspectives on mathematics and mathematics education. These foundational reflections are followed by discussions on the design of the SimCalc-technology, which influence the design of the whole research programme. The SimCalc research project is not limited to these kinds of reflections, it aims to scale up and influence mathematics education in a general way. The papers describe and analyze how SimCalc changed ordinary teaching at various levels including dynamic representations during the whole teaching-and-learning-processes. The book shows by an impressive collection of research studies, how the SimCalc-programme changed classroom discourse, enabled an equal access to mathematics by considering a high diversity. The impressive collection of research studies is followed by commentaries and reflections from outsiders, who connect the SimCalc research programme with the mathematics education debate in general.

The book continues the discussion of other books in this series focusing on diversity and equity based on a clear theoretical foundation, which makes the book fit perfectly into the series Advances in Mathematics Education. This monograph has the potential to strongly influence the debate on technology in mathematics education. It shows how technology and its theory-guided usage can provide a rich mathematical learning environment allowing equal access to mathematics for all students.

Hamburg, Germany — Gabriele Kaiser
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To the schools, teachers and students who contributed their energies and feedback to SimCalc research and development and made this possible.

To Rebecca Moniz for her steadfast service to the project and her dedication to completing this book.

> Anyone who presumes to describe the roles of technology in mathematics education faces challenges akin to describing a newly active volcano—the mathematical mountain is changing before our eyes...
> Jim Kaput, 1942–2005

Stephen J. Hegedus
Jeremy Roschelle

Contents

Part I
Philosophy & Background

Ubiratan D'Ambrosio

While still teaching at SUNY at Buffalo, in the late 1960s, I became familiar with the new program, proposed and run by Jim Kaput, at the University of Massachusetts, Dartmouth, called START. The purpose of the program was to help students to overcome, in their freshman year, deficiencies in Algebra and to help them to proceed onwards to a math, science, or engineering degree. Naturally, the students presenting deficiencies come from different cultural environments, revealing the broad social concern of the program. These were the pioneer ideas that led to the SimCalc Project, launched in 1994 with the main objective of helping students to advance in Algebra and Calculus. In 2007 I was invited to join the Advisory Board of the *James J. Kaput Center for Research and Innovation in Mathematics Education* at the University of Massachusetts Dartmouth. Later, we had the opportunity of receiving the visits of researchers from the Kaput Center at UNIBAN/Universidade Bandeirantes de São Paulo. There the visitors made it very clear that the proposals of SimCalc are of much benefit for practically every country.

The core of the SimCalc Project is the idea of Jim Kaput to let Mathematics Education spring out of change and variation, which he conceptualized as a major strand of mathematical development leading through Algebra and beyond Calculus. The seminal idea is to enliven and enrich mathematics in every grade level and simultaneously to give students access to critically important mathematical concepts much earlier. The development of specific technologies to achieve this goal is key for the success of this ambitious educational proposal. In this development, research of SimCalc must review, critically, the most relevant current educational practices, which largely reflect our past and, at the same time, venture into the future, proposing new directions for education. The same care of critically regarding the past is present in their views of the future. Thus, they avoid being trapped by the marvels suggested by the new, amazing, technologies.

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Education has always been a two-faced enterprise. The past establishes goals and methods of Education, and the other face tries to capture the future, suggesting and proposing new directions of thought and new styles of behavior for the generation which, in a few years, will take over both routines and societal innovation. History tells us that this face of Education has always been sensitive to emerging technologies. And as we enter into the 21th Century, the presence of technology, particularly techno-science, in everyday life is overwhelming. Institutions in the modern world are affected by this presence.

Technologies of communication and information have been particularly influential in new directions of society, in particular of education. The transition from orality to writing marked a new role for the teacher. From the sole repository of accumulated knowledge, the teacher became a guide and interpreter of registered knowledge. The emergence of hardware, in the broad sense, from language, oral and written, particularly documents and books, initiated a companionship between teachers and hardware. It is also remarkable how the emergence of writing strengthened individual memory, contrary to the concerns of Thamus when Theuth explained to him the discovery of writing. The conservative king was afraid that the new invention would implant forgetfulness in the souls of men, as described in Plato. Something similar occurred in Europe with the introduction of the technology of calculation of Indian and Arabic origins, which strengthened the analytic instruments of the philosophers of the late European Middle Age, thus paving the way to the Renaissance and Modern Age. We are now living new possibilities in our communicative and analytic capabilities, thanks to the powerful new technology of communication and information.

Since Middle Ages, the scholars concern with movement were, together with representations, the backbone of a new way of understanding the physical world. To understand and to explain movements, relying on experience and imagination, led to a formalism in which change and variation play a fundamental role. With the support of computational techniques and the symbolism introduced by Algebra, Calculus became the main intellectual instrument to understand and to explain the worlds we experience.

Mathematics is a set of practices and languages, rooted on culture. It is applied and extended through systematic forms of reasoning and argument. Mathematics leads to representation systems which are organized as arts, humanities, natural and social sciences. Mathematics change over time and respond specially to technologies of communication and information. Mathematics embodies strategies for the generation of knowledge: observing, comparing, classifying, computing, measuring, inferring. Although generated as individual strategies, they are socialized through communication and are culturally shared.

It is a fact that billions are spent in education worldwide. But they risk to be lost if we insist in declining educational models and practices. This big loss is unbearable for most countries, where human resources, so necessary for their future, receive an obsolete, and in most cases, useless, education. Even the more prosperous economies are very much concerned with the downgrading of their education, in spite of enormous resources available. We all agree that technology, by itself, is not the guarantee of a good education. But it is undeniable that lack of technology may hinder progress in education.

The challenges to the educator, from the cognitive dimensions to the political issues, are all dealt with by the authors of this part. They claim a *de facto* evolution of the species towards higher levels of humanity, in the sense of a species impregnated with respect, solidarity and team spirit. This is particularly noticed when they focus on the interaction of humans and technology. Refusing a common concern that technology leads to lack of humanity, researchers of SimCalc draw from many examples from the history of culture to show the opposite. Indeed, there has been an interaction of the humans and the technology they have created, and the evolution of the human species results from this interaction, to the points of a true merging of technologies in everyday life and, remarkably, in the way we think and act. These facts point to the responsibility of Education to guide this merging to the ultimate goal of humanity. This is absolutely necessary for the survival, with dignity, of civilization.

The trajectory to a humanity impregnated with respect, solidarity and team spirit meets with obstacles of a political nature. It is undeniable that some educators have a reaction to the new and favor sameness. On the other hand, there is a growing number of educators absorbing the new. Caution, necessary in every step of human action, should not hinder venturing into the new. The authors of the different chapters are well aware of the need of caution, and adopt all the required instruments of control and evaluation. Most of the technological innovation in Mathematics Education, internationally recognized, received attention of the authors and were the subject of careful research.

The description of the projects, accompanied by the results of their research and by very important remarks, will be extremely valuable for those wishing to innovate.

Introduction: Major Themes, Technologies, and Timeline

Jeremy Roschelle and Stephen Hegedus

The long-term imperative of the SimCalc project has been to democratize access to the Mathematics of Change and Variation (MCV) (Kaput, 1994)—especially algebraic ideas underlying calculus (Kaput and Roschelle, 1998)—using a combination of new dynamic technologies for representing and communicating mathematics with new curriculum materials for grades 6–13 and aligned teacher professional development.

Over time, many investigators at many institutions around the world have taken up this imperative and advanced it through their own innovations and research, linked by common themes and technologies. The resulting program of research has led to several advances in mathematics education including, but not limited to, new theoretical perspectives in the development of dynamic, representationally-rich systems, the role of new networks and the impact they have on curriculum and interaction inside classrooms, effectiveness results in large-scale experiments, as well as insights into scaling-up, diffusion and necessary professional development for effective implementation of research-developed materials and resources.

Our intention in this volume is to provide value to the field of mathematics education by bringing together the depth and breath of the diffuse, emergent research program through a series of relatively short, accessible chapters that may engage readers interested in:

- Advanced perspectives on how to design curriculum, technology and professional development to address important yet difficult mathematics concepts.

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- Best practices in integration curriculum, technology, and professional development in mathematics education.
- Rich mixed methods descriptions and findings about how students and teachers learn in technology-rich mathematics classrooms
- Effectiveness results at scale for such integrated applications of technology.
- Contributions to expanding theories of teaching and learning to account for the interplay of student identities, cognitive processes, and social interactions in the mathematics classroom.

This book is a testament to the field and all contributing authors who sustained and advanced these ideas through research and development over many years, through many funding agencies and advocacy, and in many parts of the world. It does not represent the totality of work, the contributors and adherents to such a vision, or the scholars who can report on such work but it does attempt to capture the multiple projects and voices that have evolved over the years.

This introductory chapter presents the key technologies and themes that have supported the multi-investigator, multi-institutional research program over the course of about 18 years and counting. As such, it sets the stage for the chapters that follow. Hence, we also provide a timeline, which describes how key themes, technologies, and major research thrusts emerged since the beginning of the first SimCalc project in 1994 along with major funders whom makes this historic book possible.

Finally, this book is dedicated to the scholars and research students in this book and many others who have contributed to such work. Kaput had a vision and set forward many innovative principles regarding how all students should learn with new technologies and this could only be done through the dedicated vision and innovative research and development of many people. This book aims to demonstrate how that vision was completed and aims to be sustained through the Kaput Center and its many associates around the world (see http://www.kaputcenter.umassd.edu).

1 Context

From its onset in 1994 and continuing forward in time, the SimCalc Project builds on growing foundations in mathematics education research, applied to the needs of students who are struggling with algebra and calculus. Indeed, Jim Kaput originally founded SimCalc to advance a program he had been running at the University of Massachusetts, Dartmouth since the late 1960s, called START. The purpose of the program was to enable students who came to university underprepared in mathematics to learn enough in their freshman year to proceed onwards to a math, science, or engineering degree. Thus, at its onset, SimCalc was deeply concerned with serving diverse populations. While Kaput had a university population in mind at one end of the spectrum, he was also strongly involved in discussions about how to weave algebra into the curriculum much earlier, for example in elementary school.

Kaput's grounding drew heavily on prior advances in mathematics education research, particularly those concerned with epistemology: how could fundamental, yet challenging ideas be learned? As with Papert's earlier program, designing new microworlds to allow students to interact with technology was seen as a tool for exploring how deep mathematics could be more readily learned by all students. Relevant innovations that were under development at the time included The Geometer's Sketchpad® and Cabri Geometre, Function Probe, the Geometry and Algebra Supposers, and many Logo microworlds. In conjunction with these, investigators were studying small numbers of students as they struggled to learn foundational ideas in mathematics, and describe the micro-genetic pathways that students could travel from their prior knowledge to more advanced conceptions.

Kaput was a visionary poet of what he called "the mathematics of change and variation," which he conceptualized as a major strand of mathematical development leading through algebra and beyond the conventional calculus course. At its heart, this strand was concerned with the mathematical treatment of rates of change and accumulations of quantities over time. In his poetic way, Kaput believed that the "layer cake" school curriculum in which a "calculus" course was the icing on a cake made up of successive grade level courses did not serve students well. By instead treating "rate" and related concepts as vertical strand of long-term mathematical development, Kaput and his colleagues foresaw the opportunity to enliven and enrich mathematics in every grade level and simultaneously to give students access to critically important mathematical concepts much earlier. The SimCalc project was created to take on this challenge in the broadest possible way, and to create the specific technologies that would be needed to allow new epistemologies of the mathematics of change and variation to be realized and to flourish widely.

Through the history of SimCalc, the phrase "democratizing access" has been key to the mission among participating researchers. However, the team has never defined exactly what it means. In one sense, the implicit contrast to "democratizing" is "elite"—historically, advanced mathematical topics such as calculus have been the province of more privileged groups as less well-off students tend to be filtered out of mathematics before they get to such topics. In another sense, "democratizing" refers to the mathematics itself. Kaput liked to observe the literacy not achieved merely by the printing press but also by transitioning from Latin to the vernacular, and by analogy, he saw dynamic representations as providing an opportunity to create more accessible notations for mathematics. In a third sense, "democratizing" refers to the use of mathematics, for example that understanding of rate is critical for citizens to understand the economy and participate in public discourse effectively. There are likely many additional senses.

The concluding part of the phrase was initially "the mathematics of change and variation." This phrase was carefully chosen because calculus has become a particular course and the SimCalc imperative was never to advance the teaching of that specific course; SimCalc was quite apart from university efforts at "calculus reform." Instead it was meant to point to a branch of mathematics and allow for the reconstruction of that branch as a curricular strand. As the program advanced, the focus on MCV was too opaque or narrow for many settings, and so researchers have substituted various phrases, such as "advanced mathematics" or "the mathematics pathway through algebra, calculus, and beyond."

Regardless of the exact senses and phrases used, there is a distinct family resemblance among SimCalc efforts—for example, most research projects have focused on rates, graphs, and functions in one way or another, and most research projects likewise focus on student populations that are traditionally underserved by more conventional mathematics programs.

An inspirational corollary to this key phrase has been the observation of how much change in human learning is possible in a 100-year time span. When we look at year-to-year educational change, it seems that mathematics education is barely improving. But Kaput liked to point out how much change occurred between 1900 and 2000. In 1900, only a few percent of students nationwide learned any algebra and almost none learned calculus. Yet by 2000, it became reasonable to expect all students to pass an algebra course as a graduation requirement, with a high proportion passing their first algebra course before high school. Likewise, Kaput believed that over a longer timespan, it is very reasonable to expect that all students can learn more advanced mathematics such as the mathematics of change and variation—and that given changes in society, figuring out how to "democratize access to advanced mathematics" is more important than ever.

Traditionally, algebra emphasizes symbolic expressions. The meaning of algebra is developed in school in relation to rather trivial "story problems" and expressed by drawing graphs or making tables corresponding to a symbolic function. Kaput called expressions, graphs, and tables "the Big Three" and pointed out that to most students, they were meaninglessly self-referential—they refer to each other but stand for nothing in the student's mind. An innovation in all SimCalc approaches and curricula is to build mathematical understanding NOT by starting with symbols and then re-representing them (e.g., in graphs and tables), but rather by starting with more familiar representations and developing their mathematical treatment, culminating in very compact and operational notations such as symbolic expressions. The key familiar representation in SimCalc is motion, mostly commonly in the form of a "world," which game-like graphics that cue familiar situations in which motion occurs. Along with the reference to experienced motion, telling narrative stories about motion is also critical to the SimCalc experience. These are NOT story problems (e.g., "two trains started at stations 12 km apart and travelling in opposite directions...") but rather narrative descriptions that students create of their experience of motion ("they were running a race, and blue started out going faster, but dropped his baton, and had to stop to look for it, and eventually walked slowly back to the start line."

2 Representation and Communication Infrastructures

To sell the proposal for the first SimCalc project, Kaput made a video of what the software might look like. This 1993 video has almost nothing in common with today's technologies; through the process of R&D and innovation, the design iterated in ways that were not anticipated. In the original video, a student is essentially in the cockpit of a moving vehicle with instruments that read out position and velocity as graphs. Although graphs and motion continue to be at the heart of SimCalc, the technology now focuses on graphs that students can easily edit and manipulate. Motions tend to be shown in a flat, 2D perspective not in 3D, because 3D proved not to have any advantages for learning and is hard for students to interpret. Representations are linked to establish an infrastructure that a student can work with in a meaningful way, for example, if they change the slope of a graph by a dragging action, the table and motion simultaneously updates.

Beginning in 2000, the SimCalc team deliberately opened a new dimension of technological explorations, focused on communication among devices. Theoretically, SimCalc leaders argued for the necessity of both *representational* and *connectivity* infrastructures to transform the mathematics classroom. As we will discuss in more detail in the themes sections, whereas the representational dimension mostly targeted cognition, the connectivity dimension aimed to bring in more social and participatory elements of positive mathematics classroom cultures. Connectivity was implemented in several ways: as infrared beaming among handheld Palm computers, as wired and later wireless connectivity among TI calculators and a laptop, and as "wifi" connectivity among laptops in a classroom. Connectivity continues to be featured and important aspect of today's SimCalc MathWorlds® software demonstrating impact on student learning and motivation.

3 Our Intentions

Although one cannot introduce SimCalc properly without acknowledging Kaput's singular vision and contributions, it was also his nature to bring people together around a shared mission. From a charming, ramshackle outpost on the rural southeastern coast of Massachusetts, Kaput brought the best and brightest together from around the world to work on "democratizing access to advanced mathematics." The diversity of contributors to this book reflects the nature of the enterprise—rich in perspectives, varied in approaches, touching a wide range of populations and their needs, and having in common a commitment to high quality scholarship and making a difference for math learners everywhere. The book is written by researchers and students involved in the rich tapestry of work called the SimCalc project and so it is written for such people who are continuing to build on such work and apply the themes related to the *mathematical of change and variation* and *social mathematics* more generally to a wider variety of settings.

Today, although we are saddened to no longer have Kaput with us, his communitarian vision of *how* the transformation would be realized is faithfully embodied by the ongoing work of the Kaput Center (see Fig. 1) and we are pleased that all royalties from this book are donated to support student scholarships there.

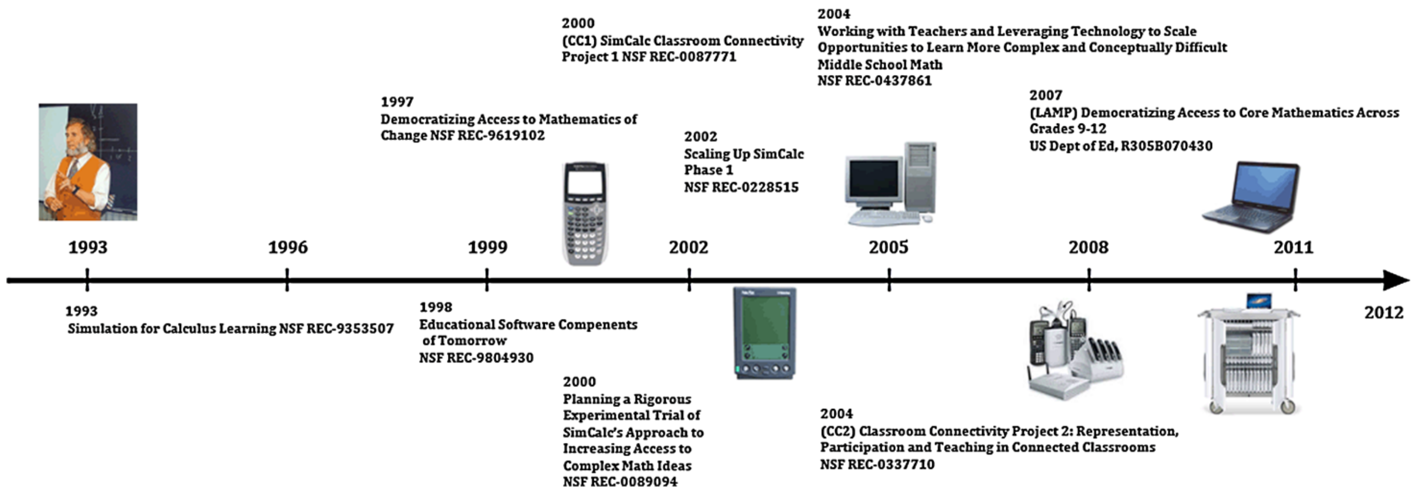

Fig. 1 Timeline of SimCalc Research and Development 1993–2012

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The Mathematics of Change and Variation from a Millennial Perspective: New Content, New Context

James J. Kaput and Jeremy Roschelle

This chapter raises three broad questions for the present day:

1. Will the movement of mathematics from static-inert to dynamic-computational media lead to a widening of mathematical genres and forms of mathematical reasoning?
2. Will mathematical activity within computational media lead to a democratization of access to (potentially new forms of) mathematical reasoning?
3. Can these changes transform our notions of a core mathematics curriculum for all learners?

But before going further, by way of starting points, we should like to give a broad view of what we take mathematics to be. We regard mathematics as a culturally shared study of patterns and languages that is applied and extended through systematic forms of reasoning and argument. It is both an object of understanding and a means of understanding. These patterns and languages are an essential way of understanding the worlds we experience—physical, social, and even mathematical. While our universe of experience can be apprehended and organized in many ways—through the arts, the humanities, the physical and social sciences—important aspects of our experience can be approached through systematic study of patterns. In addition, mathematics embodies languages for expressing, communicating, reasoning, computing, abstracting, generalizing, and formalizing—all extending the limited powers of the human mind. Finally, mathematics embodies systematic forms of

James J. Kaput is deceased.

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reasoning and argument to help establish the certainty, generality, and reliability of our mathematical assertions. We take as a starting point that all of these aspects of mathematics change over time, and that they are especially sensitive to the media and representation systems in which they are instantiated.

1 A Condensed National History of Representation

While the evolutionary history of representational competence goes back to the beginnings of human evolution (Donald, 1991, 1993; Mithen, 1996), and can be linked to the evolution of the physiology of the brain (Bradshaw and Rogers, 1993; Calvin, 1990; Lieberman, 1991; Wills, 1993), with three exceptions this history is beyond our scope. The first is simply to recognize that representational competence, reflected in spoken and then written languages, both pictographic and phonetic, in visual representations of every sort, is a defining feature of our humanity. It is reflected in our physiology, our cultures, and our technologies, physical and cognitive.

The second exception involves the two-step evolution of writing systems from the need to create quantified records (Schmandt-Besserat, 1988, 1992). As convincingly described by Schmandt-Besserat (1980, 1981, 1985), clay tokens were first used in clay envelopes to record quantities of grain and other materials in storage and commercial and tax transactions, i.e., a given number of grain-tokens represented a certain number of bushels. Before being put inside the soft clay envelopes, these tokens were pressed into the exterior, leaving an image of the envelope's contents. Over many generations, the envelope markings replaced the tokens. The envelopes evolved into tablets, and the representations led to pictographic writing. The second step for western civilization was the invention of phonetic writing. Arbitrary characters were used to encode arbitrary sounds (phonemes), giving rise to abstract expression (Logan, 1986, 1995). This supported new written structures such as codified law, for example Hammurabic code and Moses' commandments, and, when the idea reached Greece, it enabled the expression of science, mathematics, logic, and rational philosophy (McLuhan and Logan, 1977). We draw two broad, albeit unsurprising, inferences from this history. One is that quantification—mathematics—and the functioning of human society have been inextricably linked, beginning as early as the invention of writing. The second is that representational changes in the constraints and affordances of concrete media play a critical role in how we organize our worlds (Goodman, 1978).

The third major historical event to which we direct attention is the invention of the printing press. Of special interest are three consequences. First, there was a standardization of dialects and vernaculars used in spoken language in Europe, first in England with English—as opposed to the existing standard *formal* languages, namely Latin and Greek. Second, and closely related, was the democratization of literacy (Innis, 1951; McLuhan, 1962). Up until that time, there was a small collection of written works and a tiny elite who read and commented on them. Indeed, you could fit almost all of the available written classics on to a decent sized bookshelf. Along with the democratization of literacy came a critically important third event, the dramatic widening of literary forms and the rapid proliferation of original literary material based in everyday life (as opposed to narrowly academic commentary on the classics). For example, the novel was invented. Importantly, these events occurred largely outside of, and independently of, either the universities or the monasteries. A fifteenth-century monk would not today recognize the “language arts” curriculum as being about the “literacy,” which was practiced and taught before the invention of the printing press. We need to remind ourselves that Shakespeare, and virtually all fiction of the sixteenth and seventeenth centuries, were regarded as “vulgar” literature, not admitted as the subject of academic study.

More recently, we have seen the invention of dynamic visual media, film, and especially television. These have again led to a democratization of visual culture and a widening of dynamic visual forms (McLuhan, 1967). Almost from the beginning, films, for example, were not sequential representations of visual events. Film generated new art forms, much in the way that new literary art forms flourished after the development of the printing press (Arnheim, 1957). There has been a democratization of visually mediated culture (Salomon, 1979). Most people enjoy film and can understand its idioms. Most people can follow the extraordinary visual and auditory feats of contemporary television, despite the rapid sequences of images and semiotic complexity (Fiske and Hartley, 1978; Williams, 1974). This democratization of visual culture occurred without formal instruction or education, outside the academic realm. Indeed, the former masters of the visual arts had rather little ability to guide the new genres that arose in Hollywood and Madison Avenue. These genres built upon naturally occurring visual and language-interpretation capabilities widely distributed across the population, and now continue with the ready ability of ways for everyday people to author and distribute videos

The invention of manipulable formalisms, numeric and algebraic, occurred around the same time that the printing press was invented. The first of these, the Hindu-Arabic place-holder system for numbers, was intimately involved in the commercial economy of the time (Swetz, 1987). And perhaps even more important for the longer term was the rapid development of an algebraic symbol system with syntax for manipulation. This was tied to an explosion of mathematics and science development that is continuing today. Importantly, this mathematics and science, and the notation systems in which it was encoded were developed by and for an intellectual élite—far less than one percent of the population. Until very recent times, only a very tiny minority of the population who were expected to learn these symbol systems and use them productively—and thus no effort went into designing the symbol systems to be readily learnable by the majority of the population. Whereas a more accessible vernacular style of writing emerged in literature and journalism (for example), mathematics has yet to develop an equally accessible style of reading and writing algebra for everyday use by the general public. Indeed, each of these successive inventions, writing and the printing technology that democratized it, dynamic visual forms, and now interactive digital notations, are much more deeply embedded in ordinary life than is “classic” school mathematics.

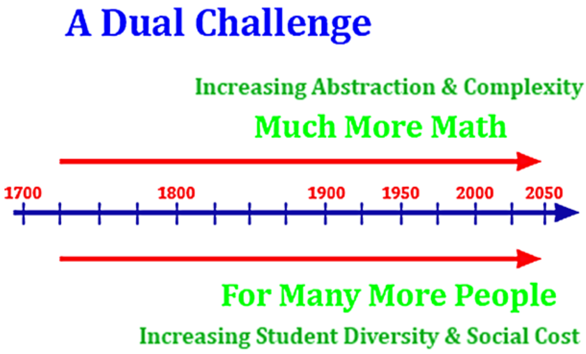

Fig. 1 A long-term trend: much more math for many more people

2 Dual Challenges: Much More Mathematics for Many More People

At the end of the twentieth century, we face a dual challenge in mathematics education at all levels, from kindergarten to adult education: we need to teach much more mathematics to many more people. The radical increase in the numbers of people who are expected to know and use mathematics is leading to a corresponding increase in student diversity and increases in the social cost of mathematics education—to near the limits for which societies are willing to pay. We need to achieve dramatic new efficiencies across the entire K-12 mathematics curriculum. These trends, as indicated in Fig. 1, have been under way for centuries and are expected to continue.

To illustrate the change in content, we recall a story from Tobias Dantzig (1954):

> It appears that a [German] merchant had a son whom he desired to give an advanced commercial education. He appealed to a prominent professor of a university for advice as to where he should send his son. The reply was that if the mathematical curriculum of the young man was to be confined to adding and subtracting, he perhaps could obtain the instruction in a German university; but the art of multiplying and dividing, he continued, had been greatly developed in Italy, which, in his opinion, was the only country where such advanced instruction could be obtained.

This story has been well corroborated by historians of mathematics, for example, Swetz (1987). While it concerns commercial mathematics, it makes the point that there was a time when even this mathematics was the province of a very élite group of specialists. Over time, the fact that widespread acceptance of the newly available notation system had large impact on the larger population's access to what we now term "shopkeeper arithmetic."

Another useful orienting statistic is derived from U.S. Department of Education (1996). In the United States, 3.5 % of the 17- to 18-year-old population cohort took high school Advanced Placement Calculus (successful completion of the associated test allows them to substitute this course for a corresponding one in the university). *This is almost exactly the percentage of students graduating from high school in the U.S. a century earlier.* Perhaps 2 % of the US population was expected to learn algebra a century ago; today in the US almost all students are expected to take an

Algebra course even before they attend high school. Similar dramatic changes in expectations regarding who can or should learn mathematics have occurred internationally, throughout both the developed and developing world.

Indeed, reflecting on these longer-term trends, the question we asked about whether a democratization of mathematical reasoning would occur, now shifts from the interrogative to the imperative: we *must* teach much more mathematics to many more people. But how can we speak of much more mathematics when the curriculum is already overflowing? And for many more people when most of those people presently end up disliking mathematics intensely?

Before replying, we ask how many people can travel 50 miles per hour? Or can fly? Or can speak and be heard a thousand miles away? Answer: most of us. Rendering much more mathematics learnable by many more people will require at least the levels of co-ordinated innovation standing behind the automobile, airplane or telephone. Let's step back a bit and examine these other innovations.

First note, the automobile involved considerably more than the invention of the internal combustion engine. Automobiles are embedded in a sophisticated system of interrelated innovations and practices that cover a wide range of systems, mechanical, hydraulic, electronic, as well as roadways, laws, and maps. Then there is the matter of educating and organizing the people to build, operate, and market them. Of course, jet airplanes, airports, navigation systems, worldwide communication systems, airline reservation systems, radar-based flight controllers, are at least as great a miracle. With a very occasional exception, all these staples of the late twentieth century operate with extraordinary efficiency in the service of quite ordinary people—*and are expected to!*

We see parallel developments now becoming possible in educational technology. Much attention has been drawn to graphical and dynamic media, with its attendant possibilities for engaging children in constructing, reasoning, and communicating across multiple representational forms. Likewise, the ubiquitous availability of social networking suggests a broad change to communication is now afoot, with potentially as far-reaching consequences. Yet new media and networking are incomplete without a third development: the possibility of mass-producing personalizable educational content. Just as transportation required Henry Ford's assembly-line-produced Model T, and communications required the dial tone, educational technology needs a wave of modularization, substitutability, and combinatoric composition. This is now becoming possible under the rubric of "component software architectures" (Cox, 1996) which allow for the mix-and-match interoperability, integration, and customization of modular functionalities: notebooks, graphs, calculators, simulations, algebraic formulae, annotation tools, etc. Component software architectures bring the possibility of constructing large complex systems through a highly distributed effort among developers, researchers, activity authors, curriculum experts, publishers, teachers, and students, among others. As we argue elsewhere (Roschelle and Kaput, 1996; Roschelle et al., 1998), the integration of media, networks, and component architecture can begin to allow us to approach educational problems on a scale that was formerly inconceivable.

With our confidence stiffened by clear success in transportation and communication, and with an understanding that the infrastructure for similar advances in educational technology is now emerging, let us now turn to our particular interest in present day reform—democratizing access to powerful mathematics.

3 Access to Powerful Mathematics Through New Representational Forms

Educational innovators have long experimented with the construction of alternative notational systems to enable learning of mathematics and science. One well-established method is to embed mathematics in computer languages (Ayers et al., 1988; diSessa et al., 1995; Hatfield and Kieren, 1972; Noss and Hoyles, 1996; Papert, 1980; Sfard and Leron, 1996). Familiar examples could include Turtle Geometry (Abelson and diSessa, 1980), mathematical programming in ISETL (Dubinsky, 1991), and spreadsheets (Neuwirth, 1995). Another method is to embed the content in activities such as computer games (Kraus, 1982; Shaffer et al., 2005). Here we argue for a representational alternative: embedding mathematics in direct manipulation of dynamic spatial forms and conversation over those forms (Kaput, 1992). Dynamic geometry is one example of alternative notational form based upon direct manipulation of spatial forms (Jackiw, 1991–2009; Goldenberg, 1997; Laborde, 1984–2009). Direct manipulation of two-dimensional vectors is another (Roschelle, 1991). For the mathematics of change and variation, our SimCalc project has chosen to focus on directly manipulable Cartesian graphs that control the action of animations.

The properties of graphs suggest interesting answers to the three major questions we posed earlier:

1. Widening of forms? Graphs already support a range of forms that is considerably wider than can be expressed in closed-form symbolic algebra (Kaput, 1994), and more specific to particular reasoning techniques. For example, as we shall describe below, graphs can easily support manipulation of piecewise defined functions, a form that is extremely cumbersome in traditional algebra.
2. Democratization of access? Graphs are already a more democratic form, appearing frequently in newspapers, television, business presentations, and even U.S. presidential campaign speeches—at least in terms of reading and interpretation, as opposed to writing and manipulating graphs. These are all places where equations are seldom found, and indeed usually taboo. As was the case with the explosion of literary forms, graphs appear to draw upon cognitive capabilities, which are more widespread or accessible than formal mathematical symbols, although not without challenges (Leinhardt et al., 1990; McDermott et al., 1987). We shall deal with the matter of writing and manipulating graphs shortly.
3. New core curriculum? Most of the basic characteristics of mathematical thinking outlined at the beginning of this chapter and highlighted in modern curriculum standards can be carried over to graphical representational forms, allowing students to begin grappling with powerful concepts earlier and more successfully. In the next millennium, graphical mathematics will need to be part of the basic

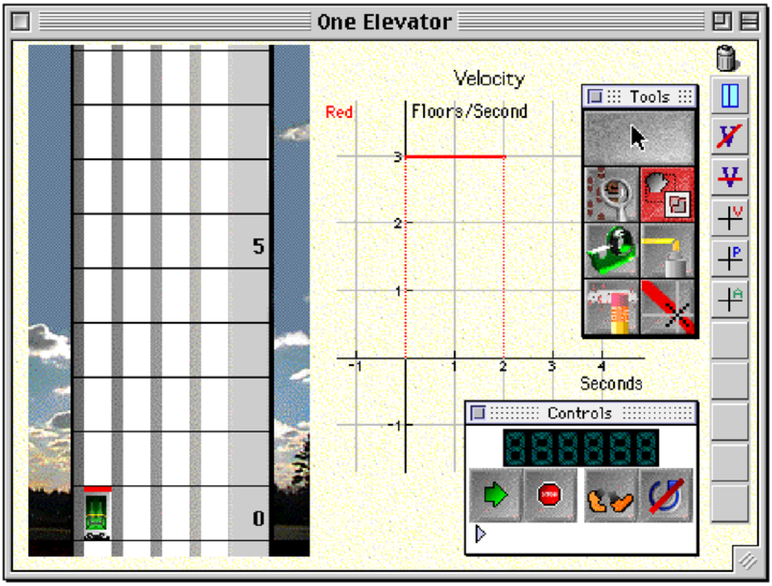

Fig. 2 An introductory elevators activity in SimCalc MathWorlds® software

mainstream experience for all students. But a major step in this direction will require a move from static graphs that are merely read and interpreted to dynamically manipulable graphs that can be linked to phenomena and simulations of various kinds. And this change must occur in concert with substantial changes in how the content is organized and experienced.

Our SimCalc Project is building upon the unique potential of manipulable graphs in our *MathWorlds* software, which supports learning about rates of change of objects in motion, along with many related concepts. Figure 2 shows a screen from an introductory activity, with a moving elevator controlled by a piecewise-defined step function for velocity. In this activity (designed by our colleague, Walter Stroup), middle-school students make velocity functions that occupy six grid squares of area. The students are asked to make as many different (positive) functions as they can, and compare similarities and differences. As mathematicians well know, all such functions will cause the elevator to move upwards 6 floors, but will vary in times and speeds. Pictured is a very simple one-piece velocity function. As we discuss elsewhere (Roschelle et al., 2000), velocity step functions also draw upon students' prior knowledge and skills: students can compute the integral by multiplying the sides of the rectangles or simply counting squares. They can readily distinguish duration (width) from speed (height), and distance travelled (area). Questions about the meaning of negative areas (below the axis) arise naturally and their resolution can be grounded in the motion of the simulated elevator.

Over many weeks with MathWorlds, students can study the properties of velocity graphs in relation to motion, then position graphs, then relations between the two, and finally (for older students) acceleration graphs. Along the way, students also work with manipulable graphs that are piecewise linear (instead of step functions), continuous instead of discontinuous, and varying arbitrarily (not just linearly). The various representations are each dynamically linked, so that students can directly observe the effects of changing a velocity graph upon position, or vice versa.

4 Relationships Among Representations Move to the Center

As mentioned above, the printing press led to increasing diversity of literary forms including, for example, the novel. We argued that computational representations are doing the same for mathematics, and that new forms of graphs are likely to become common tools for mathematical reasoning. Here we push our millennial comparison one step further. Among the new literary forms that emerged, the novel stands out as creating a more participatory experience for the reader; readers of novels are swept into a fully articulated world that at times seems as real as the familiar world. Indeed, the great achievement of successful authors is to relate experience in the reader's personal world to the new imaginary worlds. Moreover, novelists were now free to treat topics that were neither religious, nor mythical, nor heroic—contemporary life became the subject of literary experience. Of course, these new forms did not arrive without precedent; oral story-telling traditions paved the way; and contemporary novels are certainly no more constrained to common experience than film-makers are bound to reproducing common events.

This trend has its parallel in technology that brings motion experiences into the mathematics classroom, and thus ties the mathematics of change to its historical and familiar roots in experienced motion. Motion can be represented cybernetically (as an animation or simulation), as we described above in MathWorlds. And motion can also be represented physically, in experiences of students' own body movement, or objects that they move. When desired, these physical motions can be digitized and imported as data into the computer, attached to actors, repeated, edited, and so on. Below we discuss three ways in which SimCalc is using the relationship between physical and cybernetic experience to give students new opportunities to make sense of traditionally difficult concepts such as mean value, limits, and continuity.

When using MathWorlds in classrooms, especially with young children, we often begin with physical motion, unconnected to the computer at all. For example, students might be asked to walk along a line, with speeds qualitatively described as "fast," "medium," or "slow." The class can then measure the time to cover a fixed distance, beginning the slow process of building and differentiating the quantities of distance, rate, and time and their relationships. Later students move to the computer and use an activity that displays a "walking world" with animated characters whose velocities are constrained to three fixed heights, corresponding to fast, medium and slow. With the greater precision and control supported by the computer representations, students can now begin to make quantitative comparisons. Here we use the

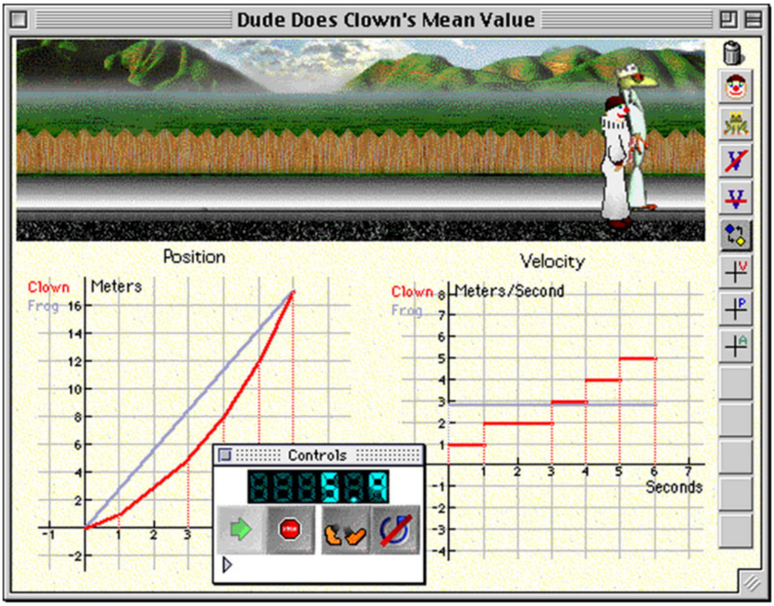

Fig. 3 Dude does Clown's mean value

kinesthetically rich experiences of the physical world to present difficult quantification challenges for students who have only the vaguest idea of what one might measure and why (Piaget, 1970; Thompson and Thompson, 1995). Animated clowns, on the other hand, are less grounded in real experience (indeed their gaits are cartoonish at best), but easier to control, measure, examine, and repeat. They provide pedagogically powerful intermediate idealizations of motion phenomena.

In later activities, physical and cybernetic experience can be connected directly through data. For example, in Roschelle et al. (1998), we describe an activity sequence in which students explore the concept of mean value—a mathematically central concept upon which much theoretical structure depends (Fleming and Kaput, 1979). Here a student's body motion is captured with a motion sensor, and input into MathWorlds, where it becomes replayable as the motion of the walking "Clown" character. The motion also appears in a MathWorlds graph as a continuously varying velocity function (assuming the walking student varied speed). Students may now construct a second animated character ("Dude") whose motion is controlled by a single constant velocity function (see Fig. 3). The challenge is to find the correct velocity to arrive at the same final location at exactly the same time—thus finding the mean value.

It is exciting for students to find the mean value of their own variable-velocity body motion. While a large body of research exists regarding student mathematization of motion, it focuses mainly on high school and college age students, and generally deals with "regular" motion that is describable algebraically (McDermott et al., 1987; Thornton, 1992). In contrast, we allow students to work with highly irregular motion, generated by their own bodies, and leading towards the important technique of approximating continuous variation with piecewise linear functions.

Once students are comfortable with the concept of mean value, they can use piecewise segments to try to approximate a varying motion. In the example given above, the two motions will only intersect at one place, the final location. But in MathWorlds students can use two piecewise linear velocity segments (each comprising half the given interval) and thus intersect at two places, the final location and the mid-duration location. This process can be repeated with more and more (and smaller and smaller) segments, making the two characters meet 2, 4, 8, 16, 32, or more times. Of course, as the number of segments increases, the approximation between the motions becomes greater, giving students a concrete sense of taking a limit. Here students see how an idealized abstraction (a linear velocity segment) can model a continuously varying real world variable (their varying body position) with as much precision as required (see Roschelle et al., 1998, for more detail).

Although the activity sequence described above was important in early teaching experiments with MathWorlds and shows how dynamic representations can fruitfully reinterpret ideas in a Calculus course, over time our work has come to emphasize position graphs before velocity graphs. When we initiated a sequence of projects in Texas to test SimCalc curriculum at scale, our entry point was 7th grade mathematics. At this level, SimCalc relates best not to mean value (a more advanced concept) but rather to the core construct of proportionality.

In traditional seventh grade teaching, proportionality is addressed through the equality of two ratios: $a/b = c/d$. Problems rely heavily on the useful but inscrutable cross-multiplication procedure, which transforms this equality to $a * d = c * b$, and the process of identifying three given quantities and calculating a fourth. While useful, this conceptualization of proportionality is narrowly useful and is not particularly fertile for students' further mathematical development. We found that SimCalc could bring the "democratizing access to the mathematics of change" theme to life at this grade level by refocusing proportionality on its relationship to rate, as in the speed of a moving object. Further, graphs could be used to visualize how speed connects changes in time to changes in position, e.g., through the graph of a line. This could then be developed into the first important mathematical function that students encounter, $y = kx$ (which is further developed into $y = mx + b$ in our eighth grade curriculum unit).

These curricula units developed a rich network of connections among representations along two dimensions, a familiar-formal dimension and a graphical-linguistic representation. For instance, in initial exercises, an animated motion (familiar, graphical) is related to a story of a race (familiar, linguistic) and a graph of the motion of the actors in the race (formal, graphical). As the curriculum progresses, algebraic expressions (formal, linguistic) are also introduced in relationship to graphs, using tables as a stepping stone. Overall, we have argued that optimal curricular sequences using new representational media should focus on connections between types of representations and should develop more formal representations as their more familiar counterparts are better understood.

As reported elsewhere in this volume, we have had considerable success in demonstrating the effectiveness of these curricular units for seventh and eighth grade mathematics through large-scale randomized experiments. We attribute some of this success to the idea that "new technology without new curriculum isn't worth the silicon it's written in." To realize the potential of dynamic representations, new curricular pathways must be envisioned and made concrete for teachers in curriculum workbooks and teacher professional development.

To summarize, we see new technologies creating a possibility to reconnect mathematical representations and concepts to directly perceived phenomena, as well as to strengthen students' understanding of connections among different forms of mathematical representation. By starting from more familiar antecedents, such as graphs and motion, both in kinesthetic and cybernetic form, and developing towards more compact and formal mathematical representations, we see an opportunity to create a new path of access to mathematics that has too often remained the province of a narrow elite.

5 Discussion: Mathematics Education at the Beginning of a New Millennium

By momentarily rising from the trenches of mathematics education reform to a larger time scale, we identified a major long-term trend: *Computational media are reshaping mathematics, both in the hands of mathematicians and in the hands of students as they explore new, more intimate connections to everyday life.* As we mentioned earlier, we can already be fairly certain this will lead to widening of mathematical forms, just as the printing press increased the range of acceptable literary forms. Already, we are seeing new forms such as spreadsheets and data visualization becoming prominent in everyday life. Computational tools are leading to new epistemological methods as professional mathematicians explore the extraordinary graphical phenomenology of dynamical systems (Stewart, 1990). Educators are rapidly inventing new, additional forms—such as mathematical programming languages and construction kits for dynamic geometry.

Based on our experiences and experimental investigations with SimCalc, we are more and more optimistic about the second question we raised: democratic access. Some representational forms, like directly editable graphs, can make difficult concepts such as mean value, limit, and continuity in calculus newly available to ordinary middle-school (10- to 12-year-old) students and can lead to fertile reinterpretations of existing concepts, such as rate and proportionality. The technology's capability to provide better links between graphical representations and phenomena (physical and cybernetic) also appear essential, as this linkage grounds concepts in familiar semantic referents.

Yet as Sherin (1996) points out, new forms also lead to changes in the meaning of the concepts; in Sherin's studies, students who learned physics by programming in a computer language learned a set of physics concepts subtly different from those learned by students using traditional algebraic symbols. Indeed, in our work with SimCalc, we are currently working out the curricular bridge from editable piecewise functions back to traditional algebra. It is by no means easy. And this is just the beginning! We want to lead students towards understandings of the larger mathematics of change and variation that includes dynamical systems because this relatively new mathematical form, with roots in Poincaré's work at the end of the previous century, is revolutionizing many sciences simultaneously as we approach the next century (Hall, 1992). However, the major long-term educational experiment with systems concepts, based in the use of *Stella*, is far from a clear success (Doerr, 1996). Other innovative approaches to systems concepts, *StarLogo* (Resnick, 1994) and *AgentSheets* (Repenning, 1994) look promising but present difficulties in linking back to commonplace notation. Nonetheless, we believe that with time and effort, innovations in computational representations will make democratic access to systems dynamics possible.

To harness this potential fully, however, reformers will need to rise to the challenge of our third question: Can these new possibilities transform our notion of a core mathematics curriculum for all learners? The technological revolutions in transportation and communications would be meaningless or impossible if core societal institutions and infrastructures remained unchanged in their wake. Today's overnight shipments and telecommuting workers would be a shock to our forebears 100 years ago, but our curriculum would be recognized as quite familiar. If we are to overcome this stasis, we must seize the opportunities implicit in new dynamic notations to reorganize the curriculum to enable extraordinary achievement from ordinary learners.

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From Static to Dynamic Mathematics: Historical and Representational Perspectives

Luis Moreno-Armella and Stephen Hegedus

1 Preface

We present new theoretical perspectives on the design and use of digital technologies, especially dynamic mathematics software and "classroom networks." We do so by taking a more contemporary perspective of what can be possible, as the notations, the mathematical experiences, and the medium with which these all work, come closer together and co-evolve. In effect, this approach takes a more applied epistemological stance to the nature of mathematics education in the future versus an epistemological tension between the contemporary mathematician and "their" mathematics, and society today.

Kaput began to take a deep appreciation of the evolution of sign systems in the mid-1990s—producing diverse perspectives on the semiotics of mathematical notations for education (Kaput, 1999; Shaffer and Kaput, 1999) and later in Moreno-Armella and Kaput (2005) and Kaput et al. (2008). In these works, he incorporated the evolutionary and cultural perspectives of Merlin Donald's two seminal works (Donald, 1991; 2001) focusing on cognition and representations. Shaffer and Kaput (1999) suggest that the "new phase" (virtual culture) is a logical next-step in the development of the evolutionary-cognitive perspective developed by Merlin Donald. This perspective considers that the evolutionary study of cognition can be conceived of as a timeline going from the mimetic culture, then mythic culture (orality) to finally, the theoretical culture, based on external memory supports. This includes

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writing as its main component. They suggest that virtual culture, based on the processing capacity of new technologies (and future ones), is the next logical step. As their perspective is mainly cognitive-evolutionary, they do not study in depth the semiotic aspects of this new stage in the evolution of cognition. We approach the study of notation systems from a historical perspective emphasizing the semiotic dimension. This leads us to a serious consideration of the dynamic structure of the reference field. This is in contrast to Shaffer and Kaput (1999), which emphasizes the processing power of the new notation system due to the computational embodiment. We build now on this work in considering the epistemological transformations due to the presence of the executability of digital semiotic representations. This has allowed us to cast light on historical stages of evolution and compare them with present stages of technological evolution in mathematics education (Kaput, 2000). In this chapter, we propose a new way of conceiving a reference field in mathematical activity. This reference field becomes possible through certain new technologies and hardware infrastructures, particularly wireless networks.

2 Perspectives on Symbols

A symbol is something that takes the place of another thing. For instance, *pencil* takes the place of those material objects we use to write. But this does not still explain how *taking the place* occurs in general. A symbol is something that someone *intends* to stand for or represent something other than itself. Symbols *crystallize* intentional actions, and are instrumental for generating and developing human cultures. We wish to use the metaphor *crystallize* instead of encapsulate, because we want to direct our attention to essential properties of a crystal, particularly its stability and its possibility for change and growth.

Generalization and symbolization are at the heart of mathematical reasoning in the SimCalc learning environment. One way a person can make a single statement that applies to multiple instances—a generalization—without making a repetitive statement about each instance is to refer to multiple instances through some sort of unifying expression that refers to *all* of them at once, in some unitary way—as a single thing. The SimCalc program of development has attempted to establish such through linked and multiple-representational software environments with integrated curriculum that attends to and exploits such affordances in order to focus the attention of learners in mathematically meaningful ways.

The unifying expression requires some kind of symbolic structure, some way to unify the multiplicity, and this is the focus of our attention to such a problem in mathematics education. Symbolization is in the service of generalization—both within individuals and historically as communal thinkers. And once a level of symbolization is achieved, it becomes a new platform on which to express and reason with generally, including further symbolization (see Kaput et al., 2008, p. 20).

Now, the difference between different modes of reference can be understood in terms of levels of interpretation (Deacon, 1997, p. 73). We should emphasize that the nature of reference is relative at its core, and what is a referent in one description may be the result of a prior symbolization. These levels of interpretation call attention to *icons*, *indexes*, and *symbols* as used by C. S. Peirce (Deacon, 1997, p. 70). Hence, where we take up the description has a lot to do with "what is a symbol" and "what is a referent" for that symbol.

The idea of crystallization does not imply a rigid and/or static structure for the reference field. Instead, with this perspective, reference fields (meanings) are *dynamic*—they grow and transform with the shared use of symbols. The reference field lodged in a symbol can be greatly enhanced when that symbol is part of a network of symbols.

Emergent meanings come to light because of the new links among symbols. For instance, the meaning of a word, in a dictionary, can be found inside the net of relations established with other words. Nevertheless, Donald (2001, p. 154), has suggested that as our early experience is gained in a non-symbolic manner, the roots of meaning can be found in our non-symbolic engine, that is, in our analogue modes of operation, as if the ultimate meaning of a symbol were an experience, an intuition. Yet, we have been able to create symbolic universes that *duplicate* our experience and provide a meta-cognitive mirror where we can see ourselves and enrich our lives and thinking. This is the case, for instance, with works of arts, novels, and scientific theories. The feeling of objectivity that comes with our symbolic creations explains the Platonic viewpoints of many scientists. In mathematics, this viewpoint translates into the belief in a pre-symbolic mathematical reality. Explaining the mathematical power embodied in Maxwell's equations for electromagnetism Hertz wrote:

> One cannot escape the feeling that these mathematical formulas have an independent existence and intelligence of their own (Kline, 1980, p. 338).

This crystallizing impact of symbols in our minds generates the belief that they are the primordial world of experiences in the first place. But if we are doing mathematics, for instance, we need *some* conviction that we are working with objects that have a real existence even if this existence is not material existence. This has been a central philosophical and theoretical focus of the mathematical tapestry of the SimCalc learning environment.

Platonism becomes acceptable only as *emotional* Platonism. However, in absence of symbolic representations, we lose the access to mathematical objects, as they are intrinsically symbolic objects. We can speak of mathematical inscriptions as the external marks of symbols but we cannot forget that symbol and reference are like a one-sided coin—each one is the condition of existence of the other. Before, we said that a symbol crystallizes an action or an intentional act. What kind of action is crystallized in a mathematical symbol? We consider this question central for the epistemology of mathematics and mathematics education, which the SimCalc project has problematized over several decades.

Later in this chapter, we analyze how new forms of mathematical activity—through dynamic media—appreciates this fundamental perspective for allowing students more direct access to mathematical structures. We will present examples of classroom activities to illustrate the dynamics of this process, but first we aim to cast light on the evolving nature of the relationships between a mathematical symbol and its reference field using some historical examples.

Incised bones like the one found in Moravia (Flegg, 1983), dated 30,000 B.C., constitute what is perhaps the first example of manmade symbols. We interpret this finding as an example of the use of a one-to-one correspondence between a concrete collection of objects (perhaps preys attributed to a hunter) and the set of incisions on the bone. This set of incisions, acquire a symbolic meaning. In fact, the act of incising a bone is an *intentional* act by means of which the bone is modified to store, manipulate, and transport information—-an incision, for instance, can represent a rabbit, a bird. On the bone, one can see after this intentional act the birth of a new symbolic world—-the *territory* of the symbol. Tokens are our next example in the production of mathematical symbols. As D. Schmandt-Besserat has written in her fascinating account on *How Writing Came About* (1996) tokens were "small clay counters of many shapes which served for counting and accounting for goods" (p. 1). Tokens served the needs of economy and their development was tied to the rise of social structures (Schmandt-Besserat, 1996, p. 7). After a few decades as trade increased, Sumerians needed a more compact way to keep track of goods than individual tokens. Thus, the tokens which according to shape, size, and number represented different amounts and sorts of commodities—were put into a sealed envelope, a container for the tokens. This process compacted information but created a new problem: to inspect the content of an envelope, it had to be destroyed. This new problem was resolved, as Schmandt-Besserat recounts (1996, p. 7) by *imprinting* the shapes of the tokens on the surface of the envelope. A mark impressed on the surface of the envelope kept an *indexical* relation with one counter inside, which figured as its referent. After another one hundred years, Sumerians realized that they could dispense with the tokens themselves by just impressing them on wet clay. In fact, transferring their conventional meaning of the tokens to those external inscriptions was enough to convey the information intended (Schmandt-Besserat, 1996, pp. 50–51). That decision altered the semiotic status of those external inscriptions. Afterwards, scribes began *to draw* on the clay the shapes of former counters. But drawing a shape versus impressing the shape of a token are extremely different activities, even if both are intentional. This gradual, emerging set of physical inscriptions worked as a meta-cognitive mirror to guide actions—both mental (interpretive actions) and physical (elaborations)—on the new inscriptions.

As Duval (2006) has explained,

> One has only to look at the history of the development of mathematics, to see that the development of semiotic representations was an essential condition for the development of mathematical thought. For a start, there is the fact that the possibility of treatment, for example calculation, depends on the representation system (p. 106).

Duval (2006, p. 107), explains as well that the crucial problem of mathematical comprehension for learners arises from the fact that the access to a mathematical object is possible only by means of semiotic representations and yet that these representations cannot be confused with the object itself. In fact, each time we produce a new system of representation for a mathematical object, that object is no longer the same object. Mathematical objects have many potential faces and each face corresponds to a certain way of operating the object. Mathematical objects are always under construction. This construction takes place within symbolic cultures, as happens with novels and sonatas, for instance. The importance of notation systems (semiotic representations) cannot be overemphasized. Reading classical mathematical texts from the remote past, one can appreciate how after translating those texts into modern notation, the problems become almost trivial. This is the case, for instance, with arithmetic problems from pre-Greek mathematical cultures. Were these trivial problems? No. Reflecting on these issues, one arrives at the conclusion that mathematical notation systems are not epistemologically neutral. It must be taken into consideration that notation (or semiotic) systems are artifacts coextensive with our thinking. We say we think with a notation system when we use it as a cultural tool. For instance, when we compute using the binary system for numbers we feel that system is outside of our mind. But if we compute with the decimal system, the feeling is quite different. It is as if this system were an intrinsic component of our mind. And it is, in fact, because a process of internalization has taken place. The system has gone from the (school) culture into our mind. It becomes coextensive with our mind. We think through it. Vygotsky considered the process of internalization—cultural artifacts becoming cognitive tools—central to his theory of cognition. He said "any higher mental function is external because it was social at some point before becoming an internal truly mental function" (Wertsch, 1985, p. 62).

3 Shifting from Static to Dynamic Media for Twenty-First Century Classrooms

The visual, gestural, and expressive capacity of the use of new technologies becomes apparent in various ways. These capacities primarily focus on the medium within which the technology user, learner or teacher (from hereon described as the user) operates. To describe this change, we introduce the idea of co-action to mean, in the first place, that a user can guide and/or simultaneously be guided by a dynamic software environment. This is basic in understanding that humans-with-media (see Borba and Villarreal, 2005) is a fundamental development in the co-evolution of technology and educational environments.

Notation systems have evolved in new ways with new mathematical explorations, hitherto impossible or impractical in the static medium. This stage of mathematical epistemology is presently situated in the education domain, and less so, if at all, in the mathematician's domain.

We suggest that the evolutionary transition from static to dynamic inscriptions, and hence new forms of symbolic thinking, can be modeled through five stages of development, each of which can still be evident in mathematics classrooms in the twenty-first century.

3.1 Stage 1. Static Inert

In this state, the inscription is "hardened" or "fused" with the media it is presented upon or within. Even though this historically has been how ancient writing was preserved (e.g., cuneiform art, bone markings) it is also the description of many textbooks and handouts from printers in today's classroom. Early forms of writing can even include ink on parchment, especially calligraphy as an art form of writing since it was very difficult to change the writing once "fused" with the paper. In this sense, it is inert.

3.2 Stage 2. Static Kinesthetic/Aesthetic

With the advance of scribable implements and the co-evolution of reusable media to inscribe upon, we enter a second stage of use, categorized by erasability. Here, chalk and marker pens allow a transparent use of writing and expression, as their permanence is temporal, erased over time. But, this new form, albeit static, affords a more kinesthetic inscription—given it is easy to move within the media of inscription—and an aesthetic process—given the use of color to differentiate between notations.

3.3 Stage 3. Static Computational

As the media within which the notation system is processed and presented changes we observe a third stage of evolution. Here presentations (e.g., graph-plotting) are artifacts of a computational response to a human's action. The intentional acts of a human are computationally refined. A simple example is a calculator where the notation system (e.g., mathematical tokens, graphs, functions) is processed within the media and presented as a static representation of the user's input or interaction with the device.

3.4 Stage 4. Discrete Dynamic

As computational affordances make the medium less static, and user interactions become more fluid, the media within which notations can be expressed becomes more plastic and malleable. The co-action between user and environment can exist. This process of presentation and examination is discrete. For example, a spreadsheet offers an environment within which a user can work to represent a set of data by different intentional acts, e.g., "create a" list, "chart a" graph, "calculate a" regression line, or is generated through parametric inputs, e.g., a spinner or a slider alters some seed value. Both of these discretize actions into observable expressions—

expressions that are co-actions between the user and the environment—yet the media is still dynamic, as it is malleable, and re-animates notations and expressions on discrete inputs.

3.5 Stage 5. Continuous Dynamic

This stage builds on the previous stage by being sensitive to kinesthetic input or co-action, to make sense of physical force, or gestural interaction through space and time. Some software allows the user to navigate through continuous actions of a mouse—the perception or properties of a mathematical shape or surface through re-orientating its perspective, e.g., what does this surface look like when I click/drag and move the object? Haptic devices can detect motion through space and time, and provide feedback force on a user's input. For example, a user could perceive the steepness of a surface through a force-feedback haptic device and move it to a point of extreme value without asking the computer to calculate relative extrema.

4 Dynamical Perspectives of Mathematics Reference Fields: Variation and Geometry

The nature of mathematical symbols has evolved in recent years from static, inert inscriptions to dynamic objects or diagrams that are constructible, manipulable and interactive. Learners are now in a position to constitute mathematical signs and symbols into personally identifiable objects, and systems of objects. The evolution of a mathematical reference field can now be an active process that learners and pedagogues can both assist in, can identify with and can actively update. Hence, the reference field has the potential to co-evolve with human symbolic thinking. We will use examples from new innovations in technology to illustrate how work in dynamic mathematical environments (mainly software but also one example which combines both software and hardware) allow new avenues for learners to be actively involved in the evolution of new reference fields.

4.1 Variation and Geometry (Co-action and ZPDA)

Mathematical objects are crystallized through diverse symbolic representations. At prior stages, we only had inert symbolic systems. Those are found in printed books, for instance, and still continue to be instrumental in mathematics at school and research levels. Crystallization is a process with social and cultural dimension. Today, mathematical objects are undergoing another level of crystallization as they migrate to screens and other media where symbols and representations are executable.

We will use examples from recent innovations in digital technology to illustrate how working in dynamic mathematical environments opens new perspectives for

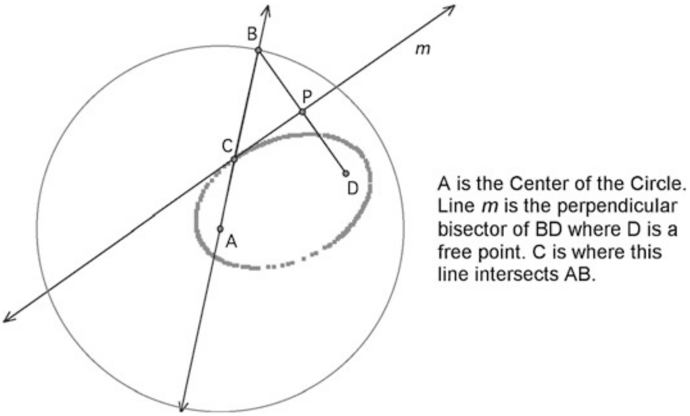


Fig. 1 Locus

learners. Dynamic geometry environments offer point-and-click tools to construct geometrical objects. These can be selected and dragged by mouse movements in which all user-defined mathematical relationships are preserved. In such an environment, students have access to conjecture and generalize by clicking and dragging hotspots on an object, which dynamically re-draws and updates information on the screen as the user drags the mouse. In doing so, the user can efficiently test large iterations of the mathematical construction.

Figure 1 attempts to illustrate this dynamism through a snapshot of such a physical action. As point B is dragged by the user the environment updates by presenting a dynamic representation of all possible iterations of the construction, or "solutions" to the constraints of the construction, i.e., the locus of point C (an ellipse in this case). The nature of the construction constrains the path of C to an ellipse. Since triangles PCB and PDC are congruent, $CB = CD$. $AC + CB =$ constant, so $AC + CD =$ constant. This is the original, static definition of the ellipse. Now, something else comes to the frontline, the enhancement of the mathematical expression, through the animation of point B. As the point C is structural to the construction (it is always updated as the intersection of the bisector m with line AB) it follows the elliptical path.

Indeed we have discretized this "physical" motion. But what we have here is an illustration of where the user has not only actively constructed the ellipse, but has the affordance of a flexible media where the diagram can be deformed with the engineering preserved, through one dynamic action. The dynamic action allows a series of constructions to be instantly created as an embedded environmental automated process. For instance, if we take the point D outside the circle, the ellipse becomes a hyperbola making tangible the intrinsic link between these conic sections. Here the system of tools are embedded and the field of reference (for the symbolic representation of the conic section) is being broadened because the structural points in the construction, due to the possibility to be re-placed in the (digital) plane, lodge new meanings into the executable structure. The dragging of structural, well-constructed objects enables the user to establish whether the mathematical constructs that underlie its engineering can be preserved upon manipulation. Once this is done, the user is enabled to flexibly explore the digital object—-which embodies a mathematical structure. This possibility translates into another dynamic perspective on geometric diagrams and is referred to as a "drag test." Such embodied actions of pointing, clicking, grabbing and dragging allows a semiotic mediation (Falcade et al., 2007; Kozulin, 1990; Mariotti, 2000; Pea, 1993) between the object and the user who is trying to make sense, or induce some particular attribute of the diagram or prove some theorem. Once again, the reference field co-evolves with the user's symbolic thinking and/or reasoning. The kinesthetic actions of the user are crystallized within the geometric diagram and they become part of the enriched, new, mathematical object. The conic is not anymore a definition whose visual trace appears magically on the page. Now, the user *sees* the conic emerging from the screen through her actions now coextensive with the executable system of representation. The user is *co-acting* with the medium, her intentionality is embedded in there and the answer arrives as a digital gesture: the conic on the screen. The plasticity of these actions, the mutual transformation of medium and user, is much more than the classical *inter*action between a user and a rigid tool. The media can keep a trace of such constructions and actions and the user is allowed to rehearse the whole event. The diagram is crystallized in the digital medium but the virtual realities of the diagram obey the rules of geometry that are preserved in the elements of the diagram, just as world objects obey the rules of physics in nature (Laborde, 2004). Again, this sense of reality that the user feels becomes an important element in her cognitive space. We call this certainty *emotional* Platonism. When an element of a diagram is dragged, the resulting re-constructions are developed by the environment NOT the user.

Formalization and rigor are relative to the media in which they take place. They have to respect the nature of these media. If we use digital semiotic representations of mathematical objects, what are the new rules *to prove* a theorem, for instance—that are considered *legal* in the new digital environment? This methodology is highly dependent on time. We can find traces of it in the works of the greatest mathematicians of the past. Euler, for instance, is the author of proofs that could not be published today. Mathematics has been continuously transforming its standards of proof.

As we have seen above, the *executability* of digital semiotic representations of mathematical objects broadens mathematical expressivity. By allowing the externalization of certain cognitive functions (graphing a function or finding the derivative, for instance), executability makes possible the co-action of the student with a digital environment. How is this to affect mathematics education in the future? Kaput et al. (2007, p. 174), observed that the inherited corpus of shared mathematical knowledge produced in interaction with pre-digital technologies is large and stable. So, we need to create early *transition strategies* to transform basic contents of this stable corpus of mathematical knowledge into the new digital semiotic supports. For instance, take the Hilbert space-filling curve—a continuous fractal curve first described by David Hilbert in 1891. In Fig. 2, we illustrate the recursive process that renders the curve as the limit of the sequence.

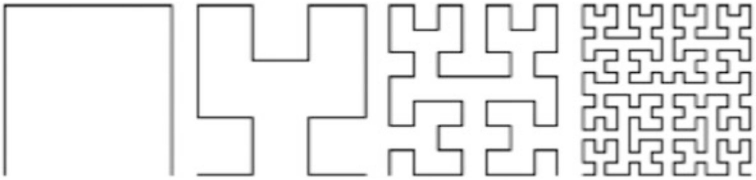


Fig. 2 Hilbert space filling curve

To prove the original theorem, following classical methodology, is an intricate task. However, when one turns the result into a digital one—writing an executable Logo procedure, for instance—we can arrive at the following version: *Given a (screen) resolution, there is a step in the recursive process that generates the curve that fills that screen.*

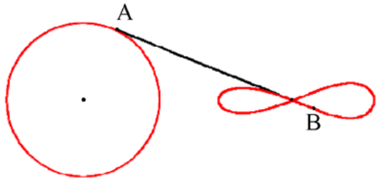
We can imagine the sequence of recursive levels that generate increasingly better approximations to the Hilbert curve, as metaphorical levels of crystallizations of this mathematical object. When the only recourse the students have to penetrate into the mathematical complexity of this object is the analytical representation, we are rather sure that it will remain hidden from their eyes. Now, the executable representation from the digital medium comes as a substantial mediation artifact for students in terms of their cognition. We do not intend to say that this version is formally equivalent to Hilbert's original version of the theorem. In fact, the digital version is a different result. The digital version opens a window into the former version and, at the same time, suggests what is in the future of the students: the meaning of the theorem. René Thom (1973) said it with these words:

> The real problem that confronts mathematical teaching is not that of rigor but the problem of the development of *meaning*... (p. 202).

In the case of Hilbert's theorem, the digital, executable representation is an artifact for developing the meaning hidden in the original analytic representation. Knowing what a mathematical object entails, we need to find and construct the web of relationships among a diversity of previous symbolic instantiations of the mentioned object. In the present example, the executability of the procedure and the role of the digital medium, make the mathematical object tangible. Knowing the resolution, we can calculate the step in the recursive process that will fill the screen. This is an unexpected activity made possible by the new instantiation of the theorem—one which might provide educational meaning. Here we are still working at the border between the paper and pencil (classical) epistemology and the new digital (applied) one. Courant and Robbins—in their classic *What is Mathematics?*—advocate the role of intuition as the driving force of mathematical achievements. And intuition becomes reinvigorated when mediated by digital media as these provide a strong visual component for mathematical thinking. Visual, dynamic perception offers an opportunity to extend mathematical interpretation and, following Thom, *meaning*.

In his *Remarks on the Foundations of Mathematics*, Wittgenstein (1983) emphasizes the role of the eye whilst describing a sketch of a mechanism for drawing

Fig. 3 Wittgenstein figure eight



curves: "when I work the mechanism *its movement proves the proposition* to me; as would a construction on paper" (italics added, p. 434).

He was thinking of this as a mechanism to draw a figure eight as shown in Fig. 3. The digital version of the mechanism makes it ostensible that it is probably more powerful than Wittgenstein originally thought. In fact, by changing the length of segment AB one obtains a beautiful family of curves, full of plasticity, and unfolding continuously on the screen. The unfolding process itself makes explicit the intimate connection among these curves a fact that, in a static medium, results invisible for most students. *Exploring through movement* becomes a new tool for the students. Let us exhibit three stages in the evolution of the figure eight coming from Wittgenstein *digital* machine (see Fig. 4).

The unfolding process takes place as the segment AB is lengthened. The mathematical object under study is not any longer a remote, static object. The immersion in the digital medium provides the students with extended resources to explore and articulate their mathematical reasoning.

We will dedicate the remainder of this chapter to explain and substantiate the thesis that dynamic, digital technologies have the potential to transform the infrastructure of the mathematics classroom, in particular, the distributed cognitive and communicative activities. As Rotman (2000) has forcefully suggested:

> Such a transformation of mathematical practice would have a revolutionary impact on how we conceptualize mathematics, on what we imagine a mathematical object to be, on what we consider ourselves to be doing when we carry out mathematical investigations, and persuade ourselves that certain assertions, certain… a "theorem" for example would undergo a sea change (p. 68–69).

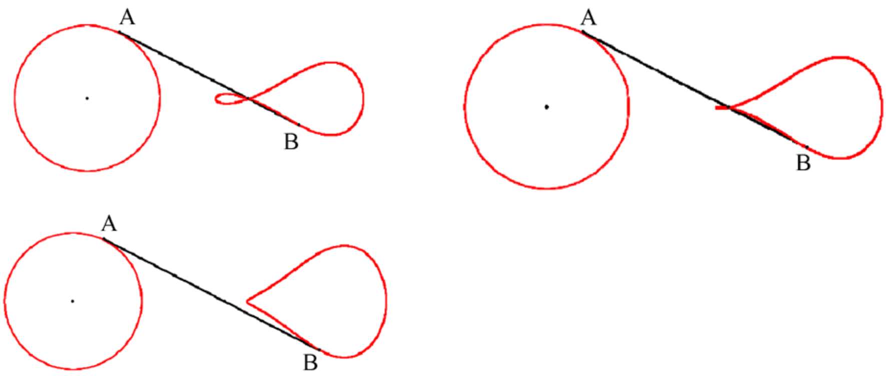


Fig. 4 Stages of evolution of the figure eight

4.2 The Case of Multiple Representations in a Networked Context

Dynamic representations or representations that can be executed either by the user or the environment are available in algebraic contexts as well as geometry. We now continue to illustrate the change in the evolution of reference fields as one shifts from a static to dynamic media and how exposure to such environments can contribute to the fusion of multiple forms of function hitherto loosely connected. SimCalc MathWorlds®—hereon called SimCalc—supports the creation of motions via linear position and velocity graphs, which are visually editable—by clicking on hotspots—as well as algebraically editable. These motions are simulated in the software so that users can see a character move whose motion is driven by the graphs they, or someone else, have constructed. Students can step through the motion and perform other operations in order to help them make inferences about an algebraic expression to represent the graphs. Software runs on hand-held devices (for example, the TI-83/84Plus graphing calculator or the Palm) as well as across computer platforms (as a Java Application). In SimCalc, users can interact with simulations of phenomena. In addition to three traditional core representations of mathematical functions (tables, graphs and formula), motion becomes a conduit to allow fusion of these forms. Motions can be created synthetically within the environment or physically through the use of a motion detector. Consider the following example: Your friend is walking at 2 ft/s for 10 seconds, you need to create a motion that starts where she ends her motion and ends where she starts. Motion data would be represented as a position time graph, but an additional feature is that your motion can be re-played in SimCalc.

Now your contribution to the environment is personally meaningful, and a fusion between traditional "engineered" forms of functional forms, i.e., graphs, and personal mathematical motions occur. Your motion is a form of semiotic embodiment since your motion is mathematical and provides a facilitator for mathematical symbolism. Re-enacting the phenomenon is yet another form of executable representations that allows users' intentions to become crystallized into new, examinable mathematical symbols. Developing understanding of core algebraic ideas such as slope as rate and linear functions ($y = ax + b$) is an important piece of mathematics in which such an environment focuses. Objects in SimCalc are referred to as actors in associated curriculum. Marks indicating where an actor is—at specified intervals of time—can be a feature that one can use in SimCalc.

The actor's motion is preserved or crystallized in this set of marks and the slope of the associated graph is also crystallized in this set of marks. Indeed, it is a new, erasable (through resetting the simulation) inscription that informs the viewer that the actors are moving at constant rates (at least from second to second) and at a speed (or rate) of 2 ft (the gap between marks) per second. The slope of the graph is this rate—2 ft/s—and so it is a representation of a rate graph (velocity) that would be associated with this motion. Links between position (accumulation) and velocity (rate) graphs is a fundamental calculus principle that is being made accessible through executable representations in SimCalc (Nemirovsky and Tierney, 2001; Nemirovsky et al., 1998). Such work by Nemirovsky and his colleagues has shown that students can make sense of dynamic time-based graphs and connect these with certain ideas and skills in arithmetic, e.g., number sentences involving addition and subtraction.

We conclude with a brief, yet synergistic example, based upon recent work on classroom connectivity. Our work (Hegedus and Kaput, 2002, 2003; Roschelle et al., 2000) has combined the power of dynamic mathematics with connectivity. Here mathematical constructions that individual students have created are aggregated into a public workspace via a communication infrastructure (for example, using internet protocols or wireless networks). The computer version can send constructions to other computers and receive constructions from various TI graphing calculators running SimCalc software (see http://kaputcenter.umassd.edu and education.ti.com).

In allowing this, students can create families of mathematical objects that interact in mathematically meaningful ways with a well-structured activity. An example of such an activity is a Staggered Race that requires the students to first attain a count-off number within a group. Such an activity exploits the naturally occurring physical set-up of the classroom by segregating the whole class into numbered groups where students within each group are assigned a count-off number. So students can have a unique identifier both in terms of their group and their place in a group. This number is critical to the establishment of structure in the activity and contextualization of the student's construction within the aggregation of the complete class of functions. In this example, each student starts at three times their count-off number but "ends the race in a tie" with the object controlled by the target function $y = 2x$ (so the target racer moved at 2 ft/s for 6 seconds and started at zero—see the bottom graph in Fig. 5). Students now need to calculate how fast they have to go to end the race in a tie.

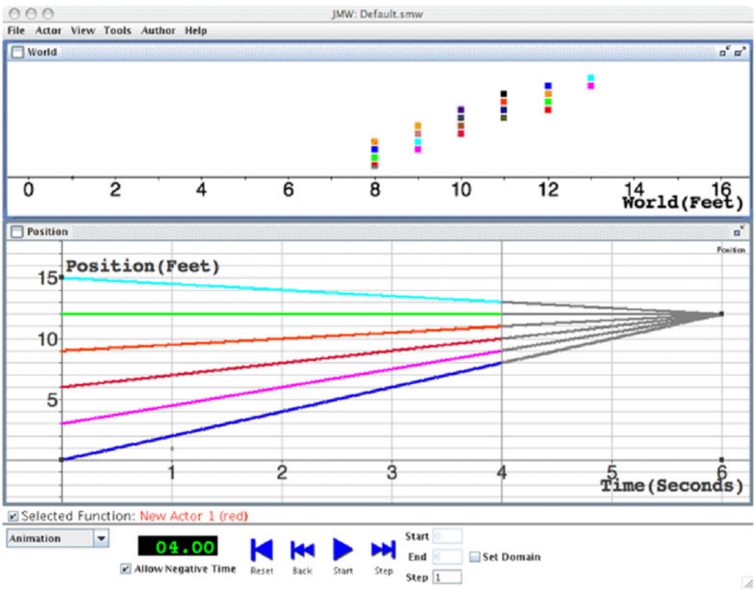


Fig. 5 Staggered start, simultaneous finish

Since they start at different positions, the slope of their graph changes depending on where they start, which in turn depends on their personal count-off number. Secondly, and more importantly, the Count-Off Numbers 4 and 5 give rise to two important slopes (or gradients). The person with Count-Off Number 4 has a graph with constant slope (or gradient), $y = 0x + 12$, since he starts at 12 ft, which is the finish line, so he does not have to move! The person with Count-Off Number 5, starts beyond the finish line (15 ft) and so has to run backwards, thus forcing the student to calculate a negative slope.

In simpler examples, the display of $y = ax$—with a varying as a slope of a family of linear functions—has been described by students as a collection of outwardly spreading motions or a "fan" of functions using a graphical representation. In addition, they use a gestural metaphor via a physical splayed-out finger representation or a set of external artifacts—i.e., a bunch of pens—to display their personal construction of a family of linear functions.

It is through such work that we propose that these digital environments include the social structure of the classroom. Groups of motions and associated graphs can be analyzed both individually and collaboratively. Here the technology serves not primarily as a cognitive interaction medium for individuals, but rather as a much more pervasive medium in which teaching and learning are instantiated in the social space of the classroom (Cobb et al., 1997). Mathematical experiences emerge from the distributed interactions enabled by the mobility and shareability of representations. The student experience of "being mathematical" becomes a joint experience, shared in the social space of the classroom in new ways as student constructions are aggregated in common representations.

Lave and Wenger (1991) have made clear the centrality of legitimate peripheral participation in learning. In the classes that we have been designing and studying, we have deliberately exploited the social situatedness of student learning and likewise the conversational resources for learning (Donald, 2001; Roschelle, 1992, 1996). However, our work has revealed, particularly in the context of currently available communication technologies, that the basic aggregation participation structures as described above have hard edges and little room for legitimate peripheral participation. A student either makes the function for uploading into the aggregate, or not, and the salient presence or absence of the student's contribution is a central rather than a marginal contributing factor to the power of the approach. Hence, we have a design tension requiring creative responses, both in task design and in pedagogy. A simple example of a small change is to have a group assign its own numbers instead of simply counting-off, which offers opportunity to discuss numbers that might be special in the construction (e.g., in the "Simultaneous Finish" situation, to choose 4 as your number so you start at the finish line).

As we discussed earlier with different examples, the reference field now "enlarges" through experience or proximal development of the participant in such an environment, until the symbol and the reference field become the same. Mathematical or theoretical referents are now very individual and personal. The pervasive medium of "connectivity allows the aggregation of dynamical objects into a "dynamic mathematical symbol." Crystallization of individual contributions into a gestalt of dynamic inscriptions occurs with rapid evolution in such an environment. Crystallization embodies the mathematical symbol (in this case, a family of functions) and it is shared across a social space. As the aggregation of individuals' construction are built, shared, and executed, a pathway is laid for mathematical reasoning, abstraction and discovery.

5 Conclusion: New Theoretical Perspectives

Mathematical thinking cannot be achieved exclusively through written symbols; the production of mathematical knowledge requires the use of the body as well, something that just recently, has been accepted, but not for all. We still see in the practice of education that many teachers and curricula designers conceive of mathematics as a purely intra-mental activity expressed in verbal form. This position has a long history. For instance, Plato wrote that "he who has got rid, as far as he can, of eyes and ears and, so to speak, of the whole body, these being… distracting elements which when they infect the soul hinder her from acquiring truth and knowledge" (Buchanan, 1976, p. 203). Mathematics, in Plato's epistemology is disembodied and would be the same even in the absence of human beings. Today, a famous follower of this way of thinking is the Fields Medal winner Alain Connes (for a fascinating discussion on this theme see Changeux and Connes, 1998).

In recent times though, as we have mentioned, the body has come to the fore in mathematics education. Research on gestures, for instance, shows this is the case. It is as if the brain were not enclosed in the head, but (not metaphorically) distributed across the body. An important insight that has taken root is that if we look deep into the meaning of a mathematical symbol—as in a process of *deconstruction*—we will find a bodily experience, an intuition. Of course this is not always an easy task because the structure of the reference field associated with a mathematical symbol is rather complex.

However, our work with digital media, especially with SimCalc, has shown that the mathematics of change and variation that in the past has been a black box for students, now can be approached in such a way that the mathematical structure behind change and motion tells a different story, a story in which the students finds mathematical understanding and identity. The classroom becomes a public scenario for discussion of ideas (closer to democracy than to the authoritarian Platonic epistemology) where students can compare their productions in an environment open to discussion with those of their classmates. A central feature of SimCalc is the potential to transform the socially disintegrated classroom into a participation space, where cognition is socially shared.

We have designed the simulation of a world to study change and variation, not through a classic analytic approach (where all is inert) but incorporating a dynamic narrative about motion and velocity. When the actor is in motion, we can simultaneously appreciate the corresponding Cartesian graph being born. It is a short cognitive distance for the student to imagine that, instead of the actor, it is herself who is walking and causing the corresponding Cartesian graph. Additionally, there is an emerging sense of mathematical identity in the design task as the student chose the analytic graph that should control the motion of the actor: The motion is controlled by the graph. *Motion is change*. This is the kernel of the grounding metaphor for the study of change. The mathematics of change and variation, of accumulation, is crystallized in the SimCalc universe.

The traditional discourse of the Calculus textbooks, supposedly delivers the opportunity to study the mathematics of change. However, instead of that, they offer a discourse whose tacit structure banishes change. This creates a rupture between the intuitively clear ideas of the mathematics of variation and change, as presented through *change is motion*, whose symbolic notation is controlled by basic motion metaphors (converging, oscillating, continuous, monotone behavior, etc.) and the formal structure whose *telos* is quite different: to create a *justification structure* based on the Arithmetization program of Weierstrass. It was Felix Klein, in 1896, who called this program the *Arithmetizing of Mathematics*. In this paper, Klein emphatically declares that, "it is not possible to treat mathematics exhaustively by the method of logical deduction alone, but that, even at the present time, intuition has its special province." A few years later, J. Pierpoint (1899) ended his paper on the arithmetization of mathematics with a feeling of melancholy clearly felt in his words:

> Built up on the simple notion of number, its truths are the most solidly established in the whole range of human knowledge. It is, however, not to be overlooked that the price paid for this clearness is appalling, it is *total separation from the world of our senses* (italics added, p. 406).

SimCalc learning environments provide students with a medium through which students can overcome the artificial difficulties that a premature arithmetization of the mathematics of change and variation can cast on the classroom. When the school does not hear the voice of the students, frequently uttered too low, the result has been to expel students from the Newtonian and Leibnizian paradise of Calculus. In response to this N. Luzin writes:

> What Weierstrass, Cantor, Dedekind did was very good. That is the way it *had* to be done. But whether this corresponds to what is in the depths of our consciousness is a very different question. I cannot but see a stark contradiction between the intuitively clear fundamental formulas of the integral calculus and the incomparably artificial and complex work of the "justifications" and their "proofs" (Demidov and Shenitzer, 2000, p. 80).

In consequence, we have been witnessing a rupture, this time between a basic set of embodied conceptual mathematics, and its apparent formalization. The deep problem here is that Weierstrass Arithmetization ideas do not correspond to the formalization of the ideas of the mathematics of change and variation as embodied in SimCalc. In 1979, Jim Kaput explained that the meaning of mathematical operations was achieved through an essential projection from our internal cognitive experience onto the timeless, abstract-structural mathematical operations (Kaput and Clement, 1979). And more recently, Merlin Donald (2001) has provided a long-term perspective that cast light on the epistemological and didactical conflict we have been making explicit:

> *Humans thus bridge two worlds*. We are hybrids, half analogizers, with direct experience of the world, and half symbolizers, embedded in a cultural web. During our evolution we somehow supplemented the analogue capacities built into our brains over hundreds of millions of years with a symbolic loop through culture (p. 157).

Primary experiences are key for the students in their learning process. These experiences provide the roots of meanings.

It is clear that these reflections contribute to the perspective that has been called *embodied cognition*. According to Donald, it is our hybrid nature or rather, our *analogue half* that provides the (implicit) instructions for moving ourselves in the world of our experiences. This would not be possible if our knowledge of this vital space came from thought exclusively. Again, Donald (2001) provides the deep insight:

> Basic animal awareness intuits the mysteries of the world directly, allowing the universe to carve out its own image in the mind... In contrast, the symbolizing side of our mind... creates a sharply defined, abstract universe that is largely of its own invention (p. 155).

It is the human body moving in its (social) space that carries the seed for the process of symbolic abstractions. This is what we have tried to awaken in the SimCalc classroom mediated by forms of digital technology that cast light on cognition.

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Intersecting Representation and Communication Infrastructures

Stephen Hegedus and Luis Moreno-Armella

1 Overview

For the past two decades, educational technology has been evolving in various ways along various research and development trajectories. Software has become more visual, interactive, and more dynamic. Hardware has evolved to allow more complex programs to be executed for work to be done at a distance (both proximally and longitudinally) through the advances of networks (in particular wireless), and to be more portable in terms of its hand-heldability. These affordances[1] impact two types of infrastructure: (1) Representational Infrastructures and (2) Communication Infrastructures. We posit that these infrastructures have at times evolved independently but when they co-evolve or intersect each other's growth pathways, then new forms of activity occur.

We begin by describing what this evolving dynamic looks like from the perspective of a mathematics classroom, or mathematics education community, and contex-

[1]By affordance we mean *a quality of an object, or an environment, that allows an individual to perform an action*. We thank one of our reviewers for offering us a succinct and meaningful definition.

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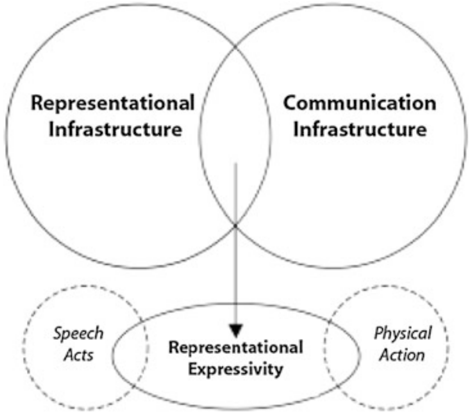
tualize our work within a program of research (Classroom Connectivity research—referred to as CC1, CC2 and LAMP in project timeline in the introductory chapter of this volume) that has investigated the impact of the integration of dynamic software environments such as SimCalc MathWorlds® and wireless networks (e.g., TI-Navigator™ Learning System) on learning, participation and motivation. This work resulted in highlighting three products from the intersection of representation and communication infrastructures, which we present in this paper.

2 Representational Infrastructures and Communication Infrastructures

Technology has offered and afforded representations and interactions between representations for a long time. These have been in terms of symbolic manipulators where computational duties are offloaded to the microprocessor and new actions are linked to traditional notation systems. But in addition, there is now support for new interactive notation systems such as programming languages underlying mathematics packages (e.g., Maple) and spreadsheets, enhanced interactivity and expressibility of new phenomena by linking traditional notation systems and representations to new ones (e.g., simulations), and finally digital communication. This last modern affordance has been translated into mathematics classrooms as a mode to enhance access: offering students the ability to see through abstract constructs or symbolic figures. But we will explain in this paper how this is one inlet to describe a massive fusion-possibility between two critical ingredients for advancing 21st century learning. The connections of such computational and visual affordances offers a deep and wide infrastructure which can be defined as a complex bedrock of functionalities, expressive and operational affordances, established through an overarching organizational structure that allows mathematically valid and viable connections. Contrast this with an infrastructural aspect of society. A road network is established to organize the flow of traffic between places of habitation and commercial enterprise. Moving water along a road would not be efficient or functional, similar to connecting a graph to a bunch of signs that made no actual mathematical sense. Once again, in contemporary society, infrastructure has not only become a material concept but a social one, and the affordances of the Representational Infrastructure (RI) allows the possibility to create a social learning network and enhanced communication; yet another inlet to a separately evolving infrastructure.

Let us now describe Communication Infrastructures (CI) separately, as much as is possible. Communication has been a critical aspect in the evolution of mankind, and in recent decades the advancement of knowledge. As symbolic species (Deacon, 1997), language and the brain have co-evolved, and since the evolution of external supports of memory some 35,000 years ago (Donald, 2001), language has been expressed through ever-changing forms of media. We will refer to *Communication* as

Fig. 1 Model of RI and CI intersecting



human actions in terms of speech or physical movement (e.g., gesture) or digital inscriptions through modern-day interfaces. Once again, a communication infrastructure is the organizational structure of the various communication inlets and outlets available in society. A digital infrastructure is composed of networks, wires, and servers to create information flow of communication acts and services to various populations. These are often thought to be large, even global, but with the advent of wireless networks, the same power and functionality can be brought inside a classroom, historically called an *intranet*. Similar to RIs, these have largely been material in nature but are also social in their constitution now, yet the primary development trajectories have been in terms of their physical implementation into society (e.g., installation of broadband connectivity across a whole town or country, or the construction of cellular antennae). Our central position is that such infrastructures have largely evolved independently even though each have obvious inlets to each other where one can enhance the functionality of the other, but for a long time we propose that these types of infrastructure have not co-evolved.

When these intersect in an educational context (see Fig. 1), the evolution of meaning is enhanced as traditional forms of expression are transformed or enabled. This is our main claim and one that we will unpack throughout the course of this chapter. At this stage, we describe in general what this intersection yields. At the heart of such convergence is a transformation of expression, and what we prefer to call *representational expressivity*, where learners can express themselves through the representational layers of software and where a participatory structure enables learners to express themselves in natural ways through speech acts (e.g., metaphors, informal registers, deixis) and physical actions (e.g., gesture or large body movements). We will also illustrate that such forms of expressivity are also loaded with intentionality; both for the learner and the designer. For the learner, an action to identify themselves in classroom dialogue and to attach themselves to a representational artifact of the technology. For the designer (software/curriculum developer, teacher or whomever) as a specific *a priori* pedagogical or epistemological decision made upon the basis of the affordances of the environment and the specific algebraic structure of the activities addressed.

3 Combining Representational Infrastructures with Communication Infrastructures in Mathematics Classrooms

3.1 Overview

We have designed and used the software SimCalc MathWorlds® to transform students' mathematical constructs into fascinating motion phenomena. The SimCalc RI has four essential elements: (1) hot-links between graphs and simulations, (2) visually editable, piecewise-definable graphs of functions, (3) hot-links between rates and totals graphs, and (4) importing physical data into the computational notation.

The implementations of these RI elements emphasize flexibility in the sense that the various elements can be used in huge varieties of combinations tuned to specific curricular objectives, student needs and pedagogical approaches. Of special interest is (3), in effect embodying the extraordinary human achievement of the fundamental theorems of calculus *inside the RI*. We regard this as analogous to other historical encapsulations of structure into a notation system. For example, the standard placeholder system for whole numbers embodies in extraordinarily compact form a hierarchical exponential structure in a way that democratized access to computation with almost arbitrarily large numbers (Shaffer and Kaput, 1999).

Second, we have integrated a CI into SimCalc's RI in the form of wireless networks in ways that can intimately and rapidly *link private cognitive efforts to public social displays*. We have integrated the use of graphing calculators from Texas Instruments and a specific wireless network, TI-Navigator, to connect students' work in SimCalc on their calculators with SimCalc on the teacher's computer. We have also done this by connecting computers to computers wirelessly or in labs. Consequently, students can each be assigned a specific mathematical goal (e.g., playing the part of a single moving character by making a graph with certain mathematical characteristics), which instantly links to public social display (e.g., the parade constituted by all characters moving simultaneously).

For example, students, in small groups of three or four, have a group number (i.e., 1 through 5) and are asked to create a position function so that the motion of a character moves for a duration of 6 seconds at a speed equal to their group number. So Groups 1, 2, and 3 create functions that can be defined more formally as $y = x$, $y = 2x$, and $y = 3x$ respectively for a domain of [0, 6], or through other representations such as graphical or tabular editing. The ability to draw graphs through stretching it across a Cartesian space offers a more informal register of functional representation for students to access through simple click-and-drag hotspots to define slope and domain. This is similar to the role of hotspots in *Dynamic Geometry*™ to stretch and drag figures and constructions. SimCalc couples graphs with animations and so the results of such actions can be observed through executable representations.

The important concept is slope as rate—an underlying concept of the mathematics of change and variation—and a family of functions is created by the class

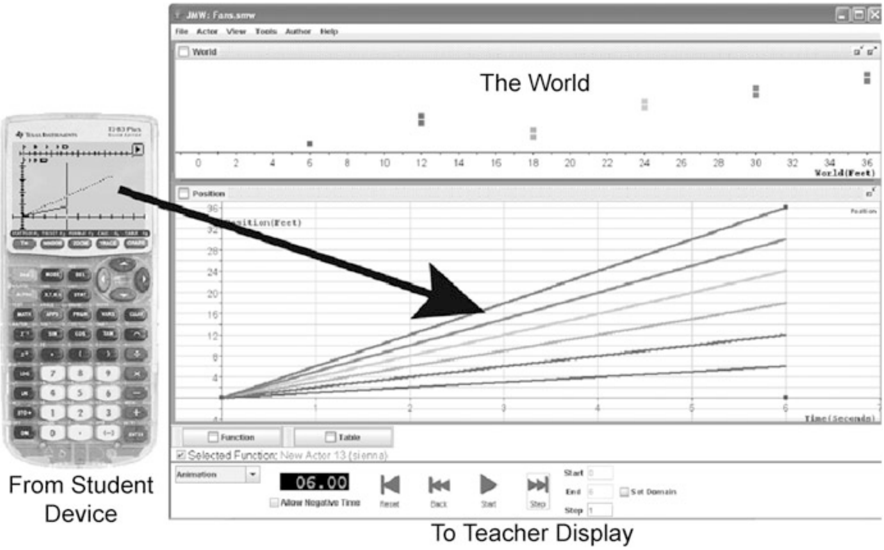


Fig. 2 Aggregating student work from calculators to a computer

via varying the parameter k, in $y = kx$, where k is group number. The variation becomes meaningful for the students; the family of functions is a result of their independent contributions (see Fig. 2). By default, student work is hidden. Student contributions are revealed at the discretion of the teacher; this allows students to conjecture and make generalizations about the whole work of the class, and about how their contributions relate to the whole class set of contributions before seeing the variation. The conjectures and generalizations are prompted via curriculum and teacher questioning strategies such as, “What do you expect to see in terms of motion (or graphs) for the whole class?” This type of mathematical activity has brought about new forms of participation and mathematical expressivity in the classroom.

This approach shifts the types of critical thinking that are possible in mathematics classrooms and transforms the role of instructional technology by integrating it into the social and cognitive dimensions of the classroom and eliminating its use as a “prosthetic device” to prop up existing teaching practices and methodologies. In addition, linking private work in a mathematically meaningful way through networks, and displaying the aggregations of whole class work, potentially enhances a student’s metacognitive ability to reflect upon their own work in reference to others (Huffaker and Calvert, 2003). For the past decade, we have been deeply engaged in producing Algebra materials that deliver upon this vision of networked classrooms. Our connected approach to classroom learning is reiterated in seminal works (Bransford et al., 2000) that highlight the potential of classroom response systems to achieve a transformation of the classroom-learning environment. Similarly others have expanded their approaches to include devices that allow aggregation of mathematical objects submitted by students (Resnick et al., 2000; Stroup, 2003; Wilensky and Resnick, 1999; Wilensky and Stroup, 1999, 2000).

We focus on three main products that result from the intersection of an RI and CI in a mathematics education context. These three products are a synthesis of the program of research that we have previously described. The context of work thus is important in framing the types of products that we focus on here. Since our work has focused on participation and motivation through synergizing the representational affordances of SimCalc with the fast, information-sharing capacity of wireless networks, we present the results of our iterative development cycles. Of course, as our paper begins, there is a more general picture where these infrastructures converge, but we wish to consolidate our theoretical piece through the results of our ongoing work. A focus on motivation through enhanced participatory activities has yielded three broad areas of advancement. First, new forms of mathematical expressivity through gesture and deixis. *Deixis* examines the properties of linguistic expressions (indexes) that cannot be interpreted without reference to a nonlinguistic context of their use (Duranti, 1997). Deixis extends to the use of gestures, movements, posture and gaze, as well as pointing acts used in collaboration with speech. Second, enhanced forms of identity and identification of ones contributions to a mathematical argument. And third, new forms of activity structures that sustain these forms of expression through coherent pedagogical strategies.

3.2 Product 1: Mathematical Expressivity

Consider the following activity: You must make a motion for your clown that goes at the same speed as Clown A (2 ft/s) for 6 seconds but starts at a distance equal to your group number of feet ahead of Clown A. This is a parade where each group starts 1 foot ahead of each other.

Through regular group-based activities, we can exploit the use of such physical structures to embed them into a representational mathematical structure. In doing so, we move from private or local thoughts to the representation of these thoughts as artifacts in a public space. Here the affordances of the SimCalc RI can layer visual effects to motivate the whole class to respond in expressive ways. The aesthetics of the collection can be represented as a whole, a parade of motions, or colored in a strategic way (e.g., color by group, individual, etc.). But we must not forget the overall gestalt of a collection of functions—a family of functions—as a powerful meta-representation of variation. Such "collections" are often left for undergraduate mathematics courses in the analysis of families of functions and differential equations. In Fig. 3 below, we can see a collection of square dots (students' contributions) parading in the animation world. There is also a collection of graphs colored by each individual and where every individual is a member of a group of 3–4 students and hence they share the same color (shown here as dots versus characters to minimize the complexity of the figure).

It can be formally represented as $Y = 2X + G$ on $[0, 6]$, here G is your group number as some constant. It can also be represented as a motion; a parade. Because of the equivalence of motion, a direct variation from such similarity would

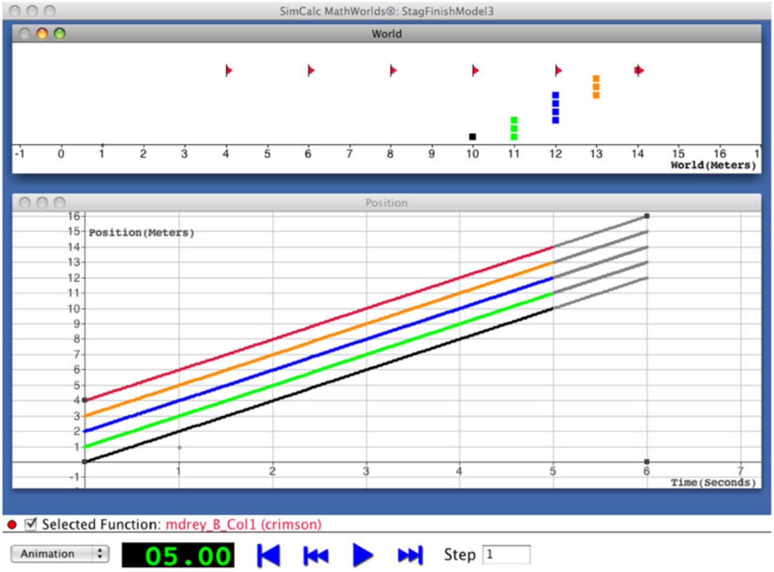


Fig. 3 Parades in SimCalc

yield an examination of the whole set of contributions. A coloring of the graphs (or clowns) would aim to structure the underlying mathematical system, parallelism, or equivalence of slope. It could also highlight differences. We use color and the literal notation (e.g., Crimson in our example above) to describe differences. In Fig. 3, a student's login "mdrey" is used to signify that student's contribution. It is colored Crimson (randomly by the software), and the teacher has stepped through the motion. This student's work has marks dropping (forward facing triangle) every second to denote a trace of the moving square's location (this is an attempt at describing motion in a static medium) and is also a representation of the changing accumulation of distance traveled per second (hence a rate). Here color can illustrate variation. Even without color, the visual effect of parallelism synergized with a uniform *parade-like* motion can be a sufficient visual gestalt to provide access for students to describe attributes of the mathematics of the whole motion.

With the ability to show and hide every representation of the contribution of every student (see Fig. 4), the teacher can scaffold the classroom discourse in a highly structured way, and in a manner that allows the structure of the underlying mathematics to emerge. For example, questions in our curriculum follow a pattern of "where are you in the world?," i.e., which colored dot or animated character represents your work? And then "what do you expect to see for Group 1?" by selecting to see just Group 1 in the World (see Fig. 4).

Followed by, "what do you expect to see for Group 1's graphs?" or "How does the work of Group 1 compare with Group 2?" In structuring the discourse in this way, students can conjecture and reason the variation as a collaborative exercise and their ideas can be tested through the executability of the environment, i.e., teachers

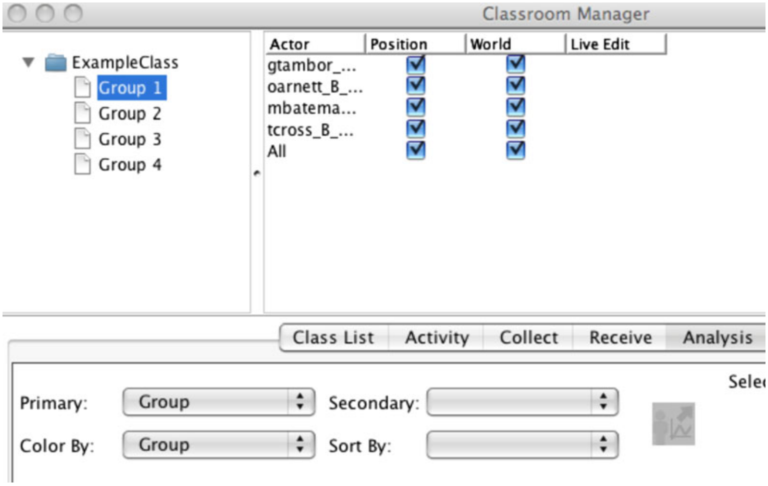


Fig. 4 Classroom management of groups

can animate the actors following a conjecture and so the feedback is from the environment. The software (in the hands of the teacher and projecting the work of every student) works with the students to support their reflective problem-solving. This is a critical juncture in how language (and particularly communication) is working synergistically with the representational affordances of the SimCalc software and the infrastructures underlying the execution of these activities. Students express themselves in vivid ways, both informally and formally, and often conjecture or react to the shape, form, and structure of the family of graphs or motions as they emerge, or symbolic expressions later in the activity. For example, students have described the graphs in this activity to be parallel, and members of a particular group's graphs to "lie on top of each other" which illustrates a basic idea of identity. A slight variation in this activity where each group makes a function to represent an actor's motion that goes at a speed equal to their group number for 6 seconds yields a family of functions that has been described with rich metaphors such as a "fan" with gestured support (see Fig. 5), and is formally encapsulated as $Y = GX$ on [0, 6] (see Hegedus et al., 2006 for more details).

3.3 Product 2: Identity Formation and Identification of Self—From Private to Public

Our preliminary analyses (Hegedus and Penuel, 2008) suggest that a key characteristic of such environments is fluid turn-taking, in which students bid for and gain the floor in classroom debates and discussions about mathematics. This type of participation structure is quite different from traditional classrooms, in which the teacher

Fig. 5 Gesturing a fan

is the primary speaker who nominates students to speak and controls the floor of discussion.

In a complementary fashion, we have found at least two kinds of identity-formation processes occurring simultaneously within these classrooms. On the one hand, fairly stable social identities (see, especially, Eckert, 1989; Wortham, 2004), often negotiated and maintained within classrooms, are enacted. These types of social identities are often stable and well-defined both inside and especially outside the classroom and this plays a role in how a student participates in a classroom discussion, whether calls for attention are upheld and how their contributions and participatory role is accepted, e.g., the "smart kid," "outspokenness," socio-economic status. These are largely shaped through cultural values and established norms within a community.

We also observe a more local identity within networked classrooms that is temporal, less stable or well-defined, and constructed through the mathematical activity made possible by a networked environment. Personification and identification are defined within a shared work space. Here identity can be virtual and public as work is projected away from a local self to a representation space managed by a teacher. Identity can also be discursive, as we have observed above where it can be enacted through deixis (e.g., deictic markers or references in dialogue) or gesture (see Hegedus et al., 2006 for further analysis).

Studies (Eckert, 1989; Gee, 1999; Wortham, 2004) suggest that the acts of positioning associated with particular social identities can inhibit participation in classrooms, and so it seems relevant that the construction of new, less stable and more flexible forms of identity in these environments can support active forms of participation and engagement of more students versus just those students engaged in a student-teacher turn taking dialogue. We wish to stress that we prefer to focus on identity as enactments of identity in terms of how students represent themselves in their use of language, physical action, and social positioning. It is not the more traditional psychological notion of identity, which is more an enduring sense of self.

We offer examples from one classroom to illustrate how identity and representation of ones self are transformed as the RI and CI that we are focused on intersect. The class has worked on the following activity:

> You and your partner will start at different positions. You are positioned *G* (your Group-number) feet away from 3 feet. The person with the odd count-off number (number uniquely assigned to each member in the group) will start to the right of 3 feet. The person with the even count off # will start to the left of 3 feet. You and your partner must meet at 3 feet at the same time. You and your partner will determine the amount of time you will travel for. The group CANNOT travel for the same amount of time, only you and your partner can. You must create a linear expression for your motions.

There is ambiguity in the problem statement, which offers a wide variety of correct responses. Given the environment is to support public engagement of each person's work, this is an important generative feature that structures the environment the students work within. Since students are in pairs traveling in different directions towards a common (for the whole class) meeting point (3 feet) but start an equal distance from this point, pairwise contributions are similar in shape, i.e., the slope of each student's graph is the inverse of the other. Each pair travels for the same amount of time, i.e., share the same domain, but they are potentially different across pairs for the whole class. So this simple task generates many similar but structurally different answers, and it is this structure, which is exposed at a public level. The transfer from private to public also initiates a projection of the self (for each student) into a social network, where general discussion and argumentation is saturated with personality and social cues. We offer some simple examples of this to illustrate how such behavior can exist. We believe this is fundamentally structured by the activity structure and sustained by the teacher utilizing the affordances of the technology (for example that she can see everyone's work at once).

The students have constructed function-based actors on their own individual graphing calculators in SimCalc and their results have been collected into the teacher's version of SimCalc. A correct version is displayed in Fig. 6. Two groups of dots (encircled) should move towards the place at 3 ft. Before these dots (or actors) are displayed (via the classroom manager) and animated, one student (AC) asks another student (KO) how many "dots" (numbers of students in the class) there will be:

AC: How many are there?
KO: They'll overlap each other.
KC: There's 19.
KO: Oh yeah. There will be 19.
AC: 19 minus Joe, that's 18.

In this example, AC replaces the number "1" with "Joe," a move that in a sense marks Joe as identified simultaneously with a mathematical object—a dot—and as a variable in a simple equation. This blending of personal and mathematical—through replacing third-person, gender-neutral descriptions of mathematical objects with personal pronouns and even names—takes place at several points in this and other discussions in classrooms observed in our study.

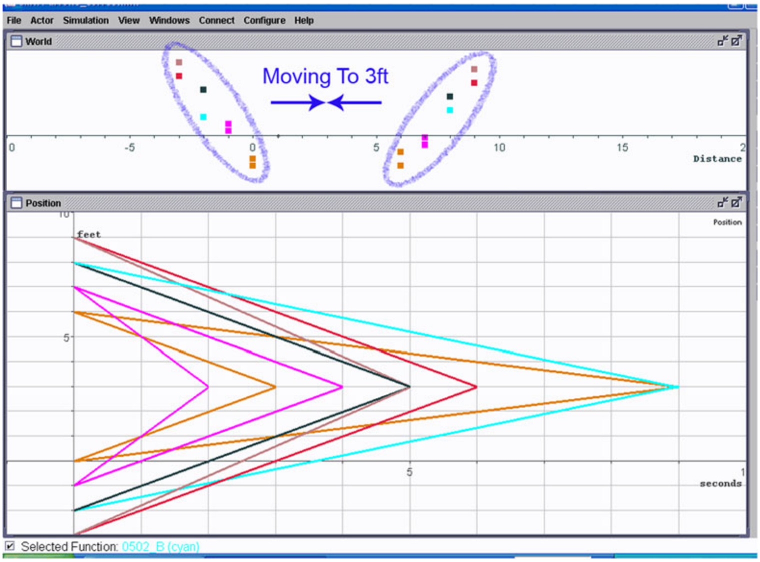


Fig. 6 The "Arrows Activity"—a correct aggregation in SimCalc

Later it is well understood that two students (one pair) are not correct as they do not end at 3 ft. This pair has confused domain and range by calculating their slopes incorrectly. The class is arguing what they believe is wrong with their functions.

JS: It's not time?
KC: It's their distance.
KO: Exactly.
KC: Ugh!

As KC realizes the mistake of the two students, it is important to once again focus on the use of deictic markers such as "their." In addition, students also assign mathematical attributes to other students in making sense and reasoning publicly.

NP: *{still at front corner of room, facing side of class}* Okay, AS, what did you have for domain? For how many seconds do you go for?
AS: Nine.
NP: Exactly. *{standing next to JS, looks to her while responding}*
JS: Jess, how many seconds do you go for?
JC: Nine.

Reference is personal, i.e., "…do *you* go for", and identifiable and reasoning is inherently mathematical.

As the class progresses, we observe more examples of identity-rich discourse which again illustrates an important effect of the shifting of dynamic representations of students' work from a private to a public workspace. We observed that in the midst of arguing about whether lines are parallel from certain contributions from across pairs (see Fig. 6 previously), the teacher (JS) attempts to get KO to explain why two lines are parallel by referring to the concept of slope. This is part of the pedagogical intentions of the teacher as she attempts to consolidate the core mathematical ideas in the activity and build on the previous analysis of errors. Instead, KO simply asserts they are parallel based on what she sees.

AC: They're not parallel are they?
KO: Yeah-huh. I told you that.
AC: But they all end. *{AC claps her hands together.}* Ugh. Geez.
{JS steps back away from them. Pause for 3 seconds.}

The teacher (JS) and AC perform two very different speech acts through their next moves. The teacher asks KO to explain her idea to her peer. AC, in turn, makes an identity attribution to KO that has the immediate effect of negating the teacher's move:

JS: Why don't instead of saying 'trust me' explain to her why she should trust you.
AC: She thinks she knows everything.
KO: See, those are two parallel lines and those are two more parallel lines.
JS: But why? You're just looking at them and saying they're parallel.

Here, classroom identities—in this case, a negative attribution to another person as someone who thinks she knows everything—have the effect of shutting down rather than allowing for deeper mathematical argumentation, much as they can in traditional classroom settings. Later in the clip, after the teacher and KO have taken turns several times to explore just how the lines are parallel, AC reasserts her claim that KO is a "Know It All," which could be interpreted as a code to the interaction, summarizing its significance for the class.

JS: Good. What are those lines gonna be? *{Pause for 2 seconds.}*
AC: Uhhh.
JS: If you were to connect...
AC: Stair steps.
JS: ...those lines?
Yeah, but if you were to connect them like this ...
{Pause for 3 seconds while JS draws the lines she is referring to.}
What are those lines going to be?
AC: Parallel. *{AC says this very quietly.}*
JS: Are they ever going to intersect?
AC: No.
JS: So, they're parallel based on their slope not just because they look like they're never going to intersect. *{JS turns and looks at KO while she says this.}*
AC: Yeah.
KO: I knew they were never going to intersect because...
JS: But you didn't say *why*.
AC: Alright Miss Know It All.
KO: I don't know how to explain things, I just know them.
AC: I know. She doesn't even work.... She shouldn't be in a group.
KO: I really don't. I'll just sit there.

Significantly, KO seems to accept a fixed identity after AC's assertion, though it is not quite AC's positioning of KO. Instead, KO seems to be saying that she cannot participate in mathematical discourse in the way the teacher originally asked; she is not someone who can explain, but someone who simply knows. AC takes another swipe at KO, saying that she doesn't work. KO's response can only be interpreted as sarcastic, an active resisting of AC's attempt to position her in this way. KO seems ready for others in class to recognize her as someone who cannot engage in mathematical argument for her ideas, but she does want to be seen as an active, engaged member of her group.

Finally, we believe that the most significant piece of this work is that the forms of identification that are used by students and teachers in our study are infused with mathematical structure, and hence informal mathematical registers are used (in terms of speech and actions) to defend and reason their work (both positively and negatively). In essence, students identify themselves with the object, or the mathematical attributes of the object, in a sense embodying the mathematical idea as a personal expression. Their thinking is a representation of their identity and how they wish others to attend to or ignore their work in a public and social setting (Penuel and Wertsch, 1995). We believe this is a fundamental feature of representational expressivity that emerges as the RI and CI are combined.

4 Activity Structures—From Intentionality to Attentionality

A communication infrastructure can yield flow, and in our work, a third product is a pedagogical model that structures curriculum activities in SimCalc classrooms and the flow of certain patterns of discourse.

Bakhtin (1981) describes how we borrow words so we can mean to others, and we populate them with our own meanings and intentions so we can signify our relationship, attitude, and identity with others. Burke (1969) explains how people using representations direct others' attention to some things, and deflect them away from others. This is called external intentionality. In directing attention, intentionality can be externalized through various forms of expression and action.

We believe evidence of change in such classrooms is a shift towards a more fluid movement across different representations, evident from classroom discourse. The model in Fig. 7 attempts to offer a structure to help analyze how flow can be directed from, and to, participants in a classroom through representational tools and actions. It is intentionality that forms, and continually transforms, our identity in these environments. This model can help generate other methods for analyzing the nature of participation in such classroom environments. It is another result of when an RI and CI intersect in ways that enhances and structures dialogue in such classrooms.

