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# Telegraph Processes and Option Pricing

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# Preface

This book gives an introduction to the contemporary mathematical theory of noninteracting particles moving at finite velocity in one dimension with alternating directions, so-called the telegraph (or telegrapher's) stochastic processes. The main objective is to give the basic properties of the one-dimensional telegraph processes and to present their applications to option pricing. The book contains both the well-known results and the most recent achievements in this field.

The model of a mass-less particle that moves at infinite speed on the real line and alternates at random two possible directions of motion infinitely many times per unit of time is of great interest to physicists and mathematicians beginning with the classical works of Einstein [1] and Smoluchowski [2]. First, A. Einstein determined the transition density of such a kind of motion as the fundamental solution to the heat equation. Then, M. Smoluchowski described this as a limit of random walks. This interpretation is used by physicists as an instrument for mathematical modelling the physical processes of mass and heat transfer. Later this stochastic process, called afterward the *Brownian motion*, was applied to explain the motion first observed by the botanist Robert Brown in 1828.

It is curious that in 1900, i.e. 5 years before Einstein, Louis Bachelier proposed and analysed the model of financial contracts based on what is now called "Brownian motion" [3] (see also [4]). A. Einstein was completely unaware of the work of R. Brown, as well as of the work of L. Bachelier. Nevertheless, Einstein's paper has had incredible influence on the science of the twentieth century, but the unusual and outstanding work of L. Bachelier has been lost from scientific interchange and it was only rediscovered in 1964. In the textbook of Feller [5] Brownian motion is named as the process of Wiener-Bachelier.

The crucial point in studying the Brownian motion was a work by Wiener [6] in which he was able to introduce a Gaussian measure in the space of continuous functions. Thus, he had given the opportunity of rigorous axiomatic constructing an extremely important stochastic process that was afterwards called the Wiener process. After the appearance of the Einstein–Smoluchowski's model governed by the heat equation, Brownian motion has been extensively used to describe various real phenomena in statistical physics, optics, biology, hydrodynamics, financial markets and other fields of science and technology. It was discovered that the theoretical calculations based on this model agree well with experimental data if the speed of the process is sufficiently big. If the speed is small, this agreement becomes worse. This fact, however, is not too surprising if we take into account the infinite-velocity nature of the Wiener process.

That is why many attempts were made to suggest alternative models in which the finiteness of both the speed of motion and the intensity of changes of directions per unit of time, could be assumed. Such a model was first introduced in 1922 by Taylor [7] in describing the turbulent diffusion, (see also the discussion between Prof. Karl Pearson and Lord Rayleigh in 1905, [8–10]). In 1926 V. Fock [11] suggested the use of a hyperbolic partial differential equation (called the *telegraph, or damped wave equation*) to describe the process of the diffusion of a light ray passing through a homogeneous medium. Later the time-grid approach was developed at length by Goldstein [12]. This naturally led to the telegraph equation describing the spatio-temporal dynamics of the potential in a transmitting cable, (without leakage) [13]. In his 1956 lecture notes, Kac, (see [14]), considered a continuous-time version of the telegraph model. Since then, the telegraph process and its various generalisations have been studied in great detail with numerous applications in physics, and, more recently, in financial market modelling. The telegraph process is the simplest example of the so-called *random evolution* (see, e.g., [15, Chap. 12] and [16, Chap. 2]).

An efficient conventional approach to the analytical study of the telegraph process, similar to that for diffusion processes, is based on pursuing a fundamental link relating various expected values of the process with initial value and/or boundary value problems for certain partial differential equations. One should note that the telegraph equation first appeared more than 150 years ago in a work by W. Thomson (Lord Kelvin) in an attempt to describe the propagation of electric signals on the transatlantic cable [17]. At present, it is one of the classical equations of mathematical physics.

The main objective of the book is to give a modern systematic treatment of the telegraph stochastic processes theory with an accent on the financial markets applications. These applications are rather new in the literature, but we believe that our approach is quite natural if we take into account the finite-velocity market motions joint with abrupt jumps (deterministic or of random values) that naturally produce heavy tails in such models.

In this book we develop a unified approach based on integral and differential equations. This approach might seem somewhat unusual for those specialists who mostly use the stochastic calculus methods in their research. We, however, believe that our approach is quite natural and could be exploited as a fruitful addition to the classical methodology.

The book consists of five chapters and is organised as follows.

In Chap. 1, for the reader's convenience and in order to make the book more self-contained, we recall some mathematical preliminaries needed for further analysis.

Chapter 2 deals with the general definition and basic properties of the telegraph process on the real line performed by a stochastic motion at finite speed driven by a homogeneous Poisson process. We derive the finite-velocity counterparts of the classical Kolmogorov equations for the joint transition densities of the process and its direction representing a hyperbolic system of two first-order partial differential equations with constant coefficients. Basing on this system, we derive a second-order telegraph equation for the transition density of the process. The explicit formulae are obtained for the transition density of the process and its characteristic function as the solutions of respective Cauchy problems. It is also shown that, under the standard Kac's condition, the transition density of the telegraph process tends to the transition density of the one-dimensional Brownian motion. The formulae for the Laplace transforms of the transition density and of the characteristic function of the telegraph process are also obtained.

In Chap. 3 we consider some important functionals of telegraph processes. We describe the distributions of the telegraph process in the presence of absorbing and reflecting barriers. First passage times and spending times of the telegraph processes are considered also. This presentation corrects some stable inaccuracies in the field.

The applications of telegraph processes to financial modelling presented in Chap. 5, require studying of the asymmetric telegraph processes. Moreover, it is crucial to add jumps to the asymmetric telegraph process. In Chap. 4 we introduce the reader to this new situation.

Chapter 5 is devoted to some contemporary applications of the telegraph processes to financial modelling. We modify the classical Black–Scholes market model exploiting a telegraph process instead of Brownian motion. As is easy to see, the simple substitution of a telegraph process instead of Brownian motion in the framework of Black–Scholes-Merton model leads to arbitrage opportunities. To get an arbitrage-free model we add a jump component to the telegraph process.

The huge literature on the mathematical modelling of financial markets began from two fundamental papers of Black and Scholes [18] and of Merton [19]. In this classical model the price of risky asset is assumed to follow a geometric Brownian motion. This assumption permits one to obtain nice closed formulae for option prices and hedging strategies.

Nevertheless, the famous Black–Scholes formula has well-known shortages. It is commonly accepted that Black–Scholes pricing formula distorts some option prices. Typically, it substantially underprices deep-in-the-money and out-of-the-money options and overprices at-the-money options, but downward (or upward) slopes are possible. To accord the Black–Scholes formula with market prices of standard European options different volatilities for different strikes and maturities are used. This trick is referred to as the *volatility smile*. In the "typical" comportment the implied volatility of deeply in-the-money and out-of-the-money options is higher than at-the-money options. Modern fearful markets are afraid of large downward movements and crashes. A smile pattern of these markets more resembles a "skew", where implied volatility increases with shortening the maturity time.

These observations provoke a growing interest in the construction of more and more complicated extensions of the Black–Scholes model. Stochastic volatility models are based on the stochastic dynamics of the Black–Scholes implied volatility. Various patterns of smiles and skews can be constructed depending on the correlation and the parameters of the volatility process. These models have some empirically approved evidences of their realistic and unrealistic features, but we believe that such an approach proposes the quantity sophistication instead of fundamental explanation of problem.

Another approach, that adds a pure jump process to Black–Scholes diffusion, can capture many volatility smiles and skews. This idea of jump-diffusion model have been proposed for better adequacy by Merton [20], and nowadays is applied to handle option pricing, especially when options are close to maturity. Similar to stochastic volatility models, jump-diffusion models increase Markov dimension of the market and form incomplete market models.

We suggest here a new model to explain market's movements. Suppose the log-returns are driven by a telegraph process, i.e. they move with a pair of constant velocities alternating one to another at Poisson times. To make the model more adequate and to avoid arbitrage opportunities the log-return movement should be supplied with jumps occurring at times of the tendency switchings.

