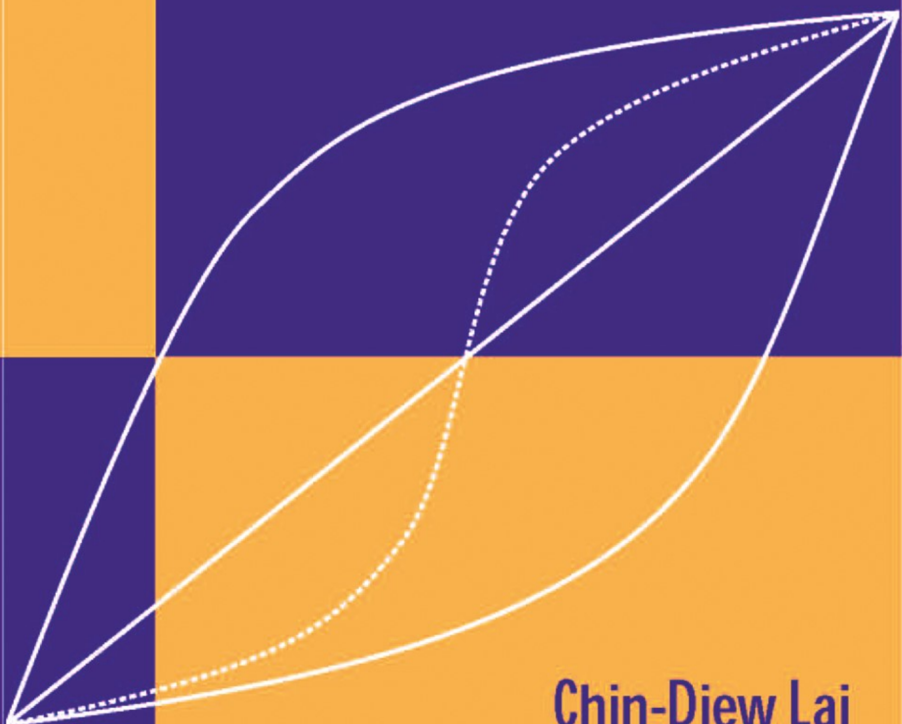
Stochastic Ageing and Dependence for Reliability



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Springer

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Foreword by Richard E. Barlow

Springer

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Stochastic Ageing and Dependence for Reliability

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This book is dedicated to
Ai Ing, Joseph, Eugene, Serena,
my brother Chin-Yii and my parents
C. D. L

Wenhong, Jessica, Harry, William
M. X

Foreword

I first met Min Xie in the 1980s at Linkoping University in Sweden. He was working with Professor Bo Bergman, the Professor in Quality at the University. As I recall, Min Xie was a very serious student of reliability theory at the time. He was very familiar with the book *Mathematical Theory of Reliability* by myself and Frank Proschan.

My first meeting with C. D. Lai was in 1999 at Massey University in New Zealand. I was impressed then by his serious interest in research.

The subject of this monograph is ageing and dependence in the context of reliability. Both of these ideas are important and controversial. Ageing is a phenomenon experienced by both machines and people. There has been a great deal of progress in understanding ageing relative to people by molecular biologists such as Giusseppe Attardi at the California Institute of Technology. Other researchers have even tried to apply ideas in mathematical reliability theory to biological ageing. Unfortunately, it seems that this is not a useful activity. This is because biological organisms are capable of self-repair and reproduction while machines at this point in time are not.

Probabilistic dependence has also been discussed at length by many mathematicians and philosophers. One of the best classical mathematical discussions can be found in *Statistical Independence in Probability Analysis and Number Theory* by Mark Kac (1959). However, this work is solely applied mathematics and leaves the subject somewhat mysterious at the philosophical level which is also the level at which applications need to be made.

From another point of view, de Finetti, in 1937, for the first time presented a rigorous and systematic treatment of the concept of exchangeability together with the fundamental result which became known as "de Finetti's representation theorem." [See Kotz and Johnson (1992)]. De Finetti's paper illuminates the conditions under which frequencies may be related to subjective probabilities (that is, probabilities based on judgment) and also formalizes this connection. It replaces the classical notion of observations assumed to be "independent and identically distributed with unknown distribution" by the concept of exchangeable observations. This helps to resolve the mystery behind the ideas of independence and dependence. De Finetti also helped in the understanding of conditional probability. Conditional dependence is closely tied to finite populations (i.e., all populations in this world) while unconditional independence is relative to conceptually infinite populations.

To illustrate, consider n binary random quantities $(x_1, x_2, , x_n)$ judged *a priori* to be exchangeable, i.e., distributed with the hypergeometric distribution with parameters (N, S) where $S = \sum_{i=1}^{N} x_i$ is unknown since in this case observations $(x_{n+1}, x_{n+2}, , x_N)$ are not available. Although N is known, S is unknown. We are interested in inference concerning S. Now $(x_1, x_2, ...x_n)$ are *a priori* dependent, conditional on S. However, if S has a prior distribution which is judged binomial with parameters N (the known population size) and

n specified, then $(x_1, x_2, , x_n)$ are *a priori* unconditionally independent, with joint probability

$$\prod_{i=1}^{n} \rho^{x_i} (1-\rho)^{1-x_i}$$

Since the binomial distribution with parameters (N, ρ) is only suitable for conceptually infinite populations, we begin to see the connection between independence and infinite populations. (In the binomial case, N would be the sample size, not the population size.) This is presented as an exercise on page 52 of Barlow (1998). It was pointed out to me by a colleague, Max Mendel. Of course once $(x_1, x_2, , x_n)$ are observed they are no longer random quantities. Any judgment concerning S would require knowledge of the problem at hand and this judgment is only partly a mathematical problem.

The present monograph deals with life distributions belonging to various classes of failure (hazard) rate functions and mean residual life functions. The so-called 'bathtub' distributions are featured prominently and a brief introduction of the Bayesian approach on ageing concepts is given. The text provides a lot of material on test procedures and bivariate life distributions, with various concepts and measures of dependence. The material concerning reliability of coherent systems with positively dependent components is very important as component lifetimes are generally dependent in practice.

The book should be considered as a very useful reference. Results of the last three decades are brought together without delving into unnecessary detail. The reader is referred to papers, which are listed in the bibliography. It covers most of results in the literature pertaining to ageing classes and bivariate life distributions; so it can be regarded as a compendium of ageing concepts. It is encyclopedic in scope, contains much information, and will be useful to researchers in reliability engineering and other disciplines.

Berkeley, September 2005 Richard E. Barlow

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Preface

Reliability is an important and challenging subject, which involves the disciplines of science and engineering. Researchers in both these fields have been working on reliability problems for several decades. The aim of this book is to summarize various ageing and dependence concepts of the lifetimes that have been widely studied in the field of reliability.

Chapter 1 provides a summary of the book and notations and acronyms are also listed for easy reference later on. Chapter 2 deals with various concepts of stochastic ageing starting with the definition of the failure rate function (or hazard function). In this book, we will use the term failure rate instead of hazard rate, which is more common in survival analysis. Part of the reason is because most abbreviations such as IFR/DFR/IFRA/DFRA, etc., contain FR which stands for failure rate. We think that more confusion will be caused if the abbreviations are changed. Moreover, the acronym 'failure rate' is more commonly used in reliability engineering, especially for non-repairable systems.

Chapters 3–7 deal with some specific concepts of ageing and lifetime distributions. In particular, we consider bathtub shaped life distributions in Chapter 3. Existing models are grouped and summarized with their properties listed. Chapter 4 considers the mean residual lifetime function which is an important measure of ageing in reliability applications. Chapter 5 deals with the Weibull distribution and its generalizations that can be flexible in modeling lifetime data. Chapter 6 considers ageing concepts for discrete distributions. Chapter 7 summarizes statistical tests of ageing.

Chapter 8 extends the univariate ageing concepts to two or more variables. A brief introduction to the Bayesian approach to multivariate ageing in terms of majorization and Schur-concavity is given.

Dependence concepts, dependence orderings and measures of dependence are dealt with in Chapter 9. This is an extensive and important topic which caught the attention of many authors in recent years. We emphasize the positive (negative) quadrant dependence as this property is verifiable and realistic in many situations. All relevant results concerning dependence are summarized in this chapter. However, most of these results are related to statistical concepts and some are theoretical probability applications. We expect further research and applications in this area to be carried out by researchers. As a follow-up, Chapter 10 discusses the reliability of coherent systems with positively dependent components. We feel that this topic is a very important one in reliability applications.

Last but not least, in Chapter 11, we list 33 data sets of failure times or survival times. This could be useful for researchers and students in their future study in this field. The book ends with a large collection of references with nearly eight hundred entries.

It is our aim to provide a comprehensive treatment of both ageing and dependence concepts with emphasis on reliability and survival analysis. Proofs of many results are omitted, especially when they are either obvious or are long. The interested readers may refer to references listed in the bibliography section for detailed proofs. The readers should, however, have some basic knowledge in probability and statistics before reading this book.

Apart from the excellent classical text by Barlow and Proschan (1981), Gertsbach (1989) is another good book on statistical reliability. There is an excellent book on ageing, written from a Bayesian point of view, by Spizzichino (2001). Also on dependence concepts and stochastic ageing, there is an excellent book by Shaked and Shanthikumar (1994). On multivariate dependence concepts, Joe (1997) has provided us an excellent monograph.

We hope that both reliability researchers and practitioners find the book useful for reference and for some new ideas. This book will also be useful for graduate students in reliability or applied probability.

This book is a summary of the work carried out by many people. It would be too long a list if we acknowledge them one by one – most of the names can be found in the reference list at the end of the book. We wish to thank, in particular, Mr. John Kimmel of Springer who had guided us through the whole project with much encouragement and professionalism. We also wish to record our our sincere thanks to several anonymous reviewers for their constructive comments. We appreciate very much the help from all of them, and other colleagues and students of us.

C. D. Lai, Massey University, New Zealand
M. Xie, National University of Singapore, Singapore
September 2005

Contents

1

Introduction

1.1 Aim and Scope of the Book

As the title suggests, the main aim of this book is to bring together different facets of ageing concepts of the lifetime of a device or a system. An ageing concept largely describes how a device ages with time. Though in most cases, ageing has an averse effect on a 'product', there are some other cases in which ageing is beneficial. These ageing concepts have a direct impact on the behaviors of two important reliability measures (i) the failure rate function and (ii) the mean residual life function. These reliability measures are important in maintenance planning, replacement planning, resource allocation and other reliability related decisions.

Looking from another angle, these ageing concepts were defined in the first place through the characteristics of these two named functions, especially their shapes. Another important aspect of a reliability study is that components of a system may not always be mutually independent so a description of how two or more component lifetimes depend on one another may be of interest. Further, it will be helpful to find some indices that quantify the degree or strength of relationships between them (according to a defined concept such as the linear dependence). Thus we will consider in this book some measures of dependence that are relevant in reliability or survival analysis.

The present book provides a comprehensive treatment of both ageing and dependence with the emphasis on reliability and survival analysis. Proofs of many results are not given, but extensive references are provided, so interested readers can refer to them. The book assumes a basic course in mathematical statistics and some familiarity of the classical reliability text by Barlow and Proschan (1981) "*Statistical Theory of Reliability and Life Testing: Probability Models*". It is intended that reliability researchers and practitioners may find the book useful for reference and new ideas. This book will also be useful for graduate students in reliability or applied probability.

1.2 Brief Overview

Chapter 1 – Introduction

This chapter provides a bird's eye view of the book which focuses on the usefulness of ageing and dependence concepts in real life, particularly in reliability engineering and survival analysis. We also give three lists of acronyms and nomenclatures that will appear frequently throughout the book.

Chapter 2 – Concepts and applications of stochastic ageing

This chapter begins with defining the failure rate function (hazard rate) which forms one of the pillars for reliability and survival analysis. We then introduce various ageing concepts based on the reliability characteristics such as the failure rate function, survival function or the mean residual life function. Their relative strengths are compared and a chain of relationships is given. Several examples of lifetime distributions together with their ageing properties are given. Special attention is given to discuss the ageing behavior of finite mixtures of life distributions. Partial orderings based on ageing concepts are also introduced. The chapter ends with a discussion on some existing or potential applications.

Chapter 3 – Bathtub shape life distributions

It is common for failure rate function (hazard rate) to have a bathtub shape. This chapter presents theoretical and practical discussion on this and presents several life distributions that can be used to model a bathtub curve. The change point (turning point) of the bathtub shaped failure rate function is also discussed as it plays an important role in establishing the optimal burn-in time of a product. Applications of bathtub models are also indicated.

Chapter 4 – Mean residual lifetime (MRL)

This chapter focuses on the use of MRL, which is an important measure in reliability applications. The traditional reliability analysis has been based on the failure rate function, but usually it is the residual life that is of great interest when one considers repair and replacement strategies. The former relates only to the risk of immediate failure whereas the mean residual life summarizes the entire remaining life distribution. We investigate how the shapes of MRL are related to the shapes of the failure rate functions; these relationships provide us a strategy to determine the optimal burn-in time and to solve other maintenance problems. In addition to burn-in time determination, the chapter also lists several other applications in diverse disciplines including demography and social studies.

Chapter 5 – Weibull and generalized distributions

The Weibull distribution is a generalization of the exponential distribution that has no ageing. However, the Weibull distribution can only be used to model increasing or decreasing failure rate distributions so it is not sufficiently flexible. Various extensions or generalizations have been added in the reliability literature to give rise to more flexible distributions. This chapter discusses various properties of the Weibull distribution and its generalizations. Reliability operations such as mixtures and formation of additive and multiplicative systems from the Weibull family are also discussed.

Chapter 6 – Ageing concepts for discrete distributions

When the failure time is in the form of discrete measurement such as on–off switching, a discrete distribution should be used. We review some common discrete failure time models together with their discrete ageing properties. As expected, most of these properties are analogous to their continuous counterparts. We also include an alternative definition of a failure rate which is closer to the continuous time failure rate than the traditional definition of a discrete failure rate.

Chapter 7 – Tests of ageing

An important problem in practice is to test the constant failure rate (hazard rate) versus other forms of ageing property. Many tests have been proposed in the past three decades and these tests will be discussed in this chapter.

Chapter 8 – Bivariate and multivariate ageing concepts

Univariate ageing concepts are generalized to bivariate and multivariate distributions. Various versions are available for the same marginal ageing concepts. Tests of bivariate ageing concepts are also briefly given. We also introduce the Bayesian approach to multivariate ageing through majorization and Schur-concavity of a joint survival function.

Chapter 9 – Dependence concepts and measures of dependence

Various types of dependence among two or more lifetime variables are considered. We pay a special attention to the so-called 'positive dependence' concept such as 'association', positively quadrant dependent, etc. Several bivariate distributions with positive dependence property are given to illustrate the theory. A chain of relations among positive ageing concepts is also presented.

Included also is a discussion of dependence orderings that give the relative strength of dependence between two pairs of lifetime random variables with respect to the same concept. For example, we say F is more positively quadrant dependent than G if the joint survival function of the former dominates the latter.

Various measures of dependence between two variables are available in the literature, e.g., Pearson's correlation, rank correlations, etc. We give an overview of these measures and show how these numerical indices vary with or are related to the dependence concepts.

Some local dependence measures, as opposed to the traditional global measures of dependence, are also introduced.

Chapter 10 – Reliability systems with dependent or independent components

Further analysis of the use of stochastic dependence in reliability studies will be presented. For example, dependence is common among the component lifetimes of a system, thus it plays an important role in redundancy improvement. In particular, we consider the reliability performance of parallel and series systems of two components with dependent component lifetimes. We discuss how the efficiency of redundancy is often determined by whether they are positively or negatively dependent.

For a system with independent components, we examine whether active spare allocation at the component level is superior (in some sense) to active spare at the system level. We also compare two k-out-of-n systems with different k or n using some partial ordering concepts.

Chapter 11 – Failure time data sets

We have collected 33 data sets of failure times or survival times which are now given in this chapter. These data sets are arranged according to the ageing classes they belong to. One of the primary aims of this chapter is to illustrate the existence of real data sets that have either bathtub or upside-down bathtub shaped failure rates. Also, the data may be a suitable testing ground for sophisticated techniques that the original author did not think of.

1.3 Acronyms and Nomenclatures

In this book, we follow a general convention regarding the shape of a function. We say that a function is increasing if it is nondecreasing. Similarly, we say a function is decreasing if it is nonincreasing.

We now provide three lists of acronyms and nomenclatures: (i) general, (ii) ageing concepts, and (iii) dependence concepts.

Table 1.1. General List

$B(a,b)$	The beta function of two parameters
cdf	Cumulative distribution function
D	A class of decreasing functions
E	Expectation
$E_1(t)$	Exponential integral function
$\mathcal{N}$	Set of all integers
$\mathcal{N}^+$	Set of all positive integers
$\Gamma(x)$	Gamma function
$F(x)$	(Cumulative) distribution function
$f(x) = F'(x)$	Density function if exists
$\in$	Belongs to a class or in a class
$H_0:$	Null hypothesis
$H_1:$	Alternative hypothesis
I	A class of increasing functions
$I(a) = \begin{cases} 0 \text{ if } a \leq 0 \\ 1, \text{ if } a > 0 \end{cases}$	Indicator function
i.i.d.	Independent and identically distributed
log	Natural log (based on e)
MTTF	Mean time to failure
μ	Mean of lifetime variable (Mean time to failure)
μ_X	Mean of the random variable X
μ'_k	kth moment about the origin (zero)
pdf	Probability density function
Pr(E)	Probability of event E to occur
R	Set of real numbers
R^+	Set of positive real numbers
T	Lifetime variable
$\tau_{k\|n}$	System lifetime of a k-out-of-n system
$X_1, X_2, ..., X_n$	Random sample from a population with distribution function F
$X_{(1)} < X_{(2)} < ... < X_{(n)}$	Order statistics from a sample of size n
$X_{i:n}$	ith order statistic of a k-out-of-n system
$[x]$	The largest integer that is less than or equal to x
$[x]^+$	The largest positive integer that is less than or equal to x
$\bar{X}$	Sample mean
U_n	U-statistic
var	Variance
$\leq_*$	Partial ordering with respect to an ageing characteristic *

Table 1.2. Ageing Concepts and Dependence List

$\bar{F}(t) = 1 - F(t)$	Survival function of a lifetime random varaible
$\bar{F}(x\|t) = \bar{F}(x+t)/\bar{F}(t)$	Conditional reliability of a unit of age t
$\bar{F}(x, y)$	Joint survival function of X and Y
$F_n(x)$	Empirical cdf
$F_X(x)$ $(F_Y(y))$	cdf of marginal random variable X (Y)
$\mu(t)$	Mean residual life function
MRL	Mean residual life
$\mu_{(1)}$	MTTF of the series system of two components
$\mu_{(2)}$	MTTF of the parallel system of two components
$R(\cdot)$	Reliability function or survival function
$r(t)$	Failure rate (hazard rate) function
T	Lifetime random varaible
T_1	Lifetime of a series system of two components
T_2	Lifetime of a parallel system of two components
τ	Change point
X	Lifetime random varaible
IFR (DFR)	Increasing (decreasing) failure rate
IFRA (DFRA)	Increasing (decreasing) failure rate average
MBT	Modified bathtub shaped
NBU (NWU)	New better (worse) than used
NBUE (NWUE)	New better (worse) than used in expectation
BT (UBT)	Bathtub shaped (Upside-down bathtub shape)
DIMRL (IDMRL)	Decreasing (increasing) then increasing (decreasing) mean residual life.
NWBUE (NBWUE)	New worse then better than used in expectation (New better then worse than used in expectation)
DMRLHA	Decreasing mean residual life in harmony average
DPRL-α(IPRL-α)	Decreasing (Increasing) α-percentile residual life

Table 1.3. Dependence Concepts List

PQD (NQD)	Positive (Negative) quadrant dependence
LTD (RTI)	Left-tail decreasing (Right-tail increasing)
SI (alias PRD)	Stochastically increasing (alias positively regression dependent)
RCSI (LCSD)	Right corner set increasing (Left corner set decreasing)
TP_2 (alias LRD)	Totally positive of order 2 (alias likelihood ratio dependent)
WPQD	Weakly positive quadrant dependent
PDO	Positive dependent ordering
RR_2	Reverse regular of order 2
$\bar{F} = 1 - F$	$\bar{F}$ survival function, F cumulative distribution function
ρ	Pearson product-moment correlation coefficient
τ	Kendall's tau
ρ_S	Spearson's rho

2

Concepts and Applications of Stochastic Ageing

2.1 Introduction

The concept of ageing is very important in reliability analysis. ‘No ageing’ means that the age of a component has no effect on the distribution of residual lifetime of the component. ‘Positive ageing’ (also known as ‘averse ageing’) describes the situation where residual lifetime tends to decrease, in some probabilistic sense, with increasing age of a component. This situation is common in reliability engineering as components tend to become worse with time due to increased wear and tear. On the other hand, ‘negative ageing’ has an opposite effect on the residual lifetime. ‘Negative ageing’ is also known as ‘beneficial ageing’. Although this is less common, when a system undergoes regular testing and improvement, there are cases for which we have reliability growth phenomenon. Though we concentrate on positive ageing in this book, it is being understood that a parallel development of negative ageing can also be carried out.

Concepts of ageing describe how a component or system improves or deteriorates with age. Many classes of life distributions are categorized or defined in the literature according to their ageing properties. An important aspect of such classifications is that the exponential distribution is nearly always a member of each class. The notion of stochastic ageing plays an important role in any reliability analysis and many test statistics have been developed in the literature for testing exponentiality against different ageing alternatives. Our aim in this chapter is to provide an overview of these developments.

By ‘life distributions’ we mean those for which negative values do not occur, i.e., $F(x) = 0$ for $x < 0$. The nonnegative variate X is thought of as the time to failure (or death) of an electrical or mechanical component (or organism), but other interpretations may be possible – an inter-event time is normally necessarily positive.

In this chapter, we focus on classes of life distributions based on notions of ageing–IFR (increasing failure rate) is perhaps the best-known, but we shall meet several others also, and study their interrelationships whenever possible.

The chapter may serve as a continuation of the ageing concepts developed in the pioneering book Barlow and Proschan (1981), which was first printed by Holt, Reinhart and Winston in 1975.

The major parts of the current chapter are devoted to

- Introducing different ageing characteristics,
- Classifications of life classes based on various ageing characteristics and establishing their interrelationships,
- Failure rates of mixtures of distributions,
- Elementary properties of these life classes,
- Partial orderings of two life distributions based on comparison of their ageing properties.

From the definitions of the life distribution classes, results may be derived concerning such things as properties of systems (based upon properties of components), bounds for survival functions, moment inequalities, and algorithms for use in maintenance policies (Hollander and Proschan, 1984).

Most readers will know that statistical theory applied to distributions of lifetime lengths plays an important part in both the reliability engineering and the biometrics literature. We may also note a third applications area: Heckman and Singer (1986) review econometric work on duration variables (e.g., lengths of periods of unemployment, or time intervals between purchases of a certain good), much of which, they say, has borrowed freely and often uncritically from reliability theory and biostatistics.

Section 2.2 gives characterizations of lifetime distributions by their survival, failure rate or mean residual life functions. In Section 2.3, we list several commonly used life distributions together with their basic properties. In Section 2.4 we give formal definitions of ten basic ageing notions and their interrelationships together with a table of summary furnished with key references. Section 2.5 discusses the properties of some of these basic ageing classes and Section 2.6 is devoted to the non-monotonic failure rate classes such as the bathtub and upside-down bathtub life distributions, which are important in reliability applications. Section 2.7 briefly presents some additional but less known ageing classes. In Section 2.8, we consider failure rates of mixtures of life distributions. This has an important application in burn-in. Section 2.9 provides an introduction to partial ordering through which the strength of the ageing property of the two life distributions within the same class is compared. Section 2.10 considers briefly the matter of relative ageing of two life distributions. Relative ageing is really a form of partial ordering. We discuss in Section 2.11 how the relationship between the sth and the $(s+1)$th equilibrium distribution can be used to describe the relationship between the shape of the failure rate and the shape of mean residual life function of a distribution. Finally in Section 2.12, we tidy up the loose ends on stochastic ageing and the section ends with some remarks concerning future research directions that may bridge the theory and applications.

Abbreviations

The following table of acronyms and abbreviations will be a useful reference. Although this has largely been given in Chapter 1, the list here gives a more exhaustive coverage for ageing concepts.

Table 2.1. List of Ageing Class Abbreviations

Abbreviation	**Ageing Class**
BT (UBT)	Bathtub shape (Upside-down bathtub shape)
DMRL (IMRL)	Decreasing mean residual life (Increasing mean residual life)
HNBUE	Harmonically new better than used in expectation
(HNWUE)	(Harmonically new worse than used in expectation)
IFR (DFR)	Increasing failure rate (Decreasing failure rate)
IFRA (DFRA)	Increasing failure rate average (Decreasing failure rate average)
$\mathcal{L}$-class	Laplace class of distributions
NBU (NWU)	New better than used (New worse than used)
NBUE	New better than used in expectation
(NWUE)	(New worse than used in expectation)
NBUC	New better than used in convex ordering
(NWUC)	(New worse than used in convex ordering)
NBUFR	New better than used in failure rate
(NWUFR)	(New worse than used in failure rate)
NBUFRA	New better than used in failure rate average
(NWUFRA)	New worse than used in failure rate average
NBWUE	New better then worse than used in expectation
(NWBUE)	(New worse then better than used in expectation)

We note that NBUFRA is also known as NBAFR.

2.2 Characterizations of Lifetime Distributions

Rather than $F(t)$, we often think of $\bar{F}(t) = \Pr(X > t) = 1 - F(t)$, which is known as the survival function or reliability function. Here, X denotes the lifetime of a component, i.e., time to first failure. The expected value of X is denoted by μ. The function

$$\bar{F}(x \,|\, t) = \bar{F}(t + x)/\bar{F}(t), \quad x, t \geq 0, \tag{2.1}$$

represents the survival function of a unit of age t, i.e., the conditional probability that a unit of age t will survive for an additional x units of time. The expected value of the remaining (residual) life, at age t, is $\mu(t) = E(X - t \,|X > t)$ which may be shown to be $\int_0^\infty \bar{F}(x \,|\, t)\, dx$. It is obvious that $\mu(0) = \mu$.

When $F'(t) = f(t)$ exists, we can define the failure rate (hazard rate or force of mortality) of a component as

$$r(t) = f(t)/\bar{F}(t) \tag{2.2}$$

for t such that $\bar{F}(t) > 0$. This can also be written as

$$r(t) = \lim_{\Delta \to 0} \frac{\Pr(t \le X < t + \Delta |\ t \le X)}{\Delta}. \tag{2.3}$$

Thus for small Δ, $r(t)\Delta$ is approximately the probability of a failure occurring in $(t, t + \Delta]$ given no failure has occurred in $(0, t]$.

It follows that, if $r(t)$ exists, then

$$-\log \bar{F}(t) = \int_0^t r(x)\, dx \tag{2.4}$$

represents the cumulative failure (hazard) rate which may be designated by $H(t)$. Equivalently

$$\bar{F}(t) = \exp\left\{-\int_0^t r(x)\, dx\right\} = \exp\left\{-H(t)\right\}. \tag{2.5}$$

A lifetime distribution can also be characterized by its mean residual life (MRL) defined by

$$\mu(t) = E(X - t \,|\, X > t) \tag{2.6}$$

through

$$\bar{F}(t) = \frac{\mu}{\mu(t)} \exp\left\{-\int_0^t \mu(x)^{-1}\, dx\right\},\ t \ge 0. \tag{2.7}$$

We will discuss MRL more fully in Chapter 4.

In short, a lifetime distribution may be characterized by $\bar{F}(t)$, the conditional survival function $\bar{F}(x \,|\, t)$, $r(t)$ or $\mu(t)$. In addition, Galambos and Hagwood (1992) have shown that a life distribution may also be characterized by the second moment of the residual life $E[(X - t)^2 \,|\, X > t]$.

Remarks on terminology

Calling the function $r(t)$ the failure rate in (2.2) could cause some confusion if this terminology is not adequately explained. The confusion aries because another 'failure rate' is also used by some authors in the context of a point process of failures. We now follow the approach of Thompson (1981) to highlight this confusion and attempt to provide a distinction between the two concepts.

Let $N(t)$ denote the number of failures in the interval $(0, t]$. Set $M(t) = EN(t)$ and let $\xi(t) = M'(t)$ and so $\xi(t)$ is the instantaneous rate of change of the expected number of failure with respect to time; thus we may call $\xi(t)$ the failure rate of the process.

Another characteristic of interest in a failure process is

$$\lambda(t) = \lim_{\Delta \to 0} \frac{\Pr[N(t, t+\Delta) \geq 1]}{\Delta}. \tag{2.8}$$

If $\lambda(t)$ exists, then for small Δ, $\lambda(t)\Delta$ is approximately the probability of failure in the interval $(t, t+\Delta]$. Assuming the simultaneous failures do not occur (which is true for most applications), $\xi(t) = \lambda(t)$, if they exist.

Clearly, $r(t)$ is not the same as $\lambda(t)$ since the $r(t)\Delta$ as defined via (2.2) is (approximately) a conditional probability of a failure in $(t, t+\Delta]$ whereas $\lambda(t)\Delta$ is not conditional on the event prior to t.

Under the framework of a stochastic point process, Thompson (1981) discussed basic ways to characterize reliability. The distinction between the failure rate of a process, useful for repairable systems, and the failure rate of a distribution, useful for nonrepairable systems is drawn.

In the point process literature, the failure rate of the process $\xi(t)$ or $\lambda(t)$ is generally known as the intensity function. In reliability modeling, this is sometimes called the 'rate of occurrence of failure (ROCOF)' for repairable systems so that it is not to be confused with the traditional failure rate concept for the lifetime distribution. For further discussion, see Ascher and Feingold (1984).

Note that in the case of a homogeneous Poisson process, the failure rate of the process is λ which is also the failure rate of the the exponential distribution. We wish to emphasize here the 'failure rate' used in this book is the failure rate of a life distribution F defined in (2.2); it is *not* the failure rate of a point process of failures.

One of the reasons for our usage of the acronym 'failure rate' instead of 'hazard rate' in this book is that IHR (DHR) is rarely used in the literature on classification of life distributions. The 'near' universal use of the ageing notions such as IFR (DFR) is consistent with our choice in calling $r(t)$ the failure rate of a life distribution.

2.2.1 Shape of a Failure Rate Function

We assume that the failure rate function $r(t)$ is a real-valued differentiable function $r(t) : R^+ \to R^+$. As usual, by increasing we mean nondecreasing and by decreasing, we mean nonincreasing. $r(t)$ is said to be

(1) strictly increasing if $r'(t) > 0$ for all t and is denoted by I;

(2) strictly decreasing if $r'(t) < 0$ for all t and is denoted by D;

(3) bathtub shaped if $r'(t) < 0$ for $t \in (0, t_0)$, $r'(t_0) = 0$, $r'(t) > 0$ for $t > t_0$, and is denoted by BT;

(4) upside-down bathtub shaped if $r'(t) > 0$ for $t \in (0, t_0)$, $r'(t_0) = 0$, $r'(t) > 0$ for $t > t_0$, and is denoted by UBT;

(5) modified bathtub shaped if $r(t)$ is first increasing and then bathtub shaped, and is denoted by MBT;

(6) roller-coaster shaped if there exist n consecutive change points $0 < t_1 < t_2 < ... < t_n < \infty$ such that in each interval $[t_{j-1}, t_j]$, $1 \leq j \leq n+1$, where $t_0 = 0, t_{n+1} = \infty$, $r(t)$ is strictly monotone and it has opposite monotonicity in any two adjacent such intervals. For detailed description of physical basis for the roller-coaster failure shaped failure rate, see Wong (1988, 1989, 1991).

Remark 1: We wish to point out that the points at which the derivative of the failure rate function $r(t)$ or the mean residual life function $\mu(t)$ changes sign are called the 'change points' in this book. The term 'change point' is used in a different context in the statistical literature.

Remark 2: Some authors include $r(t) =$ constant in the middle interval for their definitions of BT and UBT. We will incorporate this more general and yet more realistic definition in Chapter 3.

Remark 3: A MBT shape may be considered as a curve increases at the beginning and then follows a bathtub shape, see for example, Gupta and Warren (2001). So a MBT curve can be considered as a roller-coaster curve.

Remark 4: The roller-coaster failure rate curve was first promoted by Wong (1989) who observed that the failure rates of many electronic systems have generally decreasing failure rates with failure humps on them. Thus, these failure rate curves manifested a roller-coaster shape. However, we have yet to find any published failure rate data of this shape that can be used for our study. Further, there is no well-known lifetime distribution that we know of which has a failure rate function that exhibits this shape.

It is convenient to extend the above shape definitions to an arbitrary function. To that end, we say that a function $g \in$ I, D, BT, UBT or MBT accordingly as its shape has the appropriate characteristics. For example, $g \in$ BT means that g is first decreasing and then increasing.

Many of the failure rate functions have complex expressions because of the integral in the denominator and thus the determination of the shape is not straightforward. Glaser (1980) presented a method to determine the shape of $r(t)$ with at most one turning point. His method uses the density function instead of the failure rate.

Note: A turning point of a function is a point at which the function has a local maximum or a local minimum.

Define

$$\eta(t) = -\frac{f'(t)}{f(t)}. \tag{2.9}$$

We will see later that this eta function plays an important role in our study of the failure rate function $r(t)$.

The relationships between $r(t)$ and $\eta(t)$ are given by

$$\frac{d}{dt}\log r(t) = r(t) - \eta(t) \tag{2.10}$$

and

$$\left[\frac{1}{r(t)}\right]' = \frac{\eta(t)}{r(t)} - 1. \tag{2.11}$$

Here we obviously assume that $f(t)$ is a twice differentiable positive density function on $(0, \infty)$.

The above equations also suggest that the turning point of $r(t)$ is a solution of the equation $\eta(t) = r(t)$. We can also verify that $\lim_{t\to\infty} r(t) = \lim_{t\to\infty} \eta(t)$.

Theorem 2.1: (Glaser, 1980). Let $\eta(t)$ be defined as in (2.9).

(a) If $\eta(t) \in$ I, then $r(t)$ is of type I.

(b) If $\eta(t) \in$ D, then $r(t)$ is of type D.

(c) If $\eta(t) \in$ BT and (i) if there exists a y_0 such that $r'(y_0) = 0$, then $r(t)$ is of type BT and (ii) otherwise $r(t)$ is of type I.

(d) If $\eta(t) \in$ UBT and (i) if there exists a y_0 such that $r'(y_0) = 0$, then $r(t)$ is UBT and (ii) otherwise $r(t)$ is of type D.

Proof: Define the reciprocal of the failure rate by

$$g(t) = 1/r(t) = R(t)/f(t). \tag{2.12}$$

It follows that its derivative given in (2.11) may be written as

$$g'(t) = g(t)\eta(t) - 1 \tag{2.13}$$

where $\eta(t)$ is defined as above. Without going into detail, it can be shown that

$$g'(t) = \int_t^\infty [f(y)/f(t)][\eta(t) - \eta(y)]\, dy. \tag{2.14}$$

(It has been pointed out that the preceding equation implicitly requires that $f'(t)$ be integrable at infinity).

We can now proceed to prove the theorem.

(a) The assumption that $\eta'(t) > 0$ for all $t > 0$ implies, from (2.14), that $g'(t) < 0$ for all $t > 0$, which from (2.12), implies $r(t) \in$ I.

(b) $\eta'(t) < 0 \Rightarrow g'(t) > 0$ for all $t > 0$ so $r(t) \in$ D.

(c)(i) Let t_0 be the change point of η so that $\eta'(t_0)$=0. Claim $g''(y_0) < 0$. Since $g'(y_0) = 0$, it follows from (2.13) that $g''(y_0) = g(y_0)\eta'(y_0)$. Therefore, $g''(y_0) < 0 \Leftrightarrow \eta'(y_0) < 0 \Leftrightarrow y_0 < t_0$. Suppose $y_0 \geq t_0$. By (2.14) and the assumption, it is obvious that $g'(t) < 0$ for all $t > t_0$. Therefore,

$g'(y_0) < 0$, which is a contradiction. Thus $y_0 < t_0$ and $g''(y_0) < 0$. It is now obvious that there is a unique root in $(0, \infty)$ to $g(y) = 0$, i.e., $y = y_0$, and g attains a maximum at this point. This implies $r(t) \in$ BT with the turning point $t^* = y_0$.

(c)(ii) Here we have either $g'(t) > 0$ for all $t > 0$ or $g'(t) < 0$ for all $t > 0$. From (2.14) we have that $g'(t) < 0$ for all $t \geq t_0$. Therefore $g'(t) < 0$ for all $t > 0$ so $r(t) \in$ I.

(d) The proof is analogous to that of (c) and will be omitted here.

It is noted in Glaser (1980) that in the last two cases, determining the existence of y_0 leaves us with the original difficulty of evaluating the derivative of $r(t)$. However, we may simplify the problem in many situations with the following lemma.

Lemma 2.1: Let $\varepsilon = \lim_{t \to 0} f(t)$ and $\delta = \lim_{t \to 0} g(t)\eta(t)$, where $g(t) = 1/r(t)$.

- Suppose $\eta \in$ BT, then
  (a) if either $\varepsilon = 0$ or $\delta < 1$, then $r(t) \in$ I.
  (b) if either $\varepsilon = \infty$ or $\delta > 1$, then $r(t) \in$ BT.
- Suppose $\eta \in$ UBT, then
  (a) if either $\varepsilon = 0$ or $\delta < 1$, then $r(t) \in$ UBT.
  (b) if either $\varepsilon = \infty$ or $\delta > 1$, then $r(t) \in$ D.

Gupta (2001) used Glaser's theorem to determine the shapes of several lifetime distributions that include the lognormal, inverse Gaussian, mixture of inverse Gaussians, power quadratic exponential families, mixture of gammas, etc.

In the proof of Theorem 2.1 above, Glaser showed that the change point of $r(t)$ occurs before the change point of $\eta(t)$. This finding has an important impact on the relationship between the shape $\mu(t)$ and that of $r(t)$. We will follow up this matter in Section 4.5.

Extension of Glaser's Result

Gupta and Warren (2001) generalized the result of Glaser to the case where $r(t)$ has two or more turning points. To achieve this, they first gave the following theorem which relates the turning points of $r(t)$ with those of $\eta(t)$.

Theorem 2.2: Let $\eta(t)$ defined as on (2.9), i.e., $\eta(t) = -\frac{f'(t)}{f(t)}$ and $f(t)$ is a twice differentiable positive density on $(0, \infty)$. If $\eta'(t)$ has zeros at $z_1, z_2, ..., z_n$ (n finite) such that $z_1 < z_2 <, ..., < z_n$, then the the equation $r'(t) = 0$ has at most one solution on $[z_{k-1}, z_k]$ for $k = 1, ..., n$ with $z_0 = 0$. Thus $r(t)$ has at most n changes of monotonicity.

Proof: It follows from (2.14) that

$$g'(t)f(t) = \int_t^{\infty} f(y)[\eta(t) - \eta(y)]\, dy.$$

Since $f(t) > 0$ for all $t > 0$, the sign and zeros of g', and therefore of r', are completely determined by the integral of the right side of the equation. We next designate this integral by

$$s(t) = \int_t^\infty f(y)[\eta(t) - \eta(y)]\,dy. \tag{2.15}$$

We note that the zeros of s are precisely the critical (change) points of r. It can be verified that $s'(t) = \eta'(t)\bar{F}(t)$ so both the sign and zeros of s' and η' are the same. That is, both have identical monotonicity.

By the given assumption, η is monotonic on $[z_{k-1}, z_k]$. Since s and η have identical monotonicity, s is monotonic on each interval $[z_{k-1}, z_k]$, such that the expression $s(t) = 0$ has at most one solution on that interval. Using the fact that the zeros and sign of r' are determined by s, we conclude that $r'(t) = 0$ has at most one solution on $[z_{k-1}, z_k]$. Thus the proof is completed.

The following theorem of Gupta and Warren (2001) is a generalization of Glaser (1980) and is useful when the shape of η is known and the number of critical points of r is known.

Theorem 2.3:

1. Suppose $\eta \in$ UBT. Then
   (a) If r' has no zeros, then $r(t) \in$ I.
   (b) If r' has one zero, then $r(t)$ is strictly increasing except at one point or $r(t) \in$ B.
   (c) If r' has two zeros, then $r(t) \in$ BT.
2. Suppose η is bathtub then upside-down bathtub. Then
   (a) If r' has no zeros, then $r(t) \in$ I.
   (b) If r' has one zero, then $r(t)$ is strictly decreasing except at one point or $r(t) \in$ UBT.
   (c) If r' has two zeros, then $r(t)$ is bathtub then upside-down bathtub.

Proof: See Gupta and Warren (2001).

The Glaser's extension will be applicable when considering the gamma mixtures with common scale parameter in Section 2.8.

2.3 Ageing Distributions

There are many lifetime distributions that have been proposed. Below are a selected few that have appeared more frequently in the literature. We do not think a comprehensive study of these distributions is warranted in the current text as most of them can be found in Johnson et al. (1994, 1995). Thus, only basic properties that are related to reliability are briefly given below. Several of these distributions will be further studied within an appropriate context throughout Chapters 3–5.

2.3.1 Exponential

The exponential (or negative exponential) distribution is applied in a very wide variety of statistical procedures. Currently among the most prominent applications are in the field of life-testing. The density function is

$$f(t) = \lambda e^{-\lambda t}, \ \lambda > 0, t \geq 0 \tag{2.16}$$

and

$$\bar{F}(x \mid t) = \bar{F}(x), \ \text{for all } x, t \geq 0$$

which means that its survival probability over an additional period of duration x is the same regardless of its present age. It describes a component that does not age with time. In case where such a simple structure is not adequate, a modification of the exponential distribution (often a Weibull distribution) is then used.

Also, it has a constant failure rate, i.e., $r(t) = \lambda, \ \text{for all } t \geq 0$.

The exponential distribution is a special case of the gamma, Weibull, Gompertz, linear failure rate and the exponential-geometric distributions to be presented below. It is a common member of nearly every known ageing class. It plays an important role in tests of stochastic ageing which will be discussed in Chapter 7. Lastly, we note that

$$E(X) = \frac{1}{\lambda}, \quad \text{var}(X) = \frac{\alpha}{\lambda^2}.$$

In general, the rth moment about zero is

$$\mu'_r = \frac{\Gamma(r+1)}{\lambda^r}.$$

2.3.2 Gamma

The density function of a standard two-parameter gamma distribution is

$$f(t) = \frac{\lambda^\alpha t^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda t}, \ \alpha, \lambda > 0. \tag{2.17}$$

If $\alpha = 1$, (2.17) reduces to an exponential distribution discussed above. In fact, the gamma distribution can be constructed from the exponential by taking powers of the Laplace transform of the latter. If α is a positive integer, we have an Erlang distribution. Moreover, if $\alpha = \nu/2$, we obtain a chi-square distribution with ν degrees of freedom.

The gamma distribution appears naturally in the theory associated with normally distributed random variables as the distribution of the sum of squares of independent standard normal variables.

For general α, the distribution function does not have a closed form. However, when α is a positive integer, $F(t)$ may be written in a closed form as

$$F(t) = 1 - \sum_{i=0}^{\alpha-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t}, \text{ for } t \geq 0. \tag{2.18}$$

The rth moment about zero of the gamma distribution is

$$\mu_r' = \frac{\Gamma(\alpha + r)}{\lambda^r \Gamma(\alpha)}, \; r = 1, 2, ...$$

In particular,

$$E(X) = \frac{\alpha}{\lambda}, \quad \text{var}(X) = \frac{\alpha}{\lambda^2}.$$

It can be shown, by a change of variable, that

$$r(t)^{-1} = \int_0^\infty \left(1 + \frac{u}{t}\right)^{\alpha-1} e^{-\lambda u} du.$$

It follows that $r(t)^{-1}$ is increasing for $0 < \alpha \leq 1$, decreasing for $\alpha \geq 1$. Thus $r(t)$ is increasing for $\alpha \geq 1$ and decreasing for $0 < \alpha \leq 1$. The shape of $r(t)$ can be determined through Glaser's eta function easily since

$$\eta(t) = -\frac{f'(t)}{f(t)} = \lambda - \frac{\alpha - 1}{t}. \tag{2.19}$$

Here η is increasing for $\alpha > 1$, constant for $\alpha = 1$, and decreasing for $0 < \alpha < 1$ and thus the shape of $r(t)$ is confirmed as stated above.

We refer the reader to Johnson et al. (1994, Chapter 17) for other facets of this well known lifetime distribution.

Mixtures of gamma distributions will be considered in Section 2.8.2.

2.3.3 Truncated Normal

The density function of a (positively) truncated normal is given by

$$f(t) = \frac{1}{a\sigma\sqrt{2\pi}} e^{-(t-\mu)^2/2\sigma^2}, \text{ for } 0 \leq t < \infty, \tag{2.20}$$

where $\sigma > 0$, $-\infty < \mu < \infty$, $a = \int_0^\infty (1/\sigma\sqrt{2\pi}) e^{-(t-\mu)^2/2\sigma^2}\, dt$.

The mean is

$$E(X) = \mu + \frac{\sigma\phi\left(\frac{-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{-\mu}{\sigma}\right)} = \mu + \frac{\sigma\phi\left(\frac{-\mu}{\sigma}\right)}{a}$$

where $\phi(\cdot), \Phi(\cdot)$ are, respectively, the density and distribution function of the standard normal random variable. Here μ is the mean of the normal distribution. Clearly, $E(X) > \mu$ and $\text{var}(X) < \sigma^2$. If $\mu - 3\sigma \gg 0$, then a is close to 1 and $E(X) \cong \mu$ and $\text{var}(X) \cong \sigma^2$.

Davis (1952), after examining failure data for a wide variety of items, has shown empirically that items manufactured and tested under close control may be fitted with truncated normal life distributions of the form (2.20).

We note that

$$\log f(t) = -\log(a\sigma\sqrt{2\pi}) - \frac{(t-\mu)^2}{2\sigma^2},\ t \geq 0 \tag{2.21}$$

is a concave function on $[0, \infty)$ and thus F is IFR (Barlow and Proschan, 1981, p. 77).

Though the expression for $r(t)$ is complicated, Navarro and Hernandez (2004) noted that the following:

1. $r'(t) = (r(t) - (t-\mu)/\sigma^2)r(t)$,
2. $r(t) > (t-\mu)/\sigma^2$,
3. $r(t)$ increases to ∞ as $t \to \infty$,
4. $\lim_{t\to\infty} r'(t) = 1/\sigma^2$,

and other properties.

The distribution here is a singly truncated normal from below. We note that various other types of normal truncations have been investigated (see, e.g., Johnson et al. (1994, pp. 156-162).

2.3.4 Weibull

The Weibull distribution is named after the Swedish physicist Waloddi Weibull, who in 1939 used it to represent the distribution of the breaking strength of materials and in 1951 for a variety of other applications. It is perhaps the most frequently used lifetime model in the reliability literature. Hallinan (1993) gave a comprehensive review of its properties and applications. Chapter 21 of Johnson et al. (1994) is devoted to this distribution. A recent monograph by Murthy et al. (2003) gives nearly every facet regarding Weibull and its related distributions. The survival function of the two-parameter Weibull is

$$\bar{F}(t) = \exp\{-(\lambda t)^\alpha\},\ \alpha, \lambda > 0. \tag{2.22}$$

When $\alpha = 1$, the Weibull distribution reduces to an exponential distribution. In fact, if X has an exponential distribution with parameter 1, then $X^{1/\alpha}/\lambda$ has the survival function (2.22).

The rth moment about the zero of the Weibull distribution is

$$\mu_r' = \frac{\Gamma\left(\frac{r}{\alpha}+1\right)}{\lambda^r}.$$

In particular, the mean and variance are, respectively,

$$E(X) = \frac{1}{\lambda}\Gamma\left(\frac{1}{\alpha}+1\right),$$

$$\text{var}(X) = \frac{1}{\lambda^2}\left\{\Gamma\left(\frac{2}{\alpha}+1\right) - \Gamma\left(\frac{1}{\alpha}+1\right)^2\right\}.$$

An important characteristic of the Weibull distribution is that its failure rate $r(t)$ has a simple form:

$$r(t) = \alpha\lambda(\lambda t)^{\alpha-1}. \tag{2.23}$$

It follows that $r(t)$ is increasing in t for $\alpha \geq 1$ and decreasing for $\alpha \leq 1$.

Mixtures of the Weibull distribution are considered in Section 2.8 below. A more detailed study on the Weibull and its related distributions will be given in Chapter 5.

2.3.5 Lognormal

The lognormal distribution is sometimes called the antilognormal distribution. This alternative name has some logical basis in that it is not the distribution of the logarithm of a normal variable (this is not even always real) but of an exponential (that is, antilogarithm) function of such a variable. In other words, if $\log X$ has a normal distribution, then X is said to have a lognormal distribution. However, 'lognormal' is most commonly used and we will follow this practice.

The cdf of the lognormal distribution is given by

$$F(t) = \Phi\left\{\frac{\log t - \alpha}{\sigma}\right\},\ \sigma > 0, t \geq 0, \tag{2.24}$$

where $\Phi(\cdot)$ denotes the standardized normal distribution function. The density function is

$$f(t) = (t\sqrt{2\pi}\sigma)^{-1}\exp[-(\log t - \alpha)^2/2\sigma^2],\quad t \geq 0. \tag{2.25}$$

The rth moment of X about the origin is

$$\mu_r' = \exp\left(r\alpha + \frac{1}{2}r^2\sigma^2\right).$$

The failure rate function of the lognormal has been shown as

$$r(t) = \frac{(1/\sqrt{2\pi}t\sigma)\exp\left\{-(\log at)^2/2\sigma^2\right\}}{1-\Phi\left\{\log(at)/\sigma\right\}}, \tag{2.26}$$

where $a = e^{-\alpha}$.

Although the expression of $r(t)$ is quite complicated, $\eta(t)$ is however quite simple, namely,

$$\eta(t) = -\frac{f'(t)}{f(t)} = \frac{1}{\sigma^2 t}(\sigma^2 + \log t - \alpha). \tag{2.27}$$

An application of Glaser's theorem shows that $r(t)$ is UBT. Also, $\lim_{t\to 0} r(t) = 0$ and $\lim_{t\to\infty} r(t) = 0$ (Sweet, 1990). For estimation of the change point, see Gupta et al. (1997).

Chapter 14 of Johnson et al. (1994) gives a full discussion on this distribution. Its failure rate and mean residual life will be discussed further in Chapter 3 and Chapter 4, respectively.

2.3.6 Birnbaum-Saunders

Birnbaum and Saunders (1969a,b) introduced a lifetime distribution

$$F(t) = \Phi\left\{\frac{1}{\alpha}\cdot\left[\left(\frac{t}{\beta}\right)^{1/2} - \left(\frac{t}{\beta}\right)^{-1/2}\right]\right\} = \Phi\left\{\frac{1}{\alpha}\xi\left(\frac{t}{\beta}\right)\right\},\ t > 0, \tag{2.28}$$

where $\xi(t) = t^{1/2} - t^{-1/2}$, $\alpha, \beta > 0$ and $\Phi(\cdot)$ denotes the cdf of the standard normal. The density function is given by

$$f(t) = (\alpha\beta)^{-1}(2\pi)^{-1/2}\xi'\left(\frac{t}{\beta}\right)\exp\left\{-\frac{1}{2\alpha^2}\xi^2\left(\frac{t}{\beta}\right)\right\},\ t > 0. \tag{2.29}$$

Desmond (1986) noted that in this distributional form, derived by Birnbaum-Saunders (1969a,b), had been previously obtained by Freudenthal and Shinozuka (1961) with a somewhat different parametrization.

The random variable X that corresponds to (2.28) is a simple transformation of the the standard normal variable

$$X = \beta\left[\tfrac{1}{2}U\alpha + \sqrt{(\tfrac{1}{2}U\alpha^2) + 1}\right]^2.$$

The above variable arises from a model representing the time to failure of material subject to a cyclically repeated stress pattern.

It can be shown that X has a Birnbaum-Saunders distribution if

$$\frac{1}{\alpha}\left(\sqrt{\frac{X}{\beta}} - \sqrt{\frac{\beta}{X}}\right) \tag{2.30}$$

has a standard normal. From this expression, Chang and Tang (1994a,b) proposed a simple random variate generating algorithm for this distribution.

Surprisingly, the mean and variance of X are quite simple. These are given, respectively, by

$$E(X) = \beta\left(\frac{1}{2}\alpha^2 + 1\right), \quad \text{var}(X) = \beta^2\alpha^2\left(\frac{5}{4}\alpha^2 + 1\right).$$

The failure rate function $r(t)$ cannot be given explicitly. Since both the lognormal distribution and the Birnbaum-Saunders distribution can be derived from the normal distribution, we expect a similarity between the two in this respect. Indeed, a comparison between the failure rates of the Birnbaum and Saunders and the lognormal distribution was given in Nelson (1990). While the failure rate of Birnbaum and Saunders is zero at $t = 0$, then increases to a maximum for some t_0 and finally decreases to a finite positive value (i.e., $r(t) \in$ UBT) when $\beta = 1$ and $\alpha > 0.8$, the failure rate of the lognormal also has a UBT shape but decreases to zero. It was shown in Chang and Tang (1993) that $r(t) \in$ I when $\alpha \to 0$. Some recent work on this distribution can be found in Dupuis and Mills (1998), Rieck (1999) and Ng et al. (2003). The last discussed the maximum likelihood estimates and a modification of the moment estimates of the two parameters, and proposed a bias-correction method for these estimates. See Chapter 33 of Johnson et al. (1995) for a more detailed discussion on the properties of this distribution.

2.3.7 Inverse Gaussian

The name ‘inverse Gaussian’ was first applied to a certain class of distributions by Tweedie (1947), who noted the inverse relationship between the cumulant generating functions of these distributions and those of Gaussian (normal) distributions. The same class of distributions was derived by Wald (1947) as an asymptotic form of average sample number in sequential analysis and hence the distribution is also known as the Wald distribution. The inverse Gaussian distribution was popularized as a lifetime model by Chhikara and Folks (1977).

The density function of the inverse Gaussian is

$$f(t) = \sqrt{\frac{\lambda}{2\pi t^3}} \cdot \exp\left[-\frac{\lambda}{2\mu^2 t}(t-\mu)^2\right], \ \lambda > 0, t \geq 0. \tag{2.31}$$

The corresponding distribution function is

$$F(t) = \Phi\left\{\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu} - 1\right)\right\} + e^{2\lambda/\mu}\Phi\left\{-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu} + 1\right)\right\}. \tag{2.32}$$

The mean and variance of the distribution are, respectively,

$$E(X) = \mu, \quad \text{var}(X) = \frac{\mu^3}{\lambda}.$$

Again, the expression for $r(t)$ is quite complicated. However, one can verify easily that

$$\eta(t) = \frac{3\mu^2 t + \lambda(t^2 - \mu^2)}{2\mu^2 t^2}. \tag{2.33}$$

It follows from Theorem 2.1 that that $r(t)$ is UBT. Further, $\lim_{t\to 0} r(t) = 0$ and $\lim_{t\to\infty} r(t) = c \neq 0$.

For further properties see Chhikara and Folks (1989) and Chapter 15 of Johnson et al. (1994).

2.3.8 Gompertz

Gompertz (1825) derived possibly the earliest probability model for human mortality. He postulated that "the average exhaustion of a man's power to avoid death to be such that at the end of equal infinitely small intervals of time he lost equal portions of his remaining power to oppose destruction which he had at the commencement of these intervals." From this hypothesis Gompertz deduced the force of mortality or the failure rate function as

$$r(t) = Bc^t,\ t \geq 0, B > 0, c \geq 0, \tag{2.34}$$

which, when solved as a differential equation, yields the survival function as

$$\bar{F}(t) = e^{-B(c^t - 1)/\log c},\ t \geq 0. \tag{2.35}$$

The density function is easily obtained as

$$f(t) = Bc^t e^{-B(c^t-1)/\log c},\ t \geq 0, B > 0, c \geq 0. \tag{2.36}$$

It is clear that $r(t)$ increases (decreases) in t if $c > 1$ ($c < 1$). For $c = 1$, $r(t) = B$ showing that the Gompertz distribution includes the exponential as its special case.

In discussing reliability theory of ageing and longevity, Gavrilov and Gavrilova (2001) stated that while the Weibull distribution is more commonly applicable for failure times of technical devices, the Gompertz distribution is more common for biological systems.

2.3.9 Makeham

The survival function of the Makeham distribution is

$$\bar{F}(t) = \exp[-\alpha t + (\beta/\lambda)(e^{\lambda t} - 1)],\ t \geq 0, \alpha, \beta, \lambda > 0, \tag{2.37}$$

and its failure rate function is

$$r(t) = \alpha + \beta e^{\lambda t}. \tag{2.38}$$

It is clear that $r(t) \in$ I.

In the literature, the Makeham distribution is more often called the Gompertz-Makeham distribution. It is a generalization of the Gompertz distribution. Letting $c = e$ in (2.35), we clearly obtain a special case of (2.37). This distribution is widely used in life insurance, mortality studies and survival analysis in general. For a brief review, see Al-Hussaini et al. (2000).

2.3.10 Linear Failure Rate

The survival function of the linear failure rate function is given by

$$\bar{F}(t) = \exp\{-\lambda_1 t - \lambda_2 t^2/2\},\ \lambda_1, \lambda_2, t \geq 0 \tag{2.39}$$

with

$$r(t) = \lambda_1 + \lambda_2 t. \tag{2.40}$$

The linear failure rate distribution aries often in reliability literature probably because of its simple form.

This simple two-parameter model in the IFR class is a simple special case of the quadratic failure rate model (see Section 3.4.1) and a generalization of the exponential distribution in a direction distinct from the gamma and Weibull discussed in this section. While, in the IFR case, both gamma and Weibull require the failure rate to be zero at $t = 0$, the linear failure rate model has $r(0) = \lambda_1 > 0$, thus providing a gentler transition from the constant failure rate to the strict IFR property.

The linear failure rate distribution was motivated by its application to human survival data (Kodlin, 1967, Carbone et al., 1967). Its properties have been studied by several authors, notably Bain (1974) and Sen and Bhattacharyya (1995).

A basic structural property of the linear failure rate distribution of the minimum of two independent variables X_1 and X_2 having exponential (λ_1) and Rayleigh (λ_2) distributions whose survival functions are given above. This series structure provides a physical motivation in the framework of competing risks.

We will study the mixture of two linear failure rate distributions in Section 2.8.4.

2.3.11 Lomax Distribution

The Lomax distribution is also known as the Pareto of the second kind. The distribution may arise as a mixture distribution. Suppose X has an exponential distribution with parameter λ having density function

$$g(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta},\ x \geq 0,$$

then the resulting unconditional survival function of X is given by

$$\bar{F}(t) = (1 + \beta t)^{-\alpha}. \tag{2.41}$$

The λ here may be considered as the operating environment of a component which varies according to a gamma distribution.

We can easily verify that

$$E(X) = \frac{1}{\beta(\alpha - 1)}, \ \alpha > 1$$

and

$$\text{var}(X) = \frac{\alpha}{(\alpha - 1)^2(\alpha - 2)}, \ \alpha > 2.$$

The failure rate function is given by

$$r(t) = \frac{(\alpha + 1)\beta}{1 + \beta t}. \tag{2.42}$$

It is easy to see that $r(t) \in \mathrm{D}$.

The special case $\beta = 1$ corresponds to the Burr XII distribution with $c = 1, k = \alpha$ (see below).

2.3.12 Log-logistic

The probability density function and the survival function are, respectively, given by

$$f(t) = \frac{k\rho(\rho t)^{k-1}}{[1 + (\rho t)^k]^2}, \ t > 0, \rho > 0, k > 0, \tag{2.43}$$

$$\bar{F}(t) = \frac{1}{1 + (\rho t)^k}. \tag{2.44}$$

The rth moment about zero of the log-logistic distribution is

$$\mu_r' = \frac{1}{k\rho} B\left(\frac{r}{k}, 1 - \frac{r}{k}\right).$$

It is easy to verify that the failure rate function is

$$r(t) = \frac{k\rho(\rho t)^{k-1}}{1 + (\rho t)^k}. \tag{2.45}$$

It can be shown easily that $r(t) \in \mathrm{D}$ when $k \leq 1$; $r(t) \in \mathrm{UBT}$ when $k > 1$. The turning point of the failure rate function is given by

$$t^* = \frac{(k - 1)^{1/k}}{\rho}.$$

The log-logistic distribution has proved to be quite useful in analyzing survival data, see, e.g., Cox (1970), Cox and Oakes (1984), Bennett (1983), O'Quigley and Struthers (1982), and Gupta, Akman and Lvin (1999). Note that when the scale parameter $\rho = 1$, the log-logistic is also a special case of Burr XII below.

2.3.13 Burr XII

The Burr XII distribution was first introduced by Burr (1942). It includes the exponential, Weibull, and log-logistic distributions for particular limiting values of the parameters. Rodriguez (1977) and Tadikamalla (1980) explored in great detail the connection between the Burr XII distributions and other continuous distributions. Zimmer et al. (1998) and Ghitany and Al-Awadhi (2002) have discussed properties and reliability applications of Burr XII distribution having reliability function

$$\bar{F}(t) = \frac{1}{(1+t^c)^k}, \ \ k, c > 0, \ \ t > 0. \tag{2.46}$$

The corresponding density function is

$$f(t) = \frac{kct^{c-1}}{(1+t^c)^{k+1}}.$$

For $c = 1$, it becomes the Pareto distribution of the second kind (the Lomax).

The rth moment about zero of the Burr XII distribution is given by

$$\mu_r' = k\left[B\left(k - \frac{r}{c}, 1 + \frac{r}{c}\right)\right], \ \ k > \frac{r}{c},$$

where $B(p,q)$ is the beta function defined by $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$; so

$$\mu = k\left[B\left(k - \frac{1}{c}, 1 + \frac{1}{c}\right)\right]$$

and

$$\mathrm{var}(X) = k\left[B\left(k - \frac{2}{c}, 1 + \frac{2}{c}\right)\right] - \mu^2.$$

It is easy to verify that

$$r(t) = \frac{kct^{c-1}}{(1+t^c)}, \tag{2.47}$$

and

$$r'(t) = \frac{ct^{t-2}(c-1-t^c)}{(1+t^c)^2}. \tag{2.48}$$

For $c \le 1$, the slope is always negative, for $c > 1$ the slope is positive for $t^c < c - 1$ and negative for $t^c > c - 1$. Thus $r(t)$ is D for $c \le 1$ and UBT if $c > 2$. The maximum failure rate occurs at $t = (c-1)^{1/c}$.

Zimmer et al. (1998) have also shown that the Burr XII can approximate several useful reliability distributions (a fact that we have noted above). Watkins (1999) gave an algorithm for calculating the maximum likelihood estimates of the three-parameter Burr XII distribution. The algorithm exploits the link between this distribution and the two-parameter Weibull distribution, which merges as the limiting case of the former.

2.3.14 Exponential-geometric (EG) and Generalization

The exponential-geometric distribution is a special case of Marshall and Olkin's (1997) family of exponential distributions. The current name was apparently coined by Adamidis and Loukas (1998) who along with Marshall and Olkin (1997) studied its properties in detail.

The distribution may be obtained by compounding (mixing) an exponential distribution with a geometric distribution. The density function is

$$f(t) = \lambda(1-p)e^{-\lambda t}(1-pe^{-\lambda t})^{-2},\ \lambda > 0, 1 < p < 1. \tag{2.49}$$

Here, λ is the scale parameter of an exponential distribution whereas p is the proportion parameter of the geometric distribution. The reliability function is given by

$$\bar{F}(t) = (1-p)e^{-\lambda t}(1-pe^{-\lambda t})^{-1},\ t > 0 \tag{2.50}$$

and thus the failure rate function for the EG distribution is

$$r(t) = \lambda(1-pe^{-\lambda t})^{-1}. \tag{2.51}$$

It is easy to see that the above failure rate function is decreasing in t although the DFR property also follows from the results of Proschan (1963) on mixture. The initial failure rate $r(0) = \beta(1-p)^{-1}$ and the long-term failure rate $r(\infty) = \lambda$ which are both finite. In contrast, the failure rate of the Weibull distribution has $r(0) = \infty$ and $r(\infty) = 0$ when the shape parameter $\alpha < 1$. So the EG distribution could be an attractive alternative to the Weibull in the case when the long-term failure rate is finite.

The rth moment about zero is given by

$$\mu'_r = (1-p)r!(\lambda^r p)^{-1}L(p;r),$$

where $L(p;r) = \sum_{j=1}^{\infty} p^j j^{-r}$ is the polylogarithmic function which can be evaluated easily.

Adamidis and Loukas (1998) have considered the maximum likelihood estimates of the parameters p and λ and they gave an EM algorithm for the computation of these estimates.

As mentioned above, the EG distribution is a special case of the Marshall and Olkin (1997) family of distributions obtained by adding a parameter to the original survival function $\bar{G}(t)$ such that

$$\bar{F}(t) = \frac{\beta\bar{G}(t)}{1-(1-\beta)\bar{G}(t)},\ -\infty < t < \infty, 0 < \beta < \infty. \tag{2.52}$$

For our purpose, we consider only lifetime random variables so $t > 0$.

The case $\bar{G}(t) = \exp(-\lambda t)$ was studied in detail, in particular, it was shown that

$$E(X) = -\frac{\beta \log \beta}{\lambda(1-\beta)},$$

and

$$\text{mode}(X) = \begin{cases} 0, & \beta \leq 2 : \\ \lambda^{-1}, & \beta \geq 2. \end{cases}$$

The failure rate function is

$$r(t) = \lambda(1 - (1-\beta)e^{-\lambda t})^{-1}$$

which is decreasing in t for $0 < \beta < 1$ and increasing in t for $\beta > 1$.

For $\beta = 1 - p < 1$, it reduces to the EG distribution discussed above. If $\beta = 1$, it becomes the exponential distribution. In view of its flexility, it is conceivable that this 2-parameter family of distributions may sometimes be a competitor to the Weibull and gamma families.

The case where $\bar{G}(t) = \exp\{-(\lambda t)^\alpha\}$ will be considered in Section 5.5 as an extended Weibull distribution.

2.4 Basic Concepts for Univariate Reliability Classes

2.4.1 Some Acronyms and Notions of Aging

The concepts of increasing and decreasing failure rates for univariate distributions have been found very useful in reliability theory. The classes of distributions having these ageing properties are designated as the IFR and DFR distributions, respectively, and have been extensively studied. Other classes such as 'increasing failure rate on average' (IFRA), 'new better than used' (NBU), 'new better than used in expectation' (NBUE), and 'decreasing mean residual life' (DMRL) have also been of much interest. For fuller accounts of these classes see, e.g., Bryson and Siddiqui (1969), Barlow and Proschan (1981), and Hollander and Proschan (1984).

A class that slides between NBU and NBUE, known as 'new better than used in convex ordering' (NBUC), has also attracted some interest recently.

The notion of 'harmonically new better than used in expectation' (HNBUE) was introduced by Rolski (1975) and studied by Klefsjö (1981, 1982). Further generalizations along this line were given by Basu and Ebrahimi (1984a). A class of distributions denoted by $\mathcal{L}$ has an ageing property that is based on the Laplace transform, and was put forward by Klefsjö (1983c). Deshpande et al. (1986) used stochastic dominance comparisons to describe positive ageing and suggested several new positive ageing criteria based on these ideas – see their paper and Section 2.7 for details. Two further classes, NBUFR ('new better than used in failure rate') and NBUFRA ('new better than used in failure rate average') require the absolute continuity of the distribution function, and have been discussed in Loh (1984a,b), Deshpande et al. (1986), Kochar and Wiens (1987), and Abouammoh and Ahmed (1988).

We are now ready to give formal definitions of ten basic reliability classes. Some of the ageing classes in this group are by no means the most important ones ones in terms of their applications. Some members are selected for historical reasons.

2.4.2 Definitions of Reliability Classes

Most of the reliability classes are defined in terms of the failure rate $r(t)$, conditional survival function $\bar{F}(x \mid t) = \frac{\bar{F}(x+t)}{\bar{F}(x)}$, or the mean residual life $\mu(t)$. All these three functions provide probabilistic information on the residual lifetime and hence ageing classes may be formed according to the behavior of the ageing effect on a component.

The ten reliability classes mentioned above are defined as follows.

Definition 2.1: F is said to be IFR if $\bar{F}(x \mid t)$ is decreasing in $0 \leq t < \infty$ for each $x \geq 0$. It is a decreasing failure rate (DFR) distribution if $\bar{F}(x \mid t)$ is increasing in t. F is IFR (DFR) iff $-\log \bar{F}(t)$ is convex (concave). When the density exists, IFR (DFR) is equivalent to $r(t) = f(t)/\bar{F}(t)$ being increasing (decreasing) in $t \geq 0$ (Barlow and Proschan, 1981, p. 54).

Definition 2.2: F is said to be IFRA if $-(1/t) \log \bar{F}(t)$ is increasing in $t \geq 0$. This is equivalent to $\bar{F}(\alpha t) \geq \bar{F}^{\alpha}(t)$, $0 < \alpha < 0$, $t \geq 0$ (Barlow and Proschan, 1981, p. 84). (The latter is equivalent to $-\log \bar{F}(t)$ being a star-shaped function; i.e., $-\log \bar{F}(\alpha t) \leq -\alpha \log \bar{F}(t)$. For more information about this notion, see Dykstra, 1985.) It is also equivalent to $\int_0^t r(t)\, dx/t$ increasing in $t \geq 0$, because of the fact that $-\log \bar{F}(t) = H(t) = \int_0^t r(x)\, dx$. It is a decreasing failure rate in average (DFRA) distribution if $-(1/t) \log \bar{F}(t)$ is decreasing in $t \geq 0$ or $\bar{F}(\alpha t) \leq \bar{F}^{\alpha}(t)$ for all $0 < \alpha < 1$.

Definition 2.3: F is said to be DMRL if the mean remaining life function $\mu(t) = \int_0^\infty \bar{F}(x \mid t)\, dx$ is decreasing in t, i.e., $\mu(s) \geq \mu(t)$ for $0 \leq s \leq t$. In other words, the older the device is, the smaller is its mean residual life (Bryson and Siddiqui, 1969). Similarly, F is said to be IMRL if $\mu(s) \leq \mu(t)$ for $0 \leq s \leq t$.

Definition 2.4: F is said to be new better than use (NBU) if $\bar{F}(x \mid t) \leq \bar{F}(x)$, i.e., $\bar{F}(x + t) \leq \bar{F}(x)\bar{F}(t)$ for $x, t \geq 0$. This means that a device of any particular age has a stochastically smaller remaining lifetime than does a new device (Barlow and Proschan, 1981). The definition here is also equivalent to $\log \bar{F}(x + t) \leq \log \bar{F}(x) + \log \bar{F}(t) \Leftrightarrow \int_0^t r(u)\, du \leq \int_x^{x+t} r(u)\, du$.

F is said to be new worse than used (NWU) if $\bar{F}(x + t) \geq \bar{F}(x)\bar{F}(t)$ for all $x, t \geq 0$.

Definition 2.5: F is said to be new better than used in expectation (NBUE) if $\int_0^\infty \bar{F}(x \mid t)\, dx \leq \mu$ for $t \geq 0$. This is equivalent to $\int_t^\infty \bar{F}(x)\, dx \leq \mu\bar{F}(t)$. This means that a device of any particular age has a smaller mean remaining lifetime than does a new device (Barlow and Proschan, 1981). F is said to be new worse than used in expectation (NWUE) if $\int_0^\infty \bar{F}(x \mid t)\, dx \geq \mu$ for all $t \geq 0$.

Definition 2.6: F is said to be harmonically new better than used (HNBUE) if $\int_t^\infty \bar{F}(x)\, dx \leq \mu \exp(-t/\mu)$ for $t \geq 0$. There is an alternative definition in terms of the mean residual life (Rolski, 1975). This is equivalent to $1/\{\frac{1}{t}\int_0^t \mu^{-1}(x)\, dx\} \leq \mu$. Similarly, F is said to be harmonically new worse than used (HNWUE) if $\int_t^\infty \bar{F}(x)\, dx \geq \mu \exp(-t/\mu)$ for $t \geq 0$.

Definition 2.7: F is said to be a Laplace class ($\mathcal{L}$)-distribution if for every $s \geq 0$, $\int_0^\infty e^{-st}\bar{F}(t)\, dt \geq \mu/(1+s)$. The expression $\mu/(1+s)$ can be written as for $\int_0^\infty \exp(-sx)\bar{G}(x)\, dx$, where $\bar{G}(x) = \exp(-x/\mu)$. This means that the inequality is one between the Laplace transforms of $\bar{F}$ and of an exponential survival function with the same mean as F (Klefsjö, 1983c).

Definition 2.8: F is said to be new better than used in failure rate (NBUFR) if $r(t) > r(0)$ for $t \geq 0$ (Deshpande et al., 1986). F is said to be new worse than used in failure rate (NWUFR) if $r(t) < r(0)$ for $t \geq 0$.

Definition 2.9: F is said to be new better than used in failure rate average (NBAFR or NBUFRA) if $r(0) \leq \frac{1}{t}\int_0^t r(x)\, dx$ for all $t \geq 0$ (Loh, 1984ab). Note that this is equivalent to $r(0) \leq \frac{-\log \bar{F}(t)}{t}, t \geq 0$. Similarly, F is said to be new worse than used in failure rate average (NWUFRA) if $r(0) \geq \frac{-\log \bar{F}(t)}{t}$ for $t \geq 0$.

Definition 2.10: F is said to be new better than used in convex ordering (NBUC) if $\int_y^\infty \bar{F}(t \mid x)\, dt \leq \int_y^\infty \bar{F}(t)\, dt$ for all $x, y \geq 0$ (Cao and Wang, 1991).

Using Laplace transforms, Block and Savits (1980a) established necessary and sufficient conditions for the IFR, IFRA, DMRL, NBU, and NBUE properties to hold.

There are many other ageing classes have been defined and some of these 'further' classes will be given in Section 2.7. The ten classes above are chosen because they are easily understood, intuitively appealing with known applications. Their relevance to reliability theory and survival analysis have been well documented in the literature, especially the first five classes.

2.4.3 Interrelationships

The following chain of implications exists among the ten ageing classes (adapted from Deshpande et al., 1986; Kochar and Wiens, 1987):

$$
\begin{array}{ccccccccc}
\text{IFR} & \Rightarrow & \text{IFRA} \Rightarrow & \text{NBU} & \Rightarrow & \text{NBUFR} & \Rightarrow & \text{NBUFRA} & \\
\Downarrow & & & \Downarrow & & & & & \\
\Downarrow & & & \text{NBUC} & & & & & \\
\Downarrow & & & \Downarrow & & & & & \\
\text{DMRL} & & \Rightarrow & \text{NBUE} & \Rightarrow & \text{HNBUE} & \Rightarrow & \mathcal{L} &
\end{array}
$$

We note that a partial chain IFR $\Rightarrow$ IFRA $\Rightarrow$ NBU $\Rightarrow$ NBUE has long been established (Barlow and Proschan, 1981, p. 159). For completeness, a brief sketch of the proof is as follows.

(i) F is IFR if $-\log \bar{F}(x)$ is convex whereas F is IFRA if $\log \bar{F}(x)$ is a star-shaped function, i.e., if $-\log \bar{F}(\lambda x) \leq -\lambda \log \bar{F}(x)$ for $0 \leq \lambda \leq 1$ and $x \geq 0$. Since a convex function is stared-shaped so IFR $\Rightarrow$ IFRA.

$$\begin{aligned}
F \text{ is IFRA} &\Rightarrow -(1/x) \log \bar{F}(x) \text{ is increasing } x \\
&\Leftrightarrow \bar{F}(x)^{\frac{1}{x}} \text{ is decreasins in } x \\
&\Leftrightarrow \bar{F}(x+y)^{\frac{1}{x+y}} \leq \bar{F}(x)^{\frac{1}{x}} \leq \bar{F}(y)^{\frac{1}{y}} \text{ assuming } x > y \\
&\Rightarrow \bar{F}(x+y)^{\frac{x+y}{x+y}} \leq \bar{F}(x)^{\frac{x+y}{x}} \\
&\Rightarrow \bar{F}(x+y) = \bar{F}(x)\bar{F}^{\frac{y}{x}}(x) \leq \bar{F}(y)\bar{F}^{\frac{y}{y}}(y) = \bar{F}(x)\bar{F}(y) \\
&\Leftrightarrow F \text{ is NBU.}
\end{aligned}$$

(ii) Now F IFR implies that $\dfrac{\bar{F}(x+t)}{\bar{F}(t)}$ decreases in $t \geq 0$, in other words, $\dfrac{\bar{F}(x+t)}{\bar{F}(t)} \leq \dfrac{\bar{F}(x+s)}{\bar{F}(s)}$ for $t \geq s$. Integrating both sides with respect to x, we have $\mu(t) \leq \mu(s)$, i.e., F is DMRL.

(iii) Show NBU $\Rightarrow$ NBUC $\Rightarrow$ NBUE $\Rightarrow$ HNBUE $\Rightarrow \mathcal{L}$.

F is NBU $\Leftrightarrow \dfrac{\bar{F}(x+y}{\bar{F}(x)} \leq \bar{F}(y)$. Integrating both sides with respect to y so $\dfrac{\int_0^\infty \bar{F}(x+y)\, dy}{\bar{F}(x)} \leq \displaystyle\int_0^\infty \bar{F}(y)\, dy$, i.e., $\mu(x) \leq \mu$ and thus F is NBUE.

Next, F NBU implies $\bar{F}(x\,|\,t) \leq \bar{F}(x) \Rightarrow \displaystyle\int_y^\infty \bar{F}(x\,|\,t)\, dx \leq \int_y^\infty \bar{F}(x)\, dx$ for all $y \geq 0$ which implies that F is NBUC.

Letting $y = 0$, the preceding inequality reduces to the corresponding definition of NBUE showing that NBUC $\Rightarrow$ NBUE.

Next, if F is NBUE, we have $\mu(t) \leq \mu$ for all $t \geq 0$. Thus $\mu^{-1}(t) \geq \frac{1}{\mu}$. Therefore $\int_0^t \mu^{-1}(x)\, dx \geq t/\mu$ or equivalently $1/\{\frac{1}{t}\int_0^t \mu^{-1}(x)\, dx\} \leq \mu$ which implies F HNBUE.

Lastly, we want to show that if F HNBUE then $F \in \mathcal{L}$. It can be shown easily that

$$\int_0^\infty e^{-st}\bar{F}(t)\, dt = \mu - s\int_0^\infty e^{-st}\left(\int_t^\infty \bar{F}(x)\, dx\right) dt.$$

As $\int_t^\infty \bar{F}(x)\,dx \le \mu e^{-t/\mu}$ because of $F \in$ HNBUE, it follows from above equation that $\int_0^\infty e^{-st}\bar{F}(t)\,dt \ge \mu/(1+s\mu)$ so $F \in \mathcal{L}$ is proved.

(iv) We next want to show that NBU$\Rightarrow$ NBUFR $\Rightarrow$ NBUFRA.

$$
\begin{aligned}
F \text{ is NBU} &\Leftrightarrow \frac{\bar{F}(t+x)}{\bar{F}(t)} \le \bar{F}(x), x, t > 0 \\
&\Leftrightarrow \frac{\bar{F}(t+x) - \bar{F}(t)}{\bar{F}(t)} \le -F(x) \\
&\Rightarrow \lim_{x\to 0} \frac{F(x)}{x} \le \lim_{x\to 0} \frac{\bar{F}(t+x) - \bar{F}(t)}{x\bar{F}(t)} \\
&\Leftrightarrow f(0) \le \frac{f(t)}{\bar{F}(t)} \quad \text{or} \quad r(0) \le \frac{f(t)}{\bar{F}(t)} \\
&\Leftrightarrow F \text{ is NBUFR}
\end{aligned}
$$

$$
\begin{aligned}
F \text{ is NBUFR} &\Leftrightarrow r(t) < r(0) \\
&\Rightarrow \tfrac{1}{t} \textstyle\int_0^t r(x)\,dx < r(0) \\
&\Rightarrow F \text{ is NBUFRA}
\end{aligned}
$$

so we have completed the proof of the chain.

In definitions 2.1–2.10 of Section 2.4.2, if we reverse the inequalities and interchange "increasing" and "decreasing", we obtain the classes DFR, DFRA, NWU, IMRL, NWUE, HNWUE, $\bar{\mathcal{L}}$, NWUFR, NWUFRA, and NWUC. They satisfy the same chain of implications. These are sometimes referred to as the 'dual' classes and their roles are to define negative ageing effects to a device.

2.5 Properties of the Basic Ageing Classes

The properties of interest concerning ageing classes are mainly on

(1) Preservation or closure property of an given ageing class under the reliability operations of
   (a) Formation of coherent systems of independent components,
   (b) Addition of life lengths (convolution),
   (c) Mixtures of distributions,
(2) Reliability bounds,
(3) Whether any ageing class can arise from a shock model,
(4) Moment inequalities,

(5) Testing exponentiality against an ageing alternatives.

Item (4) will be dealt with in Section 2.5.5 whereas item (5) will be considered in Chapter 7 in detail.

2.5.1 Properties of IFR and DFR

The following properties of the IFR and DFR concepts can be found in Barlow and Proschan (1981), Patel (1983) and many others:

1. If X_1 and X_2 are both IFR, so is $X_1 + X_2$; but the DFR property is not so preserved.
2. A mixture of DFR distributions is also DFR; but this is not necessarily true for IFR distributions.
3. Parallel systems of identical IFR units are IFR.
4. Series systems of (not necessarily identical) IFR units are IFR.
5. Order statistics from an IFR distribution have IFR distributions, but this is not true for spacings from an IFR distribution; order statistics from a DFR distribution do not necessarily have a DFR distribution, but spacings from a DFR distribution are DFR.
6. The pdf of an IFR distribution need not be unimodal.
7. The pdf of a DFR distribution is a decreasing function.
8. If the rth moment (about zero) of a continuous life F distribution is known, the IFR lower bound on $\bar{F}(t)$ is
$$\bar{F}(t) \geq \begin{cases} e^{-\alpha t}, & \text{if } t < \mu_r^{1/r} \\ 0, & \text{if } t > \mu_r^{1/r}, \end{cases} \tag{2.53}$$
where $\alpha = [\Gamma(r+1)/\mu_r]^{1/r}$ (Barlow and Proschan, 1981, p.112). The bound is sharp.
The special case $r = 1$ is important in reliability applications as the first moment is usually easy to find or estimate.
9. Let F be DFR with mean μ, then
$$\bar{F}(t) \leq \begin{cases} e^{-t/\mu}, & \text{for } t \leq \mu; \\ \frac{\mu e^{-1}}{t}, & \text{for } t \geq \mu. \end{cases} \tag{2.54}$$

The IFR phenomenon is well understood and needs no further elaboration. To put in nontechnical terms, a device having IFR lifetime deteriorates with age, i.e., the age has an adverse effect on the device for if it has an IFR lifetime distribution.

Defying a common expectation, DFR phenomenon also occurs quite frequently. Generally speaking, a lifetime population is expected to exhibit decreasing failure rate (DFR) when its behavior over time is characterized by

- 'work hardening' (in engineering), and
- immunity in (biological organisms).

The term 'infant mortality phase' is sometimes used to described the DFR phenomenon over the early part of the life span. In a DFR population, 'age' is actually beneficial to a device or an organism. Improvement of reliability might have occurred by means of physical changes that caused self-improvement or simply it might have been due to population heterogeneity. Indeed, the DFR property is often inherent in mixtures of distributions.

2.5.2 Properties of IFRA

As $-\log \bar{F}(t)$ given in (2.4) represents the cumulative failure rate, the name given to this class is appropriate. Block and Savits (1976) showed that the IFRA is equivalent to $E^{\alpha}[h(X)] \leq E[h^{\alpha}(X/\alpha)]$ for all continuous nonnegative increasing functions h and all α such that $0 < \alpha < 1$.

This ageing notion is fully investigated in the book by Barlow and Proschan (1981). It is the smallest class containing the exponential distribution which is closed under the formation of coherent systems as well as under convolution. The IFRA closure theorem is pivotal to many of the results given in Barlow and Proschan (1981). The IFRA class is perhaps one of the more important ageing classes in reliability analysis. Curiously, interest about IFRA has seemed to wane in the recent time. It has been shown that a device subject to shocks governed by a Poisson process, which fails when the accumulated damage exceeds a fixed threshold, has an IFRA distribution (Esary et al., 1973).

One of the attractive properties that an IFRA (DFRA) distribution enjoys is that its reliability bound can be obtained in terms of its known quantile.

Theorem 2.4: Let F be IFRA (DFRA) with pth quantile ξ_p (i.e., $F(\xi_p) = p$). Then

$$\bar{F}(t) \begin{cases} \geq (\leq) e^{-\alpha t}, \text{ for } 0 \leq t < \xi_p \\ \leq (\geq) e^{-\alpha t}, \text{ for } t > \xi_p. \end{cases} \tag{2.55}$$

Proof: We note that the exponential survival probability $e^{-\alpha t}$ has the same pth quantile ξ_p as does F. Thus at least one crossing of $e^{-\alpha}$ by $\bar{F}(t)$ must occur at $t = \xi_p$. By the single crossing property of an IFRA distribution with $e^{-\lambda t}, \lambda > 0$ (Barlow and Proschan, 1981, p. 89), we conclude $\bar{F}$ crosses with the exponential survival function with the same quantile at most once from above (below).

Sengupta (1994) presented a unified derivation of the upper and lower bounds (in terms of finite moments) of IFR (DFR), IFRA (DFRA) or NBU (NWU) reliability functions. A table of bounds on $\bar{F}(t)$ based on the rth moments for various cases is also given. However, numerical methods are required to solve for the values of these bounds.

Recently, El-Bassiouny (2003) has shown that if F is IFRA, then for all integers $r \geq 0$, $k \geq 2$,

$$\nu_{(r+1)} \geq \frac{\mu'_{r+1}}{k^{r+1}} \tag{2.56}$$

where $\nu_{(r)} = E[\min(X_1, ..., X_k)]^r$, $\mu'_r = E(X_1^r)$ and $X_1, ...X_k$ are independent and identically distributed random variables.

Several moment inequalities for IFR (DFR) appeared much earlier and these will be given Section 2.5.5 below.

2.5.3 NBU and NBUE

Properties of NBU, NWU, NBUE, NWUE were also well documented in the book by Barlow and Proschan (1981). These classes of life distributions may arise from a consideration of shock models similar to those involving IFRA distributions. Barlow and Proschan (1981, Chapter 6) showed that these concepts are useful in the study of maintenance policies.

Closure property

Abouammoh and El-Neweihi (1986) show that the NBU class is closed under formation of parallel systems of i.i.d. components.

Probability bounds

The first three bounds below are found in Barlow and Proschan (1981, p. 188).

If F is NBU with $\bar{F}(t) = \alpha$ for a fixed value of t, then

$$\bar{F}(x) \begin{cases} \geq \alpha^{1/k} & \text{for} \quad \frac{t}{k+1} < x < \frac{t}{k},\ k = 0, 1, ..., \\ \leq \alpha^k & \text{for} \quad kt \leq x \leq (k+1)t, k = 0, 1, ... \end{cases}$$

The bounds are sharp.

If F is now NWU, then

$$\bar{F}(x) \begin{cases} \leq \alpha^{1/(k+1)} & \text{for} \quad \frac{t}{k+1} < x < \frac{t}{k},\ k = 0, 1, ..., \\ \geq \alpha^{k+1} & \text{for} \quad kt \leq x \leq (k+1)t, k = 0, 1, ... \end{cases}$$

The bounds are sharp.

If F is NBUE with mean μ, then

$$\bar{F}(t) \geq \begin{cases} 1 - \frac{t}{\mu}, & t \leq \mu; \\ 0, & t > \mu. \end{cases}$$

The bounds are sharp.

If F is NWUE with mean μ, then

$$\bar{F}(t) \leq \mu/(\mu + t),\ t \geq 0$$

(Haines and Singpurwalla, 1974).

The next two bounds were given by Launer (1984).

(i) If F is NBUE with mean μ and variance σ^2, then

$$\bar{F}(t) \geq \begin{cases} (\sigma^2+\mu^2-t^2)/(\sigma^2+(\mu+t)^2-t^2), & t \leq \sqrt{\mu_2'}\,; \\ 0, & t > \sqrt{\mu_2'}. \end{cases}$$

(ii) If F is now NWUE, then

$$\bar{F}(t) \leq \begin{cases} (\sigma^2+\mu^2)/(\sigma^2+(\mu+t)^2), & 0 < t \leq 2\sigma^2/\mu; \\ \sigma^2/(\sigma^2+t^2), & 2\sigma^2/\mu \leq t. \end{cases}$$

Launer (1984) obtained other bounds that are based on $E(X^r|X > t)$ and $E(X^r|X \leq t)$. In general, the usefulness of a bound depends on (i) how easy a bound it can be estimated from data and (ii) how sharp the bound is.

Replacement models

Consider a system operating over an indefinite period of time. Upon failure, repair (or replacement) is performed, requiring negligible time. The successive intervals between failures are independent, identically distributed random variables $X_1, X_2, \ldots$ of a renewal process. Let $N(t)$ denote the number of renewals (replacements) in $(0, t]$ and $M(t)$ the expected number of renewals (replacements) in $(0, t]$, i.e., $E(N(t)) = M(t)$.

Theorem 2.5: Let $E(X) = \mu < \infty$ be the mean lifetime of a component.

(a) If F is NBUE , then

$$\frac{t}{\mu} - 1 \leq M(t) \leq \frac{t}{\mu}. \tag{2.57}$$

(b) If F is NWUE, then

$$M(t) \geq \frac{t}{\mu}. \tag{2.58}$$

Proof: The following is essentially an abridged version of the proof given in Barlow and Proschan (1981, pp. 169-171).

(a) Let $S_{N(t)}$ denote the time to the kth renewal if $N(t) = k, k \geq 0$ so the $(k+1)$th renewal must occur after time t. Thus $S_{N(t)+1} - t \geq 0$. It follows from the classical renewal theory that $E(S_{N(t)+1} - t) = \mu[M(t)+1] - t \geq 0$ which gives $M(t) \geq \frac{t}{\mu} - 1$. (This is true irrespective of the ageing class it belongs to.)

On the other hand, F NBUE implies the stationary distribution $\hat{F}(t) = (1/\mu)\int_0^t \bar{F}(x)\,dx \geq F(t)$. This means the expected time to the first renewal under the the stationary renewal process is smaller than under the ordinary renewal process. Hence, $\hat{M}(t) \geq M(t)$ where $\hat{M}(t)$ denotes the expected number of renewals in a stationary renewal process which is given by $\frac{t}{\mu}$. It now follows immediately that $M(t) \leq \frac{t}{\mu}$ so the proof of (a) is completed.

(b) Suppose now F is NWUE so that $\hat{F}(t) = (1/\mu)\int_0^t \bar{F}(x)\,dx \leq F(t)$. Using the second part of the proof for (a) but with the inequality reversed, we conclude that $M(t) \geq \frac{t}{\mu}$. See Barlow and Proschan (1981, p. 171).

Chen (1994) showed that the distributions of these classes may be characterized through certain properties of the corresponding renewal functions.

Cheng and He (1989) studied the reliability bounds on NBUE and NWUE classes and Cheng and Lam (2002) obtained reliability bounds on NBUE from first two known moments.

NBU-t_0 and NWU-t_0 Classes

Without getting side-tracked from discussing the ten basic classes, we now introduce a new life class which is obtained by relaxing the conditions for NBU (NWU) class somewhat. Let $t_0 \geq 0$.

Definition 2.11: We say that a life distribution F is new better than used at t_0 (NBU-t_0) if

$$\bar{F}(x+t_0) \leq \bar{F}(x)\bar{F}(t_0), \text{ for all } x \geq 0. \tag{2.59}$$

The class was first introduced in Hollander et al. (1985). The dual notion of new worse than used at t_0 is defined analogously by reversing the first inequality in the preceding equation. Several non-parametric tests dealing with this class have been proposed in the literature. It is interesting to note that apart from the exponential, there are some other distributions that belong to the boundary members of NBU-t_0 and NWU-t_0 classes. Though not listed along with the ten classes in Section 2.4.2, the NBU-t_0 (NWU-t_0) class has been frequently discussed and referred to so it could be considered as an important ageing class in reliability. Park (2003) gave a detailed review of its properties and applications.

HNBUE and HNWUE

Klefsjö (1982) obtained the properties of HNBUE and HNWUE classes of life distributions. Basu and Ebrahimi (1986) gave a survey on these classes. Cheng and Lam (2001) gave reliability bounds on HNBUE life distributions with the first two known moments.

Though mathematically elegant, we have yet to find any meaningful or significant applications for HNBUE or HNWUE classes.

NBUC and NWUC

NBUC and NWUC classes were first defined by Cao and Wang (1991). These authors showed that neither NBUC nor NWUC is closed under mixture or formation of series systems. We have seen earlier that NBUC is sandwiched between NBU and NBUE, i.e.,

$$\text{NBU} \Rightarrow \text{NBUC} \Rightarrow \text{NBUE} \Rightarrow \text{HNBUE}.$$

Hendi et al. (1993) have shown that the new better than used in convex ordering is closed under the formation of parallel systems with independent and identically distributed components. Both Li et al. (2000) and Pellerey and Patakos (2002) showed that this closure property holds for nonidentical parallel components as well. Besides, Li et al. (2000) also presented a lower bound of the reliability function for this class based upon the mean and the variance. Hu and Xie (2002) gave a new proof of the closure property of NBUC under convolution (they corrected the errors of Cao and Wang, 1991 and Li and Kochar, 2001 regarding this closure property).

NBUFR (NWUFR)

Abouammoh and Ahmed (1988) showed that every k-out-of-n system, $1 \leq k \leq n$, has the NBUFR property.

Gohout and Kuhnert (1997) showed that NBUFR class is closed under the formation of coherent systems of independent components. El-Bassiouny et al. (2004) derived a moment inequality for the class of NBUFR (NWUFR):

$$\mu'_r \geq (\leq) \frac{f(0)\mu'_{r+1}}{r+1}, \ f(0) > 0, r \geq 0,$$

where $\mu'_s = E(X^s), s \geq 0$ and $f(t)$ is the density function of the distribution.

Laplace classes $\mathcal{L}$ and $\bar{\mathcal{L}}$

Suppose $F \in \mathcal{L}$ with mean μ, Sengupta (1995) showed that

(a) $\alpha_t \leq \bar{F}(t) \leq 1$ if $t \leq \mu$,
(b) $0 \leq \bar{F}(t) \leq 1 - \alpha_t$, if $t \geq \mu$,

where

$$\alpha_t = \inf \left\{ \alpha : \inf_{s>0} [e^{st/\mu} - (1+s)(1 - \alpha + e^{-s(1-t/\mu)/\alpha})] \geq 0 \right\}.$$

The bounds are sharp. It appears the the actual computations of bounds are nontrivial.

On the other hand, let $F \in \bar{\mathcal{L}}$ with mean μ, Sengupta (1995) showed that

$$0 \leq \bar{F}(t) \leq \begin{cases} \inf\limits_{s>0} \dfrac{s}{(s+t/\mu)(1-e^{-s})} & \text{if } t \leq 2\mu \\ \mu/t & \text{if } t \geq 2\mu \end{cases}$$

Again, the bounds are sharp.

We are of the opinion that this ageing concept has limited application.

2.5.4 DMRL and IMRL

The properties DMRL and IMRL classes will be studied in Chapter 4 in details. For time being, it suffices to state that the DMRL class is closed under formation of parallel systems of i.i.d. components (Abouammoh and El-Neweihi, 1986).

2.5.5 Summary of Preservation Properties of Classes of Distributions

One may wish to know under what reliability operations a given class of life distributions is preserved.

Here we consider the reliability operations of

(a) Formation of coherent systems,
(b) Addition of life lengths (convolution of distributions),
(c) Mixture of life distributions,
(d) Mixture of non-crossing life distributions,

applied to several basic classes of life distributions. The following table adapted from Park (2003) provides a summary of these preserving (or non-preserving) properties under reliability operations. Park's table was itself an update of the original table given by Barlow and Proschan (1981, p.104, 187).

Table 2.2. Preservation Under Reliability Operations

Class of life distribution	Formation of coherent structure	Convolution of life distributions	Mixture of life distributions	Mixture of non-crossing life distributions
IFR	Not closed	Closed	Not closed	Not closed
IFRA	Closed	Closed	Not closed	Not closed
NBU	Closed	Closed	Not closed	Not Closed
NBUE	Not closed	Closed	Not closed	Not closed
DMRL	Not closed	Not closed	Not closed	Not closed
HNBUE	Not closed	Closed	Not closed	Not closed
NBU-t_0	Closed	Not closed	Not closed	Not closed
DFR	Not closed	Not closed	Closed	Closed
DFRA	Not closed	Not closed	Closed	Closed
NWU	Not closed	Not closed	Not closed	Closed
NWUE	Not closed	Not closed	Not closed	Closed
IMRL	Not closed	Not closed	Closed	Closed
HNWUE	Not closed	Not closed	Closed	Not closed
NWU-t_0	Not closed	Not cloded	Not closed	Closed

2.5.6 Moments Inequalities

Moment inequalities for elementary ageing classes have been in the literature for many years. The following results are found in Barlow and Proschan (1981, p. 116, 187).

Theorem 2.6: Let F be a continuous distribution with known mean μ and μ'_r be the rth moment about zero. Let $\lambda_r = \mu'_r/\Gamma(r+1)$.

(i) Let F be IFRA (DFRA). Then

$$\mu'_r \le (\ge)\Gamma(r+1)\mu^r \qquad \text{for } 0 < r \le 1,$$
$$\mu'_r \ge (\le)\Gamma(r+1)\mu^r \qquad \text{for } 1 < r \le \infty. \tag{2.60}$$

The bounds are sharp.

(ii) Let F be NBU (NWU) and $\lambda_r = \mu'_r/\Gamma(r+1)$, the normalized rth moment. Then

$$\lambda_{r+s} \le (\ge)\lambda_r\lambda_s$$

for $r \ge 0, s \ge 0$.

(iii) Let F be NBUE (NWUE), then

$$\lambda_{r+1} \le (\ge)\lambda_r\lambda_1, \quad \text{for } r \ge 0.$$

Proof: We largely follow the approach of Barlow and Proschan (1981) in the proof below.

(i) First, F IFRA is equivalent to F is star-shaped with respect to G where $G(t)$ is an exponential distribution (see Barlow and Proschan 1981, p.107 and also Section 10.3 for a definition for 'F is star-shaped with respect to G'.) So $\bar{F}$ crosses $\bar{G}$ at most once. Now if $\bar{G}(t) = \exp\left(-t/\lambda_s^{1/s}\right)$, then

$$\int_0^\infty t^s dG(t) = \int_0^\infty (t^s/\lambda_s^{1/s}) \exp\left(t/\lambda_r^{1/s}\right)\, dt = \mu'_s = \int_0^\infty t^s dF(t),$$

so $\bar{F}$ crosses G exactly once. Now, it can be shown that

$$\int_0^\infty \psi(x)x^{s-1}\bar{F}(x)\,dx \le \int_0^\infty \psi(x)x^{s-1}\bar{G}(x)\,dx;\ \ s > 0 \tag{2.61}$$

if ψ is increasing. The inequality in (2.61) reverses if F is DFRA.

Now, let $0 < r < s$, it follows from the rth moment about zero of an exponential distribution that

$$\mu'_r = r\int_0^\infty x^{r-1}\bar{F}(x)\,dx = r\int_0^\infty x^{r-1}\exp\left(-x/\lambda_r^{1/r}\right)dx, \tag{2.62}$$

i.e.,

$$\int_0^\infty x^{r-1}\bar{F}(x)\,dx = \int_0^\infty x^{r-1}\exp\left(-x/\lambda_r^{1/r}\right)dx.$$

Let $\psi(x) = x^{s-r}$ and applying (2.61), we obtain

$$\lambda_s = \frac{\mu_s'}{\Gamma(s+1)} = \frac{1}{\Gamma(s+1)}\int_0^\infty sx^{s-1}\bar{F}(x)\,dx$$

$$\leq (\geq)\int_0^\infty \frac{sx^{s-1}}{\Gamma(s+1)}\exp\left(-x/\lambda_r^{1/r}\right)dx = \lambda_r^{s/r},$$

that is

$$\lambda_r^{1/r} \geq (\leq)\lambda_s^{1/s}, \text{ for } 0 < r < s.$$

Letting $s = 1$, we prove the $\mu_r' = \Gamma(r+1) \geq (\leq)\mu^r$ for $0 < r < 1$. The inequalities are reversed for $1 \leq r < \infty$.

(ii) If F is NBU, then $\bar{F}(x+y) \leq \bar{F}(x)\bar{F}(y)$ so for $r, s \geq 0$,

$$\int_0^\infty\int_0^\infty x^r y^s \bar{F}(x+y)\,dx\,dy \leq \int_0^\infty x^r\bar{F}(x)\,dx\int_0^\infty y^s\bar{F}(y)\,dy.$$

Applying (2.62), the right-hand side is $\mu_{r+1}\mu_{s+1}/(r+1)(s+1)$.

Letting $x + y = u$ and $y = v$, the left-hand side of the above equation becomes

$$\int_0^\infty\int_0^u \bar{F}(u)(u-v)^r v^s\,dv\,du = \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+3)}\mu_{r+s+2}'.$$

It now follows immediately $\lambda_{r+s+2} \leq \lambda_{r+1}\lambda_{s+1}$.

If F is NWU, then $\lambda_{r+s+2} \geq \lambda_{r+1}\lambda_{s+1}$.

(iii) Since F is NBUE, $\int_x^\infty \bar{F}(u)\,du \leq \mu\bar{F}(x)$. Thus, for all $r \geq 0$,

$$\int_0^\infty x^r\int_x^\infty \bar{F}(u)\,du\,dx \leq \mu\int_0^\infty x^r\bar{F}(x)\,dx,$$

i.e.,

$$\int_0^\infty \bar{F}(u)\int_0^u x^r\,dx\,du \leq \mu\int_0^\infty x^r\bar{F}(x)\,dx.$$

The left-hand side can be evaluated by integration by parts giving $\mu_{r+2}'/(r+2)$ whereas the right-hand side is $\mu_{r+1}'\mu$ because of (2.62). Thus $\lambda_{r+2} \leq \lambda_{r+1}\lambda_1$ is now proved.

The case for NWUE can be proved similarly. We note that an important feature of these inequalities is their simplicity.

Interest in moment inequalities for several classes has been rekindled in recent years. The main objective of these inequalities is to formulate test statistics for testing a a distribution is from a particular ageing class. Unfortunately, some of these have complicated expressions and those listed below are some special cases which we think would be of interest to the readers.

Ahmad (2001) presented moment inequalities for IFR, NBU, NBUE and HNBUE. Recently, Ahmad and Mugdadi (2004) also provided similar inequalities for IFRA, NBUC and DMRL classes. In the following, we assume that $\{X_i\}$ are i.i.d. with distribution function F; r and s are integers. We also let $\nu_{(r)} = E[\min(X_1, X_2)]^r$.

(i) F IFR (Ahmad, 2001)

$$2^{(r+2)(r-1)/2}\nu_{(r)} \geq r!\mu^r,\ r \geq 2$$

and

$$\nu_{(2r+2)} \geq \binom{2r+2}{r+1}\left(\frac{1}{2}\right)^{(2r+2)}\{\mu'_{r+1}\}^2.$$

(ii) F NBU (Ahmad, 2001)
For integer $k \geq$ and $r_i \geq 0$, $i = 1, 2, ..., k$

$$\left(\sum_{i=1}^{k} r_i + k\right)! \prod_{i=1}^{k} \mu_{r_i+1} \geq \prod_{i=1}^{k} (r_i+1)!\mu_{r_1+\ldots r_k+k}. \tag{2.63}$$

For $r_1 = r$, $r_2 = s$, $k = 2$, the above inequality reduces to

$$(r+s+2)!\mu'_{r+1}\mu'_{s+1} \geq (r+1)!(s+1)!\mu'_{r+s+2} \tag{2.64}$$

which is equivalent to (ii) of Theorem 2.6 above. Thus (2.63) is a generalization of the result (ii) of the theorem mentioned. In view of its complexity, one doubts if the above moment inequality would generate wide applications.

(iii) F NBUE (Ahmad, 2001)

$$\mu'_{r+1}\mu \geq \mu'_{r+2}/(r+2),\ \ r \geq 0.$$

(This is the same as (iii) of Theorem 2.6 above)

(iv) F HNBUE (Ahmad, 2001)

$$\mu^{r+2} \geq \mu'_{r+2}/(r+2)!,\ \ r \geq 0. \tag{2.65}$$

The proof of (2.65) is straightforward by noting that F HNBUE implies $\int_x^\infty \bar{F}(u)\,du \leq \mu e^{-x/\mu}$ so

$$\int_0^\infty x^r \int_x^\infty \bar{F}(u)\,du\,dx \leq \mu \int_0^\infty x^r e^{-x/\mu}\,dx.$$

By exchanging the order of integration and applying (2.62), the left-hand side is $\mu'_{r+2}/\left[(r+2)(r+1)\right]$. The integral on the right is related the $(r+1)$th moment (about zero) of the exponential distribution so the result follows immediately.

(v) F IFRA (Ahmad and Mugdadi, 2004)

$$\mu'_{r+1} \geq E\left\{\min\left(\frac{X_1}{\alpha}, \frac{X_2}{1-\alpha}\right)^{r+1}\right\}, r \geq 0,\ 0 < \alpha < 1, .$$

(vi) F NBUC (Ahmad and Mugdadi, 2004)

$$(r+2)!(s+1)!\mu'_{r+s+3} \leq (r+s+3)!\mu'_{r+2}\mu'_{s+1},\ \ r, s \geq 0.$$

We note that this inequality is equivalent to (2.64) which holds for NBU distributions.

(vii) F DMRL (Ahmad and Mugdadi, 2004)

$$(r+1)E[X_1\{\min(X_1, X_2)\}^r] \geq (r+2)\nu_{(2)}^{(r+1)},\ r \geq 0.$$

Abu-Youssef (2002) obtained a simple bound $\nu_{(2)} \geq (\leq)\frac{\mu^2}{2}$ if F is DMRL (IMRL).

2.5.7 Scaled TTT Transform and Characterizations of Ageing Classes

The concept of the total time on test (TTT) processes was first defined by Barlow and Campo (1975). The TTT transform has been found useful to study the ageing properties of the underlying distribution and at the same time can be applied to solve geometrically some stochastic maintenance problems.