In this model, we refer to individual students, groups of students (pairs, small groups), and the "class" as all students, teachers and the public display in the classroom. Discursive acts (e.g., directive, reflective, or reflexive), representational affordances (e.g., show/hiding a graph), or deixis (e.g., a deictic marker) have all been exemplified in our analyses. The model can be used to describe the potential flow of interaction that could occur because of an intention, an activity, a question, or any communication situation, and has the potential to be germane to a wider variety of settings. One example (from our analysis) is where attention was directed from a

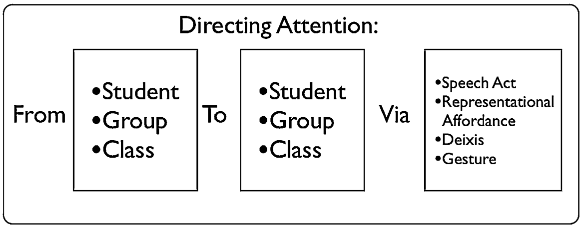


Fig. 7 Structuring dialogue in SimCalc classrooms

student to the whole class via the representational affordance (observing the whole set of actors moving in Figs. 3 or 6). Note that the director of attention could be a teacher, student, or any active participant (e.g., the shared display).

In this model, directing attention is fundamentally an identification procedure in surfacing identity in five non-ordered forms that have been evident in our classroom observations:

1. Through a physical space (e.g., the actual person).
2. Through an extended physical space (e.g., via a gesture or positional stance).
3. Through an artifact or projection of a cognitive act (e.g., a person's graph or motion).
4. Via projection into a contribution space (which we have specified as a shared space) as a representation of a personal contribution from a private/local to a public/social space.
5. Within a gestalt, i.e., a well-defined segregated place within a meaningful whole as defined by a mathematical space or structure, e.g., the mathematical activity analyzed earlier.

Indeed, Item 4 could be Item 3 because there are examples such as an utterance of the form "my dot moves like this" that could be an example of a projected cognitive act enabled by the connected environment but it has not been re-presented in a shared representational space. We expect that further work will yield more specific pedagogical actions and related teacher knowledge around effective practice as determined by measurable learning gains.

5 Conclusions

We have aimed to describe three products that occur when the SimCalc RI and CI are integrated. In our work, new forms of participation can occur when two such unique technological ingredients are integrated—dynamic, interactive software that works with multiple representations simultaneously (RI) and wireless networks to create networked classrooms (CI).

In addition, a student's identity can possibly transfer or shift during the course of a class, and depending on how the student believes their peers perceive their identity, or how they perceive their identity to be relevant in a public workspace, might direct the flow of discourse (argumentation, presentations, reaction).

Finally, we believe that the most significant piece of this work is that the forms of identification that are used by students and teachers in our study are infused with mathematical structure, and hence informal mathematical registers are used (in terms of speech and actions) to defend and reason their work. In essence, students identify themselves with the object, or the mathematical attributes of the object, thus embodying the mathematical idea as a personal expression.

Kaput (1991) felt that the interplay of natural language and social interactions, especially with respect to the status of referents in discourse where reality emerges in a social situation, was a challenging problem to analyze, and one which we would be able to address over time. It has taken many years to begin to analyze what that means in a mathematics classroom, but we believe that as learning environments begin to evolve into more dynamic, interactive and social spaces, it is necessary to build new theories to help us structure and analyze the nature of learning. Learning mathematics that stretches more deeply over a school curriculum is an accomplishment of interactions both human and digital. We have presented how it is necessary to come to a common understanding of the meaning of mathematical representations through enhanced communicative forms in the SimCalc learning environment that is now possible in 21st Century classrooms.

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Part II
Aspects of Design

Reflections on Significant Developments in Designing SimCalc Software

James Burke, Stephen Hegedus, and Ryan Robidoux

From the first SimCalc project (see timeline in introduction), researchers and software developers made decisions shaping the design of the software based on contributions from students, teachers, and an evolving understanding of what this technology made possible in the classroom. We developed the SimCalc learning environments using the principles of dynamic interactive technologies suggested by Kaput (1994).

Our development of this software over time allowed us to consider and refine its design according to observations of its use in classrooms, advances in the available technology, and evolving theories of how students can learn important mathematics. New development brought new technological affordances. Important decisions can be seen in the mathematical representations, in the configurability of the software, and in the communication infrastructure that we developed to take advantage of networked activities. Importantly, as designers and developers, we have seen technological affordances as a way to introduce new mathematical affordances into the classroom.

We focus on two main themes throughout this chapter that describe major shifts in the evolution of the SimCalc learning environment: (1) The rationale for how changes in the software were made and the decisions that drove such changes, and (2) How our thinking about the software changed in terms of new affordances for learning. We will focus on three main areas of research and how each has guided the design, development and implementation of the software: (1) Representational

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Infrastructures and how design is guided by mathematically meaningful representations, (2) Activity Structures, and (3) Classroom Connectivity and its impact on learning and participation. As researchers and software developers that make up the core design team, we prefer not to describe a litany of how the software changed over time but rather major changes to exemplify how our research was tightly integrated into our development by focusing on improving mathematical learning for many different kinds of learners. While the process of development took place over more than 15 years of continually-funded projects, an especially dramatic shift took place in the later years. SimCalc MathWorlds® (hereon referred to simply as SimCalc) had been a student-centered, 1:1 software environment that mediated meaningful student exploration and expression of mathematical ideas. It has become a mathematics-learning environment that connects the entire classroom, allowing the interrelationships among student contributions to be the basis of classroom-level discussion and thought.

As the development cycle neared its close, the notion of the importance of getting the software into the hands of users to inform design decisions was bookended by design changes that would, in turn, put the software into the hands of more students, particularly underrepresented groups. Dissemination became a priority in order to allow wider access to important mathematical affordances for educational and research purposes. Design decisions play a role in helping the software reach a larger community.

1 Innovation Research Software Development

The SimCalc design team (including researchers, software developers, and teachers) sought not only to enhance existing classroom curriculum, but also to transform it with activities that were not possible without the use of technology. We intended for students and teachers to engage in thinking about mathematics in powerful ways. Introducing SimCalc into the classroom frequently was, for us, a means to see what might be possible with mathematics learning and instruction in ways that eventually led us to support the emergence of new forms of participation.

Hegedus and Moreno-Armella (2010) has described the co-evolution of technology and user action that draws research and development together this way as a relationship between instrumentation (i.e., a shaping of a participant's actions through co-actions with a tool) and instrumentalization (i.e., the shaping of the tool itself by a user's knowledge and by the environment). This iterative design cycle led to highly configurable software that could support a wide variety of pedagogical and curricular intentions to produce mathematically meaningful lessons. And when we sought to introduce the affordances of networked devices, the communication infrastructure introduced a new connection to mathematical structure (e.g., a family of functions). We found that SimCalc lessons could structure the physical setup of the classroom (e.g., how students were placed in groups and how the work of each group was structured) in ways related to the mathematics. Students were afforded different ways to interact because the classroom itself had become mathematically structured.

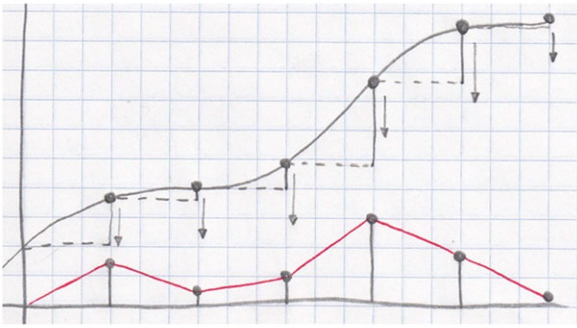

Fig. 1 Sketch of "lollipops"

2 Mathematically-Meaningful Design

2.1 Representational Infrastructure

The representations that are central to SimCalc have always been tied to thinking about how they served mathematical meaning. Even in early discussions, Kaput described his visions for representations that would connect mathematical concepts through the actions of users and through some dynamic presentation of the software. He outlined an animated method for connecting the graph of some varying quantity to a graph that approximated the change in that quantity over time. He suggested that the changes over unit time could be drawn and then animated by dropping them to the axis, where they would provide a very rough approximation of a graph of change. This idea for connecting (for example) position and velocity graphs was affectionately referred to as "lollipops" (see Fig. 1).

It was never implemented because it merely existed to serve as the beginning of a conversation not just about what we could possibly do with the available technology, but that we should always think about mathematical structure and connections as part of how we envisioned the development and use of the software as a learning environment.

2.2 Dynamically-Linked Representations

An important representational infrastructure of SimCalc is its dynamically-linked representations. This allows different "views" of the functions through the user interface that are linked so that any changes in a function are reflected in all representations (see Fig. 2). For example, the editing of a position graph is immediately and continuously reflected in a velocity graph.

The details of these representations are important to preserving them as mathematically meaningful. For example, the segments of piecewise functions in SimCalc software are considered to be open on the leftmost end and closed on the rightmost end.

In the creation of activities, many of these functions are built with segments that start and end on integer values of x. The implications of how segment boundaries are

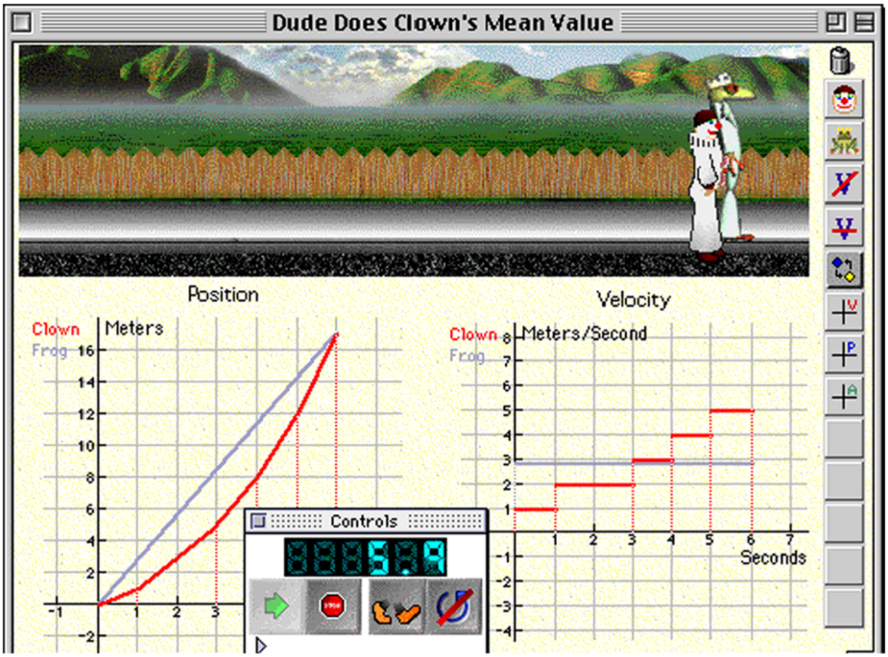

Fig. 2 Screenshot of the multiple, linked representations in Classic MathWorlds (name used prior to SimCalc MathWorlds®)

defined become apparent when a teacher makes a table to show the velocities at each integer value of x. Developing the software to display the velocity at moments when the velocity is changing instantaneously presents a problem. We resolved this problem by determining into which interval that x value falls, but the result may not be what a teacher expected. The details of the representation can be mathematically justified, but this is just one example of how ambiguity produces the need to make a decision in some seemingly minor detail of how the software works in legitimate ways.

These details are necessary, intentional, and part of the dynamic experience students can have by acting on the software. As Hegedus and Moreno-Armella (2010) note, the editing of a linear function is accomplished using very distinct types of (user) actions (and user interface elements). "Hotspots" that appear on the graph allow a student to explicitly edit the domain or range of a function through separate actions. With the editing of each variable separated, a student can see the mathematical consequences of changing one or the other (see Fig. 3).

2.3 "Ghosting"

While the idea of separate hot-spots for editing the graph representation was a core *mathematical* idea in the earliest prototypes of SimCalc software, some aspects of

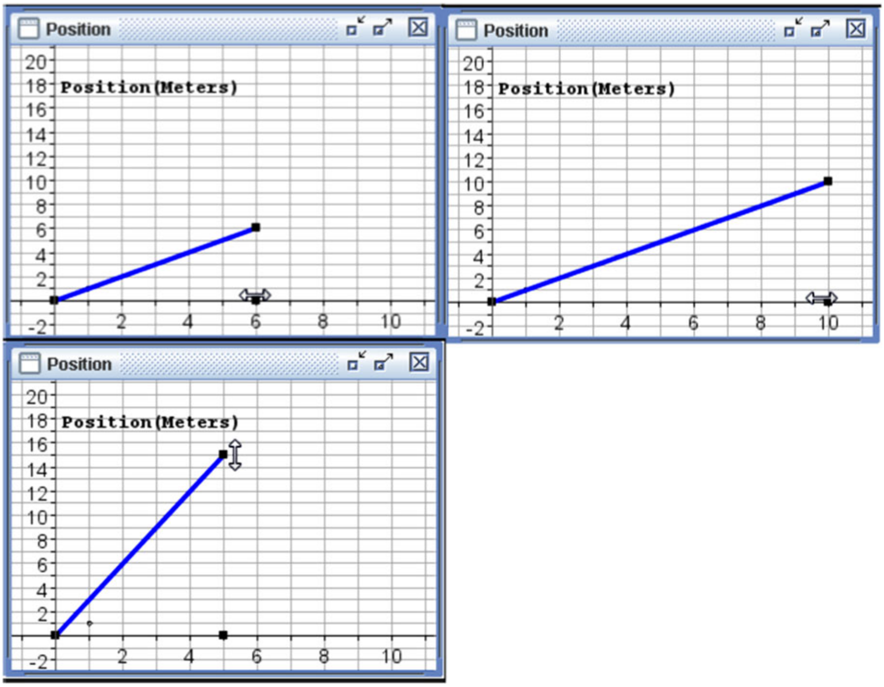

Fig. 3 This is a constant velocity position graph showing oversized hotspots used for editing the domain and range of this segment of a function. The point at (5, 0) in the *first image* can be dragged horizontally to edit domain. The point at (5, 6) in the *first image* is constrained to vertical dragging, allowing a change in the function's value at time $t = 5$, but also the slope of the function. The *other images* show the results of those two separate, possible drag actions. Note also the *cursor*, which has changed to indicate the possible drag direction

a representation developed over many iterations and was revisited in numerous discussions over extended periods of time as the software saw increasing classroom use and feedback flowed back to the project with the aim of improving student learning. One such representational detail is the "ghosting" effect of characters in the motion representation (the "world" as it is called). The idea of ghosting was created to solve the problem of using functions that were only defined over an interval or a restricted domain in an environment that animates the characters using that data. This was an important design principle that was mathematical in its intent and purpose.

In early versions of the software, it was unclear what to do with the animated characters (often referred to as "actors") when the end of a function's domain was passed during animation. Arguments can be made for a number of ways to handle this situation. A function is undefined outside of its domain, so you might represent that by having the actor appear only when the animation is within the defined time domain. The consequence of this is that actors will be appearing and disappearing while the animation clock is running. A trial of this approach proved to be confusing for students, despite its mathematically justified interpretation of "undefined."

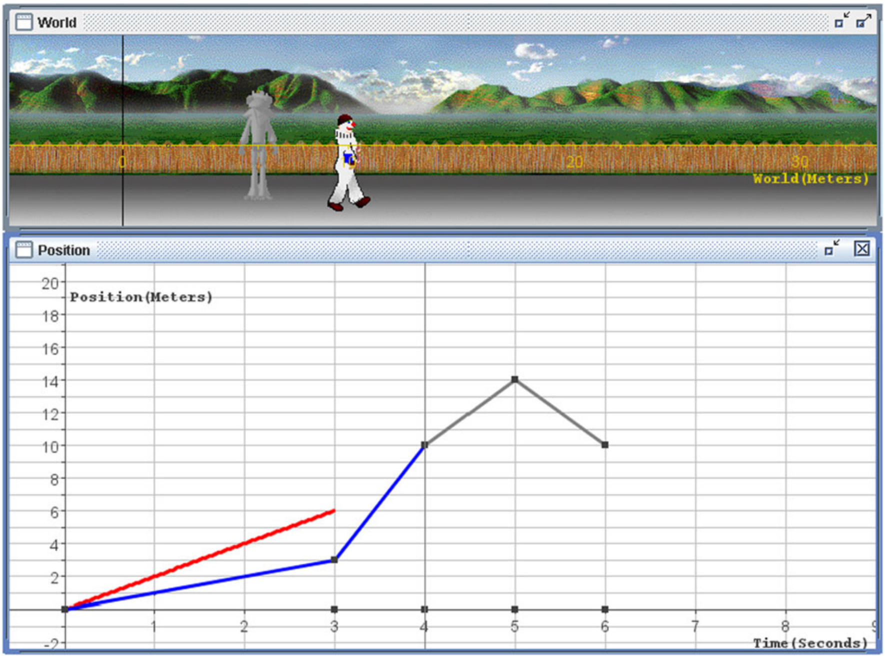

Fig. 4 An example of an actor that is "ghosted" once its domain ends at 3.0 seconds

Another approach is to have the character persist somehow. For example, the actor can simply stop at the position where its data "runs out." This simple solution is problematic because showing an actor at some position during a time when his position is undefined in the graph is inconsistent. The two representations contradict each other mathematically, and maintaining the position is arbitrary. Consider a sonar tracking system in a submarine. If you are following some sort of target and your contact with the target is interrupted, you have no current position data. It might be reasonable to assume that your target continued to move with its last known velocity instead of coincidentally stopping at its last known location. Making assumptions about missing data can depend on the context. The choice of representation should not be arbitrary.

Since many SimCalc activities involve actors getting to a specific position, students need to clearly see the end of a character's motion. This is accomplished if the actor remains at that position (rather than continue moving, as in the submarine example above). This justifies using "last known position" but does not solve the representational conflict with the graph. Our solution was to ghost the character out. Ghosting is accomplished by graying out the character's image to indicate that the data has run out, or the function of the character's motion is undefined (see Fig. 4). The additional indication allows the mathematical meaning of the end of domain to be represented even while a remnant of the actor remains as a marker; the state of being undefined is simultaneously represented while the last known position is preserved. Among the different choices of how these dynamic representations can behave, the decisions are made to be mathematically meaningful and provide the maximum visual feedback to students in ways that provide support to the goals of classes of activities.

To resolve what was both a usability problem and a problem in mathematical integrity, we looked to the structure of activities and to enriching the representation for a solution that solved both problems.

3 Activity Structures

3.1 Activities as an Organizing Principle

In the late 1990s, when the SimCalc development team was considering how to take advantage of new software environments that could cross platforms and possibly allow new ways to envision software as being composed of reconfigurable components, the notion of what a SimCalc activity was became a focal point. The "Classic MathWorlds" application (SimCalc's Mac-only product) had existed for a few short years and had the dynamic representations for which SimCalc was already known. It also supported scripting through AppleScript (Roschelle et al., 1996), allowing other researchers to alter and augment its functionality in meaningful ways, like changing the method of student input from dragging a hotspot to one that used prompts and text input boxes so the students could enter a numerical value (Olive and Lobato, 2008).

The SimCalc team began using Sun's Java environment to create applets that would be able to deliver the core SimCalc curriculum on any web browser. An applet, so-named to imply something smaller than a full application, is a program designed to run in a web browser, usually with a limited user interface specific to a particular task. A feature of applets is their somewhat isolated nature; they could be run within a browser, but the browser environment limited the applet's access to information from the user's computer. These security-driven restrictions were limiting to developers looking to connect applets. A number of applets were created based on the combinations of necessary representations (position, velocity and acceleration graphs and different animation worlds and meters). These applets were created by, literally, disassembling the curriculum units and re-grouping the activity based on what representations (Cartesian graphs, animation world, a single number-line-like meter) were needed to complete the activity. This allowed researchers to expand the activities to multiple platforms.

There were challenging limitations in this proliferation of applets, and eventually the applet approach was abandoned. However, it had forced the design team to consider activities as an important way to think about the structure of the software, and the structure of the use of the software. Central to the activity was what held it together as mathematically meaningful. For the applets, the use of specific representations was an observable surface attribute, but also reflected the mathematical structure of an activity: an activity that needed both an animation world and a velocity graph relied on the mathematical link between those representations. The intentions of a well-defined activity structure are reflected in how the software is configured for any given activity.

3.2 The Problem and Opportunity of Configurability

After SimCalc became a single application, configurability was revisited. There was also a sense that the increasing set of features of SimCalc allowed possibilities beyond one vision for implementation of a curriculum. Designers and developers of the software considered the possibility that it could become a generative environment for activity development and implementation.

The main concerns over configurability were considered in two categories of issues: the first having to do with opportunistic discovery of ideas that could be explored in the classroom and the second having to do with activity building. When working with students, Kaput would often remark that he had come across some significant, interesting, and surprising situation that resulted from students engaging in a discussion of some mathematical insight. It was natural that, as a researcher, he would want the ability to make a small change in the activity, based on his pedagogical expertise. He did this in order to provide the students opportunities to explore further, to extend the reach of a successful activity based on what the students had brought to the situation. This allowed Kaput to make observations about what was possible in the undergraduate classroom; it provided specific moments where the understanding of what was possible could be seen to co-evolve with the activity itself. This led to user-configurability becoming a central function of the SimCalc software so that researchers and teachers in the future could engage in similar actions as Kaput, both in the classroom and in preparation for a class.

An example of an early configurability feature issue under discussion was "inner windows." The applet version of SimCalc and the first desktop version presented the activity with immutable representations. For example: one position graph and an animation world. Not only would a user be unable to add a velocity graph, but the relative sizes and position of these representations were fixed. When the possibility of draggable inner windows was offered, we implemented them. This introduced configurability options with significant implications for the way an activity could unfold. With draggability came (1) re-arranging representations, (2) hiding representations, and (3) opening a new representation. Re-arranging representation allowed a teacher to do such things as place a vertical animation world next to a position graph so that students could see that a height along the y-axis corresponds to a position in the animation world. Hiding a representation allowed the teacher to create a situation in which students see one representation and discuss another one

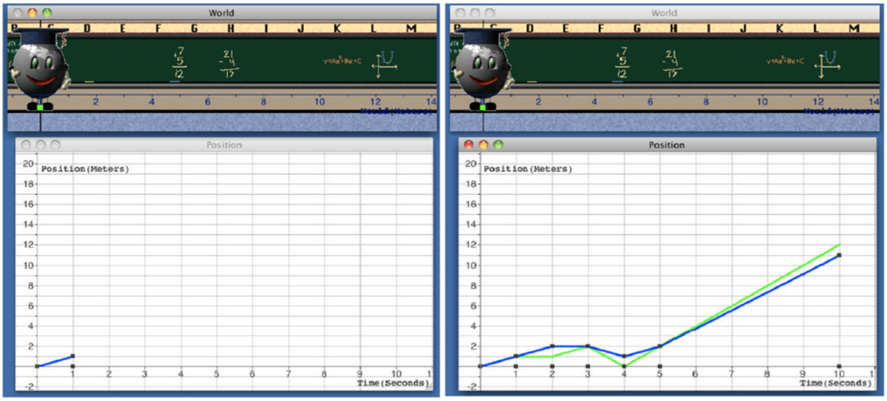

Fig. 5 On the *left* is the position graph with the (*light*) green actor's graph hidden, and, on the *right*, is the position graph where the (*light*) green actor's graph is shown along with the student-created graph, which is in (*dark*) blue. (Color figure online)

without seeing it. For instance, students watching an animation could try to describe and even argue about what the position graphs would look like. The position graph could then be revealed to spark further discussion about the predictions and encourage meaning making across multiple representations—a critical mathematical skill. Opening a new representation could allow a teacher to make a point that crosses representations to follow a student's assertion, for example. These three activity structures were enabled by a change in the control the teacher had in the configurability of the software. The second structure—hiding representations—proved to be extremely powerful. An example of this structure is illustrated in the screenshots shown in Fig. 5. On the left is the representational view at the beginning of an activity where students are asked to create a linear-piecewise position graph, shown in blue, which represents the motion of the green actor, whose graph is hidden. Once the students' graph is complete, the teacher shows the green actor's graph, as shown on the right side of Fig. 5, to begin a discussion comparing the two graphs.

Configurability brought challenges as well. The more flexibility that was built into the software, the greater the opportunity existed to mutate the activity. And while mutations can be beneficial, they can also be fatal to the intentions of the activity (Brown and Campione, 1996). Fidelity of implementation became more of an issue, and a possible argument against configurability. Configurability must recognize constraints that define the boundaries of activity within which explorations remain close to the intentions of the activity designer.

Where that boundary line is drawn was a source of recurring debate. A solution that emerged was to entrust teachers with wide latitude as authors of activities, and interpreters of an activity's intentions. Teachers were trusted to understand the intention of an activity in the belief that they are primarily focused on implementing a curriculum activity, and only secondarily interested in developing or modifying the way an activity is presented in the software.

4 Classroom Connectivity

Classroom connectivity was a major expansion of the SimCalc learning environment, connecting students and teachers through a network of computer-based devices running the software. With the advent of network connectivity, the communication infrastructure of the classroom was radically changed, as student work could now be aggregated and publicly displayed on the teacher's computer. This would impact the range of mathematical discourse across groups and individual students in the classroom. Mathematical discourse became an even more central focus for understanding how students could learn in such an environment as well as be motivated to learn more in the future as discussed further in other chapters of this book (for example, Dalton and Hegedus; and Brady, White, Davis and Hegedus, this volume) Furthermore, with these new forms of participation and communication in the classroom, new activity structures within a SimCalc classroom emerged. Not only were new activity structures designed for the affordances of connectivity, but activity structures were developed and evolved from those affordances.

4.1 Evolution of Network Connectivity Within the SimCalc Environment

The communicational infrastructure of SimCalc began as a network of graphing calculators running a pared-down version of the SimCalc computer software and evolved into a cross-platform network of personal computers running the complete, representationally-rich SimCalc environment.

The communications infrastructure emerged from efforts to use the social network within the classroom and incorporate that into the structure of an activity. Kaput had often discussed his desire to see activities that involved numerous actors in the animation world, each with a function that formed part of a family of functions that varied systematically, like a strange army whose marching revealed some mathematical variation instead of regimented uniformity. There were technical limitations that held us back from smoothly animating such an "army" of actors in the world, but those limitations lifted with time. And as Kaput's *army of animated actors* became a technical reality, the SimCalc team began to consider how students might be the ones to give that army its marching orders.

4.2 Graphing Calculators and Our Partnership with Texas Instruments

From 2000, Jim Kaput and Stephen Hegedus began to simultaneously design two versions of SimCalc software—one for the popular TI-83+ graphing calculators and a separate cross-platform computer application that extended the work discussed earlier in this chapter.

The primary concern at the time was developing something that was affordable and utilizable on a platform that was used at scale. Texas Instruments (TI) had obtained large market penetration in the sale of graphing calculators. A portable document format (APPVAR) allowed developers to create small calculator activities that could be distributed with paper activities in TI's on-line store.

This was a challenging time for the SimCalc development team, as they had to deal with multiple programming issues particularly in deploying a fairly complex representational infrastructure in a small amount of memory. Secondly, the creation of activities was done in-house due to many complexities and the lack of a simple authoring environment making scale problematic. Nevertheless, several curriculum packages were released through the TI webstore that focused on linear functions, slope as rate and averaging problems using rate graphs. The main demand was creating a compact animation that was smooth and linked to other representations including graphs and algebraic expressions. Many compromises were made due to the screen resolution and the lack of color. The development team focused on using just two actors due to such limitations that in turn constrained the kinds of activities that could be designed. But this was a time to contrast the constraints and affordances of such devices with respect to their equivalent but more expensive computer counterparts.

At the same time there was a new dawn in connectivity. People were talking more about the potential of social networks and how such potential could be introduced into classrooms. Connecting graphing calculators through a hardware and software environment was being actively discussed at TI in consultation with the SimCalc design team and other partners.

At SimCalc headquarters in Massachusetts, development in this direction was being undertaken on the desktop application. Development was extremely new and highly prototypical, with design cycles iterating every day, as new ideas for activity structures were discussed and the software development team worked on creating a communication infrastructure using simple network protocols. The SimCalc team found a local school to partner with who had a lab of networked e-Macs. The driving force behind this groundbreaking development was the search for deploying new activity structures that modified the way mathematics was thought about in classrooms and where each student could contribute something mathematically meaningful. At the time, it was unclear how much impact such insights would have on modifying the very nature of participation in the classrooms, and how this might impact learning and motivation. Aided by funding from the National Science Foundation (NSF) in 2000[1] and 2004[2], the SimCalc design team produced a network applica-

[1]PI: Kaput, J. (2000–2003). *Understanding classroom interactions among diverse, connected classroom technologies*. REC-0087771.

[2]PI: Kaput, J. & Hegedus, S. (2004–2009). *Representation, participation and teaching in connected classrooms*. REC-0337710.

tion that worked in concert with the SimCalc application on several computers and discovered a lot about what types of activities could be done under such conditions.

5 MathWorlds Server: A Glimpse into New Forms of Student Participation

The new, connected classroom prototype was designed on a push-pull model, whereby students pushed their work (i.e., constructed functions) from their SimCalc environment to a server, and the teacher would then aggregate all student contributions within her version of SimCalc. The network topology involved teachers and students working on desktop computers running SimCalc, with an additional computer dedicated to running the SimCalc MathWorlds Server application. Once a student completed a task, she would log into the server and submit her function across the network. When teachers aggregated all student work from the server, the students' representations (e.g., motion, graphs) were projected on the publically viewed display of the teacher's computer. Prior to this prototypical version of SimCalc, activities involved operating on 2–3 functions. However, with the advent of student contributions, teachers would be operating on functions to an order of magnitude based on the number of students in their class. The curriculum and software developers realized that a new level of configurability had to be built into the SimCalc environment to handle this scale up of available functions. Thus, the View Matrix was created to allow teachers to hide and show the representations contributed by students (e.g., graphs, actor motions) in their publically displayed SimCalc environment.

In 2002, the connectivity-enhanced SimCalc environment, which combined the data collection afforded by SimCalc Server and highly interactive group-based classroom activities, was piloted in a 5-week, after-school program with 7th, 8th and 9th grade students. For the first time in the development of this communicational infrastructure, students would participate in a networked classroom over an extended period of time. During this pilot intervention, the entire SimCalc design team—researchers, software developers, teachers—descended upon the after-school program, and was witness to new forms of student participation that emerged within the connected classroom (Hegedus and Penuel, 2008).

5.1 Connected MathWorlds on Multiple Platforms

In 2003, it was clear that the development trajectories of SimCalc on the TI-83Plus and on the computer were proving to be problematic. With a new burst of funding from the NSF in 2004, Kaput and Hegedus set about re-conceptualizing both software platforms within one environment, based upon TI's newly created Navigator product. Over the next few years, Navigator became a successful commercial product, and the SimCalc team worked closely with TI to develop a calculator application that could receive and send packets to the SimCalc application on the computer through semi-wireless networks, where the calculators would be connected to wireless hubs communicating via a proprietary Access Point connected to the teacher computer.

Still, major issues resulted from the use of graphing calculators in the SimCalc network—in terms of teacher usability and network stability. Teachers, on the whole, struggled with the setup of the classroom network, which involved configuring TI-Navigator hubs that were not controlled via the SimCalc software. Even when the Navigator network was properly setup, the communication between the teacher computer and the student calculators was prone to miscommunication. To increase reliability of the network, protocols were added to the software to handle various network states that arose due to this instability. For example, protocols were implemented to allow for students that had been dropped from the network to reconnect and receive all messages that were sent across the network during their absence. These protocols marked the first substantial effort to move the SimCalc environment towards a commercial-level of stability, and allowed the SimCalc team to research *Connected* SimCalc at scale during a longitudinal efficacy study in Massachusetts' high schools as reported in this book and elsewhere (see Dalton and Hegedus, this volume; Dalton et al., 2011).

By 2007, the SimCalc application for the TI-83/84Plus had been completely re-written from scratch with a brand new interface, and communication as its central core. Simultaneously, the computer version of SimCalc was re-written into a Java Application with an embedded protocol utilizing TI network infrastructure. Finally, both versions were working together and co-evolving in their development cycles. The computer version became a more parent application, which could configure and send activities to the calculator, as well as collect student work. After an aesthetic overhaul, which set the computer version of SimCalc on a potential commercialization trajectory, design challenges now shifted from specific activity functions to broader functionality, such as activity configuration across both platforms, e.g., representational control (see Sect. 5.1.1 on classroom management) and activity design that could aggregate the work of up to 32 students each working on a calculator.

Funding from the US Department of Education's Institute of Education Sciences[3] allowed the SimCalc team to conduct a large efficacy study in Massachusetts which solidified this work, allowing them to shift their development to curriculum at scale and train teachers from a wide variety of backgrounds to implement it. The SimCalc team was now in a position to implement the integrated software and curriculum project into several school systems.

Such efficacy and scale-up work was not limited to calculators though, and one great result of such work is that the co-development of a product on two platforms can improve both platforms in profoundly important ways, which would not have

[3]PI: Hegedus, S. (2007–2012). *Democratizing access to core mathematics across grades 9–12*. #R305B070430.

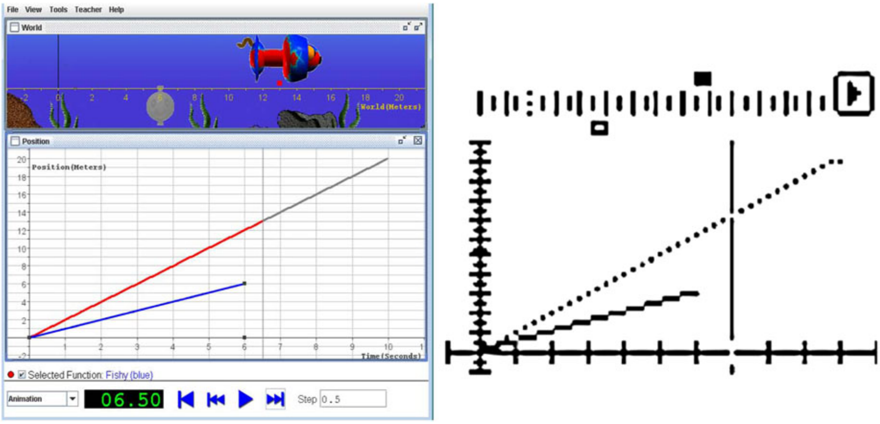

Fig. 6 Students' SimCalc environments on computers, *left*, and graphing calculators, *right*

been possible otherwise. The Java application of SimCalc was still fully functional as a stand-alone desktop application, working on both PC and Macs. This was used in a large randomized control trial in Texas (see chapters by Shecthman, Haertel, Roschelle, Knudsen and Singleton; Roschelle and Shecthman; and Vahey, Roy and Fueyo, this volume). Similarly, SimCalc on TI-83/84 calculators could work in an off-line mode if a network was not found and activities could be exported by the parent computer application for such stand-alone use.

In 2006, during a bleak period after losing Kaput, Hegedus decided to move development into a commercialization trajectory to help fuel the new Kaput Center which the University of Massachusetts had approved, and over the next 2 years both products went into a thorough testing and quality assurance phase. Many followers became part of the team to test and also to think about producing the software in multiple languages. In 2008, the first commercial version of the computer software was released with a new name that incorporated a federal trademark, SimCalc MathWorlds®, that Hegedus had applied for and obtained in the meantime and became the main tradename for all derivative products; e.g., SimCalc MathWorlds® for the TI Graphing Calculator and SimCalc MathWorlds® for the Computer (see Fig. 6).

Since 2008, the SimCalc development team has completed a networked version of the computer software, as the team saw increasing opportunities in developing countries for using smaller computers (e.g., Netbooks) in wireless networks. As such development solidified and froze its core functionality, development was focused on making the application as usable as possible by a wide variety of users in conjunction with the SimCalc curriculum materials in the future without any more external support for development. The focus, as it had been from the start, was on creating access to important mathematical ideas to many students in simple ways.

Many decisions were made in cutting features that were not deemed functional or too confusing. For example, it was decided there was no mathematical reason to constrain the use of negative time. Classroom management tools were made simpler, and network robustness improved so now over 60 SimCalc users could connect and work together—which was first trialed in Mexico—see Fig. 7.

Fig. 7 SimCalc trial in Mexico, using over 60 connected users on a single network activity

5.1.1 Classroom Management

What was central to such development in the past few years was a deep focus on enhancing learning and motivation through mathematically meaningful participation—for that management of representational affordances was critical both in the intentional design of the activities and in the enacted curriculum in the classroom.

One affordance of a connected classroom was an explosion of student data for teachers to manage during a SimCalc activity. For example, a teacher might have 25 students in her classroom who are each working on 1 or 2 functions. These functions can have multiple forms of representation. For example, the function could be animated in the world, in addition to having a graphical (position or velocity graph) and tabular representation. Therefore, given these multiple representations for each student function, within seconds one teacher might have hundreds of mathematical representations to manage—preferably in meaningful ways. The classroom management window (see Fig. 8) was developed over several years to combat such a challenge in simple and effective ways, keeping in mind that a teacher has limited time to click through a software interface during class. At the same time, the SimCalc team believed that mathematical structure should be an emergent phenomenon and the potential of the social activity space was evolving too. Therefore, it was necessary to create a system so that teachers could progressively show the work of multiple students at once in mathematically meaningful ways.

For example, the teacher could demonstrate how groups of students differed from one another in a systematic way by hiding and showing particular students' representations. Group 1 might have been using their group-number to construct an actor whose velocity was three times their group number (see Sect. 5.2 for more informa-

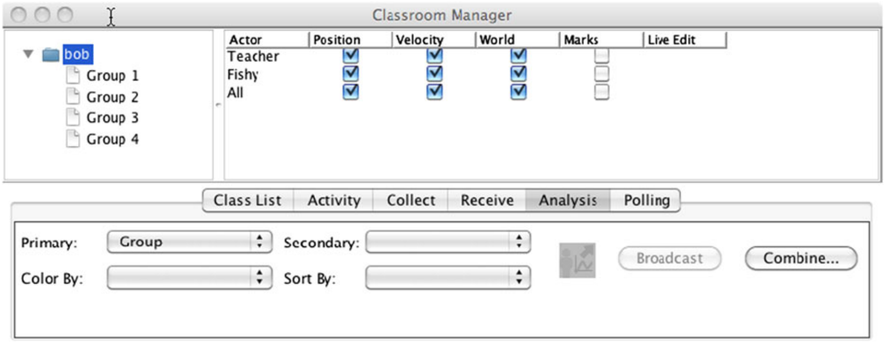

Fig. 8 The Classroom Manager interface

tion on group number). On aggregate, each group systematically varied in terms of the velocity of their actors, or the slope of their linear position graphs, e.g., $y = 3Gx$, where G is their group number. The classroom management window allowed a teacher to show or hide individual or whole group work at a click of the mouse, and progressively show other representations using a similar method, thus fusing meaning across representations. This advance in the software functionality on the desktop computer was critically important as it co-evolved with the development of the curriculum. Embedded in such curriculum were suggestions to teachers of suitable questions that proved useful in prior work for eliciting responses and building consensus within the classroom at a whole group level. The classroom management window not only allowed the teacher to do traditional network activities such as initialization and sending/collecting work, but allowed teachers the possibility for drawing attention to particular student work in progressively meaningful ways that cohered with the intentional design embedded in the curriculum.

5.2 Connectivity-Based Activity Structures

SimCalc was initially developed as a 1:1 student environment, but infusing connectivity with the dynamic, interactive features of the SimCalc learning environment created a classroom and activity space that allowed new, mathematically meaningful ways of participation for students in a mathematics classroom. From this infusion, new classroom activities exploited connectivity by designing mathematical tasks that used the natural classroom set-up of groups to offer variation in the graphs students were creating. The teacher would then aggregate the student work and choose to show all or some of their work. What followed were mathematically intense interactions among the students, and public debate within the classroom about what they saw, or where they were, in the group structure.

Emerging from this new form of student participation was an activity structure that adapted students' unique network login identifiers, which were embedded with a student's assigned group number and count-off number within the group, into a parameter within the mathematical task at hand. For example, a student might have been asked to construct a motion where their velocity was equal to their group, and their count-off number was the starting position. Thus, the students began to create linear motions, either graphically or algebraically, which were personally meaningful to them since they incorporated their login information. Teachers also used these identifiers to aggregate student contributions into a mathematically meaningful object, e.g., a family of functions. This activity structure allows for a move from a personal individual construction to a significant group structure to a class aggregation.

5.3 Innovative Tools for Formative Assessment

The SimCalc connected classroom not only allows for private, student work to be elevated to a whole-class discussion, but there is functionality that could focus students on analyzing data in the public space (i.e., teacher's display). The polling interface allows for a teacher to pose a question to the entire class that is focused on one of the linked, Cartesian representations within the SimCalc environment (i.e., a graph or the simulated world) and, from their personal device, students specify a point in the representation that they perceive as a possible solution to the question.

The teacher could pose the question to the class, "Can you show me in the simulated world at what point does Kevin's position graph intersect Jenny's position graph?" This type of questioning involves varied student solutions, and in answering the questions, students must make sense of mathematical representations they themselves did not construct. By definition of the activity, there are at least two appropriate answers to the question, because the graphs should intersect at the beginning of the simulation and at the end. However, more answers could be possible if the graphs intersect during an activity such as Sack Race. In this case, students would have to analyze the two graphs for points of intersections and then translate those points to an appropriate point in the simulated world.

6 Dissemination of SimCalc MathWorlds®

When Hegedus initiated a commercialization trajectory for SimCalc in 2006 it was the first step towards an at-scale diffusion of the software and materials. Prior to this, disseminating SimCalc for independent implementation (i.e., without influence from the SimCalc team) was limited by the software environments lack of configurability by those outside the SimCalc team. However, as part of Hegedus' commercialization effort, the SimCalc software development team created a robust authoring environment, and added the ability for it to be implemented in four languages. These additions to the software environment have allowed SimCalc to now be independently adapted and implemented in a number of countries, including research projects in Mexico and Greece. In order to allow true internationalization for the purpose of wider dissemination, the SimCalc team redesigned the localization support within the software, making it possible to "plug in" additional languages. Working together with some SimCalc personnel, new target language resource files could be created and plugged in, with greatly reduced technical intervention. This new ability allowed for the multilingual SimCalc software to extend beyond the development lifecycle.

7 Future Perspective

SimCalc software development existed in an environment of many influences. Cross-disciplinary tensions among software developers, researchers, and activity developers sparked discussion and innovation, sometimes in the smallest of details. However, a consistent focus on the mathematical structure guided the decisions underlying the representational infrastructure, the need for configurability to produce generative activity structures, and new forms of participation driven by a communicational infrastructure that formats the classroom according to the mathematical intentions of activities.

The activity structures that emerge out of the communicational and configuration affordances of the software extend the life of SimCalc beyond its development cycle. Development of SimCalc was finalized recently, making it important to establish a version that was stable and sustainable to be used by existing users and a wider variety of students and teachers around the world in the foreseeable future. The final authoring system allows teachers to strip away any part of the menu system to minimize the actions available to a student, enabling a more focused activity system and simpler interface.

It was also important for us to establish the software to be used in multiple languages and to date it is fully functional in English, French, Spanish, Portuguese, and Greek. While the software is commercially available, the curriculum that was simultaneously developed over a similar amount of time is freely available from the Kaput Center and can be adapted and distributed under a Creative Commons license. It was imperative for the dissemination of a learning environment that was made possible by several federal grants over 18 years, to be available, usable and configurable by many more researchers, teachers and students in the future.

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Designing for Generative Activities: Expanding Spaces for Learning and Teaching

Nancy Ares

1 Introduction

I address design principles guiding SimCalc's development and implementation in a range of classroom and school contexts with particular attention to generative activity. Expansion of content, opportunities to participate, and avenues for students to draw on the varied resources they bring to learning are explored and further theorized given research findings from SimCalc and other potentially generative network-supported activities. While the topics and concepts addressed in this chapter are germane to examining generativity and equity in a family of networked classroom technologies, I pay particular attention to SimCalc. As such, SimCalc provides what can be referred to as a "supporting space" for this exploration. I include explicit attention to issues of equity in generative activities and in SimCalc's goal of democratizing access to rigorous mathematics. Importantly,

> the promise of educational software is not teaching efficiency, but fundamentally altering the curriculum. Students learning only 19th century mathematics will not thrive as 21st century citizens. Educational technology must enable *more students* to engage in more sophisticated subject matter at a younger age. In ongoing field tests at all grade levels and in remedial college classes, we find that SimCalc helps students who would otherwise never reach, much less pass, a conventional calculus course" (Kaput and Roschelle, 1996, p. 97; emphasis added).

Two issues are key in the above quote. First, altering curriculum is not simply drawing on the technological affordances of connectivity supported by SimCalc and related networks. As Bu et al. (2011) note, "The interactive nature of new technologies further support and constrain the co-actions between learners and the target system (Moreno-Armella and Hegedus, 2009)" (p. 19), where dynamic, linked

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mathematical objects operate at the border between representation and communication. Students are thus able to use the "interplay between these communicational and representational affordances [to engage in] *representational expressivity*, which enables the users to express themselves through speech acts and physical actions" (Sollervall, 2011, p. 5).

Second, and of particular importance for this chapter, the attention to *more* students is critical. Comprehensive work toward equity in mathematics learning is not only about advocating for greater numbers of students to have access to SimCalc and other networks' powerful connectivity. Questions about efficacy of these systems must be answered in relation to the demographic diversity of students involved and the social, cultural resources available to students and teachers in networked classroom learning. Beyond transformation of curriculum, research findings demonstrate that particular features of connected classrooms are important mediators of activity that scaffolds, invites, and leverages the power of groups to construct powerful mathematical learning and communication (Ares, 2008; Hegedus and Penuel, 2008; Stroup et al., 2005; Vahey et al., 2007). There is not only what Moreno-Armella et al. (2008) term a mediation space involving "co-action [where] a user can guide and/or simultaneously be guided by a dynamic software environment" (p. 102). There is also opportunity to acknowledge the heterogeneity within groups of users (i.e., linguistic, cultural, gender, ideas). This heterogeneity can be a source of important resources (rather than barriers) for teaching and learning that harness the affordances of SimCalc's mediating affects as a tool-in-use (Wertsch, 1995) that participates in the creation of such spaces.

2 What Is Generativity?

I use the term generative to mean, "orchestrating classroom activity in ways that occasion productive and expressive engagement by participants, characterized by increased personal and collective agency" (Stroup et al., 2005, p. 188). The attention to agency in both individual and group arenas of activity lends itself well to explicit attention to issues of equity in mathematics classrooms, given that such learning environments are populated by individuals working together (or at odds with each other) to accomplish learning goals. Efforts among many researchers (see Kaput and Hegedus, 2002; Stroup et al., 2002, 2005) are directed at extending and reconceptualizing earlier analyses of generativity (Lesh et al., 2000; Wittrock, 1974) that focused mostly on individual or small group learning (e.g., dyads) to consider how to design for the variety and multiplicity of learners' ideas and insights. These reconceptualizations aim to support the emergence and development of mathematical reasoning in whole class and large group contexts. This approach to generativity rests on the idea that connected classrooms include multiple agents participating at richly interactive levels of engagement and agency. Attending to the diversity of identities, the plurality of conceptual structures, the shifting of collective understandings, and the evolution of disciplinary concepts found in classroom activity deepens our insights into the emergence and development of ideas. Simultaneously, it also illuminates ways designing for SimCalc and related interventions can best support the advancement of mathematical and scientific thinking for all our students (Stroup et al., 2005).

3 Affordances of SimCalc and Other Networks for Generative Design

Design features of next-generation networked systems typically include individual devices (e.g., calculators, computers) that support peer-to-peer, peer-to-group, or group-to-group communication and sharing of mathematical objects; connectivity among devices via the internet or computer servers; portability; a core set of functionality in each device (e.g., at least that of a graphing calculator), and a mixture of public and private display spaces. The public space can be a computer projection system as in SimCalc (Kaput et al., 2002) and HubNet and Participatory Simulations (Wilensky and Stroup, 1999). The private space can be the students' own individual displays on a calculator or computer. The network allows teachers, students, or others to create new activities or change the flow of a given activity. Participants can exchange both group and individual artifacts, including text, strings, numeric values, ordered pairs, lists, matrices, individual and whole-class graphs, images, sounds or video. Given the blending of public and private spaces, students and teachers act and interact in that blended space. As a result, "(meanings) are *dynamic*—they grow and transform with the shared use of symbols. . . .Emergent meanings come to light because of the new links among symbols" (Moreno-Armella and Hegedus, p. 29, this volume).

Earlier work suggests that there is a kind of "resonance" between technological affordance and generative forms of teaching and learning: "SimCalc MathWorlds® lends itself to investigations involving piecewise constant and linearly changing functions. A major design feature of the software and associated activities is the potential to tap into students' real-world intuitions, experiences, and understandings about speed and motion (especially those that involve descriptions and notations involving velocities, positions, and times)" (Schorr and Goldin, 2008, p. 137). Kaput's (1998) work on inert versus dynamic artifacts and students' acting on representations is helpful here to pinpoint unique affordances of SimCalc's public and private spaces. Traditionally in many math classrooms, the static notational system embodied in calculators is used to act on the textbook- or teacher-provided representations "to reason about and make sense of [the] situation" (Kaput, 1998, p. 258). He argued that this action on static notation sets up a one-way relationship between mathematics and experience, denying students any control of either. As Noble et al. (2001) note, "Dienes emphasized the importance of imagery and storytelling in mathematics, stating that 'symbol-manipulation in mathematics is all too often meaningless simply because there is no corresponding transformation of images' (1964, p. 105)" (p. 87). In SimCalc and related systems' activities, the dynamic notational system embodied in the calculators involves varying mathematical representations and allows students to act on:

> …such motion phenomena, e.g., velocities, positions, times and combinations of these in graphs… [that] are not only modeled by the notations that describe them, they can be *controlled* by those notations. …These kinds of affordances turn a fundamental representational relationship between mathematics and experience from one-way to bi-directions. (Kaput, 1998, as cited in Ares, 2008, p. 5)

The same can be said of the public spaces, where a group's collective and coordinated efforts to control phenomena via dynamic notational systems are displayed for all to see, fostering understanding and sharing of experiences, strategies, and use of mathematical artifacts constructed by the group and manipulated in response to their individual and collective insights.

3.1 Networks as Mediating Artifacts

Examining the ways that SimCalc supports generativity in design and activity lends itself to viewing these technologies as mediating artifacts (Vygotsky, 1987; Wertsch, 1995). Human activity and learning are profoundly influenced, or mediated, by the use of psychological and physical tools (e.g., language, computers) (cf., Cole and Engeström, 1993; Vygotsky, 1987; Wertsch, 1995). A sociocultural perspective pays attention to artifacts-in-use: "the agent of mediated action is seen as the individual or individuals acting in conjunction with mediational means" (Wertsch, 1995, p. 33). Cultural tools are never "mere" artifacts because, by virtue of people's use of them in service of achieving a goal, they inevitably shape the activity by influencing the means by which goals are achieved (Cole, 1996). Artifacts—like computers and calculators—can be examined for their mediating role in human activity and interaction, including the ways that generative designs can expand social spaces for teaching and learning to foster more equitable access to powerful mathematical tools and discourses. Furthermore, it is through such mediation that symbolic reasoning and mathematical activity are fostered.

To achieve the goal of democratizing access to rigorous mathematics learning, SimCalc builds on three lines of innovation: restructuring the subject matter; grounding mathematical experience in students' existing understandings; and providing dynamic representations (Vahey et al., 2007, p. 15). Generativity is a function of the affordances of these technologies, the nature of the activities, and the range of resources and practices invited in. As shown below, possibilities for addressing issues of equity in mathematics learning are numerous due to the affordances of SimCalc and related networked technologies; and the social spaces for learning that can be constructed, drawing on the diversity of expressive forms and invitations to participate and contribute (language, gesture, interaction and communication patterns) that are inherent in mathematics classrooms. Importantly, related to Hegedus and Moreno-Armella's (2009) notion of mediation space,

> . . .these activities are designed to be generative in that, "Learners create a space—or co-ordinated collection—of expressive artifacts and actions in relation to some shared task or set of rules. The structures that are created. . . are not determined in advance but are co-constructed by learners as their sense-making evolves and develops" (Stroup et al., 2004, p. 1403)

4 Expanded Social Spaces for Contribution and Participation

Recent theorizing about space, including classroom spaces, conceptualizes it as being socially constructed rather than "static as in classroom space with its requisite desks, tables, chairs, etc., staying largely unchanged over decades, particularly in under-resourced schools serving non-dominant students." Soja (1996), Harvey (1973/2009) and de Certeau (1997), argue that social space is actually dynamic and mutable. Regarding SimCalc classrooms and their learning goals, Noble et al. (2001) posit that

> If the purpose of a mathematical activity is not simply to learn the rules and conventions of an environment but instead is to make the environment a lived-in space for oneself, then a diverse range of actions and intentions may be legitimate parts of that mathematical activity. (p. 88)

Viewing space as a social construction and one that students can shape gives us new ways to understand the mediating role of networked classroom technologies. Hegedus and Penuel (2008) portray SimCalc and related technologies as involving

> Mathematical Performances. . . [that] emphasize individual student creations, small group constructions, or constructions that involve coordinated interactions across groups. . . and Participatory Aggregation to a Common Public Display. . . [that] involve systematic variation. . . displayed and discussed to reveal patterns, elicit generalizations, expose or contextualize special cases, and help raise student attention from individual objects to families of objects. (p. 173).

The combination of performance and participatory aggregations that leverage group contribution and participation link mathematics learning and social structures of networked classrooms in ways that can invite a diversity of expressive forms (e.g., language, gesture, mathematical objects) into the activity of learning. The focus on collective activities opens the space of participation and contribution in both the mathematics and the participation structures involved.

Students' interactions, contributions and participation in SimCalc-type activities can be seen to involve "Space-creating play. . . a central feature of generative instructional designs" (Stroup et al., 2004, p. 840). Indeed, in creating social spaces of networked classrooms:

> . . .what distinguishes playing along from playing a meaningful part includes, in some sense, the size of the space the students can explore. Playing along invokes a sense of constraint and limited possibility. . . .Playing a part, on the other hand, involves one's own explorations being juxtaposed to others', to the group's evolving notion of the domain, and to the more formalized insights of the dynamic communities of science and mathematics. (Stroup et al., 2004, p. 840)

It is in these opened, expanded spaces for performance that invitations to bring to bear resources that are often denigrated, untapped, or kept invisible are found. Students' making lived-in spaces infers ownership, creativity, safety, and familiarity—dimensions of mathematics learning not often found in under-resourced schools (Ladson-Billings, 1997).

4.1 Varieties of Contributions

Multiple modalities for interaction are available in network-mediated activity, including text, physical and electronic gestures, as well as verbal contributions to classroom dialogue (e.g., conjecture, prediction, observation, and explanation). Moreover, the collaborative character of participation in those modes of contribution invites multiple ways of belonging, as students have access to a variety of representations of phenomena (text, graphs, visual displays of emergent systems, language) and engage in inquiry-oriented discussion and analyses (Ares and Stroup, 2004). Together, the varied modes of participation and joint construction of knowledge mean there is unique potential in networked classroom technologies to draw on students' cultural and social practices to support learning in mathematics and science.

5 Expanding Spaces for Learning via Language, Interaction, and Communication

Lee and Fradd (1998) noted that communication patterns vary across cultural groups and that "students from diverse language backgrounds often have different interpretations of verbal communication and paralinguistic expression… alternative communication patterns can provide… students with powerful ways of demonstrating their knowledge and understanding" (p. 17). For instance, during a PartSims networked activity involving integers, rate, and slope, two Latinas collaborated in Spanish throughout the simulation, sitting next to two European American boys who did their own work and then compared their results in English (Stroup et al., 2005). The goals of activity were successfully reached in both pairs' interactions, expanding the important and appropriate ways of participating seen in more conventional teaching. In addition to language varieties (e.g., Spanish and English) serving as cultural resources, choosing to collaborate or working independently may also have had gendered or cultural roots. The structuring of their activity occurred through the students' use of individual and collective social, cultural, and academic resources. Thus, being able to draw on varied ways of participating made good use of important resources the students brought to the task. Such resources include language as a resource, as well as norms for a variety of classroom interactions; generative designs have central features that can be important in leveraging these resources in service of rigorous mathematical learning.

5.1 Norms for Communication, Interaction Patterns

Communal, collective efforts characterize classroom interactions supported by generative designs. SimCalc's classroom activities (i.e., Sack Race, Spreading Apart) and other networks' curricula require groups' coordinated efforts to construct and analyze mathematical objects and relations together. (See Brady, White, Davis, and Hegedus, this volume, for more curricular examples.) Classroom participation structures are transformed from the conventional focus in schools on individualistic/competitive practices (Boykin, 1986; Boykin and Ellison, 1995; Boykin et al., 2005). Culturally relevant and funds of knowledge pedagogies also feature coordination and co-construction as strategies to leverage communities' cultural practices, with "teachers encourag[ing] a community of learners rather than competitive individual achievement" (Ladson-Billings, 1997, p. 480). Potential for explicit attention to expanding engagement in powerful mathematics to include non-dominant communities often excluded in school mathematics is seen in such practices as the collective construction of stories in native Hawaiian communities (Au, 1980), *confianza* (mutual trust) and networks of relations in Mexican communities in Tucson, AZ, (Moll and Greenberg, 1990; González et al., 2001), and call and response traditions in African American churches in Chicago (Moss, 1994). Co-construction and coordination are central features of practices these communities have developed over time and in particular social, cultural, historical contexts. These features are also central to mathematical activities in SimCalc, e.g., in the Fans activity, group members must coordinate their work with their individual slope and intercept assignments to co-construct a shared understanding of slope and intercept in linear functions (Brady et al., this volume). Also, Boykin and colleagues (1986, 1995, 2005) provide extensive documentation of the role of communalism in urban African American communities. Flores (1993) traces similar practices in Puerto Rican communities in the U.S., citing the close proximity both geographically and in terms of social positioning in the U.S. of African Americans and Puerto Ricans as influences that foster cross-group influences on cultural practices. These potential congruities between network-mediated activity and under-served communities' cultural practices of communities are critical to exploit for generative designs to be truly transformative, democratic, and inclusive.

Schorr and Goldin's (2008) work on the importance of respect in SimCalc classroom activities they conducted in Newark, NJ is germane here, too, as the students involved in their work come from non-dominant, often dismissed or marginalized communities. As they point out:

> To a greater extent than in suburban environments, the interactions between urban teachers and students have been described as involving admonishment and criticism (Hart and Risley, 1995, as cited in Pogrow, 2004). The implicit message to students may be that the teacher neither respects them nor believes that they have the ability to engage with and learn mathematics, with devastating impact on their emotions, attitudes, beliefs and personal values. . . .Dance (2002), Anderson (1999), and others likewise stress the centrality of respect specifically in urban schools, where many students do not trust teachers or the classroom environments that teachers promote. (p. 134)

Norms for interaction and for the relationships among mathematics learning and

> emotions, attitudes, beliefs and values associated with intense, engaged, and vulnerable interaction in doing mathematics, ... have special relevance to our work with urban students. For example, respect is important as it pertains to an individual student's personal identity—to feelings of being valued, believed in, looked up to, or accepted by others in the mathematics classroom. (Schorr and Goldin, 2008, p. 135)

Their exploration of interactions and communication patterns in these classrooms illustrate how the participation structures that characterize SimCalc classrooms involve not only social and cultural resources (e.g., language, collaboration) but also provide space and opportunity for students to experience interactions that can help to ameliorate the limitations of the social and cultural contexts of schooling found too often in urban settings.

5.2 Linguistic Resources

Lee (2003) calls on designers explicitly to draw on community-based norms for discourse.

> ...norms for who can talk, how, when, and about what help to construct roles for participants to play. Lack of congruity with community-based norms for talk (including use of different national languages—such as Spanish; language varieties—such as African-American English Vernacular [AAVE]; or registers) has been shown to result in lack of participation in classroom talk. (Lee, 2003, p. 47)

Importantly, while classroom technologies themselves cannot open up the kinds of language use and interaction patterns invited into learning activities, features of their design and use can be examined for the potential to treat community-based linguistic resources as legitimate and powerful resources. As Hegedus et al. (2007) note:

> This flow of communication has restructured the social space of the classroom. Each student's individual contribution is a piece of the whole. ...Students are co-acting with the software environment but also with each other... The technology becomes a partner with the teacher at a public level to support the emergence of ideas, support or refute conjectures made by students and guides, as well as be guided by, the software at a local, personal level, as students interact and explore dynamic links between graphs, simulations, and each other's thoughts. (p. 1419)

Co-action involves a multitude of resources that can be important avenues for participation and contribution. Examination of the types and function of such fluid action and reaction cycles characteristic of SimCalc and other networked technologies illuminates the nature of the evolving relations among the cultural resources students bring to bear in classroom learning, uses and roles of tools that shape activity and interaction, and the emergent outcomes of those activities. As Lee (2003) notes, "With the new opportunities for forms of representation and communication afforded by new computer-based technologies, it may well be quite useful for designers to consider the implications of this work for communication opportunities within computer-based environments" (p. 44). In the excerpt below from a Gridlock Participatory Simulation (Wilensky and Stroup, 1999), students were using their individual calculators to control a traffic light in a traffic grid; their task was to coordinate their efforts to maximize traffic flow as measured by average speed of cars, number of stopped cars, and average wait time (see http://ccl.northwestern.edu/ps/guide/ for detailed description of this activity). The activities and interaction are similar to SimCalc in that individual students control some feature of the simulation and their collective efforts are displayed in public space to serve as the focus of analysis. Evidence of such opportunity is captured in the following extended field note excerpt:

David: Somebody messin' me up here.
Moniek: Who's 11? (intersections were designated by numbers)
Alicia: 18, gotta move.
Moniek: 19, change to a different stop light.
David: Well, I ain't movin' at all. I'm mad.
Leroy: Can't see those numbers.
David: Thank you, thank you, God. I'm finally moving.
. . .
Sharee: There we go... get through... I put that red light... oooh... change... I'm gonna change 7 thru 9 ... 15, 15, 15 ..." I'm gitting backed up. Number 14... let my people go through please... Thank you... go... go... go... go... yeah go... stay there... go... go... number 2, number 5. I'm good. Nobody complaining about me. I wonder why? I'm just good. I was gitting it, did you all see me... did we beat the freshman yet? If everyone here was like me, we'd be phat.[1]

The use of informal language to manage the groups' efforts relied on what Lee (2003), citing Smitherman (1977), identifies as features of African American English: "verbal inventiveness, unique nomenclature, rhythmic, dramatic evocative language, sermonic tone, [and] cultural references" (p. 54). Potential for changing norms is seen above in talk often associated with African American churches (Thank you, thank you, God; let my people go) and youth culture (I ain't movin'; we'd be phat). While the language and talk invited in to generative, connected mathematical activities may be distinctive, the goal of engaging students in rigorous mathematical discourses and practices remains, strengthened by the inclusion of varieties of learning resources available in classrooms serving non-dominant students and communities.

5.3 Identity Within a Collective

In generative design as mediated by group supportive networking, individualization is associated with seeing the uniqueness and diversity of each student's participation as making an essential contribution to the emergent sense-making taking place in the classroom.

[1]This extended excerpt is also included in Ares, 2008, p. 32.

> Students construct parts of a mathematical whole and so the focus of their attention is on the relations between their individual contribution and the whole. Thus, students' personal identities are intimately involved in their building and sharing of mathematical objects in the public space of the classroom. (Hegedus and Kaput, 2003, p. 4).

Further, given their contribution to and participation in the construction and analysis of mathematical objects, not only are their personal identities involved (individual intersections were labeled with 3-letter names), their collective sense of themselves as a coordinated group is heightened:

> The students' coordination of effort was supported by the combination of interacting with the dynamic interactive medium… communication among individuals, and individual and group strategies. The comments, "WE need HHH to change" and "look at OUR graphs," indicate a group ownership of the activity. (Ares, 2008, pp. 20–21)

Returning to the notion of making lived-in mathematical spaces, students' identifying themselves as contributors to social space and as members of a collective social and mathematical space seems important in not only expanding participation and interaction, but also in personalizing those spaces.

6 Discussion: Freedom to Participate in Democratizing Access

Diversity in the ways that students from a variety of backgrounds and cultural groups are invited to and can participate and contribute is, as has been shown, central to efforts to pursue equity via generative design of networked classrooms. Freedom to participate is critically important in terms of inviting students to be themselves, to not feel constrained to be a certain kind of person. Students in one study indicated that, during the networked activities, they felt able to relax, "because really you know, when you're doing the technology you're not really worried about it [surveillance] because it's like your time to do the technology piece… and we'll be talking like we're going home" (Ares, 2008, p. 34). Colloquial vocabulary, overlapping speech, pacing of talk, and playful use of language are characteristic of the ways many researchers have observed youth interacting in network mediated activity (Ares, 2008; Ares et al., 2009; Schorr and Goldin, 2008). These observations point to the potential to respond to Lee's (2003) call to "designers explicitly to draw on community-based norms for discourse… as anchors for instruction" (p. 47). Generative designs in networked classroom support expansion of norms for classroom discourse, which may support the construction of learning environments where school-based discourses and those of youths'

> social worlds [are] blended, making the boundaries between these worlds porous and movement between these worlds fluid. It is in this new discourse space that new forms of participation [are] legitimized, thereby extending the repertoire of resources accessible to all students. (Barton, 2007, p. 24, as cited in Ares, 2008, p. 34)

Clearly, generative design in SimCalc and related systems has the potential to support expanded access to powerful mathematics to include students and communities who have been attended to in terms of conducting implementation and design research. It also has potential to support teachers and researchers' explicit attention to and recognition of the cultural and social resources available but too-often ignored in under-resourced schools and classrooms. The goal of this chapter is to further SimCalc's commitment to democratizing access to the mathematics of change and variation, to calculus, and to enjoyment of and identification with mathematics among a broader group of students than has been the case in US schools over many, many years. These commitments are critical to 21st century learning and citizenship.

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SimCalc and the Networked Classroom

Corey Brady, Tobin White, Sarah Davis, and Stephen Hegedus

1 Introduction

Over the past fifteen years, SimCalc and other research projects have attended to emerging possibilities for research into student thinking and learning presented by the combined representation and communications infrastructure (Hegedus and Moreno-Armella, 2009) of classroom network technologies. The purpose of this chapter is to describe design principles, which serve to guide work in classroom connectivity: both as they have appeared in work within SimCalc itself and in several independent lines of inquiry among researchers in the Kaput Center network.

We begin by outlining the common architecture of the classroom networks involved in our design work. Next, we discuss five major activity structures that have emerged: three that have been instantiated in SimCalc, and two that we see as theoretically complementary. Finally, we reflect on fundamental developments in the study of group-centered learning that have been enabled by this research.

In order to gauge the depth of influence of the classroom network upon research on group learning, we conduct this discussion through the lens of Roschelle and

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Teasley's (1995) seminal work on computer-mediated collaborative problem solving. We choose this work as representing the state of the art at a moment shortly *before* our field's engagement with the classroom network. Roschelle and Teasley's analytic framework provides a toolkit for examining the phenomena of student interactions in the presence of powerful and carefully designed *representation* infrastructures offered by the computer; however, it does not contemplate the effects of a compatible *communications* infrastructure. Therefore, by considering the fit of Roschelle and Teasley's framework to the phenomena observed within the network designs we discuss, we can assess both lines of continuity and points of contrast between a learning paradigm based on a small group of (two or three) students gathered around a computer and the variety of activity structures for large and small group learning enabled in the networked classroom.

2 Overview of the Networked Classroom

For the purposes of this chapter, we consider a classroom network to be comprised of hardware, software, and a relation to curricular content. At the hardware level, it consists of computing devices (often handhelds), distributed in the classroom on a one-to-one basis (i.e., every student has a device). These devices communicate (usually wirelessly) with the teacher's computer, which is connected to a public display (usually a projector). At the software level, the student devices are programmed to provide domain- and task-specific activity interfaces that allow them to participate in mathematically significant ways. The software on the teacher computer manages communications coming from the student devices, routing these messages appropriately and/or aggregating them to form a public display of the collective activity of the classroom group. At the level of content, materials are sent to students either electronically on their devices or by other means, to structure classroom activity in a particular curricular context. Figure 1 shows a diagram of these system components.

On top of this general-purpose network architecture, a wide range of specific collaborative interactions can be designed. Indeed, flexibility and openness are critical features of the architecture, as different types of activities make fundamentally different demands of the underlying network. To give an initial sense of the scope of this design space, we here identify four dimensions of variation in these collaborative interactions. These dimensions also correspond to degrees of flexibility desirable in the underlying network.

A first dimension concerns the nature of the student devices and their activity-specific software interfaces. In some activities, a fairly elaborate private workspace is needed, where the student can independently build contributions that are later sent to the teacher. In other activities, a "thin client" that has no offline capabilities can be used to give students real-time remote control of specific elements in the upfront space. A second dimension concerns the flow of information to the teacher computer and between the teacher computer and the public display. In some activities, the messages must be transmitted immediately and continuously from the student devices to the public display, so that the aggregate can emerge reflexively—i.e., simultaneously revealing and influencing the students' ongoing work. In other

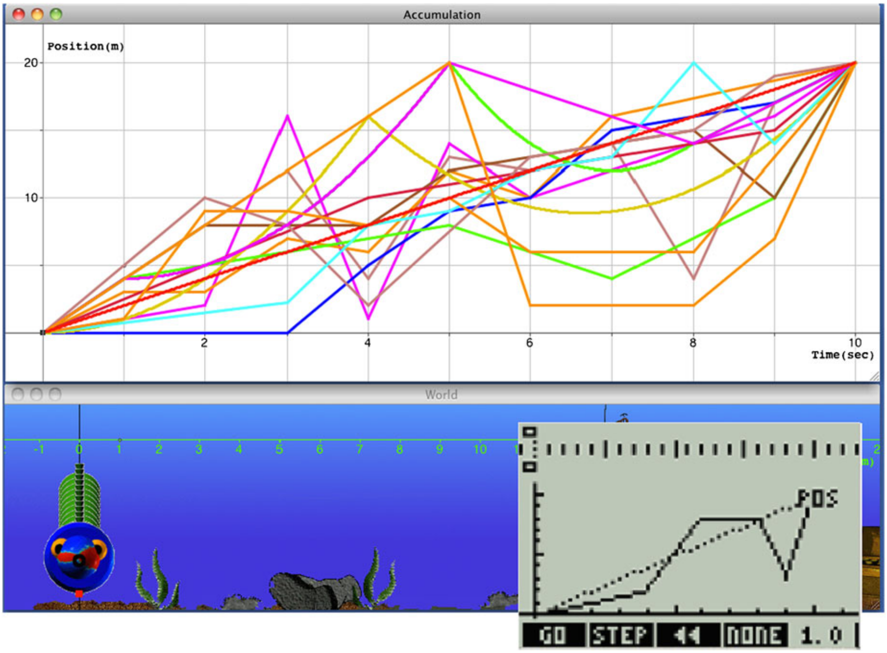

Fig. 1 Public and private spaces in SimCalc MathWorlds®. The public space runs on the teacher's computer and can be projected. The private space (running here on a graphing calculator) allows the student to prepare and test their work before contributing

designs, student contributions are displayed all at once, so that while the real-time rendering of contributions is not as critical, the network may need to support the transfer of "heavyweight" artifacts that reflect more elaborate student constructions. A third dimension deals with the traceability of mathematical objects to their student authors. In some activity designs, it is important that the contributions are anonymous in the group space, as this allows certain kinds of freedom in group discussion; in other designs, the public display should allow contributions to be traced to the groups, or even the individuals that produced them, through features such as color or through mathematical aspects that are mapped to activity roles. Finally, a fourth dimension concerns the mathematical representation of contributed objects: in some activities, student contributions are presented in the public display as they appear in the student workspace; in others, the representational form of these contributions must be dynamically manipulable by the teacher (e.g., algebraic equations sent by students might be represented by their graphs in the public display).

3 Five New Activity Structures

In this section, we describe five important *activity structures* that have emerged in the last 15 years, within the broad network design space we have indicated. We define an activity structure as a generative category of task design that specifies patterns in the use of available social and technological infrastructures to organize the actions and interactions of participants. An activity structure thus represents a *genre* of interaction, and many concrete activities can be built upon the pattern of a given activity structure.

We distinguish this notion from that of a *participation structure,* which reflects patterns of actual practice in the implementation of activities. Participation structures are thus features of discursive communities; they interact with activity structures to determine concrete features of the implementation of an activity in a given context, as the social group perceives and/or makes active use of different affordances of the task design or of available infrastructures.

The five activity structures we discuss are:

- Mathematical Performances
- Participatory Aggregation
- Generative Activities
- Small Groups
- Participatory Simulations

The first two of these have been incorporated deeply into the SimCalc curriculum and implemented at scale. The third comes from a research program that has intersected richly with SimCalc but has not been incorporated systematically into the materials used in SimCalc's large-scale experimental studies. And the last two have been pursued by colleagues in the Kaput Center network, leading to productive and influential design discussions. These five activity structures are also ordered here to produce a trajectory through the multi-dimensional design space sketched above. We hope that this serves both to illuminate this design space further and to indicate connections between the activity structures.

3.1 Mathematical Performances

This activity structure involves students as authors of personally meaningful mathematical objects and as creators of coordinated multiple representations of these objects, including narrative descriptions. One example activity in this category is *Sack Race*,[1] where students create three distinct representations of a motion phenomenon: a piecewise-linear position-time graph, an on-screen animation, and a story. In the private workspace on their individual device, each student creates a piecewise-linear position-time graph that drives the animated motion of their actor. In parallel, they draft a story that provides a third representation of their actor's motion. In building their motion, the only constraint is that it must end the race in a tie with the teacher's actor, who travels at a constant rate of 2 feet/s for the entire

[1]Download the *Sack Race* activity software/curriculum documents at: http://www.kaputcenter.umassd.edu/products/curriculum_new/algebra1/units/unit2/.

I was in a race with actor A today. He kept a steady pace at 2 ft per second for the entire race. The race was to 20 ft. I fell behind at a velocity of 4 ft per 5 sec for the first five seconds. I then got a burst of speed and went 12 ft per sec for one second. After that I had an asthma attack and went 0 ft per sec for two seconds while I took my medicine. When Actor A caught up with me I matched his speed of 2 ft per sec for the final two seconds of the race. Therefore, we tied in the race. We both went a total of 20 feet in ten seconds.

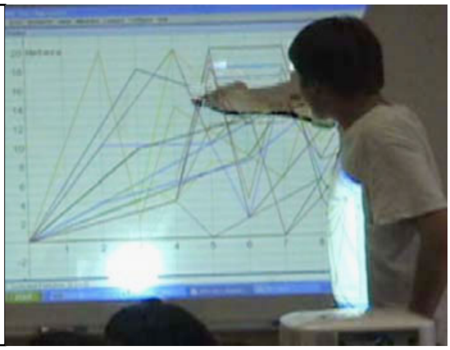

Fig. 2 One student's race narrative (*left*). At *right*, the student-author is at the board attempting to identify his own graph among the aggregate

20-foot race. The teacher-actor's graph and animation are displayed in the students' private workspace to allow direct comparison with this "target." After the students complete their motions, they are uploaded to the teacher computer, and the entire collection is displayed simultaneously. The collective race can then be run, and the teacher can select individual students to narrate their actor's race by reading the stories they have written while the animation is played. Figure 2 gives a glimpse of this activity.

In our first implementations of this activity, spontaneous applause occurred when a student's race was shown and narrated. Over time, this has become an expected response, which alerts us to deeply "theatrical" aspects of the activity: not only is there a "performative" element to the creation and sharing of these artifacts, but the class coheres as an authentic and appreciative audience.

Numerous variations and extensions of *Sack Race* are possible. Some involve adding requirements to the motion-making task. For instance, the actors might be required to arrive at a given point at a specific moment (e.g., 3 m when $t = 4$ s). Or actors might be required to run at the same speed as the teacher-actor during a specific interval (e.g., between $t = 3$ s and $t = 5$ s). Such constraints introduce graphical regularities in the aggregate, while also imposing authoring challenges in both mathematical and narrative representations. Other extensions of the activity emphasize the interplay between narrative and graphical representations. For example, the teacher may choose a graph from the aggregate, send it back out to the entire class, and ask for different stories that fit the same race. The reverse relationship can also be explored, choosing a story and asking for different motions that fit its description. In these variations, students explore the flexibility in the mapping between representation systems of graphs and stories (cf. Nemirovsky and Monk, 2000).

Of the activity structures discussed in this chapter, Mathematical Performances makes the most extended and clearly delimited use of the independent private workspace. In *Sack Race*, this feature gives students the time and latitude to explore relations between the representational systems of piecewise-linear Cartesian graphs, animated motion, and natural-language narratives. Later, the aggregation and display of the entire class's motions provides a powerful experience of the Cartesian space as a representation system capable of bearing diverse stories, since students know that each of the graphs in the aggregate both stands for and "conceals" its related story. As students share the stories for some of the graphs in the space, the class can begin to imagine *possible* stories for others.

3.2 Participatory Aggregations

This activity structure makes strategic use of the size of the class to explore families of functions that are mapped to variation of parameters in their symbolic definition. The class is divided into numbered groups, based both on the size of the class and the nature of the parameter space to be explored. Within their groups, students also receive a "count-off number," so that each student has two "personal parameters." In their private workspaces, students create functions that depend in some critical way on their personal parameters. These are aggregated, organized, and selectively displayed and discussed.

A paradigmatic example is the *Spreading Apart*[2] activity, in which students create functions of the form $y = (C/2) * x + (G - 1)$, where C is their count-off number and G their group number. Thus, in the upfront display all members of a given group have the same y-intercept, while all students with the same count-off number have parallel graphs. When the World representation is animated, the members of each group begin clustered together, while students with the same count-off numbers move in lock-step with each other, maintaining equal distance throughout the race. In the graphical space, each group's graphs make a "fan," and the class as whole consists of a series of fans, shifted vertically. Figure 3 shows the fan for group number 1.

Because each student has a unique function, graph, and animation, the *Spreading Apart* activity supports a powerful "Where are you?" task. The act of "finding yourself" has proven to be highly motivating and to engage students in the important work of conceptually coordinating the various representations (e.g., relating the "m" and "b" in the algebraic expression to the slope and y-intercept of the graph and to the initial position and velocity of the animated actor).

As with Mathematical Performances, the student initially works independently of other students and without reference to the shared public display, until the moment when the teacher aggregates all of the class work. At that point, because the teacher is in control of how and when the class contributions are shown, she is able to make strategic use of the public display to structure classroom discourse. For example, a powerful teacher strategy for engaging the private-to-public dynamic of this activity structure is to ask, "What do you expect to see?" before displaying the class's work or structurally significant subset of it. This engages students' and groups' abilities to generalize from their own experience of the activity task to the range of tasks

[2]Download the *Spreading Apart* activity software/curriculum documents at: http://www.kaputcenter.umassd.edu/products/curriculum_new/algebra1/units/unit4/.

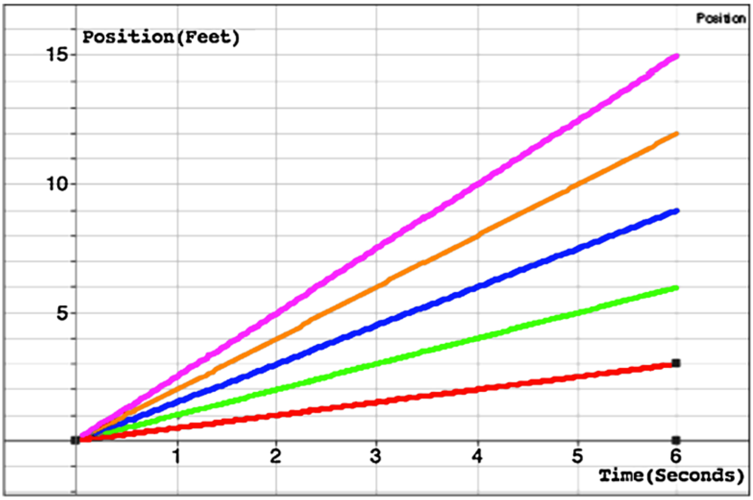

Fig. 3 *Spreading Apart* fan for five members of Group 1

parameterized by group and count-off numbers. The resulting discussion has a "low threshold" because every member of the class can provide an account of their own contribution. And yet it has high mathematical significance, because the variation across roles in the class reveals important aspects of mathematical structure.

3.3 Generative Activities

In this activity structure, each member of the class uses a task-specific environment on her individual device to create one or more mathematical objects that comply with a given condition, and then contribute these to the public space. The public display provides the class a view on the diversity of possible ways of complying with the condition, which often relates to important aspects of mathematical structure. A paradigmatic example is the $2x$ *Activity*, which asks students to make "expressions that are equivalent to $2x$" (Stroup et al., 2005) or another simple expression. Below are several contributions from one classroom's implementation (Davis, 2009):

- 123456789X−123456787X
- X100000000000−99999999998X
- 1000000X/500000
- 15153X−15151X
- 800000X/400000
- (X*X*X*X*X*X*0)+2X
- (X+X+X)/3*2/2*2
- 2(0.2X+0.2X+0.2X+0.2X+0.2X)
- 3X−X+X−X+X−X+X−X+X−X+X−X+X−X+X−X+X−X+X−X+X−X+X−X+X−X+X−X+X−X+X−X+X−X

As can be seen here, the set of student contributions illuminates an algebraic space: the *equivalence class* of $2x$ within the set of algebraic expressions. Because all of these expressions in x are equivalent, they give the same result when evaluated at any x. Thus, their graphs are coincident with $y = 2x$. The class observes the wide variety of expressions that can be created from $2x$ by applying the "rules for simplifying" in reverse; and they see that whenever these rules are applied correctly, the results have the same graph. This experience provides a strong anchor for the meaning of the algebraic manipulations involved in simplifying, factoring, collecting like terms, and so forth.

Generative Activities uphold the tradition of providing "a low threshold and a high ceiling" in learning design (e.g., Papert, 1980). On one hand, all students can participate; and for some students, submitting "$2x$" or "$2 * x$" or "$x * 2$" may indeed be a valid form of participation in the activity. On the other hand, as the examples above begin to indicate, students can engage the activity in mathematically sophisticated ways. Moreover, this "high ceiling" effect has mathematical value for the class. In creating complicated expressions equivalent to $2x$, students are led to use features of the space of algebraic equivalence such as the distributive law, identity properties of 0 and 1, and so forth (Davis, 2009). In discussion, then, the class can identify these properties as technical innovations over which particular student-experts in the community have demonstrated mastery.

Generative activities make important use of a private workspace where students can, if they wish, test their work before submitting it to the public space. In the case of the $2x$ *Activity*, the student device enables them to graph their expressions to check equivalence to $y = 2x$. This feature, along with anonymity in the public display, provides support for students to build 'adventurous' contributions that push the limits of their knowledge. At the same time, this use of the private space is continually connected with activity in the public display, allowing students to change or enhance their responses. An important driver for individual creativity is the appearance of interesting expressions sent by others, and the potential for one's own work to be made public and received as interesting. Thus, the sophistication of the entries from the class develops through a playful competition that is supported by both the technology and the teacher's intervention. Giving voice to impressive entries that emerge ("Look: this one uses the Distributive Law!") or, alternatively noting algebraic features that the class has *not* yet explored ("I see that no one has used any decimals yet..."), the teacher can prompt students to explore different aspects of the mathematical space.

Within SimCalc, Generative Activities have been used by Stroup and colleagues in curriculum based on the qualitative introduction of concepts of calculus within the algebra curriculum (Noble et al., 2004; Stroup, 2002, 2005). This work foregrounds piecewise-constant velocity graphs, offering these as means of producing motion by specifying changes in position. Single velocity x time units are concretized as "delta blocks," so that adding a block extends motion by one distance unit. One Generative Activity in this context asks all students in the class to build a motion using six blocks: the diversity of responses creates a diversity of motions, each of which represents a way of getting from distance $= 0$ to distance $= 6$. Later, students are asked to use piecewise-constant velocity functions to move their elevators to the

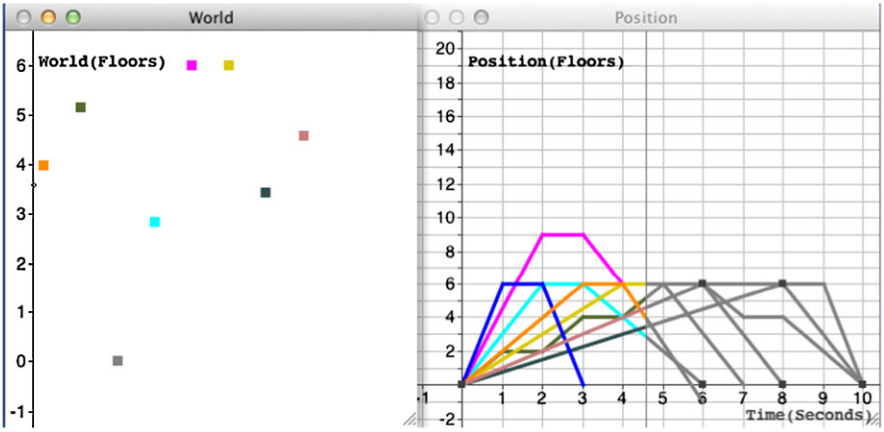

Fig. 4 Student responses to the task to move their elevator to the 6th floor, stop, and return

6th floor, stop, and return to the ground floor. These artifacts are aggregated, and the upfront space displays the position/time graphs and motions that result (see Fig. 4).

3.4 Small Groups

This family of activity designs for classroom networks emphasizes virtual mathematical objects jointly constructed or manipulated by groups of two to four students who work together in teams. Each of the designs centers on a mathematical object that can be subdivided into two, three or four components and distributed across the devices of two, three or four students. In one such example, two students each move an individual Cartesian coordinate point in order to jointly manipulate a line (White and Brady, 2010; White et al., 2012); in another, each student transforms one side of a algebraic equation they work together to solve (Sutherland and White, 2011); and in yet another, four students each examine different representations of the same function (White, 2006; White and Pea, 2011). Some of these designs make use of the public display by featuring views of artifacts associated with each student and each small group, while others simply display objects shared by the small group on the device of each team member. In each instance, these designs structure novel forms of small group collaboration by complementing face-to-face peer interaction with joint manipulation of shared mathematical objects facilitated by the overlapping representation and communication infrastructures of the classroom network.

As an illustrative case, we consider the example of *Graphing in Groups*, an environment in which all students in a class work in teams of two, with each student in the pair moving a coordinate point on her calculator screen in order to manipulate a line drawn in a group-level graphing window in a public display. Typically, two student pairs sitting together in the classroom share one of 6 to 8 group-level graphing windows arranged in a grid on the upfront display (Fig. 5). As students use

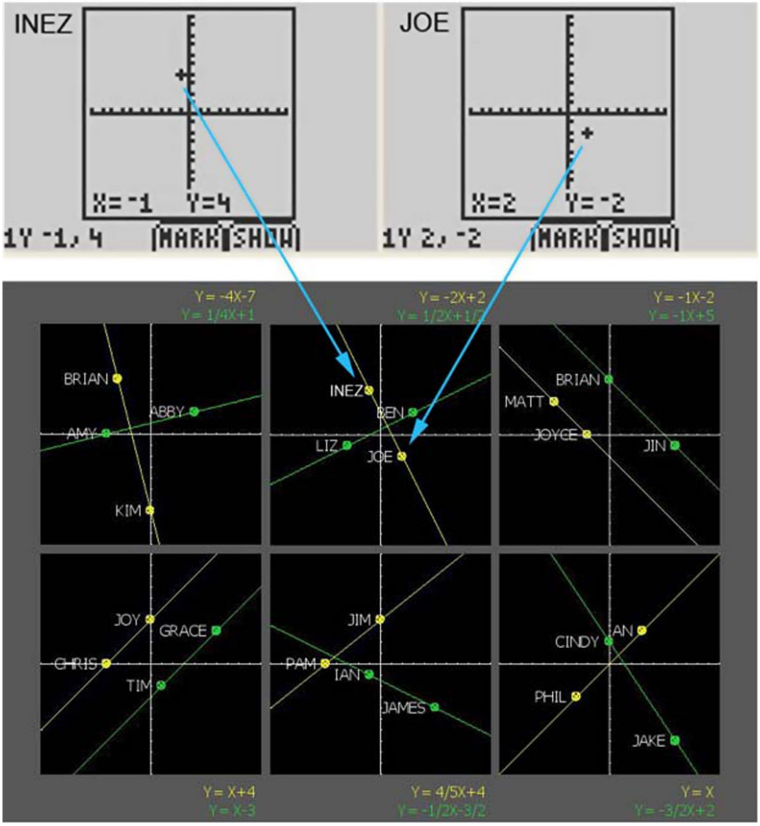

Fig. 5 Two student Cartesian coordinate locations shown calculator screens (*at top*) form a line drawn in a group-level graphing window in a Graphing in Groups public display

their calculators to move their respective point markers to new coordinate locations, they dynamically update both the line and the corresponding equation shown in the public display. *Graphing in Groups* tasks generally involve the teacher asking students to construct a line with certain characteristics: a slope of three, a y-intercept equal to negative one times the group's number, perpendicular to that of the other pair in the same window, etc. Completing these tasks is an activity that plays out primarily at the level of the small group; teammates discuss possible strategies for coordinating their respective point movements in order to construct the desired line. Those strategies often begin as exploratory investigations of the graphical space, or efforts to coordinate simultaneous movements of both students' points in relation to new task demands. Over time, students develop more systematic means of enacting familiar properties of linear functions (such as a difference-ratio characterization of slope) as movements in the virtual graphical space (White and Brady, 2010; White et al., 2012). Importantly, these small group sessions are usually punctuated by regular cycles of whole-class discussion, as the teacher pauses an activity in the middle or follows the completion of a pair-level task to lead a conversation about different solution strategies or linear function characteristics.

In this activity structure, the students' individual devices do have some attributes of a private workspace, though the tight coupling among the devices of the small group demands close coordination of action among group members to manipulate the shared object effectively. Likewise, the real-time changes in the public display as student pairs manipulate points and lines introduce a highly dynamic element to the flow of classroom activity; students are able to view the changing graphs produced by other groups even as they construct their own, and the teacher can utilize this display both to monitor the progress of all students and groups through successive tasks and as a resource for drawing student attention to alternative solutions, or illustrating connections across multiple groups. The traceability of student productions in *Graphing in Groups* is set by the teacher and variable throughout the activity; student points may be labeled either with their names in the public display (as in the example of Fig. 5) to allow student work to be monitored by the teacher or peers, or with the coordinates of the current location to emphasize Cartesian geography and provide anonymity.

3.5 Participatory Simulations

In this activity structure, each student uses their individual networked device to control a single element of a larger system that encompasses the group as a whole. Typically, participants act according to motives or rules that are independent of others in the class, and yet this "agent-level" activity gives rise to emergent phenomena that have domain significance. A common objective of activities of this type is to understand a system "from the inside" and to learn to trace the genesis of emergent phenomena back to agent-level mechanisms. After the primary agent-level experience of the simulation, classroom discussion allows the participants to step out of the system and view it "from the outside." This two-sided experience of the system gives students a pair of complementary conceptual lenses through which to view the system and its structure. In the study of complex systems, the NetLogo (Wilensky, 1999) agent-based modeling environment provides the HubNet (Wilensky and Stroup, 1999a) networked system for the purpose of producing participatory simulations. Classrooms can study the spread of disease through a population as an emergent effect arising out of the interaction of individuals in the population; or the phenomena of traffic flow arising out of the coordination of changes in stoplights (Wilensky and Stroup, 1999b, 2000).

In the domain of mathematics, a paradigmatic Participatory Simulation is the *Point Activity*, which develops the basic idea of functions. Students are given a simple interface allowing them to move a point in the Cartesian plane. They are asked to go to a location where, for example, their y-coordinate is twice their x-coordinate. In the public space, these independent behaviors cause the image of the line $y = 2x$ to emerge (see Fig. 6).

The *Point Activity* provides the classroom group with a direct experience of the graph of a function as a locus of points, each of which complies with a condition relating its y-coordinate to its x-coordinate. This point-based perspective can be leveraged to study elements of linear functions such as slope (relating to the question

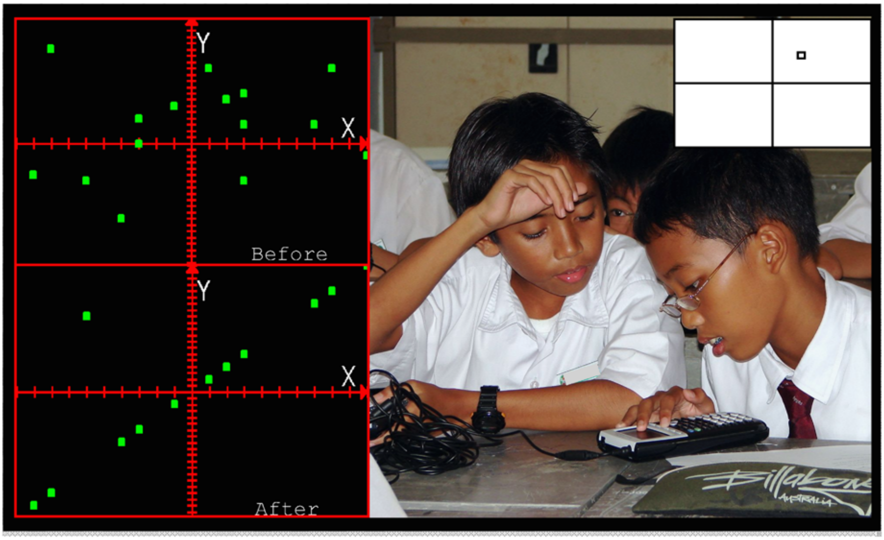

Fig. 6 Private and public display for the *Point Activity*

of how one student on the line might move to reach a partner also on the line); equivalence of functions (pairs of rules that are the same for *all* students in the class); and function intersections (pairs of rules that are only the same for only students in the class—who are identified as standing at these intersections). The *Point Activity* also engages the definition of function as a rule that associates one and only one value of Y for every value of X in the domain, allowing students to explore this definition at the personal level. (If I choose 3 as my x-value, can I follow the rule with any y-coordinate other than 6?)

In Participatory Simulations, the use of the private space is minimized. For instance, in the case of the *Point Activity*, the individual device shows only the student's own point and gives the numeric x- and y-coordinates. These *are* important features supporting the student's work in the activity (permitting her to focus on her own point and providing the numeric data required to complete the tasks); however, the student devices are not truly private spaces, as movements are immediately sent to the public display. This is in fact important for the activity, providing visibility into the entire class's thinking as it emerges, complete with hesitations and explorations.

4 Roschelle and Teasley's Framework in the Context of Ongoing Network Research

In this section, we review Roschelle and Teasley's (1995) analysis of computer supported collaborative problem solving in small groups and discuss its fit with the learning phenomena exhibited in classrooms engaging in the range of activity structures described in Sect. 3. We find that many key aspects of their perspective are still highly relevant, and that recent results can be seen as extending them along a shared trajectory of inquiry. Moreover, we find that points where their perspective seems limiting also indicate important directions in current research.

At the outset of this discussion we should recognize two ways in which Roschelle and Teasley's analysis is historically situated. First, in approaching group interactions, they make innovative use of techniques and constructs from discourse analysis and conversation analysis (e.g., Schegloff, 1991), as a means to study the processes of collaboration in action and the development of conditions that enable it. And second, in approaching phenomena of small group problem-solving, they select an influential information-processing model of individual cognition and consider ways in which this model can provide a framework of understanding distributed cognition at the group level.

Among the aspects of Roschelle and Teasley's (1995) argument that seem *most* compatible with research perspectives on the network current in the field are the basic ideas behind their focus on the Joint Problem Space and their study of features of discourse. This includes their insight that collaboration depends on participants' having the means for continuously establishing shared attention and negotiating the meaning of their actions and observations. Extending that insight leads to notions that (1) collaborative work exists in the discursive field and can be studied there; and that (2) it can be enhanced through the design of learning environments that provide additional, discipline-specific means for extending these social functions. In contrast, among the aspects of Roschelle and Teasley's argument that seem *least* compatible with today's perspectives is the premise that a single, specific model of individual functioning in problem solving could serve as an adequate framework for understanding the diversity of interactions and second-order effects that can appear in collaborative activities supported by a classroom network.

4.1 Points of Continuity

In identifying ways in which Roschelle and Teasley's work aligns well with current perspectives on network-supported collaboration, we focus primarily on their introduction of the Joint Problem Space (JPS) construct as the site of a variety of forms of discursive work by group members over the course of collaborative problem solving. In this context, we investigate ways in which Roschelle and Teasley's ideas about relations between technological environments and the JPS can in fact be seen as prefiguring key elements of classroom network design.

The metaphorical construct of the JPS reflects Roschelle and Teasley's conception of shared understanding as a critical element in collaboration. Indeed, their definition of collaboration—as "a coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem" (1995, p. 70)—demonstrates the fundamental importance they give to this notion. Moreover, they view common ground as being achieved through concerted discursive activity, including both initial work to establish the JPS and ongoing effort to maintain it in the face of changing problem solving circumstances and the evolving individual understandings of group members.

Roschelle and Teasley are sensitive to the impact that a virtual environment (their Envisioning Machine) has in inflecting and extending these discursive patterns in collaborative problem solving. For instance, they note that discourse becomes distributed across language and action as students communicate through a range of modalities including not only words and gestures, but also performative and demonstrative interactions with the software. They observe that turn-taking behavior is affected by features of the learning environment, and that collaboration contains identifiable periods when group members withdraw from each other to focus on private reflection or non-communicative interaction with the software, before returning to a public register to continue their interpersonal work. In addition, they note discourse patterns that seem to be specifically provoked or encouraged by the virtual learning environment, in this case the phenomenon of "collaborative completion" of two-part (IF-THEN) formulations of shared understanding. Finally, they identify conversational repairs and narrations as strong indicators of the social nature of some student interactions with or about the virtual environment.

These fundamental notions—that a significant component of collaboration involves ongoing coordination among collaborators of their evolving perspectives; that this effort can be understood through a metaphor of constructing a shared "space" where agreement can be produced, registered, and stored; and that this effort exhibits unique rhythms and characteristics when it occurs in the context of a discipline-specific representational infrastructure—are consistent with the views of much current research on classroom networks. In the next few paragraphs, we briefly highlight instances of resonance between these notions and findings from current research. Specifically, we consider the utility of the notion of "spaces" in network design, and we describe some findings in recent research that develop Roschelle and Teasley's discursive categories of language-and-action, turn-taking, repairs, and narrations.

4.1.1 Network Spaces

Although the JPS is an abstract and metaphorical construct for Roschelle and Teasley, the computer screen does fulfill key roles in supporting the collaborative group's work to engender and maintain this space. Collaborating students use it as a referential field and also as a communication medium in the course of their problem solving work. Nevertheless, because the Envisioning Machine's representation system embraced only the problem-and-solution context and not collaborative interactions as such, it could not act as an *embodiment* of significant aspects of the JPS.

In contrast, in the classroom network, representational infrastructure is merged with communications infrastructure (Hegedus and Moreno-Armella, 2009); thus, it becomes possible for design to attend explicitly to supporting learner behaviors that cut across these domains. In this way, learning environments can be constructed to provide private, small-group, and/or public "spaces" that support different kinds of social and representational engagement. These spaces can be layered in the course of the activity, offering different interfaces and fostering different kinds of "co-action" (Moreno-Armella and Hegedus, 2009) at different moments. Indeed, as Hegedus and Moreno-Armella, 2009 imply, the social and representational features of the network are deeply intertwined with the mathematics of the activity. That is, the network provides support for both the representation *of* student communications and contributions, while also stimulating students to communicate *about* representations.

Finally, when we imagine a classroom group engaging in collaborative activity in a supportive network environment over a long period of time, we can envision the JPS rising to the level of a *persistent structure* and shared, collective resource in some degree. Extended in time in this way, the JPS would reflect a history of patterns of agreement among collaborators, and it could be seen to intersect fruitfully with notions of sociomathematical norms (Yackel and Cobb, 1996) and the classroom micro-culture (e.g., Voigt, 1994). A full exploration of these connections is beyond the scope of this chapter, but conceptualizing the network as a designed space whose fundamental properties can be manipulated (its topologies, the visibility and appearance of elements, the anonymity or identification of agents and their productions, and so forth), opens the door to a consideration of the design and use of learning environments as the construction of sociomathematical virtual realities.

4.1.2 Language-and-Action and Turn-Taking

In network activities, classroom discourse is even more acutely affected by the technological medium than was the case in the environment studied by Roschelle and Teasley (1995). The network setting is inherently more complex and multi-voiced, with greater scale and concurrency in both social and technological spheres. It features a larger number of students each participating with individual networked devices. Thus such activity settings often feature parallel interactions between students, and between students and the private workspaces of their devices, some aspects of which are made publicly visible as "electronic gestures" (Stroup et al., 2007) in the upfront display, while others can be communicated to neighbors by holding up the device. Network discourse can involve the production and manipulation of mathematical objects that may be completely anonymous or loosely coupled to their student authors. All the while, the verbal interactions of the classroom are often characterized by spontaneous and responsive utterances of various kinds that serve to influence or coordinate behavior of local groups or the classroom as a whole.

In spite of these complications, network researchers have gained analytical traction with even the most tumultuous activities by strategically applying constructs such as turn-taking in situation-specific ways, as did Roschelle and Teasley. For instance, within the Small Groups activity structure, White and colleagues have characterized small-group interactions on the basis of turns of discourse that include verbal utterances and electronic gestures (White et al., 2012). At the whole class level, recent SimCalc work by Hegedus and Penuel (2008) has applied analysis of *adjacency-pairs* in turns to provide a quantitative measure of the qualitatively new student-centered patterns of discourse and participation that appear as the class engages with the public display. And in the context of Generative Activities, Davis and colleagues have begun to identify the movement of structural memes through the classroom's semiotic activity by studying time-stamped logfiles of networked interaction, synchronized with video data (Davis, 2011).

One point of difference between these approaches and Roschelle and Teasley's (1995) perspective is that network researchers tend to treat discourse-analytic patterns as affordances and to incorporate them explicitly and intentionally into the iterative design of activities. For instance, where Roschelle and Teasley note an oscillation in discourse patterns between periods of intensive interaction and periods of withdrawal, network researchers build activities that exploit different uses of public and private workspaces. Similarly, where Roschelle and Teasley note a tendency of collaborating pairs to execute linked components of IF-THEN statements about the Envisioning Machine, network researchers build on ideas like this "collaborative completion" in designing the roles and interfaces that individuals and groups use to produce mathematical objects. For instance, the Small Groups activity structure is specifically designed to exploit the potential for this kind of interdependence among small groups of students in constructing and manipulating mathematical objects. Work by White et al. (2012) illustrates the ways in which social and discursive structures relate to mathematical structures within this mode of distributed production. Participatory Aggregation, Generative Activities, and Participatory Simulations extend the idea of collaborative completion to structure whole activities in which the entire class group explores a mathematical structure.

4.1.3 Repairs and Narrations

Repairs and narrations enter collaborative discourse when participants detect actual or potential breakdowns in shared understanding. In Roschelle and Teasley's context, speakers were clearly identified with their contributions, so that such discourse moves were always clearly situated as coming from author or observer. In the network context, this picture is complicated by three factors. First, some activities use anonymity in the virtual space, which decouples the speaking student from her contributions. Second, some activities distribute control over a mathematical object among a group of students, so that ownership of contributions is collective rather than individual. And third, the larger number of participants creates a range of perspectives towards a given contribution, so that repairs and narrations address a collective audience rather than a single interlocutor. Nevertheless, these constructs still offer attractive approaches to classroom discourse in network activities.

Repairs take a variety of forms across the network activity structures: we begin by considering Small Groups. In this activity structure, some attempts at repairs can function as they do in the single-computer context described by Roschelle and Teasley (1995). For example, in *Graphing in Groups*, collaborating students can use the medium of language to repair discrepancies in their understandings of the problem situation. More often, however, a qualitatively different mixed-modality discourse appears. Since *all* students of the group have simultaneous access to their components of the shared mathematical object, it often happens that individuals initially attempt to proceed in solving the problem, possibly with the intention of *showing* their partners what they mean. The design of the activity immediately thwarts this attempt (since the control of the mathematical object is symmetrically distributed), but the multi-modal approach to repairs frequently continues, with students working out repairs through a combination of language and demonstrative electronic gesture. It should be noted that here the network does not necessarily make repairs more *efficient*: in fact, the mobility of the collective object often serves only to highlight the *need* for repair, making discrepancies in understanding salient and increasing the pressure on the groups to use a combination of language and action to resolve them (White and Brady, 2010).

As Roschelle and Teasley also note, the analysis of failed or degenerate collaborations in the case of Small Groups illuminates the dynamics of the activity structure and of the environment. In *Graphing in Groups*, for example, the *failure* to repair is made evident in groups or dyads where one individual seizes control of the other's computing device, or monologically commands the other to execute movements according to orders (White and Pea, 2011). Such breakdowns reveal the difficulties of achieving the shared understanding and focus necessary for coordinated action.

In activity structures that foreground whole-class discussion, the notion of a repair can be generalized to describe how the group responds to discrepant contributions. Many teachers in networked activities use the question, "What might the person who contributed this have been thinking?" or "What might the person who contributed this have been trying to do?" This style of question makes use of the anonymity of the public space to focus attention on the mathematical contribution rather than on the individual who produced the error or unexpected contribution (Davis, 2005). It also focuses the discussion on *intentions* and *conceptions* (rather than on individual *persons*) and opens a space for discussion of process, with the implicit recognition that a student's contribution may not adequately reflect her entire thought process. In this context and using phrasing such as, "They might have been trying to..." students are often quite willing to describe difficulties that they themselves have had, which they see as relating to the problematic contribution. In fact, inspection of the data after class sometimes shows that the student who contributed the erroneous contribution uses this subjunctive phrasing to make a correction to her own work (Davis, 2003). In such cases, the "cover" of anonymity in the virtual space allows the student to benefit from both the reflective learning involved in correcting his error and the social validation of publicly displaying his new understanding in class discussion.

Narrations, too, vary their form within different activity structures. In Roschelle and Teasley's context, narration always involved students augmenting their own actions with an explanatory gloss or commentary. In contrast, when contributions are anonymous the opportunity for narrations is much broader and more speculative, allowing students to experiment with *potential* explanations for contributions. In some cases, the work of *speaking for* and *explaining* objects and phenomena that appear in the public space becomes a substantial part of the discursive work of the class. Thus students are often engaged in providing explanations not only of their own work, but also of other students' creations, as well as of the relations among contributions and of patterns in the class aggregate. The teacher can explicitly trigger such meta-narrations with questions like "What might the person who created this object have been trying to do?" (Davis, 2003, 2005). Mathematically, this practice enables students to discuss shared difficulties as well as to celebrate interesting responses—particularly when the group discovers a valid but unexpected rationale.

Finally, in providing narrations about the relations among contributions or about patterns of the aggregate, students can be led to make mathematically significant generalizations. The task of explaining what *we created* is a group problem distinct from, but related to, explaining what *I did*. This occurs frequently in the Participatory Simulations activity structure, where collective phenomena emerge from individual behavior. As the class makes sense of the emergent phenomenon, their narration bridges from local, agent-based perspectives to global, aggregate explanations. Narration about the collective artifact also appears strongly in the case of Participatory Aggregations, where families of functions need to be explained as deriving from the activity of groups and individuals, as well as from the representational features of the environment and properties of the mathematics.

4.2 Points of Contrast

In identifying ways in which Roschelle and Teasley's analysis appears limited with the benefit of hindsight, we consider two basic elements of their approach, associated with (1) the relations between individual and group-centered activities, and (2) the nature of intersubjectivity as an empirical construct.

4.2.1 Individuals and Groups

One path of inquiry into collaborative learning considers group structure as an independent *layer* in the implementation of an activity, which can be analyzed separately from other factors. This approach, which we might call "Scaled-up Individualism" is implicit in Roschelle and Teasley, in that their descriptive framework depends rhetorically at least on the proposition that it is possible to extend individualistic information-processing theories of cognition focused on production systems (Newell and Simon, 1972; Young, 2001) to act as models for richly social, collaborative human problem solving. An example is the discussion of "Socially-Distributed Production" where Roschelle and Teasley (1995) find evidence that the two members of a collaborative pair may each supply a part of an IF-THEN production rule in the course of joint problem-solving and implicitly treat such manifestations as illuminating group learning processes.

On one hand, as long as such individualistic models of cognition are used in group-centered learning design explicitly as *models*—as useful over-simplifications of a reality not yet well understood which are intended to be refined and even discarded over iterative cycles of research—we may regard them as offering a legitimate, pragmatic starting point for design. However, Scaled-up Individualism makes the questionable assumption that the nature of a task remains the same regardless of the group structures (among other contextual factors) characterizing the environment where it is encountered. In this view, problem solving processes are "hardware-independent"—the same elements are posited to exist in group and individual settings, with the main difference being their distribution among the available actors.

The notion that this could be an accurate picture in an absolute sense runs counter to much of the research and theory of the last two decades. In fact, even in the "person-plus" case where an individual human agent distributes her cognition in the use of cultural tools (Pea, 1993; Perkins, 1993), there is strong evidence to believe that this distribution affects the nature of tasks in a fundamental way. In a socially distributed setting with multiple humans *and* tools (Hutchins, 1996), the differences are likely to be further amplified.

Strong adoption of the Scaled-up Individualism perspective may blind researchers to more "emergent" aspects of collaborative activities that disrupt direct analogies between individual and group learning processes. For instance, in a Generative Activity like the $2x$ activity discussed above, classrooms often explore and build expertise with fundamental algebraic structures such as inverse operations. These explorations appear in the record of the activity as patterns with increasing resonance: rather than being the contribution of any individual student, they emerge as themes in the discourse. Similarly, in the Small Group activity structure, students may discover structural properties of their shared object that become salient through coincidences in their individual, exploratory movements. In *Graphing in Groups*, for instance, if both students move their points one step to the left, they may find to their surprise that the slope of their joint line remains unchanged. This discovery can serve to motivate coordination or stimulate collaborative investigation. In general, research experience in classroom networks suggests that while initial activity designs may be guided by analogy with individual activity patterns, researchers should not expect that the patterns of exploration or learning exhibited by the group will always map cleanly to individual patterns.

4.2.2 Questions of Intersubjectivity

Roschelle and Teasley distinguish "collaborative" from "cooperative" problem solving; indeed, they introduce the construct of the Joint Problem Space in part as a means of gaining analytical purchase on the difference (1995, p. 70). This distinction remains important today. However, implied in their analysis is the idea that there is a continuum of collaboration, based on the degree to which cognitive processes are shared between group members. Though the collaborative end of this continuum is not explicitly given higher value, the authors' interest in collaboration process does lead them to select "one of the *most collaborative* dyads" for analysis (1995, p. 71, emphasis added).

Here again, our intention is not to critique Roschelle and Teasley's pioneering work. Nevertheless, subsequent research has revealed limitations in adopting the perspective that the interactive behavior of groups should be ordered on a scale of "more" or "less" collaborative. Alternative characterizations are possible and useful—in which collaborators develop or negotiate shared perspectives along some dimensions, while in others they maintain differences (or indifference) with respect to each others' viewpoints. Thus, depending on the activity structure, a group may develop one or another *style* of intersubjectivity, and it may be appropriate that these styles vary—across groups and across activities.

Moreover, in many approaches to activity design in group spaces, increased student agency is a primary objective, and diversity of thinking is regarded as an essential resource. In such settings, while it may desirable for students to recognize the validity or strengths of others' contributions, it is also important that they pursue their own individual directions. For example, in Generative Activities and Mathematical Performances, it is often preferable for students *not* to share a common approach to the task, since diversity of thinking leads the group as a collective to explore a broader space of possibility. Indeed, some of the most powerful and productive interactions in these activities can begin with a contribution that evokes surprised laughter from the rest of the group. In *Graphing in Groups,* still other styles of intersubjectivity are valuable. Here, periods of high productivity in the activity are often characterized by group members being "on the same page" to some degree, but without yet having established a completely shared perspective. Moreover, when the group does achieve a fully shared perspective, they sometimes "defeat" the activity structure by adopting formulaic divisions of labor that reduce the need for interaction (e.g., one student always moving immediately to the y-intercept of the desired function). Still other styles of perspective-sharing appear in Participatory Simulations and Participatory Aggregation. Here, students spend a good portion of the activity immersed in "local" perspectives that are parallel to but independent from their classmates. Later, the aggregate construction of the entire class provides striking evidence of the interrelation of these local worlds, which becomes a challenge for the collective interpretation of the group. Intersubjectivity is therefore purposely deferred so that it can later be developed explicitly at the whole-class level.

5 Conclusion

Roschelle and Teasley's construct of the Joint Problem Space (JPS) arose out of a desire to see problem solving as a fundamentally social phenomenon and to study the impact of representationally-rich computer environments on the social dynamics of collaboration. In the time since that seminal article, a line of inquiry in educational research around classroom networks has tapped into some of the potential suggested by the JPS. In developing this research trajectory, a common network architecture has emerged, yielding clear system requirements for a classroom network that would support the five classes of mathematically powerful activities we have described in this chapter. Moreover, each of these five activity structures can make a claim to being a general category of activities, and together they indicate a strong frontier in research-based learning design. Student work within each of the activity structures has been shown to be powerfully *thought-revealing* in the sense of

Lesh and colleagues (e.g., Lesh et al., 2007), and there is a growing body of data to characterize the unique types of learning that occur in these activities, at individual, small-group, and/or whole-class levels. Finally, design research in these areas holds strong potential to illuminate processes of learning and to help build theory that is generalizable beyond the context of the classroom network, addressing models of mathematical learning more broadly.

At the same time, research in classroom networks faces a number of important challenges today. While the architectural requirements for networks that provide technological support for the five activity structures described in this chapter are clear and increasingly supported by research data, robust systems that instantiate this representation and communications infrastructure have not yet achieved the wide distribution for which researchers hoped. Thus, whereas the representation infrastructure of dynamic mathematics environments has become increasingly common, classroom networks are still only present in a minority of mathematics classrooms. Furthermore, there is a need for additional work in the area of analytical tools and visualizations to support both researchers and teachers in conducting deeper assessment and analysis of the student interactions that occur during networked activities. "Scaled-up" Cartesian representations such as function graphs and scatter plots are sufficiently powerful to support aggregated student and group work during the activities described in this chapter, but additional innovations are needed to extend the kinds of analysis and visualization available to researchers and teachers interested in mining the wealth of data produced in the course of these activities, at individual, small-group, and whole-class levels (see, e.g., Davis, 2009). Innovations in these areas are in development both by the authors and by other researchers, but the path from innovation to widespread adoption of such analytical tools will require time and concerted effort.