Such a model looks attractive due to finite propagation velocity and the intuitively clear comportment. The jump-telegraph model captures bullish and bearish trends using velocity values, and it describes crashes and spikes by means of jump values. This model describes adequately the processes on oversold and overbought markets, when changes on the market tendencies accumulate in course of time.

At the same time, the model is analytically tractable. It allows us to get solutions for hedging and investment problems in closed form. Jumps are used in the model to avoid arbitrage opportunities, but not solely for adequacy.

The model based on the telegraph processes with jumps of deterministic values is complete as well as in the classical Cox-Ross-Rubinstein and Black–Scholes cases. It is attractive mathematically and allows us to freely modify the model to meet the needs of applications.

Under respective rescaling, the jump-telegraph model converges to Black–Scholes model. It permits us to define naturally a volatility of the jump-telegraph model depending on the velocities and jump values as well as on the switching intensities. The model based on jump-telegraph processes is characterised by volatility smiles of various shapes including frowns and skews depending on the parameters' values.

Unfortunately, some important topics remain outside this book because of the volume restrictions. In particular, the whole complex of problems related to the multidimensional counterparts of the telegraph processes is omitted. The reader interested in this multidimensional theory should address to the survey article [21] recently published in the Encyclopedia of Statistical Science, Springer, where the most important results and open problems in this field are presented.

We hope that this book will be interesting to specialists in the area of diffusion processes with finite speed of propagation and in financial modelling. We expect that the book will also be useful for students and postgraduates who make their first steps in these intriguing and attractive fields.

Finally, we would like to thank the staff of Springer, Editorial Statistics Europe, for the invitation to write this book.

March 2013 Alexander D. Kolesnik
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# Contents

# Chapter 1
# Preliminaries

**Abstract** In this chapter we recall the main mathematical notions and concepts of the required theory on probability and calculus.

**Keywords** Markov processes · Brownian motion · Stochastic integrals · Poisson process · Bessel functions · Generalised functions

## 1.1 One-Dimensional Markov Processes

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be the probability space. Let $\mathscr{B}$ be the $\sigma$-algebra of the Borel subsets of the real line $\mathbb{R}$, $\mathbb{R} = (-\infty, \infty)$, and $T > 0$ be an arbitrary positive number. A Markov process $\xi(t)$, $t > 0$, on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ is determined by the *transition probability function* $P(\Gamma, t, x, s)$, $0 \le s < t \le T$, $x \in \mathbb{R}$, $\Gamma \in \mathscr{B}$, which is defined as

$$P(\Gamma, t, x, s) = \mathbb{P}\{\xi(t) \in \Gamma | \xi(s) = x\}.$$

Function $P(\Gamma, t, x, s)$ is treated as the probability that the process $\xi$ (which at time $s$ is located at point $x$) at time $t$, $t > s$ will be located at the Borel set $\Gamma \in \mathscr{B}$. In other words, $P(\Gamma, t, x, s)$ is the probability of passing within the time $t - s$ from the point $x$ into the set $\Gamma$.

By definition, function $P(\Gamma, t, x, s)$ satisfies the following properties:

1. $P(\Gamma, t, x, s)$ is a $\mathscr{B}$-measurable function with respect to $x$ under fixed $s, t, \Gamma$.
2. Under fixed $s, t, x$ function $P(\Gamma, t, x, s)$ is a probability measure on $\mathscr{B}$ (so, $P(\mathbb{R}, t, x, s) = 1$).
3. For all $0 \le s < t_1 < t_2$, $x \in \mathbb{R}$, $\Gamma \in \mathscr{B}$, the following relation holds:

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$$P(\Gamma, t_2, x, s) = \int_{-\infty}^{\infty} P(\mathrm{d}y, t_1, x, s) P(\Gamma, t_2, y, t_1). \tag{1.1.1}$$

Relation (1.1.1) is referred to as *Chapman-Kolmogorov equation*.

One can distinguish two main classes of Markov processes in dependence of the continuity of their sample paths. The first class consists of those Markov processes whose $\mathbb{P}$-almost all sample paths are continuous. The most important representative of such class of Markov processes, the Wiener process, will be considered in Sect. 1.2.

Another class consists of the Markov processes with discontinuities of finite values (jumps), so-called the first-type discontinuities. Such processes are referred to as the *jump Markov processes*. The representative of such processes, the Poisson process, will be examined in Sect. 1.4. See also jump processes in Chap. 5.

## 1.2 Brownian Motion and Diffusion on $\mathbb{R}$

We begin our survey of subclasses of Markov processes with the important example of a Brownian motion.

**Definition 1.1** The stochastic process $w = w(t) = w(t, \omega)$, $t \ge 0$, $\omega \in \Omega$, is a real-valued standard Brownian motion (also called the Wiener process) if

(a) $w(0, \omega) = 0$ and sample paths $w = w(t, \omega)$ are continuous functions for almost all $\omega \in \Omega$;
(b) for any $t$ and $s$ ($t, s > 0$) the increment $w(t+s) - w(s)$ has the normal distribution with the mean 0 and variance $t$;
(c) for any $0 = t_0 < t_1 < t_2 < \ldots < t_m$, the variables $w(t_k) - w(t_{k-1})$, $1 \le k \le m$, are independent.

The first and the second moments, that is, the mean value and the variance of the Wiener process $w(t)$ are $\mathbb{E}\{w(t)\} = 0$, $\mathbb{E}\{[w(t)]^2\} = t$. The covariance function of $w(t)$ is given by $\mathbb{E}\{w(t)w(s)\} = \min\{t, s\}$.

It is easy to see that if $w(t)$ is a Brownian motion, then, for arbitrary $s > 0$ and $\alpha \ne 0$, the process generated by increments, $w(t+s) - w(s)$, $t \ge 0$, as well as the scaled process $\alpha^{-1} w(\alpha^2 t)$, $t \ge 0$, are the Brownian motions too. In particular, the process $\{-w(t), t \ge 0\}$ is a Brownian motion. Moreover, the process generated by increments, $w(t+s) - w(s)$, $t \ge 0$ is independent of the past, i. e. of the values $w(t')$, $t' \le s$.

The stochastic processes $\{w(t), t > 0\}$ and $\{t w(1/t), t > 0\}$ have the same distribution (see e.g. [1]).

The process $w(t)$ has the Gaussian density given by the formula

$$p(x, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right), \quad x \in \mathbb{R}, \quad t > 0. \tag{1.2.1}$$

Function $p = p(x, t)$ is the fundamental solution to the one-dimensional heat equation

$$\frac{\partial p(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2}. \tag{1.2.2}$$

The characteristic function of $w(t)$, that is, the Fourier transformation $\mathscr{F}_{x\to\xi}$ of the density $p = p(x, t)$ with respect to spatial variable $x \in \mathbb{R}$, has the form

$$\mathbb{E}\{\exp(i\xi w(t))\} = \mathscr{F}_{x\to\xi}[p(x,t)](\xi) = \exp\left(-\xi^2 t/2\right), \quad \xi \in \mathbb{R}. \tag{1.2.3}$$

The Laplace transformation $\mathscr{L}_{t\to s}$ with respect to time variable $t \geq 0$ is given by the formula

$$\mathscr{L}_{t\to s}[p(x,t)](s) = \frac{1}{\sqrt{2s}}\exp\left(-|x|\sqrt{2s}\right), \quad \mathrm{Re} s > 0. \tag{1.2.4}$$

For more detailed definitions of these integral transformations, $\mathscr{F}_{x\to\xi}$ and $\mathscr{L}_{t\to s}$, and their main properties see Sect. 1.6.