Let F be an lifetime distribution and define

$$F^{-1}(p) = \inf\{x : F(x) \geq p\},\ p \in [0, 1). \tag{2.66}$$

Let us define

$$H_F^{-1}(p) = \int_0^{F^{-1}(p)} \bar{F}(x)\, dx, p \in [0, 1]. \tag{2.67}$$

If the mean lifetime μ is finite, then

$$H_F^{-1}(1) = \int_0^{F^{-1}(1)} \bar{F}(x)\, dx = \mu. \tag{2.68}$$

The scaled total time on test transform (scaled TTT transform) is defined by

$$\phi(p) = H_F^{-1}(p)/H_F^{-1}(1) = H_F^{-1}(p)/\mu. \tag{2.69}$$

It follows from the above definition that $\phi(p), p \in [0,1]$ is equivalent to the equilibrium distribution of the probability distribution function F, if F is non-arithmetic. The curve $\phi(p)$ versus $p \in [0,1]$, is called the scaled TTT curve.

Classifications of ageing distributions

It follows from (2.67) that

$$\frac{d}{dp} H_F^{-1}(p)|_{p=F(t)} = \frac{1}{r(t)},\ t > 0, p \in [0,1]. \tag{2.70}$$

Theorem 2.7 Let F be a continuous lifetime distribution.

(i) F is IFR (DFR) if and only if $\phi(p)$ is concave (convex) in $p \in [0,1]$.

(ii) F is IFRA (DFRA) if and only if $\phi(p)/p$ is decreasing (increasing) in $p \in [0,1]$.

(iii) F is NBUE (NWUE) if and only if $\phi(p) \geq p$ ($\phi(p) \leq p$) for $p \in [0,1]$.

(iv) F is DMRL (IMRL) if and only if $(1-\phi(p))/(1-p)$ is decreasing (increasing) in $p \in [0,1]$.

(v) F is HNBUE (HNWUE) if and only if $\phi(p) \leq 1 - \exp\{-F^{-1}(p)/\mu\}$ ($\phi(p) \geq 1 - \exp\{-F^{-1}(p)/\mu\}$) for $p \in [0,1]$.

(vi) $F \in$ BT (UBT) if ϕ has only one reflection point u_0 such that $0 < u_0 < 1$ and it is convex (concave) on $[0, u_0]$ and concave (convex) on $[u_0, 1]$.

Proof: Without loss of generality, we assume $F(t)$ is absolutely continuous.

(i) The result was independently proved by Barlow and Campo (1975) and Lee and Thompson (1976). Assuming $F(t)$ is absolutely continuous so that $p = F(t)$ implies $t = F^{-1}(p)$. Using the chain rule, we can easily verify that $\phi(p)$ is concave (convex) in p implies $r(t)$ is increasing (decreasing) in t so that the result (i) is proved.

(ii) The result was also due to Barlow and Campo (1975). The proof was nontrivial.

(iii) The NBUE (NWUE) characterization was made by Bergman (1977). This follows from

$$\phi(p) \geq (\leq)\, p \Leftrightarrow \int_0^t \bar{F}(x)\,dx \geq (\leq)\, \mu F(t) \Leftrightarrow \int_t^\infty \bar{F}(x)\,dx \leq (\geq)\, \mu \bar{F}(t).$$

Langberg et al. (1980a) gave more detailed results on IFRA (DFRA) and NBU (NWU).

(iv) The result on DMRL (IMRL) was from Klefsjö (1982). We note $(1-\phi(p))/(1-p)$ is equivalent to $\mu(t)/\mu$ so the result follows immediately.

(v) The result on HNBUE (HNWUE) was also obtained by Klefsjö (1982). This follows easily from $1 - \phi(p) = \int_t^\infty \bar{F}(x)\, dx/\mu$.

(vi) The result on bathtub distributions was proved by Barlow and Campo (1975). The proof follows from (i) that F is DFR iff ϕ is convex and F is IFR iff ϕ is concave so ϕ lies below the 45°-line in its leftmost part and above the line in its rightmost part.

Result (vi) was used by Aarset (1987) and Xie (1989) to derive test statistics for testing exponetiality against BT distributions.

Note that several of test statistics that are based on the scaled TTT transform will be presented in Chapter 7.

2.6 Non-monotonic Failure Rates and Non-monotonic Mean Residual Lives

Survival and failure times are frequently modelled by increasing or decreasing failure rate distributions. While this is appropriate for many cases, it may be inappropriate if the course of a disease is such that the mortality reaches a peak after some finite period and then declines slowly. Gupta and Warren (2001) gave two such examples:

- In a study of curability of breast cancer, Langlands et al. (1979) found that the peak of mortality occurred after about three years.
- Bennett (1983) analyzed the data from Veterans Administration lung cancer presented by Prentice (1973) and showed that the empirical failure rates for both low PS and high PS groups are non-monotonic. (PS = Potassium sulfide)

Thus, we need to analyze such data sets with appropriate lifetime models that have non-monotonic failure rates $r(t)$.

We postpone a full discussion on these ageing classes to the next chapter. Here we present a brief preview only.

2.6.1 Non-monotonic Failure Rates

A failure rate function falls into one of the four categories: (a) monotonic failure rates if $r(t)$ is either increasing or decreasing; (b) bathtub type failure rate if $r(t)$ has a bathtub (BT) or an upside-down bathtub (UBT) shape; and (c) modified bathtub failure rate if $r(t)$ is first increasing and then bathtub; and (d) generalized bathtub failure rate if $r(t)$ is a polynomial, or has a roller-coaster shape or some generalization.

Lai et al. (2001) give an overview on the class of bathtub shaped failure rate (BT) distributions. We will devote a a fuller discussion on this class of life distributions in Chapter 3.

2.6.2 Non-monotonic Mean Residual Lives

Recall from (2.6), the mean residual lifetime is defined as $\mu(t) = E(X - t \,|X > t)$ which is equivalent to $\int_0^\infty \bar{F}(x \,|\, t)\, dx = \int_t^\infty \bar{F}(x)\, dx / \bar{F}(t)$.

Recall also from Definition 2.3 that F is said to be DMRL if the mean remaining life function $\int_0^\infty \bar{F}(x \,|\, t)\, dt$ is decreasing in x. That is, the older the device is, the smaller is its mean residual life and hence $\mu(t)$ is monotonic. However, in many real life situations, the mean residual lifetime is non-monotonic and thus there arise several ageing notions defined in terms of the non-monotonic behavior of $\mu(t)$.

Guess et al. (1986) also defined a life class known as the increasing then decreasing mean residual life (IDMRL). To put in simply, F is IDMRL if $\mu(t) \in$ UBT. The dual class of 'decreasing initially, then increasing mean residual life' (DIMRL) has its $\mu(t) \in$ BT. We will discuss in Chapter 4 various facets of mean residual life, in particular, how the shapes of $r(t)$ and $\mu(t)$ are interrelated.

2.7 Some Further Classes of Ageing

Ageing concepts are proliferating in the literature. In addition to those lifetime classes defined above, there are a number of other ageing classes that have been investigated over the years. Without giving details, we just present their acronyms, definitions and references below.

- IFR(2) (Increasing failure rate of second order) iff

$$\int_0^x \frac{\bar{F}(u+s)\, du}{\bar{F}(s)} \geq \int_0^x \frac{\bar{F}(u+t)}{\bar{F}(t)}\, du, \text{for all}\, x \geq 0, t \geq s.$$

  See Deshpande et al. (1986), Franco et al. (2001).
  Clearly IFR $\Rightarrow$ IFR(2) $\Rightarrow$ DMRL.

- NBU(2) (New better than used of second order) iff

$$\int_0^x \bar{F}(u)\, du \geq \int_0^x \frac{\bar{F}(t+u)\, du}{\bar{F}(t)} \text{ for all } x, t \geq 0.$$

  See Deshpande et al. (1986), Franco et al. (2001), Li and Kochar (2001), Hu and Xie (2002), Li (2004).
  Clearly NBU $\Rightarrow$ NBU(2) $\Rightarrow$ NBUE.

- HNBUE(3) (Harmonic new better than used of third order) iff

$$\int_x^\infty \int_t^\infty \bar{F}(u)\, du\, dt \leq \mu^2 e^{-x/\mu} \text{ for all } x, t \geq 0.$$

  See Deshpande et al. (1986).

  Clearly HNBUE $\Rightarrow$ HNBUE(2).

- DMRLHA (Decreasing mean residual life in harmonic average) iff

$$\left[(1/t)\int_0^t (1/\mu(x)\,dx\right]^{-1} \text{ is decreasing in } t.$$

See Deshpande et al. (1986).
It can be shown that DMRL $\Rightarrow$ DMRLHA $\Rightarrow$ NBUE.

- SIFR (Stochastically increasing failure rate)
Let Y be a random variable with cdf F and mean μ; $X_i's$ are i.i.d. exponential random variables with the same mean μ and the $X_i's$ are independent such that $X_0 \equiv 0$. Then F is said to be stochastically increasing failure rate if

$$\Pr\left(Y \geq \sum_{i=0}^{k+1} X_i / Y \geq \sum_{i=0}^{k} X_i\right) \leq \Pr\left(Y \geq \sum_{i=0}^{k} X_i / Y \geq \sum_{i=0}^{k-1} X_i\right)$$

for all $k = 1, 2, ...$ See Singh and Deshpande (1985).

- SNBU (Stochastically new better than used). With the preceding assumptions, F is said to be stochastically new better than used if

$$\Pr\left(Y \geq \sum_{i=0}^{k+1} X_i / Y \geq \sum_{i=0}^{k} X_i\right) \leq \Pr(Y \geq X_{k+1})$$

for all $k = 1, 2, ...$ See Singh and Deshpande (1985).
It has been shown that (i) IFR $\Rightarrow$ SIFR, (ii) NBU $\Rightarrow$ SNBU, and (iii) SIFR $\Rightarrow$ SNBU.

- BMRL-t_0 (Better mean residual life at t_0 class). The mean life declines during the time 0 to t_0, and thereafter is no longer greater than what it was at t_0. See Kulasekera and Park (1987).

- DVRL (Decreasing variance of residual life) iff $\sigma^2(t) \leq \sigma^2(s)$ for all $s \leq t$ where $\sigma^2(t) = \text{var}(X - t|X \geq t)$ is the variance of the residual life. See Launer (1984).

- DPRL-α (Decreasing 100α percentile residual life) iff the α-percentile residual life $q_{\alpha,F}(t)$ defined by

$$q_\alpha(t) = \inf\{x : F_t(x) \geq \alpha\}, 0 < \alpha < 1, F(0) = 0$$

is decreasing in $t \in [0, T)$. Here $F_t = 1 - \bar{F}(t+x)/\bar{F}(t), x \geq 0$. See Joe and Proschan (1984).
F is IFR $\Leftrightarrow$ F is DPRL-α for all $0 < \alpha < 1$.

- NBUP-α (New better than used with respect to the 100α percentile) iff $F(0) = 0$ and $q_\alpha(t) \leq q_\alpha(0)$ for all $t \in [0, T)$. See Joe and Proschan (1984).
It is easy to see that

* DPRL-$\alpha \Rightarrow$ NBUP-α for any $0 < \alpha < 1$.
* NBU $\Leftrightarrow$ NBUP-α for all $0 < \alpha < 1 \Rightarrow$ NBUE.

Joe and Proschan (1984) provided other chains of relationships but we will not elaborate on them here.

- IFR $*\, t_0$ (IFR after t_0) iff $\bar{F}(bx) \geq \bar{F}^b(x)$ for all $x \geq t_0 > 0$ and all $t_0/x \leq b \leq 1$. See Li and Li (1998). It is easy to follow that IFR $\Rightarrow$ IFR $*\, t_0$.
- NBU $*\ t_0$ (NBU after t_0) iff $\bar{F}(x+y) \leq \bar{F}(x)\bar{F}(y)$ for all $x \geq 0$ and $y \geq t_0 > 0$. See Li and Li (1998).
  It is easy to follow that NBU $\Rightarrow$ NBU $*\ t_0$.
- UBA (used better than aged) if $\bar{F}(x+t) \geq \bar{F}(x)e^{-t/\mu(\infty)}$ for all $x, t \geq 0$ and UBAE (used better than aged in expectation) if $\mu(t) \geq \mu(\infty)$ for all $t \geq 0$ assuming $0 < \mu(\infty) < \infty$. See Alzaid (1994), Willmot and Cai (2000) and Ahmad (2004).
  It has been shown that DMRL $\Rightarrow$ UBA $\Rightarrow$ UBAE $\Leftarrow$ DVRL.

Nearly every one of these additional ageing classes is sandwiched in between two well-known classes discussed in Section 2.4.2. There are rarely any known distributions that belong to these further classes which are not already in the established classes. Apart from the DVRL class, we do not find these ageing concepts intuitive or easily interpretable. It is conceivable that more meaningful applications may emerge in future.

2.8 Failure Rates of Mixtures of Distributions

Interest on the ageing behaviour of mixtures has a long history (Barlow and Proschan, pp. 161-164). Mixtures arise from heterogeneous populations. A typical case is where a population consists of two subpopulations (which may be referred to as components of the mixtures). Mixtures also arise when we pool data from several distributions to enlarge the sample, for example. Mixtures are important in burn-in (see Block and Savits, 1997). Although mixtures of DFR distributions are always DFR, some mixtures of IFR may also be DFR. A well-known 'border line' example by Proschan (1963) exhibits the strict DFR property of a mixture of exponential distributions that have constant failure rates. Barlow (1985) and Mi (1998a), respectively, gave a Bayesian and non-Bayesian explanation of this unexpected phenomenon. Gurland and Sethuraman (1995) considered various types of finite and continuous mixtures of IFR distributions and developed conditions for such mixtures to be 'ultimately' DFR.

The density function of a mixture from two subpopulations with density functions f_1 and f_2 is simply given by

$$f(t) = pf_1(t) + (1-p)f_2(t), \ \ t \geq 0, 0 \leq p \leq 1; \tag{2.71}$$

and thus the survival function of a mixture is also a mixture of the two survival functions, i.e.,

$$\bar{F}(t) = p\bar{F}_1(t) + (1-p)\bar{F}_2(t). \tag{2.72}$$

The mixture failure rate $r(t)$ obtained from failure rates $r_1(t)$ and $r_2(t)$ associated with f_1 and f_2, respectively, can be expressed as

$$r(t) = \frac{pf_1(t) + (1-p)f_2(t)}{p\bar{F}_1(t) + (1-p)\bar{F}_2(t)} \tag{2.73}$$

where $f_i(t), \bar{F}_i(t)$ are the probability density and survival function of the distribution having failure rate $r_i(t), i = 1, 2$.

Let $r(t)$ in (2.73) be expressed as

$$r(t) = h(t)r_1(t) + (1 - h(t))r_2(t) \tag{2.74}$$

where $h(t) = 1/(1+g(t))$, $\quad g(t) = (1-p)\bar{F}_2(t)/[p\bar{F}_1(t)]$. Clearly $0 \leq h(t) \leq 1$.

We can easily generalize the above equation to accommodate mixtures of k subpopulations giving

$$r(t) = \frac{\sum_{i=1}^{k} p_i f_i(t)}{\sum_{i=1}^{k} p_i \bar{F}_i(t)} \tag{2.75}$$

where $i = 1, 2, ...k$, $0 < p_i < 1$, $\sum_{i=1}^{k} p_i = 1, k \geq 2$.

2.8.1 Mixture of Two DFR Distributions

A mixture of two DFR is again DFR; this result has been proved by Barlow et al. (1963) and other authors but we think Gupta and Warren's (2001) approach is simpler.

On differentiation of (2.74), we can be verify that

$$r'(t) = h(t)r_1'(t) + (1 - h(t))r_2'(t) - h(t)(1 - h(t))(r_1(t) - r_2(t))^2. \tag{2.76}$$

(Navarro and Hernnandez (2004) noted that the original expression (3.2) in Gupta and Warren (2001) was incorrect.) Since $0 \leq h(t) \leq 1$, it is obvious from (2.76) that the mixtures of DFR is DFR. Also Theorem 1 of Gurland and Sethuraman (1995) can now be easily obtained.

2.8.2 Possible Shapes of $r(t)$ When Two Subpopulations Are IFR

The behavior of mixtures of IFR distributions is unintuitive. It is easy to find examples of subpopulations with increasing failure rate whose mixture can have either increasing, decreasing or other shapes.

The following are taken from Block, Li and Savits (2003a) to illustrate that a variety of shapes can occur even for a simple mixture of two IFR subpopulations.

Example 2.1 We consider two IFR Weibull distributions $f_1(t) = 2t \exp\left\{-t^2\right\}$ and $f_2(t) = 3t^2 \exp\left\{-t^3\right\}$ with $p = 0.5$. In this case $r(t)$ is IFR.

The next example show that a mixture of two IFR distribution gives rise to a DFR distribution

Example 2.2 Let $r_1(t) = 1 - \exp\{-5t\}$, $r_2(t) = 6 - \exp\{-5t\}$ with $p = 0.5$. We note that $r_1(t)$ strictly increases to 1 and $r_2(t)$ strictly increases to 6. However, $r(t) \in$ DFR strictly decreases to 1.

Example 2.3 Take $f_1(t) = \exp\{-t\}$ to be exponential, $f_2(t) = 16t \exp\{-4t\}$ to be an IFR gamma and let $p = 0.5$. In this case, $r(t) \in$ UBT, i.e., $r_m(t)$ has an upside-down bathtub shape.

Example 2.4 Let $f_1(t) = 4 \exp\{-4t\}$ be exponential, $f_2(t) = t \exp\{-t\}$ be an IFR gamma and $p = 0.5$. Then $r(t) \in$ BT.

Example 2.5 Consider two Weibull distributions, $f_1(t) = 2t \exp\left\{-t^2\right\}$ and $f_2(t) = 4t^3 \exp\left\{-t^4\right\}$ with $p = 0.5$. Both $r_1(t)$ and $r_2(t)$ increase to ∞. The mixture failure rate $r(t) \in$ MBT. This phenomenon is also noted in Jiang and Murthy (1998).

The above examples show how the shape of the mixture failure rate varies as the subpopulation failure rates and the mixing proportion $p = 0.5$ remains fixed. The next example keeps the failure rates of the subpopulation fixed while varying the mixing proportion p.

Example 2.6 Consider two increasing linear failure rates $r_1(t) = t + 1$ and $r_2(t) = 4t + 5$. Block, Li and Savits (2003a) showed that

- $r(t) \in$ MBT for $p = 0.1$,
- $r(t) \in$ BT for $p = 0.65$,
- $r(t) \in$ IFR for $p = 0.95$.

Gurland and Sethuraman (1995) gave a necessary and sufficient condition for a mixture of two IFR distributions to be DFR. However, the condition does not appear to be easily verified.

We next consider mixtures of two life distributions from the same family of distributions possibly with different parameters.

2.8.3 Mixture of Two Gamma Densities with a Common Scale Parameter

Consider the mixture of two gamma densities:

$$f(t) = pf_1(t) + (1-p)f_2(t), \tag{2.77}$$

where

$$f_i(t) = \frac{\lambda^{\alpha_i} t^{\alpha_i - 1}}{\Gamma(\alpha_i)} e^{-\lambda t}, t > 0, \ \alpha_i, \lambda > 0, i = 1, 2. \tag{2.78}$$

Assuming $\alpha_1 < \alpha_2$, Glaser (1980) was able to determine the shape of the failure rate of the distribution specified by (2.77) in all cases except for one: $\alpha_1 > 1, \alpha_2 - \alpha_1 > 0$ with $\alpha_1 - 1 < (\alpha_2 - \alpha_1 - 1)^2/4$. For this case, he conjectured that the mixture density is IFR.

Gupta and Warren (2001) generalized the result of Glaser (1980) and also disproved the above conjecture. We now summarize the results of Glaser (1980) and that of Gupta and Warren (2001) as follows.

Theorem 2.8: Assuming $\alpha_1 < \alpha_2$, the gamma mixture has the following failure rate shapes:

(i) $0 < \alpha_1 < \alpha_2 \leq 1$ implies that F is DFR,
(ii) $0 < \alpha_1 < 1 < \alpha_2$ implies that F is BT,
(iii) $\alpha_1 = 1 < \alpha_2 \leq 2$ implies that F is IFR,
(iv) $\alpha_1 = 1, 2 < \alpha_2$ implies that F is BT,
and for $1 < \alpha_1 < \alpha_2$,
(v) $\sqrt{\alpha_2 - 1} - \sqrt{\alpha_1 - 1} \leq 1$ implies that F is IFR, while
(vi) $\sqrt{\alpha_2 - 1} - \sqrt{\alpha_1 - 1} > 1$ implies that F is IFR or MBT.

Proof: The first five cases were proved by Glaser (1980) by considering the behavior of $\eta'(t)$ with $\eta(t) = f'(t)/f(t)$ as defined in (2.9). Case (vi) was incorrectly conjectured by Glaser but proved by Gupta and Warren (2001) via Theorem 2.3 above.

Remark: We have not listed the case when $\alpha_1 = \alpha_2 = 1$. This corresponds to the mixture of two exponential distributions. If we assume the scale parameters are the same as in the theorem, then the mixture has an exponential which is both IFR and DFR. However, if the two shape parameters are different, then the mixture has a DFR distribution as observed by Proschan (1963).

2.8.4 Mixture of Two Weibull Distributions

Jiang and Murthy (1998) categorized the possible shapes of failure rate function for a mixture of any two Weibull distributions in terms of five parameters. The failure rate shape can be one of eight different types including IFR, DFR, MBT, UBT and 'roller-coaster' shaped. They showed that, among other authors, this mixture distribution F cannot have a BT failure rate. They also stated that the mixture failure rates from two strictly IFR Weibull distributions with the same shape parameter can be either MBT or IFR. However, they did not explicitly classify the two possibilities. Wondmagegnehu (2004) developed the work of Jiang and Murthy (1998) further but assumed the two Weibull distributions to be strictly IFR.

Let $\bar{F}_1(t) = \exp\{-\theta_1 t^\alpha\}$ and $\bar{F}_2(t) = \exp\{-\theta_2 t^\alpha\}$ be the survival functions of two Weibull distributions so that $r_1(t) = \theta_1 \alpha t^{\alpha-1}$ and $r_2(t) = \theta_2 \alpha t^{\alpha-1}$ where $\alpha > 1$ and $\theta_2 > \theta_1 > 0$. Set $\beta = \theta_2/\theta_1$ and define

$$\omega_1 = \alpha(\beta - 1) + \sqrt{\alpha^2(\beta-1)^2 + 4(\alpha-1)^2\beta}$$

and

$$\omega_2 = 2\alpha(\beta-1)\exp\left\{\frac{(\alpha-1)(\beta+1) + \sqrt{\alpha^2(\beta-1)^2 - 4(\alpha-1)^2\beta}}{\alpha(\beta-1)}\right\}.$$

Theorem 2.9: Let ω_1 and ω_2 be defined as above and p be the mixing proportion. Further, we define

$$\xi = \frac{\omega_1}{\omega_1 + \omega_2}.$$

Then the mixture failure rate $r(t)$ has

(a) a modified bathtub (MBT) shaped failure rate when $0 < p < \xi$ and
(b) an increasing failure rate (IFR) when $\xi \le p < 1$.

Proof: From (2.73), it is easy to verify that

$$r(t) = \theta_1 \alpha t^{\alpha-1} + \frac{(1-p)(\beta-1)\theta_1 \alpha t^{\alpha-1}}{p e^{\theta_1(\beta-1)t^\alpha} + (1-p)}. \tag{2.79}$$

Letting $z = e^{\theta_1(\beta-1)t^\alpha}$, we have $t = [[1/\{\theta_1(\beta-1)\}]\log z]^{1/\alpha}$. Substituting this expression of t and $b = p(1-p)$ into (2.79) gives

$$r^*(z) = \left[\frac{bz+\beta}{bz+1}\right]\left(\frac{1}{\theta_1(\beta-1)}\log z\right)^{(\alpha-1)/\alpha}$$

where $r^*(t) = r(t)/(\theta_1\alpha)$.

Both $r(t)$ and $r^*(z)$ have the same monotonicity in the corresponding domains $\{t : t \ge 0\}$ and $\{z : z \ge 0\}$, respectively. It is now easier to study the shape of $r(t)$ via $r^*(z)$. Taking logarithm of $r^*(z)$ and then differentiating it with respect to z, we find

$$\frac{d}{dz}\log r^*(z) = \frac{K(z)}{\alpha z(\log z)(bz+\beta)(bz+1)},$$

where

$$K(z) = (\alpha-1)(bz+\beta)(bz+1) - b\alpha(\beta-1)z\log z.$$

We then examine the derivative $K'(z) = 2b(\alpha - 1)z[b - h(z)]$. The theorem can then be established, after several tedious steps, by considering the behaviour of $h(t)$. See Wondmagegnehu (2004) for a complete proof.

Wondmagegnehu (2004) also used several examples to illustrate possible shapes that the mixture failure rate can encounter when the two Weibull distributions have different shape and scale parameters.

2.8.5 Mixtures of Two Positively Truncated Normal Distributions

Navarro and Hernandez (2004) have considered the shape of the failure rate of the mixture of two positively truncated normal distributions given in Section 2.3.3. The method is based on the s-order equilibrium distribution of a renewal process defined by Fagiuoli and Pellerey (1993) and will be given in Section 2.9.1 below.

For a truncated normal distribution with parameter μ and σ, it is noted by Navarro and Hernandez (2004) that its eta function defined by the negative value of the derivative of the density over the density is given by $(t-\mu)/\sigma^2$. Let f_i be the two truncated normal densities with parameters μ_i and $\sigma_i, i = 1, 2$, and f be the density of the mixture and $\eta = -f'/f$. Also, let

$$w(t) = \frac{1}{1 + \alpha(t)},$$

where

$$\alpha(t) = \frac{1-p}{p}\frac{f_2(t)}{f_1(t)}.$$

Assuming that $\sigma_1 = \sigma_2$ and letting $\delta = \sigma_1^2/(\mu_2 - \mu_1)^2$, the authors showed that

1. If $\delta > 1/4$, then $r(t) \in$ I.
2. If $\delta \leq 1/4, w(0) \geq 0$, and $w(0)(1 - w(0) < \delta$, then $r(t) \in$ I.
3. If $\delta \leq 1/4, w(0) \geq 0$, and $w(0)(1 - w(0) \geq \delta$, $r(t) \in$ I or BT.
4. If $\delta \leq 1/4, w(0) < 1/2$, and $w(0)(1 - w(0) > \delta$, then $r(t) \in$ I or BT.
5. If $\delta \leq 1/4w(0) < 1/2$, and $w(0)(1 - w(0) \leq \delta$, then $r(t) \in$ I, BT or MBT.

Moreover, the change points of η are determined by $w(t)(1 - w(t)) = 0$.

The key ingredient of the proof for the above results hinges on the fact that if $\sigma_1 = \sigma_2$,

$$\sigma_1^2\eta'(t) = 1 - (1 - w(t))w(t)\frac{(\mu_2 - \mu_1)^2}{\sigma_1^2}$$

and $\eta'(t) \geq 0$ iff $(1 - w(t))w(t) \leq \delta$.

Navarro and Hernandez (2004) also obtained a general result on the shape of $r(t)$ when the variances are not equal and $\eta_2(t) \geq \eta_1(t)$. The proof now requires the theory of s-order equilibrium distribution of the renewal process mentioned which will be further developed in Section 2.9 and Section 2.11.

2.8.6 Mixtures of Two Increasing Linear Failure Rate Distributions

Block, Savits and Wondmagegnehu (2003) gave explicit conditions which delineate the possible shapes of the failure rate function for the mixture of two IFR linear failure rate distributions.

Let the two increasing linear failure rates be given by, respectively,

$$r_1(t) = c_1 t + d_1, \quad r_2(t) = c_2 t + d_2,$$

where, without loss of generality, we assume $c_2 \geq c_1 > 0$ and $d_2, d_1 \geq 0$. Thus the expressions for the two component survival functions are, respectively,

$$\bar{F}_1(t) = \exp\left\{-\frac{c_1 t^2}{2} - d_1 t\right\},$$

and

$$\bar{F}_2(t) = \exp\left\{-\frac{c_2 t^2}{2} - d_2 t\right\}.$$

Substituting the expressions for $\bar{F}_1(t), \bar{F}_2(t)$, $r_1(t), r_2(t)$ into (2.74), we have

$$r(t) = (c_1 t + d_1) + \frac{(1-p)[c_1(\gamma-1)t + a]}{p\exp((c_1/2)(\gamma-1)^2 + at) + (1-p)}, \tag{2.80}$$

where $\gamma = c_2/c_1 \geq 1, c_2 \geq c_1 > 0$, $a = d_2 - d_1$ and $0 < p < 1$. It turns out all the possible shapes are determined by these parameters: $\gamma = c_2/c_1$, $\delta = a/\sqrt{c_1}$ and p. Define the following parameters:

$$\alpha_1 = \frac{(\delta^2 + \gamma - 1) - \sqrt{(\delta^2 - \gamma - 1)^2 - 4\gamma}}{2\delta^2},$$

$$\alpha_1 = \frac{(\delta^2 + \gamma - 1) + \sqrt{(\delta^2 - \gamma - 1)^2 - 4\gamma}}{2\delta^2},$$

$$\alpha_3 = \frac{(\gamma-1) + \beta}{(\gamma-1) + \beta + \exp\{((\gamma+1) - \delta^2 + 2\beta)/2(\gamma-1)\}}$$

where $\beta = \sqrt{(\gamma-1)^2 + \gamma}$.

Non-crossing linear failure rates

We now consider the mixture failure rate for two non-crossing linear failure rates. Its possible shapes can be summarized as follows:

Theorem 2.10: (Block, Savits and Wondmagegnehu, 2003). Consider the mixture failure rate $r(t)$ given in (2.50) for two non-crossing linear failure rates $r_1(t) = c_1 t + d_1$ and $r_2(t) = c_2 t + d_2$ such that $c_2 > c_1 > 0$ and $d_2 > d_1 \geq 0$. Recall the expressions for α_1, α_2 and α_3 are given as above.

(i) If $\delta \leq \sqrt{\gamma} + 1$, then
  (a) $r(t) \in$ BT if $0 < p < \alpha_3$, or
  (b) $r(t) \in$ IFR if $\alpha_3 < p < 1$.

(ii) If $\sqrt{\gamma} + 1 < \delta < \sqrt{(\gamma + 1) + 2\beta}$, then
  (a) $r(t) \in$ MBT if $0 < p < \alpha_1$,
  (b) $r(t) \in$ BT if $\alpha_1 < p \leq \alpha_2$,
  (c) $r(t) \in$ MBT if $\alpha_2 < p < \alpha_3$, or
  (d) $r(t) \in$ IFR if $\alpha_3 \leq p < 1$.

(iii) If $\delta \geq \sqrt{(\gamma + 1) + 2\beta}$, then
  (a) $r(t) \in$ MBT if $0 < p < \alpha_1$,
  (b) $r(t) \in$ BT if $\alpha_1 < p \leq \alpha_2$, or
  (c) $r(t) \in$ IFR if $\alpha_2 \leq p < 1$.

Proof: The proof is rather lengthy. See Block, Savits and Wondmagegnehu (2003).

Linear failure rates with same slope

We now assume the two failure rate functions have the same slope, that is, $c_1 = c_2 = c$. Define

$$\delta = \frac{a}{\sqrt{c}} > 0, \quad \zeta_1 = \frac{1 - \sqrt{1 - 4/\delta^2}}{2}, \quad \zeta_2 = \frac{1 + \sqrt{1 - 4/\delta^2}}{2}.$$

Block, Savits and Wondmagegnehu (2003) showed that

(i) If $0 < \delta \leq 2$, then $r(t) \in$ IFR for all $p \in (0, 1)$.

(ii) If $\delta > 2$, then
  (a) $r(t) \in$ MBT if $p < \zeta_1$;
  (b) $r(t) \in$ BT if $\zeta_1 < p < \zeta_2$;
  (c) $r(t) \in$ IFR if $\zeta_2 < p < 1$.

Two failure rates with the same y-intercepts

We now consider the case $d_1 = d_2 = d$. Define

$$\xi = \frac{(\gamma-1)+\beta}{(\gamma-1)+\beta+\exp\{((\gamma+1)-\delta^2+2\beta)/2(\gamma-1)\}}$$

where $\beta = \sqrt{(\gamma-1)^2+\gamma}$. Block, Savits and Wondmagegnehu (2003) showed that

(i) $r(t) \in$ MBT if $0 < p < \xi$;

(ii) $r(t) \in$ IFR if $\xi < p < 1$.

Mixtures of crossing failure rates

Two linear failure rates may cross at the point $t_0 = -a/c_1(\gamma-1)$ so that $r_1(t) > r_2(t)$ for all $t \in [0, t_0)$ and $r_1(t) < r_2(t)$ for all $t > t_0$. Block, Savits and Wondmagegnehu (2003) delineated all possible shapes that the mixture failure rate $r(t)$ can take assuming the two linear failure rates intersects at t_0.

2.8.7 Mixtures of an IFR Distribution with an Exponential Distribution

Gurland and Sethuraman (1995) gave the following definition:

Definition 2.12: An IFR distribution is said to be MRE if its mixture with an exponential is 'ultimately' DFR for some mixing proportion p. This means that for sufficiently large t, says $t \geq t_0$, the mixture failure rate is decreasing.

Gurland and Sethuraman (1995) also provided several examples that are MRE, some of these are now listed below.

Examples

Exponential distribution

It is a well known that the mixture of two exponential distributions is DFR (Proschan, 1963).

Gamma distribution

Let the density function of the IFR gamma distribution be

$$f_1(t) = \frac{\lambda_1^\alpha}{\Gamma(\alpha)} e^{-\lambda_1 t} t^{\alpha-1}, \ \lambda_1 > 0, \alpha > 1, t > 0$$

and $f_2(t) = \lambda_2 e^{-\lambda_2 t}$. Then the gamma mixture with exponential is MRE for large t. Further, if $\lambda_1 > \lambda_2$, the mixture is DFR.

Weibull distribution

Let $\bar{F}_1(t) = e^{-\theta t^\alpha}$, $\theta, \alpha > 0, t > 0$ be the survival function of the Weibull distribution with failure rate given by $r_1(t) = \theta\alpha t^{\alpha-1}$. Gurland and Sethuraman (1995) showed that for $\alpha > 1$ the Weibull distribution is MRE. Let $r(t,p)$ be the failure rate of the mixture $p\bar{F}_1(t) + (1-p)\bar{F}_2(t)$ of a Weibull with an exponential having density $f_2(t) = \lambda e^{-\lambda t}$. Thus $r(t,p) = pr_1(t) + (1-p)r_2(t) = p\theta\alpha t^{\alpha-1} + (1-p)\lambda$. Also, there is a turning point $t_0 = t_0(p)$ such that $r(t,p)$ is decreasing for $t \geq t_0$. Using $\alpha = 3, \theta = 2.5$ and $\lambda = .25$, they gave plots of $r(t,p)$ for $p = .05, .1, .5, .7, .9, .95$. They observed that the turning point $t_0(p)$ of $r(t,p)$ decreases as p increases, because the exponential plays an increasingly important role in the mixture and, accordingly, fewer items have failed. We also observe that $r(t,p)$ tends to λ as $t \to \infty$.

The authors also gave an intuitive explanation of the MRE phenomenon having $r(t,p)$ decreasing for large t. This is because the early failure times of the mixture come from the distribution with larger failure rates, so that the larger failure times (in the tail of mixture) come from the distribution that has smaller failure rate in the tail. They also sounded a warning that the practice of pooling several IFR distributions may reverse the IFR property of the individual samples to a DFR property.

2.8.8 Failure Rate of Finite Mixture of Several Components Belonging to the Same Family

Al-Hussaini and Sultan (2001) gave a comprehensive review on reliability and failure rates of mixture models. Seven finite mixture models in which the components belonging to the same family of distributions are investigated. These are

1. Mixtures of normal components.
2. Mixtures of lognormal components.
3. Mixtures of inverse Gaussian components.
4. Mixtures of exponential components.
5. Mixtures of Rayleigh components.
6. Mixtures of Weibull components.
7. Mixtures of Gompertz components.

Three of this list have already been considered above. Plots of failure rates (with selected parameter values) of the mixture of two components are also given in Al-Hussaini and Sultan (2001). We observe that, nearly every one of these figures has an upside-down bathtub shape. This is reminiscent of the findings of Gurland and Sethuraman (1995) who stated in their Introduction section that ‘ many standard families of IFR distributions exhibit the property that the mixtures of two distributions from the same family are ultimately DFR’. We need, however, to put this statement in perspective, as we have seen in Section 2.8.1 that several other shapes are also possible for a mixture of two lifetime components.

Note: Al-Hussaini and Sultan (2001) also considered mixture models with components belonging to different families.

2.8.9 Initial and Final Behavior of Failure Rates of Mixtures

Block et at. (2001) reviewed the behavior of the failure rate of mixtures $r_m(t)$ of several components. This review mainly draws on the results of Block and Joe (1997), Block et al. (1993) and Block, Li and Savits (2003a). We now summarize them below.

Initial behavior of $r(t)$

Let the failure rate of the finite mixtures be given by

$$r(t) = \frac{\sum_{i=1}^{n} p_i f_i(t)}{\sum_{i=1}^{n} p_i \bar{F}_i(t)}. \tag{2.81}$$

Block, Li and Savits (2003a) showed that

$$r(0+) = \sum_{i=1}^{n} p_i r_i(0+) \tag{2.82}$$

assuming $r(0+) = f_i(0+)$ exists; and

$$r'(t) = \sum_{i=1}^{n} p_i f_i'(0+) + \left[\sum_{i=1}^{n} f_i(0+)\right]^2, \tag{2.83}$$

assuming both $r(0+)$ and $f_i'(0+)$ exist. Note that we have used $r_i(0+)$ to denote $\lim_{t \downarrow 0} r_i(t)$. The other limits are also defined similarly.

The authors also showed that the initial behavior of the failure rate r_ϕ of a system of several components is similar to the failure of the mixture.

Asymptotic behavior of $r_m(t)$

The asymptotic behavior of mixtures of exponentials has been studied by Clarotti and Spizzichino (1990) and more generally by Block et al. (1993).

(a) The first general result on mixtures of distributions is that, under mild conditions, the asymptotic limit of the failure rate of a mixture is the same as the limit of strongest component.

(b) The second result, due to Block and Joe (1997), is that for failure rates, which asymptotically behave like the ratios of polynomials, the eventual monotonicity of the mixture is the same as the monotonicity of the strongest component. Furthermore, if the failure rate of the strongest component is increasing, so is the failure rate of the mixture.

A similar result holds for systems except the role of the strongest subpopulation is replaced by strongest minimal path set.

Block, Li and Savits (2003a) gave a definite result concerning the limit of the failure rate of a mixture of two components. The result is now stated as below.

Theorem 2.11: Consider a mixture of two subpopulations with failure rates $r_1(t) \to \lambda \in [0, \infty)$ and $r_2(t) \to \infty$ such that

$$r(t) = \frac{pf_1(t) + (1-p)f_2(t)}{p\bar{F}_1(t) + (1-p)\bar{F}_2(t)}$$

(as given in (2.73)) converges to ξ as $t \to \infty$. Then ξ must be finite and equal to λ.

Proof: Set $\bar{F}(t) = p\bar{F}_1(t) + (1-p)\bar{F}_2(t)$. From the assumption, we deduce that

$$-\frac{\log \bar{F}(t)}{t} = \frac{1}{t}\int_0^t r(u)\, du \to \xi$$

and

$$\frac{\log \bar{F}_1(t)}{t} = \frac{1}{t}\int_0^t r_1(u)\, du \to \lambda$$

as $t \to \infty$. Since $\log \bar{F}(t) \geq \log\big(p\bar{F}_1(t)\big) = \log p + \log \bar{F}_1(t)$, we find that

$$\xi = -\lim_{t\to\infty} \frac{\log \bar{F}(t)}{t} \leq -\liminf_{t\to\infty} \left\{ \frac{\log p}{t} + \frac{\log \bar{F}_1(t)}{t} \right\} = \lambda.$$

Thus, $\xi \leq \lambda < \infty$.

On the other hand, since $r_2(t) - r_1(t) \to \infty$,

$$\frac{\bar{F}_2(t)}{\bar{F}_1(t)} = \exp\left\{ -\int_0^t [r_2(t) - r_1(t)]\, du \right\} = \mathrm{O}(e^{-Kt})$$

for all $K > 0$. In particular, $\bar{F}_2(t)/\bar{F}_1(t) \to 0$ as $t \to \infty$. It now follows that

$$\frac{f(t)}{\bar{F}_1(t)} = r(t) \left\{ \frac{p\bar{F}_1(t) + (1-p)\bar{F}_2(t)}{\bar{F}_1(t)} \right\} \to p\xi$$

where $f(t) = pf_1(t) + (1-p)f_2(t)$. As $f(t) \geq pf_1(t)$, we have

$$p\xi = \lim_{t\to\infty}\left[\frac{f(t)}{\bar{F}_1(t)}\right] \geq \limsup_{t\to\infty}\left[\frac{pf_1(t)}{\bar{F}_1(t)}\right] = p\lambda,$$

i.e., $\xi \geq \lambda$ so $\xi = \lambda$.

Corollary 2.1: Let $r(t)$ be the failure rate of a finite mixture of subpopulations with failure rates $r_i(t)$ satisfying $r_i(t) \to a_i \in [0,\infty], 1 \leq n$, as $t \to \infty$. Then either $r(t) \to \alpha = \min_{1\leq i\leq n} a_i$, or $r(t)$ does not converge.

Proof: See Corollary 2.1 of Block et al. (2003a).

2.8.10 Continuous Mixtures of Distributions

Shaked and Spizzichino (2001) gave a review on the failure rate function of a continuous mixture.

Continuous mixtures of DFR distributions

It has been shown, see for example, Barlow and Proschan (1981, p. 103), that continuous mixtures of DFR distributions is DFR. More specifically, let F_α be the distribution function with parameter α. Suppose α itself is a random variable with distribution $M(\alpha)$, then the resultant distribution (generally known as the mixture distribution) is expressed as

$$F(t) = \int_{-\infty}^{\infty} F_\alpha(t)\, dM(\alpha).$$

M is called the mixing distribution. If each F_α is DFR, then mixture distribution F is DFR irrespective of the mixing distribution. The proof utilizes the fact that the hazard transform of a mixture is concave.

For example, let $\bar{F}_\alpha(t)$ be an exponential survival function with α being a gamma random variable having density $m(\alpha) = [\beta^\gamma \alpha^{\gamma-1}/\Gamma(\gamma)]e^{-\beta\alpha}$, $0 \leq \alpha < \infty$. Then the resulting distribution has a Pareto survival function $\bar{F}(t) = (1+t/\beta)^{-\gamma}$ which is DFR. A partial converse to this result is stated by Gleser (1989) showing that a gamma distribution with scale parameter λ and shape parameter $\beta \leq 1$ (i.e., having decreasing failure rate) can be expressed as a scale mixture of exponential distributions

$$f(x) = \int_0^{\infty} g_\lambda(\gamma)\gamma e^{-\gamma x}\, d\gamma,\ \lambda > 0,$$

where

$$g_\lambda(\gamma) = \frac{(\gamma-\lambda)^{-\beta}\lambda^\beta}{\gamma\Gamma(1-\beta)\Gamma(\beta)},\ \ \gamma \geq \lambda.$$

Continuous mixtures of IFR distributions

We have seen earlier that a mixture of two IFR distributions may be IFR, DFR or other ageing classes. Gurland and Sethuraman (1994) have given some examples of finite mixtures of rapidly increasing failure rate distributions but the resultant mixture distributions ultimately having decreasing failure rate.

Lynch (1999) gave some general conditions under which the IFR property is preserved by continuous mixtures. His result can be restated as follows. Let $\{\bar{F}(t\,|\,\theta) : \theta \geq 0\}$ be a family of survival functions with univariate parameter $\theta \geq 0$. Let M be the mixing distribution on $[0, \infty)$. The resultant mixture survival function is as below:

$$\bar{F}_M(t) = \int \bar{F}(t\,|\,\theta)\, dM(\theta).$$

The main result of Lynch (1999) is that if the mixing distribution M has an IFR distribution and if $\bar{F}(t\,|\,\theta)$ is log concave in the variables (t, θ), then $F_M(t)$ is IFR.

Block, Li and Savits (2003b) showed that Lynch's result is a special case of Savits (1985) with the correct interpretation. They also showed that similar closure theorems are possible for other ageing classes such as IFRA, NBU and DMRL.

Finkelstein and Esaulova (2001) considered several types of continuous (infinite) mixtures of IFR distribution. In particular, the corresponding limiting behavior of the mixture failure rate function is analyzed for the specific case of mixing which can be interpreted in terms of the proportional hazard model. It is found that under certain assumptions the mixture failure rate decreases to zero as $t \to \infty$.

2.9 Partial Orderings and Generalized Partial Orderings

Partial orderings of two life distributions have been studied quite extensively. Essentially, we are comparing two lifetime variables X and Y in terms of their failure rates $r_F(t)$ and $r_G(t)$, density functions $f(t)$ and $g(t)$, survival functions $\bar{F}(t)$ and $\bar{G}(t)$, mean residual lives $\mu_F(t)$ and $\mu_G(t)$, or other ageing characteristics. Ageing classes can often be characterized by some partial orderings. For example, in Barlow and Proschan (1981, pp.105-107), IFR and IFRA classes are characterized by 'convex ordering' and 'star-shaped ordering', respectively. However, these partial orderings do not fit in with the main body of of our approach so we will not discuss them till Section 10.3. Several authors have studied partial orderings and stochastic dominance, for example, Desphande et al. (1986), Kochar and Wiens (1987), Singh (1989), Fagiuoli and Pellerey (1993), Shaked and Shanthikumar (1994) and several others.

We now give definitions for several of these basic partial orderings. These are selected on the basis that they are easily understood and there is a chain of implications for these orderings to indicate their relative stringency.

Definition 2.13: X is said to be greater than Y in likelihood ratio ordering ($X \geq_{\mathrm{LR}} Y$) if $f(t)/g(t)$ is increasing in $t \geq 0$.

Definition 2.14: X is said to be greater than Y in weak likelihood ratio ordering ($X \geq_{\mathrm{WLR}} Y$) if $f(t)/g(t) \geq f(0)/g(0)$ for all $t \geq 0$.

Definition 2.15: X is said to be greater than Y in failure rate ordering ($X \geq_{\mathrm{FR}} Y$) if $r_F(t) \leq r_G(t)$ for all $t \geq 0$ or $\bar{F}(t)/\bar{G}(t)$ is increasing in $t \geq 0$.

Definition 2.16: X is said to be greater than Y in stochastic ordering ($X \geq_{\mathrm{ST}} Y$) if $\bar{F}(t) \geq \bar{G}(t)$, for all $t \geq 0$.

Definition 2.17: X is said to be greater than Y in mean residual ordering ($X \geq_{\mathrm{MR}} Y$) if $\mu_F(t) \geq \mu_G(t)$, for all $t \geq 0$.

It is found that $X \geq_{\mathrm{MR}} Y$ if and only if $\int_t^\infty \bar{F}(x)\,dx / \int_t^\infty \bar{G}(x)\,dx$ is increasing in $t \geq 0$.

Definition 2.18: X is said to be greater than Y in harmonic average mean residual ordering ($X \geq_{\mathrm{HAMR}} Y$) if $\int_t^\infty \bar{F}(x)\,dx/\mu_F \geq \int_t^\infty \bar{G}(x)\,dx/\mu_G$, for all $t \geq 0$.

Definition 2.19: X is said to be greater than Y in variance residual life ordering ($X \geq_{\mathrm{VR}} Y$) if $\int_t^\infty \int_x^\infty \bar{F}(u)\,du\,dx / \int_t^\infty \int_x^\infty \bar{G}(u)\,du\,dx$ is increasing for all $t \geq 0$.

Definition 2.20: X is said to be greater than Y in convex ordering ($X \geq_{\mathrm{CX}} Y$) if $\int_t^\infty \bar{F}(x)\,dx \geq \int_t^\infty \bar{G}(x)\,dx$, for all $t \geq 0$. It is sometimes known as variable ordering (Ross, 1983). The convex ordering defined here does not appear to be equivalent to the 'convex ordering' given in Barlow and Proschan (1981, p.106).

Definition 2.21: X is said to be greater than Y in concave ordering ($X \geq_{\mathrm{CV}} Y$) if $\int_0^t \bar{F}(x)\,dx \geq \int_0^t \bar{G}(x)\,dx$, for all $t \geq 0$.

Singh (1989) gave a chain of implications between the first eight partial orderings .

It appears that the stochastic ordering and failure rate ordering are two most important ones. For applications for redundancy applications in series, parallel and k-out-of-n systems, see Boland et al. (1992, 1998), Shaked and Shanthikumar (1995) and Boland (1998). Also, a chain of implications will be presented in Table 2.3 connecting these concepts and some of these generalized partial orderings to be defined in the next subsection. We note that there are other partial orderings introduced from different view points by various authors. A summary of these partial orderings is given in Chapter 33 of Johnson et al. (1995). Navarro et al. (1997) also studied some stochastic partial orderings between two doubly truncated variables.

2.9.1 Generlized Partial Orderings

There is a proliferation of generalized partial orderings in the literature over the recent years. Several of these definitions gave rise to new ageing classes some of which are presented in Section 2.7 (e.g., IFR(2), NBU(2) and HNBUE(3)). Here, our discussion are selective rather than inclusive. Fagiuoli and Pellerey (1993) defined several additional partial orderings based on the survival function of an equilibrium distribution of a random variable.

Let X be an absolutely continuous non-negative random variable with distribution function $F(t)$ and differentiable density function $f(t)$. Let $\mu_F(t)$ denote the mean of the lifetime variable X with cdf F. The equilibrium distribution corresponding to the lifetime variable X is defined as

$$E_F(t) = \int_0^t \bar{F}(x)\, dx/\mu_F, \tag{2.84}$$

and we will denote the survival equilibrium function by the function

$$\bar{E}_F(t) = 1 - E_F(t) = \int_t^\infty \bar{F}(x)\, dx/\mu_F. \tag{2.85}$$

We now define the survival functions of the equilibrium distributions recursively:

$$\bar{T}_0(X, x) = f(x),\ \bar{T}_{-1}(X, x) = -f'(x) \tag{2.86}$$

and

$$\bar{T}_s(X, x) = \frac{\int_x^\infty \bar{T}_{s-1}(X, u)\, du}{\mu_{s-1}(X)},\ \text{for integer } s \geq 1, \tag{2.87}$$

where

$$\mu_s(X) = \int_0^\infty \bar{T}_s(X, x)\, dx, s \geq 0. \tag{2.88}$$

The functions $\bar{T}_s(X, x)$ were first introduced by Fagiuoli and Pellerey (1993). Clearly, $\mu_0(X) = 1$. It follows from (2.87) and (2.88) that $\bar{T}_1(X, x) = \bar{F}(x)$, so $\mu_1(X) = E(X)$. Note also $\bar{T}_s(X, 0) = 1, s \geq 1$. Further, $\bar{T}_2(X, x)$ is the survival function of the equilibrium distribution of X from which we deduce that $\bar{T}_s(X, x)$ is the survival function of the equilibrium distribution of a distribution with survival function $\bar{T}_{s-1}(X, x)$.

Nanda et al. (1996) have established some interesting properties regarding moments for s-order equilibrium distributions:

$$\mu_0(X)\mu_1(X)...\mu_s(X) = \frac{E(X^s)}{s!},\ \ s = 0, 1, ... \tag{2.89}$$

from which we obtain

$$\mu_s = \frac{E(X^s)}{sE(X^{s-1})},\ \ s = 1, 2, ...$$

In addition, Fagiuoli and Pellery (1993) also defined what we call the s-order failure rate function

$$r_s(X,x) = \frac{\bar{T}_{s-1}(x)}{\int_x^\infty \bar{T}_{s-1}(u\,du)} = -\frac{\frac{d}{dx}\bar{T}_s(x)}{\bar{T}_s(x)},\ s \geq 0 \qquad (2.90)$$

and $r_0(X,x) = \frac{-f'(x)}{f(x)}$ when $f'(x)$ exists. For $s = 1$, $r_1(X,x) = \frac{f(x)}{\bar{F}(x)} = r_F(x)$, the failure rate function that corresponds to X. For $s = 2$, $r_2(X,x) = (\mu_F(x))^{-1}$, the reciprocal of the mean residual life function of X.

Suppose Y is also another random variable with distribution function $G(y)$ and density $g(y)$ having similar properties as X. We can now define four partial orderings in X and Y by comparing their respective equilibrium distributions.

The following four definitions are generalizations of Definitions 2.15, 2.16, 2.20 and 2.21.

Definition 2.22: X is said to be greater than Y in s-FR ordering ($X \geq_{s-\mathrm{FR}} Y$) if $\bar{T}_s(X,t)/\bar{T}_s(Y,t)$ is increasing in $t \geq 0$. This was shown to be equivalent to $r_s(X,x) \leq r_s(Y,t)$, for all $t \geq 0$.

Definition 2.23: X is said to be greater than Y in s-ST ordering ($X \geq_{s-\mathrm{ST}} Y$) if

$$\frac{\bar{T}_s(X,t)}{\bar{T}_s(Y,t)} \geq \frac{\bar{T}_s(X,0)}{\bar{T}_s(Y,0)},\ \text{for all } t \geq 0.$$

This is equivalent to $\bar{T}_s(X,t) \geq \bar{T}_s(Y,t),\ s \geq 1$.

Definition 2.24: X is said to be greater than Y in s-CV ordering ($X \geq_{s-\mathrm{CV}} Y$) if

$$\int_0^t \frac{\bar{T}_s(X,x)\,dx}{\bar{T}_s(X,0)} \geq \int_0^t \frac{\bar{T}_s(Y,x)\,dx}{\bar{T}_s(Y,0)},\ \text{for all } t \geq 0.$$

This is equivalent to

$$\int_0^t \bar{T}_s(X,u)\,du \geq \int_0^t \bar{T}_s(Y,u)\,du,\ s \geq 1.$$

Definition 2.25: X is said to be greater than Y in s-CX ordering ($X \geq_{s-\mathrm{CX}} Y$) if

$$\int_t^\infty \frac{\bar{T}_s(X,x)\,dx}{\bar{T}_s(X,0)} \geq \int_t^\infty \frac{\bar{T}_s(Y,x)\,dx}{\bar{T}_s(Y,0)},\ \text{for all } t \geq 0.$$

This is equivalent to

$$\int_t^\infty \bar{T}_s(X,u)\,du \geq \int_t^\infty \bar{T}_s(Y,u)\,du,\ s \geq 1.$$

We observe that the following equivalence relationships between the classical and generalized partial orderings :

0-FR$\Leftrightarrow$LR, 1-FR$\Leftrightarrow$FR, 2-FR$\Leftrightarrow$MR, 3-FR$\Leftrightarrow$VR, 1-CX$\Leftrightarrow$CX,

0-ST$\Leftrightarrow$WLR, 1-ST$\Leftrightarrow$ST, 2-ST$\Leftrightarrow$HAMR, 1-CV$\Leftrightarrow$CV.

These equivalences show that the generalized orderings defined in Definitions 22–25 are indeed generalizations of those orderings given from Definition 2.13 through Definition 2.21. Although the approach of defining generalized orderings through the survival functions of the s-order equilibrium distribution of X is mathematically novel, we feel that it does not have the same intuitive appeal that is prevalent in the more basic stochastic ordering concepts. On the other hand, we will see in Section 2.11 that the concept of sth order equilibrium distribution can play an important role in fostering a link between the shapes of the two important reliability measures, namely $r(t)$ and $\mu(t)$.

2.9.2 Connections Among the Partial Orderings

We now proceed to give the relationships among some classical orderings as well as with the generalized partial orderings we have just defined. Fagiuoli and Pellerey (1993) have shown that

s-FR $\Rightarrow$ $(s+1)$-FR; s-FR$\Rightarrow$ s-ST$\Rightarrow$ s-CV; s-ST$\Rightarrow$ s-CX.

The proofs of these implications follow directly from the definitions. Fagiuoli and Pellerey (1993) further showed that $(s+1)$-ST $\Rightarrow$ s-CV.

Combining all these relationships, we may summarize them in the following table (bearing in mind 1-FR$\Leftrightarrow$ FR, 1-ST$\Leftrightarrow$ST,1-CX$\Leftrightarrow$CX, and 1-CV$\Leftrightarrow$CV):

Table 2.3. Chains of relationships between partial orderings

LR	$\Rightarrow$	FR	$\Rightarrow$	MR	$\Rightarrow$	VR	$\Rightarrow\cdots$
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$	
WLR	$\Rightarrow$ 0-CX $\Leftarrow$	ST	$\Rightarrow$ CX $\Leftarrow$	HAMR	$\Rightarrow$ 2-CX $\Leftarrow$	3-ST	$\Rightarrow$3-CX
$\downarrow$		$\Downarrow$		$\Downarrow$		$\downarrow$	
0-CV		CV		2-CV		3-CV	

2.9.3 Generalized Ageing Properties Classification

In this subsection, we discuss ageing classifications of equilibrium distributions and their relationships to the generalized partial orderings. Recall, the failure rate function that corresponds to the survival function $\bar{T}_s(X,t)$ as defined in (2.90) is:

$$r_s(X,t) = \frac{\bar{T}_s(X,t)}{\int_t^\infty \bar{T}_{s-1}(X,x)\,dx} = \frac{\frac{d}{dt}\bar{T}_s(X,t)}{T_s(X,t)}.$$

Averous and Meste (1989) proposed generalized ageing properties classification (a)–(c) below based on s-tailweight. Fagiuoli and Pellerey (1993) presented same definitions as given by Averous and Meste (1989) but based on $r_s(X,t)$ and $\bar{T}_s(X,t)$ instead. This latter approach seems to be more in line with the traditional way of defining ageing concepts through the failure rate function $r(t)$ and the survival function $\bar{F}(t)$. Thus the concepts to be presented below are in parallel to those basic concepts discussed in Section 2.4.

Definition 2.26: Let s be a non-negative integer and X be a lifetime random variable.

(a) X is said to be s-IFR (s-DFR) if $r_s(X,t)$ is increasing (decreasing) in $t \geq 0$. This is equivalent to $\dfrac{\bar{T}_s(X,x+t)}{\bar{T}_s(X,t)}$ is decreasing (increasing) in t for each $x \geq 0$.

(b) X is said to be s-IFRA (s-DFRA) if $\int_0^t r_s(X,x)\,dx/t$ is increasing (decreasing) in $t \geq 0$. or equivalently, $\left(\bar{T}_s(t)/\bar{T}_s(0)\right)^{1/t}$ is increasing in t. Note that $\bar{T}_s(0) = 1$ for $s \geq 1$.

(c) X is said to be s-NBU (s-NWU) if

$$\bar{T}_s(X,x+t)\bar{T}_s(X,0) \leq (\geq)\, \bar{T}_s(X,x)\bar{T}_s(X,t) \text{ for all } x,t \geq 0.$$

For $s \geq 1$, this becomes

$$\bar{T}_s(X,x+t) \leq (\geq)\, \bar{T}_s(X,x)\bar{T}_s(X,t).$$

(d) X is said to be s-NBUFR (s-NWUFR) if $r_s(X,t) \geq (\leq)\, r_s(X,0)$, for all $t \geq 0$.

(e) X is said to be s-NBAFR (s-NWAFR) if $\int_0^t r_s(X,x)\,dx/t \geq (\leq)\, r_s(X,0)$ for all $t \geq 0$.

(f) X is said to be s-NBUCV (s-NWUCV) if

$$\bar{T}_s(X,0)\int_0^t \bar{T}_s(X,x+y)\,dy \leq (\geq)\, \bar{T}_s(X,x)\int_0^t \bar{T}_s(X,y)\,dy \text{ for all } x,t \geq 0.$$

For $s \geq 1$, this becomes

$$\int_0^t \bar{T}_s(X,x+y)\,dy \leq (\geq)\, \bar{T}_s(X,x)\int_0^t \bar{T}_s(X,y)\,dy.$$

(g) X is said to be s-NBUCX (s-NWUCX) if

$$\bar{T}_s(X,0)\int_t^\infty \bar{T}_s(X,x+y)\,dy \leq (\geq)\, \bar{T}_s(X,x)\int_t^\infty \bar{T}_s(X,y)\,dy, \text{ for all } x,t \geq 0.$$

For $s \geq 1$, this becomes

$$\int_t^\infty \bar{T}_s(X, x+y)\,dy \le (\ge)\, \bar{T}_s(X,x) \int_t^\infty \bar{T}_s(X,y)\,dy.$$

The last two ageing classifications were due to Fagiuoli and Pellerey (1993).

We note the equivalence between the generalized ageing concepts and the classical ageing classes as follows:

0-IFR	⇔	ILR	1-IFR	⇔	IFR
2-IFR	⇔	DMRL	3-FR	⇔	DVRL
1-IFRA	⇔	IFRA	1-NBUFR	⇔	NBUFR
2-NBUFR	⇔	NBUE	2-NBAFR	⇔	HNBUE

All the abbreviations in the table above have been introduced in Sections 2.4–2.7 and this section except ILR. This is defined as X having an increasing likelihood ratio. It is also known as X having a PF_2 (Pólya frequency of order 2) density. We note that ILR ⇒ IFR. For a definition of a PF_2 density class, see for example, Barlow and Proschan (1981, p. 76).

Fagiuoli and Pellerey (1993) also established the equivalence of the generalized partial orderings between X and its residual lifetime X_t and the generalized ageing classes.

Hu et al. (2001) gave connections among some generalized orderings, and characterized s-FR and s-ST orderings in terms of residual lives as well as in terms of equilibrium distributions, respectively. The s-CX and s-CV orderings were also both characterized by the equilibrium distributions and by the Laplace transforms.

Despite of their mathematical nicety, it is unclear how these generalized orderings can be applied effectively in reliability given their apparent lack of a meaningful interpretation.

2.9.4 Applications of Partial Orderings

First and higher order stochastic dominances which are essentially partial orderings have important applications in econometrics (Whitmore, 1970). Examples of applications may be found in Barlow and Proschan (1981), Stoyan (1983) and in Ross (1983), where partial orderings are used, respectively, in reliability context, in queues, and in other stochastic processes. Singh and Jain (1989) and Fagiuoli and Pellerey (1993) have proposed an application to stochastic comparison between two devices that are subjected to Poisson shock models.

Design engineers are well aware that a system where active spare allocation is made at the component level has a lifetime stochastically larger than the corresponding system where active spare allocation is made at the system level. Boland and El-Neweihi (1995) investigated this principle in hazard rate (failure rate) ordering and demonstrated that it does not hold in general. However, they discovered that for a 2-out-of-n system with independent and identical components and spares, active spare allocation at the component level is superior to active spare allocation at the system level. They conjectured that such a principle holds in general for a k-out-of n system. Singh and Singh (1997) have proved that for a k-out-of-n system where components and spares have independent and identical life distributions, active spare allocation at the component is superior to active spare allocation at the system level in likelihood ratio ordering. This is stronger than failure (hazard) rate ordering and thus establishing the conjecture of Boland and El-Neweihi (1995).

Boland and El-Neweihi (1995) have also established that the active spare allocation at the component level is better than the active spare allocation at the system level in hazard rate (failure rate) ordering for a series system when the components are matching although the components may not be identically distributed. Boland (1998) gave an example to show that the failure rate (hazard rate) comparison is what people really mean when they compare the performance of two products. For more on the failure rate (hazard rate) and other stochastic orders and their applications, the readers should consult Shaked and Shanthikumar (1994).

Apart from the basic partial orderings such as the likelihood ratio ordering, stochastic ordering, failure rate ordering and the mean residual life ordering, we have not found too many applications for the others.

2.10 Relative Ageing

Sengupta and Deshpande (1994) have studied three types of relative ageing of two life distributions. The first of these relative ageing concepts is the partial ordering originally proposed by Kalashnikov and Rachev (1986) which is defined as follows:

Definition 2.27: Let F and G be the distribution functions of the random variables X and Y, respectively. X is said to be *ageing* faster than Y (written as $X \prec_c Y$) if the random variable $\Lambda_G(X) = -\log \bar{G}(X)$ has an increasing failure rate.

If the failure rates $r_F(t)$ and $r_G(t)$ both exist with $r_F(t) = \frac{f(t)}{1-F(t)}$ and $r_G(t) = \frac{g(t)}{1-G(t)}$, then the above definition is equivalent to $\frac{r_F(t)}{r_G(t)}$ being an increasing function of t.

Another relative ageing defined in Sengupta and Deshpande (1994) is as follows:

Definition 2.28: $X \prec_* Y$ (X is ageing faster than Y in average) if $Z = \Lambda_G(X)$ is IFRA .

We observe that there are three types of 'X ages faster than Y' depending on whether $r_F(t)$ dominates $r_G(t)$ or being dominated by $r_G(t)$ or $r_F(t)$ crosses $r_G(t)$ from below. In the case of $r_F(t) \leq r_G(t)$, then X is said to be greater than Y in failure rate ordering according to Definition 2.15.

Remarks

If X ages faster than Y, it is equivalent to Y ages slower than X.

- Suppose now X ages slower than Y, i.e., $r_F(t)/r_G(t)$ is decreasing in t, Corollary 6 of Hu et al. (2001) shows that if $X \leq_{\text{FR}} Y$, then $X \leq_{\text{LR}} Y$ (see Definition 2.13 for LR ordering). Note that $X \leq_{\text{LR}} Y \Rightarrow X \leq_{\text{FR}} Y$ without a condition.
- Suppose $r_s(X,t)/r_s(Y,t)$ is decreasing in t, Theorem 5 of Hu et al. (2001) shows that if $X \leq_{s-\text{FR}} Y$, then $X \leq_{(s-1)-\text{FR}} Y$ for $s \geq 1$. (See Definition 2.22 for s-FR ordering.) This is a converse of the result in Section 2.9.2. which says if $X \leq_{(s-1)\text{FR}} Y \Rightarrow X \leq_{(s)-\text{FR}} Y$.

In Lai and Xie (2003) some results on relative ageing of two parallel structures were established. It is observed that the relative ageing property may be used to allocate resources and for failure identification when two components (systems) having the same mean. In particular, if 'X ages faster than Y' and that they have the same mean, then $\text{var}(X) \leq \text{var}(Y)$. Several examples are given therein. In particular, it is shown that when two Weibull distributions have the same mean, the one that ages faster has a smaller variance.

2.11 Shapes of η Function for s-order Equilibrium Distributions

Recall in Section 2.9, the equilibrium distribution function of a lifetime variable X with cdf F is defined in (2.84), i.e.,

$$E_F(t) = \int_0^t F(x)\, dx/\mu$$

so the density function is given $\bar{F}(t)/\mu$. Section 2.9 also defines the survival functions of the equivalent distributions recursively via

$$\bar{T}_0(X,t) = f(t),\ \bar{T}_{-1}(X,t) = f'(t)$$

and

$$\bar{T}_s(X,t) = \frac{\int_t^\infty \bar{T}_{s-1}(X,x)\, dx}{\mu_{s-1}(X)}, \text{ for integer } s \geq 1,$$

where

$$\mu_s(X) = \int_0^\infty \bar{T}_s(X,x)\, dx, s \geq 0.$$

It follows that (2.90) and the preceding two equations that

$$r_s(X,t) = \frac{\bar{T}_{s-1}(X,t)}{\int_t^\infty \bar{T}_{s-1}(X,x)\, dx}\ s \geq 1.$$

Now let us define an s-order η function by

$$\eta_s(X,t) = \frac{\bar{T}_{s-2}(X,t)}{\int_t^\infty \bar{T}_{s-2}(X,x)\,dx} \tag{2.91}$$

We assume that $\lim_{x\to\infty} f(x) = 0$.

It is now clear that

$$\eta_s(X,t) = r_{s-1}(X,t), s \geq 1. \tag{2.92}$$

For $s = 1$, $\eta_1(X,t) = r_0(X,t) = -\dfrac{f'(t)}{f(t)}$. As we are now dealing with one variable X only so we may suppress the argument X giving

$$\eta_s(t) = r_{s-1}(t), s \geq 1. \tag{2.93}$$

Thus,

$\eta_1(t) = \eta(t) = -\dfrac{f'(t)}{f(t)},\ \ r_1(t) = r(t) = \frac{f(t)}{\bar{F}(t)},$

$\eta_2(t) = r(t),\ r_2(t) = 1/\mu(t).$

Now the relationships between the shapes of $\eta(t)$ and the shapes of $r(t)$ have already been established by Glaser (1980) (our Theorem 2.1) and by Gupta and Warren (2001) (our Theorem 2.2). The same relationships between the shapes of $\eta_2(t) = r(t)$ of the equilibrium distribution of X and of the shapes of $r_2(t) = 1/\mu(t)$ also hold. These results can be generalized to establish the relationship between $\eta_s(t)$ and $r_s(t) = \eta_{s+1}(t)$.

Based on the above observations, Navarro and Hernandez (2004) gave the following theorems:

Theorem 2.12: If $E(X^{s+1}) < \infty$ for $s = 0, 1, 2, ...$, then

(a) $\eta_s \in$ I (D) $\Rightarrow \eta_{s+1} \in$ I (D);
(b) $\eta_s \in$ BT (UBT) $\Rightarrow \eta_{s+1} \in$ BT or I (UBT or D).

Proof: It follows directly from Theorem 2.1 and equation (2.92).

Theorem 2.13: If $E(X^{s+1}) < \infty$ for $s = 0, 1, 2, ...$, then $\eta'_{s+1}(t) = 0$ has at most one solution on the closed interval $[z_{k-1}, z_k]$, where $z_0 = 0 < z_1 < ... < z_n$ are the zeros of $\eta'_s(t)$ and $\eta'_{s+1}(t) = 0$ does not have any solution in (z_n, ∞).

Proof: It follows directly from Theorem 2.2 and equation (2.93).

The relationship between $\eta_2(t)$ and $\eta_3(t)$ will be used to establish between the relationships between the shapes $r(t)$ and $\mu(t)$ in Theorem 4.2 of Chapter 4.

Mi (2004) also studied (i) the shape of $\eta_2(t) = r(t)$ when $\eta_1(t) = -f'(t)/f(t)$ has a roller-coaster shape with a finite number of change points (ii) the shape of $\eta_3(t) = 1/\mu(t)$ when η_2 has a roller-coaster shape. However, Mi's (2004) did not use the s-order equilibrium distribution approach.

Instead, he defined $\eta_1(t) = N_1(t)/D_1(t) = N_1(t)/\int_t^\infty N_1(x)\,dx$ and $\eta_2(t) = N_2(t)/D_2(t) = D_1(t)/D_2(t) = D_1(t)/\int_t^\infty D_1(x)\,dx$ and established their relationships assuming $N_1(t)$ is integrable on $[0,\infty)$.

We will revisit these results when we consider the shape of $\mu(t)$ when the shape of $r(t)$ is known in Chapter 4.

2.12 Concluding Remarks on Ageing

The study of length of life of human beings, organisms, structures, materials, etc., is of great importance in the actuarial, biological, engineering and medical sciences. It is clear that research on ageing properties (univariate, bivariate, and multivariate) is currently being vigorously pursued. Many of the univariate definitions do have physical interpretations such as arising from shock models. The simple IFR, IFRA, NBU, NBUE, DMRL etc have been shown to be very useful in reliability related decision making, such as replacement and maintenance studies.

While positive ageing concepts are well understood, negative ageing concepts (life improved by age) are less intuitive. Nevertheless, negative ageing phenomenon does occur quite frequently. There have been cases reported by several authors where the failure rate functions decrease with time. Sample examples are the business mortality (Lomax, 1954), failures in the air-conditioning equipment of a fleet of Boeing 720 aircrafts or in semiconductors from various lots combined (Proschan, 1963), and the life of integrated circuit modules (Sanuders and Myhre, 1983). Gerchak (1984) reported that "Studies conducted in various social disciplines discovered that, the longer individuals remain in a state, the lower the chances of their leaving the state in subsequent periods." In general, a population is expected to exhibit decreasing failure rate (DFR) when its behaviors over time is characterized by 'work hardening' (in engineering terms), or 'immunity' (in biological terms). Modern phenomenon of DFR includes reliability growth (in software reliability).

Non-monotonic ageing concepts have been found useful in many reliability and survival analysis such as burn-in time decision. Applications of mean residual life concepts will be given in Chapter 4 whereas applications of bathtub (upside-down) shaped ageing will be presented in Chapter 3. We also refer our readers to Barlow and Proschan (1981), Bergman (1985) and Newby (1986) for other applications.

We envisage that these concepts are of interest not only to reliability modelers but also to the mainstream reliability practitioners.

3

Bathtub Shaped Failure Rate Life Distributions

3.1 Introduction

Bathtub shaped failure rate distributions have been introduced in the preceding chapter. Recall, we say that a failure rate function $r(t) \in \mathrm{BT}$ if

$$r(t) = \frac{f(t)}{\bar{F}(t)} \tag{3.1}$$

has a bathtub or U shape, i.e., $r(t)$ decreases first, then remains approximately constant and eventually increases. A review of this class of life distributions was given by Rajarshi and Rajarshi (1988) and another by Lai, Xie and Murthy (2001). However, a significant amount of literature on this subject has appeared over the last decade. Thus there is a need for another thorough study to update the recent developments. In this chapter, we give a detailed review of many different facets and issues relating to this important ageing class.

Bathtub shaped failure curves were discussed in the engineering literature a long time ago, see, e.g., Kao (1959), Kamins (1962), Shooman (1968) and Krohn (1969). Theoretical aspects of the bathtub failure rate are much studied in recent years. For brevity, when we say F is BT, we mean F has a bathtub shaped failure rate function. In Chapter 11, several data sets that exhibit BT failure rates are given. The shapes of the mean residual life function $\mu(t)$ of BT failure rate distributions will be discussed in the next chapter.

The aim of Section 3.2 is to dispel a misconception that bathtub shaped failure rate phenomenon is a myth. In Section 3.3 we give several definitions of a bathtub shaped failure rate function. The main difference is whether $r(t)$ has one or two change points. Some basic properties are also presented. Section 3.4 provides several families of life distributions that exhibit BT failure rates. The question of how to construct a bathtub shaped failure rate distribution is considered in Section 3.5. This is followed by a literature review on estimating the change point of a BT failure rate function in Section 3.6. We then preview the relationship between a bathtub shaped failure rate distribution and its associated mean residual life in Section 3.7. In Section 3.8 we address the issue of burn-in times for bathtub shaped distribution. In section 3.9 we investigate another important class of non-monotonic failure rate distributions denoted by UBT which is a dual life class of BT. The failure rate function $r(t)$ in this class has an upside-down bathtub shape. It turns out that several well known life distributions belong to this class. For example, the lognormal distribution has been known for a long time to have this property. In Section 3.10 we extend our discussion of the traditional bathtub distributions to some generalized bathtub shaped failure rate distributions which include the modified bathtub (MBT) and the roller-coaster failure curves. Finally in Section 3.11, we outline several applications of bathtub distributions.

3.2 Bathtub Shaped Failure Rate Is Not a Myth

In Chapter 2, many statistical ageing concepts have been defined through one of the three functions (i.e., failure rate, survival function and mean residual life). Essentially, these concepts describe how a component ages with time. ‘No ageing’ means the age of a component has no effect on the distribution of the residual lifetime. ‘Positive’ ageing describes the situation where the residual lifetime tends to decrease, in some probabilistic sense, with increasing age of the component. On the other hand, ‘negative ageing’ has an opposite effect on the residual lifetime.

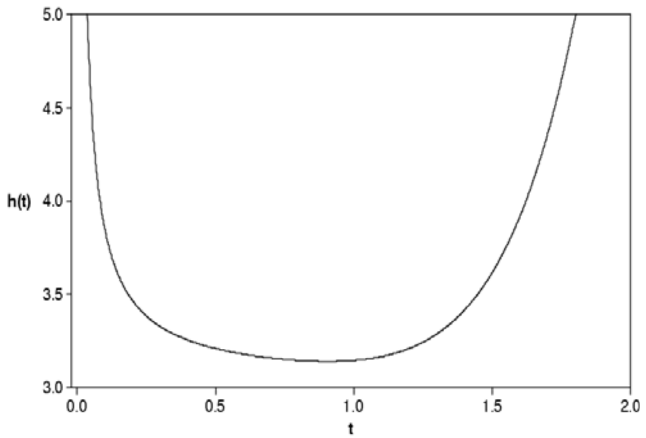
Monotonic ageing concepts are found to be popular among many reliability engineers. However, in many practical applications, the effect of age is initially beneficial (a burn-in phase where negative ageing takes place), but after a certain period, it is age adverse indicating a ‘wear-out’ phase where ageing is positive. It is now widely believed that many products, particularly electronic items such as silicon integrated circuits, exhibit a bathtub shaped failure rate function. This belief is supported by much experience and extensive data collection in many industries.

Despite of several critiques of the role of bathtub shaped failure rate distributions, see, e.g., Talbot (1977), Wong (1988) and Sherwin (1997), BT (UBT) classes have been studied more extensively than most other ageing classes.

3.3 Definitions and Basic Properties

A class of life distributions which has received considerable attention is the class of bathtub shaped failure rate life distributions. A systematic account of such distributions was given by Rajarshi and Rajarshi (1988). We say that F is BT (bathtub shaped failure rate) if its failure rate function decreases at first and then remains constant for a period and finally it increases with time. In other words, the failure rate function has a bathtub shape. This corresponds to the three distinct phases of a component or a system: early life, useful life and wear-out as shown in Figure 3.1. During the early life period, failures tend to be caused by manufacturing defects or birth defects in the case of human beings. Failures in the useful life period can be called chance failures. The wear-out region has an increasing failure rate with time because of the older the unit the more likely it is to fail.

Fig. 3.1. Bathub shaped failure rate function



As we mentioned in the preceding section, the class of lifetime distributions having a bathtub shaped failure rate function is very important because the lifetime of electronic, electromechanical, and mechanical products are often modeled with this feature. In survival analysis, the lifetime of human beings exhibits this pattern. Several real life examples of bathtub shaped life distributions can be found in Kao (1959), Krohn (1969), Lieberman (1969) and Lawless (2003).

3.3.1 Acronyms for Bathtub Shaped Failure Rate Life Distributions

The bathtub shaped failure rate life distributions, often known simply as bathtub distributions, have a failure rate curve that resembles to a bathtub shape. Such failure rate curves are also known as U-shaped or J-shaped curve and we are unsure who first coined the phrase 'bathtub' for these life distributions. Unlike other classes of ageing distributions, there appears no consistent acronyms or abbreviations. Some of the known abbreviations are: BT, BTD, BFR, BTF, BTFR, BTFRD, BTR, DI, DIB, etc. It seems to us the preferred one would be BT which we now adopt in this book.

3.3.2 Definitions

There are several variants of the definition of a bathtub shaped failure rate but they are essentially the same. The main difference is whether the assumption of having two change points is imposed. Barlow and Proschan (1981, p. 55) gave a semi-formal definition of a bathtub shaped failure rate having three phases: 'infant mortality' phase, 'useful life' phase and wear-out phase.

Definition 3.1: Let F be a cdf with a failure rate function $r(t)$ which is continuous. Then F is BT if there exits a t_0 such that (a) $r(t)$ is decreasing for $t < t_0$, (b) $r(t)$ is increasing for $t > t_0$, i.e., $r'(t) < 0$ for $t < t_0$, $r'(t_0) = 0$ and $r'(t) > 0$ for $t > t_0$. See, for example, Glaser (1980).

Here the strict monotonicity is implied in this definition. The Glaser's convention differs from us in that he used 'an increasing function' to mean a strictly increasing function instead of a 'non-decreasing function'. The bathtub curves given in this definition would probably represent some U shaped tubs rather a traditional bathtubs as there is no interval for which $r(t)$ is a constant.

If F is not absolutely continuous, we may define BT through the conditional reliability function

$$\bar{F}(x \,|\, t) = \frac{\bar{F}(t+x)}{\bar{F}(t)}, \quad \bar{F}(t) = 1 - F(t) > 0. \tag{3.2}$$

Barlow and Proschan (1981, p.54) have used the conditional reliability function to define IFR and DFR concepts. The following definition arises from such a consideration.

Definition 3.2: F is BT if there exists a t_0 such that

- $\bar{F}(x \,|\, t)$ is strictly increasing in t for $0 \le t < t_0, 0 \le x \le t_0 - t$,
- $\bar{F}(x \,|\, t)$ is strictly decreasing in t for $t_0 \le t < \infty, x \ge 0$.

Haupt and Schäbe (1997) among others have used this definition for a bathtub class.

Definition 3.1 above may be modified to allow for a more 'comfortable' bathtub shape as the one shown in Figure 3.1

Definition 3.3: A life distribution F which is absolutely continuous and having support $[0, \infty)$ is said to be a bathtub failure (BT) distribution if there exists a $t_0 > 0$ such that $r(t)$ is decreasing for $[0, t_0)$ and increasing on $[t_0, \infty)$. Mitra and Basu (1995), among many others, have adopted this approach.

In this definition, a bathtub with a 'flat' middle portion is possible though not explicitly imposed. The point t_0 is referred to as a change point of the distribution F by the these authors. We note, however, every point in the 'level flat' part can be taken as as a 'change point' according to this definition.

In Mitra and Basu (1996a), {BTs} is used to denote the family of the bathtub distributions covered by Definition 3.1 with the letter 's' standing for 'strict', whereas {BT} is used to denote the family of bathtub distributions covered by Definition 3.3, i.e., $\{BT\} = \{BTs\} \cup \{IFR\} \cup \{DFR\}$. To avoid any ambiguity, we assume in this book that every member of a BT class is strictly non-monotonic unless explicitly stated.

A more explicit definition that gives rise to curves having a definite 'bathtub shape' is as follows:

Definition 3.4: A distribution F is a bathtub shaped life distribution if there exists $0 \le t \le t_0$ such that:

(a) $r(t)$ is strictly increasing, if $0 \le t \le t_1$;

(b) $r(t)$ is a constant if $t_1 \le t \le t_2$; and

(c) strictly increasing if $t \ge t_2$.

Park (1985) and Mi (1995), among several others, have used this definition of a bathtub shaped failure rate function with two change points. Several comments on these definitions are now in order:

- In Definition 3.4 above, Mi (1995) called the points t_1 and t_2 as the change points of $r(t)$. If $t_1 = t_2 = 0$, then a BT becomes an IFR; and if $t_1 = t_2 \to \infty$, then $r(t)$ is strictly decreasing so becoming a DFR. In general, if $t_1 = t_2$, then the interval for which $r(t)$ is a constant degenerates to a single point. In other words, the strict monotonic failure rate distributions IFR and DFR may be treated as the special cases of BT in this definitions.
- In Definition 3.4, at most two change points are allowed. In other words, the points in the flat interval (t_1, t_2) are not change points according to Mi (1995) but would have been called the 'change points' according to Mitra and Basu (1995).
- Definition 3.4 may be rewritten as

$$r(t) = \begin{cases} r_1(t), & \text{for } t \le t_1, \\ \lambda, & \text{for } t_1 \le t \le t_2, \\ r_2(t), & \text{for } t \ge t_2; \end{cases} \tag{3.3}$$

where $r_1(t)$ is strictly decreasing in $[0, t_1]$ and $r_2(t)$ is strictly increasing for $t \ge t_2$. We are not aware of any well known parametric BT distributions that posses a 'flat' middle part. However, Jiang and Murthy (1997c) constructed one such distribution while studying sectional models involving three Weibull distributions. However, BT distributions with one change point are more common and several of these will be given in Section 3.1.

- We may differentiate the types of bathtub failure rates based on the asymptotic nature of $r(t)$ as t approaches 0 and infinity. The function $r(t)$ may be finite or infinite at these asymptotes.

Another definition of a bathtub shaped failure rate distribution may be defined through $\log \bar{F}(t)$.

Definition 3.5: A life distribution F having support on $[0, \infty)$ is said to be a bathtub shaped failure rate distribution if there exists a point t_0 such that $-\log \bar{F}(t)$ is concave in $[0, t_0)$ and convex in $[t_0, \infty)$. See for example, Deshpande and Suresh (1990).

Marshall and Olkin (1979, p.76) have used log concavity (convexity) of $\bar{F}(t)$ to define IFR (DFR) a long time ago so the above definition of a bathtub in terms of $\log \bar{F}$ cannot be considered a recent idea.

It is to be noted that the above definition of a BT distribution is quite general and extends the idea of distributions possessing a bathtub shaped failure rate to situations where the failure rate itself does not exists.

In studying the property of BT class, in particular its relationship with other classes, one has to be aware of which definition of BT is used.

3.3.3 Some Further Properties

Recall when we say F is BT, we mean its failure rate function $r(t) \in$ BT. Mitra and Basu (1996a) presented some basic properties concerning the bounds for the survival function and moments of a BT random variable T. Closure properties of the BT class under the formation of coherent systems, convolutions and mixtures were also dealt with.

- Suppose F is BT, then $\bar{F}(t) \leq \bar{G}(t)$ where G is exponential with mean $\{r(t_0)\}^{-1}$. Here t_0 is a change point at which $r(t)$ is minimum.
- $E(X^k) \leq \frac{\Gamma(k+1)}{r(t_0)^k}, k > 0$.
- A BT life distribution F with the kth moment (about zero) equal to $E(X^k) = \frac{\Gamma(k+1)}{\{r(t_0)\}^k}$ is necessarily an exponential.
- Convolution of distributions from a BT class is not necessarily in the BT class. In fact, even in the broader BT class that includes monotonic failure rate distributions is not closed under convolution. This is seen in the following example:

  $$\bar{F}(t) = \frac{1}{2}\left(e^{-t} + e^{-t/2}\right), t \geq 0; \quad \bar{G}(t) = e^{-t}.$$

  The failure rate function of the convolution $H = F * G$ is given by $r(t) = \frac{(t-1)e^{-t}+e^{-t/2}}{te^{-t}+2e^{-t/2}}$; accordingly, $r(0) = 0, r(2) = .5, r(4) = 0.5533$ and $r(t) \to 0.5$ as $t \to \infty$ so $r(t)$ does not have a bathtub shape.
- The mixture of BT distributions need not be BT.

- Suppose we have a competing risks model: $\bar{F}(t) = \bar{F}_1(t)\bar{F}_2(t)$ where the lifetime of each component is BT with a common turning point t_0. Then the lifetime of the system again has a BT distribution with t_0 as one of its turning points. (Note: the turning point here is defined as that given in Definition 3.3. Also a competing risks model is simply a series system.)
- A parallel system of two independent BT components need not be a BT distribution.

3.4 Families of Bathtub Shapes Failure Rate Distributions

Many parametric families of bathtub shaped life distributions have been constructed from various contexts over the last two decades. Ideally, we should classify them into groups or strata according to some common characteristics. However, this exercise seems untenable. Instead, we summarize them into two categories: (a) Lifetime distributions that have explicit expressions for failure rates and (b) distributions whose failure rate functions are unwieldy or unknown. For the latter, we give only either the probability density function or the distribution function, whichever is more convenient. The asymptotic behavior of $r(t)$ at the origin or infinity is given whenever possible. We aim to list the examples of bathtub shaped failure rates in the increasing order of sophistication from Sections 3.4.1 to 3.4.3.

3.4.1 Bathtub Distributions with Explicit Failure Rate Functions

Quadratic model and its generalization

Bain (1974, 1978) and Gore et al. (1986) considered a quadratic failure rate model with

$$r(t) = \alpha + \beta t + \gamma t^2; \alpha \geq 0, \quad -2(\alpha\gamma)^{1/2} \leq \beta < 0, \gamma > 0 \tag{3.4}$$

which has a bathtub shape. Here, $r(0) = \alpha$, $r(t) \to \infty$ as $t \to \infty$.

It is easy to verify that $\hat{r}(t) = \exp\{r(t)\}$ has a bathtub shape if $r(t)$ also has a bathtub shape. For example, BT 9 of Rajarshi and Rajarshi (1988) listed $r(t) = \exp\left\{\alpha + \beta t + \gamma t^2\right\}, \alpha, \gamma \geq 0, 0 > \beta \geq -2(\alpha\gamma)^{1/2}$ as having a bathtub shape.

In fact, any increasing function of a bathtub failure rate is itself having a bathtub shaped failure rate.

Competing risk models

(i) Murthy et al. (1973) model

$$r(t) = \frac{\alpha}{1+\beta t} + \gamma\delta t^{\delta-1}, \alpha, \beta, \gamma > 0; \delta > 2. \tag{3.5}$$

Here, $r(0) = \alpha,\ r(t) \to \infty$ as $t \to \infty$.
This may be considered as a competing risks model involving a Lomax distribution (the Pareto distribution of the second kind) and a Weibull distribution.

(ii) Hjorth (1980) model
The case when $\delta = 2$ in (3.5) is considered in Hjorth (1980). The failure rate function is $r(t) = \frac{\alpha}{1+\beta t} + 2\gamma t, \gamma \leq \frac{\alpha\beta}{2}$. Guess et al. (1998) studied the behavior of the mean residual life of the Hjorth's model.

A flexible family

Gaver and Acar (1979) have proposed a model that has a bathtub shaped failure rate given by $r(t) = \lambda + g(t) + k(t)$ where $g(t) > 0$ is a decreasing function of t with $\lim_{t\to\infty} g(t) \to 0$ whereas $k(t)$ is an increasing function of t such that $k(0) = 0$, $\lim_{t\to\infty} k(t) \to \infty$ and λ is any real number such that $r(t) > 0$. This is a popular method for constructing bathtub shaped failure rate functions. Several special cases of this family are now presented below:

- $r(t) = \lambda + \frac{\theta}{t+\varphi} + \alpha t^p, \alpha, \theta \geq 0;\ t, \varphi, p > 0$; the model is an extension of Murthy et al. (1973) and studied by Jaisingh et al. (1987). If both $g(t)$ and $k(t)$ are failure rate functions, then this model is simply a competing risks model involving three distributions.
- Canfield and Borgman (1975): $r(t) = \theta_1\alpha_1 t^{\alpha_1-1} + \theta_2 + \theta_3\alpha_3 t^{\alpha_3-1}$, $\alpha_3 > 2, \alpha_1 < 1$. $r(t) \to \infty$ as $t \to 0$ or ∞.
- Hjorth (1980): $r(t) = \frac{\alpha}{1+\beta t} + \gamma t;\ 0 < \gamma \leq \alpha\beta$. $r(0) = \alpha,\ r(t) \to \infty,$ as $t \to \infty$.
- Generalized Makeham's curve. This is listed as No. 15 in Section 5 of Rajarshi and Rajarshi (1988). It differs from the Gompertz-Makeham distribution as discussed in Section 2.3:
  $r(t) = \delta \exp(\mu t) + \alpha\beta(1+\beta t)^{-1}$, $\mu\delta < \alpha\beta^2, t > 0$. $r(0) = \delta + \alpha\beta,\ r(t) \to \infty$ as $t \to \infty$.

Additive models

These are in fact competing risk models with both components from the same family of distributions.

- Additive Weibull model
  Xie and Lai (1995) and Jiang and Murthy (1997c) considered a competing risk model involving two Weibull distributions resulting the below failure rate function:

$$r(t) = \alpha_1/\beta_1(t/\beta)^{\alpha_1 - 1} + \alpha_2/\beta_2(t/\beta_2)^{\alpha_2 - 1}, \alpha_i > 0, \beta_i > 0, i = 1, 2. \quad (3.6)$$

The function $r(t)$ has a bathtub shape when $\alpha_1 < 1$ and $\alpha_2 > 1$. The turning point t_0 given by

$$t_0 = \left\{ \frac{\alpha_1(1-\alpha_1)\beta_1^{\alpha_1}}{\alpha_2(1-\alpha_2)\beta_2^{\alpha_2}} \right\}^{\frac{1}{\alpha_2 - \alpha_1}}.$$

Also $r(0) = r(\infty) = \infty$.

- Additive Burr XII model
  Burr XII distribution has been discussed in Section 2.3. Wang (2000) considered an additive Burr XII model that combines two Burr XII distributions, one with a decreasing failure rate and another one with an increasing failure rate. The failure rate function of the additive Burr XII is given by

$$r(t) = \frac{k_1 c_1 (t/s_1)^{c_1 - 1}}{s_1[1 + (t/s_1)^{c_1}]} + \frac{k_2 c_2 (t/s_2)^{c_2 - 1}}{s_2[1 + (t/s_2)^{c_2}]}, \ t > 0 \quad (3.7)$$

  $(k_1, k_2, s_1, s_2 \geq 0, 0 < c_1 < 1, c_2 > 2)$. It was shown that $r(t)$ has a bathtub shape.

Modified Weibull model of Lai et al. (2003)

Lai et al. (2003) have recently proposed a 'modified Weibull' model which contains the Weibull distribution as its special case. The survival function is

$$\bar{F}(t) = \exp\left(-at^{\alpha}e^{\lambda t}\right), \ t > 0, \quad (3.8)$$

with parameters $a > 0, \alpha > 0$ and $\lambda > 0$. The distribution reduces to the Weibull when $\lambda = 0$. Note that $\alpha = 0$ is excluded here as it give rise to a distribution that fails to have a density because there is a positive mass at the origin. The distribution (3.8) has a relatively simple failure rate function:

$$r(t) = a(\alpha + \lambda t)t^{\alpha - 1}e^{\lambda t}. \quad (3.9)$$

We observe that if $\alpha < 1$, the term $t^{\alpha-1}$ dominates for small t, whereas the term $e^{\lambda t}$ dominates for large t, thus producing the bathtub shape. Lai et al. (2003) have established the explicit formula

$$t^* = \frac{\sqrt{\alpha} - \alpha}{\lambda} \quad (3.10)$$

where $0 < \alpha < 1$, for the turning point of $r(t)$. We may note that in this model, $r(t) \in \mathrm{I}$ for $\alpha \geq 1$.

The distribution will be revisited in Section 5.5 as a generalization of the Weibull family.

Sectional model with two Weibull distributions

Murthy and Jiang (1997) have considered two sectional models involving two Weibull distributions having failure rate functions given by

$$r(t) = \begin{cases} (\alpha_1/\beta_1)(t/\beta_1)^{\alpha_1 - 1}, & 0 \leq t \leq t_0, \\ (\alpha_2/\beta_2)\left(\frac{t-\gamma}{\beta_2}\right)^{\alpha_2 - 1}, & t_0 < t < \infty; \end{cases}$$

satisfying two conditions: $t_0 = [\beta_1^{\alpha_1}(\alpha/\beta_2)^{\alpha_2}]^{1/(\alpha_1-\alpha_2)}$, $\gamma = (1-\alpha)t_0$ where $\alpha = \alpha_2/\alpha_1$ so that $r(t)$ is continuous at t_0. We note that

(a) The authors considered a second model which is the same as above except that $\gamma = 0$. Now the shift (location) parameter γ has no influence on the shape type of the failure rate. Thus the two models are essentially the same. For $\alpha_1 < \alpha_2$, $r(t)$ would have a bathtub shape if $\alpha_1 < 1$ and $\alpha_2 > 1$.
(b) Jiang and Murthy (1997b) extended their results giving two sectional models involving three Weibull distributions. Four types of bathtub shapes are possible for each of the two sectional models.

Exponential power

Smith and Bain (1975, 1976), Dhillon (1981), Paranjpe et al. (1985), Paranjpe and Rarjasi (1986) and Leemis (1986) studied the exponential power model having failure rate given as

$$r(t) = \lambda\alpha(\lambda t)^{\alpha-1}e^{(\lambda t)^\alpha}. \tag{3.11}$$

For $\alpha < 1$, $r(t) \to \infty$ when $t \to 0$ or $t \to \infty$ so yielding a bathtub shape. In particular, a bathtub with quite a flat middle part is achieved if $\alpha = 1/2$. For $\alpha \geq 1$, $r(t) \in$ I. $\bar{F}(t)$ has a rather simple expression, i.e.,

$$\bar{F}(t) = \exp\left\{-(e^{(\lambda t)^\alpha} - 1)\right\}. \tag{3.12}$$

We note that (3.12) is a straightforward generalization (by introducing a shape parameter α) of the Gompertz-Makeham distribution discussed in Section 2.3. The density function is given by

$$f(t) = \lambda\alpha(\lambda t)^{\alpha-1}\exp\left\{-(e^{(\lambda t)^\alpha} - (\lambda t)^\alpha - 1)\right\}. \tag{3.13}$$

Dhillon (1981) has used a Weibull probability plot technique to estimate the two parameters. The maximum likelihood estimates for λ and α were also derived although numerical algorithms to compute these estimates are required.

Weibull extension

Consider the case $\lambda = 1$ in the exponential power model. Then (3.12) becomes

$$\bar{F}(t) = \exp\left\{-(e^{(t)^\alpha} - 1)\right\}. \tag{3.14}$$

Chen (2000) introduced another parameter λ to the distribution specified in (3.14) above so that the new cdf becomes

$$\bar{F}(t) = \exp\left\{-\lambda(e^{(t)^\alpha} - 1)\right\} \tag{3.15}$$

with failure rate function

$$r(t) = \lambda\alpha t^{\alpha-1}e^{t^\alpha},\ t \geq 0. \tag{3.16}$$

The parameter λ here does not alter the shape of the failure rate function so (3.16) behaves similarly to the function given in (3.11). In particular, $r(t) \in$ I for $\alpha \geq 1$ and $r(t) \in$ BT for $\alpha < 1$.

Chen (2000) has provided the exact confidence interval and the exact confidence regions for the two parameters in the model.

Xie et al. (2002) extended Chen's model by incorporating a scale parameter β into (3.15) to give

$$\bar{F}(t) = \exp\left\{-\lambda\beta(e^{(t/\beta)^\alpha} - 1)\right\}. \tag{3.17}$$

They referred to (3.17) as a modified Weibull extension. We note that the parameter β in the Weibull extension model plays a role more than a scale parameter for the Chen's model since β also appears as a scaling factor for the first 'exponent' in (3.15). The failure rate function can be easily obtained and is expressed as

$$r(t) = \lambda\beta(t/\beta)^{\alpha-1}e^{(t/\beta)^\alpha},\ t \geq 0. \tag{3.18}$$

Estimates of parameter were obtained by Xie et al. (2002) and the model was fitted to two data sets. It is found that the model compares favorably with other existing bathtub shaped models.

We will further discuss this distribution within the context of extended Weibull families in Section 5.5.

Double exponential power

Paranjpe et al. (1985) and Paranjpe et al. (1986) considered the following model having

$$r(t) = \beta\alpha t^{\alpha-1}\exp(\beta t^\alpha)\exp\left[\exp(\beta t^\alpha) - 1\right], \alpha < 1. \tag{3.19}$$

The above expression is obviously quite complex. Clearly, $r(t) \to \infty$ as $t \to 0$ or $t \to \infty$ so a bathtub shape is produced.

The survival function is, however, quite straightforward although it does involve 3-fold 'exp' :

$$\bar{F}(t) = \exp\left\{-\exp\left[\exp(\beta t^{\alpha}) - 1\right]\right\}.$$

A percentile estimate approach may be used to estimate these parameters.

3.4.2 Finite Range Distribution Families

Finite range distributions could offer a competitive alternative to the distributions with unbounded support because all the values of any empirical data are finite. However, a location parameter is often required and this could limit their usefulness for reliability applications. Also, Glaser's (1980) results for classifying the shape of a failure function $r(t)$ as given in Theorem 2.1 need to be modified (Ghitany, 2004) in this case. Ghitany (2004) has provided a set of more useful sufficient conditions to determine the shape of $r(t)$ of a finite distribution defined on $(0, b), b < \infty$ such that

- If $\eta(t) \le 0$ and $f(b) > 0$, then $r(t) \in$ I $\quad (\eta(t) = -f'(t)/f(t))$.
- If $\eta(t) \in$ D and $f(0) = f(b) = \infty$, then $r(t) \in$ BT.

Below are a few models that exhibit a bathtub shape for their failure rate functions.

- Beta distribution
  Gupta and Gupta (2000) have investigated the monotonic properties of the failure rate function of the beta distribution having pdf given by
  $$f(t) = \frac{1}{B(p,q)} t^{p-1}(1-t)^{q-1},\ 0 \le t \le 1 \tag{3.20}$$
  with $p > 0, q > 0$ and $B(p,q) = \int_0^1 y^{p-1}(1-y)^{q-1}\,dy$. The corresponding failure rate function is
  $$r(t) = \frac{t^{p-1}(1-t)^{q-1}}{B(p,q) - B_t(p,q)} \tag{3.21}$$
  where $B_t(p,q) = \int_0^t x^{p-1}(1-x)^{q-1}\,dx$.
  Ghitany (2004) has shown that $r(t) \in$ B if $p < 1$, independent of the value of $q > 0$ and thus correcting a result of Gupta and Gupta (2000).
- Power-function distribution
  Mukherjee and Islam (1983) (see also Lai and Mukherjee, 1986) proposed a finite range distribution with a bathtub failure rate:
  $$r(t) = \frac{pt^{p-1}}{\theta^p - t^p}, 0 \le t < \theta,\ p < 1, \tag{3.22}$$
  and $r(t) \to \infty$ when $t \to 0$ or $t \to \theta$, thus a bathtub is formed. It is obvious that (3.22) is a special a case of (3.21).

- Beta failure rate distribution
  (Moore and Lai, 1994) proposed another finite range distribution with a failure rate function which is an extension of a beta function of the form:

$$r(t) = c(t+p)^{a-1}(q-t)^{b-1},\ 0 < a < 1, b < -1, 0 \le t < q, c > 0, p \ge 0,$$

  $r(0) = cp^{a-1}q^{b-1},\ r(t) \to \infty$ as $t \to q$.

- Integrated beta failure rate distribution
  Lai et al. (1998) considered a lifetime distribution with its cumulative hazard (failure) function $H(t)$ being proportional to a beta function, i.e., $H(t) = t^a(1-t)^b, 0 \le t \le 1$. The failure rate function is given by

$$r(t) = t^{a-1}(1-t)^{b-1}\left\{a - (a+b)t\right\},\ 0 < t < 1, a > 0, b < 0.$$

  It is obvious that $r(t) \to \infty$ as $t \to 0$ or 1 and hence a bathtub result. A minor extension yields

$$H(t) = \int_0^t r(x)\,dx = ct^a(1-dt)^b,\ b < 0,\ 0 < a \le 1,\ 0 < t < 1/d.$$

- Govindarajula distribution
  Govindarajula (1977) considered a family of distributions having

$$r(t) = \left[\lambda(\lambda+1)t^{\lambda-1}(1-t)^2\right]^{-1}, \lambda > 1,\ 0 < t < 1.$$

  Clearly, $r(t) \to \infty$ as either $t \to 0$ or $t \to 1$ and thus forms a bathtub shape.

- Generalized Weibull family of distributions
  Mudholkar et al. (1996) considered a generalized Weibull family having failure rate specified by

$$r(t) = \frac{\alpha(t/\beta)^{\alpha-1}}{\beta\left[1 - \lambda(t/\beta)^{1/\alpha}\right]},\ \alpha, \beta > 0; \lambda \text{ real}.$$

  The range of this generalized Weibull random variable is $(0, \infty)$ for $\lambda \le 0$ and $(0, \beta\lambda^\alpha)$ for $\lambda > 0$. $r(t)$ has a bathtub shape for $\alpha < 1$ and $\lambda > 0$. $r(t) \to \infty$ as $t \to 0$ or $\beta/\lambda\alpha$. The distribution function is

$$F(t) = 1 - (1 - \lambda(t/\beta)^\alpha)^{1/\lambda}.$$

  We will revisit this distribution in Section 5.5. for other properties.

- Haupt and Schäbe distribution
  Haupt and Schäbe (1992, 1994) proposed a bathtub distribution having the failure rate defined by

$$r(t) = \begin{cases} \frac{1+2\beta}{2T\sqrt{\beta^2+(1+2\beta)t/T}\left(1+\beta-\sqrt{\beta^2+(1+2\beta)t/T}\right)}, 0 \le t \le T; \\ \\ 0, \text{otherwise}; \end{cases} \tag{3.23}$$

where $-1/3 < \beta < 1$. We note that $r(0) = \frac{1+2\beta}{2T\beta},\ r(t) \to \infty$ as $t \to \infty$. The cdf is given by

$$F(t) = \begin{cases} 1, & t \ge T \\ -\beta + \sqrt{\beta^2 + (1+2\beta)t/T}, & 0 \le t \le T \\ 0, & \text{otherwise.} \end{cases}$$

Schäbe (1994b) showed that the mean time between unscheduled removals has an upsidedown bathtub shape.

- J-shaped distribution
  Topp and Leone (1955) proposed a family of distributions with cdf given by

$$F(t) = \left(\frac{t}{\beta}\right)^{\alpha}\left(2 - \frac{t}{\beta}\right)^{\alpha},\ 0 \le t \le \beta < \infty \tag{3.24}$$

$(0 < \alpha < 1)$. The corresponding density function is

$$f(t) = \frac{2\alpha}{\beta}\left(\frac{t}{\beta}\right)^{\alpha-1}\left(1 - \frac{t}{b}\right)\left(2 - \frac{t}{\beta}\right)^{\alpha-1}. \tag{3.25}$$

Nadarajah and Kotz (2003) provided a motivation for the model based on the failure rate function given by

$$r(t) = \frac{2\alpha}{\beta}\frac{y(1-y^2)^{\alpha-1}}{1-(1-y^2)^{\alpha}}, \tag{3.26}$$

where $y = 1 - t/\beta$. The failure rate $r(t)$ in (3.26) has a bathtub shape for all $\alpha \in (0,1)$. It attains a minimum at $t = t_0$, where $y_0 = 1 - t_0/\beta$ is the root of the equation

$$(1-y)^{\alpha} = 1 - \frac{2\alpha y}{1+y},\ \alpha \in (0,1).$$

3.4.3 Bathtub Distributions with More Complicated Failure Rates

Exponentiated Weibull family

Mudholkar and Srivastava (1993) and Mudholkar and Hutson (1996) considered a Weibull distribution function that is exponentiated by a parameter θ to give a new cdf:

$$F(t) = \left[1 - \exp\left(-(t/\beta)^{\alpha}\right)\right]^{\theta},\ 0 \le t < \infty, \tag{3.27}$$

and the quantile function of the model is

$$Q(u) = F^{-1}(u) = \beta\left[-\log(1-u^{1/\theta})\right]^{1/\alpha}, \; 0 \le u \le 1.$$

For $\theta = 1$, this distribution reduces to the standard two-parameter Weibull distribution. The special case $\theta = 2$ was studied by Jiang and Murthy (1997a) as a multiplicative Weibull model.

The failure rate function is

$$r(t) = \frac{\alpha\theta}{\sigma^{\alpha}} t^{\alpha-1} e^{-(t/\beta)^{\alpha}} (1 - e^{-(t/\beta)^{\alpha}})^{\theta-1} / [1 - (1 - e^{-(t/\beta)^{\alpha}})^{\theta}]. \tag{3.28}$$

For $\alpha > 1$ and $\alpha\theta < 1$, the failure rate function has a bathtub shape. For other possible shapes and ageing properties, see Section 5.5.