Nonetheless, the growing field of classroom network learning design and research has shown the power of combining representation and communications infrastructure as a platform for mathematically-rich activities. The five activity structures we have described in this chapter combine these infrastructural affordances in different ways; together, they carve out a broad space for ongoing design work. We expect that the future holds strong potential for continued discovery both within this design space and beyond, as advances in technology and the science of learning continue develop these activity structures as well as to uncover powerful new ways of engaging in collective mathematical activity.

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Part III
Impacts from Large-Scale Research

Jinfa Cai

Quantitative research methods have been made available in the psychometric and sociometric literatures, but these research methods are not yet common knowledge in the mathematics education research community. This part shows how quantitative research methods can make important contribution to understand issues in the teaching and learning of mathematics.

This part has five chapters, reporting findings from large-scale studies of SimCalc using quantitative research methodology. Comparing to the later part on small-scale studies, the studies in this part use large data set to examine the impact of SimCalc through exploring various relationships among variables. All of the large-scale studies reported in this part have shown positive effect of SimCalc, especially on students. In addition to the positive effect of SimCalc, this set of chapters particularly addressed several important issues in large-scale studies.

Measuring Changes The overarching theme of this part is on measuring changes over time with a primary focus on students. Researchers used different outcome measures, including student learning, attitude, and classroom interaction to measure the effect of SimCalc on students. This set of studies will help readers to be aware of and understand techniques and methodological issues in the proper measurement of change in educational practice and in the proper interpretation of the change that is related to changes in student learning outcome measures.

Instrument Development In large-scale studies, it is absolutely critical to develop proper instruments to measure changes over time. This part includes information about the development of several instruments to measure what students learn and what teachers must know to support their learning, as well as several instruments to measure the interaction between learning and participation. Researchers in this part not only described detailed processes of developing these instruments, but

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also the instruments themselves. It would be even better if the researchers would had described how the instruments they developed are related to or different from other instruments in the similar nature.

Implementation and Sustainability Any educational intervention faces challenges of faithful implementation and sustainable use. Large-scaled studies reported in this part demonstrate the unique position that the large-scale studies can address these issues because large-scale studies can identify important factors in faithful implementation and in sustainable use of SimCalc materials. In this part, readers will find how local capacity should be built to support the SimCalc program and how teacher professional development should be designed to increase the likelihood of sustained use in the district.

In summary, the methodological issues of large-scale studies discussed in these chapters can be used to measure the effectiveness of other mathematical interventions at scale. Researchers in this part have used various designs, including randomized controlled experiment design and quasi-experiment design. This part shall help us to understand the issues in the search for making valid causal inferences.

SimCalc at Scale: Three Studies Examine the Integration of Technology, Curriculum, and Professional Development for Advancing Middle School Mathematics

Jeremy Roschelle and Nicole Shechtman

1 Introduction

SimCalc takes a decidedly representational approach to improving mathematics learning; dynamic representations are introduced as a tool for increasing conceptual understanding. The research base supporting a representational approach is broad but fragmented. Cognitive theory supports the approach via the multimedia principle, which has firmly established the benefits of carefully integrated presentations of the same concept in linguistic and graphical forms (Mayer, 2005). A large body of design research with dynamic tools such as SimCalc, The Geometer's Sketchpad® and Cabri Geometre reports benefits to student learning from the use of this representational technology (e.g., Heid and Blume, 2008; Hoyles and Lagrange, 2010), but this research includes few experiments. In more distantly related research, a meta-analysis that summarized findings from more than 100 research studies involving 4,000+ experimental/control group comparisons revealed that both representing knowledge graphically and using manipulatives to explore new knowledge and practice applying it had a large effect size. "The overall effect size for these techniques was 0.89, indicating a percentile gain of 31 points. The use of computer simulation as the vehicle with which students manipulate artifacts produced the highest effect size" (Marzano, 1998, p. 91).

Although Marzano's meta-analysis found positive effects in general, more specifically, the field has lacked rigorous experimental evidence of the effectiveness of a dynamic representational approach. The most prominent random-assignment experiment related to teaching mathematics with technology is the National Study of the Effectiveness of Educational Technology Interventions (EETI), which found "test

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scores were not significantly higher in classrooms using selected reading and mathematics software products" (Dynarski et al., 2007, p. xiii). The EETI did not, however, include products that take a representational approach. To our knowledge, the research reported in this paper (along with the related Dalton and Hegedus work reported in this volume) is among the first to examine a representational approach within a program of randomized controlled experimentation with a sufficiently large scope to use multilevel modeling and thereby rigorously establish the effectiveness of this approach across a diversity of settings.

2 Research Design and Methods

The research design and methods are documented elsewhere in extensive detail (see Roschelle et al., 2010). Here we provide an overview of the core elements.

2.1 Research Questions

The core research questions of the Scaling Up SimCalc Research Program were as follows:

1. Can a wide variety of teachers use an integration of technology, curriculum, and professional development to create new opportunities for middle school students to learn complex and conceptually difficult mathematics?
2. Can these findings be extended across grade levels?
3. Do student gains persist as we reduce the presence of the research and development team?

This chapter primarily focuses on the first two questions. The third question is addressed in other publications examining sustainability (see Fishman et al., (2009); Hegedus et al., 2009).

2.2 Experimental Design

To investigate all three research questions, we implemented two randomized experiments (one of which contained an embedded quasi-experiment) with pre/post measurement. The first experiment—the Seventh-Grade Experiment—began in Summer 2005 with seventh-grade content, students, and teachers. The second—the Eighth-Grade Experiment—began in Summer 2006 and was designed to extend the findings of the Seventh-Grade Experiment to eighth-grade content, students, and teachers and investigate a train-the-trainers approach to teacher professional development.

Whereas the Eighth-Grade Experiment lasted 1 year only, the Seventh-Grade Experiment lasted 2 years and followed a delayed-treatment design. The second year of the study afforded an embedded Seventh-Grade Quasi-Experiment in which control teachers (also called the delayed-treatment teachers) began to use the SimCalc replacement unit, and treatment teachers (also called the immediate-treatment teachers) continued to use it.

2.3 Components of the Treatment Interventions

This section provides an overview of the treatment interventions, which were an integration of technology, curriculum, and teacher professional development. We discuss the rationale for conceptualizing these interventions as replacement units, and the development of the focal mathematical content, the curricula, and the teacher professional development.

2.3.1 SimCalc Interventions as Replacement Units

We conceptualized the SimCalc intervention as a replacement unit for several reasons. Prior large-scale research had recommended the replacement unit strategy (Cohen and Hill, 2001) because it balances the trade-offs between ambition and specificity. The goals of the research were inherently ambitious and so, too, was the use of the representational infrastructure. Replacement units were large enough and long enough to allow real change and meaningful learning consistent with these goals. At the same time, the short, contained nature of a replacement unit limited the perceived risks of the teachers and schools in participating; allowed us to provide explicit curricular content and pedagogical guidance, and tight connections to existing standards; and enabled us to understand and manage the conditions of implementation.

2.3.2 Focal Mathematical Content

The process of identifying and refining key concepts to be covered in the curricula and assessments included review of Texas and national standards, an extensive review of Texas mathematics textbooks and the research literature in middle school mathematics, and consultation with an advisory panel of experts in mathematics and mathematics education. This process is also discussed in the chapter by Shechtman, Haertel, Roschelle, Knudsen, and Singleton, this volume.

We identified proportionality and linear function as our target mathematics. Among middle school mathematical concepts, proportionality ranks high in importance, centrality, and difficulty (Hiebert and Behr, 1988; National Council of Teachers of Mathematics, 2000; Post et al., 1993). For example, the National Council of Teachers of Mathematics (NCTM) describes proportionality and related concepts as "focal points" for learning in seventh and eighth grade (National Council of

Table 1 Mathematical conceptual frameworks: Focal knowledge, skills, and abilities for the seventh grade and eighth grade curricula and assessments

Framework	M_1 component	M_2 component
	Foundational concepts typically covered in the grade-level standards, curricula, and assessments	*Building on the foundations of* M_1, *essentials of concepts of mathematics of change and variation found in algebra, calculus, and the sciences*
Rate and proportionality for Seventh-Grade	Simple $a/b = c/d$ or $y = kx$ problems in which all but one of the values are provided and the last must be calculated Basic graph and table reading without interpretation (e.g., given a particular value, finding the corresponding value in a graph or table of a relationship)	Reasoning about a representation (e.g., graph, table, or $y = kx$ formula) in which a multiplicative constant k represents a constant rate, slope, speed, or scaling factor across three or more pairs of values that are given or implied Reasoning across two or more representations
Linear function for Eighth-Grade	Categorizing functions as linear/nonlinear and proportional/nonproportional Within one representation of one linear function (formula, table, graph, narrative), finding an input or output value Translating one linear function from one representation to another	Interpreting two or more functions that represent change over time, including linear functions or segments of piecewise linear functions Finding the average rate over a single multirate piecewise linear function

Note: M_1 and M_2 refer to the two major dimensions of each framework

Teachers of Mathematics, 2007). From a mathematics perspective, proportionality is closely related to the important concepts of rate, linearity, slope, and covariation. In addition, proportionality offers an opportunity to introduce students to the concept of a function, through the constant of proportionality, k, that relates x and $f(x)$ in the functional equations of the form $f(x) = kx$. A deep understanding of the concept of function as it relates to rate, linearity, slope, and covariation is central to progress in algebra and calculus. Mathematics education research has identified persistent difficulties in mastering these concepts and has theorized that proportionality is at the heart of the conceptually challenging shift from additive to multiplicative reasoning (Harel and Confrey, 1994; Leinhardt et al., 1990; Vergnaud, 1988).

We developed a mathematics framework for the seventh- and then for eighth-grade intervention that articulated the focal knowledge, skills, and abilities for the curricula and assessments (Table 1; note that is also described in Shechtman et al., this volume). We use the symbol M_1 to refer to the foundational concepts typically covered in the Texas state grade-level standards, curricula, and assessments. This mathematics embodies a *formula* approach to proportionality and linearity, and tends to ask students to find one number given two or three other numbers. We use the symbol M_2 to refer to mathematics that goes beyond what is tested in Texas, providing essential concepts of the mathematics of change and variation found in higher-level math and science. This mathematics embodies a *function* approach to proportionality and linear function and often asks students to consider the mapping between a domain and range and to connect such concepts as *rate* across multiple representations (e.g., k, in $y = kx$ and the slope in a graph of $y = kx$).

2.3.3 Curricula

We designed two replacement units, one for the seventh grade and one for the eighth grade. Each unit covered the relevant mathematical content as outlined in Table 1. The materials for both units were student workbooks, a teacher's guide, and corresponding SimCalc MathWorlds® files. The package was designed to be used daily over a 2- to 3-week period. It was designed to cover the requirements for proportionality in seventh grade and linear function in eighth grade while also introducing a more advanced perspective.

The seventh-grade curriculum, *Managing the Soccer Team*, addresses central concepts of rate and proportionality while also introducing functions in the form $y = kx$. Speed as rate is developed through a sequence of increasingly complicated simulations. Lessons progress through representations—from graphs, to tables, to equations—aiming to teach students to translate among all three and to connect each concept to verbal descriptions of motion or other real-world contexts. In this unit's contextual theme, students play the role of a soccer team manager—training players, ordering uniforms, planning trips to games, and negotiating their own salary.

The eighth-grade curriculum, *Designing Cell Phone Games*, addresses linear function and average rate. Linear functions are developed as models of motion and accumulation. Students learn to use different representations of these functions for problem-solving and to translate among the representations. Graphical representations are intended to enable students to solve efficiently traditionally difficult word problems about average rate. In this unit's contextual theme, students play the role of an electronic game designer who must use mathematics to make their games functional.

2.3.4 Teacher Professional Development

For each of the studies, teachers were provided with professional development opportunities to strengthen their mathematical content knowledge, learn to use the curriculum materials, and/or plan specifically how to use the materials.

In all three studies, treatment teachers attended a 3-day summer workshop introducing the respective SimCalc replacement units. The teachers worked through the SimCalc materials as learners, experiencing a complete but compressed version of the entire unit. The workshop facilitators emphasized the mathematics in the replacement unit and the mathematics knowledge needed for teaching the unit. Treatment teachers also attended a 1-day workshop in the early fall in which they made specific plans for how and when to use the SimCalc materials in their classrooms. In addition, in the Seventh-Grade Experiment Year 1, before the 3-day SimCalc material workshop, treatment teachers attended a 2-day workshop which addressed the mathematical knowledge for teaching rate and proportionality. In Year 2 of the Seventh-Grade Quasi-Experiment, these immediate treatment teachers who had already attended a SimCalc workshop attended a more advanced workshop during the summer, focusing on pedagogical techniques.

To investigate whether student gains would persist as we reduced the presence of the research and development team, we used two different teacher professional development delivery models. For the Seventh-Grade studies, two members of the SimCalc team—both highly experienced mathematics teacher educators—led all the professional development workshops. In the Eighth-Grade Experiment, we used a train-the-trainers model. As a dissertation describes in detail (Dunn, 2009), these differences in implementation models did not significantly impact teacher practice or student gains.

2.4 Control Condition

In the classroom, teachers in the control condition simply taught their "business as usual" curriculum. Data was collected around the unit in their scope and sequence that would have been replaced by the SimCalc unit had they been in the treatment group. Thus the experimental comparison examined the implementation of SimCalc versus traditional curriculum targeting similar mathematical content.

In addition, to make participation equitable and palatable for control teachers, teachers in each control condition received professional development training that was of equal professional value but did not discuss the SimCalc intervention (i.e., training in mathematical knowledge for teaching rate and proportionality in the Seventh-Grade study, and training in using technology to teach statistics in the Eighth-Grade study).

2.5 Assessment Design and Development

We developed two student assessments, one for the Seventh-Grade studies focusing on rate and proportionality and one for the Eighth-Grade Experiment focusing on linear function. Within each study, the identical assessment was administered at pre-test and post-test.

As described in detail in Shechtman et al. (this volume), to develop valid and reliable assessments, we followed models of best practices in assessment development (e.g., AERA, APA, and NCME, 1999) and drew on the tenants of Evidence Centered Design (ECD; Almond et al., 2002; Mislevy et al., 2003, 2002). The ECD framework emphasizes the evidentiary base for specifying coherent, logical relationships among all essential assessment elements. Our assessment development followed a progression of four processes, as follows.

In the first process, we established a mathematical conceptual framework and assessment blueprint. The mathematical conceptual frameworks are found in Table 1. The blueprint had four dimensions: (1) complete coverage of all the M_1 and M_2 topics with subscales for each (see Table 1), (2) alignment with the state content standards (the Texas Essential Knowledge and Skills [TEKS]), (3) various problem contexts (i.e., motion and money), and (4) a diversity of task types (about one third each of multiple choice, short response, construction of multiple mathematical representations).

In the second process, we specified the types and properties of items we would develop. We decided to build our pool of items from those already existing in released standardized tests, previously validated instruments, the research literature, the SimCalc pilot (Tatar et al., 2008), and the SimCalc curriculum.

In the third process, we developed, validated, and refined a pool of items. Using the blueprint as a guide to ensure coverage of all relevant concepts, the team drew from the instrument used in the pilot study, surveyed existing standardized tests (TAKS, NAEP, TIMSS, and other state tests) and literature for items, and created some new items. We validated and refined these items using empirical methods, each of which provided important data to help us iteratively select and refine appropriate items: (1) *formative expert panel* to review and rate items for alignment with our mathematical conceptual framework (Table 1), alignment with TEKS, and grade-level appropriateness; (2) *student cognitive think-alouds* to obtain information about item clarity and how individual students would solve the problems; (3) *field testing of a prototype instrument* with a sample of 200–300 students to characterize the technical qualities of the items and forms (using both classical test theory and item response theory); and (4) *summative expert panel review* to assess the content alignment ratings made by the formative panel and recommend refinements to the items for better alignment with the content framework.

In the fourth process, we documented the assessment processes and technical qualities. The basic test specifications of the resulting assessments were as follows. The Seventh-Grade rate and proportionality assessment had 30 items with an alpha of 0.86. The M_1 subscale had 11 items with an alpha of 0.73, and the M_2 subscale had 19 items and an alpha of 0.82. The Eighth-Grade linear function assessment had 36 items with an alpha of 0.91. The M_1 subscale had 18 items with an alpha of 0.79, and the M_2 subscale had 18 items and an alpha of 0.87.

2.6 Demographic and Implementation Measures

We also collected data on student demographics and classroom implementation. Before teaching their units, teachers were asked to fill out a roster of the students in their classroom. For each student, teachers reported gender, ethnicity, and their subjective rating of the student's prior achievement level as low, medium, or high. For each day the unit was taught, the teacher filled out a log page probing various aspects of implementation. In addition, school-level data were obtained through a database maintained and distributed by the Texas Education Agency, the state department of education. We measured several other quantitative and qualitative variables, which are reported on elsewhere.

2.7 Analysis Methods and Procedures

Given the hierarchical nature of the data, we used multilevel modeling (MLM), specifically hierarchical linear modeling, to estimate the effects of the treatment (Raudenbush and Bryk, 2002). We constructed a two-level model as follows. The first level predicted student gain scores as a function of a school-specific intercept and P student level covariates.

$$\text{Level 1 (Student):} \quad Y_{ij} = \beta_{0j} + \sum_{p \in P} \beta_{pj} X_{ij}^{(p)} + r_{ij}.$$

At Level 2, the school-specific intercept was modeled as the sum of a grand mean, a fixed effect for treatment assignment T_j, Q school-level covariates and a random deviation.

$$\text{Level 2 (School):} \quad \beta_{0j} = \gamma_{00} + \gamma_{01} T_j + \sum_{q \in Q} \gamma_{0q} W_j^{(q)} + u_{0j}.$$

As it turns out, tests for random slopes for all student-level covariates were nonsignificant, so all β_{pj} in the Level 1 equation are modeled as fixed effects (set equal to the corresponding γ_{p0}).

All models were fit using the *xtmixed* procedure within Stata version 9 and restricted maximum likelihood estimation. Continuous covariates were grand-mean centered, whereas categorical variables were represented as 0/1 indicators. In testing the impact of mediating variables, we fit multiple models, each adding a single fixed covariate (at the student or school level) and interaction with the treatment indicator to the model. We managed the risk of inflated Type I error rates by using the false discovery rate procedure of Benjamini and Hochberg (1995). This procedure ensures that fewer than 5 % of the reported statistically significant results within a logical family of comparisons will be due to Type I error.

2.8 Recruitment and Assignment to Condition

This research took place in the state of Texas, a large state with wide regional variations in the diversity of subpopulations of teachers and students. We recruited teachers through the Dana Center and regional Education Service Centers (ESCs) throughout Texas. ESCs are public organizations (affiliated with the Texas Education Agency) that provide supports for schools and districts in their region. By working with the Dana Center and with ESCs, the SimCalc project team could use the existing network of professional development service providers with strong connections to teachers and a positive track record in the eyes of Texas teachers.

We performed selection and random assignment at the school level; that is, if we accepted one mathematics teacher from a school, we would accept all applicant mathematics teachers from that school and assign them all to the same condition. We decided to recruit seventh- and eighth-grade teachers from different schools; thus no students or teachers participated in both studies.

2.9 Participants

An online appendix associated with Roschelle et al. (2010) shows the sample characteristics, illustrating the diversity of regions, teacher demographics, and student demographics (http://aer.sagepub.com/content/47/4/833/suppl/DC1). A technical report (Tatar and Stroter, 2009) examined the diversity of the seventh- and eighth-grade samples, as well as their representativeness relative to broader populations. The samples were diverse in terms of campus poverty levels, school size, and campus ethnicity. They were also diverse in terms of teachers' gender, ethnicity, years of teaching experience, highest degree obtained, and mathematical knowledge. Comparisons were made to the population in the Texas regions in which the experiments were conducted, as well as to the state of Texas as a whole. For all variables for which we had data at the regional and state levels, the ranges and means were similar among our samples and the middle school mathematics teaching population by region and in the state. Note that the low percentages of African-American teachers and students, as well as schools from large urban settings, reflect their small populations in the regions in which the experiments were conducted.

Whereas seven of the 20 geographical regions in Texas participated in the studies, of particular note is the participation of Region 1 because of its unique demographic and socioeconomic characteristics. Region 1 is in the Rio Grande Valley adjacent to the Mexican border. It has one of the highest poverty levels in the United States and is predominantly Hispanic. Region 1 participated in the Seventh-Grade Studies; however, because of a shift in local circumstances, the region did not participate in the Eighth-Grade Experiment.

2.10 Experimental Procedure

We used tightly controlled experimental procedures to minimize the possibility of bias across groups. We designed the treatment and control procedures to be almost identical with the exception of which unit was implemented. Each year, teachers attended their designated workshop(s) at their regional ESC. To ensure that they all had a consistent understanding of the research, all teachers were shown a video at the beginning of the summer workshop that explained the research project and procedures. Within each experiment, teachers received the same stipend regardless of which condition they participated in.

3 Results

Two-level MLM analyses were used in all three studies to show that the main effect was statistically significant, demonstrating that students who had the SimCalc intervention learned more than control students who had the business as usual curricula. Table 2 shows that in all three studies, although the treatment and control groups began with similar pretest scores, treatment students had significantly higher gains from pretest to posttest. In all three studies, the effect sizes were large and educationally significant, particularly for the M_2 portion of the tests. As Fig. 1 illustrates, the gains differences between the two groups in all three studies occurred mostly on the M_2 portion of the tests. Since conducting this study, we have extended these results to a large school district in Florida, obtaining similar learning gains for students in classrooms that use the seventh-grade SimCalc intervention (see Vahey, Roy, & Fueyo, this volume).

We also examined whether the intervention was effective across five policy-relevant demographic factors. We began by first examining the extent to which some students in these groups may have begun at a relative disadvantage. Within each study, we ran a series of two-level MLM models predicting student M_2 pretest scores, one for each of the five demographic factors, entering the factor independently as a covariate at the appropriate level. Overall, we found that all of the factors, except being located in Region 1, significantly predicted M_2 pretest in all three studies at a significance level of $p < 0.01$ or lower, indicating baseline disadvantages for traditionally underserved populations. Specifically, girls started lower than

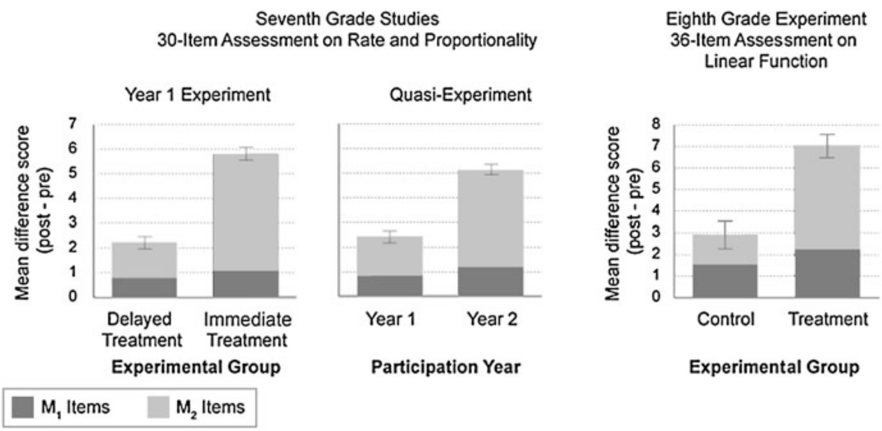

Fig. 1 Student mean difference scores (± SE of total using MLM) at the student level

Table 2 Student test scores at the student level

	n	Total score						Effect size of gain score difference
		Pretest		Posttest		Gain		
		Mean	*SD*	Mean	*SD*	Mean	*SD*	
Total score								
Seventh Year 1								
Control	825	12.7	5.7	15.0	5.7	2.2	3.8	0.63***
Treatment	796	13.2	5.7	19.0	6.0	5.8	4.0	
Seventh Quasi-Exp								
Delayed Year 1	510	12.8	5.2	15.2	5.5	2.4	3.9	0.50***
Delayed Year 2	538	12.6	5.4	17.7	6.2	5.1	3.9	
Eighth Grade								
Control	303	12.5	7.6	15.4	8.4	2.9	5.2	0.56***
Treatment	522	11.9	7.3	18.9	8.7	7.0	5.0	
M_1 subscale								
Seventh Year 1								
Control	825	7.2	2.7	8.0	2.5	0.8	2.2	0.10
Treatment	796	7.5	2.6	8.6	2.0	1.1	2.1	
Seventh Quasi-Exp								
Delayed Year 1	510	7.3	2.5	8.2	2.4	0.8	2.3	0.13*
Delayed Year 2	538	7.3	2.6	8.5	2.2	1.2	2.1	
Eighth Grade								
Control	303	7.2	3.8	8.7	4.0	1.5	2.9	0.19
Treatment	522	7.2	3.6	9.4	4.2	2.2	2.7	
M_2 subscale								
Seventh Year 1								
Control	825	5.5	3.6	7.0	4.0	1.4	2.7	0.89***
Treatment	796	5.7	3.8	10.5	4.5	4.7	3.3	
Seventh Quasi-Exp								
Delayed Year 1	510	5.4	3.4	7.0	3.8	1.6	2.8	0.69***
Delayed Year 2	538	5.3	3.5	9.2	4.5	3.9	3.2	
Eighth Grade								
Control	303	5.3	4.4	6.6	4.9	1.4	3.5	0.81***
Treatment	522	4.7	4.2	9.5	4.9	4.8	3.3	

***$p < 0.0001$; *$p < 0.05$

Note: The seventh grade assessment had 30 items and the eighth grade assessment had 36 items

boys, Hispanic students started lower than other students, students rated as low or high achieving by their teachers started lower or higher respectively than those rated

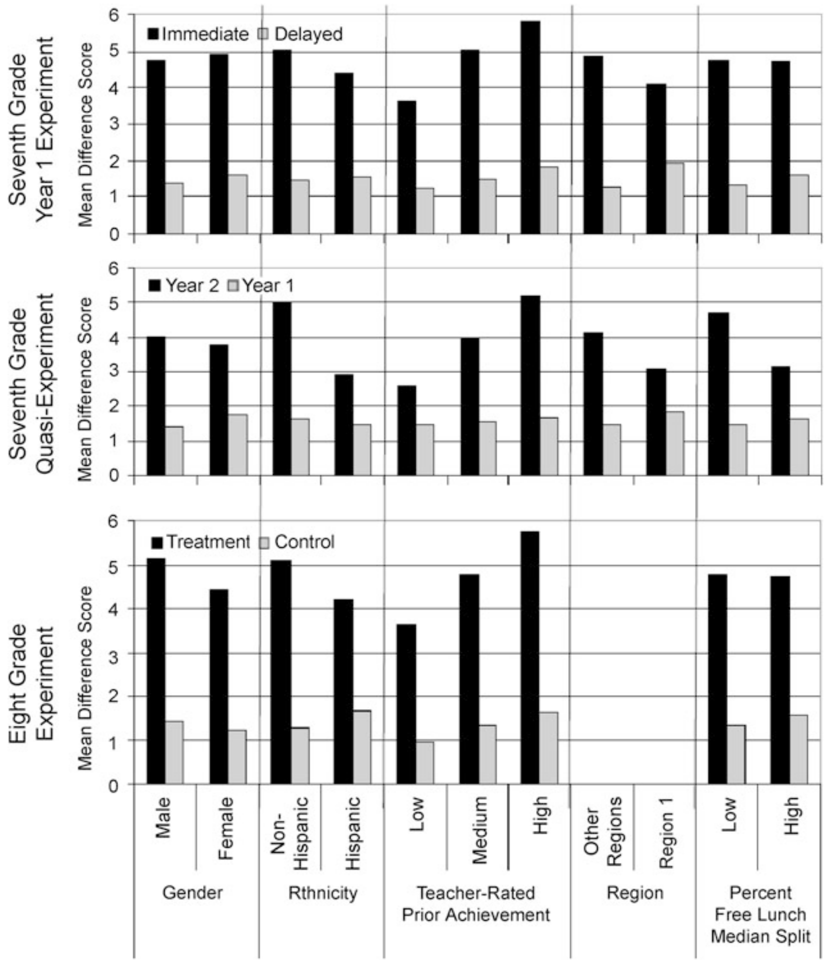

Fig. 2 Student mean difference scores (± SE of total using MLM) at the student level. Note that scores across the Seventh- and Eighth-Grade studies are not directly comparable, as the assessments were different

as medium achieving, and the higher the percentage of students qualifying for lunch programs in the school, the lower the pretest scores.

We then examined the extent to which students in these groups may have had differential gains. Within each study, we ran a series of two-level MLM models predicting student M_2 gain scores, one for each of the five demographic factors, entering the factor independently as a covariate at the appropriate level. These models also included as covariates an indicator for the experimental group and the factor by group interaction.

Figure 2 and Table 3 summarize the gain models. In the two main experiments, population factors did not predict student learning gains except for those students

Table 3 Two-level MLM models run in each study for each factor predicting M_2 gains

Model	Seventh-Grade Year 1 Experiment (Main effect is experimental condition) $N = 1{,}444$ Value	SE	Seventh-Grade Quasi-Experiment (Main effect is year 1 vs. year 2) $N = 997$ Value	SE	Eighth-Grade Experiment (Main effect is experimental condition) $N = 657$ Value	SE
Unconditional						
Intercept	3.03***	0.275	2.82***	0.252	3.26***	0.377
Level 2 variance	4.72		1.31		4.88	
Residual variance	8.07		9.16		8.82	
$\bar{\chi}^2_{01}$ †	463.03***		76.85***		174.64***	
Main effect only						
Main effect	3.55***	0.343	2.46***	0.179	3.26***	0.544
Intercept	1.34***	0.236	1.63***	0.280	1.44***	0.416
Level 2 variance	1.55		1.53		2.11	
Residual variance	8.07		7.65		8.84	
$\bar{\chi}^2_{01}$	134.21***		109.66***		51.55***	
School is in Region 1						
Main effect	3.74***	0.377	2.83***	0.204	3.26***	0.544
Reg. 1	0.35	0.611	0.62	0.760	0.00	
Reg. 1 interaction	−1.10	0.909	−1.48***	0.415	0.00	
Intercept	1.28***	0.262	1.47***	0.314	1.44***	0.416
Level 2 variance	1.55		1.62		2.11	
Residual variance	8.07		7.56		8.84	
$\bar{\chi}^2_{01}$	120.24***		110.34***		51.55***	
Free/reduced-price lunch (%)						
Main effect	3.57***	0.344	2.55***	0.177	3.27***	0.553
SES	0.53	0.877	0.52	0.975	1.03	1.847
SES interaction	−1.25	1.247	−3.53***	0.636	−2.35	2.500
Intercept	1.35***	0.236	1.59***	0.263	1.41***	0.423
Level 2 variance	1.54		1.31		2.17	
Residual variance	8.07		7.45		8.84	
$\bar{\chi}^2_{01}$	120.34***		81.42***		50.09***	
Student is Hispanic						
Main effect	3.73***	0.383	3.49***	0.259	3.63***	0.570
Hisp.	−0.25	0.282	0.10	0.292	0.40	0.506
Hisp. interaction	−0.41	0.391	−1.84***	0.351	−1.04	0.603
Intercept	1.47***	0.269	1.56***	0.293	1.33**	0.433
Level 2 variance	1.49		1.22		2.05	
Residual variance	8.05		7.41		8.82	
$\bar{\chi}^2_{01}$	123.96***		75.04***		49.43***	

Table 3 (Continued)

Model	Seventh-Grade Year 1 Experiment		Seventh-Grade Quasi-Experiment		Eighth-Grade Experiment	
	(Main effect is experimental condition)		(Main effect is year 1 vs. year 2)		(Main effect is experimental condition)	
	$N = 1{,}444$		$N = 997$		$N = 657$	
	Value	SE	Value	SE	Value	SE
Student is Female						
Main effect	3.49***	0.374	2.60***	0.246	3.35***	0.594
Female	−0.01	0.211	0.20	0.253	−0.51	0.411
Female interaction	0.14	0.307	−0.27	0.359	−0.17	0.505
Intercept	1.35***	0.259	1.53***	0.308	1.69***	0.459
Level 2 variance	1.54		1.52		2.06	
Residual variance	8.08		7.66		8.78	
$\bar{\chi}^2_{01}$	133.05***		107.46***		49.89***	
Teacher-rated prior achievement						
Main effect	3.45***	0.387	2.56***	0.261	3.02***	0.609
High group	0.47	0.273	0.36	0.315	0.48	0.490
Low group	−0.35	0.256	−0.30	0.303	−1.11*	0.505
High interaction	0.77	0.399	0.69	0.435	0.11	0.614
Low interaction	−0.81*	0.379	−0.91*	0.424	0.41	0.626
Intercept	1.33***	0.264	1.61***	0.303	1.70***	0.470
Level 2 variance	1.55		1.41		2.05	
Residual variance	7.76		7.36		8.67	
$\bar{\chi}^2_{01}$	141.98***		97.42***		45.97***	

***$p < 0.0001$; **$p < 0.01$; *$p < 0.05$

† $\bar{\chi}^2_{01}$ statistic is an adjusted chi-square statistic from a likelihood ratio test of the given model against a model without random intercepts. See Gutierrez et al. (2001) for details.

Note: Full model is $Y_{ij} = \gamma_{00} + \gamma_{01}T_j + \gamma_{02}X_{ij} + \gamma_{03}T_j * X_{ij} + r_{ij} + u_j$, where X_{ij} may be a level 1 or level 2 covariate. All models within an experiment fit are estimated on identical sets of cases

rated as low achievers. However, in the Seventh-Grade Quasi-Experiment, ethnicity, region, and percentage receiving free or reduced-price lunch in the school negatively predicted learning gains. The specific findings were as follows:

1. *Student gender.* Whereas boys started out with higher pretest scores, there were no main effects or interactions for the learning gains.
2. *Student ethnicity.* Although Hispanic students started out with lower pretest scores, there were no main effects or interactions for learning gains in the two main experiments. In the Seventh-Grade Quasi-Experiment, however, there was

an interaction such that Hispanic students using SimCalc in Year 2 had lower learning gains than their non-Hispanic counterparts.

3. *Teachers' ratings of student prior achievement levels (low, medium, and high).* In all three studies, students at all three achievement levels gained more in the SimCalc replacement unit than their peers studying the ordinary curriculum; however, there were also interactions in the seventh-grade studies (but not the Eighth-Grade Experiment) such that students in the SimCalc replacement units rated as low had lower gain scores than students rated as medium or high.
4. *Region 1 (Seventh-Grade studies only).* In the Year 1 experiment, there was no main effect and no interaction. In the Seventh-Grade Quasi-Experiment, however, there was an interaction such that Region 1 students using business as usual curriculum in Year 1 had higher learning gains than their counterparts in other regions, and students using SimCalc in Year 2 had lower learning gains than their counterparts in other regions.
5. *Percentage receiving free or reduced-price lunch.* Although this variable was a strong negative predictor of pretest scores, there was no main effect or interaction for the learning gains in the main experiments. In the Seventh-Grade Quasi-Experiment, however, there was an interaction such that in Year 2 when students used SimCalc, this variable was a negative predictor of learning gains.

4 Discussion

In these two randomized experiments and quasi-experiment, we found a causal relationship between classroom implementation of a SimCalc replacement unit and student learning of more advanced mathematics. Several findings held true across all studies. SimCalc students learned advanced aspects of the target mathematics concepts (M_2) without sacrificing gains on the mathematics measured by the state test. Indeed, for the simpler aspects of the target concepts (M_1), students of teachers who used the SimCalc replacement unit showed a trend toward greater gains that was nonsignificant in the two experiments and statistically significant in the quasi-experiment. These findings are consistent with the SimCalc program philosophy of increasing opportunities to learn advanced mathematics within the context of the topics already included in the curriculum.

In addition, we found the main effects comparing treatment and control students to be robust across demographic groups. Our sample included the more cosmopolitan Dallas-Fort Worth and Austin areas, as well as the uniquely Texan western and border regions of the state. Schools within these regions varied in poverty and prior achievement levels. Within those schools were teachers with different backgrounds, practices, beliefs, and attitudes. And within the schools were boys and girls who came from White, Hispanic, and other ethnic backgrounds and had different levels of prior achievement. Across the five demographic categories we investigated, students using the SimCalc treatment interventions outperformed their control student counterparts.

While these main effects were significant, as shown in Fig. 2, there were also differences among demographic subgroups within the treatment conditions. These differences were statistically significant in the Seventh-Grade Quasi-Experiment only. In comparisons in the Seventh-Grade Experiment Year 1 and Eighth-Grade Experiment, we found that while gender, ethnicity, and socioeconomic status were associated with students' baseline test scores, there were no statistically significant differences within these groups in learning gains. In the Seventh-Grade Quasi-Experiment, however, ethnicity, region, and socioeconomic status were associated with both baseline test scores and learning gains. We conjecture two possible explanations. An important shift in the population occurred in the Seventh-Grade Quasi-Experiment; many teachers in Region 1 dropped out. While other poor and Hispanic campuses remained in the study, these campuses may differ from the campuses in Region 1. Another possible explanation is suggested by teacher interviews: after teaching the unit a first time, teachers reported a belief that it was more appropriate for high achieving students (a belief which is not supported by our data). Teachers in the quasi-experiment were teaching with SimCalc a second time and may have oriented their teaching away from traditionally underachieving students.

As Fig. 2 also shows, learning gains were different in the treatment group among students rated as low, medium, and high in prior achievement. These trends were only statistically significant in the Seventh-Grade Studies, such that students rated as low prior achievers showed significantly lower learning gains than their medium and high achieving counterparts. These trends were not statistically significant in the Eighth-Grade Experiment (though there was a main effect across treatment and control students). This is an important issue, and we are triangulating data from many sources and conducting further research to better understand this issue. For example, in interviews, some teachers expressed a belief that SimCalc is only good for their high performing students. Findings will be reported in future articles.

As in any experiment, these findings should be interpreted with caution. First, the gains applied to more advanced (M_2) mathematics. Consequently, schools may not see benefits unless they assess more advanced reasoning. Second, the results were obtained in Texas, a state with a long record of a stable standards-based educational system and an ability to implement a train-the-trainer model across regions. Results may vary in states with different contexts. Third, although we view replacement units as a good strategy to fit within school constraints, the tested replacement units occupied only a modest amount of instructional time. We do not yet know the consequences of more extended uses of such units and do not necessarily recommend using SimCalc every day; SimCalc use may be most useful when targeted specifically at the conceptually advanced aspects of mathematics learning. Fourth, we worked with volunteer teachers and do not know how well nonvolunteer teachers would fare. Fifth, we tested an intervention that incorporated only one kind of software and not others. Other software and hardware technologies emphasize dynamic representations, including graphing calculators, dynamic geometry software (e.g., The Geometer's Sketchpad®, Cabri Géomètre), and dynamic statistics packages (e.g., TinkerPlots®, Fathom). But there are also many technologies for mathematics learning that are not included in this family. We do not know whether these results will generalize within or beyond the category of representational tools or dynamic mathematics tools.

The intervention might have greater impact with more attention on the interaction between teacher-reported achievement level and student learning gains within classrooms. In both the Seventh- and Eighth-Grade studies, teacher-reported achievement expectations correlated with student gains. In interviews after implementation of the intervention, we noted that many teachers reported a belief that these materials are more appropriate for their high-achieving students. To the contrary, our findings suggest that the materials are better than the existing materials for students in all teacher-reported achievement categories. It could be that with further professional development, teachers could learn to more effectively use these materials with students they believe are low or medium achievers. In case studies conducted within the context of our experiments, we are examining this possibility.

5 Conclusions

The slogan of the SimCalc program is "democratizing access to the mathematics of change and variation." Given the robust findings, it is fair to say that the integrated SimCalc approach provided students in a wide variety of settings with access to more advanced mathematics while providing ample opportunity for them to make progress on the basics for which schools are most accountable. "Democratizing" can have multiple meanings. Our preferred meaning is that SimCalc provides a wide variety of students a realistic opportunity to learn more advanced mathematics; this meaning was confirmed in these experiments. Another meaning would be "closing achievement gaps." Although we are concerned about achievement gaps and are conducting further analyses and research to understand students' differential performance with SimCalc, we doubt that any 2–3 week intervention can address achievement gaps rooted in deeply structural societal conditions nor do we find reasonable to hold a 2–3 week intervention accountable to that standard.

It is perhaps particularly interesting that this approach enabled students *both* to learn the basics as required by federal and state mandates *and* to learn more advanced mathematics on the pathway to Algebra, an important policy goal. If we had only measured the basic skills required in Texas, we would have obtained null results. Technology may be particularly valuable in mathematics education when educators seek to go beyond the basics. Educators who wish to go beyond the basics may be able to use representational technology to intensify instruction and thus cover both the basics and more advanced skills and concepts.

In terms of broader recommendations to the field, we see this work as suggesting that less emphasis should be placed on the value of technology alone and more on interventions that deeply integrate professional development, curriculum materials, and software in a unified curricular activity system. We select the word "activity" with care based on our observation that all elements of the SimCalc intervention align around enacting particular activities in the classroom (in contrast to a focus on lessons, assessments, or projects). Through our research, we observed the complexity and variability in implementing these activities in classrooms. More research is needed to understand the design features of curricular activities that allow for adaptation to different student populations and teaching styles without undermining effectiveness.

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Learning and Participation in High School Classrooms

Sara Dalton and Stephen Hegedus

1 Introduction

We report on the development and impact of a SimCalc MathWorlds® (hereon referred to as SimCalc) Algebra 2 package developed at the UMass Dartmouth Kaput Center. This work is part of the U.S. Department of Education, Institute of Education Sciences-funded project, entitled "Democratizing Access to Core Mathematics Across Grades 9–12." The findings presented here are the results of the third year of a four-year longitudinal project in High School algebra focused on Algebra 2 (PI: Stephen Hegedus, IES Award #: R305B070430).

Our overall aim of this chapter is to present key findings of the SimCalc Algebra 2 study using both quantitative and qualitative methods. The research questions addressed are:

1. Do students learn more from a SimCalc Algebra 2 software and curriculum package as measured by a mathematical content test, which meets the State of Massachusetts standards for mathematical learning compared to what they learn from their current classroom materials?
2. How is the student participation and interaction in a SimCalc connected classroom influenced by the features of the technology?

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Our research investigated a curriculum package across eight school districts of varying size and diversity in Massachusetts, USA. The Algebra 2 curriculum materials were developed during the first year of this four year project, piloted and refined during the second year of the project, implemented again in a larger number of randomly assigned classrooms in the third year of the project, and utilized in a subset of classrooms in the fourth year of the project as a replication study. The SimCalc Algebra 2 package consists of the SimCalc software, specific software documents paired with teacher and student written curriculum materials, and hardware (TI graphing calculators and TI-Navigator™).

Qualitative and quantitative data were collected in both randomly assigned SimCalc classes and in randomly assigned control classes across various school districts. Quantitatively, our findings support previous research that higher learning gains occurred in the SimCalc classrooms as compared to the "business-as-usual" classrooms. This also held true when we ran further analyses to control for any effect the pretest might have had on the posttest; there was still a significant amount of variance accounted for by experimental group in the favor of the SimCalc classrooms. We also present data on changes within specific content categories of the mathematics content test that was administered.

Qualitatively, we analyze two classroom case studies from two different SimCalc activities. A strength of this research is the amount of classroom video data collected from every participating classroom with a focus primarily on students but not excluding the role of the teacher. It is through the qualitative classroom video that we investigated how student participation arises in the classroom and exemplify various forms of participation that arise at the whole class level and at the small group level. While some traditional forms of participation arise in these case studies—for example, bids for the floor and identification of self—these case studies were selected to exemplify the various forms of participation that are demonstrated in a SimCalc learning environment. We use three analytic tools from discourse analysis to help us defend this argument. Additionally, we will present specifics about the SimCalc software and the structure of the intervention to give readers a clearer depiction of the package as a whole.

We argue that the representational affordances of the software, the structure of the activities, the encouraged interaction amongst students, and inclusion of connectivity affect students' learning and participation in the classroom in positive ways.

2 Literature Review

Over the past 10 years, we have studied the impact of combining two technological ingredients on learning and participation in algebra classrooms. The first is the representational affordances of the SimCalc software, which allows students and teachers to manipulate and construct mathematical functions as well as show various representations of these functions at once. The second is the new connectivity affordances of robust devices combining inexpensive hand-held devices and computers across wireless networks (Roschelle and Pea, 2002). Such a connected classroom allows representational fluidity to be distributed across the classroom and reformat the interaction patterns between students, teachers and technology. Such technology has roots in more than a decade of classroom response systems, most notably ClassTalk™ (Abrahamson, 1998, 2000), which enable instructors to collect, aggregate and display (often as histograms) student responses to questions. We have extended such work by exploring how students can express themselves in mathematically meaningful ways through representations that can be shared and replayed in a private or public mode. The activity structures that we have developed resonate deeply with broader views of learning as participation (Lave, 1988; Lave and Wenger, 1989, 1991; Matos, 2010) but the demands for individual contributions establish participation structures with hard edges and little room for legitimate peripheral participation. Our theoretical design principles format the analytical framework necessary to understand how the salient presence or absence of student work is central rather than a marginal contributing factor. Hence we focus on "instantiated" ahead of "situated" (Kirshner and Whitson, 1997) forms of cognition because it is our position that mathematical experience emerges from the distributed interactions enabled by the mobility and shareability of representations.

In addition, we build on the work of Nemirovsky and colleagues (Nemirovsky and Noble, 1997; Nemirovsky et al., 1998), which stresses the importance of the student's experience being mathematical. As students participate in mathematical ways, ownership of their constructions can become personal and deeply affective, triggering various forms of interaction after their work is shared and projected into a public display space. This joint experience that becomes shared in a social space through aggregations of student constructions is similar to others' work on participatory simulations (Resnick et al., 2000; Stroup, 2003; Stroup et al., 2005; Wilensky, 1991; Wilensky and Resnick, 1999; Wilensky and Stroup, 1999, 2000). For teachers, the shared work in a public display space can change the nature of teaching by altering how participation structures can be defined and controlled, how attention can be managed, how information flows and can be displayed, and how pedagogical choices and moves are made in real time (Hegedus and Moreno-Armella, 2009).

In this chapter, we investigate how the mathematical content and activity structures of the SimCalc Algebra 2 materials play out in the classroom. We look at both whole class discussions of collected student work and within small group interactions amongst students, as students work to construct mathematical objects. We build on Sfard's (2008) work that defines thinking as communicating and add that a students' interaction with the technology, both the graphing calculator in the hands of each student, or the shared upfront space, is also a form of socially and technologically mediated thinking and hence these actions and interactions are important to study. The technology allows a teacher to aggregate student work and project student work from the teacher computer. We posit that this plays a central participatory role and supports co-action (Moreno-Armella and Hegedus, 2009) between the students and the representational affordances of the software. We analyze the display space as co-constructed by students and examine how the students can guide what is part of the space and at the same time the display space guides the students and teachers by offering visual feedback that can mediate meaning making at a whole group level. Students exhibit agency by developing ideas, building on the ideas of others, making connections across representations, making predictions or generalizations of families of functions, exploring and discussing the properties and attributes of different types of functions, and participating in a discourse about the mathematics.

3 SimCalc Software

In the study we report on here, we used SimCalc in conjunction with Texas Instruments Navigator™ Wireless Network as described elsewhere in this book (see chapter by Brady, White, Davis, and Hegedus, this volume). SimCalc takes advantage of wireless connectivity in the classroom by enabling teachers to set up a roster of students prior to class, have students log in from their desks on graphing calculators, send activities down to students effortlessly, and collect or receive student work via this network. With work collected, a teacher can display student work in the up-front space for the whole class to analyze and discuss. However, this display is not shown all at once. Instead, aggregated work emerges in various dynamic representations at the teachers' discretion with suggestions made within the curriculum. The curricular materials encourage teachers to ask for predictions of what the student work will look like based on the activity structure in the form of open ended questions like, "what do you expect to see in the motion when I run the animation of everyone in the class?" Or, "what do you expect to see in the rate graph when I show everyone's work?"

Students typically work on the activities in groups constructing functions either symbolically, by editing a parameter in an equation, or graphically, by moving graphical "hotspots" (Moreno-Armella et al., 2008). For example there is a hotspot controlling the slope of a linear function. This hotspot keeps all other aspects of the line segment fixed while only allowing the slope of the function to vary using the up and down arrows keys on the calculator (see Fig. 1).

This is important as it allows students to see how a change affects other representations such as a simulation, an algebraic expression, or a tabular representation. Each of these representations provides feedback to the students of the change made and its effect on another representation. It is also important to note that every representation is not always available to the student. For pedagogical purposes, the curriculum designers or the teacher can restrict the representations available to the student to focus attention or stress particular relationships without overloading the students with too many representations at once (Kaput et al., 2007). The *Math Object Properties* window enables the teacher or activity designer to easily restrict representations available to view and edit in the activity.

The activities are designed to take advantage of mathematical variation within groups, across groups, and sometimes both. This provides students the opportunity to analyze their work in relation to the work of their peers and vice versa.

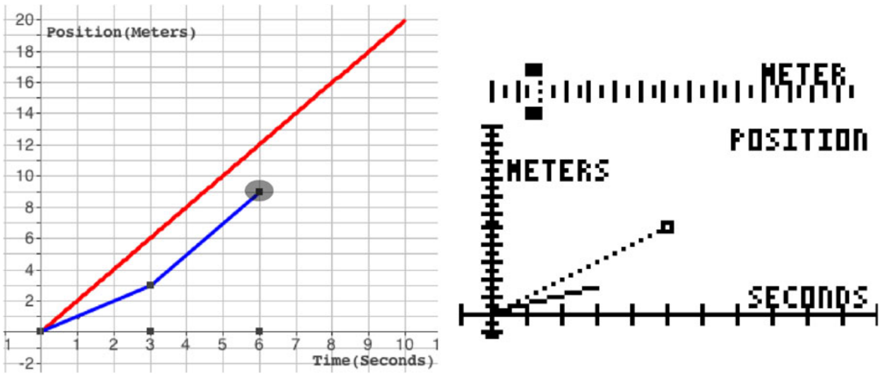


Fig. 1 Position graph in SimCalc MathWorlds® for Computers (*left*) and SimCalc MathWorlds® for TI Graphing Calculators (*right*) with hotspots highlighted

4 The Algebra 2 Package

The SimCalc curriculum philosophy of observing and understanding change across representations of functions aims to provide access to understanding the core properties of functions. This is at the heart of calculus and begins with algebra. In order to extend our previous work with linear functions (i.e., SRI 7th and 8th grade materials—see Knudsen (2010) and the Kaput Center's Algebra 1 package—see http://www.kaputcenter.umassd.edu/products/curriculum_new/), and introduce new functions such as quadratic and exponential functions, we wanted to make use of derivative functions (and representations of derivative functions) to enhance access to these core ideas. For example, most students at this level of high school know how to plot a quadratic function via a graphing tool or calculate its roots via a formula, but data in our study provides evidence that few know why it is shaped the way it is, how each coefficient in $y = ax^2 + bx + c$ affects the shape of the graph and what variation it represents, and the implications of such variation in modeling contexts. Examination of quadratic and exponential functions at greater depth is often left to coursework in calculus, a level most high school students in the U.S. will not reach.

The Kaput Center team, under the lead of author Stephen Hegedus, has developed a series of activities that introduce quadratic functions via linear velocity. These activities are based upon prior completion of coursework on linear position functions, so students can represent linear graphs as $y = mx + b$, but, in a velocity context. As such, labels (graphs, units, axis, and variables) become especially important. Still, students do not always realize such subtleties, and so simulations become an important executable representation to help them make sense of variation with "familiar" graphs. A linearly increasing velocity motion, with marks dropped every second for example (see Fig. 2), illustrates a different representation to a constant velocity motion. Moving to an algebraic form, we need to be careful and prepared to support such shifts.

The aim of our Algebra 2 curriculum materials is not to just replace existing algebra curriculum, but transform the core concepts normally covered to improve students' success with procedural and conceptual thinking in deeper, more sustainable ways. Participation in the activity is central to our approach. Critical themes that permeate our entire curriculum package include:

- Part 1. Interpreting motion
- Part 2. Graphical interpretation
- Part 3. Using aggregation to create a family of quadratics ($y = kx^2$) via parametrically varying rate graphs

5 Trajectory of Curriculum Package

The activities in this package start with individual and whole class work and then move on to group work for a sequence of aggregation activities that depend on the class being divided into groups of 3 to 5 students. As in most SimCalc materials, small group work is intended to lead to whole class discussions of student work and the existing relationships.

The SimCalc Algebra 2 curriculum package begins by comparing the motion of a car controlled by a linear function with that of a car controlled by a quadratic function. The purpose of this preliminary activity is to examine and analyze the properties of each function in various representations in order to identify differences between a constant rate and a varying rate. At the start of the activity, only the motion is visible, and various representations are introduced as the activity progresses in order to build connections amongst the various representations available in SimCalc (graphs, tables, algebraic expressions, and motion). Starting with motion provides opportunities for students to describe what they are seeing, what differences are visible, and use natural language to talk about these two different types of functions. From here, a feature called *Marks* can be turned on. With the Marks feature on, a marker is dropped on a position ruler every 1 second (or however many seconds you define, see Fig. 2). Marks provide a second visual opportunity for students to examine the differences between 2 actors, which we have called Car A and Car B.

The materials encourage the teacher to focus attention to the distances between the marks. Following this analysis of the motion, the teacher can display a Position/Time table for each car and the rate of change of position over time can be found and examined by the class. The rate of change of the rate of change in Position (or what we refer to as second differences) are also found and analyzed. The values are 0 for Car A but they are a constant value of 2 for Car B. The teacher and students can re-animate the cars in the World and connect the motion back to Car B's acceleration, the distance between the marks, and the first set of differences in the table (see Fig. 3).

The teacher can then introduce a new representation to the class: the graphical representations of the functions that control Car A and Car B's motion. First, the class can analyze the position graph and then the velocity graph. Prior to displaying the graphs, students are encouraged to make conjectures about the shape of the

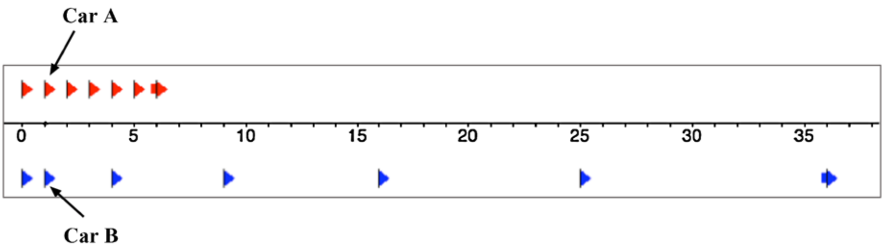


Fig. 2 The *Position Ruler with Marks* dropped for a car moving with a constant speed (*Car A*) and a car moving with a varying speed (*Car B*)

Column of 1st differences

Time	A -- Pos	
00.00	0.00	
01.00	1.00	1.00
02.00	2.00	1.00
03.00	3.00	1.00
04.00	4.00	1.00
05.00	5.00	1.00
06.00	6.00	1.00

Time	A
0.00	0.00

A Position table for Car A with the 1st set of differences

Column of 1st differences — Column of 2nd differences

Time	B -- Pos		
00.00	0.00		
01.00	1.00	1.00	
02.00	4.00	3.00	2.00
03.00	9.00	5.00	2.00
04.00	16.00	7.00	2.00
05.00	25.00	9.00	2.00
06.00	36.00	11.00	2.00

Time	B
0.00	0.00

A Position table for Car B with the 1st and 2nd sets of differences

Fig. 3 Time/Position tables, with differences shown, for *Car A* and *Car B*

functions in each graphical representation drawing on previous analyses from the table.

This work provides the basis for the continuing exploration of quadratic functions in the SimCalc package. In the second unit, students continue examining the attributes of a quadratic function in a position graph that controls the motion of a runner but through the corresponding velocity graph (rate) for the runner. This unit continues the critical analysis of the graphical space and connections between position and velocity for a quadratic function. In the third activity for this unit, students work in groups to create a motion for an actor with a systematic linearly varying rate that is dependent on their group number, G. The teacher aggregates the student work, which creates a family of quadratic functions ($y = Gx^2$) of parametrically varying rate graphs. There is a progression in this unit to build up to this construction, which focuses on how the "a" value affects $P(\text{osition}) = ax^2$ and $V(\text{elocity}) = 2ax$. Each student is contributing a component of the family of functions, thus the family of functions is a collective creation of the classroom of students (see Fig. 4).

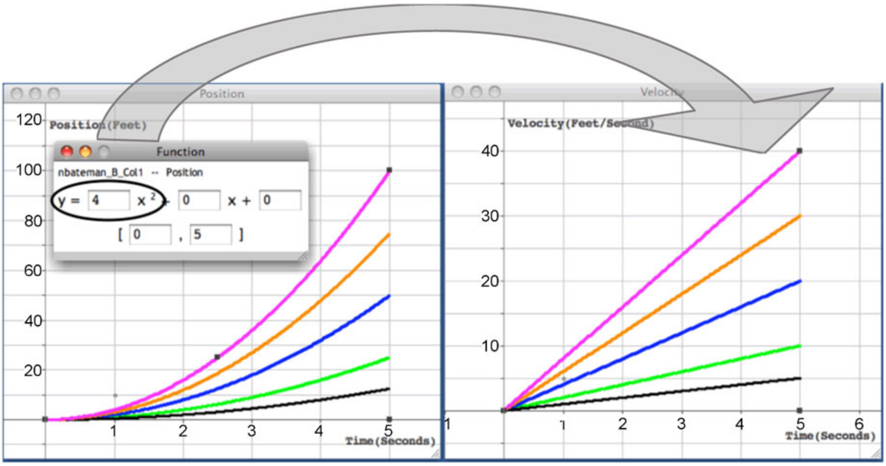


Fig. 4 Position and corresponding velocity (*rate*) graphs for a family of functions

From here, the curriculum introduces the role of the "*c*" value in the standard form of a quadratic function, $P = ax^2 + bx + c$. This unit also introduces the vertex form of a quadratic function with vertical-only shifts in the graph. A further analysis of the role of the "*a*" value in the standard form of a quadratic function delves into problems related to factoring quadratic functions. This analysis of quadratic functions is continued with a focus on investigating the "*b*" value in the standard form using various representations of a function available in SimCalc. The activities are structured to also take advantage of the number of students in a classroom by continuing to exploit the features of classroom connectivity in which students are contributing a function to a systematically varying family of quadratic functions. This investigation of the "*b*" value is continued and culminates in the systematic variation of each value, a, b, and c, at once in the context of a rocket ship flying through space. This unit ties the previous units together while continuing to investigate important properties of the quadratic function such as: symmetry, meaning of roots (solutions), maximum and minimum values, a vertex, and how each of these are represented in a position graph, a velocity graph, a table, and in the function expressions for each derivative. An example of what we mean is as follows: what does it mean for a linear velocity function to intersect the x-axis in terms of speed and velocity? What does this critical point correspond to in the position graph and what does it mean? What are the implications of such a point in the motion context of a rocket ship flying in our one-dimensional world representation?

The curriculum ends by comparing the motions controlled by different types of functions. But this time, the comparison is between the motion of a car controlled by a quadratic function and the motion of a car controlled by an exponential function. The previous analysis of quadratic functions can now be used to analyze the differences between the properties and attributes of a quadratic function and an exponential function with a concrete representation of motion always at hand. The curriculum package concludes after a focus on various representations of exponential functions in an attempt to understand exponential change.

6 Methods

6.1 Design & Participants

The participants for the study we report here were high school Algebra 2 students (15–17 years old) and their teachers from six school districts of varying achievement levels in the South Coast region of Massachusetts. Each school district had two treatment classes and two control classes, which were randomly assigned. The control classes continued using their district adopted classroom materials while the treatment classes replaced portions of their text with the SimCalc materials. The total population for this study included 268 control students and 298 treatment students. The study was designed as a cluster-randomized control trial where students were nested within classes (cluster). This allowed us to compare results within a cluster (class) and across clusters (classes).

6.2 Data Collection

Students in both SimCalc and control classes completed a mathematics content test at the start and end of the intervention (and at similar times for control classrooms). The assessment was composed of standardized test items to measure student's mathematical ability and problem-solving skills before and after the intervention across three content categories: multiple representation items (8), graphical interpretation items (5), and procedural/computational items (6). Of the 19 items on the content test, 9 were designated as conceptually simple type items (M1) and 10 were designated as conceptually complex type items (M2) as in other SimCalc studies (see Shechtman, Haertel, Roschelle, Knudsen, and Singleton, this volume). The M1 category refers to items that are typically one-step problems that are conceptually simple. An example of an M1 item or task would be to ask students to read a specific value in a table or on a graph. The M2 category refers to items that are conceptually more difficult and complex. Often M2 items are multi-step items. Examples of M2 items include reasoning across mathematical representations or comparing two or more linear functions. Graphical Interpretation (GI) items focus on the ability for a student to read a graph and interpret the relationship between two variables. Reading a graph focuses on the ability of a student to find a value for y given an input value x. It also focuses on reading and interpreting more than one value. Graphical interpretation requires students to be able to see significant trends or patterns that might be observable in other representations, e.g., is this a linear function or a quadratic function? where is the slope positive or negative?, etc.

We collected pre and posttest data from 268 control students across 16 classrooms and 298 treatment students across 15 classrooms from 6 school districts in our Algebra 2 Main Study (2009–2010). Sixteen teachers participated in the study; 5 teachers taught only control classes, 4 teachers taught only treatment classes, and 7 teachers taught at least one class of each.

Video data was collected in every SimCalc classroom once before the intervention started and twice during the intervention. The first SimCalc video was collected during the first activity where student work varied across groups and was collected by the teacher. The second SimCalc video was collected about 75 % of the way into the intervention during an activity where groups create unique functions in which the "a" value, "b" value, and "c" value in $y = ax^2 + bx + c$ vary across groups and are examined in multiple representations. The Algebra 2 control classrooms were videotaped once during an activity involving finding and interpreting solutions to quadratic functions. This activity was similar in content to the second SimCalc classroom video and our aim was to capture a snapshot of all control classrooms. For each classroom video, two cameras captured the classroom action: one camera remained stationary in the back of the classroom focused on the teacher and the upfront space. A second camera was positioned at the front of the classroom focused on the students and was operated by a trained research associate from the Kaput Center. This student camera remained stationary during whole class discussions but roamed around the classroom focused on small groups during the group work portion of class.

6.3 Data Analysis

For the student mathematics content pre/posttest, we analyzed overall change in score, change in score within complexity categories for the content test (M1 and M2), and change in score within the content categories (e.g., GI). ANCOVA was used to test how much of the difference in the posttest score between groups is accounted for on the pretest. For the classroom video data collected, we analyzed emerging classroom participation using discourse analytical methods, primarily using the three methods outlined by Hegedus and Penuel (2008) to explore different forms of participation in the classroom.

7 Results

7.1 Student Learning Gains

SimCalc students showed higher learning gains over the control students in total points on the content test ($U = 44{,}899$, $p < 0.001$, $r = 0.166$) and in several content categories, including M1 ($U = 43{,}288$, $p < 0.01$, $r = 0.130$), M2 ($U = 42{,}486$,

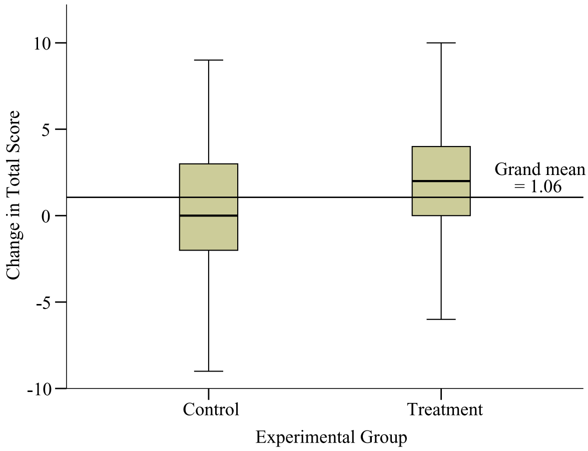


Fig. 5 *Box plot* of total change score for control and treatment non-honor students

$p < 0.01$, $r = 0.111$), and graphical interpretation ($U = 43{,}583$, $p < 0.01$, $r = 0.139$). Each of these differences represents a small effect.

When we disaggregated the data by class level—Honors or Non-honors students (as designated by the school district)—SimCalc students in the non-honor group ($n = 203$, $Mdn = 2$) had higher learning gains compared to control students in the non-honor group ($n = 234$, $Mdn = 0$), $U = 30{,}897.5$, $p < 0.001$, $r = 0.261$. This represents a medium effect size, see Fig. 5.

7.2 *Student Participation in SimCalc Classrooms*

We wished to understand potential reasons for such learning gains. Based on our theoretical framework and design principles, we focus on student participation. We will now present two case study analyses from two of our classroom observations during the Algebra 2 intervention to exemplify the various forms of participation that are modified in a SimCalc learning environment. Three analytic tools from discourse analysis that have been proven useful in analyzing participation in prior SimCalc work (Hegedus and Penuel, 2008) help us to accomplish this goal. These tools are: (1) analysis of bids for attending to and seeing phenomena as rhetorical strategies of participants; (2) explication of participant structures; and (3) analysis of how speakers use deictic markers to position themselves vis-à-vis others and classroom discourse. Such analysis was triangulated with the quantitative results in order to understand the roots of such significant learning gains in contrast to other Algebra 2 classrooms.

These three forms of discourse arise at different points in time during each class. The overarching activity structure can establish a production format since there are various expected stages of development within it. After the teacher has started the class on his or her own computer, the activity gets sent down to the students' calculators. Students then work individually or as part of a small group. The activity structures the work of each student. The teacher then collects student work, and proceeds to show and analyze various representations of each function. This cycle—*send activity*, *students construct*, *teacher collects*, *class analysis*—can be iterated various times until the class ends or the activity is completed. Our activities are designed so that this cycle is completed at least once and completion of the activity is within 90 minutes.

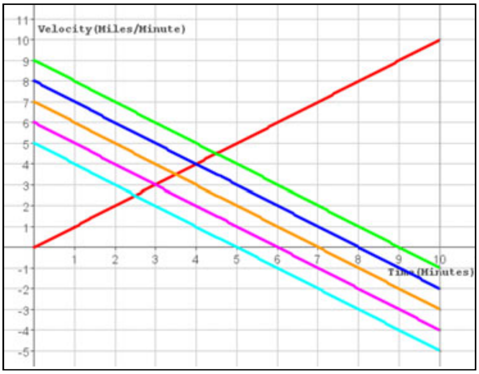
7.3 Case Study #1

This case study focuses on a particular lesson, *Blast Off*!, because of its mathematical aims: varying rate, interpreting the x-intercepts of a linear velocity function as a change in direction and associated vertex point of a parabolic graph, and solving a problem set in a motion context. In this activity, students are editing the velocity function symbolically and graphically to control the motion of a rocket. The rocket must start at a position equal to ten times the students' group number and the rocket has to decelerate at a fixed rate of one mile per minute per minute. The final constraint involves a target function; students must end at the same place as the target function, 50 miles away from 0 miles, in a time of 10 minutes. Students can watch the motion of the rocket they are editing alongside the target rocket, as well as view the velocity graph and velocity table for both rockets. Students can also choose to view a position table for their rocket in addition to the motion but no other position representations are available. Mathematically, each group of students is constructing a unique motion and velocity graph via editing $V = -1x + b$ [1] where $y = -\frac{1}{2}x^2 + bx + G$, and G is equal to group number. Students must determine their specific value for "b". Students have previously investigated the relationship between the area under a velocity versus time graph, in this lesson a line, and the total position traveled by an actor. But, this is the first activity in which students encounter a velocity graph that intersects the x-axis. The general class relationship for "b" in $V = -1x + b$ and $y = -\frac{1}{2}x^2 + bx + G$, is the quantity 10-$G$, where G is group number (see Fig. 6). For this activity students were grouped in either groups of three or four students.

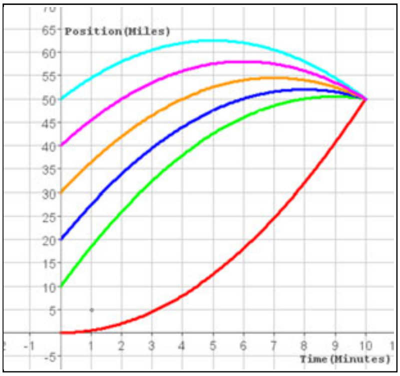
Four students (Nate, Ally, Haley, and Carrie[2]) are working on this task as Group 4. In the following three segments of group discussion, we see the students respond to one another, ask each other questions, explore, conjecture, and test ideas on their calculators. The technology has allowed students to construct functions, animate their constructions, receive feedback, re-edit their constructions, and re-animate again. This cycle is very important for students because it empowers them to see how a parameter of a function is affecting multiple representations.

[1]The activities have used V for the velocity function expression rather than y' because of issues of symbolization, which we will not address in this chapter.

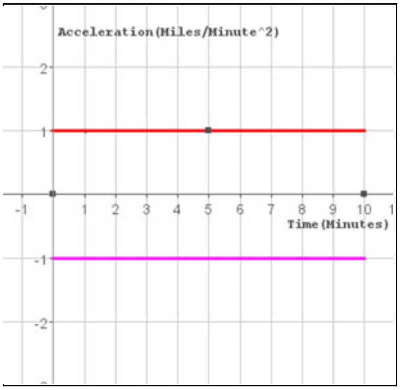
[2]All names are pseudonyms.



Velocity Graph



Position Graph



Acceleration Graph

Fig. 6 The *Velocity graph*, *Position graph*, and *Acceleration graph* for the class set of functions for the Blast Off! activity

In this first excerpt, Ally has edited her graph to be similar to Haley's, rather than run the animation, she seeks feedback from her group members.

Ally: Mine is at −2 *{Ally is referring to her y-intercept. See Fig. 7.}*
Haley: That's not right.
Ally: That's not right?
Haley: Nope.
Ally: Why isn't it right? Nate, why is it not right? Why isn't the line at negative two?
Nate: *{inaudible}*
Ally: *{While running the animation}* Why am I going backwards? Mine is wrong. Cause I'm going backwards. I'm [sic] gonna [sic] have to change that real quick.

With Ally's question "why am I going backwards?," we see two important components with relation to Ally's participation with the calculator and with her group. First, Ally comments that "she" is going backwards. She is identifying herself in the activity as an active agent who has made something happen. She is conveying a sense of ownership over the rocket, which could potentially be a driving force for

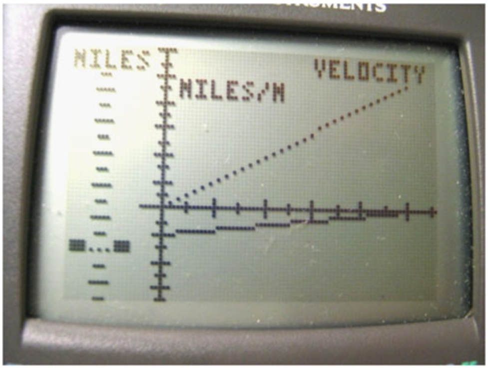


Fig. 7 Ally's graph below the x-axis with the target function above the x-axis

her continued participation in the group and in the activity. She is able to identify what is incorrect based on the motion and her inquiry about the motion prompts her further participation in the activity. Second, we see evidence that Ally does not understand the relationship between the line segment in the velocity graph and the motion representation. There seems to be confusion for Ally between positively versus negatively sloped linear velocity functions; this confusion is evident by her seeming surprise that the motion is traveling in the opposite direction than she expected it too. Although Ally receives some feedback from her group members, the feedback she receives from the motion leads to her surprise. Ally's belief about what will happen when she runs the animation conflicts with what actually happens. None of Ally's peers justify this confliction. It might be that the others in the group do not know why, or they may not be able to express why at this stage. Ally receives feedback based on the mathematical object she constructed. And while the technology does not tell her why, she can interact with her construction and receive more feedback. The SimCalc learning environment enables her to make additional changes to the symbolic representation of the function and see how that change affects the motion and graph and what is different from her previous construction.

Later in the class, Nate and Ally have stopped editing their function. They entered $V = -1x + 6$ on a domain of [0, 2] as the function for their rocket. Haley and Carrie however, continue to work on their calculators keeping their domain fixed as [0, 10]. While it is unclear as to whether Nate understands the relationship between the velocity graph and the position graph, there is evidence to suggest that Ally does not understand this relationship even though she has a solution. In the next interchange, Haley has found both solutions, both positions where the rocket is located at 50 miles, the desired ending position, since the rocket travels up and down.

Haley: Ohh, I did it! I did it! The equation is… $-1x + 6$ from 0 to 10.
Ally: That's close to this. Why? Our's is $-1x + 6$
Nate: Yeah.
Haley: It works.
Ally: How can they both work?
{Ally is referring to both function expressions, $V = -1x + 6$ [0, 2] *and* $V = -1x + 6$ [0, 10].
She is talking as though these are two symbolically different functions}
Haley: It hits 50 twice. *{Haley gestures with her index finger.}*

Fig. 8 Nate gesturing for his group members with an *arrow* tracing the position of his finger

When Haley explains why her solution is the same as Ally's she says, "it hits 50 [miles] twice" and makes a vertical gesture of the motion of the rocket with her index finger in the air. In this gesture, Haley brings her index finger up and then changes direction moving it downward to mimic the motion of the rocket in the vertical SimCalc world. Next, after looking at Haley's calculator, Nate picks up his calculator and re-edits his graph to be the same as Haley's graph. After getting feedback from the motion on his own calculator, and verbally from Haley, Nate makes a gesture of a parabola in mid air with his index finger. He sites the vertex point as the time where the velocity graph intersects the x-axis, 6 minutes.

Although involving much feedback from the technology, this last interchange was really initiated by the social component of the group work. We could say Haley continued to edit her function because she was not satisfied with the seeming conflict between what her group mates had and the component of the goal that stated the rocket should end at a time when the target rocket ended, 10 minutes. As a result of her re-editing, Haley received feedback from the SimCalc software on the calculator that seemed to enable her to make the connection of "hitting twice." Haley then provided feedback to Ally both verbally and with a gesture that the functions were equal despite different domains. After editing on his own calculator, Nate identified the relationship between the velocity graph and position graph for the group.

Ally: Cause you went by our equation before, you just changed the domain.
Haley: It just hits it twice.
{Haley makes a gesture of the vertical motion of the rocket with her index finger in the air.}
Nate: It goes
{Nate gestures a parabola in the air with his index finger, see Fig. 8} It hits twice. The vertex is like, 75 [miles].

Through analysis of student discourse, we found students disagreeing with one another, confirming conjectures, explaining to one another, and interpreting results for one another. While SimCalc served as a medium for confirmation, interpretation and disagreement via multiple representations—specifically the motion representation—the social aspect played a critical role in the groups understanding of the varying rate and the connections between velocity and position (for related work see chapter by Ares, this volume).

7.4 Case Study #2

In this second case study, we focus on *Varying Slopes*, the first activity of the SimCalc curriculum where groups have to create a different function that is collected by the teacher and discussed. The aim of this activity is to continue to introduce students to quadratic functions by examining the attributes of the function in terms of a runner moving with a linearly varying speed. In this activity, there is a target runner who moves at a linearly increasing speed of 1 ft/s every second, the position function expression is $y = x^2$. Students are introduced to such a motion via a rate graph of $V = x$ on the domain $[0, 5]$. The students control the second runner in which they create a systematically varying family of quadratic functions by focusing on rate graphs and use those, with the table and the motion, to understand the attributes of a quadratic function.

In our analysis of this activity, the group number plays an important role in creating systematic variation across the whole class. Each group uses their group number to create the slope of their velocity function or rate graph. In this activity, each group member should produce the same function so there is consolidation within the group, but across groups the functions vary in a systematic way, yielding a family of rate graphs, $V = Gx$, and associated Quadratic position functions.

The SimCalc curriculum and associated training encourages teachers to not show the work of every student at once, but gradually show student work in a fashion that can yield the overall mathematical structure in an emergent way. In addition, teachers are encouraged to focus the attention of the whole class on the overall motion of each person's actor first and to analyze differences and possible contradictions as a group. Our second case study begins at this stage.

7.4.1 Analyzing Bids

The teacher asks the class "What do you expect to see in the World?" which turns the attention of the whole class to the up-front space and initializes the coordination of each student relative to each other. In this activity, students have been focused on velocity graphs but the World is a representation of position. The velocity graph does not have information about the starting position of each student, but the World does and since everyone should be starting at 0 feet, a public display offers the first important piece of information.

The teacher reports that "someone is behind zero" but proceeds to move on. He has initialized the group space and before he runs the animation or initiates the executable representation, he makes sure that there is a set of conjectures verbalized about what form the motion will take and how the groups will structure the overall motion. Before running the animation, the teacher identifies, and points out verbally, which motion each student should have relative to each other.

T: So when I play this is everything Scott has said… is exactly what happens… we should see each group ending at a different position. Does everybody agree with that…
S: {yeah}
T: And they should be in clusters of how many actors per group?
S: 4
T: …with the exception of this group as they had just 3

This is an important move in offering a reference point in the discussion and a field of referential markers to use later on. The teacher refers to the members of the World in a variety of ways—both as actors and as groups—which map to the physical group setup of the classroom.

7.4.2 Participant Structures

Following this animation, the class agrees that not everyone has completed the activity correctly. This evaluation of approximately 16 contributions happens almost instantaneously, which exemplifies one affordance of connectivity in this form. The teacher moves quickly into analyzing what each group should be doing by moving into a different functional space by discussing what the spacing of the marks should be as described earlier on. This feature is another representation of rate that the class is familiar with and which the teacher can talk about abstractly without actually running the animation with marks on. Participation is structured by the activity space and roles have been assigned by the intentional assignment and use of personal group numbers.

The teacher continues to use the upfront space, suspending the animation to analyze a sub-set of the work of the class. Here the up-front space is the focus. The teacher asks, "how many actors should be here?" moving back from individualizing the actors as owned or created by a particular student (although he knows who they belong to) in order to "fix them." to use his language.

T: It looks like we are missing an entire group
If this group is going twice as much where should the next group end at?
S: 50
T: where are you?
T: group 2… who is group 2?
S: Oh-oh {laughter}
T: group 2 is gone
S: it stayed in our graph
S: our graph is perfect
T: I'm not saying you aren't perfect
S: yeah but why am I not up there

The final utterance is another specific form of participation in this learning environment in that the work of a student can be identified by not being visible. First, a contribution is part of the mathematical structure but it cannot be directly analyzed. Second, the student is drawn into participating by their work not being visible.

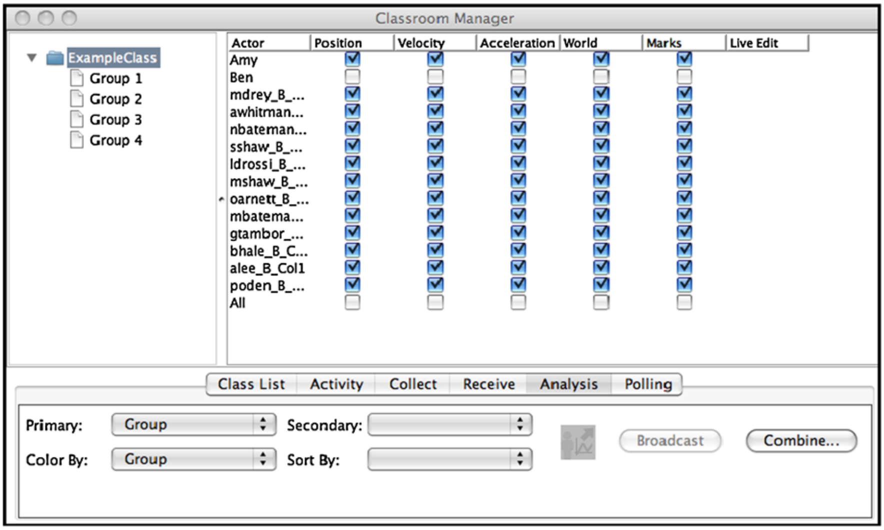


Fig. 9 SimCalc MathWorlds® Classroom Manager

7.4.3 Studying Identity

Here we highlight various utterances that sustain this episode of discourse. The group display is analyzed by the teacher and, following the isolation of missing group, a member of that missing group reacts quite strongly. The student seeks ratification of her contribution and is confused as to why her group is not part of the display projecting her own identity into the display by asking why *she is* not up there rather than her actor or graph.

The teacher moves to switching on marks in the Classroom Management window (see Fig. 9) to analyze each group.

T: Lets do this group by group…
We'll start with group 1, turn the marks on and watch them
S: You can do that?
T: I can do anything… don't you know
{1 minute passes as the teacher changes the display. Downtime}
{Display shows just group 1}
S: I know what color I am
T: When we do this we should expect that each person in group 1 should drop a mark at the exact same spot for the exact same time for the exact same place.… So lets play this even though we know the outcome
…
S: that was probably me…
…
T: we had two people who were in the right position…
S: actually one person was right
T: Did you have a table on this activity?

This cycle again shows the flexibility that the representation system affords to help the students analyze a single group, that students are open in owning parts of that system, and the teacher can employ a pedagogical routine that pre-examines the groups' motions prior to running the animation.

The class moves to completing the activity again which the teacher collects and their work is re-examined in contrast to the prior contributions.

8 Discussion of Findings

The quantitative analysis demonstrated that SimCalc had a positive effect on participating high school students in our study as evidenced by their scores on a mathematics test that aligned with the MA State Standards for mathematics. We argue that the *mechanism* for these learning gains has to do not just with SimCalc's dynamic representations but also how the activity structures of our curriculum lead to deeper student participation.

Classroom interactions (within the software environment, with students, and the teacher) offer the opportunity for students and the teacher to engage with meaningful mathematics and construct meaningful mathematical objects. This opportunity for students to interact with content in significant ways is a critical component for opportunities to learn (Gee, 2008). Students participate with the technology in two ways: in a small group setting and in a whole class discussion of the aggregation of functions. Both are important features of the SimCalc activity design and allow students to interact at a personal (or private) level via exploration with their own calculator and at a social (or public) level as students are encouraged to make mathematical connections between their own work and the work of their peers, and generalize predictions.

In terms of the SimCalc software and affordances of connectivity, each student in the classroom is given two opportunities to learn: (1) the ability to make a personal mathematical contribution to a group activity and (2) a public display to foster group discourse of these contributions. Together, mathematical experience emerges from the distributed interactions enabled by the shareability of students' contributions.

Three forms of participation in the two classroom case studies were analyzed. Identification of one's self and ownership of one's work was important to the SimCalc activity and its participants. Student work was used to foster participation in mathematical experiences. For example, each student was given the ability to construct their own function and their own representation of the motion situation, as well as given the opportunity to analyze their work in comparison to the work of another group. Bids for attending to their work and seeing phenomena and identification of one's work in relation to another's occurred at the student-to-student level and the student-to-technology level. The technology served as a representation for student thought, which could be shared and compared with other students. We saw evidence of this occurring in Case Study #1 when Ally compared her work with Haley's and asked her group, "how can they [the functions] both work?" This question prompted two of her group members to each see across representations: the velocity graph, the motion phenomena, and the graph of the position function to detail an explanation of what has occurred within the mathematics.

The SimCalc activity structure of the materials encouraged personally-meaningful interactions amongst students as well as a transition from a specific group-defined function to a more general class set family of functions, which we believe is a factor in improving student learning that we observed in their content test scores. Some key features were:

- The public display space allows a teacher to project aggregated student work.
- The design of the activity structure in conjunction with the intention of the connected classroom allowed students to participate in new was.
- The SimCalc software provided students an opportunity to create functions and explore various representations.

Together, these features are aligned with important mathematical practices that are important for increased student learning: developing ideas, building on the ideas of others, making connections across representations, making predictions or generalizations of families of functions, exploring and discussing the properties and attributes of different types of functions, and participating in a discourse about mathematics.

9 Implications for Future Research

Our ongoing work is investigating, in more detail, under what conditions students learn the most. How does the teacher influence student learning? In addition to variables related to teacher's background and prior knowledge, how is fidelity of implementation related to changes in student achievement? We have begun to examine fidelity in terms of how teacher's perceptions, values and expectations relate to student learning, their pedagogy and the mathematical affordances of the SimCalc environment. We believe it is important for future work to more carefully examine how learning and participation deeply intersect as demonstrated in the analysis we have presented here.

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Development of Student and Teacher Assessments in the Scaling Up SimCalc Project

Nicole Shechtman, Geneva Haertel, Jeremy Roschelle, Jennifer Knudsen, and Corinne Singleton

A core aspect of examining the efficacy of any intervention moving to scale is the development of valid and reliable measures of impacts on learning. While most standardized or off-the-shelf measures test foundational concepts, few capture the conceptual depth that students can reach using the SimCalc technology and curricula. Thus using such tests for outcome measures could lead researchers to overlook potentially important impacts of SimCalc interventions. Instead, we need to develop more appropriate measures of student learning that are closely aligned with the particular knowledge, skills, and abilities that the SimCalc approach affords learners to develop.

In addition, in the Scaling Up SimCalc project (see chapter by Roschelle and Shechtman, this volume), we wanted to study the knowledge that teachers must have to best support their students' learning. Investigating teacher knowledge in complement to student knowledge can inform the design and development of teacher professional development—an integral aspect of a full curricular activity system. Drawing on the work of Ball, Hill, and other researchers in mathematics teaching and learning (Ball, 1990; Ball et al., 2005; Hill et al., 2005; Ma, 1999; Shulman, 1986), our team used the construct of *mathematical knowledge for teaching* (MKT) to develop the necessary teacher-oriented assessments.

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To meet rigorous standards for reliability and validity (AERA, APA, and NCME, 1999), we developed and implemented a set of development processes for student and teacher content assessments. These processes build on best practices and contemporary methods in assessment development.

In this chapter, we provide an overview of the approach and lay out each of the processes, illustrating how they were used to develop the four assessments in the Scaling Up SimCalc project. Two assessments were for students; these tested the knowledge, skills, and abilities taught in the project's 3-week replacement units at the seventh- and eighth-grade levels. The other two assessments were for teachers, testing the mathematical knowledge necessary to teach these units. Note that greater technical detail about the assessments are available in a Scaling Up SimCalc Project technical report (Shechtman et al., 2010a).

By describing the assessments we used and how they were developed, an additional aim is to provide researchers with a methodological approach that can be used in future research and development that examines the impacts of dynamic mathematics approaches at scale.

1 Overview of Assessment Development Processes

To design the assessment development processes, we followed best practices in assessment development (e.g., AERA, APA, and NCME, 1999) and used tenets of evidence centered design (ECD) (Almond et al., 2002; Mislevy et al., 2003; Mislevy and Haertel, 2006; Mislevy et al., 2002). ECD emphasizes the evidentiary base for specifying coherent, logical relationships among (1) the complex of knowledge, skills, and abilities that are constituents of the construct to be measured; (2) the observations, behaviors, or performances that should reveal the target construct; (3) the tasks or situations that should elicit those behaviors or performances; and (4) the rational development of construct-based scoring criteria and rubrics (Messick, 1994). This evidentiary base supports both the construct and content validity of the assessment.

Figure 1 illustrates the progression of ECD processes we followed to build assessments with a strong evidentiary base.

In the initial ECD processes, domain analysis and domain modeling, the assessment's conceptual foundation is established. In domain analysis, experts in the content domain articulate the important core knowledge, skills, and abilities to be assessed. During domain modeling, the experts elaborate the structure and content of the assessment tasks to be developed. These processes provide input into the development of a test specification that serves as the blueprint for the overall assessment.

In the second process, conceptual assessment framework, the types of assessment items and their properties are specified.

The third process, assessment task development, is an iterative cycle of developing a pool of potential assessment items, refining them, collecting and analyzing validity data, using the data to refine the items, and perhaps developing more items.

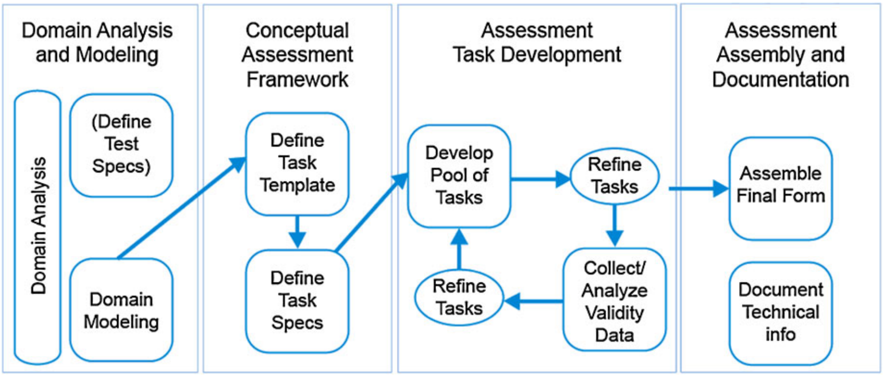

Fig. 1 ECD processes used to design, develop, validate, and document the final version of each assessment

Finally, in the fourth process, assessment assembly and documentation, the validity data are used as a guide to assemble the items to meet the test specifications in the assessment blueprint as closely as possible, and the technical documentation describing the assessment is prepared.

In the following sections, we describe the development of each of the four SimCalc assessments across the processes outlined in Fig. 1.

2 Process 1—Domain Analysis and Modeling

In ECD, the goal of the domain analysis is to establish and articulate important core knowledge, skills, and abilities (KSAs) to be assessed. For each grade level, we performed two major domain analyses. In the first, we developed a conceptual framework that specified the mathematical content that we would focus on in the project's 3-week replacement unit curriculum and the corresponding student assessment. Building on this, we developed a conceptual framework for the mathematical knowledge for teaching (MKT) that would be necessary to teach this content.

Here we describe the domain analyses and the test specifications that were derived from them.

2.1 Domain Analysis for the Curriculum and Student Assessment

We considered several perspectives in specifying the focal mathematical content for each grade level curriculum and assessment. These perspectives can be summarized as follows (see Roschelle et al., 2010 for a detailed description):

1. The mathematical concepts that students could potentially learn from the SimCalc approach, which allows students to connect their mathematical understand-

ing across familiar representations (narrative stories and animations of motion) with key mathematical representations (algebraic expressions, tables, graphs).

2. Texas seventh- and eighth-grade standards (Texas Essential Knowledge and Skills [TEKS]) and our analysis of their content as covered in current Texas mathematics instruction (the research took place in the state of Texas, U.S.).
3. Recommendations of the national standards and focal points (i.e., the National Council of Teachers of Mathematics [NCTM] in the U.S.).
4. Research knowledge about student cognition in the learning sciences and best pedagogical practices for supporting student learning of conceptually difficult mathematics.

We started by identifying the broad concepts at the intersection of these perspectives. This led to the identification of proportionality and linear function as the target mathematics. Among middle school mathematical concepts, proportionality ranks high in importance, centrality, and difficulty (Hiebert and Behr, 1988; NCTM, 2000; Post et al., 1993). In conjunction with our project's mathematics advisory board, which included three mathematicians and three mathematics educators, we developed a mathematics framework for the seventh- and then for the eighth-grade intervention that articulated the focal KSAs for the curricula and assessments. These are summarized in Table 1.

2.2 Domain Analysis for the Mathematics Necessary to Teach the Curricula

As discussed in detail in Shechtman et al. (2010b), we built on the teacher knowledge literature and prior studies' approaches to measuring MKT (Ball, 1990; Ball et al., 2005; Hill et al., 2005; Ma, 1999; Shulman, 1986) to develop the project's framework for the MKT construct. It is important to note that the nature of teacher knowledge is complex and multifaceted and that different researchers define and use the MKT construct in different ways for different purposes. The SimCalc MKT framework was developed to capture the core types of mathematical knowledge teachers would need to support students' learning of the focal KSAs in the SimCalc units, including (1) knowledge of those concepts and (2) specialized mathematics knowledge necessary during instruction to evaluate student thinking and to help students make connections within and across concepts. Our definition of MKT does not include pedagogical knowledge—knowledge of what teaching moves would support students' learning. For example, MKT includes the knowledge a teacher needs to judge the mathematical quality of the typical kinds of work that students do, but does not include knowledge of how frequently they should expect to see various kinds of work or knowledge of what are the best teaching strategies to respond to particular kinds of work. Thus, MKT is essentially mathematical knowledge, but a specialized type that teachers need to make sense of students' mathematical thinking. Figure 4 in the Appendix provides an illustrative example; the question seeks

Table 1 Mathematical conceptual frameworks: Focal KSAs for the Seventh Grade and Eighth Grade curricula and assessments

Framework	M_1 component *Foundational concepts typically covered in the grade-level standards, curricula, and assessments*	M_2 component *Building on the foundations of M_1, essentials of concepts of mathematics of change and variation found in algebra, calculus, and the sciences*
Rate and proportionality for Seventh-Grade	Simple $a/b = c/d$ or $y = kx$ problems in which all but one of the values are provided; the last must be calculated Basic graph and table reading without interpretation (e.g., given a particular value, finding the corresponding value in a graph or table of a relationship)	Reasoning about a representation (e.g., graph, table, or $y = kx$ formula) in which a multiplicative constant k represents a constant rate, slope, speed, or scaling factor across three or more pairs of values, given or implied Reasoning across two or more representations
Linear function for Eighth-Grade	Categorizing functions as linear/nonlinear and proportional/nonproportional Within one representation of one linear function (formula, table, graph, narrative), finding an input or output value Translating one linear function from one representation to another	Interpreting two or more functions that represent change over time, including linear functions or segments of piecewise linear functions Finding the average rate over a single multirate piecewise linear function

Note: M_1 and M_2 refer to the two major dimensions of each framework

mathematical interpretation of an unconventional representation, but does not ask how teachers would respond pedagogically.

The framework comprises six specific types of knowledge, skills, and abilities that teachers should know or be able to do:

1. Link and translate between precise aspects of functional representations (i.e., story, graph, table, algebra)
2. Evaluate the validity of students' mathematical conjectures
3. Differentiate between colloquial and mathematical uses of language and evaluate student statements for their mathematical precision
4. Interpret common unconventional (in many cases, mathematically correct) forms or representations that students are likely to make as they construct their understanding
5. Generate, choose, and evaluate problems and examples that can illustrate key curricular ideas
6. Make connections to important advanced mathematics beyond the unit

To determine specific MKT relevant for each curriculum unit, an expert panel of project staff and consultants examined the cells of a matrix crossing these six types

Table 2 Test specifications

Dimension	Student assessments	MKT assessments
Mathematical content	Items aligned with each of the focal KSAs in Table 1 (such that they create reliable M_1 and M_2 subscales)	Items aligned with each of the focal KSAs in Table 1
	Alignment with Texas state standards (TEKS)	Items aligned with the MKT categories that support the understanding of KSAs in Table 1
Task types	Varied across contexts (i.e., motion, money)	Multiple choice, following the model of prior MKT work in the field
	Diversity of tasks types (about one-third each of multiple choice, short response, construction of multiple mathematical representations)	In seventh-grade studies only, velocity items assessed through multiple choice and constructed response

of knowledge with the specific mathematical content covered in the curriculum and student assessments. We did not seek comprehensive coverage of this matrix, but rather used it as a tool to prompt for the various important facets of teacher knowledge.

2.3 *Test Specifications*

Based on the domain analysis, we developed test specifications that provided high-level descriptions of the mathematical content of interest and the item types to be included in the assessments (see Table 2). All assessments were paper and pencil. The student assessments were designed to be administered by teachers in their own classrooms within one class period (about 45 minutes). The teacher assessments were designed to be administered by workshop leaders or self-administered by teachers at home within about 1.5 hours.

3 Process 2—Conceptual Assessment Framework

The goals of this ECD process were (1) to characterize the types of assessment items and their properties that would be required for each assessment and (2) to develop item templates to guide the development of new items.

For the MKT assessments outlined in our test specifications, few items of these types existed, so we had to generate a pool of new items. A common approach in many large-scale assessment development processes is to create *item templates* that form the structure for assessment items and can be filled in with variable content. Drawing on previous examples of MKT assessments (e.g., Ball, 1990; Ball et al., 2005; Hill et al., 2005), we created a set of item templates that could be filled in with new MKT content. Each template could be used to produce a set of multiple-choice questions situated within the context of mathematics classrooms and teaching. Each template addressed one of the six key facets of MKT and had three parts. The first part was a particular teaching situation that evoked one of the facets of knowledge, such as a teacher presenting a problem to the class, grading papers, examining errors students made on a particular problem, or attending a professional development workshop. The second part was a mathematical question about this situation, and the third part was a list of distracters, including the correct answer.

For the student assessments, rather than develop item templates, we decided to build our pool of items from those already existing in released standardized tests, previously validated instruments, the research literature, the SimCalc pilot (Tatar et al., 2008), and the SimCalc curriculum.

4 Process 3—Assessment Task Development

Assessment task development is an iterative process, moving between item collection or development and empirical validation methods, and iterating back through the processes to revise the items. Here we describe item development and validation processes. The Appendix provides sample student and MKT items (see Figs. 2–4).

4.1 Development and Analysis of Assessment Items

For the student assessments, we collected candidate items from a variety of sources, guided by the assessment test specifications, including:

- Released standardized tests (seventh-grade Texas Assessment of Knowledge and Skills [TAKS], eighth-grade Trends in International Mathematics and Science Study [TIMSS], eighth-grade National Assessment of Educational Progress [NAEP], California High School Exit Examination [CHSEE], the eighth- and tenth-grade Massachusetts Comprehensive Assessment System [MCAS])
- Items used in early SimCalc design research and the SimCalc pilot
- The rate and proportionality literature (e.g., Kaput and West, 1994; Lamon, 1994; Lobato and Thanheiser, 2002)
- The math of change and variation literature (e.g., Carlson et al., 2002)
- Items adapted directly from the SimCalc unit

For the seventh-grade and eighth-grade assessments, the team produced initial pools of 59 and 58 items, respectively.

For the MKT assessments, to develop our initial pool of items, we held a 1.5-day "item camp," a workshop in which individuals with various types of expertise came together to collaboratively generate new assessment items. In addition to the SimCalc curriculum designer and core research team, members of the item camps included an experienced middle school math teacher, math education researchers, mathematicians, and assessment experts. Participants were provided with the test specifications, item templates, an outline of the focal KSAs (Table 1) and MKT framework, the SimCalc curriculum, and various resources such as the Texas middle school mathematics standards, assessments, and textbooks. They were then asked to use these resources to generate items that addressed all the important mathematics teachers should know to support student learning during the unit. In addition, in the seventh-grade assessment, to test mathematics beyond the unit, we included items from previous SimCalc research that assessed knowledge of connections between representations of changes in position and velocity.

For the seventh-grade and eighth-grade MKT assessments, the team generated initial pools of 45 and 57 items, respectively.

4.2 Collection and Analysis of Validity Data

In assessment validation, evidence is accumulated to provide a scientifically sound argument that the assessment items measure the constructs they are intended to measure. The intent is to provide evidence that the assessment will support the intended interpretation of test scores (AERA, APA, and NCME, 1999). Table 3 summarizes the methods of assessment validation that we used and the validity issues they each addressed. Items were also iteratively refined at each step.

Table 3 Methods used for assessment validation

Method	Validity issue
Formative and summative expert panel reviews	Does the task align with the intended content (Table 1)?
	Does the task align with the state standards?
	Is the task appropriate for the intended grade level?
Cognitive think-alouds	Does the task make sense to respondents?
	Is the language clear?
	Does the task elicit the cognitive processes intended?
	Can the task be completed in the available time?
	Can respondents use the diagrams, charts, tables as intended?
Field testing for psychometric information	Does the task elicit a range of responses from students representing different levels of mathematical understanding?
Information for individual tasks	Is the amount of variation in responses sufficient to support statistical analysis?
	What is the distribution of responses by distractor?
	Are there ceiling and/or floor effects?
	Does the task discriminate among students at different levels of the construct being assessed?
Information for overall form and subscales	Are the whole assessment and each individual subscale adequately internally consistent (i.e., reliable)?
	Are there any biases in responses among sample subgroups?

4.2.1 Formative and Summative Expert Panel Review

The first validation method was expert panel review. For each assessment, we conducted a review that had both formative and summative components. Experts were provided with the assessment items and asked to make specific judgments about the items and recommendations for how to improve them.

For the student assessments, we conducted formative and summative expert reviews separately. The formative reviews were conducted early in the item development process, soon after the initial pool of items had been developed. Each summative review was the final step in the assessment development process.

The two formative panels took place in person in Austin, Texas, in a 1-day workshop. There were two subpanels. The first subpanel comprised mathematics education researchers and assessment experts (four for the seventh-grade assessment and three for the eighth-grade assessment). This subpanel focused on making judgments of categorical concurrence, which Webb (1997) describes as an integral aspect of assessment validation that examines the extent to which the same or consistent categories of content appear in both "expectations" and assessments. The experts thus made judgments about the alignment of each item and two sources of expectations: (1) the KSAs in the SimCalc conceptual framework and (2) a focused subset of the TEKS. We also asked them to make recommendations for improving the items. The second subpanel for each assessment comprised two local curriculum and instruction experts (e.g., math supervisor, textbook contributor). This subpanel rated items relative to three aspects of grade-level appropriateness: (1) reading load, (2) computation load, and (3) graphics load. They rated each item on a 3-point scale (*appropriate, somewhat inappropriate, inappropriate*) for each aspect of grade-level appropriateness. They also made recommendations for modifications that would make the items more grade-level appropriate.

For each assessment, data were compiled at the item level and used to classify and refine the items. Items were eliminated from the pool or modified to be more suitable if they did *not* demonstrate (1) high percentage agreement among raters on categorical concurrence classifications, (2) were poorly aligned with the target TEKS, and/or (3) were rated as inappropriate for the grade level. The categorical concurrence data were also used to classify items to determine the degree of coverage of the mathematical content in our field test instrument across the KSAs.

The summative expert panels for the student assessments took place after field-testing (see below). The experts were members of our project advisory board who had worked with us to develop the mathematical conceptual frameworks. The panel members were provided with the refined items and the categorical concurrence classifications determined during the formative review. For each item, they checked off whether they agreed or disagreed with the classification. If they did not agree with it, they were to explain their decision. The summative panel members agreed with each other and with the prior ratings in almost all cases. This summative classification was used to determine content coverage in the final instruments.

We conducted only one review of the MKT assessments with both formative and summative components (findings about teacher knowledge were lower stakes in this project than those about student knowledge). Senior members of the SimCalc research team served as the experts. For each assessment, two experts aligned each item in the initial MKT item pool with the mathematical and MKT frameworks, and recommended refinements to enhance clarity and alignment. These data were used to revise the items and determine coverage of the content in these frameworks.

4.2.2 Cognitive Think-Alouds

The second validation method was cognitive think-aloud interviews. While getting direct test-taker feedback is always important in assessment development, it was essential in these studies for helping us to determine whether students and teachers would use appropriate response processes for new items targeting complex KSAs. We conducted cognitive think-alouds on all the items remaining in the item pool after the formative review (about 30 items per assessment). In a cognitive think-aloud protocol, the student is instructed to speak their thoughts out loud as they do the assessment (e.g., Ericsson and Simon, 1993). An interviewer records this monologue and asks a minimal number of probing questions as necessary but does not interfere with the test-taker's engagement with the mathematics. For each student assessment, we conducted think-aloud interviews with eight middle school students who each did a subset of the items. To provide representation across a range of achievement levels in mathematics, student volunteers were drawn from those identified by local partner teachers as low-, middle-, and high-achieving. For each MKT assessment, we conducted think-aloud interviews with three teachers known from our prior work with them to represent a range of depth of knowledge of mathematical content.

For each participant, an analyst examined videotapes or audiotapes of the think-alouds and documented the time needed to complete each item, the mathematical strategies the test-taker used, the mathematical mistakes made, difficulties in comprehending the problem because of ambiguous or unclear language, unfamiliar terminology, or confusing calculations (when applicable). This information was then summarized for each item and used to eliminate and/or modify items in the pool. We eliminated items that were too easy or too difficult for the test-takers or those for which test-takers could use a construct-irrelevant strategy to solve them (e.g., counting to solve a problem intended to measure proportional reasoning). We modified item instructions, text, and graphics as necessary to increase clarity and refine mathematical logic.

4.2.3 Field-Testing for Psychometric Information

The third validation method was field-tests of the assessments with a large sample of students/teachers for the purpose of gathering psychometric data on individual item performance and overall assessment performance. We assembled the pilot assessments using the categorical concurrence ratings for each of the remaining refined items in the item pool in alignment with the test specifications in Table 2.

We field-tested each of the two pilot student assessments with a sample of middle school students in the classrooms of local partner teachers ($n = 230$ for the seventh-grade assessment and $n = 309$ for the eighth-grade assessment). As indicated by teacher report and the wide range of scores, the students represented a range of prior achievement levels within their grade level. For each of the two MKT assessments, we conducted field testing by mailing the assessment to a national random sample of 1,000 middle school mathematics teachers (the names and addresses were purchased from an educational data service). The response rates were 17.9 % and 12.8 %, yielding 179 and 128 teachers for the seventh- and eighth-grade assessments, respectively. On key demographic variables (gender, age, teaching experience, ethnicity, region type, and first language), the teacher samples were representative of the population of teachers we expected to participate in the Scaling Up SimCalc Project; however, suburban regions, relative to rural and urban regions, were slightly oversampled. These expectations were based on analyses of Texas's Public Education Information Management System (PEIMS), a publicly available database maintained and distributed by the state department of education.

We used classical test theory (CTT) and item response theory (IRT) with the field test data to examine three critical evidentiary concerns for both the student and teacher assessments. First, we examined individual items for the range of possible responses, statistical variation, ceiling and floor effects, and the capacity of the items to discriminate among test-takers at different ability levels (using IRT parameters for a two-parameter logistic model). Second, we examined the internal consistency (i.e., reliability) of the whole form and each subscale. Third, we examined possible biases among population subgroups.

We used these data to refine the instrument for a final version. Items with low discrimination parameters (i.e., items that could not discriminate among individuals of differing ability) or ceiling/floor effects were eliminated or modified. Items were kept that were likely to contribute the most information about the test-taker's ability and to maintain representative coverage of the focal KSAs.

5 Process 4—Assessment Assembly and Documentation

Table 4 presents assessment form and aggregate item statistics. Details about the sample characteristics and how the assessments were actually used in the Scaling Up studies are described in Roschelle and Shechtman, this volume. Note in particular that the internal reliability statistics for each form and subscale were adequate to high.

6 Conclusion

In moving the SimCalc approach to scale, a key consideration was how to measure students' understanding of and teachers' MKT for the particular knowledge,

Table 4 Summary of basic test statistics for student and MKT assessments

Assessment	Whole form			M_1 subscale			M_2 subscale		
	Items	Internal reliability	Pre-test mean (SD)	Items	Internal reliability	Pre-test mean (SD)	Items	Internal reliability	Pre-test mean (SD)
Seventh-grade									
Student	30	0.86	12.9 (5.7)	11	0.73	7.3 (2.6)	19	0.82	2.6 (3.7)
MKT	24	0.80	10.0 (4.5)						
Eighth-grade									
Student	36	0.91	12.1 (7.4)	18	0.79	7.2 (3.7)	18	0.87	4.9 (4.3)
MKT	28	0.80	16.3 (5.0)						

Note: MKT subscale statistics not reported because scores always reported in aggregate

skills, and abilities that the SimCalc approach affords learners to develop. These are challenging to measure because the mathematical content tends to go beyond what is assessed in typical assessments, and few instruments are available to measure teacher knowledge. In this chapter, we have described our approach to assessment development, illustrating its implementation with the four instruments used in the Scaling Up SimCalc Project. Our intention is to provide not only documentation of the technical qualities of the assessments themselves, but also provide researchers with a methodological approach that can be used in future research and development examining impacts of dynamic mathematics approaches at scale.

Appendix: Sample Items

Fig. 2 Seventh-grade student M_1 item

If $\frac{2}{25} = \frac{n}{500}$ then n =

☐ A. 10

☐ B. 20

☐ C. 30

☐ D. 40

☐ E. 50

Annie and Bonnie are running on the same track. They practice several **45-meter** races. For each race, make a line graph that represents their position by time.

Race 1: Annie and Bonnie start at the starting line (0 meters) at the same time, and each runs at a constant speed. Annie finishes the 45-meter race 2 seconds before Bonnie.

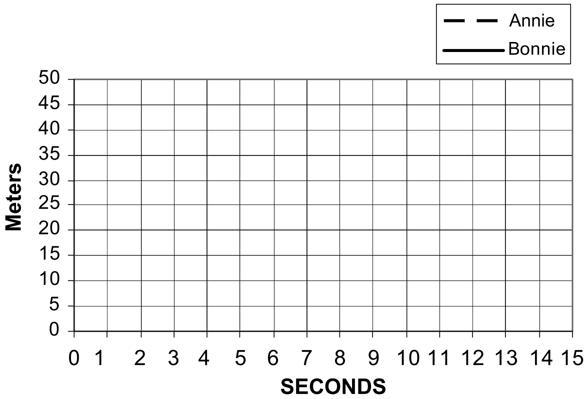

Fig. 3 Eighth-grade student M_2 item (excerpted from a series)

Here is a graph of a 50-meter dash that a student made. Notice that distance is on the x-axis and time is on the y-axis.

Which are true statements about the relationship between the line graph and the speed of the runner? (*Choose all that apply.*)

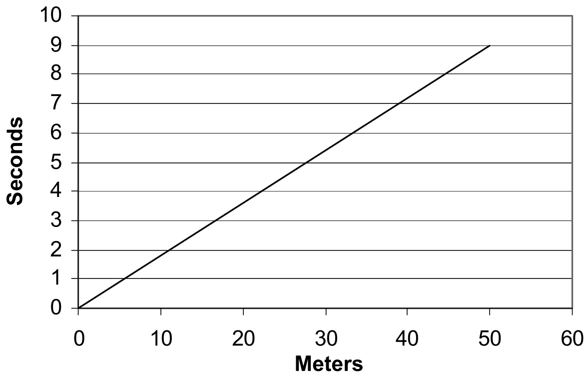

Fig. 4 Eighth-grade teacher MKT item. (Interpreting unconventional forms or representations that students are likely to make)

A. The slope of the line is 9/50 or 0.18 as was the average speed in meters per second of the runner during the dash.
B. The slope of the line is 50/9 or 5.56 as was the average speed in meters per second of the runner during the dash.
C. The slope of the line is 9/50 or 0.18, and the average speed of the runner was 50/9 or 5.56 meters per second.
D. The slope of the line is 50/9 or 5.56, and the average speed of the runner was 9/50 or 0.18 meters per second.
E. None of the above.

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Sustainable Use of Dynamic Representational Environments: Toward a District-Wide Adoption of SimCalc-Based Materials

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1 Background

SimCalc MathWorlds® (SimCalc) has been shown to be an effective tool for democratizing access to the mathematics of change (Dalton and Hegedus, this volume; Kaput, 1994; Kaput and Roschelle, 1998; Shechtman and Roschelle, this volume). In this chapter, we report on the SunBay Digital Mathematics program (SunBay Math), an effort currently in its third year in Pinellas County, Florida, that has the goal of assisting a large, urban school district in adopting SimCalc as an integral part of middle school mathematics education. We have taken a curriculum replacement units approach and built directly on the work described by Roschelle and Shechtman (this volume; see also Roschelle et al., 2010) to increase student learning of important mathematics. We have been guided by research on the sustainability of SimCalc-based replacement units (Fishman et al., 2011; Hegedus et al., 2009) to increase the likelihood that our materials become "a regular part of the instructional repertoire and does not remain a special departure from normal practice" (Fishman et al., 2011, p. 2).

In the following sections, we discuss our efforts at designing and implementing the SunBay Math program in the Pinellas County School District (PCS). PCS is a large, urban school district in Florida, with more than 100,000 K-12 students. PCS was chosen as a district partner because students in the district were underperforming in middle school mathematics, district personnel recognized the potential of

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a dynamic mathematics approach in improving middle school education, and there was overall district support for the use of technology in the mathematics classroom. There was general agreement by all stakeholders that the goal of the SunBay Math program would not be a wholesale change in the way mathematics is taught in PCS middle schools. Instead, the goals of the program would be to identify core areas that are mathematically important and difficult to teach and learn, and to address these areas with relatively short curriculum replacement units of 2 to 3 weeks in length. These units would be designed as "peak experiences," in that they would be dependent upon dynamically rich technology environments (with an initial emphasis on the use of SimCalc MathWorlds®), and would be a clear distinction from teaching and learning using traditional textbooks.

During the past 3 years, the SunBay Math program has garnered strong local support, has built substantial local capacity, has been shown to be effective in increasing student learning, and is showing strong potential to achieve the program goal of sustainable and widespread use in a large school district.

2 Design of the Program

2.1 The Initial Curriculum Unit

The research team, composed of educational researchers and college of education faculty, in collaboration with district personnel, determined that the first curriculum replacement unit would be *Managing the Soccer Team*, the seventh grade curriculum unit described in Roschelle and Shechtman (this volume). This unit was selected because it uses dynamic representations to address important next generation mathematics standards identified by the state of Florida (Florida Department of Education, n.d.) related to rate and proportionality. Initially, activities in *Managing the Soccer Team* address unit rate and proportional functions through the simple analyses of motion at a constant speed. The unit progresses incrementally until the unit ends with analyses of multi-rate functions and an informal expression of the meaning of positive, negative, and zero slope. The unit combines consumable paper materials with SimCalc MathWorlds® software files, for a structured exploration of algebraic representations through connections to real-world topics. *Managing the Soccer Team* presents soccer players running races and team buses traveling from one town to another. For example, using the SimCalc simulation shown in Fig. 1, students were prompted to use the various algebraic representations to find vehicle speeds and write stories to explain patterns of motion.

In other activities non-motion contexts were used, including saving money when buying uniforms and predicting how much fuel vehicles would use. The unit was not created as stand-alone piece of curriculum, but instead was created as part of a curricular activity system that included professional development, paper materials, and technology, all integrated to meet the needs of students, teachers, and schools (see Vahey et al., in press). More information on the curriculum can be found at http://sunbay.sri.com.