The homogeneous Brownian motion $w_\sigma := \sigma w(t)$ with arbitrary diffusion coefficient $\sigma^2 > 0$ has the marginal distributions which are defined by densities $p_\sigma = p_\sigma(x, t)$. For $t > 0$

$$p_\sigma(x,t) = \sigma^{-1}p(x/\sigma, t) = \frac{1}{\sigma\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2\sigma^2 t}\right), \quad x \in \mathbb{R}, \quad t > 0 \tag{1.2.5}$$

(see (1.2.1)), which is the fundamental solution to the heat equation

$$\frac{\partial p_\sigma(x,t)}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 p_\sigma(x,t)}{\partial x^2}. \tag{1.2.6}$$

The moments of $w_\sigma(t)$ and respective integral transforms can be found by the evident rescaling of the formulae for the standard Brownian motion. The mean and the variance of $w_\sigma(t)$ are $\mathbb{E}\{w_\sigma(t)\} = 0$, $\mathbb{E}\{[w_\sigma(t)]^2\} = \sigma^2 t$. The integral transforms (see (1.2.3) and (1.2.4)) have the form

$$\mathbb{E}\{\exp(i\xi w_\sigma(t))\} = \mathscr{F}_{x\to\xi}[p_\sigma(x,t)](\xi) = \exp\left(-\xi^2\sigma^2 t/2\right), \quad \xi \in \mathbb{R}, \tag{1.2.7}$$

$$\mathscr{L}_{t\to s}[p_\sigma(x,t)](s) = \frac{1}{\sigma\sqrt{2s}}\exp\left(-\frac{|x|\sqrt{2s}}{\sigma}\right), \quad \mathrm{Re}\, s > 0. \tag{1.2.8}$$

The peculiar property of the Brownian motion is the following: almost all sample paths of Brownian motion $\{w(t, \omega), t \geq 0\}$ are nowhere differentiable and for $\mathbb{P}$-almost all $\omega \in \Omega$ the sample paths of the Brownian motion have unbounded variation in any subinterval. Moreover, the length of any piece of Brownian trajectory is infinite, see e. g. [2].

The distributions of some important functionals of the one-dimensional Wiener process $w(t)$ are given by the following results (see [2]).

1. *Distribution of the maximum of Brownian motion*. For $x > 0$

$$\mathbb{P}\left\{\sup_{0\le s\le t} w(s) < x\right\} = \sqrt{\frac{2}{\pi t}}\int_0^x \mathrm{e}^{-z^2/(2t)}\mathrm{d}z. \tag{1.2.9}$$

2. *Distribution of the first passage time*. Let $a > 0$ be an arbitrary point on the right-half of the real line $\mathbb{R}$. Let $T_a = \inf\{t : w(t) > a\}$ be the instant of first passing through the point $a$ of the Wiener process $w(t)$. Then the random variable $T_a$ has the density (for $x > 0$):

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}\{T_a < x\} = \frac{a}{\sqrt{2\pi x^3}}\mathrm{e}^{-a^2/(2x)}, \quad x > 0. \tag{1.2.10}$$

3. *Joint distribution of the maximum and of the value of Brownian motion*. Let $a > 0$ be an arbitrary point on the right-half of the real line $\mathbb{R}$. Then for $x < a$ the following relation holds:

$$\mathbb{P}\left\{\sup_{0\le s\le t} w(s) < a, w(t) < x\right\} = \frac{1}{\sqrt{2\pi t}}\int_{x-2a}^{x} \mathrm{e}^{-z^2/(2t)}\mathrm{d}z, \quad x < a, a > 0. \tag{1.2.11}$$

4. *Arcsine law*. Consider the occupation time functional

$$\mathfrak{h}_T := \frac{1}{T}\int_0^T H(w(t))\,\mathrm{d}t, \quad T > 0, \tag{1.2.12}$$

where $H = H(x)$ is the Heaviside unit step function, i. e.,

$$H(x) = \begin{cases} 1, \ x > 0, \\ 0, \ x \le 0, \end{cases} x \in \mathbb{R}.$$

So, $\mathfrak{h}_T \in [0, 1]$ is the proportion of time spent by the Brownian motion $(w(t), 0 \le t \le T)$ on the positive semi-axis. The distribution of the random variable $\mathfrak{h}_T$ does not depend on $T$ (this follows from the self-similarity of the Brownian motion and the fact that $H(\alpha x) \equiv H(x)$ for any $\alpha > 0$) and is given by the classic *arcsine law*,

$$\mathbb{P}\{\mathfrak{h}_T < y\} = \frac{2}{\pi} \arcsin \sqrt{y}, \qquad 0 \le y \le 1, \tag{1.2.13}$$

with the probability density

$$p_{\text{arcsine}}(y) := \frac{1}{\pi\sqrt{y(1-y)}}, \quad 0 < y < 1. \tag{1.2.14}$$

We will continue the description of occupation time functionals in Chap. 3 (for the telegraph processes). Let $\mathscr{B}$ be the $\sigma$-algebra of the Borel subsets of a line $\mathbb{R}$. A real-valued Markov process $X = X(t), t \ge 0$, with the transition probability function $P(\Gamma, t, x, s)$, where $t > s \ge 0$, $x \in \mathbb{R}$, $\Gamma \in \mathscr{B}$ is referred to as the *diffusion process* on the real line $\mathbb{R}$, if the following conditions are fulfilled:

1. For all $\varepsilon > 0, x \in \mathbb{R}, t \ge 0$

$$\lim_{\Delta t \to +0} \frac{1}{\Delta t} \int\limits_{|y-x|>\varepsilon} P(\mathrm{d}y, t + \Delta t, x, t) = 0. \tag{1.2.15}$$

2. There exist functions $\mu = \mu(x, t)$ and $\sigma = \sigma(x, t)$, such that for all $\varepsilon > 0$, $x \in \mathbb{R}, t \ge 0$

$$\lim_{\Delta t \to +0} \frac{1}{\Delta t} \int\limits_{|y-x|<\varepsilon} (y - x) P(\mathrm{d}y, t + \Delta t, x, t) = \mu(x, t), \tag{1.2.16}$$

$$\lim_{\Delta t \to +0} \frac{1}{\Delta t} \int\limits_{|y-x|<\varepsilon} (y - x)^2 P(\mathrm{d}y, t + \Delta t, x, t) = \sigma^2(x, t). \tag{1.2.17}$$

Functions $\mu = \mu(x, t)$ and $\sigma = \sigma(x, t)$ defined by formulas (1.2.16) and (1.2.17) are called the *drift* and the *diffusion* coefficients, respectively.

Diffusion processes are closely related to partial differential equations of parabolic type. Let a diffusion process $X = X(t)$ be such that functions $\mu = \mu(x, t)$ and $\sigma = \sigma(x, t)$ are bounded and continuous.

We define the differential operator L,

$$\mathrm{L} f(x, s) := \frac{1}{2}\sigma^2(x, s)\frac{\partial^2 f}{\partial x^2} + \mu(x, s)\frac{\partial f}{\partial x}, \quad s \ge 0. \tag{1.2.18}$$

Here function $f = f(x, s)$ is twice continuously differentiable in $x$.

Suppose that functions $\mu = \mu(x, t)$ and $\sigma = \sigma(x, t)$ satisfy the following conditions:

(a) $\sigma(x, t) \ge \delta > 0$ for all $x$ and $t > 0$;
(b) $\mu(x, t)$ and $\sigma(x, t)$ satisfy a Hölder condition in $x$ and $t$, that is, for some $\varepsilon > 0$

$$|\mu(x,t)-\mu(x',t')|+|\sigma^2(x,t)-\sigma^2(x',t')|\le C(|x-x'|^{\varepsilon}+|t-t'|^{\varepsilon})$$

for all $x, x' \in \mathbb{R}, t, t' > 0$. For any bounded continuous function $g : \mathbb{R} \to \mathbb{R}$ and any $t > 0$ consider

$$u(x,s) := \int_{-\infty}^{\infty} g(y)P(\mathrm{d}y, t, x, s), \quad s \in [0,t], \quad x \in \mathbb{R}.$$

Then function $u = u(x, s)$ satisfies the equation

$$-\frac{\partial u(x,s)}{\partial s} = \mathrm{L}u(x,s), \quad s < t, \quad x \in \mathbb{R}, \tag{1.2.19}$$

with the terminal condition

$$\lim_{s\uparrow t} u(x,s) = g(x).$$

Suppose that the transition probability function $P(\Gamma, t, x, s)$ has the density (with respect to Lebesgue measure in $\mathbb{R}$), that is, there exists a function $p(y, t, x, s)$, such that for all $0 \le s < t, x \in \mathbb{R}, \Gamma \in \mathscr{B}$,

$$P(\Gamma,t,x,s) = \int_{\Gamma} p(y,t,x,s)\mathrm{d}y,$$

and this transition density $p(y, t, x, s)$ is sufficiently smooth with respect to $(y, t)$.