Gamma mixture family

Glaser (1980), Kunitz and Pamme (1993), and Pamme and Kunitz (1993) all considered the gamma mixture of the form:

$$f(t) = pf_1(t) + qf_2(t), \; p + q = 1, \, t \ge 0 \tag{3.29}$$

where $f_i(t) = \lambda^{\alpha_i} t^{\alpha_i - 1} e^{-\lambda t} / \Gamma(\alpha_i), \; i = 1, 2.$

A bathtub failure rate occurs for either $\alpha_1 > 2, \alpha_2 = 1$ or $\alpha_1 > 1, \alpha_2 < 1$. Section 2.8.3 shows that several other shapes are possible when the shape parameters are appropriately restricted.

Generalized gamma

Glaser (1980), McDonald and Richards (1987a,b) and Richards and McDonald (1987b) considered a generalized gamma distribution having probability density function:

$$f(t) = ct^{\alpha\upsilon - 1} \exp[-(t/\beta)^{\upsilon}], \, t \ge 0, \tag{3.30}$$

$\alpha, \beta, \upsilon > 0$ with c being the proportional constant. A direct determination of the shapes of the failure rate is difficult. These authors found the shapes via Glaser's (1980) function $\eta = -f'(t)/f(t)$.

It is found that for $\upsilon > 1, \upsilon\alpha < 1$, the failure rate function has a BT. With appropriate choices of parameter values, the model is also able to give I, D, or UBT failure rate curves.

Generalized exponential distributions

- Cubic exponential family
  Glaser (1980) and Cobb et al. (1983) considered a cubic exponential family with density given by

$$f(t) = c \exp\left[-\alpha t - \beta t^2 - \gamma t^3\right], c < \alpha, t \geq 0. \tag{3.31}$$

  BT distributions arise if (i) α and β real, $\gamma > 0$, or (ii) α real, $\beta > 0, \gamma = 0$ or (iii) $\alpha > 0, \beta = \gamma = 0$.

- Exponential family of densities
  Glaser (1980), Cobb (1981), Cobb et al. (1983) and Pham-Gia (1994) all considered the density function given by

$$f(t) = c \exp\left[-\alpha t - \beta t^2 + \gamma \ln t\right] = ct^{\gamma} \exp\left[-\alpha t - \beta t^2\right], \gamma < 0. \tag{3.32}$$

  BT distributions occur for (i) α real $, \beta > 0, \gamma > -1$ or (ii) $\alpha > 0, \beta = 0, \gamma > -1$.

Piecewise exponential family

Kunitz (1989) considered a family of extremal class of distributions based on the TTT transform $\phi(p)$ that changes once from convex to concave in (0, 1) and that possesses the Kolmogorov distance $\max\{\Delta, \varepsilon\}$ represented by the graph denoted as Fig. 3.2.

The corresponding life distributions have bathtub shapes and they are part of the family of piecewise exponential distributions.

3.4.4 A Mistaken Identity: the Mixed Weibull Family

Mixtures of Weibull distributions have been considered in Section 2.8.4. Kao (1959) considered a mixture of two Weibull distributions in the following form:

$$F(t) = pF_1(t) + qF_2(t), p + q = 1 \tag{3.33}$$

where

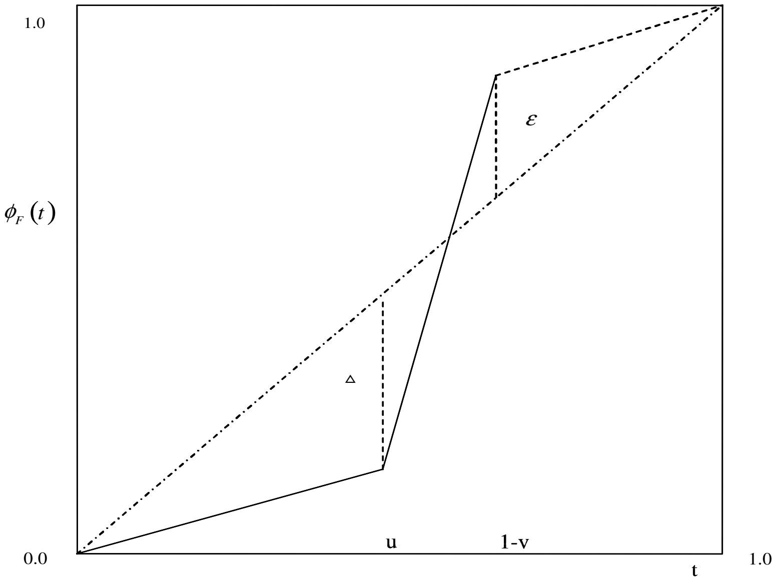
$$F_1(t) = 1 - \exp\left\{-t^{\alpha_1}/\beta_1\right\}, \, \beta_1 > 0, \, 0 < \alpha_1 < 1,$$

and

$$F_2(t) = 1 - \exp\left\{-(t - \gamma_2)^{\alpha_2}/\beta_2\right\}, \, t > \gamma_2, \, \beta_2 > 0, \, \alpha_2 > 1.$$

He claimed that this Weibull mixture has a bathtub shaped failure rate function but several other authors contradicted his result, see, e.g., Glaser (1980), Pamme and Kunitz (1993), and Jiang and Murthy (1998) who all showed that F defined above can never be a BT distribution. Unfortunately, several other books cited this particular result of Kao (1959).

Fig. 3.2. TTT-transform of a piecewise exponential distribution



3.4.5 Some Comments on the Bathtub Shapes

Almost all the parametric distributions given above do not have an interval for which $r(t)$ is a constant. However, many of these would have a nearly flat middle part if the parameters are chosen properly. For example, the additive Weibull model considered by Xie and Lai (1995) can give rise to an approximately flat bathtub shape. It appears that only a sectional model can achieve a constant in the middle part of $r(t)$ to describe the ageing behavior of a device during its 'useful' life phase.

It was pointed out by Haupt and Schäbe (1997) that for many of these bathtub shaped failure rate distributions, one often resorts to extensive iterative procedures to find estimates of parameters. Furthermore, the main characteristics of these distributions such as moments and quantiles are not available in closed forms.

3.5 Construction Techniques for BT Distributions

There are many ways of constructing bathtub shaped failure rate distributions. The following list is unlikely to be an exhaustive one.

3.5.1 Glaser's Technique

In Section 2.1, we saw Glaser's (1980) results have been instrumental to determine the shape of a failure rate function from which many ageing classes were defined. Recall, he chose a function $\eta(t)$ which fulfils the following criteria:

(a) $\eta(t) = -f'(t)/f(t)$ where $f(t)$ is a density function;
(b) there exists a $t_0 > 0$ such that $\eta'(t) < 0$ for all $t \in (0, t_0), \eta'(t_0) = 0$ and $\eta'(t) > 0$ for all $t > t_0$;
(c) there exists a $y_0 > 0$ such that $\int_{y_0}^{\infty} [f(y)/f(y_0)]\eta(y_0)dy - 1 = 0$.

Then the life distribution that possesses these conditions will have a bathtub shaped failure rate function.

3.5.2 Convex Function

The definition of a BT model obviously implies that a BT distribution can be constructed by choosing a positive convex function $r(t)$ over $(0, \infty)$ such that $\int_0^{\infty} r(t)\, dt = \infty$. The quadratic model (3.4) of Bain (1974, 1978) is an example of this constructions. The exponential quadratic model (a BT) given by $r(t) = \exp\{\alpha + \beta t + \gamma t^2\}$ suggests that a strictly increasing function of a bathtub failure rate is itself has a bathtub shape (Rajarshi and Rajarshi, 1988).

3.5.3 Function of Random Variables

This procedure is due to Griffith (1982).

Let X have an exponential distribution with mean 1, and let $\psi(\cdot)$ be a strictly increasing differentiable (except perhaps at m_1 and m_2) function on $[0, \infty)$. Further, if $\psi(\cdot)$ is convex on $[0, m_1)$, linear on (m_1, m_2) and concave on (m_2, ∞) (where possibly $m_1 = m_2$), then $\psi(X)$ has a bathtub shaped failure rate.

3.5.4 Reliability and Stochastic Mechanisms

- Series system (competing risks model)
  Suppose we have a series system of two independent components. It is well known that the failure rate of such a system is simply equal to the sum of the two component failure rates. If one of them has an IFR distribution and the other has a DFR distribution, then the system lifetime may have a bathtub shaped failure rate function. Models obtained by Murthy et al. (1973), Canfield and Borgman (1975), Gaver and Acar (1979) are of this type.

- Stochastic failure models
  Consider the life distribution of a device that is subject to a sequence of shocks occurring randomly in time according to a homogeneous Poisson process, under appropriate conditions on the probability of surviving a given number of shocks, Mitra and Basu (1996b) have shown that the lifetime of the device has a bathtub shaped failure rate.
- IDMRL classes
  They are also known as upside-down bathtub mean residual life distributions. A bathtub shaped failure rate distribution often arises from an IDMRL model.
- Stochastic differential equation models and population abundance distributions.

  Bathtub shaped failure rates may arise from mixtures of two gamma distributions, mixtures of two increasing linear failure rates distributions (Section 2.8.4), mixture of three distributions (Krohn, 1969), mixtures of two positive truncated normal distributions (Navarro and Hernandez, 2004) and others.
- Generalized Weibull
  The Weibull distribution has often been used in the reliability literature to describe failure times. Its failure rate function has a simple form but it is monotonic. In order to incorporate a non-monotonic failure rate, some form of generalization is required. For example, the exponentiated Weibull of Mudholkar and Srivastava (1993) and the modified Weibull of Lai et al. (2003) are two such examples. Section 5.5 is devoted to generalized Weibull families.

3.5.5 Mixtures

Mixtures of distributions often give rise to BT distributions. For example, Glaser (1980) showed that for $0 < \alpha_1 < 1 < \alpha_2$, the gamma mixture has a bathtub shaped failure rate. The mixture of two increasing linear failure rate distributions also results a bathtub shape for an appropriate choice of mixing proportion (Block, Li and Savits, 2003a). See Section 2.8 for other examples of BT distributions that arise from mixtures.

3.5.6 Sectional Models

Shooman (1968), Colvert and Boardman (1976), and Jaisingh et al. (1987) have all considered a bathtub shaped failure rate that is piecewise linear in three regions. For example,

$$r(t) = \begin{cases} \varepsilon_1 - \eta_1 t, & 0 < t \leq t_1, \\ \varepsilon_2, & t_1 < t \leq t_2, \\ \varepsilon_2 + \eta_2(t - t_2), & t > t_2; \end{cases} \tag{3.34}$$

subject to the conditions $t_1 = (\varepsilon_1 - \varepsilon_2)/\eta_1, \varepsilon_1 > \varepsilon_2 > 0, \eta_1, \eta_2 > 0$.

Other sectional models that give rise to BT distributions were given in Jiang and Murthy (1997b) and Murthy and Jiang (1997).

3.5.7 Polynomial of Finite Order

Jaisingh et al. (1987) and Shooman (1968) suggested a polynomial of finite order failure model: $r(t) = a_0 + a_1 t + ... + a_n t^n$. As the constants $a_i, i = 0, ..., n$ may be positive or negative, bathtub shapes can be achieved.

3.5.8 TTT Transform

In Kunitz (1989) and Haupt and Schäbe (1997), the total time on test (TTT) transform was used to construct parametric BT life distributions. Recall from Section 2.5.6, the TTT transform of a lifetime distribution F was defined as

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} \bar{F}(x)\, dx,\ 0 \le t \le 1,$$

where $F^{-1}(t) = \inf\{x : F(t) \ge t\}$. The scaled TTT transform was also defined earlier as $\phi(t) = H_F^{-1}(t)/H_F^{-1}(1)$.

Haupt and Schäbe (1997) provided the following algorithm.
Choose a twice-differentiable function $\phi(u)$ with the properties:

(1) $\phi(0) = 0, \phi(1) = 1, 0 \le \phi \le 1$,

(2) the solution $F(t)$ of the differential equation

$$\frac{\theta\phi(F(t))\, dF(t)}{1 - F(t)} = dt, \qquad \text{with}\, \theta = H_F^{-1}(1) > 0,$$

is a lifetime distribution;

(3) ϕ has only one reflection point u_0 such that $0 < u_0 < 1$ and it is convex on $[0, u_0]$ and concave on $[u_0, 1]$.

We note that any distribution whose TTT transform has the property given by the item (3) above is a BT distribution – see (vi) of Theorem 2.7.

3.5.9 Truncation of DFR Distribution

Schäbe (1994a) has constructed bathtub shaped failure rate distributions from decreasing failure rate distributions by truncations.

3.6 Change Point Estimation for BT Distributions

Change points (turning points) of $r(t)$ are important in non-monotonic failure rate distributions. Without loss of generality, we only consider those bathtub shaped failure rate functions $r(t)$ with a unique change point t^*. Clearly, $r(t)$ is minimum at this point so it may be taken as a possible optimal burn-in time. (See Section 3.9 below for this concept.) Estimating the change point of $r(t)$ for BT (and others) distributions is particularly relevant in the context of maintenance policies, since one would not wish to replace a component until its age has well passed t^*. (If there are two change points, then we should not replace the component until its age reaches the second change point.) Thus the question of estimating t^* becomes very relevant in reliability analysis. Since t^* is often a function of the model parameters, a parametric approach is often more feasible. Let τ be an empirical estimator of t^*.

Nguyen et al. (1984) considered the estimation of the turning point of a two-step piece-wise linear failure rate function. They observed a difficulty in applying the maximum likelihood methodology in this situation, proposed an estimator based on properties of a density represented as a mixture, and established its consistency. Yao (1986) and Pham and Nguyen (1990) noted that, under natural constraints, the maximum likelihood estimator exists, proved its consistency, and derived the limiting distribution. Pham and Nguyen (1993) considered estimating the turning point of a truncated bathtub shaped failure rate function. The authors proposed two semi-parametric estimators as well as a truncated maximum likelihood estimator and proved their consistency. Asymptotic distributions of estimators in the case of the aforementioned failure rate functions were investigated by Basu et al. (1988), Loader (1991), Ghosh and Joshi (1992), Joshi and MacEachern (1997), and Chen et al. (2001). A Bayesian approach to the estimation problem has been considered by Ghosh et al. (1996). Estimation of the turning point of a general bathtub shaped failure rate function was considered by Kulasekera and Lal Saxena (1991). They proposed a nonparametric approach to solve this problem by using the fact that the failure rate is the ratio between the corresponding density and survival functions. For estimating the latter two, one can use a kernel-type density estimator and the empirical distribution function, respectively. Kulasekera and Lal Saxena (1991) proved consistency and asymptotic normality of their nonparametric estimators of the turning point. Mitra and Basu (1995) considered nonparametric estimation of the change point of a more general failure rate function than those discussed above. They proved consistency of their estimator and noted that in the special case of bathtub shaped failure functions, their estimator works under less restrictive assumptions in Kulasekera and Lal Saxena (1991). Suresh (1992) also obtained two estimates of the change point; one by using the definition of BT distribution, another by a characterization of BT distribution in terms of TTT transform. Gupta, Akman and Lvin (1999) considered estimation of the turning point of the failure rate function in the case of the log-logistic model, proving that the maximum likelihood estimator exists and is consistent.

Recently, Bebbington et al. (2005a) considered a parametric approach to estimate t^* for the modified Weibull model of Lai et al. (2003). From the literature review above, we noted that the parametric (i.e., maximum likelihood based) methods sometimes encounter serious difficulties but sometimes succeed. In the case of difficulties, semi-parametric or nonparametric approaches have been used, and satisfactory results obtained. In Bebbington et al. (2005a), we investigated when a maximum likelihood based estimator works and when it encounters difficulties by identifying the region(s) of the parameter space $a > 0, b \in (0, 1)$ and $\lambda > 0$ in which the estimator works.

3.7 Mean Residual Life and Bathtub Shaped Life Distributions

3.7.1 Mean Residual Life

Recall, the mean residual life function (MRL) of a lifetime random variable is defined as

$$\mu(t) = E\left(X - t \,|\, X > t\right) = \left[\int_t^\infty \bar{F}(x)\,dx\right] / \bar{F}(t). \tag{3.35}$$

Although numerous studies have been conducted on the MRL distributions and its applications, few of them involve BT.

It is well known that the class of IFR is contained in the class of DMRL (decreasing mean residual lifetime). A DMRL distribution need not be an IFR. One may conjecture that BT distributions may be related to a class of life distributions whose mean residual life $\mu(t)$ is increasing and then decreasing, that is, the mean residual life has an upside-own bathtub shape. Indeed this relationship has been observed empirically as pointed out by Rajarshi and Rajarshi (1988) who said in their review article “It can be observed from the life-tables of human and animal populations that the shape of the empirical MRL function is upside bathtub”. Such an ‘inverse’ relationship between $r(t)$ and $\mu(t)$ may be seen from their functional relation

$$r(t) = \left[\mu'(t) + 1\right] / \mu(t). \tag{3.36}$$

(Muth, 1977). Eq (3.36) indicates that $\mu'(t) \geq -1$.

Many authors (e.g., Park 1985) have discovered that the turning point of the mean residual life function $\mu(t)$ precedes the turning point(s) of the bathtub failure rate function $r(t)$. In other words, the time at which a bathtub failure rate is a minimum does not maximize the mean residual life. The mean residual life function $\mu(t)$ in the constant failure rate region of a bathtub shaped failure curve is not constant but decreasing.

3.7.2 Bathtub Shaped Failure Rate and Decreasing Percentile Residual Life Function

The 'α-*percentile* residual life function' (α-*percentile* RLF) was first defined by Haines and Singpurwalla (1974). Joe and Proschan (1984) showed that this function may be expressed as

$$q_{\alpha,F}(t) = F^{-1}\left(1-(1-\alpha)\bar{F}(t)\right). \tag{3.37}$$

In Section 2.7, we have defined an ageing class based on this function, i.e., we say a distribution is DFRL-α, if and only if for some α, $0<\alpha<1$, $q_{\alpha,F}(t)$ decreases in t.

Launer (1993) has shown that a bathtub-shaped failure rate distribution is DPRL-α for all $\alpha_0<\alpha<1$ for some $\alpha_0 > 0$, provided there exists a t_0 with $r(t_0) \geq r(0)$.

3.7.3 Relationships Among NWBUE, BT and IDMRL Classes

In this subsection, we preview some results in Chapter 4 on relationships between BT distributions and non-monotonic mean residual life distributions.

We say A life distribution F having finite mean μ is said to be 'new worse then better than used in expectation' (NWBUE) if there exists a point t_0 such that

$$\mu(t)\begin{cases} \geq \mu, & \text{if } t < t_0, \\ \leq \mu, & \text{if } t \geq t_0. \end{cases} \tag{3.38}$$

The turning point t_0 will be referred to as a change point of the distribution F. The following theorem was due to Mitra and Basu (1994).

Theorem 3.1: Let F be a continuous and strictly increasing life distribution. If F is BT with mean μ, then it is NWBUE.

Proof: Let ϕ be the scaled TTT transform defined by (2.69). It follows from (2.68) and (2.69) that $\frac{1-\phi(p)}{1-p} = \mu(t)/\mu$ where $p = F(t)$.

Since $F \in$ BT, it follows from result (vi) of Theorem 2.7 that ϕ is convex for $p < p_0$ and concave for $p > p_0$ so

$$\phi(p) = \begin{cases} \leq p & \text{for } p < p_0, \\ \geq p & \text{for } p \geq p_0; \end{cases}$$

since $0 \leq \phi(p) \leq 1$. Let $t_0 = F(p_0)$. As F is monotonely increasing in t, it is now clear that $\mu(t)/\mu \geq 1$ for $t < t_0$ and $\mu(t)/\mu \leq 1$ for $t \geq t_0$ and thus we conclude that F is NWBUE.

Remark: In the definition above, NWBUE includes DMRL and IMRL classes. If the last two classes are excluded, a condition such as $r(0)\mu > 1$ is required for the theorem to hold.

The converse of the above theorem is false as can be seen from the following example.

In this example, we assume that the MRL of the distribution F is given by

$$\mu(t) = \begin{cases} 1, & 0 \le t < 1, \\ t, & 1 \le t < 2, \\ 4/t, & 2 \le t < \infty \end{cases}$$

so F is NWBUE with $t_0 = 4$. The corresponding failure rate function $r(t)$ can be shown to be given as:

$$r(t) = \begin{cases} 1, & 0 \le t < 1, \\ 2/t & 1 \le t < 2, \\ (t/4 - 1/t)\,, & 2 \le t < \infty \end{cases}$$

which reveals that F is not BT.

We say that a distribution F is IDMRL if $\mu(t)$ has an upside-down shape, i.e., $\mu(t) \in$ UBT. (Guess et al., 1986). Mitra and Basu (1994) showed that If F is IDMRL(t_0), then F is NWBUE(t_0') with $t_0' > t_0$, i.e., the class of IDMRL distributions is a subset of NWBUE class. We will state this result as Theorem 4.1 in Chapter 4.

Mitra and Basu (1994) have provided an example to show that a life distribution can be NWBUE without being IDMRL, i.e., the inclusion is strict.

Gupta and Akman (1995a,b) (see also Mi, 1995) showed that under some mild condition such as $r(0)\mu > 1$, a BT distribution has an upside-down mean residual life function $\mu(t)$.

Analogously, we say that a life distribution F is DIMRL if its mean residual life has a bathtub shape.

Ghitany (1998) showed that a generalized gamma distribution developed by Agarwal and Kalla (1996) has a bathtub shaped mean residual life if the parameters satisfy certain constraints.

Detailed relationships between a BT failure rate $r(t)$ and its associated $\mu(t)$ will be fully explored in Chapter 4.

3.8 Optimal Burn-in Time for Bathtub Distributions

Burn-in plays an important role in reliability engineering. Jensen and Petersen (1982) is recognized as the first textbook completely devoted to the topic of burn-in. In this section, we discuss the subject burn-in from the failure rate perspective whereas in the next chapter it will be discussed from the mean residual life angle.

3.8.1 Concepts of Burn-in

Due to the high failure rate in the early stages of component life (most notably, silicon and integrated circuits), burn-in has been widely accepted as a method of screening out failures before these components are shipped to customers or put into the field operations. That is, before delivery to the customers, the components are tested under electrical or thermal conditions that approximate the working conditions in field operation. Those components which fail during the burn-in procedure will be scrapped or repaired and only those which survive the burn-in procedure will be considered to be of good quality. These are then shipped to customers or put into field operation.

With insufficient burn-in, high initial failure rates cause high repair costs. On the other hand, with excessive burn-in, the reduced failure rate will be at the cost of increased capital and recurring costs. A major problem is to decide how long the procedure should continue. The best time to stop the burn-in process for a given criterion to be optimized is called the *optimal burn-in time*. A general background on burn-in can be found in Kuo and Kuo (1983), Kuo (1984). For an excellent review on burn-in, see Block and Savits (1997) or Leemis and Beneke (1990).

3.8.2 Burn-in and Bathtub Distributions

We now consider the optimal burn-in time for BT distributions under different criteria .

Burn-in and failure rate $r(t)$

Wang (2000) considered that one of the uses of a bathtub shaped failure rate distribution is that we can determine the optimum burn-in time when the initial failure rate is too high for the product to be released directly after production. For example, if the customers' requirement is to have the failure rate less than r_b, then the optimal burn-in time b^* can be determined by $r(b) = r_b$ where t lies outside the wear-out phase. In other words, $b^* = \inf\{t : r(t) = r_b\}$.

Maximizing the reliability for a given mission time

Let the lifetime T of a component have a continuous bathtub shaped failure rate $r(t)$. This component is required to accomplish a mission which lasts for time τ. The reliability of completing the mission is thus $\bar{F}(\tau)$. If we burn-in the component for a time b and if the component survives the burn-in, then the conditional reliability of accomplishing the mission is given by

$$\frac{\bar{F}(b+\tau)}{\bar{F}(b)} = \exp\left(-\int_b^{b+\tau} r(t)dt\right). \tag{3.39}$$

We want to determine the optimal burn-in time such that the survival probability (3.39) above will be maximized. The set of burn-in times is defined as

$$B^* = \left\{ b \geq 0 : \frac{\bar{F}(b+\tau)}{\bar{F}(b)} = \max_{t \geq 0} \frac{\bar{F}(t+\tau)}{\bar{F}(t)} \right\}. \tag{3.40}$$

Alternatively,

$$B^* = \left\{ b \geq 0 : \int_b^{b+\tau} r(s)ds = \min_{t \geq 0} \int_t^{t+\tau} r(s)ds \right\}. \tag{3.41}$$

Mi (1994b) characterized the structure of the set B^* as below:

Theorem 3.2: Let the continuous failure rate $r(t)$ have a bathtub shape with change points t_1 and t_2, and $\tau > 0$ be a given mission time.

- If $\tau \leq t_2 - t_1$, then the optimal burn-in occurs at each point of $[t_1, t_2 - \tau]$, i.e., $B^* = [t_1, t_2 - \tau]$.
- If $\tau > t_2 - t_1$, then the optimal burn-in occurs on or before the first change point t_1, i.e., $B^* = \{b^*\}$ and $b^* \in [0, t_1]$ where B^* is defined as above.

Proof: The proof hinges on a lemma which states that if there exists $0 \leq b_1 < b_2$ such that the given bathtub shaped failure rate function satisfies $r(b_1) = r(b_2)$ and $b_2 - b_1 = \tau$, then

$$\int_b^{b+\tau} r(t)\,dt \geq \int_{b_1}^{b_2} r(t)\,dt \quad \text{for } b \geq 0$$

and the inequality is strict whenever $r(b) \neq r(b+\tau)$. For the rest of the proof, see Theorem 1 of Mi (1994b).

Maximizing the mean residual lifetime MRL

Often the quality of products is taken to be the lengthen of time they give satisfactory service, i.e., their lifetime. After products are manufactured, the only way to improve this aspect of their quality is to operate them for a fixed period of time, say b. Suppose cost is not to be considered, it is reasonable to set our goal for the longest mean lifetime. Accordingly, we need to determine b such that MRL is maximized, as only those items that survives the fixed burn-in time are placed in service. In other words, we want to find b^* such that

$$\mu(b^*) = \max_{b \geq 0}\{\mu(b)\}. \tag{3.42}$$

Theorem 3.3: Assume $r(t)$ has a bathtub shape and is differentiable with two change points t_1 and t_2. Then

- $t_1 = 0$: there is no need to burn-in, that is, $b^* = 0$;
- $t_2 = \infty$ and $t_1 > 0$; then we can always choose $b^* = t_1$;
- $t_1 = t_2 = \infty$ (F is actually DFR): the cost should be considered;

- $0 < t_1 \le t_2 < \infty$: b^* equals the unique change point t^* of $\mu(t)$.

Proof: The first three results are obvious and the last will follow easily from Theorem 4.2 of Chapter 4. We also note from Theorem 4.3 that b^* must occur before t_1. See Mi (1995) for details.

Accordingly, we never need to burn-in products longer than the first change point t_1 unless F is DFR.

The above results establish a general principle which states that burn-in should occur at or before the point at which a bathtub shaped failure rate function starts increasing. Block et al. (1999) have established a framework for determining when the above principle holds.

Optimal burn-in time under age replacement with complete repair policy

We consider the age replacement policy described in Barlow and Proschan (1965). Let c_f denote the cost incurred for each failure in field operation and c_a, satisfying $0 < c_a < c_f$, the cost incurred for each non-failed item which is replaced at age $T > 0$ in field operation.

Theorem 3.4: Suppose the failure rate function $r(t)$ is differentiable and has a bathtub shape. Then under the age replacement policy with complete repair at failure, the optimal burn-in time b^* and the corresponding optimal age $T^* = T^*(b^*)$ satisfy $0 \le b^* \le t_1$ and $b^* + T^* = b^* + T^*(b^*) > t_2$, where $T^*(b^*)$ is either the unique solution of the equation given below:

$$r(b+T)\int_0^T \frac{\bar{F}(b+t)}{\bar{F}(b)}dt + \frac{\bar{F}(b+T)}{\bar{F}(b)} = \frac{c_f + k(b)}{c_f - c_a} \tag{3.43}$$

or equal to ∞ depending on whether (3.43) has a solution or not.

Here $k(b)$ is the expected cost of burn-in per item:

$$k(b) = Er(b) = c_0 \frac{\int_0^b \bar{F}(t)dt}{\bar{F}(b)} + \frac{c_s F(b)}{\bar{F}(b)}. \tag{3.44}$$

Proof: The proof is given in the appendix of Mi (1994c). It is rather long, involving consideration of several subcases.

Cha (2000) proposed a new burn-in procedure for a repairable component. During the burn-in period, the failed component is only minimally repaired rather than being completely repaired. This procedure was shown to be economical and efficient when the minimal repair method is applicable during the burn-in processes. The properties of the optimal burn-in time b^* and block replacement policy T^* were also given.

Burn-in time under block replacement and minimal repair policy

Mi (1995) has also obtained the optimal burn-in time under the block replacement and minimal repair policy. The result, as given in Theorem 2 of Mi (1995), is rather complicated.

3.8.3 Burn-in Time for BT Lifetime under Warranty Policies

Warranty for durable goods provides a peace of mind for the consumers. It is a written guarantee that states the manufacturer of that good will provide labour and parts to fix or replace the defective good when necessary. Consumers will feel safe when they purchase a good that comes with warranty. However, the warranty cost may drastically reduce profitability.

Burn-in is a common procedure to improve the quality of the products after they have been produced, but it is also costly.

Following Nguyen and Murthy (1982) and Chou and Tang (1992), we assume that cost is additive and has the following elements:

c_0 : the manufacturing cost per unit without burn-in;

c_1 : the fixed setup cost of burn-in per unit;

c_1 : the cost per unit time of burn-in per unit;

c_3 : the shop repair or replacement cost per failure;

c_4 : the extra repair or replacement cost per failure during the warranty period.

Nguyen and Murthy (1982) proposed a model to determine the optimal burn-in time for products sold with warranty. Let $T_w > 0$ be the length of the warranty period. They considered two type of warranty policies. One is the failure-free policy where all failed products are repaired or replaced by independent and identically distributed one during the warranty period. The other is the rebate policy where the consumer is refunded some amount $R_0(t)$ if the product fails at time t during the warranty period $[0, T_w]$. These authors then considered the total cost as the sum of burn-in cost and the warranty cost. By assuming that the failure rate of products has a bathtub shape with change points $t_1 = t_2$ (i.e., having a unique change point), they obtained several results regarding the optimal burn-in time minimizing the total mean cost function. Their main conclusion was that the optimal burn-in time is no later than the unique change point. In Nguyen and Murthy's model, the refund amount function $R_0(t)$ is assumed to be a decreasing linear function of t in $[0, T_w]$.

Mi (1997) also considered the same problem but relaxed the assumptions in two ways: (1) $r(t)$ may have two change points according to the Definition 3.4, (2) $R_0(t)$ is an arbitrary decreasing function of t. Mi discussed different product type-warranty policy combinations and concluded that in each case optimal burn-in time b^* that minimizes the total mean cost function $c(b)$ never exceeds the first change point t_1.

It is believed that the replacement-free policy favours the consumers at the expense of the manufacturer, and the rebate (pro-rata) policy favours the manufacturer at the expense of the consumers. Because of this, Nguyen and Murthy (1984) introduced a mixed warranty policy that initially starts with a replacement-free period followed by a pro-rata period. This will be more reasonable from both the manufacturers' and the consumers' point of view.

In addition to the above classification, warranties can also be renewable or nonrenewable. Mi (1999a) considered the burn-in problem under renewable mixed warranty policy.

Theorem 3.5: Suppose the lifetime distribution F has a bathtub shaped failure rate function $r(t)$ with change points $0 < t_1 \leq t_2 < \infty$. Then for the renewable mixed warranty with decreasing rebate function $R_0(t) \geq 0$, the optimal burn-in time b^* that minimizes the mean warranty cost $C(b)$ must satisfy $b^* \leq t_1$.

Proof: First, we note that if the underlying lifetime distribution of a new product is F and the burn-in time is b, then the burn-in device has a distribution $F_b(x) = \bar{F}(b+x)/\bar{F}(b)$. The remaining lifetime that corresponds to F_b is denoted by X_b.

Since $r(t)$ exhibits a bathtub shape so $r(b) \geq r(t_1)$ for all $b > t_1$. Now $r(b) \geq r(t_1)$ is equivalent to $X_b \leq_{\text{FR}} X_{t_1}$ (see Definition 2.15 of Section 2.9 on partial orderings of distributions). By Theorem 6 of Mi (1999a), we have $C(t_1) < C(b)$ for all $b > t_1$. This shows that $b^* \leq t_1$ for b^* to satisfy $C(b^*) = \min_{b \geq 0} C(b)$. See Mi (1999a) for missing details.

3.8.4 Optimal Replacement Time and Bathtub Shaped Failure Rate Distributions

For certain maintenance policy, a component is replaced if its age has reached a certain level. The question aries as what would be the 'optimal age'. Wang (2000) considered a scenario where the product has to be replaced by a new one when the original component has failure rate higher than a threshold value h_r. If this is taken as the criterion, then the optimal replacement age can be determined by solving the functional equation $r(t) = r_b$. (We restrict t to lie in the wear-out phase). The equation can be solved numerically using some standard algorithms. Furthermore, the hazard plot may also be used to determine the optimum replacement time.

3.9 Upside-down Bathtub Shaped Failure Rate Distributions

Another important family of life distributions is known as the upside-down bathtub shaped failure rates class which is introduced in Chapter 2 and denoted by UBT. An UBT model is defined as having a unimodal failure rate function $r(t)$ by Chang (2000). Jiang et al. (2003) considered an UBT class as the family of distributions whose failure rates are unimodal or reverse bathtub shaped. We are of the opinion that referring to UBT as a family of distributions having unimodal failure rates is not easily understood, since one rarely uses 'mode' of a function outside the context of a random variable.

For the situations where the failure is mainly caused by fatigue or corrosion, the time to failure is often represented by such UBT models.

3.9.1 UBT Models

Surprisingly, there are several well known statistical models listed in Section 2.3 belong to this class. For the convenience of the reader, we give the failure rate functions of the UBT distributions again in this subsection.

(i) Lognormal

This well known distribution has been considered in Section 2.3. Its distribution function is

$$F(t) = \Phi\left(\frac{\log t - \mu}{\sigma}\right) \tag{3.45}$$

with the failure rate function

$$r(t) = \frac{(1/\sqrt{2\pi}t\sigma)\exp\left\{-(\log at)^2/2\sigma^2\right\}}{1 - \Phi\left\{\log(at)/\sigma\right\}}, \tag{3.46}$$

where $a = e^{-\alpha}$. It is well known the lognormal distribution belongs the UBT class (see Section 2.3); $r(0) = 0$ and $r(t) \to 0$ when $t \to \infty$. Sweet (1990) gave a comprehensive analysis of the failure rate of the lognormal distribution.

(ii) Inverse Weibull

The two-parameter Weibull distribution is given by

$$F(t) = 1 - e^{-(t/\beta)^\alpha},\ \alpha, \beta > 0. \tag{3.47}$$

Let X denotes the random variable from the Weibull model. Define $Y = \beta^2/X$ so that the distribution of Y is given by

$$F(t) = \exp\left(-(\beta/t)^\alpha\right). \tag{3.48}$$

The distribution that associated with (3.48) is called the inverse Weibull distribution. Jiang et al. (2001) provided a historical development and some basic properties of this distribution. The failure rate function is given by

$$r(t) = \alpha^\beta \beta t^{-\beta-1} e^{-(\alpha t)^\beta}/(1 - e^{-(\alpha t)^\beta}). \tag{3.49}$$

In particular, it has been shown that $\lim_{t\to 0} r(t) = \lim_{t\to\infty} r(t) = 0$ and $r(t) \in$ UBT.

We will revisit this distribution in Section 5.4 for other properties of the inverse Weibull.

(iii) Inverse Gaussian

The inverse Gaussian was studied earlier in Section 2.3. The distribution function is

$$F(t) = \Phi\left\{\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu}-1\right)\right\} + e^{2\lambda/\mu}\Phi\left\{-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu}+1\right)\right\}. \tag{3.50}$$

Although the failure rate function is quite complicated, it has been shown that $r(t)$ has a UBT shape with $\lim_{t\to 0} r(t) = \lim_{t\to\infty} r(t) = 0$ (see, e.g., Chhikara and Folks, 1979). It was also shown that the time at which $r(t)$ attains its maximum is also the time when the mean residual life $\mu(t)$ reaches its minimum. Hsieh (1990) has considered various methods for estimating the critical time (change point) of $r(t)$.

(iv) Birnbaum-Saunders

This was also studied in Section 2.3 earlier. The distribution function is

$$F(t) = \Phi\left\{\frac{1}{\alpha}\cdot\left[\left(\frac{t}{\beta}\right)^{1/2} - \left(\frac{t}{\beta}\right)^{-1/2}\right]\right\},\ t > 0. \tag{3.51}$$

We note that $r(t) \to 1/(2\alpha^2\beta)$ as $t \to \infty$ (see, e.g., Chang and Tang, 1993). The function $r(t) \in$ UBT for $0.8 < \alpha < 2.2$ and $\beta = 1$. The critical point of $r(t)$ is given in Chang (1994).

A comparison between the failure rates of the Birnbaum-Saunders and the lognormal distributions is given in Nelson (1990).

(v) Log-logistic

We have also considered this distribution in Section 2.3.

$$\bar{F}(t) = \frac{1}{1+(\rho t)^k},\ t > 0, \rho > 0, k > 0, \tag{3.52}$$

and

$$r(t) = \frac{k\rho(\rho t)^{k-1}}{1+(\rho t)^k}. \tag{3.53}$$

Gupta, Akman and Lvin (1999) have shown that $r(t) \in$ UBT if $k > 1$ and the critical (turning) point of the failure rate function is given by

$$t^* = \frac{(k-1)^{1/k}}{Q}.$$

(vi) Exponentiated Weibull

$$F(t) = \left[1 - \exp\left(-(t/\sigma)^\alpha\right)\right]^\theta, 0 \le t < \infty. \tag{3.54}$$

Mudholkar et al. (1995) have shown that for $\alpha < 1$ and $\alpha\theta > 1$, $r(t) \in$ UBT. We will revisit this distribution in Section 5.5.

(vii) Dhillon's second model

Dhillon (1981) also constructed a second two-parameter system, with survival function

$$\bar{F}(t) = \exp\left[-\{\log(\lambda t + 1)\}^{\beta+1}\right], \ \beta \geq 0, \lambda > 0, t > 0. \tag{3.55}$$

The failure rate function is

$$r(t) = (\beta + 1)(\lambda t + 1)^{-1}\{\log(\lambda t + 1)\}^{\beta}. \tag{3.56}$$

For $\beta = 1$, $r(t)$ has an UBT shape (Johnson et al., 1995, p. 645).

(viii) Mixtures

The mixture of the Weibull of shape parameter $\beta > 1$ by the exponential (Gurland and Sethuraman, 1994) has a UBT shape for all mixing proportion p such that $0.05 < p < 0.95$. See Section 2.8.4 for other ageing properties of the Weibull mixtures.

(ix) Generalized gamma

The density function of the generalized gamma is

$$f(t) = ct^{\alpha v - 1} \exp[-(t/\beta)^{v}], \ t \geq 0,$$

$\alpha, \beta, v > 0$ with c being the proportional constant. It was pointed out in Section 3.4.3 that for $v < 1, v\alpha > 1$, $r(t)$ has an UBT shape.

For other examples of UBT, see Gupta (1995) and Ghitany (1998).

3.9.2 Optimal Burn-in Decision for UBT Models

Chang (2000) considered optimal burn-in problem for UBT distributions. Under a mild condition ($r(0)\mu < 1$), a UBT distribution F has a BT shaped mean residual life function and $\mu(t)$ may go to ∞ so maximizing MRL cannot be the goal of an optimal burn-in decision. However, it is required that $\mu(t) \geq \mu(0) = \mu$, where μ is denoted by MTTF as in the figure below. So an optimal burn-in time δ should occur after t_{μ}.

Chang (2000) provided a procedure that is based on the costs, the failure rate function together with a constraint imposed on $\mu(t)$ as mentioned above. He also demonstrated his procedure using the lognormal model as an example.

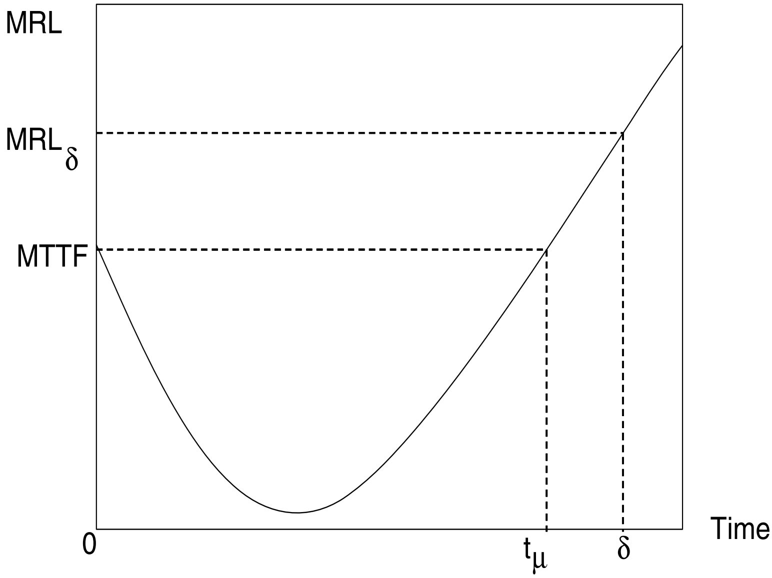
Chang (2000) also indicated that burn-in is not always necessary for products with UBT failure rate functions.

3.10 Modified and Generalized Distributions

3.10.1 Modified Bathtub Distributions

A modified bathtub shape was defined in Section 2.2.1 as a curve that is first increasing and then having a bathtub shape.

Fig. 3.3. Burn-in decision for products with UBT failure rate based on MRL



Some researchers (see, e.g., Kuo and Kuo, 1983), upon collecting failure statistical data found that component failure rates often follow a more complex pattern than those described the bathtub curves. The modifications of a bathtub curve are known as generalized/modified bathtub curves. The major difference between the traditional and the modified bathtub curves is the behavior during their infant mortality part. Jensen and Petersen (1982) suggested a two-stage model of infant mortality: the first stage has an increasing failure rate, indicating failures that rise from comparatively coarse defects, such as those from imperfect manufacturing, improper handling, or defective control processes. Such a failure rate in this first stage peaks quickly and is followed by a period of decreasing failure rate and then increasing. In another words, the failure rate function increases at the beginning and then follows by a bathtub shape. We use the abbreviation MBT to denote a modified bathtub shape and refer to a distribution F as a MBT distribution if its failure rate has a MBT shape. Jensen and Petersen (1982) were probably the first to use a MBT model for describing the failures of modern electronic devices.

Modified bathtub distributions are often found in mixtures of distribution such as mixtures of gammas and mixtures of Weibulls (see Section 2.8).

Example: Extended Weibull of Marshall and Olkin

In Section 2.2.1 we introduced the extended Weibull distribution of Marshall and Olkin (1997) constructed by adding a parameter to the survival function $\bar{G}$:

$$\bar{F}(t) = \frac{\beta \bar{G}(t)}{1 - \bar{\beta}\bar{G}(t)}, -\infty < t < \infty,\ \beta > 0, \tag{3.57}$$

where $\bar{\beta} = 1 - \beta$. In particular, they considered the case when

$$\bar{G}(t) = \exp\left\{-(\lambda t)^{\alpha}\right\},\ \alpha > 0, t > 0. \tag{3.58}$$

Upon substituting (3.58) into (3.57), we have a new three-parameter extended Weibull distribution given by

$$\bar{F}(t) = \frac{\beta \exp\left\{-(\lambda t)^{\alpha}\right\}}{1 - \bar{\beta}\exp\left\{-(\lambda t)^{\alpha}\right\}}. \tag{3.59}$$

The failure rate function is given by

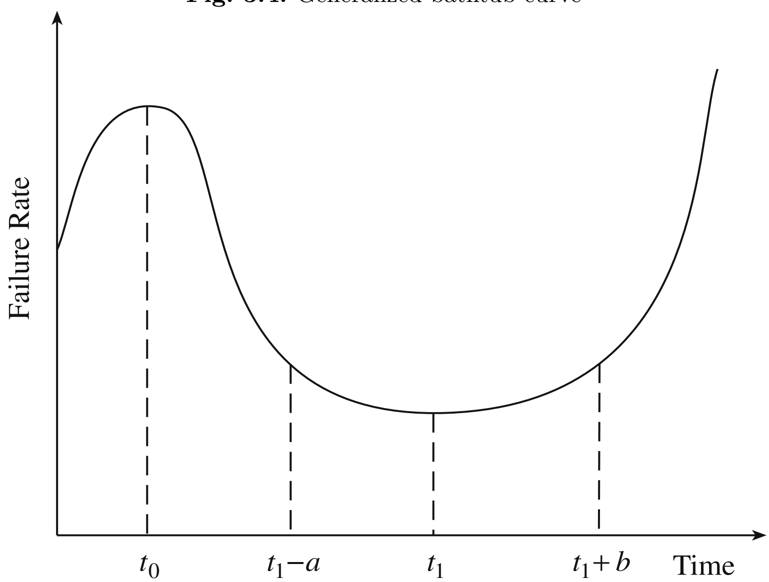
$$r(t) = \lambda\alpha(\lambda t)^{\alpha-1}/[1 - \bar{\beta}\exp\{-(\lambda t)^{\alpha}\}]. \tag{3.60}$$

Their Fig. 2 indicates that $r(t) \in$ MBT for $\lambda = 1, \alpha = 2, \beta = 0.05$ or $\beta = 0.1$.

3.10.2 Generalized Bathtub Curves

Kogan (1988) introduced a set of 'polynomial models' of failure rates that generalizes the traditional bathtub curves. One of his models has a shape like the Figure 3.4 below:

Fig. 3.4. Generalized bathtub curve



This curve has a minimum at t_1 and a maximum at t_0. The failure rate function $r(t)$ is quite flat in the interval $(t_1 - a, t_1 + b)$. Kogan (1988) constructed a polynomial model in which $a = b$ such that $r(t_1 - a) = r(t_1 + a)$. To begin with, he assumed that the failure rate function has a derivative given by

$$r'(t) = A(t - t_0)^{2n-1}(t - t_1)^{2m-1}, t \geq 0, A > 0, t_1 \geq t_0 > 0, \tag{3.61}$$

where n and m are positive integers such that $A(t_1 + a - t_0)^{2n-1}a^{2m-1} < \varepsilon$. For the special case $n = m = 1$, it is found that the failure rate equation is

$$r(t) = At^3/3 - A(t_0 + t_1)t^2/2 + At_0t_1t + r(0) \tag{3.62}$$

subject to $A(t_1 - a - t_0)a < \varepsilon$.

Notes

- In the above curve (Fig 3.4) we have a bathtub shape for $t > t_0$. It could also be considered as having a modified bathtub shape.
- A different form would appear if it also has a bathtub shape for $t < t_0$.
- Kogan (1988) gave another generalized curve which has two bathtubs jointed by a hump.
- For a mixture involving two Weibull distributions, Jiang and Murthy (1998) have shown that we can have both of the above two shapes. They have also shown that in this case, the shape for small t is the same as that for large t so that if it is increasing (decreasing) for small t then it is also increasing (decreasing) for large t.

3.10.3 Roller-Coaster Curves

A roller-coaster curve was briefly defined in Section 2.2. Wong (1988, 1989, 1990, 1991) as well as Wong and Lindstrom (1988) considered a generalization of the bathtub failure curve, called the roller-coaster curve. Essentially, the name suggests that the failure rate has a roller-coaster shape, a bathtub with one or more humps. Wong (1991) suggested some plausible physical reasons for the formation of the roller-coaster shape. The generation of the shape starts with the basic failure mechanisms, which lead to the generally decreasing failure rate. The humps could be caused by changing hazard conditions, wear-out failure distribution of flawed items, distribution of flaw sizes or residual small size flaws left in the equipment because of test and inspection limitations.

Jiang and Murthy (1997b) considered two sectional models involving three Weibull distributions. By considering different constraints on the parameters, various shapes of failure rate functions result. Two of these have a 'roller-coaster' shapes. The same authors, Jiang and Murthy (1998), also considered a mixture involving two 2-parameter Weibull distributions. They concluded that the failure rate shape can be one of eight different types. Two of these are monotonic whereas the rest can be viewed as 'roller-coaster' shaped. We also note in passing that in addition to the two models mentioned here, these two authors also considered a competing risk model (Jiang and Muthy, 1997c), and a multiplicative model (Jiang and Murthy, 1997a) involving two Weibull distributions. Bathtub shaped failure rates can arise in all these models except the mixture model.

In Section 4.5.4, we will discuss the shape of the mean residual life function $\mu(t)$ when the associated failure rate function $r(t)$ has a roller-coaster shape.

3.11 Applications

There are many applications of BT distributions, both in reliability and survival analysis. Below are a few selected examples.

- Failure times of jet engine starters

Kamins (1962) synopsized a very large set of failure times of jet engines starters in the form of histogram representation of hazard rate (failure rate). Barlow and Proschan (1981, p. 55) used this histogram as an example of real data which can be adequately modeled by a bathtub failure rate.

Siddiqui and Kumar (1991) used the finite range model of Mukherjee and Islam (1983) to fit the failure data of V600 indicator tubes used in aircraft radar sets collected by Davis (1952).

- Car failures

Xie and Lai (1995) used an additive Weibull model to analyse an actual set of car failure times data collected during a unit test (name of the brand is suppressed for the sake of confidentiality). The failure rate has a bathtub shape.

- Bus motor failures

Davis (1952) obtained a large number of datasets on bus motor failures . Two of these exhibited bathtub shaped failure rates. Bain (1974) observed that the quadratic failure model fits well to one of these whereas Smith and Bain (1975) found that the exponential power model fits well with the other set.

- Electronic tubes failures

Kao (1959) used a mixed Weibull distribution to fit the actual life testing data gathered by a small-scaled life test of some 800 6AQ5A's conducted at Cornell University.

- Electricity generators' failures

Failure data of 500 MW generators were collected over 6-year period and Dhillon (1981) found that the exponential power model fits well with this dataset.

- Biological and ecological applications:

-Birds: Paranjpe and Rajarshi (1986) fitted the exponential model and the double exponential power model to survival data of birds in Deevey (1947) and Pinder et al. (1978)

-Deer: Gore et al. (1986) used the quadratic failure rate model to study the decomposition rates of heaps of pellet of *Axis axis* (a deer species) in an animal reserve.

- Halley's mortality life data

Halley obtained the survival data of 1000 children born in the city of Breslau in Germany (now Wroclaw in Poland). The failure rate has a bathtub shape. Jaisingh et al. (1987) and Moore and Lai (1994) found some models that would fit the dataset well.

- Load-capacity (stress-strength) interference

Lewis and Chen (1994) showed that the infant mortality, constant failure rate (Poisson failures), and ageing are associated with capacity variability, load variability and capacity deterioration, respectively. Bathtub-shaped failure rate curves are obtained when all three failure types are present.

- Non-Hodgkin's Lymphoma survival

Alidrisi et al. (1991) obtained survival data of 989 patients treated for non-Hodgkin's Lymphoma from the King Faisal Specialist Hospital and Research Center in Riyadh, Saudi Arabia. The failure rate function has a definite bathtub shape.

- Preventive maintenance schemes

A preventive scheme with periodic checkup has been found to be an important device to avoid catastrophic system failure. Of late, the same scheme has been found to be useful in the clinical follow-up studies of patients suffering from chronic diseases such as hypertension, diabetes, cancers, etc.

–Gross and Clark (1975) has found that the bathtub failure rates are highly applicable for preventive maintenance schemes.

–Biswas and Abid (1991) studied the optimal time for check-up in a maintenance scheme under bathtub and Weibull type failure rates.

- Centrifugal water pumps

Pamme and Kunitz (1993) found the mixed gamma model (which has a bathtub shaped failure rate for a correct choice of shape parameters) gives a good fit to the Centrifugal Water Pumps data.

4

Mean Residual Life – Concepts and Applications in Reliability Analysis

4.1 Introduction

The mean residual lifetime (MRL) has been studied by reliabilists, statisticians, survival analysts and others. Many useful results have been derived concerning it. Given a component or a system is of age t, the remaining lifetime after t is random. The expected value of this random residual lifetime is called the mean residual life or mean remaining life. The MRL is often an important criterion for finding an optimal burn-in time for an item.

A review of its theory and applications was given by Guess and Proschan (1988). Over the last two decades, many papers have been written on this subject and thus an update has become necessary.

Let $\bar{F}$ be the survival function of an item with a finite first moment μ and X be the random variable that corresponds to $\bar{F}$ assuming $F(0) = 0$. The residual life random variable at age t, denoted by $X_t = X - t \,|\, X > t$, is simply the remaining lifetime beyond that age. The mean residual life (also known as the mean remaining life) is defined formally as $\mu(t) = E(X - t \; |X > t)$ which can be given as

$$\mu(t) = E\left(X - t \,|\, X > t\right) = \left[\int_t^{\infty} \bar{F}(x)\, dx\right] / \bar{F}(t). \tag{4.1}$$

Clearly

$$\mu(0) = \mu = E(X). \tag{4.2}$$

If F has a density f, we can then alternatively write

$$\mu(t) = \left(\int_t^{\infty} x f(x)\, dx\right) / \bar{F}(t) - t. \tag{4.3}$$

In industrial reliability studies of repair and replacement strategies, the MRL function may prove to be more relevant than the failure (hazard) rate function. The former summarizes the entire residual life distribution, whereas the latter relates only to the risk of immediate failure. In studies of human populations, demographers often refer the MRL under the names of life expectancy or expectation of life. Obviously, the MRL is of vital importance to actuarial work relating to life insurance policies.

In Section 4.2, we investigate how the MRL is related to other ageing properties. Section 4.3 gives the MRLs of several known lifetime distributions. Section 4.4 considers how the MRL may be broadly categorized into two groups: monotonic and non-monotonic MRL and thus various MRL classes are derived. Section 4.5 establishes the relation between non-monotonic MRL and non-monotonic failure rate functions. In Section 4.6, we explore the effect of burn-in to the mean residual life. We consider briefly in Section 4.7 the tests on the mean residual classes and estimates of change points. Section 4.8 discusses some several mean residual life functions that have special characteristics. We consider in Section 4.9 other residual life functions that are not MRL. In Section 4.10, we discuss some stochastic orderings of two lifetime variables based on their the mean residual lives. The relationships between three reliability orderings are reviewed. The concept of univariate mean residual life is extended to bivariate (and multivariate) mean residual life in Section 4.11. A particular form of bivariate mean residual function is discussed in some detail. Finally in Section 4.12, we outline some applications of this ageing characteristic and conclude the chapter with some comments.

4.2 Mean Residual Life and Other Ageing Properties

Theoretical properties of the MRL function $\mu(t)$ were given in Cox (1962), Kotz and Shanbhag (1980), Hall and Wellner (1981) and Bhattacharjee (1982).

It is easy to show that MRL determines the distribution uniquely, see, e.g., Gupta (1975) and Muth (1977).

As in the previous chapters, we let $r(t)$ be the failure rate function defined by

$$r(t) = \frac{f(t)}{\bar{F}(t)}, \;\; f(t) = F'(t). \tag{4.4}$$

It can be shown that

$$\mu'(t) = \mu(t)r(t) - 1. \tag{4.5}$$

We note from the above that $r(0)\mu > 1$ iff $\mu'(0) > 0$ which is equivalent to $\mu(t)$ increases at the beginning. This observation explains why the additional condition stated in Theorem 4.2 in Section 4.5.2 is essential to determine the shape of $\mu(t)$ from a BT class.

By differentiating the last equation, we can verify that

$$r'(t) = \frac{\mu''(t)\mu(t) - (\mu'(t)+1)\mu'(t)}{(\mu(t))^2}, \;\; t \geq 0. \tag{4.6}$$

Since $\log \bar{F}(t) = -\int_0^t r(x)\,dx$, it is to see that

$$\bar{F}(t) = \exp\left\{-\int_0^t r(x)\,dx\right\} \tag{4.7}$$

as given in (2.5) and

$$\bar{F}(t) = \frac{\mu}{\mu(t)} \exp\left\{-\int_0^t \mu(x)^{-1}\,dx\right\},\ t \geq 0. \tag{4.8}$$

as given in (2.7). Differentiating the last equation with respect to t, we obtain the density function that is expressed in terms of the mean residual life function $\mu(x)$

$$f(t) = \frac{\mu\,(\mu'(t)+1)}{\mu(t)^2} \exp\left\{-\int_0^t \mu(x)^{-1}\,dx\right\},\ t \geq 0. \tag{4.9}$$

Thus a life distribution is indeed determined uniquely by its MRL. In particular, F is exponential if and only if its MRL is a constant, i.e., $\mu(t) = \mu$ for all $t \geq 0$.

Limiting properties of the MRL have been studied by Meilijson (1972) and Balkema and De Haan (1974).

Thus the survival function $\bar{F}(t)$, the failure rate function $r(t)$, and the mean residual life function $\mu(t)$ are equivalent in the sense that knowing any one of them, the other two can be determined provided they exist.

4.2.1 Mean Residual Life and its Reciprocity with Failure Rate

By applying the L'Hôsptal's rule to (4.1), Calabria and Pulcini (1987) derived the relationship

$$\lim_{t\to\infty} \mu(t) = \lim_{t\to\infty} \frac{1}{r(t)}, \tag{4.10}$$

provided the latter limit exists and is finite. Then they used (4.5) to conclude that $\lim_{t\to\infty} \mu'(t) = 0$, or equivalently, that

$$\lim_{t\to\infty} r(t)\mu(t) = 1. \tag{4.11}$$

Bradley and Gupta (2003) showed that (4.11) is not true in general and (4.10) does not imply (4.11) unless one assumes that $\lim_{t\to\infty} r(t)$ is finite and strictly positive. A simple counter example is provided by Bradley and Gupta (2003). Consider the linear MRL of the Pareto distribution with $\mu(t) = A + Bt$ so $r(t) = (1 + B)/(A + Bt)$ – see (2.42). It is clear that (4.10) holds whereas (4.11) fails. We observe that in the case here, $\lim_{t\to\infty} r(t) = 0$ which is not strictly positive.

We will consider in Section 4.10 the question whether an ordering of two life distributions with respect to the mean residual life implies the same ordering with respect to the failure rate function.

4.3 Mean Residual Lives of Some Well-known Lifetime Distributions

Unlike the failure rate function $r(t)$, $\mu(t)$ often has a complicated expression for many lifetime distributions. Below are a few that are mathematically tractable. These distributions have been introduced and studied in Chapter 2.3 where reliability properties except the mean residual lives may be found. It may be argued that the MRLs should be presented there and then, but it was felt that they should be given after the concept of the mean residual life has been adequately understood. For the convenience of the reader, we repeat some of the basic formulae here.

Example 4.1: Exponential distribution

Because of the memory-less property, the mean residual life of the exponential distribution $\bar{F}(t) = e^{-\lambda t}$ is equal to its mean, i.e.,

$$\mu(t) = \frac{1}{\lambda}. \tag{4.12}$$

Example 4.2: Gamma distribution

Govil and Aggarwal (1983) have shown that the MRL of the gamma distribution with density

$$f(t) = \lambda(\lambda t)^{\alpha-1}e^{-\lambda t}/\Gamma(\alpha)$$

is given by

$$\mu(t) = \frac{\lambda^{\alpha-1}t^{\alpha}e^{-\lambda t}}{\Gamma(\alpha)\bar{F}(t)} + \frac{\alpha}{\lambda} - t. \tag{4.13}$$

Recall from Section 2.3.2, the gamma distribution F is DFR for $0 < \alpha \le 1$ and IFR for $\alpha > 1$. Since F IFR (DFR) implies DMRL (IMRL), it follows that $\mu(t) \in$ IMRL for $0 < \alpha \le 1$ and $\mu(t) \in$ DMRL for $\alpha > 1$. Of course, we can confirm the shape of $\mu(t)$ directly by differentiation.

Example 4.3: Weibull distribution

The mean residual life of the 2-parameter Weibull distribution $\bar{F}(t) = e^{-(t/\beta)^{\alpha}}$ is rather complex in general but it can be found in Nassar and Eissa (2003). However, for $\alpha = 2$ and $\beta = \sqrt{2}\sigma$, it has a rather neat expression:

$$\mu(t) = \sqrt{2\pi}\sigma\left\{1 - \Phi(t/\sigma)\right\}e^{t^2/2\sigma^2}, \tag{4.14}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal.

Since F is IFR for $\alpha > 1$, it follows that F is DMRL for $\alpha > 1$, i.e., $\mu(t) \in$ D for $\alpha = 2$.

Example 4.4: Inverse Gaussian distribution

$$F(t) = \Phi\left\{\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu} - 1\right)\right\} + e^{2\lambda/\mu}\Phi\left\{-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu} + 1\right)\right\}. \tag{4.15}$$

The MRL is given by (see, e.g., Gupta, 2001):

$$\mu(t) = \frac{(\mu - t)\Phi(\sqrt{\lambda/t})(1 - t/\mu) + (\mu - t)^2 e^{2\lambda/\mu}\Phi(-\sqrt{\lambda/t})(1 + t/\mu)}{\Phi(\sqrt{\lambda/t}(1 - t/\mu)) + e^{2\lambda/\mu}\Phi(-\sqrt{\lambda/t}(1 + t/\mu))}. \tag{4.16}$$

It accentuates how complex a MRL $\mu(t)$ can be.

Example 4.5: Pareto distribution (Lomax distribution)

The Pareto distribution (of the second kind) has survival function

$$\bar{F}(t) = \left[\frac{A}{A + Bt}\right]^{1/B+1}, \tag{4.17}$$

with a slightly different form from (2.41). It is easy to verify that (see, e.g., Oakes and Dasu, 1990)

$$\mu(t) = A + Bt, \tag{4.18}$$

i.e., it has a simple linear MRL. This may partly explain why the Pareto distribution has been widely used for modelling lifetime data.

Example 4.6: Lognormal distribution

Let erfc(t) denote the complimentary error function given by

$$\text{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2}\, dx = \frac{1}{\sqrt{2}}\Phi(\sqrt{2}t). \tag{4.19}$$

Govil and Aggarwal (1983) have shown the the mean residual life MRL is

$$\mu(t) = \frac{\sqrt{e}\text{erfc}[\log(t/e)/\sqrt{2}]}{\text{erfc}(\log t/\sqrt{2})}. \tag{4.20}$$

Example 4.7: Log-logistic distribution

The log-logistic distribution was introduced in Chapter 2. Its failure rate function is

$$r(t) = \frac{k\rho(\rho t)^{k-1}}{1 + (\rho t)^k}. \tag{4.21}$$

Gupta, Akman and Lvin (1999) have shown that

$$\mu(t) = \frac{1}{\rho k}\left[B\left(\frac{1}{k}, 1 - \frac{1}{k}\right) - B_{A(t)}\left(\frac{1}{k}, 1 - \frac{1}{k}\right)\right](1 + (\rho t)^k),\ k > 1, \tag{4.22}$$

where $A(t) = (\rho t)^k/\{1 + (\rho t)^k\}$ and $B_x(p, q) = \int_0^x y^{p-1}(1 - y)^{q-1}\, dy$.

Example 4.8: Exponential-geometric distribution

Recall from Section 2.3, the survival function of the exponential-geometric is

$$\bar{F}(t) = (1-p)e^{\beta t}(1-pe^{\beta t})^{-1},\ t \geq 0.$$

Adamidis and Loukas (1998) derived its mean residual life function as

$$\mu(t) = -(\beta p)^{-1} e^{\beta t}(1-pe^{-\beta t}) \log(1-pe^{-\beta t}),\ \ t \geq 0. \qquad (4.23)$$

4.4 Mean Residual Life Classes

Various classes of life distributions have been defined through MRL. We have dealt with some of these Section in 3.7 as a consequence of BT ageing class because $r(t)$ is very much related to $\mu(t)$. Broadly speaking, we can categorize the mean residual life classes into two groups on the basis of the behaviour of the MRL function: monotonic or non-monotonic.

4.4.1 Monotonic MRL Classes

The decreasing (increasing) mean residual life ageing concept was defined by Definition 2.3. For the ease of cross-referencing within the chapter, we give the definition here again. So in some way, the current subsection can be considered as a supplement to Section 2.4 and Section 2.5.

Definition 4.1: F is said to be DMRL (IMRL) if the mean remaining life function $\mu(t)$ is decreasing (increasing) in t. That is, the older (newer) the device is, the smaller (greater) is its mean residual life and hence $\mu(t)$ is monotonic.

Section 2.5 does not present a discussion on the properties of this class except its closure properties under various reliability operations. We note here however, the DMRL classes are closed under the formation of parallel systems as shown by Abouammoh and El-Neweihi (1986). Furthermore, Abu-Youssef (2002) derived a simple moment inequality

$$\nu_{(2)} \geq (\leq) \frac{\mu^2}{2} \qquad (4.24)$$

if F is DMRL (IMRL) where $\nu_{(r)} = E[\min(X_1, X_2)]^r$.

The above inequality was used by the author to derive a test for testing exponentiality against DMRL (IMRL).

Earlier, Bryson and Siddiqui (1969) proved that IFR (DFR) implies DMRL (IMRL) (see also the chains of implications Section 2.4). They also gave a counterexample to show that DMRL does not imply IFR. Lillo (2000) also provided another counter example as follows.

Example 4.9: A counter example

$$\mu(t) = e^{-(t-t^*)^2}(t - t^* + 1),\ \text{where}\ t^* = \frac{1-\sqrt{3}}{3},\ t \geq 0.$$

Clearly, $\mu(t) \in$ D, but $r(t)$ is not increasing.

A sufficient condition for a DMRL (IMRL) distribution that is also IFR (DFR) is obtained through equation (4.7) giving:

(i) If $\mu(t) \in$ I and is concave, then $r(t) \in$ D.

(ii) If $\mu(t) \in$ D and is convex, then $r(t) \in$ I.

4.4.2 Non-monotonic MRL Classes

There are several ageing notions may be derived from the non-monotonic behavior of $\mu(t)$.

Recall from Definition 2.3, the NBUE class can be defined via the MRL as follows:

F is said to be NBUE if the inequality

$$\int_0^\infty \bar{F}(t+x)\,dx \le \mu\bar{F}(t)$$

holds, i.e.,

$$\int_0^\infty \bar{F}(t+x)\,dx/\bar{F}(t) \le \mu.$$

This last inequality is equivalent to $\mu(t) \le \mu = \mu(0)$ for all $t \ge 0$.

Similarly F is said to be NWUE if $\mu(t) \ge \mu$ for all $t \ge 0$. Thus, DMRL implies NBUE whereas IMRL implies NWUE.

The following important non-monotonic MRL life class was briefly discussed in Section 3.7.3.

Definition 4.2: (Guess et al., 1986). A life distribution with a finite first moment is called an increasing then decreasing mean residual life (IDMRL) if there exists a turning point τ, $0 < \tau < \infty$, such that

$$\mu(s) \begin{cases} \le \mu(t), & \text{for } 0 \le s \le t < \tau; \\ \ge \mu(t), & \text{for } \tau \le s \le t. \end{cases} \tag{4.25}$$

Thus, $F \in$ IDMRL if there exists $0 < \tau < \infty$ such that $\mu(t)$ is increasing on $[0, \tau)$ and decreasing on $[\tau, \infty)$.

If $\tau \to 0$ then IDMRL becomes DMRL; on the other hand, as $\tau \to \infty$, IDMRL becomes IMRL. So our definition of the IDMRL class precludes monotonic MRL.

Remark

In their definition, Guess et al. (1986) assumed that $\tau \ge 0$.

The above non-monotonic MRL class has an obvious dual class associated with it. The dual class 'decreasing initially, then increasing mean residual life' (DIMRL) is obtained by reversing the above inequality (4.25). Again, we impose the condition that $0 < \tau < \infty$ so DIMRL will not include DMRL or IMRL.

With this more restricted definition, we can say that $\mu(t) \in$ UBT (BT) iff to $F \in$ IDMRL (DIMRL).

We note again a more general definition of a BT (UBT) shape which includes a 'flat middle part' was given in the literature, e.g., Mi (1995).

In Section 3.7.3, we also considered another ageing class NWBUE introduced by Mitra and Basu (1994) who defined this ageing notion based on the non-monotonic behaviour of MRL.

Definition 4.3: A life distribution F having support on $[0, \infty)$ (and finite mean μ) is said to be NWBUE (new worse then better than used in expectation) if there exists a point $0 < \tau < \infty$ such that

$$\mu(t) \begin{cases} \geq \mu, & \text{for } t < \tau; \\ \leq \mu, & \text{for } t \geq \tau. \end{cases} \tag{4.26}$$

Similarly, we say F is NBWUE (new better then worse than used in expectation) if there exists a point $\tau > 0$ such that

$$\mu(t) \begin{cases} \leq \mu, & \text{for } t < \tau; \\ \geq \mu, & \text{for } t \geq \tau. \end{cases} \tag{4.27}$$

Note that in the limiting case case $\tau \to 0$, (4.26) and (4.27) become $\mu(t) \leq \mu$ and $\mu(t) \geq \mu$, respectively, for all $t \geq 0$, which corresponds to NBUE and NWUE respectively.

Theorem 4.1 If F is IDMRL(t_0), then F is NWBUE(t_0'), i.e., IDMRL is a subset of NWBUE class.

Proof: Since F is IDMRL, it follows that $\mu(t)$ increases on $[0, t_0)$ and decreases on $[t_0, \infty)$. This is equivalent to $\mu(t)/\mu$ increases on $[0, t_0)$ and decreases on $[t_0, \infty)$. Therefore $\mu(t)/\mu$ increases from 1 at $t = 0$ to $\mu(t_0)/\mu > 1$ at $t = t_0$. From there, $\mu(t)/\mu$ decreases to 1 at $t = t_0' > t_0$ and $\mu(t)\mu \leq 1$ for $t > t_0'$ showing that $F \in$ NWBUE.

In a similar manner, we can show that $F \in$ DIMRL implies $F \in$ NBWUE. Thus, the authors showed that {IDMRL} $\subset$ {NWBUE} and {DIMRL} $\subset$ {NBWUE}.

Mitra and Basu (1996b) showed that the non-monotonic MRL classes NWBUE and NBWUE may also arise from the shock models.

Deshpande et al. (1986) has proposed an ageing class called 'decreasing mean residual life in harmonic average' (DMRLHA).

4.5 Non-monotonic MRL and Non-monotonic Failure Rate

Survival and failure data often cannot be modelled by monotonic failure rate distributions. This is particularly true where the course of a disease is such that the mortality rate reaches a peak after some finite period and then declines slowly. For example, in a study of curability of breast cancer, Langlands et al. (1979) found that the peak mortality occurred after about three years. Bennett (1983) analyzed the same data from the Veterans Administration lung cancer trial presented by Prentice (1973) and showed that the empirical failure rates for both low PS and high PS groups are non-monotonic. (PS here is the abbreviation for potassium sulfide.)

Gupta and Akman (1995a,b,c) showed that the non-monotonic behaviour of MRL is related to the non-monotonic failure rate functions. Their results can be summarized in Section 4.5.2 below.

4.5.1 Non-monotonic Failure Rates Life Distribution

In Chapter 2, we have briefly defined two classes of non-monotonic failure rates that are of particular interest to us. Recall from Section 2.2.1,

(i) $r(t)$ has a bathtub shape, if $r(t)$ is decreasing and then increasing. Equivalently, we say that $r(t) \in \text{BT}$ if

$$r'(t) \begin{cases} < 0, & \text{for } t < \tau; \\ = 0, & \text{for } t = \tau; \\ > 0, & \text{for } t > \tau. \end{cases} \tag{4.28}$$

(ii) $r(t)$ has an upside-down bathtub shape, if $r(t)$ is increasing and then decreasing. Equivalently, we say that $r(t) \in \text{UBT}$ if

$$r'(t) \begin{cases} > 0, & \text{for } t < \tau; \\ = 0, & \text{for } t = \tau; \\ < 0, & \text{for } t > \tau. \end{cases} \tag{4.29}$$

In the definition given above, no provision for a ‘flat’ part is given to the bathtub or upside-down bathtub curve. In fact, it has only one unique turning point. Some authors denote the classes in (i) and (ii) as DIFR and IDFR, respectively. Glaser (1980) referred to the class in (i) as type B and the class in (ii) as type U. A bathtub shaped failure rate function $r(t)$ that has two possible change points is defined in Definition 3.4.

4.5.2 Relations Between MRL and Failure Rate in Terms of Shapes and Locations of Their Change Points

There have been numerous studies conducted on the MRL functions and its applications. Over the last few years BT and UBT shaped MRL have attracted much attention.

It is well known that the IFR class is contained in the DMRL (decreasing mean residual lifetime) class. A DMRL distribution need not be an IFR. One may conjecture that a BT failure rate is related to the class of life distributions whose mean residual life functions $\mu(t)$ are increasing and then decreasing (that is, each mean residual life in the class has an upside-own bathtub shape). Indeed this relationship has been observed empirically as pointed by Rajarshi and Rajarshi (1988) who said in their review article "It can be observed from the life-tables of human and animal populations that the shape of the empirical MRL function is upside bathtub."

The following results give a general relation between the non-monotonic failure rate $r(t)$ and the non-monotonic mean residual life $\mu(t)$.

Theorem 4.2: Suppose $r(t)$ is of the type BT. Then

(i) $\mu(t) \in$ D if $r(0) \leq 1/\mu$,
(ii) $\mu(t)$ is of UBT shape if $r(0) > 1/\mu$.

On the other hand, if $r(t)$ is of the type UBT, then

(i) $\mu(t) \in$ I if $r(0) \geq 1/\mu$,
(ii) $\mu(t)$ is of the type BT if $r(0) < 1/\mu$.

Proof: A proof was given in Gupta and Akman (1995a,b). However, it seems more elegant to prove the theorem by utilizing the suggestion made after the proof of Theorem 2.12 in Section 2.11. Essentially we use the relationship between the shape of the eta function $\eta_2(t)$ of the equilibrium distribution and the shape of the failure rate function $r_2(t)$ of the same equilibrium distribution. Recall, in (2.84), we defined the equilibrium distribution of a lifetime distribution F by

$$E_F(t) = \int_0^t \bar{F}(x)\,dx/\mu_F,$$

and the survival equilibrium function by (2.85), that is,

$$\bar{E}_F(t) = 1 - E_F(t) = \int_t^\infty \bar{F}(x)\,dx/\mu_F.$$

The Glaser's eta function, as defined in (2.9) for the equilibrium distribution, is given by $\eta_2(t) = -\dfrac{E_F''(t)}{E_F'(t)} = r(t)$. The failure rate function that corresponds to E_F is $r_2(t) = \dfrac{1}{\mu(t)}$. Now the theorem follows immediately from (c) and (d) of Theorem 2.1 and Lemma 2.1. Note that in our current notation, the δ in Lemma 2.1 is defined as $\lim_{t\to 0}(1/r_2(t))\eta_2(t) = \lim_{t\to 0}\mu(t)r(t) = r(0)\mu$.

We note that the first part of the above theorem was also proved in Mi (1995) although no explicit conditions on $r(0)\mu$ are imposed. This is because his definition of a bathtub shape includes the extreme cases I or D shapes.

Remark

If follows from the remark that follows (4.5) that $r(0)\mu > 1 \Leftrightarrow \mu'(0) > 1$.

We now present the following results which determine the locations of the turning points of $\mu(t)$ when $\mu(t) \in$ BT (UBT).

Theorem 4.3: Let F be a continuous BT life distribution with a change point t^* and let $r(t)$ be differentiable. If $r(0)\mu > 1$, then $\mu(t) \in$ UBT with a unique change point $k^* \in (0, t^*]$, i.e., $k^* < t^*$.

Proof: In the proof Theorem 2.1 (c), it was shown that the change point of $r(t)$ occurs before the change point of $\eta(t)$. Since the relationship between $\eta_2(t) = r(t)$ and $r_2(t) = 1/\mu(t)$ is the same as the one between $\eta(t)$ and $r(t)$, a direct translation proves the theorem immediately. Note again that the condition $\delta > 1$ given in Lemma 2.1. is equivalent to $r(0)\mu > 1$ in our present context.

Mi (1995) has earlier also shown that the change point of a bathtub shape $r(t)$ is greater than or equal to the change point of its corresponding upside-down $\mu(t)$. Guess et al. (1998) have also proved the same theorem.

Theorem 4.4: Suppose $r(t) \in$ UBT and differentiable. If $r(0)\mu < 1$, then $\mu(t) \in$ BT with a unique change point $k^* \in (0, t^*]$, i.e., $k^* < t^*$. Otherwise (i.e., if $r(0)\mu > 1$), then it is an IMRL distribution.

Proof: The proof is analogous to the preceding one except we now invoke (d) of Theorem 2.1. See also Gupta (1995a,b), Guess et al. (1998) and Tang et al. (1999).

The preceding two theorems show that the change point of a non-monotonic $\mu(t)$ always precedes the change point of its corresponding $r(t)$. They also show us how the shape of the mean residual life function $\mu(t)$ is determined by the shape of the failure rate function. We now discuss how the shape of $\mu(t)$ will determine the shape of its counter part $r(t)$. For convenience, we relabel (4.7) as

$$r'(t) = \frac{\mu''(t)\mu(t) - (\mu'(t) + 1)\mu'(t)}{(\mu(t))^2}, \ t \geq 0. \tag{4.30}$$

Theorem 4.5: Let $\mu(t)$ be a MRL and $r(t)$ be its corresponding failure rate function. Then

(i) If $\mu(t)$ is decreasing and convex for $t \geq 0$, then $r(t)$ is strictly increasing for $t \geq 0$.

(ii) If $\mu(t)$ is increasing and concave for $t \geq 0$, then $r(t)$ is strictly decreasing for $t \geq 0$.

Proof: Recall, a function $h(t)$ is said to be convex if $h''(t) > 0$ and concave $h''(t) < 0$. Then (i) and (ii) follow immediately from (4.30).

Result (i) was also proved by Kupka and Loo (1989).

The following theorem is due to Lillo (2000).

Theorem 4.6: Suppose $\mu(t)$ has a minimum at $t = t_0$, then $r'(t) > 0$ for some interval $[t_0, t_0 + \varepsilon]$ with $\varepsilon > 0$; that is, both $\mu(t)$ and $r(t)$ are increasing at the same time in that interval.

Proof: The assumption that $\mu(t)$ has a minimum at $t = t_0$ implies that $\mu'(t_0) = 0$ and $\mu''(t_0) > 0$. It follows from the preceding equation that

$$r'(t_0) = \frac{\mu''(t_0)}{\mu(t_0)} > 0.$$

By continuity, $r'(t) > 0$ at least, for some interval $[t_0, t_0 + \varepsilon]$; i.e., $r(t)$ is increasing in that interval.

Remarks: By the an analogous proof, we can show that if $\mu(t)$ has a maximum value at t_0, then both $\mu(t)$ and $r(t)$ are decreasing in $[t_0, t_0 + \varepsilon], \varepsilon > 0$.

Mi (1995) has shown that if a device has a bathtub shaped failure rate, then its MRL is unimodal (if $r(0)\mu > 1$ though he did not specify this condition) but the converse doe not hold. He demonstrated by the following example to show that a $\mu(t)$ with a UBT shape need not have a BT shaped $r(t)$.

Example 4.10: Consider the following distribution F with MRL given by

$$\mu(t) = \begin{cases} t^2 + 1, & \text{if } 0 \le t \le 1; \\ 2t, & \text{if } 1 \le t \le 2; \\ 4\exp\left(-\frac{1}{4}(t-2)\right), & \text{if } t > 2. \end{cases}$$

It is easy to verify that $\mu(t)$ has an upside-down bathtub shape (i.e., $\mu(t) \in$ UBT) so F is IDMRL. It follows from the above equation that

$$r(t) = \begin{cases} (2t+1)/(t^2+1), & \text{if } 0 \le t \le 1; \\ 3/(2t), & \text{if } 1 \le t \le 2; \\ {}^1\!/_4 \exp\left({}^1\!/_4(t-2) - 1\right), & \text{if } t > 2; \end{cases}$$

which does not have a bathtub shape. Note however, in this example, we have $r(0)\mu = 1$.

Ghai and Mi (1999) developed sufficient conditions for unimodal MRL (i.e., $\mu(t) \in$ UBT) to imply $r(t) \in$ BT. However, their definition of a bathtub (upside-down) shaped function allows two change points as in Definition 3.4. In particular, if the two change points are the same, then $g(t) \in$ BT as defined in Section 2.2.1. They also defined a UBT curve in the following manner:

Definition 4.4: A function $k(t) \neq 0$ defined on $[0, \infty]$ has an upside-down bathtub shape if $g(t) = 1/k(t)$ has a bathtub shape. In particular, if the two change points of $1/k(t)$ are equal, then $k(t) \in$ UBT.

Note: In their original definition given in Mi (1995), $t_1 = 0$ or $t_2 = \infty$, or both $t_1 = 0$ and $t_2 = \infty$ are not precluded.

As we mentioned above, Ghai and Mi (1999) gave sufficient conditions under which the bathtub (upside-down bathtub) shaped MRL $\mu(t)$ implies that its associated failure rate has an upside-down bathtub (bathtub) shape.

Theorem 4.7: Let t_0 be the unique change point of $\mu(t) \in$ BT (i.e., F is IDMRL). Suppose there exists $\tau_0 \in [t_0, \infty)$ such that $\mu(t)$ is concave on $[0, \tau_0)$ and convex on $[\tau_0, \infty]$. If $\mu'(t)$ is convex on $[t_0, \tau_0)$, then one of the following alternatives is true for $r(t)$:

(1) $r(t)$ exhibits a bathtub shape (in the sense above) that has two change points, says $t_1 < t_2$; where $t_0 \leq t_1 < t_2 \leq \tau_0$.

(2) $r(t)$ exhibits a bathtub shape that has a unique change point, say t^*; where $t_0 \leq t^* \leq \tau_0$.

Proof: $\mu'(t) > 0$ on $[0, t_0]$ because $\mu(t)$ is UBT with change point t_0. Next, $\mu''(t) < 0$ because $\mu(t)$ is concave on $[0, \tau_0]$ which contains $[0, t_0]$.

From (4.30), it follows that $r'(t) < 0$ for all $t \in (0, t_0)$, i.e., $r(t)$ strictly decreases on $[0, t_0]$. Similarly, $r(t)$ strictly increases on $[\tau_0, \infty)$. Define $\phi(t) = r'(t)\mu(t)^2$ so from (4.30), we have

$$\phi(t) = \mu''(t)\mu(t) - [1 + \mu'(t)\mu(t)].$$

Now, let

$$J \equiv \{t_0 < t < \tau_0 : \phi(i) = 0\}.$$

We now claim that J must be a connected set in the sense that if both $s_1 < s_2$ belong to J, then $[s_1, s_2] \subseteq J$. To see this, we note that

$$\phi'(t) = \mu'''(t)\mu(t) - [1 + \mu'(t)]\mu''(t).$$

On (t_0, τ_0), $\mu''(t) \leq 0$ because

- $\mu(t)$ is concave on $[t_0, \tau_0)$,
- $\mu'''(t) \geq 0$ since $\mu'(t)$ is assumed to be convex on (t_0, τ_0).

Furthermore, (4.6) implies that $1+\mu'(t) \geq 0$ for any MRL function $\mu(t)$. Thus, $\phi'(t) \geq 0$ on (t_0, τ_0): $\phi(t)$ is increasing in $t \in (t_0, \tau_0)$. From the assumption $s_1, s_2 \in J$ and the definition of J, it follows that any $s \in [s_1, s_2]$ must satisfy $\phi(s) = 0 : s \in J$ and consequently $[s_1, s_2] \subseteq J$.

As J is a connected set and because $\phi(t)$ is continuous on (t_0, τ_0), we know that J is either a closed subinterval of (t_0, τ_0), or $J = (t_0, \tau_0)$. Three cases are considered:

Case (i): $J = [t_1, t_2]$ with $t_0 < t_1 < t_2 < t_0$.

Without giving details, it can be shown that $r(t)$ exhibits a bathtub shape with change points $t_1 < t_2$ that satisfy $t_0 < t_1 < t_2 < \tau_0$.

Case (ii): $J = (t_0, \tau)$.

We can verify in the same manner as in case (i) that $r(t)$ has a bathtub shape with change points $t_1 = t_0$ and $t_2 = \tau$.

Case (iii): $J = \{t^*\}$ is a singleton with $t_0 < t^* < \tau_0$.

In the same manner as in Case (i), it can be verified that $r(t)$ has a bathtub shape with a unique change point $t^* \in (t_0, \tau_0)$. See Ghai and Mi (1999) for missing gaps in the proof.

Theorem 4.8: Let t_0 be the unique change point of $\mu(t) \in$ UBT (i.e., F is DIMRL). Suppose there exists $\tau_0 \in [t_0, \infty)$ such that $\mu(t)$ is convex on $[0, \tau_0)$ and concave on $[\tau_0, \infty]$. If $\mu'(t)$ is convex on $[t_0, \tau_0)$, then only one of the following two alternatives is true for $r(t)$:

(1) $r(t)$ exhibits an upside-down bathtub shape (in the sense above) that has two change points, says $t_1 < t_2$; where $t_0 \le t_1 < t_2 \le \tau_0$.

(2) $r(t)$ exhibits an upside-down bathtub shape that has a unique change point, say t^*; where $t_0 \le t^* \le \tau_0$.

Proof: The proof is similar to Theorem 4.7. See Ghai and Mi (1999) for details.

Remarks

Many burn-in problems are based on the behaviour of $r(t)$ of a device. Usually it is inappropriate to burn-in an IFR device (i.e., $r(t) \in$ I). Burn-in is particularly beneficial when $r(t)$ exhibits either a BT shape or $r(t) \in$ D. Thus it is important to check whether a lifetime distribution has a BT shaped failure rate based on its lifetime data. In their Remark 2, Ghai and Mi (1999) suggested that "it would be nice if the practitioners could graph the FR (failure) rate and then visualize its trend. However, the estimation of FR is not very stable. On the other hand, since the statistical properties of estimated means are better than those derivatives (which enter into the FR), the estimated MRL is much more stable than estimated FR. Therefore, we can use the information provided by the graph of the estimated MRL to check whether the underlying lifetime distribution is unimodal (or bathtub shaped) MRL." Theorem 4.7 and Theorem 4.8 above provide us more information on whether the underlying distribution has a bathtub or upside-down bathtub shaped failure rate; thus we can apply burn-in procedure to improve the quality of products with respect to different criteria.

Examples of non-monotonic MRL with non-monotonic failure rate

The first three of following examples are found in Gupta and Akman (1995a); the others can be found in Section 2.3. See also Muth (1980).

Example 4.11:

$\mu(t) = \frac{1}{1+2.3t^2}, t \ge 0$ is a decreasing function in t (DMRL). It can be shown that

$r(t) = \frac{(1+2.3t^2)^2 - 4.6t}{1+2.3t^2}, t \ge 0$ has a BT shape and that $r(0) \le 1/\mu$.

Example 4.12:

$$\mu(t) = \tfrac{1}{4}(1-t)(1+2t), \qquad r(t) = \tfrac{5-4t}{(1-t)(1+2t)},\ 0 \le t < 1.$$

In this case, $r(t)$ is of type BT such that $r(0) > 1/\mu$ and $\mu(t)$ is of type UBT.

Example 4.13:

$$\mu(t) = 1 + t^2, \quad r(t) = \frac{1+2t}{1+t^2},\ 0 \le t \le \pi/2.$$

In this case, $r(0) \geq 1/\mu$, $r(t)$ is of type U and $\mu(t)$ is strictly increasing (IMRL).

Several well known lifetimes distributions that have bathtub shaped $\mu(t)$ (i.e., F is IDMRL) are now given below.

Example 4.14: Inverse Gaussian

$\mu(t)$ is given in Section 4.3. It is well-known that $\mu(t) \in$ BT or F is DIMRL.

Example 4.15: Lognormal

$\mu(t)$ is given in Section 4.3. It has been shown that $\mu(t) \in$ BT or F is DIMRL.

Example 4.16: Log-logistic distribution

It is shown that $r(t)$ of the log-logistic distribution is of type UBT whereas $\mu(t)$ is of type BT when the shape parameter $k > 1$. (See Gupta, Akman and Lvin, 1999).

Again, care needs to be taken as some authors, e.g., Mi (1995) included a flat part in his definition of a bathtub or upside-down bathtub shape whereas others, like Gupta and Akman (1995a,b), do not allow a flat part to be included in their definitions.

The following two examples have BT shape failure rates. We now investigate the shapes of their MRLs.

Example 4.17: Exponential power distribution

This was considered in Section 3.4 earlier having

$$r(t) = \lambda\alpha(\lambda t)^{\alpha-1}\exp[(\lambda t)^{\alpha}],\ \alpha > 0, \lambda > 0.$$

When $\alpha < 1$, $r(t) \in$ BT. In particular it has an approximately flat bathtub when $\alpha = 1/2$, see for example, Dhillon (1981). Since $r(0) = \infty$ (for $\alpha < 1$) so that $r(0)\mu > 1$. Thus by Theorem 4.2, $\mu(t)$ is of UBT shape.

Example 3.17: Hjorth Model

$$r(t) = \delta t + \theta/(1 + \beta t).$$

When $0 < \delta < \theta\beta$, $r(t) \in$ BT. See Hjorth (1980) for other details concerning this distribution. Now, $r(0) = \theta$ so by Theorem 4.2, F is DMRL if $\mu \leq 1/\theta$ and F is IDMRL if $\mu > 1/\theta$. Note that μ does not have an explicit expression so a numerical algorithm is required to compute it. Guess et al. (1998) have obtained the change points for both $r(t)$ and $\mu(t)$ for various combinations of three parameter values.

A Summary

Table 4.1 gives a summary of the shapes of $\mu(t)$ of several well known distributions (the table is similar to the one given by Tang et al., 1999).

Table 4.1. Shapes of MRL for various distributions

Distributions	IMRL	DMRL	DIMRL	IDMRL
Weibull	Yes	Yes	No	No
$r(t) = \alpha\lambda^\alpha t^{\alpha-1}$	$\alpha < 1$	$\alpha > 1$	No	No
Lognormal	No	No	Yes	No
$F(t) = \Phi\left\{\frac{\log t - \alpha}{\sigma}\right\}$				
Birnbaum-Saunders	No	Yes	Yes	No
$F(t) = \Phi\left\{\frac{1}{\alpha} \cdot \left[\left(\frac{t}{\beta}\right)^{1/2} - \left(\frac{t}{\beta}\right)^{-1/2}\right]\right\}$		$\alpha \to 0$	$\alpha > 0.6$	
Inverse Gaussian	No	No	Yes	No
$f(t) = \sqrt{\frac{\lambda}{2\pi t^3}} \exp\left[-\frac{\lambda}{2\mu^2 t}(t-\mu)^2\right]$				
Log-logistic	Yes	No	Yes	No
$r(t) = \frac{k\rho(\rho t)^{k-1}}{1+(\rho t)^k}$	$k \le 1$		$k > 1$	
Exponential power	No	Yes	No	Yes
$r(t) = \beta\alpha(\beta t)^{\alpha-1} \exp[(\beta t)^\alpha]$		$\alpha \ge 1$		$\alpha < 1$
Hjorth model ($0 < \delta < \theta\beta$)	No	Yes	No	Yes
$r(t) = \delta t + \theta/(1 + \beta t)$		$\mu\theta \le 1$		$\mu\theta > 1$

4.5.3 A General Approach Determining Shapes of Failure Rates and MRL Functions

Recall in Section 2.2.1, we have discussed the Glaser's method of determining the shape of $r(t)$ via the density function by defining

$$\eta(t) = -\frac{f'(t)}{f(t)}. \tag{4.31}$$

This idea was generalized by Block et al. (2002) by considering the ratio of two well behaved functions. Their consideration was motivated by a realization that most reliability functions of interest are naturally expressed or can be cleverly expressed as the ratio of two simple functions.

Let

$$G(t) = \frac{N(t)}{D(t)}, \quad -\infty \le a < t < b \le \infty, \tag{4.32}$$

where $N(t)$ and $D(t)$ are continuously differentiable with $D(t)$ positive and strictly monotone on (a, b). The authors showed that the shape of $G(t)$ is very much dependent on the shape of $\eta(t)$ defined by

$$\eta(t) = \frac{N'(t)}{D'(t)} \tag{4.33}$$

as well on the monotonicity of $D(t)$ together with the sign of $L(t) = \eta(t)D(t) - N(t)$. This result has an application in linking the shape of $\mu(t)$ to $r(t)$ by selecting an appropriate pair $N(t)$ and $D(t)$.

Consider a special case where

$$\eta(t) = 1/r(t) = \frac{\bar{F}(t)}{f(t)} \tag{4.34}$$

so

$$G(t) = \frac{N(t)}{D(t)} = \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x)\, dx. \tag{4.35}$$

It is clear that $\mu(t) = G(t)$ so the shape of MRL $\mu(t)$ can be determined through the failure rate function $r(t)$. In particular, Theorem 4.2 will follow immediately from the result of Block et al. (2002). On the other hand, this special case also follows from Theorem 2.12 which shows that the shape of the η function of the sth-order equilibrium distribution will determine the shape of the η function of the $(s+1)$th order equilibrium distribution.

The results of Block et al. (2002) give further insights on the shape of other reliability measures.

Recall, the residual life time of a component which has survived t units of time is $X_t = X - t|\, X > t$. In addition to the mean residual life, one may also be interested to consider the residual variance defined by

$$\sigma^2(t) = E(X_t^2) - \mu^2(t) = \frac{2}{\bar{F}(t)} \int_t^\infty \bar{F}(x)\mu(x)\, dx - \mu^2(t). \tag{4.36}$$

Assuming the variance of lifetime is finite, Block et al. (2002) have shown if a positive failure rate function $r(t)$ has a BT shape with possible two change points, then the residual variance function $\sigma^2(t)$ has an UBT shape and the change point occurs before the change point of $\mu(t)$.

Lynn and Singpurwalla (1997) viewed the burn-in concept as a process of reduction of uncertainty of the lifetime of a component. One approach to this is to minimize $\sigma(t)$. Combining this with maximizing the mean residual life leads Block et al. (2002) to consider balancing mean residual life and residual variance through minimizing the the residual coefficient of variation

$$CV(t) = \sigma(t)/\mu(t). \tag{4.37}$$

They showed that for a BT distribution, if $r(t)\mu(t)$ has a bathtub shape, so has the residual coefficient of variation $CV(t)$. The proof of this result was achieved by setting $N(t) = \bar{F}(t)\mu^2(t)$ and $D(t) = 2\int_t^\infty \bar{F}(x)\mu(x)\, dx$ so that $\eta(t) = N'(t)/D'(t) = 1 - r(t)\mu(t)/2$ in (4.32). In this case, the optimal burn-in time for the objective function $CV(t)$ occurs after the change point of $\mu(t)$. This optimal burn-in time may be before or after the first change point of $r(t)$.

4.5.4 Roller-Coaster Failure Rates and Mean Residual Lives

A roller-coaster curve was defined in Section 2.2.1. Suppose $r(t)$ has a roller-coaster shaped failure rate and we wish to establish the shape of its corresponding $\mu(t)$. Bekker and Mi (2003) gave the following results.

Theorem 4.9: Suppose that the failure rate function $r(t)$ is differentiable and has roller-coaster shape with change points $\{t_1, ..., t_k\}$. Let $\mu(t)$ be the associated MRL function of this distribution. Then the following are true:

1. $\mu(t)$ is strictly monotone on $[t_k, \infty)$ and $\mu(t)$ does not have t_k as its change point.
2. $\mu(t)$ has at most one change point in each of (t_{j-1}, t_j), $1 \le j \le k$, where $t_0 = 0$.
3. None of the t_j, $1 \le j \le k$, can be a change point of $\mu(t)$, even if it is a critical point of $\mu(t)$, i.e., $\mu'(t_j) = 0$.
4. $\mu(t)$ has at most k change points and all its change points must be in some of the open intervals (t_{j-1}, t_j), $1 \le j \le k$.

Proof: The statement of the Theorem and its proof were given in Theorem 2.4 of Bekker and Mi (2003). The nature of the proof is simple but rather tedious.

To prove (1), we need to show that if $r(t)$ strictly increases on $[\tau, \infty)$, then $\mu(t)$ strictly decreases in $t \ge \tau$; if $r(t)$ strictly decreases on $[\tau, \infty)$, then $\mu(t)$ increases in $t \ge \tau$. To see this, we recall that

$$\mu(t) = \frac{\int_t^\infty \bar{F}(x)\,dx}{\bar{F}(t)}.$$

Let

$$A(t) = r(t) \int_t^\infty \bar{F}(x)\,dx - F(t) \tag{4.38}$$

so

$$\mu'(t) = \frac{A(t)}{\bar{F}(t)}. \tag{4.39}$$

Now, assuming $r(t)$ strictly increases in $t \ge \tau$. For any $t \ge \tau$, from (4.38), we have

$$\begin{aligned} A(t) &= \int_t^\infty r(t)\bar{F}(x)\,dx - \bar{F}(t) \\ &< \int_t^\infty r(x)\bar{F}(x)\,dx - \bar{F}(t) \\ &= \int_t^\infty f(x)\,dx - \bar{F}(t) = 0. \end{aligned}$$

It follows from (4.39) that $\mu'(t) > 0$ for all $t \ge \tau$. The case when $r(t)$ strictly decreasing on $[t, \tau)$ can be similarly proved. From this result, (1) follows immediately.

To prove (2), we need to show first that if $r'(t) > 0$ in (τ_1, τ_2) and $\mu'(\tau^*) = 0$, $\tau^* \in (\tau_1, \tau_2)$, then $\mu(t)$ has a bathtub shape in (τ_1, τ_2) and achieves its minimum value on $[\tau_1, \tau_2]$ at $t = \tau^*$. From (4.38), we have

$$A'(t) = r'(t) \int_t^{\infty} \bar{F}(x)\, dx \tag{4.40}$$

so $A'(t) > 0$ for all $t \in (\tau_1, \tau_2)$ because $r'(t) > 0$ in this interval. That is, $A(t)$ strictly increases in $t \in (\tau_1, \tau_2)$. As $\mu'(t^*) = 0$, it follows from (4.39) that $A(\tau^*) = 0$. Hence $A(t) < 0$ for $t \in (\tau_1, \tau^*)$ and $A(t) > 0$ for $t \in (\tau^*, \tau_2)$. This is equivalent to say that $\mu'(t) < 0$ for $t \in (\tau_1, \tau^*)$ and $\mu'(t) > 0$ for $t \in (\tau^*, \tau_2)$. Therefore, $\mu(t)$ strictly decreases in $t \in (\tau_1, \tau^*)$, strictly increases in $t \in (\tau^*, \tau_2)$, has a bathtub shape on $t \in (\tau_1, \tau_2)$, and achieves its minimum value on $[\tau_1, \tau_2]$ at $t = \tau^*$.

In the same manner, we can show that if $r'(t) < 0$ in (τ_1, τ_2) and $\mu'(\tau^*) = 0$, $\tau^* \in (\tau_1, \tau_2)$, then $\mu(t)$ has an upside-down bathtub shape in (τ_1, τ_2) and achieves its maximum value on $[\tau_1, \tau_2]$ at $t = \tau^*$. We now conclude that in any case, $\mu(t)$ does not have any critical point in (τ_1, τ_2) other than τ^*. Letting $\tau_1 \equiv t_{j-1}$ and $\tau_2 \equiv t_j$ we prove the statement (2).

To prove (3), we need to show that if $r'(t) < 0$ for $t \in (\tau_1, \tau^*)$, $r'(\tau^*) = 0$, $r'(t) > 0$ for $t \in (\tau^*, \tau_2)$ with $\mu(t^*) = 0$; then $\mu(t)$ strictly increases in $t \in (\tau_1, \tau_2)$, i.e., τ^* is a critical point but not a change point of $\mu(t)$. Now from (4.40), we see that $A'(t) < 0$ for $t \in (\tau_1, \tau^*)$. This implies that $A(t)$ strictly decreases in $t \in (\tau_1, \tau^*)$. Note that $A(t^*) = 0$, since $\mu'(t^*) = 0$. Hence $A(t) > 0$ for $t \in (\tau_1, t^*)$. This means $\mu'(t) > 0$ for $t \in (\tau_1, \tau^*)$, or $\mu(t)$ strictly increases in $t \in (\tau_1, t^*)$. On the other hand, $A'(t) > 0$ for $t \in (\tau^*, \tau_2)$ and so $A(t)$ strictly increases $t \in (\tau^*, \tau_2)$. This in turn implies that $A(t) > 0$ and so $\mu'(t) > 0\; t \in (\tau^*, \tau_2)$. Consequently, $\mu(t)$ strictly increases in $t \in (\tau^*, \tau_2)$. Thus $\mu(t)$ strictly increases in $t \in (\tau_1, \tau_2)$. Similarly, we can show that $\mu(t)$ strictly decreases if $r'(t) > 0$ for $t \in (\tau_1, \tau^*)$, $r'(\tau^*) = 0$, $r'(t) < 0$ for $t \in (\tau^*, \tau_2)$ with $\mu(t^*) = 0$. Next, assume $j \leq k - 1$, since the case when $j = k$ is considered in (1). Let $\tau_1 \equiv t_{j-1}$, $\tau^* \equiv t_j$, and $\tau_2 = t_{j+1}$. We see from the result just proved that $t_j = \tau^*$ cannot be a change point of $\mu(t)$ for $1 \leq j \leq k - 1$. Thus, result (3) is proved.

Finally, combining results (1)–(3), we see the result (4) is true.

Tang et al. (1999) gave a sufficient condition so that a roller-coaster shaped $r(t)$ will have a roller-coaster shaped $\mu(t)$. Bekker and Mi (2003) have shown that this condition is also necessary.

Theorem 4.10: Suppose that the failure rate function $r(t)$ is differentiable and has a roller-coaster shape with change points $\{t_1, ..., t_k\}$. Let $\mu(t)$ be the associated MRL function. Then the necessary and sufficient condition for $\mu(t)$ to have a unique change point in (t_{j-1}, t_j), $1 \leq j \leq k$, is

$$[\mu(t_{j-1})r(t_{j-1}) - 1][\mu(t_j)r(t_j) - 1] < 1. \tag{4.41}$$

Proof: See Theorem 2.6 of Bekker and Mi (2003) and Tang et al. (1999).

We first show the sufficiency of the condition (4.41). Noting from (4.5), i.e., $\mu'(t) = -1 + r(t)\mu(t)$, we see that the condition (4.41) implies that $\mu'(t_{j-1})\mu'(t_j) < 0$. That is, $\mu'(t_{j-1})$ and $\mu'(t_j)$ have different signs. Thus, there exists $t^* \in (t_{j-1}, t_j)$, such that $\mu'(t^*) = 0$. By part (2) of Theorem 4.9, t^* must be a unique change point of $\mu(t)$ in (t_{j-1}, t_j).

We leave out the proof of the necessity of the condition (4.41) as it is quite involved and the reader can find the complete proof from the source references.

Following almost the same approach as Block et al. (2002) which we discussed in the Section 4.5.3, Mi (2004) defined two similar functions with each expressed as a ratio except the denominator is now an integral of the numerator. More explicitly, they are

$$\eta_1(t) = \frac{N_1(t)}{D_1(t)} = \frac{N_1(t)}{\int_t^\infty N_1(x)\,dx} \tag{4.42}$$

and

$$\eta_2 = \frac{N_2(t)}{D_2(t)} = \frac{D_1(t)}{\int_t^\infty N_2(x)\,dx}. \tag{4.43}$$

The first function $\eta_1(t)$ is analogous to $\eta(t)$ in (4.32) whereas $\eta_2(t)$ is similar to $G(t)$ in (4.31). Mi (2004) investigated the shape of $\eta_2(t)$ based on $\eta_1(t)$. Letting $\eta_1(t) = -f'(t)/f(t)$ so that $\eta_2(t) = r(t)$, he proved Theorem 2.2 which was established earlier in Gupta and Warren (2001). On the other hand, by letting $\eta_1(t) = 1/r(t) = f(t)/\bar{F}(t)$ so that $\eta_2(t) = 1/\mu(t)$, he proved Theorem 4.10 above.

4.6 Effect of Burn-In on Mean Residual Life

The burn-in concept was discussed in Section 3.8 in relation to the BT shaped failure rate distributions. The method is used to screen out defective components before they are delivered to customers or put into field operation. In this chapter, our discussion on burn-in stems from a mean residual life perspective. Since the shapes of $r(t)$ and $\mu(t)$ are intimately connected, some repetitions are unavoidable. For an earlier account of burn-in, see Jensen and Petersen (1982).

Let b be the burn-in time, which is the length of the burn-in period. The question arises as when the burn-in process should be stopped. This obviously depends on what reliability goal or criteria one wishes to achieve. The best time to stop the burn-in for a given criterion to be optimized is called the optimal burn-in time. After the burn-in (assuming the product survives the test), it is the remaining lifetime that will be of interest to most people. The mean of the remaining lifetime is the mean residual life $\mu(t)$. Suppose the cost is not to be considered, what we really want to achieve is simply the longest MRL, then the optimal burn-in time b^* satisfies the following condition

$$\mu(b^*) = \max_{b \geq 0} \mu(b) \tag{4.44}$$

which has been given earlier by (3.42). Now, b^* is the unique change point if $\mu(t) \in$ UBT. Part (ii) of Theorem 4.2 shows that a BT distribution with $r(0)\mu > 1$ will give rise to a UBT shaped $\mu(t)$.

As noted by many authors, a bathtub shaped failure rate phenomenon arises from several different modes of failure: initial failure due to initial defects, a middle constant failure rate and a wear-out failure. Here, we use Definition 3.4 for a bathtub (upside-down) shaped function that permits a 'flat' middle portion. As mentioned in Block and Savits (1997), mixing two or more different subpopulations often gives rise to bathtub shaped failure rates. It is shown in Glaser (1980) that the mixture of two gamma distributions having a common scale parameter but with different shape parameters $\alpha_1 = 1$ and $\alpha_2 > 2$ gives rise to a bathtub distribution.

It is popularly believed that as the burn-in period increases, the failure probability of a product surviving the burn-in tends to decrease, until the beginning of the constant failure rate region of the curve (Dhillon,1983, p. 23).

Park (1985) questioned this belief by examining the effect of burn-in on the mean residual life of a product. He found that the time at which a bathtub failure rate is minimum does not maximize the mean residual life $\mu(t)$. The mean residual life in the constant failure rate region of a bathtub failure rate curve is not a constant. In fact, it follows from Theorem 4.3 that the change point of $\mu(t)$ precedes that of $r(t)$.