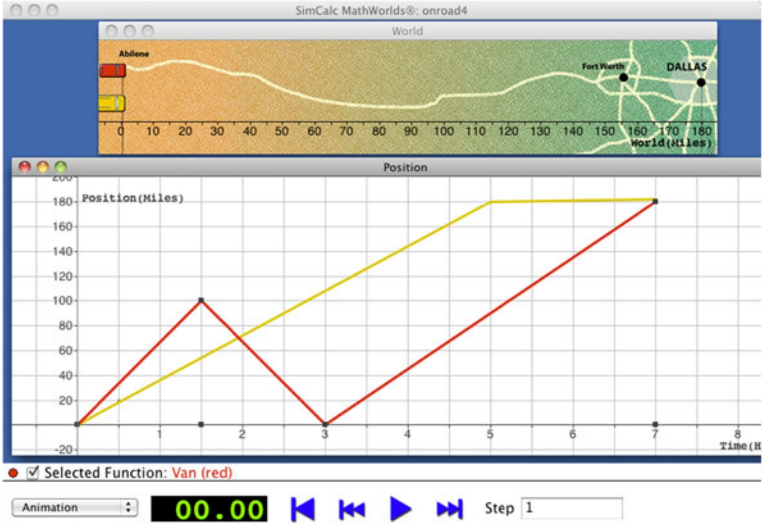


Fig. 1 Example SimCalc simulation and graph

2.2 Extension of Prior Research

While the SunBay effort to build a sustainable and district-wide program was based directly on the SimCalc randomized control trial (RCT) conducted in Texas (described in Shechtman and Roschelle, this volume), there are important differences in the goals, and hence the design, of the programs. The SimCalc RCT was designed to generate findings that were generalizable across a large population of teachers and students; as such, the RCT intentionally recruited a large number of teachers throughout Texas and did not have a critical mass of teachers in any one local area. SunBay Math made the explicit design decision to recruit a relatively small number of teachers, located in a small subset of schools in the district, and to support these teachers in forming a strong teacher community. We expected this small teacher community to provide the core of expertise in the district, and this community could then be leveraged as a resource to aid in scaling the program to the entire district (House, 1994; Kennedy, 2005; Penuel and Riel, 2007).

The RCT required randomly assigning teachers to a treatment or control group, required a significant amount of data generation and collection on the part of participating teachers, and had a monetary incentive for participating teachers so they would complete the data collection activities. While teachers participating in the RCT were enthusiastic about the project, factors from within PCS, as well as factors related to the project's philosophy about sustainability, argued against making the same demands on teachers in the SunBay project. PCS was strongly against making significant demands on teachers, especially those teachers who were not using the SunBay Math materials. Furthermore, the district did not want to have a control group (or even a comparison group), as they felt that this would cause divisions amongst the teachers. In addition, part of the project philosophy was that, to accept the intervention as part of their regular practice, teachers must not be burdened with a large number of demands that are not part of their regular practice or be rewarded with significant incentives to meet such demands. As a result, our research design was based on a one-group sample, with only minor data collection demands placed on teachers.

2.3 Additional Sustainability Challenges

We next address three important considerations identified by Fishman et al. (2011) and Hegedus et al. (2009) that we considered when moving from the research-based RCT to a large-scale sustainability effort: maintaining high-quality classroom practice outside of hothouse research environments, teacher perceptions of coherence and value, and teacher expectations related to student prior mathematics achievement.

2.3.1 Classroom Practice Outside of the "Hothouse"

Well-funded RCTs are "hothouse environments" in which support, funding, and encouragement are plentiful (Fishman et al., 2011). Once the research is over, this support typically fades, leaving teachers to fend for themselves when using the materials. Without ongoing support, teachers may unintentionally create "lethal mutations" (Brown and Campione, 1996), in which the materials are used in a way that is at odds with the designers' original intent. This may be especially likely in cases such as ours where the materials require teachers to deviate from their existing methods of teaching.

We took, as part of our challenge, the creation of an environment that was perceived by teachers to have the positive aspects of a hothouse environment, but that could be sustained without significant ongoing external investment (of course, *initial* investment was required, and, expansion to additional curriculum units would also require investment). For the program to be self-sustaining, it would have to ultimately be run by a local organization whose own sustainability model was aligned with the program goal of teacher support and professional development. SRI International (SRI), the research institute that initiated the program was not such an organization. The University of South Florida St. Petersburg (USFSP) College of Education was a natural partner, as it is the primary provider of professional development to PCS, and USFSP was interested in expanding its capacity in both middle school mathematics teacher education and the use of technology. To create a supportive environment that could address the concern of lethal mutations, USFSP and SRI began collaboration on two parallel tracks: material-specific professional development (PD) and a five-course graduate certificate to increase teacher Mathematical Knowledge for Teaching (MKT) (Hill et al., 2004) and Technological Pedagogical Content Knowledge (TPCK) (Mishra and Koehler, 2006) in the district more generally.

2.3.1.1 Material-Specific Professional Development

The SRI and USFSP teams designed a 3-day workshop as well as a series of monthly PD meetings for participating teachers. The goals of the workshop were to introduce the teachers to the SunBay Math project, the SimCalc MathWorlds® software, and the *Managing the Soccer Team* replacement unit. While the PD was based on the PD provided in the RCT, changes were made to align the experience with state and district expectations and requirements. During the monthly PD meetings, which were not part of the RCT, teachers investigated the target pedagogies, mathematics, and technologies.

The SRI and USFSP teams worked together to create a plan by which SRI International would lead the initial PD sessions, but as the program progressed, USFSP would take more responsibility for leading the PD, until USFSP would be the sole organization delivering the PD. This has resulted in SunBay PD becoming a standard USFSP offering, not an extraordinary program that requires outside support.

2.3.1.2 Program to Increase Teacher MKT and TPCK

The USFSP team also began to design a Certificate program in Middle Grades Digitally Enhanced Mathematics Education (MG DEME). The certificate is composed of five courses. Three courses focus on the foundation of higher mathematics in grades 7 through 12 and beyond: (1) algebraic thinking, (2) geometry and measurement, and (3) the processes of mathematics and data analysis. These courses integrate software such as SimCalc MathWorlds®, Geometer's Sketchpad®, GeoGebra, and TinkerPlots® with activities that exemplify how representation-rich technology environments can transform mathematics teaching and learning. Another course was designed to increase middle school mathematics teachers' technological pedagogical content knowledge based on common classroom technologies such as graphing calculators and presentation software. The final course provides middle school teachers with explorations of reading in the content area strategies. The certificate program was designed with the expectation that approximately 20 % of middle school teachers in the district would participate in the program. Ideally, each middle school in the district will have at least one teacher who has gone through the program, and therefore each school will have one local expert on technology-based approaches to teaching advanced mathematics to all students.

Taken together, these efforts were designed to provide a supportive but sustainable environment for the use of the SimCalc-based materials. By building capacity within USFSP and reducing the role of SRI, a local PD provider is now able to support the program as part of its normal operations. As the program continues, these programs could be supported by ongoing sources of funding, such as local foundations, district funds set aside for professional development, or tuition fees.

2.3.2 Perceived Coherence and Value

Coherence refers to the alignment between the innovation and the myriad other demands placed on teachers (Fishman et al., 2011; Hegedus et al., 2009). These demands include messages from school and district administration about instructional priorities, accountability requirements, and challenges resulting from the makeup of the local school population. Teachers who perceive an innovation to be coherent with other messages and demands are more likely to adopt the innovation. *Value* is a set of personal perceptions about the likely benefit of an innovation (Fishman et al., 2011; Hegedus et al., 2009). While we would expect teachers to value materials with high coherence, this valuation may not always be the case. Materials may be coherent with district requirements, but, if teachers believe that the school's existing approach is more effective than the innovation, they may not value the new materials. For teachers to commit to using a new approach to teaching, we would expect that the materials would have to have high value as well as high coherence.

One place to look for coherence is in the materials themselves. The materials used in the SunBay Math project are based on the prior Texas RCT. While that study showed that *Managing the Soccer Team* replacement unit resulted in statistically significant student learning gains on complex concepts in the areas of rate and proportionality when used in Texas (Roschelle et al., 2010), it was not clear that teachers in Pinellas County Florida would consider these materials to be coherent with their local demands. SRI and USFSP formed a review team to investigate the relevance of the materials to district standards and textbooks, to Florida's Next Generation Sunshine State Standards (NGSSS), and more recently, to the Common Core State Standards for Mathematics (Florida Department of Education, n.d.; National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010). The review team found that the materials needed minor changes to be appropriate for use in Pinellas County. Some changes were purely cosmetic—for instance, in the original unit, the soccer team traveled from one Texas city to another, and the cities were changed to cities in Florida. Some changes were more germane to the mathematics in the unit, such as ensuring that graphs and mathematics terminology were consistent with Florida content standards. Fortunately, the mathematics content was appropriate once these relatively superficial changes were made. The review team then created a mapping between the district's curriculum scope and sequence and the unit to help teachers understand how teaching the unit would help them meet their accountability requirements. Finally, USFSP and SRI worked with district personnel to modify the district pacing guide for teachers in the study, again to allow teachers to build a sense of coherence between the materials and their accountability requirements.

To further increase coherence between the district messages and the materials, we worked with the district superintendent and the mathematics coordinator for grades 6–12 to ensure that they understood how the project goals aligned with district goals. Once the district personnel saw the value of the project, they were willing to make public statements in support of the project, including appearances at the PD sessions, during which they made clear that the use of dynamic representation environments was a strategic direction for the district that could help teachers meet upcoming high-stakes assessments.

To address teachers' perception of *value*, we built upon conversations between SRI, USFSP, and district personnel, that showed general agreement that issues related to the core mathematics content in the unit, rate and proportionality, were difficult to teach using traditional means. In the PD, we made explicit the innovativeness of the SunBay Math materials and the ways in which the integration of technology and paper-based activities made this content area more engaging and accessible to students. These activities gave the design team confidence that as we move away from being a "hothouse environment," teachers can still experience the support necessary to use the materials effectively, while avoiding lethal mutations.

2.3.3 Teacher Expectations Relative to Students' Prior Mathematics Achievement

Fishman et al. (2011) and Hegedus et al. (2009) discuss the "Matthew Effect" (Walberg and Tsai, 1983), which refers to a passage in the Gospel according to Matthew in which those who have gain more, but those with little lose what they have. This has been used to describe the common finding that teachers provide opportunities to learn deep and conceptual mathematics only to high-achieving students (who also tend to be students from high socioeconomic [SES] backgrounds), and so low-achieving (and low-SES) students fall even farther behind. Unfortunately, the "Matthew Effect" was found in their analyses of sustainability based on the RCT. Their analysis of which teachers chose to continue using SimCalc after the formal RCT had ended showed that the greater the students' prior mathematics knowledge and the higher the students' SES, the more likely teachers were to continue using the materials. This was particularly disappointing because the SimCalc materials used in the RCT, which are also the basis for the SunBay Math program, were found to be effective for a wide range of students, including traditionally low-achieving students (Roschelle and Shechtman this volume; Vahey et al., 2010). Clearly, then, simply having materials that are effective is not enough to convince teachers to continue using those materials with their low-achieving students: in SunBay Math, we chose to explicitly address the Matthew Effect. However, there is a danger in focusing solely on students with low prior achievement in mathematics—teachers may then consider the materials to be part of a remedial intervention, resulting in the perception that they should be used *only* with students with low prior achievement.

The team decided that the best way to address the Matthew Effect, while also mediating the risk of being considered a remedial intervention, would be to ensure that classrooms in the first year of use would be chosen such that students with a wide range of prior achievement scores were included in the study. In addition, the project made clear to district personnel, especially the teachers, district superintendent, and K-8 mathematics coordinator, that the project was focused on a wide range of learners. We believed that only through an explicit focus on how the materials were useful to a wide range of students would teachers recognize the value of the SunBay Math materials and hence advocate for the use of the materials with all their students.

3 Implementing the Program: Year 1

3.1 Starting Up

Investigation into funding of the project found that two local foundations were willing to fund the initiative, and, in June 2009 the Helios Education Foundation, the Pinellas Education Foundation, and PCS itself provided funding for the pilot year of the SunBay Digital Mathematics program.

While the overarching goal of the program was scale-up and sustainability, all parties involved in the project agreed that the materials would have to be shown to be effective before it was worth the investment in scaling up. So, although the program was designed from the beginning to eventually scale up to a district-wide program, the initial year was focused on documenting student learning. In addition, the project had to build the infrastructure necessary to engage in the research and sustainability activities. This included recruiting the teachers, determining those classes and students who would be the subject of study, building local capacity, and determining when in the school year the SunBay Math curriculum units would be taught.

Recruitment was led by the mathematics supervisor for grades 6–12 in Pinellas County, who invited 10 schools to participate in the study. Seven of these schools accepted the invitation, and these schools represent the wide variety of schools in Pinellas County. Two of the schools were high-poverty (greater than 50 % of the students were on the free or reduced-price lunch program), and these high-poverty schools were also majority-minority schools. Two teachers per school were chosen to participate. Due to teacher transfers between the time of recruitment and the beginning of the school year, a total of 13 teachers in 7 schools participated in the first year of the study.

We used a "target class" approach to determine with which classes these teachers would use the materials. We have used this approach in prior research, including the RCT in Texas. In this approach, the research team designates a class period for each participating teacher for which they are required to use the materials: this is the target class. Teachers also have the option of using the materials with their other classes. Data is collected only on the target class for each teacher, even if the materials are used with other classes. This strategy has a number of advantages over other methods of determining the population of students to be researched. This allowed the SunBay Math project to purposefully select classrooms with a wide range of prior student mathematics achievement, while minimizing demands on participating teachers in this pilot year. To select the target classes for the SunBay Math project, we required each participating teacher to send to SRI a summary of students' prior year Florida Comprehensive Achievement Test (FCAT) scores for each of their seventh-grade mathematics classes. SRI used this data to categorize classes into one of three levels of prior student mathematics achievement: low (average FCAT score of below 2, on a scale of 1 to 5), medium (average FCAT score between 2.4 and 3.2), and high (average FCAT score higher than 3.2). The team chose the target classes to ensure a wide range of student prior mathematics achievement: four classes were classified as being low prior achievement, five classes were classified as being medium prior achievement, and four classes were classified as being high prior achievement.

3.2 Methods

To investigate student learning while addressing the district's concern of not having a control or comparison group of teachers, we conducted a replication study that did not include a control condition; instead the research team, in consultation with USFSP and the district, decided to compare our results to those from the RCT in Texas. Because the instructional materials and the assessments were essentially the same except for superficial changes, they were deemed similar enough to allow a comparison of the results of the SunBay Math students to those of the RCT. As in the RCT, we used a pretest administered immediately before the *Managing the Soccer Team* unit and an identical posttest administered immediately after the unit to determine student learning. A student's gain score was calculated by subtracting the pretest score from the corresponding posttest score.

To investigate teacher perceptions of the program, at every PD session (both the initial 3-day workshop and the monthly sessions) we had participating teachers complete an evaluation form. The form consisted of 16 questions asking teachers to rate specific dimensions on a scale of 1 to 5 (5 being the highest) and provided a set of open-ended questions.

In addition, all 13 teachers that participated in the program were interviewed to capture data that might have been difficult to surface using other sources. These one-on-one interviews were scheduled after each of the teachers completed teaching the *Managing the Soccer Team* unit and prior to the final monthly PD session. Each one-on-one interview was conducted by a member of the SunBay Math team, and teachers were asked about their experiences when teaching the unit, their impressions of the professional development, their impressions of students' learning of the mathematics in the unit, their use of technology, and lastly, school-based and district support.

3.3 Results

3.3.1 Comparison of Student Populations and Pretest Scores

There was no statistically significant difference between the SunBay Math students and the students in the RCT (both treatment and control) on pretest mean or standard deviation, as shown in Fig. 2. Thus, the pretest scores provide evidence that the prior mathematical knowledge of the Pinellas County students and the Texas students was comparable.

3.3.2 Comparison of Student Gain Scores

Figure 3 shows the gain scores for three populations of students: SimCalc Control students from the RCT (who did not use SimCalc), SimCalc Treatment students

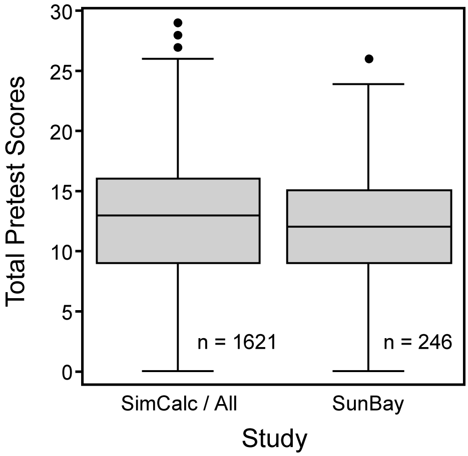


Fig. 2 Comparison of SimCalc RCT and SunBay pretest scores

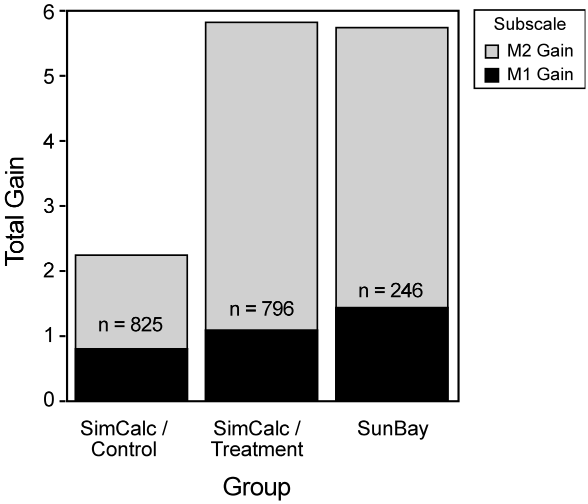
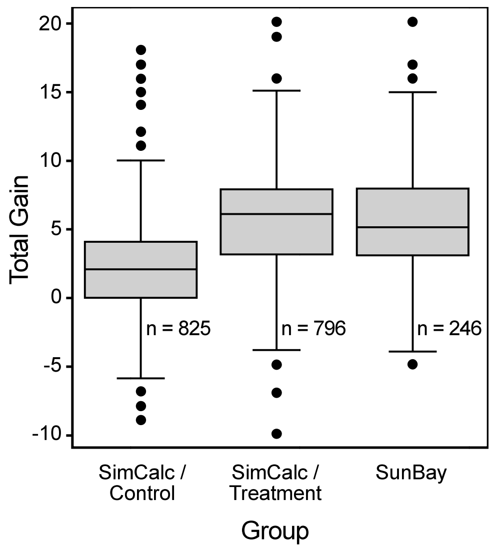


Fig. 3 Comparison of Texas and SunBay learning gains

from the RCT, and SunBay Math students. The gain scores are a measure of student learning and are calculated by subtracting each student's pretest score from their posttest score. As shown in Fig. 3, the SunBay Math gain scores were almost identical to the SimCalc Treatment gain scores, and both are significantly greater than the SimCalc Control gain scores. Figure 3 shows two different gain scores (black and gray), which represent two types of items on the test. The black part of the graph represents simple $\frac{\square}{\square} = \frac{\square}{\square}$, $y = kx$ problems, or questions calling for straightforward graph and table reading (often called "the basics"), which the SimCalc study called M1 mathematics knowledge. The gray part of the graph represents more complex proportional reasoning, such as requiring a functional approach (e.g., in which students must map between a domain and range) or requiring reasoning across two or more representations, which the SimCalc study called M2 mathematics knowledge.

Fig. 4 Comparison of Texas and SunBay distributions



While the graph indicates that the M1 gain for SunBay students is slightly greater than the M1 gain for the Texas SimCalc students, and the M2 gain is slightly less than that of the Texas SimCalc students, these differences are not statistically significant (see Roschelle and Shechtman, this volume, for more detail on results from the RCT).

In addition to comparing the gain scores of the SunBay Math and RCT students, we also compared the spread of gains. That is, we investigated the possibility that the SunBay Math student gains are somehow differently distributed across students. Figure 4 shows that this was not the case: the distribution of scores for the SunBay SimCalc students was nearly identical to that of the SimCalc Texas Treatment students.

As further confirmation that the materials were effective across the range of all teachers in both studies, Fig. 5 shows that teachers who did not use the SimCalc materials had limited learning gains (indicated by the large number of "Texas Control" teachers grouped to the left of the graph), whereas all teachers who used the SimCalc materials had greater learning gains than half the control teachers, and all PCS teachers (labeled "Florida Intervention" in Fig. 4) who used the SunBay Math materials had greater learning gains than approximately two-thirds of the control teachers.

Finally, we explicitly investigated the Matthew Effect by analyzing the relationship between prior mathematics knowledge and learning gains. For the SunBay Math students, there is a strong and statistically significant relationship between the mean classroom prior mathematics FCAT scores and scores on the SimCalc pretest (see Fig. 6). That is, classrooms with low FCAT scores also had low pretest scores. This finding is expected because the pretest assessment (which is identical to the posttest) and the FCAT have some overlap in the content being assessed. In

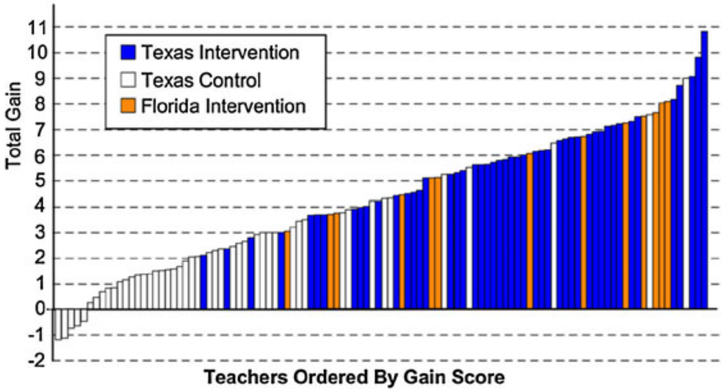


Fig. 5 The spread of mean classroom student gains shows the consistent effectiveness of the SunBay approach in both Florida and Texas

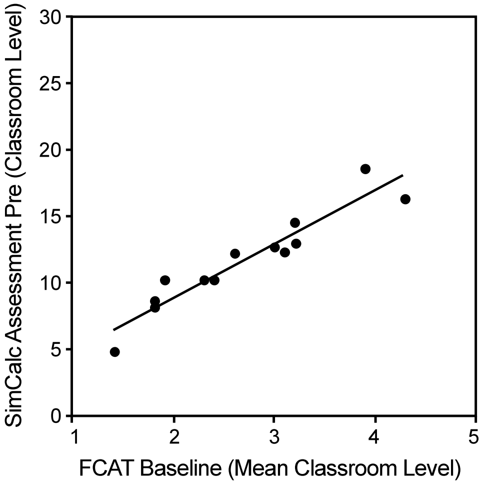


Fig. 6 Classroom FCAT levels and pretest scores are highly correlated

addition, it is worth noting that even the classrooms that score highest on the pretest had significant room for growth, as no classroom had an average score greater than 60 %.

Looking next at the relationship between learning gains and prior FCAT scores, we see that prior mathematics achievement is not a statistically significant predictor of student learning (Fig. 7). This result provides strong evidence that the SunBay Mathematics program is effective for students from a variety of prior mathematics backgrounds.

Taken together, these results provide substantial evidence that the SunBay Math materials were effective for the range of students and teachers who participated in the first year of the sustainability study.

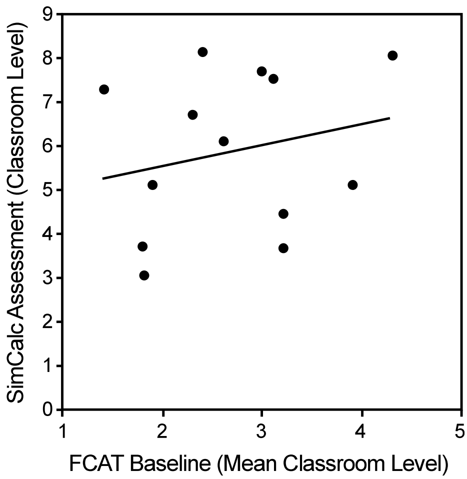


Fig. 7 Classroom FCAT levels and gain scores are not significantly correlated

3.3.3 Teacher Perceptions of Support

To investigate teacher perceptions of professional support, at each PD session, we asked teachers to rate how prepared they felt to: (1) teach the mathematics content, (2) use the materials, (3) use the software, and (4) engage in the pedagogy needed to teach the unit. The average rating was 4.54 out of 5, indicating that teachers were very satisfied with the support they received in terms of teaching the materials. In the one-on-one interviews, the teachers made the following statements about the support they received:

> The lessons that we did together, it was much easier to teach, because you already knew where it was going, and what the next lesson was. And, without going through the entire thing, if I had walked in blind, and just started with the first lesson, and taught it through, and not known where it [*Managing the Soccer Team*] was going, you know, I might not have been able to bring out as many thing to the kids, so it was good to know where it started and where it ended, and know what the progression was.

> I think that without that [initial three-day PD], the program would not have been successful, because it [PD] served as an introduction, and then we had, which we don't normally have, more extensive training prior to beginning something. When we field test something, normally we are just thrown in cold, and you have to sink or swim. I think they [the SunBay team] provided us with a pretty good foundation and an overall view of what the goals of the program were. I think they modeled, there was a lot of modeling so that you can see alternative strategies for teaching it [*Managing the Soccer Team*], and sometimes when you've taught for quite a while, you tend to get in a rut, so it is nice to see best practices and current practices modeled, and I think that was really great! They handled that really well.

> It [PD] helped us prepare because we got to look at the program, and actually go through the unit and see how it all tied together; that was I think the big thing... I remember a light bulb going off, oh, that is why they are not talking about this yet, because here it is you know 4 days later. I thought that helped immensely.

> They [the monthly PD sessions] were good; I liked the fact that they [the project team] seemed to be concerned not only just about the field test, but also opening us up to ideas

> and strategies that maybe we had forgotten or never learned. So I felt like, I learned stuff not necessarily just about SimCalc but something to help me be a better teacher.

As House (1994) pointed out, one feature that leads to a successful PD program is for teachers to use networking and interactions with other teachers as a support structure when trying new instructional approaches. This approach allows teachers to receive feedback, encouragement, and novel ideas from colleagues. As such, another aspect of the *teachers' perception of support* was the support they provided to each other during the project. As the teachers stated during their one-on-one interviews:

> It was interesting to listen to obstacles and challenges that arose when they [other teachers] were teaching the lessons and how they handled them.

> When you work with the same people for very long, you tend to become stagnant, and, you know, you run out of new ideas, but when we were out with other people... you get another perspective.

> I think it was nice to hear them [other teachers] share what was working at their school and then also what was not working.

> It was interesting to see all of the different, you know, obstacles that other people were going through, to see what other teachers were going through. I think it was good that we knew people to talk to, had people who were going through the same stuff that we were going through, to help each other out.

> Usually, pro-ed meetings could go on and on. I really liked these meetings, and I think it is because of the partnering up with the university... It [the PD] is a lot more collaborative than regular canned pro-ed. So I think a lot of it, working with the other middle school teachers was helpful, you know, cause obviously we could bounce ideas off each other, and the professors, I think, really helped with that.

3.3.4 Teacher Perceptions of the Materials: Perceived Coherence and Value

To investigate teacher perceptions of coherence and value, at each PD session we asked teachers to answer the items on a scale ranging from 1 to 5 (5 being the highest). The items asked the teachers: (1) How useful was the workshop to you as a teacher?; (2) Would you recommend the workshop to a colleague?; and (3) Overall, how satisfactory did you find the workshop? The average rating was 4.7 out of 5, indicating that teachers were very satisfied, finding the materials to be very useful and related to their work in the district. Teachers stated during one-on-one interviews:

> I think you know looking at the big ideas of the Sunshine State Standards and knowing that we are going to be using less topics for a longer period of time, I think this will help a lot. I think this gets into more depth than we have been able to cover and they [students] can see the connections.

> All of [the relevant mathematics content] is in there that we need for rate and proportionality. I thought it had lots of upper level thinking and critical thinking for going deeper into it.

> It definitely met them [the Next Generation Sunshine State Standards' of rate and proportionality]. Um, I think the great thing about the next generation is, you get to go more in depth... I think this is perfect, especially you wouldn't feel the pressure to finish it in a

> short amount of time. You can go more in depth and do more things with it; and there is so much more you can do with that.
>
> I think the unit is perfectly set up for Next Generation Sunshine State Standards. I'm so excited about these new standards! Actually, I think this surpassed what I thought… It's meaningful, you know, it's not going a mile out an inch deep, it's really delving into it, into the concepts here. It is definitely worthy [of the time].
>
> I think these [SunBay materials] have done a better job [than the regular textbook], because the new standards are becoming a bit broader in a sense, I think you are hitting it a little bit better, instead of us each year hitting 80 something benchmarks, you know, they are going to cut them down and whatever 40 that we are hitting you really go into more depth.

3.3.5 Teacher Perceptions of the Materials: The Matthew Effect

To investigate the Matthew Effect, we asked teachers in their one-on-one interviews, "*What went well with your target class?*" This allowed the teachers to discuss how the materials met the full range of their students' mathematical needs. We found that all teachers believed that the materials could be used effectively with students of differing prior achievement. Example quotes include the following:

> I have mainly 1's and 2's [students with very low FCAT scores] in my target class and they felt success, and students who never want to share were sharing every day. They were able to tell me the speed, they wanted to share their graphs, they answered questions. It was good to see the kids that don't usually get math get it.
>
> The group that was selected for me was as low as low can get, I mean, a co-taught class, 90 % of them were ESE, EBD kids with behavior problems, and they couldn't read… What I wanted to them to understand is to be able to look at a graph and tell me what kind of slope it is and explain to me do they understand what rate is, and how to solve the distance formula, and they were able to do that, so I think I was successful… working with low-level kids, I think it works.
>
> The one thing about the simulation, the children that may not read as well were not bound to just reading the book, it was more of the kinesthetic way of learning, because they were touching, and looking. So I think it gives the lower-level kid kind of an advantage.
>
> I had a whole group of kids that were engaged much more than they would be in the regular math.
>
> I would like to address the English Language Learners, which we have quite a few of, I think it leveled it a little bit for them in that they could see it… this [the unit] in particular was very leveling for them.

By the end of Year 1 of the SunBay Math program, the team had achieved its primary goals of building a strong collaborative team, showing that the materials could increase student learning for a wide range of students, and of building a committed community of core teachers who valued the program.

4 The Program: Beyond Year 1

At the time of this writing, the SunBay Math program is in its third year. While the Year 1 implementation was a success, to achieve the program goal of sustainability requires that the program continue to be a significant presence in the district for a number of years. During the second year, the focus of the program was on incremental scaling. Due to teacher turnover, about half of the first year teachers were either not teaching in the district or not teaching middle school mathematics in the second year of the program. We recruited 6 new teachers, and the program engaged a total of 12 teachers (6 continuing and 6 new teachers) and approximately 1,200 students. Six of the Year 2 teachers joined the first cohort of the MG DEME certificate program. In Year 3 of the program, we are more focused on scaling up, both in terms of the number of teachers involved and the number of units we are offering.

In this section, we report on a number of factors that indicate that the program is making progress toward achieving its goal of sustainability: (1) the project has garnered continued local and national support, (2) the district has increased its support for the program, and (3) the program has continued teacher support.

4.1 Continued Local and National Support

Once Year 1 was shown to be a success, the Helios Education Foundation and the Pinellas Education Foundation again agreed to fund the program, and the Progress Energy Foundation agreed to fund the first cohort of the MG DEME certificate program.

As a result of the strength of the results from Years 1 and 2, PCS agreed to fund teacher professional development to allow more teachers to participate in the program. Soon after the PCS announcement, the SunBay Math team was awarded a highly competitive Next Generation Learning Challenge (NGLC) grant. This grant will help the SunBay Math program achieve its goals of increasing the amount of materials, while also allowing us to pilot technology-based formative assessment and expand to more teachers. Specifically, the NGLC grant will allow the team to develop another SimCalc replacement unit (an eighth-grade linear functions unit), and will allow us to expand to additional teachers in the existing SunBay schools, as shown in Table 1.

While the program success thus far does not guarantee future success, the significant local support, combined with national recognition through the NGLC program, gives us reason to be optimistic that our scaling will continue in future years.

Table 1 Current and future teachers and students in the SunBay Math program

	Status as of July 2011	Expectation for July 2012, under the NGLC Program
Number of teachers	12	28
Number of schools	6	9
Number of students	>1,200	>2,500

4.2 Increased District Support

The school district has increased support for the program in two ways: through funding and through public statements. Up to this point, the district has provided funding for teachers to engage with the SunBay Math program. This funding came at a time when PCS, along with most other school districts in the country, was facing significant budget shortfalls. The fact that the district has considered SunBay Math materials valuable enough to continue funding is a significant sign of support.

In addition, at the beginning of the 2011–12 school year, the SunBay Math program was invited by the PCS K-8 district mathematics coordinator to present the SunBay Digital Mathematics program to all Pinellas County Schools' middle school mathematics teachers at the school district's annual district-wide training. In addition, nine teachers from the SunBay Math project presented and demonstrated SunBay Math materials at the training. This highly visible support resulted directly in increased teacher interest in the program, and the project had to inform several teachers that they would have to wait for the 2012–13 school year to participate.

4.3 Continued Teacher Support

The teachers that have used SunBay Math materials continue to be the program's biggest advocate. Four teachers met with representatives from the Pinellas Educational Foundation to express the importance of this project and to request that the foundation work to find ways to fund the program district-wide. As stated above, teachers were also enthusiastic about representing the SunBay Math program at the district's annual district-wide training. This passionate support has resulted in district word-of-mouth about the program, with several teachers in the district expressing disappointment that they are not part of the SunBay Math program.

5 Discussion and Risks

Our findings show that, up to this point, the SunBay Math program has been successful in becoming a sustainable and integrated part of middle school mathematics in PCS. That is not to say that the district has fully adopted the SunBay Math materials. Instead, we have succeeded in reaching our goal of building a small but enthusiastic teacher community. These teachers are currently the most influential advocates of the program: they have expressed their satisfaction with the program with other teachers, with local foundations, and with the mathematics district coordinator. These discussions have built pent-up demand for the SunBay Math materials, and have resulted in a general sense that the scale-up of the SunBay Math program should be a priority for the district and local community. We note that most district-wide mathematics reform efforts are met with substantially less enthusiasm by teachers, and many reform efforts are met with active teacher resistance.

By working closely with a small group of teachers, the SunBay Math program has found that teachers are its greatest advocates.

While the program has been successful thus far, we have identified several risks to the ultimate program goal of long-term and district-wide adoption. We discuss three such risks: continued funding, the technology requirements, and changes in key personnel.

The SunBay Math program has already found ways to mitigate against the risk of limited future funding. Because the overall PD and the MG DEME certificate program are now an integrated component of the USFSP professional development offerings, there is a mechanism by which teachers can be supported in the effective use of the SunBay Math materials that does not rely on significant external funding. However, future funding is still required to expand the SunBay Math curriculum replacement units beyond the small number we currently have today. For the SunBay Math program to have large-scale impact, we must still rely on future funding.

The technology requirements also pose a risk to large-scale adoption of SunBay. The SunBay Math materials have been designed to require what seems to be a relatively small technology commitment from schools: during each of a small number of 2- to 3-week units, one low-cost computer is required for every two to three students. In our initial conversation with district administrators, it seemed that this commitment would be relatively simple to meet, as the district already had a goal of one computer for every three students. However, once we got into the schools, we recognized that the technology was not always available for SunBay use. Some computers in the schools, often up to 20 %, were broken and waiting repairs. Other computers were reserved for the library or computer lab, and these rooms were often booked far in advance. Some computers were reserved for specific tasks, such as required student assessments. In the first year of implementation, we often found that our two teachers in the school had to share one laptop cart between them. While this solution is workable in the case of two teachers in a school where each teaches one or two classes with the SunBay Math materials, one laptop cart for the entire middle school mathematics department, where each teacher is using the SunBay Math materials for multiple classes, clearly will not suffice. As the SunBay Math program expands, we will be reliant on the technology infrastructure of schools also expanding.

In addition to the number of computers, the type of software being used is also a risk. Currently SimCalc is a stand-alone computer application that must be installed on each individual computer (as are other applications, such as Geometer's Sketchpad®). The primary benefit for schools is that SimCalc does not require an always-on, reliable Internet connection to be used. However, SimCalc does require system administrators being willing and able to install the software, and the software staying on the computer. Unfortunately, these conditions are not always obtainable; in some cases, the computers are configured such that only software applications that have gone through a rigorous district approval can be installed, making the installation of SimCalc difficult. Even when SimCalc can be installed, it is not uncommon for districts to erase all "non-standard" files from computer drives at certain "refresh" points during the year (e.g., winter break and summer break), requiring that SimCalc be installed multiple times during the year. We are currently investigating different models of software deployment, to determine what models may be most compatible with large-scale use in districts.

The final set of risks we discuss are risks associated with district personnel turnover. Any new district-wide initiative runs the risk of being associated with the particular district administration that introduced the initiative. As such, when the administration changes, the new administration may look to bring in a new set of initiatives. Because the SunBay Math program has actively worked with a small community of teachers and with local foundations, we believe that this is less of a risk than it might be for other initiatives. In fact, shortly before the writing of this chapter, the PCS District Superintendent left the school district. Fortunately for the SunBay Math program, this change in the administration has not had an adverse impact on the district's stance toward SunBay, as the new superintendent has expressed support for the program. This gives us reason to be optimistic that the SunBay Math program will soon move even closer to our goal of full district-wide and sustainable adoption. As we move forward with PCS, we will investigate questions that are critical to better understanding widespread adoption of technology-based materials, such as the types and amounts of support necessary for schools to adopt these materials for all mathematics teachers, the trade-offs of different technology uses (e.g., tablets and laptops), and the forms of professional development necessary to ensure that teachers and district leaders are prepared to implement SunBay in all middle school mathematics classes.

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Impact of Classroom Connectivity on Learning and Participation

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1 Introduction: History of Classroom Connectivity

We have studied the potential of combining the representational innovations of the computational medium (Kaput and Roschelle, 1998; Roschelle et al., 1998, 2000) through the development of SimCalc MathWorlds® software and curriculum (hereon collectively referred to as SimCalc) with the new connectivity affordances of increasingly robust and inexpensive hand-held devices in wireless networks (Roschelle and Pea, 2002) linked to larger computers (Kaput, 2002; Kaput and Hegedus, 2002). We chose to address the mathematics of change and variation, a core school mathematics strand (National Council for Teachers of Mathematics

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S.J. Hegedus, J. Roschelle (eds.), *The SimCalc Vision and Contributions*, Advances in Mathematics Education, DOI 10.1007/978-94-007-5696-0_12,

[NCTM], 2000) that is representationally demanding and is studied at many levels by all students, from early algebra through calculus (Kaput, 1994).

Classroom Connectivity (CC) identifies a particular type of learning environment that uses various technologies to connect student work in a variety of representational forms (see Brady, White, Davis, and Hegedus, this volume). It has roots in more than a decade of classroom response systems, most notably ClassTalk™ (Abrahamson, 1998, 2000), which enable instructors to collect, aggregate and display (often as histograms) student responses to questions. Once this is done, new levels of interaction appear in large classrooms and in various domains including mathematics in secondary schools and physics at the undergraduate level (Burnstein and Lederman, 2001; Crouch and Mazur, 2001; Dufresne et al., 1996; Hake, 1998; Littauer, 1972; Piazza, 2002; White, 2006). The review by Roschelle et al. (2003) shows remarkably consistent positive impacts across multiple domains and levels.

Our work is situated within this domain but moves beyond classroom response systems by studying the following new CC affordances:

- The mobility of multiple representations of mathematical objects such as functions as reflected in the ability to pass these bi-directionally and flexibly between teacher and students and among students.
- The ability to flexibly harvest, aggregate, manipulate, and display to the whole classroom representationally-rich student constructions, and to broadcast mathematical objects to the class.
- The opportunity to engineer classroom activity structures in concert with the mathematics to be taught and learned that engage students in new ways.
- Teachers can arrange, organize, and analyze sets of whole-class contributions at once, while students can make sense of their work in a social context, reasoning, and generalizing about their contribution with respect to their peers' work.

Curricular materials and software documents were created to be used in conjunction with SimCalc and are aligned with the new classroom connectivity affordances listed above. Our integrated software and curriculum was designed to support student participation in mathematically meaningful ways and, in doing so, aim to improve student understanding and success with mathematics. In order to investigate the impact of such an approach on learning and participation, we designed a mixed methods study, which we report here.

2 Theoretical Commitments and Research Focus

Our program of work posited classroom connectivity (CC) as a critical means to unleash the long-unrealized potential of computational media in education (Cuban, 2001), because we saw its potential impact as direct and at the communicative heart of everyday classroom instruction. The work we report on here shares insights into how those new ingredients, in combination, may provide the concrete means by which that potential may be realized by enabling students to express themselves in mathematically meaningful ways (Hegedus and Moreno-Armella, 2009).

The student experience of "being mathematical" (cf. Nemirovsky and Noble, 1997; Nemirovsky et al., 1998) becomes a joint experience, shared in the social space of the classroom in new ways as student constructions are aggregated in common representations similar to those of a participatory simulation (Resnick et al., 2000; Stroup, 2003; Stroup et al., 2005; Wilensky, 1991; Wilensky and Resnick, 1999; Wilensky and Stroup, 1999, 2000). Cognitive activity is distributed in the socio-material space (Hutchins, 1996). Similarly changed are how students interact mathematically with each other and their teacher, and, critically, how their personal identity manifests in their shared mathematical experience in the classroom (Cobb et al., 1992a,b). Renninger (2009) focuses on how identity and interest develop through interactions and refers to "the learner's self-representation as a person who pursues particular content and the processes that inform the development of this self-representation" (p. 106). Our present work extends this by understanding how identity can be projected as a mathematical representation into a public display of self, infused with personal feelings of ownership as a mathematically meaningful form of participation.

Based on such prior work, we focused on how learning and attitudes towards participation can be improved through mathematically meaningful activity structures where all students can contribute by personally constructing mathematical functions or models of scientific phenomena. We distinguished "activity structures" as properties of tasks and their descriptions from "participation structures" that are putative organizations of the classroom interaction phenomena that result from carrying out a designed task. From this basis, we focused our study through the following research question:

> How can the SimCalc connected learning environment improve learning in Algebra classrooms through enhancing mathematically meaningful participation?

In order to address our question, we operationalized *learning* as the acquisition of algebraic concepts and skills. Intersecting the learning of core algebraic ideas with students' attitudes created an opportunity to assess the impact of participation structures within such learning environments. Hence the outcome measures of our intervention were the demonstration of knowledge of algebraic concepts, and of attitudes towards working individually and collaboratively. We conclude our report with specific forms of participation under which we observed successful learning occurring in various classrooms; these conditions were correlated with students' positive attitudes regarding classroom participation compared within treatment (SimCalc) and comparison ("business-as-usual") classrooms even in the cases with greatest learning gains.

Before we present the specific details and findings of our study, we outline the design of our intervention, how our curriculum relates to the learning expectations of a standardized set of learning frameworks in a U.S. context, and what mathematically-meaningful participation means.

3 Intervention Design: Integrating Classroom Connectivity and Content

3.1 Overview of SimCalc Environments

The SimCalc software allows students to create mathematical objects on graphing calculators and see dynamic representations of these functions through the animation of actors whose motions are driven by the defined functions. Students are then able to send their work to a teacher computer. Calculators are connected to hubs that wirelessly communicate to the teacher's computer via a local access point. Due to advances in wireless communication and interactivity between desktop PCs and Texas Instruments graphing calculators, the flow of data around a classroom can be very fast allowing large iterations of activities to be executed during one class. This is not just an advance in connectivity, but in the development and application of software that maximizes such an innovation.

We created activities that allow students to generate functions in SimCalc on the TI-83 Plus or TI-84 Plus graphing calculator, which can then be collected (or "aggregated") by a teacher into the SimCalc software running in parallel on a computer using a TI-Navigator™ Wireless network. The activities were part of a curriculum developed and refined over many years that focus on core high school Algebra ideas in the United States such as linear functions, simultaneity, covariation and slope-as-rate versus slope as m in $y = mx + b$, and that utilize CC to supplement or replace existing traditional algebra curriculum (Hegedus and Kaput, 2004). Such activities accompanied the software and were structured to take advantage of the natural social setup of the classroom to create variation.

Our work defined three broad activity structures, which articulated the unique aspects of our intervention at the level that intersects curriculum through the classroom connectivity (CC) technological affordances mentioned in the introduction: (i) polling students, (ii) mathematical performances—publicizing individual constructions, and (iii) aggregation of individual constructions in a common representation system in a public display–Parametric variation. Specific details of these activities can be found elsewhere (Brady et al., this volume; Dalton and Hegedus, this volume; Hegedus and Moreno-Armella, 2009). These activities illustrate how we focus on participation in terms of mathematically meaningful work. The contribution of each student is parametric (i.e., their unique count-off number), situates them within a group or across groups, and is used to construct a mathematical motion that is part of a wide family of algebraic functions.

3.2 SimCalc Curriculum as a Replacement Unit

The SimCalc Algebra 1 curriculum, designed for this study, replaces 6 weeks of instruction on a period (45 minutes) schedule. These activities comprise a package of six units dealing with concepts such as relating graphs to events, linking representations, families of functions, rate of change, and slope as rate. The first unit introduces piecewise defined Position versus Time graphs as a way of describing and controlling motion qualitatively, as well as introducing the basic features of SimCalc. Mathematically, this unit concentrates on drawing piecewise defined Position versus Time graphs and analyzing the table representation of the functions. In the second unit, students create motions and correlating scenarios or exciting stories for two actors, A and B, where students manipulate a character's motion by changing the slope and domain of a segment, adding segments, and deleting segments to define a piecewise linear function. After creating them, students are asked to quantify their piecewise functions. Additions to their exciting stories should include specific velocities, and specific intervals of time, as well as mathematically correct grammar. The third unit aims to help students develop ways of writing constant rate situations via function expressions primarily in the slope-intercept form, $y = mx + b$. Units 4, 5 and 6 focus on varying the "m" and "b" values in $y = mx + b$ by considering m as slope as rate (or velocity) and b as starting position of each actor.

The SimCalc Algebra 1 curriculum is mapped to Massachusetts Frameworks (MA) and the National Council of Teachers of Mathematics (NCTM) Standards. Approximately 30 % of the 9–10th grade MA frameworks are addressed. The curriculum attends to various topics under the NCTM Standards (NCTM, 2000) including Numbers and Operations, Algebra, Geometry, and Measurement as well as each of the Process Standards particularly: *Problem-Solving & Metacognitive Skills*—by allowing students to solve problems across contexts, e.g., graphical interpretation, modeling; *Representations*—students use multiple-linked representations to model and interpret social phenomena; and, *Communication*—collaborative learning in mathematically meaningful ways.

4 Research Design and Instruments

4.1 Research Design

A goal of the study was to investigate the effect of a SimCalc replacement unit on learning and participation in mainstream high school classrooms through the use of repeated measures over time. First, our outcome measures illustrate learning of core algebraic concepts that are expected not just through a SimCalc intervention (hereon referred to as treatment) but also in any high school curriculum. This is accomplished through a content test. Hence, the SimCalc intervention is mapped to core curriculum that is normally covered in algebra classrooms. Secondly, we measured changes in students' attitudes to participating in mathematics classrooms through a survey instrument. Our results illustrate the effect of the SimCalc resources on these intersecting dimensions and conclude with two case studies to explore potential classroom conditions and pedagogy that can lead to such variation.

We conducted a quasi-experimental, controlled-study, repeated measures design. We wished to conduct a slightly underpowered study first to establish under what ideal conditions learning and participation can be affected, which could later be used to design an efficacy study. We used two districts—of similar type in terms of socio-economic status (SES) and achievement. All 16 ninth-grade algebra classes in these two school districts took part in the study. Eight classes received the SimCalc treatment as a replacement package. A total of five teachers used the SimCalc curriculum—some taught more than one class. The teachers involved in the study were not randomly chosen; rather they agreed to be a part of the project for various reasons such as: to earn graduate credit, a desire to try using new hand-held technology, or had seen SimCalc before and were interested in implementing it in their own classrooms. There was a similar distribution of teachers in each group with respect to number of years in teaching. Due to the quasi-experimental nature of the study, we wished to work with volunteer teachers to observe effect under ideal conditions but we did match the treatment classrooms to other factors in the comparison classrooms.

The treatment classrooms covered identical algebra topics as outlined in the districts chosen curriculum and mapped to the MA frameworks. The SimCalc curriculum introduced these core topics through different activity structures utilizing CC. Students were placed randomly into classes by the school district and, since we were collecting data in both college algebra and honors algebra classrooms, our study focused on a wide representation of students in each school. Originally, there were 160 students in the treatment group and 236 in the comparison group; this was the whole of the ninth-grade class in the two districts. All students in the study were baseline tested (as a pretest) at the start of their freshmen year prior to any intervention to compare experimental groups. Our attitude survey was also administered at this time. No significant differences in pretest scores were found. This pretest score was used as a covariate in our later analyses.

The content test was administered a second time to students in the treatment classrooms after the 6-week curriculum was completed. The same test was given to students in the comparison classrooms at a similar point in the existing curriculum sequence. We administered our attitude survey at similar points in time.

Video data were collected daily for six of the eight treatment classes and for one class in the comparison group during the course of the intervention. Video was not collected in every class due to a teacher's desire to not be video recorded, project resources, and time. We collected video data in one classroom from the comparison group because its schedule did not intersect with other classes. This proved to be the only viable comparison class in which to collect video data; the teacher was well experienced and regarded by the school as a highly successful teacher. Such convenience proved opportune for the project and answering our research question specifically, as this was the highest achieving comparison class at the end of the intervention allowing us to conduct a case study at the classroom level that we report here.

Each class was recorded with two digital cameras—one focused on the teacher and the whiteboard space where SimCalc software was projected and the other positioned at the front of the class focused on the students. This camera used a wide-angled lens to pan out and observe whole class dynamics as well as small group interactions. Both cameras were used as roaming cameras when the class was involved in small group work. We used this classroom video data as a qualitative component that reinforced our quantitative findings because quantitative surveys and/or structured qualitative (interviews) analyses do not always accurately reflect the attitude/affect of a student (Goldin, 2008; Ma and Kishor, 1997; Schorr and Goldin, 2008).

Our final results report on 324 students—187 students in the comparison group and 137 students in the treatment group. Attrition was due to students moving, absences, and switching classes. Attrition did not differ significantly between treatment and comparison participants in terms of standard variables (e.g., gender) at the whole class level.

The SimCalc software and curriculum was implemented with a focus on core high school Algebra. Teachers who used the SimCalc materials in their classrooms met over the summer to review curriculum and receive training. The treatment teachers were asked to use the intervention materials to replace their existing curriculum and the comparison teachers continued using their existing textbooks (which we refer to as "business-as-usual"). The treatment teachers were asked to not discuss the software, intervention, or curriculum with each other until at a final meeting after everyone completed the intervention.

The central research question of our study focuses on understanding the complex interaction between learning and participation. We also examine what student and teacher interactions look like when these interactions correlate in strong and weak ways through analyzing classrooms with lowest and highest learning gains. We collected data on *learning* as knowledge acquisition through a content test and changing attitudes to participate in classrooms through a survey and classroom observation (video) to analyze discourse patterns.

4.2 Measuring Learning Outcomes

Learning outcomes of students in our study were measured using an instrument that was developed by assessment developers and the research team (see http://tinyurl.com/6dgjpfp for a copy of the full test).

We constructed the instrument by following the principled assessment design approach (Mislevy et al., 2003). In this approach, we identified the specific knowledge or skills to be tested and then articulated an evidence model that specified the kinds of tasks (e.g., problem solving, application, open-ended, multiple-choice) that are likely to reveal whether students have mastered the target knowledge base and skill set. Next, in collaboration with an external advisory group, we created specific problems for students. The evidence model was refined through piloting and procedures for ensuring the technical quality of assessments.

In order to improve content validity, the instrument was primarily developed using standardized test items from high-stakes examinations, items from textbooks, research literature and NAEP items. Ten percent of the items were situated within a SimCalc activity context and covered two broad types of questions: (i) mathematics related to our content areas that students are expected to know on 10th-grade Massachusetts (MA) statewide examinations, and (ii) questions that require advanced mathematical ability that draw on deeper links between algebra and calculus. A total of 60 test items were compiled from various high stakes U.S. State assessment tests, such as the Massachusetts Comprehensive Assessment System (MCAS), as well as National Assessment of Educational Progress (NAEP) items, and Advanced Placement Calculus items. A few non-standardized items designed by the research team were included. These items had previously been used as assessment items in undergraduate SimCalc classrooms. In order to attend to improving face and content validity, all 60 items were reviewed by a panel of experts including members of the research team, mathematics professors, high school administrators, and teachers. As a result, 22 items were selected for a pilot study.

The assessment developers piloted this set of questions with similar students as in the study and revised them to ensure that the items were not too easy for students ($p > 0.80$) and that items could discriminate effectively between different performance levels. We also recruited undergraduate mathematics students, in a preservice mathematics education preparation program, to complete the test in order to set limits on how long the test should take. The final instrument was composed of 22 items. The test was designed to take no more than 45 minutes to complete—on average it took 30 minutes for our pilot participants to complete.

Twenty of the 22 test items were multiple choice and worth one point. There was one short answer question and one long answer question from a previous Massachusetts state assessment test. The short answer question was worth a maximum of two points—one point for partial credit. The long answer question was worth a maximum of four points—partial credit was also given. The scoring for both questions followed the rubric outlined by the state of Massachusetts. Three different raters scored the two questions. The three raters reached an inter-rater reliability of 92 % ($r = 0.92$). A content validation study was conducted to ensure instrument validity. A panel of four experts in mathematics education was surveyed for the study. The survey asked panel members to rate items on the content test for clarity, representativeness, inclusion in the subscales created for the assessment, and overall comprehensiveness as outlined by Rubio et al. (1999). Panel members demonstrated an inter-rater reliability of 80.9 %, supporting content validity of the instruments (see Tapper, 2011, for further details).

The content test was composed of the following content categories and focused on principled mathematical concepts and skills (NCTM, 2000): graphical interpretation (nine items), rate and proportion (five items), computational/procedural (one item), and making connections across representations or multiple representations (seven items). These proportions are aligned with coverage in the MA frameworks and intensity of coverage in our curriculum. We used these content categories in our analyses, as they were more representative of a set of general algebraic thinking skills.

4.3 Measuring Changes in Attitudes to Participate

The Student Attitude Survey was constructed under a principled assessment design approach (as described earlier) using items selected from a variety of established instruments (Longitudinal Study of American Youth, 1990; Ma and Kishor, 1997) including previous SimCalc research on affect and attitude (Goldin et al., 2007; Schorr and Goldin, 2008) and the Circumplex Scale of Interpersonal Values (Locke, 2000).

Various issues were addressed over the course of instrument development. First, the interval of the Likert-scale was changed from 1–7 to 1–5—to preserve stability given the notion of attitude towards certain activities measured by various items was less than well agreed upon amongst the age group we were observing. Second, certain items did not address affect but instead attitudes (the name of the final instrument), given there was a mixture of "attitude-towards-mathematics" questions mixed up with "sociability" questions. Later on in this paper, we present a factor analysis, which breaks the overall survey into components that relate to such broad areas in a statistically reliable way. Third, the overall structure of the survey was modified through various iterations. The order of questions changed so similar items were not in a sequence. Social desirability questions were modified or eliminated to attend to affective-response behavior. To attend to this behavior, we added some items with reversibility, i.e., we negated some items and checked for reverse responses among our respondents to ensure reliability. To this end, the valency of the potentially coupled items were adjusted to address the positivity or negativity of the terminology used.

To further explore instrument concurrent validity, the Attitude Survey was compared to a similar instrument—the *Fennema-Sherman* (F-S) *Mathematics Attitude Scale* (Fennema and Sherman, 1976). Details can be found elsewhere (see http://tinyurl.com/6dgjpfp). Slight differences were observed but were not significant.

The final 27-item attitude questionnaire was administered to all participants in both experimental conditions during the first two weeks of their freshmen year—near the same time as the content test. It took approximately five to ten minutes to complete. It was administered again to both groups at the completion of the intervention.

In order to detect sensitive changes in attitudes and make correlations with changes in difference scores on other outcome measures, it was necessary to reduce the dataset into quantifiable chunks that were meaningful in addressing the main research question. This process relied on establishing factors that we used to create four attitude factor scores to measure correlations with learning scores.

We conducted an exploratory factor analysis on items from the student attitude survey with the data collected in this study (Group 1, $n = 283$) to investigate the factor structure proposed by our theory regarding changes in attitude. We cross-validated the factor structure using a confirmatory factor analysis on a separate data set (Group 2, $n = 153$) that was collected at a later time (see http://tinyurl.com/5uw7xmn for more technical details on the analysis). We used principal components analysis (PCA) because our purpose was to compute composite scores for dimensions on the survey in order to measure change in attitude over time.

Four components explained 48 % of the total variance in our sample. The first factor, which we called *Positivity towards learning mathematics and school* ($\alpha = 0.782$), explained 20.8 % of the variance. These items collectively gave us a sense of relatively stable student beliefs and attitudes towards math and school, which we predicted would not change over the course of the short intervention. The second factor, which explained 11.4 % of the variance, was labeled *Publicizing work and related affect* ($\alpha = 0.739$). The items in this factor illustrated students' attitudes to participate within the classroom. We did expect that responses to these items would change as a result to our intervention. The next component, *Working privately* ($\alpha = 0.754$), explained 8.5 % of the variance. Our activities were centered around group work so, although changes due to the intervention could be interesting, we made no claims of the effect on student outcome. The last component, *Use of technology* ($\alpha = 0.610$), explained 7.3 % of the variance. This component was focused on attitudes towards the use of technology in class. See Table 1 for a summary of the items in the factors.

Table 1 Factors from the attitude survey

Factor	Item
Positivity towards learning mathematics and school (Attitude 1)	I do not like school.
	In middle school, my math teachers listened carefully to what I had to say.
	I think mathematics is important in life.
	In middle school, I learned more from talking to my friends then from listening to my teacher.
	I like math.
	I enjoy hearing the thoughts and ideas of my peers in math class.
	Mathematics interests me.
Publicizing work and related affect (Attitude 2)	I sometimes feel nervous talking out-loud in front of my classmates.
	I do not like to speak in public.
	When I see a math problem, I am nervous.
	I do not participate in many group activities outside school.
	I like to go to the board or share my answers with peers in math class.
	I am not eager to participate in discussions that involve mathematics.
	I feel confident in my abilities to solve mathematics problems.
	In the past, I have not enjoyed math class.
	I receive good grades on math tests and quizzes.
Working Privately (Attitude 3)	I enjoy working in groups better than alone in math class.
	I prefer working alone rather than in groups when doing mathematics.
	I learn more about mathematics working on my own.
Use of Technology (Attitude 4)	I enjoy using a computer when learning mathematics.
	When using technology for learning mathematics, I feel like I am in my own private world.
	Technology can make mathematics easier to understand.

5 Analysis

We first present our analyses of how students in the treatment and comparison classrooms compare on the content and attitude instruments pre to post (difference scores). We then disaggregate the data to illustrate how particular content categories of mathematical concepts/skills, e.g., graphical interpretation, can be improved by using SimCalc. The second part of our analysis focuses on correlations between differences in content scores and attitude in terms of the content categories and the four attitude factor scores outlined in the previous section.

The final part of our analysis looks quantitatively and qualitatively at the comparison classroom from which we collected video data and a similarly achieving treatment classroom, as measured by our mathematics content test. When we examined all the comparison classes, we found that the comparison classroom in which we collected daily video data during the intervention was the highest achieving. In order to conduct case study analysis, we paired this comparison classroom with a similar achieving treatment classroom (approximately, on average, 3 points gain out of a total 26 points from pre to post). We present evidence that there were differences between treatment and comparison groups in these high achieving classrooms. There were no correlations between learning gains and attitudinal changes in the comparison classroom, but in the treatment classroom we observed statistically significant correlations. Using discourse methods from prior work (Hegedus and Penuel, 2008), we discovered that students took speech turns with each other in a progressively increasing manner over the course of the intervention whereas there was no change with how students interacted with each other or the teacher over time in the high achieving comparison classroom. We also observed quite different styles of discourse. We offer examples that highlight important aspects of the SimCalc classroom that we believe interpret why we saw such strong learning gains intersecting with changes in attitudes to participate in mathematical activity.

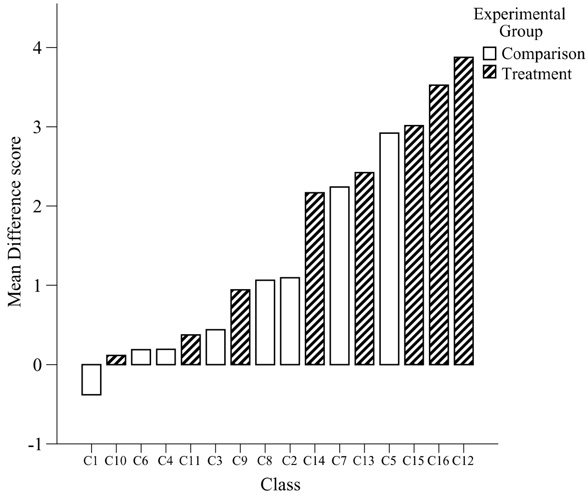
Through this analysis, we aim to support our two final conclusions as to why the SimCalc learning environment had such significant impact on learning: (i) the participatory nature of the classrooms was fundamentally different, which is illustrated in our turn-taking charts and (ii) discourse was personal and identity-laden. Students related and personified their mathematical contributions in public. There was a deep sense of ownership.

5.1 Comparing Groups

In our statistical analyses, we used parametric techniques where the assumptions for such tests were met. These included independent t-test, ANCOVA and Pearson correlation coefficients. For some of our categories, the data deviated from normality slightly. In such cases, to achieve greater statistical power, we used non-parametric counterparts including Wilcoxon Signed rank test and Spearman's rho.

Our main effect was statistically significant and showed that students in the treatment group had a higher gain on items related to linear functions, slope as rate, proportion, linear variation, seeing across representations, and graphical interpreta-

Fig. 1 Mean difference scores by individual classes



tion. Both groups were not statistically different on the pre-test, which addressed concerns about selection bias. In the treatment group, the mean gain was approximately two points out of a total of 26 points. In the comparison group, the mean gain was approximately one point. This group difference was statistically significant, $t(322) = 2.711$, $p = 0.007$. This represented a medium effect size $d = 0.34$ (or $r = 0.17$). Please note that we are actually referring to difference scores (post minus pre) but will use "gain" from now on (unless specified) as all differences were positive. We further looked at analyzing posttest scores using the pretest as a covariate for the non-honors students given the honors population was small and unequal across both groups. Using an ANCOVA, we found that the covariate—the student pretest score—was significantly related to the students posttest score, $F(1, 250) = 113.360$, $p < 0.001$ and the non-honors students who used SimCalc learned more than those who did not after controlling for the effects of pretest score ($F(1, 250) = 17.405$, $p < 0.001$, $r = 0.26$).

The majority of teachers who used the SimCalc intervention package had the most significant gains (see Fig. 1).

Figure 1 highlights variation in difference scores across classrooms in both groups. This illustrates robustness of the study (Roschelle et al., 2010), that algebraic learning is occurring across both settings, and that the content instrument is sensitive to measuring change. It also illustrates that the test is not biased to learning in a SimCalc setting alone. While there were some low gains in some SimCalc classes there was an overall intervention effect.

5.2 *Mathematical Item Analysis*

The SimCalc environment allows teachers and/or curriculum developers to show and hide various representations in an activity. This is a very valuable tool in

Table 2 Odds ratio calculation

	Post-test	Pre-test
Success	A	B
Failure	C	D

terms of facilitating classroom flow and allowing connections to be seen and made by students. An example of this would be building a tabular representation from a motion-based representation, or an executable representation (Hegedus and Moreno-Armella, 2009). The ability to work with showing or hiding representations can be extended into higher order thinking skills, i.e., write a general rule for the family of functions that is shown. A functional representation may be hidden from the students, but can later be revealed to the students at the discretion of the teacher. The activities were written to take advantage of this important feature and exploit it in meaningful ways. They were designed to guide and engage students in discovery and learning.

In particular, the treatment group had a significantly greater gain than the comparison group on multiple representation items $t(322) = -4.771$, $p < 0.001$, which represents a medium sized effect, $d = 0.53$ ($r = 0.26$). These items deal with generalizing relationships and making connections across representations. Conceptual transfer across multiple representations of a mathematical concept or object is an important theme in mathematics education and one of the NCTM process standards (NCTM, 2000).

Because of the dynamicity and linked representations that exist in the SimCalc software and the promotion of linking representations in the curriculum, we focused on the multiple representation items. For example: which function expression describes the given table of data? These items involve making mathematical connections among ideas in symbols, graphs, diagrams, and words and require a different type of thought than interpreting a graph, or determining a pattern amongst a given set of data would.

We present an item-by-item analysis of mean scores from pre to posttest highlighting questions of statistically significant gain for either group. Table 2 highlights the results of the multiple-choice items on the pre and posttest for the treatment and comparison groups. Since we are analyzing dichotomous data, we calculated the odds ratio rather than parametric or non-parametric tests. Odds Ratios (OR) are simply the ratio of odds of success in one event to odds of success in another event. Table 2 shows a 2×2 contingency table can be set up for both groups, where the ratio of odds (OR) is AD/BC.

In a controlled-design, we used a modified version, which incorporates both groups into one contingency table (see Table 3). A 95 % Confidence Interval (CI) is defined by (e^Y, e^X) where $Y = \ln(\text{OR}) - (1.96 * \text{SE})$ and $X = \ln(\text{OR}) + (1.96 * \text{SE})$. So, the estimated risk for the treatment on a particular item is OR $\times$ 100 % of the comparison with a 95 % confidence interval of (e^Y, e^X) as defined above. Hence, we wish to see ORs of significantly less than 1. A z-statistic is defined as $z = \ln(\text{OR})/\text{SE}$ for significance testing.

Table 3 Modified odds ratio

	Incorrect on post	Correct on post & incorrect on pre
Treatment	A	B
Comparison	C	D

We observed significant gains for 7 out of the 20 multiple-choice items in the treatment group relative to just one for the comparison group (see Table 4). Four of the items fell into the multiple representation content category—two were graphical interpretation items, and one was a pattern recognition item. We will extract two items that illustrate the fundamental mathematical ideas that were addressed in this intervention.

Table 4 Odds ratio by item

	OR	Content/skill	SE	CI		
				Low	Up	*p*
1	0.44	Identify correct algebraic expression that corresponds to graph given	0.502	0.164	1.177	0.051†
2	0.69	Development of an algebraic expression describing real life quantities	0.333	0.359	1.324	0.132
3	1.78	Interpretation of a Graph; Development of an algebraic understanding of a verbal expression	0.322	0.944	3.340	0.963
4	0.47	Open response; Understanding of real life quantities expressed algebraically	0.407	0.210	1.033	0.030*
5	0.61	Recognizing a relationship between two algebraic equations	0.307	0.334	1.113	0.053
6	0.19	Identify pattern in quadratic sequence to predict additional terms	0.301	0.104	0.337	0.000
7	1.36	Concept of average velocity given a position function	0.736	0.322	5.768	0.663
8	0.76	Solve for unknown variable	0.279	0.441	1.316	0.165
9	0.82	Understanding of parallelism through an algebraic expression	0.323	0.436	1.545	0.270
10	0.44	Given a quadratic function relate the area of a square to one of its sides	0.272	0.257	0.748	0.001**
11	0.60	Determine the lengths of the sides of a rectangle given its area and perimeter. System of equations: use one to solve another	0.258	0.361	0.994	0.024*
12	0.68	System of equations; Comprehension in graphical context	0.264	0.407	1.145	0.074
14	0.52	Understanding parallelism expressed algebraically with additional constraints	0.247	0.322	0.845	0.004**
15	0.67	Determine whether two lines intersect given their functions and understand the process of calculation in order to derive answer	0.267	0.397	1.130	0.067

Table 4 (Continued)

	OR	Content/skill	SE	CI		
				Low	Up	*p*
16	1.03	Relationship between speed and position	0.300	0.571	1.850	0.536
17	1.10	Recognition of a pattern between two variables that can be expressed algebraically	0.308	0.604	2.017	0.626
18	0.89	System of equations; Interpretation of a graph; Intersection of two functions	0.296	0.498	1.588	0.346
20	0.58	Transform a verbal expression into an algebraic 1 in order to develop a system of equations	0.325	0.308	1.099	0.048*
21	0.60	Manipulation of a parabolic function and how it will change graphically.	0.293	0.340	1.070	0.042*
22	0.81	Determine y-intercept of function; Formal notation	0.294	0.458	1.114	0.242

† $p < 0.10$; * $p < 0.05$; ** $p < 0.01$
Note: OR = odds ratio; SE = standard error; CI = confidence interval

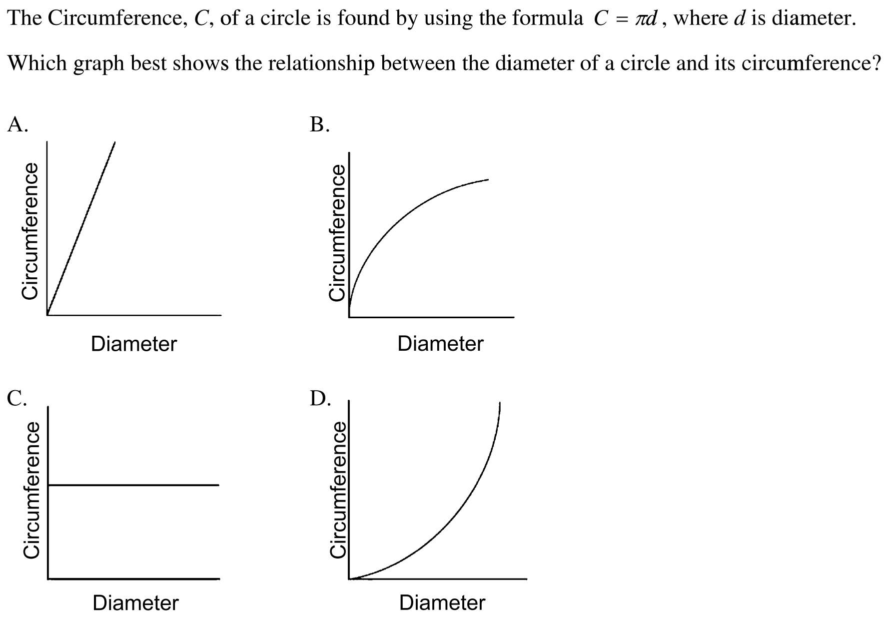
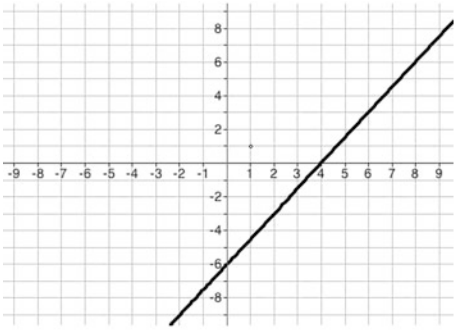


Fig. 2 Content test Item 21

Item #21 (see Fig. 2) presented a formula representing a linear relationship. Students were asked to identify the graphical representation of that linear relationship. The estimated risk for the treatment group on this item was 60 % of the comparison

Fig. 3 Content test Item 10



Which statement **best** describes the slope of the line graphed above?

A. The slope is -6.

B. The slope is $-\frac{2}{3}$.

C. The slope is $\frac{3}{2}$.

D. The slope is 4.

group, with a 95 % confidence interval of (0.34, 1.07). The formula $C = \pi * d$ had never been identified or discussed in the SimCalc curriculum. The relevant mathematical topics and problem solving skills needed were as follows: Patterns, relations and algebra; proportional reasoning; understanding of constant rate, graphically and algebraically. This item is a potential example of cognitive transfer of skills in terms of graphical interpretation across models and contexts.

Item #10 (see Fig. 3) involved interpreting a graph and identifying slope from a graphical representation by using two points on a line. Both groups had a gain on this item pre to post. Gain for the treatment group was almost two standard deviations above the mean. The estimated risk for the treatment group on this item is 44 % of the comparison group, with a 95 % confidence interval of (0.26, 0.75).

Immersing students in the SimCalc environment encourages and demands that they use multiple representations with the aim of making connections across representations that they may not otherwise recognize. Reasoning across representations is an important and difficult skill, which is explicitly focused on in the SimCalc learning environment. This has been classified as a conceptually difficult skill in other related work (see Roschelle et al., 2010).

5.3 Intersection of Learning and Attitudes

Only 214 students (80 treatment and 134 comparison) completed both content tests and attitude surveys. We use this constrained sample for the purposes of the second

Table 5 Two-tailed correlations between changes in student attitudes with changes in student outcome measures for the treatment group ($n = 80$)

	Δ MC	Δ GI	Δ R & P	Δ C/P	Δ MR	Δ Att. 1	Δ Att. 2	Δ Att. 3	Δ Att. 4
Δ Total Pts.	0.979**	0.581**	0.626**	0.280*	0.659**	0.108	0.001	−0.055	−0.126
Δ MC		0.603**	0.555**	0.306**	0.650**	0.117	−0.012	−0.048	−0.118
Δ GI			0.189†	0.070	0.108	−0.016	0.013	0.092	−0.040
Δ R & P				0.141	0.172	−0.050	0.021	−0.229*	−0.161
Δ C/P					0.080	−0.053	0.039	0.006	0.027
Δ MR						0.152	−0.108	0.005	−0.053
Δ Att.1							−0.130	0.121	0.275*
Δ Att. 2								−0.045	−0.029
Δ Att. 3									−0.069

† $p < 0.10$; * $p < 0.05$; ** $p < 0.01$
Note: Correlations are Spearman Rho. MC = Multiple Choice Items; GI = Graphical Interpretation; R & P = Rate & Proportion; C/P = Computational/Procedural; MR = Multiple Representations; Att. = Attitude

part of our analysis focusing on correlations between learning and attitudes towards participation as measured by our content and survey instruments.

Gain on the proportion and rate content category was negatively correlated with changes in attitude regarding preferences to work alone (Attitude 3), $r_s(80) = -0.229$, $p = 0.041$, for treatment students. Students who gained on this content subscale tended to agree less with statements such as "I prefer working alone rather than in groups when doing mathematics" on the post-survey compared to the pre-survey.

There were no significant correlations between gain on content categories and attitude components at the $p = 0.05$ level for two-tailed and one-tailed correlations for the comparison group. Tables 5 and 6 display all correlations between *content categories* and *attitude subscales* for the treatment and comparison groups respectively.

5.4 *Comparing High Achieving Classes from Each Experimental Group*

The final part of our analysis focuses on a comparison classroom (C5) and a similar achieving treatment classroom (C13). The comparison classroom, in which daily video data was collected, was the highest achieving comparison classroom. Both the treatment and comparison classes gained approximately 3 points, on average, from pre to post, and had similar numbers of students when we visited their class-

Table 6 Two-tailed correlations between changes in student attitudes with changes in student outcome measures for the comparison group ($n = 134$)

	Δ MC	Δ GI	Δ R & P	Δ C/P	Δ MR	Δ Att. 1	Δ Att. 2	Δ Att. 3	Δ Att. 4
Δ Total Pts.	0.983**	0.729**	0.470**	0.133	0.549**	0.018	0.015	0.095	−0.043
Δ MC		0.744**	0.387**	0.133	0.572**	0.010	0.036	0.088	−0.039
Δ GI			0.092	0.057	0.169†	−0.027	0.113	0.102	−0.003
Δ R & P				−0.011	0.008	0.071	−0.072	0.040	0.011
Δ C/P					−0.078	0.074	−0.036	−0.068	0.067
Δ MR						−0.001	−0.026	0.072	−0.099
Δ Att.1							−0.130	0.229**	0.058
Δ Att. 2								0.072	−0.161†
Δ Att. 3									−0.047

† $p < 0.10$; * $p < 0.05$; ** $p < 0.01$

Note: Correlations are Spearman Rho. MC = Multiple Choice Items; GI = Graphical Interpretation; R & P = Rate & Proportion; C/P = Computational/Procedural; MR = Multiple Representations; Att. = Attitude

rooms (C5, $n = 22$; C13, $n = 19$). Both classes were non-honors college preparation courses.

5.4.1 Correlations Between Changes in Content Test Scores with Changes in Attitudes

Changes in attitudes did not significantly correlate with one another for comparison students in C5. However, for treatment students in C13, changes in positivity towards math and school were positively related to students agreeing more strongly with statements such as "I enjoy using a computer when learning mathematics" on the post-survey, $r_s(18) = 0.601,\ p = 0.008$.

In these high achieving classrooms, there were no correlations between learning gains and attitudinal changes in the comparison classroom ($p < 0.05$). However, in the treatment classroom, we observed several statistically significant correlations (see Tables 7 and 8).

5.4.2 Turn-Taking as an Indicator of Differences in Participation

We used classroom video data to conduct discourse analysis to investigate reasons for such strong correlations between changes in learning and participation at the classroom level through analyzing differences in participation as described in our theoretical overview. One way to examine discourse is to examine the number of utterances that are made by students compared with those made by the teacher.

Table 7 Two-tailed correlations between changes in student attitudes with changes in student outcome measures for a treatment class (C13) ($n = 18$)

	Δ MC	Δ GI	Δ R & P	Δ C/P	Δ MR	Δ Att. 1	Δ Att. 2	Δ Att. 3	Δ Att. 4
Δ Total Pts.	0.979**	0.687**	0.458	0.371	0.635**	0.353	−0.360	0.166	0.185
Δ MC		0.733**	0.428	0.367	0.608**	0.363	−0.261	0.165	0.218
Δ GI			0.131	0.029	0.169	0.352	0.116	0.215	0.359
Δ R & P				0.284	0.050	0.003	−0.083	−0.214	0.246
Δ C/P					0.058	0.129	−0.442†	−0.174	0.258
Δ MR						0.140	−0.563*	0.415†	−0.288
Δ Att.1							−0.112	−0.030	0.601**
Δ Att. 2								−0.202	0.178
Δ Att. 3									−0.222

† $p < 0.10$; * $p < 0.05$; ** $p < 0.01$
Note: Correlations are Spearman Rho. MC = Multiple Choice Items; GI = Graphical Interpretation; R & P = Rate & Proportion; C/P = Computational/Procedural; MR = Multiple Representations; Att. = Attitude

Table 8 Two-tailed correlations between changes in student attitudes with changes in student outcome measures for a comparison class (C5) ($n = 18$)

	Δ MC	Δ GI	Δ R & P	Δ C/P	Δ MR	Δ Att. 1	Δ Att. 2	Δ Att. 3	Δ Att. 4
Δ Total Pts.	0.982**	0.752**	0.542*	0.609**	0.767**	0.062	−0.051	0.401†	−0.034
Δ MC		0.757**	0.442*	0.630**	0.794**	0.032	−0.014	0.349	−0.058
Δ GI			0.270	0.738**	0.435†	0.251	−0.089	0.395	0.060
Δ R & P				0.072	0.090	−0.196	−0.132	0.345	0.189
Δ C/P					0.523*	0.264	−0.293	0.183	0.107
Δ MR						0.072	−0.008	−0.029	−0.150
Δ Att.1							−0.131	−0.037	0.131
Δ Att. 2								0.184	−0.263
Δ Att. 3									0.049

† $p < 0.10$; * $p < 0.05$; ** $p < 0.01$
Note: Correlations are Spearman rho. MC = Multiple Choice Items; GI = Graphical Interpretation; R & P = Rate & Proportion; C/P = Computational/Procedural; MR = Multiple Representations; Att. = Attitude

Looking at the proportion of student-talk provides a sense of who dominates discourse; however, one also needs to know *when* the teacher is speaking to understand student-domination within the discourse. There could be an equal number of utterances made by students and teachers, but the student contributions could all be prompted by the teachers (as in a question-and-answer sequence); therefore, it is useful to consider how many adjacency pairs there are that involve teacher-student interchanges as opposed to student-student interchanges.

An evaluation of utterances and adjacency pairs is one way to understand the participation framework of the classroom. A count of student-student versus teacher-student adjacency pairs represents how discourse is structured and is potentially dominated by various participants. To evaluate this, we analyzed digital video and transcripts of our two high achieving classrooms over the course of our intervention (C5 and C13). We visited the treatment classroom 25 times during the course of the experimental intervention and we visited the comparison classroom 11 times—when similar algebra material was covered. Since the comparison classroom was meeting for twice as long each class relative to the treatment classroom (90 minutes versus 45 minutes), we plotted our counts with respect to units of time (i.e., 45-minute class period) to represent similar periods of time across our complete intervention. This allowed us to do some comparable regression analysis. Five-minute segments were randomly selected from the video and transcript corpus of data for our two high achieving classrooms. The only requirement for selection was that segments showed similar content, in a similar time, with students with similar baseline scores (entering high school freshmen). The clips were randomly selected, so there was no selection bias.

The dark plot (see Fig. 4) illustrates the treatment class (with annotated dashed regression line) and the gray plot illustrates the comparison class (with an annotated solid regression line). The comparison class (C5) shows a stable student-to-student turn taking sequence over the course of the observation period and it also represents a lower proportion than the treatment class. But more importantly, the treatment classroom exhibits a greater proportion with a steadily increasing rate over time and an extremely high correlation ($R^2 = 0.53$). This rate equates to *one* more student-to-student turn take per 5 minutes per class, which strongly suggests a steadily evolving

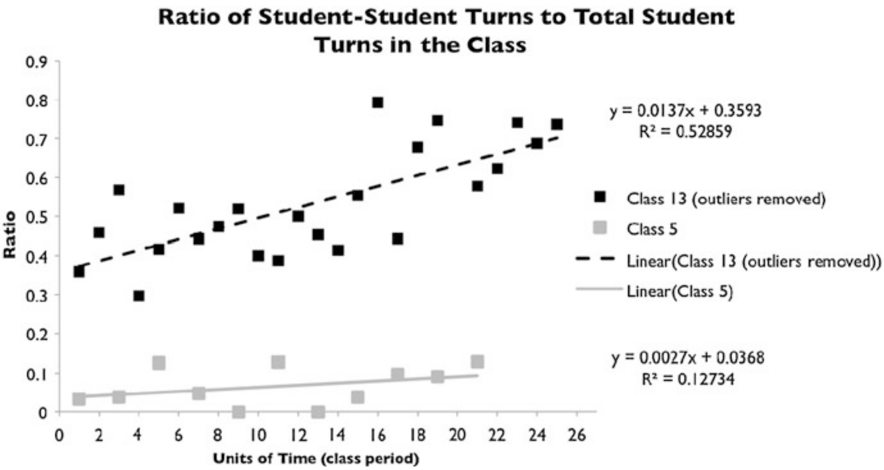


Fig. 4 Ratio of student-student turns to total student turns in the class

class norm in the discourse patterns of students as a representation of their participation. Interestingly, the teacher-to-student adjacency pairs (not plotted here) do not differ significantly between the groups, which addresses, in part, a possible reaction to these findings that teachers are letting students talk more amongst themselves at the expense of their own interaction with them.

5.4.3 Students' Personification of Mathematical Contributions

To complete the analysis with a finer level of granularity that we believe is necessary and sufficient to substantiate our claims, we exemplify the different forms of turn-taking in these two classes to unpack the steady state and progressions of discourse moves illustrated previously. These examples shed some light on the contrasting participatory natures of these classrooms with respect to the mathematical content and representation. This is one of three methods of analyzing discourse to explore different forms of participation in classrooms that were presented in detail in Hegedus and Penuel (2008). These are: (i) analyzing bids for attending and seeing to phenomena, (ii) participant structures as a frame for analyzing classroom participation, and (iii) studying identity by analyzing use of deictic markers.

Students' discourse was infused with personal value. It was evident that their contributions, through the affordances of the SimCalc environment and the network, framed the participatory nature of the class in a meaningful way. Identity was enhanced as evidenced by the specific use of deictic markers.

Talk is an arena for displaying competence and expertise, especially to differentiate oneself from another person in the setting (Goodwin, 1986). In some classrooms, students can display competence by responding correctly to teacher questions. They do not necessarily use their turns on the floor of the classroom to convince peers or their teacher of a particular point of view. However, as we have illustrated, this classroom structure appears to have been reformatted by the affordances of a connected classroom. In particular, students attempted to help their peers "see" what they were doing and contributed as they moved their mathematical product from a private to a public affair, often in ways that supported their position in an argument.

In the treatment class (C13), we observed evidence for this in such turns (which relate back to the original activity at the start of the paper). Even when some students referred to the animated actors in SimCalc in a way that might be expected—as objects that are created by individuals, rather than as objects that stand in for individuals—the identification with objects often reappeared in the exchange [S1, S2, S3—Students; T—Teacher]:

S1: Everyone's gonna have their own dot but there's gonna be… they're gonna be on top of each other. So every group, like, every…
S2: No.
S1: …if you have 3 people then there's gonna be 3 dots…
S3: He's lying.

S1: . . .then there's gonna be 2 more dots under that 1 dot.
T: So let's do, let's do a scenario. There's three people in this group. They're gonna, you're saying they're gonna be stacked on top of each other.

Here the phrase "on top of" expresses solidarity of identity and consolidation of identity, which has mathematical consequences in terms of interpreting the utterance. Even though each student's contribution was individualized as a simulation representing a function that moves in the public space, there is similarity—in fact, identical graphical forms of the same function. For a function to be identical, they must be mapped onto the same set of values on the same domain. This was informally described in this clip as being "stacked on top of each other."

Such personal terms were not evident in the comparison classroom, which focused on a similar activity involving graphical representations. Instead we saw discourse patterns such as "that graph." In addition, even when students were working collaboratively and had shared their mathematical contributions on the whiteboard to interpret a rate problem using a graphical representation, they did not use personal language. The language is within the content and context of the problem statement and no more. In our observed classroom it was the rate of apples falling from a tree, which was an example used in the textbook that the school adopted. Another such example was the distance from home plate when running the bases in an American baseball game. Both examples were used to convey the idea of slope as rate. Neither the tree nor the apples, nor the rate had any personal relevance to the studies in the class. The SimCalc materials, on the other hand, presented this idea through a motion scenario in which the students were creating an "exciting" race in which *their* direction and speed changed at least once during the race.

One student (S1) explains:

> In the middle of September, you're startin' to pick them so it doesn't take the whole month to pick apples off a tree so they go down extremely sharply then it stays dormant through the coldest part of the winter. Then it starts going up gradually, summer. Then there's fall and they kinda stay on the tree, they get extremely. . . whatever.

A second student (S2) followed up with her graph and a description:

> Okay, well, um. The apples start growing like in September, err, I don't know. So they grow up to here and then they stop growing and then they either are picked or fall off.

Deictic markers and indicators such as "the apples," "them," "they," "it," and "I don't know"—which could refer to not knowing when apples start growing on trees—were used. When the teacher asked the class if the graphs were similar or different, S1 says "different" and then adds "similar." The teacher asked him why, and S1 said: "because they both have the gradual rise and the gradual fall. . . one has the gradual rise and gradual fall, the other one has a. . ." Again, note his use of the terms "they," "one has," and "the other one has." This is interesting since one of the graphs was his own—he came up with a graph for apples falling off a tree and it was his own created description of his graph.

There was no personal investment of their self or identity and whilst this class did well on questions on graphical interpretation, we attribute the lack of correlations with their success in learning as measured by the content test with this basic syntactic and semiotic feature of the class. The students did not own the representations. Whilst students knew the representations and knew how to operate with them, there was no personal value or need for investment in this classroom exercise. Finally, some students had drawn piecewise linear functions and some students had drawn piecewise quadratic functions or quadratic functions, but the students appeared to be talking similarly about their graphs: "apples growing," "apples falling," or "getting picked." There was no discussion about the rate of growth or rate of depletion of apples.

6 Conclusions

Many activities deliberately bring to bear student identification with their construction to serve very specific learning goals, e.g., coordinating multiple representations and objects (Sfard, 1991; Thompson and Sfard, 1994). And they do so by "lifting" such traditional activity structures from the closed, individual cognitive space mediated by a single-device technology to the social space of the classroom (Stroup et al., 2002). Similar cognitive and perceptual actions are at work in identifying and relating features across representations. But these actions take place in a greatly enhanced context—a social space—with all the resources of the social context and conversation available to come into play (Roschelle, 1992, 1996), especially for "irregular" items. A frequent occurrence involves students collectively, in rapid and energetic cascades of interchange across several students, publicly "sleuthing" to determine, for example, the ownership of an outlier position graph ("Who was speeding?"). More importantly, the searches typically involved intimate mixes of logic and data interpretation of exactly the sort, which are prized in curriculum standards. Further, personal ownership drives attentional focus for students to locate themselves, and, through task and representation design (e.g., "Where are you?"), we ensure that the act of doing so pulls intense attention to the desired mathematical features.

Students do not need to belong to a community or *become* participants, but are actively involved in a mathematical activity space from the start that structures the classroom interaction. Wenger (1998) also denies any participation to the non-living—citing computers as an example of something that does not participate. As we have described, our position embraces a particular form of technology, which is a significant participant in representing multiple voices and feedback in a public space that projects students' personal work and promotes their identity. Their work can be identified in a public space with contrasts and comparisons being possible at a whole class level. Our definition of identity is less broad than Wenger. Students' identities are within the various mathematical representations they can share publically and can be infused with personal feelings of ownership (not membership) as a mathematically meaningful form of participation. It is in this way that students can relate and personify their mathematical contributions in public.

It is a striking result that specific intersecting patterns in learning and attitude change are positive in SimCalc classrooms versus similar algebra classrooms. Profound changes occur in the participation frameworks of such classrooms—in particular with the discourse patterns. And, peer-to-peer discourse not only increase in frequency of occurrence, but increase steadily over time. We believe that the intersection of a dynamic representational infrastructure in cohort with a fluid communication infrastructure is the most specific reason for such change.

The potential of CC can be realized only if we understand it sufficiently to inform the design and improvement of (i) its technologies; (ii) classroom activities, teaching practices, and forms of assessment that optimally exploit it; and (iii) the preparation and support of teachers to utilize this new constellation of technologies, activities, practices, and assessments. This will require a new, highly interdisciplinary domain of educational research—one that is now in its early stages—uncovering the new phenomena to be investigated, formulating issues, descriptive languages, candidate theories, research agendas, and building research communities to extend and elaborate the inquiry.

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Part IV
Impacts from Small-Scale Research

Mathematical Discourse as a Process that Mediates Learning in SimCalc Classrooms

Jessica Pierson Bishop

While the infrastructure of the SimCalc MathWorlds® (hereon referred to as SimCalc) technology provides unique opportunities for students to wrestle with complex and conceptually difficult mathematics in developmentally appropriate ways, the degree to which students engage in "doing" mathematics varies widely across classrooms. Some classrooms provide minimal opportunities for students to conjecture, justify, and exemplify, whereas in other classrooms these activities are an indispensable part of their day-to-day operations. One of the primary mechanisms by which students can be encouraged to engage in authentic mathematical activities is through the use of appropriately challenging discourse. In this chapter, I describe discursive norms and patterns observed across multiple SimCalc classrooms in order to better understand the interaction between discourse and technology in the broader classroom environment.

SimCalc technology and curriculum has particular affordances embedded into its design. These affordances include: (a) animations of real-world situations that bring the phenomena of motion to life, (b) dynamic linking of multiple representations that provide opportunities to explore relationships between graphs, tables, equations, and real-world scenarios in ways that non-interactive, static representations do not afford, (c) representational forms other than the formal, symbolic notational system that allow mathematical ideas to be presented in ways that leverage visual and graphical representations of a concept, and (d) visually editable graphs that allow students to manipulate problem parameters or the graph itself and discover the implications of these changes through immediate feedback. However, the existence of these affordances is not a guarantee that students are prepared to, or necessarily will, take advantage of them. Other factors mediate whether, and how, technology is used and to what degree students can leverage certain features of the technology.

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For example, some classrooms may not be designed to take advantage of the possibility for student-directed exploration made possible by the software. Perhaps resources are allocated such that the teacher is the only one with access to the software. Or perhaps participant structures and classroom norms allow little freedom for students to explore and conjecture; instead activities and interactions are tightly controlled and funnelled through the teacher. Or students may simply not recognize the significance or meaning of a particular affordance without appropriate scaffolding from someone more knowledgeable.

Cohen and Ball incorporated these ideas into their model of classroom instruction. They describe instruction as what the teacher and students do together—their interactions with one another and with educational materials around mathematical content (Cohen and Ball, 2001; Cohen et al., 2003). Learning, then, is a "function of the interaction among these elements [teachers, students, resources], not the sole province of any single one, such as teachers' knowledge and skill or curriculum" (Cohen and Ball, 1999, p. 2). Or, I might add, technology. Clearly technology plays an important role in learning, but its influence is mediated by a number of features of classroom environments. In this chapter, I will consider one particular factor mediating students' engagement with technology and the underlying mathematical concepts—classroom discourse. The following section outlines the conceptual framework for the study, offers a rationale for the discursive focus in this paper, and provides an overview of research related to the discursive constructs that emerged as significant in this study.

1 Conceptual Framework

1.1 Discourse and the Interactive Work of Teaching

The primary mode of interaction in schools involves discourse. In many ways, discourse is the heart of the classroom. Whether discourse itself is the main activity (e.g., engaging in a mathematics discussion) or the means to engage in the activity (e.g., creating a mathematical model), most of the doing in classrooms is discursive. Further, different opportunities to learn mathematics are created by habitualized, moment-to-moment choices teachers make in scaffolding everyday conversations in math classes. Over time, ways of responding *in-the-moment* are internalized and eventually become recognized and accepted ways of doing things in the classroom community (Gutierrez, 1994; Kennedy, 2005). Thus, variation in the normative discursive routines and kinds of mathematical discourse across classrooms is likely to affect the ways in which teachers and students engage with resources, including technology, and mathematics itself.

My approach to classroom discourse is consistent with an interactionist perspective of learning wherein meanings emerge for individuals as they interact socially within a community (Blumer, 1969; Cobb and Bauersfeld, 1995; Cobb and Yackel, 1996; Voigt, 1994). Though *individuals* make statements and pose questions, those statements and questions exist and have meaning within a larger classroom *community* with a unique set of norms, expectations, and routines. The classroom culture and even the mathematical content within that community are emergent properties that are jointly constructed by participants (Cobb and Bauersfeld, 1995; Voigt, 1994). Teachers and students mutually influence one another's discourse so that acceptable patterns of interacting are not imposed by the teacher but negotiated collectively. In this sense, discourse both reflects and creates not only the classroom microculture, but mathematical learning itself.

1.2 Key Discursive Constructs—Intellectual Work and Mathematical-Connectedness

Lemke (1990) claims that the common patterns of discourse pervasive in American classrooms reinforce the social norm that "they [the students] are there to listen to the teacher, not each other... what matters officially in the classroom is that each individual student pays attention to the teacher" (p. 78; see also research on Initiation-Response-Evaluation [IRE] patterns as described by Mehan, 1979 and Wells, 1999). Traditional models of schooling and many of the familiar, taken-for-granted practices of education position the teacher as the subject-matter expert whose job is to provide clear explanations of difficult mathematical topics that most children will probably never understand. The students' primary job is to remember what they have been told. These traditional and well-established models of schooling give rise to an interesting question: What happens if classroom responsibilities are shifted and *students* are expected to explain, elaborate, and evaluate for themselves?

Research indicates that encouraging students to make conjectures, generate ideas, verbalize their thinking, and provide justification for claims can support coherence and clarity in thinking, help students organize and restructure new information into prior experiences, create opportunities for participation and engagement, and increase metacognitive awareness (Chi et al., 1989; Piaget, 1952; vanZee and Minstrell, 1997; Vygotsky, 1978; Webb, 1991). Consequently, one of the discursive constructs I explored in this study was the intellectual work required of students. I use the term *intellectual work* to reflect the cognitive work set in motion and required of students within a given turn of talk. Intellectual work can be thought of in terms of the cognitive activities in which students are asked to engage during real-time, classroom conversations. Over time, if students routinely make sense of mathematics, struggle with complex problems, reason for themselves, generate multiple solution paths, and communicate their understandings, the potential for deeper mathematical understanding increases. Higher levels of intellectual work are invitations to do exactly this, to engage deeply with the processes and content of mathematics.

The second discursive construct that emerged from this study was mathematically-connected discourse. Hiebert and Grouws (2007) identify "explicit attention to *connections* [emphasis added] among ideas, facts, and procedures" as a feature of classroom interactions that facilitates conceptual understanding of mathematics (p. 391). They describe the discourse arising from this kind of teaching as coherent, connected, and structured around big mathematical ideas. I use the term *mathematically-connected discourse* to describe the coherent selection, sequencing, and discursive enactment of curricular activities in order to achieve specific mathematical goals. In this study, mathematically-connected discourse reflects the degree to which classroom activities, as enacted through discourse, support the primary mathematical activity of interpreting graphs of motion and their relationships to rate and variation.

2 Methods

2.1 Participants

This study was conducted as part of a larger program of research, Scaling Up SimCalc (Roschelle et al., 2010; Tatar et al., 2008). The Scaling Up study implemented a new, technology-rich curricular unit on rate and proportionality in middle school classrooms across the state of Texas. It used realistic problem contexts and simulations of motion to teach the foundational mathematical concepts of variation and covariation. The data for this paper comes from video footage of the same lesson being taught across 13 seventh-grade classrooms during the 2005–2006 academic year. All teachers were implementing the curricular unit and using SimCalc for the first time. The video-recorded lesson occurred during the middle of the unit and addressed rates of change through piecewise linear graphs of motion.

2.2 Data and Analysis

To facilitate a focused analysis of the mathematical discourse across classrooms, all the video recordings were transcribed. I then analyzed transcripts, video footage, and field notes through an open coding process (Strauss and Corbin, 1998) to determine common themes and categories, focusing specifically on teacher and student discourse surrounding the use of and engagement with SimCalc. In particular, my goal was to identify characteristics of discourse that both enhanced and limited the ways that the technology was used across classrooms. Given my discursive focus, I briefly describe my use of the term. I define *discourse* to be the spoken and written words, representations, and gestures of classroom participants as they use language to communicate, interact, and act (Johnstone, 2002; see also Gee, 2005; Wetherell, 2001). Though discourse includes written text and nonverbal communication, in this study, I restrict discourse primarily to conversation.

2.2.1 Coding Intellectual Work

My coding of intellectual work took into account the distinction between giving and requesting information as well as the cognitive action required of the participant.

Table 1 Intellectual work coding scheme

	Request	Give
Low-level		
Moves that provide basic information such as reading values off graphs/charts, performing calculations, giving an interpretation with no justification or evidence, or requesting these activities from others. Also includes positive or negative evaluations of others' comments.	So look at the graph, how many hours did it take the bus to get to Dallas? The distance was 180 miles and it took us 3 hours. What was the speed? Look at your x-axis, what is it showing you?	It took 2 hours to get to Dallas. Yes, 2 hours. That's right. Distance equals rate times time. So you have to figure out your distance. The bus is speeding up. Well, it maybe it changes directions.
High-level		
Moves facilitating engagement in mathematical argumentation and justification, engaging with another's thinking in a sophisticated way, or requesting these kinds of activities from others.	How do you know that the bus had stopped? How does the graph show that the van turned around?	The bus stopped because the distance stayed the same but the time kept going (student makes horizontal gesture). It's [the line] going down. The distance, the distance is less, and time is still going.

First, I developed codes to track the flow of information (is the speaker giving information or requesting information). Then, within the *give* and *request* categories, I defined two levels of intellectual work (*high* and *low*). High levels of intellectual work extend student thinking and include discursive moves such as providing justifications, examples, conjectures, explanations, and challenges; making connections across representations; generating problems and scenarios (contextualizing); or requesting these activities from students (Hiebert and Wearne, 1993; Pierson, 2008; Stein et al., 1996; Webb et al., 2006). Low levels of intellectual work include requesting or giving basic information (a recalled formula, definition, etc.), values read from a chart or graph, the result of a calculation, or making evaluative comments. Note that the high and low categories of intellectual work are defined and assigned with respect to the cognitive activity of *students*. For example, if a teacher's turn of talk is assigned a code reflecting a high level of intellectual work (i.e., high request or high give), this does not mean that the request itself involved a high level of intellectual work for the teacher, but that this move *positions the respondent* for a high level of intellectual work. Table 1 displays the categories of intellectual work with relevant examples. (See Pierson, 2008 for a more detailed description of the coding scheme.)

I coded each transcript turn by turn indicating the level of intellectual work present and whether the move was a give or a request through the use of four intellectual work codes—high request, low request, high give, and low give. Note that a single turn of talk can receive multiple intellectual work codes. In the next section, I report the results of my analysis through the use of illustrative excerpts from SimCalc classrooms.

3 Findings

The primary finding from my study was that discourse mediated how teachers and students interacted with the SimCalc technology and, therefore, the underlying mathematics. In the following sections, I identify and describe how two characteristics of discourse—intellectual work and mathematical connectedness—can enhance or limit the affordances of the SimCalc technology by providing examples of these discursive constructs in action.

3.1 Intellectual Work—Limiting or Enhancing the Affordances of Technology?

To illustrate how discourse can mediate students' interactions with technology, I share two whole-class discussions of the same task from different classrooms. Both classes are considering the position-time graph of a trip from Abilene to Dallas and its corresponding simulation of motion as shown in Fig. 1.

Students in Teacher L's classroom have run the simulation and described both the graph and the motion of the vehicles as "curving." Based on these comments, Teacher L suspects that some students may view a position-time graph as having "map-like" properties wherein the vehicles' paths are represented by the lines and the bus is literally turning after 2 hours. She is attempting to correct the students' interpretation by drawing their attention back to what the graph represents (see Table 2).

A careful inspection of this sequence reveals that the students' primary discursive responsibilities are to respond to basic questions and accept the teacher's interpretation for the given position-time graph. Opportunities for students to conjecture and reason about graphs of motion are minimal; instead, students are asked to remember formulas and recall facts related to position-time graphs. Overall, the intellectual work required of students in this excerpt is relatively low. For example, the students in this class are told that certain shapes and orientations on the position-time graph (e.g., "curving" in this exchange) represent different types of motion but are not asked to explain, nor are they given an explanation for why this is the case. Using "the graph bends" to justify the claim that an object is slowing down does not address the meaning embedded in the graphical representation. The question of why

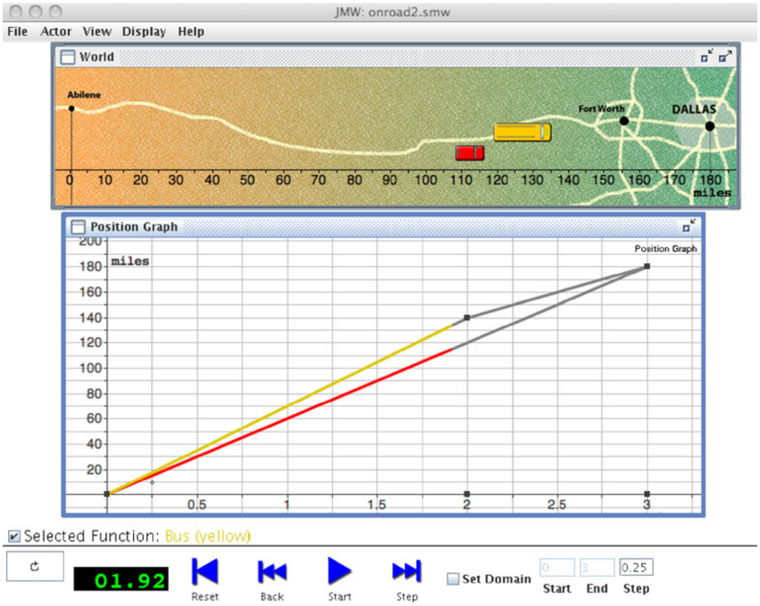

Fig. 1 Screenshot of a SimCalc MathWorlds® road trip simulation

a "bend" or "curve" represents a change in speed remains. It is easy to imagine a student repeatedly associating "bending" or "curving" with a change of speed and never understanding why the association exists. Contrast that with a student who notices that when going over one unit on the x-axis both before and after the "bend" (i.e., holding horizontal distance the same), the distance traversed vertically changes. Because the vertical distance decreases, the position-time graph is not as steep and appears to "bend" down. In terms of speed, the distance the vehicle covered in one hour changes from 60 miles to 40 miles which results in a decrease in speed.

Encouraging students to interpret the meaning behind graphical representations in this fashion is precisely what SimCalc is designed to do. However, to realize this goal, teachers must step up the intellectual work they require of their students by posing questions that can help students to make connections among the different representations of the motion situation. How exactly does one do this? In the next transcript excerpt (see Table 3), we see a teacher use higher levels of intellectual work to deepen students' understanding of rate and variation as she leverages some of the unique affordances of SimCalc. Students in Teacher E's class are also considering the graph of motion in Fig. 1. They have opened the simulation and, as they are running it, have been asked to offer possible connections between the simulation and the graph of motion.

One noteworthy aspect of this exchange is the way that Teacher E uses intellectually demanding discursive moves to make connections between the simulation and the graph of motion. Students first made observations based on the simulation,

Table 2 Transcript excerpt from teacher L's classroom: Interpreting graphs of motion

Turn	Speaker	Text	Level of intellectual work
1	Teacher L	What are we measuring on this graph? We're measuring miles and time. And when we put them together, what do we, what do we get when we put the miles and the times together?	Low give & Low request
2	Student 1	Um, how long like.	Low give
3	Teacher L	No. What do we do? Miles per hour. What is that?	Low request
4	Student 2	Uh you divide the hours by miles. . . .	Low give
5	Teacher L	What did we get when we did that, when we did their, their meters per second? What do we call that?	Low request
6	Student 2	A formula?	Low give
7	Teacher L	We used a formula, but wasn't that the speed? . . . So these lines are telling us about the speed. So if, if this graph's measuring our speed what happened to the bus's speed? What happened to the bus's speed and what happened to the van's speed?	Low give & High request (request interpretation)
8	Student 3	The bus started to get in front but whenever it curved the van sped up and caught it.	Low give (no justification of interpretation)
9	Teacher L	Now you said the bus started out in front, but they both started in the same place didn't they?	Low request
10	Student 3	Yes ma'am but when–	Low give
11	Teacher L	–No 'cuz the lines have nothing to do with which direction they went. The lines are their speed. So what do we see, what is the difference between the bus's speed and the van's speed?	Low give & High request (request comparison)
12	Student 4	They started at the same time but whenever the race started the bus pulled out in front.	Low give
13	Teacher L	OK so the bus was ahead of the van. Is that what you're saying, the bus was going faster than the van?	Low give & Low request
14	Student 4	Yes.	Low give
15	Teacher L	OK then what happened when the change in the bus's line?	High request
16	Student 4	It got slower.	Low give
17	Teacher L	It got slower. It slowed down. So do you see that that bend in the graph doesn't say that the bus turned in any way. It just said that it slowed down because it's speed changed. The graph changed. Does everybody see that? Can you see in the simulation where the bus changes its speed?	Low give & Low request

Table 2 (Continued)

Turn	Speaker	Text	Level of intellectual work
18	Student 5	I think he, it looked like he curved at Ft. Worth.	Low give
19	Teacher L	No. He didn't curve, he's still going the same route as the van.	Low give
20	Student 6	He doesn't even curve on this thing.	Low give
21	Teacher L	No there's no curve. The curve is where he changes his speed.	Low give

and, because the position-time graph was created simultaneously, they could connect changes in physical movement as depicted in the simulation to attributes of the graph itself. She then used students' observations as the basis to extend their reasoning about the relationships between motion and the corresponding position-time graph representations. For example, Teacher E used more intellectually demanding discourse when she asked students, "How do you know that the bus slowed down according to your graph?" and then later when she problematized a possible interpretation of a graph as a map and pushed students to provide explicit justification for why they disagreed with this common conception of position-time graphs.

Additionally, notice who is doing the intellectual work and mathematical reasoning in this exchange. Students are asked to make observations, connect the observed features of motion to graphical representations, consider alternative interpretations of motion, and justify their thinking. It is the *students* who are doing the mathematics, and in doing so they are refining their collective thinking about speed, rate, variation, and representations of these phenomena. Some might argue that this subtlety is unimportant as long as a correct, mathematically robust explanation is provided. However, not only are the activities of observing, conjecturing, justifying, and considering counterclaims central to the discipline of mathematics into which these students are being enculturated (Ernest, 1991; Kline, 1980; Lakatos, 1976), but engaging in these activities leads to deeper, more connected learning (Hiebert and Wearne, 1993; Kawanaka and Stigler, 1999; Pierson, 2008; Stein et al., 1996). Teacher E skillfully used discourse to help her students leverage the affordances of the SimCalc technology in order to connect visual traits of the graph with the motion observed in the simulation.

3.2 Mathematically-Connected Discourse—Making Conceptual Connections Embedded in Technology Apparent

SimCalc has affordances that can make conceptual connections more obvious; however, it relies on skillful teachers to sequence instruction, organize classroom activity, select specific tasks, pose questions, and facilitate conversations to help students make these desired mathematical connections. Earlier I introduced the term

Table 3 Two transcript excerpts from teacher E's classroom: Interpreting graphs of motion

Turn	Speaker	Text	Level of intellectual work
1	Teacher E	I want you to run that simulation and see what happened.	Low give
2	Abel	Uh–	
3	Nolan	Oh, they–	
4	Teacher E	Mara?	Low request
5	Zane	They uh–	
6	Teacher E	Okay, hold on Zane, let's let Mara answer this one.	Low give
7	Mara	The bus went at a constant speed but the red slowed down but sped back up.	Low give
8	Teacher E	Mara you feel like the bus stayed at a constant speed but the van didn't?	Low request (request confirmation)
9	Nolan	No, the bus went–	Low give
10	Abel	Which one is the bus?	Low request
11	Teacher E	(laughing) The bus is the big yellow one.	Low give
12	Nolan	The van (with emphasis) was at a constant speed.	Low give
13	Teacher E	The van was at a constant speed? It's kind of hard to tell. And you might have to get your position controls so you can see it.	Low request
14	Mara	Oh! The red one did stay at a constant speed.	Low give
15	Teacher E	The red one, which is the van, stayed at a constant speed. And what happened to the bus?	Low give & High request
16	Mara	The bus slowed down.	Low give
17	Teacher E	It slowed down. Can you look at your graph. Is that what you see on your graph?	Low give & High request
18	Students	Uh-huh. Yeah.	Low give
19	Teacher E	How do you know that the bus slowed down according to your graph? Kelsy?	High request
20	Kelsy	The, if you look at the van, it's like a constant rate and it stays straight. And the bus, like, leans forward.	High give

mathematically-connected discourse to describe the degree to which classroom activities and the mathematics content itself are coherently connected to the foundational mathematical ideas for a given topic through the medium of discourse. Data indicated that the mathematical discourse across SimCalc classrooms varied in its connectedness and coherence.

We see an example of mathematically-connected discourse in Teacher E's classroom (see Table 3). Generally speaking, she framed discussions using core mathematical concepts to organize the lesson's trajectory. Discussions had a sense of cohesion, and procedures and formulas were introduced as a natural part of answering central questions related to the big ideas of rate, variation, and speed. This resulted in longer sequences of talk where she and the students continued to work together

Table 3 (Continued)

Turn	Speaker	Text	Level of intellectual work
1*	Teacher E	Leans forward. That's not the bus turning the corner, or is it? Is the bus changing direction? (pause) This angle, where this is actually two lines (points to position-time graph), does it show that the bus went off this direction? (Teacher gestures upward motion following the path of the first linear segment in the van's path.) And then came this direction (teacher gestures horizontally)?	High request
2*	Students	Some students indicate disagreement	Low give
3*	Teacher E	No, do you disagree?	Low give (request confirmation)
4*	Student 1	Yes.	Low give
5*	Teacher E	Why do you disagree?	High request
6*	Student 1	Because it doesn't, it's not showing you the direction it's showing you how many miles they traveled.	High give

*Note that Turns 1–6 are from a different episode

to understand a task or problem (note that the entire interaction in Table 3 is one sequence). She provided students sufficient time to grapple with problems and resisted decreasing the cognitive demand of the task by breaking problems down into a series of smaller steps or procedures that students might have experienced success with more quickly. Teacher E expected her students to play an active role in discussion and knowledge-generation. She probed student thinking and allowed their observations and conjectures to be the impetus for further class discussion. Additionally, the connectedness reflected in Teacher E's discourse was supported by the way she blended classroom activities in an instructionally sound manner using simulations, calculations of speed, and generating real-world scenarios to support the primary mathematical activity of interpreting graphs of motion. She explicitly referenced previous tasks and problems to help students notice potential connections and provide a sense of lesson and unit coherence. In these ways, she leveraged the affordances of the technology to support students' mathematical thinking.

Unsurprisingly, in some classrooms, the discourse was less mathematically-connected. For example, Teacher A was more procedure-oriented and preferred to present mathematics as a series of small, manageable tasks and calculations. Connections to the big mathematical ideas of rate and variation were largely absent. This was reflected in shorter sequences of talk, shorter turns of talk for students, and less uptake of student ideas (because few student ideas were present to take up). As an example, consider the excerpt in Table 4 from a whole-class discussion in Teacher A's class. They are discussing the graph of motion in Fig. 1.