If the limiting relations (1.2.15), (1.2.16) and (1.2.17) are fulfilled uniformly with respect to $x \in \mathbb{R}$ and functions $\mu = \mu(x, t)$ and $\sigma = \sigma(x, t)$ have two partial derivatives with respect to $x$, which are bounded and satisfy a Hölder condition (with respect to $x$), then there exist continuous derivatives

$$\frac{\partial p(y,t,x,s)}{\partial t}, \quad \frac{\partial}{\partial y}(\mu(y,t)p(y,t,x,s)), \quad \frac{\partial^2}{\partial y^2}(\sigma^2(y,t)p(y,t,x,s)),$$

and the density $p(y, t, x, s)$ satisfies the equation

$$\frac{\partial p(y,t,x,s)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(y,t)p(y,t,x,s)) - \frac{\partial}{\partial y}(\mu(y,t)p(y,t,x,s)), \tag{1.2.20}$$

for all $y$ and $t$.

Equation (1.2.19) is in the backward variables $(x, s)$. Hence this equation is called *backward Kolmogorov equation*. Equation (1.2.20) (in the forward variables $(y, t)$) is referred to as *forward Kolmogorov equation*. It is also known as *Fokker-Planck equation*. The differential operator L is called the generator of diffusion process $X$.

Equations (1.2.19) and (1.2.20) are parabolic which implies the "parabolic" comportment of diffusion processes, i. e. the infinite propagation speed and lack of memory. In contrast, in the next chapter we will introduce and study a stochastic process with *finite speed* which is characterised by a bounded propagation velocity and is described by hyperbolic PDEs (so-called *Cattaneo system* and *telegraph equation*).

## 1.3 Stochastic Integrals and Itô's Formula

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P})$ be the filtered probability space. Suppose $w(t)$, $t \geq 0$ is an $\mathfrak{F}_t$-adapted Brownian motion. Let $X = X(t)$ be a stochastic process on this probability space. Our goal is to define the stochastic integral $\int_0^t X(s)\mathrm{d}w(s)$. It is known that the trajectories of $w$ are of unbounded variation with probability 1. Thus the integral $\int_0^t X(s)\mathrm{d}w(s)$ cannot be defined as a Stieltjes integral.

First, consider the integral of a *simple adapted process*. Process $X = X(t)$ is called the simple adapted if there exists a partition $0 = t_0 < t_1 < \cdots < t_n = t$ and random variables $\xi_0, \xi_1, \ldots, \xi_{n-1}$, such that $X(t_0) = \xi_0$ and $X(t) = \xi_i$ for $t \in (t_i, t_{i+1}]$, $i = 0, 1, 2, \ldots, n-1$, where $\xi_i$ is $\mathfrak{F}_{t_i}$-measurable and $\mathbb{E}|\xi_i|^2 < \infty$, $i = 0, 1, \ldots, n-1$. That is

$$X(t) = \sum_{i=0}^{n-1} \xi_i \mathbb{1}\{t \in (t_i, t_{i+1}]\}. \tag{1.3.1}$$

If $X = X(t)$ is a simple adapted process, the *stochastic integral* or *Itô integral* is defined by

$$\int_0^t X(s)\mathrm{d}w(s) = \sum_{i=0}^{k-1} \xi_i \left(w(t_{i+1}) - w(t_i)\right) + \xi_k \left(w(t) - w(t_k)\right), \quad t \in (t_k, t_{k+1}].$$

The stochastic integrals of simple processes have the following properties, which can be carried over to the stochastic integrals of general adapted processes.

1. *Linearity.* For any simple adapted processes $X$ and $Y$ and constants $\alpha$ and $\beta$

$$\int_0^t (\alpha X(s) + \beta Y(s))\,\mathrm{d}w(s) = \alpha \int_0^t X(s)\mathrm{d}w(s) + \beta \int_0^t Y(s)\mathrm{d}w(s).$$

2. For the indicator of an interval $(a, b]$

$$\int_0^t \mathbb{1}\{s \in (a, b]\}\mathrm{d}w(s) = w(b) - w(a), \quad \int_0^t \mathbb{1}_{\{s\in(a,b]\}} X(s)\mathrm{d}w(s) = \int_a^b X(s)\mathrm{d}w(s).$$

3. *Zero mean.* $\mathbb{E}\left(\int_0^t X(s)\mathrm{d}w(s)\right) = 0.$
4. *Isometry.*

$$\mathbb{E}\left(\int_0^t X(s)\mathrm{d}w(s)\right)^2 = \int_0^t \mathbb{E}\,(X(s))^2\,\mathrm{d}s.$$

*Proof* Properties 1 and 2 immediately follow from the definition. Since $\xi_i$ is $\mathfrak{F}_{t_i}$-measurable (depends on the values of $w(t)$, $t \le t_i$) and the increments $w(t_{i+1}) - w(t_i)$ are $\mathfrak{F}_{t_i}$-independent, then

$$\begin{aligned}\mathbb{E}\left(\int_0^t X(s)\mathrm{d}w(s)\right) &= \sum_{i=0}^{k-1} \mathbb{E}\left\{\xi_i \mathbb{E}\left(w(t_{i+1}) - w(t_i) \mid \mathfrak{F}_{t_i}\right)\right\} \\ &\quad + \mathbb{E}\left\{\xi_k \mathbb{E}\left(w(t) - w(t_k) \mid \mathfrak{F}_{t_k}\right)\right\} \\ &= \sum_{i=0}^{k-1} \mathbb{E}\left\{\xi_i\right\} \mathbb{E}\left(w(t_{i+1}) - w(t_i)\right) \\ &\quad + \mathbb{E}\left\{\xi_k\right\} \mathbb{E}\left(w(t) - w(t_k)\right) = 0,\end{aligned}$$

proving property 3.

The isometry property 4 can be proved in the similar way by conditioning on $\mathfrak{F}_{t_i}$, see [2]. □

To define Itô integral for more general processes the following result is applied.

**Theorem 1.1** *Let* $X = X(t)$ *be a regular (continuous) adapted process such that* $\mathbb{E}\left(\int_0^t |X(s)|^2 \mathrm{d}s\right) < \infty$. *Then*

- *There exists a sequence* $\{X^n(\cdot)\}$ *of simple processes such that*

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^t \left|X^n(s) - X(s)\right|^2 \mathrm{d}s\right) = 0.$$

- *The sequence of integrals* $\int_0^t X^n(s)\mathrm{d}w(s)$ *converges in probability. The limit is said to be the integral* $\int_0^t X(s)\mathrm{d}w(s)$.
- *The stochastic integral defined above satisfies properties* 1–4.

*Proof* Only an outline of the proof is given. Let $\{t_i^n\}_{i=0}^n$ be a partition with $\max_i(t_{i+1}^n - t_i^n) \to 0$, as $n \to \infty$. First, define the sequence $\{X^n(\cdot)\}$ of simple processes by means of (1.3.1) with

$$\xi_i = \xi_i^n = (t_{i+1} - t_i)^{-1} \int_{t_i}^{t_{i+1}} X(s)\mathrm{d}s, \quad i = 0, 1, 2\ldots, n-1.$$

Second, the Itô integral $\int_0^t X(s)\mathrm{d}w(s)$ is approximated by the sequence of sums

$$\sum_{i=0}^{n-1} X(t_i^n)\left(w(t_{i+1}^n) - w(t_i^n)\right),$$

where $\{t_i^n\}$ is a partition with $\max_i(t_{i+1}^n - t_i^n) \to 0$ as $n \to \infty$.