Mi (1995) presented an optimal burn-in policy to maximize the MRL when the underlying life distribution has a bathtub shaped failure rate with change points τ_1, τ_2. We bear in mind that in this paper, he has assumed a BT class to include both IFR and DFR classes. His conclusions were given by Theorem 3.3 and we list them here again for the convenience of the reader:

- If $\tau_1 = 0$ (hence F is IFR), then set $b^* = 0$ and obviously there is no need to burn-in.
- If $\tau_1 > 0$ but $\tau_2 = \infty$, we can always choose $b^* = \tau_1$.
- If $0 < \tau_1 \leq \tau_2 < \infty$, the optimal burn-in time b^* must be equal to the unique change point of $\mu(t)$ and $b^* \in [0, \tau_1]$.
- If $\tau_1 = \tau_2 = \infty$ (F is DFR), cost should be taken into consideration in determining b^*.

Cha (2000) proposed a new burn-in procedure for a repairable component. During the burn-in period, the failed component is only minimally repaired rather than being completely repaired. This procedure was shown to be economical and efficient when the minimal repair method is applicable during a burn-in process. The properties of the optimal burn-in time b^* and block replacement policy T^* were also given.

For other detailed literature reviews of burn-in models and methods, see for example, Leemis and Beneke (1990), Tang and Tang (1994), and Kececioglu and Sun (1997).

4.6.1 Optimal Burn-in Criteria

An early quantitative approach for determining optimal burn-in periods was given by Kuo (1984). There are various criteria for 'optimality' depends the objective of the user.

(i) Traditionally, the failure rate is essential to burn-in decision because of its interpretation as the force of mortality (Chang, 2000).
(ii) Mi (1994b) used the maximization of a survival probability for a given mission time as a criterion for an optimal burn-in time.
(iii) Mi (1994c) determined the optimal burn-in time by minimizing the costs associated with maintenance policies and burn-in.
(iv) Mi (1995) presented an optimal burn-in policy to maximize the MRL.
(v) Nguyen and Murthy (1982) and Mi (1997) obtained burn-in time by minimizing the total costs that include warranty cost and burn-in.
(vi) Block et al. (2002) obtained a criterion for burn-in that balances mean residual life and residual variance. The objective function to be minimized is the residual coefficient of variation $CV(t)$ defined by (4.37).

4.6.2 Optimal Burn-in for Upside-down Bathtub Distributions

Chang (2000) discussed optimal burn-in decision for products with a unimodal failure rate function that has an upside-down shape so that $\mu(t)$ will have a bathtub shape. For example, the inverse Gaussian and lognormal distributions have this property. Chang (2000) formulated a total cost function which is expressed in terms of burn-in cost, failure cost during burn-in and warranty cost. So the optimal burn-in time which yields the minimal expected total cost can be obtained by solving a non-linear programming problem. The author concluded that the burn-in is not always necessary and economical for products with an upside-down failure rate distributions.

4.7 Tests and Estimation of Mean Residual Life

Given a data set of survival times, we often wish to develop tests of $H_0 : F$ is exponential (i.e., MRL is constant) versus the alternatives $H_1 : F$ belongs to a MRL class. Recall, we may broadly categorize the MRL classes into (i) Monotonic MRL and (ii) Non-monotonic MRL. We shall consider the two classes separately below.

4.7.1 Tests for Monotonic Mean Residual Life

In Section 7.6, We will consider the question of statistical tests for testing exponentiality against DMRL (DMRL) and other monotonic MRL classes. A summary of these tests will be given in Table 7.3.

4.7.2 Tests of Trend Change in Mean Residual Life

Recall, for a non-monotonic mean residual life a trend change must have occurred. To detect the whereabout of this turning point τ and finding an estimating for τ are important problems in survival analysis.

Mi (1995) proved that the trend change of MRL functions occurs before the bathtub shaped failure rate changes its trend in general. Gupta and Akman (1995a) and Guess et al. (1998) derived conditions under which DIFR (IDFR) implies IDMRL (DIMRL). (Recall, F is DIFR (IDFR) iff $r(t) \in$ BT (UBT))

Let τ be the turning point and ρ be the proportion of the population that 'dies' at or before the turning point τ, i.e., $\tau = F^{-1}(\rho) = \inf\{x|F(x) \geq \rho\}$. An $L-$statistic, a linear combination of order statistics, is used as a test statistic.

A fuller discussion on tests of non-monotonic MRL ageing concepts will be given in Section 7.7. A table of summary of these tests will be presented in Table 7.4.

4.7.3 Estimation of Monotonic Mean Residual Life

Consider the function $\mu(t) = \left[\int_t^\infty \bar{F}(x)\,dx\right]/\bar{F}(t)$ which is the mean residual life at age t.

Yang (1978) proposed the estimator $\widehat{M}_n(x) = I\left(X_{(n)} - x\right)\int_x^\infty \bar{F}_n(u)/\bar{F}_n(x)$.

Kochar et al. (2000) considered an estimator each for IMRL and DMRL in the following manner.

Let $0 \equiv X_{(0)} \leq X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ be the order statistics from a random sample from F with support $[0,T]$ for some finite T or $[0,\infty)$. Let $\bar{F}_n$ denote the empirical survival function. Define an estimate of $\mu(x)$ by

$$\widehat{M}_n(x) = I\left(X_{(n)} - x\right)\int_x^\infty \bar{F}_n(u)du/\bar{F}_n(x). \tag{4.45}$$

Their estimators for $\mu(t)$ from DMRL and IMRL classes are, respectively,

$$M_n^*(x) = I\left(X_{(n)} - x\right)\inf_{y\leq x}\widehat{M}_n(y), \tag{4.46}$$

and

$$M_n^{**}(x) = I\left(X_{(n)} - x\right)\sup_{y\leq x}\widehat{M}_n(y). \tag{4.47}$$

Kochar et al. (2000) proposed a projection-type estimator for estimation of a monotonic mean residual life.

Mi (1994a) proposed an estimator for the classes of life distributions that have decreasing, increasing or upside-down MRL. This estimator also has the uniformly strong consistency property.

Estimation of truncated MRL

Consider a truncated mean residual life defined by

$$e_M(t) = \frac{\int_x^M \bar{F}(u)du}{\bar{F}(t)},\ t \geq 0. \tag{4.48}$$

Ghorai et al. (1982) proposed an estimator by modifying the classical product-limit estimator of Kaplan and Meier (1958).

Na and Kim (1999) proposed a smooth nonparametric estimator of a truncated MRL based on a randomly censored sample. This estimator was constructed from the estimator derived as the maximum likelihood estimate of the cumulative failure rate in the class of distributions having piecewise linear failure rate functions between each pair of uncensored observations. The asymptotic properties of this estimator was also derived.

Smooth estimation of MRL

A smooth estimator based on a modified weighting scheme is proposed by Chaubey and Sen (1999) for $\mu(t)$.

Na and Kim (1999) proposed a spline smooth estimator of $e_M(x)$ based on a randomly right censored sample.

4.7.4 Estimation of Change Points

Ebrahimi (1991) proposed a procedure to estimate the change point τ of MRL for a lifetime model with a truncated upside-down bathtub shaped MRL , i.e., $\mu(t)$ is a non-decreasing function for $t < \tau$ and is a constant for $t \geq \tau$.

Mitra and Basu (1995) also considered change point estimation for the IDMRL and NWBUE classes.

Gupta et al (1999) have obtained the maximum likelihood estimate for the change point of the MRL based on a log-logistic model.

4.8 Mean Residual life with Special Characteristics

4.8.1 Linear Mean Residual Life Function

Hall and Wellner (1981) introduced a family of survival distributions with linear mean residual life function $\mu(x) = A + Bx$ $(B > -1, A > 0)$, namely

$$\bar{F}(x) = [A/(A + Bx)]^{1/B+1}.$$

The distribution above was given earlier as Example 4.5. When $B > 0, B = 0$ and $-1 < B < 0$, the above definition gives respectively a Pareto, an exponential and a rescaled beta distribution.

Earlier, Morrison (1978) has shown that the gamma distribution is the unique mixing distribution of exponentials that leads to a linearly increasing mean residual lifetime function given by $\mu(t) = A + Bt$. Although not specifically mentioned, it is clear from the context that $A > 0$ and $B > 0$. Also, it is well known that the gamma mixture of exponentials is simply a Pareto and thus Morrison's result is indeed a special case of Hall and Weller (1984).

Two further characterizations based on (4.46) were given by Oaks and Dasu (1990).

Another characterization of the class is given by Korwar (1992) who utilized the coefficients of skewness and kurtosis of the residual life distribution.

4.8.2 Proportional MRL and its Generalization

Oaks and Dasu (1990) proposed a new family of semi-parametric models requiring proportional MRL.

Two survival functions $\bar{F}_0(x)$ and $\bar{F}_1(x)$ are said to have proportional mean residuals if, in an obvious notation,

$$\mu_1(x) = \theta\mu_0(x), \text{ for all } x \geq 0, \ \theta > 0. \tag{4.49}$$

By inversion formula (4.9),

$$\bar{F}_1(x) = \bar{F}_0(x) \left\{ \int_x^\infty \bar{F}_0(y) dy / \mu_0 \right\}, \ \mu_0 = \mu_0(0). \tag{4.50}$$

Maguluri and Zhang (1994) extended the above model to a regression model with explanatory variables.

Gupta and Kirmani (1998) studied the proportional MRL further and its relationship with the proportional hazard model was explored.

Zahedi (1991) also proposed a proportional mean remaining life model motivated by the proportional hazard model of Cox (1972).

4.9 Other Residual Life Functions

Recall, the residual life random variable at age t, is defined by $X_t = X - t \,|X > t$. Apart from the MRL which is the expected value of X_t, there are several other associated functions that are of interest.

4.9.1 Residual Life Distribution Function

The distribution function of X_t is often denoted by RLD with the survival function

$$\bar{F}(x \,|\, t) = \Pr\left(X > t + x \,|\, X > t\right). \tag{4.51}$$

Barlow and Proschan (1981, p. 53) referred to the above function as the conditional reliability of a unit of age t. Several of the ageing classes presented in Section 2.4.2 can be characterized by this function.

Now, it is well known that $\bar{F}(x\,|\,t) = \bar{F}(x)$ if and only if X is exponentially distributed. This identity is also known as the 'lack of memory property' or 'memoryless property'. Further, using the RLD one can infer the properties of the original distributions also; see for example, Gupta and Kirmani (1990).

4.9.2 Variance Residual Life Function

Another function which has also generated some interest in the recent years is the variance residual life function defined as

$$\sigma^2(t) = \mathrm{var}(X - t\,|X > t), \tag{4.52}$$

see for example, Launer (1984) and Gupta et al. (1987). An alternative expression for the residual variance in (4.52) is given by (4.36). Theses authors studied the monotonicity of the above function and showed that $\sigma^2(t)$ is increasing (decreasing) according to whether

$$\psi(t) = \gamma^2(t) = \sigma^2(t)/\mu^2(t) \geq 1(\leq 1). \tag{4.53}$$

The function $\gamma(t)$ is simply the residual coefficient of variation defined by $CV(t)$ of (4.37).

Gupta and Kirmani (1998) gave a comprehensive treatment on this subject. Gupta and Kirmani (2000) showed that $\psi(t)$ characterizes the distribution and presented some examples.

4.9.3 Percentile Residual Life Function

The 'α-percentile residual life function' (α-percentile RLF) was first defined by Haines and Singpurwalla (1974). Joe and Proschan (1984) showed that this function may be expressed as

$$q_{\alpha,F}(t) = F^{-1}\left(1 - (1-\alpha)\bar{F}(t)\right),\ t \geq 0. \tag{4.54}$$

Recall, in Section 2.7, we have defined a distribution F to be DPRL-α (IPRL-α) if and only if for some α, $0 < \alpha < 1$, the α-percentile RLF $q_{\alpha,F}(t)$ is decreasing (increasing) in t. See also Section 3.7.2 for its relationship with a BT distribution.

4.10 Mean Residual Life Orderings

In Section 4.2, it was pointed out that the three reliability measures, namely, the survival function $\bar{F}(t)$, the failure rate function $r(t)$ and the mean residual life function $\mu(t)$ are equivalent to one another in the sense that knowing any of one them, the other two can be obtained. The question arises as whether an ordering with respect to a reliability measure implies the same ordering with respect to another reliability measure.

Consider two life time random variables X and Y with survival functions $\bar{F}$ and $\bar{G}$, respectively. We now consider the following three partial orderings:

- Stochastic ordering: $\bar{F}(x) \geq \bar{G}(x)$ for all $x \geq 0$ ($X \geq_{\mathrm{ST}} Y$);
- Failure rate ordering: $r_F(t) \leq r_G(t)$ for all $x \geq 0$ ($X \geq_{\mathrm{FR}} Y$) meaning that at the same age, the system whose lifetime is Y is more likely to instantaneously fail than the one whose lifetime is X;
- Mean residual life ordering: $\mu_F(t) \geq \mu_G(t)$ for all $t \geq 0$ ($X \geq_{\mathrm{MR}} Y$).

These three orderings have been considered in detail in Chapter 2.9. The last ordering (MR ordering) is particularly important in analyzing maintenance policies and renewal processes. It has been shown that $X \geq_{\mathrm{FR}} Y \Rightarrow X \geq_{\mathrm{ST}}$ as well as $X \geq_{\mathrm{FR}} Y \Rightarrow X \geq_{\mathrm{MR}} Y$

Gupta and Kirmani (1987) have proved that the following three conditions are equivalent:

(i) $r_F(t) \leq r_G(t)$;
(ii) $\bar{F}(t)/\bar{G}(t)$ is increasing in t;
(iii) $\bar{F}(x\,|\,t) \geq \bar{G}(x\,|\,t)$ where $\bar{G}(x\,|\,t) = 1 - G(x\,|\,t)$, $\bar{F}(x\,|\,t) = 1 - F(x\,|\,t)$ are, respectively the residual survival functions of X and Y as defined in (4.51).

The equivalence of (i) and (ii) are obvious, in fact, either of them define $X \leq_{\mathrm{FR}} Y$ according to Definition 2.15. The authors further proved that

1. $r_F(t) \leq r_G(t) \Rightarrow \mu_G(t) \leq \mu_F(t)$. (We have already noted this implication above). The converse is not true in general.
2. Suppose $\mu_G(t) \leq \mu_F(t)$ and $\mu_G(t)/\mu_F(t)$ is a non-decreasing function for all $x \geq 0$, then $r_F(t) \leq r_G(t)$.

Another partial ordering that is based on the mean residual life was introduced in Kochar and Wiens (1987) as follows:

We say that a distribution F is more decreasing mean residual life than another distribution G if $\mu_F(F^{-1}(u))/\mu_G(G^{-1}(u))$ is nonincreasing in $u \in [0,1]$. Aly (1993) considered the problem of testing for the mean residual life ordering defined above.

4.11 Multivariate Mean Residual Life

Consider two lifetime variables X and Y having joint survival function $\bar{F}(x,y) = \Pr(X > x, Y > y)$.

A popular way to define a bivariate mean residual life function is by the vector $(\mu_1(x,y), \mu_2(x,y))$, with the components of the vector given by

$$\begin{aligned} \mu_1(x,y) &= E\left(X - x | X > x, Y > y\right) \\ \mu_2(x,y) &= E\left(Y - y | X > x, Y > y\right), \end{aligned} \tag{4.55}$$

see for example, Arnold and Zahedi (1988) and Nair and Nair (1989).

A complicated form of bivariate mean residual life function is given by Shaked and Shanthikumar (1993).

Earlier, Johnson and Kotz (1975) defined a two-dimensional hazard rate $(r_1(x,y), r_2(x,y))$ which they called the hazard gradient of (X,Y) where

$$\begin{aligned} r_1(x,y) &= -\tfrac{\partial}{\partial x} \log \bar{F}(x,y) \\ r_2(x,y) &= -\tfrac{\partial}{\partial y} \log \bar{F}(x,y). \end{aligned} \tag{4.56}$$

Roy and Gupta (1996) showed that

$$\begin{aligned} r_1(x,y) &= \frac{1 + \frac{\partial}{\partial x}\mu_1(x,y)}{\mu_1(x,y)} \\ r_2(x,y) &= \frac{1 + \frac{\partial}{\partial y}\mu_2(x,y)}{\mu_2(x,y)}. \end{aligned} \tag{4.57}$$

We will revisit these functions in Section 8.7.

Anderson et al. (1992) defined a time-dependent association measure between two survival functions via the conditional expected residual life:

$$\phi_1(x,y) = \frac{E\left(X - x | X > x, Y > y\right)}{E(X - x \,| X > x)}. \tag{4.58}$$

We note that having values of $\phi_1(x,y)$ very different from 1 would indicate strong influence of Y on X and therefore strong (positive) association between X and Y. $\phi_2(x,y)$ can be defined analogously in terms of $\mu_2(x,y)$ and the MRL of X. Both $\phi_1(x,y)$ and $\phi_2(x,y)$ may be considered as a local dependence function, a concept to be considered in Section 9.11.

4.11.1 Characterizations of Multivariate Survival Distributions Based on Mean Residual Lives

Define the variances and the coefficients of variation of bivariate residual life, respectively by

$$\begin{aligned} V_1(x,y) &= \mathrm{Var}\left(X - x | X > x, Y > y\right) \\ V_2(x,y) &= \mathrm{Var}\left(Y - y | X > x, Y > y\right) \end{aligned} \tag{4.59}$$

and

$$\begin{aligned} C_1(x,y) &= \left\{V_1(x,y)\right\}^{1/2} / \mu_1(x,y) \\ C_2(x,y) &= \left\{V_2(x,y)\right\}^{1/2} / \mu_2(x,y). \end{aligned} \tag{4.60}$$

Roy and Gupta (1996) showed that the residual coefficients of variation defined above can be used to characterize a bivariate exponential distribution, a bivariate Lomax distribution and a bivariate finite range distribution. See also Sankaran and Nair (1993a) and Gupta and Kirman (2000). For other extensions, see for example, Ma (1996, 1998) and Asadi (1999).

4.11.2 Bivariate Decreasing MRL

Four multivariate generalizations of univariate DMRL were proposed by Buchanan and Singpurwalla (1977) of which we mention only the following two in the bivariate case.

A bivaraite distribution with survival function $\bar{F}(x, y)$ is said to be

1. bivariate DMRL-I (BMRL-I) if for all $t \geq 0$ for which $\bar{F}(t, t) > 0$, $\int_t^\infty \int_t^\infty \bar{F}(x, y)\, dx\, dy / \bar{F}(t, t)$ is nonincreasing in t, together with a similar condition for the two marginals.
2. bivariate DMRL-II (BMRL-II) if for all $t \geq 0$ for which $\bar{F}(t, t) > 0$, $\int_t^\infty \bar{F}(x, x)\, dx / \bar{F}(t, t)$ is nonincreasing in t, together with a similar condition for the two marginals.

Ghosh and Ebrahimi (1982) discussed shock models that lead to BMRL-I and BMRL-II, respectively.

Five decreasing bivariate (multivariate) mean residual life classes were defined by Zahedi (1985) based on the bivariate mean residual life function $(\mu_1(x, y), \mu_2(x, y))$ defined in (4.54) above.

Sen and Jain (1991b) and Bandyopadhyay and Basu (1995) have developed tests for bivariate exponentiality against bivariate decreasing mean residual life alternatives. See Chapter 8 for other details concerning bivariate DMRL.

4.12 Applications and Conclusions

Guess and Proschan (1988) gave an extensive coverage of possible applications of the mean residual life. There are many other studies on MRL data. We may broadly list the following aspects:

- Survival analysis in biomedical sciences

In biomedical setting researchers analyze survivorship studies through the MRL function $\mu(t)$. Bjerkedal (1960) gave a data set on guinea pigs' resistance to virulent tubercle bacilli which exhibits a bathtub shape MRL.

- Life insurance

Life length of human: High infant mortality explains the initial IMRL. Deterioration and ageing explains the DMRL stage. Obviously, MRL is of vital importance to actuarial work relating to life insurance policies. In fact, actuaries apply MRL to setting rates and benefits for life insurances.

- Maintenance and product quality control

To eliminate the initial failures, products are often burn-in before they leave the factory until they reach a low failure rate. One helpful tool for analysing burn-in is to model the ageing process of a device by the MRL (see. e.g., Mi, 1995).

- Economics and social studies

In economics, MRL is applied for investigating landholding, optimal disposal of an asset.

Guess and Proschan (1988) has postulated a situation whereby IDMRL model may be applicable to social studies.

Length of time employees stay with certain companies: An employee with a company for four years has more time and career invested in the company than an employee of only two months. The MRL of the four-year employee is likely to be longer than the MRL of the two-month employee. After this initial IMRL (this is called 'inertia' by social scientists), the process of ageing and retirement yield DMRL period.

Also, the mean residual life $\mu(t)$ is used to model the length of hospital stay of surgical patients.

- Demography

In studies of human populations, demographers will be interested in life expectancy or expectation of life which is simply the mean residual life concept in disguise.

- Product technology

Chinnam and Baruah (2004) have applied the mean residual life to a cutting tool monitoring problem.

In conclusion, the mean residual life function is a very useful reliability measure. In many ways, MRL is a more intuitive concept than the failure rate function. Graphs of MRL provide useful information not only for data analysis but also for presentation. Gertsbakh and Kordonsky (1969) noted that estimation of MRL is more stable than estimation of the failure rate.

5

Weibull Related Distributions

5.1 Introduction

The Weibull distribution is one of the best known lifetime distributions. It adequately describes observed failures of many different types of components and phenomena. Over the last three decades, numerous articles have been written on this distribution. Hallinan (1993) presented an insightful review by presenting a number of historical facts, and many forms of this distribution as used by the practitioners and possible confusions and errors that arise due to this non-uniqueness. Johnson et al. (1994) devoted a comprehensive chapter on a systematic study of this distribution. More recently, a monograph written by Murthy et al. (2003) contains nearly every facet concerning the Weibull distribution and its extensions. Lai et al. (2005) also provided a bird's eye view of this vast subject. Section 2.3.4 has briefly introduced the Weibull distribution. In this chapter, we angle our discussion towards reliability aspects of the Weibull and its related distributions. Consequently, many other important properties of these distributions and their applications will be glossed over. On the other hand, some repetitions of the earlier materials will be essential in order to give a fuller picture on the Weibull and its related distributions.

In Section 5.2, we first define a two-parameter Weibull distribution and then consider some basic properties such as moments, parameter estimation and failure rate function. A three-parameter Weibull distribution is introduced in Section 5.3. We study some graphical methods for estimating parameters and other reliability properties of this model. Three models that are derived from a simple transformation of the Weibull variable are studied in Section 5.4. We devote Section 5.5 to study five Weibull derived models which are extensions and generalizations of the 2-parameter Weibull. It is found that these models are quite flexible as they are able to gives rise to various shapes of failure rate including BT and UBT curves. In Section 5.6, we extend our consideration from a single Weibull to two or more Weibull variables forming mixtures, series and parallel structures as well as sectional models. Section

5.7 deals briefly Weibull models with varying scale parameters. Section 5.8 gives a preview of some discrete Weibull distributions which are analogues of a continuous two-parameter Weibull model. Some bivariate models with Weibull marginals are briefly considered in Section 5.9. Finally in Section 5.10, we outline various selective applications, especially those in the reliability context. Because of the vast literature, we are unable to cite all the source authors and we apologize in advance for any omissions in this regard.

5.2 Basic Weibull Distribution

5.2.1 Two-parameter Weibull Distribution and Basic Properties

The two-parameter distribution with survival function

$$\bar{F}(t) = \exp\{-(\lambda t)^{\alpha}\}, \ \alpha, \lambda > 0$$

was considered in Section 2.3.3 as an example of a lifetime distribution. Some of its reliability properties were given there and then. Since the theme of this chapter is on the Weibull and its related distributions, some of these properties will be reviewed here for the ease of cross-referencing.

Distribution function

The distribution function of the standard two-parameter Weibull distribution (Weibull, 1951) has an alternative form (due to a different parametrization) given as

$$F(t) = 1 - \exp\left[-\left(\frac{t}{\beta}\right)^{\alpha}\right], \ \alpha, \beta > 0, t \geq 0. \tag{5.1}$$

The present form seems more natural when a location parameter is added to the distribution as given in the next section. The parameters α and β are usually called the shape and scale parameters, respectively. The corresponding reliability function is given by

$$\bar{F}(t) = \exp\left[-\left(\frac{t}{\beta}\right)^{\alpha}\right], \ t \geq 0. \tag{5.2}$$

We observe that both the reliability function and the failure rate function as given in (5.10) below have simple forms which give the Weibull a head start over other lifetime models.

Two well known special cases are the exponential distribution ($\alpha = 1$) and the Rayleigh distribution ($\alpha = 2$).

If X has a two-parameter Weibull distribution, then $\log X$ is an extreme value distribution with location parameter $\log(\beta)$ and scale parameter $1/\alpha$, i.e., the pdf of $Y = \log X$ is:

$$f(y) = \alpha \exp\left\{\alpha\left(y - \ln \beta\right) - \exp\left[\alpha\left(y - \ln \beta\right)\right]\right\}. \tag{5.3}$$

Skewness and Kurtosis

The distribution is positively skewed for small value of α. The skewness index $\sqrt{\beta_1}$ decreases and equals zero for $\alpha = 3.6$ (approximately). Thus, for values of α in the vicinity of 3.6, the Weibull distribution is similar in shape to a normal distribution. The coefficient of kurtosis β_2 also decreases with α and then increases, reaching a minimum value of about 2.71 when $\alpha = 3.35$ (approximately). See, e.g., Johnson et al. (1994, pp. 631-635) and Mudholkar and Kollia (1994).

Probability density function

The probability density function that corresponds to (5.1) is given by

$$f(t) = \left(\frac{\alpha}{\beta}\right)\left(\frac{t}{\beta}\right)^{\alpha-1} \cdot \exp\left[-\left(\frac{t}{\beta}\right)^{\alpha}\right]. \tag{5.4}$$

The value of α has strong effects on the shape of the probability density function. For $0 < \alpha \leq 1$, the probability density function is a monotonic decreasing function and is convex as t increases. For $\alpha > 1$, the density function has a unimodal shape .

The mean, variance and moments

The kth moment about the origin may be obtained via its special case without a scale parameter defined by $X' = X/\beta$. Now it is easy to show that

$$\mu'_k = E(X'^k) = \Gamma\left(\frac{k}{\alpha} + 1\right),\ k = 1, 2, ... \tag{5.5}$$

In particular, the mean and variance are, respectively;

$$E(X') = \Gamma\left(\frac{1}{\alpha} + 1\right) \tag{5.6}$$

and

$$\operatorname{var}(X') = \Gamma\left(\frac{2}{\alpha} + 1\right) - \left\{\Gamma\left(\frac{1}{\alpha} + 1\right)\right\}^2 \tag{5.7}$$

(Johnson et al., 1994, p.632). The last three expressions, though in a slightly different form, have already been given in Section 2.3.4. The coefficient of variation is:

$$\nu = \sqrt{\frac{\Gamma\left(2/\alpha + 1\right)}{\Gamma^2\left(1/\alpha + 1\right)} - 1}. \tag{5.8}$$

The kth moment of X is obtained by

$$E(X^k) = \beta^k \mu'_k = \beta^k \Gamma\left(\frac{k}{\alpha} + 1\right). \tag{5.9}$$

Failure rate function

The failure rate function of the two-parameter Weibull distribution is given by

$$r(t) = \left(\frac{\alpha}{\beta}\right)\left(\frac{t}{\beta}\right)^{\alpha-1} \tag{5.10}$$

(cf. (2.23)). The shape parameter α also has a strong influence on the shape of the Weibull failure rate. It is obvious to see that $r(t)$ is a decreasing function in t when $0 < \alpha < 1$, constant when $\alpha = 1$ (the exponential case), and an increasing function when $\alpha > 1$. In view of the monotonic behaviour of its failure rate function, the Weibull distribution often becomes suitable when the conditions for 'strict randomness' of the exponential distribution are not satisfied, with the shape parameter α having a value depending upon the fundamental nature being considered. Thus the Weibull model is flexible and can be used to model IFR or DFR ageing distributions. A moderate value of α within 1 to 3 is appropriate in most situations (Lawless, 2003). However, the very monotonic shape of its failure rate has also become a limitation in reliability applications because for many real life data $r(t)$ exhibits some form of non-monotonic behavior. For this reason, several generalizations and modifications of the Weibull distribution have been proposed to meet the need of having various shapes.

Mean residual life function

Nassar and Eissa (2003) have shown that the MRL function has the following form

$$\mu(t) = \beta e^{\tau}\Gamma\left(1 + 1/\alpha\right)\left[1 - \Gamma_{\tau}(1/\alpha)/\Gamma(1/\alpha)\right],\ \tau = (t/\beta)^{\alpha}. \tag{5.11}$$

Here $\Gamma_{\tau}(r) = \int_0^{\tau} x^{r-1}e^{-x}dx$ is the incomplete gamma function and and $\Gamma(\cdot)$ is defined by $\Gamma(r) = \int_0^{\infty} x^{r-1}e^{-x}dx$.

For $\beta = \sqrt{2}$ and $\alpha = 2$,

$$\mu(t) = \sqrt{2\pi}e^{t^2/2}\left(1 - \Phi(t)\right),\ t > 0 \tag{5.12}$$

where $\Phi(\cdot)$ denote the cdf of a standard normal variable.

As IFR (DFR) implies DMRL (IMRL), it follows that $\mu(t)$ decreases (increases) in t for $\alpha > 1$ $(0 < \alpha \leq 1)$.

5.2.2 Parameter Estimation Methods

Many different methods can be applied to estimate the parameters of the Weibull distribution. Generally, these methods can be classified into two main categories, the graphical techniques and the statistical methods. Some frequently used graphical methods include methods using the empirical cumulative distribution plot (Nelson, 1982), Weibull probability plot (Nelson, 1982; Kececioglu, 1991; Lawless, 2003), hazard rate plot (Nelson, 1982) and so on.

Maximum likelihood estimation

Suppose there are r components in a sample of N components failed in a sample testing. Given the failure data following the Weibull distribution, t_1, t_2,..., t_r are the lifetime of r failed components; let t_r be the censoring time for the rest $n-r$ components. The likelihood function of the standard two-parameter Weibull distribution has the form:

$$L(\alpha,\beta) = \frac{N!}{(N-r)!}\left(\frac{\alpha}{\beta^\alpha}\right)^r \prod_{i=1}^{r} t_i^{\alpha-1} \exp\left\{-\frac{1}{\beta^\alpha}\left[\sum_{i=1}^{r} t_i^\alpha + (N-r)t_r^\alpha\right]\right\}. \tag{5.13}$$

Thus the log-likelihood function is given by:

$$\begin{aligned}\log L(\alpha,\beta) &= \log\left[\tfrac{N!}{(N-r)!}\right] + r\left(\log\alpha - \alpha\log\beta\right) \\ &+(\alpha-1)\sum_{i=1}^{r}\log t_i - \left\{\tfrac{1}{\beta^\alpha}\left[\sum_{i=1}^{r} t_i^\alpha + (N-r)t_r^\alpha\right]\right\}.\end{aligned} \tag{5.14}$$

Take the first derivative with respect to both parameters and then set them to zero, we obtain

$$\frac{\sum_{i=1}^{r} t_i^\alpha \log t_i + (N-r)t_r^\alpha \log t_r}{\sum_{i=1}^{r} t_i^\alpha + (N-r)t_r^\alpha} - \frac{1}{\alpha} - \frac{1}{r}\sum_{i=1}^{r}\log t_i = 0 \tag{5.15}$$

and

$$\beta = \left\{\frac{1}{r}\left[\sum_{i=1}^{r} t_i^\alpha + (N-r)t_r^\alpha\right]\right\}^{1/\alpha}. \tag{5.16}$$

Solving the equation (5.15), we can find the maximum likelihood estimate (MLE) of the shape parameter α, and the estimation of the scale parameter β can then be obtained from (5.16). For estimation procedures for grouped data, see, e.g., Nelson (1982), Lawless (2003), Cheng and Chen (1988) and Rao et al. (1994).

Other estimation methods

The least squares estimation (LSE) and the MLE are two common methods to estimate the Weibull parameters. While the MLE method is preferred by many researchers because of its good theoretical properties, the LSE method, especially when used in conjunction with graphical methods such as the Weibull probability plot (WPP) to be discussed in the next section, is most widely used by practitioners. This is due in part to its computational simplicity.

By equating the first three moments of the Weibull distribution to the first three sample moments and solving it, it is possible to find the moment estimators of α, β and τ (the location parameter).

A minimax optimization procedure for estimating the Weibull parameters with the Kolmogorov-Smirnov distance used as the objective was proposed by Ling and Pan (1998).

There are several other parameter estimation methods for the Weibull and we refer our readers to the monograph by Murthy et al. (2003) for details.

Bias

It is worth noting that when dealing with complete and small samples, the MLE and LSE estimates of the Weibull parameters, especially the shape parameter, are long known to be significantly biased (Montanari et al., 1997). The bias is also significant for the heavy censoring cases which are common in field conditions.

There are several methods for correcting the bias of the MLE of the Weibull parameters, mostly in the area of dielectric breakdowns studies. We refer our readers to Ross (1994, 1996) and Hirose (1999) for a review of various bias correction formulas for the shape parameter.

5.2.3 Relative Ageing of Two 2-Parameter Weibull Distributions

We say that X ages faster than Y if the ratio of the failure rate of X over the failure rate of Y is an increasing function of t. The concept of relative ageing is a particular form of partial ordering between two lifetime random variables. This concept was defined in Section 2.10.

Suppose we have two independent Weibull random variables X and Y with distribution functions $F(x)$ and $G(y)$, respectively, given by

$$F(x) = 1 - \exp\{-(x/\beta_2)^{\alpha_2}\}, \quad G(y) = 1 - \exp\{-(y/\beta_1)^{\alpha_1}\}. \tag{5.17}$$

This ratio of the failure rate of X to the failure rate of Y is given by

$$\frac{\beta_1\alpha_2}{\beta_2\alpha_1} \times \frac{\beta_1^{\alpha_1-1}}{\beta_2^{\alpha_2-1}} t^{\alpha_2-\alpha_1} \tag{5.18}$$

which is an increasing function of t if $\alpha_2 > \alpha_1$.

Suppose $E(X) = E(Y)$, i.e., $\beta_2\Gamma(1+1/\alpha_2) = \beta_1\Gamma(1+1/\alpha_1)$. Lai and Xie (2003) have shown that $\mathrm{var}(X) \le \mathrm{var}(Y)$. Two Weibull distributions can also be partially ordered with respect to the shape parameter in 'convex ordering' in the sense as in Barlow and Proschan (1981, p. 105). (See also Section 10.3.2 for a brief definition.)

5.3 Three-parameter Weibull distribution

Introducing a location parameter to a two-parameter Weibull distribution will result a three-parameter Weibull distribution. This more general but also more flexible distribution has cdf given by

$$F(t) = 1 - \exp\left\{-\left[\frac{t-\tau}{\beta}\right]^{\alpha}\right\}, t \geq \tau. \tag{5.19}$$

The three parameters of the distribution are given by the set $\theta = \{\alpha, \beta, \tau\}$ with $\alpha > 0, \beta > 0$ and $\tau \geq 0$; where β is a scale parameter, α is the shape parameter that determines the appearance or shape of the distribution and τ is the location parameter. An alternative form of a three-parameter Weibull distribution can be expressed as

$$F(t) = 1 - \exp\left\{-\lambda\left(t-\tau\right)^{\alpha}\right\}, t \geq \tau. \tag{5.20}$$

Here, the parameter λ combines both features of scale and shape. Clearly, $\lambda = \beta^{-\alpha}$. For $\tau = 0$, this becomes a two-parameter Weibull distribution. Murthy et al. (2003) referred to this special case as the *standard* Weibull model, however, Johnson et al. (1994) called a standard Weibull when $\beta = 1$ (or $\lambda = 1$) together with $\tau = 0$ in the above equations.

Density function

The probability density function of the three-parameter Weibull is

$$f(t) = \alpha\beta^{-\alpha}(t-\tau)^{\alpha-1}\exp\left\{-\left[\frac{t-\tau}{\beta}\right]^{\alpha}\right\}, \ t \geq \tau. \tag{5.21}$$

Mode and Median

The mode is at $t = \beta\left(\frac{\alpha-1}{\alpha}\right)^{1/\alpha} + \tau$ for $\alpha > 1$ and at τ for $0 < \alpha \leq 1$. The median of the distribution is at $\beta(\log 2)^{1/\alpha} + \tau$.

Moments

The kth moment of X defined by the density function (5.21) can be easily obtained from the relationship $X = \beta X' + \tau$ with $\mu'_k = E(X'^k)$ given by (5.5). In particular, the mean and variance of the three-parameter Weibull random variable are, respectively,

$$E(X) = \beta\Gamma\left(\frac{1}{\alpha}+1\right) + \tau \tag{5.22}$$

and

$$\mathrm{var}(X) = \alpha^2\Gamma\left(\frac{2}{\alpha}+1\right) - \alpha^2\left\{\Gamma\left(\frac{1}{\alpha}+1\right)\right\}^2. \tag{5.23}$$

Weibull probability plot

The Weibull probability plot (WPP) can be constructed in several ways (Nelson and Thompson, 1971). In the early 1970's a special paper was developed for plotting the data in the form $F(t)$ versus t on a graph paper with log-log scale on the vertical axis and log scale on the horizontal axis. A WPP plotting of data involves computing the empirical distribution function which can be estimated in different ways with the two standard ones being

- $\hat{F}(t_i) = i/(n+1)$, the "mean rank" estimator, and
- $\hat{F}(t_i) = (i-0.5)/n$, the "median rank" estimator.

Here, the data consists of successive failure times t_i, $t_1 < t_2 < ... < t_n$. For censored data (right censored or interval), the approach to obtain the empirical distribution functions needs to be modified, see for example, Nelson (1982).

These days, most computer reliability software packages contain programs to produce these plots automatically from a given data set. A well known statistical package MINITAB provides a Weibull probability plot under **Graph** menu >> **Probability Plot.**

We may use an ordinary graph paper or spreadsheet software with unit scale for plotting. Taking logarithms twice of both sides of each of the cdf in (5.19) yields

$$\log(-\log \bar{F}(t)) = \alpha \log(t-\tau) - \alpha \log \beta. \tag{5.24}$$

Let $y = \log(-\log \bar{F}(t))$ and $x = \log(t-\tau)$. Then we have a linear equation

$$y = \alpha x - \alpha \log \beta. \tag{5.25}$$

The plot is now on a linear scale. We can now see that a WPP can indicate a straight line if the assumption of a Weibull population for the data set concerned is plausible. The least squares estimates derived from (5.25) can be used as an initial estimates of the Weibull parameters. Thus the Weibull probability plot is a favoured tool by many reliability engineers.

Weibull hazard plot

The hazard plot is analogous to the probability plot, the principal difference being that the observations are plotted against the cumulated hazard (failure) rate rather than the cumulated probability value. Moreover, this is designed for censored data.

Let $H(t)$ denote the cumulative hazard rate (also referred to as hazard function), then $\bar{F}(t) = \exp(-H(t))$ so

$$H(t) = -\log \bar{F}(t) = \left(\frac{t-\tau}{\beta}\right)^{\alpha} \tag{5.26}$$

or equivalently

$$H(t)^{1/\alpha}/\beta = (t-\tau). \tag{5.27}$$

Let $y = \log(t-\tau)$ and $x = \log H(t)$, then we have

$$y = \log\beta + \frac{1}{\alpha}x. \tag{5.28}$$

Rank the n survival times (including the censored) in ascending order and let K denote the reverse ranking order of the survival time, i.e., $K = n$ for the smallest survival time and K =1 for the largest survival time. The hazard is estimated from $100/K$ (a missing value symbol is entered at a censored failure time). The cumulative hazard is obtained by cumulating the hazards. The Weibull hazard plot is simply the plot arising from (5.28). See Nelson (1972) for further details.

Both WPP and the Weibull hazard plot, in addition to providing simple straight line fitting for parameter estimation, they also have a role in model validation which is important in any engineering analysis.

Order statistics

Let $X_1, X_2, \cdots, X_n$ denote n independent and identically distributed three-parameter Weibull random variables. Furthermore, let $X_{(1)} < X_{(2)} < \cdots \leq X_{(n)}$ denote the order statistics from these n variables. The kth order statistic $X_{(k)}$ from a sample of n observations corresponds to the lifetime of a $(n-k+1)$-out-of-n system of n independent and identically distributed Weibull components. The probability density function of $X_{(1)}$, is given by

$$\begin{aligned} f_1(t) &= n[1-F(t)]^{n-1}f(x) \\ &= \tfrac{n\alpha}{\beta}\left(\tfrac{t-\tau}{\beta}\right)^{\alpha-1} e^{-n[(t-\tau)/\beta]^{\alpha}},\ t \geq \tau \geq 0. \end{aligned} \tag{5.29}$$

It is obvious that $X_{(1)}$ is also distributed as a Weibull random variable, except that α is replaced by $\beta n^{-1/\alpha}$. The density function of $X_{(r)}$ $(1 \leq r \leq n)$ is

$$\begin{aligned} f_r(t) = &\tfrac{n!}{(r-1)!(n-r)!}\left(1-e^{-[(t-\tau)/\beta]^{\alpha}}\right)^{r-1} e^{-[(t-\tau)/\beta]^{\alpha}(n-r+1)} \\ &\times\alpha\beta^{-\alpha}(t-\tau)^{\alpha-1},\ t \geq \tau \geq 0. \end{aligned} \tag{5.30}$$

It can be shown that

$$E\left[(X_{(r)})^k\right] = \sum_{i=0}^{k} \tau^i \beta^{k-i} \omega_{(r)}^{k-i} \tag{5.31}$$

where

$$\omega_{(r)}^k = \frac{n!}{(r-1)!(n-r)!}\Gamma\left(1+\frac{k}{\alpha}\right)\sum_{i=0}^{r-1}\frac{(-1)^r\binom{r-1}{i}}{(n-r+i+1)^{1+(k/\alpha)}}.$$

Failure Rate Function

The failure rate function for the three-parameter Weibull is

$$r(t) = \frac{f(t)}{\bar{F}(t)} = \frac{\alpha}{\beta}\left(\frac{t-\tau}{\beta}\right)^{\alpha-1}, \ t \geq \tau. \tag{5.32}$$

It is obvious that $r(t)$ given above is similar to its corresponding function for the two-parameter case as given in (5.10) except the function is now defined over $[\tau, \infty)$ instead of $[0, \infty)$.

There are many extensions, generalizations and modifications of the Weibull distribution. They arise out of the need to model some empirical data sets which cannot be adequately described by a three-parameter Weibull model. For example, the monotonic property of the Weibull's failure rate function which is unable to capture the behavior of a data set that has a bathtub shaped failure rate. Xie et al. (2003) reviewed several Weibull-related distributions that exhibit bathtub shaped failure rates. Plots of mean residual life from several of these Weibull derived models were given in Lai et al. (2004). For simplicity, we simply refer these Weibull-related models as Weibull models.

5.4 Models Derived from Transformations of Weibull Variable

We consider in this section four models derived from the Weibull variable by a simple transformation, either linear or non-linear. These models are now given as follows:

5.4.1 Reflected Weibull Distribution

Suppose X has a three-parameter Weibul distribution, then $T = -X$ has a reflected Weibull whose distribution function is

$$F(t) = \exp\left\{-\left(\frac{\tau-t}{\beta}\right)^{\alpha}\right\}, \ \alpha, \beta > 0, \ -\infty < t < \tau. \tag{5.33}$$

This is also known as type 3 extreme value distribution (Chapter 22, Johnson et al., 1995). The density function is given by

$$f(t) = \left(\frac{\alpha}{\beta}\right)\left(\frac{\tau-t}{\beta}\right)^{\alpha-1} \exp\left\{-\left(\frac{\tau-t}{\beta}\right)^{\alpha}\right\}, \ \alpha, \beta > 0, -\infty < t < \infty. \tag{5.34}$$

The failure rate function is given by

$$r(t) = \left(\frac{\alpha}{\beta}\right)\left(\frac{\tau - t}{\beta}\right)^{\alpha-1} \frac{\exp\left\{-(\frac{\tau-t}{\beta})^{\alpha}\right\}}{1 - \exp\left\{-(\frac{\tau-t}{\beta})^{\alpha}\right\}}. \tag{5.35}$$

Strictly speaking, the reflected Weibull is not suitable for reliability modelling unless $\tau > 0$ and $(\tau/\beta)^{\alpha} \geq 9$ so that $\Pr(0 < T < \tau) \approx 1$.

5.4.2 Log Weibull Distribution

This is an extreme value distribution derived from the logarithmic transformation of the two-parameter Weibull having distribution function as given in (5.1). The transformed variable has distribution function given by

$$F(t) = 1 - \exp\left\{-\exp\left(\frac{t-a}{b}\right)\right\},\ -\infty < t < \infty, \tag{5.36}$$

where we have let $a = \log\beta$, $b = 1/\alpha$. This is also known as type 1 extreme value distribution or the Gumbel distribution. In fact, it is the most commonly referred to in discussions of extreme value distributions (Johnson et al., 1995, Chapter 22). The density function is already given in (5.3)–though in a different parametrization, i.e.,

$$f(t) = \frac{1}{b}\exp\left(\frac{t-a}{b}\right)\exp\left\{-\exp\left(\frac{t-a}{b}\right)\right\},\ -\infty < t < \infty.$$

The failure rate functions is given by

$$r(t) = \frac{f(t)}{1 - F(t)} = \frac{1}{b}\exp\left(\frac{t-a}{b}\right). \tag{5.37}$$

Again, we note that since the distribution is defined over the whole real line, it has limited roles for reliability application.

5.4.3 Inverse (or Reverse) Weibull Model

Let X denote the 2-parameter Weibull model with distribution function $1 - e^{-(t/\beta)^{\alpha}}$. Define T as follows:

$$T = \frac{\beta^2}{X}. \tag{5.38}$$

Then T has a distribution function given by

$$F(t) = \exp(-(\beta/t)^{\alpha}),\ \alpha, \beta > 0, t \geq 0. \tag{5.39}$$

Alternatively, we may express (5.39) as

$$F(t) = \exp(-(t/\beta)^{-\alpha}),\ \alpha, \beta > 0, t \geq 0. \tag{5.40}$$

The density function is

$$f(t) = \alpha\beta^{\alpha}t^{-\alpha-1}e^{-(\beta/t)^{\alpha}}. \tag{5.41}$$

The inverse Weibull is also known as type 2 extreme value or the Fréchet distribution. (Johnson et al., 1995, Chapter 22). Erto (1989) has discussed the properties of this distribution and its potential use as a lifetime model. The maximum likelihood estimation and the least squares estimation of the parameters of the inverse Weibull distribution have been discussed by Calabria and Pulcini (1990).

The failure rate function is given by

$$r(t) = \frac{\alpha\beta^{\alpha}t^{-\alpha-1}e^{-(\beta/t)^{\alpha}}}{1-e^{-(\beta/t)^{\alpha}}}. \tag{5.42}$$

It can be shown that the inverse Weibull distribution generally exhibits a long right tail and its failure rate function is similar to that of the log-normal and inverse Gaussian distributions.

Jiang et al. (2001) showed that the failure rate function is unimodal (upside-down) with the mode at $t = t_M$ given by the solution of the following equation

$$\frac{z(t_M)}{1-e^{-z(t_M)}} = 1 + 1/\alpha \tag{5.43}$$

where $z(t) = (\beta/t)^{\alpha}$ and

$$\lim_{t\to 0} r(t) = \lim_{t\to\infty} r(t) = 0. \tag{5.44}$$

This is in contrast to the standard Weibull model for which the failure rate is either decreasing (for $\alpha < 1$), constant (for $\alpha = 1$) or increasing (for $\alpha > 1$).

The inverse Weibull transform is given by

$$x = \log t, \quad y = -\log\left(-\log(F(t)\right). \tag{5.45}$$

The above transformation was first proposed by Drapella (1993). The plot y versus x is called the inverse Weibull probability plot (IWPP) plot. Under this transform, the inverse Weibull model as given in (5.39) also yields a straight line relationship

$$y = \alpha(x - \log\beta). \tag{5.46}$$

5.5 Modifications or Generalizations of Weibull Distribution

A common factor among the generalized models considered below is that the Weibull distribution is a special case of theirs. Either $F(t)$ or $r(t)$ of the distribution of interest is related to the corresponding function of the Weibull distribution in some way. A common feature of these distributions is that the mean does not have a simple expression although their failure rate function $r(t)$ is able to model more diverse problems than the Weibull does.

5.5.1 Extended Weibull Distribution

Marshall and Olkin (1997) proposed a modification to the standard Weibull model through the introduction of an additional parameter ν $(0 < \nu < \infty)$. The model is given through its survival function function

$$\bar{F}(t) = \frac{\nu \bar{G}(t)}{1-(1-\nu)\bar{G}(t)} = \frac{\nu \bar{G}(t)}{G(t)+\nu \bar{G}(t)} \tag{5.47}$$

where $G(t)$ is the distribution function of the two-parameter Weibull and $\bar{F}(t) = 1 - F(t)$.

The case when G is an exponential distribution function has been considered as the exponential-geometric in Section 2.3.14.

When $\nu = 1$, $\bar{F}(t) = \bar{G}(t)$ so the model reduces to the standard Weibull model.

Using (5.1) as G in (5.47) the distribution function is given by

$$F(t) = 1 - \frac{\nu \exp[-(t/\beta)^{\alpha}]}{1-(1-\nu)\exp[-(t/\beta)^{\alpha}]}. \tag{5.48}$$

Marshall and Olkin (1997) called this the extended Weibull distribution. The mean and variance of the distribution cannot be given in a closed form, but they can be obtained numerically. The model may be considered as a competitor to the three-parameter Weibull distribution defined in (5.19).

The resulting density function associated with (5.48) is given by

$$f(t) = \frac{(\alpha\nu/\beta)(t/\beta)^{\alpha-1}\exp[-(t/\beta)^{\alpha}]}{\{1-(1-\nu)\exp[-(t/\beta)^{\alpha}]\}^2} \tag{5.49}$$

and the corresponding failure rate function is

$$r(t) = \frac{(\alpha/\beta)(t/\beta)^{\alpha-1}}{1-(1-\nu)\exp[-(t/\beta)^{\alpha}]}. \tag{5.50}$$

Marshall and Olkin (1997) carried out a partial study of the hazard (failure rate) function. It is increasing when $\nu \geq 1$, $\alpha \geq 1$ and decreasing when $\nu \leq 1$, $\alpha \leq 1$. If $\alpha > 1$, then the failure rate function is initially increasing and eventually increasing, but there may be an interval where it is decreasing. Similarly, when $\alpha < 1$, the failure rate function is initially decreasing and eventually decreasing, but there may be an interval where it is increasing.

From their Fig. 2 we observe that $r(t) \in$ MBT for $\beta = 1, \alpha = 2, \nu = 0.05$ or $\nu = 0.1$.

In view of the complexity of the mean residual life function $\mu(t)$ of the Weibull distribution, we anticipate that the mean residual life function of the extended Weibull is worse in terms of complexity. Lai et al. (2004) have provided several plots of MRL for different combinations of parameter values from this distribution.

5.5.2 Exponentiated Weibull Distribution

Mudholkar and Srivastava (1993) proposed a modification to the standard Weibull model through the introduction of an additional parameter ν $(0 < \nu < \infty)$. The distribution function is

$$F(t) = [G(t)]^{\nu} = [1 - \exp\{-(t/\beta)^{\alpha}\}]^{\nu}, \quad \alpha, \beta > 0, t \geq 0, \tag{5.51}$$

where $G(t)$ is the standard two-parameter Weibull distribution. The support for F is $[0, \infty)$.

The model was introduced as an example that can achieve a bathtub shaped failure rate function in Section 3.4.3. Here we will give a fuller study of this distribution.

When $\nu = 1$, the model reduces to the standard two-parameter Weibull model. When ν is an integer, the model is a special case of the multiplicative model to be discussed in Section 5.6.3. The distribution has been studied extensively by Mudholkar and Hutson (1996), Jiang and Murthy (1999) and more recently Nassar and Eissa (2003). The density function is given by

$$f(t) = \nu\{G(t)\}^{\nu-1} g(t), \tag{5.52}$$

where $g(t)$ is the density function of the standard two-parameter Weibull distribution. So

$$f(t) = \frac{\alpha\nu}{\beta^{\alpha}} t^{\alpha-1} e^{-(t/\beta)^{\alpha}} \left(1 - e^{-(t/\beta)^{\alpha}}\right)^{\nu-1}. \tag{5.53}$$

Two special cases worth noting:

(i) For $\alpha = 1$, the pdf is

$$f(t) = \frac{\nu}{\beta} e^{-t/\beta} \left(1 - e^{-t/\beta}\right)^{\nu-1} \tag{5.54}$$

which is the exponentiated exponential distribution studied by Gupta et al (1998).

(ii) For $\alpha = 2$, we obtain the two-parameter Burr type X distribution with pdf:

$$f(t) = \frac{2\nu}{\beta^2} e^{-(t/\beta)^2} \left(1 - e^{-(t/\beta)^2}\right)^{\nu-1}. \tag{5.55}$$

We also note that from (5.53),

$$f(0) = \begin{cases} 0 & \text{if } \alpha\nu > 1, \\ \beta^{-\alpha} & \text{if } \alpha\nu = 1, \\ \infty & \text{if } \alpha\nu < 1. \end{cases} \tag{5.56}$$

The value of $f(0)$ will have an impact on the shape of its MRL $\mu(t)$ (see equation (4.5) and Theorem 4.2) which we will investigate following our study of its failure rate function.

The failure rate function is given by (Mudholkar and Hutson, 1996)

$$r(t) = \frac{\alpha\nu(t/\beta)^{\alpha-1}[1-\exp(-(t/\beta)^{\alpha})]^{\nu-1}\exp(-(t/\beta)^{\alpha})}{1-[1-\exp(-(t/\beta)^{\alpha})]^{\nu}}. \tag{5.57}$$

For small t, Jiang and Murthy (1999) have shown that

$$r(t) \approx \left(\frac{\alpha\nu}{\beta}\right)\left(\frac{t}{\beta}\right)^{\alpha\nu-1}. \tag{5.58}$$

In other words, for small t, $r(t)$ can be approximated by the failure rate of a two-parameter Weibull distribution with shape parameter $(\alpha\nu)$ and scale parameter β.

For large t , i.e., $t \to \infty$, the term $\dfrac{\exp(-(t/\beta)^{\alpha})}{1-[1-\exp(-(t/\beta)^{\alpha})]^{\nu}}$ in (5.57) converges to $1/\nu$ by applying the L'Hospital's rule. It is now clear that (5.57) converges to

$$r(t) \approx \left(\frac{\alpha}{\beta}\right)\left(\frac{t}{\beta}\right)^{\alpha-1} \quad \text{for large } t. \tag{5.59}$$

In other words, for large t, $r(t)$ can be approximated by the failure rate of a two-parameter Weibull distribution with shape parameter α and scale parameter β.

Mudholkar et al. (1995), Mudholkar and Hutson (1996) and Jiang and Murthy (1999) have all considered the shapes of $r(t)$ and its characterization in the parameter space. The shape of $r(t)$ does not depend on β and varies with α and ν. The characterization on the (α, ν)- plane is as follows:

- $\alpha \le 1$ and $\alpha\nu \le 1$: $r(t)$ monotonically decreasing.
- $\alpha \ge 1$ and $\alpha\nu \ge 1$: $r(t)$ monotonically increasing.
- $\alpha < 1$ and $\alpha\nu > 1$: $r(t)$ has an upside-down bathtub shape.
- $\alpha > 1$ and $\alpha\nu < 1$: $r(t)$ has a bathtub shape.

The mean residual life of the exponentiated Weibull has been given by Nassar and Eissa (2003) for positive integer ν as:

$$\begin{aligned}\mu(t) = \nu \textstyle\sum_{j=0}^{\nu-1}(-1)^j\binom{\nu-1}{j}/\left[(j+1)\bar{F}(t)\right]\Big\{\beta\Gamma\left(\frac{1}{\beta}+1\right)\Big[(j+1)^{-1/\alpha} \\ -(j+1)^{-1}\Gamma_{(j+1)\tau}\left(\frac{1}{\beta}\right)/\Gamma\left(\frac{1}{\beta}\right)\Big] + te^{-(j+1)\tau}\Big\} - t,\end{aligned} \tag{5.60}$$

where $\tau = (t/\beta)^\alpha$. Thus, the mean of the distribution for $\nu \in \mathcal{N}^+$ is

$$\mu = \mu(0) = \nu\Gamma\left(\frac{1}{\alpha}+1\right)\sum_{j=0}^{\nu-1}(-1)^j\binom{\nu-1}{j}/\left[(j+1)^{-1/\alpha}\right] \tag{5.61}$$

From the general theory that connecting BT (UBT) and IDMRL (DIMRL) distributions (see Section 4.5 for details), Nassar and Eissa (2003) have established that the shape of $\mu(t)$ as follows:

Let $\mu(t)$ be given as (5.60), then

(i) $\mu = \alpha$ if and only if $\alpha = \nu = 1$.
(ii) $\mu(t)$ is DMRL (IMRL) if $\alpha \geq 1$ and $\alpha\nu \geq 1$ ($\alpha \leq 1$ and $\alpha\nu \leq 1$).
(iii) $\mu(t)$ is DIMRL with a change point t_m if $\alpha < 1$ and $\alpha\nu > 1$.
(iv) $\mu(t)$ is IDMRL with a change point t_m if $\alpha > 1$ and $\alpha\nu < 1$.

Nadarajah and Gupta (2005) obtained the kth moment about the origin as

$$\mu_k' = \nu\beta^k\Gamma\left(\frac{k}{\alpha}\right)\sum_{i=0}^{\infty}\frac{(1-\nu)_i}{i!(i+1)^{(k+\alpha)/\alpha}}, \quad k > -\alpha, \tag{5.62}$$

where $(1-\nu)_i = (a)_i = a(a+1)...a(a+i-1)$.

If ν is an integer, then

$$\mu_k' = \nu\beta^k\Gamma\left(\frac{k}{\alpha}\right)\sum_{i=0}^{\nu-1}\frac{(1-\nu)_i}{i!(i+1)^{(k+\alpha)/\alpha}}, \quad k > -\alpha, \tag{5.63}$$

which was established by Nassar and Eissa (2003). Equation (5.63) follows from (5.62) because $(1-\nu)_i = 0$ for all $i \geq \nu$.

5.5.3 Modified Weibull Distribution

Lai et al. (2003) proposed a modified Weibull which is an extension of the two-parameter Weibull distribution. The distribution function is given as in (3.8)

$$F(t) = 1 - \exp(-at^\alpha e^{\lambda t}),\ \ t \geq 0,$$

where the parameters $\lambda > 0$, $\alpha > 0$ and $a > 0$. The distribution is considered in Section 3.4.1 as an example that can give rise to a bathtub shaped failure rate function. Here we give a fuller account of this model.

For $\lambda = 0$, it reduces to a Weibull distribution.

The density function of the modified Weibull is given by

$$f(t) = a(\alpha + \lambda t)t^{\alpha-1}e^{\lambda t}\exp(-at^\alpha e^{\lambda t}). \tag{5.64}$$

The failure rate function given in (3.9) is reproduced here for convenience

$$r(t) = a(\alpha + \lambda t)t^{\alpha-1}e^{\lambda t}. \tag{5.65}$$

The shape of the failure rate function depends only on α because of the factor $t^{\alpha-1}$ and the remaining two parameters have no direct effect on the shapes.

(i) For $\alpha \geq 1$:

1. $r(t)$ is increasing in t, implying an increasing failure rate function, thus F is IFR.
2. $r(0) = 0$ if $\alpha > 1$ and $r(0) = a$ if $\alpha = 1$.
3. $r(t) \to \infty$ as $t \to \infty$.

(ii) For $0 < \alpha < 1$:

1. $r(t)$ initially decreases and then increases in t, implying a bathtub shape for the failure rate function.
2. $r(t) \to \infty$ as $t \to 0$ and $r(t) \to \infty$ as $t \to \infty$.
3. The change point t^*, i.e., the turning point of the failure rate function, is as given in (3.10):

$$t^* = \frac{\sqrt{\alpha} - \alpha}{\lambda}.$$

   The interesting feature is that t^* increases as λ decreases. The limiting case when $\lambda = 0$ reduces to the standard Weibull distribution.

Like most generalized Weibull distribution, the mean of this distribution does not have a closed form.

A simple method for parameters estimation through a WPP (Weibull probability plot) was given in Lai et al. (2003). More recently, Bebbington et al. (2005a) have suggested an empirical estimator for the turning point t^* and investigated its performance using a real data set as well as a simulation study. In a follow-up study, Bebbington et al. (2005b) have developed a methodology to estimate the optimal burn-in time for this model based on the criterion that the $\mu(t)$ attains its maximum at this point. Theoretical results are accompanied with simulation studies and applications to real data. Further, a statistical inferential theory was developed for the difference between the minimum point of the corresponding failure rate function and the maximum point of the mean residual life function.

5.5.4 Modified Weibull Extension

Another extension of Weibull was given by Xie, Tang and Goh (2002) who called it a 'modified Weibull extension'.

In Section 3.4.1 we have discussed how this model is related to other earlier models. The distribution function and the failure rate functions are given respectively by (3.17) and (3.18):

$$F(t) = 1 - \exp\left\{-\lambda\beta\left[e^{(t/\beta)^\alpha} - 1\right]\right\}, \quad t \geq 0, \alpha, \beta, \lambda > 0;$$

$$r(t) = \lambda\alpha(t/\beta)^{\alpha-1}\exp\left[(t/\beta)^\alpha\right].$$

The distribution approaches to a two-parameter Weibull distribution when $\lambda \to \infty$ with β in such a manner that $\beta^{\alpha-1}/\lambda$ is held constant.

For $\lambda = 1$, the above distribution is the exponential power distribution considered in Section 3.4.1 and studied by Smith and Bain (1975, 1976). The case with scale parameter $\beta = 1$ was considered by Chen (2000) who also considered the estimation of parameters.

The shape of $r(t)$ depends only on the shape parameter α.

For $\alpha \geq 1$:
- (i) $r(t)$ is an increasing function;
- (ii) $r(0) = 0$ if $\alpha > 1$ and $r(0) = \lambda$, if $\alpha = 1$;
- (iii) $r(t) \to \infty$ as $t \to \infty$.

For $0 < \alpha < 1$:
- (i) $r(t)$ is decreasing for $t < t^*$ and increasing for $t > t^*$ with

$$t^* = \beta(1/\alpha - 1)^{1/\alpha}. \tag{5.66}$$

  This implies that the failure rate function has a bathtub shape;
- (ii) $r(t) \to \infty$ for $t \to 0$ or $t \to \infty$;
- (iii) The change point t^* increases as the shape parameter α decreases.

For this model, the mean residual life function $\mu(t)$ does not have a closed form although its turning point μ_0 can be obtained numerically. In fact, $\mu(t)$ attains its maximum value at $t = \mu_0$. Xie et al. (2004) have given plots of $\mu(t)$ and $r(t)$ for $\alpha = 0.45, 0.5$, and 0.6 with $\beta = 100, \lambda = 2$.

Nadaraja (2005) derived explicit algebraic formulae for the kth moment (about zero) of the distribution when $1/\alpha$ is a non-negative integer.

5.5.5 Generalized Weibull Family

This model introduced in Section 3.4.2 was derived by Mudholkar and Kollia (1994) and Mudholkar et al. (1996) from the basic two-parameter Weibull distribution and involves an additional parameter. The quantile function for the new model is given by

$$\begin{aligned} Q(u) &= \beta\left[\tfrac{1-(1-u)^\lambda}{\lambda}\right]^{1/\alpha}, \lambda \neq 0 \\ &= \beta[-\log(1-u)]^{1/\alpha}, \lambda = 0 \end{aligned} \tag{5.67}$$

where the new parameter λ is unconstrained so that $-\infty < \lambda < \infty$. This implies

$$F(t) = \left[1 - \left(1 - \lambda\left(\frac{t}{\beta}\right)^{\alpha}\right)^{1/\lambda}\right], \ \alpha, \beta > 0, \tag{5.68}$$

where the support is for $F(t)$ is $(0, \infty)$ for $\lambda \leq 0$ and $(0, \beta/\lambda^{1/\alpha})$ for $\lambda > 0$. So the model can be regarded as a finite range distribution listed in Section 3.4.2.

Note that the model reduces to the basic two-parameter Weibull when $\lambda \to 0$.

The proposed model was called the generalized Weibull family in Mudholkar et al. (1996) who has applied the model to the two-arm clinical trials considered by Efron (1988).

Characterization of failure rate

This generalized family not only contains distributions with BT and UBT failure rate shapes, but also allows for a broader class of montonic failure rates. The failure rate associated with (5.68) is given by

$$r(t) = \frac{\alpha(t/\beta)^{\alpha-1}}{\beta\left(1 - \lambda(t/\beta)^{\alpha}\right)}. \tag{5.69}$$

The following classification was from Mudholkar et al. (1996):

1. $\alpha < 1$ and $\lambda > 0$: $F \in$ BT
2. $\alpha \leq 1$ and $\lambda \leq 0$: $F \in$ DFR
3. $\alpha > 1$ and $\lambda < 0$: $F \in$ UBT
4. $\alpha \geq 1$ and $\lambda \geq 0$: $F \in$ IFR
5. $\alpha = 1$ and $\lambda = 0$: F exponential (i.e., $r(t)$ is a constant)

It was also shown that the family (5.68) is closed under proportional hazards relationships, that is, for any $\nu > 0$, $\bar{F}(t)^{\nu}$ is also a member of the family (5.68).

Moreover, we have

1. Let $\theta = \beta/\lambda^{1/\alpha}$ and $\alpha = p$, (5.69) reduces to the failure rate function (3.22) of a beta distribution.
2. For $\lambda \leq 0$, (5.69) reduces to (2.47), the failure rate of the Burr XII distribution.

The maximum likelihood estimates of the model parameters were obtained by Mudholkar et al. (1996).

5.5.6 Generalized Weibull Distribution of Gurvich et al.

It has been pointed by Nadarajah and Kotz (2006) that several of the modifications of Weibull distributions discussed in this section can arise from a representation suggested by Gurvich et al. (1997). This distribution did not arise from a reliability perspective but from the context of modelling random length of brittle materials. The distribution function of this class is given by

$$F(t) = 1 - \exp\{-aG(t)\}, \tag{5.70}$$

where $G(t)$ is a monotonically increasing function in t such that $G(t) \geq 0$.

(i) $G(t) = t^\alpha \exp(\lambda t)$, it reduces to the model of Lai et al. (2003).

(ii) $G(t) = \exp(t/\beta)^\alpha - 1$, it reduces to the model of Xie et al. (2002).

(iii) $G(t) = \exp((t-a)/b))$, it reduces to the log Weibull.

Several other Weibull related distributions are also contained in this family.

5.6 Models Involving Two or More Weibull Distributions

These are univariate models derived from two or more Weibull distributions. We assume that life component $i, i = 1, 2, .., n$ has a Weibull distribution with cdf F_i having scale parameter α_i and shape parameter β_i. Four such models are now presented below.

5.6.1 *n*-fold Mixture Model

Mixtures of two Weibull distributions have been considered in Section 2.8.4. In this subsection, we recap some of the results.

Let p_i be the mixing proportion of the ith subpopulation so the distribution of the mixture model is represented by

$$F(t) = \sum_{i=1}^{n} p_i F_i(t), \quad p_i \geq 0, \sum_{i=1}^{n} p_i = 1. \tag{5.71}$$

We assume here the n Weibull random variables involved in the mixture are independent. Gurland and Sethurama (1995) have considered a mixture of the Weibull distribution with failure rate $\lambda\alpha t^{\alpha-1}$ and the exponential distribution with failure rate λ_1. For $\alpha > 1$, the Weibull distribution is IFR. They found that the resulting mixture distribution is 'ultimately' DFR. In fact, the mixture with the exponential has resulted a failure rate with an upside-down bathtub shape.

Jiang and Murthy (1995b, 1998) have categorized the possible shapes of the failure rate function for a mixture of any two Weibull distributions in terms of five parameters. In particular, they stated that the mixture failure rates from two strictly IFR Weibull distributions with the same shape parameter β can either have a modified bathtub shape (MBT) or an increasing failure rate. However, they did not classify the two possibilities.

Wondmagegnehu (2004) has fine-tuned the results of Jiang and Murthy (1998) with a definite discriminant between the two classes based on the value of the mixing proportion p with $p_1 = p$ and $p_2 = 1 - p$ as given in (5.71). In the case where the two Weibull shape parameters are different, i.e., $\alpha_1 \neq \alpha_2$ and both $\alpha_1, \alpha_2 > 1$, he used examples to illustrate all possible shapes that the mixture failure rate can encounter.

We also note that a mixture of two Weibull distribution cannot give rise to a BT shaped failure rate.

Reliability approximation using a finite Weibull mixture distributions

Bučar et al. (2004) have shown that the reliability of an arbitrary system can be approximated well by a finite Weibull mixture with positive component weights only, without knowing the structure of the system, on condition that the unknown parameters of the mixture can be estimated. To support the main idea, they have presented five examples for demonstration. In order to estimate the unknown component parameters and components weights of the Weibull mixture, some of the already known methods were applied and the EM algorithm for the m-fold mixture was derived. The fitted distributions obtained by different methods were compared to the empirical ones by calculating the AIC and δ_c values. The authors concluded that the suggested Weibull mixture with an arbitrary but finite number of components is suitable for lifetime data approximation.

For other ageing characteristics of Weibull mixtures, see Gupta (1995).

5.6.2 *n*-fold Competing Risk Model

Again we assume here the n Weibull random variables involved in the n-fold competing risks model are independent. The distribution function is given by

$$F(t) = 1 - \prod_{i=1}^{n} (1 - F_i(t)). \tag{5.72}$$

The above model represents the lifetime distribution of a series system of n independent Weibull components. Note that the failure rate function of a series system is the sum of the failure rates of its n components. The special case $n = 2$ is more widely studied.

Additive Weibull model

Jiang and Murthy (1997b) have given a parametric study of a competing risks model involving two Weibull distributions. Earlier, Xie and Lai (1995) studied a 2-fold competing risks model which they called an additive Weibull model in view of its additive property of the two failure rate functions. The reliability function is very simple and can be given by

$$\bar{F}(t) = \exp\left\{-(t/\beta_1)^{\alpha_1} - (t/\beta_2)^{\alpha_2}\right\}, \; \alpha_1, \alpha_2, \beta_1, \beta_2 > 0. \tag{5.73}$$

The failure rate function is also very simple and was given earlier by (3.6), i.e.,

$$r(t) = \alpha_1/\beta_1(t/\beta_1)^{\alpha_1-1} + \alpha_2/\beta_2(t/\beta_2)^{\alpha_2-1}, \; t \geq 0.$$

It is easy to see that F is IFR if both shape parameters are greater than 1, i.e., if $\alpha_1 > 1$ and $\alpha_2 > 1$; and DFR if $\alpha_1 < 1$ and $\alpha_2 < 1$.

For $\alpha_1 < 1$ and $\alpha_2 > 1$, $r(t)$ then has a bathtub shape. This is because the second term in the preceding equation which dominates for small t is decreasing. For large t, the first term of $r(t)$ dominates and is an increasing function. An important feature of this model is that $r(t)$, with an appropriate choice of parameter values, can achieve a fairly flat middle part of the life phase so it is able to model a population with a stable useful period. There is however, an undesirable aspect of this model which yields rather low values of $r(t)$ in the middle period of the lifetime. This can be rectified by adding a 'lift' factor r_0 to $r(t)$ as was proposed in Lai et al. (2004). The resulting new system corresponds to a series system of two Weibull components augmented by an exponential component with scale parameter r_0. This additional component is arranged in series along with the first two components. Since the exponential is a special case of the Weibull distribution, the modified additive model corresponds to a 3-fold competing risks model involving Weibull distributions.

It is obvious that that $r(t)$ cannot achieve an UBT shape.

5.6.3 *n*-fold Multiplicative Model

The n-fold multiplicative model also known as the complementary risk model in the literature, has the distribution function given by

$$F(t) = \prod_{i=1}^{n} F_i(t). \tag{5.74}$$

If all the n component lifetimes are independent and identically distributed as a Weibull distribution, then (5.74) is equivalent to (5.51), the exponentiated Weibull distribution. It is easy to see that (5.74) corresponds to the distribution of a system of n independent components that are arranged in parallel. The multiplicative model involving two Weibull distribution was studied in detail by Jiang and Murthy (1997a). It was shown that there are only four possible shapes of the failure rate function: (i) decreasing, (ii) increasing (iii) UBT or (iv) MBT depending on the values of the scale and shape parameters of the two Weibull variables. It is interesting to note that, like the case of mixture, a BT shaped failure rate cannot be achieved.

5.6.4 *n*-fold Sectional Model

The distribution function in a n-fold sectional model has n segments joined together as follows:

$$F(t) = \begin{cases} k_1 F_1(t), & 0 \le t \le t_1, \\ 1 - k_2 \bar{F}_2(t), & t_1 < t \le t_2, \\ \cdots\cdots & \\ 1 - k_n \bar{F}_n(t), & t > t_{n-1}. \end{cases} \tag{5.75}$$

where the sub-populations $F_i(t)$ are the two- or three-parameter Weibull distributions and the t_i's (called partition points) are an increasing sequence. Jiang and Murthy (1997b) and Jiang et al. (1999) considered the case where $n = 2$ in detail whereas Jiang and Murthy (1997c) considered the case where $n = 3$. Both cases can achieve a BT shaped failure rate; the former has a 'V' shape bathtub whereas the latter can attain a flat bottom provided $\alpha_2 = 1$ where α_i is the shape parameter of the ith Weibull distribution. In the case $n = 3$, Jiang and Murthy (1997c) reported that the failure rate can be one of twenty different shapes which can be classified into five types – (i) I (ii) D (iii) BT (iv) UBT and (v) roller-coaster shape. Thus, the sectional models involving three Weibull distributions can be used to model a variety of reliability data.

Lai et al. (2004) also gave a plot for the mean residual life for each case.

5.6.5 Model Involving Two Inverse Weibull Distributions

Although this model does not quite fit in with the rest of this section, we think it is nevertheless appropriate to include it here because of a similar theme being discussed.

Recall in Section 5.4.3, the inverse Weibull is obtained by inverting the Weibull random variable. Its distribution function is given by $F(t) = \exp\{-(\beta/t)^{\alpha}\}$, $\alpha, \beta, t > 0$. Jiang et al. (2001) have studied three models (mixture, competing risk and multiplicative) involving two inverse Weibull distributions.

Mixture model

It was shown that since the failure rate of the two sub-populations are unimodal (UBT), the possible shapes for the mixture model are: (i) unimodal (UBT) and (ii) bimodal (two local maxima).

Competing risk model

Interesting enough, the shapes of the failure rates in a competing risks model are almost the same as those for the mixture model, i.e., either $r(t)$ is UBT or bimodal.

Multiplicative model

After examination of the failure rate function plots for a range of parameter values, Jiang et al (2001) found that the failure rate function is always UBT, so is its density function.

5.7 Weibull Models with Varying Parameters

In this models the scale parameter β is a function of some supplementary variable S. In reliability applications S represents the stress on the item and the life of the item (a random variable with distribution F) is a function of S. The shape parameter is unaffected by S and hence a constant. See Section 2.6 of Murthy et al. (2003).

Arrhenius model

The relationship is given by

$$\beta(S) = \exp(\gamma_0 + \gamma_1 S). \tag{5.76}$$

The model was discussed in Jensen (1995).

Power model

The relationship is given by

$$\beta(S) = \frac{e^{\gamma_0}}{S^{\gamma_1}}. \tag{5.77}$$

The power model can also be found in Jensen (1995).

Weibull proportional hazard models

The relationship is given by

$$r(t) = \psi(S) r_0(t), \tag{5.78}$$

where $r_0(t)$ is called the baseline hazard for a two-parameter Weibull distribution. The only restriction on the scalar function $\psi(\cdot)$ is that it be positive. Many different forms for $\psi(\cdot)$ have been proposed. One such is the following:

$$\psi(S) = \exp(b_0 + \sum_{i=1}^{k} b_i s_i). \tag{5.79}$$

(See Kalbfleisch and Prentice, 1980).

5.8 Discrete Weibull Models

Here the lifetime variable X can only assume non-negative integer values and this defines the support for $F(t)$. Chapter 6 will give a detailed study on this subject. Here, we merely list three discrete analogues of the Weibull distribution.

Model-1

(Nakagawa and Osaki, 1975).

$$F(t) = \begin{cases} 1 - q^{t^\alpha} & t = 0, 1, 2, 3 \cdots, \\ 0 & t < 0. \end{cases} \tag{5.80}$$

Model-2

(Stein and Dattero, 1984).

The cumulative hazard function is given by

$$H(t) = \begin{cases} ct^{\alpha-1} & t = 1, 2, \cdots, m, \\ 0 & t < 0. \end{cases} \tag{5.81}$$

where m is given by

$$m = \begin{cases} [c^{-\{1/(\alpha-1)\}}]^+ & \text{if } \alpha > 1, \\ \infty & \text{if } \alpha \le 1; \end{cases} \tag{5.82}$$

and $[\ ]^+$ represents the integer part of the quantity inside the square brackets.

Model-3

(Padgett and Spurrier, 1985).

$$F(t) = 1 - \exp\{-\sum_{i=1}^{t} r(i)\} = 1 - \exp\{-\sum_{i=1}^{t} ci^{\alpha-1}\}, \quad t = 0, 1, 2, ... \tag{5.83}$$

We refer our readers to Section 6.7.1 for more discussions.

5.9 Bivariate models

These models to be presented below are multivariate extensions of the univariate case so that the distribution function is given by an n-dimensional distribution function $F(t_1, t_2, \cdots, t_n)$. In this section we discuss only some selective bivariate models; multivariate extensions of these models can be developed in a similar way although it generally involves more parameters. For a fuller coverage, we refer our readers to the monograph by Murthy et al. (2003).

5.9.1 Marshall and Olkin (1967)

This is obtained from the power law transformation of the well known bivariate exponential distribution (BVE) studied in Marshall and Olkin (1967). The joint survivor function with Weibull marginals are given as:

$$\bar{F}(t_1, t_2) = \exp\{-[\lambda_1 t_1^{\alpha_1} + \lambda_2 t_2^{\alpha_2} + \lambda_{12} \max(t_1^{\alpha_1}, t_2^{\alpha_2})]\}, \tag{5.84}$$

where $\lambda_i > 0, \alpha_i \geq 0, \lambda_{12} \geq 0; i = 1, 2$. This bivariate Weibull reduces to the bivariate exponential distribution when $\alpha_1 = \alpha_2 = 1$.

Lu (1992) considered Bayes estimation for the above model for censored data.

5.9.2 Lee (1979)

A related model due to Lee (1979) involves the transformation $Z_i = T_i/c_i$ and $\alpha_1 = \alpha_2 = \alpha$. The model is given by

$$\bar{F}(t_1, t_2) = \exp\left\{-\left[\lambda_1 c_1^{\alpha} t_1^{\alpha} + \lambda_2 c_2^{\alpha} t_2^{\alpha} + \lambda_{12} \max(c_1^{\alpha} t_1^{\alpha}, c_2^{\alpha} t_2^{\alpha})\right]\right\}, \tag{5.85}$$

where $c_i > 0, \lambda_i > 0, \lambda_{12} \geq 0$.

Yet another related model due to Lu (1989) has the survival function:

$$\bar{F}(t_1, t_2) = \exp\left\{-\lambda_1 t_1^{\alpha_1} - \lambda_2 t_2^{\alpha_2} - \lambda_0 \max(t_1, t_2)^{\alpha_0}\right\}, \tag{5.86}$$

where $\lambda's > 0, \alpha_i \geq 0;\ i = 0, 1, 2$. This can be seen as a slight modification (or generalization) of the Marshall and Olkin's bivariate exponential distribution due to the exponent in the third term having a new parameter.

5.9.3 Lu and Bhattacharyya (1990)-I

A general model proposed by Lu and Bhattacharyya (1990) has the form:

$$\bar{F}(t_1, t_2) = \exp\left\{-(t_1/\beta_1)^{\alpha_1} - (t_2/\beta_2)^{\alpha_2} - \delta h(t_1, t_2)\right\}, \tag{5.87}$$

where $\alpha_i > 0, \beta_i \geq 0, \delta \geq 0; i = 1, 2$.

Different forms for the function of $h(t_1, t_2)$ yield a family of models. One form for $h(t_1, t_2)$ is the following:

$$h(t_1, t_2) = \left[(t_1/\beta_1)^{\alpha_1/m} + (t_2/\beta_2)^{\alpha_2/m}\right]^m, \ m > 0. \tag{5.88}$$

This yields the following survival function for the model:

$$\bar{F}(t_1, t_2) = \exp\left\{-(t_1/\beta_1)^{\alpha_1} - (t_2/\beta_2)^{\alpha_2} - \delta[(t_1/\beta_1)^{\alpha_1/m} + (t_2/\beta_2)^{\alpha_2/m}]^m\right\}. \tag{5.89}$$

5.9.4 Morgenstern-Gumbel-Farlie System

The Morgenstern-Gumbel-Farlie system of distributions (Hutchinson and Lai, 1990, Section 5.2 and Kotz et al., 2000, Section 44.13) is given by

$$\bar{F}(t_1,t_2)=\bar{F}_1(t_1)\bar{F}_2(t_2)\left\{1+\gamma\left[1-\bar{F}_1(t_1)\right]\left[1-\bar{F}_2(t_2)\right]\right\},\ -1<\gamma<1. \tag{5.90}$$

(See also Example 9.1.) With $\bar{F}_i(y_i)=\exp\{-y_i^{\alpha_i}\},\alpha_i>0$, this yields a bivariate Weibull model with the marginals being a standard Weibull in the sense of Johnson et al. (1994).

5.9.5 Lu and Bhattacharyya (1990)-II

A different type of bivariate Weibull distributions due to Lu and Bhattacharyya (1990) is given by

$$\bar{F}(t_1,t_2)=\left[1+\left[\{[\exp(t_1/\beta_1)^{\alpha_1}]-1\}^{1/\gamma}+\{\exp[(t_2/\beta_2)^{\alpha_2}]-1\}^{1/\gamma}\right]^{\gamma}\right]^{-1}. \tag{5.91}$$

This model has a random hazard interpretation, but for no value of γ, the model yields independence between the two variables.

5.9.6 Lee (1979)-II

Lee (1979) proposed the following bivariate Weibull distribution

$$\bar{F}(t_1,t_2)=\exp\{-(\lambda_1 t_1^{\alpha_1}+\lambda_2 t_2^{\alpha_2})^{\gamma}\} \tag{5.92}$$

where $\alpha_i>0,\ 0<\gamma\le 1,\ \lambda_i>0,\ t_i\ge 0,\ i=1,2$. This is also known as the logistic model with Weibull marginals in the multivariate extreme value context.

The model was used by Hougaard (1986) to analyze a tumor data. A similar model was also proposed and studied in Lu and Bhattacharyya (1990).

We note that it is easy to generate a bivariate Weibull distribution by a marginal transformation, a popular method for constructing a bivariate model with specified marginals (Lai, 2004).

5.10 Applications of Weibull and Related Models

There are numerous applications for Weibull and Weibull related distributions in all aspects of life so it would be futile, perhaps unhelpful, to list them all here. Since our theme is pitched towards reliability, we select a few applications that are relevant in reliability context. These are arranged in alphabetical order as follows:

1. Adhesive wear in metals – Queeshi and Sheikh (1997).
2. Aircraft windshield failures data– Murthy et. al (2003).
3. Analysis of survival data from clinical trials – Carroll (2003).
4. Bus motor failures data – Davis (1952), Mudholkar et al. (1995)
5. Carbon fibers and composites failures – Durham and Padgett (1997).
6. Cleaning web failure times in photocopiers – Murthy et al. (2004).
7. Device failure times – Aarset (1987).
8. Failure of coatings recoil compressive failure in high performance polymers – Almeida (1999), Newell et al. (2002).
9. Failures of brittle materials – Fok et al. (2001).
10. Failure probability prediction of concrete components – Li et al. (2003).
11. Fatigue of bearings – Cohen et al. (1984).
12. Fracture strength of glass – Keshevan et al. (1980).
13. Material strength – Weibull (1939).
14. Product warranty – Blischke and Murthy (1994, 1996); Murthy and Djamaludin (2002).
15. Pitting corrosion and pipeline reliability – Sheikh et al. (1990).
16. Throttle failure times – Carter (1986), Murthy et al. (2003).
17. Yield strength of steel, fatigue life of steel – Weibull (1951).
18. Die cracking in the assembly and reliability testing of flip-chip (FC) packages – Zhao (2004).
19. Large-scale multiprocessor systems – Al-Rousan and Shaout (2004).
20. Cavitation erosion resistance – Meged (2004).
21. Fracture strength data obtained from ASTM D3039 tension tests of 19 identical carbonepoxy composite specimens – Birgoren and Dirikolu (2004).

Since the Weibull is also an extreme value distribution, it is also frequently used to model environmental data such as rains and floods. For example,

(i) Annual flood discharge rates – Mudholkar and Hutson (1996)
(ii) Flood frequency – Heo et al. (2001)
(iii) Wind speed data – Al-Hasan and Nigmatullin (2003).

6

An Introduction to Discrete Failure Time Models

6.1 Introduction

An important aspect of lifetime analysis is to find a lifetime distribution that can adequately describe the ageing behavior of the device concerned. Most of the lifetimes are continuous in nature and hence many continuous life distributions have been proposed in the literature. On the other hand, discrete failure data arise in several common situations, for example:

- Reports on field failures are collected weekly, monthly, and the observations are the number of failures, without specification of the failure times;
- A piece of equipment operates in cycles and the experimenter observes the number of cycles successfully completed prior to failure. A frequently referred example is a copier whose life length would be the total number of copies it produces. Another example is the number of on/off cycles of a switch before failure occurs;
- An experimenter often discretizes or groups continuous data.

Interests in discrete failure data came relatively late in comparison to its continuous analogue. The subject matter has to some extent been neglected. It was only briefly mentioned by Barlow and Proschan (1981). For earlier works on discrete lifetime distributions, see Ebrahimi (1986), Padgett and Spurrier (1985), Salvia and Bollinger (1982) and Xekalaki (1983).

In Section 6.2, we define a discrete version of the survival function, the failure rate function, and the mean residual life of a discrete random variable X. Section 6.3 deals with the discrete analogues of ageing classes such as IFR, NBU, NBUE, DMRL, etc. We also provide a chain of implications among these classes. We briefly introduce some more advanced ageing classes in Section 6.4 but no detailed study is given. Both monotonic failure rate and non-monotonic mean residual life functions play an important role in modeling reliability data so discrete BT (UBT) and IDMRL (DIMRL) are also defined in Section 6.5. It is shown in Section 6.6, that discrete ageing classes play important roles in Poisson shock models. Section 6.7 contains a number of discrete lifetime models and their ageing properties are studied. Most of these examples are derived from their continuous counterparts. A discussion on the merits of these examples is presented in Section 6.8. We outline in Section 6.9 some reliability and maintenance applications of discrete lifetime models. Section 6.10 highlights some undesirable properties associated with the traditional definition of the failure rate function. Finally in Section 6.11, we present an alternative definition of a discrete failure rate concept proposed more recently. In this new definition, the ageing concepts so derived are closer to their continuous analogues.

6.2 Survival Function, Failure Rate and Other Reliability Characteristics

Let the random variable X with support $\mathcal{N}^+ = \{1, 2, ...\}$ be the discrete lifetime of a component and denote by $f(k)$ the probability of a failure occurs at time k, i.e.,

$$f(k) = \Pr\{X = k\}, \; k = 1, 2... \tag{6.1}$$

The above definition implicitly implies that a device (component) can fail only at times in $\mathcal{N}^+$. The results follow will also apply to a random variable Y with support $\{a, a+1, ..\}$ for some $a \in (-\infty, \infty)$. All that requires is to consider the transformed variable $X = Y - a + 1$. In particular, the results which follow can be applied to the random variables Y with support in $\{0, 1, 2...\}$ by a simple translation $X = Y + 1$.

The reliability (survival) function that corresponds to X is given by

$$R(k) = \Pr\{X > k\} = \sum_{j=k+1}^{\infty} f(j), \;\; k = 1, 2, ..., \tag{6.2}$$

noting that $R(0) = 1$. The survival function may be defined over the whole non-negative real line by

$$R(t) = R(k) \quad \text{for } 0 \le k \le t < k + 1, \tag{6.3}$$

where $t \in [0, \infty)$. Apply this definition whenever $R(t)$ is referred to in this chapter.

The failure rate function $r(k)$ is defined as

$$r(k) = \Pr(X = k \,|\, T \ge k) = \frac{\Pr(X = k)}{\Pr(X \ge k)} = \frac{f(k)}{R(k-1)}, \tag{6.4}$$

provided $\Pr(T \ge i) > 0$. The above equation may be expressed as

$$r(k) = \frac{R(k-1) - R(k)}{R(k-1)}. \tag{6.5}$$

We note that the discrete time failure rate function given here is believed to be first defined in Barlow et al. (1963). It gives the conditional probability of the failure of a device at time k, given that it has not failed by $k-1$. We also note that, in contrast to the continuous counterpart, $r(i) \leq 1$, for every integer $i \geq 0$. The failure rate function defined above uniquely determines a discrete distribution. It is important for readers to note that some authors such as Salvia and Bollinger (1982) and Guess and Park (1988) have defined the survival function $R(k)$ as $\Pr(X \geq k)$ instead of $\Pr(X > k)$ as defined in (6.2) above; so beware of various possible notations and definitions.

Although (6.4) is widely used in the literature, there are a few problems associate with this definition. We will discuss in detail concerning these problems and provide an alternative definition toward the end of the chapter.

Shaked et al. (1995) gave the necessary and sufficient conditions for a sequence $\{r(k), k \geq 1\}$ to be a failure rate:

(a) For all $k < m$, $r(k) < 1$ and $r(m) = 1$. The distribution is defined over $\{1, 2, ..., m\}$, or

(b) For all $k \in \mathcal{N}^+ = \{1, 2, ..\}$, $0 \leq r(k) \leq 1$ and $\sum_{i=1}^{\infty} r(i) = \infty$. The distribution is defined over $k \in \mathcal{N}^+$ in this case.

The mean residual life (or mean remaining life) at time k is defined as

$$\mu(k) = E\left(X - k \,|\, X \geq k\right) \tag{6.6}$$

which can be rewritten as

$$\begin{aligned}\mu(k) &= E\left(X - k \;|X \geq k\right) = E\left(X \mid X \geq k\right) - k \\ &= \sum_{j=k}^{\infty} \tfrac{(j-k)P(X=j)}{P(T \geq k)} = \tfrac{0+1f(k+1)+2f(k+2)+...}{R(k-1)} \\ &= \sum_{j=k}^{\infty} \tfrac{R(j)}{R(k-1)} \\ &= \textstyle\int_k^{\infty} R(t)\,dt / R(k-1).\end{aligned} \tag{6.7}$$

The last equality follows obviously from the fact that $R(k) = \int_k^{\infty} R(t)\,dt$ where $R(t)$ is interpreted by equation (6.3) for positive t. Equation (6.7) also shows that the discrete mean residual lifetime is defined analogously to its continuous time counterpart.

Let μ be the mean lifetime of a device, i.e., $\mu = E(X)$. Then it follows from (6.6) and (6.7) that

$$\mu = \mu(0) = R(0) + R(1) + R(2) + R(3) + ... = \sum_{j=0}^{\infty} R(j), \tag{6.8}$$

since $R(-1) = 1$.

Lemma 6.1:

$$\begin{aligned} R(k) &= \prod_{1 \le i \le k} (1 - r(i)) \\ &= (1 - r(1))\,(1 - r(2)) \ldots (1 - r(k)) \,. \end{aligned} \tag{6.9}$$

Proof:

$$\begin{aligned} &1 - r(1) = 1 - f(1) = \Pr(X > 1) = R(1) \\ &1 - r(2) = 1 - \tfrac{f(2)}{R(1)} = \tfrac{R(1) - f(2)}{R(1)} = \tfrac{R(2)}{R(1)}, \\ &\ldots\ldots, \\ &1 - r(k) = \tfrac{R(k)}{R(k-1)}, \quad \text{so} \\ &\prod_{1 \le i \le k} (1 - r(i)) = R(1) \times \tfrac{R(2)}{R(1)} \times \ldots \times \tfrac{R(k)}{R(k-1)} = R(k), \end{aligned}$$

so the lemma is proved.

The expression in (6.9) is consistent with the one appears in Mi (1993). It now follows that

$$f(k) = R(k-1)r(k) = r(k)\,(1 - r(1))\,(1 - r(1)) \ldots (1 - r(k-1)) \tag{6.10}$$

and thus

$$\mu = E\,(X) = \sum_{k=0}^{\infty} R(k) = \sum_{k=0}^{\infty} \prod_{j=0}^{k} (1 - r(j)),\; r(0) = 0 \tag{6.11}$$

if it exists.

Remarks

(a) If the support of F is now $\mathcal{N} = \{0, 1, 2, \ldots\}$, we then extend our definitions and results in (6.1)–(6.9) to include the case $k = 0$. We note that $R(0)$ would then no longer equal to 1 although $\mu(0)$ is still equal to the mean μ with the same expression; i.e.,

$$\mu = \mu(0) = R(0) + R(1) + \ldots \tag{6.12}$$

We need to be aware of different versions of definitions concerning discrete lifetime distribution.

(b) There is an alternative definition for MRL which is a slight modification of the one given above:

$$\mu(k) = E(X - k \,|\, X > k), \tag{6.13}$$

see for example, Berenhaut and Lund (2002). Using the same arguments given earlier, we can verify that

$$\mu(k) = \int_k^{\infty} R(t)\,dt / R(k) = \sum_{j=k}^{\infty} R(j)/R(k) \tag{6.14}$$

which has an identical expression as its continuous counterpart, with $R(t)$ being defined by (6.3). We note that the two expressions of $\mu(k)$ given by (6.7) and (6.14) have the same shape. To avoid a possible confusion, we take $\mu(k)$ given in (6.7) as our definition for the discrete MRL function unless otherwise stated.

(c) Define

$$H(k) = \sum_{j=1}^{k} r(j),\ k \geq 1 \tag{6.15}$$

which can be considered as a discrete time cumulative hazard function. If $\{r(k), k \geq 1\}$ is small, we can see from (6.9) that the reliability function can be approximated by

$$R(k) \approx \exp\{-(r(1) + ... + r(k))\} = \exp(-H(k)).$$

It now follows from (6.10) that the probability function can also be approximated as

$$f(k) \approx r(k) \exp(-H(k-1)).$$

These approximations correspond to the well known representations of $R(t)$ and $f(t)$ for the continuous lifetime random variables. However, the fact that $r(t)$ is required to be small for these approximations to hold indicates the definition of $r(t)$ in (6.4) may not completely resemble its continuous counterpart.

(c) Salvia and Bollinger (1982) have shown that a necessary and sufficient condition that $\{f(k)\}$ defines a proper probability function is $H(k) \to \infty$ as $k \to \infty$.

The discrete version of ageing concepts given below are not fully equivalent to the usual ageing concepts for continuous distributions. We will elaborate these differences in subsequent sections.

6.3 Elementary Ageing Classes

Generally speaking, the discrete ageing classes are defined analogously to their continuous counterpart. So the definitions given below come with no real surprise.

6.3.1 IFR and DFR

Definition 6.1: F is IFR (increasing failure rate) if $r(k)$ is increasing in $k = 1, 2, ...$ (Salvia and Bollinger 1982).

From (6.9), we see that $R(k)/R(k-1) = 1 - r(k)$. Hence it obvious that the above definition is equivalent to $R(k+1)/R(k)$ decreasing in $k = 1, 2, ...$;

so it is analogous to the condition that defines the continuous IFR property (Barlow and Proschan, 1981, p. 98).

Definition 6.2: F is DFR (decreasing failure rate) if $r(k)$ is decreasing in $k = 1, 2, ...$ (cf. Definition 2.1)

This is equivalent to $R(k+1)/R(k)$ increasing in $k = 1, 2, ...$. This is obviously also equivalent to the condition: $R(k+1)^2 \leq R(k)R(k+2)$ (Langberg et al. 1980b).

Borrowing the idea from the continuous counterpart (Lemma 5.9 of Barlow and Proschan, 1981, p. 77), Gupta et al. (1997) used the log concavity (convexity) as a sufficient condition for a discrete lifetime distribution to be IFR (DFR). Recall,

- A distribution is log concave if and only if $f(k+2)f(k) > [f(k+1)]^2$, $k \geq 0$. This is equivalent to $\left\{\frac{f(k+1)}{f(k)}\right\}$ decreasing in k.
- A distribution F is log convex if and only if $f(k+2)f(k) < [f(k+1)]^2$, $k \geq 0$. This is equivalent to $\left\{\frac{f(k+1)}{f(k)}\right\}$ increasing in t.

Let $\eta(k) = 1 - \frac{f(k+1)}{f(k)}$. Gupta et al. (1997) defined

$$\Delta\eta(k) = \eta(k+1) - \eta(k) = \frac{f(k+1)}{f(k)} - \frac{f(k+2)}{f(k+1)}$$

and they showed that

(i) F is IFR if it is log concave, i.e, if $\Delta\eta(k) > 0$.

(ii) F is DFR if it is log convex, i.e, if $\Delta\eta(k) < 0$.

(iii) If the sequence $\left\{\frac{f(k+1)}{f(k)}\right\}$ is constant, $k \geq 0$; (this is equivalent to $\frac{f(k+1)}{f(k)} = \frac{f(k+2)}{f(k+1)}$) then $f(k) = c^k f(0)$, where c is a constant.
Three distributions are possible.

(a) F is geometric with $f(k) = p(1-p)^k$, $k = 0, 1, 2, ...$
In this case, we have a constant failure rate.

(b) F is uniform with $f(k) = c$, $k = 0, 1, 2, ..., m$.
In this case, we have IFR.

(c) $f(k) = \dfrac{c^k}{1 + c + c^2 + ... + c^m}$, $i = 0, 1, 2, ..., m$.
In this case, we have IFR.

Gupta et al. (1997) also identified several IFR and DFR distributions based on the shape of the sequence $\left\{\frac{f(i+1)}{f(i)}\right\}$.

Remark

The above definition $\eta(k) = 1 - \frac{f(k+1)}{f(k)} = -\frac{f(k+1)-f(k)}{f(k)}$ is equivalent to Glaser's eta function $\eta(t) = -\frac{f'(t)}{f(t)}$ defined in (2.9) and $\Delta\eta(k) = \eta(k+1) - \eta(k)$ is equivalent to $\eta'(t)$ in Theorem 2.1. So the results of Gupta et al. (1997) are parallel to Glaser's (1980).

Properties of IFR class

For an increasing failure rate (IFR) distribution,

(i) Salvia and Bollinger (1982) gave an upper bound for the reliability function by applying the condition $r(1) \leq r(2) \leq ...$ to (6.9) so that

$$R(k) \leq (1 - r(1))^k \approx \exp(-r(1)k). \tag{6.16}$$

(ii) Using the above inequality and (6.11), Salvia and Bollinger (1982) also showed that

$$\mu \leq [1 - r(1)]/r(1). \tag{6.17}$$

(iii) Salvia (1996) showed that the MRL is bounded by

$$1 - r(k) \leq \mu(k) \leq [1 - r(1)]/r(1). \tag{6.18}$$

Properties of DFR class

Langberg et al. (1980b) commented that "...Discrete DFR life distributions govern (a) in the group data case, the number of periods until failure of a device governed by a DFR life distribution; and (b) the number of seasons a TV show is run before being canceled. Thus DFR life distributions are of great importance despite of their relative neglect in the reliability literature".

Some of the properties of a DFR class are now given below.

(i) The inequalities in (6.16) and (6.17) are reversed for DFR distributions.

(ii) Langberg et al. (1980b) also showed that the class of discrete DFR life distributions is a convex class so that $\theta F_1 + (1 - \theta)F_2$ is DFR if both F_1 and F_2 are DFR. This is parallel to the results pertaining to the continuous counterpart. They identified the extreme points of this convex class.

(iii) They also showed how to represent any discrete DFR distribution as a mixture of these extreme points of the convex class.

(iv) If the mean $\mu = E(X)$ exists, Salvia (1996) showed that

$$[1 - r(1)]/r(1) \leq \mu(k) < \mu(k-1)/[1 - r(1)]. \tag{6.19}$$

(v) Consider a classical recurrent event (renewal) sequence $\{u_n\}$ defined through the generating function:

$$U(z) = \sum_{n=0}^{\infty} u_i z^i = (1 - \mathcal{F}(z))^{-1}, \; u_0 = 1. \tag{6.20}$$

It can be verified that the resulting sequence satisfies the relationship:

$$u_k = \sum_{i=1}^{k} f(i) u_{k-i}. \tag{6.21}$$

where $\mathcal{F}(z) = \sum_{i=1}^{\infty} f(i) z^n$ is the probability generating function of the random variable X with support in $\mathcal{N}^+ = \{1, 2, ...\}$. Berenhaut and Lund (2002) studied the geometric convergence of $u_n \to u_\infty$. They have shown that if F is DFR, then

$$|u_n \to u_\infty| \leq \frac{1}{\mu(r-1)} r^{-n}, \;\; n \geq 0, \, r \in (1, R_F) \tag{6.22}$$

where R_F is the radius of convergence of $\mathcal{F}$. Letting $r \uparrow r_F$ in (6.22) gives

$$|u_n \to u_\infty| \leq \frac{1}{\mu(R_F - 1)} R_F^{-n}, \, n \geq 0. \tag{6.23}$$

6.3.2 IFRA and DFRA

Recall from Chapter 2.4, we say that a continuous distribution F is IFRA if $-(1/t) \log R(t)$ is increasing in $t \geq 0$ which is equivalent to $H(t)/t$ increasing in t. The equivalence of the two definitions in the continuous time case arises from the relationship $H(t) = -\log R(t)$. The discrete analogues of these two definitions of ageing concepts are:

Definition 6.3: F has increasing failure rate IFRA1 if $1 \geq R_0 \geq R_1 \geq ...$ such that $R(k)^{1/k}$ is decreasing in k. Similarly, F has decreasing failure rate (DFRA1) if $R(k)^{1/k}$ is in creasing in k (Esary et al., 1973). The condition is equivalent to $-\frac{1}{k} \log R(k)$ is increasing in k.

Definition 6.4: F has increasing failure rate IFRA2 if $\left\{\frac{H(k)}{k}\right\}$ is an increasing sequence in k; $H(k) = r(1) + r(2) + ...r(k)$ is defined as in (6.15). (cf. Definition 2.2).

It has been noted by several authors, for example Lawless (2003), that Definitions 6.3 and 6.4 are not equivalent because $-\log R(k)$ and $H(k)$ are not the same for the discrete case. Applying the arithmetic-geometric inequality to (6.9), it is easy to show IFRA1 $\Rightarrow$ IFRA2; but the converse is not true. A counter example was provided by Shaked et al. (1995). We will later provide a chain of relationships between various discrete ageing concepts.

6.3.3 NBU (NWU)

Recall from Section 2.4 we say that a continuous distribution F is NBU if $R(x+t) \leq R(x)R(t)$ for $x, t \geq 0$. This has been shown to be equivalent to $\int_0^t r(u)\,du \leq \int_x^{x+t} r(u)\,du$.

The discrete analogues of the preceding definitions are now stated below:

Definition 6.5: F is NBU1 (new better than used) if

$$R(j+k) \leq R(j)R(k), \quad j,k = 0,1,2,... \tag{6.24}$$

(Esary et al., 1973).

Definition 6.6: F is NBU2 (new better than used) if

$$\sum_{i=1}^{k} r(i) \leq \sum_{i=j+1}^{j+k} r(r),\ k \geq 1. \tag{6.25}$$

(cf. Definition 2.4).

Shaked et al. (1995) have shown by counter examples that none of (6.24) and (6.25) implies the others.

We have seen that all the discrete ageing classes presented so far are defined through $r(k)$ or $R(k)$. In fact, this is true for most discrete ageing classes.

The dual of NBU1 may be called NWU1 which is defined by reversing the inequality of (6.24). Similarly, NWU2 class may be defined by reversing the inequality in (6.25).

Berenhaut and Lund (2002) showed that if F is NWU1, then the renewal sequence $\{u_n\}$ defined in (6.21) satisfies

$$|u_n - u_\infty| \leq \frac{1}{(R_F/c-1)} \left(R_F/c\right)^{-n}, n \geq 0, \tag{6.26}$$

where $c = (1+\sqrt{5})/2$.

6.3.4 NBUE

Definition 6.7: F is NBUE (new better than used in expectation) if

$$R(k)\sum_{j=0}^{\infty} R(j) \geq \sum_{j=k}^{\infty} R(j),\ k = 1,2... \tag{6.27}$$

We also observe that this is equivalent to $\sum_{j=k}^{\infty} R(j)/R(k) \leq \mu$, $k = 1,2...$, which is analogous to the similar property of the continuous NBUE notion. (cf. Definition 2.5).

By summing both sides of (6.24) over $i = 0$ to ∞, it is clear that NBU1 $\Rightarrow$ NBUE.

We say that F is NBWE if the inequality in (6.27) is reversed.

6.3.5 DMRL and IMRL

Definition 6.8: F is DMRL (decreasing mean residual life) if the sum $\sum_{j=k}^{\infty} R(j)/R(k)$ is decreasing in k. (Esary et al., 1973, Ebrahimi, 1986).

Now, $\sum_{j=k}^{\infty} R(j)/R(k) = \mu(k)$ according to our definition in (6.13). So, the above definition is equivalent to the the sequence $\{\mu(k)\}$ being decreasing in k. Thus, the acronym 'DMRL' seems appropriate for this definition. Similarly, F is IMRL (increasing mean residual life) if $\mu(k)$ is increasing in k or $\sum_{j=k}^{\infty} R(j)/R(k)$ is increasing in k. (cf. Definition 2.3).

A necessary and sufficient condition for a distribution to belong to a DMRL was given by Ebrahimi (1986) as follows:

Theorem 6.1: F is DMRL iff there exists a decreasing sequence $\{a_n > 0\}$ such that for all n, the failure rate function can be expressed as

$$r(k) = 1 - \frac{a_k}{1 + a_{k+1}}, \quad k = 0, 1, 2, ... \tag{6.28}$$

Similarly, F is IMRL (increasing mean residual life) if (6.28) holds for an increasing sequence $\{a_n\}$.