Sequences 1–5 were orchestrated to accomplish Teacher A's goal of calculating the speeds for the van and bus. Notice that this larger goal was not revealed to the

Table 4 Transcript excerpt from teacher A's classroom: Interpreting graphs of motion

Turn	Speaker	Text
		Sequence 1
1	Teacher A	Can you see when you replay the simulation when the bus started to slow down?
2	Student 1	Yes.
3	Teacher A	Right before it got to the end, correct? OK at what?
4	Student 1	140.
		Sequence 2
5	Teacher A	OK. So now let me ask you some questions. What, how long did it take them for this trip? Raise your hand. How long did it take 'em for this trip? Chris.
6	Chris	Three hours.
7	Teacher A	How do you know it went three hours?
8	Chris	Cuz it ends, the line ends at three.
9	Teacher A	Very good, cuz the line for the bus or the van?
10	Mult S's	Both.
11	Teacher A	Both end at what?
12	Mult S's	Three hours.
13	Teacher A	Three hours. So you know it took them three hours.
		Sequence 3
14	Teacher A	Now, was the speed of the two vehicles the same?
15	Mult S's	No.
16	Teacher A	No. You know that by how?
17	Student 2	They're not the same.
18	Teacher	Because of the (pause) graph right? And because of running the simulation. They didn't stay right beside each other, did they?
		Sequence 4
19	Teacher A	OK, umm, how far did they go in 3 hours? Raise your hand. How far did they go in three hours? Charles?
20	Charles	180
21	Teacher A	How'd you know that?
22	Charles	** (unintelligible)
23	Teacher A	180 miles where? (pause – no student response). It's on the graph, right?
		Sequence 5
24	Teacher A	OK. And let's see if you can calculate the speed. See if you can calculate the speed for the van and then calculate the speed for the bus and then also calculate the speed for the bus when it started to slow down since you guys have told me it started to slow down.

students until the last sequence (see Turn 24). Instead, it remained obscured in a series of known-answer and, from the students' perspective, loosely related questions whose answers provided the necessary components for calculating speed. Generally speaking, we might describe the lesson organization as a piecemeal collection of tasks and procedures with few explicit connections made among the activities or to the big ideas of variation, function, and rate.

Also noteworthy was the peripheral role technology played in this exchange. Presumably, if the lesson had been taught without the SimCalc software there would be few, if any, differences in the discourse or task selection and sequencing. This is because the technology was primarily used to verify calculations of speed. In Teacher A's class, the technology was often in the background so that students could, in the teacher's words, "calculate and do some math" (personal communication, January 23, 2006).

Additionally, as seen in this excerpt, little control or responsibility is given to the students as the teacher generated and controlled the topic of discussion. There was little room for student ideas to shape the discussion. Teacher A heard student responses but did not incorporate them substantially into the flow of the lesson. In fact, there was very little of substance for her to incorporate since students' roles were limited to paying attention, performing basic calculations, recalling formulas, and answering yes/no or short-answer questions. One might wonder, though, if this kind of scripted activity and interaction could, in some cases, lead to more connected mathematical discourse, particularly if the 'script' was based on the core, conceptual ideas for a topic. I would distinguish *scriptedness* and *planfulness*. Although both involve deliberate and intensive planning for the activities and related discourse, the difference is the extent to which a teacher is comfortable improvising based on his or her assessment of student needs in-the-moment. Planful discourse is purposefully designed to respond to and incorporate student ideas into the discourse, thereby adding to the connectedness and coherence of the discourse. Scripted discourse does not improvise and respond based on student ideas because of an inability or unwillingness to do so. (I realize that others might define and use these terms differently.) A scripted interaction, even when following a good "script," is not interaction in the true sense of the word, and trying to fit a student's comments into a predetermined script diminishes the coherence and connectedness of the discourse.

To summarize, I found that the presence of intellectually demanding discourse and the degree to which discourse was mathematically-connected mediated the ways in which students and teachers made use of technology to reason about the mathematics of rate and change. In particular, the more mathematically-connected and intellectually demanding the discourse, the more opportunities students had to take advantage of the affordances the SimCalc MathWorlds® technology offered.

4 Discussion and Implications

In each of the classrooms described in this chapter, students learned. In fact, all three of these classrooms produced higher mean learning gains, in the larger Scaling Up study, than their control group counterparts who did *not* use SimCalc (Teacher A's mean gain score: $\bar{x} = 5.62$, Teacher E's mean gain score: $\bar{x} = 10.25$, and Teacher L's mean gain score: $\bar{x} = 3.88$, control group's mean gain score: $\bar{x} = 1.50$). It seems that providing students opportunities to interact with linked, dynamic representations of quantitative relationships, most of which involve the phenomena of motion, is positively related to student learning. Even though the effect of the technology is robust ($p < 0.0001$, student-level effect size of 0.63; see Roschelle et al., 2010), each of these teachers experienced varying degrees of success in helping students to fully leverage the affordances SimCalc provided. In this chapter, I provided evidence that the effectiveness of SimCalc can be mediated by the mathematical discourse in which teachers and students engage. However, SimCalc is one instance of many existing and possible tools that can support student engagement in conceptually-oriented mathematics activities in the classroom. It seems likely that the discourse moves identified here are not limited to SimCalc but would apply across many such technologies.

4.1 Purposefully Planning for Productive Discourse

The intellectually demanding, mathematically-connected discourse in Teacher E's classroom was not a fortuitous event that happened merely by chance. Nor was her discourse scripted so that the outcome was predetermined. The reality is that this conversation was possible, in part, because Teacher E was familiar with the technology, tasks, and the underlying mathematics she wanted her students to learn. Because of this, she was able to recognize the potential mathematical ideas in students' contributions and weave those ideas together in order to meet her mathematical objectives. Thus, one implication I draw from this study is the importance of purposefully planning for productive discourse.

At first glance, facilitating conversations based on students' ideas and using those ideas to extend and deepen mathematical understanding seems daunting. How does one know what question to ask next, especially when a student's comment is unclear and you have only a few seconds to respond? Smith and Stein (2011) encourage teachers to view the sometimes overwhelming demands of responding to students in-the-moment through the lens of planning. They argue that planning helps teachers to better manage the improvisation required to skillfully facilitate real-time mathematics discussions. Anticipating possible student contributions, considering ways to respond to these contributions, and then deciding how to sequence and connect student ideas in a way that helps them reach their mathematical goals for the day, decreases the need for on-the-spot improvisation (Smith and Stein, 2011). The result is that mathematical conversations feel more manageable. Given this, I see two primary avenues to support teachers in planning for productive discourse in technology-rich classrooms—designing discursive supports into the curriculum itself and explicitly attending to discourse during professional development.

Designing instruction that can use discourse in powerful ways to leverage the affordances of a given technology starts with clear instructional goals. How can a teacher make decisions about which strategy to highlight or which connection to make without knowing the mathematical goal she wants to help her students reach? Thus, one way to design for powerful mathematical discourse is to identify specific mathematics goals for each lesson in the curriculum itself. For example, in existing SimCalc teacher materials, seven broad mathematical goals are listed in the unit introduction, and a "big idea" is given for each lesson. However, the big idea often does not clearly link back to the initial mathematical goals or may not be sufficiently explicit with respect to the tasks for that lesson. This curricular feature would help provide direction to teachers as they decide which strategies to highlight, which ideas to focus on, and how to effectively use the features of the technology for specific problems. During professional development, this feature could be highlighted, and teachers could be asked to personalize goals for particular lessons.

Additionally, for each task the curricular materials could provide multiple student responses, both correct and incorrect, along with the corresponding mathematical idea related to each response (see Smith and Stein, 2011). This information would familiarize teachers with unexpected and alternative approaches to tasks so that the first time they encountered new strategies would not be during the lesson itself. This type of curricular support could assist teachers in knowing what mathematical idea to highlight or problematize for a given strategy. During professional development, teachers might engage in producing this information themselves by anticipating possible responses to a subset of tasks, identifying the corresponding mathematical ideas in each response, and writing questions that might help students to notice connections across responses and to the mathematics itself.

And finally, sample classroom discussions around a set of tasks could be included in the curriculum and incorporated into professional development experiences. An example of a real classroom conversation gives one a sense for how he or she might put all of this information together using mathematically-connected and intellectually demanding discourse. However, these examples should not be viewed as scripts to be recited but as one of many discursive possibilities. Additionally, during professional development sessions, video of excerpts included in the curriculum materials could be used to compare and contrast whole-class discussions of the same task. Professional development facilitators might encourage teachers to reflect on differences in the video clips, particularly focusing on the selection and sequencing of tasks and students' solution strategies as well as the intellectual work reflected in both the teachers' and students' discourse.

In summary, technology can provide opportunities for students to grapple with fundamental mathematics ideas. Discourse, though, plays a mediating role in a class's ability to leverage the unique affordances of a given technology. Both technology *and* discourse are resources whose most compelling effects are seen in the hands of a skillful teacher. Perhaps the true mediating factor in the classroom is the teacher herself.

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Scaling Up Innovative Mathematics in the Middle Grades: Case Studies of "Good Enough" Enactments

Susan B. Empson, Steven Greenstein, Luz Maldonado, and Jeremy Roschelle

The reality for those who wish to reform mathematics instruction in schools is that a tension exists between achieving innovation and reaching scale. Reformers concerned with scale aim to reach a large number of students in many types of classrooms. Those concerned with innovation aim to maximize the possibilities inherent in a new curriculum. When an ambitious, innovative curriculum is implemented on a large scale, the question is, "What quality of implementation is realistic and 'good enough' to achieve the substantive goals?" rather than the question often asked in small-scale design research, "What quality of implementation is possible and most transformative given the potential of new technologies?"

We address the former question using case studies of the classroom implementation of a SimCalc-based curriculum replacement unit for seventh-grade mathematics focused on rate and proportionality in the context of linear functions. The randomized experimental study from which the cases were drawn documented a significant main effect on student achievement (Roschelle et al., 2010b). Student achievement gains were robust across demographic and regional variation, and attributable to a combination of factors that the experiment treated as integrated, including professional development, the curriculum materials, and the technology. The experimental

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study was not designed to document the complexity of instruction as it unfolded or the multiplicity of resources that supported student engagement, and so inferences about how instruction resulted in gains in student outcomes are limited. Our goal was to examine teachers' enactments of the SimCalc replacement unit and to identify those factors that appeared to successfully support student engagement and learning. Because we assume that any set of curriculum materials will be enacted in a variety of ways and that these enactments will have different impacts on student outcomes, we also ask just what constitutes "good enough" instruction in a scaled up intervention?

We use the idea of "good enough" enactments to signal an affinity with the idea of good enough parenting (Winnicott, 1971). Rather than using it to evaluate the teacher, we use it to refer to the set of possibilities created for students in a teacher's enactment of the curriculum replacement unit. Teachers' enactments will naturally vary. We argue, however that to be good enough, they need to provide resources that support opportunities to learn for all students.

1 Framework

1.1 Learning Resources

A set of curriculum materials can be provided to teachers, but as a resource it is not "self-acting" (Cohen et al., 2003). It takes teachers' and students' personal and collective resources to enact the curriculum and promote learning. We use *learning resources* to refer to what teachers and students use to enact the curriculum and engage with the content. This definition is consistent with Grubb's (2008) definition of resources as "those practices and programs within schools and classrooms (including the human resources) that improve valued outcomes" (p. 106). Learning resources fall into several categories and are created and used synergistically by teachers and students as they engage in tasks. These categories include material resources, such as curriculum workbooks and computer simulations; personal resources, such as teacher knowledge or student understanding and disposition; collective resources, such as activity structures or interactional norms; and abstract resources such as instructional coherence over time (Cohen et al., 2003; Gresalfi, 2009; Grubb, 2008).

Different configurations of resource use during lessons are possible, thereby providing different opportunities for student engagement. For example, one teacher may use cooperative group work to engage students in complex problem solving tasks using a configuration of resources that include complex tasks designed to require multiple types of input, routines for interacting with peers, and the teacher's management of the groups. Another teacher may use one-on-one dialogue with individual students to examine and extend their thinking about mathematical ideas using a configuration of resources that include tasks designed to be solved independently of the teacher, the teacher's repertoire of questioning to probe and extend students' thinking, and teachers' knowledge of students and the mathematics they understand.

The content of the replacement unit used by teachers in the experimental study was based on SimCalc MathWorlds® and used computer simulations of motions linked with graphs, tables, and equations to teach the big ideas of rate and proportionality with a focus on linear function. To explore students' access to learning resources during the curriculum unit, we focused on the development of concepts across lessons and the quantity and quality of connections among ideas that were made by teachers and students within lessons.

1.2 Making Connections and Expending Effort

We used *connectedness* as a lens through which to examine the nature of mathematics content as it was presented and developed during instruction. Connectedness is associated with learning with understanding and the coherence of instruction (Bransford et al., 1999). Cohen and colleagues (2003) emphasized that, "coordinating instruction… depends on making connections among teachers' and students' ideas, among students' ideas, among both over time, and between both and elements in the environment" (p. 126). Consistent with this claim, Hiebert and Grouws (2007) synthesized evidence from a variety of studies to argue that teaching for understanding is associated with explicit attention by students and teachers to making "connections among ideas, facts, and procedures" (p. 391).

Hiebert and Grouws (2007) also argued that to develop conceptual understanding, students must have opportunities to expend intellectual effort to make connections. Students might make such connections while they are listening to a teacher's explanation or watching a computer simulation, but the opportunities to learn presented by activities that do not afford students' active participation tend to be taken up by a only small number of students. We therefore expected *good enough* teaching to support a plurality of students to expend effort to make the connections that supported their sense-making.

2 Methods

We selected three case study teachers from a total of 37 in the treatment group for Year 2 of the experimental study. We chose teachers who had a range of achievement gains in Year 1 and who had a range of Mathematics Knowledge for Teaching (MKT) as measured by an MKT assessment administered to all teachers at the conclusion of the first summer's workshop (Shechtman et al., 2010).

The materials and software were developed for ease of use by teachers who were working with the technology for the first time. A four-day teacher training focused on understanding the content, using computers, and experiencing the unit as learners.

All case-study teachers were observed teaching five lessons (out of 10) from the unit. One lesson sometimes stretched over two or three class sessions. All lessons were video-recorded.[1]

We focused our analysis in particular on three lessons that involved significant new content and where we expected to see the greatest opportunity for learning. These lessons included "A Race Day," which involved the introduction to using position-time graphs to represent linear motion; "On the Road," which involved students' first experiences with piece-wise linear graphs including graphs representing backwards motion and stopped motion (Fig. 1); and "Salary Negotiations," which involved the transfer of content involving motion in a new context, earning money.

We collected data on student outcomes to measure the impact of students' access to learning resources. All students were given a written test before and after the unit. Midway through the unit, we interviewed a subset of seven students from each class representing a range of prior achievement levels about their views on the unit and their participation.

We transcribed all lessons and segmented the transcripts into episodes. We created *Content Maps* to represent the content and connectedness of classroom talk in these episodes. Like concept maps (Novak, 1990), Content Maps are composed of links and nodes. Nodes include mathematical ideas, assertions, propositions, procedures, predictions, reasons, and story lines. Links between nodes represent relationships that are explicitly stated or implied in what participants say or do. Nodes and links are organized in clusters that corresponded to instructional episodes focusing on a task. Unlike concept maps, which are intended to represent the knowledge structures of an individual, Content Maps are based on classroom interactions and are meant to represent the enacted content—the substance of what teachers and students talked about and otherwise attended to. In deciding what to include in our maps, we looked for content that was valued or emphasized by the teacher or a student. For example, in the following exchange in which the class is discussing the graph that appears in Fig. 1b, students are oriented to the content shown in bold (nodes) by a variety of talk moves that alert students to its value. Table 1 associates these talk moves with nodes that would appear in a Content Map of this exchange.

T: I already heard Mara say that the van traveled at this **constant rate of speed**. Now I heard several people say that. . . . And she said that the bus did what?
. . .
Abel: It would have been going, like, you said a **straight line** (using pencil to point to horizontal graph on the computer to teacher) is like this.
T: Ah, remember if we have a **horizontal**—remember **they're all straight**.

[1] All data collection, summaries, and transcripts were completed by mathematics education doctoral students Theodore Chao, Steven Greenstein, Luz Maldonado, and Jessica Pierson Bishop at the University of Texas Austin.

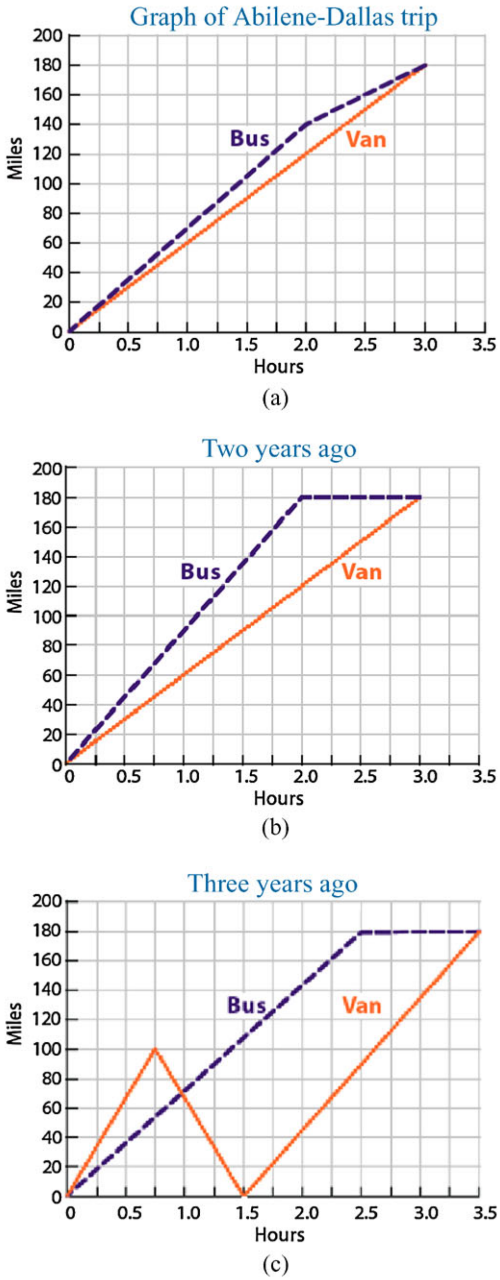

Fig. 1 The graphical component of a task in which **(a)** the *bus* and *van* slow down after two hours; **(b)** the *bus* stopped after two hours; and **(c)** the *van* returned to Abilene before making its way to Dallas

Table 1 Corresponding discourse move for each node

Line #	Node	Supporting talk move
1–2	Constant rate of speed	Revoice: "I heard Mara say… Now I heard several people say that"
5–9	Straight line, they're all straight	Draw attention: "Ah, remember…"
11–12	Horizontal line means stopped	Monitor content and remind of correct claim in response to student's ambiguous claim: "If we have a horizontal line… stopped"
16–17	Speed declined	Confirm and reinforce: "…I love that"

10	Student:	Maybe it slowed down to wait for cars.
11	T:	If we have a **horizontal line**, that's **when they're stopped**. What
12		do you think happened? At what time did that happen?
13	Neil:	At 2 hours.
14	T:	At 2 hours, what happened?
15	Neil:	The bus, uh, uh.
16	Abel:	**Speed declined**.
17	T:	The **speed declined**. I love that.
18	Zane:	I wish we could get there in 2 seconds.
19	T:	If we could get there in 2 seconds?
20	Zane:	Or 3.
21	T:	It feels like it takes forever, particularly, I tell you, from Abilene to
22		Dallas does feel that way.

There are several ways in which discourse—usually in the form of moves made by the teacher—orients students to content, including by confirming or reinforcing what is correct, monitoring content, selecting content, controlling access to content, taking up student contributions, repetition, and reinforcing desired practices and processes (Mercer, 2000). As we created Content Maps, we examined our transcripts to determine what content was expressed, valued, and reinforced through discourse and then created nodes to represent that content. If two nodes are not linked, then there were no connections made between them in classroom talk. The frequency with which a link was reiterated is shown in the thickness of the link.

To test reliability of our maps, each member of the research team created a Content Map of a lesson previously mapped by another team member. We compared the maps by identifying primary and secondary nodes for each cluster, according to the number of links to these nodes. In cases of ties, we listed both (or all) nodes as primary or secondary. We then counted the number of matching nodes, aiming to achieve 80 % or higher in matches. In all cases, the percentage matched was 86 % or higher.

We augmented our analysis of Content Maps with analyses of classroom transcripts, student interviews, and teacher interviews to examine other factors that appeared to influence students' access to learning resources.

3 Good Enough Curriculum Enactments

Our questions concerned identifying the resources for learning that were used during the replacement unit and identifying similarities and differences in configurations of resource use from one classroom to the next. We found that student learning was supported by different configurations of resources. In some configurations, the teacher was more central to supporting engagement, and in others, the use of materials figured more prominently. In one case, students' frequent use of the curriculum materials and computer simulations appeared to offset difficulties the teacher had expressing and managing the content.

We present three cases. Each case represents the use of a different configuration of resources. Table 2 includes information on the three case-study teachers and student gains in their classrooms in Years 1 and 2 of the study.[2]

Ms. Garfield and Mr. Simmons, in particular, present a surprising combination of features. Ms. Garfield's MKT score was significantly below average, yet her mean class gain was above average for both years. In contrast, Mr. Simmons's MKT score was among the highest of all the treatment teachers, yet his mean class gain was consistently below average. Further, the standard deviation of students' achievement gains in his class was the largest among our cases and significantly above the average standard deviation of the treatment sample, suggesting differences in students' learning were more pronounced in this classroom. We unpack these features below. Most notable were instances in which high MKT was not necessarily put to effective use as a learning resource, and interacting with the computer simulation appeared to compensate for weaker MKT resources in other components of instruction. Teachers' MKT was reflected in particular in the Content Maps of their lessons. However, as we detail in our analyses below, Content Maps also provided information about the quality of the connections that the teacher and students were making.

Table 2 Information about the three case study teachers

Teacher	Region	MKT*	Track	Student achievement			
				Mean pretest**	SD pretest	Mean gain	SD gain
All 48 teachers	various		n/a	13.57	5.60	5.24	3.87
Driver	suburban	16	accelerated	19.42	3.27	5.21	2.82
Garfield	rural/oil	7	non-tracked	13.36	6.00	6.45	2.94
Simmons	rural/farm	18	non-tracked	8.05	3.34	3.90	4.79

*Total possible was 24. Average MKT score at completion of Year 1 workshop was 13.1

**Total possible on assessment was 30

2Each teacher's mean achievement gain is consistent with the phenomenon of regression to the mean between Years 1 and 2 of the study. That is, in Year 2 of the study, each mean gain was closer to the mean of the entire sample than in Year 1. None of the mean achievement gains appear to be a result of a ceiling effect.

3.1 Coherence, Connectedness, and Computers: Case of Ms. Driver

Ms. Driver was a 28-year veteran mathematics teacher who taught in a mid-sized city in north Texas, USA. She held a certificate to teach Grades 4–8 mathematics. Ms. Driver's MKT score—16 out of 24—was above the average of 13.1 for the treatment sample, suggesting relatively strong content knowledge. Her students' average gain in Year 2 was about average. She taught in a high-ability classroom, which was reflected in her students' pretest scores.

Ms. Driver represents a case in which teacher knowledge and practices and the use of the computer technology all served as resources for students' engagement. We noted in particular the coherence of instruction with respect to rich, whole-group discussions of the big idea of rate.

3.1.1 Connectedness and Coherence of Whole-Group Instruction

Whole-group recitation in Ms. Driver's classroom provided students with multiple opportunities to make connections involving the big ideas of the unit. Content Maps of Ms. Driver's lesson showed how rate, in particular, was linked to related concepts, examples and counter-examples, narratives, and procedures for calculating speed and slope. Further, they revealed a strong coherence in the development of this content over time.

To illustrate, in the second lesson, "A Race Day," the majority of whole-group discussion focused on "speed." Simulations of two girls running races were presented and students were asked to compare and then calculate their speeds. The clusters of talk involving "speed" in the Content Map of this lesson show that the teacher and students verbalized several interconnected meanings (Fig. 2). Specifically, speed was defined as the "ratio of distance traveled in a certain time" and was linked to "unit rate" which was, in turn, linked to examples of rates, including miles per hour, meters per second, and feet per second. In the same cluster, speed was calculated several times by dividing the distance by the number of seconds to yield a "rate of speed." By explicitly using and interrelating a mathematically robust definition, several examples and procedures within the same episode of talk, instruction provided opportunities for students to interconnect and deepen their understanding of speed.

Whole-group discussion in a later lesson, "On the Road," exemplified the coherence of instruction. Ms. Driver again focused on speed but framed it in relationship to new content, including the constructs of rate and slope, providing students with opportunities to deepen their understanding by interconnecting speed with more sophisticated concepts. Specifically, she used the phrase "unit rate" to refer to speed and explicitly linked the concept to "slope" (Fig. 3). The link between unit rate and ratio was articulated several times, in definitional statements—such as speed is "distance in time"—and in characterizing unit rates as a type of "per change" unit. As before, their discussion included a constellation of procedures, category exemplars, and interpretations, such as motion, consumption, and cost.

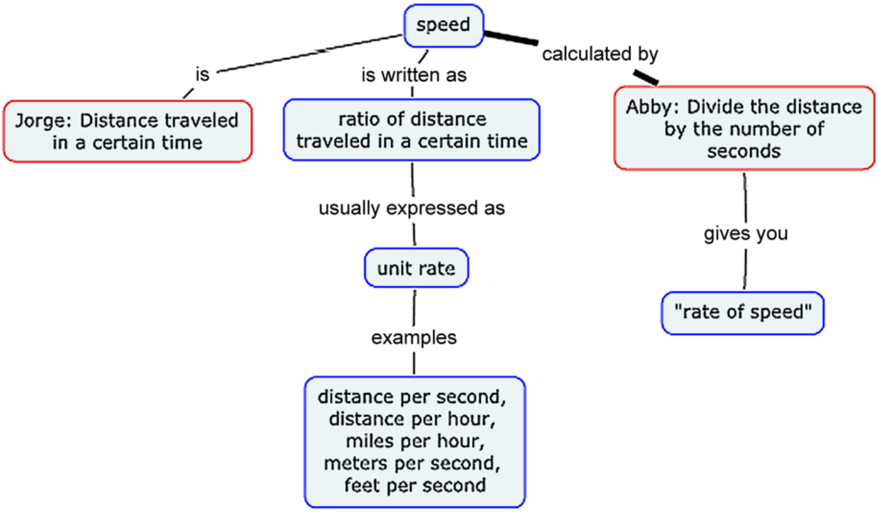

Fig. 2 Cluster from Ms. Driver's Content Map for discussion of speed in "A Race Day" lesson

3.1.2 Pressing for Reasons and Facilitating Argumentation

A second feature of Ms. Driver's instruction that provided opportunities for students to expend effort and make connections was the practice of pressing for reasons for students' claims. The quality of these elicitations distinguished her from the other case-study teachers and provided students with opportunities to make and deepen connections within a single content strand. Ms. Driver tended to use these practices in the context of developing new and sometimes challenging content but not necessarily as part of routine recitation.

For example, in "On the Road," Ms. Driver addressed the most challenging idea of this lesson—interpreting a line segment with a negative slope (Fig. 1c)—by pressing students for explanations that linked a downward sloping position-time graph with backwards motion. One student reasoned that the van must have turned around because the "distance decreased." Another argued that the speed for the first hour and a half must be equal to the speed returning to the origin during the second hour and a half because they covered the same amount of distance in the same amount of time. Other students provided counter-arguments to a claim that a sideways V-shaped segment represents the van's return to the origin by arguing that "he can't go back in time" and "he can't be in two places at one time." Ms. Driver did not consistently press for reasons, however, and at times talked substantially more than students.

3.1.3 Simulation Software

Reflections by target students on the unit and how they benefitted from it corroborate our finding that Ms. Driver's instruction provided adequate access to learning re-

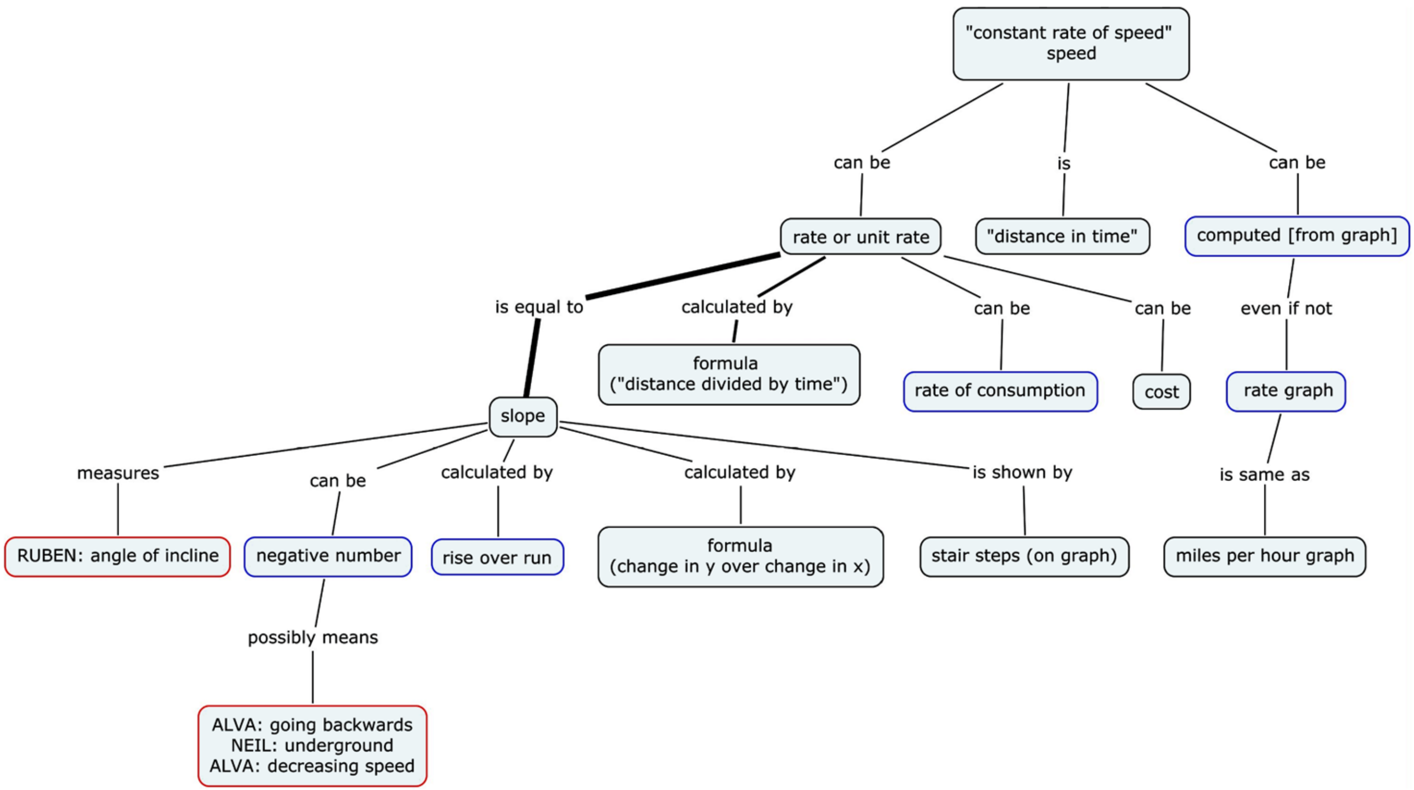

Fig. 3 Cluster from Ms. Driver's Content Map for discussion of constant rate of speed in "On the Road" lesson

sources. In particular, students reported that engaging with the simulation was challenging and intellectually satisfying. Six out of the seven target students described the use of MathWorlds as "fun." More importantly, this enjoyment was accompanied by the view that interacting with the simulation helped them to understand the material and to appreciate its applications. For example, Jorge[3] said, "It's more fun [than usual mathematics instruction] because we're learning more stuff." Nadi said, "Well, what we used to do in math class, we would, we would look at graphs and write the information down from them. But in the MathWorlds thingie, when you do graphs, you like, you dissect it. Like you look at each part, what it means, what it's showing, how to explain it."

3.2 Social Relationships and Simulations: Case of Ms. Garfield

Ms. Garfield taught in a small town in west Texas. She had been teaching for 18 years and had an elementary generalist certification for grades 1–8. She had taught mathematics every year. Ms. Garfield's MKT score—7 out of 24—was significantly lower than the average MKT score in the treatment sample and the lowest score of our case study teachers. Yet Ms. Garfield's students had above average gains in both years of the study, a puzzling exception to the claim that high student achievement depends on high teacher content knowledge (e.g., Hill et al., 2005).

Ms. Garfield and her students were part of a close-knit community and it was not unusual for them to interact outside of school. She was familiar with many of her students' families and their situations.

At first glance, the learning resources in Ms. Garfield's instruction were not obvious. Her Content Maps showed a focus on procedural explanations. The emphasis in whole-group discussion was often on answering questions quickly and correctly—chorally as a group. Some of these observations were consistent with Ms. Garfield's relatively low MKT score. However, further analysis revealed an instructional system in which memorable narratives, social cohesion, and adequate time to work with partners and the computer simulation enhanced students' opportunities to expend effort to make mathematical connections. We unpack these findings in the following sections.

3.2.1 Lack of Connectedness Within and Across Lessons

Ms. Garfield's Content Maps revealed a focus on declarative knowledge and procedures. Classroom talk was teacher-driven and many of the critical connections were made by the teacher rather than by students.

[3]The number of letters in a student's pseudonym indicates achievement level on pretest: 3 for low, 4 for medium, 5 for high.

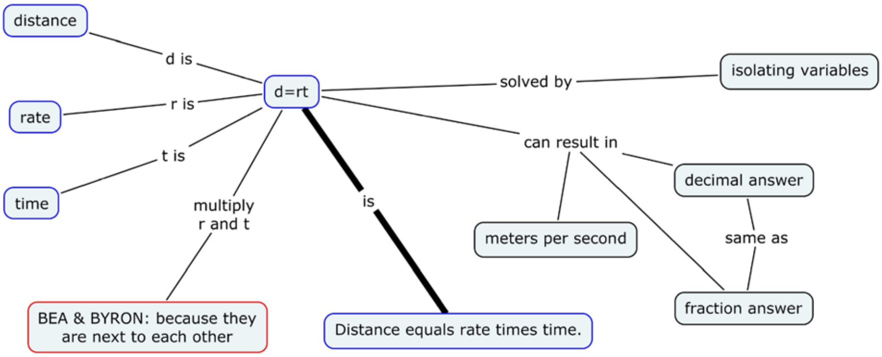

Fig. 4 Clusters from Ms. Garfield's Content Map for discussion of rate formula in "On the Road" lesson

In contrast to Ms. Driver, speed and motion were discussed in terms of how to calculate them, with few connections to the big idea of rate. Rather than making conceptual connections, Ms. Garfield focused on the mechanics of computations. She also animated this content with examples from students' daily lives.

For example, in "A Race Day," the distance formula, $d = rt$, was by far the most talked about content followed by "speed." A good deal of this talk focused on how to apply the distance formula and syntactic aspects of the coordinate plane. Students were asked to chorally repeat the distance formula several times. Speed was defined rather opaquely as "moving [while] time is passing." Ms. Garfield provided examples of people walking at different speeds in the park and had students embody the idea by walking across the room while watching the clock.

Speed was addressed descriptively and procedurally in later lessons, as well. When varying rates including slowing, resting, and returning and their associated piecewise graphs were introduced in "On the Road" (Fig. 1), Ms. Garfield focused on relationships of association, such as a "line bending" means the bus "slowed down" (Fig. 4). The horizontal line segment in Fig. 1b was interpreted as "waiting," because the bus "got more time but didn't go further." There was no further discussion of either of these new situations.

Lessons in which students were called on to create narratives that explained what happened in a graph tended to elicit greater than usual participation from a wider range of students, including Cal and Bea, two low-prior-knowledge students in Ms. Garfield's class. For example, in "On the Road," when students were asked for conjectures as to why one vehicle returned to the origin, they enthusiastically offered all kinds of reasons, such as forgetting sunglasses, money, or food. These narratives may have served as memorable examples for some of the key learning goals of the unit. Evan, a student who was quiet during whole-group recitation, apparently continued to think about these examples once the unit was over. His mother reported to Ms. Garfield that as they were driving to another town, Evan used the distance they had to travel and the speed at which they were traveling to figure out how long the trip would be.

If whole-group recitation provided few opportunities for students to make connections among concepts, facts, and procedures, where did these opportunities appear?

3.2.2 Partner Time and Computer Use

In comparison to participation in whole-group instruction, working with partners to interact with the computer simulation appeared to provide students with more opportunities to engage with the content of the unit and to make connections between ideas, facts, and procedures. Students in Ms. Garfield's class worked with partners for substantial periods of time in 10 out of the 12 sessions that we observed. Usually this work involved interacting with each other and the computer simulation. The small size of the class allowed all students to sit in the first two rows at computers, allowing Ms. Garfield to interact with them individually. Consequently, students did not need to choose between orienting their bodies and attention towards the computer or towards the teacher, as was the case in Mr. Simmons's classroom.

This arrangement also facilitated informal conversations among students as they worked in pairs, and Ms. Garfield actively fostered a classroom culture in which students were expected to assist each other in ways that are consistent with the tenets of cooperative learning (Webb and Palinscar, 1996). For example, when a third of the class was absent on the first day of the unit, Ms. Garfield allowed the students to fully investigate the simulation software. As the students worked, Ms. Garfield reminded them that they were going to be the "teachers" the next day for those students who had been absent. The next day, Ms. Garfield paired students who had been present with students who had been absent to share the discoveries they had made about the software.

Six out of the seven target students singled out partner work as helpful to their learning. All three students, who were identified as having low prior knowledge, reported partner work as beneficial, and one of them—Cal—said that while he sometimes found the teacher's explanations confusing, he felt comfortable relying on other students to help him out. Drew said that working together helped him to clarify his understanding. He reported that he worked well with Bobby and described how they would often "work it out and, like, see if whoever got a different answer."

3.3 Access Restricted: Case of Mr. Simmons

Mr. Simmons was in his fourth year of teaching, all at the same high-poverty school located in an agricultural region in south Texas. Before participating in a one-year alternative credentialing program to teach mathematics in grades 4–8, Mr. Simmons was a technical professional. Like Ms. Driver, Mr. Simmons's MKT score—18 out of 24—was above average and suggestive of strong content knowledge. In contrast, his students' average gain of 3.55 in Year 2 was well below the average gain of 5.31 for the treatment sample and the lowest within our set of case-study teachers.

The most salient feature of Mr. Simmons's curriculum enactment involved how he managed students' access to learning resources. By constraining the ways in which students participated in whole group discussion and restricting students' access to explorations of computer simulations, he essentially reduced students' opportunities to make meaningful connections involving the big ideas of the replacement unit.

3.3.1 Differential Levels of Participation

Whole-group recitation in Mr. Simmons's class was dominated by a small group of students. These six students, all male, took more opportunities to engage with the content than the other 20 students who talked less frequently, if at all, during this part of the lesson. The mean number of turns of talk per day among this group was more than 11 times the frequency with which other students contributed.

Differential participation of this sort was not unusual in our cases. For example, a small group in Ms. Garfield's classroom also tended to participate more frequently in whole-group instruction than other students. However, the difference in relative frequencies of participation between the two groups was more pronounced in Mr. Simmons's classroom than in the other teachers' classrooms.

3.3.2 Connections as Associations

Participation in whole-group discussion in Mr. Simmons's class tended to be limited to short responses to closed questions. As a result, relationships between ideas in Mr. Simmons's classroom were better characterized as associations than as well-developed conceptual relationships.

The simple connections among ideas illustrated in the following episodes typify these relationships. In the second lesson, "A Race Day," the majority of whole-group discussion in Mr. Simmons's classroom focused on "speed." Speed was defined by the teacher as equal to "distance divided by time" and as "meters over seconds." Speeds were calculated using these formulas and the results were tied to units suggested by students, including miles per hour and meters per second. Similarly, in "On the Road," Mr. Simmons's treatment of slope (Fig. 5) was nearly equivalent to his treatment of speed. Once slope was established as synonymous with speed, the discussion became about speed and a formula for calculating it. The "spine" in the map that extends the length of the cluster is darkened six times to indicate the number of times in the lesson that Mr. Simmons presented slope as "distance divided by time" and then used the "endpoint technique" to calculate the slope of a segment of a piecewise linear graph using its endpoints.

By limiting the treatments of speed and slope to procedural applications, the instruction provided fewer opportunities for students to develop an enriched understanding. Ms. Driver's treatment of speed provides an explicit contrast.

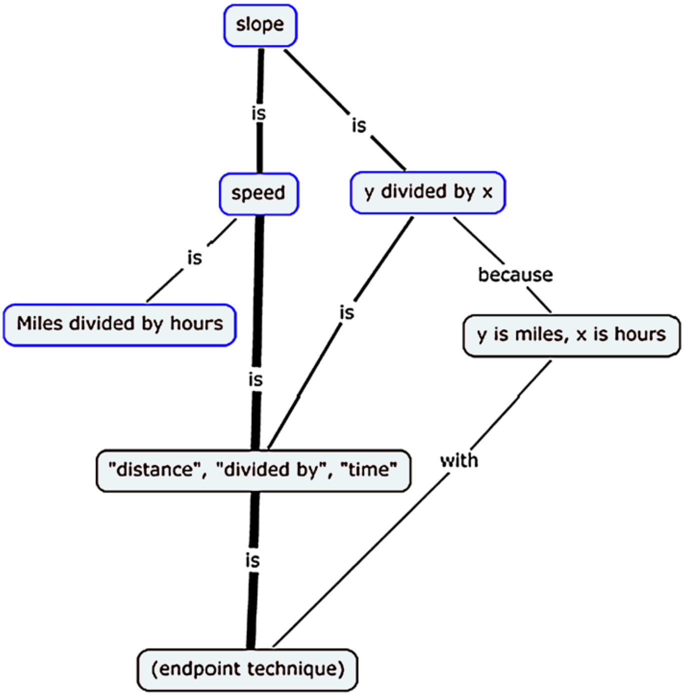

Fig. 5 Cluster from Mr. Simmons's Content Map for discussion of speed in "On the Road" lesson

3.3.3 Use of Computer Simulations

During whole-group discussion, students sat at tables while Mr. Simmons operated a computer, LCD projector, and overhead projector at a podium. This was in contrast to both Ms. Driver's and Ms. Garfield's enactments in which students had access to computers during whole-group recitation and were observed on occasion to autonomously operate the simulation to explore or answer questions. In fact, Mr. Simmons reported that he told the students, "Forget about the computers and come to the table and then just pay attention." When it was time to work with the simulation, Mr. Simmons's students moved from the tables to sit in front of individual desktop computers.

For the first 8 of the 10 days we observed, class was conducted in the lab and was split between whole-group recitation and computer activity. On three of these days, Mr. Simmons interrupted students' interactions with the software and instructed them to return to the tables. Once he told them, "Some of you are confused, so let's go back to the tables and I'm going to show you something on the [computer]." On the ninth day, the class was moved from the computer lab to Mr. Simmons's classroom—which contained just one computer and LCD projector—where the remainder of the unit was implemented. Mr. Simmons explained the rationale for his decision: "The reason why we are here right now, guys, and we didn't go to the lab today… yesterday, I saw a lot of you that were not getting this program, or maybe you were doing something else, OK? You were spending, like, maybe, half an hour on just the first… problem."

Similarly to students in other classrooms, all seven target students reported that the computer simulations were helpful to them. One said that the computers "help us learn and help us do things. They make it easier." Another said that computers "make it easier to participate." Still another said, "it's more entertaining at the computers" and "you get bored" when you are away from them. Despite their enjoyment of using the computer and their ability to learn from it, Mr. Simmons, in contrast with Ms. Driver and Ms. Garfield, limited students' access to computers and, when they were working at the computers, managed their interactions with the software in ways that curtailed opportunities for them to make connections.

4 Discussion

What constitutes a good enough curriculum enactment? Our cases help us begin to answer this question by identifying how learning resources were created and managed in each of the classrooms. Content Maps revealed differences in how teachers' expressed MKT in instructional talk and in the kinds of content connections that the teachers emphasized. Other resources that were related to the quality of students' engagement included the computer simulation and social relations.

Both Ms. Driver and Ms. Garfield had high gains, but used learning resources in different ways. Ms. Driver had high MKT and pressed for reasons in full group discussions. Mathematical conversations in her classroom featured rich connections. In contrast, Ms. Garfield had lower MKT and full group discussions in her classroom had less variety and depth of connections. Students had to work harder to extract meaning from these discussions. However, Ms. Garfield supported students' use of the software and interactions with their peers as resources by providing ample time, a facilitative classroom layout, and a supportive culture for engagement. Although Ms. Driver had a somewhat more teacher-directed approach to instruction and Ms. Garfield had a somewhat more materials-centered approach (software and workbook), both teachers had classrooms in which students had high mean learning gains.

The case of Mr. Simmons provides a contrast. Students in his classroom had a low mean gain despite the teacher's high MKT. The reasons appeared to have to do with students' restricted access to learning resources. Although students sensed the intellectual potential of the materials and Mr. Simmons sensed the students' conceptual struggles with the mathematics, Mr. Simmons tended to respond by reiterating the same procedural connections and limiting students' time using the software.

Further, the learning that occurred was unevenly distributed. The standard deviation of learning gains in Mr. Simmons's classroom was 4.8, compared to 2.8 for Ms. Driver's class and 2.9 for Ms. Garfield's class (and 3.9 for the sample). Some of Mr. Simmons's students had large gains, about 25 % had no gains, and another

25 % had small gains.[4] There were learning resources in Mr. Simmons's classroom, but they appeared to be accessed by, at most, about half the students. By almost any standard this is not good enough (e.g., Roschelle et al., 2010a).

These case studies do not exhaust the possible configurations of learning resources described in prior small-scale design research (Roschelle et al., 2008). For example, some configurations involve projecting student work with the software to a classroom display and more extensively involving students in clarifying, refining, consolidating, and justifying their ideas. While both Ms. Driver and Ms. Garfield had this potential, we did not see student work with the software move from individual to full group contexts. Another possibility hinted at in Ms. Garfield's case would be a more explicit focus on organizing students to work in groups. Full class discussion might be less influential, but more attention to managing opportunities for students to create and explore connections in the context of group work would be necessary (e.g., Cohen and Goodlad, 1994).

Another notable aspect of the contrast in these cases concerns the contribution of teachers' mathematics knowledge for teaching. Although Ms. Driver and Mr. Simmons had high MKT, their classrooms featured remarkably different configurations of learning resources. Assessing how a teacher uses what she or he knows about a given topic to direct tasks and interact with students, such as we did with Content Maps, may be a more accurate characterization of teacher knowledge as a learning resource than a teacher's score on a content knowledge assessment.

Across all three case studies, we observed patterns in classroom interactions that resonate with how the developers of SimCalc viewed its potential as a learning resource. For example, the developers believed that giving students ample opportunity to use the software is critical. This belief is tested most convincingly in Ms. Garfield's case—as student interaction with the software seemed to be the primary learning resource. In Ms. Driver's case, student interaction with the software functioned as a complementary resource to full class discussions, and in Mr. Simmons's case, lack of support for student use of the software contributed to weaker gains for many students.

5 Conclusion

These case studies were drawn from a larger empirical project that established that the new curricular materials and professional development were effective in most classrooms and for a wide range of teachers and students. Within this context of positive results, the role of these case studies has been to fill in details of how learning gains were achieved—what did "good enough enactments" with these new materials look like?

When we examined classrooms that had good enough enactments to obtain strong mean learning gains with the new materials, we found that the more success-

[4] Although the class's relatively low mean pretest score may seem to have put the class at a disadvantage for learning gains, the experiment found no significance differences in mean gain scores across differential pretest scores (Roschelle et al., 2010b).

ful enactments were not all alike. The main theoretical idea that we have advanced to explain these findings is that the same software and workbook materials can be realized in different ”configurations” and that those configurations that exemplify good enough enactments support students to make connections and to expend intellectual effort to make sense of the content of the unit. Both more teacher-centered and more student-centered configurations sufficed and seemed to emerge on the basis of the teachers' level of mathematics knowledge and their pedagogical beliefs and practices. In contrast, it is plausible that what less successful enactments have in common is that they limit access to learning resources for all but a small minority of students.

An important implication is that "scale up" may require more attention to supporting teachers to use the same materials in different configurations. More specifically, beyond basic recommendations that students have adequate access to technology and enough time to complete curricular tasks, it may be counter-productive to over-specify a preferred way of teaching. Instead, recommendations for particular configurations need to be sensitive to teachers' level of mathematical knowledge, their pedagogical beliefs, and their instructional practices. Measures of "implementation fidelity" need to be revised to allow for a wider range of viable variation, if not replaced by measures that acknowledge curriculum implementation as a mutually adaptive system (McLaughlin, 1976) admitting several possible configurations of learning resources.

In any case, no case-study teacher implemented the curriculum in ways that fully embodied the SimCalc developers' vision. Given the complexities of instruction and variations in teaching approaches, it would be unrealistic to expect all or even most teachers to reproduce on a large scale the "ideal" teaching documented in small, proof-of-concept studies. Scaling up instructional innovations will entail trade-offs. Recognizing the inevitability of these trade-offs can help educators support teachers to embrace and implement innovative curriculum materials and to recognize that the interface between innovations and teachers can be organized around multiple "good enough" configurations of learning resources rather than the idea that there is a one-size-fits-all best approach.

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Changing from the Inside Out: SimCalc Teacher Changes in Beliefs and Practices

John Tapper

1 Overview

This chapter examines ways teacher beliefs and practices changed as a result of using SimCalc as a replacement for traditional content in high school algebra classes. SimCalc is a part of our investigation in three ways:

1. as SimCalc MathWorlds®, software that runs on calculators and computers;
2. as the SimCalc approach, which we operationalize as the instructional practices that come from the inquiry-based curriculum and materials that accompany SimCalc MathWorlds®; and
3. as the SimCalc intervention, a specific application of software, curriculum, and approach used by the Kaput Center for Research and Innovation in STEM Education for a specific research study.

Interviews with SimCalc teachers in the IES-funded study, "Democratizing Access to Core Mathematics Across Grades 9–12[1]" (hereon called Democratizing Access study) led to unexpected insights into instructional practices—and teacher beliefs that informed those practices. We found that working with SimCalc, over time, changed some of teachers' routine instructional practices and challenged them to reconsider their views about how children learn math. These transformations were different across teachers but patterns emerged that could be linked to the amount of time teachers spent in the project, and to the instructional approaches inherent in SimCalc.

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2 SimCalc's Theory of Change

Berube et al. (2010) suggest that a curriculum's theory of change (ToC) comes from the designers' "intentions and expectations." These intentions include assumptions about how learners understand and develop mathematical concepts. Taken together, intentions, assumptions about learning, and the nature of mathematics form a theory about the way curriculum—through the lens the teachers brings to it—will influence math learning. A ToC in a curriculum puts a theory about teaching and learning into practice. Travers (1993) called the constellation of theory, intention, and curriculum, "the intended curriculum."

The SimCalc ToC is built around the affordances that come from interactions between students, mathematics, and technology. SimCalc is a dynamic visualization environment that makes use of graphing calculators and connectivity technology. It links representations of functions to each other and to simulations, allowing graphical and algebraic editing of piecewise-defined functions. In particular, the environment allows students' work to be aggregated to a teacher computer that can be projected publicly, allowing each student to be part of a larger set of varying mathematical objects.

While teachers in our study held a wide variety of views of what SimCalc was and how it should be implemented, the intervention itself (in its iteration as part of the *Democratizing Access* study) was designed with certain goals in mind. The most obvious of these (to an observer) was that SimCalc was created to leverage meaningful communication about key mathematical concepts between students, the algebra curriculum, and the technology in the program. We came to think of this communication as the *learning conversation*. For a variety of reasons, teachers using SimCalc were not always aware of the affordances in the learning conversation, but this understanding seemed to change over time.

While it can be argued that technology is simply a tool for accessing algebra content, this assertion would misrepresent the role of SimCalc in the learning process. The technology can (and should) be conceptualized in SimCalc as a participant in learning because students pose conjectures, ask questions, and test hypotheses that could only be answered in a dynamic virtual environment that can respond to these probes (Means et al., 2003; Stroup et al., 2002). Technology acted as a participant in the learning conversation when emerging student ideas were challenged by results that SimCalc delivered. This role of technology in the learning conversation is defined in the ToC. The curriculum played a role in this conversation by initiating the topics around which discourse could grow. Learning, from the perspective of SimCalc design, was an iterative learning conversation through which students developed ever more sophisticated concepts in the domain of algebra. While not every teacher was faithful to this intention, teachers reported that SimCalc moved them continuously toward a learning conversation and (often) away from previous notions of classroom discourse.

SimCalc pedagogy leveraged both constructivist (Fosnot, 1996; Steffe and Gale, 1995; von Glasersfeld, 2002) and constructionist (Harel and Papert, 1991) elements of pedagogy. The notion that students need to build their own understanding of key concepts based on their experiences with SimCalc explorations draws on a constructivist view that thinking is transformed by the process of creation. In this view learners develop an understanding of linear functions (a key concept for algebra) by creating, manipulating, and describing them. The SimCalc approach is situated in a constructivist pedagogy because the curriculum is designed to place learners into situations where they learn concepts through exploration.

SimCalc is also emphatically a constructionist pedagogy as the learning conversation, the framework that holds it together, is dependent on discussion of the artifacts of student exploration. The curriculum and technology in SimCalc are never used for the sole purpose of individual exploration but always for rich conversations about the meanings of students' constructions, both conceptual and virtual. The learning conversation cannot exist in classes without the constructions that are the work of the class using the SimCalc curriculum and approach. The use, or lack thereof, of artifacts of exploration was one of the characteristics of SimCalc teachers' early use of the curriculum. The ability of teachers to make use of mathematical artifacts to help students develop deep conceptual understanding of algebra concepts—like slope and intercept—was closely linked to prolonged use of SimCalc software and curricula.

Implementing constructionist curricula can be demanding for teachers. A curriculum that promotes student understanding through the interaction of students with meaningful mathematical artifacts demands that teachers prepare thoroughly. Teachers must also find ways to include the diverse learners in their classrooms in the essential learning conversations, and confront their own tendencies to control the flow of information (Han and Bhattacharya, 2001). SimCalc teachers struggled with these issues as they implemented the intervention. Changes in implementation of constructionist curricula were among the most important changes we observed in SimCalc teachers over time.

Chapman (2006) suggests that reflection is a key element in the constructionist learning process. Often students do not revisit artifacts (real or conceptual) to reflect deeply on learning. This lack of reflection limits the benefits that students derive from the learning conversations in their classes. In our retrospective analysis of teachers' interactions with SimCalc we have found a similar pattern: Those teachers who engaged in reflection over time (in our case two or more years) tended to have deeper insight into both the SimCalc theory of change and the mathematics upon which it was built. The focus of this chapter is the insights that teachers reveal as a result of that reflection.

3 Data Collection Methods

The SimCalc intervention for Algebra 1 and Algebra 2 was used in high-school classrooms (and a couple of middle school classrooms) in Southeastern Massachusetts, USA, over a four-year period as part of a research study that investigated changes in content knowledge and motivation. The Democratizing Access study involved the use of SimCalc curricula for Algebra 1 and Algebra 2 courses. As part of the data collection for this study, we conducted yearly interviews with teachers implementing SimCalc. Interviews were conducted at the end of each school year for the last three years of the study and last approximately 45 minutes. A total of 18 teachers were interviewed. Twelve teachers were interviewed more than once. Some of the schools and teachers only participated in the study for one or two years, while others were in the study for the whole duration of the project.

During interviews, teachers were probed about their retrospective perceptions of the impact of the SimCalc intervention on instruction, learning, communication and engagement, and the use of the technology and intervention curriculum. Interviews were semi-structured, allowing for follow up questions not included in the script. All teachers were asked a set of questions created by the research team. Sample questions included:

- "Has using SimCalc influenced the math content you use? If so, in what ways?"
- "Algebra uses important algorithms and formulas to compute solutions. The quadratic formula and the slope-intercept formula are two examples. Has SimCalc had an impact on understanding and using algebraic procedures and algorithms?"

A phenomenographic approach to analysis of these interviews was used initially to record participant perceptions and organize them into conceptual categories (Goransson et al., 1998; Marten, 1988). Analyses of phenomenographic data were summarized in yearly reports (see Tapper, 2010). We used a constant comparison method with raw data from interviews and phenomenographic analysis from the yearly reports to compare cases, identify unifying themes, and create a cohesive theory.

The initial purpose of the interviews was evaluative, that is, they were conducted as part of an external evaluation for grant review. During the interviews, teachers reflected on their instructional practices and the effects of these on students. Over time, these retrospective reflections increasingly centered on changes teachers made in their instruction as a result of the impact SimCalc had on their students.

4 Towards a Teaching Continuum: An Analytical Framework

To describe and organize the changing interaction between teachers and SimCalc, we have constructed a framework to characterize the salient features of three stages in the interaction: initial, developing, and experienced. The initial stage is characterized by the ways SimCalc interacted with teachers' default pedagogical stance. The developing stage showed marked similarities in the ways teachers responded to SimCalc. Finally, in the experienced stage, teachers described the ways that SimCalc challenged some of their beliefs and how they struggled to reconcile this new understanding with the demands of their current teaching. We present these three stages as a framework for researchers to monitor and analyze changes in perceptions of SimCalc teachers over time. We document the movement of teachers toward a greater understanding of the SimCalc ToC and pedagogy.

4.1 Initial Stage of the Teacher-SimCalc Learning Conversation

> It was very different at first... By the time I really was starting to get used to it I wish I could have started it again and I tried it from scratch.

Instructional change is never swift. Implementation with SimCalc was not an exception to this truism. While teachers who volunteered for the project were given approximately seven, 3-hour sessions of professional development on the SimCalc software and approach, my interviews with teachers demonstrated that they tended to see the algebra intervention through the lenses of their own pedagogical stances. Teachers universally reported that SimCalc was interactive, inquiry-based, and different from their regular curriculum. How they interpreted each of these assertions, seemed to depend on the beliefs they had already possessed about how mathematics is taught and learned.

One teacher, Marcia, described her initial teaching with SimCalc as, "very different" from her regular algebra instruction. Marcia's usual instruction followed a fairly typical representational pedagogy (Cobb et al., 1992) of demonstrating formulas and procedures for students to replicate and then having students practice them. When using SimCalc, she took what she believed to be the intention of the program and filtered it through her particular pedagogical preferences. The result was that lessons began with teacher presentation, contained periods when the students were given calculators, problems and told to "go for it," and ended with teacher explanation of the correct answers.

While students in Marcia's class had access to SimCalc problems and resources, the resources were predominantly used as practice problems to support the concepts Marcia wanted them to learn. There was no real learning conversation because students did not "converse" with the technology or curriculum. They did not pose conjectures that could be tested, or shared as public artifacts of learning for the purpose of conceptual development. Observations of students during periods when they were supposed to be working together to solve problems using SimCalc showed that they frequently waited until Marcia was there to engage them. Marcia's demonstrated preference was to ask convergent questions directed at the correct answer. In several video episodes, Marcia even wrote the correct answer on student papers for each student with which she was working. These actions seemed to demonstrate a pedagogical preference for executing the correct procedure to find the correct answer. While this may be a worthwhile pedagogy, it is at odds with the intention of SimCalc to engage students in a learning conversation.

In our interview, Marcia expressed concern that allowing students to explore problems without a predetermined outcome opened the possibility for students to work fruitlessly. She repeatedly worried aloud that, in using SimCalc, her students were missing out on important parts of the curriculum.

> [We spent our time] discussing as a group first what exactly is going on here and what we need to do rather than let them do full discovery because then I felt like we wouldn't accomplish enough in the time constraint that I needed to. So what does full discovery look like? He has, he has the handout, go for it read it yourself, try to understand what you are doing. My kids were not at that level to be able to handle that and I have, you know I had

> a college algebra class I think would have been a little chaotic and I only have a forty-five, we have a forty-seven minute class so we would have wasted a lot of time I thought.

While some might argue that divergence between the intended curriculum and the enacted curriculum (Schmidt et al., 1993) occurred because SimCalc was not a good fit for Marcia's instructional style, even teachers with more constructivist tendencies did not, though, tend to leverage student conversations with each other—a critical element in the SimCalc ToC. Some of the teachers who identified themselves as having constructivist pedagogy placed inquiry into a framework of teacher-posed questions and student responses. The elements of conjecture and investigation, important for the learning conversation, were not part of this view of inquiry.

Craig identified himself as a teacher who believed in the importance of students constructing their own understanding of mathematics. The emphasis on exploring concepts before teaching conventions, however, "upset the normal instructional flow in my classroom." Craig also said that, "We pose questions to them (students). We believe in inquiry. But SimCalc is just too much. On the whole I think their understanding is good. There are just too many gaps." Craig expressed his dissatisfaction with his own understanding of the way SimCalc approached algebraic learning. He also shared what would become a common theme in teacher comments, student unhappiness with a departure from a representational approach:

> I think that for myself and for probably you (gesturing to colleague) as well, had the training been over the course of a year for teachers to really understand where it was and where it was going and how we could fill in those gaps of the quadratic formula or factoring, I think the program would be much more solid. By the end of where we are now, by the end of May, I had kids saying, "Let's just go back to a lecture". Getting and not having some of that from years past, I think that's the hardest for kids to acclimatize themselves to.

Like Marcia, Craig brought the activities of SimCalc into his own preferred pedagogy and found that the fit was not good. Unlike Marcia, though, Craig continued to use SimCalc over the next two years. Over this time, Craig's understanding of the learning conversation that underlies SimCalc learning, and his experience with it, changed his beliefs about both the program and math learning.

One key feature of the SimCalc intervention was the ability for the teacher to "harvest" the representations students created on their calculators. When the group found solutions, they sent their solutions to the teacher's computer. The teacher then chose which solutions to share with the class and use for discussion. Since SimCalc is grounded in constructionism, the intention was for the products of inquiry to become artifacts for public discussion and the creation of deeper conceptual understanding. While this was the intention in the curriculum, student representations in the initial stage of SimCalc use also tended to reflect prior pedagogical preferences.

Many teachers in their first year with SimCalc reported being relatively careful with the student graphs they chose to share with the group. The majority of teachers, in their first year with SimCalc, reported, for example, that they did not use shared student work as a catalyst for discussion. A typical assertion from teachers in the initial phase was that students were not able or willing to talk about their work publically.

> I tended to be more choosy with what I put up… I mean I collect everything… you know but… there were still kids who were very, very not interested in having, saying about what they are thinking.

Some teachers said that they chose student work to share publically that would demonstrate the correct solution. Others reported consistently showing all the students' work (rather than selecting samples). Even some of these teachers reported that they continued to use the display of student thinking as a chance for students to demonstrate the correct solutions or procedure, rather than to engage students in a conversation about their thinking:

> You know it is interesting when you collect student's data. You know how you don't want it all to be correct because you want to talk about it? And I always thank the kids when they make mistakes and I make a big deal about it because I think we learn more from mistakes than if we see everything done correctly because the mistakes allow you to talk about it.

In the initial implementation of SimCalc, teachers, who demonstrated a tendency toward representational pedagogy[2] (demonstration and repeated practice), tended to use SimCalc as a resource to support the instruction of algebra concepts. These teachers used the functions produced by SimCalc experiences as demonstrations of ideas that they wanted students to learn, rather than as explorations from which students could make meaning. For teachers with a pedagogical stance that favored more student construction of understanding, SimCalc explorations were more faithfully implemented. However, the construction of deeper conceptual understanding, as a result of the learner conversation, was not always realized.

4.2 Developing Stage of the Teacher-SimCalc Learning Conversation

> I feel more comfortable teaching SimCalc the second time around. I know what to expect.

In the second year of our interviews with teachers, several features characterized the teachers' relationship with SimCalc. Implementation of SimCalc, while still delivered through individual teacher preferences, looked more similar between teachers than in Year 1. During these developing stage interviews, teachers reported that SimCalc curriculum was created "backwards." By this, teachers meant that SimCalc put exploration of math concepts and problems solving before work on discrete skills and algorithms, while textbooks typically start with these skills. Teachers had a variety of opinions on whether this was helpful or problematic. They also began, during the interviews, to point to the ways different students responded to their SimCalc experience.

[2]Teachers sometimes identified themselves as embracing constructivist pedagogy, rather than a more traditional representational view. Video samples of their teaching often did not support these claims. In all likelihood, what the teacher believed to be a constructivist stance was different from the understanding of the researcher.

4.2.1 SimCalc Implementation Looks More Similar in Developing Stage Teachers

As teachers discussed their work with SimCalc in their second year of implementation, pedagogy varied less than in the previous year. Some of this was because some of the schools and/or teachers who participated dropped out. There were a variety of reasons for the attrition of schools in the study. In one case, attrition was caused by a change in school leadership. In another, the school felt that participation in the study was not in the best interest of students and staff. Even with the loss of some schools, the potential for variety in implementation still existed among the remaining schools.

While teachers reported some differences in pedagogy between schools, interviews also revealed a number of common observations. These common observations included the necessity to offer students the opportunity to share their work publically, the importance of student comments on their work, and the realization that not all students were comfortable with the role of constructing mathematical understanding.

Many of the teachers commented on their growing realization that when students commented on their work, they gained understanding they could not get any other way. During interviews, teachers made explicit links between discussions around the public display of student work and student learning and engagement.

> Even for kids who are unsure… They did the work. They said, 'what if…?' I was able to show the student work, ask the group, pose the question and not answer it. *They* did answer it.

> I ask questions before I put the graphs up. 'What do you expect to see?' If something comes up that we didn't expect to see, we have to figure out what's going on. Sometimes the students have already made conjectures about what the class data will look like. We put everything up to see if it's correct. It's fun to figure out why, if things are off.

> They're trying to figure out how theirs is different from everyone else—what's the difference? What are the changes? They don't see only their picture, they see their picture in relation to others.

Several teachers also suggested that some of their students had difficulty becoming involved in conversations about their graphs. According to the teachers we interviewed, much of this had to do with struggling students expectations about mathematics from their previous math courses. These students had difficulty, according to several of the teachers, because their SimCalc experience was at odds with their expectations.

> These are kids who are not accustomed to any opportunity to do it on their own. These are kids who have been the traditional—I don't mean traditional is bad—but their idea of school is, see it, do it, see it do it. Pete—repeat. Drill and kill. That's all they've done. So there isn't that conceptual foundation of what are we doing. Where are we going? Why are we doing this?

4.2.2 The SimCalc Curriculum is "Backwards"

Another recurring comment from teachers in their second year of SimCalc implementation was that the curriculum was "backwards."

> In my textbook we start by introducing the concepts and then build up to working on problems. SimCalc starts with the problems and then, sometimes, moves to the skills.

> We start with a problem that the kids have to solve in the World. Usually these come at the end of a unit or on a test. I've noticed that the AP (Advanced Placement) tests ask application questions like the ones in SimCalc. The ones in the book are always practice versions of the skill we've just learned.

The recognition that SimCalc began by building key concepts through exploration was the beginning of teachers' understanding of the theory of change in the curriculum. Though not every teacher in the program embraced the idea that beginning with exploration was positive, all acknowledged that SimCalc was created with this intention in mind.

One of the interviewees went so far as to suggest that he believed the way SimCalc engaged students in mathematical thinking was superior to the model used by his textbooks. However, he felt that, although SimCalc held more promise for developing mathematical understanding, developing concepts through inquiry would not allow him to cover the content required of him by his schools and department.

4.2.3 Teachers Note That Students Sometimes Resist the SimCalc Approach

One of the more interesting reports from teachers in Year 2 was that there was some pushback from students about using SimCalc. This resistance was characterized as difficulty dealing with a different kind of mathematics class and a concern that an inquiry approach to learning might have a negative effect on grades.

One of the first comments we received about student resistance to SimCalc came when one of the interviewed teachers talked about students in her honors algebra class. She told us that, towards the end of the SimCalc intervention, students had begun to complain about their difficulties whenever they did not quickly find the answer to a problem. In some instances, according to teachers, parents had called the school with concerns that using SimCalc would disadvantage their children on state math exams. In that class, from the teacher's perspective, students were asked to understand mathematical content from more than a procedural point of view. The teacher believed that students were not used to demonstrating their depth of understanding and it was undermining their ability to count on good grades in mathematics because they could no longer simply repeat the teacher's work.

We heard from all the teachers in the second year interviews that students were sometimes frustrated because the learning conversation was so different from the way they learned math in previous classes.

> It's like they said, we know how to get an A if you teach us the way we're used to.

> They (the students) like the routine of the regular math class. They understand what's required. They're going to feel more comfortable with what they've done for 13 years.

> When we find new ways that are more engaging, it takes them out of their comfort zone.

> Because they weren't doing as many practice problems they might not have thought about it as doing math. Toward the end some of them said, 'When are we going to go back to the textbook.'

> It's like those kids we sometimes talk about. They know how to get A's and B's but they might not really understand.

Teachers in this study identified student frustrations clearly. This could be seen as evidence that they were beginning to notice the conflicts between the kind of in-depth learning conversation that SimCalc required and the type of representational instruction to which they and their students were accustomed. In discussing student conflicts with new pedagogy, teachers also considered whether the pedagogy "demanded" by SimCalc offered deeper conceptual learning than their current practices. Reflecting on student behaviors and beliefs appeared to cause teachers to reflect on their own behaviors and beliefs—something that played a key role in the final year of the program.

4.3 Experienced Stage of the Teacher-SimCalc Learning Conversation

> (My opinion about SimCalc) has definitely changed from the first year when it was, oh my god, this is tough. But that was <u>my</u> discomfort.

The final stage in the experience continuum was available to teachers who had more than two years of experience with SimCalc. All of these teachers taught at least one section of both the Algebra 1 and Algebra 2 curricula. Several in the group taught four or more SimCalc classes. Teachers who had participated in the SimCalc program for three or more years developed a better understanding of the SimCalc theory of change, communicated very similar insights into their students' lack of deep understanding of math concepts and reported that their experience with SimCalc had changed their teaching.

Experienced SimCalc teachers reported instructional practices that were more in line with the intentions of the SimCalc learning conversation. We were particularly interested when teachers mentioned pedagogy that supported student inquiry, or the discussion of public artifacts to arrive at new mathematical understanding. Both of these practices were mentioned in every teacher interview. In several cases, for example, teachers reported more attention to preparing questions to ask students, than to material they would cover.

> I've grown as a teacher. I am much more focused on asking appropriate questions than I am on providing information to students on how to solve a problem. I think that the use of the technology is as absolute... What have to use technology. I've used graphing calculators. This is a step beyond that.

> I'm kind of like a math psychologist. I can tell what they're thinking and what I should ask them.

Teachers reported concerns about how well their students were able to make use of the math they were learning. Teachers suggested that, perhaps, too much attention to memorization was detrimental to student understanding of concepts. Several teachers said that "usual" math instruction did not prepare students well for the AP exams.

> I think it (SimCalc) gives what those kids were lacking when they entered AP calc, was just an understanding of concepts beyond memorization. They'd memorized a lot of trig information but they don't understand trigonometry as an idea, as a concept. Kids don't understand quadratics when they're doing them out of the textbook. The real world is motion and acceleration like its (SimCalc) saying.

The experienced teachers reported using harvested graphs as points for conversation, rather than demonstration. For these teachers, the "up front" space where graphs were displayed had taken on a different role for instruction. Rather than being a place to review what students had done, up front displays of graphs became the place where artifacts of mathematical thinking were displayed for the purpose of generating the learning conversation. This was a marked difference in the teachers' approach to SimCalc and evidence of closer alignment to the SimCalc ToC.

> A lot of the time the class would all want to see their work. I'd put everything up and ask the group, OK which one of these makes sense to you and which ones don't.

> Sometimes I get kids who purposely put the wrong ones (graphs) up there. I ask them (the whole class) what's going on here? How could we get an answer like that?

One of the most notable changes in teachers' reflections on their teaching from the initial stage of SimCalc experience to the experienced stage was the improvement in how articulate teachers were about students' understanding (Nardi et al., 2005). Initial interviews with teachers revealed general assertions about students' math abilities. Teachers used terms, like "fast" or "slower" "higher" or "lower," to describe students' mathematical ability. In interviews in the final year, teachers tended to focus on the depth of student understanding and their ability to make use of the mathematics they learned.

> I saw the uncomfortableness with students trying to figure out, 'How am I going to do this? What do I have to do? Just TELL me. Just tell me.' 'Am I doing it right?' was often the question.

A majority of the teachers interviewed noted that students did not make meaning from mathematics. These teachers asserted that this was not an artifact associated with SimCalc, but rather something they had begun to notice in all their math classes. The experience of teaching with a curriculum that stressed the meaning of mathematical operations, combined with the self reflection that is often part of participation in a research study, seemed to have made teachers aware of the degree to which their students did not "think mathematically."

While experienced teachers suggested that SimCalc "forced" them to focus on the meaning of algebraic operations and likewise made them create communication structures that enhanced understanding, they were not ready to adopt SimCalc wholesale as a means for improving understanding of algebra. Teachers cited departmental restrictions, or concerns about how students in SimCalc would score on state tests as reasons they could only use SimCalc as a supplementary resource. All but one of the teachers also expressed a belief that "SimCalc is not for everyone." The evidence of teachers' experience with SimCalc led most of them to the conclusion that many of their students were too steeped in traditional pedagogy to benefit from an inquiry approach to math learning and/or had a disposition that would only allow them to learn mathematics if it were first demonstrated to them.

4.4 Changing Beliefs

During the *Democratizing Access* study, SimCalc curricula, software, and professional development offered math teachers the opportunity to explore new pedagogical practices and to ask themselves, "What works best?" The chance to compare new and old ways of thinking about math learning side-by-side is rare in professional development. As teachers reflected on their experience retrospectively, they gave voice to the way their knowledge, beliefs, and practices evolved. Teachers reported, though, that they would not continue to use a constructivist/constructionist approach in future classes without SimCalc software and curricula. We found that further investigation is needed regarding the tension between what experienced SimCalc teachers identified as practices leading to deep learning for their students and the agency teachers had for implementing those practices.

Teachers reported a variety of reasons for their beliefs that the constructivist/constructionist pedagogy built into the SimCalc ToC might not work in everyday instruction. In some cases teachers said that a "SimCalc approach" would be too different from the practices in other classes. These teachers felt that students in their classes would be getting a significantly different experience than in others. This was seen (at one site) as problematic. Other teachers reported concerns that parents (or students) might object to this new kind of instruction, that textbooks do not support inquiry learning, or that the math department initiatives at their school would make a SimCalc approach to algebra difficult to implement.

Rather than refute the factuality of claims that implementing a SimCalc approach to algebra in the long-term is problematic, we would rather focus on potential approaches for bridging teacher beliefs in the effectiveness of this approach with current classroom practices. Peterson (1991) reports that instructional practices are the result of complex systems of beliefs, knowledge, and personal theories. Interaction with SimCalc appears to have operated on these beliefs, knowledge, and theories. What is missing from a more complete transformational model of teacher practice is deliberate and prolonged reflection and interaction between teachers in the process of change and the support of school leaders for innovation that leads to deeper student understanding.

5 Conclusions

As we researched this intervention, we expected that student learning and motivation would change as a result of experiences with SimCalc. What was less expected was the long-term effect on teachers as a result of their participation in the project. Teacher interviews showed that professional growth can be an iterative process if teachers reflect on the nature and results of their instructional practice with an innovative resource.

In working with SimCalc over time, teachers described the ways the curriculum and technology changed their teaching: the learning conversation built into SimCalc migrated to teachers. Teachers' learning conversations involved SimCalc, their students' experiences and understanding, and teachers' own beliefs about teaching and learning mathematics. In this dynamic conversation—which seemed to need time to develop—teachers found that they were able to look at their own practices through a new lens and gain deeper insights into how students understand mathematics.

Manouchehri (1998) and Reys et al. (1998) suggest that ongoing discussion of instructional practices, and student responses, is key to changing pedagogy in the long-term. The *Democratizing Access* study was not a study of professional development, but rather of the efficacy of the SimCalc software, curriculum, and approach. One bridge to implementing the instructional practices that experienced SimCalc teachers developed might be ongoing professional development to support and enhance teachers' new insights into constructivist/constructionist practices. Hearing about, and reflecting on, the experiences of self and others, supports continuing professional growth (Reys et al., 1998). If SimCalc teachers from the *Democratizing Access* study continued to meet with the goal of refining instructional practices developed while participating in the study, they might develop greater agency for continuing to implement these practices. Confirming the usefulness of instructional techniques, and the observed growth in student understanding, can be powerful motivation for changing practice.

Support for developing pedagogy related to SimCalc use from school leaders could also have a positive effect. In their work with schools in Canada, Leithwood and Jantzi (1990) found that collaborative work between innovative teachers and school principals is beneficial to institutionalizing change. Conversations about the norms and expectations, like ones that teachers identified as challenging to new math practices, are key to supporting changes in instructional practice.

From interviews with SimCalc teachers over time, we learned that teaching the SimCalc intervention helped teachers develop new instructional strategies that they reported led to deeper student engagement and conceptual understanding of algebra. Although teachers felt a lack of agency to bring those strategies into the mainstream of instructional practice at their high schools, the change in their knowledge and beliefs have created the potential for more long-term change in practice. If provided with the opportunity for ongoing reflection with other SimCalc teachers, and with support from school leaders, what began as an experiment may end in long-term changes in teaching.

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Connection Making: Capitalizing on the Affordances of Dynamic Representations Through Mathematically Relevant Questioning

Chandra Hawley Orrill

1 Introduction

Learning is, at its core, about making connections between different ideas in such a way that underlying structures emerge (e.g., Bransford et al., 1999; diSessa, 2006; Hiebert and Carpenter, 1992). To successfully make connections between and among mathematical ideas, students need scaffolding that supports connection making (e.g., Franke et al., 2009). Questioning serves this scaffolding role in classrooms. Questioning allows the teacher to support students in seeing mathematics as a set of related ideas that can be drawn upon to address novel situations rather than only as a set of discrete operations (National Council of Teachers of Mathematics [NCTM], 2000). However, questioning strategies in the mathematics classroom are often uncomfortable for teachers who have not experienced mathematics in this way as teachers or as learners (e.g., Cohen and Ball, 1990). Because of this, even well-intentioned teachers may not ask the important questions necessary for connection making.

In this chapter, questioning is considered from the perspective of how questions in the SimCalc classroom rely on representations and the mathematical entailments of the questions. Building from existing literature (e.g., Franke et al., 2011, 2009; Kazemi and Stipek, 2001), four teachers' questioning strategies during a single Algebra 1 lesson called *Sack Race* were considered. *Sack Race*[1] is a classic SimCalc activity in which students create motion by editing graphs, then writing stories about their character's performance in a sack race. The goal of this exploratory study was to investigate interactions among the representational affordances of the technology,

[1] Download the *Sack Race* activity software/curriculum documents at: http://www.kaputcenter.umassd.edu/products/curriculum_new/algebra1/units/unit2/.

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the mathematics, and the questions the teachers asked that were the most promising for engaging students with mathematical ideas and increasing student learning. The chapter concludes with a discussion of implications of this research for teacher learning.

2 Framework

The theoretical framework for this analysis considers how mathematics and representation interact through questioning. This extends from the small, but growing body of literature suggesting that the nature of discourse in mathematics classrooms is critical to students' learning (Franke et al., 2009), specifically that the role of mathematics in questioning is critical. In the SimCalc environment, there is a unique opportunity to consider not only the role of the mathematics in the questions, but also the role of the dynamic representation, thus offering additional insights into the ways that questioning can shape students' engagement with mathematics.

In mathematics education, promising research on effective classroom discourse has focused on the unique role mathematics plays in the questions that can be asked. For example, Wood et al. (2006), found two different over-arching questioning strategies in classrooms they were observing. The first was a line of questioning that led to strategy sharing and reporting, while the second pattern pushed beyond that to include questions of *why* and *how.* Questions in this second category, termed inquiry/argument by the authors, necessitate engagement in analysis of mathematical ideas in ways that reporting results does not. In their analysis of 42 lessons across 5 classrooms of 7- to 8-year-olds, the researchers found that the complexity of ideas expressed by the students was related to the kind of questioning strategies they experienced.

Kazemi and Stipek (2001) analyzed fourth and fifth grade classrooms to determine whether they were *high* or *low press*. High press classrooms were those in which students had to provide mathematical arguments rather than descriptions, as well as consider relationships among ideas. In the high press classrooms, errors led to reconceptualizations. High press classrooms were deemed to provide students with more opportunities for conceptual learning.

Pierson (2008) also considered discourse in her study of SimCalc as it was used with seventh grade students. She found correlations between teachers' responsiveness to student comments and student achievement. She also found that classrooms with high levels of intellectual work were correlated to student achievement. Combined, these results led her to conclude, "There is tremendous and often unrealized power in the ways teachers talk with their students" (p. 125).

Given the clear importance of the idea of *press*, argumentation, and intellectual work, it stands to reason that specific aspects of discourse in the mathematics classroom matters (e.g., Franke et al., 2007). Achieving discourse that rises to these levels, however, relies on teachers' abilities and willingness to ask questions that engage students in higher-level reasoning. However, teachers continue to struggle to ask such questions, thus the need for research to identify and characterize more effective questioning strategies.

This chapter explores the kinds of questions that lead to high press (Kazemi and Stipek, 2001); inquiry/argumentation (Wood et al., 2006); or achieve high levels of intellectual work (Pierson, 2008) which, in this study, are considered *higher-order questions*. SimCalc provides an environment that is rich with opportunity for promoting higher-order thinking in the mathematics classroom. To do so requires teachers to coordinate the mathematics and the representations to promote students' connection making. However, there is clear evidence presented in the studies cited above that crafting rich participatory learning experiences can be daunting for some teachers without using innovations such as SimCalc. Thus, it is important to understand what happens to participation when the complexity and opportunities of the classroom are increased through the inclusion of interactive technology designed to support representation and communication (Hegedus and Penuel, 2008). These include questions that move students beyond fill-in-the-blank kinds of answers. Instead, students need to be asked to explain, predict, or assess the mathematics ideas with which they are engaging.

3 Methods

For this exploratory study, videotapes from four teachers' classrooms were analyzed to determine whether there were, in fact, differences in the ways that teachers scaffolded student learning through questioning when using SimCalc. The teachers were purposively selected from a random sample of teachers participating in a larger study[2] to represent a range of classroom questioning techniques. Two of the teachers, Kevin and Barb (all names are pseudonyms) had classrooms in which there appeared to be high levels of student engagement, high levels of attention to the simulation and graphs, and rich discussions. Conversely, in initial screenings, Cecilia and Deborah seemed to have classes in which students were less actively engaged in the activities and in which the teacher maintained a stronger control. In all of the data presentation tables, the teachers are ordered from the lowest to the highest levels of higher-order questioning.

One lesson, Sack Race, was the focus of analysis. This lesson occurs fairly early in the SimCalc Algebra 1 curriculum.[3] The lesson is focused on aligning a story to a graph. The students create a virtual race so that their runner, Runner B, ties with Runner A (the computer). Then, they are asked to write a story that recounts the details of the race including important mathematical information such as the

[2]The project was funded by the U.S. Department of Education, Institute of Education Sciences (IES), Grant No. R305B070430.

[3]To find out more information on the Algebra 1 curriculum and download the associated software/curriculum documents, visit: http://www.kaputcenter.umassd.edu/products/curriculum_new/algebra1/.

Table 1 Summary of questions for each teacher

	Number of questions asked	Minutes of whole class instruction	Questions per minute
Deborah	115	51	2.3
Cecilia	148	51	2.9
Barb	167	49	3.4
Kevin	160	38	4.2

distances, starting and ending positions, velocities, and times. The entire segment lasts 20 seconds and students are asked to find imaginative ways to explain their runner's paths. This lesson was selected for analysis because it is the first lesson in which the students had an opportunity to link what happens in the World (where students see their runners moving) to what happens in the Graph. Because much of the lesson is focused on analyzing students' stories, the lesson provides clear opportunities for the teachers to ask the students questions—both to understand the students' stories and to help the students make initial connections between the Graph and the World.

Every question asked during each teacher's whole-class instruction was transcribed and coded. Interactions between teachers and small groups—outside of whole-class instruction—were not considered in this study. The coding scheme was created to capture the kind of question being asked, the way in which the representation was included in that question, and the way in which mathematics was included in the question. Teacher questioning data are included in Table 1. Most questions were coded for question type, where type was an indicator of the way in which the question was posed (thus, a hybrid of form and purpose). Questions that were purely logistical (e.g., asking a the class who has not yet read their story) were only coded in the Other category and not for question type. Only those that included a role for the representation and/or mathematics were included in those categories. Each question was coded for a maximum of one question type, one use of representation, and one link to mathematics (see Table 2 for the explanation of each code). Two additional categories were noted in the analysis: questions concerned with teaching students how to use the technology (e.g., how to add a line segment) and questions concerned with classroom logistics (e.g., asking a student to read a Sack Race story to the group). Questions focused on teaching the technology were not coded for representation or for mathematics as their focus was on specific use of the technology. However, logistical questions might include representation or mathematics elements and were coded accordingly.

Assessment results were also considered to determine whether there was any potential relationship between teacher questioning and student performance. Two assessments were administered to the students during the course of the study (see Dalton et al., 2011 for a complete description of the instrument). Each assessment was given as a pretest and again as a posttest as the relevant units were taught. For the purposes of this analysis, we consider only the students' performance on Test 1, which aligns to the content and focus of the Sack Race activity. The assessment

Table 2 Codes used analysis

Question type	
Self-answered	A question the teacher answers in the process of asking without allowing the students an opportunity to answer. For example, "That means heading back where? To where we started?"
Fill-in-the-blank	A question with a single correct answer. For example, "How long will the runners run?"
Who is this	A question that asks the students to identify which characters represent which students in the World. For example, "Who is this? Who was over here?"
Follow-up	A question asked in response to a student's question or answer (other than those follow-ups that asked for justifications). For example, when a student indicates that he wants the runner to go "wicked fast," the teacher adjusts the graph and asks, "Is that wicked fast enough?"
Open-ended	A question that might have a number of appropriate answers. For example, "What might have happened to the guy when he ran backwards?"
Assess idea	A question asking students to decide whether an idea in the conversation is acceptable or appropriate. For example, "Do you agree?"
Justification/ argument	A question that requires the student to provide a rationale or argument for a point. For example, "How do you know?"
Use of representation	
Analyze something on screen	The question requires students to make sense of something they can see on the projected screen. For example, asking which runner is faster while looking at the graph.
Make a prediction	The question requires students to predict what the graph or the motion of the runners will look like once it is projected. For example, after setting up a graph, the teacher asks, "What's going to happen in the animation?"
Make a connection	The question explicitly requires students to make connections between this lesson and earlier SimCalc lessons, between the Graph and the World display, or between mathematical ideas they have been discussing and what is happening in the Graph or World. For example, asking students what they might see in the World based on what they did with their calculators.
Link to mathematics	
Recall	The question requires students to answer with vocabulary or previously-known facts. For example asking "What is that called when the line goes like this (indicating horizontal with arm)?"
Apply	The question requires students to apply a mathematical idea to the situation. For example, "Which graph shows the runner moving faster?" Requires that the students analyze the graph by applying their understanding of slope and how it relates to speed.
Analyze/evaluate	The question requires students to make sense of what is happening in the graph. For example, having to explain what is happening to the runner if the line is going down requires analysis of the graph.
Reflect	The question requires students to reflect upon the mathematics that is the focus of the lesson. For example, "Mathematically, what are you supposed to understand from this?"

Table 3 Snapshot of student performance on Test 1

	Pretest mean	Gain on posttest	SD pre	SD post
Deborah	11.24	0.38	3.129	3.263
Cecilia	18.52	1.4	2.646	2.646
Barb	15.13	0.13	4.893	3.865
Kevin	13.47	2.47	4.19	2.774

was comprised of 22 items worth a maximum of 26 points. Most of the items were multiple-choice items, but there was one short answer and one open response item. Table 3 shows a snapshot of the class-level performance on this assessment.

4 Findings

In the initial observations of the videos, some key differences in the ways the teachers used the representations and in the ways they engaged students in the scenario were evident. For example, Deborah and Cecilia never asked students to identify themselves on screen except when their actor was the only actor showing. Instead, Deborah and Cecilia's interactions with their classes were largely focused on students correctly identifying aspects of the mathematical ideas and of the assignment criteria. Conversely, Kevin and Barb readily engaged students in conversations that were simultaneously situated in the stories represented on-screen and in the mathematical idea, thus linking them together. Even before systematic analysis, it was clear that these classrooms differed in important ways in terms of the patterns of interaction. Through the analysis, some interesting patterns emerged that provided insight into ways that innovations, like SimCalc, could be used as the basis for questioning in support of student learning. In this section, relevant findings are presented in terms of the kinds of questions asked and the ways in which mathematics concepts interacted with representations in those questions.

4.1 Question Types: Prevalence

Table 4 describes the results of coding the questions asked by each teacher. Each question was coded in four independent dimensions: question type, representation, mathematics, and other. Each question is coded into a maximum of one category (e.g., fill-in-the-blank) for each dimension. The percentages shown in Table 4 refer to the frequency of a particular code within a dimension, for example, 73 % of all of the questions Deborah asked were fill-in-the-blank questions. The uncategorized row for each dimension shows the number of questions asked by the teacher that were not included in any of the categories for that dimension. For example, a logistical question such as whether all of the students had read their Sack Race story would

Table 4 Questions asked during whole class instruction by each teacher

	Deborah	Cecilia	Barb	Kevin
Question type				
Self-answered	4 (3 %)	0	4 (2 %)	1 (1 %)
Fill-in-the-blank	84 (73 %)	98 (66 %)	116 (70 %)	70 (44 %)
Who is this	0	0	14 (8 %)	5 (3 %)
Follow-up	6 (5 %)	25 (17 %)	17 (10 %)	29 (18 %)
Open-ended	9 (8 %)	16 (11 %)	12 (7 %)	28 (18 %)
Assess idea	1 (1 %)	1 (1 %)	2 (1 %)	11 (7 %)
Justification/argument	2 (2 %)	7 (5 %)	2 (1 %)	7 (4 %)
Uncategorized	9 (8 %)	1 (1 %)	0	9 (6 %)
Representation				
Analyze something on screen	59 (51 %)	33 (22 %)	76 (46 %)	58 (36 %)
Make a prediction	9 (8 %)	18 (12 %)	5 (3 %)	15 (9 %)
Make a connection	10 (9 %)	23 (16 %)	25 (15 %)	27 (17 %)
Uncategorized	38 (33 %)	74 (50 %)	61 (37 %)	60 (38 %)
Mathematics				
Recall	32 (3 %)	23 (16 %)	22 (13 %)	8 (5 %)
Apply	39 (34 %)	27 (18 %)	36 (21 %)	16 (10 %)
Analyze/evaluate	6 (5 %)	12 (8 %)	3 (2 %)	9 (6 %)
Reflect	0	0	0	1 (1 %)
Uncategorized	38 (33 %)	87 (59 %)	106 (63 %)	126 (79 %)
Other				
Teaching the technology	31 (27 %)	31 (21 %)	20 (12 %)	17 (11 %)
Logistics	39 (34 %)	53 (36 %)	51 (31 %)	43 (27 %)
Uncategorized	44 (38 %)	64 (43 %)	90 (54 %)	99 (62 %)

not be coded under question type because it is only coded under other as logistics. Note, though, that every question asked was coded for at least one dimension.

Consistent with informal observations, important differences emerged through the analysis. For example, Barb and Kevin asked noticeably more higher-order questions such as open-ended questions, questions assessing an idea, and questions requesting a justification. This suggests that their students had more opportunity to engage in the kinds of rich mathematical discussions characterized in the literature as leading to better mathematics thinking.

Barb and Kevin fostered ownership through the questions they asked. Their questions focused on connecting students to their characters in the World by asking specific questions about characters. Barb and Kevin asked questions such as, "Whose guy is this?" or "Who is that running backwards?" These questions fostered student ownership and, sometimes, opened up other conversations. For example, when Kevin noticed a runner ending in the wrong place in the World, he asked a connection-making question, "What could you have done to fix this?" In

Table 5 Relationship of representation questions to mathematics questions

	Questions about:			
	Representations	Mathematics	Representations and mathematics	Representation & recall of mathematics
Deborah	77 (67 %)	77 (67 %)	59 (51 %)	17 (15 %)
Cecilia	74 (50 %)	61 (41 %)	40 (27 %)	12 (8 %)
Barb	106 (63 %)	61 (37 %)	18 (11 %)	3 (2 %)
Kevin	100 (63 %)	34 (21 %)	23 (14 %)	3 (2 %)

contrast, Deborah and Cecilia only asked questions about what was happening on-screen with no connection to the students in the classroom. Without focusing on which actors represent particular students in the World, questions can only be asked in the abstract, which removes the benefits of having the actors on the screen.

Assessing ideas and reflecting on learning are important for promoting higher-order thinking. Kevin differentiated himself from the others in these two categories. He asked more assess ideas questions such as, "Did we accomplish what we were trying to accomplish?" which required students to determine whether they agreed that a goal had been met. Kevin was also the only teacher in this sample to ask a reflection question. His assess ideas and reflection questions held students responsible for gauging their own learning progress, thus fostering ownership. These questions also required the students to engage with other ideas in the classroom to determine whether those ideas were sound. Engagement in argumentation-rich activity such as assessing the ideas of others was all but missing in the other three classrooms.

SimCalc offers a unique environment in which questions that relate to representations can be pursued. This is because of the public nature of the representations and the ways in which SimCalc supports both traditional and nontraditional representation. For example, graphs are a traditional representation displayed at the same time as the World, which is a nontraditional representation. Asking questions that promote students' sense making about mathematics in a representation-rich environment should foster meaningful connection making. Interestingly, Deborah asked relatively more questions that linked mathematics and representations than the other teachers in this study (Table 5). This may be because of the limited number of questions she chose to ask students about their stories and the relationships of the stories to the mathematics. Upon further analysis of Deborah's questioning, it was clear that while her questions connected the mathematics to the representation, the nature of the questions tended to be recall-level. She asked relatively fewer open-ended questions or questions that engaged higher-order thinking. She also asked fewer representation questions requiring predictions or connection making. This suggests that in Deborah's classroom, the students were not engaged in applying their learning to new situations, rather they were focused on completing just the task at hand. Based on this analysis and her classroom outcomes, it seems that simply focusing on the relationship between the representation and the mathematics is not adequate. The question type also matters.

4.2 Question Types: Descriptive Analysis

To understand the question type analysis, additional descriptive analysis is necessary. Such analysis allows for the moving beyond considering question type alone to understanding the role the questioning, as a whole, played in the classroom. For example, below, short segments from Deborah and Kevin's classes are presented as examples of the difference in questions that is evident only when looking across all three categories (question type, mathematics, and representation). In both classrooms, the teachers used questioning to ensure their students understood what was being displayed and the point of the lesson. For each teacher, a one-minute period during the launch of the lesson is presented. Deborah used questioning to focus students on one aspect of Sack Race at a time with little focus on the relationship between the representations and the mathematics. She also provided students with little opportunity to elaborate on their answers. For example, in minute 13 of class, she asked 5 questions in a row that were typical of her questioning pattern:

1. "What is it called when the line goes like this [indicating horizontal]?"
2. "What's the velocity of that person when they are stopping?"
3. "What else will we see? Will the graphs have anything in common?"
4. "Are there any going backwards?"
5. "What's their velocity going to be if they're going backwards?"

This series of questions required no more than looking at the graph or recalling definitions. For example, in Question 1, she was looking for the word "horizontal" to label a segment. Then, she asked a question that required students to understand that a stopped person had a velocity of zero. The third question started out open-ended, but ended up being a simple yes/no question. Question 4 required students to look at the graph or recall the animation to tell what they see. While this does require analysis, it is not a mathematically rich question. Finally, the fifth question is like the second in that it required students to recall that negative velocities relate to backwards motion.

Kevin's questions were more varied in the thinking required and the ways they interacted with the mathematics and the representations that were needed. From minute 5:40 to 7:00, Kevin asked his students the following questions:

1. "How long does the red guy go for?"
2. "What else do you notice about the red line?"
3. "He comes back? Are you sure?"
4. "What about the red line? What is the red motion guy going to do?"
5. "What's going to happen in the animation?"
6. "What are we looking for? What are some words we've used before?"
7. "Can you tell from the graph what the speed and velocity will be?"

Only two questions, 1 and 7, were asked as fill-in-the-blank. Questions 2 and 4 required the students to look at the representation and, in their own words, report what was important. Question 3 followed up on an incorrect student response while

Question 6 explicitly engaged students in thinking about how their previous activities in this class could inform the activity. Finally, Question 7 required students to apply mathematical understanding to determine whether the representations could support them in this way. The students were required to do more than reply to recall-level questions.

In short, the analysis showed a difference in the teachers' approaches to questioning during this lesson. Notably, the number of mathematics and representation questions was relatively small (ranging from 11–51 % of the questions asked across teachers). This was likely because of the nature of this particular lesson and the coding scheme used. For example, to a large degree in Kevin and Barb's classroom and to a lesser degree in Cecilia's classroom, many questions focused on the stories the students wrote and the way those connected to the World. While there were certainly mathematical connections in these questions (e.g., "Why did your guy run backwards?" or "What's going to happen in the animation?") the questions were not explicitly mathematical and were not coded as such, though they were valuable for meeting the goals of the lesson.

4.3 Interactions Between Key Questioning Categories

Based on the qualitative analysis and initial observations of the video, it was clear that to understand patterns of questioning in these classrooms, the interactions of certain question attributes needed to be considered. For this study, the points of intersection most likely to engage students in higher-order thinking were considered (see Table 6). Specifically, all of the questions that were coded in both categories (e.g., Open Question and Use Representation to Make a Predication) were counted for each pairing. The goal was to understand how teachers used facets of the rich SimCalc environment in their questioning.

This analysis further differentiated Deborah from the others and showed that Kevin differed somewhat from Cecilia and Barb. Deborah generally did not ask students to make connections in any kind of open-ended or higher order way. Instead, her representation questions were focused on fill-in-the-blank answers and analysis of something being shown on the screen (e.g., "What direction is that according to our picture?"). In contrast, Kevin asked some questions that focused attention on the representations while also pushing students beyond fill-in-the-blank. Interestingly, this differentiation did not show up in the analysis of the mathematics questions in which all of the teachers were relatively similar. It also did not appear in the representation and mathematics analysis.

In short, the analysis that seemed to differentiate among these teachers focused on the nature of the questions about representations being asked. Of particular note, Kevin asked the most higher-order questions that dealt with technology, but asked the fewest focused on teaching the technology. This suggested that he and his students had a relatively high comfort level with the technology. Building from this, it is plausible that the teachers' comfort levels with the technology or their perceptions

Table 6 Questions asked that were coded into both named categories

	Deborah	Cecilia	Barb	Kevin
Question type × Representation				
Open × Predict	0	3 (2 %)	0	2 (1 %)
Open × Connect	3 (2 %)	4 (3 %)	3 (2 %)	10 (6 %)
Assess × Connect	0	0	0	2 (1 %)
Justify × Connect	0	2 (1 %)	1 (1 %)	2 (1 %)
Question type × Mathematics				
Open × Apply	2 (2 %)	5 (3 %)	4 (2 %)	1 (1 %)
Open × Analysis	2 (2 %)	1 (1 %)	0	0
Assess × Apply	0	0	0	1 (1 %)
Justify × Apply	1 (1 %)	2 (1 %)	1 (1 %)	3 (2 %)
Justify × Analysis	0	3 (2 %)	0	0
Representation × Mathematics				
Connect × Apply	6 (5 %)	8 (5 %)	12 (7 %)	4 (3 %)
Connect × Analysis	1 (2 %)	3 (2 %)	2 (1 %)	2 (1 %)

about students' comfort levels with technology may shape aspects of the dialogue in the classroom. The findings here suggest that further analysis of interactions among question categories might help differentiate teachers and explain differences in student outcomes.

5 Discussion

Why does questioning matter? As pointed out in the theoretical framework, students in classrooms that feature higher-order questions promoting student reasoning about mathematics support higher levels of reasoning and argument (e.g., Wood et al., 2006). Further, we know that SimCalc was built on a vision of shared experience, of engaging in a cycle of learning that includes making predictions, testing them, and reflecting, and connecting the mathematical world to the students' experiences (Roschelle et al., 2010). In this study, questioning mattered because of the role it played in allowing students' ideas to become public. Where there were relatively fewer questions asked, such as in Deborah's class, it was not possible to ask rich follow-up questions. Without this kind of discourse tool, students' ideas were not to the foreground for examination by others. In contrast, Kevin's class had more open-ended and higher-order questions as well as high levels of student idea sharing. This allowed a culture that supported high level of student-initiated interaction. In fact, there were not only many more questions and other utterances initiated by the students, but also more instances of Kevin providing guidance on how the students should interact with each other's ideas. For example, when one student told another to "shut up" during the whole class episode, Kevin immediately remarked about the inappropriateness of the situation, thus fostering an environment in which students are safe to explore ideas. In short, using the connection-making questions opened opportunities for discussion, allowed students to engage in activities of mathematical argument, and promoted reflection.

Even in the rich environment created by the introduction of SimCalc, the overall levels of higher-order questioning remained relatively low. The relative number of questions linking mathematics with the representations in ways that promoted higher-order thinking also remained low. That may be a by-product of the particular lesson because of its focus on writing and sharing stories, which were not coded as linking representations to mathematics except where questions were raised about the graphs. However, it may also be that while these teachers understood that they should ask questions, they needed additional guidance on the kinds of questions that might be most productive in their classroom. Based on this analysis, it appears that teachers were not planning for asking questions that required students to make predictions or to make connections between representations, between mathematics and the technology, or between the SimCalc activity and the mathematics they know. More explicit attention to these kinds of questions would support greater and clearer gains in student reasoning.

6 Conclusion

This chapter opened with questions about how teachers would mediate the interactions between technology and mathematics in the questions they posed in their classrooms. The analysis suggested that technology-focused, representationally-rich classrooms have many opportunities for higher-order questioning. However, the four teachers in this study varied in their use of higher-order questions. This limited students' opportunities to make connections between the representations in the SimCalc environment, between the SimCalc activities and their other mathematics learning, and between mathematics and other aspects of their lives.

The analysis for this study focused on classifying questions in terms of the kind of question, the role of representations in the question, and the connection to mathematics. The coding scheme used for this analysis provided a promising tool for better understanding questioning. Based on the analyses presented here from this exploratory study of only four teachers, the interactions between question type, role of representations, and links to mathematics seemed to matter. This is evidenced in two ways. First, the differences between Kevin's class and Deborah's class provide a compelling glimpse into the different discourse that can exist related to the questions the teacher asks. Second, while there were not clear gains on the posttest attributable to participation in higher-level thinking classrooms, there was a decrease in standard deviation from the pretest to the posttest in those classrooms. In short, it may not be that questioning alone matters or that particular questions matter. Rather, the ways in which teachers help students make connections between their experiences and the content of interest might be what matters for promoting learning (e.g., diSessa, 2006). This coding scheme uncovered categories that seem to be important in separating teachers.

Based on the findings in this exploratory study, more work looking at the interaction of these three facets may yield important results. This could be done expanding the analysis presented above to include more SimCalc lessons and/or more SimCalc teachers so that trends could emerge across class periods. Looking at the interactions may also benefit from more sophisticated approaches such as epistemic network analysis (Shaffer et al., 2009), which provides statistical methods for looking at connections between categories. Such a tool could allow more patterns to emerge.

Better understanding of the role of questioning in supporting student connection making between mathematics and representations could lead to more effective guidance for teachers. This is important because, in practice, teachers vary in their degree of adherence to intended curricula (Remillard, 2005). Finding new ways of supporting teachers in understanding how and why to promote connection making could, perhaps, increase the amount of higher-level thinking occurring in SimCalc classrooms. And, that, in turn, could lead to different kinds of learning experiences for students.

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“They Need to Be Solid in Standard Skills First”: How Standards Can Become the Upper Bound

Margaret Dickey-Kurdziolek and Deborah Tatar

1 Introduction

The National Council of Teachers of Mathematics, the Mathematical Association of America, and the National Research Council in the United States have all published reports that convey the importance of students developing deep and interconnected understandings of mathematical concepts. Furthermore, international studies of math performance show that American students are behind their international peers in math achievement beginning in middle school and are less likely to master more complex and conceptually difficult mathematics (Schmidt et al., 2001; Suter, 2002). These same studies have suggested that shallow curriculum content is one possible cause of the learning gap. Countries that demonstrate superior student mathematical learning gains have one thing in common: curriculum units that explore particular topics thoroughly and deeply (Schmidt et al., 2001; Suter, 2002). This suggests that American mathematics classes could benefit from introducing more conceptually difficult topics that encourage students to engage in deep mathematical thinking. According to Kaput (1994), classrooms must be communities in which mathematical sense-making of the kind we hope to have students develop is practiced.

To facilitate the growth of students' mathematical understanding, activities must be designed and used that expose students to meaningful tasks that are difficult yet encourage the exploration of mathematical ideas. Incorporating these activities into classroom curriculum is a challenge that can be addressed by well-developed educational software and curriculum. Such educational programs hold the promise of helping teachers scaffold complex math concepts for their students.

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SimCalc MathWorlds® (hereon called SimCalc) is an example of one such mathematics education program. SimCalc embodies an approach that emphasizes giving students access to algebraic concepts graphically, dynamically and in relationship to simulations before and along with algebraic functions, thereby allowing students to experience the mathematical constructs of algebra and calculus as dynamic, motion-based events. SimCalc has been used, with success, in a number of small-scale studies over the past 15 years, and more recently, the Scaling-Up SimCalc study has demonstrated that a wide variety of students in a wide variety of classroom contexts can benefit from the use of SimCalc.

While the history of SimCalc's success suggests that students benefit from its use, the nature and degree of its impact hinges on school and teacher cooperation and adoption. Particularly in the United States, teachers and administrators must see SimCalc resources as a means to facilitate student learning of state adopted standards. In this report, we examine teachers' perceptions regarding their use of SimCalc in relation to meeting state-adopted curriculum standards and preparing students for standardized exams. We find that while teachers largely observed that SimCalc pushed beyond the state standards goals, they did not always view this as coherent with their goal of preparing students for statewide high-stakes assessment exams.

2 Tension Between the Ideal and Reality of Mathematics Education

We can envision teachers and students as actors in classroom environments that are nested within larger communities. For example, classrooms can be seen as nested within schools, that are nested within communities, that are nested within states, and so on. Each of the encompassing environmental levels effect the "ecosystems" or environments contained within them. Furthermore, at each of these environmental levels, we find articulations of an ideal vision of mathematics education as well as an enacted reality of mathematics education. For example, at the state level, we see the ideal purpose of mathematics education articulated through the adoption of standards while the enacted reality of mathematics education includes the utilization of standardized exams with inherent limitations. At the district and school environmental level, we see the ideal articulated through the adoption of goals and standards, yet the reality includes the struggle to allocate limited resources and funding. The differences between these ideal visions and enacted realities have been the subject of learning sciences research and political debate for decades.

Teachers and students experience this tension first hand through the course of managing classroom activities and their relationships with one another. Pais (2009) described the tension between the ideal and the enacted from a teacher's perspective. In his essay, he describes the "strange things" that have to do with "the presence of thirty children with wills, fears, desires, problems, and families" (p. 53) that are not accounted for in any articulation of what mathematics education should be for yet the reality of mathematics instruction must account for. He describes how the curriculum his school adopted explicitly mentioned the "importance of working with students on topics of mathematics and society" (p. 54), but that the "always present" high-stakes tests made it impossible to stray from a very narrow and specific mathematical content. Pais describes schools as a place of conflict, where the fissure between the ideal and the reality of teacher practice resides.

In some ways, SimCalc represents an ideal model of mathematics education that has been proven to work in the reality of classroom spaces. This has been demonstrated through the Scaling-Up SimCalc study (see Roschelle and Shechtman, this volume) in which we obtained statistically significant results indicating that students in the treatment condition had higher learning gains than their peers in control condition classrooms. The Scaling-Up SimCalc results can be simply articulated as a treatment (SimCalc) having a direct effect on outcome measures (student gains). However, while true, this account is incomplete. Students' relationship to much of the material and content SimCalc provided is filtered through teachers, in the sense that teachers received the SimCalc materials and were ultimately the ones who decided how and when the students accessed those resources. Teacher practice, as Pais (2009) described, resides within a space of tension and conflict between the ideal and reality of education. Therefore, when teachers were asked to incorporate SimCalc into their teaching practice, they were ultimately asked to find a place for SimCalc within that conflict. When we consider what and how resources are incorporated into classroom environments, we see that teachers are ultimately the gatekeepers to resource adoption. In the terms of Zhao and Frank (2003), teachers are the "keystone species" in the classroom ecosystem.

Zhao and Frank (2003) presented an ecosystem model to explain factors influencing technology adoption and use in classrooms. All actors in the ecosystem interact with one another and those interactions are vital to any actor's "survival" in the environment. However, in order for students to interact with meaningful content, they rely on their teacher to make the content accessible. Also, in order for any resource (technological or otherwise) to "survive" in the classroom climate, the teacher has to recognize its value and make it available for student use. With this metaphor, the teacher is the keystone species, computer use is a "living" species, and the introduction of new resources, such as external educational innovations, can be seen as the "invasions of exotic species" (p. 811). Zhao and Frank argue that classroom ecosystems, like biological ecosystems, exist in a state of homeostasis—where the environment is in balance and each species has their role, or niche, in the hierarchy. Therefore, invading species, such as new educational technologies or educational interventions, are unlikely to survive or last unless they are compatible with the established teaching and learning environment.

In their evaluations of the ecosystem model of classrooms, Zhao and Frank (2003) found that teacher-niche in the school ecosystem, as well as their relationship to other "species" in the ecosystem influenced their use of technology. Teachers who perceived pressure from colleagues were more likely to use computers only for their own purposes and were especially resistant to using technology that would require a reconfiguration of their teaching practices. While teachers who received help from colleagues, and had opportunities to experiment with software, were more likely to use computers with their students than for their own purposes. Remarkably, the perceived relative advantage of student use of technology had no statistically significant effects on what technologies were used in the classrooms. This illustrates that teacher rationale for using technology depends most directly on their own uses and needs, supporting their classification as keystone species in the environment. Zhao and Frank (2003) conclude that innovations cannot be implemented without a regard to the internal social structures of schools, especially teacher-level factors, and expect to survive in the classroom context. An "evolutionary rather than revolutionary" (p. 833) approach to change in school computer use is needed.

The empirical results from the Scaling-Up SimCalc study, as well as the repeated success of SimCalc in over a decade of varied small-scale studies, suggest that we can expect to find significant student learning gains when teachers are given the SimCalc package (professional development, curriculum, and the SimCalc software). While some may see this point in SimCalc's history as a point of finality or conclusion, we see this as the beginning. Now that we have found a set of learning resources that can be used with success, how can we ensure that it *will* be used with success?

There are obvious questions in relation to long-term adoption and spread, such as, "will the teachers in our study continue to use SimCalc in the future?" and "how can we 'spread the word' about SimCalc?". These questions of adoption and spread highlight the importance of teacher perceptions for an innovation's success. As Eugene Judson (2006) put it, "[w]hen establishing any classroom innovation, it is the teacher who is the key determinant of implementation" (p. 583). In order to tell a story about SimCalc's long-term prospects for widespread success, we need to know how teachers define and see SimCalc in relation to their objectives in teaching mathematics. We need to understand teacher perceptions of what SimCalc is, what it is good for, and when it should be used. While increased student learning is the central goal of the SimCalc project, we must recognize that teachers act as gatekeepers to classroom resources, or are the "keystone" species in the classroom ecosystem (Zhao and Frank, 2003).

3 Research Questions: Do Teachers Need to See Students as "Solid in Their Standard Skills" First?

The SimCalc researchers who developed the replacement unit intended for the curriculum to be more advanced than the usual seventh grade curriculum yet still on par with an average seventh graders ability. Our Year 1 and Year 2 learning gain results suggest that students in treatment classrooms gained just as much (if not more) than their peers on the "simple" portion of the test, which were questions similar to those found on the yearly TAKS exam. Furthermore, students in treatment classrooms gained significantly more than their peers on the "complex" portion of the test. This suggests that students in treatment classrooms were able to learn the mathematics outlined on their state standards while simultaneously engaging with more conceptually difficult mathematics.

When we turn to the corpus of phone interviews conducted with every participating teacher, we see that most teachers discussed the SimCalc curriculum in relation to their state standards and/or standard exams (76 out of 95), and many teachers (48 out of 95) also expressed that SimCalc contained "more" than what was typically addressed in their usual curriculum or in the state standards for seventh grade. Below is an example of a teacher describing the conceptual difficulty of the SimCalc unit as compared to their usual unit on rate and proportionality:

> I saw the whole unit as being pretty pre-algebra unit, not a pre pre-algebra unit. It struck me as a regular 8th grade, advanced 7th grade type curriculum, even towards the beginning part. And, I feel pretty good that we're teaching aligned with what the Essential Knowledge and Skills [TEKS] are for the 7th grade, as far as 7th graders. But, I love the stretch. I don't want to say I don't want to do that. It's just that I might do the more required earlier because we've got to make sure their solid in all their standard skills first.
> –Year 1 Interview, Immediate Treatment Teacher

While teachers largely recognized and described the SimCalc unit as being more conceptually difficult than their usual curriculum, they did not always express confidence that this was within the realm of their students' abilities. Several, like the teacher quoted above, expressed in their interviews that students needed to be "solid in all their standard skills" before tackling the conceptually more difficult content of the SimCalc unit. Furthermore, some teachers made curriculum changes in response to what they felt was within the capability of their students and within the scope of the statewide seventh grade mathematics standards. Specifically, 45 of the 68 teachers who completed both years of the study discussed making changes to the SimCalc curriculum in the future and 38 (of the 68) discussed omitting and/or reducing sections of the SimCalc curriculum.

The teacher perceptions of the scope and aim of seventh grade mathematics, as well as how the teachers see the scope of seventh grade mathematics in relation to the SimCalc intervention, may well impact the success of the SimCalc project. If teachers do not see the SimCalc resources as being in line with their views of what seventh grade mathematics entails, then they may make decisions regarding their curriculum that either (1) does not include the use of SimCalc at all or (2) represent a "mutation" of the SimCalc curriculum that does not lead to increased student learning gains (Brown and Campione, 1996). In this chapter, we turn to the teacher interviews to answer the following two questions:

1. How did teachers in our study describe the fit of SimCalc resources and instruction in relation to their view of the state standards and standardized exams?
2. How do the teachers' perception of the scope of seventh grade mathematics impact their decisions regarding SimCalc curriculum and resource use?

While the answers to these questions may help us to understand the potential adoption of SimCalc in the future, they more broadly represent an issue to be addressed if we hope to encourage the exploration of more conceptually difficult mathematics in K-12 education.