For details of the proof see e.g. [2].

One of the main tools of stochastic calculus is Itô's formula which gives the rule of the change of variables.

**Theorem 1.2** *If $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function, then for any $t \ge 0$*

$$f(w(t)) = f(0) + \int_0^t f'(w(s))\mathrm{d}w(s) + \frac{1}{2}\int_0^t f''(w(s))\mathrm{d}s. \tag{1.3.2}$$

More generally, Itô's rule can be formulated in the following form.

**Theorem 1.3** *If function $F : \mathbb{R} \times [0,\infty) \to \mathbb{R}$, $F = F(x,t)$ is continuously differentiable in the second variable $t$ and twice continuously differentiable in the first variable $x$, then for any $t \ge 0$*

$$\begin{aligned} F(w(t),t) =& F(0,0) + \int_0^t \frac{\partial F}{\partial x}(w(s),s)\mathrm{d}w(s) \\ &+ \int_0^t \left[\frac{\partial F}{\partial t}(w(s),s) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(w(s),s)\right]\mathrm{d}s. \end{aligned} \tag{1.3.3}$$

The proofs see in e.g. [2].

## 1.4 Poisson Process

Consider the following example of Markov process. The counting Poisson point process will be used in the forthcoming analysis of the telegraph random process and its applications. In this section we give a definition of the Poisson point process and recall its most important properties which we will substantially be relying on.

Consider a homogeneous stochastic flow $\{\xi(t), t \ge 0\}$ of events that occur at random time instants $\tau_1 < \tau_2 < \ldots$, $\tau_0 = 0$. By definition, a *counting process* is

$$N(t) := \max\{n : \tau_n \le t\}, \quad N(0) = 0.$$

**Definition 1.2** The stochastic flow $\{\xi(t), t \ge 0\}$ (and the counting process $N = N(t)$ as well) is referred to as the *Poisson process*, if it possesses the following properties:

1. *Stationarity.* The distribution of the number of events in any time interval $(t, t+\tau)$ depends only on its length $\tau$ but not on $t$.
2. *Lack of memory.* The distribution of the number of events in any time interval $(t, t+\tau)$ is independent of the distribution of the past, i.e. before time $t$. This implies that the conditional distribution of these events in $(t, t+\tau)$, under any assumptions regarding the number of these events that have occurred before the moment $t$, coincides with the unconditional probability.
3. *Ordinarity.* Counting process $N(t)$ obeys the following infinitesimal properties,

$$\begin{aligned} &\mathbb{P}\{N(t+\Delta t)-N(t)=0\}=1-\lambda\Delta t+o(\Delta t),\\ &\mathbb{P}\{N(t+\Delta t)-N(t)=1\}=\lambda\Delta t+o(\Delta t),\\ &\mathbb{P}\{N(t+\Delta t)-N(t)\geq 2\}=o(\Delta t),\quad \Delta t\to 0, \end{aligned} \tag{1.4.1}$$

where $\lambda > 0$ is some positive number.

Let us split $(0, t]$ into "small" intervals $((k-1)t/n, kt/n]$. Due to (1.4.1) in this interval only one event of the stochastic flow occurs with probability $p_n = \lambda t/n + o(1/n)$. Then the number of occurrences $N_n(t)$ in $(0, t]$ is binomially distributed:

$$\mathbb{P}\{N_n(t)=k\}=\binom{n}{k}p_n^k(1-p_n)^{n-k}.$$

Hence, as $np_n \to \lambda t$, the latter converges in distribution to the Poisson distribution with parameter $\lambda t$,

$$N_n(t)\xrightarrow{d} Po(\lambda t),\quad n\to\infty.$$

Similarly, for disjoint intervals $[t_{k-1}, t_k]$, $k = 1, \dots, d$, we have by independence (see point 2 of Definition 1.2)

$$\begin{aligned} &\mathbb{P}\left\{N_n(t_1)=k_1,\ \ N_n(t_2)-N_n(t_1)=k_2,\dots N_n(t_d)-N_n(t_{d-1})=k_d\right\}\\ &\qquad\to\frac{(\lambda t_1)^{k_1}}{k_1!}\mathrm{e}^{-\lambda t_1}\cdot\dots\cdot\frac{(\lambda(t_d-t_{d-1}))^{k_d}}{k_d!}\mathrm{e}^{-\lambda(t_d-t_{d-1})}. \end{aligned}$$

Therefore the Poisson process $N = N(t), t \geq 0$ is a stochastic process with independent increments and

$$\mathbb{P}\{N(t)-N(s)=k\}=\frac{[\lambda(t-s)]^k}{k!}\mathrm{e}^{-\lambda(t-s)},\quad k\geq 0,\quad 0\leq s<t,\lambda>0. \tag{1.4.2}$$

Parameter $\lambda$ is called the *intensity* of $N = N(t), t \geq 0$.

The time intervals $\eta_k := \tau_k - \tau_{k-1}$, $\tau_0 := 0$ between arrivals of two successive Poisson events are independent exponentially distributed random variables:

$$\mathbb{P}\{\eta_k>t\}=\mathrm{e}^{-\lambda t},\quad t\geq 0.$$

Arrival times $\tau_n$ have an Erlang distribution with the density (see e.g. [3])

$$p_{\tau_n}(t) = \frac{(\lambda t)^{n-1}}{(n-1)!}\lambda \mathrm{e}^{-\lambda t}, \quad n = 1, 2, \ldots \tag{1.4.3}$$

To describe the conditional distribution of arrival times $\{\tau_k\}_{k=1}^{\infty}$ notice that for any $t > 0$ and any $n \in \mathbb{N}$

$$\{\tau_1, \ldots, \tau_n \mid N(t) = n\} \ \stackrel{d}{=} \ \left\{Z_{(1)}, \ldots, Z_{(n)}\right\},$$

where $Z_1, \ldots, Z_n$ are independent and uniformly distributed on $[0, t]$ random variables, and $\{Z_{(1)}, \ldots, Z_{(n)}\}$ is the order statistics. Hence

$$\mathbb{P}\{\tau_1 \in \mathrm{d}t_1, \ldots, \tau_n \in \mathrm{d}t_n \mid N(t) = n\} = \frac{n!}{t^n}\,\mathrm{d}t_1 \ldots \mathrm{d}t_n.$$

In particular, for $n = 1$ we have the uniform distribution,

$$\mathbb{P}\{\tau_1 \in \mathrm{d}t_1 \mid N(t) = 1\} = \frac{1}{t}\,\mathrm{d}t_1.$$

## 1.5 Modified Bessel Functions

In this section we present a brief survey of modified Bessel functions $I_\nu(z)$ and their properties which will be used in this book.

The modified Bessel function $I_\nu(z)$ satisfies the Bessel equation

$$z^2\frac{\mathrm{d}^2u}{\mathrm{d}z^2} + z\frac{\mathrm{d}u}{\mathrm{d}z} - \left(z^2 + \nu^2\right)u = 0, \tag{1.5.1}$$

and it has the following series representation

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\nu+1)}\left(\frac{z}{2}\right)^{2k}, \quad \nu \in (-\infty, +\infty). \tag{1.5.2}$$

From series representation (1.5.2) it follows that

$$I_0(0) = 1, \quad I_\nu(0) = 0, \quad \nu > 0.$$

In this book only the case of integer $\nu$, $\nu = n$, and half-integer $\nu = n \pm \frac{1}{2}$, $n = 0, 1, 2, \ldots,$ will be used.