To prove the theorem, we need the following lemma from Ebrahimi (1986).

Lemma 6.1: Let $\{a_n\}$ and $\{b_n\}$ be two converging positive sequences and let $c > 0$. If $a_n/(c + a_{n+1}) \equiv b_n/(c + b_{n+1})$ for all n, then $a_n \equiv b_n$ for all n.

Proof: The proof of the lemma involves essentially:

Case I: $a \equiv b \neq 0$ and

Case II: $a \equiv b = 0$

where $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$.

Proof of Theorem 6.1: Let $\bar{r}(k) = 1 - r(k)$. Suppose F is DMRL so $\mu(k)$ is a decreasing in k. Now

$$\begin{aligned}
\mu(k) &= \sum_{j=k}^{\infty} R(j)/R(k-1) = \sum_{i=k+1}^{\infty} R(i-1)/R(k-1) \\
&= \sum_{i=k+1}^{\infty} \prod_{j=k}^{i-1} R(j)/R(j-1) \\
&= \sum_{i=k+1}^{\infty} \prod_{j=k}^{i-1} \bar{r}(j) \quad \text{by (6.9)} \\
&= \bar{r}(k) + \bar{r}(k) \sum_{i=k+2}^{\infty} \prod_{j=k+1}^{i-1} \bar{r}(j) \\
&= \bar{r}(k)\,(1 + \mu(k+1))
\end{aligned}$$

Thus,

$$r(k) = 1 - \frac{\mu(k)}{1 + \mu(k+1)}. \tag{6.29}$$

So the choice of $a_j \equiv \mu(j)$ gives the required representation of $r(j)$.

Suppose the representation $\bar{r}(k) \equiv \dfrac{a_k}{1 + a_{k+1}}$ holds. Since a_j decreases, $\lim_{n\to\infty} a_n \equiv a$ exists and it is non-negative. Therefore $\lim_{j\to\infty} \dfrac{a_j}{1 + a_{j+1}} < 1$. By the ratio test, $\mu(j)$ converges and it can be shown that $\lim_{j\to\infty} \mu(j) = b$. It now follows from Lemma 6.1 that $a_k \equiv \mu(k)$ for all k. This completes the proof of the theorem.

Berenhaut and Lund (2002) also showed that the geometric bounds (6.21) and (6.22) also hold for distributions which are IMRL.

Practically speaking, the above theorem does not appear to help us much in identifying a discrete DMRL (IMRL) distribution. This is because the condition given in the theorem is more or less the necessary and sufficient condition for F to be DMRL (IMRL). Note that (6.29) holds for any discrete distribution and $a_k = \mu(k)$ is the only solution to the equation $r(k) = 1 - \frac{a_k}{1+a_{k+1}}$.

Recall from Section 6.3.1, F IFR iff $R(k+1)/R(k)$ is decreasing in k. From this characterization, we have

$$\begin{aligned}
\mu(k) = \sum_{j=k}^{\infty} R(j)/R(k-1) &= \sum_{i=k+1}^{\infty} R(i-1)/R(k-1) \\
&= \sum_{i=k+1}^{\infty} \prod_{j=k}^{i-1} R(j)/R(j-1) \\
&\geq \sum_{i=k+1}^{\infty} \prod_{j=k}^{i-1} R(j+1)/R(j) \quad \text{since } F \text{ IFR} \\
&\geq \sum_{i=k+2}^{\infty} \prod_{j=k}^{i-1} R(j+1)/R(j) \\
&= \sum_{i=k+2}^{\infty} \prod_{j=k+1}^{i-1} R(j)/R(j-1) = \mu(k+1)
\end{aligned}$$

Hence $\mu(k) = \sum_k^{\infty} R(j)/R(k)$ is decreasing if F is IFR. Thus, IFR $\Rightarrow$ DMRL. However, the converse is not true in general. Ebrahimi (1986) has given a sufficient condition for $F \in$ IMRL to imply $F \in$ IFR.

Example 6.1

Define $a_0 = 1$, $a_1 = 4/6$, $a_2 = 3/5$ and $a_n = 1/n$ for all $n \geq 3$. It is easy to verify that $r(k) = 1 - \frac{a_i}{1+a_{i+1}}$ is DMRL but it is not IFR.

6.3.6 Relationships Among Discrete Ageing Concepts

So far, we have defined several discrete ageing concepts and some of which have two versions. We now wish to provide a link to show the relative strengths of these concepts.

First, we show IFR $\Rightarrow$ IFRA1. The proof here is essentially from Ross et al. (1980). Let $\bar{r}(k) = 1 - r(k)$ which decreases in k because F is IFR. It follows from (6.9) that $R(k) = \prod_{i=1}^{k} \bar{r}(k)$. We need to show that $R^{1/k}(k) = \{\prod_{i=1}^{k} \bar{r}(k)\}^{1/k}$ is decreasing in k.

Without loss of generality, we consider $k = 3$ only and the general case will follow easily. $\bar{r}(1)\bar{r}(2)\bar{r}(3) \leq \bar{r}(1)\bar{r}(2) \times \sqrt{\bar{r}(1)\bar{r}(2)} = \{\bar{r}(1)\bar{r}(2)\}^{2/3}$ because $\bar{r}(3)^2 \leq \bar{r}(1)\bar{r}(2)$. So $\{\bar{r}(1)\bar{r}(2)\bar{r}(3)\}^{1/3} \leq \{\bar{r}(1)\bar{r}(2)\}^{1/2}$ and hence F is IFRA1.

The reverse is not true. A counterexample is now given below.

Example 6.2

Consider a sequence $\{r(k), k \geq 1\}$ defined by:

$$r(1) = 0.1,\ r(2) = 0.4, \text{ and } r(k) = 0.3 \text{ for all } \ k \geq 3,$$

so F is not IFR. However, we have $R(1) = 0.9, R(2) = 0.54$ and for all $k \geq 3$, $R(k) = 0.54 \times 0.7^{k-2}$. It is now easy to show that $R(k+1)^{1/k+1}/R(k)^{1/k} = (49/50)^{\frac{1}{k(k+1)}} \leq 1$ so the sequence $\{R(k)^{1/k}\}$ is decreasing and hence IFRA1 according to Definition 6.3.

We have also shown earlier that IFRA1 $\Rightarrow$ IFRA2. Further, it is easy to verify that IFRA1 $\Rightarrow$ NBU1 and IFRA2 $\Rightarrow$ NBU2. We have also proved that IFR $\Rightarrow$ DMRL in the preceding subsection.

To sum up these interrelationships, we give the following flow chart to link various discrete ageing concepts:

$$\begin{array}{lllllll} \text{IFR} & \Rightarrow & \text{IFRA1} & \Rightarrow & \text{NBU1} & \Rightarrow & \text{NBUE} \\ \Downarrow & & \Downarrow & & & & \\ \text{DMRL} & & \text{IFRA2} & \Rightarrow & \text{NBU2} & & \end{array}$$

The table above is analogous to the one given for the continuous ageing classes in Section 2.4.3.

6.4 More Advanced Ageing Classes

Let $\{f(k), k \in \mathcal{N}\}$ be a discrete distribution with support in $\mathcal{N} = \{0, 1, 2..\}$ and $R(k) = 1 - F(k)$ where $F(k) = \sum_{i=0}^{k} f(i)$.

The following definitions are parallel to those continuous analogues defined in Section 2.7 and Section 2.4.

Definition 6.9: (Fagiuoli and Pellerey, 1994). A discrete distribution $\{f(k), k \in \mathcal{N}\}$ is said to be:-

- Discrete IFR(2) if, for all $i \geq 0$, $\sum_{j=k}^{k+i} R(j)/R(k)$ is non-increasing in k;
- Discrete NBU(2) if, for all $i, k \geq 0$, $R(k)\sum_{j=0}^{i} R(j) \geq \sum_{j=k}^{k+i} R(j)$;
- Discrete NBUC if, for all $i, k \geq 0$, $R(k)\sum_{j=i}^{\infty} R(j) \geq \sum_{j=k+i}^{\infty} R(j)$;
- NBUFR if, for all $k \geq 0$, $R(k+1) \leq R(k)R(1)$;
- NBAFR if, for all $k \geq 0$, $R(k)^{1/k} \leq R(k)R(1)$.

If $\{f(k), k \in \mathcal{N}^+\}$, we then change $i \geq 0$ and/or $k \geq 0$ above to $i \geq 1$ and/or $k \geq 1$ to accommodate the change of the support. The first two are analogous to the continuous counterparts given in Section 2.7 whereas the last three are similar to the corresponding ones in Section 2.4. For details concerning these definitions of discrete ageing, we refer our readers to Fagiuoli and Pellerey (1994).

6.5 Non-monotonic Models

The family of non-monotonic models may be divided into two classes. We first need to define the concepts of bathtub shape and upside-down bathtub shape of a discrete sequence of real numbers:

Definition 6.10: A sequence $\{a_i, i \geq 0\}$ of real numbers is said to have a bathtub shape or an upside-down bathtub shape if there exits integers $1 \leq n_1 \leq n_2 < \infty$ such that

$$a_0 > a_1 > a_2 > ....a_{n-1} > a_{n_1} = ... = a_{n_2} < a_{n_2+1} < ...$$

or

$$a_0 < a_1 < a_2 < ....a_{n-1} < a_{n_1} = ... = a_{n_2} > a_{n_2+1} > ...$$

See Guess and Park (1988) or Mi (1993) for this definition.

Non-monotonic failure rate distributions

Definition 6.11: F is BT (bathtub shaped failure rate) if the failure rate $r(k) = f(k)/R(k-1)$ is decreasing initially and then increasing in k. Guess and Park (1988) used an alternative abbreviation DIFR to denote this class of distributions.

Remark: Mi (1993) showed that $\{r(i),\ i \geq 1\}$ has a bathtub shape iff the sequence $\{f(i)/R(i),\ i \geq 1\}$ has a bathtub shape.

Definition 6.12: F is UBT (upside-down bathtub shaped failure rate) if $r(k)$ is increasing initially and then decreasing in k. It is also known as an IDFR distribution in Guess and Park (1988).

Non-monotonic mean residual life distributions

Definition 6.13: F is IDMRL (DIMRL) if the MRL sequence $\{\mu(k)\}$ is increasing (decreasing) initially and then decreasing (increasing) in k. (cf. Definition 4.2).

Using a similar proof to that of Theorem 6.1, Guess and Park (1988) showed that F is IDMRL (DIMRL) iff there exists an upside-down (bathtub) shaped sequence $\{a_n\}$, $a_n > 0$ such that $\bar{r}(k) = 1 - r(k) = a_n/(1 + a_{n+1})$ for $n = 0, 1, 2, ...$

6.5.1 BT Failure Rate and DIMRL

Guess and Park (1988) saw the relationship between the bathtub shaped failure rate and the non-monotonic behavior of the mean residual life. Although it is often that a BT distribution gives rise to a DIMRL distribution, the converse is not true in general. A simple example is presented in Guess and Park (1988) where the mean residual life increases, then decreases; however, the failure rate also increases, drops suddenly at one cycle, then increases.

Example 6.3: Guess and Park (1988)

Consider the sequence

$$a_n = \theta^2(|n - 40| + \theta) + \gamma, n = 0, 1, 2, ..,$$

with $\theta = 62.71, \gamma = 61.71$. $r(k)$ is computed via $r(k) = 1 - \frac{a_k}{1+a_{k+1}}$, for $k = 0, 1, 2...$; and $\mu(k)$ is computed through (6.7) and (6.9).

Similarly, one can construct a BT distribution F that is not IDMRL. This phenomenon is parallel to the continuous lifetime case. Mi (1993) has given a sufficient condition for a BT distribution that is also IDMRL. The following theorem is analogous to Theorem 4.3.

Theorem 6.2: Let F be a lifetime distribution having support $\{1, 2, ...\}$. Let $\{r(k), k \geq 1\}$ be the failure rate of F. If $\{r(k)\}$ has a bathtub shape with change points n_1 and n_2, $n_1 \leq n_2 < \infty$; then for the sequence $\{\mu(k), k \geq 1\}$ of mean residual life there are three cases:

(i) if $f(1) < \dfrac{1}{1+\mu^*}$, then $\mu(k)$ strictly decreases, where $\mu^* = \int_1^\infty R(t)\, dt$;

(ii) if $f(1) = \dfrac{1}{1+\mu^*}$, then

$$\mu(1) = \mu(2) > \mu(3) > ...;$$

(iii) if $f(1) > \dfrac{1}{1+\mu^*}$, then $\{\mu(k)\}$ has an upside-down bathtub shape with a unique change point, denoted by m_0, and $m_0 \leq n_1$, or two change points $m_0 - 1$ and $m_0 \leq n_1$.

Proof: The proof below follows closely as that of Theorem 2 of Mi (1993).

Recall, $\mu(i) = \dfrac{\int_i^\infty R(t)\,dt}{R(i-1)}$. Then it is easy to show that

$$\mu(i+1) - \mu(i) = \frac{A(i)}{R(i-1)}, \tag{6.30}$$

where

$$A(i) = \frac{f(i)}{R(i)} \int_{i+1}^{\infty} R(t)\,dt - R(i). \tag{6.31}$$

By the remark that follows Definition 6.11, we see that $\{f(i)/R(i)\}$ also has a BT shape with change points n_1 and n_2 since the sequence $\{r(i),\, i \geq 1\}$ does. It is now easy to check that for any $i \geq n_1$,

$$\begin{aligned} \frac{f(i)}{R(i)} \int_{i+1}^{\infty} R(t)\,dt &< \int_{i+1}^{\infty} \frac{f(t)}{R(t)} R(t)\,dt \\ &= \sum_{j=i+1}^{\infty} f(j) = R(i), \text{ for } i \geq n_1, \end{aligned} \tag{6.32}$$

where $f(t) = f(j)$ if $t \in [j, j+1)$. Thus $A(i) < 0$ for $i \geq n_1$.

Next, we show that

$$\begin{aligned} \Delta A(i) &= A(i+1) - A(i) \\ &= \left(\frac{f(i+1)}{R(i+1)} - \frac{f(i)}{R(i)} \right) \int_{i+1}^{\infty} R(t)\,dt,\ i \geq 1. \end{aligned}$$

As $\{f(j)/R(j)\}$ has a BT shape, it follows from the preceding equation that $\{A(i),\, i \geq 1\}$ also has a BT shape with the same change points n_1 and n_2.

Now consider the three cases.

Case (i): $f(1) < 1/(1+\mu^*)$. It can be proved that this condition is equivalent to $A(1) < 0$. To see this, we note from (6.31) that if $A(1) < 0$, then from (6.32)

$$\frac{f(1)}{R(1)} \int_{2}^{\infty} R(t)\,dt < R(1)$$

which implies

$$\frac{(1-f(1))^2}{f(1)} > \int_{2}^{\infty} R(t)\,dt = \mu^* - (1 - f(1)),$$

and consequently $[1 - f(1)]/f(1) \geq \mu^*$; i.e., $f(1) < 1/(1+\mu^*)$.

As $A(i) < 0$ for all $i \geq n_1$, it follows that $A(i) < 0$ for all $i \geq 1$ ($A(i)$ has a BT shape) so by (6.30), $\mu(i)$ strictly decreases.

Case (ii): $f(1) = 1/(1+\mu^*)$. From the argument used in Case (i), we see that this condition is equivalent to $A(1) = 0$. Since $\{A(i)\}$ has a BT shape, it follows that $A(i) < 0$ for all $i \geq 2$ so by (6.30),

$$\mu(1) = \mu(2) > \mu(3) > \cdots;$$

i.e., $\{\mu(k),\ k \geq 2\}$ is strictly decreasing.

Case (iii): $f(1) > 1/(1+\mu^*)$. Again from Case (i) we can show that the condition is equivalent to $A(1) > 0$. Define the integer m_0 as follows:

$$m_0 \equiv \sup\{i \geq 1 : A(i) \geq 0\}.$$

Since $A(k) < 0$ for all $k \geq n_1$ (thus in this case $n_1 > 1$), we must have $1 \leq m_0 \leq n_1 - 1$. Now it is obvious that $A(k) \geq 0$ for $k = 1, 2, ..., m_0$ and the equality may hold only at $k = m_0$; thus

$$\mu(1) < \mu(2) \cdots < \mu(m_0 - 1) < \mu(m_0) \leq \mu(m_0 + 1).$$

Furthermore, it is true that $A(k) < 0$ for all $k > m_0$. To see this we note that by (6.31) and (6.32), we have $\lim\limits_{k\to\infty} A(k) \leq 0$. This, along with the fact that $\{A(k), k \geq 1\}$ has a BT shape with change points n_1 and n_2, shows that $A(k) < 0$ for all $k > m_0$. Therefore, by (6.30) we obtain

$$\mu(1) < \mu(2) \cdots < \mu(m_0 - 1) < \mu(m_0) \leq \mu(m_0 + 1) > \mu(m_0 + 1) > \cdots;.$$

i.e., the sequence $\{\mu(k), k \geq\}$ has an UBT shape with a unique change point $k_0 = m_0 + 1 \leq n_1$, or two change point $k_0 - 1 = m_0$ and $k_0 \leq n_1$.

The converse of the above theorem is not necessarily true. The preceding example given in Guess and Park (1988) is a counter example. Mi (1993) showed that under an additional condition the converse is true.

Notes:

- To avoid a possible confusion, we have changed Mi's original notation μ defined by $\int_1^\infty R(t)\,dt$ to μ^*. It is clear from (6.7) that $\mu^* = \mu(1)R(0) = \mu(1)$. From (6.7) and (6.8), we see that $\mu(1) = R(1)+R(2)+R(3)+... = \mu - R(0)$ with $\mu = E(X)$, the mean time to failure. Thus $\mu = 1+\mu(1) = 1+\mu^*$ since $R(0) = 1$. So the expression $1/(1+\mu^*)$ in the theorem above may be replaced by $1/(1+\mu(1)) = 1/\mu$. Thus, Mi's condition for the behavior of MRL function $\mu(k)$ life can be replaced by $f(1) < 1/\mu$ (or $r(1) < 1/\mu$ as $f(1) = r(1)$) which seems to be consistent with the condition given in Gupta and Akman (1995a,b) for the continuous case.
- If the support for F were $\{0, 1, 2, ...\}$ instead of $\{1, 2, ...\}$ then $f(1)$ would be replaced by $f(0) = r(0)$ whereby the condition (ii) would reduce to $r(0) > 1/\mu$ which is exactly the same as that of Gupta and Akman (1995a,b).

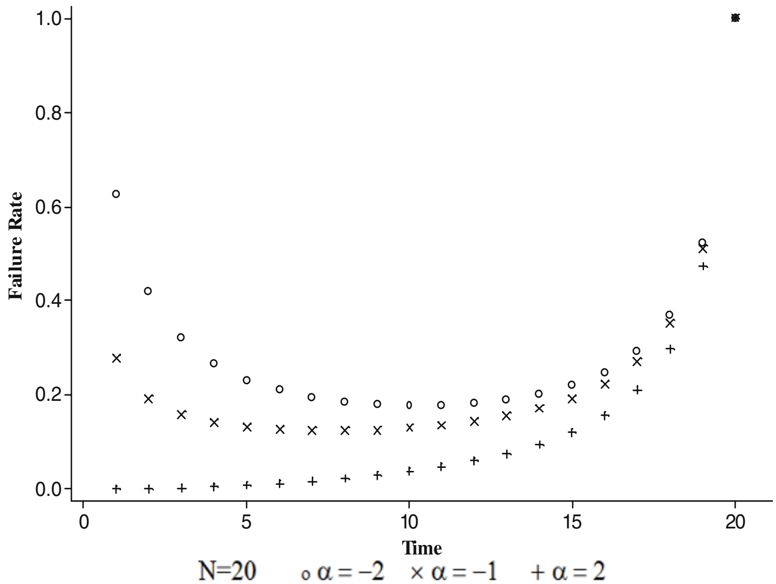
Example 6.4: A discrete bathtub distribution

(Lai and Wang, 1995). A simple discrete distribution was proposed with

$$f(k) = \frac{k^{\alpha}}{\sum_{x=0}^{N} x^{\alpha}}, \; k = 0, 1, 2, ..N$$

which has a finite range.

Fig. 6.1. Failure rate function of a finite range distribution



Lai and Wang (1995) have shown that

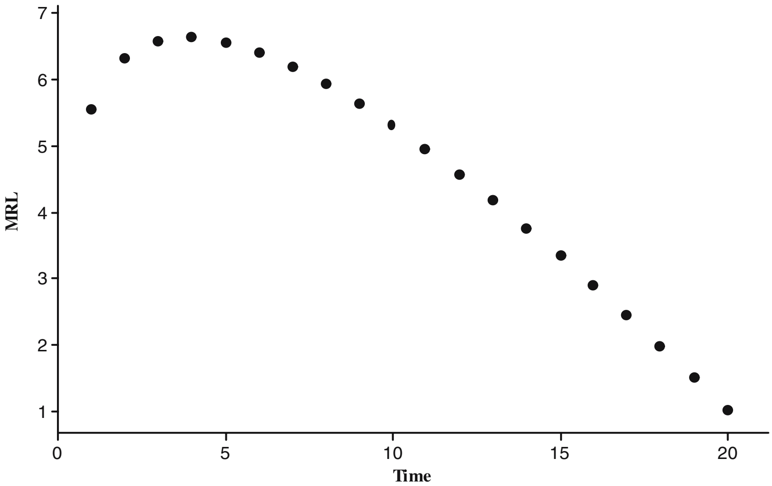
1. F is IFR if $\alpha \geq 0$,
2. F has a bathtub shape failure rate if $\alpha < 0$.

At $N = 20$, we have plotted the failure rate function $r(k)$ in Fig. 6.1 for three values of α. For $\alpha = -1$ and $N = 20$, the mean residual life $\{\mu(k)\}$ is plotted in Fig. 6.2 which clearly displays an upside-down bathtub shape.

6.5.2 UBT Failure Rate and DIMRL

Tang et al. (1999) gave sufficient conditions under which a UBT distribution would be DIMRL. The result is a complement to that of Mi (1993) given in Theorem 6.2 above. It is also analogous to the second part of Theorem 4.2.

Fig. 6.2. Mean residual life function of a finite range distribution



Theorem 6.3: Let F be a life time distribution with support set $\{1, 2, ...\}$ and mean μ. If the failure rate sequence $\{r(k), k \geq 1\}$ has an upside-down bathtub shape with a change point k_0, then for the sequence $\{\mu(k), k \geq 1\}$ of MRL there are three cases:

1. $f(1) < 1/(1+\mu^*)$: Then $\{\mu(k), k \geq 1\}$ has either a BT shape with a unique change point $m_0 \leq k_0, k_0 \geq 1$, or two change points at $m_1 = m_0 - 1$ and $m_2 \leq k_0$. Here, $\mu^* = \int_1^\infty R(t)\,dt$.
2. $f(1) > 1/(1 + \mu^*)$: Then $\{\mu(k), k \geq 1\}$ is strictly increasing.
3. $f(1) = 1/(1 + \mu^*)$: Then $\mu(1) = \mu(2) < \mu(3) < ....$

Proof. The proof is almost a mirror image of Theorem 6.2. See Tang et al. (1999).

Comment

We have changed the notation μ in Theorem 3 of Tang et al. (1999) to μ^* to avoid a possible confusion.

6.5.3 Discrete IDMRL (DIMRL) and BT (UBT) Failure Rate

Mi (1993) has found sufficient conditions under which discrete upside-down bathtub shaped (UBT) MRL implies its associated failure rate function having a BT shape. His Theorem 3 is now presented as below:

Theorem 6.4: Let $\{\mu(k)\}$ be a mean residual life sequence for a discrete life distribution F. Suppose $\{\mu(k),\ k \geq 1\}$, has a UBT shape with a unique change point m_0 and the sequence $\{\Delta\mu(k), k \geq 1\}$, where $\Delta\mu(k) = \mu(k+1) - \mu(k)$, has a BT shape with change point $m_0 + 1$. Then the sequence $\{r(k),\ k \geq 1\}$, of the failure rate of F has a BT shape with a unique change point $k_0 = m_0$ or $k_0 = m_0 + 1$.

Proof: It follows from (6.29) that

$$r(k) = 1 - \frac{\mu(k)}{1 + \mu(k+1)}.$$

Rearranging the terms gives

$$r(k) = \frac{[\mu(k+1) - \mu(k)] + 1}{\mu(k+1) + 1} = \frac{\Delta\mu(k) + 1}{\mu(k+1) + 1}. \tag{6.33}$$

Thus,

$$r(k+1) - r(k) = \frac{\mu(k+2)\Delta^2\mu(k) - (\Delta\mu(k+1))^2 - \Delta\mu(k)}{[\mu(k+2) + 1][\mu(k+1) + 1]},\ k \geq 1, \tag{6.34}$$

where $\Delta^2\mu(k) \equiv \Delta\mu(k+1) - \Delta\mu(k)$. Let the numerator of (6.34) be defined by

$$B(k) = \mu(k+2)\Delta^2\mu(k) - (\Delta\mu(k+1))^2 - \Delta\mu(k).$$

Obviously the sign of $r(k+1) - r(k)$ is the same as that of $B(k)$.

For any $k \leq k_0 - 1$ we have $\Delta^2\mu(k) < 0$ since the sequence $\{\mu(j),\ j \geq 1\}$ has a BT shape with change points $k_0 + 1$. Meanwhile $k \leq k_0 - 1$ implies $\Delta\mu(k) > 0$ since $\{\mu(j),\ j \geq 1\}$ has an UBT shape with change point k_0 by assumption. Hence $B(k) < 0$ for $k \leq k_0 - 1$ and consequently from (6.34) we see that

$$r(k+1) - r(k) < 0, \quad \text{for } 1 \leq k \leq k_0 - 1;$$

i.e.,

$$r(1) > r(2) \cdots > r(k_0 - 1) > r(k_0). \tag{6.35}$$

For $k \geq k_0 + 1$, a similar argument to the above yields

$$\Delta^2\mu(k) > 0, \quad \text{for} \quad k \geq k_0 + 1. \tag{6.36}$$

Note that the identity (6.33) yields

$$\Delta\mu(k) + 1 > 0, \quad \text{for} \quad k \geq 1. \tag{6.37}$$

Hence

$$\begin{aligned} -(\Delta\mu(k+1))^2 - \Delta\mu(k) &= \Delta\mu(k+1)[\Delta\mu(k+1) + 1] + \Delta^2\mu(k) \\ &> \Delta^2\mu(k) > 0, \quad \text{for} \quad k \geq k_0 + 1, \end{aligned} \tag{6.38}$$

where the first inequality holds because of (6.37) and our assumption about the sequence $\{\Delta\mu(j),\, j \geq 1\}$, and the second inequality follows from (6.36). Considering (6.34) and (6.38), we conclude that $r(k+1) - r(k) > 0$, for all $k \geq k_0 + 1$, i.e.,

$$r(k_0 + 1) < r(k_0 + 2) < \cdots. \tag{6.39}$$

If $r(k_0) < r(k_0+1)$, then the inequalities (6.35)and (6.39) show that the failure rate sequence $\{r(i),\, i \geq 1\}$ has a BT shape with a unique change point k_0; if $r(k_0) > r(k_0+1)$, then a unique change point k_0+1; and if $r(k_0) = r(k_0+1)$, then two change points k_0 and k_0+1. This completes our proof.

Bekker (2002) obtained sufficient conditions under which BT shaped MRL implies that the associated failure rate function has an UBT failure rate in the discrete case.

6.5.4 Discrete Bathtub-shaped Failure Rate Average

Mi (1993) also defined a discrete bathtub-shaped failure average ageing concept.

Definition 6.14: A discrete distribution F is said to have a bathtub shaped failure rate average if the sequence $\{\bar{R}(k)^{1/k}, k \geq 1\}$ has a bathtub shape.

Theorem 6.5: Suppose F is discrete BT having support $\mathcal{N}^+ = \{1, 2, ..\}$ and $r(k)$ has change points $n_1 > 1$ and $n_2 \geq n_1$. Then F has a BT shaped failure rate average which has a unique change point, denoted by m_0, and $m_0 \geq n_2$, or two change points m_0 and $m_0 + 1$.

Proof: The proof depends on a lemma which states that the sequence, $\bar{a}(k) = (1/k)\sum_{i=1}^{k} a(i)$, of the averages of an UBT shape sequence $\{a(k),\, k \geq 1\}$ also has an UBT shape.

From (6.9), we have

$$R(n) = \prod_{i=1}^{n}(1 - r(i)).$$

It follows at once that

$$n \log(R(n))^{1/n} = \sum_{i=1}^{n} \log(1 - r(i)).$$

Now the assumption that the sequence $\{r(k),\, k \geq 1\}$ has a BT shape implies that $\{\bar{r}(k),\, k \geq 1\}$ has an UBT shape with the same change points n_1 and n_2, where $\bar{r}(k) = 1 - r(k)$. Applying the above mentioned lemma immediately gives the result. See Theorem 1 of Mi (1993).

For non-monotonic failure rate functions, it would be desirable to consider estimation problem of change points. We will not get into detail on this subject but simply note that Mi (1994d) has considered the estimation problem for a discrete failure rate model with a single change point, i.e.,

$$r(k) = \begin{cases} a, & \text{if } k \leq \tau; \\ b, & \text{if } k \geq \tau + 1. \end{cases} \tag{6.40}$$

He obtained a strongly consistent estimator for the change point.

6.6 Preservation under Poisson Shocks

Suppose a device is subject to shocks occurring randomly as events in a Poisson process with constant intensity λ. Suppose further that the device has probability $\bar{P}_k$ of surviving the first k shocks. Then the survival function of the device at time t is given

$$\bar{H}(t) = \sum_{k=0}^{\infty} \bar{P}_k \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \ t \geq 0. \tag{6.41}$$

For derivation of (6.41), see for example, Barlow and Proschan (1981). For convenience, let $\bar{P}_k = R(k-1)$. The ageing properties of the discrete life distribution $\{R(k), k \in \mathcal{N}\}$ are well reflected in the corresponding properties of the continuous life distribution $H(t)$ through the sequence $\{\bar{P}_k\}$. In other words, if the discrete distribution defined through $\bar{P}_k$ belongs to the ageing class C, then the continuous distribution specified by $H(t)$ above also belongs to the class C. This is shown by Esary et al. (1973) for IFR, IFRA, DMRL, NBU and NBUE classes, by Klefsjö (1981, 1983b,c) for HNBUE and $\mathcal{L}$, by Singh and Deshpande (1985) for SIFR and SNBU, by Abouammoh and Ahmed (1988) for NBUFR, and by Fagiuoli and Pellerey (1994) for IFR(2), NBU(2), NBUC, NBUFR.

6.7 Examples of Discrete Time Failure Models

We now present several probability distributions used in reliability for modelling lifetime of nonrepairable systems. The basic properties of each model is given. Bracquemond and Gaudoin (2002) classified these distributions into two families. We now add on another two. The first class consists of discrete life distributions derived from usual continuous lifetime distributions and the second class contains distributions derived by pre-specifying the failure rate functions. The third class is based on the characterizations of $\{f(k+1)/f(k)\}$ as advocated in Gupta et al. (1997). The last family consists of distributions arise from the Polya urn schemes.

To determine the shape of the failure rate function, we define

$$\eta(k) = -\frac{f(k+1) - f(k)}{f(k)} = 1 - \frac{f(k+1)}{f(k)}, \tag{6.42}$$

which is similar to $\eta(t) = -\frac{f'(t)}{f(t)}$ in (2.9), defined by Glaser's (1980) for the continuous case. Like its predecessor, $\eta(k)$ can be used to characterize a distribution or determine the shape of $r(k)$. Instead of using $\eta(k)$ directly, we use $\frac{f(k+1)}{f(k)}$ for our analysis.

6.7.1 Common Discrete Lifetime Distributions Derived from Continuous Ones

There are several lifetime distributions which are the discrete analogues of their continuous counterparts. In what follows, we give only simple properties of each distribution, namely, the probability function, the reliability function, the ageing class it belongs to; and a reference where one may find a detailed study.

Geometric distribution

(Johnson et al., 1992).

- $f(k) = p(1 - \ p)^k$, $k \geq 0, 0 < p < 1$ is a constant.
- $R(k) = (1-p)^k$.
- $r(k) = p$.

The failure rate is a constant so the distribution is both IFR and DFR. Thus it is a discrete analogue of the exponential distribution. The mean time to failure MTTF $= q/p$.

Negative binomial

(Johnson et al., 1992).

- $f(k) = \binom{n+k-1}{n-1} p^n (1-p)^k$, $k \geq 0, n > 0, 0 < p < 1$.
  Similar to the gamma distribution, $R(i)$ and $r(i)$ of the negative binomial are in complex forms.
- $$\frac{1}{r(k)} = 1 + \frac{1}{\binom{n+k-1}{n} p^\alpha q^k} \left[1 - \sum_{i=0}^{k} \binom{n+i-1}{i} p^\alpha q^i \right], \ q = 1-p,$$

  (Gupta et al., 1997).

However, it is easy to verify that the sequence $\left\{\frac{f(k+1)}{f(k)} = 1 + \frac{\alpha-1}{k+1}, k \geq 1\right\}$ is decreasing in k if $\alpha > 1$ and it is increasing in k if $\alpha < 1$. Thus F is IFR for $\alpha > 1$ and DFR for $\alpha < 1$.

The negative binomial is the discrete analogue of the gamma; it reduces to the geometric if $\alpha = 1$.

Type I discrete Weibull distribution

(Nakagawa and Osaki, 1975). This is an analogue of the continuous Weibull distribution with

$$R(t) = e^{-t^{\alpha}}, \alpha > 0,\ t \geq 0.$$

For this discrete distribution,

- $f(k) = q^{(k-1)^{\alpha}} - q^{k^{\alpha}}, k \geq 1,\ 0 < q < 1.$
- $R(k) = q^{k^{\alpha}}.$
- $r(k) = 1 - q^{k^{\alpha}-(k-1)^{\alpha}}.$

The random variable X here denotes the number of 'shocks' survived by a system and q is the probability of surviving more than one 'shock'. As for the continuous Weibull distribution, α is the shape parameter. The distribution is IFR for $\alpha > 1$, DFR for $0 < \alpha < 1$, and for $\alpha = 1$, it reduces to the geometric distribution.

Type II discrete Weibull distribution

(Stein and Dattero, 1984). This is a distribution with a finite support with m being the upper bound.

- $f(k) = \left(\frac{k}{m}\right)^{\alpha-1} \prod_{j=1}^{k-1} \left[1 - \left(\frac{j}{m}\right)^{\alpha-1}\right],\ 1 \leq k \leq m.$
- $R(k) = \prod_{j=1}^{\min(k,m)} \left[1 - \left(\frac{j}{m}\right)^{\alpha-1}\right].$
- $r(k) = \left(\frac{i}{m}\right)^{\alpha-1}, 1 \leq k \leq m.$

Clearly, the distribution is IFR for $\beta > 1$ and DFR for $0 < \alpha < 1$.

Type III discrete Weibull distribution

(Padgett and Spurrier, 1985). The model is flexible with respect to choice of failure rate, analogous to the Weibull distribution in the continuous case.

- $f(k) = \left(1 - e^{-c(k+1)^{\alpha}}\right) e^{-c\sum_{j=0}^{k} j^{\alpha}}, k = 0, 1, 2, .., c > 0, -\infty < \alpha < \infty$
- $R(k) = e^{-c\sum_{j=0}^{k+1} j^{\alpha}}.$
- $r(k) = 1 - e^{-c(k+1)^{\alpha}}.$

The monotonicity of the failure rate depends on the value of the shape parameter α. For $\alpha = 0$, the distribution reduces to the geometric distribution. For $\alpha > 0$, it is IFR and for $\alpha < 0$, it is DFR.

'*S*' distribution

(Bracquemond and Gaudoin, 2002). This is a discrete analogue of a continuous 'S' distribution (Soler, 1996) that models the lifetime of a device subjected to random stress. Let us consider a system such that, on each 'demand', a shock can occur with probability p and not occur with probability $1 - p$. Let π be the probability that the system survives the first demand given that a shock has occurred. Without giving details on how the distribution was derived, we present the following:

- $f(k) = p(1 - \pi^k) \prod_{j=0}^{k-1} (1 - p + \pi^j),\ k \geq 1, 0 < p \leq 1, 0 \leq \pi < 1.$
- $R(k) = \prod_{j=1}^{k} (1 - p + p\pi^j).$
- $r(k) = p(1 - \pi^k).$

It is clear that F is IFR since $\pi < 1$. If a shock occurs at each demand, then $p = 1$ and we obtain a very simple expression for the failure rate: $r(k) = 1 - \pi^k$. This is in fact a special case of the type III discrete Weibull distribution with $\beta = 1$ and $c = 1 - \log \pi$.

Discrete power series distribution

(Lai and Wang, 1995). This is a finite range discrete life distribution, an analogue of the continuous finite range distribution

$$f(t) = \frac{pt^{p-1}}{\theta^p}, \quad 0 \leq t \leq \theta; p, \theta > 0$$

discussed in Lai and Mukherjee (1986).

- $f(k) = \frac{k^\alpha}{\sum\limits_{x=0}^{N} x^\alpha},\ k = 0, 1, ..., N.$
- $R(k) = \sum_{x=k+1}^{N} \frac{x^\alpha}{c(N,\alpha)}$, where $c(N, \alpha) = \sum_{x=1}^{N} x^\alpha$.
- $r(k) = \frac{k^\alpha}{\sum_{x=k}^{N} x^\alpha}, i = 1, ..., N.$

Discrete geometric-Weibull distribution

This is a discrete analogue of the continuous geometric Weibull distribution introduced by Zacks (1984). Let $A^+ = \max(0, A)$, $\alpha > 0$ and $\tau > 0$ is a change point in the continuous case with distribution function given by:

$$F(t) = 1 - e^{-\lambda t - [\lambda(t-\tau)^+]^\alpha}, t > 0,$$

where $x^+ = \max(0, x)$.

The discrete analogue of the above is, for $\alpha, \tau \in \mathcal{N}^+$,

- $f(k) = e^{-\lambda(k-1) - [\lambda(i-\tau-1)^+]^\alpha} - e^{-\lambda k - [\lambda(k-\tau)^+]^\alpha}$, $k \geq 1$.
- $R(k) = e^{-\lambda k - [\lambda(k-\tau)^+]^\alpha}$.
- $r(k) = 1 - e^{-\lambda + \lambda^\alpha \left[[(k-\tau-1)^+]^\alpha - [(k-\tau)^+]^\alpha\right]}$.

Exponential-geometric distribution

This was considered in Rezaei and Arghami (2002). It does not appear to be related to its namesake in Section 2.3.14. It may be be regarded as, however, a discrete analogue of the Gompertz distribution discussed in Section 2.3.8.

- $$f(k) = \alpha^{k-1} \exp\left\{\frac{-\beta(k-1)(k-2)}{2}\right\} - \alpha^k \exp\left\{\frac{-\beta k(k-1)}{2}\right\}, \quad k = 1, 2, ...$$

  where $0 < \alpha \leq 1, \beta \geq 0, (1-\alpha) + \beta > 0$.
- $R(k) = \alpha^k \exp\left\{\frac{-\beta k(k-1)}{2}\right\}$.
- $r(k) = 1 - \alpha \exp\{-\beta(k-1)\}$.

6.7.2 Distributions Derived from Simple Failure Rate Functions

A simple discrete IFR

(Salvia and Bollinger, 1982).

- $r(k) = 1 - c/(k+1)$, $k = 0, 1, ..., 0 \leq c \leq 1$.
- $f(k) = (k - c + 1)\, c^k / \,(k+1)!$
- $R(k) = c^{k+1}/(k+1)!$.

A simple discrete DFR

(Salvia and Bollinger, 1982).

- $r(k) = c/k + 1,\ k = 0, 1, ...; 0 \le c \le 1.$
- $f(k) = \ c(1 - c)\ (2 - c)\ ...(k - c)/(k + 1)!.$
- $R(k) = \dfrac{(1 - c)(2 - c)..(k - c)}{k!}.$

The failure rate of this distribution is a discrete analogue of a continuous Pareto distribution.

Generalized Salvia and Bollinger IFR model

(Padgett and Spurrier, 1985).

- $r(k) = 1 - c/(\alpha k + 1),\ k\ =\ 0,\ 1,\ ....,\ 0 \le c \le\ 1, \alpha \ge 0.$
- $f(k) = (\alpha k - c + 1)\, c^k / \prod_{j=0}^{k} (\alpha j + 1)!$
- $R(k) = c^k / \prod_{j=0}^{k-1} (\alpha j + 1),\ k = 0, 1, 2, ...$

Generalized Salvia and Bollinger DFR model

(Padgett and Spurrier, 1985).

- $r(k) = \frac{c}{\alpha i + 1}\quad , k = 0, 1, ...; 0 \le c \le 1, \alpha > 0.$
- $f(k) = c \prod_{j=0}^{k-1} (\alpha j + 1 - c) / \prod_{j=0}^{k} (\alpha j + 1), k \ge 0.$
- $R(k) = \prod_{j=0}^{k-1} (\alpha j + 1 - c)(\alpha j + 1).$

Example of Discrete DMRL

Define $a_0 = 1,\ a_1 = 4/6,\ \ a_2 = 3/5 \quad a_n = \frac{1}{n},$ for $n \ge 3$.
Then it is easy to verify that $r(i) = 1 - \frac{a_i}{1 + a_{i+1}}$ is DMRL but not IFR.

- $r(0) = 0.4, \quad r(1) = \frac{7}{12} = .583,\ r(2) = \frac{11}{20} = .55,\ r(n) = \frac{n^2 + n - 1}{n(n+1)}$ which is increasing for $n \ge 3$.
- $R(0) = 0.6,\ R(1) = 1/4,\ R(2) = 9/80$,...
- $f(0) = 0.4,\ f(1) = 0.3498,\ f(2) = 0.1375, ...$

6.7.3 Determination of Ageing from Ratio of Two Consecutive Probabilities

This subsection is largely based on the work by Gupta et al. (1997).

Extended Katz family

This family is characterized by the ratio

$$\frac{f(k+1)}{f(k)} = \frac{\alpha + \beta k}{\gamma + k}, \alpha > 0, \beta < 1, \gamma > 0, \tag{6.43}$$

see, for example, Gurland and Tripathi (1975) and Tripathi and Gurland (1979). Gupta et al. (1997) have shown that F is IFR if $\alpha - \beta\gamma > 0$ and DFR if $\alpha - \beta\gamma < 0$.

Special cases are:

(i) Poisson distribution (Johnson et al., 1992; Lawless, 2003).

$$f(k) = \frac{e^{-\lambda}\lambda^k}{k!},\ k = 0, 1, 2..., \lambda > 0$$

with

$$\frac{f(k+1)}{f(k)} = \frac{\lambda}{1+k}.$$

It is clear that $\left\{\frac{f(k+1)}{f(k)}\right\}$ decreases in k so that F is IFR.

$$\frac{1}{r(k)} = 1 + \frac{k!}{\lambda^k}\left[e^{\lambda} - 1 - \sum_{j=1}^{k}\frac{\lambda^j}{j!}\right].$$

(See Gupta et al., 1997.)

(ii) Binomial

$$f(k) = \binom{n}{k} p^k(1-p)^{n-k}, \quad 0 < p < 1, k = 0, 1, ..., n$$

with

$$\frac{f(k+1)}{f(k)} = \frac{n\theta - k\theta}{1+k}, \quad \theta = \frac{p}{1-p}.$$

Here, $\alpha - \beta\gamma > 0$ which implies F IFR.

(iii) Negative binomial

This was also considered in the previous subsection.

$$f(k) = \binom{n+k-1}{k} p^n(1-p)^k,\ 0 < p < 1, n > 0,\ k = 0, 1, 2, ..$$

$$\frac{f(k+1)}{f(k)} = \frac{(n+k)(1-p)}{1+k},$$

giving $\alpha - \beta\gamma = (n-1)(1-p)$. So

if $n > 1$, we have IFR;

if $n < 1$, we have DFR;

if $n = 1$, we have a geometric distribution and hence a constant failure rate.

Log series distribution

For a log-series distribution, we have

$$\frac{f(k+1)}{f(k)} = \frac{\theta k}{1+k}, \quad k = 0, 1, 2, .., 0 < \theta < 1,$$

see Johnson et al. (1992, p. 290). Since $\left\{\frac{f(k+1)}{f(k)}\right\}$ is increasing in k, it follows from Section 6.3.1 that F is DFR .

Waring distribution

The probability function of the Waring distribution is

$$f(k) = \frac{(c-a)(a+k-1)!c!}{c(a-1)(c+k)!}, \; k + 0, 1, 2, .., c > a > 0.$$

(Johnson et al. 1992, p. 278).

The probability ratio is given by

$$\frac{f(k+1)}{f(k)} = \frac{a+k}{c+k+1}$$

which is increasing in k and thus F is DFR.

Distribution of cluster size

The ratio of the probability functions of the so called ‘cluster size’ distribution is

$$\frac{f(k+1)}{f(k)} = \frac{k-\alpha}{(k+1)(n+1)}, k = 0, 1, 2, .., \; 0 < \alpha < 1, \; n \geq 0,$$

see Proposition 4.4 of Lee and Whitmore (1993). Again, $\{\frac{f(k+1)}{f(k)}\}$ is increasing in k and hence F is DFR.

6.7.4 Polya Urn Distributions

Johnson et al. (1992) stated that numerous distributions can be built from urn representations. The urn scheme originally considered by Eggenberger and Polya (1923) is the following. An urn contains W white balls and R red balls. After each drawing of a ball, a replacement policy is chosen. Polya distributions are the number of times a red ball is drawn in N drawings. Inverse Polya distributions are the number of drawings required to obtain a specified number r of red balls (Johnson and Kotz, 1977, p. 192). Thus, there are as many distributions as possible replacement policies. For example, if after each drawing, only the chosen ball is returned in the urn, the Polya distribution reduces to the binomial distribution, and the corresponding inverse Polya distribution is the negative binomial distribution (geometric distribution for $r = 1$).

An inverse Polya distribution with $r = 1$ may used to describe the discrete lifetime of a device. Clarotti et al. (1997) have explicitly used a Polya distribution for modelling a discrete lifetime. The generalized Salvia and Bollinger distribution (Padgett and Spurrier, 1985) is also a member of inverse Polya family. The IFR and DFR inverse Polya distributions and Eggenberger-Polya distribution are also studied in the review by Bracquemond and Gaudoin (2002) who also provides other detailed discussions.

6.8 Discussion on Discrete Failure Time Models

Bracquemond and Gaudoin (2002) gave a comprehensive review on discrete lifetime distributions considered in their paper. We now summarize their discussion and conclusion here. As for the continuous case, it is important to select an appropriate model to fit a set of discrete reliability data. If the studied phenomenon is such that a constant failure rate is acceptable, the geometric distribution is appropriate. But if the observed device is ageing (positive or negative ageing), what distributions should be chosen? Bracquemond and Gaudoin (2002) give two criteria:

Criteria 1:

These are:

(i) Simplicity of expression of the reliability functions.

(ii) Flexibility or ability to describe various situations.

(iii) Physical basis of the distribution and interpretation of the parameters.

Criteria 2

The quality of parameter estimates in a model.

The authors drew the following conclusions (some of which we may not agree).

1. The negative binomial is interesting only as the analogue of the gamma distribution. Its parameters have no practical interpretation.
2. The type I Weibull distribution is very simple, flexible according to the value of β; and its parameters have physical meaning. The maximum likelihood estimator is satisfactory, except for data of high order of magnitude.
3. The type II Weibull distribution should not be used because of its bounded support.
4. The type III Weibull distribution is simple, flexible, and its parameters have an interpretation (not as obvious as the type I Weibull distribution). The quality of the estimator is not very good.

5. The 'S' distribution is simple and has an interesting physical meaning. But there are some numerical and identifiability problems in the parameter estimation which are not solved.
6. The discrete truncated logistic distribution is slightly more complex than the Weibull distributions, is not flexible (only IFR), and its parameters have no practical interpretation. The distribution may be defined through its reliability function given by

$$R(k) = \frac{e^{-\frac{k-c}{d}} + e^{-\frac{k}{d}}}{1 + e^{-\frac{k-c}{d}}}, \quad k \geq 1, c \text{ real}, d > 0.$$

7. The geometric–Weibull distribution is clearly of great practical interest and should be studied.
8. The IFR Polya distribution has a very interesting interpretation and its failure rate has a simple expression. However, it is not flexible and the estimation is not very satisfactory.
9. The Eggenberger-Polya distribution defined by its probability function

$$f(k) = \frac{1}{(1+d)^{h/d}} \times \frac{(h/d)_{k-1}}{(k-1)!}, \left(\frac{d}{d+1}\right)^{k-1}, \ k \geq 1,$$

where we have used the notation $(a)_k = k(k+1)...(n+k-1)$. Though $f(k)$ has a very complex expression, but it is very flexible and the parameter h is easily interpreted. Its main advantage is that it is the only distribution for which the estimators have an explicit expression. Further,
   - if $h = d$, then the failure rate is constant and the model reduces to the geometric distribution with parameter $1/(1+d)$,
   - if $h < d$, the distribution is log convex and thus DFR, and
   - if $h > d$, the distribution is log concave and thus IFR.

In all three cases, $\lim_{k\to\infty} r(k) = 1/(1+d)$.

6.9 Applications of Discrete Failure Time Models

Discrete failure rates arise in several common situations in reliability theory where clock is not the best scale on which to describe lifetime. Below are quotes from Shaked et al. (1995) saying " ...For example, in weapons reliability, the number of rounds fired until failure is more important than age in failure. This is the case also when a piece of equipment operates in cycles and the observation is the number of cycles successfully completed prior to failure. In other situations a device is monitored only once per time period and the observation then is the number of periods completed prior to the failure of the device."

Similar to its continuous counterpart, discrete ageing concepts are also very useful in practical applications. We merely list a couple examples below.

Burn-in

Mi (1993) established the relationship between the discrete bathtub shaped failure rate distributions and the upside-down mean residual life (IDMRL).

Suppose we consider the optimal burn-in time for obtaining the longest mean residual life in field operation. From the above Theorem 6.2 we see that if the failure rate sequence has a bathtub shape with change points n_1 and n_2, then the optimal burn-in time need not exceed the first change point n_1. See the continuous analogue given in Theorem 3.3.

Shock models

This has been discussed in Section 6.6.

Infection control in hospital

In terms of infection control of a serious disease in a hospital, what is important to the infection control clinicians is knowledge about what is likely to happen under normal conditions. Therefore advance warning of possible epidemics or elevated level of infection should be detected as soon as possible in order to impose additional infection control measures. In the context of the time to colonization of Methicillin-resistant *Staphylococcus aureus* (MRSA) patients in a Brisbane hospital, Ismail and Pettitt (2004) estimated the failure rate function $r(k)$ which they considered to have a much more useful description of the time progress of an infection than the survival function $R(k)$. The failure rate $r(k)$, the probability of a patient colonized with MRSA on day k given that they are not colonized before day k, can be estimated by the ratio of the number of patients colonized on day k to the number of patients at risk on day k. The authors provided a Bayesian nonparametric estimate as well as smoothing the failure rate for the MRSA data.

Imperfect repair modelling

Consider an item with discrete lifetime X with support in $\mathcal{N}$. Let a be the largest time epoch for which $\Pr(X = a) > 0$. In many applications, if the item fails at some epoch t $(t \neq a)$, then a repair of the item is attempted. If the repair is successful, the item is brought to the functioning state, but it is only as good as a similar item which has not failed by time t. Such a repair is called a '*minimal repair*.' See Barlow and Proschan (1965, pp. 96–98) and Blumenthal et al. (1976) for early development on continuous time minimal repair.

Shaked et al. (1995) considered a discrete time imperfect repair model in which an item undergoes a repair upon failure. With probability $p \in (0, 1)$ the repair is unsuccessful and the item is scrapped (and is usually replaced by a new one). With probability $1 - p$ the repair is minimal and it is assumed that a repair takes negligible time. If the item fails at the time epoch then it is not repair.

Let $T_p, p \in (0,1)$, denote the time until unsuccessful repair in the imperfect model. Clearly, the distribution of T_p is determined by p and by the distribution of X. The following theorem summarizes the result.

Theorem 6.6: If X is IFR (respectively IFRA2, NBU2), then $T_p, 0 < p < 1$, is IFR (respectively IFRA2, NBU2).

Proof: The result was given as Theorem 5.2 of Shaked et al. (1995).

Let $r_p(k)$ be the failure rate of T_p. Using the same argument as its continuous analogue (Brown and Proschan, 1983), one can see that $r_p(k) = pr(k)$. Since p is a constant so the shape of $r_p(k)$ is the same as $r(k)$.

Since the conditions for discrete IFR, IFRA2 and NBU2 are all expressed in terms of $r(t)$ as given by Definitions 6.1, 6.4 and 6.6, respectively, it is obvious that X and T_p are of the same class.

6.10 Some Problems of Usual Definition of Discrete Failure Rate

The definition of the failure rate $r(k)$ given in (6.4) is different from its continuous counterparts in several aspects. It is the conditional probability of failure at $X = k$ given the device has not failed by $k-1$ and thus $r(k) \leq 1$. In contrast, it is the product $r(t)\Delta$ (Δ small) that is approximately the probability of immediate failure conditional on $X > t$ for the continuous case. Hence $r(t)$ can be unbounded in some situation but $r(k)$ is always finite. Moreover, Xie, Gaudoin and Bracquemond (2002) commented that $r(k)$ cannot grow exponentially which is often the case for components in the wear-out lifetime period.

Interesting enough, continuous time failure rate function is often estimated by a probability which is always less than or equal to 1. In practice, the failure rate for the continuous time data is estimated by the proportion of devices failed in an interval per unit time, given that they have survived to the beginning of the interval, that is,

$$\hat{r}(t) = \frac{\text{number of devices failed per unit time in the interval}}{\text{number of devices survived at}\, t},$$

see, e.g., Lee (1992, p. 11).

Curiously, the failure rate defined by (6.4) is not additive for a series system of independent components. Let $r_i(k)$ be the failure rate of the ith component so that the system failure rate is

$$\begin{aligned} r(k) &= \frac{R(k-1)-R(k)}{R(k)} = \frac{\prod_{i=1}^{n} R_i(k-1) - \prod_{i=1}^{n} R_i(k)}{\prod_{i=1}^{n} R_i(k-1)} \\ &= 1 - \prod_{i=1}^{n} \frac{R_i(k)}{R_i(k-1)} = 1 - \prod_{i=1}^{n} [1 - r_i(k)] \neq \sum_{i=1}^{n} r_i(k). \end{aligned}$$

A major problem of $r(k)$ has already been briefly discussed in Section 6.3.2, that is, the cumulative hazard function $H(k) = \sum_{i=1}^{k} r(i)$ is not equivalent to $-\log R(k)$ as in the continuous case. Thus,

$$H(k) = \sum_{i=1}^{k} r(i) \neq -\log R(k). \tag{6.44}$$

Equation (6.44) is the real cause for the two versions of discrete IFRA and NBU being nonequivalent. The above nonequivalence phenomenon has been noted by several authors. This has prompted some authors a desire to find an alternative definition to be discussed below.

6.11 Alternative Definition of Failure Rate and Its Ramification

Because of the problems associated with the common definition of the failure rate $r(k)$, several authors including Roy and Gupta (1999), Bracquemond et al. (2001), and Xie, Gaudoin and Bracquemond (2002) have presented an alternative definition of a discrete failure rate function. In order to make a distinction, the alternative failure rate function considered in this section will be denoted by $r^*(k)$.

Definition 6.15: For discrete distribution with reliability function $R(k)$, the alternative failure rate function $r^*(k)$ is defined as

$$r^*(k) = \log \frac{R(k-1)}{R(k)}, k = 1, 2, .. \tag{6.45}$$

The rationale and background to the introduction of this definition is as follows. For continuous distribution, the failure rate function is defined as:

$$r(t) = \frac{f(t)}{R(t)} = -\frac{d}{dt} \log R(t).$$

Instead of using $R(k-1) - R(k)$ for $f(k)$ which leads to the expression in (6.5), we could use $\log R(k-1) - \log R(k)$ for $-d[\log R(t)]/dt$ above and define the failure rate as

$$r^*(k) = -[\log R(k) - \log R(k-1)] = -\log \frac{R(k)}{R(k-1)} = \log \frac{R(k-1)}{R(k)}.$$

This justifies the definition of failure rate function (6.45). Clearly, $r^*(k)$ is not bounded in this case.

We note that Roy and Gupta (1999) used this function and named it 'the second failure rate function'. Xie et al. (2002) have devoted their paper to study this function. Here in this section, we largely follow their approach.

6.11.1 The Relationships between $r(k)$ and $r^*(k)$

We have defined several discrete ageing concepts in terms of $r(k)$ such as IFR (DFR), IFRA2, NBU2, and BT (UBT). Xie et al. (2002) have noted that most of the results concerning IFR (DFR) are still valid for $r^*(k)$ because of a simple relationship between the two definitions of failure rates $r(k)$ and $r^*(k)$:

$$r^*(k) = -\log \frac{R(k)}{R(k-1)} = -\log \frac{R(k) - R(k-1)}{R(k-1)} = \log[(1 - r(k)], \quad (6.46)$$

or

$$r(k) = 1 - e^{-r^*(k)}, \quad (6.47)$$

showing that the two concepts $r(k)$ and $r^*(k)$ have the same monotonic property, i.e., $r^*(k)$ is increasing (decreasing) if and only if $r(k)$ is increasing (decreasing). Hence, we may define the discrete IFR (DFR) in terms of $r^*(k)$ instead.

6.11.2 Effect of Alternative Failure Rate on Ageing Concepts

Earlier, we have established that

$$-\log R(k) \neq H(k),\ H(k) = r(1) + r(2) + ... + r(k).$$

The above inequality has caused the nonequivalence between the two definitions IFRA1 and IFRA2. We now define an alternative cumulative hazard function to be expressed in terms of $r^*(k)$ instead of $r(k)$:

$$H^*(k) = r^*(1) + r^*(2) + ... + r^*(k). \quad (6.48)$$

It now follows from (6.48) that

$$H^*(k) = \log \frac{R(0)}{R(1)} + \log \frac{R(1)}{R(2)} + ... \log \frac{R(k-1)}{R(k)} = \log \frac{R(0)}{R(k)} = -\log R(k). \quad (6.49)$$

Thus, IFRA1 $\Leftrightarrow$ IFRA2 under the new definition of the failure rate $r^*(k)$.

Also from (6.24) that

$$R(j)R(k) \geq R(j+k),\ j, k = 1, 2, ...$$

which is equivalent to $R(k) \geq \frac{R(j+k)}{R(j)}$. On replacing $R(j)$ by $e^{-H^*(j)}$, we obtain

$$\sum_{i=1}^{k} r^*(j) \leq \sum_{i=j+1}^{j+k} r^*(i), j, k \geq 1$$

showing that NBU1 $\Leftrightarrow$ NBU2 if the failure rate is defined by $r^*(k)$. We may also define the mean residual life function in terms of $r^*(j)$ such that

$$\begin{aligned} \mu(k) &= \sum_{j=k}^{\infty} \frac{R(j)}{R(k-1)} \\ &= \sum_{j=k}^{\infty} e^{-(H(j)-H(k-1))} \\ &= \sum_{j=k}^{\infty} e^{-\left(\sum_{k}^{j} r^*(i)\right)}. \end{aligned} \tag{6.50}$$

It follows that if $r^*(k+1) \geq r^*(k)$, then

$$\begin{aligned} \mu(k) = \sum_{j=k}^{\infty} e^{-(H(j)-H(k-1))} &\geq \sum_{j=k}^{\infty} e^{-(H(j+1)-H(k))} \\ &= \sum_{j=k+1}^{\infty} e^{-(H(j)-H(k))} = \mu(k+1). \end{aligned}$$

This is obvious as we have shown in (6.47) that $r(k)$ and $r^*(k)$ have the same monotonic property.

6.11.3 Additive Property for Series System

Suppose we have a series system of n independent components with failure rate $r_j^*(k), j = 1, 2, ..., n$, then the system reliability is given by

$$\begin{aligned} R(k) &= \prod_{j=1}^{n} R_j(k) = \prod_{j=1}^{n} \exp\left\{-\sum_{i=1}^{k} r_j^*(i)\right\} = \exp\left\{-\sum_{j=1}^{n}\sum_{i=1}^{k} r_j^*(i)\right\} \\ &= \exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{n} r_j^*(i)\right\} = \exp\left\{-\sum_{i=1}^{k} r^*(k)\right\} \end{aligned}$$

where $r^*(i)$ is the failure rate function for the system. It is now clear that

$$r^*(i) = \sum_{j=1}^{n} r_j^*(i). \tag{6.51}$$

Hence, the failure rate defined in (6.45) is additive for series systems, and this well known and widely used property is now valid for discrete distributions under this alternative definition.

6.11.4 Examples

We now give two examples to demonstrate how the alternative failure rate function $r^*(k)$ can be calculated.

Example 6.5: Discrete Pareto distribution

The discrete Pareto distribution has the following reliability function

$$R(k) = \left(\frac{d}{k+d}\right)^c, \ c, d > 0, k \geq 1. \tag{6.52}$$

Then the alternative failure rate is

$$\begin{aligned} r^*(k) &= \log \tfrac{R(k-1)}{R(k)} = \log \tfrac{(d/k-1+d)^c}{(d/k+d)^c} \\ &= c \log \tfrac{k+d}{k-1+d} = c \log \left[1 + \tfrac{1}{k-1+d}\right] \end{aligned}$$

which is a decreasing in k.

Example 6.6: Discrete logistic distribution

The continuous version of the logistic model (Johnson et al., 1995, Chapter 23) is useful when the failure rate is increasing but does not increase very fast at the beginning. The discrete version has a reliability function given by

$$R(k) = \frac{e^{-(k-c)/d} + e^{-k/d}}{1 + e^{-(k-c)/d}}, \ c, d > 0, k \geq 1. \tag{6.53}$$

It can be verified that

$$r^*(k) = \log \frac{R(k-1)}{R(k)} = \log \left(1 + \frac{e^{1/d} - 1}{1 + e^{-(k-1-c)/d}}\right)$$

and this is an increasing function in k and hence F is IFR. This result is consistent with its continuous analogue.

7

Tests of Stochastic Ageing

7.1 Introduction

We have seen in the last few chapters that many ageing concepts have been defined and studied in the literature. A reasonable starting point for analyzing a reliability or survival data set is to determine which ageing class it belongs to. Thus, tests of stochastic ageing play an important role in any reliability study. In this chapter, we present available tests for testing exponentiality against various ageing classes.

The chapter is divided as follows. Section 7.2 briefly reviews the reliability properties that characterize the exponential distribution. In Section 7.3 we give a general sketch of a test procedure for an ageing class. Section 7.4 forms the main body of the chapter outlining the known statistical tests for univariate ageing classes, particularly the basic ageing classes defined in Section 2.4. Reliability and survival data sets are often censored so Section 7.5 lists the known tests of ageing for censored data. Section 7.6 is devoted to the discussion of the tests on DMRL and IMRL which are essentially the only monotonic mean residual life classes. IDMRL (DIMRL) and NWBUE (NBWUE) are non-monotonic mean residual life classes that have a trend change at a point τ. Whether the change point τ is known or whether the proportion p of the population that dies before τ is known will have an impact on the type of test statistics to be used for these classes. These and other issues are examined in Section 7.7. Bathtub distributions feature strongly in this book, particularly in Chapter 3. Tests of exponentiality versus bathtub are considered in Section 7.8. For completion, three miscellaneous tests are also discussed briefly in Section 7.9. Section 7.10 concludes the chapter on how to choose a test statistic.

7.2 Exponential Distribution

The exponential distribution has been widely used as a lifetime model for many reliability situations. It plays a pivotal role in ageing concepts and tests of ageing and its basic properties are listed in Section 2.3.1. This distribution can be characterized in many ways, see, for example, Galambos and Kotz (1978). Because of its important role in testing for an ageing class, we shall discuss briefly the characterization of the exponential distribution through two stochastic ageing properties.

Let X have an exponential distribution function $F(t) = 1 - e^{-\lambda t}$, for $t \geq 0$. A property of the exponential distribution which makes it especially important in reliability theory and application is that the remaining life of a used exponential component is independent of its initial age (i.e., having the memoryless property). In other words,

$$\Pr\{X > t + x | X > x\} = e^{-\lambda t}, \ t \geq 0 \tag{7.1}$$

(independent of x). This property tells us that a used exponential component is essentially 'as good as new'.

Let $\bar{F}(t) = 1 - F(t) = e^{-\lambda t}$, then (7.1) can be written equivalently as

$$\bar{F}(t + x) = \bar{F}(t)\bar{F}(x). \tag{7.2}$$

Let G denote the distribution function of a non-negative continuous random variable and let $\bar{G} = 1 - G$. Then G is an exponential distribution if and only if

$$\bar{G}(t + x) = \bar{G}(t)\bar{G}(x). \tag{7.3}$$

That is, the functional equation (7.3) characterizes the exponential distribution.

The failure rate function of the exponential lifetime is given by

$$r(t) = \frac{f(t)}{\bar{F}(t)} = \lambda. \tag{7.4}$$

Thus, the exponential distribution has a constant failure rate. In fact, it is the only life distribution that has this property. In other words, the exponential distribution is also characterized by having a constant failure rate and thus it sits at the boundary between the IFR and DFR classes. Stephens (1986) gave a comprehensive review on goodness-of-fit tests for the exponential distribution with explicit attention to IFR and DFR alternatives.

7.3 A General Sketch of Tests

In developing tests for different classes of life distributions, it is almost without exception that the exponential distribution is used as a strawman to be knocked down. As the exponential distribution is always a boundary member of an ageing class C (in the univariate case), a usual format for testing is:

H_0 (null hypothesis): F is exponential versus

H_1 (alternative): $F \in C$ but F is not exponential.

If one is merely interested in testing whether F is exponential, then no knowledge of the alternative is required; in such a situation, a two-tailed test is appropriate. Ascher (1990) discussed and compared a wide selection of tests for exponentiality. Power computations, using simulations, were done for each procedure. In short, he found

(i) certain tests performed well for alternative distributions with non-monotonic failure (hazard) rates while others fared well for monotonic failure rates, and

(ii) of all the procedures compared, the score test presented in Cox and Oakes (1984) appears to be the best if one does not have a particular alternative in mind.

In what follows, our discussion will focus on tests with alternatives being specified. In other words, our aim is not to test exponentiality but rather to test whether a life distribution belongs to a specific ageing class. A general outline of these procedures is:

(i) Find an appropriate measure of deviation of F (under H_1) from the null hypothesis of exponentiality.

(ii) Based on this measure, some sort of U-statistics are proposed.

(iii) Large-sample properties such as the asymptotic normality and consistency are proved.

(iv) Pitman's asymptotic relative efficiencies are usually calculated for the following families of distributions (with scale factor =1, $\alpha \geq 0,\ x \geq 0$):

  (a) The Weibull distribution $\bar{F}(x) = \exp\{-x^{\alpha}\}$.

  (b) The linear failure rate distribution $\bar{F}(x) = \exp\{-x - \alpha x^2/2\}$.

  (c) The Makeham (or Gompertz-Makeham) distribution

$$\bar{F}(x) = \exp[-\{x + \alpha(x + \exp(-x) - 1)\}].$$

  (d) The gamma distribution with density $f(x) = \dfrac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x}$.

Basic reliability properties of these ageing distributions have been given in Section 2.3. All these distributions are IFR or DFR depending on the value of α; hence they all belong to a wider class. Moreover, all these distributions reduce to the exponential distribution when $\alpha = 0$ or $\alpha = 1$.

7.3.1 Estimation of Survival, Failure Rate and Mean Residual Life Functions

Some of the tests presented in this chapter involving empirical survival, failure rate or MRL functions. There is a vast literature on estimation of these quantities but we will not review them here because this subject matter falls outside the theme of this book. Below is a list of references pertinent to each known method.

Estimation of survival and density functions

- Kernel method – Devroye (1989).
- Bayesian approach – Ahsanullah and Ahmed (2001).
- Minimax estimation – Yu and Phadia (1996).
- Smooth estimation – Chaubey and Sen (1996).

See Chaubey and Sen (1996) for a brief review.

Estimation of failure rates

- Nonparametric methods – A survey by Singpurwalla and Wong (1983a).
- Kernel method – Singpurwalla and Wong (1983b).
- Bayesian nonparametric methods – Ho and Lo (2001).

Estimation of MRL

- Smooth estimator – A modified weighting scheme is used to develop a smooth estimator of $\mu(t)$ by Chaubey and Sen (1999). See also Na and Kim (1999).
- Nonparametric estimate – Ghorai et al. (1982), Ghorai and Rejtö (1987), Mi (1994a) and Li (1997).
- Bayesian method – Tiwari and Zalkikar (1993).
- Use of weight function – Csörgő and Zitikis (1996).
- Local linear fitting technique – to estimate the corresponding mean residual life function. The limiting behaviour of the obtained estimator is presented. (Abdous and Berred, 2005).
- Projection type estimators – estimation of a monotonic mean residual life by Kochar et al. (2000).

7.4 Statistical Tests for Univariate Ageing Classes

Definitions of the ageing classes considered in this section can be found in Sections 2.4–2.7.

We shall now present several statistical procedures for testing exponentiality of a life distribution against different alternatives. The alternatives to be considered here are IFR, IFRA, NBU, NBUE, HNBUE, NBUC, NBU-t_0, DPRL-α and NBUP-α. Tests for the remaining basic classes will be discussed in the subsequent sections. It is perhaps understandable that most proposed tests are applicable to only one or two classes. However, Ahmad (2001) and Ahmad and Mugdadi (2004) recently developed tests based on moments inequalities for testing exponentiality against IFR, IFRA, NBU, NBUE, HNBUE, NBUC, or DMRL. Earlier, Lai (1994) gave a review on tests of several univariate ageing classes.

7.4.1 Some Common Bases for Test Statistics

The test statistics are usually based on some properties that characterize an ageing class under H_1. A measure $\triangle(F)$ of departure of F from the exponential distribution is often derived. The following are some of the common ones:

- Moments inequalities – Section 2.5.5.
- Inequalities for survival functions.
- Order statistics.
- Partial ordering with the exponential.
- Scaled TTT statistic – Section 2.5.6.

7.4.2 IFR Tests

An obvious alternative to the constant failure rate function is a monotonic failure rate function (IFR or DFR). Many of the real life data exhibit this characteristic. Hence testing of exponentiality against IFR/DFR alternatives are widely studied, and in fact, they are among the first proposed tests in the literature.

TTT-plot

There are several tests for IFR and IFRA alternatives. The most popular one (as we understand it) is the ‘total time on test’ procedure. It is a popular test because of the graphical interpretation. Let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ be the order statistics of the sample and define

$$\tau(X_{(i)}) = \sum_{j=1}^{i} (n-j+1)\left(X_{(j)} - X_{(j-1)}\right),\ i = 1, 2, ..., n. \tag{7.5}$$

$\tau(\mathrm{X}_{(i)})$ is called the total time on test up to the ith order statistic. Note that $\tau(\mathrm{X}_{(i)})$ is the empirical TTT transform that corresponds to the TTT-transform function defined by

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} \bar{F}(u)\,du$$

and discussed in Section 2.5.6. The scaled TTT-transform is given by

$$\phi(t) = H_F^{-1}(t)/H_F^{-1}(1).$$

The so called TTT-plot (Barlow and Campo, 1975) is the empirical version of the above obtained by plotting

$$U_j = \tau(X_{(j)})/\tau(X_{(n)}) \text{ against } j/n, \text{ for } j = 1, 2, ..., n, \tag{7.6}$$

and then connecting the plotted points by straight lines. Based on U_j, Klefsjö (1983a) formed different test statistics for testing exponentiality against IFR, IFRA, NBUE and DMRL. In particular, the test statistic for IFR alternative is given by

$$A_2 = \sum_{j=1}^{n} \alpha_j U_j \tag{7.7}$$

where $\alpha_j = \frac{1}{6}\{(n+1)^3 j - 3(n+1)^2 j^2 + 2(n+1)j^3\}$.

A quick diagnostic check for IFR (DFR) property is to inspect whether the scaled TTT-transform curve is concave (convex) lying above (below) the diagonal line. A graphical display of such curves is given by Fig. 7.1 in Section 7.8.1.

Earlier, Barlow and Doksum (1972) proved that a test which rejects exponentiality in favour of IFR when the signed area between the TTT plot and the diagonal is large, is asymptotically minimax.

U_n-test

A test designed for testing IFR solely was proposed by Ahmad (1975, 1976). The so called U_n-test is a special form of Hoeffding's U-statistic, see Hoeffding (1948). More specifically,

$$U_n = \frac{4}{n(n-1)(n-2)(n-3)} \sum \psi[\min(X_{\gamma_1}, X_{\gamma_2}), (X_{\gamma_3} + X_{\gamma_4})/2] \tag{7.8}$$

where the summation $\sum$ extends over all combinations $1 \leq \gamma_i \leq n, 1 \leq i \leq 4$ such that $\gamma_1 \neq \gamma_2$, $\gamma_1 \neq \gamma_4$, $\gamma_2 \neq \gamma_3$, $\gamma_2 \neq \gamma_4$, $\gamma_1 < \gamma_2$, and $\gamma_3 < \gamma_4$, with ψ being the indicator function such that

$$\psi(a, b) = \begin{cases} 1 & \text{if } a > b, \\ 0 & \text{otherwise.} \end{cases} \tag{7.9}$$

H_0 is rejected in favour of H_1 for large values of U_n. The test was shown to be unbiased and consistent against H_1. The asymptotic normality of the test statistic was proved. The asymptotic relative efficiency against other IFR tests was studied for the Weibull and the linear failure rate distributions, and it was shown that the test performs well.

$\triangle^*_{\mathbf{IFR}}$-test

Belzunce et al. (1998) considered first a test for the stochastic order and then used the measure of deviation for the stochastic order to form the basis of test for IFR (DFR) ageing classes. Define

$$\triangle_k^n = \sum_{l=k+1}^{n} \left(\frac{n-l+1}{n-k} \right)^2 \cdot (X_{(l)} - X_{(l-1)}),\ k = i, j \tag{7.10}$$

which is the empirical analogue of a measure of deviation from exponentiality toward IFR, DFR alternatives. The test statistic is

$$\triangle_{\mathrm{IFR}}(n) = \frac{1}{n^2} \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \left(1 - \frac{i}{n}\right)^2 \left(1 - \frac{j}{n}\right)^2 (\triangle_i^n - \triangle_j^n). \tag{7.11}$$

For large or small values of $\triangle_{\mathrm{IFR}}(n)$, reject H_0 in favour of H_1. The $\triangle_{\mathrm{IFR}}(n)$ test is not scale-invariant, but the test statistic

$$\triangle^*_{\mathrm{IFR}} = \frac{\triangle_{\mathrm{IFR}}(n)}{\bar{X}}$$

is scale-invariant.

7.4.3 IFRA Tests

The IFRA class, an extended class of IFR, is closed under formation of coherent systems. The class was studied in detail in Barlow and Proschan (1981). Testing of exponentiality again IFRA alternatives are summarized in the following.

J_b-test

Deshpande (1983) developed a test specifically for IFRA class. The test grew out of the measure of deviation of F from the exponential, which was based on the fact that $\bar{F}(bx) \geq \{\bar{F}(x)\}^b$ $(x > 0,\ 0 < b < 1)$ with strict inequality for some x when F is IFRA.

Define

$$\mu(F) = \int_0^\infty \bar{F}(bx)\, dF(x). \tag{7.12}$$

If F is exponential, then $\mu(F) = (b+1)^{-1}$, whereas for all other F's belonging to IFRA, $\mu(F) > (b+1)^{-1}$. Hence $\mu(F) - (b+1)^{-1}$ is a measure of deviation of F from the exponential.

Let

$$h_b(x, y) = \begin{cases} 1 & \text{if } x > by, \\ 0 & \text{otherwise}; \end{cases} \tag{7.13}$$

where $0 < b < 1$. Define J_b as the U-statistic based on h_b:

$$J_b = \frac{n}{(n-1)} \sum h_b(X_i, X_j) \tag{7.14}$$

where $\sum$ denotes the summation over $1 \leq i \leq n,\ 1 \leq j \leq n$ such that $i \neq j$. Large values of the statistic J_b indicate the alternative hypothesis to be accepted. The calculation J_b is as follows: Multiply each observation by b. Arrange $X_1, ..., X_n$ and $bX_1 ..., bX_n$ together in increasing order of magnitude. Then

$$S = \sum_{i=1}^{n} R_i - \frac{n}{2}(n+1) - n \tag{7.15}$$

is the number of pairs of (X_i, bX_j), for $i \neq j$, such that $X_i > bX_j$.

Then

$$J_b = \{n(n-1)\}^{-1}\ S \tag{7.16}$$

is the statistic for testing exponentiality against IFRA.

Deshpande (1983) showed that J_b tests are U-statistics and hence asymptotically normally distributed. The asymptotic relative efficiency with respect to Hollander–Proschan statistic and the cumulative TTT statistics are reasonably high for $b = 0.5, 0.9$. Bandyopadhyay and Basu (1989) removed this restriction and showed that the results are valid for any $0 < b < 1$.

Q_n-test

Kochar (1985) proposed a test statistic for IFRA which is denoted by Q_n:

$$Q_n = \sum_{i=1}^{n} J\left(\frac{i}{n+i}\right) / n\bar{X} \tag{7.17}$$

where

$$J(u) = 2(1-u) \cdot [1 - \log(1-u)] - 1, \tag{7.18}$$

and $X_{(i)}$'s are the order statistics such that $X_{(1)} < X_{(2)} < ... < X_{(n)}$. The test is to reject H_0 for large values of Q_n. The author showed that the test has good asymptotic relative efficiencies and is consistent for testing H_0 against H_1.

Link's test

Link (1989) developed a test which is related to Deshpande's J_b given in (7.16). More specifically, Link's test can be expressed in a simple form:

$$\Gamma = \frac{2}{n(n-1)} \sum_{i<j} X_{(i)} / X_{(j)} \tag{7.19}$$

where $X_{(i)}$ are the order statistics. The null hypothesis is rejected for large values of Γ. This test is easy to implement and the author showed that Γ compares favourably with its competitors in efficiency and power comparisons.

TTT-plot

The test statistic for IFRA alternative is

$$B = \sum_{j=1}^{n} \beta_j U_j \tag{7.20}$$

with $\beta_j = \frac{1}{6}\{2j^3 - 3j^2 + j(1-3n-3n^2) + 2n + 3n^2 + n^3\}$. The test is to reject H_0 in favour of H_1 for large values of B.

See also Ahmad (1994) for a further discussion on this test.

$\hat{\triangle}_{r+1}$-test

Recently El-Bassiouny (2003) proposed a test based on the moment inequality (2.56) for testing exponentiality against IFRA alternatives. The proposed invariant test statistic is

$$\hat{\triangle}_{r+1} = \frac{\hat{\delta}_{r+1}}{\bar{X}^{(r+1)}}$$

where

$$\hat{\delta}_{r+1} = \frac{2}{n(n-1)} \sum\sum_{i<j} \left\{ \min(X_i^{r+1}, X_j^{r+1}) - \frac{X_i^{r+1}}{2^{r+1}} \right\}. \tag{7.21}$$

We note that $\hat{\delta}_{r+1}$ is a classical U-statistic. $\hat{\triangle}_1$ and $\hat{\triangle}_2$ are used to test the IFRA alternatives. The author showed that his tests are superior to other tests that were used for comparison.

7.4.4 NBU Tests

The NBU and NBUE classes of distributions are defined in Section 2.4.1 and their basic properties are given in Section 2.53. These ageing concepts are useful in replacement and maintenance related studies (Barlow and Proschan, 1981, Chapter 6). Maintenance policies are followed to reduce the incidence of system failure or to return a failed system to the operating state. In this subsection tests against NBU class are described and those against NBUE can be found in the next subsection.

J_n-test

Hollander and Proschan (1972) proposed a test statistic defined by

$$J_n = 2[n(n-1)(n-2)]^{-1} \cdot \sum \psi(X_{\gamma_1}, X_{\gamma_2} + X_{\gamma_3}) \tag{7.22}$$

where ψ is the indicator function defined by (7.9) and the sum $\sum$ is over all $n(n-1)(n-2)/2$ triples (γ_1, γ_2, γ_3) of three integers such that $1 \leq \gamma_i \leq n, \gamma_1 \neq \gamma_2, \gamma_1 \neq \gamma_3$ and $\gamma_2 < \gamma_3$. The null hypothesis F exponential is rejected in favour of NBU alternative if J_n is small. For simplicity, we may refer to this test as the J-test.

δ-test

Koul (1978a) generalized the method of Hollander and Proschan (1972) by defining a non-negative, nondecreasing, right continuous function $\phi(x), 0 \leq x < 1$, such that $\phi(0) = 0$ and $\int\int \phi(uv)\, du\, dv < \infty$. He then proposed a class of test statistics:

$$\delta_n = n^{-2} \sum_{i}^{n} \sum_{j}^{n} \phi(S_{ij}/n) \tag{7.23}$$

such that

$$S_{ij} = \sum_{k=1}^{n} \psi(X_k, X_i + X_j),\ 1 \leq\ i,\ j \leq\ n \tag{7.24}$$

where ψ is the indicator function defined as before, i.e.

$$\psi(a, b) = \begin{cases} 1 \text{ if } & a > b, \\ 0 \text{ if } & a = b. \end{cases}$$

By choosing $\phi(u) = u$, one gets the J_n test of Hollander and Proschan (1972) discussed above. The test is rejected when δ_n is small. Koul (1978a) studied the case $\phi(u) = u^{\frac{1}{2}}$ in detail. He found that this particular statistic has an asymptotic Pitman efficiency relative to the J_n-test equal to

- 1.873 at $\bar{F}(x) = \exp(-x - x^2\alpha/2),\ \alpha > 0,\ x \geq 0$
- 1.054 at $\bar{F}(x) = \exp(-x^\alpha), \alpha > 1,\ x \geq 0$, and
- 0.899 at the gamma survival function $\bar{F}(x)$ with shape parameter α.

The consistency and asymptotic normality of this class of tests were proved under fairly broad conditions on the underlying entities.

Generalized J-test

The J-test proposed by Hollander and Proschan (1972) has also been generalized by Alam and Basu (1990) with the following motivation. So far all the tests (H_0: F exponential vs H_1: F is NBU and not exponential) were based on a fixed sample of size n. However, there are situations where evaluation of a test item is time-consuming or the cost of evaluating a test item is prohibitive. For these reasons and possibly other physical constraints, it is useful to make a decision with a smaller sample size. It is well known that a two-stage test procedure has the ability of reducing the average sample size (in comparison with single sampling procedure), while at the same time not reducing the power of the test too much.

Using the J-test of Hollander and Proschan together with the two-stage test procedure, Alam and Basu (1990) gave the following decision rule for testing NBU:

Stage 1: Take a sample X_1, ..., X_{n_1} of size n_1.

If $J_{n_1} < C_1$, reject H_0.

If $J_{n_1} > C_2$, accept H_0.

If $C_1 \leq J_{n_1} \leq C_2$, continue to Stage 2.

Stage 2: A second sample $X_{n_1+1}, ..., X_{n_1+n_2}$ of size n_2 is taken.

If $J_{n_1+n_2} < C_3$ reject H_0, otherwise accept H_0.

(Note that J_{n_k} is defined as that given in (7.22)).

For a given significance level α one needs to find the joint distribution of J_{n_1} and $J_{n_1+n_2}$ (which is difficult for small or moderate sample size) in order to determine C_i, $i = 1, 2, 3$. Hence they derived the asymptotic joint distribution and two tables for finding C_1, C_2 and C_3 were constructed when (i) $\alpha = 0.01$ and (ii) $\alpha = 0.05$. See Alam and Basu (1990) for a more detailed discussion.

S-test

Another test, derived by Deshpande and Kochar (1983), is based on a linear combination of the U-test of Ahmad (1975, 1976) and the J-test of Hollander and Proschan (1972). The desired test statistic is given by

$$S = U - J \tag{7.25}$$

and reject H_0 if S is too large. Pitman's asymptotic relative efficiencies of the S-test relative to V-test of Proschan and Pyke (1972), the U-test of Ahmad (1975) and the J-test of Hollander and Proschan (1972) were calculated.

L_n-test

Kumazawa (1983) proposed a class of statistics for testing NBU. The test statistic is now described below. First, define a function ϕ such that

(i) for $0 \leq x \leq 1$, $\phi(x) \geq 0$, non-decreasing, right continuous, $\phi(0) = 0$ and

(ii) $\int \phi(u^2)\, du < \infty$. The test statistic obtained is

$$L_n(\psi, m) = n^{-1} \sum_{i=1}^{n} \phi(n^{-1} \cdot S_i^{(m)}) \tag{7.26}$$

where

$$S_i^{(m)} = \sum_{j=1}^{n} \psi[X_{(j)}, mX_{(i)}] \tag{7.27}$$

for $1 \leq i \leq n$ and $X_{(1)} < X_{(2)} < ... < X_{(n)}$ are the order statistics of the X_i's and ψ represents the indicator function defined as before. Kumazawa (1983) proved that the L_n-test is consistent, unbiased and asymptotically normal. The Pitman's efficiencies were computed and compared with some other statistics. It was found that the L_n-test performed well against other tests.

$\triangle^*_{\mathrm{NBU}}$-test

In addition to the $\triangle^*_{\mathrm{IFR}}$ test, Belzunce et al (1998) also proposed a test for NBU (NWU) ageing classes. The test statistic is

$$\triangle_{\mathrm{NBU}}(n) = \frac{1}{n}\sum_{i=1}^{n-1}\left(1-\frac{i}{n}\right)^2(\triangle_i^n - \triangle_j^n) \tag{7.28}$$

where $\triangle_i^n$ is defined as in (7.10). The scale-invariant statistic is

$$\triangle^*_{\mathrm{NBU}}(n) = \frac{\triangle_{\mathrm{NBU}}(n)}{\bar{X}}.$$

For large or small values of $\triangle^*_{\mathrm{NBU}}(n)$, reject the the null hypothesis in favour of the alternatives.

7.4.5 NBUE Tests

K^*-test

Several tests for NBUE alternatives have been proposed. Hollander and Proschan (1972) derived a test statistic K^* which was shown to be equivalent to the total time on test statistic discussed in Section 7.4.2. It is given by the expression

$$K^* \equiv K/\bar{X} \tag{7.29}$$

where $K = \frac{1}{2n^2}\sum_{i=1}^{n}\left(\frac{3n}{2} - 2i + \frac{1}{2}\right)\cdot X_{(i)}$, and $X_{(1)} < X_{(2)} < ... < X_{(n)}$ are the order statistics. Significantly large values of K^* suggest NBUE alternatives; significantly small values of K^* suggest NWUE alternatives.

D-test

Koul (1978b) introduced what he called the D-test for NBUE alternatives, but it appears rather complex and difficult to compute.

CV-test

de Souza Borges et al (1984) introduced a simple test using the sample coefficient of variation $S/\bar{X}$ where

$$S^2 = \sum (X_i - \bar{X})^2/n. \tag{7.30}$$

Under H_0, $S/\bar{X}$ is asymptotically normal with mean 1 and variance $\frac{1}{n}$. The CV test rejects H_0 (F is exponentially distributed) in favour of H_1 (NBU) for large values of $|\sqrt{n}(S/\bar{X}-1)|$. In particular, large negative values of $\sqrt{n}(S/\bar{X}-1)$ can be viewed as evidence towards the NBUE distribution, and large positive values as evidence towards NWUE.

Kanjo's test

Let $\Delta_i = X_{(i)} - X_{(i-1)}, D_i = (n-i+1)\Delta_i, i = 1, 2, ..., n$. The test statistic Kanjo (1993, 1994) proposed is

$$V_n = \sum_{j=1}^{n} \frac{j}{n} D_j/(n\bar{X}). \tag{7.31}$$

It was pointed out in Kanjo (1994) that it is equivalent to the well known total time on test statistic V_n^*.

7.4.6 HNBUE

T_n-test

Doksum and Yandell (1984) have shown that the test based on the statistic

$$T_n = \frac{1}{n} \sum_{i=1}^{n} \exp(-X_i/\bar{X}) \tag{7.32}$$

which is asymptotically most powerful test in the class of all similar tests for testing exponentiality against the Makeham distribution with survival function

$$\bar{F}(t) = \exp\left[-\{\lambda t + \theta(\lambda t - \exp(-\lambda t) - 1)\}\right], \quad (\lambda > 0, \theta \geq 0, t \geq 0).$$

Singh and Kochar (1986) showed that the test based on T_n is consistent for testing H_0: F exponential vs H_1: F HNBUE. A small value of T_n indicates F being HNBUE.

TTT-test

Klefsjö (1983a,b) has shown that the total time on test statistic given in (7.5) is also consistent for testing H_0 against a much wider class HNBUE.

E_n-test

Kochar and Deshpande (1985) showed that the exponential score statistic E_n, originally suggested by Barlow and Doksum (1972) as a possible test for testing exponentiality against IFRA alternative, is also consistent against a wider class of HNBUE. Here,

$$E_n = \frac{1}{n} \sum_{i=1}^{n-1} \log\left(1 - \tau(X_{(i)})\right) \tag{7.33}$$

where τ is defined as in (7.5).

V^*-test

Hendi et al. (1998) introduced an exact test for the HNBUE alternative. The scale invariant test statistic is

$$V^* = \frac{1}{\bar{X}} \sum_{j=1}^{n} X_{(j)} \left(\frac{j}{n}\right) \frac{1}{n} \tag{7.34}$$

($X_{(i)}$, the order statistics) which is similar to the one introduced by Hollander and Proschan (1975, 1980).

$T_{n,a}$-test

Klar (2000) presented a class of tests for exponentiality against HNBUE alternatives which is based on the difference between the integrated distribution function $\Psi(t) = \int_t^\infty \bar{F}(x)\,dx$ and its empirical counterpart. The test statistic can be represented as

$$T_{n,a} = \frac{1}{na^2} \sum_{j=1}^{n} e^{-aY_j} - \frac{1}{a^2(1+a)}, \ \ a > 0 \tag{7.35}$$

where $Y_j = X_j/\bar{X}_n$, for $j = 1, ..., n$.

For $n = 1$, $T_{n,1}$ is equivalent to the T_n test given in (7.32). Jammalamadaka and Lee (1998) also considered the test statistic $T_{n,1}$.

7.4.7 NBU-t_0

Recall in Section 2.5.3, we say a distribution F is new better than used of age t_0 (NBU-t_0) if

$$\bar{F}(x + t_0) \leq \bar{F}(x)\bar{F}(t_0) \tag{7.36}$$

for all $x \geq 0$. Define

$$A = \{F : \bar{F}(x + t_0) = \bar{F}(x)\bar{F}(t_0), \text{ for all } \ x \geq 0\}. \tag{7.37}$$

Note that A is the class of boundary members of NBU-t_0 and NWU-t_0. Hollander et al. (1986) have shown that in addition to the exponential, there are two other distributions that are contained in A.

T-test

Hollander et al. (1986) proposed a test of the null hypothesis H_0: F is in A against H_1: F is NBU-t_0. Their test statistic is

$$T = \{n(n-1)\}^{-1} \sum \psi(X_{\alpha_1}, X_{\alpha_2} + t_0) - (2n)^{-1} \sum_{i=1}^{n} \psi(X_i, t_0) \tag{7.38}$$

where $\sum$ is the sum over all $n(n-1)$ sets of two integers (α_1, α_2) such that $1 \leq \alpha_1$, $\alpha_2 \leq n$, $\alpha_1 \neq \alpha_2$ and ψ is an indicator function defined by (7.9). The null hypothesis is rejected in favour of NBU-t_0 if $-n^{1/2}T$ is large.

The authors commented that the test based on T was not intended to be a competitor of J_n (given by (7.22)) or the total time on test statistic defined by (7.5). The latter statistics were designed for smaller classes of alternatives. T was designed for the relative large class of alternative A given by (7.37). They also calculated the asymptotic relative efficiencies from which they concluded that the T test will often be preferred when F is NBU-t_0 and is not a member of the smaller classes such as the NBU class.

T_k-test

Ebrahimi and Habibullah (1990) proposed a class of tests statistics, indexed by a positive integer k, for testing the null hypothesis that F belongs to A against the alternative hypothesis that F is NBU-t_0.

The statistic is given by

$$T_k = T_{1k} - T_{2k} \tag{7.39}$$

where

$$T_{1k} = \left\{1 / \binom{n}{k}\right\} \sum_{1 \leq i \leq j \leq n} \psi(X_i, X_j + kt_0)$$

and

$$T_{2k} = \left[1 / \left\{2 \binom{n}{k}\right\}\right] \sum \psi_k(X_{i_1}, \ ..., X_{i_k})$$

where the sum is over all combinations of k integers $(i_1, \ ..., \ i_k)$ chosen out of $(1, ..., n)$. Here, $\psi(a, b)$ is defined as in (7.9), and

$$\psi_k(a_1, ..., a_k) = \begin{cases} 1, \text{ if } \min a_i > t_0, \\ 0, \text{ otherwise.} \end{cases} \tag{7.40}$$

If $k = 1$, T_k reduces to the test statistic T defined by (7.38).

From the table of Pitman's asymptotic efficiency of T_k relative to T they concluded the following:

(1) Although the value of k varies, there is always a $k > 1$ for which T_k performs better than T for small t_0.

(2) If the alternative is NBU-t_0 but not NBU, then for all t_0, there is a $k > 1$ for which T_k performs better than T.

Ahmad's test

Let t_0 be the pth percentile so that $\bar{F}(t_0) = 1 - p$. Ahmad (1998) proposed a test statistic

$$\begin{aligned}\hat{\gamma}_k(F_n) &= \tfrac{1}{2}(1-p)^k - \int_0^\infty \bar{F}_n(x + kX_{([np])})\, dF_n(x) \\ &= \tfrac{1}{2}(1-p)^k - \textstyle\sum_{i=1}^n \sum_{j=1}^n \psi(X_i, X_j + kX_{([np])})\end{aligned} \qquad (7.41)$$

where ψ is the indicator function defined in (7.9) and $[x]$ is the largest integer that is less than or equal to x. The details on asymptotic properties were given in Ahmad (1998).

7.4.8 NBUC Tests

The NBUC class is defined by Definition 2.10. Using the moment inequalities for NBUC given in Section 2.5.5, Ahmad and Mugdadi (2004) obtained a test statistic $\hat{\delta}^{(2)}$ for the NBUC class:

$$\hat{\delta}^{(2)} = \frac{1}{\bar{X}^3}\left\{\frac{1}{n(n-1)}\sum_{i\neq j}\sum (X_i X_j^2 - X_i^3)\right\}. \qquad (7.42)$$

The authors claimed that the proposed test is simple and efficient.

7.4.9 NBUFR (NWUFR) Test

El-Bassiouny et al. (2004) constructed a test procedure for testing exponentiality against NBUFR (NWUFR) defined by Definition 2.8. It is a classical U-statistic that was derived from the difference between a moment inequality and the moment itself. It was shown that the proposed statistic has a high asymptotic relative efficiency with respect to tests of other classes for commonly used alternatives.

7.4.10 DPRL-α and NBUP-α Tests

Joe and Proschan (1983) developed tests for testing the null hypothesis of exponentiality against alternatives representing decreasing 100α-percentile residual life and the property 'new better than used with respect to the 100α-percentile'. These classes were defined in Section 2.8. As usual, let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ be the order statistics.

Tests for DPRL-α alternative: $W_{1:n}$-test

The test statistic is given by

$$W_{1:n} = \tfrac{1}{4}\sum_{i=0}^{n-1} B_1\left(\tfrac{i}{n}\right)\cdot(X_{(i+1)} - X_{(i)}) \qquad (7.43)$$

where

$$B_1(t) = \begin{cases} -(1-t)^2[(1-t)^2 - 1], & \text{for } 0 \le t < \alpha; \\ (1-t)^2[2((1-\alpha)^{-4} - 1)(1-t)^2 - ((1-\alpha)^{-2} - 1)], & \text{for } \alpha < t < 1. \end{cases}$$

Reject H_0 for large values of $W_{1:n}$.

Test for NBUP-α alternative: $W_{2:n}$-test

The test statistic is

$$W_{2:n} = \frac{1}{2}F_n^{-1}(\alpha) - 1/2\sum_{i=1}^{n}[B_2((i-1)/n) - B_2(i/n)], \tag{7.44}$$

where F_n is the empirical distribution function and

$$B_2(t) = \begin{cases} -\frac{1}{2}[(1-t)^2 - 1], & \text{for } 1 \leq t < \alpha; \\ \frac{1}{2}((1-\alpha)^{-2} - 1)(1-t)^2, & \text{for } \alpha < t \leq 1. \end{cases}$$

7.4.11 Summary of Tests of Basic Ageing Classes

The following two tables give an overview of the tests for various ageing classes discussed so far. Recall, all the statistical procedures were constructed to test exponentiality of a life distribution against different alternatives ageing classes. Two functions are used throughout the following tables of summary:

- ψ, the indicator function defined as

$$\psi(a,b) = \begin{cases} 1 & \text{if } a > b, \\ 0 & \text{otherwse;} \end{cases}$$

- ϕ, an increasing function.

The order statistics $X_{(1)} < X_{(2)} < ... < X_{(n)}$ from the random sample $X_1, X_2, ..., X_n$ are usually involved. Many of the tests discussed so far are U-statistics. In order to assess the 'goodness' of a test, the Pitman's asymptotic relative efficiency is usually evaluated for comparison with other existing test statistics that were derived to test for the same ageing alternative.

These tables of summary have four columns that give the test name, test statistic formula, ageing alternatives and some key references. The tables aim to provide the readers a quick and easy reference to the available tests for various ageing classes.

Table 7.1. Tests on Univariate Ageing (Part 1)

Test Name	**Basic Statistic**	**Ageing Alter-natives**	**Key References**
TTT Plot	$\tau(X_{(i)}) = \sum_{j=1}^{i} (n-j+1)(X_{(j)} - X_{(j-1)})$, $j = i, ..n.$	IFR	Klefsjö (1983a)
		IFRA NBUE DMRL	
	$U_i = \tau(X_{(i)})/\tau(X_{(n)}), T = \sum_{j=1}^{n} a_j U_j$ Each test has a different set of $a_i's$	HNBUE	Klefsjö (1983b)
U_n	$U_n = [4/n(n-1)(n-2)(n-3)] \times \sum \psi\{\min(X_{\gamma_1}, X_{\gamma_2}), (X_{\gamma_3} + X_{\gamma_4})/2\}$ sum over $\Omega = \{\gamma_i \neq \gamma_j\} \cap \{\gamma_1 < \gamma_2\} \cap \{\gamma_3 < \gamma_4\}$	IFR	Ahmad (1975, 1976) Hoeffding (1948)
J_b	$J_b = \frac{n}{(n-1)} \sum \psi(X_i,\ bX_j), 0 < b < 1$	IFRA	Deshpande (1983)
Q_n	$Q_n = \sum_{i=1}^{n} J\left(\frac{i}{n+i}\right)/n\bar{X}$, $J(u) = 2(1-u) \cdot [1 - \log(1-u)] - 1$	IFRA	Kochar (1985)
Link	$\Gamma = \frac{2}{n(n-1)} \sum_{i<j} X_{(i)}/X_{(j)}$	IFRA	Link (1989)
V^*	$V^* = V/\bar{X}, V = n^{-4} \sum_{i=1}^{n} C_{i,n} X_i, C_{i,n} = \frac{4}{3}i^3 - 4ni^2 + 3n^2 i - \frac{1}{2}n^3 + \frac{1}{2}n^2 - \frac{1}{2}i^2 + \frac{1}{6}i$	DMRL	Hollander and Proschan (1975, 1980)
J_n	$J_n = 2[n(n-1)(n-2)]^{-1} \sum \psi(X_{\gamma_1}, X_{\gamma_2} + X_{\gamma_3})\ 1 \leq \gamma_i \leq n$, $\gamma_1 \neq \gamma_2, \gamma_1 \neq \gamma_3, \gamma_2 < \gamma_3$	NBU	Hollander and Proschan (1972)
δ	$\delta_n = n^{-2} \sum_{i}^{n} \sum_{j}^{n} \phi(S_{ij}/n), \phi$ increasing; $S_{ij} = \sum_{k=1}^{n} \psi(X_k, X_i + X_j)$	NBU	Koul (1977, 1978a)
Genealized J	J test with a two-stage test procedure	NBU	Alam and Basu (1990)
S	$S = U - J$	NBU	Deshpande and Kochar (1983)
L_n	$L_n(\psi, m) = n^{-1} \sum_{i=1}^{n} \phi(n^{-1} \cdot S_i^{(m)})$, $S_i^{(m)} = \sum_{j=1}^{n} \psi[X_j, mX]$	NBU	Kumazawa (1983)

Table 7.2. Tests of Univariate Ageing (Part 2)

Test Name	Basic Statistic	Ageing alternatives	Key References
K^*	$K^* = K/\bar{X}, K = \frac{1}{2n^2}\sum_{i=1}^{n}\left(\frac{3n}{2} - 2i + \frac{1}{2}\right)\cdot X_{(i)}$	NBUE	Hollander and Proschan (1972)
CV	$S/\bar{X}, \quad S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/n$	NBUE NWUE	de Souza Borges and Proschan (1984)
T_n	$T_n = \frac{1}{n}\sum_{i=1}^{n}\exp(-X_i/\bar{X})$	HNBUE	Singh and Kochar (1986)
E_n	$E_n = \frac{1}{n}\sum_{i=1}^{n-1}\log(1-\tau(X_{(i)})),$ $\tau(X_{(i)}) = \sum_{j=1}^{i}(n-j+1)(X_{(j)} - X_{(j-1)})$	HNBUE	Kochar and Deshpande (1985)
T	$T = \{n(n-1)\}^{-1}\sum\psi(X_{\alpha_1}, X_{\alpha_2}+t_0)$ $-(2n)^{-1}\sum_{i=1}^{n}\psi(X_i, t_0)$	NBU-t_0	Hollander et al. (1985)
T_k	$T_k = T_{1k} - T_{2k},$ $T_{1k} = \sum_{i=j=n}\psi(X_i, X_j + kt_0)/\frac{2}{n(n-1)}$ $T_{2k} = \frac{1}{2}\sum\psi_k(X_{i_1}, ..., X_{i_k})/\binom{n}{k},$ $\psi_k(a_1, ..., a_k) = \begin{cases} 1, \text{if } \min a_i > t_0, \\ 0, \text{ otherwise.}\end{cases}$	NBU-t_0	Ebrahimi and Habibullah (1990)
$W_{1:n}$	$W_{1:n} = \frac{1}{4}\sum_{i=0}^{n-1}B_1\left(\frac{i}{n}\right)(X_{(i+1)} - X_{(i)}),$ $B_1(t) = \begin{cases} -\bar{t}^2[\bar{t}^2 - 1], & 0 \leq t < \alpha \\ \bar{t}^2(\bar{\alpha}^{-2} - 1)[2(\bar{\alpha}^{-2}+1)\bar{t}^2 - 1], & \alpha < t < 1,\end{cases}$ $\bar{t} = 1-t, \ \bar{\alpha} = 1 - \alpha$	DPRL-α	Joe and Proschan (1983, 1984)
$W_{2:n}$	$W_{2:n} = \frac{1}{2}F_n^{-1}(\alpha) - \sum_{i=1}^{n}[B_2((i-1)/n) - B_2(i/n)],$ $B_2(t) = \begin{cases} -\frac{1}{2}[(1-t)^2 - 1], & 1 \leq t < \alpha; \\ \frac{1}{2}((1-\alpha)^{-2} - 1)(1-t)^2, & \alpha < t \leq 1.\end{cases}$	NBUP-α	Joe and Proschan (1983, 1984)

Table 7.3. Tests of Univariate Ageing (Part 3)

Test Name	**Basic Statistic**	**Ageing alter-natives**	**Key References**
Stoch compa -rison	Based on comparing residual life at different times		
	$\triangle_{\text{IFR}}(n) > 0$	IFR	Belzunce et al. (1998)
	$\triangle_{\text{IFR}}(n) < 0$	DFR	
	$\triangle_{\text{NBU}}(n) > 0$	NBU	
	$\triangle_{\text{NBU}}(n) < 0$	NWU	Belzunce et al. (1998)
	Test statistics based on testing stochastic order $\triangle_{\text{IFR}}(n) = \frac{1}{n^2}\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\left(1-\frac{i}{n}\right)^2\left(1-\frac{j}{n}\right)^2 \times(\triangle_i^n - \triangle_j^n)$ $\triangle_{\text{NBU}}(n) = \frac{1}{n}\sum_{i=1}^{n-1}\left(1-\frac{i}{n}\right)^2(\triangle_i^n - \triangle_j^n)$		

7.5 Tests of Aging Properties When Data Are Censored

In this section, we shall briefly discuss tests of exponentiality against other alternatives when data are randomly right censored. There are at least two types of censoring, details of which may be found in Lawless (2003) and Nelson (1982).

Randomly right censored data are often the only information available in a life testing model or in clinical data. This experimental situation can formally be modelled as follows: Let $X_1, X_2, ..., X_n$ be independent identically distribution (i.i.d.) non-negative random variables having continuous life distribution function F. Let $Y_1, Y_2, .., Y_n$ be i.i.d random variables having a continuous distribution function G which is unknown. The Y_i's are treated as the random times to the right censorship. It is assumed that X's and Y's are independent. The available observations consist of the pairs $\{Z_i, \delta_i\},\ i = 1, ..., n$ where

$$Z_i = \min(X_i, Y_i) \text{ and } \delta_i = \psi(Y_i, X_i) \tag{7.45}$$

(ψ is the indicator function defined in (7.9), $\delta_i = 1$ would mean the ith object is not censored, whereas $\delta_i = 0$ means that the ith object is censored by Y_i on the right.)

Let $Z_{(1)} < ... < Z_{(n)}$ denote the ordered Z's and $\delta_{(1)}$, ..., $\delta_{(n)}$ be the δ's corresponding to $Z_{(1)}, ..., Z_{(n)}$, respectively. Most of the available tests for the censored data are based on the so-called Kaplan-Meier estimator defined by

$$\hat{F}_n(x) = 1 - \prod_{\{i: Z_{(i)} \le x\}} \left(\frac{n-i}{n-i+1} \right)^{\delta_{(i)}}, \tag{7.46}$$

see for example, Kaplan and Meier (1958) or Lee (1992). Without getting into details we shall now give several references on tests for different alternatives:

IFRA: Wells and Tiwari (1991), Pearn and Nebenzahl (1992)
NBU: Chen et al. (1983a) and Kumazawa (1987)
NBUE: Koul and Susarla (1980)
NBAFR (NBUFRA): Tiwari and Zalkikar (1994)
NBU-t_0: Hollander et al. (1985), Park (2003), Sen and Srivastava (2003)
DMRL: Chen et al. (1983b,c), Lim and Park (1993, 1997)
DPRL-α: Joe and Proschan (1983)
NBUP-α: Joe and Proschan (1983)

7.6 Tests of Monotonic Mean Residual Life Classes

It was pointed out in Chapter 4, the mean residual life (MRL) is often a more important reliability characteristic than the failure rate function $r(t)$. In this section, we consider only monotonic $\mu(t)$, the mean residual life function associated with F, so the life classes concerned are DMRL, IMRL and DMRLHA (decreasing mean residual life in harmonic average). The definitions of the first two classes were given in Section 2.4 whereas the last was in Section 2.7. Many of the tests discussed in Section 7.4 are also applicable in testing for exponentiality against DMRL (IMRL).