4 Phone Interview Data and Analysis

In both years of the seventh grade study, all of the participating teachers were interviewed within ten days of their administration of the posttest to their students. We used a semi-structured interview protocol, which started with general questions about their teaching experience ("how did the rate and proportionality (SimCalc) unit go?" "what went well?" "what went poorly?"). From that point, the interviewer would lead the teacher to discuss topics in the following six categories: teaching experience, mathematics, technology, students, colleague collaboration, administrative support, and participating in research. There were also additional "wrap-up" questions at the end of the interview. An outside corporation transcribed all of the interviews.

4.1 Phone Interview Data

During the first year of the study, 95 interviews were collected: 48 interviews with delayed treatment (control) teachers and 47 interviews with treatment teachers (Table 1). The average control interview lasted 50 minutes while the average treatment interview lasted 59 minutes.

After the first year of the study, 27 of the Year 1 participants decided to discontinue their participation and did not complete the second year of the study. We gathered a total of 68 interviews with the remaining participants: 31 interviews with delayed treatment teachers and 37 interviews with immediate treatment teachers (Table 1). The average Year 2 delayed treatment interview lasted 59 minutes and the average Year 2 immediate treatment interview lasted 57 minutes.

4.2 Analysis

An important question to ask before beginning the analysis of our data is, "what exactly can the interviews tell us?" Specifically to this research, what data is and is not in teacher phone interviews? It has been recorded that a teacher's self report of

Table 1 Number of interviews collected in each condition per year

	Year 1	Year 2	Total interviews collected in each condition across years
Delayed treatment (control)	48	31	79
Immediate treatment	47	37	84
Total interviews collected in each year	95	68	Total interviews collected overall: 163

their teaching pedagogy can differ greatly from what they do in practice (Judson, 2006). Clifford Geertz (1973) described culture, and the subject of anthropological inquiry, as "stories people tell themselves about themselves" (p. 448). These phone interviews, accordingly, are the stories teachers told us about themselves. These stories tell us what happened in the classroom through the teachers' eyes, as well as the teachers' opinions on the classroom events and environment.

To collect and categorize the teachers' stories, we developed a set of categories that were used to code the interview transcripts. These categories served as a means to collect similar opinions expressed by a variety of teachers. To ensure that the categories accurately captured our data, the categorization-scheme was developed from the goals and reasoning in conducting the interviews, and from questions directly asked in phone interview protocol. Once all of the phone interview transcripts had gone through the initial broad category-coding phase, each category was then brought to light and broken down further into smaller sub-codes for in depth analysis. Instead of initially imposing important themes upon the interviews, the coding scheme evolved over time as the interview transcripts were analyzed. This process allowed for us to identify the most pertinent emerging themes.

Throughout the course of the interview transcript analysis, a total of 64 separate sub-codes—nested within 13 broader categories—were identified. The categories included discussion topics such as classroom management, teaching philosophy, school and community description, teacher classification of students, project perceptions, technological resources, administration and collegial support, mathematics, and instructional decision-making. (For a more complete description and analysis of the interview data, please refer to Kurdziolek, 2007.) In this report, we will focus specifically on teachers' discussion of the Texas curriculum standards and standardized tests as they relate to their teaching of SimCalc.

5 Findings

Throughout the course of the interviews, and in talking about the mathematical content of the SimCalc unit or their usual unit on rate and proportionality, the teachers in our study would discuss what they saw to be the scope of seventh grade mathematics. Furthermore, they would describe the scope of seventh grade mathematics as it had been communicated to them through multiple sources—such as their state-adopted standards, colleagues, and administrators. These views on what seventh grade mathematics is, or should be, varied and were not always in agreement. The teachers in our study, like most across the United States, are held accountable for teaching the state-mandated standards to their students, report to their administration on enactment of those standards, and often take cues from colleagues and administrators on how to enact standards-based instruction in their classrooms. Since in the Scaling-Up SimCalc study we were not only concerned with the immediate success of our intervention, but also in the continued use and success of the SimCalc materials, the teachers' discussion of the scope of seventh grade mathematics was important for us to record. In the following sections, we outline (1) the teachers discussion of SimCalc with regards to curricular standards, and (2) the instructional decisions made in response to SimCalc's relation to standards.

5.1 Teachers' Discussion of SimCalc in Relation to Curricular Standards

We restrict our current discussion to teachers' description of standards and standardized exams in relation to their implementation of SimCalc. Specifically, we review teacher discussion of important TEKS-related to rate and proportionality, the degree to which SimCalc covered, or did not cover, important seventh grade standards, and the teachers' concerns regarding student performance on standardized exams.

5.1.1 SimCalc: More than or "Beyond" what Is Typically Addressed in Seventh Grade Standards

A total of 59 (out of 95) teachers mentioned standards (TEKS) and/or testing (TAKS) at least once in either their Year 1 or Year 2 (or both) post-unit phone interviews. The majority of these teachers (48 out of 95) expressed that the SimCalc curriculum contained "more" than what was typically addressed in their usual curriculum for seventh grade. Below is an excerpt from a Year 1 interview conducted with an immediate treatment teacher.

> INTERVIEWER: Sometimes when teachers are teaching a new curriculum, they get into the middle and discover something is a lot more confusing than they thought initially? Did this ever happen to you?
>
> INTERVIEWEE: Because I don't teach slope, I had forgotten how to do run over rise and figure out slope so that's something I had to go to an 8th grade teacher and re-learn because I just hadn't used it in years. So, got to that part and I had to re-learn run over rise and teach that and so just some concepts that I normally don't cover, which is not required on the TEKS. So, that kind of—I kind of panicked about and re-learned that real quick and they got the concept of run over rise by going to a stairway and lifting their foot and putting it on the step and learning run over rise in that way.
>
> –Year 1 Interview, Immediate Treatment Teacher

In this quote, the teacher says that the SimCalc unit presented slope, which is typically part of eighth grade rather than seventh grade curriculum. The teacher states that she "kind of panicked" and sought out the help of a colleague so she could "re-learn" slope for the purposes of teaching that particular lesson.

While the teacher quoted above did not express an opinion as to whether the inclusion of specific mathematic topic outside the scope of seventh grade standards was favorable or unfavorable in her eyes, other teachers in our study did. In some cases, teachers discussed SimCalc's reach beyond typical topics found in the Texas seventh grade standards as favorable or positive. Below is an example quote from a teacher expressing the sentiment that SimCalc curriculum was more than, or went beyond, what they typically taught and that this was a positive aspect of utilizing the SimCalc curriculum.

INTERVIEWER: Regarding the math content, is it consistent with the directions your school has been moving towards and the things people are worried about? Like, as far as TAKS tasks and...
INTERVIEWEE: Yes.
INTERVIEWER: And what about the concept of slope? Even though, it's not normally a seventh grade concept, how is that viewed? That you guys are covering it—that you're using it with your seventh graders?
INTERVIEWEE: I think the exposure is awesome because—okay, they were presented to it in seventh grade. Eighth grade, if they're presented again, the concept is a little bit stronger. So, those kids that have algebra in eighth grade, this will be awesome. Now, those kids that have algebra in ninth grade? When they finally see it the third time, they're going to rock the boat. They are going to blow it out of this world because it's there. You know, like I said, repetition is the only way to learn.
–Year 1 Interview, Immediate Treatment Teacher

In the quote above, the teacher remarks that exposing students to conceptually difficult material not required in the seventh grade standards will ultimately help the students succeed in mathematics in later years.

In other cases, teachers described the "stretch" of the SimCalc curriculum as potentially problematic, unfavorable, unnecessary, and/or time consuming. Below is an example of a teacher discussing the tradeoffs she saw with regards to teaching more than what was required of her by the TAKS:

INTERVIEWER: So in years past you guys, you have just shown them how to set up the ratio, how to cross multiply.
INTERVIEWEE: Yeah and divide.
INTERVIEWER: And divide and not really look at the graphs or the tables and then if they can find the ratio and make their comparisons, they can see the pattern?
INTERVIEWEE: Yeah.
INTERVIEWER: So this year was very different for you.
INTERVIEWEE: Yes it was.
INTERVIEWER: And how do you feel about it? I mean just I don't know. What are your thoughts on it? Did you like it better? Do you have some issues with it?
INTERVIEWEE: I like part of it better because when the TAKS test they take in high school has a lot of graphing on it that we don't even get into until they are freshman. Because the middle school's test, like they have to know how to read on the quadrant. Like there will be point 'J' and they will say this is quadrant 3 or 2. Kind of like that. So then in the freshman year, it seems like our scores always drop really bad because simply there is all this graphing there and they want to know what the slope is and what's happening with lines. And so in that sense that they have been exposed to more of the graphing and they understand what, I think they have an idea what's going on. So to me that was a good thing. I guess the other part of me is like... when we take the test in April, there is not going to be anything on that from this unit. So then part of me is like 'is this a waste of their time for this year's test or will it actually help them?' I mean I know it definitely is better that they see something now that will make it easier on these kids in the ninth grade test. But I guess when I

look at the little tiny picture, then I am like 'was that worth the days I have been on that part of it on the whole?'

–Year 2 Interview, Delayed Treatment Teacher

In her interview, this teacher discusses the value of teaching with SimCalc as it relates to her students' performance on the yearly TAKS test. In her quote, we see that she sees SimCalc's potential to be beneficial in terms of her students long-term success, but she questions what, if any, effect SimCalc instruction will have on her students exam scores for the current year.

5.1.2 SimCalc Curriculum and Coverage of Important TEKs

In all of the post-unit phone interviews, teachers were asked to discuss specific mathematics topics that they covered with their students. Throughout the course of these discussions, 15 (out of 95) teachers discussed a specific TEK or mathematics topic within the scope of "rate and proportionality instruction" they felt was left out or missing from the SimCalc curriculum. These topics included scale drawings, similar figures, indirect measurement, and percent proportions. These are all specific topics students must demonstrate understanding of on the yearly TAKS exam. Below is an excerpt from a Year 2 interview conducted with an immediate treatment teacher in which the teacher describes topics that were not covered in the SimCalc curriculum.

INTERVIEWER: Now in terms to these nitpicky things as you call them, can you characterize some of those or do you know what kinds of things those are that SimCalc kind of leaves out or doesn't cover?

INTERVIEWEE: There wasn't a whole lot like as far as percentages, I think there would maybe one and maybe it was because we didn't get that, we didn't get through the entire book. I know there was one activity in the workbook that dealt with percentages, yeah it was "suiting up" and we didn't get that far. It was like page 55 and there is 60 pages in the book so we didn't get to look at the percentage aspect when it ties into proportion. So that's something I am going to have to come back and teach, the percent of change, percent proportions like finding 60 % of something or 30 % or 60 % of something, things like that I am going to have to come back and teach.

INTERVIEWER: Anything else off the top of your head?

INTERVIEWEE: Yeah scale factors as far as setting up a proportion and seeing scale factors. There wasn't a whole lot of the A over B equals C over D kind of thing but I do have to come back on some of that. And I think if I would have allowed more time on the unit, I could have tied more stuff into it.

INTERVIEWER: Right, right could have tied it to the SimCalc material.

INTERVIEWEE: Yeah SimCalc.

INTERVIEWER: Now just so I understand this. These things are things that are on the TAKS test that's in your regular curriculum, is that right that you didn't feel like they were covered?

INTERVIEWEE: Yes.

–Year 2 Interview, Immediate Treatment Teacher

In this teacher's view, there were a number of topics related to proportionality that SimCalc did not cover. She explicitly mentions percent proportions, scale factors, and solving "$a/b = c/d$" kind of problems. Also, this particular teacher did not teach the entire SimCalc unit with her students because otherwise she would not have had enough time to teach the rest of the TEKs she needed to cover.

Two of the immediate treatment teachers in their Year 1 interviews reported that what SimCalc covered and defined as "proportionality" was different than what they or their district defined as proportionality.

INTERVIEWEE: It was a pretty good experience. My kids didn't get as much out of it as I thought but I think a lot of it has to do with the difference between what our district considers proportionality and what was actually taught as proportionality in the unit.

INTERVIEWER: Okay. So, what would—could you give me an example of the difference?

INTERVIEWEE: For instance, the unit was more or less just being able to relate tables and graphs and finding the proportions inside a table and taking that information and then applying it to the real life. Whereas our district wants you to be able to take it even further than that and apply it to the real world and changing from metric—into the metric system and changing in a customary system and all of that and so my kids really struggled with being able to take the graph and apply it to what the District needed us to do.

–Year 1 Interview, Immediate Treatment Teacher

INTERVIEWER: Is there anything you wanted your students to learn that they didn't learn?

INTERVIEWEE: Well, we didn't actually do that much with proportions, I didn't feel like. And that is something that I tried to really get into in the seventh grade so that—because there's so many word problems that they can work using a proportion.

–Year 1 Interview, Immediate Treatment Teacher

In the first quote, the teacher conceptualizes the SimCalc curriculum as "just being able to relate tables and graphs and finding the proportions inside a table and taking that information and then applying it to the real life." This, she states, is insufficient in terms of the district standards. In the second quote the teachers says that with the SimCalc curriculum they "didn't actually do that much with proportions" and that was a central component of seventh grade mathematics. This perception is surprising since the Scaling Up SimCalc materials were explicitly designed to instruct students on the mathematics of proportional relationships in terms of rate and change. However, these quotes suggest that for these teachers, and potentially others, the presentations of proportionality in the SimCalc materials was so different from how they typically see it, and conceptualize it, that they did not perceive SimCalc as related to proportionality at all.

5.1.3 TAKS Exam Performance Concerns

As mentioned previously, a total of 59 (out of 95) teachers mentioned standards (TEKS) and/or testing (TAKS) at least once in either their Year 1 and/or Year 2 post-unit phone interviews. Of these teachers, 23 explicitly mentioned concerns that they, or their administration, had in relation to student performance on the annual benchmark and TAKS exams. These concerns included the amount of time the SimCalc unit took to teach (and therefore, the reduction of time spent on other topics), whether students would need to review or be "re-taught" proportionality before the exam, and whether the students would be able to translate what they learned in the SimCalc unit to what was on the test. The following quote represents an exemplar of typical a teacher discussion of the SimCalc curriculum in relation to their exam performance concerns.

INTERVIEWER: How about the math content that the study has? Is it consistent with the directions your school has been moving towards and things people are worried about?

INTERVIEWEE: Well, there are a few other things. In the way that it's tested, that's mainly it, I think. The material is there. I think we would have to add a few more things to it about similarities, then the proportions, the rest of the objectives—but I think it's okay. It's mainly how it would be tested because of course the tests are very, very important. So, if it were in the format, certain things in the format that the TAKS test would be in, I think that would be more not pleasing but it would be more helpful. It would be easier for the kids to make the connection of what they're learning and then show how they're going to tested because, if they've learned it this way, but it's not tested in that format, I think that's something that we need to go through and come up with to use that material but the format that we're going to be tested in and that would be combined, it would help.

–Year 1 Interview, Immediate Treatment Teacher

In the quote above, we see that not only was the teacher concerned about the content of the SimCalc curriculum, but also the format in which content was presented to the students. To this teacher, preparing students for the TAKS exam includes preparing them for the format of questions. Pais (2009) could describe this as an example of the conflict between "ideal" and "reality" of the mathematics classroom: Ideally teachers would focus on student learning of the mathematics, but the reality (at least for this teacher) includes preparing students for the specific format of a test. This is one articulation of the reality of the mathematics classroom that potentially should be accounted for in future iterations of SimCalc design and research if it stands in the path of teacher adoption.

5.2 *Instructional Decisions Based on Teacher Perception of SimCalc and its Relation to Standards and Testing*

In the previous sections, we presented examples of how teachers from the Scaling-Up SimCalc study viewed and described SimCalc curriculum in relation to state standards (TEKS) and state standardized exams (TAKS). Largely, teachers in our study saw SimCalc curriculum as "more" or beyond what they typically taught. However, some teachers expressed concern that the SimCalc curriculum did not cover important TEKS related to rate and proportionality, and some experienced pressure (potentially from their administration) to hurry through SimCalc instruction so they could move on to other TEK topics. The teachers' perceptions of SimCalc's fit within what they see, and what their administrators see, as the scope of seventh grade mathematics ultimately influenced the teachers' decisions regarding future SimCalc instruction.

As mentioned previously, 45 of the 68 teachers who completed both years of the study discussed making changes to the SimCalc curriculum in the future. These changes included adding additional worksheets or student activities to SimCalc instruction, reordering particular SimCalc activities, and changing the timing of SimCalc instruction within the school year. There were a few observations regarding the teachers' future instructional decisions that could prove problematic for the future adoption and success of SimCalc. Eight out of the 68 teachers who completed both years of the study (9 out of the total 95 in the study) discussed adding a "pre-unit" on "basic proportionality" prior to beginning SimCalc instruction. A significant number of teachers (38 of the 68/42 out of 95) discussed omitting and/or reducing sections of the SimCalc curriculum. Furthermore, 3 out of 68 (or 4 out of 95) teachers explicitly said they would only use SimCalc materials with certain groups of students, such as their "advanced" or "gifted and talented" students.

In the following sections, we will discuss teachers' reasoning for making some of these notable instructional decisions. In particular, we review teacher discussion of adding a "pre-unit" to their SimCalc instruction and skipping or omitting sections of the SimCalc curriculum. We also review one example of a teacher discussing the conditions under which she would discontinue using SimCalc altogether. These instructional decisions, and the rationale behind them, highlight the importance for teachers' perceptions on innovation adoption and use.

5.2.1 Pre-unit Instruction

As mentioned previously, 8 out of the 68 teachers who completed both years of the study (9 out of the total 95 participating teachers) discussed adding a "pre-unit" prior to beginning SimCalc instruction. The pre-units were typically described as units on "basic skills" or "traditional proportionality." The following quote is an exemplar of a teacher discussing their rationale behind teaching a pre-unit.

INTERVIEWEE: I think we found after teaching it last year that this year we needed to teach them some of the basic proportionality and just how to setup a proportion, how to work it out, how to read a word problem that we could use a proportion to solve and we didn't find that connection is closing the project. And again I think it's TAKS test related and it's our benchmark related. We knew those were the kinds of questions that these kids would be tested over the benchmarks as well as on the TAKS and so we thought like we needed to spend sometime going over some of those basic skills with them as well.

INTERVIEWER: So were there things that you introduced before the unit or before certain parts of the unit or how did you know when to integrate them based on how things went last year?

INTERVIEWEE: Last year we did it after the unit, this year we chose to do it before the unit because our benchmark exam was before the unit. So we thought like we needed to get that part of the material in before we started the project.

–Year 2 Interview, Immediate Treatment Teacher

In this quote, the teacher describes the "basic skills" that her students need for solving the kinds of questions they will see on the TAKS exam. She decided to teach a pre-unit on these basic skills before teaching with SimCalc. This particular teacher also goes on to say, later in her interview, that she did not teach the entire SimCalc unit. She stopped her SimCalc instruction after seven instructional days while the SimCalc unit itself was designed to be a 10-day unit.

5.2.2 Omitting Sections of SimCalc Curriculum

Several teachers (38 of the 68/42 out of 95) discussed omitting and/or reducing sections of the SimCalc curriculum in either their Year 1 or Year 2 interview. When teachers discussed why they chose to omit portions of the SimCalc curriculum, they most frequently sited a lack of time, the necessity of moving on to other TEKS, or the complexity of the SimCalc unit itself as their rationale. The following quote provides and example of all three of these rationale types:

INTERVIEWEE: Anyway that's what I was trying to say is that some of those later lessons included more complex things that I didn't feel like we had time to look at because I needed to focus on what they had to actually know for the rest of the school year.

INTERVIEWER: And how closely did you follow the curriculum and individual lessons?

INTERVIEWEE: I had done it really closely. I mean I really didn't skip anything until the very end.

INTERVIEWER: And that was the salary negotiations you mentioned that.

INTERVIEWEE: It was the salary negotiation part and then there was, we looked at the miles per gallon but we skipped like the thing before that... Like I didn't assign homework for them, we did all that stuff in class because there would be no way I would get those books back. And if we could just get the basic idea without diving into it more, then that's where we kind of stopped at and moved on to the next thing.

–Year 2 Interview, Immediate Treatment Teacher

In this quote, the teacher explains that she didn't skip any part of the unit "until the very end." She states that the later lessons contained more "complex things" that she felt they did not have time to spend doing since she had to prepare her students for "what they had to actually know for the rest of the school year." She goes on to say that if her students could "just get the basic idea" without diving into it further then they would stop and move on, presumably to the next lesson.

In this quote and in others, the teachers expressed a notion of what was necessary to teach, or what was within the boundaries of seventh grade mathematics instruction. This was often expressed generally as "the point that needed to be made" or "what they actually had to know" for the rest of the school year. None of the teacher quotes indicate that they found the units they skipped to be "sub par" or of no instructional value, rather their rationale centered on having limited time and specific targets to teach—targets that SimCalc went "beyond." For all of these teachers, they decided to omit or skip sections of SimCalc curriculum when those sections went beyond the perceived boundaries of seventh grade mathematics. This suggests that two environmental factors can lead to the reduction of SimCalc use: perception of time spent and time needed for the instruction of mathematics, as well as the perceived "scope" of instruction as communicated by standards.

5.2.3 Administrative Support for Teacher Participation

While the teachers themselves discussed their personal concerns over the SimCalc unit and its usefulness in preparing their students for standardized exams, the teachers also reflected on their administrator's concerns about TAKS test preparation. The following is an exemplar quote that demonstrates how a particular teacher interpreted and described their administrator's concerns.

INTERVIEWER: How about the principal?
INTERVIEWEE: Well, he really didn't have much interest either. He did sign the permission slip though. First thing he said was, 'well what you are just covering right now is probability and rate. I mean, in Texas, that's one of the TEKS out of all those things.' The TAKS is composed of more objectives than all the objectives you cover right now. So, how long is it going to take you? Pressure. Finish it. Finish it. Go onto other objectives in the TAKS. A little bit of pressure by him. Either complete it or they're going to score low. You just hope there's only one objective or two objectives when there's other four or five objectives to cover.
–Year 1 Interview, Immediate Treatment Teacher

Overall, many teachers described their administrators as supportive (or at least, apathetic or unaware) of their participation in the Scaling-Up SimCalc study. However, a few teachers described their administration as actively unsupportive of their participation. Several teachers, such as the one quoted above, discussed pressure from their administration to quickly finish their SimCalc instruction and move on to other topics. In one extreme case, a teacher reported in her first year interview that she would not continue her participation if she was at the same school or had the same administration (interview excerpt below).

INTERVIEWER: Did they [colleagues and administrator] all feel the same way or did any of them feel differently?
INTERVIEWEE: No, they were all at first really excited about it. It was just the fact that—and they lost interest because we took so long to get through it because of the fact that the breaks and—with anything with a child, there's got to be continuity and we didn't have the continuity this year. And because of that, I'm not going to teach it next year if I'm at that school because you've got to have the administration's support and I don't feel we got it. Now, if I go to another school, yes, I will let them know that I've already been trained, already been through it one year—can I go through the second year. Any principal that worth their salt would let you do it.
...(later in the same interview)....

INTERVIEWER: Were you excited about learning the SimCalc?
INTERVIEWEE: Yes, I was. I was happy that I was selected. I thought it was a great honor.
INTERVIEWER: Do you still feel the same way about it now?
INTERVIEWEE: I would teach it again in another school. I will not teach it in this same school.
–Year 1 Interview, Immediate Treatment Teacher

Unfortunately, this teacher did not continue her participation in the Scaling-Up SimCalc study past the first year.

6 Discussion and Conclusions

We can envision the classroom environment nested within the larger communities of which the students and teachers are members. For example, classrooms can be seen as nested within schools, that are nested within communities, that are nested within the state, and so on. Each of the encompassing environmental levels effect the "ecosystems" or environments contained within them. In the case of the teachers participating in the Scaling-Up SimCalc study, they discussed the purpose and scope of seventh grade mathematics with regards to the different environments of which they are members. Specifically, they discussed the purpose and scope of seventh grade mathematics with regards to what they saw in their own classrooms, what was communicated to them through the colleagues and administrators in their school, and what was presented to them in the form of state-level standards and assessments. In this report, we have focused on the teachers' discussion of the state standards and assessment as they reflected on their use of SimCalc materials and rate and proportionality instruction.

While the Scaling-Up SimCalc materials were explicitly designed to be within seventh grade mathematics students' abilities and in line with the mathematics topics identified by the Texas standards (TEKS), the teachers in our study often described SimCalc as being outside the usual bounds of seventh grade mathematics or "more" than what was typically taught in seventh grade. Teachers in our study also discussed important TEKS related to rate and proportionality that the SimCalc study did not cover. This included what was largely described as "basic proportionality" or "$a/b = c/d$" proportionality. What the teachers described as "basic proportionality" could be likened to what is typically needed to pass proportionality questions on annual TAKS exams, and what in the Scaling-Up SimCalc study was deemed the "simple" portion of the pre and posttests.

Students in classrooms utilizing SimCalc resources largely out performed their peers in the control condition on the pre and posttests of the study. When we focus our attention to student performance on the "simple" portion of the test, or the portion of the test that resembled state standardized exams, we see that SimCalc students gained just as much as (or more than) their no-SimCalc peers. While there was no detectable loss due to the SimCalc intervention, several treatment teachers still expressed their concerns regarding the scope of the SimCalc materials and their students' performance on annual TAKS exam.

Not only did teachers in our study discuss their concerns related to student performances on TAKS exams and the "stretch" of SimCalc's reach beyond the scope of the TEKS, they also made instructional decisions based on these observations. Several felt that their students needed to be "solid in basic skills" before moving to more complex mathematical ideas. This resulted in some teachers deciding to teach a "pre-unit" on basic proportionality before beginning SimCalc instruction. Additionally, several teachers discussed the possibility of using SimCalc materials only within the short span of time between TAKS exam administration and the end of the school year. Most troubling was the large number of teachers who discussed cutting SimCalc instruction short in order to review what they considered important TEKS not covered by the unit.

In recent years, the education boards in every state in the United States have adopted sets of standards in order to facilitate the selection of curriculum and content areas for major subject areas and grade levels. However, while the intended message of the state-adopted standards and assessment may be seen as inclusive and extendable at the statewide policy level, we have shown here that the message may be interpreted and enacted by teachers as boundary lines defining the scope of what is and is not part of seventh grade mathematics instruction. Zhao and Frank (2003) would describe this phenomenon in terms of the "keystone species" of the classroom (the teacher) dealing with stressful environmental factors—environmental factors that in the case of the Scaling Up SimCalc study led some teachers to make instructional decisions that ultimately restricted if not prohibited SimCalc's chances for "survival." Pais (2009) would describe this as one of the "fissures" between ideal articulations of classrooms and what the reality of classrooms must entail. In an ideal classroom, teachers would perceive both the value of SimCalc resources as well as their fit within state guidelines and standards. However, the reality of classrooms include, in some cases, teachers spending time preparing students for the specific format of their high-stakes exam as well as its content (reported in Sect. 5.1.3), dealing with uncooperative colleagues and administration (reported in Sect. 5.2.3), and presenting mathematical ideas (such as proportionality) only in terms of what is presented on the exam (reported in Sect. 5.1.2). Furthermore, when the environmental pressures prove to be overwhelming, and/or the perceived relative advantage of an innovation is low, the reality of the classroom includes teaching more of what has been described as "basic skills" in lieu innovative instruction that goes beyond (reported in Sects. 5.2.1 and 5.2.2).

While teachers and students can benefit from more and better resources to use, in some cases, the reality of the classroom space makes using such resources prohibitive. In this chapter, we see the interpretation and enactment of standards instruction as an environmental factor that could critically reduce SimCalc's potential for impact. Future iterations of design and research should attend to and account for teacher perceptions of SimCalc's "fit" within the classroom ecosystem to reduce barriers to SimCalc's adoption. Furthermore, by attending to the fissures between "ideal" and "reality" in classrooms, we can move towards more robust theories of innovation design and classroom practice.

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Part V
International Contributions

Developing and Enhancing Elementary School Students' Higher Order Mathematical Thinking with SimCalc

Demetra Pitta-Pantazi, Paraskevi Sophocleous, and Constantinos Christou

1 Introduction

Nowadays, there is a greater demand for our schools to produce graduates who are highly creative and critical, and able to demonstrate more complex thinking processes (European Commission, 2011). The mathematics curricula of a number of countries stress that students must develop the mathematical knowledge and skills necessary to further their education, careers and everyday lives, as productive and independent members of society (Ministry of Education and Culture in Cyprus, Cyprus Pedagogical Institute, and Development Programs Service, 2010; National Council of Teachers of Mathematics [NCTM], 2000). Despite this realization, the results of several studies show that students have limited abilities in problem solving, lack conceptual understanding, and have inadequate critical thinking skills (Henningsen and Stein, 1997; Hiebert and Carpenter, 1992). It can be argued that we have not yet sufficiently developed the kinds of environments that facilitate students' content knowledge, critical, creative and complex thinking.

According to a number of researchers, such environments can be developed through the use of technology (Clements et al., 2008; Jonassen, 2000; Jonassen et al., 2008; Slangen et al., 2008). Clements et al. (2008) suggest that the various environments that support learning with the use of technology can enhance students' learning in new dynamic ways. Technology, as a mindtool, might offer students the opportunity to develop higher order thinking (Heid and Blume, 2008; Pea, 1987). However, it is not yet clear which pedagogical principles are necessary to maximize

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the potential that technology has to offer to mathematical thinking and learning (Lagrange et al., 2003).

In this chapter, we describe the design of six lessons, using SimCalc MathWorlds® (hereon called SimCalc), which were aimed to enhance elementary school students' higher order mathematical thinking. We illustrate the way in which this software and the organization of the lessons supported the development of elementary school students' higher order thinking in mathematics. To this end, Sect. 2 offers an overview of the advances that the use of technology might offer, and in particular the potential SimCalc creates for higher order mathematical thinking. Then, we concentrate on the pedagogical role of technology in the teaching of mathematics. Section 3 considers the purpose of this chapter and related research questions, while Sect. 4 provides information about the design of the lessons. The outcomes of the implementation of the proposed lessons are presented in Sect. 5, and in Sect. 6 we draw some conclusions and consider implications for teaching.

2 Theoretical Background

2.1 Technology as a MindTool: Changes in Students' Thinking and Learning in Mathematics

According to Jonassen (2000), the use of new technologies supports students' learning in a way that is meaningful to them. However, technology by itself cannot bring change or positive results in the learning and understanding of mathematics (Heid and Blume, 2008). It appears that it may be beneficial for students to be engaged in active, constructive, authentic activities and cooperative learning, and to be offered opportunities to explore situations and interpret the results of their interventions. This can be promoted by the use of technology as mindtools. Mindtools are technological tools and learning environments which have been developed or adopted so as to be used as intellectual partners, requiring students to think critically or use higher order thinking (Jonassen, 2000). Clements et al. (2008) and Tall et al. (2008) suggested that the use of appropriate technological tools gives students the opportunity for reflection, and allows them to understand in greater depth mathematical ideas and procedures. It is therefore important to investigate the types of thinking that it is possible to develop through the use of cognitive technology in mathematics.

Using the Integrated Thinking Model by Iowa Department of Education (1989), Jonassen (2000) suggested that cognitive technology can promote higher order thinking. More specifically, he suggested that students might develop more complex processes of thinking in a technological environment, which includes:

> The goal-directed, multi-step, strategic processes, such as designing, decision making and problem solving. This is the essential core of higher order thinking, the point at which thinking intersects with or impinges on action (Iowa Department of Education, 1989, p. 7).

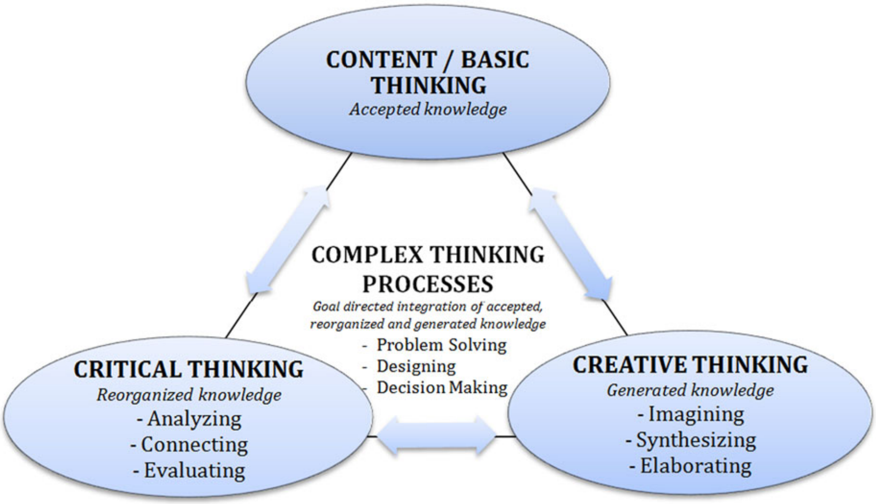


Fig. 1 Integrated Thinking Model (Iowa Department of Education, 1989)

For someone to reach higher order thinking, a combination of content/basic knowledge, critical thinking and creative thinking is necessary (see Fig. 1). These three components should be interrelated and dependent on each other. According to Jonassen (2000) and NAACE and BECTA (2001), these three components can be developed and supported by cognitive technology.

Students' content/basic knowledge is the knowledge that they can retrieve directly from what they have learned (Jonassen, 2000). This knowledge refers to both procedural and conceptual knowledge, i.e., what and why an operation is executed. Critical thinking is the ability to reorganize the knowledge using the processes of analyzing, connecting, and evaluating in accepted knowledge (Iowa Department of Education, 1989). The process of analyzing involves breaking a whole into meaningful parts, recognizing patterns and understanding interrelationships. Connecting refers to the ability to find similarities in and differences between things, and construct relationships within and between systems. The evaluation process involves making judgments based on criteria and information, and investigating the implications or results of a hypothesis in order to confirm or reject it (Iowa Department of Education, 1989). Creative thinking involves "using and going beyond the accepted and reorganized knowledge to generate new knowledge" (Iowa Department of Education, 1989, p. 7). More specifically, creative knowledge is the new knowledge brought about by imagining, synthesizing and elaborating processes (Iowa Department of Education, 1989). Creative thinking involves imagining processes, which require original ideas through intuition, visualization, prediction, and fluency. It also involves synthesizing skills, which depend on the ability to combine parts to form a new whole using analogies, to summarize key ideas succinctly, to hypothesize and to plan a process. Elaborating refers to the ability to develop an idea fully by expansion, extension and modification (Iowa Department of Education, 1989). Finally, complex thinking, as mentioned above, combines the skills and knowledge types of the other three kinds of thinking, to produce an integration of accepted, reorganized and generated knowledge. In other words, complex thinking is not a separate kind of thinking, but incorporates critical and creative thinking skills in various ways. This kind of thinking includes processes such as problem solving, designing and decision making. Problem solving is the process of using systematic methods to reach a goal. Designing refers to the invention of any type of creation to fit some goal or purpose, while decision making is defined as the ability to choose from alternatives systematically (Iowa Department of Education, 1989).

This Integrated Thinking Model has been used in different ways in the research field of educational technology. For example, Jonassen (2000) used this model to evaluate different technological tools as mindtools. Michko et al. (2003) used the dimensions of critical thinking as criteria to analyze the potential of specific mathematical software. Slangen et al. (2008) used this model to develop a checklist in order to investigate the type of thinking skills that eight grade students applied when using the microworld Techno-Logica. They found that students tended to apply the evaluating skills rather than the connecting, synthesizing and analyzing skills. However, they suggested that further investigation is needed to examine the type of thinking skills stimulated by various microworlds.

Jonassen (2000) suggests that SimCalc, as well as a number of other microworlds, allow students to develop their critical, creative, and complex thinking processes. He also suggests that compared with other technological tools, microworlds engage students in more critical and creative procedures, such as the recognition of patterns, inductive reasoning, hypothetical reasoning, and flexible manipulation of situations. In particular, it is suggested that SimCalc can develop students' critical thinking by offering them the possibility to assess available information, decide upon the criteria for selecting information, recognize patterns, identify causal relationships and use logical reasoning to support their answers inductively or deductively (Dalton and Hegedus, this volume; Jonassen, 2000; Jonassen and Carr, 2000). For example, students who work in SimCalc can assess and connect information presented to them in different representations of the microworld: the world, the table and the distance-time graph or velocity-graph (Dalton and Hegedus, this volume). Furthermore, SimCalc can develop students' creative thinking by offering an environment where students can expand their thinking to something new: it has the fluency and flexibility to provide multiple solutions to a problem. For example, students can expand the given graphs in SimCalc to fit specific instructions. Such activities were used by Dalton and Hegedus (this volume) in their implementation in high school Algebra 2 classrooms. In particular, students had to edit the velocity function symbolically and graphically to control the motion of a rocket to fit given instructions. Finally, SimCalc supports the development of complex thinking by engaging students in situations which require competence in problem solving, designing solutions, and decision making (Jonassen, 2000; Mousoulides, this volume). For example, students can design a graph in SimCalc to fit specific instructions or to find the solution to a problem. In a similar way, complex modeling activities were used by Mousoulides (this volume). In particular, he invited elementary school students to construct models in the SimCalc environment to fit specific instructions. In addition to this, Bishop (this volume) and Orrill (this volume) underlined that despite the opportunities offered by SimCalc environment to solve complex problem solving activities in a meaningful way, SimCalc offers students the opportunity to be engaged in a more productive discourse in the classroom.

2.2 Pedagogical Role of Technology in Mathematics Teaching and Learning

A technological tool is not sufficient in itself to become a mindtool. Heid and Blume (2008) argue that the effects of technology on teaching and learning are the result of a range of configurations of technologies, teachers' and students' actions, as well as the nature and organization of the curriculum and mathematical content. Based on this idea, Pierce and Stacey (2010) described the ways in which technology may be used to bring pedagogical advantages in terms of three levels—subject, classroom organization, and tasks.

Subject level refers to the opportunities that technology offers to "provoke or support new or changed goals or teaching methods for a mathematics course as a whole and to provide its users with new insights into the subject matter that they are teaching" (Pierce and Stacey, 2010, p. 10). Technology might be used to alter the balance in teaching skills, concepts and applications, and to build metacognition abilities and higher order thinking skills.

The second level of Pierce and Stacey's (2010) pedagogical map, the classroom level, focuses on the changes that occur in the interpersonal dimension of the classroom when compared with the traditional classroom. Pierce and Stacey (2010) argue, for example, that the use of technology changes classroom social dynamics, with teachers facilitating rather than dictating, and it also encourages group work, with students working collaboratively and engaging in mathematical discussion. In addition to this, technology engages students in expressive and exploratory activities (Doerr and Pratt, 2008), where students with the same technological tool can create their own construction or investigate using a prepared environment.

At the task level of their pedagogical map, Pierce and Stacey (2010) present different uses of technology, which may enhance mathematical activities in the classroom. It appears that the functionalities of some software provide an opportunity to use real world data or to explore regularity and variation. Moreover, some software offers the opportunity to elaborate simulations and/or link different representations, which promotes students' understanding.

3 Purpose and Research Questions

Taking into consideration both the possibilities that SimCalc may offer as a mindtool, as well as Pierce and Stacey' s (2010) pedagogical map, the aim of this study was to examine whether SimCalc could be used as a mindtool to enhance students' higher order thinking in mathematics. Our aim was to design a learning environment and analyze the ways in which SimCalc enhanced students' content knowledge and critical, creative, and complex thinking in mathematics. In particular, we investigated the following research questions:

- Is SimCalc MathWorlds® software a useful mindtool to facilitate elementary school students' high order thinking in mathematics?
- Can we observe active high order mathematical thinking in the SimCalc environment?

4 Designing a Learning Environment with SimCalc

Many researchers stress that it is high time to search for the ways in which computers bring added value to and can transform education, supporting the development of different types of thinking and abilities (Heid and Blume, 2008; Papert, 2006). In the following section, we analyze the learning environment that we designed with SimCalc (Kaput and Roschelle, 1998). We will present the aims of the lessons, their structure and examples of activities employed.

4.1 Aims of the Lessons with SimCalc

The aims of these lessons addressed the four components: content knowledge, critical thinking, creative thinking, and complex thinking. The underlying assumption is that content knowledge is achieved through students' involvement with the interpretation and invention of linear graphs. Critical thinking is built with the requirements for students to find similarities and differences between different graphs and link the various representations in the SimCalc environment. Creative thinking is targeted through multiple solution tasks, and finally complex thinking is required in designing solutions to fit a given purpose. In particular, the aims of the lessons were to help students to: (a) interpret distance-time graphs and velocity-time graphs and identify similarities and differences between them; (b) link verbal descriptions with distance-time and velocity-time graphs; (c) provide multiple solutions to mathematical problems in the SimCalc environment; and (d) design graphs using SimCalc to fit given instructions.

4.2 Description of the Learning Environment with SimCalc

To design the learning environment using SimCalc, we adopted Pierce and Stacey's (2010) pedagogical map and concentrated on the subject, classroom organization,

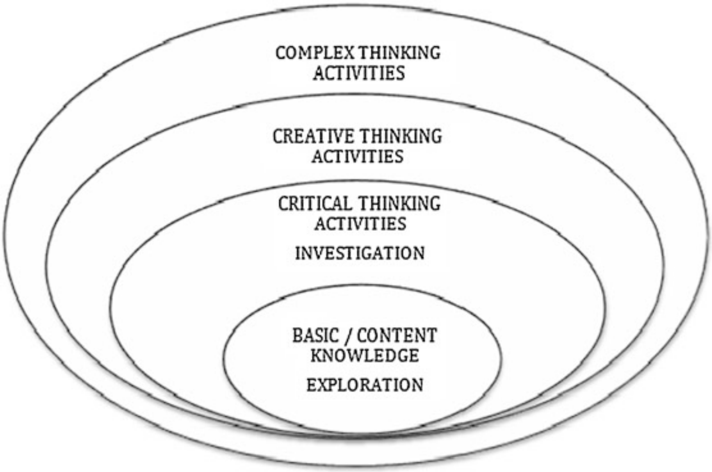


Fig. 2 Structure of the learning environment with SimCalc

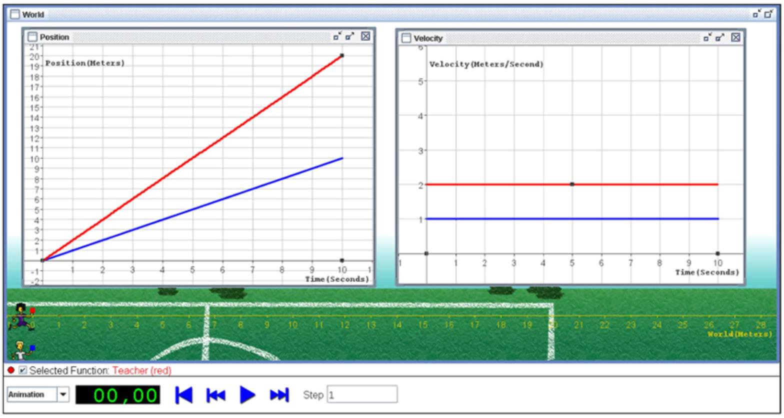
and tasks levels. By subject level, we mean the mathematics that students should learn, including mathematical thinking skills. We define classroom level as the way in which we organized the classroom setting, i.e., the role of students, teachers, and technology in the classroom. Finally, by task level, we refer to the advantages of SimCalc and the way in which these are incorporated in the activities of the learning environment.

4.2.1 Subject Level

Figure 2, offers a description of the way in which we designed our lessons in an attempt to address the subject. Our aim was that by the end of the six lessons, students would have developed their content knowledge of distance-time and velocity-time graphs, and also critical, creative, and complex thinking about these mathematical concepts. Thus, at the beginning of the lesson, students were asked to explore a real life situation. They were shown a video of a race from the 2008 Olympic Games without any commentary. Students were asked to find a way to present this race in a diagram. In the next stage, students were asked to think critically—analyzing the meaning of different representations presented in SimCalc—and investigate the similarities and differences between them (see, for example, "Distance-Time Graph Versus Velocity-Time Graph," Fig. 3). Once these activities were completed, students were asked to work with tasks that required creative thinking. In particular, students were asked to elaborate given distance-time graphs, to visualize the verbal instructions they were given, and to produce as many graphical solutions as possible (see, for example, "Time for Creation!!!," Fig. 3). Lastly, students were engaged in complex thinking activities. They were asked to sketch a graph with SimCalc in order to fit specific instructions (see, for example, "Article in the Sport Section of

DISTANCE-TIME GRAPH VERSUS VELOCITY-TIME GRAPH

Open the file «Runners4». This file contains two graphs. Describe what you see on the screen.



(a) What are the similarities and differences between these two graphs?

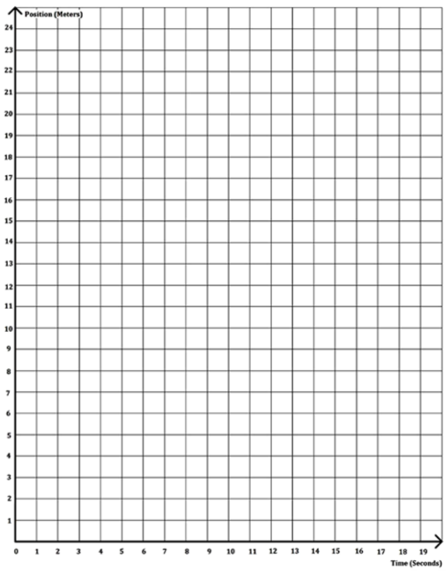
(b) Drag the two graphs and write your observations. For instance, what happens on the distance-time graph, when the velocity of a runner increases?

TIME FOR CREATION!!!

What changes can you make in the SimCalc environment so that the blue runner finishes first in the 20 m race?

Describe the way you worked.

Draw on the grid the graph that you constructed in the SimCalc environment.



Draw graphs in which the blue runner finishes the 20 m race first. Draw as many solutions as possible.

Fig. 3 Critical thinking, creative thinking and complex thinking activity in SimCalc

ARTICLE IN THE SPORT SECTION OF A NEWSPAPER

Three cars participated in a 20 km car race.

The first car finished the race in 8 minutes.

Although the second car led the race for the first 4 minutes, it eventually finished second in 10 minutes.

The third car started after 2 km, and stopped for 2 minutes due to a technical problem. It finished the race in 15 minutes.

OUR RACE

Describe your race and draw it in the SimCalc environment. Give your description to one of your fellow students to check.

Fig. 3 (Continued)

a Newspaper," Fig. 3) and to design their own race using SimCalc, and ask their partner to describe it (see, for example, "Our Race," Fig. 3).

4.2.2 Classroom Organization Level

With regard to the classroom organization level, students were asked to work in pairs. For the majority of the lesson, students were working without any guidance from their teacher. They were reading, exploring and completing the activities that they had been given in collaboration with their partner. The teacher only interfered if students needed clarification about a task. The teacher also addressed the whole class for a few minutes at the beginning and at the end of the lesson. This was done in order to give students the opportunity to present their results, make connections and comparisons between the various solutions, and exchange ideas and experiences gained while they were working with SimCalc.

4.2.3 Task Level

Finally, at the task level (Pierce and Stacey, 2010), students were given the opportunity to use real data about races—velocity, distance, and time. They had the opportunity to simulate these situations and explore the similarities and differences between simulations, graphs and tables. Students could link and make comparisons between the various representations and learn from the feedback provided by the computer. In addition, students were also asked to "teach" the computer to simulate a narrative story in a visual form.

5 Implementing the Designed Learning Environment with SimCalc in an Elementary School Classroom

5.1 Participants and Setting

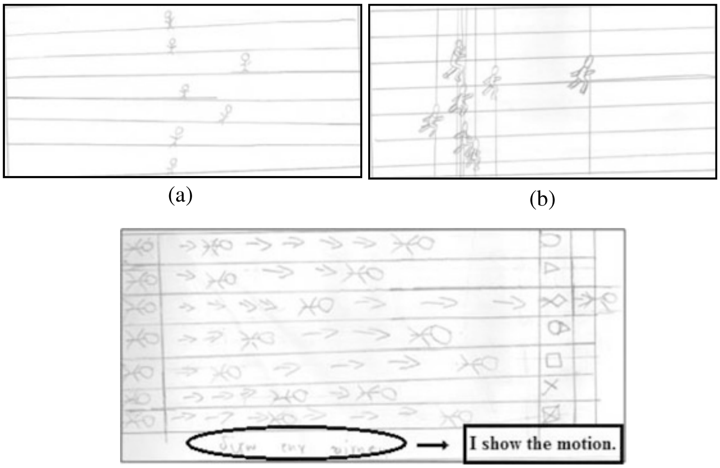
Fifteen 5th and 6th grade students (10 and 11-year-olds, 6 boys and 9 girls) of a rural elementary school in Cyprus participated in the study. The students participated in six 40-minute lessons taught by one of the authors. These lessons were conducted during a 3-week period. The participants had never used SimCalc or any similar computer program beforehand.

To answer the research questions, we collected students' written responses and the electronic SimCalc files which they produced for each activity. For the purposes of the current chapter, all responses and collected material were translated into English. In addition to this, detailed observation notes were taken of students' actions and discussions, both with their peers and their teacher. Particularly, an observation protocol was used which was based on the Integrated Thinking Model (Iowa Department of Education, 1989). We recorded students' statements that showed: (a) critical thinking processes in terms of analysis, connection and evaluation (e.g., students' statements which suggested construction of hypothesis, prediction of outcomes, comparison of similarities and differences, analysis of situations, generalization, connection between concepts and procedures, logical thinking, verification of arguments etc.); (b) creative thinking processes in terms of imagining, synthesizing and elaborating (e.g., statements that showed students synthesis of data in order to provide answers and original ideas etc.); and (c) complex thinking processes (e.g., statements that showed the way that students solved complex thinking tasks). For each of the above processes, researcher indicated the task which students were engaged with and the time that these processes occurred.

In the following section, we will present the results of the implementation of the lessons with SimCalc on students' higher order thinking in mathematics. For the data analysis, the constant comparative method was used (Maykut and Morehouse, 1994). This method combines "inductive category coding with a simultaneous comparison of all units of meaning obtained" (Glaser and Strauss, 1967, as cited as Maykut and Morehouse, 1994, p. 134). In particular, we were informed by the literature about the components of conceptual understanding of linear graphs and the various types of processes involved in critical, creative and complex thinking. Based on these, we created categories, organized the data collected from multiple sources, and presented the results of our analysis.

5.2 Students' Performance

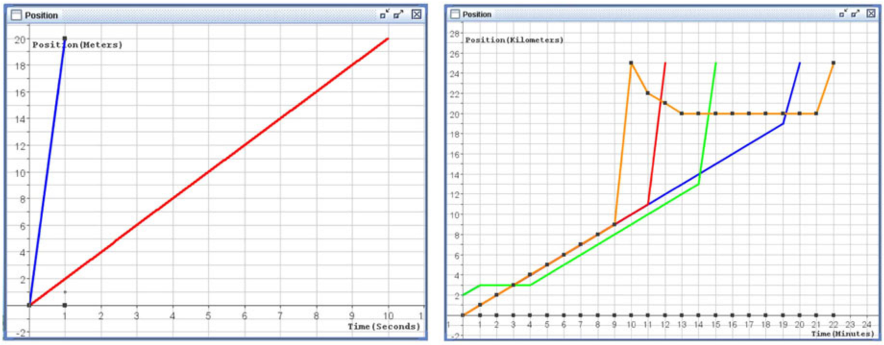
With regard to students' performance, at the end of the six lessons, we compared students' initial inventions of a distance-time graph and their final inventions, as



(a) (b)

(c)

Fig. 4 Students' diagrams of Olympic Games race prior to the use of SimCalc



(a) (b)

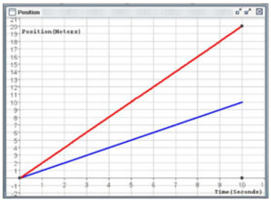
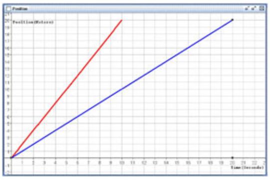
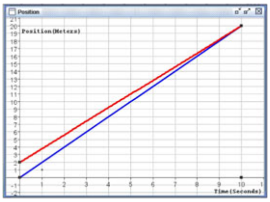
Fig. 5 One student's invention of distance-time graphs

well as students' initial interpretations of the distance-time graphs with their final interpretations. Students' responses and actions were grouped into the above categories using the constant comparative method analysis.

At the beginning of the first lesson, students were asked to represent the Olympic Games race, which they had watched on video, in a diagram. Six of the students presented the race in a static form (see Fig. 4a), five tried to show the motion of the athletes, drawing bent arms and legs (see Fig. 4b), and four tried to show the motion by presenting multiple figures for every runner (see Fig. 4c).

After students' engagement with SimCalc, all of them were able to invent distance-time graphs to represent a race of two or more athletes, both on paper and in the SimCalc environment. One student's drawings of distance-time graphs are displayed in Fig. 5. Figure 5a shows, as the student said, "the blue runner finishing

Table 1 Students' initial interpretations of distance-time graphs

 (a)		 (b)		 (c)	
Students' initial interpretation of distance-time graph (a)	Number of students	Students' initial interpretation of distance-time graph (b)	Number of students	Students' initial interpretation of distance-time graph (c)	Number of students
"I think that the graph shows the two runners' times and the number of meters they ran."	8	"I think that the graph shows that the blue runner is much further ahead than the red runner."	7	"I think that the graph shows that the red and the blue runners finished their race."	8
"I think that the graph shows the way in which the red and the blue athletes ran"	3	"I think that the graph shows the time that the two runners needed to finish the race."	3	"I think that the graph shows that the red runner started his race from the 2 meter mark."	3
"I think that the graph shows the movement of two runners."	2	"I think that the graph show the number of meters run by two runners."	3	"I think that the graph shows that the two athletes ran equally fast."	2
"I think that the graph shows that the blue line is on the 10th position, but the red line is much further away."	2	"I think that the graph shows that the red athlete ran 10 meters, while the blue athlete ran 20 meters."	2	"I think that the graph shows that the two runners started from two different points and finished at the same point."	2

a 20 meter race in 1 second, while the red runner needs 10 seconds to complete the race". Figure 5b shows another graph designed by the same student, which according to him is "a 25 kilometer race with four cars."

At the beginning of the study, students were also asked to interpret three distance-time graphs. Their initial interpretations of these graphs are presented in Table 1. It appears that, at the beginning of the lessons, students were not able to give accurate interpretations of the distance-time graphs. Most of them interpreted the graphs as if they were static pictures. For the distance-time graph (a), most students (8 out of 15) concentrated on the names of the axes, and claimed that they showed the two runners' times and the number of meters they ran. The other students claimed that this graph showed the movement of two runners. For the distance-time graph (b), seven students hypothesized that the blue runner came first, since his line graph was "bigger." The others interpreted the graph as a picture—they concentrated on its characteristics (meters, position, seconds). For the distance-time graph (c), eight students said that the two athletes finished their race. We hypothesize that they may have reached this conclusion because the two line graphs intercepted. It is a classic graph interpretation given by students. Three students concentrated only on the starting point of the red runner, and two described the two runners beginning from two different points but finishing at the same point (they probably saw the black points on the graph), while two students maintained that the two athletes ran equally fast since they finished at the same point.

After students' engagement with SimCalc, they were able to interpret distance-time graphs. This is illustrated by the responses they provided when asked to write a description of the races that these graphs demonstrated. It is noteworthy that all students carried out this task correctly after the lessons with SimCalc (see, for example, Fig. 6).

5.3 Students' Higher Order Thinking in Mathematics

The second research question was about whether we could observe active higher order mathematical thinking in the SimCalc environment. The answer to this question is positive. In the following section, we demonstrate evidence of students' critical, creative, and complex thinking using the constant comparative method analysis of data. In particular, we looked carefully at the data collected from the classroom (students' responses, actions, and discussions) and identified thinking processes that fit into the three categories of thinking: (a) critical thinking processes in terms of analysis, connection and evaluation, (b) creative thinking processes in terms of imagining, synthesizing and elaborating, and (c) complex thinking processes.

5.3.1 Critical Thinking

During the lessons with SimCalc, students employed critical processes to respond to the tasks, namely analysis, connection, and evaluation of information.

5.3.1.1 Analysis

At the beginning of the first lesson, students were asked to watch the video of the Olympic race and explain "why" the runner came first. Six students claimed that, "This athlete ran faster than the rest," five students argued that "He trained very well and he was confident," two students commented that "This runner had more support from the spectators," and two students said that "His strides were bigger than those of the other runners." From these responses we can see that some students referred to velocity, whereas others had other ideas.

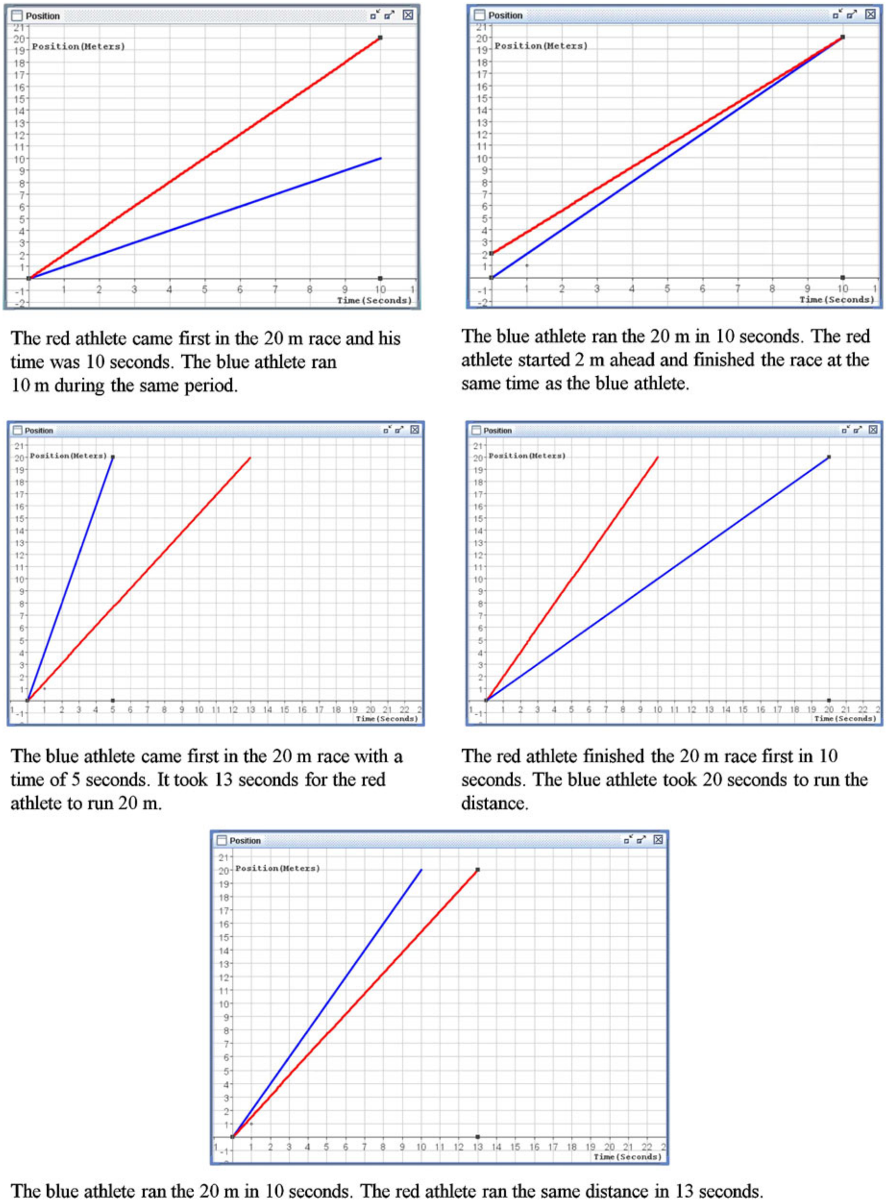


Fig. 6 A student's answer in the assessment activity

As the lesson unfolded, students' explanations became more mathematical and also more specific. For example, in one of the tasks students were asked to talk about a 20 m race with two athletes that they saw in the SimCalc simulation. All students simply said that the red runner finished the race first. Eleven claimed that, "This athlete came first, because he was running faster than the other runner," while four students added, "The red athlete came first and had a large distance difference from the blue athlete." As time elapsed students analyzed what they saw in the simulation using the various functionalities of SimCalc, and their responses became more specific. All students argued that, "The red athlete came first in the 20 m race. His time was 10 seconds. The blue athlete only managed to run 10 m during these 10 seconds." To facilitate students' understanding and highlight the difference between the two athletes, the teacher invited the students to use the *marking* option and describe what they saw in the simulation. Nine students commented that "The red athlete runs 2 m every second, while the blue runner runs 1 m every second. For example, when the red athlete has run 6 meters, the blue runner has run 3." In other words, the *marking* option gave them the opportunity to break the race into parts and see the pattern of the motion of the two runners. The other six students were not able to understand this pattern completely. They stated that: "The red athlete did long strides, while the blue athlete did short strides. This is why the red athlete covered twice the distance that the blue athlete covered."

When students were asked to discuss the numbers presented in the SimCalc table, they again started with more general comments. As time elapsed, they were able to identify patterns and understand interrelationships. All students' initial comments about the table were that "the table shows time and meters." Once the teacher suggested that they should pay more attention to the information provided on the computer screen, students started making connections between the data in the table and the simulation. All students were able to identify some patterns. Five students stated that: "The two athletes run at different speeds. The red athlete covers twice the distance that the blue athlete covers in the same time." Four argued that, "The time is the same for the two athletes. When the blue athlete has run 4 meters, the red athlete has run 8 meters," and six students commented that, "We noticed that the red athlete did strides of two meters, while the blue athlete did strides of one meter." In other words, all the students attended to the relationships between the simulation and the information in the table, but only five of them reached a conclusion about the different rates of the two athletes.

5.3.1.2 Connection

In another task, students were asked to identify the similarities and differences between three different distance-time graphs (see figures (a), (b), (c) in Table 1). All students claimed that the three distance-time graphs showed the distances that two athletes covered in a specific period of time, and all of them noticed that in figure (c) "the two athletes finished at the same time, while in the other two graphs the red athlete finished first."

When students were asked to compare these distance-time graphs with their respective velocity-time graphs (Fig. 3), they were able to say that the distance-time and the velocity-time graphs showed the same race but provided different information. They claimed that one of the graphs showed the distance that the athletes covered, whereas the other showed the athletes' velocity. Three pairs of students pointed out that "In the velocity-time graph the *lines* are horizontal and straight. This does not appear in the distance-time graph." They also noticed that "In the distance-time graph the athletes' *lines* started from 0, while in the velocity-time graph the two runners' *lines* started from 2 and 1." It appeared that these students' comments were more descriptive than showing any evidence of conceptual understanding. For this reason, the teacher suggested that the students use the *marking* option and look for connections between the marks and the velocity shown on the graph. Six students commented that "The red athlete had velocity 2, since he covered 2 meters every second, while the blue athlete had velocity equal to 1, since he covered 1 meter every second." Moreover, two pairs of students noticed that the athletes' velocity was constant. They argued that "the distance that the runners covered each second did not change."

All students were able to understand the relationship between the two graphs. This was achieved by dragging the line of one of the graphs and observing the changes appearing on the other graph. Indicative of this was the following dialogue between two boys:

GS: Move further up the red straight line in the velocity-time graph. Move it to number eight. What happens to the simulation?
AP: I think that the red athlete will run faster now. I will press "play" to see it.
GS: You are right. When we drag the line upwords in the velocity-time graph the athlete runs faster than before... See, the red line in the distance-time graph also moves (He pointed with his hand).
AP: Yes. But the red athlete continues and finishes the race at 20 m.
GS: I think that we can conclude that every action on one graph influences the appearance of the other graph.

5.3.1.3 Evaluation

During these lessons all students frequently checked their hypotheses, explanations, and connections by "exploring the microworld." One student said that he was "watching very carefully what was happening on the screen to check my answers." In another incident, a student asked his partner: "How many seconds does the red athlete need to run 20 m in this graph (pointing to the distance-time graph) and how many seconds does he need in this graph (pointing to the velocity-time graph)?" "What are the distances that every athlete runs in this graph and in this graph? (He pointed to both graphs)." His partner responded that both graphs referred to the same distance. We could argue that discussions like these suggest that all students became more engaged in the activities, and examined the information presented to them critically.

5.3.2 Creative Thinking

During the lessons with SimCalc, students employed creative processes to respond to the tasks, namely: imagining, synthesizing, and elaborating.

According to Leikin and Lev (2007), multiple solution tasks are a useful instrument to develop and also measure students' creative thinking. This is the reason we asked students to provide multiple solutions in two of the tasks presented to them. In one of these tasks, students were asked to draw distance-time graphs showing that the blue runner finishes the 20 m race first and the red runner second. The second creativity task required the students to add an actor (a car) in a given 2-car race (see Figs. 8a, 8b, 8c, 8d), and have this new actor overtake the previous two actors in the race. The two creativity tasks were presented to the students in two 40-minute lessons, which took place one week apart.

5.3.2.1 Imagining

From the students' behavior in the classroom, it appears that they all tried to imagine what they needed to do before designing the graphs. As the teacher moved around the students and asked them to describe what they had to do, all of them stated that they had to draw lines in such a way as to show that the blue runner would finish the 20 m race first. Students were moving their hands along the screen to indicate the directions of the two lines, frequently maintaining that there were many different solutions. Indicative of this is the following comment by one of the students: "I am sure that there are more than 20 graphs that I could sketch to show that the blue runner finished the race first." In the next section, we analyze the procedure students used to provide more than one solution in the two creativity tasks.

5.3.2.2 Synthesizing

With regard to students' creative processes in the first creativity task, we identified two strategies used. The first is illustrated in Fig. 7a. In particular, we observed that eight students initially moved the *graph* of the blue runner slightly away from the horizontal axis but did not change the *graph* of the red runner. Then, they dragged the *graph* of the blue runner up a little towards the y-axis and then even closer to it. When the teacher asked a pair of students what they observed, they claimed that, "When the blue runner runs very fast, the angle created by his "line" graph and the horizontal axis becomes bigger."

The second strategy is illustrated in Fig. 7b. In particular we observed that the rest of the students (seven) initially moved the blue line and covered the red line and then they dragged the red line "slightly below the *graph* of the blue runner." As one of the students said: "First, we set the blue runner to cover 20 m in 10 seconds. Then we moved the red runner slightly below the blue runner's line."

All students drew at least four different distance-time graphs to show the 20 m race with the blue runner winning and the red runner coming second, using one of the two strategies described above.

5.3.2.3 Elaborating

Elaboration processes were obvious in students' solution processes in the second creativity task. In this task, all students originally created an actor similar to those

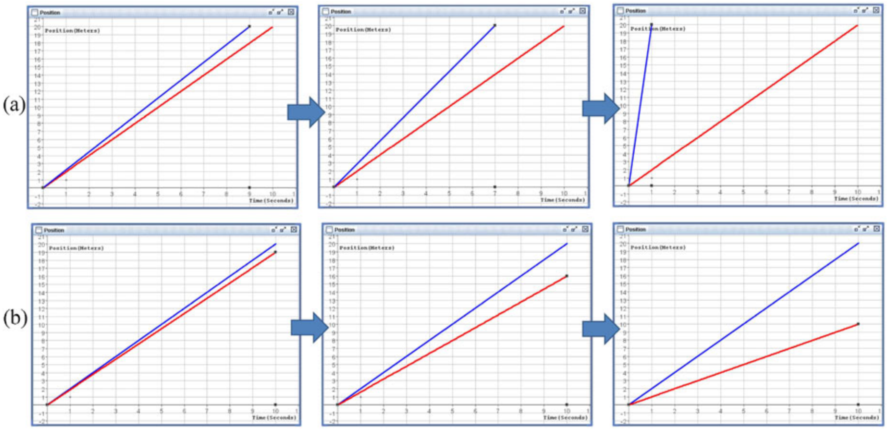


Fig. 7 The two strategies students used to draw distance-time graphs where the blue runner finishes the 20 m race before the red runner

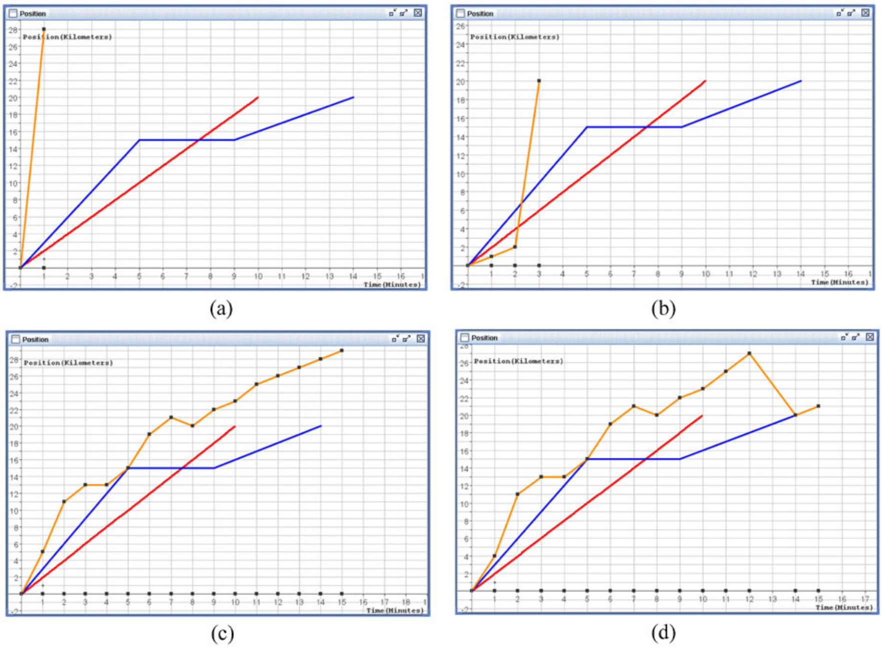


Fig. 8 Creations of a new actor who finishes the race first

that they had already observed. Indicative of this was one student's comment to her partner: "To finish first, the graph needs to be "steeper," nearer to the vertical axis. We need to do it in the same way as we did before" (in the first creativity task, see Fig. 8a). As students became more familiar with SimCalc and appeared to gain a better understanding of the graphs, they started presenting more elaborated solutions. For example, three pairs of students set their car to initially move with small velocity and then increased its velocity and overtake the other two cars (see Fig. 8b). Two other pairs of students decided to set their cars to move forward for a few seconds, stop for a while, then move forward again and then backwards for some seconds (see Figs. 8c and 8d).

5.3.3 Complex Thinking Processes

In the final two activities, students were asked (a) to design a distance-time graph to fit given data, and (b) to design their own race in SimCalc. These activities require complex thinking processes, since students need to apply a combination of critical and creative approaches to provide solutions for them.

In the first design task, students identified the goal of the task and then analyzed the data that they had been given. Then one student in each pair drew a graph on the empty grid of SimCalc, which their partner then checked. The following dialogue between two students exemplifies the thinking involved:

MC: It is easier to create a car that travels 20 km in 8 minutes.
KX: OK. I will do it.... What does it mean when it says that the second car "in the first 4 minutes"? Do we draw it here? (He points at point (0,4) on the distance axes)
MC: No! According to the instructions the second car is leading the race for the first 4 minutes. It is not written that the car starts from the 4th km. Then, this car moves for 10 minutes.
KX: OK, like this (he draws the graph)?
MC: Yes. I will press play, to check if we are correct.
KX: It is cool!

These students were able to analyze the instructions step by step, design a graph to fit the given data, and check their solution by activating the simulation. In other words, they applied complex thinking processes to provide their solutions, since they used critical thinking processes (analyzing, evaluating) and creative thinking processes (synthesizing).

In the activity where students were asked to create their own race and describe it, four of them created three races, four more created two races and six created one race. Students' designs were quite elaborate and they used existing SimCalc Worlds, such as Cars, Fishy and Rockets. Their objects did not simply move forward; They also moved backwards or stopped for some time. During this task students produced races with many "changes." A pair of students put it nicely: "we designed a "crazy" race."

The processes that students followed to design their own races were common, but their products were very different. First of all, they set their goal, then they planned their design and simultaneously sketched it on the SimCalc grid. Once having finished, they evaluated their design by activating the simulation and checking whether it was presenting the story that they had in mind. This SimCalc functionality allowed students to co-act and verify their responses. Below is a dialogue between two boys:

VA: I want to design a car race.
RO: Me too. I want my car to stop for a while.
VA: Yes!!! We should sketch it in a similar way to the one we saw before.
RO: The car which will finish the race first we have to draw it near this axis (The boy points on the y-axis on the screen).

For every race that students designed a graph, they also had to write a story. All students were able to connect the visual forms (graphs and simulated motions) with narrative stories. One such story, written by a pair of students, is presented below.

> The green fish starts at 6 m, then goes for 2.5 m and continues for 15 m. Then he stops for 1 second. Then he continues to 19 m and in 9 seconds he reaches his destination which is at 24 m. The blue fish goes straight to 24 m, then he returns to 15 m. He stops for 2 seconds and then continues for 3 m. He stops for 1 second. Then he goes to 24 m in 7 seconds. The red fish starts at 10 m. Then he goes for 5 m, stops for one second and then goes to 14 m. Then he goes for 3 m and stops for 1 second. Then he starts again to reach his destination at 24 m. He returns to 20 m, he makes a stop for one second and then returns to 24 m. This is the crazy ride we thought of!

6 Conclusions

The purpose of this chapter was to examine whether SimCalc could be used as a mindtool, and to describe and analyze the ways in which it might develop and enhance elementary school students' higher order thinking in mathematics. The results of the study showed that SimCalc can be used as a mindtool in elementary schools. SimCalc provides students with the opportunity to be engaged in activities, which may facilitate not only their content knowledge, but also their higher order thinking.

After the implementation of lessons with SimCalc, students showed evidence of development in their content knowledge, critical, creative, and complex thinking. With regard to content knowledge, students were able to interpret distance-time and velocity-time graphs flexibly and discuss their similarities and differences. The SimCalc environment also gave students the opportunity to develop their critical abilities. These seem to have been developed though the possibilities that SimCalc offered them to analyze visual (graphs and simulated motion) and linguistic (numbers and algebraic symbols) forms, to connect various types of representations, and to evaluate their responses by the feedback provided. Furthermore, SimCalc appears to enhance students' creative abilities, allowing them to expand their thinking to new answers, and offering them the fluency and flexibility to provide more than one solution. Finally, it developed students' complex thinking by allowing them to design solutions.

The findings from our study provide evidence that SimCalc offers students the potential to be actively engaged in authentic activities, and to construct knowledge in a meaningful way. Until now, these types of activities were absent from the Cypriot mathematics classrooms. More intensive implementation of SimCalc is needed in the Cypriot mathematics classrooms. It will be interesting to investigate if the positive effects of SimCalc appear both in quantitative and qualitative form. In particular, our future work involves the investigation of the effects of SimCalc on elementary school students' thinking compared to students' thinking in regular classrooms.

As illustrated in this chapter, in order to achieve higher order thinking, it appears to be necessary to consider not only the subject to be taught, but also classroom organization and tasks. For this, it will be interesting if future research concentrates in more detail on the way that the mathematical subject is taught, classroom organization, and the impact of the various types of tasks. In particular, do different types of learning environments (subject, class organization and tasks) with SimCalc activate different aspects of thinking? It will be necessary to carry out further research on SimCalc to discover the specific effects of these learning environments. For example, do learning environments with SimCalc, which address the topic of proportional reasoning activate higher order thinking skills? In addition to this, it will be interesting to investigate the degree of impact on different student variables (e.g., students' efficacy, students' argumentation).

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Adapting SimCalc to Different School Mathematics Cultures: A Case Study from Brazil

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1 Introduction

When SimCalc is initialised, four windows appear: a position-time graph, a table of values, an algebraic expression of function, and the "World" (see Fig. 1). The first three of these representations are very familiar to mathematics teachers in Brazil. They are found in Brazilian Mathematics textbooks, as well as in the software graphing packages that are available for use in schools. This is not the case, however, for the World. The central character, or characters, in the World are known as actors—they can be fish (two fish are the actors shown in Fig. 1), clowns, cars or some other such creatures—and they move in time, according to the function represented. In this way, the actors bring motion to the concept of function, a representation that is likely to be much less familiar, if familiar at all, to teachers in Brazil—and completely new to their students.

If teachers are to make use of the SimCalc software in their mathematics classes, they first need to make their own sense of the behaviour of the actors in the World. But what might this sense-making process involve and what kinds of interpretations might the teachers develop for these dynamic representations? Perhaps an important step is a matter of integrating all four of the representations, analysing what they have in common and how they can be connected. SimCalc integrates them on a single screen, enabling the user to see the simultaneous changes that occur when the data displayed in one of the windows is modified. Such integration should be part of teachers' work in classroom when they are dealing with functions.

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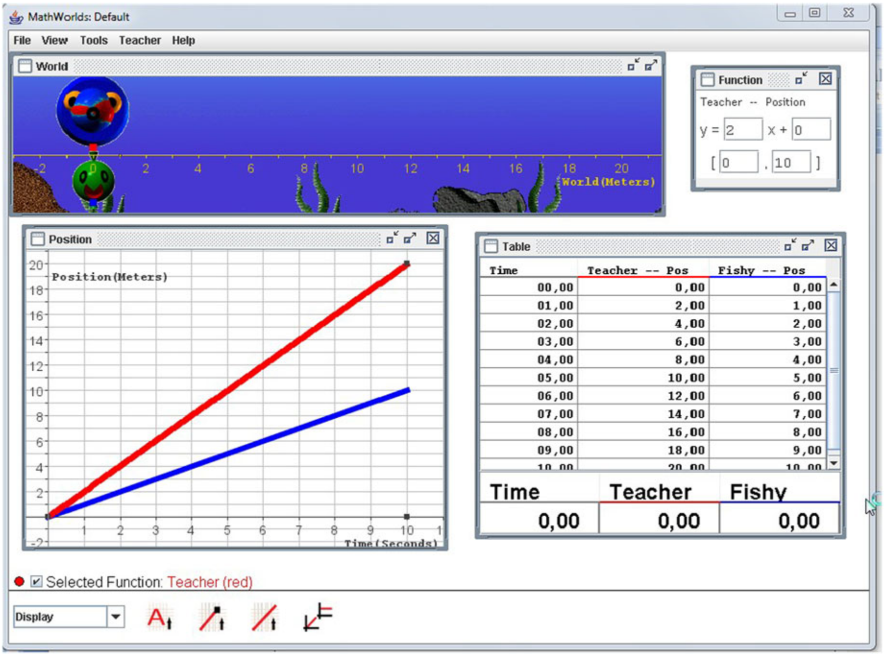

Fig. 1 SimCalc and its four windows

A second question is how we, as researchers, might interpret the teachers' attempts to make sense of all the information presented on the SimCalc screen. Since the screen brings information of various different natures—visual, numeric and symbolic, as well as dynamic movements, all "controlled" by an underlying formalism—we decided to use as our theoretical lens the framework of three worlds of mathematics (Tall, 2004, 2008), because it integrates embodiment, symbolism and formalism. That is, this theoretical framework is based on the premise that there are at least three different kinds of objects in mathematics, constituting three different worlds, the *conceptual embodied* world, the *proceptual symbolic* world and the *formal axiomatic* world.

The *conceptual embodied* world is the world of perceptions, in which the individual observes an object, its properties and makes sense of it by describing it. In the *proceptual symbolic* world, mathematical entities are symbolised, and actions can be performed upon them by the use of procedures that may be flexibly seen as both a procedure and the product of this procedure, the concept, in the duality of *procepts* (Gray and Tall, 1994). The *formal axiomatic* world is the world of axioms, properties, definitions and theorems that make it possible to construct the body of mathematics by using formal proof.

Perhaps the best way to exemplify this framework is by using it to present our view of interacting with SimCalc and by considering the relationships of each of the four windows that are presented when SimCalc is initialised to the three worlds of mathematics.

The world window presents a scenario with one or more actors that move according to the function it represents. Such movement, the actor itself and the scenario are, in our view, part of embodied world, since making sense of changes in this window involves users in analysing what they see, visualising how the position and the speed of the actor change over time and extracting properties from these perceptions of the proposed function by observing and describing the within window behaviour. Actors may leave dots (called Marks) behind, while they "walk," to mark their position in each step, which can also be part of this visual analysis of the behaviour of the functions these actors represent.

When there is more than one actor in the World, and the functions that are represented in their actions have different domains, an actor stops and turns grey at the points for which the function it represents are not defined, while the others continue to walk, swim or drive across the screen. This might give the impression that an actor has been temporarily "turned off," an interesting metaphor perhaps to help make an embodied sense of the behaviour of functions with different domains, which could subsequently be revisited in the light of more formal characteristics.

The position window, or more specifically the Cartesian position-time graph displayed within it, also has characteristics of the embodied world (Lima and De Souza, 2007). Since it presents a visual representation of a function, it is possible to perceive—bodily—aspects of the function's behaviour, such as, for instance, maximum and minimum points. This would be true of any position-time graph, but the SimCalc representation includes also a dynamic element: It is possible to "follow" the actor's movement in the graph by observing a vertical line that passes through the graph, showing the actor's position as the graph is played out dynamically over time, and offering another way of feeling the movement associated with the function in question. This vertical line also serves as a visual trace connecting the World and position window; making this connection involves evoking the characteristics of the formal world which dictate how Cartesian graphs are to be constructed.

The graph presented in the position window might also be representative of a procept (Gray and Thomas, 2001), and hence, an inhabitant of the symbolic world. That is, the position-time graph might be treated as simultaneously representing the *pro*cess of linking independent and dependent variables, and the con*cept* of function. Furthermore, because the graph is plotted according to mathematical rules that are part of the formal world (Lima and De Souza, 2007), the graph and the actors within the World are also ruled by the formal world.

Considering the symbolic world, it is also possible to find example characteristics in the function window. This window shows the algebraic expression that determines what is displayed in the graph and World. As it represents the function algebraically, it has symbolic characteristics, representative again of the *pro*cess by which function pairs might be calculated as well as the relation between the image and domain—a central feature of the con*cept* of function. Finally in the table window, the displayed table of values can also be argued to be part of symbolic world, since the table brings specific values for both independent and dependent variables that were calculated, based either on the algebraic expression, or collected from the graph, and translated into numbers.

Like the World and position window, the function and table windows also carry within them parts of the formal world. For instance, interpreting the algebraic expression requires a very sophisticated level of thought on the part of SimCalc users. It is written using x for the independent variable and y for the dependent variable. This is the usual terminology and one with which mathematics teachers and their students are familiar, but the axes in the position window do not have the same names, at least in the default case. The default is for the horizontal axis to be labelled "Time (Seconds)" and the vertical axis "Position (Meters)." Of course, there may be a "natural" response from the user of the software that the representations are of the same function, but, for those who may not be so familiar with position graphs, it is quite complex to relate "x" to "Time (Seconds)" and "y" with "Position (Meters)." This relation demands formal characteristics and may not be straightforward to all. The same happens with the table of values that have columns for time and position of each actor in the World.

Analysing the SimCalc screen and tools in the light of the three worlds of mathematics made it possible for us to realise that SimCalc does not only integrate four different representations of the same function, but, at least potentially, the windows also integrate characteristics from all three worlds of mathematics. The possibility to dynamically combine these four representations may be important for the teaching of functions and enrich the user's view of the concept. Those, like the Brazilian mathematics teachers who worked with us, already familiar with conventional representations of functions, might start to make sense of the World, by searching within it for characteristics of the concept of function they know from using the other representations. In this way, the integration, and the enrichment of their ideas about function, can be made by connecting different worlds of mathematics.

2 Teachers' Exploration of SimCalc

In an attempt to examine teachers' reactions to the SimCalc software and especially to investigate their ideas about the possible role of the world window as a means of introducing a new manifestation of functions and their behaviour to their students, we worked together with a group of mathematics teachers studying for post-graduation qualifications in mathematics education at Bandeirante University of São Paulo, in Brazil. As part of their studies, they were participating in a course which focussed on the use of digital technologies in mathematics classrooms. The work with SimCalc was carried out as part of this course. In total, fourteen mathematics teachers participated. Our aim was that they would get to know some of the characteristics of the software and how it works, and that they would be encouraged to reflect upon how they might utilise the software in their own classes (amongst the group were teachers of middle school, high school and university mathematics). In particular, after a period of familiarisation, they were expected to work in groups to design a SimCalc activity for a (school-level) classroom of their choice. Of the fourteen participants, three attending this course had previous experiences with the SimCalc software, as they intend to use it in their research, but the others had not previously seen or used the software.

Although the teachers attending this course were interested in the use of technologies in classrooms, either for their research or for their practise as teachers, they all reported that they did not currently use it systematically in their teaching. Reasons for this were varied, but most frequently related to difficulties associated with the access to computers in the institutions in which they worked. Schools in São Paulo do not always have ideal conditions for working with technologies. They have a computer lab, but often with only somewhere between 10 to 15 computers, not usually enough for the whole class, making it necessary to work with 3 or 4 students per computer. Internet connections also are not always very reliable or even available in the laboratories.

To explore the SimCalc software, the teachers first worked with an activity which involved analysing the movement of an elevator travelling at different speeds. The activity culminated in the creation, by each student group, of an actor in the World that would go from “hell” to “heaven” (or from “heaven” to “hell”), using any kind of function, and any of SimCalc’s tools. To tackle these activities, the participants worked in pairs or triplets.

The first activity was based on an activity developed by the Kaput Center. We chose this activity because it involved aspects that were likely to be familiar to the participating teachers, as well as aspects which were probably much less familiar. On the unfamiliar side, the activity involves a piecewise function, which is a type of function which receives little attention in the Brazilian mathematics curriculum. On the other hand, the function consisted of three linear pieces, which are extremely familiar entities in the Brazilian mathematics curriculum. The idea was that by combining the familiar and the unfamiliar, the teachers would begin to interpret the role of the world window, which initially was the only window displayed on the screen. In an activity reminiscent of the “black-box” tasks of dynamic geometry (Kynigos, 2004), in which the behaviour of dynamic geometry figures are explored, observed and then reproduced, the teachers were asked to create their own actor which represented a second elevator with identical behaviour to the one provided. To do this, the teachers had to think about how the action of the elevator should be represented on a position-time graph.

In the activity of creating an actor—a rocket—who would pass from “heaven” to “hell,” all group discussions quickly turned to considerations of functions and the notion of positive and negative infinity. The metaphorical association of heaven and hell with the infinities occurred spontaneously with no explicit discussion between groups. Interestingly this association has previously been reported in relation to the narratives students construct to make sense of dynamic graphs with asymptotes which tend to positive and negative infinity (Sinclair et al., 2009). Perhaps we can interpret this association as a kind of connection involving all three worlds of mathematics—the formal, the embodied and the symbolic: formal because the notion of limit is inherent; symbolic since the teachers had to think of how they might represent symbolically the process of moving from very high values on the vertical axis of the graph to very low values; and embodied—or at least almost embodied—by allowing heaven and hell to represent “physical” places, located high in the sky or deep in the depths (of course, this too is a metaphorical association).

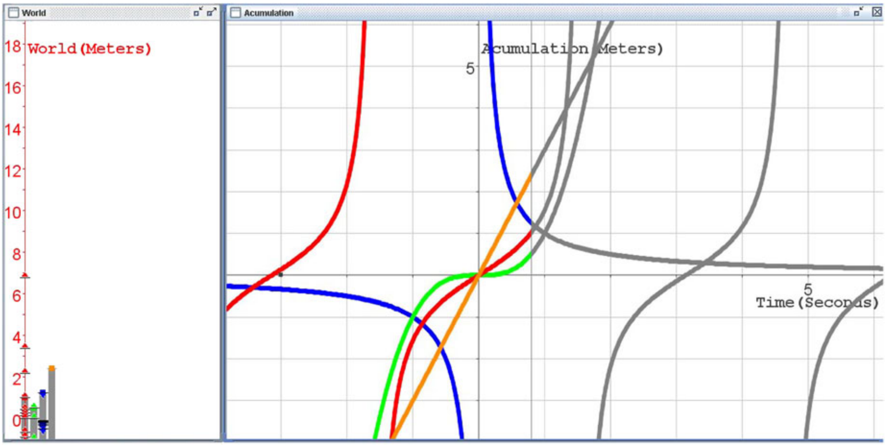

Fig. 2 Students' functions from hell to heaven

In total, three different functions which satisfied the required conditions were constructed: the reciprocal function, the tangent function, and the function $y = x^3$, as shown in Fig. 2. As they compared their different solutions, the teachers became involved in discussing similarities and differences in the voyages of their respective actors—how some went from heaven to hell, while others apparently traversed the opposite direction or moved very slowly around zero, while others were very fast a little after zero.

Following this activity, some of the teachers became interested in thinking about what would happen in the World for other types of functions, such as, for example, trigonometric ones. To this end, they created an actor, choosing a periodic function. They were working with the Fishy world, and they were particularly interested in how the fish turns around when it has to swim from one side to the other. This effect gave them the idea that the fish was pacing forwards and backwards, as if it was worried. Changes in the function, and the resulting changes in the fish's movement, gave raise to new stories for these teachers. Depending on the period of the function, the actor would swim faster or slower, bringing about the idea that the fish was more or less worried, while the amplitude of the function determined how far the fish would go. The interesting thing about these stories is that while they bring elements that are not strictly mathematical, these elements motivated the teachers to look more deeply at the functions they were creating, and to make mathematically valid connections between the shape of the functions and the movements of the actors.

This example shows the impact that this new representation had for the teachers. It caught their interest, and their discussions were always triggered by the actor's movements. Indeed, for some of the teachers the world window was becoming an essentially important part of the software that should be integrated into any learning activity. This brings us to the second part of our work, in which the teachers worked on designing SimCalc activities for use in their own classrooms. The starting point for these activities was the curriculum material developed by the São Paulo State's Secretariat of Education for use in São Paulo schools.

3 The Curriculum Proposal of the State of São Paulo

In 2008, São Paulo State public schools began to use a new curriculum, the *Curriculum Proposal of the State of São Paulo*. The aims of its developers are

> to support the work done in state public schools and to contribute to the improvement of its students' learning quality (São Paulo, 2008, p. 8)

and to

> assure to all a common base of knowledge and competences, so that our schools function as a network. (São Paulo, 2008, p. 8)

The material is composed of documents regarding both the principles of the curriculum and its management in schools, respectively entitled the *Presentation Guide*, and *Orientations for Management of the Curriculum in School*, along with material for the development of each discipline's contents within the classrooms of all state schools in São Paulo: the *teacher's booklet* and the *student's booklet*.

The two booklets for the mathematics discipline were designed by a group of mathematics educators, taking into consideration previous studies conducted by the State of São Paulo Secretariat of Education, as well as the experiences of state public schools teachers (São Paulo, 2008). In the teacher's booklet there are suggestions for activities to guide the development of teachers' work with students in classrooms. The student's booklet contains the same activities.

3.1 Teacher's and Student's Booklets and the Concept of Function

The student's booklets are used by middle school students from 6th to 9th grades (11–14 years old), and the 1st, 2nd and 3rd years of high school (15–17 years old). Each school year has four volumes of booklets, each one used bimonthly. In addition to these booklets, students also have mathematics textbooks provided by the government.

The teacher's booklets start with a general orientation about the booklets and how they are to be used, which is a text common to all booklets, independent of the discipline in question. The teacher's booklets also include information regarding the basic contents of each volume; and the proposed learning situations. At the end of each booklet, there is also a section called *Resources to Enrich Teachers' and Students' Perspectives and Understanding About the Theme*, and another on *Considerations Regarding Assessment*.

In relation to the teaching of function, in the sixth and seventh grade, students (11–12 years old) are taught about direct and indirect proportionality between two quantities, by emphasising the relationship between those quantities and coordinates of points in a Cartesian plane. In the 8th and 9th grade (13–14 years old), the ideas of number sets, inequalities, location of points in Cartesian planes, and analysis of graphs are discussed. It is in the ninth grade (14 years old) that the notion of function is formally introduced. In the second volume of the mathematics booklets for this school year, students are introduced to basic notions of functions, ideas of variation, first and second-degree polynomial functions and their graphs, and algebraic expressions of functions and tables of their values. They return to the concept of function a year later, in the first year of high school (1st year, Volume 2), when relations between quantities, proportionality and polynomial functions are revisited. In the same school year (1st year, Volume 3), students are also taught about exponential and logarithmic functions. Trigonometric functions are taught in the 2nd year of high school (2nd year, Volume 1); and in the 3rd year of high school (3rd year, Volume 3), graphs from all those functions are studied in terms of growth, values of the function in particular intervals, translations and reflections (São Paulo (Estado) Secretaria da Educação, 2008b, pp. 52–59).

Considerations presented for teachers in their booklets make it clear that the designers of the mathematics activities in the proposal intend that relations between different representations of a function—for instance the graph, algebraic expression and table of values—will form the foundation for the study of functions. Also, it is intended that teachers will build upon the material presented to students in previous school years. When presenting the plotting and analysing of graphs, teachers are supposed to involve their students in comparing graphs and discussing what changes in shape the change of parameters bring to the graph of a function. This shows the designers' concerns in integrating various registers of functions in the teaching of the concept.

It is also suggested that the teacher could use software to explore the relationships between these representations, although this is left as an open option. Considering this, the question arises as to whether SimCalc might prove to be a suitable tool to be used with activities related to the concept of function as proposed in the curriculum that the teachers who we were working with are using in their classrooms. In theory, the answer should be yes, since it is possible to integrate the means of representing all the functions discussed in the document (and more) in a single screen. In addition, the curriculum guidelines also stress how it is expected that the mathematics teacher will integrate mathematical content from other disciplines, such as Physics. SimCalc would appear to be an excellent tool for this purpose, since, as the actor in the World moves, there is a chronometer to measure how long the movement lasts, and it is possible to analyse position and velocity graphs. But will teachers share this view? And will they see fruitful ways of using SimCalc in the context of the activities proposed in the booklets—activities they are currently obliged to work upon with their students?

3.2 Technology and the Proposal

SimCalc, like some other digital technologies, brings, to the study of functions, dynamic representations that differ from the conventional static paper-and-pencil based ones. For this reason, we thought it would be useful to find out what suggestions were given to teachers regarding the use of technology in the curriculum proposal for São Paulo and whether such new possibilities are considered. What we found was that the curriculum guides do not offer advice on the use of technology in the teaching of any specific content area of mathematics. Our next step was to look for any suggestions regarding the use of technology for teaching in the mathematics teacher's booklets. In the section of the booklet that deals with general orientations, it did mention the use of different materials for the learning of mathematics:

> Whenever it is possible, within each booklet, available materials (texts, software, sites, videos, among others) in sintonicity with the proposed approach, that can be used by the teacher to enrich his/her lessons, are presented (São Paulo, 2008, p. 8).

Considering this, we looked for suggestions about the use of any software in all teacher's booklets that involve the study of functions: 9th year Volume 2; 1st year of High School, Volumes 2 and 3; 2nd year, Volume 1; and 3rd year, Volume 3. We found some indications in all of them. Student's booklets from 1st year Volume 2, and 2nd and 3rd years of High School also mention the use of software.

Looking more closely at these indications, however, we realised they are of a rather superficial nature, with suggestions only at the level of what software might be used (the ones mentioned are Graphmatica and Winplot) and to what ends (to plot graphs and analyse translations and dilations). None of the learning situations presented in the booklets come "ready" to be used with software, and there are no instructions or suggestions about how to design an activity for the learning of functions with the use of software.

Although the material from the Secretariat of Education makes relatively few mentions regarding the use of technology, this same Secretariat has provided each school with a computer lab of 10 or 15 computers. This would suggest that, to some degree, it is expected that teachers would integrate the use of the new proposal's material with computer labs available to them. Returning to our study, with this in mind, our proposal for the design of an activity using SimCalc involved adapting or being inspired by the student's booklets activities. In this way, we were interested in understanding the challenges involved for the teachers in adapting the material they are obliged to teach for use with a software which, as well as the various representation of functions explicitly cited in the curriculum material, brings a new form of thinking about expressions of functions.

4 The Teachers' Activities

In order to work on the design of the SimCalc integrated function activities, the teachers grouped themselves as they wished. As shown in Table 1, a total of six groups were formed, composed of one, two or three teachers.

When the activities were ready, the teachers from each group presented their work to the whole group and discussed their choices, the content behind the adapted activity, and how they believed the students to whom the activity was addressed (probably their own students) would deal with the proposed activity. These presentations were video recorded with the agreement of the participants.

Table 1 Teachers grouped for the design of activities

Group	Teachers
G1	Priscilla and Edson
G2	Laura, William and Miriam
G3	John and Peter
G4	Celia and Iran
G5	Felicity
G6	Brian, Charles and Janine

In possession of their activities, we have analysed them in the light of the three worlds of mathematics, looking for what kind of characteristics were privileged in their design. We have also analysed which aspects of the software the teachers intended to be used in the course of their activities. It is important to stress that, to our knowledge, the teachers did not know the theoretical framework, and were not aware that we would use it to analyse their work.

The first activity was presented by Priscilla, from G1. G1 chose an activity regarding the movement of a car, from Volume 2 of the 1st year student's booklet. In the booklet's activity, a graph representing the velocity of a car over time was presented. This was a graph of a piecewise function in three pieces: a line with a positive slope for 10 seconds, a constant for another 10 seconds and a line with a negative slope in the last 10 seconds. We found it interesting that this group had managed to find an activity which most closely resembled their own first experience with SimCalc—and one of the very few occasions in which attention was directed to a piecewise function in this curriculum.

In their adapted activity, the idea was that the student would first construct a position-time graph of an actor who moved according to the definition of a piecewise function. The teachers designed an actor in a SimCalc file that would move forward 10 meters in 10 seconds, remain stationary for 10 seconds then move backwards 10 meters in 10 seconds. That is, the behaviour of the actor they created was governed NOT by the piecewise function of the *velocity*-time graph presented in the booklet, but by a function whose *position*-time graph was visually identical. The teachers hid the graph they had created, in order to ask students to create a new actor to behave just like theirs. In this activity, the teachers intended for their students to use animation, analyse position and velocity graphs, and create their own actor. They also indicated that once different student groups created their own actors, they would make use of SimCalc's connectivity possibilities and combine all the functions into one file—again perhaps influenced by the "Heaven and Hell" activity, in which this facility had also been used.

In the description that accompanied their activity, the teachers suggested that their students would be dealing with the concept of average velocity (something indicated in the original booklet activity). But, although there are some questions regarding the velocity of the car, the velocity had a minor role in the SimCalc activity they were proposing to the students. Their activity aimed solely at the analysis of the actor's behaviour in the World. For the student dealing with this activity, a successful outcome depends on observing the graph, deciding what kind of function that movement represents, and creating an actor using this function. In this way, the activity integrates the World and position-time graph, although essentially from only an embodied point of view—that is, the symbolic characteristics of the graph may become subordinated to the embodied in this task. Perhaps one way of motivating connections between the embodied and symbolic world would be to include analysis of the associated velocity-time graph (the graph originally shown in the paper-and-pencil activity). However, it was only during the discussions that accompanied the presentation of this task that the teachers of G1 perceived the difference between their task and the one presented in the booklet.

Further analysing this activity in the light of the three worlds of mathematics, we understand that G1's attempt was initiated by characteristics of the embodied world, which could be used to model formal characteristics. In theory, if students were able to create the actor which correctly modelled the behaviour of a piecewise velocity-time graph then, they could use the software to analyse position-time graphs as well as explore the window displaying the algebraic expression—relating all three worlds of mathematics. However, although G1's activity made use of various features of the SimCalc software—the teachers created an actor, hid its representations, used "marks on," and it was their intention to use the connectivity of the software to collect and compare students' productions—the task they created reduced the complexity of its paper-and-pencil inspiration. This is because "downgrading" from velocity-time graphs to position-time graphs makes the connection between the graph and movement much more direct (perceptually speaking). The teachers were not originally aware that they had made this change. Even when one of the teachers, Pricilla, decided to try out the activity with her students, she remained unaware of the change. Her first evaluation of the students' responses was that they did not understand velocity, since they argued that the horizontal section of the graph indicated that their actor was stationary (which it did in the SimCalc version of the task). Priscilla was expecting that her students would associate the horizontal section with movement at a constant velocity (the intention in the paper-and-pencil version of the task).

Perhaps one of the problems was that, in choosing to construct an activity that closely resembled the one the teachers had previously experienced with SimCalc, the aims of the SimCalc activity and the activity of the booklet became confused. Priscilla knew the correct response to the paper-and-pencil activity, and seemed to be more salient to her when her students worked on the task versus actually seeing it on the SimCalc screen. In the event, the unplanned mismatch between the two versions of the activity perturbed the members of G1 somewhat. However, it is worth stressing that an activity involving the use of actors to create particular velocity-time graphs is not only possible, it could offer a means by which students could be motivated to mobilise and connect all three mathematical worlds.

William presented G2's activity to the whole class. The group was inspired by the idea of comparing exponential and linear growth presented in the student's booklet for the 1st year of High School (Volume 3), and used the concept of interest to

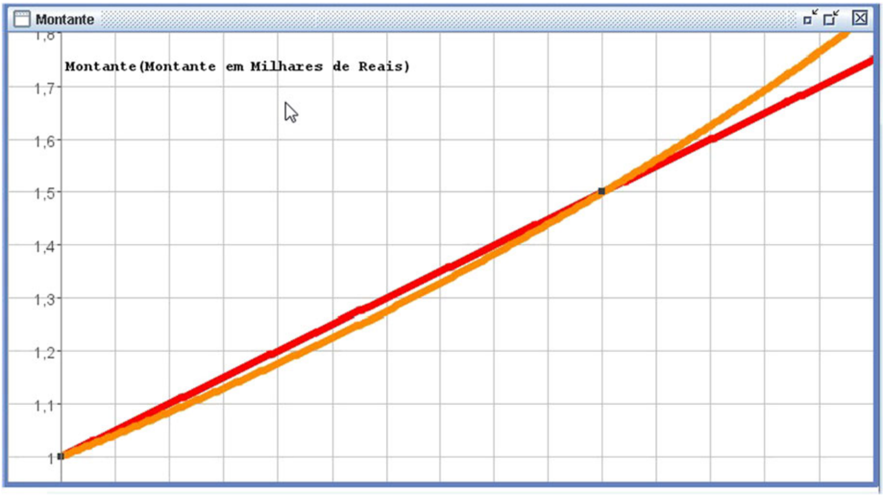

Fig. 3 G2's activity

discuss the difference of such growth. To accomplish this, the group intended to create a conflict to challenge the common sense belief that compound interest is always greater than simple interest. With this in mind, they designed an activity in which they created a situation that required the student to think what choice of interest would be best for a short term loan: simple or compound. Their conjecture was that most would initially choose simple interest. They prepared two actors in a SimCalc file (see Fig. 3), each one representing one kind of interest. Their aim was to show that when the loan is for a period of less than one month, the compound interest would gain a better return for the person who took the loan. However, to be able to present a situation in which the difference was clearly visible, they needed to use a 50 % interest with a loan of R$1,000.00—a rate of interest which, as they stressed, is rather unusual!

It seemed that what they wished to show was that in the interval (0, 1), the exponential growth was smaller than the linear growth, and they tried to connect this with compound and simple interests. To work with this situation in a classroom, they argued that it would be important for students to be familiar with the concepts of interest, and with linear and exponential functions. The first question of the activity was for the students to choose between the two kinds of interest for a short term loan. Then, students were asked to work on SimCalc, to look at the actors' movements and reconsider their first choice. After that, they were asked to provide algebraic expressions for both movements.

In our view, this activity starts by discussing formal characteristics of interests (or growth of two different functions), and moves to embodied characteristics of functions, given by the movement of the two actors. After observing this moving, the teachers intended for the students to transform what they analysed in the World and the graphs into algebraic expressions that inhabit the symbolic world. The task hence attempted to involve students in analysing the whole situation—graphs, expressions and movements—and to integrate all three worlds of mathematics in the same activity.

Regarding the use of the software, this group used it mainly for the purposes of visualisation, and there were no requirements that the students create anything in SimCalc for themselves: the idea was to explore a teacher-given model. Members of G2 explained that they wanted the World to be used as a means of analysing the difference between the movements of two actors. This embodied situation needed some adjustments in the software. Since they used one second to represent one month, and they were interested in looking at what was happening between 0 and 1, they had to change the steps from 1 to 0.1, and to turn "marks on." By doing this, the teachers felt it would be easier for students to see what they intended for them to see. In addition, the students would be able to use SimCalc to visualise the algebraic behaviour used to create the actors and their graphs. As they presented their ideas, the members of G2 explained how they had specifically looked for an activity in the booklets that involved analysing the behaviour of functions over time. Though they felt the activity they developed did exploit the dynamic representation of the World, they highlighted some concerns about the task that they had designed. First they were slightly uncomfortable with the fact that in order for the differences in behaviour between the functions be visible, the values for interest had to be completely unrealistic. And, they felt the task presumed a high degree of previous knowledge about the behaviour of different functions and expressed a desire of creating a task that could contribute more to the construction of this knowledge, rather than relying on its mobilisation.

Just like G1, the members of G2 also made the World the star of the activity. Their intention was to explore the movement in this window in order to discuss different growth patterns. To accomplish this, some other of SimCalc's resources, such as marks and the vertical line that follows the graph according to the movement of the actor were exploited in ways that made this activity different from one presented in a paper and pencil environment.

Not all groups, however, attributed such a central role to the world window. Some of them privileged other tools such as the graph window. G3 was one such case. They selected two activities from the student's booklet for the 1st year of high school (Volume 2). In Activity 1, students were asked to plot, in SimCalc, the graphs of eight functions in the form $f(x) = ax^2$, with $a = \pm 1, \pm 2, \pm 10, \pm 1/10$. No specific questions were asked. The students were required only to plot the graphs. Activity 2 involved launching a rocket and a missile. The trajectories of these objects were presented in the activity sheet on a graph y-position-x-position, as is also the case in the booklet from which the activity was drawn (Fig. 4). In this case, the x-axis did not represent time. In the activity presented in the booklet, it was explained that the missile was fired in order to intercept the rocket before it reached the ground and the functions representing the trajectories of both objects were given and the task for the students was to determine the height above the ground that the objects would intercept. The same objective was given in the SimCalc version.

In relation to Activity 1, the members of G3 justified their choice, arguing that the SimCalc tools would enable students to plot the graphs quickly and easily, and

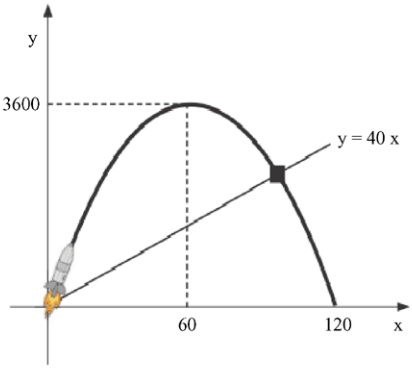

Fig. 4 Figure of G3's Activity 2 (São Paulo (Estado) Secretaria da Educação, 2008b, p. 51)

hence concentrate on comparing their properties. It seems that G3, in choosing Activity 1, was attempting to relate symbolic and embodied worlds in order to connect to characteristics of the formal world: that is, by plotting graphs from their algebraic expression, the students might be able to identify formal aspects related to how changes to the coefficient of x^2 influence the shape of the graph. What is missing from this activity is an obvious role for the World. Indeed, this kind of activity is a classic one for use with more traditional graphing packages.

In Activity 2, Peter and John intended for their students to make use of the given functions to create actors to represent the missile and the rocket so they could see, in the SimCalc representations, where the missile would intercept the rocket in its fall. They themselves created these actors in order to present the activity to the other groups (Fig. 5).

As they presented the second activity to the whole group, a heated discussion ensued: the graph and the animation suggested that the missile would be launched at the same time of the rocket, as if it was known a priori that the rocket would fall. It was only after the original booklet activity was re-consulted that it became clear that this was not the case at all. At first, a general worry emerged: since the graph in the booklet was not a postion-time graph, but a graph of vertical-horizontal distance, perhaps SimCalc was not an appropriate tool for its solution. This stimulated a debate about the pros and cons of using software in which the x-axis is designed to represent time. During this debate, the issue shifted from limitations of the software to questions of task design—perhaps instead of investigating the x and y coordinates of given trajectories, the task could be modified to that of determining the possible trajectories given specific launching times or exploring the heights at which the interception might occur under particular conditions. Figure 6 presents a possible example.

In terms of the three worlds framework, this activity certainly does have an embodied situation to be discussed. There is a story to be analysed and understood, but the graph presented initially gave the misleading idea that both rocket and missile were leaving the base at the same time. What was most interesting in the discussion that accompanied the whole group's consideration of Activity 2 was the questioning of how to deal with the fact that the SimCalc x-axis represents time, something that

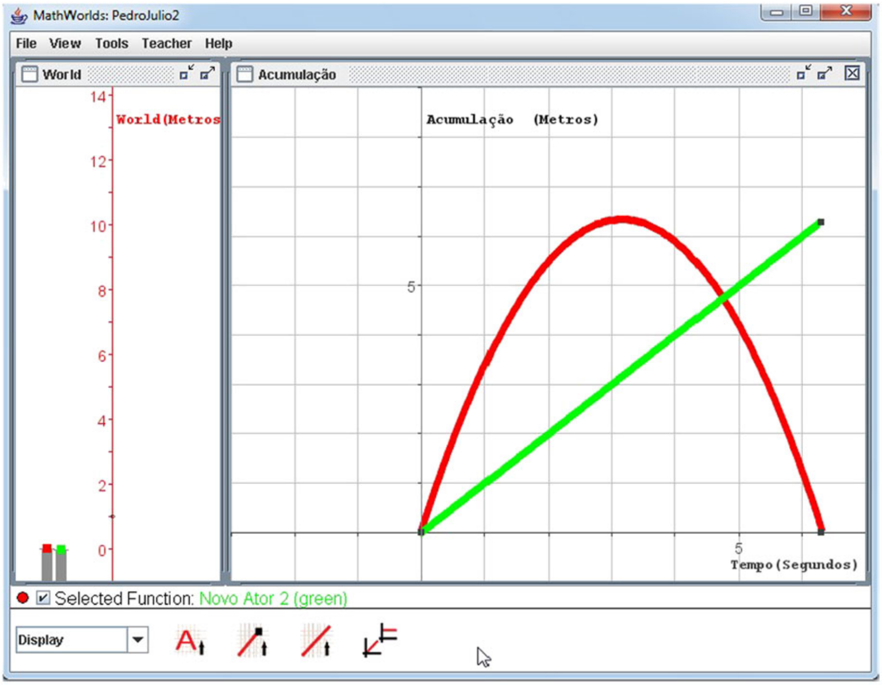

Fig. 5 G3's SimCalc file

was not the case for the majority of the function activities presented in the booklets. In Activity 2, once this had been realised, it was possible to create more sensible graphical representations, but this was not always the case, as an example to follow will illustrate.

Both of G3's activities required that actors be created, by using the algebraic expressions of the functions, and the important representation for them was the graph window. For Activity 2, functions were also entered by their expressions, but it was necessary for students to decide the domain of each function—the rocket and the missile—in order to make the missile leave the base at the right time to put the rocket down. It is possible that teachers of G3 were trying to use SimCalc as a plotter, like Graphmatica or Winplot, and had not appropriated all of the peculiarities of SimCalc.

G4 chose an activity involving the concept of proportionality in relation to the movement of a braking bus, adapted from 9th year student's booklet, (Volume 2). In the student's booklet, the situation involved a quadratic function relating velocity and distance ($d = 10v^2$), and, by looking at a table of values, students were asked to find the constant of proportionality. Since it is not possible to directly construct a velocity-distance graph in SimCalc—rather the graph comes about as a consequence of the actor's movement—Celia and Iran decided to construct an actor, a red bus that moves for 7 seconds at a speed of 5 m/s, to represent a linear function relating

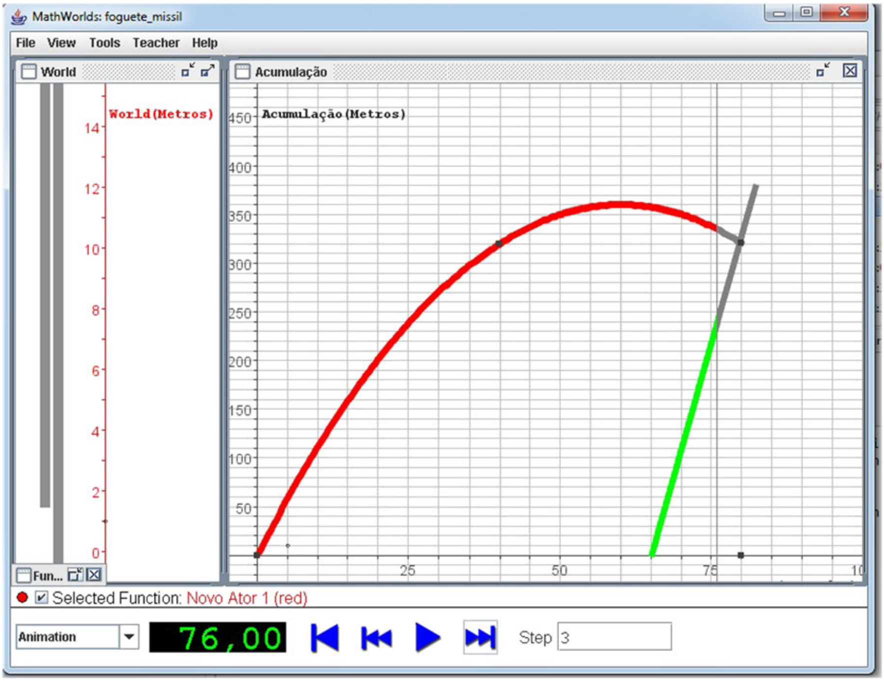

Fig. 6 Missile reaching the rocket

distance and time ($d = 5t$). In this way, Celia and Iran brought questions of time to an activity, which previously related only velocity and distance. This turned out to make quite a considerable difference to what was expected of the student. In the case of the SimCalc activity, the function used was more elementary than the one in the original task. The work of the students involved observing the movement of the actor in the World and to completing a table of values in order to determine the constant of proportionality in the linear function. Perhaps to preserve the connection with velocity, the students were also required to plot the velocity-time graph ($v = 5$).