The important particular cases of modified Bessel functions (1.5.2) are given by

$$I_0(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{(k!)^2}, \quad I_1(z) = I_0'(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+1}}{k!(k+1)!}$$
$$\text{and} \;\; I_2(z) = -\frac{2}{z} I_1(z) + I_0(z). \tag{1.5.3}$$

One can easily check also that for $\nu = 1/2$ formula (1.5.2) yields:

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sinh z. \tag{1.5.4}$$

From (1.5.3) the useful estimates follow

$$I_0(z) \le e^z, \quad \frac{I_1(z)}{z} \le \frac{1}{2} e^z, \quad z \ge 0. \tag{1.5.5}$$

Indeed,

$$I_0(z) = \sum_{k=0}^{\infty} \left( \frac{(z/2)^k}{k!} \right)^2 \le \left( \sum_{k=0}^{\infty} \frac{(z/2)^k}{k!} \right)^2 = e^z$$

and

$$\frac{I_1(z)}{z} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} \left( \frac{z}{2} \right)^{2k} \le \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left( \frac{z}{2} \right)^{2k} = \frac{1}{2} I_0(z) \le \frac{1}{2} e^z.$$

We also note the integral representation of the form (see [4, 9.6.18, p.376])

$$I_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu + 1/2)\sqrt{\pi}} \int_{-1}^{1} (1 - \xi^2)^{\nu - 1/2} \cosh(\xi z) d\xi, \quad \nu > -\frac{1}{2}. \tag{1.5.6}$$

In particular , if $\nu = 0$,

$$I_0(z) = \frac{1}{\pi} \int_{-1}^{1} (1 - \xi^2)^{-1/2} \cosh(\xi z) d\xi = \frac{1}{\pi} \int_{-1}^{1} (1 - \xi^2)^{-1/2} e^{\xi z} d\xi. \tag{1.5.7}$$

The asymptotic behaviour of Bessel functions at infinity is given by the formula (see [4, 9.7.1, p. 377]):

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} (1 + O(1)), \quad z \to +\infty. \tag{1.5.8}$$

Formula (1.5.8) expresses the fact that the modified Bessel function $I_\nu(z)$ tends to infinity, as $z \to +\infty$, like $z^{-1/2} e^z$ and the first term in the asymptotic expansion of this function does not depend on index $\nu$.

*Remark 1.1* The Bessel function $J_\nu(z)$ of real argument is defined as the solution of

$$z^2 \frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + z \frac{\mathrm{d}u}{\mathrm{d}z} + \left(z^2 - \nu^2\right) u = 0. \tag{1.5.9}$$

Function $J_\nu(z)$ is given by the following series representation

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k}. \tag{1.5.10}$$

It is easy to see that functions $I_\nu$ and $J_\nu$ are connected with each other by the relation

$$I_\nu(z) = e^{-i\nu\pi/2} J_\nu(e^{-i\pi/2} z), \quad -\pi < \arg z \le \frac{\pi}{2}. \tag{1.5.11}$$

For $\nu = n$ integer, formula (1.5.11) takes the form

$$I_n(z) = i^{-n} J_n(iz), \quad n = 0, 1, 2, \ldots \tag{1.5.12}$$

## 1.6 Generalised Functions and Integral Transforms

In this book our needs in generalised functions (distributions) are extremely modest, but seeking the completeness of the presentation we describe briefly some definitions and results, which will be used below. Consider the linear space $\Phi$ of functions $\varphi = \varphi(x), x \in \mathbb{R}$ supplied with the topology defined by the convergence of the sequences $\{\varphi_n\}$ of elements of this space. This space will be called a *fundamental space*, its elements are *fundamental* or *test* functions.

The conjugate (dual) space $\Phi'$ of *continuous functionals* defined on $\Phi$ is called a space of generalised functions (distributions), see [5]. The sequence of generalised functions $f_n$ is said to be convergent to generalised function $f$, if for any $\varphi \in \Phi$

$$< f_n, \varphi > \to < f, \varphi >, \quad \text{as } n \to \infty.$$

### Examples of fundamental spaces

- *The space $\Phi = \mathscr{D}(\mathbb{R})$ of all infinitely differentiable functions $\varphi = \varphi(x), x \in \mathbb{R}$* with compact support.

The topology is defined by the convergence of sequences in $\mathscr{D}$. The sequence $\varphi_n \in \mathscr{D}$ converges to zero, if all the functions $\varphi_n(x)$ vanish for $|x| > R$ (with the common $R$) and the sequence $\{\varphi_n\}$ converges uniformly to zero together with their derivatives, for any $\alpha = 0, 1, 2, \ldots$

$$\varphi_n^{(\alpha)}(x) \rightrightarrows 0, \quad |x| \le R,$$

where $\varphi_n^{(\alpha)}$ is the derivative of order $\alpha$.

Such a topology can be defined as a countable set of the norms

$$\|\varphi\|_k = \sup_{|x|\le R, 0\le\alpha\le k} |\varphi^{(\alpha)}(x)|, \quad k = 0, 1, 2, \ldots$$

The conjugate space $\mathscr{D}'$ of continuous functionals defined on $\mathscr{D}$ is named as the Gelfand space of generalised functions.

Locally integrable function $f = f(x)$, $x \in \mathbb{R}$ (that is, integrable on any compact set in $\mathbb{R}$) generates some generalised function. The expression

$$< f, \varphi >= \int_{-\infty}^{\infty} f(x)\varphi(x)\mathrm{d}x \tag{1.6.1}$$

is a linear continuous functional on $\mathscr{D}(\mathbb{R})$. Such distributions are called *regular*. Note that, since the function $\varphi(x)$ has a compact support, the integration in (1.6.1) is doing, in fact, on a compact set.

Another example is given by the $\delta$-function. For any $x_0 \in \mathbb{R}$ define the continuous linear functional on $\mathscr{D}(\mathbb{R})$

$$< \delta(x - x_0), \varphi >= \varphi(x_0). \tag{1.6.2}$$

- *The space $\Phi = \mathscr{S}(\mathbb{R})$ of all infinitely differentiable functions $\varphi = \varphi(x)$, $x \in \mathbb{R}$, which tend to zero for $|x| \to \infty$ together with the derivatives of all orders faster than any power of $1/|x|$.*

The convergence of sequence $\{\varphi_n\}$ to zero means $\forall \alpha, \beta \in \mathbb{Z}_+$

$$|x|^\beta \varphi_n^{(\alpha)}(x) \rightrightarrows 0, \qquad \text{as } n \to \infty$$

or, equivalently, the topology is defined by the sequence of norms,

$$\|\varphi\|_k = \sup_{\beta,\alpha\le k} |x^\beta \varphi^{(\alpha)}(x)|, \quad k = 0, 1, 2, \ldots$$
$$\alpha, \beta = 0, 1, 2, \ldots$$

The conjugate space $\mathscr{S}'(\mathbb{R})$ is named the space of *tempered distributions*. The space $\mathscr{S}(\mathbb{R})$ is called the Schwartz space.

Readily, $\mathscr{D}(\mathbb{R}) \subset \mathscr{S}(\mathbb{R})$ and hence $\mathscr{D}'(\mathbb{R}) \supset \mathscr{S}'(\mathbb{R})$.

Let us define some linear operations. We can justify the new operations first with regular distribution 1.6.1. For example, if $f$ is locally integrable and $\beta = \beta(x)$ is continuous function then for any $\varphi \in \mathscr{D}(\mathbb{R})$ we have the identity

$$\int_{-\infty}^{\infty} [\beta(x) f(x)]\, \varphi(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\, [\beta(x)\varphi(x)]\, \mathrm{d}x.$$

So the following definition looks reasonable. If $\beta = \beta(x)$ is an infinitely differentiable function then the operation of multiplication of generalised functions by function $\beta$ is defined by the equality: for any fundamental function $\varphi$

$$< \beta f, \varphi >=< f, \beta\varphi >$$

(in the case of tempered distributions infinitely differentiable function $\beta$ is assumed to be bounded).

In the same manner we can define the differentiation. Let function $f = f(x)$ be locally integrable and locally differentiable (i.e. the derivative $f'(x)$ exists a.e.). Then integrating by parts we have

$$\int_{-\infty}^{\infty} f'(x)\varphi(x)\mathrm{d}x = -\int_{-\infty}^{\infty} f(x)\varphi'(x)\mathrm{d}x \quad \text{for any test-function } \varphi.$$

For arbitrary distribution $f$ its derivatives are defined as follows: for any test function $\varphi$

$$< f^{(\alpha)}, \varphi >= (-1)^{\alpha} < f, \varphi^{(\alpha)} >, \quad \alpha = 0, 1, 2, \dots \tag{1.6.3}$$

All generalised functions are *infinitely differentiable* in this sense.