7.6.1 DMRL

V^*-test

Hollander and Proschan (1975, 1980) considered a test statistic which they called the V^* test for testing exponential distribution versus decreasing mean residual life alternatives. The V^* test is based on a linear function of the order statistics $X_{(1)} < X_{(2)} < ... < X_{(n)}$ from the sample, given by

$$V^* = V/\bar{X} \tag{7.47}$$

where $V = n^{-4} \sum_{i=1}^{n} C_{i,n} X_{(i)}$ with

$$C_{i,n} = \frac{4}{3} i^3 - 4ni^2 + 3n^2 i - \frac{1}{2} n^3 + \frac{1}{2} n^2 - \frac{1}{2} i^2 + \frac{1}{6} i \ .$$

Significantly large values of V^* indicate decreasing mean residual life alternatives, significantly small values of V^* suggest increasing mean residual life alternatives. Langenberg and Srinivasan (1979) obtained the exact null distribution for the test statistic V^*.

$V_n(k)$-test

Bandyopadhyay and Basu (1990) also proposed a test for testing exponentiality against DMRL. Their test procedure can be described as below:

Define:

$$D_k(x;F) = \bar{F}(x)\bar{F}(kx) \cdot [\mu(kx) - \mu(x)], 0 < k < 1, \tag{7.48}$$

where $\mu(x)$ is the mean residual life of F defined by (4.1). Also define

$$d(F;k) = \int_0^\infty D_k(x;F)\, dF(x) \tag{7.49}$$

which represents a measure of deviation from exponentiality towards DMRL distributions. Now, under H_0 (F exponential), $\delta(F;k) = 0$ and under H_1 (F is DMRL), $\delta(F;k) > 0$. Let the estimator $\delta(F_n, k)$ of $\delta(F;k)$ be the test statistic where F_n is the empirical distribution function. Define another statistic

$$V^*(k,n) = [n(n-1)(n-2)]^{-1} \sum \phi_k(X_{i_1}, X_{i_2}, X_{i_3}) \tag{7.50}$$

where the sum $\sum$ is taken over all permutations $\{i_1, i_2, i_3\}$ of 3 distinct integers chosen from $\{1, 2, ..., n\}$, $n \geq 3$, $\phi_k = \phi_1(x_1, x_2, x_3; k) - \phi_2(x_1, x_2, x_3; k)$, where

$$\phi_1(x_1,\ x_2,\ x_3; k) = (x_1 - kx_3)\psi(x_1, kx_3)\psi(x_2, x_3) \tag{7.51}$$

and

$$\phi_2(x_1,\ x_2,\ x_3; k) = (x_1 - x_3)\psi(x_1, x_3)\psi(x_2, kx_3). \tag{7.52}$$

As before, the indicator function $\psi(a, b) = 1$ or 0 depending $a > b$ or not. It was shown that the two statistics $\delta(F_n, k)$ and $V^*(k,n)$ are asymptotically equivalent and large positive values of $V^*(k,n)$ favour the alternative hypothesis. However, the distribution of $V^*(k,n)$ is not scale invariant. In order to make the test scale invariant, the authors used the test statistic:

$$V_n(k) = V^*(k,n)/\bar{X} \tag{7.53}$$

where $\bar{X}$ is the sample mean. The authors found that the asymptotic relative efficiencies of $V_n(k)$ with respect to V^* test of Hollander and Proschan (1975, 1980) are reasonably high.

For a generalization, see Bergman and Klefsjö (1989).

Ahmad's U-test

Ahmad (1992) proposed

$$\delta(F) = \int_0^\infty \left(2t\bar{F}(t) - \nu(t)\right)\, dF(t). \tag{7.54}$$

as a measure of the degree of 'DMRL-ness' where $\nu(t) = \int_t^\infty \bar{F}(u)\,du$. Based on this measure, he derived a new U-statistic

$$U_n = \sum_{i<j} \{\phi(X_i, X_j) + \phi(X_j, X_i)\} / \left\{2\binom{n}{2}\right\}. \tag{7.55}$$

The new test is easier to compute and performs better than several alternatives than previous tests.

Lim and Park (1993) generalized Ahmad's test to accommodate the situation where the data is incomplete due to random censoring.

$\tilde{T}_n^k$-test

Motivated by Ahmad (1992), Lim and Park (1997) considered a generalized measure of deviation of F from constant MRL in favour of DMRL alternatives:

$$\Delta_k(F) = \int_0^\infty \bar{F}^k(t)\,dt - \int_0^\infty \bar{F}^{k-2}(t)\nu(t)\,dF(t), \tag{7.56}$$

where $\nu(t) = \int_t^\infty \bar{F}(u)\,du$. It is clear that for $k = 2$, (7.56) reduces to (7.54). Their test statistic is given by $T_n^k = \Delta_n^k(F_n)$. The scale invariant version of T_n^k is

$$\tilde{T}_n^k = T_n^k / \bar{X}_n. \tag{7.57}$$

The above test is applicable to both complete and censored data, and it includes Ahmad's (1992) and Lim and Park's (1993) as its special cases.

$\hat{\triangle}_n$-test

Abu-Youssef (2002) introduced a scale-invariant test based on the moment inequality (4.24) with

$$\triangle_n = \frac{\hat{\delta}_n}{\bar{X}^2} \tag{7.58}$$

where

$$\hat{\delta}_n = \frac{2}{n(n-1)} \sum\sum_{i \neq j} \left[\min(X_i^2, X_j^2) - \frac{1}{2} X_i X_j\right].$$

It is shown that the proposed test has high relative efficiency for some commonly used alternatives and it also enjoys a good power.

$\hat{\delta}^{(3)}$-test

Ahmad and Mugdadi (2004) proposed a test based on the moment inequality given in Section 2.5.4 with test statistic

$$\hat{\delta}^{(3)} = \frac{1}{\bar{X}}\left\{\bar{X} - \frac{2}{n(n-1)}\sum_{i\neq j}\sum \min(X_i, X_j)\right\}. \tag{7.59}$$

The proposed test is simpler than most other tests and was shown to have very good efficiencies.

7.6.2 DMRLHA Test

Sen and Srivastava (1999) proposed a test for testing exponentiality against DMRLHA alternative. The test statistic D_n^* is the ratio of two U-statistics and is asymptotically normally distributed. Consistency, asymptotic unbiasedness and Pitman's efficiency results of the test developed have been obtained. They have shown that the D_n^*-test can also be used to detect whether a repairable unit in the long run depicts an increasing failure rate average property. The test statistic is:

$$D_n^* = \frac{D_n}{S^2} \tag{7.60}$$

and

$$D_n = \frac{1}{4(^nP_3)}\sum \phi_1(X_{i_1}, X_{i_2}, X_{i_3}) - \frac{1}{^nP_3}\sum \phi_2(X_{i_1}, X_{i_2}, X_{i_3}),$$

where we sum over all the $^nP_3$ permutations of 3 integers $\{i_1, i_2, i_3\}$ chosen from $\{1, 2, .., n\}$, with

$$\phi_1(X_1, X_2, X_3) = (2X_1 - X_3)\psi(2X_1 - X_3)(2X_2 - X_3)\psi(2X_2 - X_3),$$

and

$$\phi_2(X_1, X_2, X_3) = X_1(X_2 - X_3)\psi(X_2 - X_3).$$

Table 7.3 is a summary of tests for testing against monotonic MRL .

7.7 Tests of Non-monotonic Mean Residual Life

So far we have considered various tests for several classes with monotonic failure rates and monotonic mean residual life functions. Here in this section, we consider testing exponentiality against IDMRL and DIMRL. The definitions of these classes were given in Section 4.4.2. Since these life distributions have a change point τ which may or may not be known, we divide these tests into two categories:

- Either the change point τ or its corresponding quantile p is known.
- Neither p nor τ are known.

Table 7.4. Tests on Monotonic MRL

Test Name	**Basic Statistic /Special feature**	**Ageing Alter-natives**	**Key References**
TTT Plot	$\tau(X_{(i)}) = \sum_{j=1}^{i} (n-j+1)(X_{(j)} - X_{(j-1)})$, $j = i, ..n, U_i = \tau(X_{(i)})/\tau(X_{(n)}), T = \sum_{j=1}^{n} a_j U_j$. Each test has a different set of a_i's.	DMRL (IMRL)	Klefsjö (1983a)
V^*	$V^* = V/\bar{X}, V = n^{-4} \sum_{i=1}^{n} C_{i,n} X_i$, $C_{i,n} = \frac{4}{3}i^3 - 4ni^2 + 3n^2 i - \frac{1}{2}n^3 + \frac{1}{2}n^2 - \frac{1}{2}i^2 + \frac{1}{6}i$	DMRL	Hollander and Proschan (1975,1980)
V^*	Exact distribution of V^* obtained	DMRL	Langenberg and Srinivasan (1979)
V^c	V^c is a generlaization V^* for randomly censored model.	DMRL (IMRL)	Chen et al. (1983b,c)
V_{jk}	Generalization of V^* with $V_{11} = V^*$	DMRL (IMRL)	Bergman and Klefsjö (1989)
U_n	$U_n = \sum_{i<j} \{\phi(X_i, X_j) + \phi(X_j, X_i)\} / \left\{ 2 \binom{n}{2} \right\}$ $\phi(X_1, X_2) = (3X_1 - X_2)\psi(X_2 - X_1)$ The measure of DMRL is based on the first derivative of the MRL.	DMRL	Ahmad (1992)
δ_n^c	$\delta_n^c = \delta_n/\hat{\mu}_n, \delta_n = U_n, \quad \hat{\mu} = \int_0^\infty \bar{F}_n(x)\, dx$. It is the generalized U_n test for randomly censored data	DMRL	Lim and Park (1993)
TTT-like	Total time on test-like functions	DMRL	Aly (1990)
Scaled TTT	Test based on the scaled TTT defined by $\varphi_F(t) = \int_0^{F^{-1}(t)} \bar{F}(x)\, dx / \bar{F}(x)$	DMRL ordering	Aly (1993)
$D_n^* = \frac{D_n}{S^2}$	$D_n = \frac{1}{4({}^nP_3)} \sum \phi_1(X_{i_1}, X_{i_2}, X_{i_3}) - \frac{1}{{}^nP_3} \sum \phi_2(X_{i_1}, X_{i_2}, X_{i_3})$ $\phi_1(X_1, X_2, X_3) = (2X_1 - X_3)\psi(2X_1 - X_3) \times (2X_2 - X_3)\psi(2X_2 - X_3)$ $\phi_2(X_1, X_2, X_3) = X_1(X_2 - X_3)\psi(X_2 - X_3)$	DMRLHA	Sen and Srivastava (1999)

7.7.1 IDMRL (DIMRL) Test When Turning Point τ Is Known

Guess et al. (1986) considered the following procedures:

Let the two alternative hypotheses be given by H_1: F is IDMRL (and not constant MRL) and H_1': F is IDMRL (and not constant MRL).

Consider a distance function between the mean residual life at two points s and t defined by

$$D(s,t) = \bar{F}(s)\bar{F}(t)\left\{\mu(t) - \mu(s)\right\} \tag{7.61}$$

and a parameter induced by it:

$$T(F) = \int_0^\tau \int_0^t D(s,t)\, dF(s)\, dF(t) + \int_\tau^\infty \int_\tau^t D(s,t)\, dF(s)\, dF(t). \tag{7.62}$$

From (7.62) we see that $T(F)$ is a weighted measure of the degree to which F satisfied the IDMRL property.

Define $T_n = T(F_n)$ where F_n is the empirical distribution from the random sample $X_1, ..., X_n$ having cdf F. The IDMRL test procedure rejects H_0 in favour of H_1 at the approximate level α if $\tilde{T}_n = n^{1/2}T(F_n)/\hat{\sigma}_n \geq z_\alpha$ ($\hat{\sigma}_n^2$ is a consistent estimator of the variance of a statistic induced by T and F). The DIMRL test rejects H_0 in favour of H_1' at the approximate level α if $\tilde{T}_n \leq -z_\alpha$.

7.7.2 IDMRL Test When the Proportion p Is Known

Let $p =$ the proportion of the population that dies at or before the turning point τ. As usual, let $X_{(1)} < X_{(2)} ... < X_{(n)}$ denote the order statistics from a random sample of F and let $\tau = F^{-1}(p)$. We also define

$$j^* = \begin{cases} np, & \text{if } np \text{ is an integer} \\ [np] + 1, & \text{if } np \text{ is not an integer,} \end{cases}$$

where $[x] =$ the largest integer less than or equal to x.

Further, define $V_n = n^{-4} \sum_{k=1}^n c_{kn} X_{(k)}$ where c_{kn} depends on p as well as on k and n.

Three cases were considered:

- $k < j^*$,
- $k = j^*$, and
- $k > j^*$.

Define $V_n^* = V_n/\bar{X}_n$ and the standardized quantity $\tilde{V}_n = n^{1/2}V_n^*/\sigma(p)$ is used as the test statistic. Reject H_0 in favour of H_1: IDMRL if $\tilde{V}_n \geq z_\alpha$. On the other hand, we reject H_0 in favour of H_1': DIMRL if $\tilde{V}_n \leq -z_\alpha$. A table of critical values of the test statistic was given in Guess et al. (1986).

Lim and Park (1995) provided a competitor to the known procedures such as those given above and the Aly (1990) test. Based on the empirical powers of these tests against lognormal alternatives, they claimed that their test outperforms the others for most sample sizes and most values of p (the proportion of the population that dies at or before τ) and that all three tests achieve high power when p is very small or very large.

The test statistic against IDMRL is obtained by

$$\begin{aligned} T_n = n^{-2} \sum_{i=1}^{k-1} (4i - 3n) X_{(i)} - 2(1-p)^2 F_n^{-1}(p) \\ + n^{-2} \sum_{i=k}^{n} (-4i + 3n + 2pn) X_{(i)} \end{aligned} \tag{7.63}$$

where $k = [np]^+ =$ the smallest integer that is greater or equal to np. The scale invariant test statistic is $T_n^* = T_n / \bar{X}_n$. In this proposal, one rejects H_0 in favour of H_1 at the approximate level α if $\sqrt{3n} \cdot T_n^* \geq z_\alpha$. Analogously, one rejects H_0 in favour of H_1' if $\sqrt{3n} \cdot T_n^* \leq -z_\alpha$.

Lim and Park (1998) considered another test statistic

$$\begin{aligned} L_k = \tfrac{1}{n(k-1)} \left\{ \sum_{i=1}^{n} X_{(i)} \left[1 - k^2 \left(\tfrac{n-i}{n} \right)^{k-1} \right] \right\} - 2(1-p)^k F_n^{-1}(p) \\ + \left\{ \sum_{i=m}^{n} X_{(i)} \left[1 + k^2 \left(\tfrac{n-i}{n} \right)^{k-1} - 2(1-p)^{k-1} \right] \right\} \end{aligned} \tag{7.64}$$

where $m = [np]^+$, $k \geq 2$.

The scale invariant version of test is the statistic defined as $L_k^* = L_k / \bar{X}$. When $k = 2$, the test reduces to that of Lim and Park (1995).

7.7.3 Tests of IDMRL When Both p and τ Are Unkown

In practice, information regarding the change point is usually lacking thus there is a need for devising tests without known p or τ.

Aly (1990) considered a test for IDMRL using the test statistic

$$t_2 = \sup_{0<p<1} n^{1/2} \triangle(p, F_n) / \bar{X} \tag{7.65}$$

where

$$\triangle(p, F) = -\int_0^p h_1(y)\, dy + 2 \int_p^1 h_2(y)\, dF^{-1}(y). \tag{7.66}$$

Here

$$h_1(y) = (1-y)\{1 + \log(1-y)\}, \quad 0 \leq y \leq 1,$$

and

$$h_2(y) = h_1(y) - (1-y)\log(1-p), \quad 0 \leq y \leq 1.$$

Hawkins et al. (1992) constructed two tests based on estimates of functionals that distinguish F being exponential from F being IDMRL. These functionals are, for $F \in$ IDMRL and tests 1 and 2, respectively,

$$\phi_1(F) = \sup\{\psi_t^{(1)}(F) : 0 \le t \le F^{-1}(1-\varepsilon)\},$$

$$\phi_2(F) = \sup\{\psi_t^{(2)}(F) : t \ge 0\},$$

where $\varepsilon > 0$ is a small fixed number,

$$\psi_t^{(1)}(F) = \mu(t) - \mu,$$

$$\psi_t^{(2)}(F) = \int_0^t \left\{\mu(x)f(x) - \bar{F}(x)\right\} \bar{F}(x)\,dx - \int_t^\infty \left\{\mu(x)f(x) - \bar{F}(x)\right\} \bar{F}(x)\,dx.$$

The two test statistics are defined as

$$T_n^{(i)} = n^{1/2}\bar{X}_n^{-1}\phi_i(F_n),\ i = 1, 2. \tag{7.67}$$

Monte Carlo power comparisons of their tests with those of Guess et al. (1986) tests indicate that test 2 generally dominates test 1 and compares well with the latter tests when τ occurs below the 75th quantile of F. When τ exceeds the 75th quantile, neither test 1 nor test 2 clearly dominates the other, and neither compares well with those of Guess et al. (1986).

Na and Lee (2003) also devised a family of test statistics motivated by the behavior of the derivative of MRL:

$$\mu'(t) = \frac{f(t)\nu(t) - \bar{F}^2(t)}{\bar{F}(t)^2}, \tag{7.68}$$

where $\nu(t) = \int_t^\infty \bar{F}(x)\,dx$. Thus, $\mu(t)$ is decreasing (increasing) if and only if $f(t)\nu(t) \le (\ge)\bar{F}(t)^2$. They then considered a measure of deviation from the null hypothesis H_0 in favor of H_1 given by

$$T_j(F) = \sup\{\phi_j(x; F) : x \ge 0\},\ j \ge -1,$$

where

$$\phi_j(x; F) = \int_0^x \bar{F}^j(t)[f(t)\nu(t) - \bar{F}^2(t)]\,dt + \int_x^\infty \bar{F}^j(t)[f(t)\nu(t) - \bar{F}^2(t)]\,dt.$$

They showed that the derivative of $\phi_j(t; F)$ has the same sign as $\mu'(t)$ and hence F is IDMRL but not exponential if $\phi_j(x; F)$ is strictly increasing (decreasing) for $x < \tau$ $(x > \tau)$ and $T_j(F) = \phi_j(\tau; F)$. The proposed statistics are defined by

$$T_j^* = \frac{\sqrt{n}\,T_j(F_n)}{\bar{X}}, \tag{7.69}$$

where $F_n(t)$ denotes the empirical distribution function. The asymptotic null distributions of the test statistics were derived and their asymptotic critical values were obtained based on Durbin's approximation method.

7.7.4 Tests for NWBUE Class

The NWBUE (NBWUE) class was defined by Definition 4.3 in terms of the mean residual life $\mu(t)$. Recall, a distribution $F \in$ NWBUE has a change point τ such that $\mu(t) \geq \mu$ for $t < \tau$ and $\mu(t) \leq \mu$ for $x \geq \tau$. Mitra and Basu (1995) considered the problem of estimating the change point τ. Hawkins and Kochar (1997) have introduced a test which is complicated and its asymptotics rely heavily on the theory of the empirical scaled total time on test transform.

Anis and Mitra (2005) developed a simple test based on the property that the rth moment of $F \in$ NWBUE is dominated by the rth moment of G where

$$G(t) = \begin{cases} 0, & t < \tau \\ 1 - e^{-(t-\tau)/\mu}, & t \geq \tau; \end{cases}$$

and F is the exponential distribution if F and G share a common rth moment for some $r > 0$. Here μ is the mean of both F and G. They then developed a measure of deviation from the exponential for a NWBUE distribution based on the difference of their rth moment. The test was shown to be consistent and the asymptotic distribution of the test statistic has been obtained. The performance of the test against other alternatives has been studied by means of simulations.

A summary of tests of non-monotonic MRL is now given in Table 7.4 below for ease of reference.

7.8 Tests of Exponentiality Versus Bathtub Distributions

In this section, we discuss tests of an important class of life distributions that do not have monotonic failure rates. Recall in Definition 3.4, we say that F is a BT (bathtub shaped failure rate) distribution if its failure rate function decreases at first and then remains constant for a period and finally it increases with time. The BT distribution considered below has only one turning point, i.e., the BT shape is defined according to Section 2.2.1.

7.8.1 Test Based on Total Time on Test (TTT) Transform

Bergman (1979) suggested a test based on the TTT-(total time on test) transform for testing exponentiality against bathtub shaped distributions. The concept of TTT-transform was introduced in Section 2.5.6. Aarset (1985) derived the exact distribution of this test under the null hypothesis of exponentiality.

Xie (1989) used also the TTT-plot for testing exponentiality against BT alternative:

$$T = \sum_{j=1}^{n} a_j U_j \quad , \quad U_i = \tau(X_{(i)})/\tau(X_{(n)}) \tag{7.70}$$

where $\tau(X_{(i)})$ is defined by (7.5) with appropriately chosen a_i .

Table 7.5. Tests on Exponentiality Versus Trend Change in MRL

Test Name	**Basic Statistics/Special characteristic**	**Ageing Alternatives**	**Key References**
IDMRL(τ) Procedure	τ is assumed known. Based on T_n test which is asymptotically equivalent to V^*	IDMRL	Guess et al. (1986)
IDMRL(p) Procedure	τ is unknown but p is known Use V^* test	IDMRL	Guess et al. (1986)
T_n^*	Test based on U_n of Ahmad's (1992). ρ is assumed known.	IDMRL (DMRL)	Lim and Park (1995)
L_k^*	Test based on generalizing T_n^*. ρ is assumed known.	IDMRL (DMRL)	Lim and Park (1998)
T_1^n, T_2^n	Unknown change point τ and unknown proportion ρ, $T_n^{(i)} = n^{1/2}\bar{X}_n^{-1}\phi_i(F_n)$, $i = 1, 2$, ϕ_i being functionals.	IDMRL	Hawkins et al. (1992) Na and Lee (2003)
L_k^*	For both censored and uncensored data. Motivated by Ahmad's U_n test. p is assumed known but τ unknown.	IDMRL (DIMRL)	Lim and Park (1998)

As we pointed out in Section 7.4.2 that τ is the empirical version of the TTT-transform that corresponds

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} \bar{F}(u)\,du,$$

and thus U_j is the empirical scaled TTT-transform that corresponds to

$$\phi(t) = H_F^{-1}(t)/H_F^{-1}(1).$$

Now for $0 \le u \le 1$, $\frac{d}{dt}H_F^{-1}(u) = \frac{1}{r(F^{-1}(u))}$, i.e.,

$$\phi'(u) = \frac{1}{r(F^{-1}(u))H_F^{-1}(1)}$$

where $r(\cdot)$ is the failure rate function. Since $F(t)$ is increasing in t, it follows that $F^{-1}(u)$ is also increasing in u. We note that as

$$\phi''(u) = \frac{r'(F^{-1}(u))(1-u)}{r^3(F^{-1}(u))},$$

it is obvious ϕ is concave for F being IFR and convex for F being DFR. For a bathtub distribution, Barlow and Campo (1975) showed that (see Theorem 2.7 (vi)), $F \in \mathrm{BT}$ if ϕ has only one reflection point u_0 such that $0 < u_0 < 1$ and it is convex on $[0, u_0]$ and concave on $[u_0, 1]$.

We thus expect the empirical TTT-transform cure of a bathtub distribution to have an s-shape, that is, we anticipate the TTT-plot to lie below the 45^0-line in its leftmost part and above the line in its rightmost part (Aarset, 1987; Kunitz, 1989). The curves in Figure 7.1 below summarize the situations

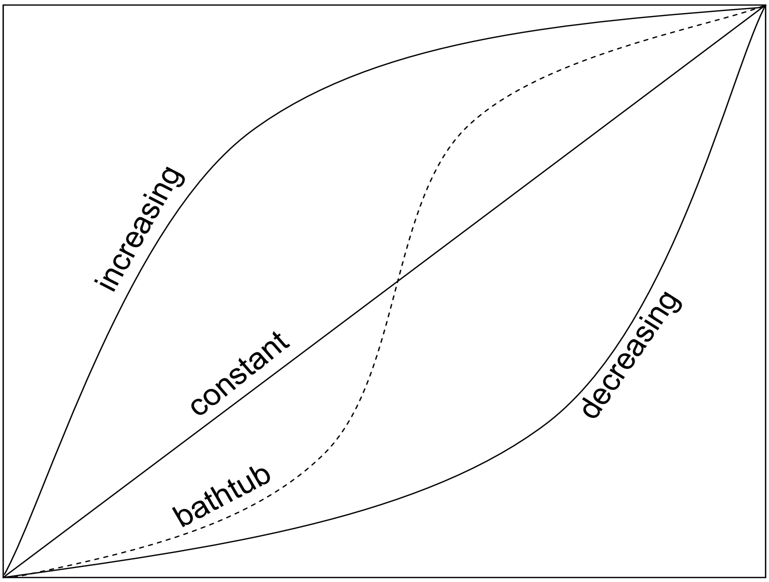


Fig. 7.1. Plot of ϕ_F against time t

Aarset (1987) also derived another test that is based on a TTT-plot which is equivalent to the well known Cramér-von-Mises test statistic. The proposed test statistic is

$$R_n = \sum_{i=1}^{n} = U_j \left(U_j - (2j-1)/n\right) + n/3$$

where U_j is defined by (7.70). The null hypothesis is rejected when R_n is large.

A Monte Carlo power comparison of the two tests were performed by Kunitz (1989).

Xie (1987) also used the total time on test concept for testing exponentiality against some partially monotone alternatives. Such ageing behaviours may arise in practical problems related to age replacements, burn-in and accelerated life testing.

7.8.2 Park's Test for BT

Apart from tests based on TTT transforms, there are other methods available for testing constant failure rate against bathtub shape. For example, Park (1988) used the following statistic:

Let $X_{(1)} < X_{(2)} ... < X_{(n)}$ denote the order statistics corresponding to the random sample from F and let F_n be the empirical cdf. The test statistic is

$$\begin{aligned} T_n \equiv & \sum_{i=1}^{k-1} n^{-2}[4i - (2+p)n]X_{(i)} \\ & + \sum_{i=k}^{n} n^{-2}[-4i + (3+p)n]X_{(i)} - F_n^{-1}(p)(1-p)(1-2p) \end{aligned} \tag{7.71}$$

where $k \equiv [np]^+, F_n^{-1}(p) = X_{(k)}$. (Recall, $[x]^+$ denotes the smallest integer greater than or equal to x, $p = F(t_0)$ = the proportion of population that dies at or before the change point t_0).

Instead of using T_n, Park (1988) used the following scale invariant test

$$T_n^* = T_n/\bar{X}, \quad \bar{X} = \sum_{i=1}^{n} X_i/n. \tag{7.72}$$

The test rejects H_0 in favour of $H_1 : F$ being BT at the approximate α level if $\sqrt{n}T_n^*/\sigma_0 \geq z_\alpha$, where z_α is the upper α-quantile of the standard normal distribution. We note in passing that $\sigma_0 = 1/3 - p + p^2$.

Although our main aim in this section is to study the testing procedures for BT, it was shown in Theorem 4.2 that under the condition $\mu r(0) > 1$, $F \in$ BT implies that F is IDMRL. We have discussed several test procedures for the IDMRL class in Section 7.7. In particular, we note that Guess et al. (1986) have developed two testing procedures for H_0: F is exponential versus H_1: F is IDMRL; one when the change point is known and the other with unknown change point but known p, the proportion of the population that die before the change point t_0. For other test statistics, see Lim and Park (1995, 1998).

7.8.3 Graphical Tests for BT Failure Rate Distributions

Apart from the TTT-plot mentioned earlier, there are other graphical tools for identifying bathtub shaped life distributions. For example, Kunitz and Pamme (1991,1993) proposed a plotting technique named by the 'ageing plot' which is a generalization of the TTT-plot. Kunitz and Pamme (1991) considered another plotting technique which is well known in reliability literature, namely, the so-called Weibull probability plot (WPP) defined in Section 5.3. These authors claimed that "convex shapes in Weibull probability plot might indicate an underlying distribution with a trend change from DFR- to IFR-behaviour (bathtub-shaped hazard rates)."

Pamme and Kunitz (1993) discussed the combined application of graphical tools and parametric estimation to identify a bathtub shaped failure distribution.

For another graphical test, see Kulasekera and Saxena (1991).

7.9 Other Miscellaneous Tests

In this section we consider some tests that do not fit in with the rest but nevertheless worth mentioning.

7.9.1 Test of Change Point of Failure Rate

Matthews and Farewell (1982) considered the testing for a constant hazard against a change point alternative of a model specified by the failure rate function

$$r(t) = \begin{cases} \lambda, & \text{for } \; t \leq \tau \\ \rho\lambda, & \text{for } t > \tau. \end{cases} \tag{7.73}$$

Let the loglikelihood function be denoted by $L(\lambda, \rho, \tau)$ and the loglikelihood test statistic for testing $\tau = 0$ (no turning point) is $\Delta_0 = L(\hat{\lambda}, \hat{\rho}, \hat{\tau}) - L(\hat{\theta}, 1, 0)$, where $\hat{\lambda}$, $\hat{\rho}$, and $\hat{\tau}$ are the maximum likelihood estimators and $\hat{\theta}$ is the maximum likelihood estimator of the simple exponential model. The authors concluded that $2\Delta_0 \sim \chi_1^2$. See also Yao (1986) for other details.

7.9.2 Aly's Tests for Change Point

A different type of null hypothesis has been proposed as follows. Let $[x]$ denote the integer part of x. Aly (1998) proposed a test for testing H_0: X_i have an unknown common distribution function F versus H_1: At most one change point such that X_i, $i \leq [n\lambda]$, have common distribution function F_1 and X_i, $[n\lambda] < i \leq n$, have common distribution F_2. Here $\lambda \in (0, 1)$ and $\mu_1(t) \geq \mu_2(t)$, $\mu_i(t)$, $i = 1, 2$ being the respective MRL. Three nonparametric test statistics $q_{1,n}$, $q_{2,n}$ and $q_{3,n}$ were proposed and their limiting distributions were obtained.

7.9.3 Testing Whether Lifetime Distribution Is Decreasing Uncertainty

Ebrahimi (1997) proposed a distribution-free test the hypothesis H_0 that uncertainty about the residual lifetime of a component does not change, i.e., F is exponential against the alternative hypothesis H_1 that uncertainty decreases over time.

We do not think the concept of uncertainty is sufficiently developed so we do not think the above test would generate much interest at present.

7.10 Final Remarks

Which test should we use if we have several competitors for testing exponentiality against the same ageing alternative? This is obviously a good question although there is no definite answer. Methods can be compared based on statistical point of view, but one has to consider the ease of implementation and interpretation. Statistically, there is a general broad principle we may apply for any statistical test. When a new statistic is proposed, consistency, unbiasedness and asymptotic normality are usually established. If there are already other existing tests for the same ageing alternatives, Pittman's asymptotic relative efficiency and power were computed and compared. Invariably, the performance and comparison are often studied by means of simulations. We may add that a test with good power characteristic or high efficiency may not attract attention unless it can be implemented easily. Many of the tests presented in this chapter are, in our opinion, are difficult to use. More research in this area should probably be carried out.

A more important question to ask in practice is what alternatives we are interested in. Often, monotonic failure rates are obvious choices. The bathtub-shaped failure rate could be of interest to practitioners as well. Other ageing classes are of less importance although, except possibly for NBU/NBUE classes which are useful in replacement models and decision making.

8

Bivariate and Multivariate Ageing

8.1 Introduction

Various univariate classes of ageing have been introduced and studied in Chapters 2-4. Also, statistical tests of these univariate ageing concepts have been considered in Chapter 7. Naturally, one would desire to generalize these concepts to multivariate lifetimes because a complex system usually consists of several components which are working under same environment and hence their lifetimes are generally dependent. Indeed, many such bivariate and multivariate ageing concepts have already appeared in the literature for a long time. Two pioneering pieces of work on multivariate ageing concepts were Harris (1970) and Thompson and Brindley (1972). Thus, a discussion on these and other concepts and their tests is indeed warranted. The following aspects of bivariate ageing concepts will be considered in this chapter:

- Bivariate reliability classes
- Bivariate IFR
- Bivariate IFRA
- Bivariate NBU
- Bivariate NBUE and HNBUE
- Bivariate decreasing mean residual life
- Bayesian notions of multivariate ageing
- Tests of bivariate ageing
- Conclusions

In this chapter, a letter 'B' is sometimes added as a prefix to a traditional ageing class. This is used to indicate a bivariate extension to an existing univariate ageing concept. Thus BIFR stands for bivariate increasing failure rate. Since there are often several possible definitions of a bivariate ageing class, only the most 'natural' or popular version is given this prefix.

Since a multivariate version is often an easy extension of a bivariate case, we will not give a separate treatment in this chapter.

8.2 Bivariate Reliability Classes

In dealing with multicomponent systems, one wants to extend the whole univariate ageing properties to bivariate and multivariate distributions. Consider, for example, the IFR concept in the bivariate case. Let us use the notation $\bar{F}(x_1, x_2)$ to mean the probability that item 1 survives longer than time x_1 and item 2 survives longer than time x_2. We note in particular, $\bar{F}(x_1, x_2) \neq 1 - F(x_1, x_2)$. Instead, the the joint survival function $\bar{F}(x_1, x_2)$ is related to the joint distribution function $F(x_1, x_2)$ through

$$\bar{F}(x_1, x_2) = \Pr(X > x_1, Y > x_2) = 1 - F_X(x_1) - F_Y(x_2) + F(x_1, x_2).$$

Then a possible definition of bivariate IFR is that $\bar{F}(x_1 + t, x_2 + t)/\bar{F}(x_1, x_2)$ decreases in x_1 and x_2 for all $t \geq 0$. But various other definitions of bivariate IFR are also possible, and a multiplicity of possible definitions also occur with the bivariate extensions of the other univariate ageing concepts. The non-unique definition on one hand provides the researchers ample opportunities for further development, on the other hand confuses the users.

Although we discuss only bivariate ageing concepts in this chapter, most of these concepts are readily extendable to higher dimensions.

Multivariate version of IFR, IFRA, NBU, NBUE, DMRL, HNBUE and of their duals have been defined and their properties have been developed by several authors. For an earlier bibliography of available results, see for example, Block and Savits (1981a,b), Basu et al. (1983), and Hutchinson and Lai (1990).

8.2.1 Different Alternative Requirements

In dealing with multicomponent systems, it is of great interest to obtain suitable bivariate and multivariate extensions of the univariate ageing properties mentioned above. In each case, however, several definitions are possible, because of different requirements imposed by various authors. This can be illustrated in the papers by Buchanan and Singpurwalla (1977) and Esary and Marshall (1979).

The multivariate definitions of the classes (IFR, IFRA, NBU, NBUE, and DMRL) given by Buchanan and Singpurwalla (1977) were motivated by the following requirements:

- The definitions should be based upon conditions imposed on the joint survival function, rather than on the corresponding random variables.

- The definitions should coincide with those that are accepted for a single variable.

- The definitions should lead to a chain of implications which is analogous to the chain of implications connecting the corresponding univariate classes as given in Chapter 2.

- The arguments which prompt the definitions should be natural extensions of those used for the various univariate definitions.

Specifically with regard to IFRA, Esary and Marshall (1979), in contrast to these requirements, gave six multivariate IFRA definitions which are not based upon requirements on the joint survival function, but on generalizations of those reliability properties which explain why the univariate IFRA concept is so important (e.g., that coherent systems have IFRA life distributions whenever their components have independent IFRA life distributions). Further, Esary and Marshall (1979) did not intend to meet the third requirement of Buchanan and Singpurwalla (1977).

We note that if the marginal distributions of bivariate V distributions are also univariate V, Buchanan and Singpurwalla (1977) called F jointly V, where V denotes IFR, IFRA, NBU, NBUE, or DMRL. Suresh (2001) showed that four of these strong bivariate ageing distributions (i.e., all but excluding bivariate IFRA) satisfy a simple moment inequality $E(X^2Y^2) \leq 4\left\{E(XY)\right\}^2$.

8.3 Bivariate IFR

The univariate IFR class of distributions was well studied in Section 2.5. Harris (1970) introduced the quantity

$$\bar{F}(x_1+t_1, x_2+t_2)/\bar{F}(x_1, x_2) \tag{8.1}$$

which is seen to be a direct bivariate analogue of the univariate conditional survival function $\bar{F}(x+t)/\bar{F}(x)$. Equation (8.1) may be interpreted as the joint probability of surviving additional t_i units ($i = 1, 2$) given that component i has survived until time x_i.

Recall from Definition 2.1, a univariate distribution F is IFR if $\bar{F}(x+t)/\bar{F}(x)$ decreases in x for every t. Then, one definition of F being a bivariate increasing failure rate distribution is that (8.1) decreases in $x_1 \geq 0,\ x_2 \geq 0$, for all $t_1 \geq 0,\ t_2 \geq 0$.

There are several variants of the above, and we shall list only four of them here.

1. $\bar{F}(x+t, x+t)/\bar{F}(x,\ x)$ decreases in $x \geq 0$ for all $t \geq 0$.

2. $\bar{F}(x+t_1, x+t_2)/\bar{F}(x, x)$ decreases in $x > 0$ for all $t_1, t_2 \geq 0$.

3. $\bar{F}(x_1+t, x_2+t)/\bar{F}(x_1, x_2)$ decreases in $x_1, x_2 \geq 0$ for all $t \geq 0$.

4. $\bar{F}(x_1+t_1, x_2+t_2)/\bar{F}(x_1, x_2)$ decreases in $x_1, x_2 \geq 0$ for all $t_1, t_2 \geq 0$.

Version (3) is due to Brindley and Thompson (1972); (4) is due to Harris (1970) and (1), (2) are due to Marshall (1975a). Many other versions are possible, but it appears that version (3) is perhaps the most important among them. One reason for its popularity is because this condition can be interpreted as the joint lifetime distribution of a series system of two components of different ages decreases stochastically as the ages of the components increase. Other reasons for its importance, according to Barlow and Proschan (1981, pp. 152–154), are:

- The marginal lifetimes X and Y have the univariate IFR property.
- And so does $\min(X, Y)$.
- There exists a sensible definition of multivariate IFR such that it is satisfied by the union of two mutually independent sets having property (3).
- The ratio $\bar{F}(x_1+t, x_2+t)/\bar{F}(x_1, x_2)$ depends on t but not on x_1 or x_2, if and only if the distribution is the bivariate exponential of Marshall and Olkin (1967).

Because of its prominence, we may use BIFR to designate version (3) of bivariate IFR concept.

A notion upon which other versions of bivariate IFR have been based is a generalized form of the univariate hazard rate (failure rate) $r(x) = f_X(x)/\bar{F}_X(x)$. Basu (1971) defined the bivariate failure (hazard) rate to be

$$r(x_1, x_2) = f(x_1, x_2)/\bar{F}(x_1, x_2). \tag{8.2}$$

But Johnson and Kotz (1973) opined that a multivariate generalization ought appropriately to be a vector, not a scalar. So they defined

$$r_i(x_1, x_2) = \frac{\partial}{\partial x_i} \log \bar{F}(x_1, x_2) = \frac{\partial}{\partial x_i} \bar{F}(x_1, x_2)/\bar{F}(x_1, x_2) \tag{8.3}$$

for $i = 1, 2$. The quantity $\boldsymbol{r}(x_1, x_2) = (r_1(x_1, x_2), r_2(x_1, x_2))$ is called the hazard gradient (see, e.g., Block, 1977a, Marshall, 1975b, and Johnson and Kotz, 1975). According to the idea of Johnson and Kotz (1973), then, bivariate IFR would imply that for all (x_1, x_2), r_1 is an increasing function of x_1, and r_2 is an increasing function of x_2. Block (1977b) showed that F has property (3) if and only if $\boldsymbol{r}(x_1+t, x_2+t)$ increases in t for all $x_1, x_2 \geq 0$.

For further developments with regard to both $f(x_1, x_2)/\bar{F}(x_1, x_2)$ and $\boldsymbol{r}(x_1, x_2)$, see, for example, Shanbhag and Kotz (1987).

Savits (1985) also defined a multivariate IFR class based on an extension of the characterization of univariate IFR in terms of the log concavity of the survival function $\bar{F}$. More precisely, a non-negative random vector $\boldsymbol{T}$ is said to have a multivariate increasing failure rate distribution if and only if $E[h(\boldsymbol{x}, \boldsymbol{T})]$ is log concave in $\boldsymbol{x}$ for all functions $h(\boldsymbol{x}, \boldsymbol{t})$ which are log concave in $(\boldsymbol{x}, \boldsymbol{t})$ and are non-decreasing and continuous in $\boldsymbol{t}$ for each fixed $\boldsymbol{x}$.

Shaked and Shanthikumar (1991b) also defined a multivariate IFR based on multivariate stochastic comparisons of residual lifetimes.

For Bayesian approach of multivariate IFR see Section 8.11 and Section 9.7 for details.

Remarks

- Apart from those discussed above, there are several other bivariate and conditional failure rate functions described in Cox (1972) and Shaked and Shanthikumar (1987).
- A multivariate ageing concept differs from the univariate one which can be described in general by the shape of failure rate function or the mean residual life function. This is because a univariate life distribution is uniquely determined by these functions if they exist. Unfortunately, this is not the case for bivariate ageing since there are various ways to define a bivariate failure rate function. Consider the definition of a bivariate failure rate $r(x, y)$ by Basu (1971) as given in (8.2):

$$\begin{aligned} f(x, y) &= \frac{\partial^2 F(x, y)}{\partial x \partial y} \\ &= r(x, y) \cdot \bar{F}(x, y). \end{aligned}$$

  The solution for the above equation has not yet been found so one cannot build a bivariate reliability model from the failure rate function as is done in the univariate case.
- It is obvious that version (4) is stronger than version (3) which is stronger than both (1) and (2).
- In this chapter, we generally do not discuss the relative strength of different versions of a bivariate ageing concept.
- Negative bivariate ageing concepts can be defined by reversing the appropriate inequalities and changing 'decreasing' to 'increasing' and vice versa.

8.4 Bivariate IFRA

The univariate IFRA class was discussed in Section 2.5. As with the bivariate IFR, several possible definitions for bivariate IFRA have been proposed by Esary and Marshall (1979) by extending the properties that characterize the univariate IFRA. Block and Savits (1980b) also defined a bivariate IFRA condition designated by BIFRA which was an extension to their particular characterization of the univariate IFRA given in Block and Savits (1976). Let $\boldsymbol{T}$=(X, Y) be a nonnegative vector with survival function $\bar{F}(t_1, t_2) = \Pr(X > t_1, Y > t_2)$. The following is their definition of a bivariate IFRA.

Definition 8.1: The vector $\boldsymbol{T}$ (or its distribution function F) is said to be bivariate IFRA if

$$E^{\alpha}[h(X, Y)] \leq E[h^{\alpha}(X, Y)/\alpha], \tag{8.4}$$

for all continuous nonnegative increasing functions h and all α such that $0 < \alpha < 1$. Although mathematically uncomplicated, the above definition seems to lack a meaningful physical interpretation. Since the class of all continuous nonnegative increasing functions in R is a subclass of all continuous nonnegative increasing functions in R^2, it is clear that the two component lifetimes are individually IFRA.

Other conditions for bivariate IFRA are given below using the same enumeration format as Esary and Marshall (1979) and Block and Savits (1982) (i.e., having condition Σ listed between conditions C and D):

A. $\bar{F}^{\alpha}(t_1, t_2) \leq \bar{F}(\alpha t_1, \alpha t_2)$, where again $0 < \alpha \leq 1$, for all $t_1 \geq 0,\ t_2 \geq 0$.

B. $\boldsymbol{T}$ is such that each monotone system formed from $\boldsymbol{T}$ is univariate IFRA.

C. $\boldsymbol{T}$ is such that there exist independent random variables $X_1, X_2, ..., X_k$ and monotone life functions τ_1 and τ_2, such that $T_i = \tau_i(X_1, X_2, ..., X_k)$.

Σ. $\boldsymbol{T}$ is such that there exist independent IFRA random variables $X_1, X_2, ..., X_k$ and nonempty sets S_i of $\{1,\ 2...,\ k\}$ such that $T_i = \sum_{j \in S_i} X_j$ for $i = 1, 2$.

D. $\boldsymbol{T}$ is such that there exist independent IFRA random variables $X_1, X_2, ..,$ X_k and nonempty sets S_i of {1, 2,..., k} such that $T_i = \min_{j \in S_i} X_j$ for $i = 1, 2$.

E. $\boldsymbol{T}$ is such that $\min(T_1, T_2)$ is IFRA.

F. $\boldsymbol{T}$ is such that $\min(\alpha_1 T_1, \alpha_2 T_2)$ is IFRA for all $\alpha_1 \geq 0, \alpha_2 \geq 0$.

Conditions A, B, C, D, E, and F have been given by Esary and Marshall (1979), and condition Σ was given by Block and Savits (1979). See also Buchanan and Singpurwalla (1977). Note that T_i in conditions C, E, and D represents the lifetime of a subsystem, whereas X_i denotes the lifetime of a basic component.

The following relationships hold between BIFRA in the sense of Block and Savits (1980b) as defined by (8.4) and the seven conditions above:

$$
\begin{array}{ccccc}
D & & \Sigma & & F \\
\Downarrow & & \Downarrow & & \Updownarrow \\
C & \Rightarrow & \text{BIFRA} & \Rightarrow & A \\
 & & \Downarrow & & \Downarrow \\
 & & B & \Rightarrow & E
\end{array}
$$

With one possible exception, the above chain is complete, i.e., no more implications are possible. The exception concerns whether $\Sigma \Rightarrow C$ holds, though Block and Savits (1982) conjectured that it does not.

An important property common to all these versions of bivariate IFRA is that the marginals are univariate IFRA (Block and Savits, 1982).

We note that condition A is the second weakest version among those listed above. It is a simple extension of an alternative definition of the univariate IFRA in Definition 2.2. Condition E is the weakest and the bivariate exponential of Marshall and Olkin (1967) given by (8.12) satisfies this condition because $\min(T_1, T_2)$ is exponential (Block, 1977c). Block and Savits, (1980b) showed that the bivariate exponential of Marshall and Olkin as well as several others are BIFRA.

Shaked and Shanthikumar (1986) gave a review on bivariate IFRA. A paper that includes another notion of bivariate IFRA is that of Mukherjee and Chatterjee (1988). See also Shaked and Shanthikumar (1988).

8.5 Bivariate NBU

The univariate NBU is defined by Definition 2.4. There are also several versions of bivariate NBU based on the inequality

$$\frac{\bar{F}(x_1 + t_1, x_2 + t_2)}{\bar{F}(x_1, x_2)} \leq \bar{F}(x_1, x_2), \;\; x_1, x_2, t \geq 0. \tag{8.5}$$

This can be interpreted as the conditional survival probability of two components of different ages being less than the corresponding survival probability $\bar{F}(x_1, x_2)$ of two new components.

Consider the following five versions:

1. $\bar{F}(x + t, x + t) \leq \bar{F}(x, x)\bar{F}(t, t)$ for all $t, \; x \geq 0$.

2. $\bar{F}(x + t_1, x + t_2) \leq \bar{F}(x, x)\bar{F}(t_1, t_2)$ for all $t_1, t_2, x \geq 0$.

3. $\bar{F}(x_1 + t, x_2 + t) \leq \bar{F}(x_1, x_2)\bar{F}(t, t)$ for all $t, x_1, x_2 \geq 0$.

4. $\bar{F}(x_1 + t_1, x_2 + t_2) \leq \bar{F}(x_1, x_2)\bar{F}(t_1, t_2)$ for all $t_1, t_2, x_1, x_2 \geq 0$.

5. $\bar{F}(x_1 + t_1, \; x_2 + t_2) \leq \bar{F}(x_1, x_2)\bar{F}(t_1, t_2)$ for all $t_1, t_2, x_1, x_2 \geq 0$ which satisfy $(x_i - x_j)(t_i - t_j) \geq 0$ for $i, j = 1, 2$.

The first four were given by Buchanan and Singpurwalla (1977), and condition (5) by Marshall and Shaked (1979). Version (3) has perhaps the greatest intuitive appeal. In Basu and Ebrahimi (1984b), version (3) is labeled as BNBU-I whereas version (1) is labelled as BNBU-II. Clearly, the bivariate exponential distribution (8.12) of Marshall and Olkin (1967) designated by BVE is the boundary member of BNBU-I and BNBU-II.

Another definition of bivariate NBU given by Marshall and Shaked (1982) is as follows:

A bivariate vector $\boldsymbol{T}$ has a bivariate NBU distribution if

$$\Pr[\boldsymbol{T} \in (a+\beta)A] \leq \Pr(\boldsymbol{T} \in aA)\Pr(\boldsymbol{T} \in \beta A), \tag{8.6}$$

for every $\alpha \geq 0, \beta \geq 0$ and every region A in the positive quadrant for which X exceeds some value X_A and Y exceeds some value Y_A. Marshall and Shaked (1982) gave several conditions equivalent to it, and they showed that it is implied by Block and Savits' (1980b) bivariate IFRA given in (8.4).

Ghosh and Ebrahimi (1983) derived some multivariate NBU and NBUE distributions from shock models.

For several other definitions of multivariate NBU, see Marshall and Shaked (1986a), who also established relationships between them. For a review, see Marshall and Shaked (1986b).

8.6 Bivariate NBUE and HNBUE

Less attention have been given to bivariate NBUE, DMRL, or HNBUE concepts than to the preceding three bivariate ageing classes. It was pointed out by Block and Savits (1981a) that various versions of bivariate NBUE can be obtained by integrating versions 1-4 of bivariate NBU given in the last section. For instance, the third one will become the condition

$$\int_0^\infty \bar{F}(x_1+t, x_2+t)\,dt \ \leq \bar{F}(x_1,x_2)\int_0^\infty \bar{F}(t,t)\,dt \tag{8.7}$$

for all $x_1, x_2 \geq 0$. This can be interpreted as a used series system of two components of ages x_1 and x_2, respectively, having a smaller remaining life than a new system. Version (4) of bivariate NBUE is given by (8.19)

For convenience, we use BNBUE to denote the the bivariate NBUE concept given by (8.7) above.

As to bivariate HNBUE, Basu et al. (1983) gave eight definitions, the first group of four being due to Klefsjö (1980) and based upon a multivariate version of the condition $\int_x^\infty \bar{F}(t)dt \leq \mu \exp\left(-x/\mu\right), x \geq 0$, and the second group of four being based upon a multivariate version of the mean residual life definition. Basu et al. (1983) gave some closure properties for the second group of classes. See also Basu and Ebrahimi (1986).

A popular version of bivariate HNBUE to be designated by BHNBUE is now defined as

Definition 8.2: F is BHNBUE if

$$\int_0^\infty \bar{F}(x_1+t, x_2+t)\,dt \le \int_0^\infty \bar{F}_0(x_1+t, x_2+t)\,dt,\ \ x_1, x_2 \ge 0 \tag{8.8}$$

where $\bar{F}_0$ is the survival function of the BVE defined by (8.15).

8.7 Bivariate Decreasing Mean Residual Life

Univariate DMRL is defined by Definition 2.3 and it is shown that DMRL $\Rightarrow$ NBUE. Now the bivariate NBUE condition (8.7) can be written as $\int_0^\infty \bar{F}(x_1+t, x_2+t)\,dt/\bar{F}(x_1, x_2) \le \int_0^\infty \bar{F}(t,t)\,dt$. Then a moderately strong bivariate DMRL can be defined as the conditional distribution

$$\int_0^\infty \bar{F}(x_1+t, x_2+t)\,dt/\bar{F}(x_1, x_2) \text{ decreases in } x_1, x_2 \ge 0.$$

Four other versions of bivariate DMRL can be derived in a similar manner from the the corresponding four versions of bivariate NBUE, see Buchanan and Singpurwalla (1977) for details.

Arnold and Zahedi (1988) introduced a new multivariate remaining life function which is different from that of Buchanan and Singpurwalla (1977). Based on this new definition, Zahedi (1985) put forward four classes of DMRL distributions; the relations among them were also established, but their relationships with the ideas of Buchanan and Singpurwalla (1977) were not given. The first two of these four definitions are given below.

Definition 8.3: F is said to BDMRL-I if

$$\mu_i(x_1+t_1, x_2+t_2) \le \mu_i(x_1, x_2), t_1, t_2 \ge 0,\ i = 1, 2; \tag{8.9}$$

where $\mu_1(x_1, x_2) = E(X - x_1 \,|\, X > x_1, Y > x_2)$ and $\mu_2(x_1, x_2) = E(Y - x_2 \,|\, X > x_1, Y > x_2)$ are the conditional mean residual life functions introduced previously in Section 4.11.

It is well known, see Gupta (2003) for example, that $\mu_1(x_1, x_2)$ and $\mu_2(x_1, x_2)$ jointly determine $\bar{F}(x_1, x_2)$. Moreover, these conditional mean residual life functions and the hazard gradient defined in (8.3) are connected by the relation

$$r_1(x_1, x_2) = \frac{1 + (\partial/\partial x_1)\mu_1(x_1, x_2)}{\mu_1(x_1, x_2)}, \quad r_2(x_1, x_2) = \frac{1 + (\partial/\partial x_2)\mu_2(x_1, x_2)}{\mu_2(x_1, x_2)}.$$

Definition 8.4: F is said to BDMRL-II if

$$\mu_1(x_1+t, x_2) \le \mu_1(x_1, x_2); \quad \mu_2(x_1, x_2+t) \le \mu_2(x_1, x_2),\ t \ge 0. \tag{8.10}$$

Zahedi (1985) showed that BDMRL-I $\Rightarrow$ BDMRL-II, but the converse is not true.

Bandyopadhyay and Basu (1995) showed that the equality in (8.9) holds if and only if X and Y are independent. Also, they showed that equality in (8.10) holds if and only if F follows the Gumbel's (1960) type I bivariate exponential distribution given by

$$\bar{F}(x_1, x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \delta x_1 x_2\}, \ \lambda_1, \lambda_2, \delta > 0. \tag{8.11}$$

We now give a list of references to each of the bivariate ageing concepts in Table 8.1 below.

Table 8.1. Bivariate versions of univariate ageing concepts

Ageing	References
Bivariate IFR	Harris (1970), Brindley and Thompson (1972), Basu (1971), Marshall (1975a), Marshall and Olkin (1967), Block (1977b), Johnson and Kotz, (1973, 1975), Savits (1985), Shanbhag and Kotz (1987), Bassan and Spizzichino (2001), Bassan et al. (2002).
Bivariate IFRA	Block and Savits (1980a), Esary and Marshall (1979), Buchanan and Singpurwalla (1977), Block and Savits (1982), Shaked and Shanthikumar (1986), Marshall and Shaked (1979) Mukherjee and Chatterjee (1988).
Bivariate NBU	Buchanan and Singpurwalla (1977), Marshall and Shaked (1982, 1986a,b).
Bivariate NBUE	Block and Savits (1981a,b), Hanagal (1998).
Bivariate DMRL	Buchanan and Singpurwalla (1977), Arnold and Zahedi (1988), Zahedi (1985), Bassan et al. (2002).
Bivariate HNBUE	Basu et al. (1983), Klefsjö (1980), Basu and Ebrahimi (1986).

8.8 Tests of Bivariate Ageing

In Chapter 7, we discussed various tests for different univariate ageing alternatives.

Let X and Y denote the lifetimes of two components having joint distribution function $F(x_1, x_2)$ and joint survival function $\bar{F}(x_1, x_2)$.

In testing bivariate ageing properties, there are two problems facing us. (i) Which bivariate exponential distribution is the null distribution? (ii) Which version of bivariate ageing property in a given class we are dealing with? (Remember there are several versions of bivariate IFR, bivariate NBU, etc.)

It seems to us all the authors used the bivariate exponential distribution of Marshall and Olkin (1967) (denoted by BVE) as the null distribution. This joint distribution has the survival function given by

$$\bar{F}(x_1, x_2) = \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)],\ x_1,\ x_2 \geq\ 0 \qquad (8.12)$$

where $\lambda_1, \lambda_2 > 0, \lambda_{12} \geq 0$.

This bivariate distribution is chosen presumably because of (a) it was derived from a reliability context, (b) it has the property of bivariate lack of memory:

$$\bar{F}(x_1 + t, x_2 + t) = \bar{F}(x_1, x_2)\bar{F}(t, t). \qquad (8.13)$$

Clearly, the BVE is the boundary member of BNBU-I and BNBU-II. From condition (3) in Section 8.3, it is also clear that the BVE is a boundary member of BIFR (bivariate increasing failure rate). Without getting into details, we now list several tests for testing H_0: F is the BVE versus the alternative hypothesis H_1: F belongs to one of the classes given in the forthcoming subsections below.

H_1: F is bivariate increasing failure rate (BIFR)

Both Sen and Jain (1991c) and Bandyopadhyay and Basu (1991) developed some type of U-statistic for testing F being the BVE versus F being BIFR as defined by version (3) in Section 8.3, i.e.,

$$\bar{F}(x_1 + t, x_2 + t)/\bar{F}(x_1, x_2) \quad \text{decreases in } x_1, x_2 \geq 0 \text{ for all } t \geq 0. \qquad (8.14)$$

H_1: F is bivariate increasing failure rate in average (BIFRA)

Recall, we have referred to BIFRA as the bivariate IFRA defined by Block and Savits (1980b). However, it seems to us the weaker version (condition A) of the bivariate failure rate average definitions specified by

$$\bar{F}^{\alpha}(t_1, t_2) \leq \bar{F}(\alpha t_1, \alpha t_2), 0 < \alpha \leq 1, \quad \text{for all } t_1 \geq 0,\ t_2 \geq 0, \qquad (8.15)$$

is easier to verify. This version is designated as condition A in Section 8.4.

Basu and Habibullah (1987) proposed a test statistic for testing BVE against condition A. The test was based on the measure

$$\Delta_{\alpha}(F) = \int_0^{\infty}\int_0^{\infty} [\bar{F}^{1/\alpha}(\alpha x, \alpha y) - \bar{F}(x, y)]\, dF(x, y),$$

where $F(x, y) = \Pr(X \le x, Y \le y)$. Hanagal and Ramanathan (1998) also developed some form of U-statistic to test the hypothesis that the lifetime distribution is BVE against the alternative (8.15) when the sample is either univariate or bivariate randomly censored.

H_1: F is a bivariate new better than used (BNBU)

Basu and Ebrahimi (1984b) proposed two tests for testing F is a bivariate new better than used (BNBU-I and BNBU-II). The first test statistic is called the J_n which is the extension of the J_n-test of Hollander and Proschan (1972) (see Section 7.4.4) to the bivariate case.

Sen and Jain (1991a) also proposed a test statistic based on a U-statistic for testing BVE against BNBU-I (version (3) of bivariate NBU), i.e., F satisfies the inequality

$$\bar{F}(x_1 + t, x_2 + t) \le \bar{F}(x_1, x_2)\bar{F}(t, t),\ t, x_1, x_2 \ge 0. \tag{8.16}$$

Both Basu and Ebrahimi (1984) and Sen and Jain (1991a) used the caterpillar data of Barlow and Proschan (1977) to illustrate their methods.

On the other hand, Hanagal (1998) also proposed a test for testing BVE against version (4) of bivariate NBU, i.e., F satisfies

$$\bar{F}(x_1 + t_1, x_2 + t_2) \le \bar{F}(x_1, x_2)\bar{F}(t_1, t_2),\ x_1, x_2, t_1, t_2 \ge 0. \tag{8.17}$$

The test is also a U statistic based on the measure

$$\Delta(\bar{F}) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \left\{\bar{F}(x, y)\bar{F}(t_1, t_2) - \bar{F}(x + t_1, y + t_2)\right\} dx\, dy\, dt_1 dt_2. \tag{8.18}$$

Let $T_1 = \sum_{i=1}^n X_i Y_i$ and $T_2 = \sum_{i=1}^n X_i^2 Y_i^2$ and set $\hat{\Delta}(\bar{F}) = T_1^2 - T_2/4$. Reject BVE in favor of bivariate NBU of (8.17) for very large values of $\hat{\Delta}(\bar{F})$.

H_1: F is a bivariate new better than used in expectation (BNBUE)

A test statistic, denoted by T_n, was proposed by Sen and Jain (1991b) for testing against the BNBUE alternative of (8.7).

Hanagal (1998) also derived a test for testing F BVE against F bivariate NBUE (given in Buchanan and Singpurwalla, 1977) which satisfies the inequality

$$\int_0^\infty \int_0^\infty \bar{F}(x_1 + t_1, x_2 + t_2) dx_1 dx_2 \le \bar{F}(x_1, x_2) \int_0^\infty \int_0^\infty \bar{F}(t_1, t_2) dt_1 dt_2. \tag{8.19}$$

The above inequality derived from integrating version (4) of bivariate NBU with respect to t_1, t_2.

Using the same statistic $\hat{\Delta}(\bar{F})$ as above, BVE is rejected in favour of (8.19) if $\hat{\Delta}(\bar{F}) > 0$.

H_1: F is bivariate harmonic new better than used in expectation (BHNBUE)

Basu and Ebrahimi (1984a) and Sen and Jain (1990) each proposed a test based on U_n and D_n statistics for testing against BHNBUE alternative. Hanagal (1997) proposed another test which he called the V_n test. A simulation study indicated that V_n test performs better than the former two tests.

H_1: F is bivariate decreasing mean residual life (BDMRL)

Bandyopadhyay and Basu (1995) proposed a class of tests for testing against BDMRL-II. These are U-statistics and hence are asymptotically normally distributed.

A test statistic, also denoted by D_n, was proposed by Sen and Jain (1991b) for testing against bivariate DMRL alternative defined by

$$\mu_F(x_1, x_2) = \frac{1}{\bar{F}(x_1, x_2)} \int_0^\infty \bar{F}(x_1 + t, x_2 + t)\, dt \geq \mu_{F_0}(x_1, x_2), \tag{8.20}$$

where $F_0(x, y)$ has a bivariate exponential distribution of Marshall and Olkin (1967) (BVE) given in (8.12).

8.8.1 Summary on Tests of Bivariate Ageing

It is our impression that tests for bivariate and multivariate stochastic ageing are more difficult than for the univariate case, however, they would probably offer a greater scope for applications. We anticipate that more research will be conducted in this area. Table 8.2 below gives relevant references to each of the bivariate ageing tests.

Table 8.2. Tests of Bivariate Ageing

Bivariate Version	Key References on Bivariate Tests
IFR	Sen and Jain (1991c), Bandyopadhyay and Basu (1991).
IFRA	Basu and Habibullah (1987), Hanagal and Ramanathan (1998).
DMRL	Sen and Jain (1991b), Bandyopadhyay and Basu (1995).
NBU	Basu and Ebrahimi (1984b), Sen and Jain (1991a), Hanagal (1998).
NBUE	Sen and Jain (1991a), Hanagal (1998).
HNBUE	Basu and Ebrahimi (1984a), Sen and Jain (1990), Hanagal (1997).

8.9 Discrete Bivariate Failure Rates

The discrete failure rate was defined by (6.4) and an alternative version by (6.45) in Chapter 6. In this section, we discuss the failure rates in the bivariate setting and they can be generalized to multivariate case as given by Shaked et al. (1995).

Suppose we have two discrete random variables X and Y each with support lies in $\mathcal{N}^+$. Denote its joint probability function by

$$f(j,k) = \Pr(X = j, Y = k),\ j,k \in \mathcal{N}^+. \tag{8.21}$$

Shaked et al. (1995) defined bivariate conditional failure rate function of (X,Y) as follows

$$r_1(k) = \Pr(X = k, Y > k \,|\, X \geq k, Y \geq k),\ \ k \in \mathcal{N}^+. \tag{8.22}$$

$$r_2(k) = \Pr(Y = k, X > k \,|\, X \geq k, Y \geq k),\ \ k \in \mathcal{N}^+. \tag{8.23}$$

$$r_{12}(k) = \Pr(X = k, Y = k \,|\, X \geq k, Y \geq k),\ \ k \in \mathcal{N}^+. \tag{8.24}$$

$$r_1(k \,|\, j) = \Pr(X = k \,|\, X \geq k, Y = j),\ \ k > j. \tag{8.25}$$

$$r_2(k \,|\, i) = \Pr(Y = k \,|\, X = i, Y \geq k),\ \ k > i. \tag{8.26}$$

provided the conditions in the above conditional probabilities have positive probabilities. Otherwise we set these functions to be 1.

The intuitive meaning of these functions is as follows. The function r_1, r_2 and r_{12} describe the initial failure rates, that is, the failure rates before a failure of any component. Suppose that one component failed at time i (j) and that the other component stayed alive at that time. Then conditional on $X = i$ (or $Y = j$), the failure rate of the life component at time $k > i$ (or $k > j$) is given by $r_2(k \,|\, i)$ (or $r_1(k \,|\, j)$).

The failure rates given in (8.22), (8.23), (8.25) and (8.26) are the discrete analogues of the bivariate conditional failure rate functions described in Cox (1972) and Shaked and Shanthikumar (1987). A discrete analogue of the bivariate failure rate, defined as $f(x,y)/\bar{F}(x,y)$ in (8.2) by Basu (1971), can be given as $r(j,k) = f(j,k)/\bar{F}(j-1,k-1)$.

Example 8.1: Bivariate geometric distribution

Let X_1, X_2 and X_3 be three independent geometric random variables on $\mathcal{N}^+$ with parameters θ_1, θ_2 and θ_3, respectively. Consider $X = X_1 + X_3$, $Y = X_2 + X_3$. The joint distribution of X and Y is called by Esary and Marshall (1973), the narrow sense bivariate geometric distribution. It is the discrete bivariate analogue of the BVE of Marshall and Olkin (1967). Shaked et al. (1995) found the conditional failure rates of this bivariate geometric distributions as follows.

1. $r_1(k) = \theta_1(1-\theta_2)(1-\theta_3)$, $k \in \mathcal{N}^+$.
2. $r_2(k) = (1-\theta_1)\theta_2(1-\theta_3)$, $k \in \mathcal{N}^+$.
3. $r_{12} = \theta_3 + \theta_1\theta_2(1-\theta_3)$, $k \in \mathcal{N}^+$.
4. $r_1(k \mid j) = \theta_1 + \theta_3 - \theta_1\theta_3$, $k > j$.
5. $r_2(k \mid i) = \theta_2 + \theta_3 - \theta_2\theta_3$, $k > i$.

8.10 Applications

Given the complexity, one would not anticipate too many real applications to this bivariate or multivariate ageing concepts. Below are a few real or possible applications.

8.10.1 Maintenance and Repairs

Bivariate imperfect repair

The concept of an imperfect repair was discussed in Section 6.0. The present application of a bivariate ageing concept to a bivariate imperfect repair was described in detail by Shaked et al. (1995). Two of several possible models of discrete bivariate imperfect repair are described below.

(A) Two items, with original lifetimes X and Y, start to function at time 1. Upon failure an item undergoes a repair. With probability p, the repair is unsuccessful and the item is scrapped. With probability $1-p$, the repair is minimal. If both items fail at the same time epoch, then, each of them, independent of each other, is successfully minimally repaired, and with probability p, and is scrapped with p.

(B) This model is the same as the above except that if both items fail at the same time epoch, then with probability $1-p$, both items are successfully minimally repaired, and with probability p, both are scrapped.

Using (8.22)–(8.26), the joint distribution function of the two times to scrap can be obtained for both models.

Maintenance and warranty

Chen and Popova (2002) proposed a new maintenance policy which minimizes the total expected servicing cost for an item with two-dimensional warranty. A two-dimensional warranty is characterized by a region in which a two-dimensional plane with one axis representing age and the other one usage. For example, when you buy a car it usually comes with 3 years or 36,000 miles warranty. They assumed that the bivariate failure rate function $r(x_1, x_2)$ either has an additive form $r(x_1, x_2) = \beta_1 x_1^w + \beta_2 x_2^w$ or a multiplicative form $r(x_1, x_2) = \exp\left(\beta_1 x_1^w + \beta_2 x_2^w\right)$ where $\beta_1, \beta_2, w > 0$.

Availability modelling

Yang and Nachlas (2001) proposed two classes of bivariate models for reliability and availability. These are the models based on stochastic functional relationships between the two variables and the models that represent the variables as two statistically dependent entities. The first class assumes a bivariate failure rate as the sum of two univariate failure rates whereas the second class has failure rate function defined by (8.2).

8.10.2 Warranty Polices

Univariate warranty polices were discussed in Section 3.8.3 in relation to the BT distributions.

Yang and Zaghati (2002) used a two-dimensional reliability model for modelling warranty data with time in service and mileage for automobile as the two variables.

Baik et al. (2004) discussed a two-dimensional failure modelling for a system where degradation is due to age and usage. Of course, degradation leads to failure. In this model, the bivariate failure rate is defined as in (8.2). The model utilizes a minimal repair policy so that the restored system is identical to what it was before failure. An application of this model to a manufacturer's servicing costs for a two-dimensional warranty was given.

8.10.3 Failure Times of Pumps

First failure times of transmission (X) and transmission pump (Y) on Caterpillar Tractors have been considered and analyzed by Barlow and Proschan (1977), Basu and Habibullah (1987), Sen and Jain (1991c), Hanagal (1997) and others.

8.11 Bayesian Notions of Multivariate Ageing

Having presented several classical definitions of multivariate ageing, the chapter will be incomplete without briefly presenting the Bayesian approach to modelling ageing. Bassan and Spizzichino (1999) commented that "These 'traditional' notions are specifically appropriate in the case when the dependence is mainly due to physical interaction among the units, interpreted as components of the same systems. Our notions are motivated from the analysis of situations where the (Bayesian) dependence due to learning about some unobservable quantity cannot be neglected."

The Bayesian approach of multivariate ageing has its source from majorization and Schur functions from inequality theory. Proschan (1975) used these ideas to obtain bounds, comparisons and inequalities in reliability and life testing. Barlow and Mendal (1992) used these ideas to define multivariate IFR for exchangeable random variables. The monograph by Spizzichino (2001) further developed these ideas.

8.11.1 Motivations and Historical Development of Bayesian Approach

Among several other aspects of reliability theory, Barlow (2003) outlined a motivation to search an alternative probabilistic approach to define ageing concepts.

Suppose we index a life distribution, say the exponential distribution using the mean life θ, i.e.,

$$F(x|\theta) = 1 - e^{-x/\theta}, \text{ for } x \geq 0.$$

In many applications, we usually do not know the mean lifetime. Suppose, however, we have prior knowledge concerning θ that can be prescribed by a probability density function $p(\theta)$. Then the unconditional distribution of the lifetime is given by

$$\int_0^\infty F(x|\theta)p(\theta)\,d\theta = \int_0^\infty \left[1 - e^{-x/\theta}\right] p(\theta)\,d\theta.$$

This unconditional distribution has a decreasing failure rate function (Proschan, 1963). In general, IFR distributions are not closed under mixing. This motivated Professor Richard Barlow and his colleagues to find a new probabilistic approach to characterize ageing under the Bayesian paradigm.

8.11.2 Concepts of Ageing and Schur Concavity

Before we define Schur concavity of a function, we need to define the concept of majorization.

Definition 8.5: The vector $\boldsymbol{y} = (y_1, y_2, ..., y_n)$ majors $\boldsymbol{x} = (x_1, x_2, ..., x_n)$, written as $\boldsymbol{x} \leq \boldsymbol{y}$, if $\sum_{i=1}^n x_{[i]} \leq \sum_{i=1}^n y_{[i]}$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Here $x_{[1]} \geq x_{[2]} ... \geq x_{[n]}$ and $\{y_{[i]}\}, i = 1, 2, ..., n$ are similarly defined (see Marshall and Olkin, 1979, pp. 7, 64).

The concept of majorization measures 'similarity' of vectors. The Bayesian's view of ageing is a relative concept. 'Similar' lifetimes should be more probable than 'diverse' lifetimes, since physical and chemical processes leading to ageing suggest more similarity with respect to lifetimes.

Definition 8.6: A function $\phi(\cdot)$ is Schur-concave iff $\boldsymbol{x} \leq \boldsymbol{y} \Rightarrow \phi(\boldsymbol{x}) \geq \phi(\boldsymbol{y})$.

Instead of just a single lifetime, we now consider a collection of exchangeable lifetimes $X_1, X_2, ..., X_n$ so they can be considered as 'similar'. Barlow and Mendel (1992) argued that if the n exchangeable units are ageing relative to lifetime, then one possible probability property capturing this notion is that the joint survival function $\bar{F}(x_1, x_2, ..., x_n) = \Pr(X_1 > x_1, X_2 > x_2, ..., X_n > x_n)$ is Schur-concave. Mathematically, Spizzichino (1992) proved that $\bar{F}(x_1, x_2, ..., x_n)$ is Schur-concave, if and only if, for ant $t > 0$ and $x_i < x_j$,

$$\Pr(X_i > x_i+t|X_1 > x_1, ..., X_n > x_n) \geq \Pr(X_j > x_j+t|X_1 > x_1, ..., X_n > x_n). \tag{8.27}$$

Spizzichino's mathematical results say that among any two items from n similar items (i.e., exchangeable items) that have survived a life test, the 'younger' is the 'best' if and only if the joint survival function is Schur-concave. This is an intuitive restatement of the IFR idea only now for for the conditional joint survival function. This shows the role of Schur-concavity in subjective multivariate ageing. See also Barlow and Spizzichino (1993).

Now, if $\bar{F}(x_1, x_2, ..., x_n|\theta)$ is Schur-concave, then it is still Schur-concave unconditionally since Schur-concavity is defined in terms of an inequality of $\bar{F}$. Thus, using Schur-concavity of the joint survival function as a definition of ageing, the difficulty with the univariate IFR definition (univariate IFR not closed under mixing) is overcome. To put in more simply, multivariate Schur-concavity is preserved under mixtures.

We now show the relationship between Schur-concavity and the classical IFR definition. If $\{X_i\}, i = 1, 2, ..., n$, are independent and identically distributed, then $\bar{F}(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} \bar{F}_X(x_i)$ which is Schur-concave iff $\log \bar{F}_X(x_i)$ is concave where finite (Marshall and Olkin, 1979, p. 74). Since univariate F_X is IFR iff $\log \bar{F}_X(x_i)$ is concave (Definition 2.1), we see that the Bayesian definition of multivariate IFR coincides with the classical definition in this case.

Example 8.2

Let X and Y be i.i.d. conditionally on a positive random variable Θ, with

$$\bar{F}_X(x|\Theta = \theta) = \exp\{-\theta x^2\}, \quad \bar{F}_Y(x|\Theta = \theta) = \exp\{-\theta y^2\}$$

and

$\Pr(\Theta > \theta) = \exp\{-\beta\theta\}$. Thus, both conditional distributions of X and Y are Weibull with shape parameter $\alpha = 2$ and hence they are both IFR. So $\bar{F}(x, y|\theta) = \bar{F}_X(x|\theta)\bar{F}_Y(y|\theta)$ is schur-concave. Now, the joint unconditional survival function

$$\bar{F}(x, y) = \int_0^\infty F_X(x|\theta)\bar{F}_Y(y|\theta)\beta e^{-\beta\theta} d\theta = \frac{\beta}{\beta + x^2 + y^2}, \; x, y \geq 0, \beta > 0$$

is clearly also Schur-concave.

8.11.3 Bayesian Notions of Bivariate IFR

The above ideas of Spizzichino (1992) were further extended to a more general analysis of Bayesian multivariate ageing by Bassan and Spizzichino (1999) and Bassan et al. (2002). Their approach was based on using different stochastic comparisons of residual lifetimes of units having different ages given the same history specified by the set D.

Let $\mathcal{L}(X|D)$ denote the distribution of X conditional on D with $\mathcal{L}$ standing for probability law.

Recall in Definition 2.1, we say that F is IFR iff $\bar{F}(x|t) = \frac{\bar{F}(x+t)}{\bar{F}(t)}$ is decreasing in t for each x, i.e.,

$$\text{F IFR} \Leftrightarrow \mathcal{L}(X - t|X > t) \geq_{\text{ST}} \mathcal{L}(X - t'|X > t'), \quad \text{for all } t \leq t'.$$

In the univariate residual lifetimes, the failure rate order and stochastic order are equivalent so the above is also equivalent to

$$\mathcal{L}(X - t|X > t) \geq_{\text{FR}} \mathcal{L}(X - t'|T > t'), \quad \text{for all } t \leq t'.$$

Bassan and Spizzichino (1999) and Bassan et al. (2002) extended this idea of comparing the residual lifetime to define two notions of bivariate IFR as follows.

Definition 8.7: An exchangeable pair of random variables (X, Y) with survival function $\bar{F}$ is bivariate IFR if, for $t_1 \leq t_2$,

$$\mathcal{L}(X - t_1|X > t_1, Y > t_2) \geq_{\text{ST}} \mathcal{L}(Y - t_2|X > t_1, Y > t_2). \tag{8.28}$$

Definition 8.8: An exchangeable pair of random variables (X, Y) with survival function $\bar{F}$ is bivariate IFR in the strong sense (s-BIFR) if, for $t_1 \leq t_2$,

$$\mathcal{L}(X - t_1|X > t_1, Y > t_2) \geq_{\text{FR}} \mathcal{L}(Y - t_2|X > t_1, Y > t_2). \tag{8.29}$$

We note that (8.28) holds if and only if the joint survival function $\bar{F}(t_1, t_2)$ is Schur-concave. On the other hand, a bivariate distribution F is (s-BIFR) if and only if

$$R(t) = \frac{\bar{F}(x + t, y)}{\bar{F}(y + t, x)} \quad \text{is increasing in} \, t \, \text{, for } 0 \leq x < y.$$

Taking logs and then differentiating both sides with respect to t, we find the above condition is equivalent to

$$r_{X|Y}(y + t|Y > x) \geq r_{X|Y}(x + t|Y > y), \tag{8.30}$$

where $r_{X|Y}(\cdot|Y > y^*)$ denotes the conditional failure rate of X given $Y > y^*$.

Unless X and Y are independent, the two Bayesian multivariate ageing concepts given by Definition 8.7 and Definition 8.8 are not equivalent. The latter implies the former, but the converse is not true as illustrated by the counter example presented in Bassan and Spizzichino (1999). The same article also shows that the BVE given in (8.12) with $\lambda_1 = \lambda_2$ satisfies the s-BIFR condition.

Condition to yield marginal IFR

As was observed by Bassan and Spizzichino (1999), the marginal ageing property of a Bayesian bivariate IFR need not IFR ageing unless the two component lifetimes are independent. Bassan et al. (2002) have shown that if X is right-tail decreasing in Y (RTD($X|Y$), see Section 9.2.7 for the definition), i.e., $\Pr(X > x|Y > y)$ decreasing in y for all x, then F s-BIRF implies both X and Y are IFR. This follows from letting $x = 0$ in (8.30) so that

$$r_X(y+t) \geq r_X(t|Y > y) \geq r_X(t)$$

for all $y > 0$ by the RTD property. Thus, X is IFR and so is Y because of the exchangeability.

8.11.4 Bayesian Bivariate DMRL

Bassan et al. (2002) also introduced a weak and a strong version of bivariate DMRL ageing, in analogy to Definitions 8.7 and 8.8.

Definition 8.9: An exchangeable pair (X, Y) is said to have bivariate DMRL distribution if, for $t_1 < t_2$,

$$E(X - t_1|X > t_1, Y > t_2) \geq E(Y - t_2|X > t_1, Y > t_2). \tag{8.31}$$

We note that (X, Y) satisfies (8.31) if and only if one (and hence all) of the following equivalent conditions holds:

- $$\int_{t_1}^{\infty} \bar{F}(x, t_2)\, dx \geq \int_{t_2}^{\infty} \bar{F}(x, t_1)\, dx, \text{ for } t_1 < t_2,$$
- $$\int_{t_1}^{\infty} \int_{t_2}^{\infty} \bar{F}(x, y)\, dxdy \text{ is Schur-concave in } (t_1, t_2),$$
- $$\mu_{X|Y}(t_1|Y > t_2) \geq \mu_{X|Y}(t_2|Y > t_1), \text{ for } t_1 < t_2,$$

  where $\mu_{X|Y}(\cdot|Y > t_2)$ denotes the conditional mean residual life function of the distribution of X given $Y > t_2$.

Since the stochastic ordering implies the mean residual ordering (see Table 2.1), it follows that bivariate IFR of Definition 8.7 implies bivariate DMRL of Definition 8.9.

Definition 8.10: We say that an exchangeable pair (X, Y) is bivariate DMRL in the strong sense (s-BDMRL) if, for all $t_1 < t_2$, the following inequality holds:

$$\mathcal{L}(X - t_1|X > t_1, Y > t_2) \geq_{\text{MR}} \mathcal{L}(Y - t_2|X > t_1, Y > t_2), \text{ for } t_1 < t_2. \tag{8.32}$$

The above definition is equivalent to the following

- $$\mu_{X|Y}(x+t_1|Y>y) \geq \mu_{X|Y}(x+t_2|Y>t_2), \text{ for } t_1<t_2 \text{ and all } x>0$$

- $$\frac{\int_{x+t_1}^{\infty} \bar{F}(u,t_2)\,du}{\int_{x+t_2}^{\infty} \bar{F}(u,t_1)\,du} \quad \text{is increasing in } x>0.$$

Definition 8.9 and Definition 8.10 are equivalent only when X and Y are i.i.d. The latter is a stronger notion than the former. Bassan et al. (2002) showed that the following bivariate Burr XII distribution

$$\bar{F}(x,y) = [1+x^3+y^3]^{-2}, \; x,y \geq 0$$

satisfies (8.31) but not (8.32).

8.11.5 Other Bayesian Bivariate Ageing Concepts

Other multivariate ageing concepts were also derived based on the inequality

$$\mathcal{L}(X-t_1|X>t_1, Y>t_2) \geq_* \mathcal{L}(Y-t_2|X>t_1, Y>t_2), \text{ for } 0<t_1 \leq t_2, \tag{8.33}$$

where $\geq_*$ can be one of the usual orderings such as the stochastic, failure rate or likelihood ratio orderings. For example, Bassan and Spizzichino (2000) derived a bivariate NBU based on this methodology. Bassan and Spizzichino (2001) stressed that their multivaraite notions of ageing are based on one-dimensional stochastic comparisons of residual lifetimes. Other notions of multivariate ageing, on the contrary, are based on multivariate stochastic comparisons (see Shaked and Shanthikumar (1991a,b, 1994). Others, such as the classical approach discussed in Sections 8.2-6, do not involve at all comparisons of lifetimes.

Bassan and Spizzichino (2001) also defined a notion of multivariate IFRA but not in terms of Schur-concavity. The IFRA notion of ageing seems to be very different from the IFR notion of ageing or its generalization in terms of Schur-concavity.

We also note that among so many classical multivariate ageing definitions, we find only Savits's (1985) version of IFR has something not so remotely related to the Schur-concavity. This reflects the Bayesian approach to ageing is quite different from the traditional one.

Bassan and Spizzichino (2001) described several notions of multivariate ageing for exchangeable lifetimes by means of the properties of the level sets of the joint survival function. These properties are characterized in terms of dependence concepts (see Chapter 9) of a suitably defined distribution. In fact, the joint survival function $\bar{F}$ is Schur-concave if and only if the level sets

$$A_k = \{\boldsymbol{x} | \bar{F}(\boldsymbol{x}) \geq k\} \tag{8.34}$$

is Schur-concave.

We will discuss Bayesian multivariate ageing notions and their relationships to dependence concepts in Section 9.7.

8.12 Conclusions

Univariate ageing concepts and tests of ageing classes are well understood but the same cannot be said for bivariate and multivariate extensions. Usually, in defining (traditional) multivariate notions of ageing a requirement is that the 1-dimensional marginals have the ageing property which is being extended. At present, it seems that many of the bivariate definitions given in this chapter lack clear physical interpretations, though it is true that some can be derived from shock models - for details, see for example, Marshall and Shaked (1979), Ghosh and Ebrahimi (1983) and Savits (1988). Unlike the univariate case where there is a chain of implications among various ageing concepts (Section 2.4.3), we know of no chain of implications among various bivariate ageing classes. The exception is the four "very strong" versions of bivariate ageing defined by Buchanan and Singpurwalla (1977) and considered by Suresh (2001). This lack of chains is not surprising given there are many possible versions for a bivariate aging concept. The several possible bivariate extensions to a univariate ageing concept would undoubtedly cause confusion for the users. It is our impression that tests for bivariate and multivariate stochastic ageing are more difficult than for the univariate case, however, they would probably offer a greater scope for applications. Some of these applications have been indicated in the Section 8.10.

In the Bayesian subjective approach, it is neither necessary nor appropriate to insist on this requirement on the 1-dimensional marginals. This approach offers a clear and definite alternative which may prove to be more useful although we have not seen how the method is implemented in a practical situation.

We anticipate that research in bivariate and multivariate ageing will continue to grow but one hopes a clearer 'structure' will emerge in future.

9

Concepts and Measures of Dependence in Reliability

9.1 Introduction

The concept of dependence permeates throughout our daily life. There are many examples of interdependence in the medicine, economic structures and reliability engineering, to name just a few. A typical example in engineering is that all outputs from an equipment will depend on the inputs in a broader sense which include material, equipment, environment and others. Moreover, the dependence is not deterministic but of stochastic nature. Here in this chapter, we limit the scope of our discussion to the dependence notions that are relevant to reliability analysis.

In the reliability literature, it is usually assumed that the component lifetimes are independent. However, components in the same system are used in the same environment or share the same load, and hence the failure of one affects the other components. We also have the case of so-called common cause failure and components might fail at the same time. The dependence is usually difficult to describe, even for very similar components. From light bulbs in an overhead projector to engines in an aeroplane, we have dependence, and it is essential to study the effect of dependence for better reliability design and analysis. There are many notions of bivariate and multivariate dependence. Several of these concepts were motivated from applications in reliability.

We may have seen abbreviations like these PQD, SI, LTD, RTI, etc over the last three decades. As one probably would expect, they refer to some form of positive dependence between two or more variables. We shall try to explain them and their interrelationships in the present chapter. Positive dependence means that large values of Y tend to accompany large values of X, and similarly for small values. Discussion of concepts of dependence involves refining, by means of definitions and deductions, this basic idea.

In this chapter, we focus our attention on a relatively weaker notion of dependence, namely, the positive quadrant dependence between two variables X and Y. We think that this easily verified form of positive dependence is more relevant in the subject area under discussion. Also, as might be expected, the notions of dependence are simpler and their relationships are more readily established in the bivariate case than the multivariate ones.

Hutchinson and Lai (1990) devoted a chapter to review concepts of dependence for a bivariate distribution. More recently Joe (1997) gave a comprehensive treatment of the subject on multivariate dependence. Thus, our goal here is not to provide another review; instead we shall focus our attention on the positive dependence concepts, in particular, the positive quadrant dependence. For simplicity, we confine ourselves mainly to the bivariate case although most of our discussion can be generalized to the multivariate situations. An important aspect of the current chapter is the availability of several examples that are employed to illustrate the concepts of positive dependence.

In the present chapter we aim to present several dependence concepts, positive dependence orderings and measures of dependence with the following structure:

- Positive dependence, a general concept;
- Important conditions describing positive dependence. (We state some definitions, and examine their relative stringency and their interrelationships);
- Positive quadrant dependence: conditions and applications;
- Examples of positive quadrant dependence;
- Dependence and Bayesian multivariate ageing;
- Positive dependence orderings;
- Pearson's product-moment correlation coefficient;
- Rank correlations; and
- Local measures of dependence.

We refer the readers to Table 1.3 for a list of some important acronyms that will appear extensively in this chapter.

9.2 Important Conditions Describing Positive Dependence

Concepts of stochastic dependence for a bivariate distribution play an important part in statistics. For each concept, it is often convenient to refer to the bivariate distributions that satisfy it as a family, or a group. Here in this chapter, we are mainly concerned with positive dependence. Although negative dependence concepts do exist, they are often obtained by negative analogues of positive dependence via reversing the appropriate inequity signs.

Various notions of dependence are motivated from applications in statistical reliability (see, e.g., Barlow and Proschan, 1981). The baseline, or starting point, of a reliability analysis of systems is independence of the lifetimes of the components. As noted by many authors, it is often more realistic to assume some form of positive dependence among components of a system. In discussing the relationships between two variables in reliability applications, one need not confine to the lifetime of two components. For example, in warranty analysis, the longevity is often defined in terms of two variables, say age (X) and usage (Y). For example, when you buy a new car, it usually comes with a 3 year or 60,000 km warranty, whichever comes first. It is obvious that age and usage are positively correlated.

Around about 1970, several works discussed different notions of positive dependence between two random variables, and derived some interrelationships among them – for example, Lehmann (1966), Esary et al. (1967), Esary and Proschan (1972), Harris (1970), and Brindley and Thompson (1972), among others. Yanagimoto (1972) unified some of these notions by introducing a family of concepts of positive dependence. Some further notions of positive dependence were introduced by Shaked (1977, 1979, 1982).

For concepts of multivariate dependence, see Block and Ting (1981) and a more recent text by Joe (1997).

9.2.1 Six Basic Conditions

The following basic conditions describe positive dependence; these are listed in increasing order of stringency. Lehmann (1966) is generally recognized as the first that formalized some early notions of bivariate dependence.

1. Positive correlation, $\mathrm{cov}\,(X, Y) \geq 0$.

2. For every pair of increasing functions a and b, defined on the real line R, $\mathrm{cov}[a(X), b(Y)] \geq 0$. Lehmann (1966) showed this condition is equivalent to
$$\Pr(X > x, Y > y) \geq \Pr(X > x)\Pr(Y > y), \tag{9.1}$$
or equivalently,
$$\Pr(X \leq x, Y \leq y) \geq \Pr(X \leq x)\Pr(Y \leq y). \tag{9.2}$$
The equivalence follows from the well known equality $\bar{F}(x, y) = 1 - F_X(x) - F_Y(y) + F(x, y)$. We say that X and Y are positively quadrant dependent (PQD) if and only if the above inequalities hold. Several families of positively quadrant dependent distributions will be introduced in Section 9.4 below.

   We also note that there is a geometric interpretation for the copula of PQD random variables–the graph of the copula must lie above the graph of the independent copula (Nelsen, 1999, p. 156). See Section 9.4.3 for the definition of a copula.

3. Esary et al. (1967) introduced the term ‘association’ for describing a dependence condition. We say that X and Y are (positively) associated if for every pair of functions a and b, defined on R^2 which are increasing in each of the arguments (separately),

$$\mathrm{cov}[a(X,Y), b(X,Y)] \geq 0. \tag{9.3}$$

We note in passing a direct verification of this dependence concept is difficult in general. However, it is often easier to verify one of the alternative positive dependence notions which imply association.

4. Y is right-tail increasing in X (written as RTI(Y|X)) if

$$\Pr(Y > y|X > x) \text{ increasing in } x \text{ for all } y. \tag{9.4}$$

Similarly, Y is left-tail decreasing in X (written as LTD(Y|X)) if

$$\Pr(Y \leq y|X \leq x) \text{ decreasing in } x \text{ for all } y. \tag{9.5}$$

The definitions above were due to Esary and Proschan (1972). Nelsen (1999, pp. 156-158) also provided geometric interpretations of the graph of the copula when the random variables are either RTI($Y|X$) or LTD($Y|X$).

5. Y is said to be stochastically increasing in x for all y (written as SI(Y $|X$)) if for every y, $\Pr(Y > y|X = x)$ is increasing in x. Similarly, we say that X is stochastically increasing in y for all x (written as SI($X|Y$)) if for every x, $\Pr(X > x|Y = y)$ is increasing in y. Note that SI($Y|X$) is often simply denoted by SI . Some authors (e.g., Lehmann, 1966) refer to this relationship as Y being positively regression dependent on X (abbreviated by PRD) and similarly X being positively regression dependent on Y. SI can be interpreted as the conditional survival probability of one component increases as the life-length of the other increases.

6. Let X and Y have a joint probability density function $f(x, y)$. Then f is said to be totally positive of order 2 (TP_2) if for all $x_1 < x_2$, $y_1 < y_2$,

$$f(x_1, y_1)f(x_2, y_2) \geq f(x_1, y_2)f(x_2, y_1). \tag{9.6}$$

(Shaked, 1977). It is easy to show that if f is TP_2, then F and $\bar{F}$ (survival function) are also TP_2, i.e.,

$$F(x_1, y_1)F(x_2, y_2) \geq F(x_1, y_2)F(x_2, y_1), \tag{9.7}$$

and

$$\bar{F}(x_1, y_1)\bar{F}(x_2, y_2) \geq \bar{F}(x_1, y_2)\bar{F}(x_2, y_1),\ x_1 < x_2,\ y_1 < y_2. \tag{9.8}$$

It is easy to see that either F TP_2 or $\bar{F}$ TP_2 implies F PQD.

The density f being TP_2 is also known as 'X and Y are likelihood ratio dependent' (LRD), or 'X and Y are positively likelihood ratio dependent' (PLRD). To avoid a possible confusion, the last acronym will not used in this book.

Remarks

- The LRD was first defined by Lehmann (1966) whereas the acronym TP_2 was defined in a more general setting by Karlin (1968, p. 15).
- Suppose X and Y are lifetime variables such that Y is stochastically increasing in X (SI($Y|X$)), i.e., $\Pr(Y > y|X = x)$ is increasing in x for every y. Then $\Pr(Y > y|X = x_2) - \Pr(Y > y|X = x_1) \geq 0$ for $x_2 \geq x_1$. Integrating this expression with respect to y, we have

$$E(Y|X = x_2) - E(Y|X = x_1) \geq 0, \text{ for } x_2 \geq x_1.$$

  In other words, if F is SI($Y|X$), then the regression curve $E(Y|X = x)$ is also increasing in x. Similarly, if F is SI($X|Y$), then $E(X|Y = y)$ is increasing in y.
- Jogdeo (1975, 1982) summarized four of the above basic conditions.
- We have not included a detailed discussion on multivariate dependence concepts in this book although many of the conditions given above can be extended easily to a multivariate setting.

9.2.2 The Relative Stringency of the Conditions

It is well known that these concepts are interrelated (see, e.g., Barlow and Proschan, 1981, Chapter 5 and Joe, 1997). The six conditions we listed above can be arranged in an increasing order of stringency. That is, (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). More precisely

$$\begin{array}{ccccc} & & \text{RTI} & & \\ & \nearrow & & \searrow & \\ \text{TP}_2 \Rightarrow \text{SI} & & & & \text{Association} \Rightarrow \text{PQD} \Rightarrow \text{cov}(X,Y) \geq 0 \\ & \searrow & & \nearrow & \\ & & \text{LTD} & & \end{array}$$

Some of the proofs for the links of the chain of implications are not straightforward whereas some others are obvious. Our proofs below are essentially those given in Barlow and Proschan (1981, pp. 143–144).

Step 1. Suppose f is TP_2, then for $x_1 < x_2$, $y_1 < y_2$,

$$f(x_1, y_1)f(x_2, y_2) \geq f(x_1, y_2)f(x_2, y_1).$$

Divide both sides of the inequality by $f_X(x_1)f_X(x_2)$ and integrate with respect to y_1 from $-\infty$ to y and with respect to y_2 from y to ∞, we have

$$\Pr(Y \leq y|X = x_1) \Pr(Y > y|X = x_2) \geq \Pr(Y \leq y|X = x_2) \Pr(Y > y|X = x_1).$$

Adding $\Pr(Y > y|X = x_1)\Pr(Y > y|X = x_2)$ to both sides, we obtain $\Pr(Y > y|X = x_2) \geq \Pr(Y > y|X = x_1)$ so F is $\mathrm{SI}(Y|X)$, i.e., (6) $\Rightarrow$ (5).

Step 2. Suppose F is $\mathrm{SI}(Y|X)$ and $x_1' \leq x_2'$. Be definition,

$$\int_y^\infty \frac{f(x_2', t)\,dt}{f_X(x_2')} \geq \int_y^\infty \frac{f(x_1', t)\,dt}{f_X(x_1')}$$

so

$$\int_y^\infty f(x_2', t)\,dt f_X(x_1') \geq \int_y^\infty f(x_1', t)\,dt f_X(x_2').$$

Integrating both sides with respect to x_1' from x_1 to x_2 and with respect to x_2' from x_2 to ∞ giving

$$\int_{x_2}^\infty \int_y^\infty f(x,t)\,dt\,dx \int_{x_1}^{x_2} f_X(x)\,dx \geq \int_{x_1}^{x_2} \int_y^\infty f(x,t)\,dt\,dx \int_{x_2}^\infty f_X(x)\,dx.$$

Adding $\int_{x_2}^\infty \int_y^\infty f(x,t)\,dt\,dx \int_{x_2}^\infty f_X(x)\,dx$ to both sides of the above inequality we obtain

$$\Pr(X > x_2, Y > y)\Pr(X > x_1) \geq \Pr(X > x_1, Y > y)\Pr(X > x_2)$$

so

$\Pr(Y > y|X > x_2) \geq \Pr(Y > y|X > x_1)$ and hence (5) $\Rightarrow$ (4).

Step 3. The proof of $\mathrm{SI}(Y|X) \Rightarrow$ 'Association' is very long, so we refer the reader to Esary and Proschan (1972), Lemma 1 and Lemma 2 and the theorem.

Step 4. Assume X and Y are associated. Let $a(X,Y) = 1$ if $X > x$, 0 otherwise; let $b(X,Y) = 1$ if $Y > y$, 0 otherwise. Then a and b are increasing functions of X and Y so $\mathrm{cov}[a(X,Y), b(X,Y)] \geq 0$ by the definition of 'association'. This is equivalent to $\Pr(X > x, Y > y) \geq \Pr(X > x)\Pr(Y > y)$, i.e., X and Y are positively quadrant dependent. Thus (3) $\Rightarrow$ (2).

Step 5. Assume X and Y are PQD. It is easy to see that PQD implies positive correlation by applying the Hoeffding's lemma, which states:

$$\mathrm{cov}(X,Y) = \int_{-\infty}^\infty \int_{-\infty}^\infty [F(x,y) - F_X(x)F_Y(y)]\,dx\,dy \tag{9.9}$$

(This identity is often useful in many areas of statistics). Thus (2) $\Rightarrow$ (1).

9.2.3 Associated Random Variables

Recall in Section 9.2.1, we say that two random variables X and Y are associated if $\mathrm{cov}[a(X,Y), b(X,Y)] \geq 0$. Obviously, this expression can be represented alternatively by

$$E[a(X,Y)b(X,Y)] \geq E[a(X,Y)]E[b(X,Y)], \tag{9.10}$$

where the inequality holds for all real functions a and b that are increasing in each component and are such that the expectations in (9.10) exist.

Barlow and Proschan (1981, p. 29) considered some practical reliability situations for which the components lifetimes are not independent, but rather are associated:

(a) Minimal path structures of a coherent system having components in common;

(b) Components subject to the same set of stresses;

(c) Structures in which components share the same load, so that the failure of one component results in increased load on each of the remaining components.

We note that in each case the random variables of interest tend to act similarly. In fact, all the positive dependence concepts share this characteristic.

An important application of the concept of 'association' is to provide probability bounds for system reliability. Many such bounds are presented in Esary et al. (1967), also in Section 3 of Chapter 2 and Section 7 of Chapter 4 of Barlow and Proschan (1981). The so called mini-max bounds on the reliability of a coherent system of associated components will be given in (9.15).

Interesting enough, the dependence concept 'association' was not even considered in Chapter 5 of Nelsen (1999) which deals with dependence between variables. We are of the opinion that the condition (9.10) that defines 'association' is simply too difficult to check directly. What one normally does is to verify one of the dependence conditions in the higher hierarchy that implies 'association'. We believe the situations mentioned above where the dependence between two components were prescribed by 'association' can also be adequately described by a weaker positive dependence concept PQD.

9.2.4 RCSI and LCSD

Among the positive dependence concepts we have introduced so far, f be TP_2 is the strongest. A slightly weaker notion introduced by Harris (1970) which is called the right corner set increasing (RCSI) meaning

$\Pr(X > x_1, Y > y_1 | X > x_2, Y > y_2)$ is increasing in x_2 and y_2 for all x_1 and y_1. Similarly, we say that F is left corner set decreasing (LCSD) if $\Pr(X \leq x_1, Y \leq y_1 | X \leq x_2, Y \leq y_2)$ is decreasing in x_2 and y_2 for all x_1 and y_1.

Shaked (1977) showed that $\mathrm{TP}_2 \Rightarrow$ RCSI. The proof largely follows from the fact that if f is TP_2 then from (9.8)

$$\bar{F}(x_1,y_1)\bar{F}(x_2,y_2) \geq \bar{F}(x_1,y_2)\bar{F}(x_2,y_1),\ x_1 < x_2,\ y_1 < y_2.$$

In fact, this inequality characterizes the RCSI property (Nelsen, 1999, Theorem 5.2.15). Note that if the joint distribution is absolutely continuous, then TP_2 is equivalent to RSCI or LCSD.

By choosing $x_1 = -\infty$ and $y_2 = -\infty$, we see that RCSI $\Rightarrow$ RTI. Further, Shaked (1977) showed that (see also Gupta 2003) X and Y are RSCI if and only if $r_1(x, y)$ is decreasing in y for all x and hence $r_2(x, y)$ is decreasing in x for all y where $(r_1(x, y), r_2(x, y))$ is the hazard gradient defined in (8.2)-(8.3).

9.2.5 WPQD

Alzaid (1990) introduced a weak PQD concept which shares most of the properties PQD. X and Y are are said to be weakly positive quadrant dependent of type 1 (WPQD1) if

$$\int_x^\infty \int_y^\infty [\Pr(X > u, Y > v) - \Pr(X > u)\Pr(Y > v)]\, dvdu \geq 0, \text{ for every } x, y;$$

and weakly positive quadrant dependent of type 2 (WPQD2) if

$$\int_{-\infty}^x \int_{-\infty}^y [\Pr(X > u, Y > v) - \Pr(X > u)\Pr(Y > v)] dvdu \geq 0 \text{ for every } x, y.$$

It is obvious that PQD implies both WPDQ1 and WPQD2 and both of the latter two imply $\mathrm{cov}(X, Y) \geq 0$.

9.2.6 Positively Correlated Distributions

Positive correlation is the weakest notion of dependence between two random variables X and Y. We note that it is easy to construct a positively correlated bivariate distribution. For example, such a distribution may be obtained by simply applying a well-known trivariate reduction technique described as follows:

Set $X = X_1 + X_3$, $Y = X_2 + X_3$, with $X_i, (i = 1, 2, 3)$ being mutually independent, then the correlation coefficient of X and Y is

$$\rho = \mathrm{var} X_3 / \left[\mathrm{var}(X_1 + X_3)\mathrm{var}(\mathrm{X}_2 + \mathrm{X}_3)\right]^{1/2} > 0. \tag{9.11}$$

For example, let $X_i \sim \text{Poisson}(\lambda_i), i = 1, 2, 3$. Then $X \sim \text{Poisson}(\lambda_1 + \lambda_3), Y \sim \text{Poisson}(\lambda_1 + \lambda_3)$ with $\rho = \lambda_3 / \left[(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)\right]^{1/2} > 0$.

X and Y constructed in this manner are also positively quadrant dependent, see Example 1(ii) Lehmann (1966, p. 1139).

9.2.7 Summary of Interrelationships

We may summarize the links among these dependence notions by the following chain of relations (in which Y is conditional on X whenever there is a conditioning):

$$\begin{array}{ccccccccccc}
\text{RCSI} & \Rightarrow & \text{RTI} & \Rightarrow & \text{ASSOCIATION} & \Rightarrow & \text{PQD} & \Rightarrow & \text{WPQD} & \Rightarrow & \text{cov} \geq 0 \\
\Uparrow & & \Uparrow & & \Uparrow & & & & & & \\
\text{TP}_2 & \Rightarrow & \text{SI} & \Rightarrow & \text{LTD} & & & & & &
\end{array}$$

There are other chains of relationships between various concepts of dependence. A more comprehensive chain of implications is now given below:

$$\begin{array}{ccccccccc}
\text{LRD}\,(\text{TP}_2) & \Rightarrow & \text{SI}(Y|X) & \Rightarrow & \text{RTI}(Y|X) & \Leftarrow & \text{RCSI} & \Leftrightarrow & \bar{F}\ \text{TP}_2 \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Uparrow \\
\Downarrow & & \text{LTD}(Y|X) & \Rightarrow & \text{PQD} & \Leftarrow & \text{RTI}(X|Y) & & \Uparrow \\
\Downarrow & & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
F\ \text{TP}_2 & \Leftrightarrow & \text{LCSD} & \Rightarrow & \text{LTD}(X|Y) & \Leftarrow & \text{SI}(X|Y) & \Leftarrow & \text{LRD}\,(\text{TP}_2)
\end{array}$$

Note that in the preceding chain, 'left corner set decreasing' (LCSD) is equivalent to F is TP_2 and 'right corner set increasing' (RCSI) is equivalent to $\bar{F}$ is TP_2. There is no known direct relationship between SI and RCSI (or LCSD).

A strong positive dependence concept defined in terms of conditional failure (hazard) rate $r(x|Y = y)$ associated with X given $Y = y$ was given in Shaked (1977). It was shown that TP_2 implies $r(x|Y = y)$ decreasing in y for every x which in turn implies both $\text{SI}(X|Y)$ and RCSI.

Negative Dependence

Some concepts of negative dependence were first introduced by Lehmann (1966), and further developed by others such as Block et al. (1982). All of these can be obtained by negative analogues of positive dependence, e.g., when the inequality signs in (1), (2), (4), (5) and (6) (as listed in Section 9.2.1) are reversed, we obtained negative dependence concepts. Thus, the duals of of (2), (4), (5) and (6) are respectively called NQD, RTD (LTI), SD, and RR_2 (Reverse regular of order 2). Also the negative analogue of RCSI is RCSD. However, 'association' has no simple negative analogue by reversing the inequality sign of (3) although a negative association concept was defined by Joag-Dev and Proschan (1983). Further, a chain of implications analogous to the one above can also be given.