G5's activity was of the same kind as G4's. Felicity adapted an activity regarding the movement of a car, from the 1st year student's booklet (Volume 2) which again involves the concept of proportionality. With this activity, her goal was that the students would analyse the idea of direct proportionality as a function (essentially what G4 ended up exploring), by interacting with several different representations, such as the algebraic expression, position graph and table of values. On paper, Felicity presented her students with a table of values relating time and distance, asked them to find the algebraic expression by which the variables were related, and to plot the graph of the function identified. In order to build the table of values, Felicity, in fact, gave the solutions she was hoping for, to the students, in a SimCalc file with an actor that would move 90 km/h and with the algebraic expression, position and velocity graphs displayed. Hence, the only activity left for the student was to observe what was shown in each window.

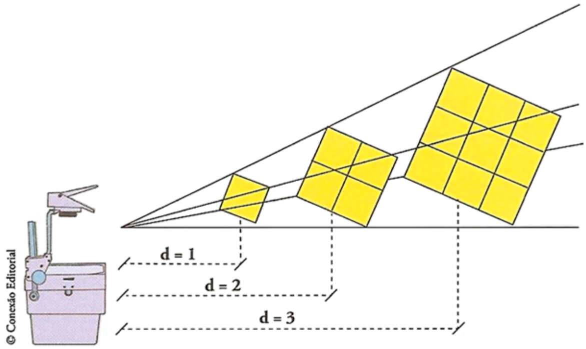

Fig. 7 Figure for G4's activity as in the booklet (São Paulo (Estado) Secretaria da Educação, 2008a, p. 42)

The teachers in G4 and G5 created activities, both centred around the concept of proportionality, which closely resembled each other—despite having been based on different activities in their original form in the curriculum booklets. Both activities involved a lot of symbolic characteristics and, since a table of values was given in the activity sheets in both cases, it would be possible for students to find the algebraic expression that related the independent variable to the dependent without actually using SimCalc. The last item of G4's activity, which asks for the velocity graph, does bring a potential role for the SimCalc software, since by plotting the velocity graph the students could see how, in the case of the given linear function, the constant of proportionality was equal to the velocity.

For these two groups, the use of SimCalc was mainly confined to plotting—or consulting—graphs. Our view is that rather little was gained by using the software. Indeed, it might be argued that the transference from paper-and-pencil to SimCalc was associated with a reduction in the complexity of the mathematical demands of the tasks as they were presented in their original paper-and-pencil form. It is also the case that the world window did not make much difference for either activity—and could be ignored by the students. In terms of the three worlds of mathematics, these activities principally inhabit the symbolic world, and the possibilities of integrating this with the other worlds are not yet obviously exploited.

Brian, Charles and Janine, from G6, chose a different activity to bring to the discussion. They selected an activity from Volume 2 of 9th year student's booklet, which they attempted to tackle almost without adaption. Instead, their idea was to solve the activity as presented in the booklet with the aid of SimCalc. As it turned out, this was by no means straightforward. The booklet activity involved the relationship between the area of a figure being projected and the distance of the projector from the screen. Figure 7 shows the situation as presented in the booklet. Members of G6 indicated that their aim also was for students to explore the given quadratic function. They devised an activity sheet in which a table of values was displayed and students were asked to generalise the table and decide what algebraic expression represented the relationship between area and distance.

The figure presented on the sheet developed by the teachers of G6 was a little different from the one in booklet (see Fig. 8). The values presented in the tables were of the areas of squares with sides measuring 1 m, 2 m, and 3 m, while the

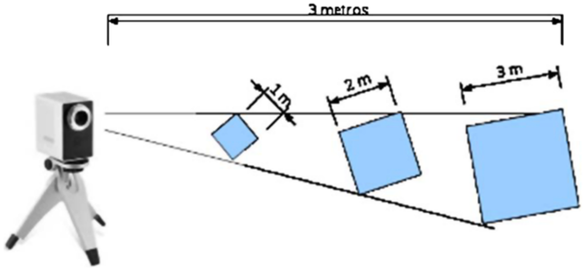

Fig. 8 Figure in G4's activity sheet

figure in the booklet showed the projector and the distance between it and the screen where the projection was being made.

Alongside this sheet, the members of G6 created a SimCalc file with two actors: one to represent the distance of the projector from the screen, and the other to represent a quadratic function corresponding to the area of the squares. Their intention was that the students would relate both functions by looking at their graphs. Although G6 attempted to make use the World in their activity, their attempt was not as successful as those of G1 and G2. Looking at the actors' movement (or the two functions graphed in the position window) did not offer any particular insights that would help relate the quantities represented. Only the symbolic characteristics from the activity were needed to solve the situation, and it was not useful to look for these on the SimCalc screen—they had to be known in order to create the actors. The members of G6 described their activity as an unsuccessful attempt to use SimCalc. However, it is important to emphasise that they had wanted to utilise the World to aid the explorations of the functions (unlike G4 and G5). The problem was that, perhaps as a result of the task they had chosen to adapt, they were just not able to use the World in a meaningful way.

In the end, the characteristics privileged in this activity are, once again, mainly symbolic ones. Students were asked to complete a table of values, analyse it and find an algebraic expression relating area and distance. The movement represented in the World was not helpful in relating the symbolic characteristics with embodied ones. Furthermore, just as in some of the previous activities, it would be possible to respond to the questions presented in the task simply by looking at the activity sheet on paper, with no need of interactions with SimCalc.

In Table 2, we present a summary of the analysis we made from the adapted activities. We show our understanding of the aim of the activity, SimCalc tools that would be necessary to work with the activity, and the mathematical worlds involved.

Before finishing this section, it is important to stress that the six groups of teachers described the task of creating a SimCalc activity as extremely challenging. They felt that they were still very much at the stage of beginners in terms of their familiarity with the software and its possibilities, something they referred to continually during the presentations of the proposed activities. We note, in particular, that the features of the software associated with connectivity, which were especially new to the teachers, were hardly integrated. On the positive side, in relation to another "new" feature, all groups were attracted by the possibility, offered by the world window, of

Table 2 Activities, SimCalc tools and mathematical worlds

Group	Activity	SimCalc tools	Mathematical worlds
G1	Constructing an actor to reproduce a given position-time graph	World; position-time graph; create an actor; marks on.	Embodied
G2	Comparing simple and compound interest	World; position-time graph; algebraic expression; marks on; steps (visualisation purposes)	Formal; embodied; symbolic.
G3	1. Perceptual comparison of graphs	1. Position-time graph; create an actor by the algebraic expression	1. Embodied
	2. Intercepting a missile with a rocket	2. Create an actor; position-time graph	2. Embodied
G4	Observing the movement of an actor to find a constant of proportionality	Task could be solved without using SimCalc	Symbolic
G5	Analysing the idea of direct proportionality as a function	Task could be solved without using SimCalc	Symbolic
G6	Relating the area of a figure being projected with the distance of the projector from the screen	Task could be solved without using SimCalc	Symbolic

bringing a dynamic quality to functions and their representations. However, most of the teachers felt rather insecure as to how best to monopolise on this possibility.

One major difficulty the teachers reported was how the World—and indeed the SimCalc software as a whole—seemed to privilege working only with functions in which the independent variable was associated with time. In some cases, this had led them to search specifically for examples in the students' booklets, which corresponded to this condition. They did not find many and those that they did find did not translate directly to the SimCalc environment (like the rocket launching activity of G3) or could be translated in a way that left the software a little redundant (G4 and G5). It was the association of the x-axis with the variable time that was cited most frequently as a disadvantage of using the software and apparently none of the groups thought about how the movement displayed in the World might be "detached" from contexts specifically associated with time. For example, in their own exploration of the software, the behaviour of function, devoid of "real world" contexts, had been experienced as an interesting phenomenon in its own right. But, this was not reflected in the activities they constructed, despite the fact that some teachers listed, amongst the advantages brought by the software, the possibility to compare movement and graphs. Indeed, the only activity specifically concerned with comparing different functions in a purely mathematical context emphasised similarities in the visual properties of classes of functions and made very little use of dynamism.

Easy visualisation was, in fact, an advantage that the teachers associated with using the software—a feature that they also linked to encouraging the making and exploring of conjectures. The teachers also saw the connectivity options of SimCalc as a positive aspect, even though it was still too new for them to know how to exploit it.

5 Reflections

In this paper, we attempted to view the possibilities for teaching and learning functions with SimCalc through the lens of the three worlds of mathematics. Our considerations come from a Brazilian context and more specifically involve a group of post-graduate students, all of whom were teachers of mathematics and most of whom worked in schools from the public system of the State of São Paulo. Our goal was to examine the potential of SimCalc in the making connections between manifestations of function in embodied, symbolic and formal forms. In particular, we are interested in examining the appropriation by teachers of the representation displayed in the World, which is a dynamic representation in which a computational actor moves across the screen according to an underlying function—defined either by means of an algebraic expression or by an interactively constructed position-time graph.

We have described how access to this world encouraged the teachers to build narratives—or stories—in which they made sense of the behaviour of particular functions by linking them simultaneously with imaginary situations which took them beyond the realm of mathematics while still being connected to it. That is, notions such as Heaven and Hell were associated with positive and negative infinites, the incessant to-ing and fro-ing of trigonometric fish was interpreted as a sense of worry and so on. In this way, as the teachers explored the software for themselves, they were encouraged to forage into all three of Tall's mathematical worlds.

However, these teachers have become much more used to dealing with symbolic characteristics and with software that are used to plot conventional graphical representations. Perhaps because of these previous encounters with functions, along with their rather limited experience with the SimCalc software, not all of them were able to design activities, which meaningfully integrated the animations of movement within the World in models of change and variation. Indeed, this difficulty may have been exacerbated by the mathematics curriculum currently in use in all public schools overseen by the State of São Paulo. Although the philosophy behind these materials is one in which connections between different representations of functions are valued and encouraged, the activities presented in the booklets for teachers and students seem to further reinforce activity within only the symbolic world of mathematics. When activity is confined to only one of the three worlds, the danger is that it becomes procedural in nature—where learners are able to interpret processes also as objects in a proceptual manner. The making of connections within a world, in this case the symbolic world, may be associated with mathematics learning, but what frequently happens is that learners seek meaning for the process in the embodied world without maintaining connections to mathematics (for more details, see Lima and Tall, 2008). Given this scenario, one of the gains of working with a new means of expressing mathematics is the opportunity to become aware of one's own "vices," that is, to avoid developing ways of working with functions that become so routine that reflecting on anything that deviates from the usual no longer occurs. There were certainly moments in the discussions of the proposed activities during which the teachers had to engage in "rescuing" buried ideas about function. On the other hand, we also evinced a certain unconscious tendency to reduce the mathematical complexity of paper-and-pencil tasks as they were modified for the SimCalc environment. Again, the teachers only became aware that they had manipulated the complexity of the activities during the whole group discussions. We are not able to pinpoint precisely what motivated this tendency, perhaps it was their unfamiliarity with the software that led them to somehow try to make things easier for their students, or perhaps too this also related to the presence of routine ways of working with function that were difficult to break out of.

The challenge for the teachers who participated in this study was twofold. First they need to be convinced that the dynamic possibilities of the SimCalc software could contribute to the process of making sense of functions. Our impression is that most of the teachers, even if not all, could incorporate the dynamic representations effectively into activities adapted from the mathematics curriculum they currently teach. The second challenge relates exactly to this curriculum: the teachers need to be convinced that this software has a role in supporting students to appropriate the views of function as it demands. We are not sure that the majority of teachers were ready to attribute such legitimacy to the software. In fact, we might go further and question whether the curriculum developers themselves have considered how digital representations change the ways by which the concept of function might be learned. This raises an important question for further reflection—to what extent are all of us involved the mathematics education community in Brazil prepared to change the curriculum to really monopolise on the kinds of interactivity and dynamism offered by software such as SimCalc?

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Mathematical Modeling with SimCalc: Enhancing Students' Complex Problem Solving Skills Using a Modeling Approach

Nicholas G. Mousoulides

1 Introduction

In recent decades the world has become governed by complex systems—in communications, engineering, finance, health, and in education. For all citizens, and especially for students, an appreciation and understanding of complex systems is crucial for making effective decisions about life, future studies, and roles as community members (Jacobson and Wilensky, 2006; Lesh, 2006). This radical increase of use of complex systems in the economy and society created a worldwide demand for new mathematical solutions to complex problems and has led to an appreciation of the power of cross-disciplinary research within the mathematical sciences and with other disciplines. Handling, explaining and predicting the behavior of such systems cannot only focus on their components, but mostly should focus on the behavior that arises from their interconnectedness (English, 2011; English and Mousoulides, 2011).

The study of these complex systems also received emphasis in recent years. At the school level, there is emphasis on interdisciplinary problem solving to fulfill the economy and work force's demands for school graduates that are able to possess flexible and creative mathematical problem solving abilities and to effectively use technological tools in working collaboratively in demanding projects (English, 2011). It is becoming increasingly recognized that future-oriented problem solving experiences in mathematics and science require interdisciplinary contexts and approaches. Interdisciplinary problem solving that involves core ideas from engineering, mathematics, technology, and science can empower students to tackle the many real-world problems society faces now and in the future (English, 2011; English and Mousoulides, 2011).

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With regard to technology, the ideas presented here are in line with the recommendations from Roschelle and Kaput (1996) and Ares (this volume) who proposed that educational technology should enable more students to engage in more sophisticated subject matter at a younger age. In this chapter, I first discuss the need for education that encourages innovation, especially in dealing with complex systems, and propose technology based interdisciplinary modeling activities as a means to enhance students' learning in the mathematical sciences and related STEM fields, and as a means to develop students' general competencies for success beyond school. I then address how SimCalc MathWorlds® (hereon called SimCalc), a conceptual technological tool, enriches student explorations and understandings while solving complex modeling problems. Students' results in the "*CAN.Be.Gr8 Planet*," an interdisciplinary modeling activity, are presented followed by a discussion on the role of technology-supported, engineering modeling activities in complex problem solving at the elementary school level.

2 Theoretical Framework

The advent of digital technologies changes not only the world of work but also the classroom environment (English, 2011; Roschelle et al., 2000). Due to the availability of increasingly sophisticated technology, classroom practices and pedagogies need to follow the changes in the way mathematics and science are being used in work place settings. These new tools not only change the way mathematics and science are used, but also stress the need for school graduates to develop a significant number of new competencies (English et al., 2008). Among these competencies, those identified as key elements of students' success are: (a) Problem solving skills, including working collaboratively in multidisciplinary teams on complex real world problems, (b) designing, analyzing, handling, explaining, and predicting the behavior of complex systems, (c) selecting, operating, analyzing, and transforming complex data sets, (d) developing personal and interpersonal skills, like communication, goal setting, working in groups, and leadership, and (e) effectively and creatively applying algebraic and spatial reasoning, in making sound judgments, in decision making problems (English, 2011; English et al., 2008).

The situation discussed above suggests that we need to rethink the nature of the mathematical and scientific problem solving experiences, without underestimating the necessity to introduce the study of complex systems in the school subjects of mathematics, science and technology (English and Sriraman, 2010). In particular, what is needed is a greater recognition to the complex learning that children are capable of doing, especially when provided with appropriate tools and contexts (*Curious minds*, 2008; English, 2004; English and Mousoulides, 2011; Lee and Ginsburg, 2007; Mousoulides, 2011; Perry and Dockett, 2008). Results from our previous work (e.g., English and Mousoulides, 2011; Mousoulides and English, 2009) revealed that young students, even at the lower elementary school level, have access to a wide range of powerful ideas and processes and they can use them effectively in solving complex problems, traditionally addressed to high school students.

Students have opportunities to elicit their own mathematical and scientific ideas, usually following a cyclic process of problem interpretation, selecting and manipulating problem information, identifying variables and operations that may lead to new data, and creating meaningful representations using the available technological tools (Lesh et al., 2007; Mousoulides et al., 2007).

These findings further underline the challenge to find ways to utilize the powerful mathematical and scientific potential students develop in the early years as a springboard for further development of the mathematical and scientific competencies, required for success in complex problem solving. Following English's (2011) suggestions to address this challenge, we need to recognize that learning is based within contexts and environments that we, as educators shape (Lehrer and Schauble, 2007) and create learning activities that are of a high cognitive demand and can promote active processing (*Curious minds*, 2008; Silver et al., 2009). One means that proved successful in developing such appropriate contexts and environments is SimCalc. The SimCalc environment has particular affordances embedded into its design, making it an appropriate tool to assist students' investigations in modeling and solving complex, real-world problems. In particular, SimCalc's environment includes rich animations of real-world situations (e.g., car racing, rocket trajectory), dynamic linking of multiple representations that provide opportunities to explore relationships between graphs, tables, equations, and animations, and editable representations (graphs) that allow students to manipulate problem parameters (Pierson-Bishop, this volume).

In the following section of the chapter, I give consideration to these suggestions, by proposing mathematical modeling as an appropriate approach to promoting complex learning through intellectually challenging interdisciplinary tasks.

2.1 Interdisciplinary Model Eliciting Activities

Mathematical models and modeling have been defined variously in the literature (e.g., Blum and Niss, 1991; Greer et al., 2007). I adopt here the perspective that models are "systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behavior of some other familiar system" (Doerr and English, 2003, p. 112). A cyclic process could represent modeling. This cyclic process of modeling includes the following steps: a problem situation is interpreted; initial ideas (initial models, designs) for solving the problem are called on; a fruitful idea is selected and expressed in a testable form; the idea is tested and resultant information is analyzed and used to revise (or reject) the idea; the revised (or a new) idea is expressed in testable form; etc. The cyclic process is repeated until the idea (model or design) meets the constraints specified by the problem (Zawojewski et al., 2008).

Mathematical modeling has been considered to be an effective medium to prepare students to deal with unfamiliar situations by thinking flexibly and creatively and to solve real-world problems (English, 2006; Lesh and Doerr, 2003). Following the calls from professional organizations (National Council of Teachers of Mathematics, 2000; National Research Council, 2001), a modeling perspective in problem solving can provide students with purposeful activities along with skillful questioning to promote the understanding of relationships among mathematical ideas. These recommendations can be pushed further and modeling activities can be used as a way to cultivate students' critical thinking and critical literacy. The effectiveness of mathematical modeling is also addressed in a number of related research studies, in which results revealed that students' work with modeling activities could help them to build on their existing understandings, allow for students' multiple interpretations and approaches, promote intrinsic motivation and self-regulation, and engagement in thought-provoking, multifaceted, complex problems (English, 2006; Lesh and Doerr, 2003; Mousoulides, 2011; Zawojewski et al., 2008).

Following Lesh and colleagues' perspective (this volume), a models and modeling perspective and SimCalc share the tradition of investigating. Students direct their investigations towards the nature of new and "flexible" types of "mathematical thinking" that is needed beyond school, the ways that these concepts and abilities develop, and new ways development can be cultivated, documented, and assessed (Lesh, English, Sevis, and Riggs, this volume). Furthermore, MMP and SimCalc share the same important goals, aiming at providing *democratic access to powerful ideas* and *foundations for the future in mathematics thinking and learning* (Hegedus and Roschelle, 2012; Kaput and Nemirovsky, 1995).

In adopting a modeling perspective, students are presented with complex real world activities that involve model development, and in which students repeatedly express, test, and refine or revise their current ways of thinking as they endeavor to create models that provide significant solutions—solutions that comprise core ideas and processes that can be used in structurally similar problems (Lesh and Doerr, 2003). The problems necessitate the use of important, yet underrepresented, mathematical processes such as constructing, describing, explaining, predicting, and representing, together with quantifying, coordinating, and organizing data (Mousoulides, 2011). The problems are designed so that multiple solutions of varying mathematical and scientific sophistication are possible and students with a range of personal experiences and knowledge can participate. The products students create are documented in various forms, including written symbols and formulae, interconnected representations, technological artifacts, and designs. These products are shareable, reusable, and modifiable models that provide, at the same time, teachers with a window into their students' conceptual understandings. Furthermore, these modeling activities might assist students in building their communication (oral and written) and teamwork skills, both of which are essential to success beyond the classroom (English and Mousoulides, 2011).

Recent studies reported that the availability of technological tools, such as SimCalc, can influence students' explorations, model development, and therefore improve students' mathematical understandings in working with modeling activities (Lesh et al., this volume; Lesh et al., 2007; Mousoulides, 2011; Mousoulides et al., 2008). Mousoulides and colleagues (2007, 2011) reported that students' work with a spatial geometry software broadened students' explorations and visualization skills through the process of constructing visual images and these explorations assisted students in reaching models and solutions that they could not probably do without using the software. In concluding, authors reported that the inclusion of appropriate software in modeling activities could provide a pathway in better understanding how students approach complex real world based mathematical tasks and how their conceptual understanding develops (Mousoulides and English, 2011; Mousoulides et al., 2007).

This chapter builds on, and extends previous SimCalc research (e.g., Hegedus and Moreno-Armella, this volume; Pitta-Pantazi, Sofokleous, and Christou, this volume; Roschelle and Hegedus, this volume; Roschelle et al., 2000; Roschelle and Shechtman, this volume) by examining how *SimCalc*, a conceptual dynamic tool for algebra (Kaput, 1992), can provide a pathway in better understanding how students approach and solve a real-world, complex problem, how the software's features and capabilities influence students' explorations and model development, and how students interact with the software in developing technology-based solutions for a complex interdisciplinary problem.

3 The Present Study

3.1 The Purpose of the Study

The purpose of the study was twofold: first to examine students' modeling and mathematization processes as they worked on a complex modeling problem using SimCalc and second to investigate how its interactive components assisted and enhanced student explorations and investigations in creating several different models for solving the problem.

The problem addressed in this study, "*CAN.Be.Gr8 Planet,*" required students to send a rocket to a recently discovered planet. During the activity, students constructed different models for accelerating a rocket to escape earth's gravity and fly off into space. Specifically, students were provided with background information on the basics of rocket science, appropriate for elementary school students, and the context of the modeling activity. Based on the provided information, students were invited to construct different models using SimCalc, as to succeed in sending a rocket to the CAN.Be.Gr8 planet.

3.2 Participants, Modeling Activity, and Procedure

One class of eighteen nine-year-olds (10 females and 8 males) and their teacher worked on the "*CAN.Be.Gr8 Planet*" modeling activity as part of their participation in a longitudinal 3-year study. The study focuses on exploring students' development of models and problem solving skills, while working on interdisciplinary engineering-based modeling problems. The students were from a public K-6 elementary school in the urban area of Nicosia, Cyprus. Although students had not met such problems before as the mathematics curriculum in Cyprus rarely includes any modeling activities, due to their participation in the research project, students were familiar with working in groups, developing models for solving quite complex problems, and presenting and documenting their results (English and Mousoulides, 2011). Specifically, students worked on the "*CAN.Be.Gr8 Planet*" during the second year of their participation in *Primas* research project. Primas, a longitudinal four-year study, was co-funded by the European Commission, under the FP7 framework in Science in Society call. Primas activities focus on enhancing students' inquiry skills in mathematics, science, and engineering (e.g., decision making) and on exploring students' development of modeling competences.

Prior to working on the activity, students spent one 40-minute session to familiarize themselves with the SimCalc software and its dynamic features. SimCalc introduces novel dynamic, graphical, and symbolic notations and representations (Kaput, 1992) to provide tools that engage students' conceptual resources, enable mathematical communication, and support growth towards more sophisticated understandings (Hegedus and Roschelle, 2012; Kaput, 1992). During this first session, students familiarize themselves with SimCalc while working on the "*Warm Up*" and "*Rockets to the Moon*" activities.[1] These activities provided, at the same time, opportunities for the students to explore the software and build an understanding of velocity as a rate. The problem was implemented by the author, two postgraduate students, and the classroom teacher. Students worked for four 40-minute sessions to find a solution for the problem presented in the "*CAN.Be.Gr8 Planet*" activity. The problem assigned to students was the following:

> Two months ago, National Aeronautics and Space Administration (NASA) announced the discovery of a new planet, named CAN.Be.Gr8. Today, NASA invited school students to participate in their next competition. During last missions to the moon and space, NASA discovered that although the design of their rockets was fine, the engine system used to produce thrust could be improved. In order to fulfill these needs, students are encouraged to explore the motion of rockets, using the SimCalc MathWorlds® software, and propose the best possible method for sending a rocket to the CAN.Be.Gr8 planet.

The design of the "CAN.Be.Gr8 Planet" modeling activity followed the design principles for developing modeling activities that are based on the work of teachers and researchers and that have subsequently been refined by Lesh and his colleagues (this volume, Lesh and Doerr, 2003; as well as English and Mousoulides, 2011). These design principles for modeling activities are integrated with the design principles of the SimCalc environment (Kaput and Blanton, 2002), in further improving the design and development of SimCalc-based, model eliciting activities. The first is the Personal Meaningfulness Principle. It is important that students can relate to and make sense of the problem situation that is presented—such situations should reflect real-life scenarios that build on students' existing knowledge and experiences. Fur-

[1] To obtain the software and curriculum documents for these activities, please contact kaputcenter@umassd.edu.

thermore, if such problems address current topics or themes in existing curricula, they are less likely to be treated as "add-ons" in already over-crowded school programs. The "CAN.Be.Gr8 Planet" modeling activity required students to interpret the activity meaningfully from their own different levels of mathematical ability and prior knowledge in mathematics and science and reflected a topic that regularly appears on the news (satellites, GPS systems, rocketry).

Engineering-based problems adopting a models and modeling perspective should also require students to develop a model that addresses the underlying structural characteristics (key ideas and their relationships) of the engineering situation being addressed (the *Model Construction Principle*). This principle, also ensures that the solution to the activity requires the construction of an explicit description, explanation, procedure, or justified prediction for a given mathematically significant situation. In the "CAN.Be.Gr8 Planet," students had to identify the structural characteristics and key elements of a rocket's motion (distance, velocity, acceleration, time) and use these characteristics in modeling a rocket's movement to the recently discovered planet.

It is important that students' model constructions be documented so that their thinking and reasoning can be externalized in a variety of ways including simulations, graphs and diagrams, and tables of data (the *Model Documentation Principle*). Furthermore, the models students construct need to involve a detailed description and explanation of the steps taken in constructing their models. In the "CAN.Be.Gr8 Planet" activity, students were encouraged to write a letter to NASA, presenting in an analytic form the way they worked in solving the problem, by making, if needed, comparisons between the different models that had developed.

The *Self-Assessment Principle* maintains that students should be provided with sufficient criteria for determining whether their final model is an effective one and adequately meets the client's (NASA's) needs (e.g., a rocket that moves in an appropriate way, taking into account escape velocity and other problem requirements). Such criteria should also enable students to progressively assess and revise their creations as they work the problem. Finally, the *Model Generalization Principle* (Share-Ability and Re-Usability and Effective Prototype) emphasizes that the models students create should be applicable to other related problem situations (e.g., motion problems). Others could use students' solutions beyond the immediate situation, which are as simple as possible, yet mathematically and scientifically significant.

The purpose of the activity was to provide students with opportunities to explore the relationships between variables (distance, velocity, acceleration, and time), explore families of linear functions, build an understanding of velocity as a rate, explore and develop links between graphical, tabular and symbolic representations as to explore concepts in motion, and to develop appropriate models for solving the rocket problem. During the first session, which lasted around 20 minutes, students were presented with a number of articles and videos from NASA's website, with an aim to familiarize students with background information on rockets (http://www.nasa.gov/audience/foreducators/rocketry/). These articles and videos followed by readiness questions, aimed to familiarize students with the context of the activity. Among the readiness questions were the following: "*What is a rocket*?"

"*What are the main components of a rocket?*" "*Explain in your own words how and why rockets fly.*" During the modeling stage of the activity (90–100 minutes) students worked in groups of three, using SimCalc, to solve the "*CAN.Be.Gr8 Planet*" problem. In the last session, students presented their models in whole class presentations for reviewing and discussing with their peers. Finally, a whole class discussion focused on the key mathematical and scientific ideas and processes, and on the SimCalc constructions that were developed during the modeling activity.

3.3 Data Sources and Analysis

The data sources were collected through audio- and video-tapes of the students' responses to the modeling activity, together with the SimCalc software files, student worksheets, and researchers' field notes. Data were analyzed using interpretative techniques (Miles and Huberman, 1994). Audio and video records helped us to identify the unique ways in which the software facilitated students' work in developing a model for solving the rocket problem, as well as the sequence of the modeling processes and strategies used by students during the solution of the problem. Detailed analysis of all data was used to develop categories of the mathematization and modeling processes of students, and to identify developments in the model creations with respect to the ways in which the students: (a) interpreted and understood the problem, and (b) used and interacted with the software capabilities and features in solving the modeling problem.

In the next section, I summarize the model developments of the student groups in their attempts to solve the "*Rocket to the CAN.Be.Gr8 Planet*" activity.

4 Results

Six groups of students worked on the problem. Each group consisted of three students. Two groups of students succeeded in developing appropriate models for solving the problem, using a quite sophisticated approach. Two more groups succeeded in understanding the core question of the problem and in handling the problem related data and requirements, and in exploring the software's capabilities and functions for developing a model for the rocket problem. However, those two groups were not successful enough in providing a coherent solution to the rocket problem, and they partially solved the problem. The last two groups of students that participated in the activity failed to understand and therefore did not provide any appropriate models for solving the problem.

Students in these two groups faced a number of difficulties in understanding the core question of the rocket problem, which was modeling the rocket's trip to the space. At first, students' efforts focused on exploring the different representations provided by the software. Students failed in connecting the different representations (animation world and the graphs) provided in the software's environment and they consequently failed in connecting the requirements of the problem with the software's environment. The latter is in line with the findings of other studies (e.g., Moreno-Armella and Hegedus, this volume; Mousoulides, 2011), which indicate that making connections between the different representations is not an easy task for students. Further, the difficulties these students faced focused on an understanding of the relation between distance and velocity. This might have happened because students could not observe the proportional relation between the two variables and their interconnectedness, probably due to their limited understandings in graphs and in the basic principles of kinematics (not part of the Science curriculum at the elementary school level). Consequently, students spent most of their time on randomly modifying the speed (constant velocity) of various rockets, without connecting their efforts with the problem requirements.

The model developments of the four groups of students that adequately solved the rocket problem are presented next. Those successful approaches correspond to three models that are presented in the next session. Results for each model are presented in terms of: (a) the modeling processes and (b) the mathematization processes. The results of the modeling processes appeared in students' work are presented with regard to the steps of the modeling procedure (Description, Manipulation, Prediction of the Problem, and Solution Verification). The respective students' mathematical developments are presented in cycles of increased sophistication of mathematical thinking.

4.1 Model A

Two groups of students partially solved the problem. With regard to the modeling processes, they failed to fully understand the problem, which was to successfully build an understanding of the rocket velocity and how this might change as the rocket travels from earth to space. Students in both groups understood the core question of the problem but they did not succeed in connecting the core question with the provided information and data. Specifically, students were provided with information on the 'escape velocity,' which is the velocity needed by a vehicle to escape earth's gravity and information on 'orbital velocity,' which is the velocity needed to balance between gravity's pull on the rocket (satellite) and the inertia of the satellite's motion.

Following their work with the "*Rockets to the Moon*" preliminary activity, students understood that a rocket could reach space, using different, yet constant, speed. During problem manipulation, both groups of students created more than three rockets, traveling to space at different speeds. When asked to modify their rockets, as to travel at different speeds, but covering the same distance, students experienced a number of difficulties. Specifically, students failed to create appropriate connections between the velocity-time and distance-time graphs and the rocket world. At first, when asked to model two or more rockets each reaching a distance

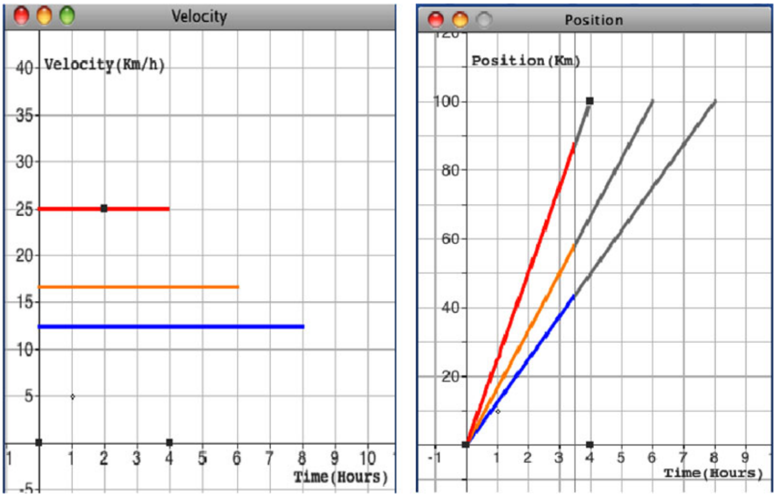

Fig. 1 Three rockets traveling at different speed cover the same distance

of 100 km, by modifying the velocity-time graph, students in both groups failed, although using for several times a trial and error approach. Researcher and classroom teacher encouraged them to use the distance-time graph to get the rockets to a distance of 100 kilometers. Students in both groups successfully created two rockets traveling at 100 km (see Fig. 1) using the distance-time graph. However, when students were prompted by the researcher to send one more actor (rocket) at a distance of 100 km, by only using the velocity-time graph, they failed in doing so, probably because they did not make all necessary links between the two graphs.

One of the reasons students' solutions were not that successful was the fact that students did not succeed in manipulating the relations between velocity, time and distance. More specifically, students failed to intuitively approach the function $velocity * time = distance$. Some students identified the relation between velocity and time, when distance was constant, but they could not transfer this knowledge in the software's environment. For instance, students in one of these two groups reported that the more the velocity, the less time needed to cover the distance of 100 km. However, velocity-time graph manipulation was not easy, and students' work resulted in rocket behaviors like those presented in Fig. 2.

Students failed to make the appropriate connections between the two graphs and the rocket world. As a consequence, the appropriateness of their models was limited and they could not provide a coherent solution for the problem. There were debates, for example, on how to modify a rocket's behavior as to meet activity's requirements (specific velocity and/or distance), but without success. A possible reason was that students might lack the necessary mathematical concepts to better mathematize the real problem and therefore to construct better models.

At a second stage, students were prompt by the classroom teacher to create a new rocket, following NASA's requirements for a rocket to escape earth's gravity (final velocity 40.000 km/h) or to balance earth's gravity (27.000 km/h at 242 km).

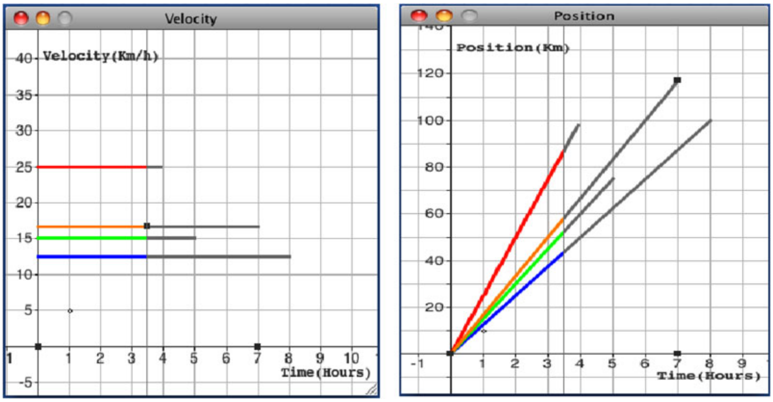

Fig. 2 Student approaches to create an actor using the velocity-time graph

Students in both groups failed to use a systematic approach to reach both the required velocity and distance, without using a non-constant velocity approach. Consequently, students did not actually verify their solution within the context of the real problem, which required students to work with acceleration-based approaches. What was evident in students' work was the difficulty to distinguish between faster motion (increased velocity) and accelerated motion, and they therefore failed to use an accelerated rocket in their solutions.

With regard to the mathematical developments, students' work in these two groups was limited to lower levels of sophistication of mathematical thinking. Students commenced the "*CAN.Be.Gr8 Planet*" activity by first exploring the rocket (animated) world and exploring the main characteristics of the two graphs. Only limited mathematical thinking was displayed in students' unsystematic work. This was also evident in students' comments: *"Look! My rocket is faster than yours."* When prompt to link the Rocket World to the graphs, students failed to distinguish between the two graphs or to link one or both graphs to the animation. In general, most of their comments were rather *descriptive: "This is my line (point to the corresponding colour). Here is the second one (points to the second graph). Look! They have the same colour.*"

When prompted to examine the relationship between the graphs and the Rocket World, the only connections they managed to document were the link between the speed of the rocket and its motion, and the link between the end point in the distance-time graph and the end point in the animation world. During the second cycle of mathematical developments, students' work could be characterized by the identification of the notion of velocity. Students successfully discussed the notion of velocity as the rate at which the rocket changed its position. Although this was a quite significant accomplishment, students failed to connect it with the requirements of the real problem, and therefore failed to develop a better solution for the rocket problem. A possible reason for this was students' difficulties to link the various representations provided by the software, and especially to explicitly link the velocity-time and the distance-time graphs. The latter was among the major differences between the work on these two groups and the work of the third group presented in the next session.

4.2 Model B

This group's work was quite similar to the work of Model A's groups with regard to the first part of the modeling activity—namely the creation of different rockets going to the moon. What was different in this group's work was their modeling and mathematical developments with regard to the identification of the various connections between the two graphs and between the graphs and the animation world. Further, this group successfully used the software's capabilities for constructing various rockets and finally developed successful models for the NASA competition.

Students' first attempts focused on drawing connections between the two graphs. Although they could not explicitly discuss and draw on the relations between the two graphs, they managed to identify the relation between the velocity graph and the rocket's position in the animation world. In contrast to student work in the previous model, students in this group identified and clearly stated that the distance covered by the rocket could be calculated by multiplying velocity with time. However, although prompted, students failed to connect the distance with the area under the velocity graph. Students also failed to make the connection between the velocity (as a rate) and the slope of the distance graph. The latter was among the achievements of the group's work presented next.

This group's proposed model for the next NASA rocket consisted of a two-part velocity function. What was evident in students' model was the ability to work with piecewise functions and change the velocity of the rocket. However, there are some constrains in this model, namely the lack of a third segment in the velocity-time graph, representing the decrease in the rocket velocity when the rocket runs out of fuel, but still moves away from earth. The researcher challenged students to carefully examine the behavior of their rocket at the end of its trip, and to propose a modification in their rocket, so the rocket would not suddenly stop, but rather slow down. Students attempted to modify their rocket, but without success, as they continued increasing rocket speed at a slower rate (acceleration). Their final model is presented in Fig. 3.

4.3 Model C

The model presented by the next group was far more sophisticated than the Model A and Model B. Students in this group used SimCalc's functionalities and tools to explore the relations between the notions of velocity, distance, and acceleration.

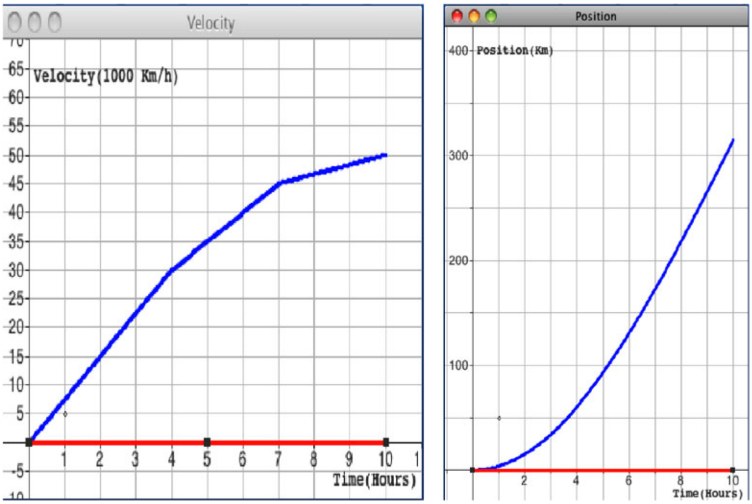

Fig. 3 Students' final model for the NASA competition

Students managed to connect the dynamic representations of the software's environment and they explicitly identified all necessary relations for developing a comprehensive model, appropriate enough for solving the rocket problem. Similar to previous group's work, students in this group easily explored the relations between the animation world and the position-time and the velocity-time graphs. In the case of constant velocity, students also found appropriate connections between the two graphs (velocity/time and distance/time). Without having access to the distance/time graph, the students set a hypothesis for the rocket's position before running the experiment. Also, when prompted to calculate the velocity of a rocket by only using the position graph, the students intuitively discussed the notion of slope. The first two parts of students' work were quite similar to students' efforts presented for Model B. Therefore, only the mathematical understandings and the third part of students' work—namely the model development of an accelerated rocket—are presented next.

Given that students rarely meet similar activities in their textbooks, students' work in identifying trends and relations in their data and graphs was impressive. Initially, students, without using any formulae, observed that they could calculate the position of the rocket using the velocity graph. Students stated a hypothesis and then ran four different experiments, as to make sure that their hypothesis was correct. They ended this first exploration by documenting that "you can find the velocity of the rocket by dividing the total distance by time."

Their next efforts focused on calculating the position, using the velocity graph. This task appeared to be more difficult than the calculation of the velocity, using the position graph. Students focused their efforts on identifying the relation, without connecting this relation to the area below the velocity/time graph. When students concluded that they could find the position by multiplying velocity by time, the re-

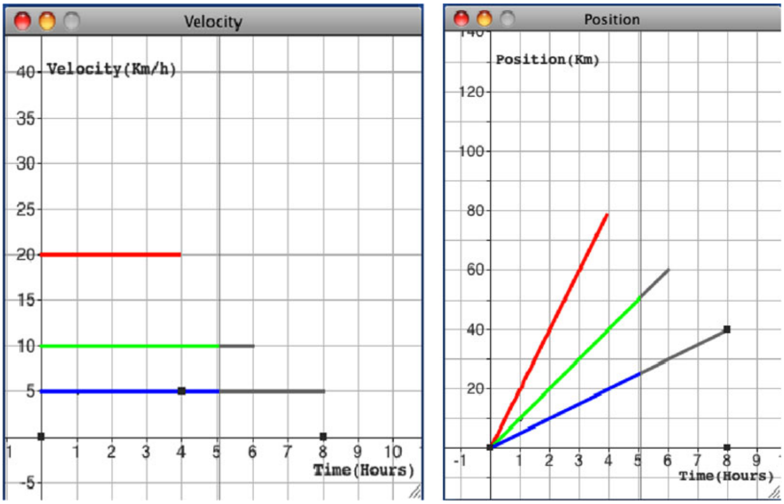

Fig. 4 Students exploring the relations between the position-time and velocity-time graphs

searcher prompted them to think in terms of the area of the different shapes appear in the graph. This hint was not enough, as students spent a lot of time without reaching an appropriate formula. On the contrary, they finally succeeded in 'sending' a rocket to the position of 80 km, without using the position graph (see Fig. 4).

The second dimension of student work that clearly distinguished this group from other groups was the final model for the NASA rocket. Based on their previous explorations, students spent a lot of time discussing, exploring and using trial and error to reach the correct piecewise function that could represent the velocity of their rocket. Although straightforward, this approach was not easy, especially when working with the position graph. Since students were not familiar with curves (see, for instance, Fig. 5), they decided to work with the velocity graph to model the behavior of their rocket.

Students' first attempt resulted in an accelerated rocket. During a discussion with the researcher, students were encouraged to refine their model so that the rocket could function in a more realistic way. Specifically, he prompted students to take into account the real-world data, indicating that at a certain point of the rocket's trip, the velocity would decrease when the engines of the rocket stopped producing thrust. Students realized that the velocity would reach zero at the same time the rocket escaped or balanced the earth's gravity.

In concluding, students designed a *three-piece function* to model rocket's behavior (see Fig. 5). Reflection on the appropriateness of their model was a distinct characteristic of students' work on this group (in contrast to students' solutions in the other groups). Although it was apparent that the students' model was quite appropriate and adequately solved the problem, students questioned the appropriateness of their model by extensively discussing how they could further improve their model. These developments were not only appropriate for improving their model, but they

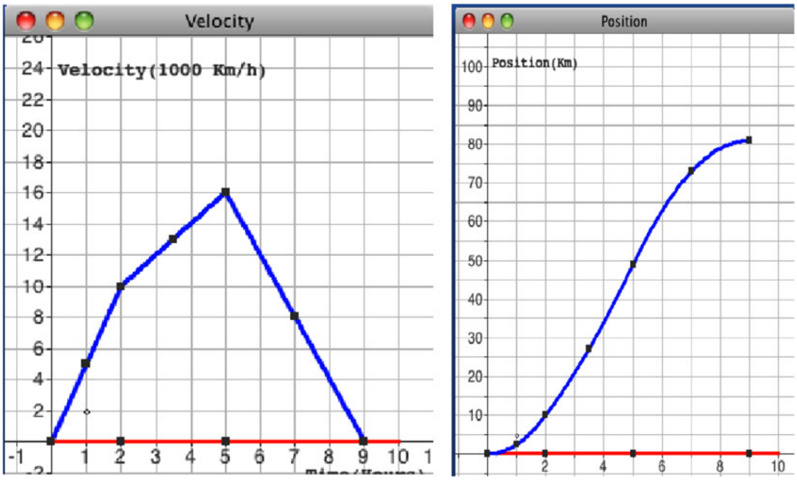

Fig. 5 A refined model for NASA rocket competition

were also important with regard to "*students' modeling awareness*." Following the guidelines of the *Model Generalization Principle* (see earlier in this chapter), an appropriate model should be appropriate not only for a specific problem, but rather for a large number of structurally similar problems (for instance elementary kinematics problems). Consequently, students refined their last model by discussing how the rocket's graphs would change if the rocket stayed in earth's atmosphere. Although they expressed their willingness to explore this new dimension, they failed to do so, probably due to their limited understanding on negative numbers (in the case rocket would change direction and return to earth).

5 Discussion

Based on the premise that young learners are capable of dealing with complex problems and contemporary technological tools, in this chapter, I have emphasized the need to design and implement interdisciplinary problem solving activities that draw on the domain of engineering. I have argued that the inclusion of engineering-based problems within the elementary school mathematics and science curricula can engage students in creative and innovative real-world problem solving and can increase their awareness of the role of mathematics, science, engineering, and technology in their environment. These activities place greater recognition on students' learning capabilities, and show young students are increasingly complex learners who are capable of dealing with cognitively demanding tasks.

Students' results in the Rocket modeling activity, especially those presented in Models C and B are in line with findings from other researchers (see the contributions from Dickey-Kurdziolek and Tatar, this volume and Moreno-Armella and

Hegedus, this volume). Modeling explorations in the environment of SimCalc enable students to use the interplay between the communicational and representational affordances to engage in representational expressivity, which enables students to better express their models and solutions. A significant finding of the present study is the role cognitive tools, like the SimCalc, might play in students' model development and problem solving. Computer-based learning environments for mathematical modeling, at the school level, are a seductive notion in mathematics education. Based on the findings presented, which are in line with findings from previous studies (English and Mousoulides, 2011; Mousoulides et al., 2007), we can claim that the capabilities of dynamic tools, such as SimCalc, can positively influence students' explorations and understandings in complex problem solving. Results showed that students, even at the age of nine, were able to successfully work with a complex mathematical modeling activity on Rocketry, when appropriate technological tools and representations were available. When working on the problems presented in this chapter, students progressed through a number of modeling cycles, from focusing on subsets of information through to applying mathematical operations in dealing with the data sets, and finally, identifying some trends and relationships. In doing so, students successfully employed the animation world, the velocity and position graphs in the environment of the software and succeeded in making connections between the real world and the world of mathematical modeling.

An interesting aspect of this study lies in the students' engagement in self evaluation, through the use of software's features and tools: at least four groups were constantly questioning the validity of their solutions and wondering about the representativeness of their models (see Self-Assessment Principle earlier in this chapter). This helped them progress from focusing on partial data to generalizing their solutions and identifying trends and relationships to create better models. Two groups progressed to even more advance models, by displaying surprising sophistication in their mathematical thinking. The students' developments took place in the absence of any formal instruction and without any direct input from the classroom teacher.

In quite complex problems, like the one presented here, students cannot simply follow a predefined procedure or strategy. On the contrary, they have to be creative, explore alternatives, reflect on their own developments, and take advantage of available tools and resources. This "*reflective and more flexible*" thinking was a distinct characteristic of the last group that was presented in Model C.

There are some more interesting insights into young learners' abilities and thinking, insights that are needed to guide classroom instruction and assessment. Although the results of this study cannot be generalized, it can be claimed that before any formal instruction on the laws of motion and algebra, students at the early elementary school can hold and successfully employ complex problem solving skills and intuitive thinking. The results provide some evidence that young learners, when presented with appropriate technology-based modeling tasks, have the ability to do and understand scientific inquiry and explore the position and motion of objects, understand complex patterns and functions, connect different representations, organize, display and examine data, and make judgments on decision-making problems.

The results from the modeling activity presented here revealed that SimCalc provides unique opportunities for students to deal with complex and conceptually difficult mathematical ideas in developmentally appropriate ways, while solving complex modeling problems. However, only a limited number of the software's capabilities were exploited here. Next steps should focus on the software's particular features of connectedness, which appeared to be important mediators to scaffold and leverage the effectiveness of groups to construct powerful mathematical learning and communication in solving complex problems (Ares, this volume; Hegedus and Penuel, 2008).

Clearly, more research is needed to examine the extent to which technology-based modeling activities, like the "*CAN.Be.Gr8 Planet,*" influence the development of abilities and thinking of elementary school students in complex problem solving and to identify the critical steps in students' development of these competencies. Such research would result in a more pervasive description of students' problem solving thinking and could be even more useful in informing instruction in elementary school.

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Approaching Calculus with SimCalc: Linking Derivative and Antiderivative

Patricia Salinas

1 Introduction

When browsing through a standard calculus textbook, it is easy to get an idea about what is involved in teaching calculus. As a matter of fact, a logical order that establishes derivative before integral can be noticed. These notions remain apart until about halfway through the book, when their deep connection is stated with the Fundamental Theorem of Calculus.

The usual treatment given to these notions relegates the Fundamental Theorem to a strictly technical role: calculating antiderivatives in order to evaluate a definite integral. It seems that the connection between these notions serves the purpose of an algorithmic process necessary for integral calculus.

College students in Mexico receive a Differential Calculus course and the next semester, Integral Calculus. When the derivative is introduced, students receive the classic geometrical representation with a curve and its tangent line at a specified point. In turn, when the integral is presented, they receive the classic geometrical representation of the area under a curve between two given points. The geometric interpretation of finding the tangent line of the graph of a function and computing the area under a function are the most common ways to give meaning to the fundamental notions of derivative and integral, respectively. Figure 1 shows common images associated with the introduction of these notions.

Once our students get to know the Fundamental Theorem, they know how to operate the algebraic representation of derivative. They have learned how to calculate, by heart, the derivative of a variety of functions. Applying the Fundamental Theorem of Calculus to an algebraic representation of a function means that once the derivative of the function is calculated, if an antiderivative is taken, we obtain the original algebraic representation of the function (for a specified initial value). Also,

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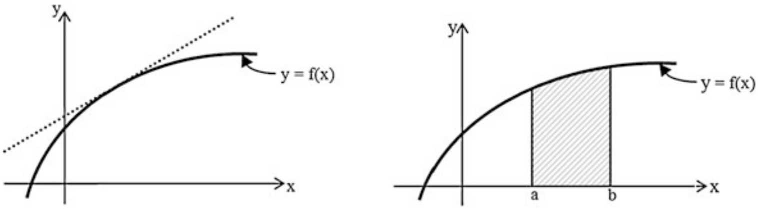


Fig. 1 Geometrical representations for derivative and integral

taking an antiderivative of a function and then obtaining its derivative returns it to the original function. It is easy to deal with this relationship confined to the algebraic representation of the functions, but in terms of the geometric representations in Fig. 1, the relationship is not clear.

Recently in Mexico, the high school level has followed a trend to increase its academic period to three years following the 9 years of primary and secondary education. It is noticeable that calculus courses are introduced at this level in an algebraic-accentuated perspective—one that accepts the standard order of first Differential Calculus and then Integral Calculus. Also, incorporating calculus into the high school curriculum using American textbooks is a custom that conditions the way calculus is conceived by teachers and students.

On the other hand, our educational institution in northern Mexico has undertaken an academic process of redesigning the courses at university level, promoting the use of technology and addressing other issues. This has been an opportunity to analyze the problem of having courses of differential and integral calculus repeated in both high school and university levels. We seek to do this by taking a deeper look into the problem of learning calculus. Salinas and Alanís (2009) and Salinas et al. (2010) provide elements for assessing the research and the curriculum design that has been conducted to respond to this situation. They bring an approach to calculus where derivative and integral appear simultaneously from the beginning of the first course. A wide research project has been supported by the institution with the intention to offer students meaning for calculus topics that are useful for applied courses they will further take in their major areas.

In this chapter, I share the way in which SimCalc has been appreciated as a mediator favoring this approach in the classroom. The Fundamental Theorem of Calculus is embedded in SimCalc and offers an environment where students have the experience of visualizing the graphical representation for derivative and antiderivative linked together. The image including the derivative and antiderivative graphical representations, suggests taking them simultaneously and making their relationship an object of study in the classroom.

2 Literature Review

Dealing with the learning of mathematics, Duval's framework (2006a, 2006b, 2008) highlights the influence of the different semiotic representations when analyzing cognitive processes. Here we are concerned with numeric, algebraic, graphical, and linguistic representations. His contribution is useful in trying to deepen our intentions to provide students with an appropriate avenue for comprehension. Today, when dealing with didactical design, we must take into account that we have several systems of semiotic representations that must be coordinated during mathematical activity.

Duval sets two types of transformations of semiotic representations: *treatment* and *conversion*. The first one refers to the changes made within the same representation, while the latter refers to the ability to change a representation—including the transformation of linguistic statements. For the learning of mathematics, both processes—treatment and conversion—are independent sources of cognitive problems. Apparently, changing representations (conversion) is more complex than making changes within the same kind of representation (treatment). Problems reported on students' difficulties with this process give the impression that it constitutes a cognitive leap, not governed by rules and basic associations, and, also, is not subject to coding. In classrooms, the conversion process often appears as a trick that cannot be well learned and that is not taught (Duval 2002, 2006a, 2006b, 2008).

Duval has broadly identified different levels of cognitive processes when dealing with mathematics representations. His analyses reveal that it is not possible to expect that by showing the numeric, algebraic, graphical and linguistic representations of a notion together, the problems with its learning will disappear. Students should be able to discern significant elements not only in the representation where the thinking process starts, but also in the representation you want them to reach. "This condition is particularly strong when cognitive representations are linguistic or visual, and not purely symbolic" (Duval, 2008, p. 11). This is exactly the situation we want to address through the use of SimCalc—linguistic and visual representations. We suggest that confronting students with the graphical semiotic representation (for derivative and antiderivative) together with the simulation will offer a better opportunity to place the linguistic description of the graphs and the movement.

The way we understand that SimCalc affects the calculus classroom is influenced by the theoretical perspective of Moreno-Armella and Hegedus (2009)—illustrated by means of an effective integration that transforms communication. They argue that "with the new advances in design of dynamic media, the accessibility of mathematical ideas and the nature of symbolization are transformed with the creation of *new* forms of symbol-mediated experience" (p. 509). The characteristic they refer to as *executability* of representation transforms the kind of interaction a student can have with the embedded mathematics.

SimCalc is a continuous dynamic media offering *co-action;* which means that the student, through the *hotspots,* guides an action with an *intentional attitude* upon the environment, from which the student is being guided. This is possible because the interaction that SimCalc allows becomes a sustainable bi-directional process where it is possible to identify the relationship stated by the Fundamental Theorem (Hegedus and Moreno-Armella, 2010).

We suggest that SimCalc in the classroom, through its visual scenario—including the link between derivative, antiderivative, and the simulation they represent—will give rise to an interaction that encourages an intentional handling in order to acknowledge a relationship between these three elements that could be stated in a general way.

Generalization and *symbolization* are central cognitive acts of mathematical reasoning. Moreno-Armella et al. (2008) state that producing a generalization is to refer to all multiple instances by means of a unifying expression, as if they all were just one thing. But the sort of symbolic structure this expression requires makes us realize that symbolization serves generalization, now part of the field of reference of the symbol. We suggest that by using SimCalc, a process of generalization can be triggered.

By considering educational goals, we must be certain that the use of computers can transform mathematical knowledge. Noss and Hoyles (2004) have been looking for new theoretical and methodological tools to cast light on the learning process related to technology integration. By means of a *situated abstraction,* they refer to a way by which a community of students can develop a common discourse and coincide with their teacher in talking about the same mathematical abstractions, thus making the students' expressions gain some mathematical legitimacy. SimCalc is a truly transformative example of software "in terms of their potential for changing not simply how mathematics is learned, but what mathematics *can* be learned" (Noss et al., 2009, p. 493).

Exploration with computational tools allows students to reorganize strategies for problem solving. First, they can make some situated observations that may relate to a certain property, theorem or formula where the environment facilitates its identification. This constitutes a *situated proof*, the result of a systematic exploration purposely exploited inside a computational environment in order to "prove" mathematical relationships. Throughout the historical development of mathematics, some kind of natural swing between *inductive* and *deductive* approaches is recognized, so, proof does not affect a theorem that was conceived in the past without the modern standards of rigor. Maybe looking at mathematical results emerging from human activity could bring more appropriate elements to the classroom's cultural environment. "The theorem is the embodied idea: the proof reflects the level of understanding of successive generations of mathematicians" (Moreno-Armella and Sriraman, 2005, p. 133).

By means of a situated proof, we suggest that SimCalc may function as the mediator for the establishment of generalizations about the relationship between derivative and antiderivative; those generalizations refer to differential calculus theorems that relate the sign and behavior of derivative and the behavior of the function graph (antiderivative).

In order to understand the nature of the *mediation* role of computing tools in learning, Moreno-Armella and Sriraman (2010) distinguish *tool* from *instrument* using the established idea that any cognitive activity is a mediated activity. The importance of the way a tool can mediate the cognitive processes of a user by transforming cognition becomes clear, because *intentional* actions are performed giving rise to new dialectical interactions between the user and the tool. When this is so, we are dealing with a computer as an instrument; one that could affect the cognition of the user by allowing the reorganization of ideas, like written language does.

"Computers had made feasible a new way of looking at symbols, looking *through* them, and transforming the resources of mathematical cognition" (p. 215).

We are clear that in the classroom, the intentional actions that students may experience are subject to the conditions they are faced with each time. Certainly, we can expect that the use of SimCalc promotes the students' interaction with the software, where the mathematical content is embedded, but the teacher's research attitude becomes a relevant element of this social environment in order to identify the signs arising and stating that some learning is taking place. We suggest that integrating SimCalc through a didactical sequence according to the approach to calculus we are using in our institution in Mexico, the emergence of signs will allow a dialogic discourse, one that embodies the assumption that there is more than one perspective and it is worthwhile to *think together* and discuss—"Knowledge is most fully achieved in the dialogue between people who are together trying to solve a problem, construct an explanation, or decide on a course of action" (Wells, 2007, p. 264).

At the convergence of representational and communication infrastructures, Hegedus and Moreno-Armella (2009) place a kind of transformation of expression that they prefer to call representational expressivity. On one hand, computational and visual affordances allow mathematically valid and viable connections to interact with the learner. On the other hand, communication is shown by means of human actions in terms of speech or physical movement (gestures) and also with digital inscriptions through the interface. In this way, learners can express themselves with intentionality, showing an intrinsic motivation to participate in the classroom.

This expression, shared from a local to the public place of the classroom, brings the opportunity for students to consolidate identities and defend and reason their work. Some informal mathematics registers, in terms of speech and actions, encourage students to identify themselves with the mathematical attributes of the object through the embodiment of the mathematical idea as a personal expression (Hegedus and Penuel, 2008).

In this chapter, I will share our experience utilizing SimCalc in a Mexican classroom paying particular attention to the semiotic means to which students make sense of the relationship between the graphs of the derivative and antiderivative. Hopefully this work outlines the importance of understanding the role of discourse in the classroom and of the reasoning expressed by signs and body actions.

3 Incorporation of SimCalc into the Classroom

In this section, I will share some basic elements that clarify the way our approach privileges the meaning of notions and procedures in order to solve problems related to change and variation by addressing the prediction of the values of a magnitude. Then, I will describe a sequence of activities that integrate SimCalc in the classroom with the visual scenario for the prediction of the position of an object moving over a straight line.

Supported by systematic observations and analyses according to a qualitative approach to these educational events, I describe the actions we have taken for the development of the sequence. I describe the dynamic symbolization process taking place in the classroom between the students and the teacher in order to support the learning of differential calculus theorems. We use this sequence of activities in the regular first mathematics course in our institution in Mexico, following the approach with the textbook *Applied Calculus* (Salinas et al., 2011).

3.1 Some Basic Elements

The approach to calculus mentioned considers prediction as a thread around which calculus is structured and identified as the mathematics of change. We offer students plenty of situations in which it is important to predict, however, didactically, the position of an object moving in a straight line is the suitable scenario to support cognitive processes. In this way, the velocity of an object will be similarly addressed as the idea of the rate of change and, in turn, the position of the object corresponds to cumulative change. Through SimCalc, we can graphically generate this scenario and surround it by the transversal educational goal of symbolizing and generalizing. The position graph is the "metaphor" of a magnitude M that is changing with respect to another, firstly rooted in time, but eventually becoming any other magnitude on which M depends.

It is important to state some basic elements about the way we address the situation. For a specific magnitude $M(t)$, we simultaneously consider its rate of change $r(t)$ and its accumulation of changes $\sum \Delta M$ over a given interval of time $[a, b]$. This offers a new way of dealing with the Fundamental Theorem of Calculus because it is implied as the simple idea of thinking that the change of the magnitude can be approximated by $\Delta M = M(t + \Delta t) - M(t) \approx r(t)\Delta t$, which assumes that the rate of change, $r(t)$, stays constant on the small interval, $[t, t + \Delta t]$. This idea—possible because of the continuity property for the derivative—supports the idea that calculus is concerned with the problems of prediction.

With this approach, we carry the numerical approximation process of adding the different changes the magnitude accumulates over the small intervals where interval $[a, b]$ has been divided. Then, we take to the "extreme" the sums through a spreadsheet—using the benefits of the technological media—making the intervals of variation for time smaller and thinking in a "limit" process whose outcome is a numeric value for the change in magnitude.

By means of an induction process, carried out through the spreadsheet, the interrelationship between the natural power function $f(x) = x^n$ and its derivative $f'(x) = nx^{n-1}$ is associated algebraically. This process is kept on the algebraic semiotic representation, and its performance is algorithmic in such a way that students can obtain the derivative and the antiderivative of any polynomial function. Once the numeric approach brings the answer algebraically to the prediction question for these functions, we propose the introduction of the graphical semiotic representation to visualize the kind of behavior the magnitude shows and the relation it has with its rate of change (Salinas et al., 2011).

3.2 SimCalc and Uniform Rectilinear Motion

Using SimCalc, we introduce the scenario of motion to the classroom in our institution in Mexico. The character, "Ryan," moves with a constant velocity. The SimCalc document reflects a design where the arrangement of graphs and numeric values are considered to manage the situation the best way possible when changing the conditions for the motion. Simultaneous graphs of velocity and position are handled in exactly that order—so the teacher promotes the presence of velocity immersed in the position graph. When working with this document, the teacher emphasizes numerically that the slope of the position graph is related to the constant velocity (see Fig. 2).

Looking at the velocity graph, one can see the "height" at any value of time, thus, indicating the constant velocity value. Meanwhile, since velocity is the rate of change of position over change in time, looking at the position graph, we can sketch a triangle where the numeric operation of dividing Δx over Δt provides the constant velocity value. Both visual perceptions (height on velocity graph and triangle on position graph) should be carried out on the same time value for both graphs, even though this feature stands for the complete image.

At this point, the teacher asks students to interact with the SimCalc document through the hotspot on the velocity graph, and to dynamically visualize different motions of constant velocity. A collective production of signs emerges in our classroom indicating that the numeric data for velocity relates to the character motion to the left or to the right. Students begin to use gestures to show what do with the velocity graph. Using the teacher's computer, which is projected in front of the room, a student guides the hotspots in order to reach the challenges proposed by the activity.

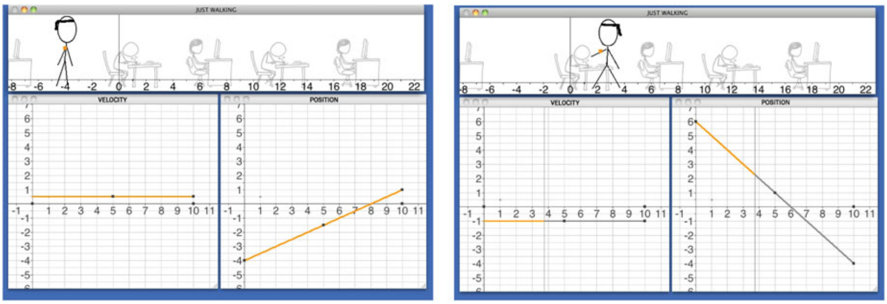


Fig. 2 Pictures that evoke motion with constant velocity

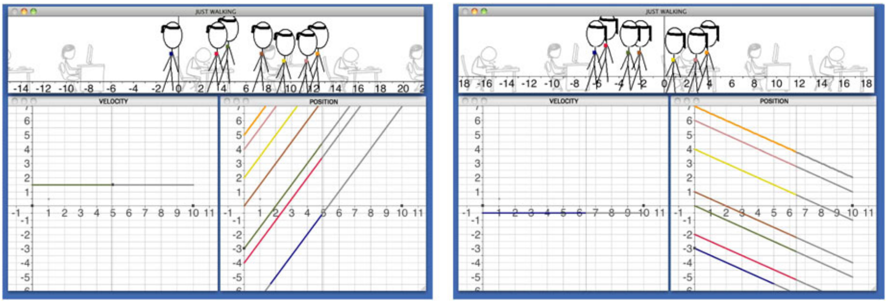


Fig. 3 One velocity graph corresponding to several position graphs with a different initial position value

At this time, we see the visual expression that Fig. 2 reveals through symbolization—every time the teacher asks for a motion to the right, the velocity should be positive, and a motion to the left indicates that the velocity should be negative. Upon looking at the position graph, the teacher identifies an increasing and a decreasing graph related to the positive or negative sign of constant velocity. Before ending the activity, the teacher gives the algebraic representation for position function and velocity function: $v(t) = v_0$ and $x(t) = x_0 + v_0 t$. Figure 3 emphasizes the identification done during a discussion about a velocity and a position graph where one velocity graph corresponds to several position graphs with different initial positions—creating a "marching" performance.

3.3 SimCalc and a Rectilinear Motion by Intervals

Next, we give students the same SimCalc document but allow the velocity graph to be editable. This gives students the opportunity to create a motion where velocity remains constant for a short period of time, and then changes in the next period of time, and so on. In the classroom, students work in pairs with one laptop. Students are asked to create four motions: (1) the actor, Ryan, must go to the right, but progressively faster, (2) Ryan must go to the left, also progressively faster, (3) Ryan must move to the right, but now progressively slower, and (4) Ryan must move to the left, also progressively slower. Each pair of students produces one motion, which is then collected in the teacher's computer and shared with the class. Even though the graphs do not look the same, students are able to identify commonalities in each of the motions and identify the linguistic description for the motion interpreting the features of the velocity graph. Figure 4 provides an example of the work collected from a group of students.

As can be seen in Fig. 4, the position graph consists of line segments—joined one by one—creating the visual perception of a curve. It is here that the concavity property of a curve begins to inform students about the features of the motion. Here

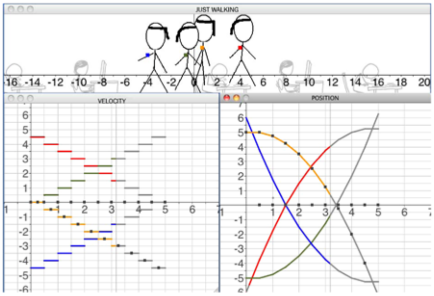


Fig. 4 Image with the collection of the four kinds of motion

again, the dialogic discourse we observed offered evidence that students began to make sense of the relationship between the graphs of velocity and position.

This same SimCalc document can potentially show students how it is possible to conceive a linear velocity as the "extreme" case of uniform motions by intervals. The teacher edits the document reducing the time intervals, and the co-action, provided by SimCalc reflected on the position graph, gives the visual perception of a linear velocity and a corresponding position graph—where the slopes of the line segments allow us to conceive the shape of a curve (see Fig. 5).

3.4 SimCalc and Motion with Linear Velocity

Once students support the idea that velocity changes uniformly through its graphical representation, we offer them the next SimCalc activity, *Ryan's Stories*. Again, students work in pairs in a new document designed so that they can edit and add constant or linear velocity segments. In this activity, students are asked to create the motion simulation based upon 15 stories, which summarize the motions done before and include others as well. Students have access to the text document containing the linguistic description of each of these stories. After the students create each motion, they must take a screenshot of the SimCalc's image and edit the text document to add the image corresponding to the motion's description. The features of velocity and position graphs should be associated with their own words for each of the stories.

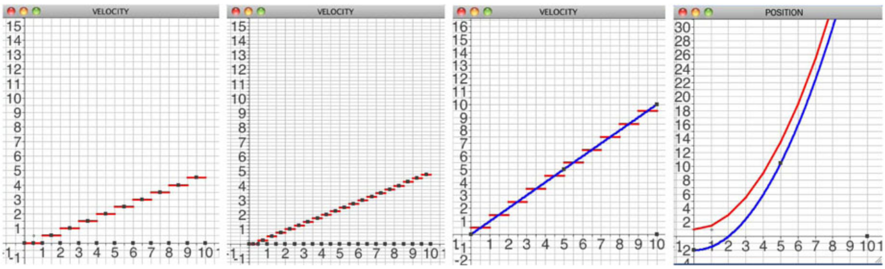


Fig. 5 Conceiving a linear velocity through uniform motions by intervals

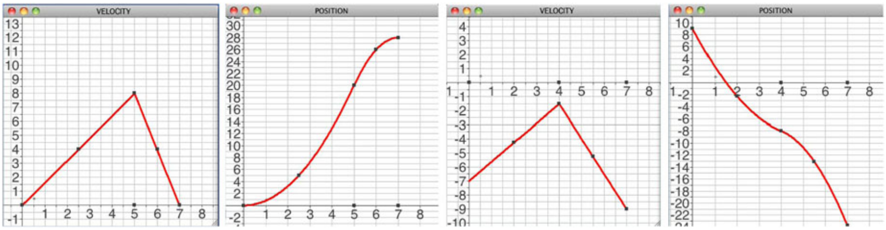


Fig. 6 Ryan moves according to different given descriptions where motion stays always to the right, or to the left, but not with the same features

During the class, while students work with SimCalc, the teacher encourages them to notice these features. It is worth noting that it is not easy for students to look at a graph and think about its positive values, or the increasing behavior it has even when it stays in the negative zone of the vertical axis. Issues like that relate to the cognitive process of conversion that should be considered in order to strengthen the experience through SimCalc. That is why the activity requires the teacher's assistance to guide the teams' reflections. For homework, students are asked to report whether they are able to find some kind of pattern in their complete document. For example, looking at the images where the velocity graph is positive, is there something in common with the corresponding position graphs? And if the feature we wish to address is the increasing behavior of velocity, do position graphs have something in common? Figure 6 shows two screenshots where the position graph is affected by the location of the velocity graph. The descriptions ("story") of the motions in Fig. 6 are: Ryan moves to the right progressively faster, and then continues to the right progressively slower, and Ryan moves to the left progressively slower, and without stopping, decides to continue to the left progressively faster.

During the next class, the teacher discusses students' work in such a way as to achieve the establishment of mathematical theorems situated in the motion scenario provided by SimCalc. The relationship between positive/negative signs of velocity and increasing/decreasing behavior of the position graph is once again proven, and the increasing/decreasing behavior of the velocity graph is related to the concave upward/downward behavior of the position graph. Even with the limitations of having combinations of linear velocity segments, it is possible to associate the change of concavity on the position graph to the existence of a maximum or minimum value for the velocity graph.

Also, the stories that the students perform in SimCalc include the situations where the maximum or the minimum value on the position graph corresponds to the velocity graph crossing the time axis from positive to negative values or from negative to positive values, respectively. Figure 7 visually states a sign that has been identified on students' writings, where they evoke the situation for the maximum and minimum value of a magnitude in terms of its rate of change. The results relate to theorems in the chapter of "Applications of the Derivative" in any standard textbook. We refer to the test for relative extrema (maximum and minimum) and test for concavity to obtain the inflection points of a given function's graph.

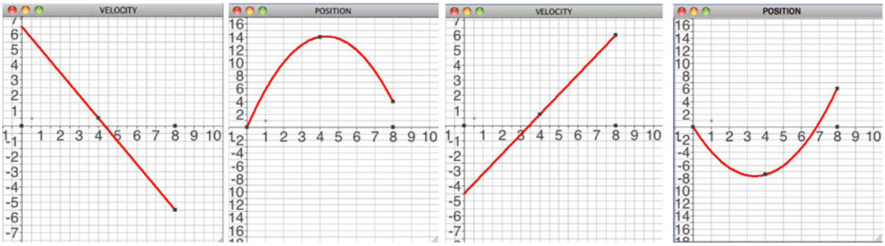


Fig. 7 Ryan moves according to different descriptions; he moves to the right (or left) and decides to return to the left (or right)

3.5 Toward the Establishment of Symbolization

The previous experience with SimCalc prepares students for activities that can be performed without the software. An example of such an activity is: Two different motions over a horizontal axis are performed by a particle, upon which the velocity (meters/second) is changing with respect to time, according to the graphs. At the start, ($t = 0$), the particle's position was 1 meter (see Fig. 8).

On each coordinate system (see Fig. 9), students are asked to draw the corresponding graph of position function $x = x(t)$ in such a way that one can describe from it the motion behavior performed by the particle that is modeled through that function.

It is possible to qualitatively recognize the graphical behavior that the position function should have, but it is also possible to obtain its algebraic representation because of the process of antiderivation—which was established previously in this approach. We are proposing to bring together both of the graphs—velocity and

Fig. 8 Particle's motion with an initial position of 1 meter

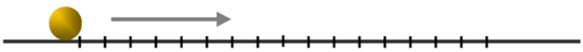
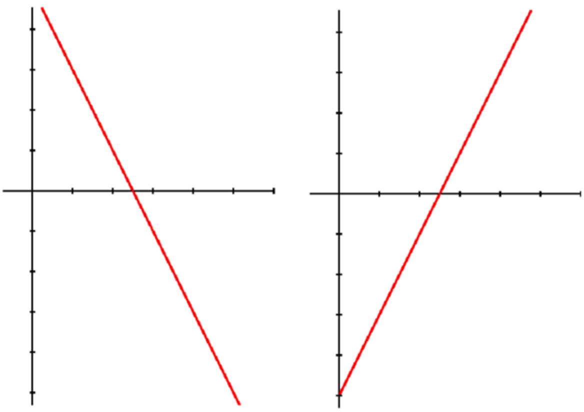


Fig. 9 Velocity graphs representing different motions



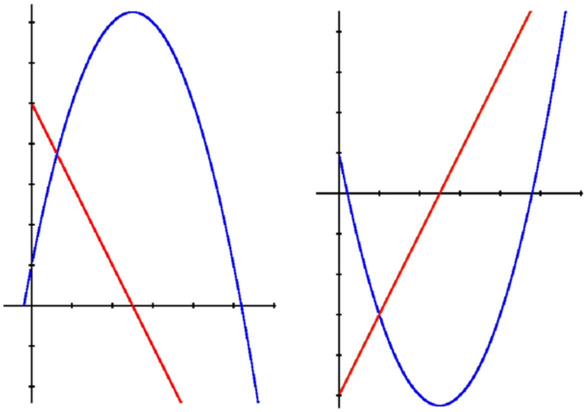


Fig. 10 Velocity and position graphs together, symbolizing the relationship between velocity and position

position—in order to state the relationship between the two graphs visually and associate numeric values in the same coordinate system.

The solution to this problem can be seen in Fig. 10. Students argue that position graph for the first velocity goes “up and down” and the second “the opposite.” Upon asking for reasons, the students describe that the motion is to the right and then to the left. At this time, the teacher challenges these informal descriptions and encourages the students to go further and try to include the way the motion is done: how is it done, progressively faster or slower? Finally, the algebraic representation for position is obtained as the antiderivative for the velocity function, which is a straight line with initial value 1 and crosses the horizontal axes at time 2.5. The concavity data can be identified through the velocity graph; increasing graph for velocity reveals position graph is concave upward; and decreasing velocity graph reveals concave downward position graph.

We believe that students’ cognitive processes are strengthened through this type of activity where the images could be regarded as signs that bring some kind of symbolization in the service of the generalization that we are asking to take place. Figure 10 is a referent for the theorems mentioned above. Here, SimCalc has been used to create a different kind of symbolization.

3.6 Generalization

By splitting these images (look at scissors in Fig. 11), we can identify four typical behaviors for the magnitude—represented by the x variable.

In the four final images in Fig. 11, the axes were introduced to promote the identification of four key behaviors where position (function) and velocity (derivative) are interrelated. This visual insight of four disjoint images that were jointed before, symbolizes the results about the qualitative behavior of a magnitude.

At this stage, we should refer to magnitude as it relates to position and the velocity as it relates to its rate of change; or else, position as the function and velocity as the derivative.

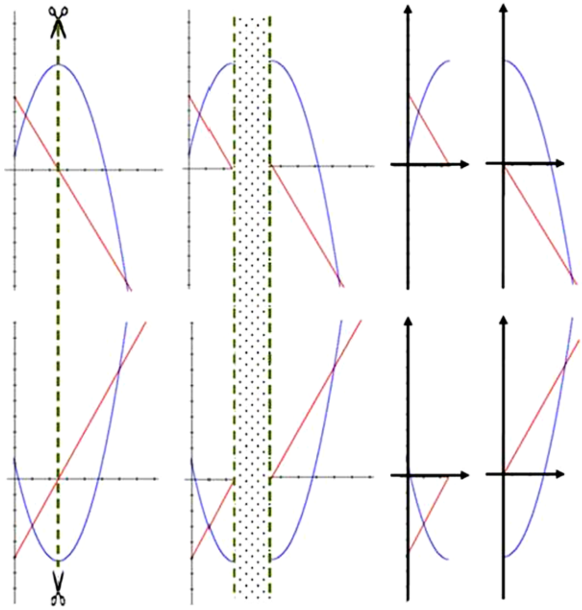


Fig. 11 Obtaining four final images with position and velocity graphs in the same coordinate system

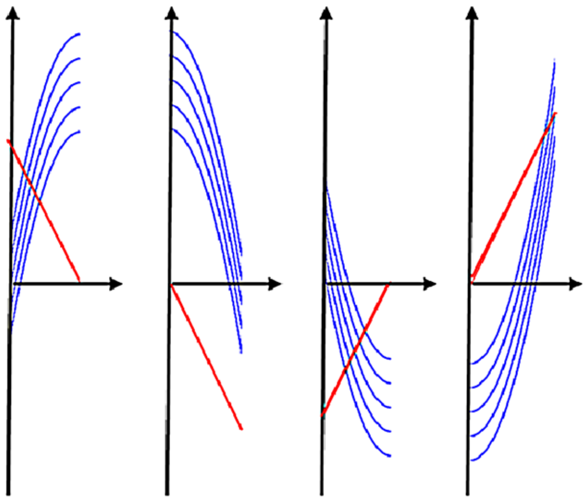


Fig. 12 The four different behaviors of a magnitude associated with different linguistic representation and with different behavior for derivative and function

Figure 12 shows the generalization that becomes a visual referent for the relationship between a magnitude and its rate of change.

Here we have the following interpretations: magnitude increasing progressively slower, magnitude decreasing progressively faster, magnitude decreasing progressively slower, and magnitude increasing progressively faster. And we can visually prove four results: (1) an increasing graph representing the function corresponds with a positive graph of the derivative function, (2) a decreasing graph representing the function corresponds with a negative graph of the derivative function, (3) a concave upward graph representing the function corresponds with an increasing graph for the derivative function, and (4) a concave downward graph representing the function corresponds with a decreasing graph for the derivative function.

4 Reflections About Dialogue in the Classroom

Every time we used SimCalc in the classroom, we paid particular attention to the semiotic means to which the students resorted in order to make sense of the graphs' behavior. The activities we have been discussing concentrate on graphical interpretation, namely, the linguistic representation of the magnitude behavior. In fact, we are concerned with the identification of four graphical shapes that allow students to represent different kinds of behavior that a magnitude might have.

We have been systematically observing how students perform a drawing in the air with their finger to represent the graphical shape for the behavior of a magnitude that increases progressively slower, for example; or how they bend their hand to represent the behavior of a magnitude that decreases progressively faster. Students in Fig. 13 show this kind of gesture, drawing with their finger or bending their hand when they have the graph of the rate of change in front and they are asked to express the magnitude's behavior.

Once we identified that these gestures emerge naturally, we promoted their use in general in the classroom to help students internalize the action of relating the behaviors between the derivative and function. Every time a student used it, we asked him/her to repeat that sign for the rest of the class. Figure 13 shows two events of gesture. Note the attentiveness of the students to the sign performance.

We consider that this body gesture synthesizes the cognitive process completed when asking for the graphical representation of a magnitude whose rate of change is represented by its graphical representation. This kind of body language has been a resource that students use when trying to explain what happens under different conditions—the four conditions provided in Fig. 12. They get used to symbolizing that a magnitude increases (decreases) progressively faster (slower) with their hands.

Figure 14 presents the image that students were shown in class when they performed the gestures in Fig. 13. The gestures were performed to visualize the graph

Fig. 13 Signs and gestures shown by students, drawing in the air and bending their hand

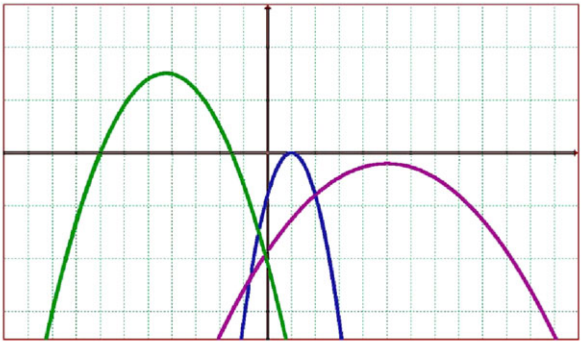


Fig. 14 Picture presented in class with three derivative graphs to think on the corresponding antiderivatives in a qualitatively way

that is obtained as the antiderivative of each of these graphs that represent the derivative of a function.

We use Fig. 14 as a group activity. The objective is to make a linguistic description of the differences in the behavior of the functions whose derivatives are given. The experience with this activity enables us to conjecture that favorable conditions for an active process of "knowing together" could be created in the classroom. Once a solution is expressed with gestures, we invite students to deepen their visual perceptions and address the differences between the first graph and the other two graphs, as well as what makes the second and third graphs different when experiencing the visualization of the three antiderivatives. One can notice a special location for the inflection point in the function with respect to the derivative graph.

Finally, we give the students a printout of the image and we ask them to produce their drawings individually. While they do this, the uncertainties due to the lack of the algebraic representation for the derivatives are seen, so we give the students the quadratic functions and the numeric value they have to assign to 0. As homework, students must perform the algebraic treatment necessary in order to bring to class the graphs with all the numeric information marked in a complete image.

5 Thinking with SimCalc

Executable representations serve to externalize cognitive processes done beforehand by people who did not have the technological device. That happens every time we graph a function with a computer, however, the challenge for the learning of mathematics lies in using that executability to generate a way to transform graphs into a resource for thinking. This is a matter regarding the features of the software and for the teacher, who must design a way to interact with the software for the students.

By means of a situated abstraction and through SimCalc, we try to develop the students' establishment of general propositions in terms of the environmental language where they experience the mathematical exploration. In doing so, we designed a SimCalc document to meet our intentions.

The existence of "hot-links" between the velocity, position, and simulations become key elements of SimCalc's features, which in addition to editable piecewise-definable graphs of functions, allows the creation of a visual scenario inviting the learner to become a part of the performance. We encourage the idea of using computer software as an instrument that works for students to confront them with their own cognition, transforming the way they interpret mathematical knowledge. SimCalc offers a valuable semiotic potential in the construction of knowledge.

The picture proposed here, with the movement simulation and graphs of velocity and position (in that order), helped us avoid conflicts of conversion in terms of Duval's framework. Both graphs must be interpreted in the same kind of graphical representation, where velocity and position are visualized as the "height" of the vertical line arising from the horizontal axis.

The teacher's attitude is essential to foster the evolution of signs that are rooted in the activity performed with SimCalc. It is not easy for all students to generate their own individual production of signs, but the teacher can take advantage of students expressing, out loud, their thoughts.

SimCalc within our classroom has allowed the incorporation of a social activity. With the students, we make use of the system of signs and the semiotic processes that contribute to the creation of an environment for socialization where we engage in a discussion full of meanings and gestures. In this way, students become conversant with calculus and make sense of its notions and procedures.

Having a motion scenario to deal with these ideas, brings the opportunity to transfer the meaning to other scenarios. If the prediction situation deals with a magnitude (a function) with a known rate of change (the derivative of the function), then the association of magnitude with "position" and its rate of change with "velocity" allows the interpretation of the magnitude's behavior. As a result, theorems corresponding to the Applications of Derivatives, the final chapter in a Differential Calculus course, can now be situated at the beginning of the discourse on calculus—giving the graphical representation the power to interpret behaviors.

Allowing SimCalc in the classroom provides an opportunity to transform the learning of calculus, which has been isolating its fundamental notions. This is the opportunity to deal with the Fundamental Theorem of Calculus from the beginning.

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Part VI
Extensions, Commentaries, & Future Visions

You Can Lead a Horse to Water…: Issues in Deepening Learning Through Deepening Teaching

John Mason

1 Introduction

At issue is the elaboration of ways of working with learners so as to maximize the opportunities learners have for engaging substantively with the topics being taught. How to teach well is not an engineering problem about promulgating specific practices or actions. Human beings do not take to social engineering. Rather, how to teach well is a human problem of promoting and supporting flexibility of response to situations as they develop. This involves an interplay between the:

- cognitive components (appropriate challenge; pertinent mathematical constructs; multiplicity of modes of thinking; connections to both the familiar and the as yet unfamiliar; encounters with pervasive mathematical themes, common heuristics and use of human powers for mathematical purposes);
- behavioral components (internalized actions being called upon; new actions being encountered; suitable language introduced and called upon); and
- affective components (personal dispositions, interests and engagement; origins of the problems the topic addresses; places where the topic, constructs and techniques have proved fruitful);

bearing in mind as well that affect provides the energy or impulse to encounter and engage with the cognitive, through appropriate behavior. At the heart of these actions lies attention: its focus and its form or structure. Jessica Bishop (this volume) asks, "What happens… if… responsibilities are shifted and *students* are expected to explain, elaborate, and evaluate for themselves?" (p. 235).

This chapter is one attempt to address that question, building on insights from Bishop's chapter and others, and data from some of those chapters.

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2 Metaphors

Lakoff and Johnson (1980) continued the enterprise begun by Jakobson (1951) to revivify sophisticated Greek grammatical insights concerning the role of metaphor and metonymy in human actions and interactions. Metonymies usually act below the surface of consciousness, triggering affective associations, which direct the flow of interaction between people and the flow of energies within them. Metaphors bring to mind thoughts and actions through structural resonance with past experience. Some metaphors are "frozen" in that the metaphoric content has either evaporated or is overlooked; others lie at the core of "commonplaces" (St. Maurice, 1991), which evoke resonance and unconsidered agreement among audiences. A prime example is "deepening learning." This phrase acts as a commonplace, since it evokes positive agreement and acceptance as a "good thing." It also invokes a spatial metaphor associated with substance and essence, in contrast to "broadening learning," which might unhelpfully be associated metonymically with "surface learning" (Marton et al., 1997). The gerund "deepening" evokes a process, as distinct from a qualification such as "deeper." It also contrasts with "higher learning" or "heightening learning," which might be associated with more abstract, less practical learning. Knowing a great deal about very little is usually contrasted with knowing a little about a great deal. Our aim as educators is surely to acknowledge that, "every stick has two ends"; that grasping one end or the other is at best unwieldy, while seeking a balance between breadth and depth affords access in both directions. This is an instance of Aristotle's "golden mean."

Being aware of frozen metaphors affords access to questioning hidden assumptions: for example, classifying some students as having "only" the goal to "get through" the course, to "acquire" such tools and techniques as might be required for their particular future intended practices. Of course the power of the commonplace-provoked sentiment is that all students would benefit from "understanding more fully," for although behavior can be trained, it is awareness that can be educated, and it is awareness that enables checks and offers flexibility when novel situations or contexts are encountered (Mason and Johnston-Wilder, 2004a, 2004b).

The chapter title is intended to evoke the adage that "you can lead a horse to water, but you can't make it drink" in the form "you can challenge and disturb students, you can expound and explain to them, but you can't make them think, and you can't make them learn." You can, however, create conditions in which their natural, mathematical sense-making powers are (likely to be) invoked. Various constructs are used to probe the import of "create conditions," both for a system such as SimCalc and for teachers with or without that assistance.

3 Richness

A current commonplace in mathematics education aligned with *deepening learning* is the phrase *rich tasks*, evoking a contrast with "poor" or "impoverished" or perhaps "watered down" tasks. The sentiment is one of potential, of opportunity and affordances. However, tasks are simply tasks, usually either ink on paper or verbal instruction. What matters is what *activity* arises from undertaking a task (Christiansen and Walther, 1986). The nature of that activity will depend strongly on what students attend to, and also how they attend to it. . . which will be discussed more fully later. The gap between the author's intention, the teacher's presentation, and the task as interpreted by students has been studied in some detail (Gravemeijer, 1994; Stein et al., 2007); the issue of student engagement and motivation has been ubiquitous in educational writing for thousands of years. I suggest that it is not the task that is either "rich" or "impoverished," and not even the activity. Activity provides experience through participation in action. That experience may be dominated by desire to get the task done as quickly as possible or with the minimum of effort (which can have positive but also negative results); it may be dominated by "going through the motions," even to the extent of simply appearing to be engaged; and it may be dominated by engagement at some level of intensity and commitment, among other foci.

With unwitting resonances with Hegel's remarks on learning from history (Hegel, 1975), it seems evident that. "one thing we do not learn from experience is that we do not often learn from experience alone" (Mason, 1994, p. 179).

Connected with this is a succinct summary of an insight of Franz Brentano (1988) and similar ideas of Immanuel Kant (Dainton, 2010), that "a succession of experiences does not add up to an experience of that succession" (Davis and Mason, 1988, pp. 488), matching an aphorism of William James (1950) that "a succession of feelings does not add up to a feeling of succession" (p. 628).

Something more is required in order to "learn from experience(s)." Resonant with Schön (1983), reflection both *in* and *on* practice are—if not required—at least helpful to a majority of students. Put another way, the shift from acting on objects to becoming aware of acting, and then again to becoming aware of those actions as objects themselves, sometimes requires intervention. Various authors have addressed this issue, using constructs such as *reflective abstraction* (Piaget, 2001; see also Simon and Tzur, 2004), *procepts* (Gray and Tall, 1994), *reification* (Sfard 1991, 1994), and *reflexive situation* (Brousseau, 1997) among others. A major role for teachers is to prompt students to withdraw from action and to reflect upon that action: what was effective and what not so effective; what powers were used, why and how and what brought them to mind; what themes and other connections were encountered; what would be useful to have come-to-mind in the future in a similar situation; what constitutes a "similar situation"; etc. Chandra Orrill (this volume, p. 292) states that only one of the four teachers she observed asked questions which either initiated or even simplify indicated an explicitly reflective stance, so overt reflection cannot be assumed to be a part of every teachers' pedagogical practice. Furthermore, questions are often projected into the teacher-student-mathematics space in such quick succession that there is no room for interactions such as explaining by students to each other, exploring, expressing or even exercising.

To achieve learner immersion in mathematical themes and thinking, and to achieve the development of powers to enable both critical and creative mathematical thinking in novel situations as espoused for example by Pitta-Pantazi, Sophocleous, and Christou (this volume, p. 319) requires more than rich tasks, more than complex activity arising from engaging with and in tasks, more even than rich experience of such activity. It requires some form of reflective stance by the student so as to inform future practice (Mason and Johnston-Wilder, 2004a).

4 Interaction and Ways of Thinking

Jim Kaput designed SimCalc so as to maximize a range of two-way interactions: student-student, student-graphs, student-formulae, student-simulation, and hence for the student to coordinate graphs, formulae and situations. Such coordination is a form of modeling as elaborated on by Lesh, English, Sevis, and Riggs (this volume, p. 419). This leads me to ask what forms of interaction are possible in a mathematics classroom?

Elsewhere (Mason 1979, 2002) I have, following the theory of Systematics (Bennett, 1966), suggested six forms of interaction based on each of the teacher, student and content taking on one of the three roles necessary for an effective action (initiating, responding and reconciling-mediating). The six modes of interaction form a framework for considering possible modes of interaction, and are for convenience known as the six Ex's: Expounding, Explaining, Exploring, Examining, Expressing, Exercising.

Each of these interactions takes place within a milieu and a domain. The milieu includes institutional affordances and constraints, including classroom and institutional social norms and demands. The domain includes the focal world(s) or spaces of the participants. Usually this consists of the mental worlds in which people dwell and from which they express their insights, but the presence of virtual screen-worlds provides a more explicitly taken-as-shared world of experience, namely the world of phenomena acted out, and interacted with, on a screen (Mason, 2007).

The key feature for our considerations here is the mediating or reconciling role of one of the impulses or agents so as both to permit the other two to be in relationship, and to sustain them in relationship, leading to a result that can partake in further actions. For example, *expounding* is best when the presence of students (actual or virtual) brings the teacher into contact with the mathematics in a fresh and meaningful way, as the students are drawn into the world of experience of the teacher. The phenomenon of planning a lesson and being aware of a rich range of possibilities speaks to this. By contrast, *explaining* is used here to describe the action in which it is the mathematics that brings the student and teacher into contact as the teacher tries to enter the world of the student. The tendency for teachers to slip from *explaining* to *expounding* ("Ah, that's where the difficulty lies...") is but one of the many common phenomena captured by this six-fold structure to interaction. Software designers are just as prone to expounding (presenting things clearly), whereas explaining (in the sense here) is really a teacher responsibility.

Seeing *examining* as an action highlights the qualities of the mathematics being that which brings the student to approach the teacher in order to check their own criteria against that of an expert (rather than usual assessment practices involving the regurgitating of memorized procedures applied to routine tasks). Seeing *expressing* as an action acknowledges that the student experiences desire to express (to themselves, to peers, to a teacher or even more widely) in which the mathematical content or insight initiates, the student responds, and the teacher or other audience mediates and facilitates the action. When teachers complain that students do not ask questions or initiate discussion, it means that expressing is not being experienced. Prompting withdrawal from action and reflection upon that action is one form of *expressing* as an interaction.

These modes take place within a classroom and wider institutional milieu. By treating students as full participants and enabling all six modes to be experienced, the issue of respectfulness, highlighted particularly by Ares (this volume, pp. 92–93) is more likely to be taken into account. By developing a conjecturing atmosphere in which every statement is treated as a conjecture to be tested mathematically or in experience, and where disagreement is phrased in terms of “inviting a modification of a conjecture,” complete respect for all participants by all participants can be modeled, enacted, and experienced. Balancing occurrence of all six modes also contributes to the provision of spaces, both public and private in which students can exercise their powers (Ares, this volume). As Ares puts it, the exploration of interactions and communication patterns such as those by Schorr and Goldin (2008)

> illustrate how the participation structures that characterize SimCalc classrooms involve not only social and cultural resources (e.g., language, collaboration) but also provide space and opportunity for students to experience interactions that can help to ameliorate the limitations of the social and cultural contexts of schooling found too often in urban settings. (p. 135)

Students expressing themselves as a mode of interaction is much valued, to the extent of forming a commonplace, but difficult to sustain in a content-heavy syllabus, and it can be difficult to know how to deal with what gets expressed. As Hegedus and Moreno-Armella (this volume) put it, “the evolution of meaning is enhanced as traditional forms of expression are transformed or enabled (p. 49)”. Multiple media afford opportunities not only for multiple expressions but for deepening appreciation of that which is trying to be expressed through making connections. SimCalc-centered classroom networks encourage student constructions to move from the local and private to the public and shared display. Students often show a preference for learning from each other’s explaining and expounding. Being able to contrast and compare their work to that of others in mathematically meaningful ways provides access to participation in modes of interaction involving different agents (other students rather than official teachers). Appreciation and understanding are experienced as integration and coordination of multiple expressions, not the elimination of the many in favor of some one. diSessa (1993) captured this neatly when he valued software that was both expressive and manipulable, enabling users to express their thinking without great obstacles, and to manipulate those expressions so as to form more complex expressions that actually perform actions. Put another way, *representational expressivity* together with *participatory structure* (Hegedus and Moreno-Armella, this volume), such as is experienced in a conjecturing atmosphere, are essential requirements for effective software to contribute to effective learning in the world of the social.

The scope, range, and variety of questions and prompts used by a teacher provide a world of experience which can profoundly influence what students take to be the

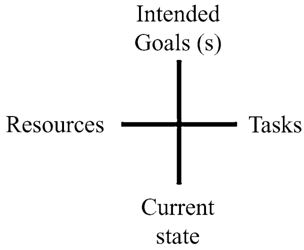

Fig. 1 Two axes of an activity

enterprise and the ways of working in mathematics. In Watson and Mason (1998), we collected a variety of prompts and questions designed to promote mathematical thinking beyond the superficial, based on some probes used by Zygfryd Dyrslag and translated for us by Anna Sierpinska (1994). This is a contribution to one aspect of Bishop's notion of "purposefully planning for productive discourse" (Bishop, this volume, p. 246), which lies at the core of professional development.

Action is essentially dynamic and unpredictable and results may not be manifested immediately, certainly when human beings are involved. Teaching involves actions that take place *in* time, whereas learning is something that takes place *over* time (Griffin, 1989). Looking for immediate consequences of engaging in discourse or working with software is bound to disappoint. Becoming stuck in routine habits—whether in the ways of working on mathematics in a classroom, or in the modes of interaction used in a classroom, or even in the ways of talking about mathematics with each other—is to lose the dynamic affordances of action and to revert to mechanicality and habit which are trained, ingrained and enculturated. This helps account for the observation of Bishop (this volume) that "the existence of... affordances is not a guarantee that students are prepared to, or necessarily will, take advantage of them" (p. 233). An action cannot be guaranteed because of the complex mix of factors that can influence the initiation, continuation, and successful fulfillment. Not only are there constraints arising from past experience and expectations of all the agents, including the milieu but in the sense of my title, no teacher act, nor institutional act or policy, can cause learning. Actions have to be invoked or triggered simply so that transformations in student awareness (learning) are possible.

In order to see what is possible with a specific technology such as SimCalc, it is necessary to extend beyond the triads of action. Bishop (this volume) observed that "the intellectual work required of students naturally varies based on the current goal, the nature of the task/activity, students' prior knowledge, and issues related to affect and motivation" (p. 235). In Systematics, this is expressed—at least in part—through seeing activity as a four-fold structure, built around a motivational axis and a means axis (see Fig. 1).

The motivation axis encompasses a tension or gap between current state and goal state (which may be variously interpreted or even experienced by different participants). The means axis encompasses a tension or gap between resources and tasks. There are four sub-triads, each constituting actions, and it is in the balance of these that effective activity takes place. For example, the resources need to be adequate and appropriate for the tasks to afford access to the desired goals, the tasks need to call upon available resources effectively, and the tasks need to be appropriate so as to afford the possibility of spanning the gap between current and goal states.

In this light, SimCalc represents a resource, which, with suitable tasks, is intended to provoke learners to move from a mechanistic, calculational perspective of mathematics as arithmetic, to a view of mathematics as a mode of enquiry, interpretation and justification through invoking student use of mental imagery and other powers required to co-ordinate graphical, symbolic, and tabular presentations of relationships. A core feature is the expressivity in conjunction with manipulability, so that interaction is two-way.

To explore the potential of something, Systematics calls upon a five-fold structure, in which the essence or core of something is revealed through the least and most it can be, and between what feeds or serves it, and what it feeds or serves. In the case of software such as SimCalc, the least it offers is a collection of complex tasks involving interpretation, negotiation of meaning, and analysis of reasoning. The least its use can fulfill is to challenge students, to awaken them to the possibilities of mathematics as a way to appreciate and to express relationships between mutually dependent changing quantities, in this case connected to motion. If tasks are effective, activity productive and engaging, and reflection stimulated, then the use of SimCalc can awaken students to the very nature of mathematical thinking through experience of and reflection on the shifts of attention involved in co-ordinating (re)presentations and perceiving those relationships as instances of more general properties.

SimCalc offers even more, for it manifests visually different components of what Tall and Vinner (1981) call the *concept images* and what Gattegno called the *awarenesses* associated with modeling motion graphically and algebraically. In addition to the graphical and tabular possibilities, there is the facility to "drop marks" on a number line at uniform time intervals. It is in the multiplicity of sense-making, experience-related awarenesses which, being manifested on screen, can not only be drawn to students' attention explicitly, but through multiple windows, serve as stimuli to promote shifts of attention between these components of understanding, that provides SimCalc's power as a learning environment.

5 Attention

A longstanding and ubiquitous conjecture of mine is that it is helpful to become aware not only of possible differences in what a teacher is attending to compared to what students are attending to, but also in how they are attending. Even when people are attending to the same thing, they may be attending differently (Mason, 1982, 1989, 1998). When this happens, it is likely that communication is at best impoverished, if not breaking down altogether, reducing the effectiveness of whatever mode of interaction is taking place.

Some of the potential of SimCalc can be explored by looking at extracts from transcripts (appearing elsewhere in this volume) and asking, simply on the basis of verbal evidence, what the teacher and the students might be attending to, and what form of interaction is being activated. I have drawn particularly on Bishop's chapter (this volume) because there is sufficient detail in her observations to make it possible to get a glimpse of students' focus of attention.

From Table 3 of Bishop (this volume, pp. 242–243)

19	Teacher E	How do you know that the bus slowed down according to your graph? Kelsy?	High demand

The teacher is clearly attending to what Kelsey had just said ("the bus slowed down"), although we do not know what else is not being expressed explicitly concerning the teacher's awareness of possible confusions, issues in interpreting of line graphs, and so on. In addition, we do not know whether Kelsy is attending to the image of the bus slowing down, or to a graphical presentation of (a mathematical model of) that action. Coded as "high demand" by Bishop, the teacher's intervention may be seeking information about what students (Kelsy) are attending to in relation to "bus" and "slowed down." The teacher may also be attending to this coordination, or only to the desirability for such a coordination. In other words, the teacher may have an answer in mind already, or, when Kelsey answers, may discover that a certain response was expected (Love and Mason, 1992).

What is being sought could be a co-ordinating action between discourse and graph, or even the generating of a narrative from a graph. Earlier in the episode from which this is taken there are moments when students confuse the graph colour and the vehicle it signifies. Their focus shifts without maintaining colour-signifying coordination. If this is not sorted out, then all subsequent discourse becomes highly problematic mathematically. The intervention could be considered to be a form of *model eliciting activity* (Lesh et al., this volume), because the "high demand" request is for interpreting or modeling the slowing down of the bus in terms of the graph (though couched in terms of questions about the model, that is, the graph).

20	Kelsy	The, if you look at the van, it's like a constant rate and it stays straight. And the bus, like, leans forward.	High give

Attention shifts to the van, but used metaphorically to mean the graph of the van's distance against time, and this analysis is validated by the "it stays straight," with "it," referring to the graph, not the van. Attention then shifts to the bus, again metaphorically, but the use of the indefinite pronoun creates ambiguity in the description "leans forward." Is the student attending to the graph, to the relation between the graphs associated with the van and the bus, or perhaps, the bus driver in some curious mixture of metaphor and metonymy. The structure of attention seems to be centered, in natural language expression of recognized relationships between graphs, or perhaps between vehicle motions. Of course the teacher is aware both of properties of graphs and of their instantiation in this specific setting. This a way of attending which it is intended the students achieve. In-the-moment or in-flight (McNair 1978a, 1978b) reactions and responses by the teacher are likely to assume that the learners are attending to what the teacher attends to when cued by the students' words and gestures, and also attending in the same way as the teacher. Often apparent student confusion arises because this assumption turns out to be a conjecture in need of modification!

1	Teacher E	Leans forward. That's not the bus turning the corner, or is it? Is the bus changing direction? (pause) This angle, where this is actually two lines (points to position-time graph), does it show that the bus went off this direction? (Teacher gestures upward motion following the path of the first linear segment in the van's path.) And then came this direction (teacher gestures horizontally)?	High demand

Teacher attention immediately shifts to the verbal expression "leans forward." This might be a diversion from attending to mathematical relationships, or it might be a probing of the relationship being attended to by Kelsey when uttering "leans forward." Alert to possible misinterpretations of graphs, the teacher may have a possible instance come-to-mind and may choose to express (or finds him/herself expressing) what it is not… typical of questioning on the fly, when something starts being expressed, and the realization of its origins and role come to mind. Evidence for this is the switch to "or is it?" (Love and Mason, 1992). The teacher then recovers ground by rehearsing the same doubt in a different form ("bus went off"). The teacher appears to be attending to (the possibility that Kelsey is attending to) the straight-line segments of the graph as directions of travel. This shift in discourse by the teacher has the potential to set up a disturbance or conflict for the learner: is the teacher focusing my attention because that is where it should be, or am I expected to disagree? In the subsequent transcript, some students disagree.

Solely on the basis of the transcript, without access to voice tones, pausing, eye-lines, posture or gesture, it seems reasonable to conjecture that the teacher is offering a world of his/her perception or attention, to the student, characteristic of expounding, rather than trying to enter the world as perceived by the student which is a characteristic of explaining. Sensitivity to epistemological obstacles (Bachelard, 1980; Cornu, 1991, p. 159) can appear to be forms of *explaining* but the student experience may be more *expounding*, since their attention is being shifted to match that of the teacher.

The "high demand" and "high give" categorizations used by Bishop for this interchange is reflected in the shifting of attention of the teacher, and the cognitive demand made with the alternative interpretation or modeling of the mathematics. Attending to the relationship between the graph and what it presents in the SimCalc animation—recalled as images and a sense-of motion—the interaction works at coordinating the relationship between graph and situation, constantly eliciting models. Sometimes the mathematics models the situation and sometimes the situation (as narrative) models the mathematics.

By way of contrast, the following extract from Bishop, Table 2 (this volume, pp. 240–241) also shows shifting and directing of attention, but in a different way.

1	Teacher L	What are we measuring on this graph? We're measuring miles and time. And when we put them together, what do we, what do we get when we put the miles and the times together?	Low give & Low demand
2	Student 1	Um, how long like.	Low give
3	Teacher L	No. What do we do? Miles per hour. What is that?	Low demand

The teacher is attending to the meaning of the axes (distance, or at least "miles") and time, and how these relate to, combine, or are coordinated in order to give speed. The wording "put together" is (probably unintentionally) ambiguous, because it could be that the teacher is attending to the graph, and wants students to attend to the graph as distance coordinated (literally) with time, although the subsequent intervention indicates that it is the measure of speed that is to be focused on. But where on the graph or in the simulation is the speed experienced?

7	Teacher L	We used a formula, but wasn't that the speed? . . . So these lines are telling us about the speed. So if, if this graph's measuring our speed what happened to the bus's speed? What happened to the bus's speed and what happened to the van's speed?	Low give & High demand (request interpretation)

The speeds can only be discerned by attending to the slopes of the line segments, but there has been nothing specific for students to focus on in order to be able to "read" or otherwise interpret relationships of speeds. The teacher offers one focus of attention, then repeats it and augments it, suggesting that attention has shifted to a relationship. The format of the questioning might indeed direct student attention to a relationship, or simply to two distinct tasks. One student tries to develop a narrative.

8	Student 3	The bus started to get in front but whenever it curved the van sped up and caught it.	Low give (no justification of interpretation)

The student is discerning the graph (re)presenting the bus, and the van, or perhaps is referring to memory of the SimCalc simulation of the movements. The student then falls into the classic trap of interpreting the line as a map-position rather than as distance coordinated with time. The line becomes "path" rather than co-ordination of distance and time.

The teacher has focused on "started" but in so doing misinterprets the student's "started to get" as "started initially." Teacher attention has shifted to the start, so the discourse shifts, and the students have to discern the shift of focus.

10	Student 3	Yes ma'am but when–	Low give
11	Teacher L	–No 'cuz the lines have nothing to do with which direction they went. The lines are their speed. So what do we see, what is the difference between the bus's speed and the van's speed?	Low give & High demand (request comparison)

What sense are students to make of "the lines are their speed"? The teacher is caught in a metonymy: the slope of the lines is the speed; "the lines" is used to refer to the associated quality of speed. The students still do not have anything specific to refer to, to discern, in order to relate the speeds of the two vehicles. Meanwhile the discourse of "turning" and "map positioning" remains in the background.

Similar analysis can be developed for the subsequent interaction. What is demonstrated clearly is that being unaware of a metonymy and of what is actually to be attended to, the discourse is likely to be at best confusing and at worst uncontrollable by students. Being aware of what you are attending to—*as* what you want students to attend to—is vital for effective interaction.

In the language of the six Ex's, in this interaction the teacher is in a mixture of *expounding* mode—trying to draw students into the teacher's world of perception; testing students—while trying to operate in *explaining* mode—entering the world of the students. Asking "testing" questions feels like *examining* in its common usage, but misses the point that in the interaction mode *examining* it is the student who presents themselves to be tested in order to check that their own criteria are coming into alignment with the expert's criteria. As Bishop comments, there is minimal opportunity for students regarding other modes such as *exploring*, for engaging in mathematical (as opposed to pedagogic) reasoning, or even *expressing* what they are thinking.

In Sequence 3 of Table 4 (Bishop, this volume), there is another example of the obstacles arising when teacher and students are discerning different things.

		Sequence 3
14	Teacher A	Now, was the speed of the two vehicles the same?
15	Mult S's	No.
16	Teacher A	No. You know that by how?
17	Student 2	They're not the same.
18	Teacher	Because of the (pause) graph right? And because of running the simulation. They didn't stay right beside each other, did they?

Testing comprehension, the teacher performs a pedagogical move to make explicit a socio-mathematical norm or classroom rubric, that it is not sufficient to know what *is* the case, because you also have to be able to justify it by reasoning on the basis of properties. But then the teacher dilutes this by referring to the graph but not to what it is about the graph that is being discerned, related and seen as an instantiation of a property, namely comparison of slopes. Instead there is parallel reference to the graph and to the simulation. Presumably the teacher is attending to the coordination of these two, to the way in which each models the other. Whether there is sufficient detail for students to pick up on that, coordination and mutual modeling cannot be discerned from the transcript presented.

One of the classic traps when evaluating or assessing student understanding—even in the weak version of the *examining* mode in which the teacher imposes the interaction of the students—is to switch from listening-to, to listening-for (Davis, 1996). Here "listening" is taken metaphorically to include observing behavior and reading written work. Once it has been decided what language patterns are expected, what techniques are required, what forms of reasoning are necessary, what behavior is expected, it becomes much harder to "really listen" in the fullest sense. The didactic tension (Brousseau 1984, 1997; see also Laborde, 1989) as I formulate it, is almost always in play: "the more clearly the teacher indicates the behavior being sought, the easier it is for the students to display that behavior without generating it, or experiencing it deeply."

Take, for example, an extract from the chapter Pitta-Pantazi et al. (this volume, p. 337):

> …What does it mean when it says that "the second car in the first four minutes…"? do we draw it here [points to [0, 4]]?
> No! according to the instructions the second car is leading the race for the first 4 minutes. It is not written that the car starts from the 4th km. Then this car moves for 10 minutes.
> OK, like this [he draws the graph]? (p. 337)

The first student shows that he is attending to a phrase whose meaning (whose instantiation in the graph as model) is not clear. Pointing to the [0, 4] coordinate point suggests that attention is dominated by the "4 minutes" perhaps obscuring or back-grounding "first." The dynamic nature of the situation being modeled is temporarily lost. The second student not only discerns the details of the instruction but also of the first student's sense of the role of the 4 minutes, and is able to relate them and to agree to one and deny the other. His attention seems to be on the movement throughout, the original situation to be modeled. The second student acknowledges a shift of focus, and proceeds to draw a graph, presumably correctly, and to check it via the animation.

An important component of Bishop's *Planning for Participation Discourse* involves locating relevant and potentially misleading foci of attention and experiencing freshly for oneself so as to be sensitized to student shifts and non-shifts of attention.

6 Deeping Understanding

Understanding is taken here to mean clarity in articulating narratives that link different mathematical objects (such as graphs, formulae and situations). It is a complex collection of acts of modeling and interpreting relationships, together with reasoning on the basis of properties, which are perceived as being instantiated in those relationships. Understanding can be displayed in any of the interaction modes, but perhaps most particularly in *expressing* and *examining,* when students withdraw from action and reflect upon those actions. Thus to broaden understanding is to discern more finely, become aware of, and more articulate about, connections and relationships between discerned objects. To deepen understanding is to become aware of relationships in the particular as instances of properties that can hold in many situations, together with relationships or properties that are consequences of these properties. This perspective is closely aligned with the notion of horizontal and vertical mathematization (Freudenthal, 1991; Treffers, 1987; Treffers and Goffree, 1985).

Teachers want students to develop flexibility in modes of thinking, in what and how they attend, and in what resources they call upon. To be flexible, for example between using graphs and using formulae, requires some degree of confidence and facility in each mode. It also involves co-ordinating them, recognizing relationships between them, and perceiving relationships as instantiations of the act of coordinating graphs and formulae. It is a reasonable conjecture that the greater the teacher's own flexibility in and between modes, the more likely it is that students will develop similar flexibility.

Burke, Hegedus, and Robidoux (this volume) describe how Jim Kaput's desire to be able to respond flexibly to student thinking that arose in class led to developments in the software so that users could similarly respond flexibly. If it is not easy or even possible to develop a line of thinking, a conjecture or an exploration, then students are likely to get the impression of mathematics as a fixed and formatted subject—rather than as a creative and engaging discipline, a form of enquiry. Having different pedagogic strategies come to mind in the moment. For example, having different modes of interaction with students come to mind, which are supported by the resources at hand, and acting upon these, makes the milieu in which students are embedded and participate, mathematically rich, thereby affording the opportunity to deepen understanding by deepening teaching possibilities.