We need also the integral transforms. For arbitrary regular function $f$ define the Fourier transformation,

$$\mathscr{F}_{x\to\xi}[f](\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{i\xi x} f(x)\mathrm{d}x, \quad \xi \in \mathbb{R}.$$

By this definition and Fubini's theorem for any integrable function $f$ and for any test-function $\varphi \in \mathscr{S}(\mathbb{R})$ we have

$$\int_{-\infty}^{\infty} \mathscr{F}_{x\to\xi}[f](\xi)\varphi(\xi)\mathrm{d}\xi = \int_{-\infty}^{\infty} \varphi(\xi)\left\{\int_{-\infty}^{\infty} \mathrm{e}^{i\xi x} f(x)\mathrm{d}x\right\}\mathrm{d}\xi$$
$$= \int_{-\infty}^{\infty} f(x)\left\{\int_{-\infty}^{\infty} \mathrm{e}^{i\xi x}\varphi(\xi)\mathrm{d}\xi\right\}\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\mathscr{F}_{\xi\to x}[\varphi](x)\mathrm{d}x.$$

It is easy to demonstrate that the space $\mathscr{S}(\mathbb{R})$ is close with respect to Fourier transformation: if $\varphi \in \mathscr{S}(\mathbb{R})$ then $\widehat{\varphi} \in \mathscr{S}(\mathbb{R})$. Here $\widehat{\varphi} = \widehat{\varphi}(x) = \mathscr{F}[\varphi](x) = \int_{-\infty}^{\infty} \mathrm{e}^{i\xi x}\varphi(\xi)\mathrm{d}\xi$.

We define the Fourier transform $\mathscr{F}[f] = \widehat{f}$ of tempered distribution $f \in \mathscr{S}'(\mathbb{R})$ as

$$< \widehat{f}, \varphi >=< f, \widehat{\varphi} >, \quad \varphi, \widehat{\varphi} \in \mathscr{S}(\mathbb{R}). \tag{1.6.4}$$

The inverse Fourier transform of the test-function $\varphi$ is given by the formula

$$\mathscr{F}^{-1}[\varphi](x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \mathrm{e}^{-i\xi x} \varphi(\xi) \mathrm{d}\xi, \tag{1.6.5}$$

and as is easy to see that $\mathscr{F}^{-1}[\widehat{\varphi}] = \varphi$.

Defining the inverse transformation in $\mathscr{S}'(\mathbb{R})$ as

$$< \mathscr{F}^{-1}[f], \varphi >=< f, \mathscr{F}^{-1}[\varphi] >$$

we obtain the one-to-one correspondence $f(x) \longleftrightarrow \widehat{f}(\boldsymbol{\xi})$ between tempered generalised functions $f \in \mathscr{S}'$ and their Fourier transforms $\widehat{f}$.

Let us now give some important properties of the Fourier transformations of generalised functions.

1. *Fourier transformation of the derivatives*: for arbitrary tempered distribution $f \in \mathscr{S}'$

$$\mathscr{F}_{x\to\xi}[f^{(\alpha)}(x)](\boldsymbol{\xi}) = (-i\xi)^{\alpha} \mathscr{F}_{x\to\xi}[f(x)](\xi), \quad \alpha = 0, 1, 2, \ldots$$

   Here $f^{(\alpha)}$ is the $\alpha$-th derivative.
2. *Shift of the Fourier transformation*: for arbitrary tempered distribution $f \in \mathscr{S}'$

$$\mathscr{F}_{x\to\xi}[f(x)](\xi + \xi_0) = \mathscr{F}_{x\to\xi}[\mathrm{e}^{i\xi_0 x} f(x)](\xi).$$

3. *Fourier transformation of the similarity*: for arbitrary tempered distribution $f \in \mathscr{S}'$

$$\mathscr{F}_{x\to\xi}[f(cx)](\boldsymbol{\xi}) = \frac{1}{|c|} \mathscr{F}_{x\to\xi}[f(x)]\,(\xi/c)\,.$$

We will also need a definition of another integral transformation of a one-dimensional generalised function, namely the Laplace transformation. It is a continuous integral transformation defined on the complex plane $\mathbb{C}$.

Let $f$ be a locally integrable function on the line such that $f(t) = 0$ for $t < 0$. Moreover, we assume that $|f(t)| < A\mathrm{e}^{at}$ for $t \geq T$ for some positive constants $A, T$ and $a$. In other words, the absolute value of the function $f$ should not increase at infinity faster than some exponential function with a fixed rate. Under this condition the integral

$$\mathscr{L}_{t\to s}[f(t)] = \widetilde{f}(s) = \int_{0}^{\infty} \mathrm{e}^{-st} f(t) \mathrm{d}t, \quad s = \sigma + i\xi, \tag{1.6.6}$$

exists for $\mathrm{Re}\, s = \sigma > a$.

Formula (1.6.6) determines a continuous transformation of the real-valued function $f$ defined on the real line $\mathbb{R}$ into a complex function $\widetilde{f}(s) = \mathscr{L}_{t\to s}[f](s)$ defined in the right half-plane $\mathbb{C}^+$ of the complex plane $\mathbb{C}$. The transformation (1.6.6) is referred to as the *Laplace transformation* of the locally integrable function $f$. The function $\widetilde{f}(s)$ is the holomorphic (analytical) function in the right half-plane $\mathrm{Re}s = \sigma > a > 0$ such that $\widetilde{f}(s) \to 0$, as $\sigma \to \infty$, uniformly with respect to $\xi$.

The integral relation (1.6.6) in terms of Fourier transformation takes the form:

$$\mathscr{L}_{t\to s}[f](s) = \mathscr{F}_{x\to\xi}[\mathrm{e}^{-\sigma t} f(t)](-\xi), \quad \sigma > a. \tag{1.6.7}$$

Denote by $\mathscr{S}'_+(a)$ the set of all tempered distributions $f \in \mathscr{S}'$ such that $f|_{t<0} = 0$ and $\mathrm{e}^{-\sigma t} f(t)$ is a tempered distribution for all $\sigma > a$. Formula (1.6.7) defines the Laplace transform of any tempered generalised function $f \in \mathscr{S}'_+(a)$.

Note that there exists the one-to-one correspondence $f(t) \longleftrightarrow \widetilde{f}(s)$ between generalised function $f \in \mathscr{S}'_+$ and its Laplace transform $\widetilde{f}(s)$.

Obviously, the Laplace transformation (1.6.6) (or (1.6.7)) is a linear operation, that is, if $f_1(t) \longleftrightarrow \widetilde{f_1}(s), \sigma > a_1$, and $f_2(t) \longleftrightarrow \widetilde{f_2}(s), \sigma > a_2$, then

$$\lambda f_1(t) + \mu f_2(t) \longleftrightarrow \lambda \tilde{f_1}(s) + \mu \tilde{f_2}(s), \quad \sigma > \max(a_1, a_2).$$

Let us now give some important properties of the Laplace transformation.