9.3 Positive Quadrant Dependent (PQD) Concept

We have presented several notions of bivariate dependence that are well known in the literature. The notion of positive quadrant dependence (PQD) appears to be more straightforward and easier to verify than other notions. The rest of the chapter mainly focuses on this dependence concept. The definition of PQD, which was first given in (9.1) is now formally defined as follows:

Definition 9.1: Random variables X and Y are positively quadrant dependent (PQD) if the following inequality holds:

$$\Pr(X > x, Y > y) \geq \Pr(X > x)\Pr(Y > y), \text{ for all } x \text{ and } y$$

which is is equivalently to

$$\Pr(X \leq x, Y \leq y) \geq \Pr(X \leq x)\Pr(Y \leq y), \text{ for all } x \text{ and } y.$$

Intuitively, X and Y are PQD if the probability that they are simultaneously small (or simultaneously large) is at least as great as it would be were they independent. Restricting ourselves to lifetime variables, we may interpret the last inequality as the probability represented by the joint density surface bounded by the quadrant $[0, x] \times [0, y]$ is as great as the product of the two probabilities represented by the areas under the marginal curves within the respective intervals $[0, x]$ and $[0, y]$.

The reason why (9.1) constitutes a positive dependence concept is that X and Y here are more likely to be large or small together compared with the independent case.

If the inequality in probability of (9.1) is reversed, then X and Y are negatively quadrant dependent (NQD).

PQD is shown to be a stronger notion of dependence than the positive (Pearson) correlation but weaker than the 'association' which is a key concept of positive dependence in Barlow and Proschan (1981), originally introduced by Esary et al. (1967).

Consider a system of two components that are arranged in series. By assuming that the two components are independent when they are in fact positively quadrant dependent, we will underestimate the system reliability. For parallel systems, on the other hand, assuming independence when components are in fact positively quadrant dependent, will lead to overestimation of system reliability (see Corollary 10.2 for a proof). This is because component B would probably fail earlier if component A fails. This dependence, from a practical point of view, may limit the effectiveness of adding parallel redundancy. Thus a proper knowledge on the extent of dependence among the components in a system will enable us to obtain a more accurate estimate of the reliability characteristic in question.

PUOD and PLOD

Unlike other bivariate dependence concepts which can be readily extended to the corresponding multivariate dependence of n variables, PQD is not the case. This is because (9.1) and (9.2) are equivalent only for $n = 2$. For $n > 2$, we say that $X_1, X_2, ..., X_n$ are positively upper orthant dependent (PUOD) if

$$\Pr(X_1 > x_1, X_2 > x_2, ..., X_n > x_n) \geq \prod_{i=i}^{n} \Pr(X_i > x_i) \tag{9.12}$$

and they are positively lower orthant dependent (PLOD) if

$$\Pr(X_1 \leq x_1, X_2 \leq x_2, ..., X_n \leq x_n) \geq \prod_{i=i}^{n} \Pr(X_i \leq x_i). \tag{9.13}$$

It is easy to prove (see, e.g., Theorem 3.2, Chapter 2 of Barlow and Proschan, 1981) that 'association' implies both PUOD and PLOD.

9.3.1 Constructions of PQD Bivariate Distributions

Let $F(x, y)$ denote the distribution function of $(X,\ Y)$ having continuous marginal cdf's $F_X(x)$ and $F_Y(y)$ with marginal pdf's $f_X = F_X'$ and $f_Y = F_Y'$, respectively. For a PQD bivariate distribution, the joint distribution function may be written as

$$F(x, y) = F_X(x)F_Y(y) + w(x, y) \tag{9.14}$$

with $w(x, y)$ satisfying the following conditions:

(i) $w(x, y) \geq 0$.

(ii) $w(x, \infty) \to 0, w(\infty, y) \to 0,\ w(x, -\infty) = 0,\ w(-\infty, y) = 0$.

(iii) $\dfrac{\partial^2 w(x, y)}{\partial x \partial y} + f_X(x)f_Y(y) \geq 0$.

Note that if both $X \geq 0$ and $Y \geq 0$, then the condition in (ii) may be replaced by

$$w(x, \infty) \to 0, \quad w(\infty, y) \to 0, \quad w(x, 0) = 0, \quad w(0, y) = 0.$$

Lai and Xie (2000) used these conditions to construct a family of PQD distributions with uniform marginals. It is likely that there are other methods available for constructing PQD distributions.

9.3.2 Applications of Positive Quadrant Dependence Concept to Reliability

The notion of association is used to establish probability bounds on reliability systems (see, e.g., Chapter 3 of Barlow and Proschan, 1981). Given a coherent system of n components with minimal path sets P_i, $(i = 1, 2, ..., p)$ and minimal cut sets K_j, $(j = 1, 2, ..., k)$. Let T_i denote the lifetime of the ith component and thus $p_i = \Pr(T_i > t)$ is its survival probability at time t. It has been shown (Barlow and Proschan, 1981, pp. 35–38), that if components are independent, then

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i \leq \text{System Reliability} \leq \coprod_{j=1}^{p} \prod_{i \in P_j} p_i \,. \tag{9.15}$$

If, in addition, components are associated, then we have an alternative set of bounds

$$\prod_{i=1}^{n} p_i \leq \max_{1 \leq r \leq p} \prod_{i \in P_s} p_i \leq \text{System Reliability} \leq \min_{1 \leq s \leq k} \coprod_{i \in K_s} p_i \leq \coprod_{i=1}^{n} p_i \,, \tag{9.16}$$

where $\coprod_{i=1}^{n} x_i = 1 - \prod_{i=1}^{n}(1 - x_i)$.

As independence is a special case of 'association', the bounds given in (9.16) are also applicable for a system with independent components; although in this case, one cannot conclude that (9.16) is tighter than (9.15) or vice-versa.

If follows from (9.16) that if we calculate the reliability of a series system (i.e., the coherent system that has only minimal path set), assuming the component independent when in fact they are associated, we will underestimate the system. The reverse is true for parallel system (where the coherent system has only one minimal cut set). One can find other details related to bounds on reliability of a coherent system with associated components in the text by Barlow and Proschan (1981).

A closer examination would readily reveal that if we calculate the reliability bound of a series (parallel) system assuming the components independent when in fact they are PUOD (PLOD), we also underestimate (overestimate) syste reliability. This result will be demonstrated in Sections 10.4 and 10.5.

We note in passing that the concept of positive quadrant dependence is widely used in statistics, for example: -

- Partial sums (Robbins, 1954);
- Order statistics (Esary et al., 1967);
- Analysis of variance (Kimball, 1951);
- Contingency tables (Douglas et al., 1990);

and others.

9.4 Families of Bivariate Distributions That Are PQD

Since the PQD concept is important in reliability applications, it is imperative for a reliability practitioner to know what kinds of PQD bivariate distributions are available for reliability modelling. In this section, we list several well known PQD distributions some of which were originally derived from a reliability perspective. Most of these PQD bivariate distributions can be found, for example, in Hutchinson and Lai (1990).

9.4.1 PQD Bivariate Distributions with Simple Structures

The distributions whose PQD property can be established easily are now given below.

Example 9.1

Farlie-Gumbel-Morgenstern bivariate distribution (Farlie, 1960):

$$F(x,y) = F_X(x)F_Y(y)\left[1+\alpha\left(1-F_X(x)\right)\left(1-F_Y(y)\right)\right],\ 0<\alpha\leq 1. \quad (9.17)$$

For convenience, the above family may simply be denoted by F-G-M.

This general system of bivariate distributions is widely studied in the literature. It is easy to verify that X and Y are positively quadrant dependent if $\alpha > 0$. Consider a special case of the F-G-M system where both marginals are exponential. The joint distribution function then has the form (see, e.g., Kotz et al., 2000, pp. 51–52):

$$F(x,y) = (1-e^{-\lambda_1 x})(1-e^{-\lambda_2 y})\left[1+\alpha e^{-\lambda_1 x-\lambda_2 y}\right],\ 0<\alpha\leq 1. \quad (9.18)$$

Clearly,

$$\begin{aligned} w(x,y) &= F(x,y) - F_X(x)F_Y(y) \\ &= \alpha e^{-\lambda_1 x-\lambda_2 y}(1-e^{-\lambda_1 x})(1-e^{-\lambda_2 x}), 0<\alpha\leq 1, \end{aligned}$$

satisfies the conditions (i)-(iii) in Section 9.3 and hence X and Y are PQD.

Mukherjee and Sasmal (1977) have worked out the properties of a system of two exponential components having the F-G-M distribution. The properties are such things as the densities, means, moment generating functions, and tail probabilities of $\min(X,Y)$, $\max(X,Y)$, and $X+Y$, these being of relevance to series, parallel, and standby systems, respectively.

Lingappaiah (1983) was also concerned with properties of the F-G-M distribution relevant to the reliability context, but with gamma marginals.

Building upon a paper by Philips (1981), Kotz and Johnson (1984) considered a model in which component 1 and 2 were subject to ‘revealed’ and ‘unrevealed’ faults, respectively, with (X,Y) having a F-G-M distribution, where $X=$ time between unrevealed faults and $Y=$ time from an unrevealed fault to a revealed fault.

Example 9.2

Bivariate exponential distribution:

$$F(x,y) = 1-e^{-x}-e^{-y}+\left(e^x+e^y-1\right)^{-1}. \quad (9.19)$$

This distribution is not well known but it has a very simple structure. However, both marginals are exponential which is used widely in reliability applications. This bivariate distribution function can be rewritten as

$$\begin{aligned} F(x,y) &= 1 - e^{-x} - e^{-y} + e^{-(x+y)} + \left(e^x + e^y - 1\right)^{-1} - e^{-(x+y)} \\ &= F_X(x)F_Y(y) + \left(e^x + e^y - 1\right)^{-1} - e^{-(x+y)}. \end{aligned}$$

Now $\left(e^x + e^y - 1\right)^{-1} - e^{-(x+y)} = \frac{(e^x-1)(e^y-1)}{(e^x+e^y-1)e^{(x+y)}} = \frac{(1-e^{-x})(1-e^{-y})}{(e^x+e^y-1)} \geq 0$ and therefore F is PQD.

Example 9.3

Bivariate Pareto distribution:

$$\bar{F}(x,y) = \left(1 + ax + by\right)^{-\lambda}, \ a, b, \lambda > 0.$$

(See Mardia, 1970, p. 91). Consider a system of two independent exponential components which share a common environment factor η that can be described by a gamma distribution. Lindley and Singpurwalla (1986) showed that the resulting joint distribution has a bivariate Pareto distribution. It is very easy to verify this joint distribution is PQD. For a generalization to multivariate components, see Nayak (1987).

Example 9.4

The Durling-Pareto distribution (Bivariate Lomax):

$$\bar{F}(x,y) = \left(1 + x + y + kxy\right)^{-a}, a > 0, \, 0 \leq k \leq a + 1. \tag{9.20}$$

Obviously, it is a generalization of Example 9.3 above. See Hutchinson (1979) for details.

Consider a system of two dependent exponential components having a Gumbel's type I bivariate distribution

$$F(x,y) = 1 - e^{-x} - e^{-y} + e^{-x-y-\theta xy}, \ x, y \geq 0, 0 \leq \theta \leq 1$$

and sharing a common environment that has a gamma distribution. Sankaran and Nair (1993b) have shown that the resulting bivariate distribution is specified by (9.20).

It follows from (9.20) that

$$\begin{aligned} &\bar{F}(x,y) - \bar{F}_X(x)\bar{F}_Y(y) \\ &= \frac{1}{(1+x+y+kxy)^a} - \frac{1}{\{(1+x)(1+y)\}^a} \\ &= \frac{1}{(1+x+y+kxy)^a} - \frac{1}{(1+x+y+xy)^a} \end{aligned}$$

which is nonnegative for $0 \leq k \leq 1$. Hence, F is PQD if $0 \leq k \leq 1$.

9.4.2 PQD Bivariate Distributions with More Complicated Structures

Example 9.5

Marshall and Olkin's bivariate exponential distribution:

The BVE of Marshall and Olkin (1967) was earlier given by (8.12) with survival function

$$\Pr(X > x, Y > y) = \exp\{-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)\}, \lambda' s \geq 0. \qquad (9.21)$$

It has become a widely used bivariate exponential distribution over the last three decades. Marshall and Olkin's bivariate exponential distribution was derived from a reliability context and it is often denoted by BVE.

Suppose we have a two-component system subjected to shocks that are always fatal. These shocks are assumed to be governed by three independent Poisson processes with parameters λ_1, λ_2 and λ_{12}, according as the shock applies to component 1 only, component 2 only, of both components, respectively. Then the joint survival function is given by (9.21).

Barlow and Proschan (1981, p. 129) showed that X and Y are PQD.

Example 9.6

Bivariate distribution of Block and of Basu (1976):

$$\begin{aligned}\bar{F}(x,y) &= \tfrac{2+\theta}{2}\exp[-x-y-\theta\max(x,y)] \\ &\quad -\tfrac{\theta}{2}\exp[-(2+\theta)\max(x,y)],\ \theta, x, y \geq 0.\end{aligned}$$

This was constructed to modify Marshall and Olkin's bivariate exponential which has a singular part. It is in fact a reparameterization of a special case of Freund's (1961) bivariate exponential distribution. The marginal is $\bar{F}_X(x) = \frac{1+\theta}{2}\exp[-(1+\theta)x] - \frac{\theta}{2}\exp((1+\theta)x)$ and a similar expression for $\bar{F}_Y(y)$. It is easy to show that this joint distribution is PQD.

Example 9.7

Kibble's (1941) bivariate gamma distribution:

The joint density function is

$$\begin{aligned}f_\rho(x,y;\alpha) &= f_X(x)f_Y(y)\exp\{-\rho(x+y)/(1-\rho)\} \\ &\quad \times\tfrac{\Gamma(\alpha)}{1-\rho}(xy\rho)^{-(\alpha-1)/2} I_{\alpha-1}\left(\tfrac{2\sqrt{xy\rho}}{1-\rho}\right),\ 0 \leq \rho < 1; \qquad (9.22)\end{aligned}$$

with f_X, f_Y being the marginal gamma probability density functions with the same shape parameter $\alpha > 0$. Here $I_\alpha(\cdot)$ is the modified Bessel function of the first kind and the αth order.

Lai and Moore (1984) showed that the distribution function is given by

$$F(x, y; \rho) = F_X(x)F_X(y) + \alpha \int_0^{\rho} f_t(x, y; \alpha + 1)\, dt.$$

Since $\alpha \int_0^{\rho} f_t(x, y; \alpha + 1)dt \geq 0$, it follows that $F(x, y) \geq F_X(x)F_Y(y)$.

For the special case when $\alpha = 1$, the Kibble's gamma becomes the well known Moran-Downton bivariate exponential distribution. Downton (1970) presented a construction from a reliability perspective. He assumed that the two components C_1 and C_2 receive shocks occurring in independent Poisson streams at rates λ_1, λ_2, respectively, and that the numbers N_1 and N_2 shocks needed to cause failure of C_1 and C_2, respectively, have a bivariate geometric distribution.

For applications of Kibble's bivariate gamma, see, e.g., Hutchinson and Lai (1990).

Example 9.8

Bivariate exponential distribution of Sarmanov:

Sarmanov (1966) introduced a family of bivariate densities of the form:

$$f(x, y) = f_X(x)f_Y(y)\{1 + \omega\phi_1(x)\phi_2(y)\} \tag{9.23}$$

where $\int_{-\infty}^{\infty} \phi_1(x)f_X(x)dx = 0$, $\int_{-\infty}^{\infty} \phi_2(y)f_Y(x)dy = 0$ and ω satisfies the condition that $1 + \omega\phi_1(x)\phi_2(y) \geq 0$ for all x and y.

Lee (1996) discussed four main properties of the Sarmanov family, two of which are of particular interest to us.

(a) The conditional distribution of Y given $X = x$ is

$$\Pr(Y \leq y|X = x) = F_Y(y) + \omega\phi_1(x) \int_{-\infty}^{y} f_Y(t)\phi_2(t)\, dt.$$

(b) The regression of Y on X is

$$E\,(Y|X = x) = \mu_Y + \omega\nu_Y\phi_1(x)$$

where $\nu_X = \int_{-\infty}^{\infty} t\phi_1(t)f_X(t)\, dt$, $\nu_Y = \int_{-\infty}^{\infty} t\phi_2(t)f_Y(t)\, dt$.

(c) Further, it was shown that f is TP_2 if $\omega\phi_1'(x)\phi_2'(y) \geq 0$ for all x and y, and RR_2 if $\omega\phi_1'(x)\phi_2'(y) \leq 0$ for all x and y. Here ϕ_1' and ϕ_2' are derivatives of ϕ_1 and ϕ_2, respectively.

Lee (1996) derived a bivariate exponential distribution with joint density given below:

$$f(x, y) = \lambda_1\lambda_2 e^{-(\lambda_1 x + \lambda_2 y)} \left\{1 + \omega \left(e^{-x} - \tfrac{\lambda_1}{1+\lambda_1}\right)\left(e^{-y} - \tfrac{\lambda_2}{1+\lambda_2}\right)\right\}, \tag{9.24}$$

where $\frac{-(1+\lambda_1)(1+\lambda_2)}{\max(\lambda_1\lambda_2, 1)} \leq \omega \leq \frac{(1+\lambda_1)(1+\lambda_2)}{\max(\lambda_1, \lambda_2)}$; $\phi_1(x) = e^{-x} - \frac{\lambda_1}{1+\lambda_1}$ and $\phi_2(y) = e^{-y} - \frac{\lambda_2}{1+\lambda_2}$.

$$F(x,y) = \\ (1-e^{-\lambda_1 x})(1-e^{-\lambda_2 y}) + \frac{\omega\lambda_1\lambda_2}{(1+\lambda_1)(1+\lambda_2)}\left(e^{-\lambda_1 x} - e^{-(\lambda_1+1)x}\right)\left(e^{-\lambda_2 y} - e^{-(\lambda_2+1)y}\right) \\ \geq F_X(x)F_Y(y)$$

whence X and Y are shown to be PQD if $0 \leq \omega \leq \frac{(1+\lambda_1)(1+\lambda_2)}{\max(\lambda_1,\lambda_2)}$.

Example 9.9

Bivariate normal distribution:

The bivariate standard normal has a density function given by

$$f(x,y) = \left(2\pi\sqrt{1-\rho^2}\right)^{-1} \exp\left[-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right], \quad -1 < \rho < 1. \tag{9.25}$$

X and Y are PQD for $0 \leq \rho < 1$, and NQD for $-1 < \rho \leq 0$. We note that ρ is the correlation coefficient between X and Y. This result follows straightaway from the following lemma:

Lemma 9.1

Let (X_1, Y_1) and (X_2, Y_2) be two standard bivariate normal distributions, with correlation coefficients ρ_1 and ρ_2, respectively. If $\rho_1 \geq \rho_2$, then $\Pr(X_1 > x, Y_1 > y) \geq \Pr(X_2 > x, Y_2 > y)$.

The above is known as the Slepian inequality (Gupta, 1963, p. 805).

By letting $\rho_2 = 0$ (thus $\rho_1 \geq 0$), we establish that X and Y are PQD. On the other hand, letting $\rho_1 = 0$ (thus $\rho_2 \leq 0$), X and Y are then NQD.

9.4.3 PQD Bivariate Uniform Distributions

A copula is a multivariate distribution function whose marginals are uniform over $(0,1)$. In the two-dimensional case, a copula $C(u,v)$ is simply a bivariate uniform distribution. Any continuous bivariate distribution F with specified marginals can be represented by a copula through the marginal transformations $U = F_X(X)$ and $V = F_Y(Y)$:

$$F(x,y) = C(F_X(x), F_Y(y)). \tag{9.26}$$

The last equation is generally known as the Sklar's Theorem. For a formal definition and properties of a copula see Nelsen (1999, Chapter 2).

Unlike F with non-uniform marginals, there is a clearer geometrical interpretation for PQD copulas. If U and V are PQD, then the graph of the copula of X and Y lies on or above the graph of the independence copula Π. There are similar geometric interpretations of the graph of the copula when the two random variables satisfy one or more of the tail monotonicity properties–interpretations which involve the shape of regions determined by the horizontal and vertical sections of the copula. See Nelsen (1999, pp. 157–158) for further details.

Joe (1997, p. 19) considered the concepts of positive quadrant dependence (PQD) and the concordance ordering (more PQD) to be discussed in Section 9.6 below as basic to the parametric families of copulas in determining whether a multivariate parameter is a dependence parameter.

There are many examples of copulas that are PQD, for example:

Example 9.10

Ali-Mikhail-Haq family:

$$C(u,v) = \frac{uv}{1-\theta(1-u)(1-v)},\ \theta \in [0,1]. \tag{9.27}$$

(Ali et al., 1978.) It is clear that the copula is PQD. In fact, it can be shown that it is a copula that corresponds to the Durling-Pareto distribution given in Example 9.4.

Nelsen (1999, p 152) has pointed out that if X and Y are PQD, then their copula C is also PQD. Nelsen's book provides a comprehensive treatment on copulas and a number of examples of PQD copulas can be found therein.

Generalised F-G-M family of copulas

The so-called bivariate F-G-M (Farlie-Gumbel-Morgenstern) distribution given in Example 9.1 earlier was originally introduced by Morgenstern (1956) for Cauchy marginals. Gumbel (1960) investigated the same structure for exponential marginals.

It is easy to show that the F-G-M copula is given by

$$C_\alpha(u,v) = uv[1+\alpha(1-u)(1-v)],\ 0 \le u,v \le 1, -1 \le \alpha \le 1. \tag{9.28}$$

It is clear that the F-G-M copula is PQD for $0 \le \alpha \le 1$.

It was Farlie (1960) who extended the construction by Morgenstern and Gumbel to

$$C_\alpha(u,v) = uv[1+\alpha A(u)B(v)], 0 \le u,v \le 1, \tag{9.29}$$

where $A(u) \to 0$ and $B(v) \to 0$ as $u,v \to 1$ and $A(u)$, $B(v)$ satisfy certain regularity conditions ensuring that C is a copula. Here, the admissible range of α depends on the functions A and B.

If $A(u) = B(u) = 1-u$, we then have the classical one parameter F-G-M family (9.28).

Huang and Kotz (1999) considered the following two types:

(i) $A(u) = (1-u)^p,\ B(v) = (1-v)^p,\ p > 1, -1 \le \alpha \le \left(\frac{p+1}{p-1}\right)^{p-1},$

(ii) $A(u) = (1-u^p),\ B(v) = (1-v^p),\ p > 0, -(\max\{1,p\})^{-2} \le \alpha \le p^{-1}.$

We note that copula (ii) was investigated earlier by Woodworth (1966). Bairamov and Kotz (2002) introduced further generalizations such that

(iii) $A(u) = (1-u)^p,\ B(v) = (1-v)^q,\ p > 1,\ q > 1\,(p \neq q)$,

$$-\min\left\{1, \left(\frac{1+p}{p-1}\right)^{p-1}\left(\frac{1+q}{q-1}\right)^{q-1}\right\} \le \alpha \le \min\left\{\left(\frac{1+p}{p-1}\right)^{p-1}, \left(\frac{1+q}{q-1}\right)^{q-1}\right\}.$$

(iv) $A(u) = (1-u^n)^p, B(v) = (1-v^n)^q,\ p \ge 1; n \ge 1$,

$$-\min\left\{\frac{1}{n^2}\left(\frac{1+np}{n(p-1)}\right)^{2(p-1)}, 1\right\} \le \alpha \le \frac{1}{n}\left(\frac{1+np}{n(p-1)}\right)^{p-1}.$$

Bairamov et al. (2001) considered a more general F-G-M model:

(v) $A(u) = (1-u^{p_1})^{q_1},\ B(v) = (1-v^{p_2})^{q_2},\quad p_1, p_2 \ge 1;\ q_1, q_2 > 1$, satisfying the following inequalities

$$-\min\left\{1, \frac{1}{p_1 p_2}\left(\frac{1+p_1 q_1}{p_1(q_1-1)}\right)^{q_1-1}\left(\frac{1+p_2 q_2}{p_2(q_2-1)}\right)^{q_2-1}\right\} \le$$

$$\alpha \le \min\left\{\frac{1}{p_1}\left(\frac{1+p_1 q_1}{p_1(q_1-1)}\right)^{q_1-1}, \frac{1}{p_2}\left(\frac{1+p_2 q_2}{p_2(q_2-1)}\right)^{q_2-1}\right\}.$$

Motivated by a desire to construct positive quadrant distributions, Lai and Xie (2000) derived a new family of F-G-M copulas that possesses the PQD property with

(vi) $A(u) = u^{b-1}(1-u)^a,\ B(v) = v^{b-1}(1-v)^a,\ a, b \ge 1;\ 0 \le \alpha \le 1$ so that the copula parametrized by α is

$$C_\alpha(u, v) = uv + \alpha u^b v^b (1-u)^a (1-v)^a,\ a, b \ge 1, 0 \le \alpha \le 1. \tag{9.30}$$

Bairamov and Kotz (2003) have shown that the range of α in (9.30) can be extended and they also provided the ranges of α for which the copulas (i)-(v) are PQD. These feasible ranges are now summarized by the Table 9.1 below.

9.5 Some Related Issues on Bivariate Dependence

9.5.1 Examples of Bivariate Positive Dependence Stronger than PQD Condition

So far, we have presented only the families of bivariate distributions that are PQD, a weaker notion of the positive dependence discussed in this chapter. We now introduce some bivariate distributions that also satisfy more stringent conditions, bearing in mind the stochastically increasing (SI) condition is also known as positively regression dependent (PRD) condition.

Table 9.1. Range of dependence parameter α for some positive quadrant dependent F-G-M copulas.

Type	Range of α for which the copula is PQD
Copula (i)	$0 \le \alpha \le \left(\frac{p+1}{p-1}\right)^{p-1}$.
Copula (ii)	$0 \le \alpha \le p^{-1}$.
Copula (iii)	$0 \le \alpha \le \min\left\{\left(\frac{1+p}{p-1}\right)^{p-1}, \left(\frac{1+q}{q-1}\right)^{q-1}\right\}$, $p > 1,\ q > 1$.
Copula (iv)	$0 \le \alpha \le \frac{1}{n}\left(\frac{1+np}{n(p-1)}\right)^{p-1}$.
Copula (v)	$0 \le \alpha \le \min\left\{\frac{1}{p_1}\left(\frac{1+p_1q_1}{p_1(q_1-1)}\right)^{q_1-1}, \frac{1}{p_2}\left(\frac{1+p_2q_2}{p_2(q_2-1)}\right)^{q_2-1}\right\}$.
Copula (vi)	$0 \le \alpha \le \frac{1}{B^+(a,b)B^-(a,b)}$, B^{+1}, B^- are some functions of a and b.

TP_2: The bivariate normal distribution

The bivariate normal density is TP_2 if and only if their correlation coefficient $0 \le \rho < 1$ (see, e.g., Barlow and Proschan, 1981, p. 149). Abdel-Hameed and Sampson (1978) have shown that the bivariate density of the absolute normal distribution is also TP_2.

TP_2: Sarmanov's bivariate exponential distribution

In Example 9.8, the result in (c) indicates that a member of the Sarmanov's family is TP_2 if $\omega\phi_1'(x)\phi_2'(y) \ge 0$. For exponential marginals, $\phi_i(t) = e^{-t} - \lambda_i/(1+\lambda_i)$ decreases in t so $\omega\phi_1'(x)\phi_2'(y) \ge 0$ iff $0 < \omega \le \frac{(1+\lambda_1)(1+\lambda_2)}{\max(\lambda_1,\lambda_2)}$.

SI: Marshall and Olkin's bivariate exponential distribution

X and Y of Marshall and Olkin's bivariate distribution are associated due to having a variable in common in the construction procedure. In fact, Barlow and Proschan (1981, p. 132) showed that Y is stochastically increasing in X (SI) which in turn implies 'association'.

SI: F-G-M bivariate exponential distribution

Rödel (1987) showed that for a F-G-M distribution, X and Y are SI if $\alpha > 0$. The following is a direct and easy proof for the case with exponential marginals such that $\alpha > 0$:

$$\begin{aligned}\Pr(Y \le y|X = x) &= (1 - \alpha(2e^{-x} - 1)(1 - e^{-y})) + \alpha(2e^{-x} - 1)(1 - e^{-2y}) \\ &= (1 - e^{-y}) + \alpha(2e^{-x} - 1)(e^{-y} - e^{-2y}).\end{aligned}$$

Thus,

$$\Pr(Y > y|X = x) = e^{-y} - \alpha(2e^{-x} - 1)(e^{-y} - e^{-2y})$$

which is clearly increasing in x for every y from which we conclude that X and Y are positively regression dependent if $\alpha > 0$.

SI: Kibble's bivariate gamma distribution

Rödel (1987) showed that Kibble's bivariate gamma distribution given as Example 9.7 in Section 9.4.2 is also SI which is a stronger concept of positive dependence than PQD.

SI: Sarmanov's bivariate exponential distribution

The conditional distribution that corresponds to (9.23) is

$$\Pr(Y \le y|X = x) = F_Y(y) + \omega\phi_1(x)\int_0^y \phi_2(z)f_Y(z)dz,$$

where

$$\phi_i(x) = e^{-x} - \frac{\lambda_i}{1+\lambda_i}, i = 1, 2.$$

It follows that, for positive ω, $\Pr(Y > y|X = x) = e^{-\lambda_2 y} - \omega\phi_1(x)\int_0^y \phi_2(z)f_Y(z)dz$ increases in x because $\int_0^y \phi_2(z)f_Y(z)\,dz \ge 0$ and $\phi_1(x)$ decreases in x; so Y is stochastically increasing in x if $0 \le \omega \le \frac{(1+\lambda_1)(1+\lambda_2)}{\max(\lambda_1,\lambda_2)}$.

$\int_0^y \phi_2(z)f_Y(z)\,dz \ge 0$ follows from $\phi_2(z)f_Y(z)$ being a decreasing function, $\phi_2(0)f_Y(0) > 0$ so $\int_0^y \phi_2(z)f_Y(z)\,dz \ge \int_0^\infty \phi_2(z)f_Y(z)\,dz = 0$.

SI: The bivariate exponential distribution of Example 9.2

The distribution function is

$$F(x, y) = 1 - e^{-x} - e^{-y} + (e^x + e^y - 1)^{-1}$$

It can shown easily that

$$\Pr(Y \le y|X = x) = 1 - \frac{e^{2x}}{(e^x + e^y - 1)^2}$$

and hence

$$\Pr(Y > y|X = x) = \left\{\frac{e^x}{(e^x + e^y - 1)}\right\}^2$$

which is increasing in x for all y so Y is SI in X.

Remark

The bivariate dependence concept SI is a relatively strong concept which is second only to TP_2 in our summary table which does not include the decreasing conditional failure rate notion. It seems the condition SI is relatively easy to verify in general. Besides, the condition is also easy to interpret as its meaning is intuitively comprehensible.

RCSI: BVE of Marshall and Olkin

Consider Example 9.5 given by (9.21). By using $\max(x_1, y_1) + \max(x_2, y_2) \leq \max(x_2, y_1) + \max(x_1, y_2)$ for $0 \leq x_1 \leq x_2$ and $0 \leq y_1 \leq y_2$, we can show that $\bar{F}(x_1, y_1)\bar{F}(x_2, y_2) \geq \bar{F}(x_2, y_1)\bar{F}(x_1, y_2)$ and hence RCSI by the characterization property in Section 9.2.4

RCSI: Bivariate distribution of Block and Basu

Consider the distribution of Block and Basu (1976) considered in Example 9.6. Gupta (2003) showed that (X, Y) is RCSI.

RTI: Durling-Pareto distribution (bivariate Lomax)

Lai, Xie and Bairamov (2001) showed that X and Y are right-tail increasing if $k \leq 1$ and right-tail decreasing if $k \geq 1$. From the chains of relationships in Section 9.2.7, it is known that right-tail increasing implies 'association'. Thus X and Y are associated if $k \leq 1$.

9.5.2 Examples of NQD and Other Negative Ageing

Although the main theme of this current chapter is on positive dependence, it is a common knowledge that negative dependence does exist in various reliability situations. For the acronyms of negative ageing, see the end of Section 9.2.7. Several bivariate distributions discussed in Section 9.4 above, namely, the bivariate normal, F-G-M family, Durling-Pareto distribution and bivariate exponential distribution of Sarmanov are NQD when the ranges of the dependence parameter are appropriately specified.

Example 9.11

Gumbel's bivariate exponential distribution (also known as the Gumbel's type I bivariate exponential distribution):

$$F(x, y) = 1 - e^{-x} - e^{-y} + e^{-(x+y+\theta xy)}, 0 \leq \theta \leq 1.$$

$$F(x, y) - F_X(x)F_Y(y) = e^{-(x+y+\theta xy)} - e^{-(x+y)} \leq 0,\ 0 \leq \theta \leq 1;$$

showing that F is NQD. It is well known (see Kotz et al., 2000, p. 351) that $-0.40365 \leq \text{corr}(X, Y) \leq 0$.

This is an example where X and Y can only be negatively dependent.

Example 9.12

In Example 9.4 above, we can see easily that the Durling-Pareto distribution is NQD if $1 < k \leq a + 1$.

Example 9.13

The bivariate normal as given in Example 9.9 is NQD if $-1 < \rho \leq 0$. In fact, it is also SD and RR_2 for negative ρ.

Example 9.14

Follow from Example 9.8 and Section 9.1, see that the Sarmanov's bivariate exponential distribution is RR_2 and NQD for $\frac{-(1+\lambda_1)(1+\lambda_2)}{\max(\lambda_1\lambda_2,1)} \leq \omega \leq 0$.

Example 9.15

Lehmann (1966) presented the following situations for which negative quadrant dependence occurs:

- Consider the rankings of n objects by m persons. Let X and Y denote the rank sum for the ith and the jth object, respectively. Then X and Y are NQD.

- Consider a sequence of n multinomial trials with s possible outcomes. Let X and Y denote the number of trials resulting in outcome i and j, respectively. Then X and Y are NQD.

9.5.3 Concluding Remarks on Concepts of Dependence

Concepts of stochastic dependence are widely applicable in statistics. Given some of these concepts have arisen from reliability contexts, it seems rather unfortunate that not many reliability practitioners have caught up with this important subject. This observation is transparent since the assumption of independence is still prevailing in the majority of reliability analyses. Among the dependence concepts, the correlation is still the most widely used concept in applications. 'Association' is advocated and studied in Barlow and Proschan (1981). On the other hand, PQD is a relatively weaker condition and it is easier to verify and to interpret. A stronger positive dependence notion SI is also easier to verify in general. It is interpreted as the conditional survival probability of one component increases as the lifetime of the other component increases.

On reflection, this apparent lack of wider applications of the dependence concepts may due in part to the fact that many of the proposed dependence models are often not readily applicable. One would hope that in the near future, more applied probabilists and reliability engineers would get together forging a partnership to bridge the gap between the theory and applications of stochastic dependence.

9.6 Links Between Dependence Concepts and Bivariate Ageing Notions

It is conceivable that bivariate ageing concepts would be somehow related to the dependence properties between two lifetime variables. But the question of whether there is a strong link between the existing non-Bayesian ageing concepts and the existing dependence concepts remains unanswered. The following two examples appear to indicate there is some relationship between the two.

Example 9.16

Consider the BVE of Marshall and Olkin (1967) given in Example 9.5. Eq (8.14) shows that $\bar{F}(x_1 + t, x_2 + t) = \bar{F}(x_1, x_2)\bar{F}(t, t)$. Hence

$\bar{F}(x_1 + t, x_2 + t)/\bar{F}(x_1, x_2)$ decreases in $x_1, x_2 \geq 0$ for $t \geq 0$ and so F is BIFR according to the definition given in Section 8.3. Also the BVE of Marshall and Olkin is BIFRA as shown by Corollary 4.2 of Block and Savits (1980b). In fact it satisfies most of the bivariate ageing concepts discussed in Chapter 8.

On the other hand the BVE also satisfies the SI$(Y|X)$ (Example 9.5) and the RCSI (see Section 9.5.1). So there seems to have some link between the two.

Example 9.17

Consider Kibble's bivariate gamma distribution discussed in Example 9.7. Block and Savits (1980b, Section 4) showed that it is BIFRA. On the other hand, Section 9.4 and Section 9.5 show that it satisfies the PQD and SI conditions, respectively.

It follows from Section 9.2.3 that the bivariate IFRA conditions C, D and Σ can be easily shown to imply 'association' (Block and Savits, 1982). On the other hand, the same authors also showed that the conditions A ($\equiv$ F), B, E and especially BIFRA do not even imply PDQ which is one of the weakest positive dependence notion.

We conclude this section with following quotes from Block and Savits (1982) on this issue of relationship between ageing and dependence concepts. "The opinion which is now generally held is that various concepts of positive dependence are not intimately related to useful definitions for nonparametric multivariate life classes. In other words, if a multivariate lifetime has an increasing failure rate average, then it need not follow that the lifetime be positively dependent in some sense. In fact, if such a definition implies positive dependence, then it is probably too strong". Two decades have now gone by, we feel that the same conclusion is still be valid. More research in this area should probably be carried out.

9.7 Dependence Concepts and Bayesian Multivariate Ageing

In Section 9.2 we defined several bivariate dependence concepts and in Chapter 8 we discussed bivariate and multivariate ageing concepts as well as the Bayesian multivariate ageing notions. We now ask ourselves again 'is there any link between these two notions: the multivariate dependence notion and the multivariate ageing notion?' In the last section, we see there is no strong relation as far as non-Bayesian ageing is concerned. In Section 8.10 we discuss Bayesian notions of multivariate ageing without considering dependence concepts. More recently, some definite attempts have been made by Bayesians to link the two was attempted recently.

Bassan and Spizzichino (2001, 2003) and others have proposed some Bayesian multivariate ageing via some dependence concepts discussed in this chapter. The background to this approach is provided by Barlow and Spizzichno (1993) who defined a two-dimensional function

$$h(x,y) = \bar{G}^{-1}(\bar{F}(x,y)) \tag{9.31}$$

where $\bar{G}$ is the univariate survival function and $\bar{F}$ is the joint survival function of the exchangeable variables X and Y. The function h has the same 'level curves' as $\bar{F}$. See (8.34) for the definition regarding a level set.

Bassan and Spizzichino (2001) defined a bivariate ageing function

$$B(u,v) = \exp\left\{-\bar{G}^{-1}(\bar{F}(-\log u, -\log v))\right\}, \; u, v \in [0,1]. \tag{9.32}$$

It now follows from (9.31) that (9.32) can be written as

$$B(u,v) = \exp\left\{-h(-\log u, -\log v)\right\}. \tag{9.33}$$

Bassan and Spizzichino (2001) pointed out that the function B shares many properties of a copula, but it need not be 2-increasing. More specifically,

$$B(0,v) = B(u,0) = 0, \quad B(u,1) = u, \quad B(1,v) = v, \quad 0 \le u, v \le 1,$$

but

$$B(u+h, v+k) + B(u,v) - B(u+h,v) - B(u,v+k)$$

need not be positive.

Definition 9.2: Let X and Y be two exchangeable random variables with bivariate ageing function B defined in (9.32).

1. (X, Y) is said to be bivariate new better than used (bivariate NBU) if B is PQD.
2. (X, Y) is said to be bivariate increasing failure rate (bivariate IFR) if B is LTD.
3. (X, Y) is said to be bivariate PF_2 if B is SI.

Note that a univariate distribution is said to be PF_2 if it has a log-concave density. Also a PF_2 distribution is necessary to be IFR but the converse is not true. Hence PF_2 is a stronger ageing condition than IFR (Barlow and Proschan, 1981, pp. 76–77).

Bassan and Spizzichino (2001, 2003) showed that (X, Y) is bivariate NBU if and only if

$$\Pr(X > t | Y > r) \geq \Pr(Y > t + r | Y > r). \tag{9.34}$$

This relationship can be given the following interpretation: conditional on a same history according to which one item is new and another item is aged at least r, the new item is preferred. The notion bivariate NWU is defined by reversing the above inequality (Bassan and Spizzichino, 2003).

It is also shown in Bassan and Spizzichino (2001) that (X, Y) is bivariate IFR if and only if

$$B(us, v) \geq B(u, vs),\ u \geq v,\ 0 < s < 1. \tag{9.35}$$

This is found to be equivalent to the joint survival function $\bar{F}$ to Schur-concave. So the bivariate IFR here is the same as the weaker version defined by Definition 8.7

Note: We have not used the abbreviation BNBU or BIFR to avoid a possible confusion with the classical bivariate ageing defined in Section 8.4 and Section 8.5.

We note that the above Bayesian approach taken by these authors was not directly linking the traditional ageing concepts with positive dependence; rather, they defined new multivariate ageing through positive dependence concepts. These new definitions are not intuitive appealing as far as we are concerned.

Example 9.18

Consider the bivariate Burr distribution considered by Bassan and Spizzichino (2003) given by

$$\bar{F}(x, y) = \frac{\beta}{\beta + x^2 + y^2}, \beta > 0, x, y \geq 0. \tag{9.36}$$

It is easy to see that $\bar{F}$ is Schur-concave, so $B(us, v) \geq B(u, vs),\ u \geq v,\ 0 < s < 1$. In fact, $B(u, v) \geq B(u)B(v)$ so B is PQD hence (X, Y) has bivariate NBU by Definition 9.2. It is also obvious that $\bar{F}(x,y) \geq \bar{F}(x)\bar{F}(y)$ so that X and Y are also PQD.

It seems that while the traditional approach of multivariate ageing lacks connection with dependence concepts, the Bayesian approach has integrated dependence concepts into some of the definitions of multivariate ageing.

9.8 Positive Dependence Orderings

Consider two bivariate distributions having the same pair of marginals F_X and F_Y. It is assumed that the marginal variables of each joint distribution are positively dependent. Naturally, we would like to know which of the two bivariate distributions is more positively dependent. In other words, we wish to order the two given bivariate distributions by the extent of their positive dependence between the two marginal variables with higher in ordering meaning more positively dependent. Here in this section the concept of positive dependence ordering is introduced. The following definition is found in Kimeldorf and Sampson (1987).

Definition 9.3: A relation $\ll$ on a family of all bivariate distributions is a positive dependence ordering (PDO) if it satisfies the following ten conditions:

P(0) $F \ll G \Rightarrow F(x,\infty) = G(x,\infty)$ and $F(\infty,y) = G(\infty,y)$;

P(1) $F \ll G \Rightarrow F(x,y) \leq G(x,y)$ for all x, y;

P(2) $F \ll G$ and $G \ll H \Rightarrow F \ll H$;

P(3) $F \ll F$;

P(4) $F \ll G$ and $G \ll F \Rightarrow F = G$;

P(5) $F^- \ll F \ll F^+$; where $F^+(x,y) = \min[F(x,\infty), F(\infty,y)]$ and $F^-(x,y) = \max[F(x,\infty) + F(\infty,y) - 1,\ 0]$;

P(6) $(X,Y) \ll (U,V) \Rightarrow (a(X),Y) \ll (a(U),V)$ where the $(X,Y) \ll (U,V)$ means the relation $\ll$ holds between the corresponding bivariate distributions.

P(7) $(X,Y) \ll (U,V) \Rightarrow (-U,V) \ll (-X,Y)$;

P(8) $(X,Y) \ll (U,V) \Rightarrow (Y,X) \ll (V,U)$; and

P(9) $F_n \ll G_n, F_n \to F$ in distribution, $G_n \to G$ in distribution $\Rightarrow F \ll G$, where F_n, F, G_n, G all have the same pair of marginals.

Joe (1997) gave a comprehensive treatment of dependence orderings. Section 3.6 of Drouet Mari and Kotz (2001) also contains a good summary on this subject. In what follows, we let F and G be the joint distributions of (X,Y) and (X',Y'), respectively, having the same marginals F_X and F_Y.

9.8.1 More PQD

Tchen (1980) defined a bivariate distribution G to be more positively quadrant dependent (more PQD) than a bivariate distribution F having the same pair of marginals (written as $(X, Y) \leq_{\mathrm{PQD}} (X', Y')$) if $G(x, y) \geq F(x, y)$ for all $(x, y) \in R^2$ or equivalently, $\bar{G}(x, y) \geq \bar{F}(x, y)$ for all $(x, y) \in R^2$. It was shown that PQD partial ordering is a PDO. In Section 10.4, we will discuss an application of 'more PQD' to the effectiveness of parallel redundancy when two component lifetimes are PQD.

Note that 'more PQD' is also known as 'more concordant' in the dependence concepts literature.

Example 9.19

Generalized F-G-M copula

Lai and Xie (2000) constructed a new family of PQD dependent bivariate distributions which is a generalization of the F-G-M copula :

$$C_\theta(u, v) = uv + \theta u^b v^b (1-u)^a (1-v)^a, a, b \geq 1, 0 \leq \theta \leq 1. \tag{9.37}$$

Let the dependence ordering be defined through the ordering of θ. It is clear from (9.37) that when $\theta < \theta'$, then $C_\theta(u, v) \leq C_{\theta'}(u, v)$.

Example 9.20

Bivariate normal with positive correlation coefficient ρ.

The Slepian inequality in Section 9.4.2 says $\Pr(X_1 > x, Y_1 > y) \geq \Pr(X_2 > x, Y_2 > y)$ if $\rho_1 \geq \rho_2$. Thus a more PQD ordering can be defined in term of the positive correlation coefficient ρ.

Example 9.21

Ali-Mikhail-Haq family (see Example 9.10) has

$$C_\theta(u, v) = \frac{uv}{1 - \theta(1-u)(1-v)}, \ \theta \in [0, 1].$$

It is easy to see that $C_{\theta'} \geq C_\theta$ if $\theta' > \theta$, i.e., $C_{\theta'}$ is more PQD than C_θ.

9.8.2 More SI

The distribution G is said to be more (positively) regression dependent (or more stochastically increasing, written as $(X, Y) \leq_{\mathrm{SI}} (X', Y')$), if, for real y and y',

$$\begin{gathered} \Pr(Y \leq y | X = x) \leq \Pr(Y' \leq y' | X' = x) \quad \Rightarrow \\ \Pr(Y \leq y | X = x') \leq \Pr(Y' \leq y' | X' = x'), \text{ for any } x' > x. \end{gathered} \tag{9.38}$$

(Yanagimoto and Okamoto, 1969). The ordering was also expressed in terms of quantiles of the conditional distributions. A slight modification of the above definition was given by Capéraà and Genest (1990). It is clear that if $(X,Y) \leq_{\mathrm{SI}} (X',Y')$ and X and Y are independent, then Y' is stochastically increasing in X'.

Note: Since we have not assumed X and Y (hence X' and Y') are exchangeable it is perhaps necessary to replace the above ordering notation by $\leq_{\mathrm{SI}(Y|X)}$ showing the conditioning.

Example 9.22

Clayton's (1978) bivariate distribution has the form

$$F(x,y) = \left[F_X(x)^\theta + F_Y(y)^\theta - 1\right]^{-1/\theta}, \ \theta \geq 0.$$

Fang and Joe (1992) showed F is increasing with respect to $\geq_{\mathrm{SI}}$ as θ increases.

9.8.3 More Associated

G is said to be more associated than F, written as $(X,Y) \leq_{\mathrm{AS}} (X',Y')$, if there exist increasing functions ϕ and ψ such that for x_1, x_2 in the support of X and y_1, y_2 in the support of Y,

$$\left.\begin{array}{l} x_1 \leq x_2 \\ y_1 \leq y_2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \phi(x_1,y_1) \leq \phi(x_2,y_2) \\ \psi(x_1,y_1) \leq \psi(x_2,y_2) \end{array}\right.$$

$$\left.\begin{array}{l} \phi(x_1,y_1) < \phi(x_2,y_2) \\ \psi(x_1,y_1) > \psi(x_2,y_2) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x_1 < x_2 \\ y_1 > y_2 \end{array}\right.$$

$$(X',Y') \sim (\phi(X,Y), \psi(X,Y)).$$

(Schriever, 1987). In the special case $\phi(x,y) = x,\ G$ is more regression dependent than F (as defined above). Fang and Joe (1992) imposed more conditions on ϕ and ψ so that the modified ordering satisfies more properties of PDO in Definition 9.3. These equivalent forms of 'more associated' and 'more SI' orderings are more easily checkable for some bivariate distributions. For several parametric bivariate families, the dependence orderings are shown to be equivalent to the ordering of the parameter. We note also that if X and Y are independent, then G is 'more associated' than F is equivalent to X' and Y' are associated.

Example 9.23

A special case of Marshall and Olkin's BVE is given by

$$\Pr(X > x, Y > y) = \exp\{-(1-\lambda)(x+y) - \lambda \max(x,y)\}, \ x,y \geq 0, 0 \leq \lambda \leq 1. \tag{9.39}$$

Fang and Joe (1992) showed that the distribution is increasing with respect to 'more associated' ordering as λ increases but not with respect to 'more SI'.

Block et al. (1990) studied 'more associated' ordering in detail for bivariate empirical distributions.

9.8.4 More TP_2

Kimeldorf and Sampson (1987) have defined a TP_2 ordering as follows: Let $I \times J$ be a rectangle, and $G(I, J)$ and $F(I, I)$ be the associated probabilities. We write $I_1 < I_2$, if all $x \in I_1$ and for all $y \in I_2$, $x < y$. We say that $(X, Y) \leq_{\mathrm{TP2}} (X', Y')$ if for all $I_1 < I_2$ and for all $J_1 < J_2$,

$$\begin{aligned} &F(I_1, J_1)F(I_2, J_2)G(I_1, J_2)G(I_2, J_1) \\ &\leq F(I_1, J_2)F(I_2, I_1)G(I_1, J_1)G(I_2, J_2). \end{aligned} \tag{9.40}$$

When F is the product of the marginals F_X and F_Y, then the above condition reduces to g being TP_2 where g is the joint density of (X', Y'). Kimeldorf and Sampson (1987) showed that the TP_2 ordering is a PDO.

Example 9.24

Kimeldorf and Sampson (1987) showed that the F-G-M copula that corresponds to Example 9.1 and given by (9.28):

$$C_\alpha(u, v) = uv + \alpha uv(1 - u)(1 - v),\ 0 \leq u, v \leq 1,\ -1 \leq \alpha \leq 1$$

can be ordered by the relation (9.40). Note however, this ordering holds for $-1 \leq \alpha \leq 0$ even though X and Y are RR_2 for $\alpha < 0$.

Genest and Verret (2002) have shown that in this ordering, the bivariate normal with given means and variances can be ordered by their correlation coefficient although very few other distributions meet the condition (9.40).

Capéraà and Genest (1990) also defined an ordering G 'more LRD' than F. Although the dependence concepts LRD and TP_2 are the same when the joint density function exists, this latter definition is not equivalent to the earlier one.

9.8.5 Relations Among Different Partial Orderings

The following chain of implications holds for dependence orderings:

$$\geq_{\mathrm{SI}} \Rightarrow \geq_{\mathrm{AS}} \Rightarrow \geq_{\mathrm{PQD}}$$

(Yanagimoto and Okamoto, 1969; Schriever, 1987). Kimeldorf and Sampson (1987) showed that $\geq_{\mathrm{TP2}} \Rightarrow \geq_{\mathrm{PQD}}$. However, Capéraà and Genest (1990) showed that $\geq_{\mathrm{TP2}} \not\Rightarrow \geq_{\mathrm{SI}}$. We are unsure if $\geq_{\mathrm{TP2}} \Rightarrow \geq_{\mathrm{AS}}$.

9.8.6 Other Positive Dependence Orderings

G is said to be more positive definite dependent (PDD) than F, if

$$\operatorname{cov}(a(X), a(Y)) \geq \operatorname{cov}(a(X'), a(Y')),$$

for all continuous function a for which covariances exist (Rinott and Pollack, 1980).

Averous and Dortet-Bernadet (2000) used the approach of Capéraà and Genest (1990) to order bivariate distributions by their degree in the LTD (left-tail decreasing) and RTI (right-tail increasing) sense.

In conclusion, orderings of bivariate random variables seem to be a fruitful and inexhaustible area of research which attracts both theoretical and applied statisticians.

9.8.7 Multivariate Dependence Ordering

In Section 9.3, we have defined the multivariate dependence concepts PLOD and PUOD which are equal to PQD in the bivariate case. Joe (1997, p. 37) defined the orderings 'more PLOD' and 'PUOD' and 'more POD' as follows.

Let $\boldsymbol{X}= (X_1, X_2, ..., X_n)$ and $\boldsymbol{Y}= (X'_1, X'_2, ..., X'_n)$ be two random vectors with joint distributions F and G, respectively.
G is said to be more PLOD than F if

$$\Pr(X_1 \leq x_1, X_2 \leq x_2, ..., X_n \leq x_n) \leq \Pr(X'_1 \leq x'_1, X'_2 \leq x'_2, ..., X'_n \leq x'_n) \tag{9.41}$$

and G is more PUOD than F if

$$\Pr(X_1 > x_1, X_2 > x_2, ..., X_n > x_n) \leq \Pr(X'_1 > x'_1, X'_2 > x'_2, ..., X'_n > x'_n) \tag{9.42}$$

G is more POD (more concordant) than F if both (9.41) and (9.42) hold. For other properties of the multivariate concordance see Joe (1990).

Denuit et al. (2002) have ordered two vectors of multivariate random sums of positive random variables by PLOD, PUOD and POD. Their results lead to some actuarial applications. Belzunce and Semeraro (2004) also considered some dependence properties and orders among multivariate distributions and studied their preservation under mixtures. Their results were applied to reliability and risk theory.

Jogdeo (1978), Lindqvist (1988), and Belzunce et al. (2002) proved preservation results under mixtures for positive association whereas Shaked and Spizzichino (1998), Scarsini and Spizzichino (1999) and Khaledhi and Kochar (2001) proved preservation results for some other multivariate notions of positive dependence.

There are several other types of positive dependence ordering in the literature and we recommend the reader to consult the text by Joe (1997). Shaked and Shanthikumar (1994) gave a comprehensive treatment on stochastic orderings and applications.

9.9 Measures of Dependence

Dependence properties and measures of dependence are interrelated. Having discussed the former, we now consider the latter. There are three prominent global measures of dependence: Pearson's product-moment correlation coefficient, Kendall's tau and Spearman's rho. Using a more modern terminology, we refer the latter two as measures of association.

If X and Y are not totally dependent, then it may be helpful to find some quantities that can measure the strength or degree of dependence between them. If such a measure can be expressed as a scalar, it is often more convenient to refer to it as an index. We may ask what conditions ought an index to satisfy or what desirable properties should have in order to be useful. Rényi (1959) proposed a set of seven conditions for this purpose, and showed that the maximal correlation (see, for e.g., Hutchinson and Lai, 1990, p. 183) fulfils all of them. Lancaster (1982) has modified and enlarged Rényi's set of axioms to nine conditions.

Let $\delta(X, Y)$ denote an index of dependence between X and Y. The following, apart from the last condition, is Lancaster's version of Rényi's conditions. Condition (9) is taken from Schweizer and Wolff (1981) instead of Lancaster (1982), as the latter is expressed in highly technical language.

1. $\delta(X, Y)$ is defined for any pair of random variables, neither of them being constant with probability 1. This is to avoid trivialities.

2. $\delta(X, Y) = \delta(Y, X)$. But notice that while independence is a symmetric property, total dependence is not, as one variable may be determined by the other, but not vice versa.

3. $0 \le \delta(X, Y) \le 1$. Lancaster said that this is an obvious choice, but not every one will agree.

4. $\delta(X, Y) = 0$ if and only if X and Y are mutually dependent. Notice how strong this condition is made by the 'only if'.

5. If the functions a and b map the spaces of X and Y in one-to-one manner, respectively, onto themselves, then $\delta(a(X), b(Y)) = \delta(X, Y)$. The condition means that the index remains invariant under one-to-one transformation of the marginal random variables.

6. $\delta(X, Y) = 1$ if and only if X and Y are mutually completely dependent.

7. If X and Y are jointly normal, with correlation coefficient ρ, then $\delta(X, Y) = |\rho|$.

8. In any family of distributions defined by a vector parameter θ, $\delta(X, Y)$ must be a function of θ.

9. If (X, Y) and $(X_n, Y_n), n = 1, 2, ...$, are pairs of random variables with joint distribution F and F_n, respectively, and if $\{F_n\}$ converges to F, then $\lim_{n\to\infty} \delta(X_n, Y_n) = \delta(X, Y)$

Comments on the above conditions:

- A curious feature of the list is its mixture of the trivial and/or unhelpful with the strong and/or deep. We would say that (1), (3), (7) and (8) fall into the first category (unless there are subtle consequences to them that elude us), whereas (2), (4), (5) and (6) fall into the second. We are unsure about (9).
- Summarising, conditions (2), (5), (4) and (6) say that we are looking for a measure that is symmetric in X and Y, is defined by the ranks of X and the ranks of Y, attains 0 only in the case of independence, and attains 1 whenever there is mutual complete dependence.
- Condition (3) is too restrictive for correlations, as the the range of these is traditionally from -1 to $+1$.
- (6) is stronger than the original condition which says $\delta(X,Y) = 1$ if either $X = a(Y)$ or $Y = b(X)$ for some functions a and b, i.e., if X and Y are functionally dependent. Rényi intentionally left out the converse implication, i.e., $\delta(X,Y) = 1$ only if X and Y are functionally dependent, as he felt it to be too restrictive. The strengthening from functional dependence to mutual complete dependence is possibly due to Lancaster himself.
- Condition (7) is not appropriate to rank correlations - it should be replaced by δ being a strictly increasing function of $|\rho|$, as is done by Schweizer and Wolff (1981).
- Schweizer and Wolff (1981) claimed that at least for nonparametric measures, Rényi's original conditions are too strong.
- The chief point about these axioms is not their virtues or demerits, either individually or as a set, but that they make us think about what we meant by dependence and what we want from a measure of it. They provide a yardstick against which to measure the properties of different measures.

As we have indicated earlier, the present chapter mainly discusses three measures of dependence: Pearson's product moment correlation, Kendall's tau and Spearman's rho. None of the three satisfies all the axioms given above.

9.10 Pearson's Product-Moment Correlation Coefficient

Pearson's product-moment correlation coefficient is a measure of the strength of the linear relationship between two random variables, its definition being

$$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\text{var}(Y)}}, \tag{9.43}$$

$\text{cov}(X,Y) = E\{[X - E(X)][Y - E(Y)]\}$ being the covariance of X and Y, and $\text{var}(X)$ and $\text{var}(Y)$ being the variances of X and Y. If either of the variables is a constant, the correlation coefficient is undefined.

It is clear that $|\rho(X,Y)| \le 1$; equality occurs only X and Y are linearly dependent; ρ takes the same sign as the slope of the regression line. Suppose the marginals $F_X(x)$ and $F_Y(y)$ are given. Then ρ can take all values in the range -1 to $+1$ if and only if constants α and β exist such that $\alpha x + \beta y$ has the same distribution as Y, and the distributions are symmetrical about their means (Moran, 1967)

If X and Y are independent, then $\rho(X, Y) = 0$. However, zero correlation does not imply independence - hence, condition (4) of the Section 9.9 is not satisfied. (Between uncorrelatedness and independence lies 'semi-independence'. This means that $E(Y|X) = E(Y)$ and $E(X|Y) = E(X)$. See Jensen, 1988.) As is well known, adding constants to X and Y does not alter $\rho(X, Y)$, and neither multiplying X and Y by constant factors with the same sign. As $\rho(X, Y)$ may be negative, condition (3) is violated. Furthermore, $\rho(X, Y)$ is not invariant under monotonic transformations of the marginals, so condition (5) is not satisfied. And as $\rho(X, -X) = -1$, the 'if' part of condition (6) is not satisfied. Condition (7) and (8) are obviously satisfied. Condition (9) is satisfied – we can prove this by using the continuity theorem for two-dimensional characteristic functions (Cramér, 1999, p. 102) and the expansions of such characteristic functions in terms of product moments (Bauer, 1972, pp. 264–265).

As to estimating the correlation in a sample of n bivariate observations $(x_1, y_1), ...(x_n, y_n)$, the usual formula is

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2(y_i - \bar{y})^2}}, \tag{9.44}$$

where $\bar{x}$ and $\bar{y}$ are the respective sample means.

If $(x_1, y_1), ...(x_n, y_n)$ are n independent pairs of observations from a bivariate normal distribution, r is a maximum likelihood estimate and approximately unbiased estimator of ρ. A disadvantage of r is that it is very sensitive to contamination of sample of outliers. Devlin et al. (1975) have compared r with various other estimators of ρ with respect to their robustness; see Ruppert (1988) for ideas on multivariate "trimming" (i.e., removal of extreme values).

$\rho(X, Y)$ will be abbreviated to ρ whenever there is no ambiguity.

9.10.1 Robustness of Sample Correlation

The distribution of r has been thoroughly reviewed in Chapter 32 of Johnson et al. (1995). While the properties of r for the bivariate normal are clearly understood, the same cannot be said about non-normal bivariate populations. Cook (1951), Gayen (1951) and Nakagawa and Niki (1992) obtained expressions for the first four moments of r in terms of the cumulants and cross-cumulants of the parent population. However, the size of the bias and the variance of r are still rather hazy for general bivariate nonnormal populations when $\rho \neq 0$, since the cross cumulants are difficult to quantify in general. Although several nonnormal populations have been investigated, the messages on the robustness of r are conflicting (Johnson et al., 1995, p. 580).

Hutchinson (1997) noted that the sample correlation is possibly a poor estimator. Using the bivariate lognormal as a case study on the robustness of r as an estimate of ρ, Lai et al. (1999) found that for smaller sample sizes, r has a large bias and large variance when $\rho \neq 0$ with skewed marginals; thus they supported the claim that r is not a robust estimator. Researchers should be remindful of the underlying assumptions of the population before reporting the size of of r. Edwardes (1993) showed that in the case of the BVE of (9.21), Kendall's t statistic (defined by $\hat{\tau}$ in (9.48)) is superior to r in estimating the correlation coefficient of this joint distribution.

9.10.2 Interpretation of Correlation

Rodriguez (1982) described the historical development of correlation, and stated that although Karl Pearson was aware that high correlation between two variables may be due to a third variable, this was not generally recognized until Yule's (1926) paper. One aspect of difficulty of interpreting correlation is that it is still all too easy to confuse it with causation.

Rodriguez argued that, in order to interpret a calculated correlation, an accompanying probability model for the chance variation in the data is necessary, the two most common ones being as follows.

- The bivariate normal distribution. In this case, r estimates the parameter ρ; confidence intervals may be constructed for ρ and hypothesis tests carried out.
- The simple regression model, $y_i = \alpha + \beta x_i +$ random error. Here, r^2 represents the proportion of total variability (as measured by the sum of squares) in the $y's$ which can be explained by the linear regression. That is,
$$r^2 = \frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} \tag{9.45}$$
where $\hat{y}_i$ is the predicted value of y_i, calculated from the estimated regression equation. In the regression context, the x_i are often chosen, not random, and thus there is no underlying bivariate distribution for r to be an estimate of a parameter in.

Even so, said Elffers (1980), "It can be difficult (i) to decide when a particular value of ρ indicates association strong enough for a given purpose, and (ii) in a given situation, to weigh the losses involved in obtaining more strongly associated variables against the gains." Elffers therefore put forwards functions of the correlation that can be interpreted as probability of taking a wrong decision in certain situations.

Though they are elementary, it perhaps worth emphasizing these four points.

- $r = 0.0$ does not mean that there is no relationship between the x's and y's. A scatterplot might reveal a clear (though nonlinear) relationship.

- And even if the correlation is close to 1, the relationship may be obviously nonlinear, either to the eye when plotted directly, or because a transformation reveals a relationship that is incompatible with linearity. If X is uniform distribution over the range 8 to 10, and Y is proportional to X^2, then the correlation between X and Y is approximately 0.999 (Blake, 1979, Example 6.18).
- Lots of different-looking sets of points can all produce the same value of r (see Chambers et al., 1983, Section 4.2, for eight scatterplots, all having r=0.7).
- The value of r calculated from a small sample may be totally misleading if not viewed in the context of its likely sampling error.

We may add that for some bivariate distributions, ρ may not even exist. For example, the bivariate Pareto distribution (see e.g., Chapter 52 of Kotz et al., 2000), ρ does not exist when $0 < c \leq 2$.

In view of the above, the computation of r should be accompanied by the use of such devices as scatterplots. When the data are not from a bivariate normal population, r provides only limited information about the observations. Barnett (1985), citing two scatterplots in Barnett (1979) has expressed the view that for highly skew bivariate distributions, such as those with exponential marginals, the ordinary correlation coefficient is not a very useful measure of association.

History of correlation coefficients

Drouet Mari and Kotz (2001) devoted their Chapter 2 to describe the historical development of 'independent event' and the correlation coefficient; and they also conducted a brief tour of its early applications and misinterpretations. The readers should find this account of the early development of statistical dependence useful.

9.11 Rank Correlations

Kendall's tau (τ) and Spearman's rho (ρ_S) are the best known rank correlation coefficients. Essentially, these are measures of correlation between rankings, rather than between actual values, of X and Y; they are unaffected by any increasing transformation of X and Y, whereas the Pearson product-moment correlation coefficient ρ is unaffected only by linear transformations.

Let (x_i, y_i) and (x_j, y_j) be two observations from (X, Y) of continuous random variables. The two pairs (x_i, y_i) and (x_j, y_j) are said to be concordant if $(x_i, y_i)(x_j, y_j) > 0$ and discordant if $(x_i, y_i)(x_j, y_j) < 0$.

9.11.1 Kendall' tau

Kendall's tau is defined as the probability of concordance minus the probability of discordance:

$$\tau = \Pr[(X - X')(Y - Y') \geq 0] - \Pr[(X - X')(Y - Y') \leq 0], \tag{9.46}$$

where (X, Y) and (X', Y') are two independent pairs of random variables from a common cdf F.

The above equation is equivalent to

$$\tau = \text{cov}[\text{sgn}(X' - X), \text{sgn}(Y' - Y)] \quad \text{where sgn} = \text{sign}.$$

Alternatively, τ may be defined as

$$\tau = 4 \int \int F(x, y) f(x, y)\, dx\, dy - 1. \tag{9.47}$$

The sample version of τ is known as the Kendall's t statistic defined as

$$\hat{\tau} = \frac{c - d}{c + d} = (c - d) / \binom{n}{2} \tag{9.48}$$

where c denotes the number of concordant pairs and d the number of discordant pairs from a sample of n observations from (X, Y). $\hat{\tau}$ is an unbiased estimator of τ.

Since τ is invariant under any increasing transformations, it may be defined via the copula C of X and Y; see, for e.g., Nelsen (1999, p. 129) or Section 9.4.3:

$$4 \int_0^1 \int_0^1 C(u, v) c(u, v)\, du\, dv - 1 = 4E(C(U, V)) - 1. \tag{9.49}$$

The table below adapted from Edwardes (1993) gives Kendall's τ's together with Pearson's ρ's for some well known bivariate distributions with the exponential integral defined as $E_1(x) = \int_x^\infty \exp(-z) z^{-1}\, dz$.

$\mathbf{TP_2}$ and Kendall's $\boldsymbol{\tau}$

Recall in Section 9.2.1, an absolutely continuous distribution function F is said to be totally positive of order 2 (TP_2) if the joint density $f(x, y)$ satisfies $f(x_2, y_2) f(x_1, y_1) - f(x_2, y_1) f(x_1, y_2) \geq 0$ for all $x_1 < x_2$ and $y_1 < y_2$.

Nelsen (1992) proved that $\frac{\tau}{2}$ represents an average measure of total positivity for the density f defined by

$$T = \int_{-\infty}^{\infty} \int_{-\infty}^{y_2} \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} [f(x_2, y_2) f(x_1, y_1) - f(x_2, y_1) f(x_1, y_2)] dx_1 dy_1 dx_2 dy_2,$$

for all $x_1 < x_2$ and $y_1 < y_2$, showing that τ is indeed a measure of a strong dependence concept.

Table 9.2. Pearson's ρ and Kendall's τ for some well known bivariate distributions

Distribution	Example	ρ	τ
Bivariate normal	9.9	ρ	$\frac{2}{\pi}\sin^{-1}(\rho)$
F-G-M copula	9.1	$\alpha/3$	$2\alpha/9$
F-G-M exponential	9.1	$\alpha/4$	$2\alpha/9$
BVE	9.5	$\frac{\lambda_{12}}{\lambda_1+\lambda_2+\lambda_{12}}$	$\frac{\lambda_{12}}{\lambda_1+\lambda_2+\lambda_{12}}$
Gumbel's type I exponential	9.11	$\exp(1/\theta)E_1(1/\theta)/\theta$ -1	$-\exp(2/\theta)E_1(2/\theta)$

9.11.2 Spearman's rho

As with Kendall's τ, the population version of the measure of association known as Spearman's rho (denoted by ρ_S) is based on concordance and discordance. Let $(X_1, Y_1), (X_2, Y_2)$ and (X_3, Y_3) be three independent pairs of random variables with a common distribution function F. Then ρ_S is defined to be proportional to the probability of concordance minus the probability of discordance for the two pairs (X_1, Y_1) and (X_2, Y_3), i.e.,

$$\rho_S = 3\Big(\Pr[(X_1 - X_2)(Y_1 - Y_3) > 0] - \Pr[(X_1 - X_2)(Y_1 - Y_3) < 0]\Big). \quad (9.50)$$

Equation (9.50) is really the grade correlation and can be expressed in terms of a copula:

$$\begin{aligned}\rho_S &= 12\int_0^1\int_0^1 C(u,v)\,du\,dv - 3\\ &= 12\int_0^1\int_0^1 uv\,dC(u,v) - 3\\ &= 12E(UV) - 3. \qquad (9.51)\end{aligned}$$

Rewriting the above equation as

$$\rho_S = \frac{E(UV) - \frac{1}{4}}{\frac{1}{12}}, \quad (9.52)$$

showing that Spearman's rank correlation between X and Y is simply the Pearson's product moment correlation coefficient between the uniform variates U and V.

Quadrant dependence and Spearman's ρ_S

Recall in Section 9.2.1, a pair (X, Y) is said to be positively quadrant dependent (PQD) if $F(x, y) - F_X(x)F_Y(y) \geq 0$ for all x and y, and negatively quadrant dependent (NQD) when the inequality is reversed (see Section 9.2.7 for details). Nelsen (1992) considered that the expression $F(x,y) - F_X(x)F_Y(y)$ measures 'local' quadrant dependence at each point of $(x,y) \in R^2$. It is well known (see Schweizer and Wolff, 1981, and the references cited therein) that the population version of Spearman's rho is given by

$$\rho_S = 12 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F(x,y) - F_X(x)F_Y(y)]\, dF_X(x)\, dF_Y(y). \tag{9.53}$$

It follows from the above equation that $\frac{1}{12}\rho_S$ represents an average measure of quadrant dependence, where the average is taken with respect to the marginal distributions of X and Y. It is easy to see from (9.1) that when X and Y are PQD, $\rho_S \geq 0$. Similarly, if X and Y are NQD, $\rho_S \leq 0$.

The sample Spearman's correlation for a sample of size n is

$$R = \frac{12}{n(n^2-1)} \sum_i \left(r_i - \frac{n+1}{2}\right)\left(s_i - \frac{n+1}{2}\right) \tag{9.54}$$

where $r_i = \text{rank}(x_i)$, and $s_i = \text{rank}(y_i)$.

Another common form for R is

$$R = 1 - \frac{6\sum_i d_i^2}{n(n^2-1)}, \tag{9.55}$$

where $d_i = r_i - s_i$. R is not unbiased, and the expectation of R of a sample of size n is $E(R) = \frac{(n-2)\rho_S + 3\tau}{(n+1)} \to \rho_S$ as $n \to \infty$.

If the distribution of (X,Y) is a bivariate normal with correlation ρ, then $\rho_S = \frac{6}{\pi}\sin^{-1}\frac{\rho}{2}$ and $\tau = \frac{2}{\pi}\sin^{-1}\rho$.

The copula that corresponds to Marshall and Olkin's bivariate exponential (as given in Example 9.5) is known as the distribution of Cuadras and Augé (1981) :

$$C(u,v) = \begin{cases} u^{1-c}v & \text{for}\, u \geq v, \\ uv^{1-c} & \text{for}\, u < v. \end{cases}$$

Cuadras and Augé (1981) determined that Pearson's correlation is $3c/(4-c)$. Since the marginals are uniform, so ρ_S is the same. It may also be shown that $\tau = c/(2-c)$ and hence $\rho_S = 3\tau/(2+\tau)$.

For the F-G-M distribution (as given in Example 9.1) with uniform marginals, it can be shown that $\rho_S = \alpha/3$ and $\tau = 2\alpha/9$; hence $\rho_S = 3\tau/2$.

Chapter 13 of Hutchinson and Lai (1990) presented rank correlations for several other bivariate distributions in addition to those mentioned above.

Remarks

- Independence of X and Y implies that $\tau = \rho_S = 0$, but the converse implication does not hold.
- τ and ρ_S are both restricted to the range -1 to $+1$, attaining these limits for perfect negative and perfect positive relationships, respectively.

- If X and Y are positively quadrant dependent, then $\tau \geq 0$ and $\rho_S \geq 0$.
- If two distributions F and F' have the same marginals and F is more concordant (more PQD) than F' (i.e., $F \geq F'$), then τ and ρ_S are at least as great for F as for F' (Tchen, 1980, Corollary 3.2). See Section 9.8 for dependence orderings.
- It has already been said that the sample correlation r is very sensitive to outliers; the sample counterparts of τ and ρ_S are less so, but Gideon and Hollister (1987) proposed a statistic that is more resistant to the influenced of outliers.
- For a review that includes rank correlations, see Nelsen (1999).

9.11.3 The Relationship between Kendall's tau and Spearman's rho

While both Kendall's tau and Spearman's rho measure of the probability of concordance between two variables with a given distribution, the values of ρ_S and τ are often quite different. In this section we will determine just how different ρ_S and τ can be. We begin by giving explicit relationships between the two indices for some of the distributions we have considered; these are summarized in Table 9.3 below

Table 9.3. Relationship between ρ_S and τ

Distribution	Relationship
Bivariate normal	$\rho_S = \frac{6}{\pi}\sin^{-1}(\frac{1}{2}\sin\frac{\pi\tau}{2})$
F-G-M	$\rho_S = 3\tau/2$
Marshall & Olkin (BVE)	$\rho_S = 3\tau/(2+\tau)$
Raftery family	$\rho_S = 3\tau(8-5\tau)/(4-\tau)^2$

We may now ask what is the relation between τ and ρ_S for other distributions, and can this relation be used to determine what is the shape of an empirical distributions? (By "bivariate shape", we mean the shape remaining once univariate shape has been discarded by ranking.)

Various examples indicate a precise relation between the two measures does not exist for every bivariate distribution but bounds or inequalities can be established. We shall now summarize some general relationships below (see Kruskal, 1958):

- $-1 \leq 3\tau - 2\rho \leq 1$ (first set of universal inequalities),

- $\frac{1+\rho}{2} \geq \left(\frac{1+\tau}{2}\right)^2$, and $\frac{1-\rho}{2} \geq \left(\frac{1-\tau}{2}\right)^2$ (second set of universal inequalities).

Combining the preceding two sets of inequalities yields a slightly improved set

$$\frac{3\tau-1}{2} \leq \rho_S \leq \frac{1+2\tau-\tau^2}{2}, \tau \geq 0 \text{ and } \frac{\tau^2+2\tau-1}{2} \leq \rho_S \leq \frac{1+3\tau}{2}, \tau \leq 0. \tag{9.56}$$

Another relationship worths noting (see for example, Nelsen, 1999):

$$E(W) = \frac{1}{12}(3\tau - \rho_S),$$

where $W = F(X,Y) - F_X(X)F_Y(Y)$ which corresponds to a measure of quadrant dependence. So $E(W)$ is the 'expected' measure of quadrant dependence.

A figure depicting ρ_S as a function of τ can be plotted for which the pair (τ, ρ_S) lies within a shaded region bounded by four constraints given in the preceding set of inequalities. Such a figure with bounds for ρ_S and τ can be found in Nelsen (1999, p. 104).

These bounds are remarkably wide: for instance, when $\tau = 0, \rho_S$ can range between -0.5 and $+0.5$. Daniels (1950) commented that the assumption that τ and ρ_S describe more or less the same aspect of a bivariate population of ranks may be far from true, and suggested circumstances in which the message conveyed by the two indices is quite different. ("The worse discrepancy...occurs when the individuals fall into two groups of about equal size, within which corresponding pairs of ranks are nearly all concordant, but between which they are nearly all discordant"–Daniels, 1950, p.190). But Fieller et al. (1957) did not think this would happen very often, saying that although after transforming the margins to normality, the resulting bivariate distribution will not necessary be the bivariate normal, "we think it likely that in practical situations it would not differ greatly from this norm", adding "This is a field in which further investigation would be of considerable interest".

For a give value of τ, how much do distributions differ in their values of ρ_S? Table 13.1 of Hutchinson and Lai (1990) shows that although ρ_S could theoretically take on a very wide range of values, for the distributions considered the values are all very similar. The distributions that are most different from the others are Marshall and Olkin's, with its singularity in the pdf. at $y = x$, and Kimeldorf and Sampson's, with its oddly shaped support. With these exceptions, at $\tau = 0.5$, ρ_S lies in the range .667 to .707, even though it could theoretically takes any value between .250 and .875. This suggests the following question to close with: is there some class of bivariate distributions which includes nearly all of those occur, for which only a narrow range of ρ_S (for given τ) is possible? For instance, if every quantile of y for given x decreases with x, and vice versa (i.e., X and Y are PRD), can bounds for ρ_S in terms of τ be found? Hutchinson and Lai (1990) has conjectured that when X and Y are PRD (SI), $\rho_S \leq 3\tau/2$.

Nelsen (1999, pp. 168-169) has constructed a polynomial copula:

$$C(u,v) = uv + 2\theta uv(1-u)(1-v)(1+u+v-2uv)$$

for which $\rho_S > 3\tau/2$ if $\theta \in (0, 1/4)$. Hence the above conjecture was proved to be false.

The above mentioned table shows us the bounds of ρ_S in terms of τ appear to be much narrower than implied by (9.56). In fact, as pointed out in Capéraà and Genest (1993) that quite a few well known bivariate distributions with their ρ_S and τ being of the same sign have their $|\rho_S| \geq |\tau|$.

This overwhelming circumstantial evidence raises the question of identifying, by means of necessary and sufficient conditions on the joint distribution $F(x,y)$, the weakest possible type of stochastic dependence between X and Y that will guarantee either $\rho_S > \tau \geq 0$ or $\rho_S < \tau \leq 0$.

Capéraà and Genest (1993) have provided a partial answer to this question and we now summarize their results in the following.

Let X and Y be two continuous random variables, then

$$\rho_S \geq \tau \geq 0, \tag{9.57}$$

if Y is left-tail decreasing and right-tail increasing in X. The same inequality holds if X is left-tail decreasing and right-tail increasing in Y.

Also, $\rho_S \leq \tau \leq 0$ if Y is left-tail increasing and right-tail decreasing in X. The same inequality holds if X is left-tail increasing and right-tail decreasing in Y.

If (X, Y) is PQD (positively quadrant dependent), then

$$3\tau \geq \rho_S \geq 0,$$

(see Nelsen, 1999, p. 153). Now, it has been shown in Section 9.2.7 that both left-tail decreasing and right-tail increasing imply PQD. It now follows from (9.57) that

$$3\tau \geq \rho_S \geq \tau \geq 0,$$

if Y is simultaneously LTD and RTI in X or X is simultaneously LTD and RTI in Y. However, Nelsen (1999, p.158) gives an example showing that positive quadrant dependence alone is not sufficient to guarantee $\rho_S \geq \tau$.

9.11.4 Other Concordance Measures

Gini index

The Gini measure of association may be defined through the copula C:

$$\gamma_C = 4\left[\int_0^1 C(u, 1-u)\,du - \int_0^1 [u - C(u,u)]\,du\right]. \tag{9.58}$$

(See Nelsen, 1999, p. 146).

Blomqvists β

This coefficient β, also known as the quadrant test of Blomqvist (1950), evaluates the dependence at the 'center' of a distribution where the 'center' is given by $(\tilde{x}, \tilde{y})$, with $\tilde{x}$ and $\tilde{y}$ being the medians of the two marginals. For this reason, β is often called the medial correlation coefficient. Note that $F(\tilde{x}) = G(\tilde{y}) = \frac{1}{2}$.

Formally, β is defined as

$$\beta = 2\Pr[(X - \tilde{x})(Y - \tilde{y})] - 1 = 4H(\tilde{x}, \tilde{y}) - 1, \tag{9.59}$$

which shows $\beta = 0$ if X and Y are independent.

Since $H(\tilde{x}, \tilde{y}) = C(\frac{1}{2}, \frac{1}{2})$, $\beta = 4C(\frac{1}{2}, \frac{1}{2}) - 1$.

It was pointed out in Nelsen (1999, pp. 148-149) that although Blomqvist's β depends on the copula only through its value at the center of $[0,1] \times [0,1]$, it can nevertheless often provide an accurate approximation to Spearman's ρ_S and Kendall's τ, as the following example illustrates:

Example 9.25

Let $C(u, v) = \frac{uv}{1-\theta(1-u)(1-v)}, \theta \in [-1, 1]$ be the copula for the Ali-Mikhail-Haq family. It can be shown that that the expressions for ρ_S and τ involve logarithms and dilogarithm function. However, it is easy to verify that $\beta = \frac{\theta}{4-\beta}$. If we reparametrize the expressions for ρ_S and τ by replacing θ by $4\beta/(1+\beta)$, and expand each of the results in a Maclurin series, we obtain $\rho_S = \frac{4}{3}\beta + \frac{44}{75}\beta^3 + \frac{8}{28}\beta^4 + \ldots$; $\tau = \frac{8}{9}\beta + \frac{8}{15}\beta^3 + \frac{16}{45}\beta^4 + \ldots$. Thus $\frac{4\beta}{3}$ and $\frac{8\beta}{9}$ are reasonable second-order approximation to ρ_S and τ.

9.12 Local Measures of Dependence

We have seen that ρ_S is an average measure of the PQD dependence and τ is an average measure of total positive dependence. Kotz et al. (1990) presented an example to show that a distribution with a high ρ_S may not be a PQD. This example is given again in page 171 of Drouet Mari and Kotz (2001). Thus, global measures have some drawbacks. Drouet Mari and Kotz (2001, p. 149) gave the following rationale for defining a local index (measure) of dependence:

"These indices (global measures) are defined from the moments of the distribution on the whole plane and can be zero when X and Y are not independent. One needs therefore the indices which measures the dependence locally. In the case when X and Y are survival variables, one needs to identify the time of maximal correlation: for example the delay before the first symptom of a genetic disease by members of the same family will appear. The pairs (X, Y) and (X', Y') can have the same global measure of dependence but may possess two different distributions H and H': a local index will allow us to compare their variation in time. The variations with x and y of some local indices allow us to characterize certain distributions and conversely choosing a shape of variation for an index allows us sometimes to choose an appropriate model."

9.12.1 Definition of Local Dependence

The following definitions can be found in Drouet Mari and Kotz (2001):

Definition 9.4: If $V(x_0, y_0)$ is an open neighbourhood of (x_0, y_0), then a distribution $F(x, y)$ is PQD in the neighbourhood $V(x_0, y_0)$ provided

$$\bar{F}(x, y) \geq \bar{F}_X(x)\bar{F}_Y(y), \text{ for all } (x, y) \in V(x_0, y_0).$$

If $V(x_0, y_0) = (x_0, \infty) \times (y_0, \infty)$, we then say F is remaining PQD. We use the term 'remaining' here to indicate a part in R^2 beyond a certain point (x, y). In the same manner, a local or remaining LRD can be defined.

9.12.2 Local Dependence Function of Holland and Wang

The following concepts were introduced by Holland and Wang (1987a,b) and motivated from the contingency table for two discrete random variables. Consider an $s \times s$ contingency table with cell proportions p_{ij}. For any two pairs of indices (i, j) and (k, l), the cross-product ratio is

$$\alpha_{ij,kl} = (p_{ij}p_{kl})/(p_{il}p_{kj}),\ 1 \leq i, k \leq (r-1), 1 \leq j, l \leq (s-1). \tag{9.60}$$

Yule and Kendall (1937, Section 5.15) and Goodman (1969) suggested consideration of the following set of cross-product ratios:

$$\alpha_{ij} = (p_{ij}p_{j+1j+1})/(p_{ij+1}p_{i+1j}),\ 1 \leq i \leq (r-1), 1 \leq j \leq (s-1). \tag{9.61}$$

Also, let $\gamma_{ij} = \log \alpha_{ij}$. Both α_{ij} and γ_{ij} measures the association in the $2\times$ 2 subtables formed at the intersection of the pairs of adjacent rows and columns. They are invariant under multiplications of rows and columns.

Now let us go back to the continuous case. Let $R(f) = \{(x, y) : f(x, y) > 0\}$ be the region of non-zero probability density function which has been partitioned by a very fine rectangular grid. The probability content of a rectangle containing the point (x, y) with sides dx and dy is approximately $f(x, y)\,dx\,dy$. This probability may be viewed as one cell probability of a large two-way table, the cross-product ratio in (9.60) may be expressed as

$$\alpha(x, y; u, v) = \frac{f(x, y)f(u, v)}{f(x, v)f(u, y)}, \quad x < u, y < v, \tag{9.62}$$

assuming that the four points are in $R(f)$. The function in (9.62) is called the cross-product ratio function.

For a LRD (TP_2) distribution, we have $\alpha(x, y; u, v) > 1$. The logarithm of $\alpha(x, y; u, v)$, denoted by

$$\theta(x, y; u, v) = \log \alpha(x, y; u, v), \tag{9.63}$$

has been used by Holland and Wang (1987b) to derive a local measure of LRD.

Based on (9.63), Holland and Wang (1987a,b) defined a local dependence function

$$\gamma(x, y) = \lim_{dx, dy \to 0} \frac{\theta(x, y; x + dx, y + dy)}{dx dy} = \frac{\partial^2}{\partial x \partial y} \log f(x, y) \tag{9.64}$$

assuming the partial derivative of the second order exists.

The above local dependence function can be rewritten as

$$\gamma(x, y) = \lim_{dx, dy \to 0} \frac{1}{dx dy} \log \Big(\frac{f(x, y) f(x + dx, y + dy)}{f(x + dx, y) f(x, y + dy)} \Big). \tag{9.65}$$

Holland and Wang (1987b) showed that $\gamma(x, y) \geq 0$ (for all x for all y) is equivalent to $f(x, y)$ belonging to TP_2 or X and Y are LRD. Hence $\gamma(x, y)$ is an appropriate local measure of LRD (TP_2) dependence.

9.12.3 Properties of $\gamma(x, y)$

We shall assume that $R(f)$ is a rectangle, R^2 may also be regarded as a rectangle for this purpose. (If $R(f)$ is not a rectangle, then the shape of $R(f)$ can introduce dependence between X and Y of a different nature from local dependence – we will take up this issue in the next section.) Note also, Drouet Mari and Kotz (2001, p. 189) regarded $\gamma(x, y)$ as a local measure of LRD dependence although it was refereed to as the local dependence function in Holland and Wang (1987a,b).

The following is a list of the properties

- $-\infty < \gamma(x, y) < \infty$.
- $\gamma(x, y) = 0$ for all $(x, y) \in R(f)$ if and only if X and Y are independent. $\gamma(x, y)$ reveals more information about the dependence than other indices; recall, for example, that the product-moment correlation ρ may be zero without being independent.
- $\gamma(x, y)$ is symmetric.
- $\gamma(x, y)$ is marginal free, thus changing the marginals has an unchanged $\gamma(x, y)$, in particular, $\dfrac{\partial^2 \log c(u, v)}{\partial u \partial v} = \gamma(x, y)$, $F_X(x) = u, F_Y(y) = v$ where c is the density of the associated copula.

- Holland and Wang (1987b) mentioned that when $\gamma(x,y)$ is a constant, any monotone function of that constant will be a 'good' measure of association, But when $\gamma(x,y)$ changes sign in $R(f)$, most measures of association will be inadequate or even misleading.
- $\gamma(x,y)$ is a function only of the conditional distribution of Y given X, or of X given Y.
- If X and Y have a bivariate normal distribution with correlation coefficient ρ, then $\gamma(x,y) = \frac{\rho}{1-\rho^2}$, a constant. Conversely if $\gamma(x,y)$ is a constant, Jones (1998) pointed out that the density function $f(x,y)$ should have the form $a(x;\theta)b(y;\theta)\exp(\theta xy)$.

Jones (1996) has shown, using a kernel method, that $\gamma(x_0, y_0)$ is a local version of the linear correlation coefficient.

9.12.4 Clayton-Oakes Association Measure

Clayton (1978) and Oakes (1989) defined the following association measure which we consider to be a local dependence function:

$$\theta(x,y) = \frac{\bar{F}S_{12}}{S_1 S_2},$$

where $S_{12} = \dfrac{\partial^2 \bar{F}(x,y)}{\partial x \partial y}$, $S_1 = \dfrac{\partial \bar{F}(x,y)}{\partial x}$, and $S_2 = \dfrac{\partial \bar{F}(x,y)}{\partial y}$. For a motivation for this definition, see Clayton (1978).

The following properties of $\theta(x,y)$ are related to the positive dependence concept RCSI, a weaker concept than LRD (TP_2). (Recall, we say X and Y are RCSI if $\Pr(X > x, Y > y \mid X > x', Y > y')$ is increasing in x' and y' for all x, y – see Section 9.2.4.) It is shown in Gupta (2003) that

- $\theta(x,y) > 1$ if and only if X and Y are RCSI.
- $\theta(x,y) = 1$ if and only if X and Y are independent.

9.12.5 Local ρ_S and τ

We can restrict ρ_S and τ to an open neighbourhood $V(x_0, y_0)$ of (x_0, y_0), and define (see Drouet Mari and Kotz, 2001, p.172) local ρ_S and τ as

$$\rho_{S,(x_0,y_0)} = \frac{12 \int\int_{V(x_0,y_0)} (C(u,v) - uv)\, du\, dv}{\int\int_{V(x_0,y_0)} dudv}, \tag{9.66}$$

and

$$\tau_{(x_0,y_0)} = \frac{4 \int\int_{V(x_0,y_0)} C(u,v) dC - 1}{\int\int_{V(x_0,y_0)} dC} \tag{9.67}$$

noting that $F_X(x) = u, F_Y(y) = v$ for all $(x, y) \in V(x_0, y_0)$. We may interpret $\rho_{S,(x_0,y_0)}/12$ as the average on the local PQD property whereas $\tau_{(x_0,y_0)}/2$ as the average on the local LRD (TP_2).

When $V(x_0, y_0) = (x_0, \infty) \times (y_0, \infty)$, it is straightforward to estimate $\tau_{(x_0,y_0)}$ by counting the remaining concordant and discordant pairs, and to estimate the variance of this estimator from n_0, the number of remaining observations.

9.12.6 Local Correlation Coefficient

Suppose the standard deviations of X and Y are, respectively, σ_X and σ_Y. Let $\mu(x) = E(Y|X = x)$, $\sigma^2(x) = \text{var}(Y|X = x)$ and $\beta(x) = \dfrac{\partial \mu(x)}{\partial x}$, then the local correlation of Bjerve and Doksum (1993) is defined as

$$\rho(x) = \frac{\sigma_X \beta(x)}{\left(\sigma_X \beta(x)\right)^2 + \sigma(x)^2}. \tag{9.68}$$

If (X, Y) has a bivariate normal distribution, then $\beta(x) = \beta$ and $\sigma(x) = \sigma$. The local correlation coefficient $\rho(x)$ has the following properties:

- $-1 \leq \rho(x) \leq 1$.
- X and Y being independent implies $\rho(x) = 0$ for all x.
- $\rho(x) = \pm 1$ for almost all X is equivalent to Y being a function of X.
- In general $\rho(x)$ is not symmetric, but it is possible to construct a symmetrized version.
- $\rho(x)$ is scale-free but not marginal-free, i.e., linear transformations of X and Y, $X^* = aX + b$ and $Y^* = cY + d$, with c and d having the same sign, leave $\rho(x)$ unchanged, but the the transformation $U = F_X(X)$ and $V = F_Y(Y)$ result in $\rho(u)$ which is different from $\rho(x)$.

Note that if $\rho(x) \geq 0$ for all x, then F is PRD. We can therefore define a local PRD when $\rho(x)$ is positive in a neighbourhood of (x_0, y_0).

9.12.7 Local Linear Dependence Function

Bairamov et al. (2003) have defined a 'local linear dependence function' $H(x, y)$ which provides a local point of view on dependence at a point (x, y). Let $\mu(x) = E(Y|X = x)$ and Let $\mu(y) = E(X|Y = y)$, then

$$H(x, y) = \frac{E(X - \mu(y))E(Y - \mu(x))}{\sqrt{E(X - \mu(y))^2 \cdot E(Y - \mu(x))^2}}.$$

Here $H(x, y)$ is obtained from the expression of the linear correlation coefficient by replacing the expectations $E(X)$ and $E(Y)$ by the conditional expectations $\mu(x) = E(Y|X = x)$ and $\mu(y) = E(X|Y = y)$, respectively.

After some algebraic manipulation the expression can be rewritten as:

$$H(x,y) = \frac{\rho + \varphi_X(y)\varphi_Y(x)}{\sqrt{(1+\varphi_X^2(y))(1+\varphi_Y^2(x))}} \tag{9.69}$$

where $\varphi_X(y) = \frac{E(X)-\mu(y)}{\sqrt{\text{var}Y}}$ and $\varphi_Y(x) = \frac{E(Y)-\mu(x)}{\sqrt{\text{var}X}}$.

9.12.8 Applications of Several Local Indices in Survival Analysis

In the field of survival analysis, there is a need for time-dependent measures of dependence, to identify, for example in medical studies, the time of maximal association between the interval from remission to relapse and the next interval from relapse to death, or to determine the genetic character of a disease by comparing the degree of association between the lifetimes of monozygotic twins (Hougaard, 2000).

The following applications are described in Section 6.3.8 of Drouet Mari and Kotz (2001):

- Covariance function of Prentice and Cai (1992),
- The conditional covariance rate of Dabrowska et al. (1999).

9.13 Conclusion

In this chapter, we present several concepts of multivariate dependence, in particular bivariate positive dependence. While there does not appear to have relationship between the traditional concepts of dependence and ageing concepts, there is a strong relationship between positive dependence and rank correlations such as Kendall's τ and Spearman's ρ_S. Also the relationship between these two non-parametric measures of association can be interpreted by positive dependence concepts such as PQD and LTD, RTI. (PQD implies $3\tau \geq \rho_S \geq 0$; whereas simultaneous LTD and RTI implies $3\tau \geq \rho_S \geq \tau \geq 0$).

Although the concept of local dependence and measure of local dependence are not fully developed yet, they would likely provide us more information about the dependence between two time intervals as in the medical studies example mentioned in the last section.

10

Reliability of Systems with Dependent Components

10.1 Introduction

In this chapter, we consider reliability properties of a system of components rather than an individual component. There are several simple system structures that appear often in the literature. These are: (i) series, (ii) parallel redundant, (iii) k-out-of-n and (iv) standby (cold) redundant. Traditionally, it is assumed that the component lifetimes are independent; however, this assumption is not realistic in many practical applications. Thus, bivariate and multivariate distributions are required for modelling reliability of systems with two or more components.

In Section 10.2, we give several bivariate distributions together with various properties of their order statistics. Section 10.3 discusses the effectiveness of redundancy in general and Section 10.4 studies various aspects of parallel systems, including how the stochastic dependence between lifetime variables affects the mean lifetime of a parallel system of two components. Section 10.5 examines the reliability properties of a series structure. Ageing classes for series and parallel systems with two dependent components are considered in Section 10.6. It is followed by Section 10.7 which discusses k-out-of-n systems. Three partial orderings are considered for comparing two such reliability systems. A general discussion on consecutive k-out-of n:F systems is given in Section 10.8. Next, Section 10.9 discusses how to allocate spares optimally to a k-out-of-n system. This is then followed by a study on standby redundant systems in Section 10.10. Finally, we suggest some future research directions concerning reliability systems in Section 10.11.

10.2 Bivariate Distributions for Modelling Lifetimes of Two Components

Because of component lifetimes of a system may be dependent on each other, we need probability models that prescribe the dependence structures among these lifetime variables. For simplicity, we start with a system of two components so only a bivariate distribution will be necessary to link the two pre-specified marginals of interest.

Let F_X and F_Y denote the distribution functions of the component lifetimes X and Y, respectively. Also, let $T_1 = \min(X, Y)$ and $T_2 = \max(X, Y)$ denote the lifetimes of the series and parallel systems of two components, respectively. The reliability properties of the order statistics T_1 and T_2 are of special interest. In what follows, we present several joint distributions either in the form of the joint survival function $\bar{F}(x, y)$ or the joint density function $f(x, y)$ whichever is more convenient to us. The first five distributions have already appeared in Section 9.4 and Section 9.5. For each model, the correlation coefficient is also given as a measure of strength of their linear dependence. It is followed by the density functions of T_1 and T_2 denoted by $f_{(1)}(t)$ and $f_{(2)}(t)$, respectively. The mean times to failure of T_1 and T_2, denoted by $\mu_{(1)}$ and $\mu_{(2)}$, respectively, will also be included. For derivations of these expressions, we refer the readers to the source references cited.

For notational convenience, we will use $F_{(i)}(t)$ and $R_{(i)}(t)$ to denote the distribution function and the survival function of T_i, respectively, for $i = 1, 2$. Since

$$R_{(1)}(t) = \bar{F}(t, t) = \Pr(X > t, Y > t) = 1 - F_X(t) - F_Y(t) + F(t, t), \quad (10.1)$$

which implies

$$\begin{aligned} R_{(1)}(t) = \Pr(X > t, Y > t) &= \bar{F}_X(t) + \bar{F}_Y(t) - (1 - F(t, t)) \\ &= \bar{F}_X(t) + \bar{F}_Y(t) - R_{(2)}(t), \end{aligned}$$

i.e.,

$$R_{(1)}(t) + R_{(2)}(t) = \bar{F}_X(t) + \bar{F}_Y(t). \quad (10.2)$$

It is now easy to verify that

$$f_{(1)}(t) + f_{(2)}(t) = f_X(t) + f_Y(t). \quad (10.3)$$

It now follows at once that

$$\mu_{(1)} + \mu_{(2)} = E(T_1) + E(T_2) = E(X) + E(Y) = \mu_X + \mu_Y. \quad (10.4)$$

The last identity shows that the sum of the mean time to failure of the series structure and the mean time to failure of the parallel structure remains constant irrespective of bivariate distributions provided the marginals remain unchanged.

It is also clear that if X and Y are independent, then T_1 is IFR (DFR) if both X and Y are.

10.2.1 Examples of Bivariate Distributions Useful for Reliability Modelling

The following distributions are commonly found in the statistics and reliability literature.

Gumbel's type I bivariate exponential distribution

The joint survival function of this distribution is

$$\bar{F}(x, y) = P(X > x, Y > y) = e^{-x-y-\theta xy}, \quad x, y \geq 0,\, 0 \leq \theta \leq 1 \tag{10.5}$$

(Gumbel, 1960). The correlation coefficient is

$$\rho = -1 + \int_0^\infty \frac{e^{-y}\, dy}{1 + \theta y} = -1 + \theta^{-1} e^{\theta^{-1}} E_1(\theta^{-1}) \tag{10.6}$$

where E_1 is the exponential integral function defined by

$$E_1(x) = \int_x^\infty t^{-1} e^{-t}\, dt. \tag{10.7}$$

The correlation is, of course, zero for $\theta = 0$, and it decreases to -0.40365 as θ increases to 1.

The density functions of T_i are, see Kotz et al. (2003) for example,

$$f_{(1)} = 2(1 + \theta t) e^{-2t-\theta t^2}, \tag{10.8}$$

$$f_{(2)} = 2e^{-t} - 2(1 + \theta t) e^{-2t-\theta t^2}. \tag{10.9}$$

The mean lifetimes of the respective systems are, respectively

$$\mu_{(2)} = 2 - e^\eta \sqrt{\pi\eta}[1 - \Phi(\sqrt{2\eta})],\ \eta = \frac{1}{\theta}, \tag{10.10}$$

and

$$\mu_{(1)} = e^\eta \sqrt{\pi\eta}[1 - \Phi(\sqrt{2\eta})],\ \eta = \frac{1}{\theta}. \tag{10.11}$$

Here, Φ denotes the cdf of the standard normal.

We have also shown in Example 9.11 of Chapter 9 that X and Y are negatively quadrant dependent.

F-G-M bivariate exponential distribution

The F-G-M distribution with exponential marginals has joint density

$$f(x,y) = \lambda^2 e^{-\lambda(x+y)}[1+\alpha(2e^{-\lambda x}-1)(2e^{-\lambda y}-1)], \quad x,y \geq 0, \qquad (10.12)$$

$(-1 \leq \alpha \leq 1)$; see for example, Chapter 44 Section 13 of Kotz et al. (2000). The distribution is given as Example 9.1 and it was shown that X and Y are SI (PRD) if $\alpha > 0$. The correlation coefficient is

$$\rho = \frac{1}{4}\alpha, \ -1 \leq \alpha \leq 1.$$

The density functions of T_1 and T_2 are, respectively,

$$f_{(1)}(t) = 2\lambda(1+\alpha)e^{-2\lambda t} - 6\lambda\alpha e^{-3\lambda t} + 4\lambda\alpha e^{-4\lambda t}, \qquad (10.13)$$

and

$$f_{(2)}(t) = 2\lambda e^{-\lambda t} - 2\lambda(1+\alpha)e^{-2\lambda t} + 6\lambda\alpha e^{-3\lambda t} - 4\lambda\alpha e^{-4\lambda t}; \qquad (10.14)$$

and the two means are

$$\mu_{(1)} = \frac{\mu}{2}(1+\frac{2}{3}\rho); \qquad \mu_{(2)} = \frac{\mu}{2}(3-\frac{2}{3}\rho), \quad \mu = \frac{1}{\lambda}. \qquad (10.15)$$

Marshall and Olkin's bivariate exponential

This widely known bivariate exponential distribution (BVE) has survival function given as:

$$\bar{F}(x,y) = \exp\left[-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)\right], x \geq 0, y \geq 0, \qquad (10.16)$$

(Marshall and Olkin, 1967) where λ's are nonnegative parameters. The BVE is given as Example 9.5 in the preceding chapter and it is found that X and Y are also SI (PRD). Note that X and Y are independent iff $\lambda_{12} = 0$. The correlation coefficient is

$$\rho = \frac{\lambda_{12}}{\lambda_1+\lambda_2+\lambda_{12}}.$$

It is easy to verify that

$$f_{(1)} = \lambda e^{-\lambda t}, \ t \geq 0,$$

and

$$f_{(2)} = (\lambda_1+\lambda_{12})e^{-(\lambda_1+\lambda_{12})t} + (\lambda_2+\lambda_{12})e^{-(\lambda_2+\lambda_{12})t} - \lambda e^{-\lambda t} \qquad (10.17)$$

where $\lambda = \lambda_1+\lambda_2+\lambda_{12}$. (See, for e.g., Baggs and Nagaraja, 1996 and Franco and Vivo, 2002). It is easy to see that

$$\mu_{(1)} = \frac{1}{\lambda}; \quad \mu_{(2)} = \frac{1}{(\lambda_1 + \lambda_{12})} + \frac{1}{(\lambda_2 + \lambda_{12})} - \frac{1}{\lambda}. \tag{10.18}$$

If $\lambda_1 = \lambda_2$, then

$$E(T_2) = \frac{3}{2\theta} - \frac{\rho}{2\theta}, \quad \theta = \lambda_1 + \lambda_{12}. \tag{10.19}$$

Moran-Downton's bivariate exponential

The density function of the Moran-Downton's bivariate exponential in the standard form was given by Downton (1970):

$$f(x,y) = \frac{1}{1-\rho} \exp\left\{\frac{-(x+y)}{1-\rho}\right\} I_0\left(\frac{2\sqrt{xy\rho}}{1-\rho}\right), \ 0 \le \rho \le 1, x, y \ge 0, \tag{10.20}$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind of zero order. The distribution is a special case $\alpha = 1$ of Kibble's bivariate gamma listed as Example 9.7 in the previous chapter. The correlation coefficient for this model is simply $0 \le \rho \le 1$; the density functions of T_i are, respectively,

$$\begin{aligned} f_{(1)}(t) &= \exp(-t) + \exp\left(-\tfrac{2t}{1-\rho}\right) I_0\left(\tfrac{2\rho^{1/2}t}{1-\rho}\right) \\ &\qquad - \exp(-t) \textstyle\int_0^t \exp\left\{-\tfrac{x(1+\rho)}{1-\rho}\right\} I_0\left(\tfrac{2\rho^{1/2}x}{1-\rho}\right) dx, \end{aligned} \tag{10.21}$$

$$f_{(2)}(t) = 2e^{-t} - f_{(1)}(t),$$

and their respective means are

$$\mu_{(1)} = 1 - \frac{1}{2}(1-\rho)^{1/2}, \quad \mu_{(2)} = 1 + \frac{1}{2}(1-\rho)^{1/2}. \tag{10.22}$$

Bivariate Lomax distribution (Durling-Pareto)

The bivariate Lomax distribution originally studied in Durling (1975) is a simple bivariate distribution that can be applied to a reliability context. It is also known as the Durling-Pareto distribution given as Example 9.4 earlier. A special case of this distribution has been used by Lindley and Singpurwalla (1986) to model two independent exponential component lifetime variables that are affected by an environmental factor. This distribution also arises through the characterizations based on some ageing properties such as the failure rate and mean residual life, see Roy (1989) and Ma (1996). It has also been derived by Sankaran and Nair (1993b) as a joint distribution of a two-component dependent system having Lomax marginals. The bivariate Lomax distribution has the following survival function:

$$\bar{F}(x,y) = \Pr(X > x, Y > y) = (1 + x + y + kxy)^{-a}, \quad x \ge 0, y \ge 0, \tag{10.23}$$

($0 \le k \le a+1, a > 0$). The correlation coefficient was given in Lai, Xie and Bairamov (2001):

$$\rho = \frac{(1-k)(a-2)}{a^2} F(1,2;a+1;(1-k)), 0 \le k \le (a+1), \tag{10.24}$$

where $F(a,b:c;z)$ is the Gauss' hypergeometric function (see, for example, Chapter 15 of Abramowitz and Stegun, 1964).