In looking for student understanding, traditional questions involving calculations can at best reveal something of student facility but as is well known by any teacher—students can get correct answers while understanding superficially (the didactic tension again), and sometimes students who do understand get incorrect or unexpected answers because their thinking takes them beyond the assumptions of the question setter.

Pitta-Pantazi et al. (this volume) report effective use of more discerning probes

> In another task students were asked to identify the similarities and differences amongst three different distance-time graphs [see their chapter]. Students answered that the three distance-time graphs showed the distances that two athletes run in a specific time interval. When students were asked to compare these distance-time graphs with their respective velocity-time graphs, they were able to say that the distance-time and the velocity-tine graphs showed the same race but provided different information. They claimed that one of the graphs shows the distance that the athletes run whereas the other graph shows the athletes velocity (p. 333)

Here we have summaries of what was said by students in response, but without sufficient detail, to determine what features students were attending to that they were expressing verbally, and how they justify their reasoning by reference to properties being instantiated as relationships in the particular graphs. By being aware of the importance of attention and the different ways of attending, more specific evidence could be given of the depth of student understanding. Invoking *expressing* and *examining* could provide more evidence. Planning for appropriate attention-shifts, productive discourse, and suitable experience on which to reflect is the basis for effective design.

7 Conclusions

Teacher actions and reactions can have a profound influence on student experience. In the presence of on-screen-locally-networked stimuli, such as SimCalc provides, the system can have a similar impact. The combination of system and teacher provides a potent combination when they run in harmony.

Bishop (this volume) concludes from her study that purposeful planning for productive discourse is an important part of lesson preparation. (pp. 246). The same applies to the milieu afforded by systems such as SimCalc. As Hegedus et al. (this volume) notes, the SimCalc system can amplify affordances for providing reactions, both from the mathematical phenomenon and from peers as well as the teacher. Preparation is not simply an assembling of a sequence of tasks, however well designed and formulated, but rather a preparation of the teacher, in terms of being alerted to and aware of what would be helpful to have *come-to-mind* in the moment, when it is needed. But alignment and harmony between system values, teacher values, and propensities seems vital.

At the heart of the matter is fostering students' use of their own mathematical powers and access to (awareness of) pervasive mathematical themes. This is supported and sustained when students' make connections to other topics and concepts. This, in turn, is achieved by making use of a range of mathematically oriented and didactically structured questions and prompts—some of which can perhaps be embedded in the system. At a more subtle level, being aware of possible states of attention, general pedagogic strategies, which include facility in arranging for different modes of enquiry and action-balanced activities, and particular didactic tactics enable the system and the teacher to engage with students and influence what and how the students are learning.

Particularly important is self-awareness by the teacher and the system designers of, for example, the desire to shift into exposition rather than holding back, to intervene more heavily than is necessary (that is, trying to do for learners only what they cannot yet do for themselves). What matters most is the spirit and ethos, the atmosphere of engagement with mathematics—and this depends strongly on what *comes-to-mind* in the moment. What does come-to-mind arrives through structural resonance (metaphor) and affective associations (metonymies).

From a Systematics perspective, purposely planning for productive discourse is one contribution to lesson preparation for expanding the opportunities for deepening understanding. Gauging the task affordances, the balances between resources and tasks in relation to the gap between current and goal state is also essential for engaging and productive activity. Alerting oneself to possible shifts of attention involved in getting to grips with the topic (epistemological obstacles) through epistemological analysis (Brousseau, 1997) is also essential. Of course, discourse plays a major role in how students are encouraged and provoked to do this—augmented and amplified by a correspondingly appropriate classroom ethos. Withdrawing from activity and promoting reflection on that activity are also vital for many if not most students.

The success of a teaching episode depends on the shifting focus and structure of attention, of both teacher and students, on what comes-to-mind (again for both students and particularly, the teacher) in the way of mathematically cognitive, pedagogically enactive, and dispositionally affective choices made in the moment. Above all, engagement in action followed by reflection upon that action is necessary for most students to learn efficiently.

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Modeling as a Means for Making Powerful Ideas Accessible to Children at an Early Age

Richard Lesh, Lyn English, Serife Sevis, and Chanda Riggs

1 Introduction

Our research and development activities are based on a *models & modeling perspective* (MMP) on mathematics problem solving, learning, and teaching (Lesh, 2007; Lesh and Doerr, 2003; Lehrer and Lesh, 2011). This chapter reports results from a project in which children in a typical first grade classroom worked in teams of three on *model-eliciting activities* (*MEAs*[1]) which were designed to be simulations of "real life" problem solving situations that children are likely to encounter in their everyday lives outside of schools in the 21st century. Although these *MEAs* were simulations of situations in which some important type of mathematical thinking is useful beyond school, they were set in the context of children's stories—because

[1]Explicit principles for designing *model-eliciting activities* have been published in a variety of recent publications (e.g., English, 2009; English and Mousoulides, 2011; Lesh et al., 2000). And, these standards also have been specially adapted for the development of teacher-level MEAs (Zawojewski et al., 2009) or MEAs for older students in fields such as engineering (Haljmarson and Lesh, 2008).

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young children's "realities" are not necessarily the same as those of adults. The primary criteria that were used to assess the "realness" of tasks were: (a) do the children try to make sense of the problem using their own "real life" experiences—instead of simply trying to do what they believe some authority (e.g., their teacher) considers to be correct (even if it doesn't make sense to the children)? and (b) when the children are aware of several different ways of thinking about a given problem, are they themselves able to assess the strengths and weaknesses of these alternatives—without needing to ask their teacher or some other authority? When the preceding two criteria are satisfied, results of our work have consistently shown that most average ability children are able to go from "first-draft thinking" to "*N*th-draft thinking" without heavy guidance from an outside authority. Furthermore, even though their first-draft responses might have been unimpressive, their *N*th-draft responses are often very impressive indeed.

Results reported here provide an existence proof showing some powerful conceptual tools that primary school children are capable of producing which: (a) involve some remarkably advanced "big ideas" from much later in the K-12 mathematics curriculum, and (b) are generalizable in the sense of being sharable (with other people), and reusable (in other situations). In particular, our goal is to demonstrate that primary school children also are able to develop powerful ways of thinking about problem solving situations which involve: (a) interactions among two or more different types of functions (or types of quantities) and (b) issues such as maximization, minimization, stabilization, compensation, which traditional curriculum materials have treated as if they needed to be postponed until after students have been taught Calculus. However, our results also suggest that such achievements are unlikely to occur unless most of the design "specs" for MEAs are satisfied. Among other things, these design "specs" state that: (a) children must clearly recognize the need for the kind of thinking that is desired, (b) their sense-making must be based on extensions of their own personal knowledge gained through "real life" experiences, and (c) their thinking must be expressed in the form of artifacts or tools whose usefulness (power, sharability, reusability) can be tested by students themselves (Kelly et al., 2009).

2 The Relevance of Our Work to Themes Emphasized in the Kaput Center

The research reported in this paper involves links among four themes that have played central roles in both MMP research and research associated with the Kaput Center: representational fluency, foundations for the future, democratic access to powerful ideas, and capitalizing on developmental perspectives to make powerful ideas accessible to children at an early age. For example, in *SimCalc*-related research, attention focused on the development of students' thinking about *the mathematics of change and variation*. Signature insights involved the development of children's understandings about interactions between (a) continuously changing quantities and (b) accumulating quantities—i.e., understandings that eventually evolve into the *Fundamental Theorem of Calculus*. But, beyond topics associated with Calculus, a Kaputian research agenda generally has focused on investigating what it means to "understand" many of the most important concepts in K-16 mathematics—and on how these understandings develop. In particular, this research agenda has investigated how early conceptions (as well as misconceptions) of "big ideas" often begin to develop much earlier than most of research perspectives have expected.

The preceding points are especially noteworthy because Kaput himself did not begin his work with the (fallacious) assumption that: I am a mathematician; therefore, I understand all that needs to be known about (a) what it means to understand the most important "big ideas" in K-12 mathematics, (b) what early conceptions (and misconceptions) of relevant concepts look like, (c) how and why these understandings develop and change, (d) how critical aspects of development can be documented and assessed, and (e) what concepts, and what levels and types of understanding, are needed to support new kinds of mathematical thinking that are increasingly important beyond school in a technology-based age of information? For example, even in his earliest research and development activities, some of the most important characteristics of Kaput's research focused on "representational fluency" (see Fig. 1) and its roles in the development of students' thinking in Algebra and Calculus (Harel and Kaput, 1991; Kaput, 1987, 1989a, 1989b, Kaput, 1999; Kaput and Goldin, 1996). In particular, Kaput's early research and development activities often focused on what he referred to as the "big three" representational modes: tables, graphs, equations—and translations within and among them (see the three ovals at the top of Fig. 1). Similarly, a large share of early MMP research focused on K-8 mathematics and on what Kaput sometimes referred to as the "little five" modes of representation: written symbols, spoken language, pictures & diagrams, experience-based metaphors, and concrete manipulate-able models (see the five ovals at the bottom of Fig. 1). As Fig. 1 suggests, however, by the 1980s both Kaput and MMP had migrated toward technology-based representational media, and toward issues aimed at providing both: (a) democratic access to powerful ideas and (b) foundations for the future in mathematics thinking and learning.

Distinctive characteristics of this later research agenda increasingly emphasized both "situated" and "social" aspects of mathematical understanding. For example, in the case of situated understandings, our research increasingly emphasized that, from beginners to experts, and from pure to applied mathematical thinking, mathematical thinking tends to be organized around experience at least as much as it is organized around abstractions (Lesh et al., 2007). And, in the case of socially mediated understandings, we recognized that learners and problem solvers are not just isolated individuals. They may be (for example) learning communities, or teams of specialists with continually evolving tools for communication, collaboration, and conceptualization (Kaput, 1989a).

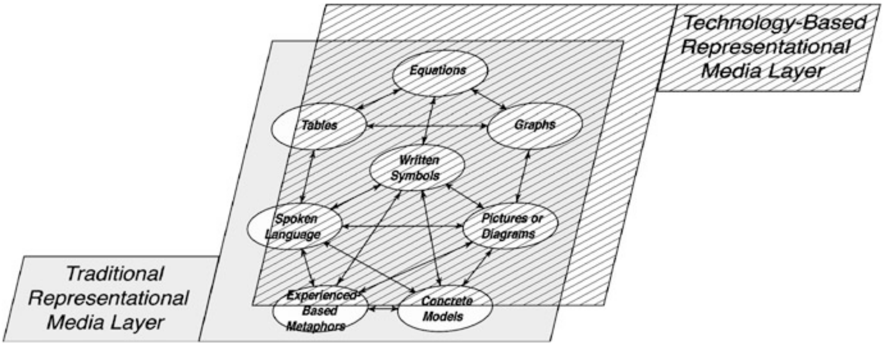


Fig. 1 A merged Kaput-Lesh diagram for thinking about representational fluency

3 What Other Theoretical Foundations Underlie MMP Research & Development Activities?

Another of the most important assumptions underlying MMP research is based on the recognition that, in virtually every area of human endeavor where learning scientists have investigated the development of expertise, exceptionally capable people not only *DO* things differently but they also *INTERPRET* (see, hear, feel, taste) things differently. For example, just as exceptionally capable teachers (or chess players, or chefs, or business managers) recognize patterns where others see only fragmented pieces of information, children also need to develop powerful models for making sense of situations where quantification or some other types of mathematization processes are needed.

Concepts are used to conceptualize situations; and, in elementary mathematics, conceptualizing is about quantifying, coordinatizing, dimensionalizing, or in other ways mathematizing situations. In other words, conceptualizing is about describing or designing—or, in short, modeling. So, when MMP research investigates *what kinds of elementary mathematical thinking is most useful beyond school*, we look beyond counting and calculating situations to also ask: *What kinds of situations do children need to be able to describe using the natural numbers:* $1, 2, 3, \ldots$ *(or other basic ideas involving topic areas such as geometry, algebra, calculus, probability, or statistics)?* We recognize that, in modern societies, outside of school classrooms, young children are surrounded by situations in which numbers refer to many things in addition to counts and in which numbers are used for many purposes in addition to calculations. For example, numbers may refer to locations (addresses, positions, or coordinates), composite units (units of units), actions (exchanges, transformations), continuous measurements (lengths, distances, areas), quantities that have both a magnitude and a direction (vectors), signed quantities (negative numbers), exchange rates or ratios, and a wide range of “ness” quantities (orange-ness, rough-ness, sweet-ness), “per” quantities (raisins per cookie), “ity” quantities (density, probability), or accumulating quantities. Furthermore, many of these situations involve information that is given in concrete, graphic, or tabular forms—so that SimCalc’s longstanding emphasis on representational fluency is highlighted (Lehrer and Lesh, 2011). Many situations involve interactions among several different kinds of quantities or "agents," whose interactions cannot be described using a single arithmetic sentence, or input-output rule. Instead, such situations may involve feedback loops, second-order effects, and issues such as maximization, minimization, accumulation, equalization, modularization, stabilization, which traditional curriculum materials have treated as if they must be postponed until after Calculus has been taught.

MMP uses the term *models* to refer to the interpretation systems that problem solvers develop to make sense of the preceding kinds of problem solving or decision-making situations (Kaput, 1991). What is a model? According to MMP, an entry-level definition is: *A model is a system for describing another system for some specific purpose* (Lesh and Doerr, 2003). What is the most important distinguishing characteristic that makes mathematical models different than other types of models—such as those used in chemistry, biology, history, or music? *Mathematics is the study of structure.* So: *Mathematical models focus on the structural* (*or systemic*) *properties of the systems-as-a-whole that they are used to describe*. Sometimes, these properties of the system-as-a-whole are referred to as *emergent properties* of the system. But, in any case, in mathematics, such properties of systems-as-a-whole include not only properties such as symmetry, invariance, and centrality; they also include properties which often appear as "undefined terms" within the axiom systems that define different kinds of mathematical structures. Examples of such "undefined terms" include "points" and "lines" in the axioms that define *Euclidean Geometry*; and, they include "identity elements" or "inverse elements" within the axiom systems that define metric spaces or counting numbers (e.g., see *Peano's Postulates*).

What does it mean to be an "undefined term" in a mathematical system? It means that *all* of the mathematical meanings of these terms come from the systems-as-a-whole in which they reside. So, the psychological counterpart of this claim can be seen in Piaget-inspired research showing what children's thinking is like before their interpretations of these "undefined terms are based on relevant systems of operations, relations, and patterns.

An important point to emphasize here is that models are not facts; nor are they skills. Yet, MMP considers them to be among the most important types of knowledge that students need to develop in order to be able to use mathematics in real life situations beyond school. This is because models are used to conceptualize (i.e., mathematize) situations in ways so that mathematical tools can be used. Consequently, MMP research expects models to provide the conceptual systems that underlie the most important concepts that citizens of the 21st century need to develop. So, a primary goal of MMP research is to clarify the nature of these continually adapting models.

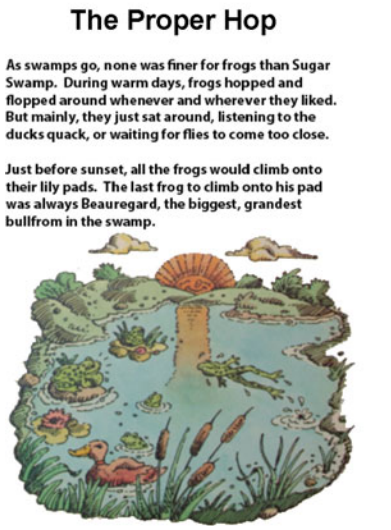
What are *model-eliciting activities* (*MEAs*). In MMP-based research, *MEAs* are problem solving situations which are designed precisely to cultivate, document, and assess the development of models related to the most important "big ideas" in K-12 mathematics (Doerr and Lesh, 2010). Unlike most "problems" that children encounter in school textbooks or tests, *MEAs* are designed explicitly to be tools for research. In particular, they are designed to provide rich contexts for investigating: (a) what it means to "understand" the half-dozen-to-a-dozen most important concepts or abilities within any given course or grade level in the K-12 curriculum, (b) how these understandings develop, and (c) how development can be cultivated, documented, and assessed. So, *MEAs* are designed to be situations in which important aspects of children's thinking can be observed directly—by researchers, by teachers, and (most importantly) by children themselves. Furthermore, because *MEAs* focus on conceptual change, and not just computational proficiency, they are designed to optimize the chances that significant conceptual adaptations will occur within single 30–90 minute problem solving sessions. In fact, in properly designed *MEAs*, children's thinking often develops from Piaget-like stage# N to stage# ($N + 2$ or $N + 3$) within sufficiently brief periods of time so that many of the most important processes that give rise to development can be observed directly (Harel and Lesh, 2003; Lesh and Harel, 2007; Lesh and Kaput, 1998; Lesh and Yoon, 2004).

This learning effectiveness of *MEAs* should not be surprising. Models are, at their simplest, purposeful descriptions. So, if children recognize the need for a specific kind of description, if they are able to make sense of the situation using the same kind of sense-making abilities that they use outside school, and if their thinking is expressed in the form of tools and artifacts which can be tested in ways that also test underlying ways of thinking, then they are likely to go through several first-draft, second-draft, and *n*th-draft ways of thinking within relatively brief periods of time. Consequently, during the past decade, more than fifty research reports have been published which included analyses of transcriptions in which the modeling cycles that students went through often resembled Piaget-like stages of development for relevant concepts (see Hamilton et al., 2008; Lesh and Doerr, 2003, 2011; Lesh et al., 2010, 2007; Lehrer and Lesh, 2011; Lesh and Zawojewski, 2007; Zawojewski et al., 2009).

The preceding results demonstrate that, even though *MEAs* are designed for research purposes, they also are proving to be remarkably effective learning activities—especially if attention shifts beyond low-level facts and skills toward an emphasis on deeper and higher-order understandings associated with the most important elementary-but-deep concepts in the elementary school curriculum (Lesh and Zawojewski, 2007; Lesh and Doerr, 2003). In fact, MEAs are proving to be doubly effective because they contribute to both teacher development and student development. One reason this is true is because MEAs are designed to make important aspects of students' thinking visible, and because helping teachers become insightful connoisseurs of students' thinking is one of the most effective ways to improve teaching practices (Zawojewski et al., 2008).

4 An Example of MEAs Leading to Extraordinary Achievements by Young Children

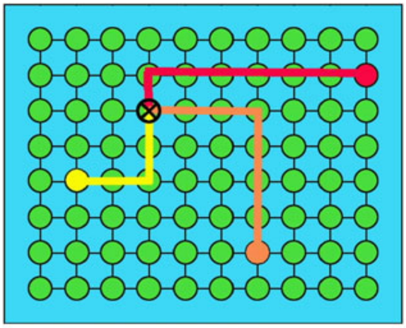
Figures 2a–2f describes a *Proper Hop Activity* that is typical of the story-based problems that have been emphasized in MMP research focusing on children in grades K-2. Like all *MEAs*, the children described here worked on the *Proper Hop Activity*



(a) The first page of a story about Beauregard

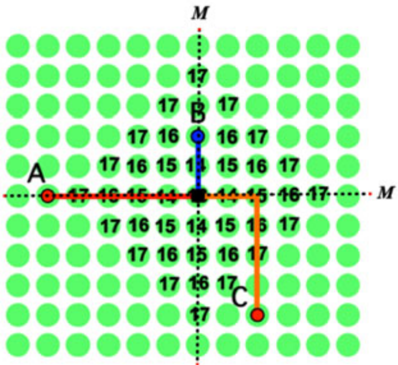
(b) Acting out "Proper Hops" in class.

The "Proper Hop" Story is about Beauregard Frog and the "proper hops" that all frogs must use to get from one lily pad to another in Sugar Swamp. The story describes why, in Sugar Swamp, frogs are only allowed to jump horizontally or vertically to adjacent lily pads. So, Beauregard's problem is that he wants his "home lily pad" to be located at a place that minimizes the sum of the distances to the three

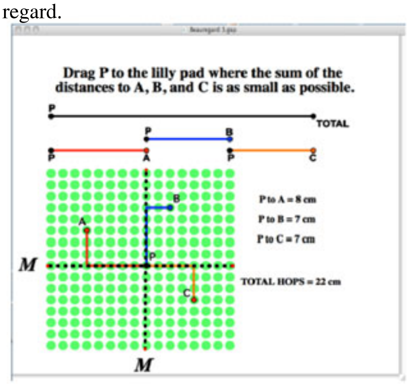


(c) Three paths from Beauregard's lily pad (X) to where his three friends live.

(d) Patterns start to emerge as this group records information to use in a letter to Beauregard.

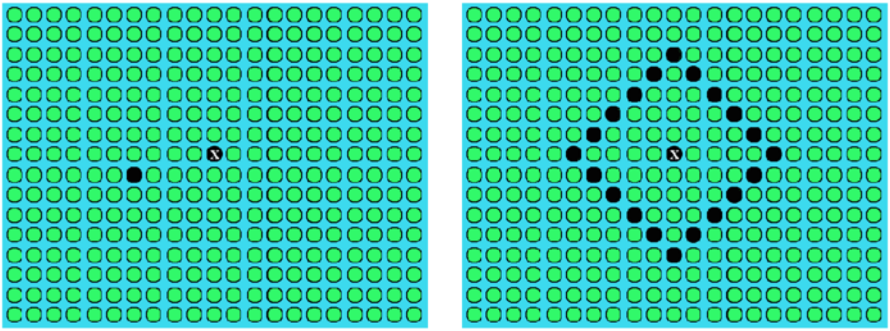


(e) In the final versions of the information shown in Figure 2d, several patterns emerged about distances from one to several homes.



(f) In follow-up teacher-led discussions, students gaves "thumb signals" (up, down, feft, right) to help the teachers to find a best place to locate Beauregard's pad.

Fig. 2 The Proper Hop activity

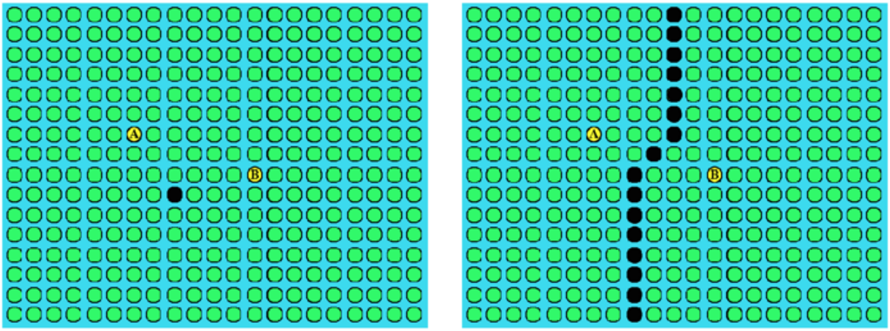


(a) (b)

Fig. 3 (**a**) As a warm-up for the Proper Hop activity, students are asked to put a mark (X) to show the location of the possible home pad for Beauregard. Then, put another marker 5 hops way from Beauregard. The student must find all the other pads that are exactly 5 hops from Beauregard. (Note: This same basic task can be repeated for other locations of X and other numbers of hops.) (**b**) Displays the solution to the task described in (**a**)

in groups of three; and, they usually stayed intensely engaged for at least one hour. In fact, after class, many children continued to work even longer; and, the teacher found it easy to create her own follow-up activities to emphasize the basic skills emphasized in her school's accountability testing programs. Furthermore, because the *Proper Hop* activities were designed to be *thought-revealing activities*, she was able to focus on specific strengths and weaknesses of individual children.

The following points are especially significant to emphasize about children's responses to the *Proper Hop Activity*. First, the activity is a minimization problem that involves integrating several interacting counting, measuring, coordinatizing, and systematizing problems. So, the results clearly showed that primary school children were able to develop insightful ways to deal with such problems. Second, most children's solution processes involved using numbers to describe a variety of different types of mathematizable "objects" (locations of lily pads, hops along paths, distances between lily pads, equivalent paths, and patterns similar to the kinds of locus of points that occur in analytic geometry). So again, the results clearly showed that primary school children were able to "think mathematically" about relationships among several such properties. Third, each group worked on versions of the problem where the locations of the homes were different for Beauregard's three friends. Therefore, because the children knew that they needed to write a letter to Beauregard explaining their procedure, they realized that their procedure needed to be powerful (in the given situations), sharable (with other people) and reusable (in a variety of similar situations). So, the ways of thinking that the children developed were generalizable. Fifth, because the method that is illustrated in Figs. 2d and 2e quickly spread throughout the class for recording information to "*explain things to Beauregard,*" the children's final solutions shifted beyond dealing with *pieces of information* toward noticing *patterns of information*—like the diamond-shaped pat-



(a) (b)

Fig. 4 (**a**) The activity asks students to put markers at two points A and B, and put another marker at a point that is the same distance from both point A and point B. The child's task is to find more such points. (Note: Some locations for A and B don't have any points that are the same distance from both of them.) (**b**) Displays all of the points that satisfy the task described in (**a**)

tern of numbers in Fig. 2e. Furthermore, in the teacher-led whole-class activities (see Figs. 3 and 4) that followed the *Proper Hop Activity*, several children noticed (and everybody understood) that the "best location" for Beauregard was always at the intersection of the two dotted lines in Fig. 2f. In other words, the children understood that the best location for Beauregard's pad will always be at the intersection point for the horizontal median and the vertical median (even though they didn't use this kind of formal terminology). So, even though the "letters" that the children wrote to Beauregard were primitive in terms of objective communication, the goal of "writing letters to Beauregard" was to make it as clear as possible to the children that their tools and ways of thinking needed to be sharable and reusable, which means memorable and transferrable.

5 Two More Multiple-Concept MEAs for Young Children

Figure 5a shows one result from an MEA that is based on a story about a horse named Isabelle who loves to eat apples in the shade of apple trees. The children's task is to write a letter to Tom, Isabelle's owner, describing how, for a given orchard, the largest number of trees can be enclosed inside a fence of a given length—where the fence is a string of soda straws on a loop of string. The children's solution must take into account the fact that, each month, when Isabelle has eaten all of the apples in a given area, Tom must move the location of his fence to a new location where the trees are distributed differently. So, their solution needs to be sharable and reusable. Similarly, Fig. 5b shows an activity that is based on a story about *Fussy Rugbugs* (whose homes are represented as colored post-it notes). The children's task is to describe how to locate the most rugbug homes within a given loop of string, when

(a) (b)

Fig. 5 (**a**) A group of children's solution to the MEA about Isabella the horse who loves to eat apples in the shade of apple trees. (**b**) A group of children's solution to the MEA, *Fussy Rugbugs*

the rugbugs insist that their rugs must be put together so they "just touch" but "never overlap."

Notice that both of the preceding problems involve issues of maximization and minimization, and that both problems again involve relationships between perimeters and "areas" (or "counts"). Furthermore, both problems require children to produce results that are not only powerful in the specific situation, but that are also sharable and reusable in other similar situations. So, even though the solutions that children develop are highly situated (i.e., shaped by the context and purposes of the problem), the results represent more than context-specific knowledge. Sharability and reusability again indicates generalizable achievements.

6 Results

For all three of the problems described in this paper, every child in the class participated in a group in which the "last draft" responses to the problems ended up being successful at developing solutions that embodied quite adult-like ways of thinking about the situations that were given. Furthermore, in follow-up interviews and quizzes, every child in the class was able to explain his or her group's results; and, in follow-up problems, these children also were able to generalize their understandings to problem situations that would have been inaccessible before their MEA experiences.

Of course, the letters that children produced were not expected to be objectively communicable to others. Nonetheless, in this study, the school had made a school-wide commitment to focus on writing. By asking the children to work together with help from their teacher, they were able craft letters which emphasized to them that: (a) their goal was to produce a tool that provides solutions to more than a single situation, and (b) "someone else" needs to use the tool that they produced. In other words, the tool that they produced needed to be sharable and reusable.

7 Conclusions

The goals of this study were to provide "existence proofs" showing what's possible in mathematics learning activities for primary grade children. The results that will be emphasized in this section focus on six issues that tend to be emphasized in modern curriculum standards documents: (a) the importance of focusing on conceptual knowledge as well as factual and procedural knowledge; (b) the importance of focusing on a small number of "big ideas"; (c) the importance of focusing on usefulness outside of school; (d) the importance of focusing on higher-order processes such as modeling; (e) the importance of focusing on research-based learning progressions; and (f) the importance of focusing on accountability.

7.1 Concerning Conceptual Understandings

As we stated earlier, mathematical *concepts* are tools for *conceptualizing*. Conceptualizing is about describing and designing; and, in mathematics and science, describing and designing are about modeling. So, when MMP researchers investigate the nature of conceptual understandings in primary school mathematics, we look beyond counting and computing skills to also ask: *What kinds of situations should primary school children be able to use whole numbers to describe?*

This paper has given examples of tasks in which average ability primary school children used numbers to describe situations that involve not only counts (i.e., the cardinality of sets of discrete objects) but also locations (i.e., ordinality for sequences of objects, or coordinates for systems of objects), actions (e.g., operations or transformations), continuous measures (e.g., lengths, areas, volumes), signed quantities (i.e., positive and negative quantities), directed quantities (i.e., vectors), a variety of "per," "ness," or "ity" quantities, patterns, and so on. Furthermore, we investigate problem solving situations which cannot be understood using only a single arithmetic sentence and which often involve issues such as maximization, minimization, stabilization, or equalization,

Our results bear witness to the fact that, even though children's first-draft responses usually were unimpressive, their nth-draft responses were often impressive—if the children themselves are able to assess the usefulness of their own responses. Furthermore, because important aspects of students thinking tend to be visible in children's responses to MEAs, readers who wish to replicate our work will have no difficulty observing, documenting, and assessing the kind of conceptual strengths and weaknesses we have described.

7.2 Concerning a Small Number of Big Ideas

Even if attention is focused on only computation-related understandings, it has been known since the seminal work of William Brownell in the 1940s that "varied practice" is far more effective than "routine practice" (focusing on drills that are repeated again and again). Brownell identified three kinds of varied practice. The first type involves mixed activities in which attention shifts among several skills—rather than emphasizing just one. This is effective partly because "understanding" involves more than knowing *how* to do something; it also involves knowing *when* to do it. The second type of varied practice involves practicing skills in a full range of situations in which they are intended to be useful. This is effective partly because useful skills need to be flexible, not rigid. And, the third type of varied practice involves using skills during complex activities—similar to the way excellent chefs not only know how to use each of the tools sold in chef's catalogues, but they also know how to orchestrate the use of these tools during the development of complex meals.

This paper has given examples of activities in which primary school children are capable of making sense of a wide range of situations in which numbers are used to describe quantities and quantitative relationships that involve much more than simple counts. This means that, if a goal is to increase the usefulness of school skills beyond schools, then the half-dozen to a dozen "big ideas" that should be emphasized in the K-2 curriculum can (and should) focus on using numbers to describe a wide variety of situations—not just those involving counts of discrete sets of objects.

7.3 Concerning Emphasizing Future-Oriented Goals

During the past 25 years, enormous changes have been occurring in the kinds situations in which new levels and types of "mathematical thinking" are needed outside of school classrooms. So, one question that should be asked about any curriculum framework is: *What is being said that could not have been said 50 years ago?* If the answer is *nothing*, then it is very unlikely that serious attention has been given to the new kinds of mathematical thinking that are needed beyond school in a technology-based age of information.

Look at a modern daily newspaper, like USA Today. In sections ranging from sports to business, readers need to be able to make sense of mathematical descriptions that involve graphs, tables, diagrams, metaphors, and mathematical ideas that are embodied in other media. Furthermore, the things being described often include systems involving global economies, knowledge industries, intelligent tools, and hyphenated sciences, and a variety of other situations that cannot be understood using only a single function going in one direction.

We have shown that primary school children are quite capable of dealing with a wide range of problem solving or decision making situations that involve several interacting agents (or functions, or arithmetic sentences), feedback loops, second-order effects, as well as issues such as maximization, minimization, stabilization, or others that once were thought of as being unmanageable until students had been taught Calculus. Furthermore, many such problems are accessible to quite young children using elementary arithmetic tools rather than Calculus.

7.4 Concerning Higher-Order Mathematical Practices such as Modeling

Modeling is becoming widely recognized as one of the most powerful and important kinds of abilities that should be emphasized in school mathematics. Yet, modeling is often conceived as involving nothing more than *"applying mathematics that (students have already been taught) to solve problems arising in everyday life."* This conception of modeling completely ignores the perspective that: (a) models for mathematizing experiences may be, in themselves, among the most important goals of the K-16 mathematics curriculum, and (b) the development of these models represents an important part of what it means to develop "conceptual" understandings of even traditionally emphasized concepts and processes, and (c) proficiency at model development is directly related to a range of higher-order competencies that most curriculum frameworks claim to emphasize (Lesh and Zawojewski, 2007).

Three types of metacognitive understandings are easy to observe as primary school children work on the kind of MEA's described in this paper. One involves thinking about (the process of) thinking. Another involves thinking about oneself as a learner or problem solver. And, a third involves thinking about the nature of mathematics or problem solving experiences.

For thinking about thinking, the children in our studies clearly learned that: (a) the problems we gave were going to take full class periods to solve (i.e., 30–60 minutes), and (b) without asking the teacher, they themselves would be able to judge "*Am I done yet?*" and "*Is what I'm doing useful?*"

For thinking about oneself as a learner or problem solver, the children in our studies clearly learned that productive teams often involve individuals who shift among a variety of roles (recorders, planners, checkers, etc.). So, many children who had never exhibited much interest or ability in mathematics were enthusiastic about participating in the MEAs described in this paper. And, most of the students remained highly engaged for the entire problem solving session.

For thinking about mathematics, or about the nature of problems in which "mathematical thinking" is useful, the children in our studies clearly learned that doing things "correctly" (in ways that make the teacher smile) is different than doing things that are sensible or productive (in ways that they themselves could judge). However, because their experiences working on our problems were so different than their past experiences with mathematics and problem solving, most of the children talked about their experiences with MEAs as if they were not "mathematics" (as they had understood mathematics to be).

7.5 Concerning Accountability

MEA's are designed to be thought-revealing activities (1993). So, teachers are able to observe and document many levels and types of understandings that are seldom addressed using exercises in traditional textbooks and tests. In fact, even in the newest Common Core State Curriculum Standards (CCSC), almost none of the deeper or higher-level goals are "operationally defined" in ways that are measurable. Make sense of problems. Reason abstractly. Construct viable arguments. Model with mathematics. Use appropriate tools strategically. Attend to precision. Look for structure. Look for regularity. What does it mean to "understand" any of these practices? How can these understandings be documented and assessed? After characterizing such goals using lofty-sounding-but-vague-language, curriculum standards documents tend to reduce such goals to lists of facts and skills. But, in MMP research and development, the mastery of nearly all such goals is considered to be closely associated with the development of a toolkit filled with powerful, sharable, and reusable models associated with the most important concepts in any given course or grade level. Furthermore, documenting the power, sharability, and reusability of these tools is often as straightforward as assessing the power, sharability, and reusability of a spreadsheet that has been created to for paying income taxes, purchasing a home or a car, or analyzing statistics in athletic contests. This is because tools that are designed to be powerful (in a given situation), sharable (by other people), and reusability (in other similar situations), have characteristics that are not shared by lower quality tools.

7.6 Concerning Research-Based Learning Progressions

Similar to the wave and particle models for describing the behaviors of light in physics, MMP supports two significantly different descriptions of learning and problem solving in school mathematics. On the one hand, Piaget-inspired mathematics educators have produced detained and extensive descriptions of the conceptual systems that children must develop in order to "think operationally" (or "systemically) about many of the most important "big ideas" in the K-16 mathematics curriculum. They have described many of the most important characteristics of pre-operational thinking; and, they have described intermediate stages between pre-operational and operational thinking. However, ever since Piaget's original studies of children's mathematical thinking), these descriptions of cognitive development have resulted in learning progressions that lead to exceedingly pessimistic views about possibilities for accelerating development. Furthermore, whereas the strength of these learning progressions is that they clarify the importance of structure in mathematics learning, both Vygotsky's notion of *zones of proximal development* and Piaget's notion of *decalage* imply that: (a) tasks characterized by the same structure are often significantly different in difficulty, and (b) the difficulty of a single task can be changed significantly by changing only seemingly superficial (non-structural) characteristics of the task. So, a strength of Piaget-inspired "constructivist" conceptions of cognitive development results from it's emphasis on structure, a weakness is that knowledge is organized around experience at least as much as it is organized around abstractions (i.e., cognitive structures). In modern research in the learning sciences, this latter perspective on cognition often is referred to as *situated cognition*; and, in research focusing on situated cognition, significant conceptual changes often occur quite rapidly.

MEAs focus on model development and, in mathematics, the models that students develop tend to be both situated and structural. That is, structure is an important characteristic of the models that students construct. Yet, these conceptual systems often develop evolve quite rapidly; they usually are strongly shaped by the context of the tasks; and, the tools that are developed often integrate concepts and procedures associated with a variety of textbook topic areas. Yet, model development is not just context-specific learning. This is because the models that students develop are expressed as tools, which are designed to be powerful, sharable, and reusable. So, the underlying models that students develop tend to be both situated and structural, and both specific and general.

Overall, MMP research suggests that it is at best a half-truth to imagine that the way mathematics concepts develop is primarily via a process of putting together and integrating lower-level concepts and processes. In many cases, children's early conceptions of important ideas involve the development of models (or ways of thinking) that integrate (and gradually begin to differentiate) constructs associated with a variety of different textbook topic areas. So, cognitive development often resembles the kind of "star burst" displays that can be seen in firecracker extravaganzas during national holidays in many countries.

8 Concluding Remarks

This paper emphasizes the shared MMP and SimCalc tradition of investigating: (a) the nature of new types of "mathematical thinking" that is needed beyond school, (b) the levels and types of mathematical concepts and abilities that are needed in the preceding situations, (c) what it means to "understand" the preceding concepts and abilities, (d) the ways that these concepts and abilities develop with special attention being given to early conceptions and misconceptions of these concepts and abilities, and (e) new ways development can be cultivated, documented, and assessed in ways that don't reduce "understanding" to the "mastery" of lists of low-level facts and skills?

A basic principle underlying this entire agenda is that measurable goals for instruction should be established through research rather than through political consensus-building proclamations of the type that have characterized the development of all recent curriculum standards documents.

Clearly, the kinds of mathematical thinking that will be needed by citizens of the 21st century is quite different than what has been offered in traditional curriculum materials where the primary goal of each course appears to be about preparing students for the next course in programs that gave little attention to the kinds of mathematical understandings and abilities that are needed outside of mathematics classrooms (Kaput, 1995b; Lesh et al., 2007).

The results described in this paper are especially intended to emphasize the fact that the seeds of many of the most powerful, useful, sharable, and reusable concepts in the K-12 mathematics curriculum are accessible to children at much earlier ages than traditional curriculum materials have assumed (Blanton and Kaput, 2000, 2001a, 2001b, Blanton and Kaput, 2002, 2004; Hegedus, in press; Kaput, 1995a, 1998; Kaput and Blanton, 2001, 2005; Lehrer and Lesh, 2011; Lins and Kaput, 2004). So, capitalizing on these affordances is one of the most effective ways to provide both *democratic access to powerful ideas* and *foundations for the future in mathematics thinking and learning*—which have always been the most important goals of both MMP and SimCalc research agendas (Hegedus and Roschelle, 2012; Kaput and Nemirovsky, 1995).

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The Kaputian Program and Its Relation to DNR-Based Instruction: A Common Commitment to the Development of Mathematics with Meaning

Guershon Harel

This paper is about two programs: the *Kaputian program for mathematics of change and variation* and *DNR-based instruction in mathematics* (*DNR*, for short). It outlines some of the central goals and characteristics of the *Kaputian program* and the *DNR theoretical perspective*, and further discusses how the former is juxtaposed with the latter. The purpose of this juxtaposition is to point to the common commitment of the two programs to the development of mathematics with meaning. Specifically, the two programs are rooted in the empirically-founded perspective that quantitative reasoning and gradual development of computation fluency anchored in meaning must be a central focus of instruction. Collectively, the paper aims at stepping up the long-standing call for the adoption of this perspective in school mathematics, especially now when the Common Core State Standards Initiative (http://www.corestandards.org/the-standards/mathematics) is underway.

1 Characteristics of the Kaputian Program

The ultimate goal of the *Kaputian* program has been *democratizing access to the mathematics of change*, a ubiquitous phrase in Kaput's writings. The following quote reflects Kaput's concern about equitable access to mathematical knowledge, in general, and that of the mathematics of change and variation, in particular:

> Mathematics of change and variation is essential to understanding ubiquitous phenomenon in science, engineering, and economics. Its concepts are needed to participate in the physical, social, and life sciences of the twenty-first century, and hence they are needed to make informed decisions in one's personal and political lives. But, Calculus is offered at the end of a long series of prerequisites, filters out 90 % of the population. Students from economically poorer neighborhoods and families are underrepresented among the 10 % of the

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> students who do take this course. And even the 10 % who do have nominal access to calculus courses develop mostly symbol manipulation skill but little understanding. (Kaput, 1994b)

Kaput's commitment to equitable access to mathematics manifest itself in numerous research and implementation projects throughout his career. In what follows, I will discuss one of the many facets of this commitment: the role of *mathematical notation* and *representation*. These terms will be defined as the section unfolds.

Kaput viewed the traditional mathematical notation as an obstacle to an equitable access to calculus. He believed that this notation evolved among intellectual elites "to serve the conceptual and communicative purposes of knowledge producers, without regard to the needs of the wider population to whom we now attempt to teach it." (Kaput, 1994a, p. 385). Kaput aimed at creating a new notational system which would be accessible to a wider range of the student population—a system that is compatible with the conceptual development of the learner. He wanted to shift the focus of instruction from formalism to quantitative meaning, and rejected the common instructional practice of "you don't really know it unless you can express it formally." (Kaput, 1994a, p. 385). He argued:

> …the mathematics of change and variation, as represented by calculus in most current curricula, is accessible only to those who have survived a long series of algebraically oriented prerequisites. The net result of this prerequisite structure is that, at least in the US, 10 percent of the population has contact with the mathematics of change and variation, and most of those are at the college level. Moreover, most of their contact is with the notation of calculus rather than its conceptual core.

The Kaputian program aimed at building on and extending what others started: experimenting with the construction of alternative notational systems to connect mathematical representation and concepts directly, whereby enabling students acquire the conceptual core of the mathematics of change and variation.

It is a mistake, however, to think that Kaput intended to completely or even partially alter the traditional notational system; rather, his goal was to increase the teachers' sensitivity to the correspondence between the symbolic representation of a concept and the level of conceptualization of that concept by an individual. From this perspective, the subject matter of calculus is reformed and restructured with the objective that students learn it *"before, during, and after algebra."* (Kaput, 1994a, p. 393) The calculus curriculum in a *Kaputian* program, thus, evolves through a gradual process of formalization and elaboration from early grades up to adulthood.

A crucial element in this *Kaputian* perspective is the development of an interactive media. While building on early educational development of computer technology, the *Kaputian* program ventured in a new approach—one that puts the complexity of authentic human experience at the center of the developmental effort.

> …Representational uses of technology in mathematics education… have been primarily used to assist activity and movement on the island [of formal mathematics], not to connect with the mainland of real human experience. (Kaput, 1994a, p. 383)

In contrast, the *Kaputian* program aimed at:

> [electronic technology that enables] students to act on traditional mathematical notations in more natural ways, as when in a computer environment, for example, one uses a pointing

> device and graphical interface to act directly on coordinate graphs by sliding, bending, reflecting, and so forth (as with *Function Probe*, Confrey, 1992). This is a subtle exploitation of the rich knowledge based in kinesthetic experience to act on mathematical notations, and hence to effect mental operations on mathematical objects, that is, functions.

In sum, the *Kaputian* program envisions dynamic and interactive properties of electronic media that utilize naturally developing human perceptual and conceptual powers.

Underlying this perspective is Kaput's distinction between two inseparable sources of mathematical experience: mental and physical, where ultimately the former is the result of the latter. Following Piaget (1985), he often characterizes this distinction in terms of the representing world (the signified, or external) and the represented world (the signifier, or internal) (Kaput, 1987). The structuring of students' mathematical experience should, according to this perspective, begin by structuring physical actions, followed with gradual structuring of "notational objects whose properties and relationships are defined mathematically and are mediated only indirectly by their physical properties. Competence in the use of these notation systems is built upon mental structures and operations that embody the mathematics and, again, relate only indirectly to the perceptual features of the systems." The implication of this perspective to the learning and teaching of calculus is then that it is imperative to conceive the mathematics of change and accumulation of quantity as rooted in and generated from everyday experience and, in addition, as the experience through which representational strategies (e.g., algebra) should emerge and be learned.

The conceptual core of the mathematics of change and variation is anchored in one's ability to reason directly about covariation of quantities. Algebraic representations of such reasoning are increasingly "built up from situations through the use of numerical tables, graphs, and other less stringently formal means before the writing of algebraic equations." (Kaput, 1994a, p. 384). These algebraic representations serve, in turn, as a source for quantitative reasoning:

> ...the development of algebra as an action notation system made possible the inheritance of powerful means of quantitative reasoning, and Leibniz' notations for calculus likewise made available an immensely powerful system of thought. In some sense, the most potent intellectual contributions, leading to cultural inheritances, are embedded in these "ways of worldmaking," to borrow Nelson Goodman's phrase (Goodman, 1978). Some of the most important work of the masters is embodied and handed down, not in the form of facts or even theorems and principles, but rather in the syntax of the representation systems that they enable us to think with.

It is appropriate to mention here the work Kaput and I did on the roles of mathematical notation (Harel and Kaput, 1991). In this publication, we discussed how "using [traditional] mathematical notations, complex ideas or mental processes can be chunked and thus represented by physical notations which, in turn, can be reflected on or manipulated to generate new ideas" (p. 88), and how such notation "either help encapsulate mathematical concepts as entities or supplant conceptual entities in reasoning processes" (p. 90). Among the various examples we discussed, one is particularly relevant to calculus. It pertains to degree of elaboration of symbols. The extent to which a notation is elaborated is determined by the extent to which it corresponds to one's conceptual development: what is elaborated for one person may appear tacit for another. For example, two different symbols are usually used to represent the composition of two functions f and g, $g: f(g(x))$ and $(f \circ g)(x)$. The symbol $f(g(x))$ is amenable to the understanding of a function as a process, and depends on the prior knowledge of input-output relations expressed using the standard $f(x)$ notation. It expresses the process in which the two functions are composed: the input x in the function-machine g produces the output $g(x)$, where $g(x)$ now acts as an input of in the function-machine f to produce the output $f(g(x))$. On the other hand, the symbol $(f \circ g)(x)$ describes an operation between two functions, f and g, which produces a third one, $(f \circ g)(x)$. This symbol, thus, describes f and g as inputs in the function-machine $\circ$. As such, it is understood as an operation between two objects, f and g. The distinction between elaborated symbols and tacit symbols has important consequences for learnability and usability.

In all, Kaput's work on the mathematics of change and variation may be viewed as a research program—a program for which Kaput paved the foundations and offered a path for progress. Such a program can be characterized as one that pays a serious attention to: *equity, quantitative meaning, gradual development* (from elementary school onward), *advanced-technology-based curriculum that is grounded in classroom context*, and *consistent epistemology*. For this reason it is appropriate to talk about the *Kaputian* program rather than Kaput's program. A *Kaputian* program is a research program whose goal is to offer a mathematics-of-change-and-variation curriculum that has these characteristics. More specifically, using a narrative adopted from Kaput's own words, a *Kaputian* program examines the nature of the mathematics content of calculus, its objectives, methodologies, and representation; it attempts to deeply understand the experiences, resources, and skills students can bring to the subject matter of the mathematics of change; it seeks to create the conditions in which students experience growth in their capability to solve and understand ever more challenging problems; it researches the means for representing important calculus ideas in way that reflects their origins in the study of change; it reassesses the proper place of calculus in the curriculum path that so many students seem unable or unwilling to complete; it looks at possibilities for an approach to the mathematics of change for all students; it begins the exposure to the mathematics of change in elementary school and builds gradually toward the formal system that is now identified as calculus; and it gives ordinary children the opportunities, experiences, and resources they need to develop profound understanding and skill with mathematical of change and variation. In this respect, SimCalc is a model that instantiates all of the *Kaputian* characteristics. In particular, SimCalc is a model that exploits the capability of novel dynamic, graphical notations and representations to provide tools that engage students' conceptual resources and linguistic resources, and support growth towards more sophisticated understandings, including more formal notations and forms of reasoning.

The five characteristics listed above are inextricably linked, and it is beyond the scope of this paper to discuss them thoroughly and in separation from each other. They manifest themselves in the work of Kaput (e.g., Kaput, 1994a) and his colleagues (e.g., Roschelle et al. (2000) and Roschelle et al. (2007b)).

2 Relation of the Kaputian Program to DNR-Based Instruction[1]

DNR-based instruction in mathematics (*DNR*, for short) is a theoretical framework for the learning and teaching of mathematics. It aims to serve as a framework that provides a language and tools to formulate and address critical curricular and instructional concerns. *DNR* has been developed from a long series of teaching experiments in elementary, secondary, and undergraduate mathematics courses, as well as teaching experiments in professional development courses for teachers at each of these levels. Briefly, *DNR* can be thought of as a system consisting of three categories of constructs: *premises* (explicit assumptions underlying the *DNR* concepts and claims), *concepts* (referred to as *DNR* determinants), and claims. These claims include instructional principles: assertions about the potential effect of teaching actions on student learning. Not every instructional principle in the system is explicitly labeled as such. The system states three foundational principles: the *duality principle*, the *necessity principle*, and the *repeated-reasoning principle* (to be formulated below); hence, the acronym *DNR*. The other principles in the system are derivable from and organized around these three principles. For a fuller discussion on *DNR* see Harel, 2008a,b.

In what follows, I will focus on two common features of the *Kaputian* program and *DNR*: the uncompromising attention to quantitative reasoning and the instructional treatment to algebraic notation.

As was mentioned earlier, the conceptual core of the mathematics of change and variation in the *Kaputian* program is one's ability to reason directly about covariation of quantities. *Quantitative reasoning* is defined in *DNR* as *a way of thinking by which one reasons with quantities and about relations among quantities. It entails the habits of creating a coherent image of the problem at hand; considering the units involved; continually attending to the meaning of quantities, in addition to how to compute them; and having multiple images of a concept and being flexible in transitioning among them.* The attention to quantitative reasoning by the *Kaputian* program and *DNR*, as well by other programs, most notably the research program set by Patrick Thompson (1993), is the result of the overwhelming evidence indicating that students across the grades do not learn to reason quantitatively; their mathematical reasoning is mostly quantitative-reasoning free. They manipulate symbols but the manipulation is often divorced from quantitative referents (Stigler et al., 1999; Stigler and Hiebert, 1999), and they focus on what operation the teacher expects them to choose rather than what operations are logically entailed (Sowder, 1988). The lack of attention to quantitative reasoning accounts for this phenomenon as well as for many other troubling occurrences in students' understanding of key concepts across the grades. Relevant to this paper is the example that rate of change, a fundamental concept in calculus, is not well understood by undergraduate students (Hackworth, 1995). Another example, relevant to algebra, is that the equal sign—perhaps the most ubiquitous symbol in school curricula—is largely interpreted by

[1] Part of this section is an unpublished research perspective on the concepts of number and quantity, which was solicited by the founders of the Common Core Mathematics Standards initiative.

middle school students as a command to perform an operation rather than as a relation between two quantities, and no improvement in equal sign understanding has been found across the middle grades (Knuth et al., 2006). This and other studies also show—not surprisingly—a strong correlation between students' equal sign understanding and their performance in algebra, especially in solving equations (Booth, 1989; Freiman and Lee, 2004).

The concern here goes beyond the understanding of the equal sign and the ability to solve equations. The question of concern by the *Kaputian* program and by *DNR* is how to achieve *computational fluency*? In *DNR*, *computational fluency is a way of thinking comprised of two inseparable abilities: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize—to pause as needed during the manipulation process in order to probe into the referential meanings for the symbols involved in the manipulation.* The literature on students' difficulty with algebra led me and others, especially *Kaputian* scholars, to the conclusion that a necessary condition for the development of computational fluency (as defined here) is uncompressing attention to quantitative reasoning throughout the curriculum.

In passing, I mention that these *DNR* definitions of *quantitative reasoning* and *computational fluency* were adopted almost verbatim by the Common Core Standards (regrettably without adequate attribution), as can be seen in following quote from the Common Core Mathematics Standards.

> Reason Abstractly and Quantitatively
> Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. (http://www.corestandards.org/the-standards/mathematics/introduction/standards-for-mathematical-practice/)

Results from studies on the implementation of SimCalc (e.g., Roschelle et al., 2000, 2007a, 2007b), as well as other studies, show that students can be taught to acquire this ability. For example, students in the middle school should be expected to have developed the ability to reason quantitatively about additive situations. Studies show that this can be done (Carpenter and Moser, 1984; Carpenter et al., 1982; Fuson et al., 1997; Thompson, 1993). High-school students should be expected to reason multiplicatively and understand related phenomena such as exponential growth. Studies show that students can be taught to acquire these ideas (Confrey, 1994; Confrey and Smith, 1995). Calculus students should be expected to understand rate of change, accumulation functions, and the Fundamental Theorem of Calculus. Studies show that students can be taught to acquire these concepts (Carlson et al., 2003; Thompson and Silverman, 2008).

How should students be taught to reason quantitatively? This is one of the central research questions of the *Kaputian* program—it has been addressed theoretically and examined empirically through programs such as the SimCalc. In what follows, I will outline *DNR*'s guiding framework for this question:

A common feature of the successful quantitatively-based curricula mentioned earlier is that they were designed on the basis of results of detailed conceptual analyses that connect concepts and skills to ways of thinking—those that have been acquired and those to be acquired. The fundamental premise behind these analyses—and this is the epistemological reason for their effectiveness—is that, in quantitative reasoning, concepts are formed in continual dependence on their natural foundations, and their mathematical meanings are abstracted from natural, concrete experiences. The underlying approach of these analyses can be abstracted into a general principle, called the *duality principle* (represented by the letter *D* in *DNR*) because of the dual nature between its two assertions:

(a) Students at any grade level come with a set of ways of thinking (practices, dispositions, beliefs, etc.), some desirable and some undesirable, that inevitably affect the way they will understand concepts and skills we intend to teach them, and
(b) Students develop desirable ways of thinking only through proper understanding of concepts and skills.

This principle may seem obvious until one observes that mathematics curricula grounded in its premises are rare. The principle entails that long-term curricular planning is essential, and absence of such planning can have harmful consequences, because the ways of thinking students acquire now will affect the quality of the concepts and skills they will learn later. The principle also entails the need to take into account students' current ways of thinking in designing curriculum and instruction, because these determine what students can and cannot learn and the quality of what they will learn.

In a curriculum that is based on the duality principle, desirable ways of thinking do not wait until students take advanced mathematics courses. Consider, for example, the concept of fraction. In current mathematics teaching, even when students learn mathematics symbolism in context, the context is usually limited. For example, the most common interpretation of fraction among students is the *part-whole* interpretation (m/n means "m out of n objects"). Many students never move beyond this limited interpretation and, as a result, encounter difficulties in developing meaningful knowledge of fraction arithmetic (Lamon, 2001) and beyond (Pustejovsky, 1999). Seldom do students get accustomed to the other interpretations of fractions: as ratios, operators, quotients, and measures (Behr et al., 1992; Post et al., 1991). While this range of interpretations is one of the predominant factors contributing to the complexities of teaching and learning fractions, it can also be a source for desirable—indeed, crucial—ways of thinking, such as (1) mathematical concepts *can* be understood in different ways, and (2) *it is* advantageous to change interpretation in the process of solving problems. These ways of thinking will be needed in the development of future concepts. Indeed, without them, students are bound to encounter difficulties in other parts of mathematics. In calculus, for example, depending upon the problem at hand, one would need to interpret the phrase "derivative of a function at a," or the symbol $f'(a)$, as "the slope of a line tangent to the graph of a function at a" or "the $\lim_{h\to 0}(f(a+h)-f(a))/h$" or "the instantaneous rate of change at a" or "the slope of the best linear approximation to a function near a." Likewise, in solving linear algebra problems, it is often necessary—or at least advantageous—to convert one interpretation into another interpretation by using the equivalence among problems on systems of linear equations, matrices, and linear transformations.

Likewise, within a curriculum that is designed on the basis of the duality principle, and consistent with the *Kaputian* program, algebraic reasoning and (informal) proving are not delayed until one learns "Algebra" and "Geometry." Rather, they emerge in mathematical activities at all levels. Thinking in terms of ways of thinking in curriculum design, one is compelled to recognize the algebraic nature of arithmetic and the role of early arithmetic to laying foundations for processes of conjecture and proof (Blanton and Kaput, 2002; Mason et al., 1985; Rico et al., 1996).

Conceptual analyses that lead to successful curricula take—often implicitly—a particular stance on the meaning of *learning*. Learning is viewed as a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium (Piaget, 1985; Thompson, 1985). The implication for instruction of this view is the *necessity principle* (represented by the letter *N* in *DNR*):

> For students to learn what we intend to teach them, they must have a need for it, where 'need' refers to *intellectual need*.

Intellectual need is different from *motivation*. Motivation has to do with people's desire, volition, interest, self-determination, and the like. Intellectual need, on the other hand, has to do with disciplinary knowledge born out of people's current knowledge through engagement in problematic situations conceived as such by them. Relevant to curriculum design, the necessity principle entails that new concepts and skills should emerge from problems understood and appreciated as such by the students, and these problems should demonstrate to the student the intellectual benefit of the concept at the time of its introduction. The following example is relevant to quantitative reasoning.

Textbooks often introduce the idea of building equations to solve word problems through trivial, one-step addition or multiplication word problems (Harel, 2009). This approach is contrived, and is unlikely to intellectually necessitate this idea since students can easily solve such problems with tools already available to them. To make this point clearer, it is worth presenting an alternative approach—one that is more likely to intellectually necessitate algebraic tools to solve word problems. In this approach, students first learn to solve non-trivial word problems, such as the following, quantitatively, without any explicit use of variables and equations:

> Towns A and B are 280 miles apart. At noon, a car leaves A toward B, and a truck leaves B toward A. The car drives at 80 m/h and the truck at 60 m/h. When will they meet?
> Students can do so by, for example, reasoning as follows:
> After 1 hour, the car drives 80 miles and truck 60 miles. Together they drive 140 miles. In 2 hours, the car drives 160 miles and the truck 120 miles. Together they drive 280 miles. Therefore, they will meet at 2 PM.

Through this kind of reasoning, students develop the habits of attending to quantities and building coherent images for the problems—habits they often lack. These problems can then be gradually modified—in context, as well as in quantities—so as to make them harder to solve with arithmetic tools alone, thereby necessitating the use of algebraic tools. For example, varying the distance between the two towns through the sequence of numbers, 420, 350, 245, 309, results in a new sequence of problems with increasing degrees of difficulty. Students still can solve these problems with their arithmetic tools but the problems become harder as the relationship between the given distance and the quantity 140 (the sum of the two given speeds) becomes less obvious. For example, for the case where the distance is 245 miles, the time it takes the vehicles to meet must be between 1 and 2 hours, and so one might search through the values: 1 hour and 15 minutes ($80(75/60) + 60(75/60) = 245$), 1 hour and 30 minutes ($80(90/60) + 60(90/60) = 245$), 1 hour and 45 minutes ($80(105/60) + 60(105/60) = 245$), and find that the last value is the solution to the problem. This activity of varying the time needed can give rise to the concept of variable (or unknown) and, in turn, to the equation, $80x + 60x = 140$. Granted, this is not the only approach to intellectually necessitate the use of algebraic tools for solving word problems. However, whatever approach is used, it is critical to give students ample opportunities to repeatedly reason about the problems quantitatively and with their available arithmetic tools. The goal is for students to learn to build coherent mental representations for the quantities involved in the problem and to intellectually necessitate the use of equations to represent these relationships. An added value of this approach is the development of computational fluency as this term was defined earlier.

Even if concepts and skills are intellectually necessitated, there is still the task of ensuring that students internalize, organize, and retain this knowledge. This concern is addressed by a third principle, called the *repeated-reasoning principle:*

> Students must practice reasoning in order to internalize, organize, and retain ways of understanding and ways of thinking.

Research has shown that repeated experience is a critical factor in these cognitive processes (Cooper et al., 1996). Repeated reasoning, not mere drill and practice of routine problems, is essential to the process of internalization—a conceptual state where one is able to apply knowledge autonomously and spontaneously—and reorganization of knowledge. The sequence of problems must continually call for reasoning through the situations and solutions, and they must respond to the students' changing intellectual needs.

In all, both the *Kaputian* and *DNR* are rooted in the perspective that quantitative reasoning and gradual development of computation fluency anchored in meaning must be a central focus of instruction. In *DNR*, an explicit attention is given to the intellectual need of the student. To address this need, a subjective approach to knowledge is necessary, since the construction of new knowledge does not take place in a vacuum but is shaped by one's current knowledge—a view central in the *Kaputian* program as well. This fundamental, well-documented fact has far-reaching instructional implications, and is the basis for both *DNR* and the *Kaputian* program.

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The Evolution of Technology and the Mathematics of Change and Variation: Using Human Perceptions and Emotions to Make Sense of Powerful Ideas

David Tall

1 The Changing Nature of Technology

> Anyone who presumes to describe the roles of technology in mathematics education faces challenges akin to describing a newly active volcano—the mathematical mountain is changing before our eyes... (Kaput, 1992, p. 515)

This quote from Jim Kaput, written two decades ago, is just as apt today as it was then. As I began preparing this chapter, my iPad rang and my three-year-old grandson came on line, demanding to speak to Tufty, our cat. Modern technology provides us with astounding ways of operating and communicating that were unimaginable not long ago. Yet while three-year-old children are embracing technology in a natural way, educators are having great difficulty in coming to terms with how to use it in teaching and learning.

The difficulty is not hard to diagnose. The speed of change of technology is so much faster than the possibilities of curriculum change, which, in turn, must take account of the rate of cultural change. Thus, while a child may pick up an iPad, with software carefully designed for ease of use, and discover ways to use it for personal benefit, the curriculum designer must take time to reflect deeply on the complex issues that arise in our society and change over the longer term. Kaput succinctly formulated his own version of the situation in the following quote from near the beginning of this book where I have added italics to highlight important aspects that will be reflected upon in this chapter.

> While our universe of experience can be apprehended and organized in many ways—through the arts, the humanities, the physical and social sciences—important aspects of our experience can be approached through systematic study of patterns. In addition, *mathematics embodies languages for expressing, communicating, reasoning, computing, abstracting, generalizing, and formalizing—all extending the limited powers of the human mind.* Finally, *mathematics embodies systematic forms of reasoning and argument to help establish the*

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> *certainty, generality, and reliability of our mathematical assertions*. We take as a starting point that all of these aspects of mathematics change over time, and that *they are especially sensitive to the media and representation systems in which they are instantiated*. (Kaput and Roschelle, this volume, p. 13)

Having written a calculus text (Fleming and Kaput, 1979), Kaput went on to rail against the standard approaches to algebra and calculus that had the effect of steadily reducing the number of individuals that make sense of the subject, seeking instead a natural way to allow the wider population to gain the insight necessary for them to function as citizens with full democratic rights and access to knowledge. He saw this through the full development of “expressing, communicating, reasoning, computing, abstracting, generalizing and formalizing” (Kaput and Roschelle, this volume, p. 13) but he also realized that these aspects change and “are especially sensitive to the media and representations in which they are instantiated” (Kaput and Roschelle, this volume, p. 14).

This applies not only to the technology that we use and develop, but also to our own personal development based on our previous experiences in life that may support or hinder our grasp of new ideas. I will illustrate this fundamental issue by recalling a difference of vision that occurred between Jim Kaput and myself that I now see in terms of his own insightful vision of the changing nature of technology as “the mathematical mountain is changing before our eyes” (Kaput, 1992, p. 515).

2 A Challenging Difference and a Resolution Using Technology

Over the years Jim Kaput and I met in various parts of the world to share ideas. Though our goals in building from personal experiences to increasingly sophisticated ideas were broadly consistent, our own personal developments caused us to focus on different aspects. His experience with the “big three” representations using expressions, graphs, and tables saw him focusing on ways of making links between them using technology.

He found the particular technology of “pointing and clicking” a mouse could quickly draw a curve graphically to represent a real life story to give a new foundation for these fundamental ideas of the calculus prior to the use of expressions or tables. It encourages a much more general notion of function than is possible in traditional calculus, which essentially focuses on the symbolic manipulation of regular expressions using the “rules of calculus” to derive the rate of change or to integrate to find the growth of a changing quantity.

My concern was more elemental. I wanted to “see” and “feel” change in a human sense through drawing a graph by the dynamic continuous movement of a finger or the use of a pencil.

We also differed in the extent of our vision. Jim focused on the wider democratic and social issues and had no desire to follow through to the formal development of traditional mathematical analysis, which he saw as the province of a privileged few. I wished to understand the full journey through the human development of mathematics itself, from the early experiences of the child to its eventual formalization and on to the frontiers of mathematical research. I also wished to develop a framework that predicted and explained *why* students followed such different paths of development in mathematics where the traditional curriculum seemed to steadily deny access to more and more learners.

Jim's use of pointing and clicking gave him a notion of "piecewise linear" graphs which could be described precisely, based on his own formal experience of mathematics, such as calculating a piecewise linear approximation to the area under a curve. My own very different experience, included teaching non-standard analysis to undergraduates who had already met standard analysis, where I could prove a formal theorem that "a differentiable function," when "locally magnified by an infinite scaling factor," would "look like a straight line" (Tall, 1981, 2009).

Both of us were imprisoned in our own cultural experiences, which proved to be obstacles in our attempts to communicate our ideas to each other. I saw his piecewise linear functions as a fine mathematical idea that was well-known to mathematicians and worked well for calculating areas. Yet, for me, it had the flaw in calculating derivatives that the graph of its rate of change consisted of discrete horizontal line segments. Thus his vision took natural "continuous" change and represented its "rate of change" in a form that is certainly not continuous in any intuitive human sense.

Meanwhile my idea of "infinite magnification" of an "infinitesimal" part of a graph flew against current trends in mathematical analysis where infinitesimals were seen to be an aberration of the past that had been replaced by the inscrutable but mathematically sound notion of the epsilon-delta definition of a limit. Even though I produced software that allowed the user to magnify graphs on the screen to see what is termed "local straightness," my insights were seen as an interesting starting point to a calculus industry that remained wedded to its traditional development based on an "intuitive" version of the formal limit concept with technology "added on."

On reflection, I can now see how both our visions may be explained in terms of his general theoretical perspective and how the changing technology has affected conceptions that are "especially sensitive to the media and representations in which they are instantiated."

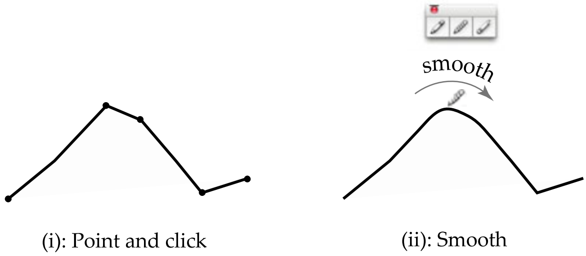
In the chapter Burke, Hegedus, and Robidoux (this volume), the long history of SimCalc has shown how theoretical ideas have to be adapted to fit with the changing technology, just as the technology changes to fit new ideas. Theoretical frameworks and technological innovations co-evolve (Hegedus and Moreno-Armella, this volume).

The iPad, which appeared only in 2010, was initially misunderstood with derisory comments from the cognoscenti such as Bill Gates who declared, "It's a nice reader, but there's nothing on the iPad I look at and say, 'Oh, I wish Microsoft had done it.' " (Bill Gates, 2010).

Now in its fourth iteration, the iPad boasts a "retina display" so that what one sees on the screen, held at a comfortable distance, is at the maximum level of accuracy that can be seen by the receptors on the human retina.

The iPad also offers radical new modes of operation. One of these is to draw a graph with a movement of a finger, and another is to control the display of an already

Fig. 1 Smoothing a graph



drawn graph—which has an appropriate method of computation, such as a function formula—to picture it at any desired scale.

The drawing of a graph with a finger on the current iPad lacks precision, although it might be possible to draw a graph more accurately with a yet-to-be designed combination of enactive finger pointing and more precise action using a pen or mouse. This idea, favored by Bill Gates, exists on Wacom Bamboo tablets and is beginning to appear on touch screens. But even here, the precision of drawing is limited to that of a retina display and graphs need to be imagined as being suitably smooth for it to have a continuously changing derivative.

Adobe Illustrator, using a mouse or tablet interface on a computer, already has the facility for drawing a graph with a pencil and then selecting another tool to smooth parts of the graph (Fig. 1).

I can see Jim in my imagination now, as he would turn up at a conference in the 80s and 90s to show the latest software, such as Excel in its earlier incarnations, that allowed him to imagine linking together symbolism and visualization in creative ways. Only now he might be looking at the iPad, drawing not with a point and click mouse, but with a finger, or with a more accurate pen, and then smoothing out the graph he had drawn. He might also organize his input to touch specific points on the graph to type their actual values, or touch a part of the curve between specific points to input a formula. He might smooth the graph as in Illustrator, so that it becomes locally straight, or if he wished, he could use techniques already existing in Illustrator to draw a corner with different left and right tangents.

Such software would enable the learner to use a finger or pointer to draw a suitably accurate representation of a suitably smooth graph, and to "crystallize" it (in the sense of Moreno-Armella and Hegedus, this volume) from a dynamic movement into a static picture where now its rate of change could again be dynamically continuous.

Visually a differentiable function is "locally straight" in the sense that, if the graph—through a point where the function is differentiable—is magnified, it will successively look less and less curved until, under high magnification, its graph looks like a straight line (Tall, 1985). We already have multi-touch technology such as the iPad where the user can touch the screen with finger and thumb and move them apart to cause the screen to be magnified. If this is programmed to keep the horizontal and vertical scales the same, then the slope of the curve can be seen under high magnification as the slope of a highly magnified segment that is visually a straight line. If two windows are available, one to show the graph and another to show the magnified part of the graph, it is possible to trace the finger along the graph and *see* the changing slope of the magnified part.

On the other hand, if the program offers a separate facility to stretch the graph horizontally and not vertically, then a continuous graph will "pull flat," as I have long advocated (Tall 1986, 2009, 2012). In this way, continuity and differentiability have simple interpretations in terms of different kinds of change of scale.

3 The Evolution of Ideas Using Technology

This single example of different views of a complex issue illustrates a profound fundamental aspect of the evolution of ideas. As mere mortals we can only focus on a small number of factors and the differing ways in which we do so affect, and are affected by, the development of technology. It is not enough just to reflect on the nature of the changing technology or on the nature of mathematics as we see it at the time and on children's growing conceptions and misconceptions. We need also be confident enough to reflect on the validity of our own ideas as our cultures evolve and technology changes.

Harel (this volume) has characterized Kaput's work on the mathematics of change and variation as follows:

> In all, Kaput's work on the mathematics of change and variation may be viewed as a research program—a program for which Kaput paved the foundations and offered a path for progress. Such a program can be characterized as one that pays a serious attention to: *equity, quantitative meaning, gradual development* (from elementary school onward), *advanced-technology-based curriculum that is grounded in classroom context,* and *consistent epistemology*. (Harel, this volume, p. 440)

This formulates what he terms "the Kaputian program" as a broader research enterprise focusing on the mathematics of change and variation. It must be taken in conjunction with Kaput's ideas expressed earlier that seek to address the whole framework of building powerful mathematical ideas developing from the child's personal experience through modes of "expressing, communicating, reasoning, computing, abstracting, generalizing, and formalizing—all extending the limited powers of the human mind" (Kaput and Roschelle, this volume, p. 13).

Kaput's reference to the limited powers of the human mind applies not just to the children we teach or to the politicians who set the legal agenda for the mathematics curriculum, they also apply to us "experts" and to Kaput himself. The amazing feature of his program is that it contains within it a vision that foresees the need to modify and evolve our own theories, even his own.

His program operates at two levels: the specific level of how we plan and deliver the curriculum using technological facilities such as SimCalc and the meta-level in which we constantly refresh and evolve the ways in which we think about how individuals make sense of mathematics.

Kaput's theory is a grand design, much bigger than the specifics of SimCalc. Though the implementation of SimCalc is vast in terms of the number of research studies that have been devoted to it and it focuses on issues beyond the mathematical content alone, in practice it has so far concentrated on a new vision of school algebra and the early development of mathematical variability and change.

Kaput crucially extends his vision of mathematics in school saying "finally, mathematics embodies systematic forms of reasoning and argument to help establish the certainty, generality, and reliability of our mathematical assertions" (Kaput and Roschelle, this volume, p. 13). This potent vision extends his program through to the frontiers of mathematical thinking at the highest levels of mathematical research.

4 Extending the Kaputian Program

In the 1990s, the Advanced Mathematical Thinking group of PME worked together to extend mathematics education to the formal mathematics experienced at university in which Harel and Kaput (1991) extended ideas to more formal aspects of functions and calculus. In particular, they distinguished between the pointwise, local and global aspects of the calculus. Formal mathematics focuses initially on the limit concept at a specific point involving local behavior near that point and then extends definitions of continuity and differentiability by varying that point over the whole domain, leading to further formal distinctions between pointwise continuity and uniform continuity over an interval.

An embodied approach works locally and *dynamically*, shifting a finger *over an interval* in time and space as the moving finger leaves a trace of the underlying variation. It does *not* build from a technical definition of continuity *at a single point* to then apply this pointwise definition to every individual point over an interval. Dynamic continuity is a single gestalt, shifting attention *along* an interval as dynamically changing quantities vary together. It is peculiarly well-suited to the use of dynamic interactive technology.

Reflecting on many aspects of learning over many years, with the help of colleagues and students, has led me to extend childhood experiences of perception and action to ideas of advanced mathematical thinking as used at a more formal level (Tall, 2006; Tall et al., 2001). This builds on human perceptions and actions and their consequences in terms of symbolism and proof. It is based on the conceptual embodiment of our perception and action where our actions—such as counting, measuring, adding, subtracting, evaluating, differentiating, integrating—may be symbolized and compressed into operational symbolism and then formalized in various ways. Formal thinking is expressed in terms of definitions and deductions.

However, as Kaput says insightfully, "mathematics embodies systematic forms of reasoning and argument" (Kaput and Roschelle, this volume, p. 13) and the forms of reasoning are different in different contexts. They may be verbal expressions of embodied principles in Euclidean geometry (such as congruence which embodies the idea of placing one triangle precisely on top of another, or parallel lines where a line is shifted dynamically maintaining corresponding angles and related properties such as those of alternate angles). The principles may be based on observed regularities of arithmetic that are formulated as "rules" to act as a basis for algebraic proof. Later they may be reformulated once more in terms of set-theoretic definitions of axiomatic systems and reasoning in terms of formal proof.

This framework has been applied to SimCalc by Lima, Healy, and Campos (this volume). They interpret SimCalc as relating real life activities to the mathematical worlds of conceptual embodiment of dynamic graphical representations. Reasoning is verbalized and communicated in terms of conceptual embodiment and operational symbolism, but not yet in terms of axiomatic formalism.

Their analysis includes a significant new reflection on what happens as learners encounter new contexts where their previous experience in terms of ideas that they have "met before" may be supportive or problematic. Using the terminology of Lima and Tall (2008) and McGowen and Tall (2010), supportive "met-befores" encourage generalization while problematic met-befores impede the learner in making sense of the new situation.

This is coupled with an analysis of emotional reactions using the goal-oriented theory of Skemp (1979), where previous success can increase confidence and encourage students to meet conflict with a determination to overcome difficulties and seek the pleasure of making sense. Alternatively, problematic aspects may lead either to a desire to satisfy external requirements to learn procedures to pass examinations, or worse, to a spiral in which failure leads to avoidance of doing mathematics which in turn leads to more failure and increasing anxiety (Baroody and Costlick, 1998). This link between cognitive success or failure and emotional pleasure or anxiety sheds new light on the nature of the long-term decrease of the number of learners who make sense of mathematics until only a small proportion end up even attempting to succeed in the more sophisticated ideas of the calculus.

It gives a broader view of the whole enterprise of mathematical thinking consonant with Kaput's program for understanding "mathematical change and variation." It suggests that technology may be used to give insight into dynamically continuous change through crystallizing the rate of change graphically based on the idea of changing slope built on the notion of local straightness. This may then be expressed symbolically not only in terms of algebraic expressions but more importantly in terms of local linearity which has the potential to develop formally in terms of the traditional definition of limit. It also offers a new vision of the limit concept consistent with the Kaput program which can be constructed from meaningful experiences of conceptual embodiment and operational symbolism.

Modern technology enhances human abilities to make sense of dynamic change through the interactive ability to physically control the variability of motion crystallized as manipulable representations of graphs. It also has the internal capacity to process expressions numerically and symbolically to enable the learner to see the effects of their actions and to share their ideas through human and technological interaction. However, although technology can be used to compute numerically, manipulate symbolically and represent ideas visually, so that it offers the human mind possibilities for future developments, it does not yet have the human capacity to imagine new conceptions and to create new theories. It is therefore important to recognize those aspects that can be supported by technology and those that humans need to develop by using their own mental facilities.

My own personal view is that we need to understand more about how individual human thinkers build mathematical ideas in increasingly sophisticated contexts and how their interpretation of new contexts is affected by fundamentally different emotional reactions to supportive and problematic changes in meaning. We also need to consider these changes not only in terms of the children's own learning but also in ourselves as teachers, mathematics educators and theory builders.

5 Building on the Confidence of Success

The transition to new ways of thinking needs to take into account the emotional reactions that feed back into learners' attitudes that can develop cycles of success encouraging more determination to solve new problems or of failure building anxieties that impede future development. This suggests the need to take account of the *success* of student's thinking processes at one level and to use the *confidence* that it generates in one situation to be able to realize what is necessary to succeed in new situations.

One possibility in the mathematics of variation and change is to build from the confidence in using piecewise linear physical drawing in SimCalc to shift to the use of a locally straight approach to develop more sophisticated levels of insight in the calculus.

However, the successive changes of meaning—from distances varying in time to the change of distance with respect to time (velocity), then velocity changing in time (acceleration) and acceleration changing in time (jerk)—gives a succession of different meanings that may impede the generality of the mathematics of change. For instance, in the case of simple harmonic motion, the distance is $x = \sin t$ velocity is $\cos t$, acceleration is $-\sin t$ and "jerk" is $-\cos t$. In what sense can the smooth trigonometric function $-\cos t$ be considered a jerk?

The concept of rate of change itself may be better served by the rate of change of a locally straight graph that may be seen by looking along the graph to see the derivative. If the derivative is again locally straight, then the process may be repeated for higher derivatives as long as they are also locally straight.

This extends the Kaput program to local straightness in functions of a single variable. This can be generalized to "local flatness" of a function of many variables to deal with multi-dimensional calculus, and the same ideas extend to integration, differential equations, partial derivatives, so that the relationship between continuity and local straightness is the foundation of the whole of calculus at all levels (Tall, 1985, 1986, 1989, 2009, 2012).

In the historical vision of the calculus, curves were imagined as polygons with an infinite number of infinitesimal sides. But unlike Kaput, who focused on the finite version of this idea using polygonal curves, I imagined a dynamic version under arbitrary magnification, where I see locally straight curves as looking straight

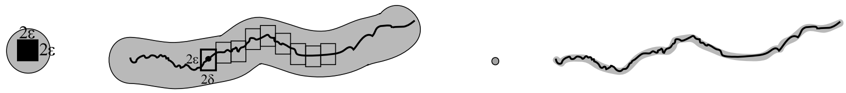


Fig. 2 Drawing a continuous graph

everywhere when highly magnified. Unlike the original vision of such great mathematicians as Barrow, Newton, and Leibniz who saw curves with infinitesimal sides and corners that turned through an infinitesimal angle, I encourage today's learners to use technology to see a differentiable function to look straight under high magnification *everywhere*, with *no* corners.

This is the resolution of a three-and-a-half century conundrum that is now rationalized in the embodied vision of modern technology. It also happens, by chance, that going back in history to my own PhD supervisor, and then to his supervisor, and back to mentors of mentors before PhDs were invented, I am one of thousands of mathematicians alive today who can trace his ancestry back 14 generations to Sir Isaac Newton and one further generation to Barrow, who also inspired Leibniz. This is a tenuous link, but it is gratifying to see the vision of the originators of the calculus as a simple dynamic idea that can be democratically shared with the wider population in our technological society.

Building on the dynamic idea of a continuous function that can be drawn with the stroke of a pencil on a sheet of paper, it is simple to translate the embodied idea of continuity to the formal epsilon-delta definition and vice-versa—to show that using a pencil whose point makes a mark covering a square of side length 2ε it is possible to draw a formally continuous function f from a point $(a, f(a))$ to another point $(b, f(b))$ as a (thick) dynamic pencil line that *covers* the graph of the theoretical function. (Figure 2. Full details are given in Tall, 2012.)

Starting with a *continuous* function, it may be integrated to give a locally straight area function whose derivative is continuous. Integrating a second time gives a function that is differentiable twice with a second derivative that is continuous. Integrating n times gives a function that can be differentiated n times to give a function that is continuous. If we define a C^n function to be a function whose nth derivative is continuous, then we see a continuous function as the case $n = 0$ at the root of a whole hierarchy of increasingly smooth functions. At the apex, we may see C^∞ functions that are infinitely differentiable. A simple case is a polynomial that may be differentiated as often as desired and this suggests a generalization from polynomials to power series.

However, as in Kaput's program, we need to develop "systematic forms of reasoning and argument to help establish the certainty, generality, and reliability of our mathematical assertions" (Kaput and Roschelle, this volume, p. 13). Contrary to the natural expectation that infinitely differentiable functions are expressible as power series, we find that there are counter-examples. For instance, the function $f(x) = e^{-1/x^2}$, where we take $f(0) = 0$, has graph that is so very flat at the origin that all its higher derivatives are zero, so that the power series associated with it is zero while the function itself is not. To be able to cope with such ideas requires a systematic form of reasoning that establishes the reliability of our assertions with clearly defined assumptions. It is for this reason that the calculus requires extension to more formal systematic modes of thinking such as those in mathematical analysis.

While this level of operation is certainly not necessary for the majority of the population, it is essential that we—who reflect on the full range of the mathematics of change and variation—encourage teachers in Science, Technology, Engineering and Mathematics (STEM) to have a grasp of the bigger picture.

6 Views of Calculus Appropriate to the Needs of the Individual

Kaput's view of democracy in terms of making sense of mathematics for the wider community can now be seen in its widest sense—to take account of how individuals play diverse roles in society, each offering his or her own contributions to make the whole so much greater than the sum of its parts. In *How Humans Learn to Think Mathematically* (Tall, 2013), I study the development of mathematical thinking as individuals mature from newborn children to adults in a wide spectrum of differing ways. This mathematical development builds from perception and action, through the use of symbolism and natural language, to successively more sophisticated forms of mathematical thinking.

In school, mathematics is seen as a blend of what I term *conceptual embodiment*, involving the static and dynamic physical and mental pictures of objects and their properties, and *operational symbolism*, which begins with actions on objects such as counting, measuring and sharing that are encapsulated into thinkable objects such as whole numbers and generalized as fractions, negatives, decimals, rationals and irrationals, real and complex numbers, and the generalized arithmetic of algebra. For a small minority, there is a development in university to the *axiomatic formalism* of formal definition and proof.

At every stage, there is a divergence in performance in different individuals as some aspects of previous personal experience feature as supportive met-befores in generalizing to a new situation, while other aspects are problematic met-befores that impede conceptual development (Tall, 2013, Chapter 3).

As human perceptions and operations become more sophisticated through the development of human reasoning, I use van Hiele's (1986) ideas to see mathematics developing broadly through levels of *recognition*, *description*, *definition*, and *deduction*. I see this more as a broad development that encourages children to make sense of mathematics in a meaningful way, rather than performing a micro-analysis of various levels to be used in assessment that often provokes teachers to teach to the test. Broadly speaking, over time, I suggest that three major stages of mathematics occur: *practical mathematics*, *theoretical mathematics*, and *formal mathematics* (Tall, 2013, Chapter 1).

Practical mathematics occurs in the geometry of space and shape, through *recognition* and *description* of visual and spatial concepts. In arithmetic it occurs through the practical activities of number and measurement, including the *recognition* and *description* of properties of arithmetic of whole numbers, fractions, decimals, and negative numbers (which may be introduced in a practical fashion, before or after fractions).

Theoretical mathematics occurs in geometry with the introduction of *definitions* of figures and their practical constructions using a straight edge and a pair of compasses to draw lines and circles. It continues with the *deduction* of theorems in Euclidean geometry. In arithmetic, the shift to theoretical mathematics occurs as observed properties of arithmetic are used in the *definition* of properties of whole numbers such as even, odd, prime, composite and the *deduction* of theorems such as the fact that there is an infinity of primes and that every whole number is expressed uniquely as a product of primes. In algebra, the "rules of arithmetic" are used as *definitions* leading to the *deduction* of various algebraic identities using an algebraic form of proof.

Calculus blends together both embodiment and symbolism in a theoretical approach based on local straightness. It builds on embodiment and symbolism through the perception and *recognition* of the dynamic changing slope of a graph and the *description* of the slope function to *see* the slope functions.

However, to be able to compute the derivatives of composite functions such as $e^x \sin(x^2)$, which quickly become too complicated to guess by just looking, it becomes necessary to give a more coherent theoretical *definition* of the limit concept to be able to develop the rules of differentiation to be able to compute derivatives symbolically (Tall, 2012). This definition of a derivative may be formulated in a simple way, as the stabilized picture of the practical derivative $(f(x+h) - f(x))/h$ through the variable point x and for small values of h, as h is taken increasingly small.

It is therefore possible to have a *theoretical* approach to the calculus that does not introduce the concept of limit until it is seen by the learner to be a necessary construct to make sense of computing derivatives.

While the vast majority of the population can make sense of a *practical* approach to the calculus—as found in SimCalc or in a subsequent development—those who need mathematics in technical applications may be well-served by a *theoretical* approach to calculus and only the small minority who need to make sense of mathematical analysis may require a *formal* approach.

The precise nature of practical and theoretical approaches will change as the available technology evolves and affords new ways of making sense of the dynamic notions of continuity and the mathematics of change and variation. SimCalc is a pioneering beginning that has evolved as the technology has evolved. But where will it go in future?

It is time for a vision that builds on the evolution of ideas over the millennia and the use of dynamic interactive technology to *see* and *sense* the ideas of the local rate of change and of dynamic growth building from human perception and action to the frontiers of mathematical research.

The framework presented here, as an extension of the Kaput program, encompasses the full range of mathematical development while addressing the wider issues of individual freedom and democracy. It encourages the whole population to gain access to the practical mathematics required for mathematical literacy in a democratic society, for those requiring a more technical approach to build a theoretical form of mathematics of use in applications, and for the pure mathematician to retain an unyielding belief in the necessity of formal mathematics.

It builds on the vision of Jim Kaput to reveal the potential to move into the future, to take advantage of new technologies that give the learner a natural dynamic interface to manipulate enactive imagery and to communicate ideas socially using new technological modes of representation and communication.

7 Reflections

Looking back on this chapter in particular, and this book as a whole, I have chosen to consider the bigger picture of the Kaput program that is instantiated in the SimCalc program from which a much wider evolution of mathematical change and variation may grow. In using the framework of Kaput to review the practicalities of SimCalc and suggesting new developments for the future, I trust that the reader will not think that I fail to show respect to his memory.

On the contrary, it is the very robustness of the overall Kaput program that enables reflective criticism to encourage us to evolve from current ideas into a future as yet unknown. I affirm that much of my own development has benefited from his profound insights. Indeed, every paper of his that I have read—whether I understood or agreed with everything he said at the time—has contained quotable pearls of wisdom that have profoundly affected my own personal development.

Jim Kaput was the first person to alert me clearly to the active volcano of technology where the mathematical mountain is changing before our eyes. He first made me aware of the relevancy of different symbol systems, though, at the time, I did not fully understand his ideas with any clarity. He was also prescient in the way that he saw even his own insights would need to change as ideas evolve. His profound overall program contains, within it, the elements for this necessary evolution.

In practice, Kaput developed the SimCalc software to represent ideas in the mathematics of variability and change and to give democratic access to profound mathematical ideas not expressible within the standard curriculum. His program also extended beyond his remit for working with multiple representations of change and variability, to move on to "systematic forms of reasoning and argument to help establish the certainty, generality, and reliability of our mathematical assertions" (Kaput and Roschelle, this volume, p. 13).

It is appropriate to close this chapter with Jim Kaput's own vision, as follows:

> We see new technologies creating a possibility to reconnect mathematical representations and concepts to directly perceived phenomena, as well as to strengthen students' understanding of connections among different forms of mathematical representation. By starting from more familiar antecedents, such as graphs and motion, both in kinesthetic and cybernetic form, and developing towards more compact and formal mathematical representations, we see an opportunity to create a new path of access to mathematics that has too often remained the province of a narrow elite. (Kaput and Roschelle, this volume, p. 23)

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Conversation About SimCalc, Its Evolution and Lessons Along the Way

Eric Hamilton and Nora Sabelli

1 Why Do We Consider SimCalc a "Successful" Technology Worthy of the Volume? How Does Its "Success" Compare to that of Other NSF Curriculum Projects?

Eric Hamilton: SimCalc opened new ways to see the roles that learners play in their own learning. It exposed new possibilities and it equipped learners with tools that allowed them to drill into some large mathematical ideas that were beyond the typical reach of modern mathematics curriculum.

One significant development in learning technologies has involved participatory content development, by which learners become actively engaged, especially through simulations, in creating what they are learning about. This takes many forms that have multiplied since the advent of SimCalc, but few approaches have been as dynamic or as interesting as those appearing in SimCalc. By giving learners control of mathematical objects in simulations, SimCalc gave rise to new forms of learner agency. It helped learners shift the understanding of themselves as actively affecting mathematics—by placing mathematical objects into a manipulable social space. That meant new ways for learners to see themselves as mathematics "doers" and to see mathematics as responsive to their actions—and those of their peers. Of course, this required new representational systems given to kids and developed by kids. The systems were dynamic and mediated a new view—literally and metaphorically—of mathematics.

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Nora Sabelli: What is missing in your answer, from my point of view, is how the framing of calculus implied by SimCalc spoke to the education constituencies outside mathematics education and even outside education. The significance of SimCalc to them lies in two aspects of the role that "the mathematics of change" plays in education: one is as an opening to a deep mathematical way of thinking in ways that are consequential outside the discipline and not limited to the specific needs of a mathematics curriculum, and another is as a gatekeeper to higher education. SimCalc uses the first to open the second. Its transformative nature can be seen in the importance of helping teachers understand the content takes that SimCalc's implementation had in later work, and by a pedagogy that does not start by purely abstract concepts of calculus. Jim Kaput corrected me more than once when I used terms like "calculus" to describe SimCalc; he wanted to use rather the label "the mathematics of change." I tie Jim's approach to his learning about calculus pedagogy from the remedial College mathematics courses he often ran.

Eric Hamilton: Yes, your point is well-taken. The SimCalc formulation spoke to constituencies outside of mathematics education. It was a sort of vindication for people who suspected that they could have and should have experienced mathematics differently. Jim looked at mathematics through the gatekeeper lens you mention, whether in SimCalc or those courses, it is crucial to understanding his legacy. As he saw it, if youngsters could have more access to mathematics, and acquire and use advanced mathematical ideas proficiently, it was like giving them the keys to the gate. In some ways, the ensemble of simulations, mathematics of motion, social space for mathematics learning—was all part of a full-throated effort to make mathematics immediately accessible to all learners. It was like throwing out all the stops and finding whatever levers for seeing, learning and doing mathematics were possible.

But what was it like for you, coming into the NSF's education directorate as a chemist, trying to get your own thoughts around what sort of research would have the most impact in the field and for the most students? You had not been there long before SimCalc came to you, were you?

Nora Sabelli: When Jim brought the SimCalc idea to NSF, I was a new program officer working with advanced technologies. (For those that don't know me, I am a computational chemist, which is basically somebody that uses computers to invert matrices.) The initial arguments that Jim made for the affordances of the new forms of agency, did not fit well with the mathematics curriculum ideas targeted for the grades where the project would end up making most sense, nor did it conform to the expected innovation in the requirements for advanced technology uses in education. The rationale for the project was neither curriculum nor technology oriented; rather it was both intellectual—in terms of what mathematics learning should entail—and policy-oriented—in terms of what barriers to further study imply for society. Basically, many of the arguments being expressed now for broadening STEM education were at the basis of SimCalc, long before they were shown to be necessary.

SimCalc had to be "rescued" from curriculum programs and seen as pedagogical research that responded to needs beyond the mathematics curriculum per se. Do not underestimate the fact that many people did not believe that everyone could learn calculus—many still do not believe it. Also, your comments below on the problems with technology show why it could not be considered purely as a technology project. The feasibility and impact of the original ideas had to be demonstrated as high-risk, high-gain "research" in the AAT program before they were accepted as worthy of funding and a viable part of the mathematics "curriculum."

I still remember that in one of the first AAT Principal Investigators Meeting at NSF that I attended, SimCalc immediately got the attention of several senior scientists looking at other projects—biologists, if I remember correctly—who saw in its ideas "the makings of a real winner" that would impact all STEM education, and that may have led to parallel thinking in other disciplines. It would be interesting to know if and how this shift in focus at the start, since the proposal had to be rewritten, impacted the development of the ideas and curriculum materials.

2 Undoubtedly, SimCalc's Success Is Based on the Powerful and Relevant Idea It Embodies. Is the Power of This Idea that Led SimCalc to a 17-year History of Support? What Allowed SimCalc to Survive the Gauntlet of Different Awards, Reviews and Expectations? Has Its Success Depended on the Time It Has Taken to Implement and Test the Core Ideas?

Eric Hamilton: SimCalc was hobbled and limited by technological constraints in those early years, as I saw it, and these limitations contributed to a longer timeline than Jim ever wanted. Jim and the team were always struggling with hand-held bandwidth, memory size, and limitations in the programming and visualization tools of the day. These matters are important. They underscore that placing deep mathematical ideas with the grasp of learners requires certain technological affordances. Making ideas accessible drove the technology, in contrast to a more common pattern of educational technology driving the learning. The technology had to be able to support learners so that their intuitions and experience could give them power to deal with the mathematics of change. The idea transcended the specifics of the technology, but the technology was a key to the power of the idea. In other words, the SimCalc concept was larger than the technological containers available at the time. Or maybe it is better to say that the SimCalc framework was larger, because the framework structured many different powerful concepts that the SimCalc team and then SimCalc users connected.

Nora Sabelli: I understand your answer, but it does not really explain why SimCalc, in difference to many other projects, continued to push ahead and not be deterred by the continuous changes in its technology core. If the answer to "what it took to get SimCalc funded" is that there had to be a research-oriented home at NSF to see its potential, then the answer to this question may lay either with the expectation that research is an iterative process that builds on its short-term successes and failures, rather than a development that ends with a deliverable product, or with the commitment of the team to "solve a problem" rather than conducting a research or an R&D project.

If you think about it, a project like the 17-year SimCalc one, requires multiple (evolving) types of expertise and research interests—and therefore methods. The existence of teams capable of deploying this expertise and evolving their focus is not a given. In fact, NSF could play a more active role in promoting such evolution of work that builds on a strong intellectual component.

3 Is SimCalc's Long Time Frame also Related to Its Attempt to 'Radicalize' the Curriculum and the Mathematical Vision Behind It?

Eric Hamilton: I think that the long-term time frame is more associated with the technological work-arounds. The challenges that the SimCalc team faced were not only in enacting a new paradigm for teaching, learning, and mathematics in technology, it was also in creating the technology that could enact that vision, and the strive to make it make sense to a constituency of policy-makers, teachers, students, and funders.

Nora Sabelli: I would say that the second part of your answer is crucial... the striving to make sense to those charged with implementation. It is what I meant by Jim wanting to "solve a problem" and not "run a project." The audience for any evidence of a project's success, and the audience for a transformative view of the mathematics curriculum are different. Most likely, the evidence provided by SimCalc at the start and at the end of its long life are quite different, while in both cases based on the interplay between deep goals and more specific learning measures.

Eric Hamilton: I would add a couple of comments in reply. I remember in 1991, just after Luther Williams came to lead the Education and Human Resources Directorate, that Luther gave an important speech at a PI meeting. Everyone wanted to hear more about changes they felt he would be making, and PIs were more than a little intimidated. I was recently funded to run a center to address minority underrepresentation in STEM fields in Chicago. Luther spoke clearly, saying in exact words that are not hard to remember, "we are here to solve a problem, not to fund programs." Maybe that does not seem like such a radical concept now, but at the time, those were considered fighting words by grantees in the audience. In my naïve way, I did not understand how much Luther was challenging the way things were. Instead, I felt a powerful resonance—like the tumblers of a combination lock falling into place. Luther's pronouncement eventually played out in many ways, but he shared Jim's determination to use money to solve problems instead of to fund programs. It is fair to say that Luther cherished deeply and was profoundly informed and moved by Jim's vision and determination.

In this respect, some comments Dick Lesh of Indiana University made to me about Jim are also appropriate. Dick said he was always impressed by how Jim responded when he learned something new about how kids think or learn, or when he had to back out of one path and try another. There was a sense almost of giddiness that he had gone one step further in learning about kids thinking, and was a step closer to solving the problem of how to help kids navigate to success in acquiring and manipulating mathematics. Instead of trying to get teachers and students to use his program the way he conceptualized it, he was more interested in learning how to create pathways that would be most productive. If anything, as Dick would point out, because Jim was a bona fide research mathematician, he had plenty of reason to plow ahead with the way he thought kids should learn mathematics. But he knew that knowing mathematics was not the same thing as understanding how youngsters learn mathematics. In pushing forward with SimCalc, Jim Kaput was a deliberate and humble observer, "delighted," to use Dick Lesh's word, every time he saw something new about how kids acquire and use important mathematical ideas. That continual freshness about discovery helped to fuel SimCalc and keep it as a vibrant part of the EHR portfolio.

Nora Sabelli: To add to Dick's second point—Jim being a mathematician, is very important for science education, but not necessarily in the way education research uses scientists' deep understanding of their discipline as it is continuously evolving. We tend to use scientists, with a few meritorious exceptions, on the basis of curriculum needs, not on the basis of breaking barriers to deep and complex ideas. Chemistry, my discipline, plays a gatekeeping role for STEM in college.

4 Much of SimCalc's History Takes Place After the Main Technological Hurdles Were Conquered. Where the Technological and Pedagogical Advances Simultaneous? Was the Process of Having to Submit Many Proposals an Intellectual Help or a Hindrance? One Could Argue that Rather than a Hindrance, the Need to Rethink Approach and Research Steps Could Have Been of Intellectual Help

Eric Hamilton: All of the above. Well-written Requests for Proposals (RFPs) make ideas stronger and sharper. Nora, I was always impressed when I moved from the systemic reform division at NSF to the education research division to see how you framed ideas in an RFP—leaving the field both led and with more capacity to lead. At its best, a good RFP can contain guidance that is knowledgable, moving and practical. Good RFPs find ways to entice both new and current "performers," to use NSF's term, to come forward with fresh conceptualizations and questions responsive to whatever the evolving conditions of a research community.

Getting the right mix of projects, and getting them properly reviewed, is always the challenge for a funding agency, however. I think NSF generally has held the view that a diversified portfolio of research investments, not only across disciplines but also of varying size, would furnish the most promising yield of advances and voices in the research community. This still leaves open questions about the blend of large and small projects, with tradeoffs associated with each. When should an agency focus resources on a few large projects over a long period of time, versus encouraging a larger number of small projects? SimCalc was always in the mix of these types of conversations at NSF.

Another issue that is easier to observe in retrospect than in real-time goes back to the mismatch between concept and technology—what I think was a primary obstacle in SimCalc development. It is difficult to predict trendlines for hardware, software, and human resource costs in learning technology research investments. The mobile devices and networking tools available to Jim and the SimCalc team were evolving vehicles for the enduring concept of engagement in deep and participatory mathematics. Those vehicles came at a high overhead. A great deal of time and money was spent programming around limitations that eventually disappeared. A great deal of energy was devoted to a corporate partnership involving a bountiful good will but also cultural differences between academia and business—differences that were never fully resolved. I came to NSF's education research division shortly after SimCalc started making the rounds there.

My context was very different than yours. For several years before coming to the agency, I had been funded by NSF to run Young Scholars Projects where we drew middle and high school students with no more than a beginning algebra background into deep ideas of the infinitesimal, of summed rectangles, of tangent slopes, and exponential functions. We had seventh through tenth grade students engaged in multiple unexpected discovery events involving calculus. We did not call it calculus, and we did not tell kids they were learning derivatives or integrals, because our interest was not in differentiating or integrating per se but rather in introducing students to a path of producing their own numerical method tools to understand mathematical behavior.

Numerical methods (as rendered in the circuitry of a calculator), of course, is the very path Jim and SimCalc relied on to put mathematical ideas in reach of more kids. In our case, we drew youngsters into the experience of writing computer programs that, for example, summed large numbers of rectangles or that experimented with the delta term of a secant slope approximation. These were sublime discovery experiments that traded on intuition, though perhaps not quite as sublime as Jim's formulations of the mathematics of change for kids that age and younger. But the mathematics was very visible and arrived at powerfully with cheap desktop computers that kids built from the bottom up.

Computers in the 1990s simply were far more powerful than TI graphing calculators. Jim was on a moral mission, to make mathematics affordable and reachable. He never wavered from his conviction that the right vehicle was a mobile rather than desktop technology. But there was a great practical tradeoff. I was frustrated that he devoted so many cognitive cycles dealing with technology limitations. We never had a conversation about SimCalc without references he made to bandwidth and coding challenges. I had come out of a computer science department, so perhaps these discussions and observing how quickly technology could get out of the way after bringing kids to new mathematics shaped my view that SimCalc was limited by the constraints of mobile devices. I saw the technology challenges perhaps more prominently than others at NSF. The path that the SimCalc team took to overcoming those challenges successfully to produce a scalable approach to the mathematics of variation is a tremendous legacy. My guess is that you may concur with some of these observations but point out other issues that you see missing in this analysis.

Nora Sabelli: Though what you say is all true, and I agree with most of it, there are structural issues that perhaps should be discussed, since they may point to what is missing in the research portfolio, not only of NSF but of education research in general. But first, a comment on the readiness of technology. The technology "to think" with in education exists in the work of the disciplines we want to teach, independently and collectively. So there is no real reason not to think about introducing deep STEM concepts; though there may be a delay in translating them into robust materials and software, often the basic new ways of looking at the concepts do not depend solely on the technology. A good example of the way I look at the interplay between technology and concepts in science is illustrated by Elliot Soloway's Modelit software—unfortunately I think a victim of technology advances. What Modelit did for modeling in science was to move from quantitative analysis to qualitative analysis—does the oxygen level in water go up or down when the temperature goes up? By a little or by a lot? The computer does the programming, and the concepts, not the numbers, are emphasized.

Education research is still to a some degree imbued with linear thinking and with the twin concepts of "dissemination" and "scaling up" successful materials, both of which imply that if something—curriculum or technology—is worthwhile, it will be picked up by somebody else and carried to the finished line. It forgets that both the initial R&D and follow-up scaling-up need research support. Linear thinking focuses on the developers of materials, while the more up-to-date nonlinear thinking adds implementers' needs to the mix. Materials are "adopted" by practice, but also "adapted" to local conditions. Research gives general answers, but all education is local.

The linear view is not limited to education, see for example the emphasis being placed in medicine on "translation research," that is to say, *research on the process of translation to practice itself*. The ideas are starting to take hold in education as implementation research, which involves collaborations between researchers and practitioners, including the policy, organizational and contextual situations in which practice takes place.

In fact, the process of developing appropriate scaling-up research strategies that focus on "education for all" (or the democratization of mathematics, as Jim would have said) is still going on with an emphasis on "implementation research" in many fields beyond education. So one lesson about long timelines for research intended to transform practice is the time it takes to build on the importance of the interplay between pedagogies and ideas.

You may remember a short-lived interagency program (IERI, Interagency Education Research Initiative) that originated in the Office of Science and Technology Policy of the White House. IERI had two characteristics that speak to my structural concerns: it looked for already promising materials to focus on their adaptation to practice contexts (the translational research aspect) and had a award length of five years, to allow for sufficient in depth experimentation. In fact, SimCalc received an IERI award for its second phase. It would be interesting to know if the award was for five years; if it was, how the five year life affected the project.

5 Is Another of the Lessons That Curriculum Development About Deep Ideas and not About Specific Curricular Needs Should Be Treated as Research, and a Strategy that NSF Needs to Employ More Often as Science Becomes More Interdisciplinary?

Nora Sabelli: I believe that NSF needs to re-think curriculum development at least as much as this question suggests, and start developing the powerful ideas enabled by new advances in the sciences—the barriers between scientific disciplines and between them and mathematics are not serving the needs of the future very well.

SimCalc and the mathematics of change are well positioned to help do this. Among the original group of projects generated by the SimCalc team was work by TERC that introduced physical models (such as trains in train tracks) that paralleled SimCalc simulations. But the curricular separation between physics and mathematics did not generate enough interest to allow for exploration and building on the real (physics) versus ideal (mathematics) nature of the concepts involved. I am still sorry that this aspect was not studied.

Eric Hamilton: In fact, with new tools coming online, I believe it is timely to consider placing curriculum development more firmly in the hands of teachers and students. There is a great deal of room for using current technology to emancipate and empower teachers and students to play a more participatory role in curriculum development. Yes, of course, move across grades. And move across people we expect to be the curriculum makers.

Nora Sabelli: Can you say why you expect that placing curriculum development more firmly in the hands of teachers and students would help with respect to a more integrated curriculum

There is complementarity between new ideas coming from content advances and from cognitive research, and new pedagogical approaches coming from the experiences of practice. I do not see this as an issue of top down versus bottom up. Both are equally important, in my view, but for different reasons. Bottom up, unfortunately, is often dependent of policies that mistrust teachers and begrudge them the time needed to exercise their profession. And, of course, on how teachers are educated.

Eric Hamilton: Agree!

Nora Sabelli: One way of looking at your question is that the three areas have to be in place for real, sustainable and meaningful advances to take place. I agree that what happens in the classroom needs to recognize the crucial role of the appropriation by teachers in its development. But I do not expect the teachers by themselves to fully conceptualize content in novel ways that reflect the advances being made in content areas, not expect them to have the time to do the cognitive research on pedagogy that learning scientists do. I would like to see teacher as highly regarded professionals like the physicians whose main job is to diagnose the needs of the patient, use her knowledge of current research to devise a plan of action, and closely monitor the success of the treatment. In education terms, I expect the materials to be "educative" for the teacher, as Joe Krajcik would say, or, as Louis Gomez proposed, to be templates that make visible the design criteria so that teachers can effectively adapt it to their students' needs. Fidelity of implementation implies fidelity to the design principles and lesson learned from use, but cannot bypass adaptation to local contexts.

Another way of looking at the question, more policy and funding oriented, is to read it as top down via agency goals and RFPs, versus bottom up as field-initiated research. Again, I see both as needed, though in practice they are not as integrated as they should be. SimCalc may be a unique example where the top down—rethinking math education—and the bottom-up—instantiating it—were merged. But I don't see similar examples on other areas that need it. Take the quintessential interdisciplinary topic of nanoscience, where NSF has issued education RFPs. The process of "rethinking" the science curriculum is different from the processes of teaching nanoscience as a topic and integrating nanoscience into disciplinary curricula, and may not arise in response to an RFP, or in the review process for submissions to that RFP unless specifically sought. That process is taking place instead in the science frameworks movement and may, in effect, be reflected in the new standards derived from them; we just have to wait and see. There has to be a more top-down push for "science" as opposed to separate scientific disciplines—and I consider mathematics as one of the disciplines.

Eric Hamilton: Let me respond to your question about teachers and students being more involved in content and curriculum development.

I do think that there is a lot of implicit top-down structure and subordination in curriculum development that places teachers—and students—on the "down" side, as in curriculum is handed down for teachers to teach and students to learn. It is not a way of doing things that fully fits the times, I believe. We live in an odd season of more tools than we have ever had to produce or generate content (exciting), more reliance on borrowed or depleted fiscal resources than in generations (depressing), and more teacher angst and dissatisfaction with accountability demands they often consider demeaning and always consider limiting (depressing). There is also more knowledge produced by our global society than ever to "pack in" to the school experience (challenging). More digital tools to engage learners, individually and socially to reveal mathematical and scientific structure (exciting) and more digital authoring tools than ever (also exciting). There are many ways that all of these variables can shake out, but a path I see is one whereby both teachers and students become far more integrated into the process of customizing the development of digital curriculum artifacts to meet local needs within a broader accountability and standards context. For example, the Common Core State Standards in Mathematics (CCSSM) have been increasingly adopted around the country. Whether or not CCSSM reflects lessons from the work of Jim Kaput and others is an important question. But there is no question that with frameworks of this nature, and with media-making tools, teachers are taking more control over producing curriculum materials for students to match the needs of their students and the accountability requirements they face. My argument—and the subject of research we carry out in Los Angeles—is that we should not only acknowledge an ascendant role of teachers in generating curriculum content, but actively furnish support and a creative space for them, even as they work with visionaries like Jim. We are finding that when teachers engage students in helping to produce curriculum materials, all kinds of positive and truly delightful dynamics emerge. Effectively, the tools add a massive infusion of problem-solving capital, imagination, unanticipated approaches to recurrent problems, and evolving technological expertise. And teachers find themselves far more empowered, and students find themselves more empowered, when they became engaged in a more active role in curriculum making. We are at the very early stages of watching what happens when we give permission for different agency and identity for teachers and students, and it is exciting. I believe this sense of creative agency in producing mathematical ideas is of kindred spirit to SimCalc. As far as that last term you used, "integrated curriculum," though, I cannot comment. I see a long path before we can reach an equilibrium on curriculum.

6 How Relevant Is the Strategy of Scaling-Up by Small Steps to Its Success? Was the Choice of Partners and Implementation Sites Instrumental to SimCalc's Success?

Eric Hamilton: I think that the scale-up strategy of small steps evolved in a set of conditions during the SimCalc developmental years that are not necessarily so applicable currently. The testing that SRI undertook, I believe, eventually was crucial—it provided the empirical results needed to convince skeptics and to make developers more attuned to making approaches work across a cross-section of populations. But I resist the notion of planned scale-up, though, in that it still has a top-down feel to it. In contrast, I believe that most innovation is a blend of emergent practice that cannot be externally organized by a funder and evolving policy structures that change every month. The miss ratio on all the scale-up funding by NSF was too high, I think. SimCalc was one of the few successes.

Nora Sabelli: I don't really know. I think it was of help to SimCalc, since it allowed for planning work and proposals in response to advances in field conversation about methodology and adaptation versus plain scaling-up. But this may have been an advantage while the time lost in terms of research flexibility may have been more relevant. I am a chemist, and I know in my field that the goals and methods of research evolve with the research. I also know that the first year of any research project is spent on building the team and calibrating the equipment—in the case of education, assessments and survey protocols, not counting parental agreements. This leaves the second year's work to be oriented towards obtaining data, and the third year to analyzing that data. But in education, one year of data is not ideal, and having to recompete with data that has not been subjected to iterations may stop promising transformative projects in their tracks. We know that it takes at least two years for teachers to become comfortable enough with new pedagogies to deploy them well.

7 In Summary, Why Is SimCalc Worth Its $20M Investment? How Can We Choose Technologies and Curricula that Merit Such an Investment?

Nora Sabelli: In the years since NSF began funding SimCalc, the agency has invested well over two billion research dollars in STEM education research, and the investment in SimCalc represented less than 1 % of that amount. *Its impact on the conceptual frameworks for research, for altering and deepening the learning of mathematics through affordances of technology, are far greater than such a figure might suggest.*

Choosing technologies and curricula to merit an investment is what you call the holy grail of the program staff at agencies that fund this work. There are no magic bullets or easy answers, but there are some pathways that are important. *But focusing on a powerful and intellectually central idea, one that is at the same time an educational gatekeeper, like the mathematics of change, and expanding its access is a good starting criterion.* SimCalc benefitted from a political focus on mathematics. Other NSF-funded work (in science), that could have paralleled SimCalc, suffered by being "off the radar" for schools.

Eric Hamilton: Scale-up implies intentionality to alter the behavior of large systems and the people in them. In that sense, it is unmistakably an impositional intentionality that takes on and tries—usually in vain, in turned out—to surmount the tendency of systems to resist change. NSF has devoted large sums to efforts that could not attain planned goals because the nature of systems and system change has (still) not been fully understood.

The portfolio analyses that NSF has undertaken in recent years should be juxtaposed with an analysis of "self-organizing scale-up" constructs that have arisen in a social media age, including viral media, crowd-sourcing, and rapidly communicated and improvised constructs such as classroom-flipping.

Nora Sabelli: I agree with your views, but think that there should be a policy-oriented study with questions appropriate to what we ask: how to choose technologies and curricula that merit such an investment? While I was at NSF, we supported a discussion meeting with members of the Education Commission of the States, mostly State legislators in education committees, and researchers using technology to advance education. Jim Kaput was one of the researchers invited. There were no presentations, just conversations about needs and about what research could and could not do. The most salient outcome of the meeting was, in my view, a comment from a legislator that the most important thing he would take back was that, if well implemented, technology can help everybody learn calculus. The details did not matter, it was the possibility of democratizing mathematics that resonated. None of the other examples received a similar response; there was interest but not an “aha!” moment for them. Which justifies Jim’s ideas that solving a problem should be the goal, and emphasizing the point. And that often we aim too low.

Eric Hamilton: My final thought on the scale-up question begins with the thought that the different conceptual and technological tools he marshaled gives creative thinkers like Jim far more running room than he would have had in a pre-digital era. Beyond that, his scale-up impact is not written in the language of electrons and technology but in terms of fairness, access to mathematical power, and the human quest to learn.

Nora Sabelli: Agreed.

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