The survival function for T_1 is $\bar{F}_{(1)} = \bar{F}(t,t) = (1+2t+kt^2)^{-a}$; so

$$f_{(1)}(t) = 2a(1+kt)(1+2t+kt^2)^{-(a+1)}, \tag{10.25}$$

and

$$f_{(2)}(t) = a(1+t)^{-(a+1)} - f_{(1)}(t).$$

The mean lifetimes of the respective systems are

$$\mu_{(1)} = \frac{\pi}{4} - \frac{1}{2} = 0.29; \quad \mu_{(2)} = 2 - 0.29 = 1.71, \text{ for } a = 2, k = 2. \tag{10.26}$$

For the case where $a = 1, 0 \le k \le 2$,

$$\mu_{(1)} = \int_0^\infty \left(1+2t+kt^2\right)^{-1} dt = \begin{cases} \frac{-1}{2\sqrt{1-k}} \log \frac{1-\sqrt{1-k}}{1+\sqrt{1-k}}, & 0 < k < 1 \\ 1, & k = 1 \\ \frac{1}{\sqrt{k-1}} \left(\frac{\pi}{2} - \arctan \frac{1}{\sqrt{k-1}}\right), & 1 < k \le 2. \end{cases} \tag{10.27}$$

When $a = 2$, the following relation holds:

$$\int_0^\infty \left(1+2t+kt^2\right)^{-2} dt = \frac{1}{2\,(1-k)} + \frac{k}{2(k-1)} \int_0^\infty \left(1+2t+kt^2\right)^{-1} dt.$$

Numerical computation is required for other values of a and k. Lai, Xie and Bairamov (2001) have shown that X and Y are PQD for $0 \le k \le 1$ and NQD for $1 \le k \le (a+1)$.

Type B bivariate extreme value distribution

This is also known as the logistic model. The distribution was first briefly studied by Gumbel (1960) and then by Hougaard (1986) who derived it from a survival analysis context. The joint cdf is

$$\bar{F}(x,y) = \exp\left[-\left(\frac{x}{\theta_1}\right)^r - \left(\frac{y}{\theta_2}\right)^r\right]^{1/r}, \; r \ge 1, \; x, y \ge 0, \; \theta_1, \theta_2 > 0. \tag{10.28}$$

The correlation coefficient was obtained in Tawn (1988) giving

$$\rho = \frac{[\Gamma(1/r)]^2}{r\Gamma(2/r)} - 1. \tag{10.29}$$

Baggs and Nagaraja (1996) have shown that

$$f_{(1)}(t) = \lambda e^{-\lambda t}, t \geq 0$$

and

$$f_{(2)}(t) = \frac{1}{\theta_1} e^{-t\theta_1} + \frac{1}{\theta_2} e^{-t\theta_2} - \lambda e^{-\lambda t}, \tag{10.30}$$

where $\lambda = [\theta_1^{-r} + \theta_2^{-r}]^{1/r}, r > 0$. Clearly, the two means are given by

$$\mu_{(1)} = \frac{1}{\lambda}, \qquad \mu_{(2)} = \frac{1}{\theta_1} + \frac{1}{\theta_2} - \frac{1}{\lambda}. \tag{10.31}$$

Further, since $(x^r + y^r)^{1/r} \leq (x + y)$ for postive x and y; so $\bar{F}(x,y) \geq \bar{F}_X(x)\bar{F}_Y(y)$ and thus X and Y are PQD.

Arnold and Strauss' exponential conditionals distribution

This bivariate exponential distribution studied in Arnold et al. (1999, p. 80) is not well known. Its joint density function is

$$f(x,y) = \frac{k(c)}{d^2} \exp\left\{-\frac{x}{d} - \frac{y}{d} - \frac{cxy}{d^2}\right\}, \ c, d > 0 \tag{10.32}$$

where

$$k(c) = \frac{ce^{-1/c}}{E_1(-1/c)}.$$

The correlation coefficient is

$$\rho = \frac{c + k(c) - k^2(c)}{k(c)(1 + c - k(c))} < 0.$$

Here E_1 is defined as in (10.7) which is equivalent to $-$Ei given in Arnold et al. (1999). Navarro et al. (2004) showed that

$$f_{(1)}(t) = \frac{2cde^{-(1+cdt)^2/c}}{(1 + cdt)E_1(-1/c)} \tag{10.33}$$

and

$$f_{(2)}(t) = \frac{2cd}{(1 + cdt)E_1(-1/c)} \left\{e^{-(c^{-1}+dt)} - e^{-(1+cdt)^2/c)}\right\}. \tag{10.34}$$

They also showed that

$$\mu_{(1)} = -d/c + 2dk(c)e^{1/c}\sqrt{\pi/c^3}\,\Phi(-\sqrt{2/c}) \tag{10.35}$$

and

$$\mu_{(2)} = 2dk(c) - \mu_{(1)}.$$

10.2.2 Other Bivariate Distributions

The distributions for parallel and series systems based on other bivariate distributions are given by following authors:

- Bivariate and multivariate normal: Gupta and Gupta (2001).
- Freund's (1961) bivariate exponential: Baggs and Nagaraja (1996).
- Friday and Patil's bivariate exponential: Baggs and Nagaraja (1996) and Franco and Vivo (2002).
- Raftery's models: Baggs and Nagaraja (1996) and Franco and Vivo (2002).

10.3 Effectiveness of Redundancy for Reliability System

10.3.1 Redundancy

Redundancy is a common method to increase reliability in an engineering design. There are many papers dealing with optimal allocation of redundancy in reliability systems, see e.g., Coit and Smith (1996), Levitin et al. (1998), Kuo and Prasad (2000) and Ng and Sancho (2001). Motivated to enhance reliability, Mi (1998b) considered the question of which component should be 'bolstered' or 'improved' in order to stochastically maximize the lifetime of a parallel system, series system, or, in general, a k-out-of n system. Boland et al. (1988, 1991) investigated the redundancy importance of components in complex systems. However, the objective of adding redundancy is to increase the reliability of the system and the effect on the system reliability should be carefully investigated. The effectiveness of adding redundancy depends on the reliability of each component reliability and its ageing properties.

There are various types of redundancy:

- parallel redundancy (also known hot or active redundancy),
- standby redundancy (also known as cold redundancy). (The spare component is put to use upon the failure of the original component), and
- redundancy at component level versus system level.

Xie and Lai (1996) studied the effectiveness of adding a single component to a parallel system consisting of several independent components. Let μ_n denote the mean lifetime of a parallel system of n independent identically distributed components. They have shown that

$$\mu_{n+1} - \mu_n \leq \mu_n - \mu_{n-1}. \tag{10.36}$$

Thus the gain in mean life time from an additional component in a parallel redundancy decreases as the number of parallel components increases.

10.3.2 Effectiveness of Parallel Redundancy of Two Independent and Identical Components

Xie and Lai (1996) have defined the effectiveness of a parallel system of two identical and independent components by

$$e_p(X) = (\mu_{(2)} - \mu_X)/\mu_X, \tag{10.37}$$

where μ_X denotes the mean of X. For example, if both X and Y are independent and identically distributed having a common Weibull distribution with shape parameter α, then

$$e_p(X) = 1 - 2^{-1/\alpha}. \tag{10.38}$$

We see that e_p is smaller if $\alpha > 1$ and larger if $\alpha < 1$. Note that $\alpha > 1$ implies the Weibull distribution is IFR and $\alpha < 1$ implies that it is DFR. In fact, Xie and Lai (1996) have shown that this result is true in general, i.e., parallel redundancy of two independent DFR component is more effective than two independent IFR components. Before we state and prove the next theorem, we need to define two partial orderings between two distributions that were used quite substantially in Barlow and Proschan (1981). As stated at the beginning of Section 2.9, their definitions would be postponed until now.

Let F_X and F_Y be continuous distributions, F_Y is strictly increasing on its support. We say that F_X is convex with respect to F_Y if $F_Y^{-1}F_X(x)$ is a convex function in x on the support of F_X. F_X is star-shaped with respect to F_Y if $(1/x)F_Y^{-1}F_X(x)$ is increasing for $x \geq 0$. Also the first partial ordering is stronger than the second. See Barlow and Proschan (1981, pp. 106-107).

Theorem 10.1: If the distribution of a component is IFR (DFR), then the effectiveness of a parallel systems (as defined above) of two i.i.d. components is less (greater) than 1/2.

Proof: Let X have distribution F_X with mean μ_X and Y have distribution F_Y with mean μ_Y. We now divide the proof into two steps although it is essentially taken from Xie and Lai (1996).

(i) We show that if F_X is convex with respect to F_Y, then $e_p(X) \leq e_p(Y)$. Define $Z = (\mu_Y/\mu_X)X$ which implies $\mu_Z = \mu_Y$ and $F_Z(z) = F_X\left((\mu_X/\mu_Y)z\right)$. As F_X is convex with respect to F_Y it is clear that F_Z is also convex with respect to F_Y. It then follows that F_Z is also star-shaped respect to F_Y and hence $\bar{F}_Z$ crosses $\bar{F}_Y$ at most once.

As $\mu_Z = \mu_Y$, it now follows from Theorem 7.6 of Barlow and Proschan (1981, p. 122) that $\mu_{(2)}$ based on F_Z is less than the corresponding one that is based on F_Y so the $e_p(Z) \leq e_p(Y)$. On the other hand, it is easy to see that $e_P(X) = e_p(Z)$ so $e_P(X) \leq e_p(Y)$.

(ii) Let X be the lifetime of the IFR component. Results 5.4 of Barlow and Proschan (1981, p.107) show that F_X is convex with $F_Y(t) = 1-\exp(-t/\mu_X)$.

We note from (10.38) above that the effectiveness for the exponential component is 1/2 and thus we complete the proof the theorem.

10.3.3 Parallel Redundancy of Two Independent but Nonidentical Components

How does the diversity of components in a parallel system affect the system failure rate? Boland et al. (1994) have considered a 2-component parallel system having exponential components with parameters λ_1 and λ_2 such that $\lambda_1 + \lambda_2$ is fixed. They showed that the more diverse the parameters (λ_1, λ_2) are, the better is the system in the failure rate sense (i.e., the smaller failure rate). This phenomenon was also observed by Barlow and Proschan (1981, p.83). Boland et al. (1994) also noted that this type of result does not extend to parallel systems with more than three components.

10.3.4 Dependence Concepts and Redundancy

In Xie and Lai (1996), as in many other related studies on parallel redundancy, it is assumed that the components are independent. This assumption is rarely valid in practice. In fact, in reliability analysis, the component lifetimes are usually dependent. Two components in a reliability structure may share the same load or may be subject to the same set of stresses. This will cause the two lifetime random variables to be related to each other, or to be dependent. Usually the failure times of the components tend to be longer or shorter at the same time, indicating the existence of some form of positive dependence.

When the components are dependent, the problem of effectiveness in adding an active component to a system may be different from the independent case. In particular, we are interested to investigate how the degree of the correlation will affect the increase in the mean system lifetime. In general, a system becomes more complex once we assume the component lifetimes are dependent. Thus we take a realistic approach to consider only some simple forms of dependence such as the cases when the component lifetimes are either positively dependent or negatively dependent so that they can be modelled by a bivariate or a multivariate lifetime distribution.

We have seen in the preceding chapter that there is a multitude of notions of bivariate dependence defined in the literature. Several of the more common dependence concepts are given in Section 9.2. We have studied in detail the notion of positive quadrant dependence (PQD) and concluded that it is more straightforward and easier to verify than other notions we are aware of. In the next two sections, we consider how this dependence property affects the efficiency of a parallel or a series system.

10.4 Parallel Systems

Parallel redundancy is a common approach to increase system reliability and mean time to failure. When studying systems with redundant components, it is usually assumed that the component lifetimes are independent; however this assumption is seldom valid in practice. As mentioned earlier, the effectiveness of adding a dependent component may be quite different from adding an independent one. In this section we investigate how the degree of correlation affects the increase in the mean lifetime for parallel redundancy when the two components are positively quadrant dependant. Gains in MTTF of a 2-component parallel system are compared for four bivariate exponential distributions. Various bounds for the mean system life time are also derived. The results are useful to reliability analysts as well as to designers who are required to take into account the possible dependence among the components .

10.4.1 Mean Time to Failure of a Parallel System of Two Independent Components

Assuming X and Y are i.i.d., it is easy to follow from (10.4) that

$$\mu_{(2)} = 2\mu_X - \int_0^\infty \bar{F}^2(t)\, dt, \tag{10.39}$$

where $F_X(t) = F_Y(t) = F(t)$. If the two components are not identical, the above equation may be replaced by

$$\mu_{(2)} = \mu_X + \mu_Y - \int_0^\infty \bar{F}_X(t)\bar{F}_Y(t)\, dt. \tag{10.40}$$

10.4.2 Mean Lifetime of a Parallel System with Two PQD Components

As before, we let $T_2 = \max(X, Y)$ which denotes the system lifetime of a parallel system of two components. Recall, (X, Y) is said to be positive quadrant dependent if

$$\Pr(X \le x, Y \le y) \ge \Pr(X \le x)\Pr(Y \le y). \tag{10.41}$$

The condition in (10.41) may be restated alternatively as:

$$\Pr(X > x, Y > y) \ge \Pr(X > x)\Pr(Y > y). \tag{10.42}$$

Since $\Pr(X > x, Y > y) = \bar{F}(x, y) = 1 - \Pr(X \le x) - \Pr(Y \le y) + \Pr(X \le x, Y \le y)$, it is clear that expressions (10.41) and (10.42) are equivalent. $\mu_{(2)}$ of three PQD distributions will be presented at the end of a subsequent subsection.

Kotz et al. (2003) showed that if X and Y are identically distributed PQD nonnegative random variables, then

$$\mu_{(2)} = E(T_2) \leq 2\mu_X - \int_0^\infty \bar{F}^2(t)\,dt, \tag{10.43}$$

where $F(t) = F_X(t) = F_Y(t)$ is the common distribution function and $\bar{F}(t) = 1 - F(t)$ is the common survival function.

Note that the right-hand side of Equation (10.43) is simply the expected lifetime of the parallel system when the components are independent. In other words, the gain in mean lifetime of the system by parallel redundancy is smaller for PQD components than for the independent ones. For ease of referencing, we use $\mu^I_{(1)}$ and to $\mu^I_{(2)}$ to denote the mean lifetimes of the series and parallel system, respectively, when the two component lifetimes are independent. So (10.43) may be expressed as $\mu_{(2)} \leq \mu^I_{(2)}$ if X and Y are positively quadrant dependent.

Bounds on reliability of a component can be obtained when the ageing class is known. For components with increasing failure rate (IFR), we have from (2.53) (see also Barlow and Proschan, 1981, p.113) that

$$\bar{F}(t) \geq \begin{cases} e^{-t/\mu} & \text{for } t < \mu, \\ 0 & \text{for } t \geq \mu; \end{cases} \tag{10.44}$$

a simple lower reliability bound based on the the first moment.

For IFR parallel components that are PQD, it was shown in Kotz et al. (2003) that an upper bound for the mean time to failure is given by

$$\mu_{(2)} \leq 2\mu - \frac{\mu}{2} + \frac{\mu e^{-2}}{2} = \frac{\mu}{2}\left(3 + e^{-2}\right). \tag{10.45}$$

We note in Section 9.4 that the Moran-Down bivariate exponential with $\rho \geq 0$, the Marshall and Olkin's with $0 \leq \rho < 1$, the F-G-M with $0 \leq \rho < 1$ as well as the bivariate Lomax with $0 \leq k \leq 1$ are all PQD distributions which may be used for modelling PQD components.

Nonidentical components

If we now assume that X and Y are not identically but retain the PQD property, it is easy to see that the upper bound (10.43) can be generalized to

$$\mu_{(2)} = E(T_2) \leq \mu_X + \mu_Y - \int_0^\infty \bar{F}_X(t)\bar{F}_Y(t)\,dt, \tag{10.46}$$

see Navarro and Lai (2005). If X and Y are exponentially distributed, the above inequality may be written as

$$\mu_{(2)} \leq \mu^I_{(2)} = \mu_X + \mu_Y - \frac{\mu_X \mu_Y}{\mu_X + \mu_Y}. \tag{10.47}$$

10.4.3 Mean Lifetime of a Parallel System with Two NQD Components

Recall from Section 9.2.7, X and Y are negatively quadrant dependent if the inequality in (10.42) is reversed.

Kotz et al. (2003) also showed that if X and Y are NQD nonnegative identically distributed random variables, then the inequality in (10.46) is reversed, i.e.,

$$\mu_{(2)} \geq \mu_X + \mu_Y - \int_0^\infty \bar{F}^2(t)dt. \tag{10.48}$$

They also showed that for two identical DFR components that are NQD, then the inequality becomes

$$\mu_{(2)} \geq \mu \left(\frac{3}{2} - \frac{e^{-2}}{2} \right). \tag{10.49}$$

Example of bounds: Bivariate Lomax distribution case.

The joint distribution is given by (10.23) with marginal cdf $(1+x)^{-a}$. It was shown in Kotz et al. (2003) that X and Y are PQD for $0 \leq k \leq 1$ and they are NQD for $1 \leq k \leq a+1$.

The marginal mean is 1 when $a = 2$. For $k > 1$, we only consider the case $k = 2$ so that X and Y are NQD. The system mean is

$$\begin{aligned} \mu_{(2)} &= 2\mu_X - \textstyle\int_0^\infty \bar{F}(t,t)\,dt \\ &= 2\mu_X - \textstyle\int_0^\infty \left(1 + 2t + 2t^2\right)^{-2} dt \\ &= 2 - \left(-.5 + .5 \times \tfrac{\pi}{2}\right) = 2.5 - \tfrac{\pi}{4} = 1.71 \end{aligned}$$

Applying the inequality (10.48), we get a lower bound

$$\mu_{(2)} \geq 2\mu_X - \int_0^\infty \bar{F}^2(t)dt = 2 - \frac{1}{3} = 1.67$$

which has a relative error of 2.3% if the bound is taken as an estimate.

We observe that the Lomax distribution (Pareto distribution of the second kind) is DFR. Thus, our result here is in line with the earlier result for independent case whereby adding (in parallel) a DFR component is more efficient than adding an IFR component. If X and Y are NQD and not identically distributed, then

$$\mu_{(2)} = E(T_2) \geq \mu_{(2)}^I = \mu_X + \mu_Y - \int_0^\infty \bar{F}_X(t)\bar{F}_Y(t)\,dt. \tag{10.50}$$

Further, for NQD exponential components,

$$\mu_{(2)} \geq \mu_{(2)}^I = \mu_X + \mu_Y - \frac{\mu_X \mu_Y}{\mu_X + \mu_Y}. \tag{10.51}$$

10.4.4 Relative Efficiency from Different Joint Distributions

For a two-component parallel redundancy system with given cdf's F_X and F_Y, different joint distributions can be used to model the dependence structure. A measure of efficiency of a system design can be expressed in terms of the increase in the mean lifetime of the system for each bivariate distribution. Thus the question arises as to which joint distribution would yield a higher efficiency. We also investigate how the correlation of X and Y will influence the efficiency of the system.

Main results

Consider two nonnegative random variables X and Y which denote, respectively, the lifetime of two components arranged in parallel. Assuming that they are PQD, we shall now investigate the type of joint distribution that would provide a higher efficiency measured by the value of $\mu_{(2)}$, the MTTF of a parallel system of two components.

Theorem 10.2: Suppose $F(x, y)$ and $F^*(x, y)$ have the same pair of marginal distributions F_X and F_Y. If $\bar{F}(t, t) \geq \bar{F}^*(t, t)$, i.e., $R_{(1)}(t) \geq R^*_{(1)}(t)$, then $R_{(2)}(t) \leq R^*_{(2)}(t)$. In particular, $\mu_{(2)} \leq \mu^*_{(2)}$.

Proof: The first part of the proof follows directly from (10.2) given as

$$R_{(1)}(t) + R_{(2)}(t) = R^*_{(1)}(t) + R^*_{(2)}(t) = \bar{F}_X(t) + \bar{F}_Y(t).$$

Next, it is obvious that $R_{(2)}(t) \leq R^*_{(2)}(t)$ implies $\mu_{(2)} \leq \mu^*_{(2)}$.

Note $R_{(2)}(t) \leq R^*_{(2)}(t)$ is equivalent to $T_2 \leq_{\mathrm{ST}} T_2^*$. See Definition 2.16.

Partial orderings of two positively dependent distributions were discussed in Section 9.8, particularly the ordering with respect to PQD. We now use this ordering to compare the mean lifetimes of two parallel systems, each has component lifetimes X and Y.

Corollary 10.1: Suppose F and F^* are both PQD distributions. If F is more PQD than F^*, i.e., $\bar{F}(x, y) \geq \bar{F}^*(x, y)$ for all x, y, then $\mu_{(2)} \leq \mu^*_{(2)}$. In particular, the parallel redundancy with X and Y being independent results in the maximal gain in mean lifetime whereas the case when $X \overset{a.s.}{=} Y$ provides the least gain (assuming they are identically distributed).

Proof: The stated result here is essentially the same one as given in Theorem 2 of Kotz et al. (2003). From (10.1) and (10.2), we find

$$\begin{aligned}
\mu_{(2)} &= \mu_1 + \mu_2 - \int_0^\infty \bar{F}(t,t)\,dt \\
&= \mu_1 + \mu_2 - \left(\int_0^\infty \bar{F}(t,t)\,dt - \int_0^\infty \bar{F}_Y(t)\bar{F}_Y(t)\,dt + \int_0^\infty \bar{F}_X(t)\bar{F}_Y(t)\,dt \right) \\
&= \mu_1 + \mu_2 - \int_0^\infty \bar{F}_X(t)\bar{F}_Y(t)\,dt - \left[\int_0^\infty \left\{ \bar{F}(t,t) - \bar{F}_X(t)\bar{F}_Y(t) \right\} dt \right] \\
&= \mu^I_{(2)} - \left[\int_0^\infty \left\{ \bar{F}(t,t) - \bar{F}_X(t)\bar{F}_Y(t) \right\} dt \right]
\end{aligned} \tag{10.52}$$

where $\mu^I_{(2)}$ denotes the mean of the parallel system lifetime when the two components are independent.

For PQD distributions, the square bracket

$$B = \left[\int_0^\infty \left\{ \bar{F}(t,t) - \bar{F}_X(t)\bar{F}_Y(t) \right\} dt \right] \tag{10.53}$$

is always larger than 0.

Now, it is also obvious that B is larger for $\bar{F}$ than for $\bar{F}^*$ provided the stated assumption given in the corollary is valid. When X and Y are independent, $B = 0$ so that $\mu_{(2)}$ is maximized and is denoted by $\mu^I_{(2)}$.

If the two component lifetimes are identical and completely dependent, i.e., $X \overset{a.s.}{=} Y$, it follows from (10.53) that B is maximal. In fact, it follows from (10.1) that $\mu_{(2)} = \mu_X$. This should be expected since two parallel components behave as if they are of one component if $X \overset{a.s.}{=} Y$. Thus from (10.52), $\mu_{(2)}$ is minimum when the two components are completely dependent.

Corollary 10.2 If X and Y are PQD, then
(i) $\mu_{(2)} \le \mu^I_{(2)}$ and
(ii) $\mu_{(1)} \ge \mu^I_{(1)}$

where $\mu^I_{(1)}$ and $\mu^I_{(2)}$, respectively, denote the mean times to failures of the series and parallel systems of two independent components.

Proof: (i) is obvious from (10.53) whereas (ii) follows from (10.4).

Theorem 10.3: If F is more PQD than F^*, then $\rho \ge \rho^*$ where ρ and ρ^* are the corresponding correlation coefficients under F and F^*, respectively.

Proof: This is a simple consequence of Hoeffding's lemma:

$$\operatorname{cov}(X,Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\bar{F}(x,y) - \bar{F}_X(x)\bar{F}_Y(y) \right] dx dy. \tag{10.54}$$

In conclusion, if F is more PQD than F^*, then $T_1 \ge_{\mathrm{ST}} T_1^*$ which is equivalent to $T_2 \le_{\mathrm{ST}} T_2^*$. The latter partial ordering implies $\mu_{(2)} \ge \mu^*_{(2)}$.

A similar result but with a reversed inequality holds for the more NQD case. Since negative dependence is not common in reliability and the proof is similar, we omit the detail here.

We note that the converse of Theorem 10.3 is not true in general. However, for some distributions, the correlation coefficient is a dependence parameter so the bivariate distributions can be ordered by its values. In other words, two bivariate distributions can be partially ordered by the magnitude of their correlation coefficients.

Example 10.1:

Marshall and Olkin's bivariate exponential distribution

Recall, the survival function of the BVE is given by (10.16):

$$\bar{F}(x, y) = \exp\{-\lambda_1 x - \lambda_2 - \lambda_{12} \max(x, y)\}, \ x, y \geq 0.$$

In Section 10.2.1, the Pearson's product moment correlation is given as $\rho = \frac{\lambda_{12}}{\lambda_1+\lambda_2+\lambda_{12}}$. Without loss of generality, we now assume $\lambda_1 = \lambda_2$ and reparametrize the distribution by letting $\theta = \lambda_1 + \lambda_{12}$ and $\rho = \frac{\lambda_{12}}{\lambda_1+\lambda_2+\lambda_{12}}$, (10.16) can be rewritten as

$$\bar{F}(x, y) = \exp\left[-\theta(x + y) + \frac{2\theta\rho\{(x + y) - \max(x, y)\}}{(1 + \rho)}\right], \ x, y \geq 0, \quad (10.55)$$

where $\theta > 0$ and $0 \leq \rho < 1$. Note that θ is the scale parameter for both X and Y and it is independent of ρ. It is now obvious that

$$\begin{aligned}&\bar{F}(x, y) - F_X(x)F_Y(y)\\&\quad = \exp\{-\theta(x + y)\} \exp\left[\frac{2\theta\rho\{(x + y) - \max(x, y)\}}{(1 + \rho)} - 1\right], \ x, y \geq 0\end{aligned} \quad (10.56)$$

which increases with ρ.

Example 10.2:

Bivariate F-G-M distribution

The survival function of the bivariate F-G-M distribution with marginals F_X and F_Y is given by

$$\bar{F}(x, y) = \bar{F}_X(x)\bar{F}_Y(y)\left[1 + \alpha F_X(x)F_Y(y)\right], \ -1 < \alpha < 1, x, y \geq 0. \quad (10.57)$$

For the purpose under discussion, we assume $0 \leq \alpha < 1$. For the exponential marginals, the joint density is given in (10.12) and $\rho = \frac{\alpha}{4}$ so that $F(x, y) - F_X(x)F_Y(y)$ increases with ρ provided $\alpha > 0$.

Example 10.3:

Moran-Downton bivariate exponential distribution

The joint density function is given as in (10.20), i.e.,

$$f(x,y) = \frac{1}{1-\rho} \exp\left\{\frac{-(x+y)}{1-\rho}\right\} I_0\left(\frac{2\sqrt{xy\rho}}{1-\rho}\right), \ 0 \le \rho \le 1, x, y \ge 0,$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind of zero order. The correlation coefficient for this model is simply $0 \le \rho \le 1$. It was shown in Note 2 of Lai and Moore (1984) that $F(x,y) - F_X(x)F_Y(y)$ increases with ρ.

10.4.5 MTTF Comparisons of Three PQD Bivariate Exponential Distributions

We shall now compare the MTTFs of two-component parallel systems under three different well known bivariate exponential models for which X and Y are positively quadrant dependent. The three distributions concerned are

(i) F-G-M bivariate exponential with positive α,

(ii) Marshall and Olkin's bivariate exponential (BVE) and

(iii) Moran-Downton's exponential.

Using the results obtained from Section 10.2, the mean times to failure of the above three models, as expressed in terms of their correlation coefficients after rescaling to unit mean for their marginals, are given in Table 10.1. The rescaling of the marginal means to 1 is required for ease of comparison of the three models.

Table 10.1. $\mu_{(2)}$ and the range of correlation for three bivariate exponential distributions.

Bivariate Distribution	Mean Lifetime $\mu_{(2)}$	Range of ρ
F-G-M	$1.5 - \rho/3$	$0 \le \rho < 1/4$
Marshall and Olkin	$1.5 - \rho/2$	$0 \le \rho < 1$
Moran-Downton	$1 + \frac{\sqrt{(1-\rho)}}{2}$	$0 \le \rho < 1$

Note that for each bivariate exponential model listed in the table, $\mu_{(2)}$ decreases as ρ increases. This is expected as we have shown in the preceding subsection that these distributions become more PQD as ρ increases.

Comparison of $\mu_{(2)}$ for a given ρ

One can easily verify that $\frac{3}{2} - \frac{\rho}{2} \le 1 + \frac{1}{2}(1-\rho)^{1/2}$, for all $0 \le \rho \le 1$ and therefore the Moran-Downton model yields a higher mean time to failure than the Marshall and Olkin's for a given ρ. Further, it is also easy to show that $\frac{3}{2} - \rho/3 \le 1 + \frac{1}{2}(1-\rho)^{\frac{1}{2}}$, for $0 \le \rho \le \frac{3}{4}$. Since the maximum range of ρ for the F-G-M is $0 \le \rho < 1/4$, it follows that the Moran-Downton model also yields a higher mean time to failure than the F-G-M model for a given ρ within this range. On the other hand, it is clear that the F-G-M model gives a higher MTTF than Marshall and Olkin's model for $0 \le \rho < 1/4$. These conclusions would be of an important factor when investigating or designing parallel systems.

We also wish to compare the effect of correlation on the mean system lifetime for the above three bivariate distributions. By differentiating $\mu_{(2)}$ in Table 10.1 with respect to ρ, we see that the Moran-Downton model yields the smallest rate of reduction in the mean system lifetime as a function of ρ, for $0 \le \rho < 1/4$. This study is useful both in reliability analysis and also for design when the effect of component dependence expressed through ρ should be taken into consideration.

10.4.6 Efficiency of Redundancy by NQD Components

As NQD is a mirror image of PQD, we can obtain results concerning efficiency of the parallel redundancy in an opposite direction to that of PQD given earlier.

Corollary 10.3: Let F and F^* be two NQD distributions with the same marginals. If F is more NQD than F^*, then $\mu_{(2)} \ge \mu^*_{(2)}$.

Proof: F is more NQD than F^* iff $\bar{F}(x,y) \le F^*(x,y) \le F_X(x)F_Y(y)$.. The result now follows directly from (10.52) and (10.53).

Mean lifetime under Gumbel's bivariate exponential distribution

Another simple bivariate exponential distribution which has attracted much attention in the literature is the Gumbel's type I bivariate exponential distribution (with exponential marginals), see Gumbel (1960). Its joint survival distribution is given by (10.6), i.e.,

$$\bar{F}(x,y) = P(X > x, Y > y) = e^{-x-y-\theta xy}, x, y \ge 0, 0 \le \theta \le 1.$$

Clearly, this distribution is NQD for $0 \le \theta \le 1$. In fact, there exists no value of θ for which F could be PQD. Therefore, this bivariate distribution is rather different from the other exponential distributions discussed in the previous section because of this property. It can be shown, see for example, Kotz et al. (2003):

$$\mu_{(2)} = 2 - e^{1/\theta}\sqrt{\frac{\pi}{\theta}}\left[1 - \Phi\left(\sqrt{2/\theta}\right)\right], \tag{10.58}$$

where $\Phi(\cdot)$ is the standard normal distribution function.

The correlation coefficient between X and Y is as given in (10.6), i.e., $\rho = -1 + \theta^{-1}e^{\theta^{-1}}E_1(\theta^{-1})$ where $E_1(z)$ is an exponential integral which is tabulated in Abramowitz and Stegun (1964, pp. 239-241). As $\theta \to 0$, $\rho \to 0$ (X and Y are independent when $\theta = 0$) and as θ increases, ρ decreases, reaching a minimum value of –0.40365 for $\theta = 1$. This distribution is thus useful for models with negative correlations between two variables.

Unlike the situation of the above three bivariate exponential distributions, we were unable to express $\mu_{(2)}$ explicitly in terms of ρ here. However, a plot of numerical values of $\mu_{(2)}$ against ρ as given in Figure 1 of Kotz et al. (2003) shows, that it is almost linearly decreasing in ρ. Table 10.2 provides some numerical values of $\mu_{(2)}$ for some selected values of ρ.

Table 10.2. Table of $\mu_{(2)}$ versus ρ

ρ	0.00	-0.10	-0.25	-0.40
$\mu_{(2)}$	1.500	1.527	1.570	1.615

10.5 Series Structures

The failure rate function of a series system of n independent components is simply expressed as the sum of the failure rate functions of the components. However, removing the assumptions of independence alters the simple nature of the system.

Chapter 2 of Barlow and Proschan (1981) clearly shows that if we calculate the reliability of a series system assuming components are independent when in fact they are associated but not independent, we will underestimate system reliability. This statement is also true if we assume the lifetimes of the n components $X_1, X_2, .., X_n$ are positively upper orthant dependent (PUOD) (a multivariate positive dependence concept sense as defined in Section 9.3), see example, Joe (1997, pp. 20-21). The inequality is, as given by (9.12)

$$\Pr(X > t_1, X_2 > t_2, .., X_n > t_n) \geq \prod_{i=1}^{n} \Pr(X > t_i). \tag{10.59}$$

The left hand side of the above equation is simply the reliability function of a series system of n components when $t_i = t$.

We now restrict our discussion to the 2-component series structure. The mean lifetime of such a system is denoted by $\mu_{(1)}$. We have indirectly discussed its properties while studying $\mu_{(2)}$ in the preceding section. Recall from (10.4), the identity $\mu_{(1)} + \mu_{(2)} = \mu_X + \mu_Y$ always holds regardless whether X and Y are independent or not. So it is a 'zero-sum' game for the two 'players' $\mu_{(1)}$ and $\mu_{(2)}$. It is a gain (loss) for $\mu_{(1)}$ if X and Y are PQD (NQD). Here, we have interpreted 'loss' as a loss in the sense of efficiency in the parallel redundancy. On the other hand, it is a loss (gain) for $\mu_{(2)}$ if X and Y are PQD (NQD). Effectively, one's increase in its mean lifetime causes a decrease in the other one's mean lifetime.

Also, $\mu_{(1)} = \int_0^\infty \bar{F}(t,t)\,dt$, it is easy to see that $\mu_{(1)} \geq (\leq)\, \mu_{(1)}^I$ if X and Y are PQD (NQD). If both components are exponentially distributed then

$$\mu_{(1)} \geq (\leq)\, \mu_{(1)}^I = \frac{\mu_X \mu_X}{\mu_X + \mu_Y}, \tag{10.60}$$

if PQD (NQD) assumption holds.

Chao and Fu (1991) studied the reliability of a large series system under a Markovian structure with component lives being discrete and finite.

10.5.1 Series and Parallel System of n Positive Dependent Components

Suppose n component lifetimes are positively lower orthant dependent (PLOD) as defined by (9.13). Then it is clear that the MTTF of the parallel system is smaller than its corresponding system with n independent components. Unlike the 2-component case, one cannot infer that the mean lifetime of the n-component series system is greater than its counterpart with n independent components. If, however, the lifetimes are also PUOD as defined by (9.12), then the positively dependent series system gives a larger MTTF than the independent case. Even in the situation where the component lifetimes are both PUOD and PLOD, we cannot conclude that it is a zero-sum situation. For illustration, suppose we have 3-component system with component lifetimes X, Y and Z. Let T_i be the ith order statistic so $E(X) + E(Y) + E(Z) = E\,(T_1) + E\,(T_2) + E\,(T_3)$. Here, T_1, T_2 and T_3 denote the lifetimes of the 3-component series system, 2-out-of-3 system and the 3-component parallel system, respectively. If the three component lifetimes are both PUOD and PLOD, then the MTTF of the series system will gain but the parallel system will lose in comparison to the independent case but we are unclear about the 2-out-of-3 system. The amount of reduction experienced by the n-component parallel system due to dependence will not be the same amount gained by the n-component series system. This is because $E(T_2)$ will also be influenced by the dependence among the three components. Thus, more research in this aspect is warranted.

10.6 Ageing Classes for Series and Parallel Systems with Two Dependent Components

In the case of a series system of two independent exponential components, the system failure rate is constant so it is both IFR and DFR. On the other hand, the failure rate of a parallel system is also IFR if the two exponential components are identical. However, if the two exponential components are unlike, it is neither IFR nor DFR but it is IFRA because the system is coherent (Barlow and Proschan, 1981, pp. 82-84).

10.6.1 Ageing Class

A generalized hyperexponential (GH) distribution of n components if its density $f(t)$ can be represented as

$$f(t) = \sum_{i=1}^{n} a_i \lambda_i e^{-\lambda_i t}, \tag{10.61}$$

where $\lambda_i > 0$ and a_i is a real number. It is shown that some of the densities $f_{(1)}$ and $f_{(2)}$ discussed in Section 10.2 have GH distributions.

For $n = 2$, the pdf of a generalized GH is given by

$$f(t) = a_1 \lambda_1 e^{-\lambda_1 t} + a_2 \lambda_2 e^{-\lambda_2 t}. \tag{10.62}$$

It is easy to see that $a_1\lambda_1 + a_2\lambda_2 \geq 0$ is a necessary and sufficient condition for (10.62) to be a probability density function (Bartholomew, 1969). Baggs and Nagaraja (1996) gave the following result.

Theorem 10.4: Suppose a GH distribution of two components with probability density function given in (10.62). If $\lambda_1 < \lambda_2$, then the distribution is

(i) IFR if $a_2 < 0$ or $a_1 < 0$, both subject to $a_1 + a_2 = 1$,

(ii) DFR if $0 < a_2 < 1$ and $a_1 + a_2 = 1$.

Proof: Integrating both sides of (10.62) from 0 to ∞, is clear that $a_1 + a_2 = 1$ since $f(t)$ is assumed to be a density function.

Now the failure rate function of the 2-component GH distribution is

$$r(t) = \frac{a_1 \lambda_1 e^{-\lambda_1 t} + a_2 \lambda_2 e^{-\lambda_2 t}}{a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t}}$$

which is increasing in $t \geq 0$ iff $r'(t) \geq 0$. It can be shown easily that this is equivalent to $-a_1 a_2 < 0$.

(i) If $a_2 < 0$, then $a_1 > 0$ so $-a_1 a_2 > 0$ and hence the distribution is IFR.

(ii) If $0 \leq a_2 < 1$, then $a_1 > 0$ so $-a_1 a_2 < 0$ and thus $r'(t) < 0$. In fact, it is well known that a convex mixture of two exponential distribution is DFR.

The situation for a GH distribution of three components is more complicated as indicated in Baggs and Nagaraja (1996). Franco and Vivo (2002) gave the following result:

Theorem 10.5: Let X be a generalized mixture of three exponential distributions with pdf $f(t) = \sum_{i=1}^{3} a_i\lambda_i e^{-\lambda_i t}$ such that $\lambda_i > 0$, $a_1 > 0$, a_2 and a_3 are real. Let $s = a_1a_2\lambda_1\lambda_2(\lambda_1 - \lambda_2)^2 + a_1a_3\lambda_1\lambda_3(\lambda_1 - \lambda_3)^2 + a_2a_3\lambda_1\lambda_2(\lambda_2 - \lambda_3)^2$, then the following results hold:

(i) If $a_i > 0$ for $i = 1, 2, 3$, then X is DFR.

(ii) If $a_2 > 0, a_3 < 0$, then X is DFR when $s \geq 0$. Anyway, it cannot be IFR.

(iii) If $a_2 < 0, a_3 < 0$, then X is IFR when $s \leq 0$. Anyway, it cannot be DFR.

(iv) If $a_2 < 0, a_3 > 0$, then X cannot be DFR. Moreover, X is IFR when either $s \leq 0$ and $\tau \leq 0$, or $g(t) \leq 0$, where

$$\tau = \frac{1}{\lambda_3 - \lambda_1} \log \frac{a_3\lambda_3(\lambda_3 - \lambda_2)}{a_1\lambda_1(\lambda_3 - \lambda_1)}$$

and

$$\begin{aligned} g(t) = {} & a_1a_2\lambda_1\lambda_2(\lambda_1 - \lambda_2)^2 e^{-(\lambda_1+\lambda_2)t} + a_1a_3\lambda_1\lambda_3(\lambda_1 - \lambda_3)^2 e^{-(\lambda_1+\lambda_3)t} \\ & + a_2a_3\lambda_2\lambda_3(\lambda_2 - \lambda_3)^2 e^{-(\lambda_2+\lambda_3)t}. \end{aligned}$$

Proof: The proof is quite long, see Theorem 3 of Baggs and Nagaraja (1996) and Corollary 3.5 of Franco ad Vivo (2002).

T_1 and T_2 of the F-G-M bivariate exponential, T_2 of the Marshall and Olkin's bivariate exponential and T_2 of type B bivariate extreme values have shared a common property, i.e., they all have a GH distribution. The ageing properties of T_i from these and those other bivariate distributions discussed in Section 10.2 are now given below.

- F-G-M bivariate exponential: The series system cannot be IFR because $s > 0$. The parallel system is a GH of four components so the classification above cannot be applied (Franco and Vivo, 2002).
- Marshall and Olkin's bivariate exponential: The series system is exponential so it is both IFR and DFR. The parallel system is neither IFR nor DFR if the two exponential components have different parameters. However, if the two components are identical, it follows from (10.17) that T_2 has a GH distribution of two components so T_2 is IFR by Theorem 10.4. In the case the two component lives are independent and identical, then T_2 being IFR has already been established by Barlow and Proschan (1981).
- Type B extreme value distribution: The series system has an exponential distribution so is both IFR and DFR. The parallel system is neither IFR nor DFR.

- Moran-Downton bivariate exponential (assuming both components having unit mean): Wang et al. (2003) have shown that T_1 cannot possibly be IFRA while it is possible to be DFRA. However, it is possible that T_2 is IFRA. Numerical calculations suggests T_1 is DFRA and T_2 is IFR for $0 \leq \rho < 1$.
- Bivariate Lomax distribution (Durling-Pareto): It follows from (10.25) that the failure rate function for T_1 is $2a(1 + kt)(1 + 2t + kt^2)^{-1}$ which is decreasing in t for $0 \leq k \leq 2$ for which the series system is DFR but the ageing class for the parallel system is yet to be established. (Recall, X and Y are independent if $k = 1$.)
- Arnold and Strauss' exponential conditionals model: Navarro et al. (2004) showed that T_1 is IFR when $0 \leq c \leq 3.862$, BT when $c > 3.82$. T_2 is not DFR. Numerically, it has an upside-down bathtub shaped failure rate.
- Raftery's three bivariate exponential models: Baggs and Nagaraja (1996) have shown that T_2 is IFR under the three models. However, T_1 has a failure rate that may be constant, increasing, decreasing or non-monotonic.
- Gupta and Gupta (2001) have shown that the distributions of the minimum and maximum of a multivariate normal retain the IFR property.

10.7 k-out-of-n Systems

Let $X_1, X_2, .., X_n$ denote n independent component lifetimes of a system with $X_{i:n}$ being their ith order statistic. A k-out-of-n system functions if and only if at least k components function. When $k = n$, the systems has a series structure whereas for $k = 1$, it reduces to a parallel structure. Thus, a k-out-of-n system is more general than either a series or a parallel structure. Hence, the result for this system holds for both series and parallel structures.

The lifetime of this system is simply given by the order statistic $X_{n-k+1:n}$. To simplify the notations, we denote this order statistic by $\tau_{k|n} = X_{n-k+1:n}$. So in this notation, $\tau_{n|n}$ and $\tau_{1|n}$ denote the lifetimes of the series and parallel systems, respectively.

10.7.1 Reliability of a k-out-of-n System

Consider a k-out-of-n system in which n lifetime components are independent and identically distributed having a common cdf F and pdf f. Then it can be shown easily that the reliability of such a system is given by

$$R_S(t) = \frac{n!}{(n-k)!(k-1)!} \sum_{j=0}^{n-k} \frac{\binom{n-k}{j}(-1)^{n-k-j}}{n-j} (\bar{F}(t))^{n-j}. \qquad (10.63)$$

Special cases

(1) Series system ($k = n$)

$$R_S(t) = (\bar{F}(t))^n.$$

(2) Parallel system ($k = 1$)

$$R_S(t) = n \sum_{j=0}^{n-1} \frac{\binom{n-1}{j}(-1)^{n-j-1}}{n-j} (\bar{F}(t))^{n-j}.$$

Reliability of a *k*-out-of-*n* system sharing a common environment

In assessing the reliability of a system of n components, it is rarely possible to test the entire system under the actual operational environment. Instead, the component reliabilities are often determined by life tests conducted under controlled environment which is generally harsher or gentler than the operational environment. Therefore, when considering the reliability of a k-out-of-n system, we may need to take into accounts these environmental effects.

Suppose the environment factor Z has a distribution with moment generating function M_Z, Gupta (2002) has shown that the reliability of the system under consideration is given by

$$R_S^*(t) = \sum_{j=0}^{n-k} \binom{n}{j} \binom{n-j-1}{k-1} M_Z[-(n-j)H(t)] \tag{10.64}$$

where $H(t)$ is the cumulative hazard corresponding to F.

Gupta (2002) has discussed two examples: (i) Z has a gamma distribution and (ii) Z has an inverse Gaussian distribution.

10.7.2 Ageing properties of a *k*-out-of-*n* system

We now discuss the ageing class of distribution to which the distribution of the system belongs to. The results are now summarized by the following theorem which was given by Gupta (2002).

Theorem 10.6: Let F be the parent distribution of the independent and identically distributed components.

(1) If F is IFR (IFRA), then the distribution of the k-out-of-n system is IFR (IFRA).
(2) If F is DFR (DFRA), then the distribution of the k-out-of-n system is not necessarily DFR (DFRA).
(3) If F is NBUE, then the distribution of the parallel system (1-out-of-n) is NBUE.

(4) If F is DMRL, then the distribution of the parallel system (1-out-of-n) is DMRL.

Proof: The proof is based on the fact that the lifetime of the k-out-of-n system is simply the order statistic $X_{n-k+1:n}$. Now (1) and (2) follow from Takahasi (1988); (3) and (4) from Abouammoh and El-Neweihi (1986).

Takahasi (1988) also showed that if $X_{k:n} = \tau_{n-k+1|n}$ is DFR, then $X_{k-1:n} = \tau_{n-k+2|n}$ is also DFR.

Remarks

(i) The identical distribution of the component lifetimes are necessary. For example, a parallel system of two nonidentical exponential components is not IFR (Barlow and Proschan, 1981, p. 83)

(ii) The examples in Section 10.6 show that the independence assumption in this theorem is also necessary. For example, T_1 is no longer IFR if the two components lifetimes have a F-G-M bivariate exponential distribution.

10.7.3 Comparative Studies of Two k-out-of-n Systems

A comparison between systems A and B is often required for reliability design or planning. However, one needs to know what do we mean when we say system A is 'better' than system B. For example, a designer may need to consider allocation of spares at component level versus system level. For a meaningful comparison, we resort to the concept of a partial ordering. The readers may recall several definitions of partial orderings given in Section 2.9, particularly concerning stochastic ordering, failure rate ordering and likelihood ratio ordering. For ease of referencing, here we give these definitions again with the obvious notations. We say:

(i) X is said to be smaller than Y in likelihood ratio ordering ($X \leq_{\mathrm{LR}} Y$) if $f(t)/g(t)$ is increasing for all $t \geq 0$.

(ii) X is said to be smaller than Y in failure rate ordering ($X \leq_{\mathrm{FR}} Y$) if $r_F(t) \geq r_G(t)$ for all $t \geq 0$ or $\bar{F}(t)/\bar{G}(t)$ is increasing in $t \geq 0$.

(iii) X is said to be smaller than Y in stochastic ordering ($X \leq_{\mathrm{ST}} Y$) if $\bar{F}(t) \leq \bar{G}(t)$, for all $t \geq 0$.

It follows from Section 2.9 that $\leq_{\mathrm{LR}} \Rightarrow \leq_{\mathrm{FR}} \Rightarrow \leq_{\mathrm{ST}}$.

Comment: The reader should note that many authors refer to the failure rate ordering as the hazard rate ordering.

Boland (1998) has argued that when people say product A is superior to product B, they probably mean that the failure rate of product B is greater than that of product A. Thus, using the failure rate as a means to compare basic systems seems justifiable. We also note from the definition (iii) that $X \leq_{\mathrm{ST}} Y$ implies that $E(X) \leq E(Y)$.

We now wish to compare two k-out-of-n systems of independent components by partial orderings.

Stochastic ordering of k-out-of-n systems

Let $Y_1, Y_2, .., Y_n$ be another set of independent components with their order statistic $Y_{i:n}$. The lifetime of this second k-out-of-n system is denoted by $\tau'_{k|n} = Y_{n-k+1:n}$.

It was shown, see for example, Boland and Proschan (1994) that $\tau_{k|n} \leq_{\text{ST}} \tau'_{k|n}$ if $X_i \leq_{\text{ST}} Y_i$ for all $i = 1, 2, ..., n$. In fact, this is true for any coherent system with life function τ. This follows from the fact that the structure function of a coherent system is increasing with each component reliability. Hence the system with the stronger set of components will stochastically live longer (Boland and Proschan, 1994).

Failure rate ordering of k-out-of-n systems

(a) Lynch et al. (1987) and Singh and Vijayasree (1991) established that if both X's and Y's are i.i.d. such that

$$X_i \leq_{\text{FR}} Y_i, \tag{10.65}$$

then

$$X_{k:n} \leq_{\text{FR}} Y_{k:n}, \;\; k = 1, 2, ..., n \tag{10.66}$$

which also implies $\tau_{k|n} \leq_{\text{FR}} \tau'_{k|n}$. Boland et al. (1994) have shown by counter examples that if both X's and Y's are nonidentical, then (10.66) does not hold.

(b) Boland and Proschan (1994) have shown that if both X's and Y's are not identically distributed but $X_i \leq_{\text{FR}} Y_j$, $i, j = 1, 2, .., n$, then (10.66) also holds.

(c) Shaked and Shanthikumar (1995) showed that (10.66) holds under conditions weaker than those of Boland and Proschan (1994). (Essentially, they required $X_i \leq_{\text{ST}} Y_{\pi_i}, i = 1, 2, ..., n$, for some permutation $\boldsymbol{\pi}$ of $(1, 2, .., n)$).

Comparisons of k-out-of-n systems with respect to failure rate ordering

We now consider the case where two k-out-of-n systems do not have the same k or n.

It is clear that $\tau_{n|n} \leq_{\text{ST}} \tau_{n-1|n} \cdots \leq_{\text{ST}} \tau_{1|n}$ or equivalently $X_{(1)} \leq_{\text{ST}} X_{(2)} \leq_{\text{ST}} \dots \leq_{\text{ST}} X_{(n)}$. Boland et al. (1998) have shown that this result extends to failure rate ordering in general, i.e.,

$$\tau_{n|n} \leq_{\text{FR}} \tau_{n-1|n} \leq_{\text{FR}} \cdots \leq_{\text{FR}} \tau_{1|n}. \tag{10.67}$$

Further, Boland et al. (1994) also obtained the following results:

Theorem 10.7: Suppose we have n independent and identically distributed component lifetimes $X_1, ..., X_n$.

(a) $\tau_{k+1|n} \leq_{\mathrm{FR}} \tau_{k|n}$ for all $k = 1, 2, ..., n-1$.

(b) If $X_i \leq_{\mathrm{FR}} X_n$ for all $i = 1, 2, ..., n-1$, then $\tau_{k|n-1} \leq_{\mathrm{FR}} \tau_{k|n}$ for $k = 1, 2, ..., n-1$.

(c) If $X_n \leq_{\mathrm{FR}} X_i$ for all $i = 1, 2, ..., n-1$, then $\tau_{k|n} \leq_{\mathrm{FR}} \tau_{k-1|n-1}$ for $k = 2, ..., n$.

Proof: The proof is quite involved. Part (a) is from Theorem 3.1, Part (b) from Theorem 3.3 and part (c) from Theorem 3.4, all of Boland et al. (1994).

Hu and He (2000) have also proved the result (c) of the above theorem without imposing the condition $X_n \leq_{\mathrm{FR}} X_i, i = 1, 2, .., n-1$ while assuming only X_i be independent but not necessary identically distributed.

Comparisons of k-out-of-n systems with respect to reversed failure rate ordering

The reversed failure (hazard) rate of a lifetime random variable X is defined as

$$\nu_F(t) = f(t)/F(t), \ t > a, \tag{10.68}$$

where $a = \inf\{t : F(t) > 0\}$.

We say that X is smaller than Y in the reversed failure rate ordering ($X \leq_{\mathrm{RF}} Y$) if $\nu_F(t) \leq \nu_G(t)$ where G is the distribution function of Y. See for example, Shaked and Shanthikumar (1994, p. 24), for more details concerning the reversed failure rate order.

Block et al. (1998) established analogous results of Boland et al. (1994) with respect to the reversed failure rate ordering.

Theorem 10.8: Assuming the n component lifetimes are independent and identically distributed, we have

(a) $\tau_{k+1|n} \leq_{\mathrm{RF}} \tau_{k|n}$ for all $k = 1, 2, ..., n-1$.

(b) If $X_n \leq_{\mathrm{RF}} X_i$ for all $i = 1, 2, ..., n-1$, then $\tau_{k|n} \leq_{\mathrm{RF}} \tau_{k-1|n-1}$ for $k = 2, ..., n$.

Proof: The proof is similar to Boland et al. (1994). See Block et al. (1998) for more detail.

Hu and He (2000) established an ordering in the reversed failure rate of the following form.

Theorem 10.9: Let $X_1, X_2, ..., X_n$ be independent (but not necessarily identically distributed) lifetimes. Then

$$\tau_{k|n-1} \leq_{\mathrm{RF}} \tau_{k|n}, \ \text{for } k = 2, ..., n.$$

Proof: The proof is very long and we refer the reader to Hu and He (2000) for details.

Let $X_{k:n}$ denote the kth order statistic from $X_1, X_2, ..., X_n$. The above theorem may be expressed more conveniently in terms of partial ordering as

$$X_{i:n} \leq_{\mathrm{FR}} X_{i:n-1} \quad \text{and} \quad X_{i:n-1} \leq_{\mathrm{RF}} X_{i+1:n}$$

Stochastic comparisons of parallel systems of heterogenous exponential components

Let $X_1, X_2, .., X_n$ be n independent exponential random variables with X_i having failure rate $\lambda_i, i = 1, 2, ..., n$. Also, let $Y_1, Y_2, .., Y_n$ be another set of independent exponential random variables with λ_i^* as the failure rate of Y_i. Further, let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n)$ and $\boldsymbol{\lambda^*} = (\lambda_1^*, \lambda_2^*, ..., \lambda_n^*)$. Pledger and Proschan (1971) showed that if $\boldsymbol{\lambda}$ majors $\boldsymbol{\lambda^*}$, then $\tau_{n|n} \equiv \tau'_{n|n}$ (i.e., identically distributed) and $\tau_{k|n} \geq_{\mathrm{ST}} \tau'_{k|n}$ for all $k = 1, 2, ..., n-1$. (See Definitions 8.5 and 8.6 for the concepts of majorization and Schur-concavity).

For the special case $n = 2$ and $k = 1$ (i.e., a 2-component parallel system), Boland et al. (1994) partially strengthened the above results from stochastic ordering to failure rate ordering. They proved that the failure rate of a parallel system of two independent exponential components is Shur-concave in (λ_1, λ_2), the component failure rates. More precisely, suppose $\lambda_1 + \lambda_2$ is fixed, the more diverse of the $\lambda' s$, the smaller the failure rate of the parallel system $\tau_{1|2}$ is. They also concluded with a counter example that this result cannot be generalized for arbitrary n.

We now let $Y_1, Y_2, .., Y_n$ be independent exponential random variables with Y_i having failure rate $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i / n$. Let $\tau'_{k|n}$ denote the lifetime of the k-out-of-n system of the exponential components represented by Y's. Dykstra et al. (1997) showed that $\tau_{1|n} \geq_{\mathrm{FR}} \tau'_{1|n}$. This gives a convenient upper bound on the failure rate of of $\tau_{1|n}$. They also showed that the distribution of $\tau_{1|n}$ is more dispersed than $\tau'_{1|n}$ in the sense that the difference between any two quantiles of the distribution $\tau_{1|n}$ is greater than the difference between the corresponding quantiles of the distribution of $\tau'_{1|n}$. In other words, we have $\mathrm{var}(\tau_{1|n}) \geq \mathrm{var}(\tau'_{1|n})$.

Returning to the special case $k = 1, n = 2$ again, Dykstra et al. (1997) strengthened the result of Boland et al. (1994) from failure rate ordering to likelihood ratio ordering. It was shown that

$$(\lambda_, \lambda_2) \text{ majors } (\lambda_1^*, \lambda_2^*) \Rightarrow \tau_{1|n} \geq_{\mathrm{LR}} \tau'_{1|n}.$$

In the light of the counter example of Boland et al. (1994) discussed in a prior paragraph, the present result of a parallel system of two exponential components cannot be extended beyond the case $n = 2$.

10.7.4 Ageing Properties Based on the Residual Life of a k-out-of-n System

Kochar and Kirmani (1995) studied stochastic orderings of normalized spacings from DFR distributions where the spacing $X_{n-k+1:n} - X_{n-k:n}$ represents the lifetime between $(n-k)$th and $(n-k+1)$ failures and also may be considered as an additional lifetime to be gained on using a $(k-1)$-out-of-n rather than a k-out-of-n system. Likewise, Kirmani (1996) studied stochastic orderings of spacings from increasing mean residual life (IMRL) distributions.

The residual life of a k-out-of-n system, given that the $(n-k)$th failure has occurred at time t, is given by the conditional random variable

$$RLS_{k,n,t} \equiv (X_{n-k+1:n} - X_{n-k:n} \,|\, X_{n-k:n}), \tag{10.69}$$

and the residual lifetime of the k-out-of-k system (a series) is denoted by

$$LS_k = X_{1:k} - \alpha \tag{10.70}$$

where $\alpha = \inf\{\text{Support}(X)\}$.

Langberg et al. (1980a) provided the following characterizations:

F is IFR (DFR) $\Leftrightarrow RLS_{k,n,t} \geq_{\text{ST}} (\leq_{\text{ST}}) RLS_{k,n,t'}$ for all $t' \geq t \geq \alpha$ and
F is NBU (NWU) $\Leftrightarrow LS_k \geq_{\text{ST}} (\leq_{\text{ST}}) RLS_{k,n,t}$ for all $t \geq \alpha$, $1 \leq k < n$.

Belzunce et al. (1999) studied the situations given above when the stochastic order is replaced by some well known ageing and variability orders.

Li and Chen (2004) investigated the ageing properties of $RLS_{k,n,t}$ with independent but non-identical components.

10.7.5 Dependent Component Lifetimes

Conventionally, it is assumed that the failure of any component of a k-out-of-n system does not affect the remaining ones. In practice, the failure of a component will somewhat influence the remaining components. For example, the breakdown of an aircraft's engine will increase the load on the remaining engines so that their lifetimes may be shortened. For this reason, sequential order statistics have been introduced (see, e.g., Cramer and Kamps, 2001), as an extension to the (ordinary) order statistics, to model sequential k-out-of-n systems where the failures of components would possibly affect the remaining ones. It seems some simplification is required for this methodology to be understood more widely.

Navarro et al. (2004) study reliability properties of k-out-of-n system under (i) multivariate Gumbel's type 1 exponential distribution, (ii) multivariate Arnold and Strauss exponential distribution, and (iii) multivariate normal distribution. We note that (i) and (ii) are multivariate versions of (10.5) and (10.32), respectively.

Navarro et al. (2005) studied stochastic comparisons among coherent systems (including k-out-of-n) with identical but possibly dependent components using signatures. See also Hu and Hu (1998) for comparisons of order statistics between dependent and independent random variables.

Rychlik (2001) obtained upper bounds for the means of k-out-of-n systems with dependent IFR, DFR, IFRA and DFRA components expressed in terms of the mean and variance of the common component life distribution.

10.8 Consecutive k-out-of n:F Systems

The consecutive k-out-of-n:F system has n independent components that are linearly connected in such a way that the system fails if and only if at least k consecutive components fail. If the n components are arranged in a circle, the resulting system is known as the circular consecutive k-out-of-n:F system. In this section, we consider only the linear system unless otherwise stated.

Similarly, a consecutive k-out-of-n:G system consists of an ordered (linearly connected) sequence of n independent components such that the system operates if and only if at least k consecutive components operate. Clearly, there is a relationship between the consecutive k-out-of-n:F and G systems in the one dimensional case (though not true for two-dimensional case which will not be discussed in this book). Kuo et al. (1990) showed that a consecutive k-out-of-n:G system is a mirror image of a consecutive k-out-of-n:F system. For this reason, we restrict our discussion to the latter one.

For a comprehensive survey of reliability studies of this and related systems see Chao et al. (1995), Chang et al. (2000) and Mokhlis (2001).

There are two main advantages of using a consecutive k-out-of-n:F system for reliability modelling:

- it usually has a much higher reliability than the series system,
- it is often less expensive than the parallel system.

Example 10.4: Telecommunication system introduced by Chiang and Niu (1981)

A sequence of n microwave stations transmit information from place A to place B. The microwave stations are equally spaced between places A and B. Each microwave station is able to transmit information to a distance up to k microwave stations. This system fails if and only if at least k consecutive microwave stations fail. (Chiang and Niu, 1981).

Example 10.5: Oil pipe system of Chiang and Niu (1981)

A system for transporting oil by pipes from A to point B has n pumps. Pump stations are equally spaced between A and B. Each pump station can transport the oil a distance of k pump stations. If one pump is down, the flow of oil could not be interrupted because the next station could carry the load. However, when at least k consecutive pumps station fail, the oil flow stops and the system fails.

Example 10.6: Railway station introduced by Kuo et al. (1990)

A railway station has n lines that receive and send trains. Consider an over size train that requires k consecutive lines in order to enter into the station without delay. Then the reliability that the train enters the station without delay is itself the probability that at least k consecutive lines which are not in use are available. This is an example of linear consecutive k-out-of-n:G systems.

10.8.1 Reliability and Lifetime Distribution

The consecutive k-out-of-n:F and its related systems have caught the attention of many engineers and researchers because of its high reliability and low cost.

Reliability evaluation

Much attention has been devoted to compute the reliability $R(k, n; p)$ of the system where the components are i.i.d. with the same reliability p and failure probability $q = 1 - p$. $R(k, n; p)$ contains a summation of binomial coefficients so it is difficult to calculate in general. The system reliability can be evaluated through recursive equations or by approximations.

An exact formula for computing the reliability of a linearly connected system was given by Lambiris and Papastavridis (1985) as

$$R(k, n; p) = \sum_{j=0}^{n} \binom{n - jk}{j} (-1)^j (pq^k)^j - q^k \sum_{j=0}^{n} \binom{n - jk - k}{j} (-1)^j (pq^k)^j. \tag{10.71}$$

A popular method is to imbed the consecutive k-out-of-n:F in a Markov chain. In particular, Fu (1986) successfully introduced a $(k+1)$-state Markov chain which simplified the probability structure of the system considerably. Subsequently, Fu and Hu (1987), Chao and Fu (1989, 1991) developed a simple formula for general case of independent but not necessarily identically distributed components with reliability p_i:

$$R(k, n; p_1, p_2, ..., p_n) = \boldsymbol{\pi}_0 \cdot \prod_{i=1}^{n} M_i \cdot \boldsymbol{u}' \tag{10.72}$$

where $\boldsymbol{\pi}_0 = (1, 0, ..., 0)$, $\boldsymbol{u}= (1, 1, ..., 1, 0)$ and

$$M_i = \begin{pmatrix} p_i & q_i & 0 & \dots & 0 & 0 \\ p_i & 0 & q_i & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ p_i & & & \dots & 0 & q_i \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Aki (2001) used the method of conditional probability generating functions (developed in statistical distribution theory of runs) to compute the exact reliability of a consecutive k-out-of-n system with i.i.d. components.

Lambiris and Papastavridis (1985) also derived an exact formula for the system reliability of the circular consecutive k-out-of-n system with i.i.d. components. It was given by

$$\begin{aligned} R_C(k,n;p) = &\sum_{j=0}^{n} \binom{n-jk}{j} (-1)^j (pq^k)^j \\ &+k\sum_{j=0}^{n} \binom{n-jk-k-1}{j} (-1)^j (pq^k)^{j+1} - q^k,\ k \le n. \end{aligned} \tag{10.73}$$

For a survey of these developments, see Chao et al. (1995), Mokhlis (2001).

Reliability bounds

Simple upper and lower bounds were also been proposed by several authors. These bounds are often employed when the value of n or k is so large that the exact computation of the reliability is not achievable.

Two well known bounds are:

1. The inequalities of Fu (1985)

$$(1-q^k)^{n-k+1} \le R(k,n:p) \le (1-pq^k)^{n-k+1},\ q = 1-p. \tag{10.74}$$

2. Bounds based on the application of the Poisson approximation (Barbour et al., 1992 and Barbour et al., 1995)

$$\nu - \varepsilon \le R(k,n;p) \le \nu + \varepsilon \tag{10.75}$$

where

$$\nu = e^{-pq^k(n-k+1)} - q^{k+1}e^{-pq^k(n-2k)},$$

$$\varepsilon = (1-e^{-pq^k(n-k+1)} + q^{k+1}(1-e^{-pq^k(n-2k)}))(2k+1)pq^k$$

Muselli (2000) obtained the following improved reliability bounds:

$$(1-q^k)^{(n-k)/h_L+1} \le R(k,n:p) \le (1-q^k)^{(n-k)/h_U+1} \tag{10.76}$$

where

$$h_U = h_U(k,p) = \frac{1-q^k}{p}, \tag{10.77}$$

$$h_L = h_L(k,p) = \frac{(1-q^k)^{k/\{(1-q^k)^k/p\}}}{p}, \tag{10.78}$$

and $n \geq k \geq \max(q/p, 1)$.

Lifetime distribution of consecutive k-out-of-n:F systems

Explicit formula was given for the lifetime distribution of a consecutive k-out-of-n:F system in Aki and Hirano (1996). It is a linear combination of distributions of order statistics of the lifetimes of n i.i.d. components. The cases where X_i follows the exponential, the Weibull and the Pareto were derived.

10.8.2 Structure Importance of Consecutive k-out-of-n Systems

Reliability importance was defined by Birnbaum (1969) as a partial derivative of system reliability with respect to component reliability.

Lin et al. (1999) obtained, through the relationship with the Fibonacci sequence of order k a closed form solution of structure importance for each component.

10.8.3 Algorithms for Determining Optimal Replacement Policies for Consecutive k-out-of-n Systems

Costs arise when a system fails and when components are replaced. Flynn and Chung (2002) developed a branch and bound algorithm for computing optimal replacement policies in consecutive k-out-of-n systems to minimize the long-run average undiscounted cost per period. Extensive computational experiments find this algorithm is effective when $n \leq 40$ or k is near n; however, the computation can be intractable when $n > 40$ and $2 \leq k < n-15$. This deficiency prompted Flynn and Chung (2004) to find a heuristic algorithm which is highly effective for each n and k tested.

10.8.4 Ageing Property

Cui et al. (1995) have proved that if k is fixed in a consecutive k-out-of-n system with independent and identically distributed increasing failure rate (IFR) lifetimes, there exists n_k for which the system does not preserve IFR when $n \geq n_k$. They also provided the complete solutions for the cases $k = 4$ and $k = 5$. However, the method is tedious for larger k.

The results by Cui et al. (1995) suggest that the system will eventually lose IFR property for a given k when n increases. Cui (2002) presented a bound $\tilde{n}_k$ for n_k such that when $n > \tilde{n}_k$, the system does not preserve IFR. Further, $\tilde{n}_k$ has an explicit expression.

10.8.5 Consecutive-*k*-out-of-*n*:F System with Markov Dependence

So far, we have assumed the component lifetimes of a k-out-of-n:F system are mutually independent. In practice, the failure probability of a component increases as the number of failed components increases as illustrated by the example in Fu (1986). If the failure probability of component i depends upon only component $(i-1)$, then we say the components are Markov dependent. Fu and Hu (1987) computed the reliability of such a system when the number of components is large; whereas Papastavridis and Lambiris (1987) computed the reliability of such a system via recurrence relation. Ge and Wang (1990) gave a direct and exact method to compute the reliability of this Markov-dependent system.

Consecutive *k*-out-of-*n*:F repairable system with exponential distribution and (*k*-1)-step Markov dependence

Lam and Ng (2001) have introduced a general consecutive k-out-of-n:F repairable system with exponential distribution and $(k-1)$-step Markov dependence. The lifetime of a component is also an exponential random variable, its parameter depends on the number of consecutive failed components that precede the component. The repair time is also an exponential random variable. A priority repair rule on the basis of the system failure risk is adopted. A linear consecutive 3-out-of-4:F system and a circular consecutive 3-out-of-4:F system were investigated.

10.9 On Allocation of Spares to *k*-out-of-*n* Systems

The problem of where and how to allocate redundant components is an interesting and important problem in reliability theory and its applications. Much work has been reported in the literature, see for example, El-Neweihi et al. (1986), Boland et al. (1988, 1991, 1992), Shaked and Shanthikumar (1992), El-Neweihi and Sethuraman (1993), Singh and Misra (1994) and Mi (1998b, 1999b). Much of the past research in this subject centered on the k-out-of-n systems because the problem becomes complicated when one deals with an arbitrary coherent system.

Active redundancy with a spare to a k-out-of-n system

An active (also known as hot or warm) redundant spare works simultaneously with one of the components in the system, while a standby spare only begins to operate when the component for which it is standing ceases to function. We will consider the latter type of redundancy in Section 10.10.

Suppose we have a k-out-of-n system where the components are ordered stochastically (without loss of generality $X_1 \leq_{\mathrm{ST}} X_2 ... \leq_{\mathrm{ST}} X_n$), Boland and Proschan (1994) considered two scenarios of active redundancy:

1. There exists an independent component with lifetime X available as a 'common' spare which can be placed in redundancy with any component in the system.
2. There are a set of independent components with lifetimes $Y_1, Y_2, ..., Y_n$ (where $X_i = Y_i$ in the stochastic sense, ie., they have a common distribution) available for redundancy such that the 'like' spare having lifetime Y_i can only be used as a spare for the ith component.

Let $X \vee Y = \max(X, Y)$ denote the resulting lifetime of an active redundancy. Boland et al. (1992) and Boland and Proschan (1994) showed that

$$\tau_{k|n}(X_1, ..., X_{i-1}, X_i \vee X, X_{i+1}, ..., X_n)$$

is stochastically decreasing in $i = 1, 2, ..., n$. It follows therefore that such a stochastically ordered k-out-of-n system, it is always stochastically preferable to perform active redundancy of a 'common' spare on weaker components. This finding is perhaps not surprising for series system where it is known the weakest component is the most important. It is more surprising for more general k-out-of-n systems ($1 < k < n$) since stochastically weakest component is not necessarily be the most important at all points of time.

Boland and Proschan (1994) went on to consider the situation where one is to allocate one 'like' component actively to the k-out-of-n system. It was shown that in general, there is no optimal selection for the 'like' active redundancy allocation. They found however, for the series system, $\tau_{n|n}(X_1, ..., X_{i-1}, X_i \vee Y_i, X_{i+1}, ..., X_n)$ is stochastically decreasing in i whereas for the parallel system, $\tau_{1|n}(X_1, ..., X_{i-1}, X_i \vee Y_i, X_{i+1}, ..., X_n)$ is stochastically increasing in i.

Optimal active redundancy allocation of r spares in k-out-of-n system

In a related but a different problem, Mi (1998b) considered a k-out-of-n system consisting of $n + r$ ($1 \leq r \leq n$) available components of which r will be used for active redundancy. In other words, from the given $n + r$ components, r components are selected to be used as active redundancy, another r components to receive active redundancies (i.e., these r components are bolstered).

The problem of which r components should be used for active redundancy, and where to allocate them in order to maximize the lifetime of the resulting k-out-of-n system was studied.

Let $C_1, C_2, ..., C_{n+r}$ denote the $(n+r)$ components and X_i be the lifetime of C_i, $i = 1, 2, ..., (n+r)$, such that $X_1 \leq_{\mathrm{ST}} X_2 \leq_{\mathrm{ST}} ... \leq_{\mathrm{ST}} X_{n+r}$. The main result of Mi (199b) indicates that the optimal active redundancy allocation is to place C_i in redundancy with $C_{2r-i+1}(1 \leq i \leq r)$. In other words, under the usual stochastic ordering '$\leq_{\mathrm{ST}}$' the first r weakest components should be used for active redundancy and allocate in reverse order to the the next r weakest components.

Allocation of spares at component level versus at system level

Design engineers are well aware that a system where active spare allocation is made at the component level has a lifetime stochastically larger than the system where active spare allocation is made at the system level, see for example, Barlow and Proschan (1981, p. 23). Boland and El-Neweihi (1995) considered this principle in failure rate ordering and demonstrated it does not hold in general. However, they showed that for a 2-out-of-n system, with independent and identical original components and spares, active spare allocation at the component level is superior (in the failure rate ordering sense) to active spare at the system level. They conjectured that such a principle holds for general k-out-of-n when all the components and all the spares are identically distributed. Singh and Singh (1997) proved that a k-out-of-n system, when components lives are independent and identical, active spare allocations at the component level is superior to active spare allocation at the system level in likelihood ratio ordering. Since $\geq_{\mathrm{LR}} \Rightarrow \geq_{\mathrm{FR}}$, Singh and Singh (1997) have in fact proved the conjecture of Boland and El-Neweihi (1995), and in fact a stronger result.

10.10 Standby Redundant System

The standby redundant components neither degrade nor fail while in standby. When inserted into the system as active components, their state is new. Thus, 'standby' redundant is also known 'cold' redundant.

10.10.1 Standby Redundancy in *k*-out-of-*n* Systems

We assume in this subsection that all the component lifetimes are independent. Let X be the lifetime of the original component and X^* the lifetime of the standby component which may be similar but not necessarily identically distributed. The total lifetime is $X + X^*$ and its density is the convolution of the two densities.

Assuming the component lifetimes are ordered in the likelihood ratio sense, i.e., $X_1 \leq_{\text{LR}} X_2 \leq_{\text{LR}} ... \leq_{\text{LR}} X_n$, Boland and Proschan (1994) considered the allocation of a common standby spare to a series or parallel and showed that for a series system $\tau_{n|n}(X_1, ..., X_{i-1}, X_i + X, X_{i+1}, ..., X_n)$ is stochastically decreasing in i and $\tau_{1|n}(X_1, ..., X_{i-1}, X_i + X, X_{i+1}, ..., X_n)$ is stochastically increasing in i. On the other hand, if a 'like' standby component with lifetime Y_i is allocated, then the parallel system $\tau_{1|n}(X_1, ..., X_{i-1}, X_i + Y_i, X_{i+1}, ..., X_n)$ is stochastically increasing in i. Here, we have also assumed that Y_i are ordered in the likelihood ratio sense.

For most results in reliability theory concerning parallel systems of components, there is a 'dual' result for series system. Curiously, this does not seem to be the case for the last result. Boland and Proschan (1994) gave a counter example.

10.10.2 Standby Redundancy at Component Versus System Level

A question aries as to what extent is standby redundancy at the component level better (either in stochastic or failure rate ordering) than standby redundancy at the system level?

Boland and El-Neweihi (1995) have shown that when considering standby redundancy and stochastic ordering, redundancy at the component level is better than redundancy at the system level for series systems, while the reverse is true for parallel systems. What does it suggest about more general k-out-of n system? We do not have a definite answer at present. However, the above authors showed that for the 2-out-of 3 system with three i.i.d. exponential components, standby redundancy at the system level is better than at the component (in the stochastic order sense).

What can we say about standby component and system redundancy for the failure rate ordering? Boland and El-Neweihi (1995) also showed that for n i.i.d. exponential components that are arranged in series, standby redundancy at the component level is better (in failure rate sense) than standby system redundancy. Thus, more research is needed to enlighten us concerning the efficiency of redundancy for a general k-out-of-n system with respect to failure rate ordering when components are not exponentially distributed.

10.10.3 Dependent Components

Lai (1985) has considered a 2-component standby redundant system in which the lifetime follows the Moran-Downton bivariate exponential distribution as given in (10.20).

Let $T_s = X + Y$ where the subscript denotes 'standby'. Thus T_s represents the lifetime of the standby system. Assuming equal means, i.e., $\mu_X = \mu_Y = \mu$, Lai (1985) showed that the density of T_s is

$$f_{T_s}(t) = \frac{\mu}{\sqrt{\rho}} \sinh\left\{\frac{\sqrt{\rho}\mu t}{1-\rho}\right\} \exp\left\{-\frac{\mu t}{1-\rho}\right\}. \tag{10.79}$$

The survival function of this standby system is

$$\bar{F}_{T_s}(t) = (\cosh\sqrt{\rho}\gamma t + \sqrt{\rho}^{-1}\sinh\sqrt{\rho}\gamma t)e^{-\gamma t};\ \gamma = \frac{\mu}{1-\rho}. \tag{10.80}$$

The mean and variance are, respectively

$$E(T_s) = \frac{2}{\mu}; \qquad \mathrm{var}(T_s) = \frac{2(1+\rho)}{\mu^2}. \tag{10.81}$$

The failure rate function is

$$r_s(t) = \mu\left\{1 + \sqrt{\rho}\coth\gamma t\right\}^{-1},\ \gamma = \frac{\mu}{1-\rho}. \tag{10.82}$$

The function $r_s(t)$ is increasing in t so $F_{T_s} \in$ IFR.

10.11 Future Directions

There are several directions of possible research.

(i) The traditional studies on the reliability allocation and system design optimization may be extended to the case of dependent components when redundancy is applied to existing components.

(ii) In this chapter, we focuss mainly on system reliability characteristics with two dependent components. It is possible to extend our study to multiple components although it is well known that parallel redundancy is only effective for small number of components. Also, as indicated in Section 10.5.1, dependence in a multiple-component system needs to be explored further. Obviously multivariate distributions are required when dealing with multiple dependent components and we refer our readers to Kotz et al. (2000) for a rich source of information on multivariate modelling.

(iv) The sequential k-out-of-n system modelled by sequential statistics as discussed in Section 10.7.5 seems to offer a practical alternative to the traditional k-out-of-n system. The new methodology is built upon the belief that a damage caused by failures will increase stress on the remaining active components. Further research is needed to simplify the approach so that it can be more widely understood.

(iii) As seen in Section 10.10.2, except in the case with independent exponential components, the effectiveness of standby redundancy to a system is still unclear when the comparison is made either in the sense of stochastic ordering or in the failure rate ordering. The situation will be even more complex if the components lifetimes are positively dependent without assuming exponentiality. Thus more research is warranted on this subject.

(v) Estimation of dependence based on failure rate data and testing of dependence could be of interest.

11

Failure Time Data

11.1 Introduction

This chapter provides data sets that are known to belong to a particular ageing class. We believe that many readers will welcome having a number of data sets reproduced here as academic staff like to give their students data that is real rather than contrived. Only the numbers are extracted here, and the readers should consult the original source for detailed analysis done. In several cases, the data sets have also been analyzed by other researchers.

In Section 11.2, a rough guide on how to select a model from many plausible models such as the Weibull models discussed in Chapter 5. We discuss briefly in Section 11.3 how survival functions and failure rate functions can be estimated from a data set as well as the structure of our data presentation. Sections 11.4–11.8 list various data sets according to their ageing classifications. Finally in Section 11.9 we refer the readers to other sources of survival and reliability data which may be useful to them.

11.2 Empirical Modelling of Data

We have discussed in Chapter 5 a large number of Weibull-related models which were simply referred to as Weibull models. They exhibit a wide range of shapes for density and failure rate functions which make them suitable for modelling complex failure data sets. The question arises as which model is suitable to fit a particular 'Weibull like' data set. Here in this section, we provide a general guideline on empirical modelling.

It is well accepted that empirical modelling usually involves three steps: model selection, estimation of model parameters and model validation. In the context of Weibull models, a selection procedure may be based on WPP plots. This is possible because of the availability of WPP or generalized WPP plots for all the Weibull models in Sections 5.4 and 5.5. Of course, the shape of the density and failure rate functions will also be valuable in the selection step.

An add-on advantage of the WPP plots is that they provide crude estimates of model parameters. These serve as a starting point for steps 2 and 3.

It has been suggested that an alternative method to estimate model parameters is through a least squares fit. Basically speaking, it involves selecting the parameters to minimize a function given by

$$J(\theta) = \sum_{i=1}^{n} (y(t_i;\theta) - y_i)^2, \tag{11.1}$$

where $y(t_i;\theta)$ using vector parameter θ and y_i is the corresponding value obtained from the data. The optimization can be carried out using any standard optimization packages. The least squares method not only furnishes us parameter estimates, it also helps to select a Weibull model. If one of the potential candidates has a value for $J(\theta)$ which is considerably smaller than that for the other models, then undoubtedly it can be accepted as the most appropriate model for modelling the given data set. If two or more Weibull models give rise to roughly the same value of $J(\theta)$, one would need to examine additional properties of the WPP plots to decide on the final model. Other approaches such as bootstrap and jackknife may be employed for the final selection. For more on this, see Murthy et al. (2003) and Murthy et al. (2004).

A couple of comments on step 3 of our empirical modelling may be in order. There are many statistical tests for validating a model. These generally require data that is different from the data used for model selection and parameter estimation. A smaller data set may pose a problem, as there will be no separate data left after model selection and parameter estimation (Murthy et al., 2003, p. 288, 290). In this situation, the same data is used for both estimation and validation but we need to take into account the loss in the degree of freedom. For a general approach on parameter estimation and model validation, we refer the readers to the book by Meeker and Escobar (1998) for further details.

11.3 Data Presentation and General Comments on Reliability Estimation

The data in this chapter appear in three formats:

(i) Ungrouped data presented as exact values.

(ii) Ungrouped data with some being right censored.

(iii) Data presented as frequency tables.

Some brief comments may be warranted on how to obtain numerical estimates and to determine the shapes of the failure rates from the data sets.

- For censored and ungrouped data, the survival function may be estimated by Kaplan and Meier's (1958) product-limit method:

$$\hat{S}(t) = \prod_{t_{(r)} \leq t} \frac{n-r}{(n-r+1)}, \tag{11.2}$$

where $t_{(r)}$ is uncensored.

- When the data are in group form, one may obtain the percentiles using the method discussed by Kendall, Stuart, and Ord (1987, pp. 50-51).
- An approximate trend of failure rate may be seen from the curvature of the cumulative hazard function $H(t)$ versus failure time plot– see Nelson (1982, Chapter 4) and Section 5.3. Plots of the cumulative hazard give us useful information about the shape of the failure rate; note, for example, that $H(t)$ is linear if $r(t)$ is constant, and convex if $r(t)$ is monotonic.
- Non-monotonic failure rates such as the bathtub or upside-down bathtub shaped failure rates may by identified by a total on time test (TTT) plot, see, for example, Aarset (1987).
- Section 3.4 of Lawless (2003) gives nonparametric estimates of failure rate functions as well various numerical smoothing techniques.

The data sets provided in this chapter that have either monotonic or non-monotonic failure rates. So they belong to various ageing classes. We have endeavored to group them according to their ageing characteristics.

A plus sign after a failure time in the tables below indicates a censored observation.

11.4 IFR Data

First bus-motor failure data–Table 11.1

Data source: Davis (1952).
The data was reanalyzed by Mudholkar et al (1995).

Table 11.1. First bus-motor failures (1,000 miles)

Class interval	Observed frequency
0-20	6
20-40	11
40-60	16
60-80	25
80-100	34
100-120	46
120-140	33
140-160	16
160-up	4

Second bus-motor failure data–Table 11.2

Data source: Davis (1952).
Reanalyzed by Mudholkar et al (1995).

Table 11.2. Second bus-motor failure (1,000 miles)

Class interval	Observed frequency
0-20	19
20-40	13
40-60	13
60-80	15
80-100	15
100-120	18
120-up	11

Aluminium coupon failure–Table 11.3

Lifetimes of 101 strips of aluminium coupon.
Data Source: Birnbaum and Saunders (1958).

Table 11.3. Lifetimes of 101 strips of aluminium coupon

370	706	716	746	785	797	844	855	858	886	930	960	988	990	1000
1010	1010	1016	1018	1020	1055	1085	1102	1102	1108	1115	1120	1134	1140	1199
1200	1200	1203	1222	1235	1238	1252	1258	1262	1269	1270	1290	1293	1300	1310
1313	1318	1330	1355	1390	1416	1419	1420	1420	1450	1452	1475	1478	1481	1485
1502	1505	1513	1522	1522	1530	1540	1560	1567	1578	1594	1602	1604	1608	1630
1642	1674	1730	1750	1750	1763	1768	1781	1782	1792	1820	1868	1881	1890	1893
1895	1910	1923	1940	1945	2023	2100	2130	2215	2268	2240				

Electric carts failures–Table 11.4

Time to first failure of 20 electric carts data.
Data source: Zimmer et al. (1998).

The survival function is estimated by $\hat{R}(t_i) = 1 - \frac{i}{21}$. Cumulated hazard plot gives a straight line indicating the the failure time distribution could be either IFR or DFR so an exponential distribution will fit well.

Batteries failure data–Table 11.5

Lifetimes (in cycles) of sodium sulphur batteries (Batch 2).
Data source: Ansell and Ansell (1987).
Data reanalyzed by Phillips (2003).

Table 11.4. Time to first failure of 20 electric carts

t_i	0.9	1.5	2.3	3.2	3.9	5.0	6.2	7.5	8.3	10.4
$\hat{R}(t_i)$	0.952	0.905	0.857	0.809	0.761	0.714	0.667	0.619	0.571	0.524
t_i	11.1	12.6	15.0	16.3	19.3	22.6	24.8	31.5	38.1	53.0
$\hat{R}(t_i)$	0.476	0.429	0.381	0.333	0.286	0.238	0.190	0.142	0.095	0.048

Table 11.5. Failure times (in cycles) of 20 batteries

76	82	210	315	385	412	491	504	522	646+
678	775	884	1131	1446	1824	1827	2248	2385	3077

Other reported references

Failure times of 112 patients with multiple myeloma–Carbone et al. (1967).

11.5 DFR Data

Times to breakdown of an insulating fluid–Table 11.6

Times to breakdown of an insulating fluid between electrodes at voltage of 34 kV (minutes).
Data source: Nelson (1982, p. 105).
Data reanalyzed by Zimmer et al. (1998).

Leukemia-free survival times–Table 11.7

Leukemia-free survival times (in months) of 51 autologous transplant patients.
Data source: Ghitany and Al-Awadhi (2002).

Pressure vessels–Table 11.8

Time to failure (in hours) for 20 pressure vessels.
Data source: Keating et al (1990).

Pooled air conditioning failure data of airplanes–Table 11.9

Data source: Proschan (1963).

The data set consists of successive failure intervals of each member of a fleet of 13 Boeing 720 jet planes. The pooled data of 213 observations were first analyzed by Proschan (1963) and further discussed in Dahiya and Gurland (1972), Gleser (1989) and Adamidis and Loukas (1998). The failure interval containing a major overhaul was omitted from the listing since the length of that failure may be affected by major overhaul. Those values are represented by $**$ in the table.

Table 11.6. Insulating fluid failure

i	t_i	$\log(t_i)$	$\hat{R}(t_i)$
1	0.19	-1.66	0.95
2	0.78	-0.25	0.90
3	0.96	-0.04	0.85
4	1.31	0.27	0.80
5	2.78	1.02	0.75
6	3.16	1.15	0.70
7	4.15	1.42	0.65
8	4.67	1.54	0.60
9	4.85	1.58	0.55
10	6.50	1.87	0.50
11	7.35	1.99	0.45
12	8.01	2.08	0.40
13	8.27	2.11	0.35
14	12.06	2.49	.030
15	31.75	3.46	0.25
16	32.52	3.48	.020
17	33.91	3.52	0.15
18	36.71	3.60	0.10
19	72.89	4.29	0.05

Table 11.7. Leukemia-free survival times (in months) of 51 autologous transplant patients

0.658	0.822	1.414	2.500	3.322	3.816	4.737	4.836+	4.934
5.033	5.757	5.855	5.987	6.151	6.217	6.447+	8.651	8.717
9.441+	10.329	11.480	12.007	12.007+	12.237	12.401+	13.059+	14.474+
15.000+	15.461	15.757	16.480	16.711	17.204+	17.237	17.303+	17.644+
18.092	18.092+	18.750+	20.625+	23.158	27.730+	31.184	32.434+	35.921+
42.237+	44.638+	46.480+	47.467+	48.322+	56.086			

Table 11.8. Pressure vessels failure

0.75	1.7	20.8	28.5	54.9	126	175	236	274	290
363	458	776	828	871	970	1278	1311	1661	1787

Table 11.9. Interval between failures of the air conditioning system

Plane number												
7907	7908	7909	7910	7911	7912	7913	7914	7915	7916	7917	8044	8045
194	413	90	74	55	23	97	50	359	50	130	487	102
15	14	10	57	320	261	51	44	9	254	493	18	209
41	58	60	48	56	87	11	102	12	5		100	14
29	37	186	29	104	7	4	72	270	283		7	57
33	100	61	502	220	120	141	22	603	35		98	54
181	65	49	12	239	14	18	39	3	12		5	32
	9	14	70	47	62	142	3	104			85	67
	169	24	21	246	47	68	15	2			91	59
	447	56	29	176	225	77	197	438			43	134
	184	20	386	182	71	80	188				230	152
	36	79	59	33	246	1	79				3	27
	201	84	27	**	21	16	88				130	14
	118	44	**	15	42	106	46					230
	**	59	153	104	20	206	5					66
	34	29	26	35	5	82	5					61
	31	118	326		12	54	36					34
	18	25			120	31	22					
	18	156			11	216	139					
	67	310			3	46	210					
	57	76			14	111	97					
	62	26			71	39	30					
	7	44			11	63	23					
	22	23			14	18	13					
	34	62			11	191	14					
		**			16	18						
		130			90	163						
		208			1	24						
		70			16							
		101			52							
		208			95							

Coal-mining disasters data–Table 11.10

Data source: Maguire et al. (1952).

The data set gives the intervals in days between successive coal-mining disasters in Great Britan for the period 1875–1951. A disaster is defined as involving the dealth of 10 or more men. Data analyzed by Cox and Lewis (1978, p. 4) and Adamidis and Loukas (1998).

Table 11.10. Intervals in days between successive coal-mining disasters

378	286	871	66
36	114	48	291
15	108	123	4
31	188	457	369
215	233	498	338
11	28	49	336
137	22	131	19
4	61	182	329
15	78	255	330
72	99	195	312
96	326	224	171
124	275	566	145
50	54	390	75
120	217	72	364
203	113	228	37
176	32	271	19
55	23	208	156
93	151	517	47
59	361	1613	129
315	312	54	1630
59	354	326	29
61	312	1312	217
1	275	348	7
13	78	745	18
189	17	217	1357
345	120	120	
20	644	275	
81	467	20	

11.6 NBU Data

Chronic granulocytic leukemia patients data–Table 11.11

Data on survival times of 43 patients suffering chronic granulocytic leukaemia.
Data source: Bryson and Siddiqui (1969).
Data reanalyzed in Hollander and Proschan (1975).

Table 11.11. 43 patients suffering chronic granulocytic leukemia survival times

7	47	58	74	177	232	273	285	317	429
440	445	455	468	495	497	532	571	579	581
650	702	715	779	881	900	930	968	1077	1109
1314	1334	1367	1534	1712	1784	1877	1886	2045	2056
2260	2429	2509							

11.7 Bathtub Shaped Failure Rates Data

Failure times of 50 devices (Aarset data)–Table 11.12
The original data set of 50 failure times of devices are ranked in the table below.
Data source: Aarset (1987).
Lai et al. (2003) used a modified Weibull model to fit this data set.

Table 11.12. Aarset data

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

Third bus-motor failures–Table 11.13
Data source: Davis (1952).
Data reanalyzed by Mudholkar et al. (1995)– see their Table 5.

Table 11.13. Third bus-motor failures

Class interval (1,000 miles)	Observed frequency
0-20	27
20-40	16
40-60	18
60-80	13
80-100	11
100-up	16

Fourth bus-motor failures–Table 11.14
Data source: Davis (1952).
Data reanalyzed by Mudholkar et al. (1995).

Table 11.14. Fourth bus-motor failures

Class interval (1,000 miles)	Observed frequency
0-20	34
20-40	20
40-60	15
60-80	15
80-up	12

External leakage of centrifugal pumps–Table 11.15

Table of first external leakage of 32 centrifugal pumps.
Data source: Pamme and Kunitz (1993).
Data analyzed by Pamme and Kunitz (1993)

Table 11.15. External leakage of 32 centrifugal pumps

666	687	1335	2044	2195	2281	2708	2764	2940	2970
2972	3004	3564	3955	4133	4230	4805	5200	5384	5766
6222	6267	6714	6794	7398	7532	7659	8696	8740	9213
9740	12213								

Car failure data–Table 11.16

This is an actual set of failure time data collected during unit testing.
Data source: Xie and Lai (1995).
Data set was analyzed in Xie and Lai (1995) using an additive Weibull model.

Table 11.16. Car failures data

Time interval	1	2	3	4	5	6	7	8	9
Number of failures	53	29	29	36	13	25	22	16	18
Time internal	10	11	12	13	14	15	16	17	18
Number of failures	8	22	11	13	5	5	4	1	1

Failure times of 406 units of a hydromechanical device–Table 11.17

Data source: Hjorth (1980).

Halley's mortality data–Table 11.18

Data source: Halley's (1693).
Data re-tabulated in Jaisingh et al. (1987). This data set was also discussed in Nelson (1982, pp. 17-18).

Leukemia-free survival times of allogenic transplant patients–Table 11.19

The table gives Leukemia-free survival times (in months) of 50 allogenic transplant patients.
Data source: Ghitany and Al-Awadhi (2002).

Table 11.17. Failure times of 406 units of a hydromechanical device data

Working hours	No. of failures	No. of censored units
0	4	-
1-5	4	-
6-10	1	-
11-20	2	-
21-40	2	1
41-60	7	1
61-100	14	6
101-150	24	11
151-200	9	56
201-250	14	110
251-300	6	20
301-350	1	19
351-400	3	37
401-450	2	36
451-500	-	10
501-600	-	6

Table 11.18. Halley's mortality data

t	$f(t)$	$F(t)$	$\bar{F}(t)$	$r(t)$
0	-	0	1.000	-
0-5	.290	.290	.710	.058
5-10	.057	.347	.653	.016
10-15	.031	.378	.622	.010
15-20	.030	.408	.592	.010
20-25	.032	.440	.560	.011
25-30	.037	.477	.523	.013
30-35	.042	.519	.481	.016
35-40	.045	.564	.436	.019
40-45	.049	.613	.387	.022
45-50	.052	.665	.335	.027
50-55	.053	.718	.282	.032
55-60	.050	.768	.232	.035
60-65	.050	.818	.182	.043
65-70	.051	.869	.131	.056
70-75	.053	.922	.078	.081
75-80	.044	.966	.034	.113
80-85	.034	1.00	0.00	.200

Table 11.19. Leukemia-free survival times (in months) of 50 allogenic transplant patients

0.030	0.493	0.855	1.184	1.1283	1.480	1.776	2.138	2.500
2.763	2.993	3.324	3.421	4.178	4.441+	5.691	5.855+	6.941
6.941+	7.993+	8.882	8.882	9.145+	11.480	11.513	12.105+	12.796
12.933+	13.849+	16.612+	17.138+	20.066	20.329+	22.368+	26.766+	28.717+
28.717+	32.928+	33.783+	34.221+	34.770+	39.539+	41.118+	45.033+	46.033+
46.941+	48.289+	57.041+	58.322+	60.625+				

Appliance failure data–Table 11.20

The data below show the number of cycles to failure for a group of 60 electrical appliances in a life test. The failure times have been ordered for convenience. Data source: Page 112 of Lawless (2003).
Data were grouped and failure rate function was estimated by three methods: Adaptive regression spline, smoothing spline and natural cubic spline. The plot by the first method clearly indicates a bathtub shape.

Table 11.20. Appliance failure data

14	34	59	61	69	80	123	142	165	210
381	464	479	556	574	839	917	969	991	1064
1088	1091	1174	1270	1275	1355	1397	1477	1578	1649
1702	1893	1932	2001	2161	2292	2326	2337	2628	2785
2811	2886	2993	3122	3248	3715	3790	3857	3912	4100
4106	4116	4315	4510	4584	5267	5299	5583	6065	9701

500 MW generator's failure data–Table 11.21

Data contains 36 times to the first failure of 500 MW generators collected over 6-year period. The empirical cumulative hazards (see Section 5.3 for computation) are included in the table for completion sake.
Data source: Dhillon (1981). A bathtub shaped failure distributions was fitted to the data set by Dhillon (1981).

Load-haul-dump-A machine failures data (LHD-A data)–Table 11.22

The table consists of 44 failure times (in hours) which refer to all subsystems of the machine, i.e., engine, hydraulic and air-conditioning subsystems, brakes, transmissions, tyres and wheels, body and chassis.
Data Source : Kumar et al (1989). Also analyzed by Pulcini (2001).

Table 11.21. 36 MW generators' times to first failure

Item Number	Failure Time	Cumulative Hazard
1	58	0.028
2	70	0.059
3	90	0.086
4	105	0.116
5	113	0.147
6	121	0.179
7	153	0.212
8	159	0.247
9	224	0.283
10	421	0.320
11	570	0.359
12	596	0.399
13	618	0.441
14	834	0.485
15	1019	0.531
16	1104	0.579
17	1497	0.629
18	2027	0.682
19	2234	0.738
20	2372	0.797
21	2433	0.860
22	2505	0.927
23	2690	0.998
24	2877	1.075
25	2879	1.158
26	3166	1.249
27	3455	1.349
28	3551	1.460
29	4378	1.585
30	4872	1.728
31	5085	1.895
32	5272	2.095
33	5341	2.345
34	8952	2.678
35	9188	3.178
36	11399	4.178

Table 11.22. LHD-A data

16	39	71	95	98	110	114	226	294
344	555	599	757	822	963	1077	1167	1202
1257	1317	1345	1372	1402	1536	1625	1643	1675
1726	1736	1772	1796	1799	1814	1868	1894	1970
2042	2044	2094	2127	2291	2295	2299	2317	

Times between failures of a 180-ton rear dump truck–Table 11.23

Table 11.23. Time between failures (1000's of hours) of a 180-ton rear dump truck

0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.02
0.02	0.02	0.03	0.04	0.06	0.08	0.10	0.10
0.12	0.12	0.12	0.13	0.14	0.15	0.15	0.15
0.16	0.16	0.17	0.18	0.18	0.19	0.20	0.21
0.22	0.23	0.25	0.26	0.28	0.28	0.30	0.32
0.34	0.36	0.38	0.39	0.41	0.41	0.42	0.43
0.44	0.44	0.45	0.45	0.50	0.53	0.56	0.58
0.58	0.61	0.62	0.62	0.62	0.64	0.66	0.70
0.70	0.70	0.72	0.77	0.78	0.78	0.80	0.82
0.83	0.85	0.86	0.96	0.97	0.98	0.99	1.05
1.06	1.07	1.18	1.35	1.36	1.42	1.55	1.59
1.65	1.73	1.77	1.79	1.80	1.91	2.09	2.14
2.15	2.15	2.31	2.33	2.36	2.36	2.43	2.45
2.50	2.51	2.58	2.64	2.68	3.08	3.94	4.12
4.33	4.42	4.53	4.88	4.97	5.11	5.32	5.55
6.63	6.89	7.62	11.41	11.76	11.85	12.36	13.22

Data source: Coetzee (1996). The original data is given in actual observed times (in hours).
Analyzed by Coetzee (1996) and Pulcini (2001).

Other reported but untabulated data sets with bathtub shaped failure rates

Survival data of 898 patients treated for non-Hodgkin's lymphoma data was analyzed in Alidrisi et. al (1991).
Failure times data of an electronic device (142 observations with 55 being censored. Data analyzed in Haupt and Schäbe (1992) but not tabbulated.

11.8 Upside-down Bathtub Shaped Failure Rates Data

Arm A data on head-and-neck-cancers patients–Table 11.24

The table contains Arm A data on the survival times (in days) of 51 head-and-neck-cancers patients.
Data source: Efron (1988).
Data reanalyzed by Mudholkar et al. (1995).

Table 11.24. Survival times (in days) for patients head-and-neck-cancer

7	34	42	63	64	74+	83	84	91
108	112	129	133	133	139	140	140	149
154	157	160	160	165	173	176	185+	218
225	241	248	273	277	279+	297	319+	405
417	420	440	523	523+	583	594	1101	1116+
1146	1226+	1349+	1412+	1417				

Lung cancer patients survival data–Tables 11.25-27

The table contains Veterans Adminstration lung cancer trial data. A subgroup of 97 patients with no prior therapy. The data represent the days of survival for lung cancer patients since therapy.
Data source: Prentice (1973).
Data reanalyzed by Gupta, Akman and Lvin (1999) using a log-logistic model. The table was then subdivided into two groups.

Table 11.25. 97 lung cancer patients data

72	228	10	110	314	100+	42	144	30	384	4	13	123+
97+	59	117	151	22	18	139	20	31	52	18	51	122
27	54	7	63	392	92	35	117	132	162	3	95	162
216	553	278	260	156	182+	143	105	103	112	87+	242	111
587	389	33	25	357	467	1	30	283	25	21	13	87
7	24	99	8	99	61	25	95	80	29	24	83+	31
51	52	73	8	36	48	7	140	186	19	45	80	52
53	15	133	111	378	49							

Table 11.26. Low PS ($X_1 \leq 50$) – 35 observations

10	314	144	4	123+	59	151	18	20	18	7	63	392
35	3	216	242	33	25	1	25	21	13	7	80	29
248	48	7	19	45	80	15	49					

Table 11.27. High PS ($X_1 > 50$) – 62 observations

72	228	110	100+	42	30	384	13	97+	117	22	139	31	52
51	122	27	54	92	117	132	162	95	162	553	278	260	156
182+	143	105	103	112	87+	111	587	389	357	467	30	283	87
24	99	8	99	61	25	95	83+	31	51	52	73	36	140
186	52	53	133	111	378								

Acute nonlymphoblastic leukaemia data–Table 11.28

The table contains times from remission to relapse for 84 patients with acute nonlymphoblastic leukaemia.
Data source: Glucksberg et al. (1981).
This was reanalysed in Ebrahimi (1991). Censored observations have been dropped in Ebrahimi (1991). Only ordered remission durations for 51 patients were listed here.

Table 11.28. Remission to relapse times for 51 leukaemia patients

24	46	57	57	64	65	82	89
90	90	111	117	128	143	148	152
166	171	186	191	197	209	223	230
239	247	254	258	264	269	270	273
284	294	304	304	332	341	393	395
487	510	516	518	518	534	608	642
697	955	1160					

Flood discharge rates in Iowa–Table 11.29

Data source: United States Water Resources Council (1977).
Data analyzed in Mudholkar and Hutson (1996). Rate unit is in (ft^3/s).

Table 11.29. The consecutive annual flood discharge rates (1935-1973) of the Floyd River at James, Iowa

1935-1944	1460	4050	3570	2060	1300	1390	1720	6280	1360	7440
1945-1954	5320	1400	3240	2710	4520	4840	8320	13900	71500	6250
1955-1964	2260	318	1330	970	1920	15100	2870	20600	3810	725
1965-1973	7500	7170	2000	829	17300	4740	13400	2940	5660	

Guinea pigs survival data–Table 11.30

The table only contains one set of of the guinea pigs survival times (in days) of guinea pigs infected with virulent tubercle bacilli.

Data source: Bjerkedal (1960) contains analyzes of two studies (Study M and Study P). The observed survival times, by study and regimen, were listed in their Table 6. Ghai and Mi (1999) mentioned that Bjerkedal (1960) gave a real data set which exhibits an upside-down bathtub shaped mean residual life function but we are unsure of which one of the data sets has this property.

Table 11.30. Survival times of guinea pigs infected with virulent tubercle bacilli

18	36	50	52	86	87	89	91	102	108	114	114	115
118	119	120	149	160	165	166	167	167	173	178	189	209
212	216	273	278	279	292	341	355	367	380	382	421	421
432	446	455	463	474	505	545	546	569	576	590	603	607
608	621	634	634	637	638	641	650	663	685	688	725	735

Repair times for an airborne communication transceiver–Table 11.31

Data source: The following maintenance data set was reported in Von Alven (1964) and Chhikara and Folks (1977) on active repair times (in hours) for an airborne communication transceiver:

Table 11.31. Repair times for an airborne communication transceiver

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7	0.7	0.8	0.8
1.0	1.0	1.0	1.0	1.1	1.3	1.5	1.5	1.5	1.5	2.0	2.0	2.2
2.5	2.7	3.0	3.0	3.3	3.3	4.0	4.0	4.5	4.7	5.0	5.4	5.4
7.0	7.5	8.8	9.0	10.3	22.0	24.5						

Bearing lifetimes data–Table 11.32

Lifetimes (millions of revolutions) of 23 ball bearings are given below.
Data source: Dumonceaux and Antle (1973).
The data set was reanalyzed by Jiang et al. (2003).

Failure of electronic devices–Table 11.33

Lifetimes of 18 electronic devices are given below.
Data source: Wang (2000).

Table 11.32. Bearing lifetimes

17.88	28.92	33.00	41.52	42.12	45.60	48.48	51.84
51.96	54.12	55.56	67.80	68.64	68.64	68.88	84.12
93.12	98.64	105.12	105.84	127.92	128.04	173.40	

Table 11.33. 18 electronic devices' lifetimes

5	11	21	31	46	75	98	122	145
165	196	224	245	293	321	330	350	420

11.9 Other Sources of Survival and Reliability Data

Lee (1992) contains several survival data sets from various sources.

- Exercise Table 2.2: Life table for the total population (of 100,000 life births), in the United States, 1959–1961.
- Table 3.1: Survival data for 30 resected melanoma patients.
- Table 3.4: Tumor-free times (days) of 90 rates on three different diets.
- Table 3.5: Life table for male patients with localized cancer of rectum diagnosed in Connecticut, 1935–1934 and 1945–1954.
- Table 4.7: A life table analysis of 2418 males with angina pectoris.
- Exercise Table 4.2 A life table analysis of females with angina pectoris.
- Table 6.3: Calculations of survivorship functions for Group 2 of rats exposed to DMBA.
- Table 6.4: Lifetimes of 101 strips of aluminum coupon.

Nelson (1982) contains a few real failure and survival times data sets.

- Page 17: Halley's mortality table.
- Page 105: Times to breakdown of an insulating fluids.
- Page 111: Connection strength data.
- Page 113: Class-H insulation life data.
- Page 121: Appliance cord data.
- Page 124: Insulation fluid times to breakdown with censoring.
- Page 133: Fan data and hazard calculations.
- Page 138: Transformer failure rates and hazard calculations.
- Page 141: Turn failure data and hazard calculations.
- Page 144: Winding data and cumulative hazards.
- Page 186: Hazard calculations for individual coils.
- page 529: Cycles to snubber data.

Lawless (2003) contains a lot of lifetime data sets throughout the text.Its Appendix G "Data Sets" gives detailed backgrounds of several sets of data.

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