1. *Laplace transformation of the derivatives*: for arbitrary generalised function $f \in \mathscr{S}'_+(a)$

$$\mathscr{L}_{t\to s}[f^{(n)}(t)](s) = s^n \mathscr{L}_{t\to s}[f(t)](s), \quad \sigma > a, \quad n = 0, 1, \dots .$$

2. *Shift of the Laplace transformation*: for arbitrary generalised function $f \in \mathscr{S}'_+(a)$

$$\mathscr{L}_{t\to s}[e^{\lambda t} f(t)](s) = \mathscr{L}_{t\to s}[f(t)](s - \lambda), \quad \sigma > a + \mathrm{Re}\lambda.$$

3. *Laplace transformation of the similarity*: for arbitrary generalised function $f \in \mathscr{S}'_+(a)$

$$\mathscr{L}_{t\to s}[f(kt)](s) = \frac{1}{k} \mathscr{L}_{t\to s}[f(t)]\,(s/k)\,, \quad \sigma > ka.$$

The main example of the generalised function which will be exploited throughout all the book is the $\delta$-function. Due to definition (1.6.2), for any test function $\varphi$ we have

$$< \delta(ct \pm x), \varphi >_x = \varphi(ct), \quad < \delta(ct \pm x), \varphi >_t = \frac{1}{c}\varphi(x/c). \tag{1.6.8}$$

Notice that $\delta$-function is the derivative of the unit step function. Applying definition of the derivative (1.6.3) to the Heaviside unit step function we obtain

$$< \theta'(x - x_0), \varphi > = - < \theta(x - x_0), \varphi' > \tag{1.6.9}$$
$$= - \int_{x_0}^{\infty} \varphi'(x)\mathrm{d}x = \varphi(x_0) =< \delta(x - x_0), \varphi > .$$

Hence $\theta'(x - x_0) = \delta(x - x_0)$.

Moreover, for any test-function $\varphi$

$$< \delta^{(\alpha)}(x - x_0), \varphi >= (-1)^{\alpha}\varphi^{(\alpha)}(x_0).$$

Fourier and Laplace transformations of $\delta$-function can be expressed directly by respective definitions and basic properties. By definitions (1.6.4) and (1.6.2),

$$< \widehat{\delta},\ \varphi >=< \delta, \widehat{\varphi} >= \int_{-\infty}^{\infty} \varphi(\xi)\mathrm{d}\xi.$$

Hence $\widehat{\delta} = 1$.

We are interested also in the Fourier and Laplace transformations of $\delta(x \pm ct)$. As is easy to see,

$$\mathscr{F}_{x\to\xi}\delta(x \pm ct) = \mathrm{e}^{\pm ict\xi}, \quad \mathscr{L}_{t\to s}\delta(x \pm ct) = \frac{1}{c}\mathrm{e}^{\pm sx/c}.$$

*Remark 1.2* Occasionally instead of $< \delta, \varphi >$ we will write down

$$\int_{-\infty}^{\infty} \delta(x)\varphi(x)\mathrm{d}x = \varphi(0).$$

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# Chapter 2
# Telegraph Process on the Line

**Abstract** We define the classic Goldstein-Kac telegraph process performed by a particle that moves on the real line with some finite constant speed and alternates between two possible directions of motion (positive or negative) at random homogeneous Poisson-paced time instants. We obtain the Kolmogorov equations for the joint probability densities of the particle's position and its direction at arbitrary time instant. By combining these equations we derive the telegraph equation for the transition density of the motion. The characteristic function of the telegraph process is obtained as the solution of a respective Cauchy problem. The explicit form of the transition density of the process is given as a generalised function containing a singular and absolutely continuous parts. The convergence in distribution of the telegraph process to the homogeneous Brownian motion under Kac's scaling condition, is established. The explicit formulae for the Laplace transforms of the transition density and of the characteristic function of the telegraph process, are also obtained.

**Keywords** Telegraph process · Kolmogorov equations · Transition density · Characteristic function · Rescaling · Laplace transform

We begin with the classical case of random evolution, so-called Goldstein-Kac telegraph process on the real line. It is performed by a particle which starts at time $t = 0$ from the origin and moves with some finite constant speed $c$ on the line $(-\infty, \infty)$, taking an initial direction of the motion (positive or negative) with equal probabilities $1/2$. The motion is controlled by a homogeneous Poisson process of a constant rate $\lambda > 0$. When a Poisson event occurs, the particle instantaneously takes the opposite direction and keeps moving with the same speed $c$ until the next Poisson event occurs, then it takes the opposite direction again, and so on.

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## 2.1 Definition of Process and the Structure of Distribution

We start our consideration with a general definition of a two-state Markov process.

On the filtered probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P})$ consider the Markov process $\varepsilon = \varepsilon(t) \in \{0, 1\},\ t \geq 0$ with transition intensity $\lambda,\ \lambda > 0$,

$$\mathbb{P}\{\varepsilon(t + \Delta t) \neq \varepsilon(t)\} = \lambda\Delta t + o(\Delta t), \qquad \Delta t \to +0. \tag{2.1.1}$$

Process $\varepsilon = \varepsilon(t)$ is assumed to be adapted to the filtration $\{\mathfrak{F}_t\}_{t\geq 0}$. The initial state $\varepsilon(0)$ is a random variable with the symmetric distribution, $\mathbb{P}\{\varepsilon(0) = 0\} = \mathbb{P}\{\varepsilon(0) = 1\} = 1/2$.

Definition (2.1.1) means that the point process of switching times $\tau_1 < \tau_2 < \ldots$ of Markov process $\varepsilon = \varepsilon(t)$ has independent and exponentially distributed increments: $\mathbb{P}\{\tau_{n+1} - \tau_n > t | \mathfrak{F}_{\tau_n}\} = \mathbb{P}\{\tau_{n+1} - \tau_n > t\} = \mathrm{e}^{-\lambda t},\ t \geq 0, \tau_0 = 0$.

Let $N = N(t),\ t \geq 0$ be a counting Poisson process, $N(t) = \max\{n : \tau_n \leq t\}$. From (2.1.1) it follows that the distribution of $N(t)$ has the form (see 1.4.2)

$$\mathbb{P}\{N(t) = k\} = \frac{(\lambda t)^k}{k!}\,\mathrm{e}^{-\lambda t}\,, \qquad t > 0,\ \ k = 0,\ 1,\ 2,\ \ldots \tag{2.1.2}$$

To define the Goldstein-Kac telegraph process it is convenient to use the direction process $D = D(t) := (-1)^{\varepsilon(t)},\ t \geq 0$. This is a two-state stochastic process taking the values $D(t) = +1$ (resp. $\varepsilon(t) = 0$) if the particle moves in the positive (forward) direction at time $t$, and $D(t) = -1$ (resp. $\varepsilon(t) = 1$) if it moves in the negative (backward) direction at this time. Notice that $D(t) = D(0)(-1)^{N(t)},\ t \geq 0$. The initial direction $D(0)$ is a random variable such that $\mathbb{P}\{D(0) = +1\} = \mathbb{P}\{D(0) = -1\} = \frac{1}{2}$.

Let $c > 0$ denote the constant speed. Then the particle's position $X(t)$ at arbitrary time $t > 0$ is given by the formula

$$X(t) := c\int_0^t D(s)\mathrm{d}s = cD(0)\int_0^t (-1)^{N(s)}\,\mathrm{d}s. \tag{2.1.3}$$

We define also the following two processes (with fixed initial directions, $D(0) = \pm 1$):

$$X^{\pm}(t) := \pm c\int_0^t (-1)^{N(s)}\,\mathrm{d}s. \tag{2.1.4}$$

Note that the distribution of $X(t)$ has two atoms at points $\pm ct$, which correspond to the case when no one Poisson event occurs till time $t$ and, therefore, the particle does not change its initial direction. It means that $N(t) = 0$ (with probability $\mathrm{e}^{-\lambda t}$). Therefore

$$\begin{aligned}\mathbb{P}\left\{X^+(t) = +ct\right\} &= \mathbb{P}\left\{X^-(t) = -ct\right\} = \mathrm{e}^{-\lambda t},\\ \mathbb{P}\left\{X(t) = +ct\right\} &= \mathbb{P}\left\{X(t) = -ct\right\} = \tfrac{1}{2}\,\mathrm{e}^{-\lambda t}.\end{aligned} \tag{2.1.5}$$

It is easy to see that

$$\mathbb{P}\{X(t) = x\} = 0 \qquad \forall x,\ x \neq \pm ct.$$

Hence the distribution function $F(x,\ t) = \mathbb{P}\{X(t) < x\},\ x \in (-\infty, \infty),\ t > 0$, is continuous on $\mathbb{R}^2_+ \setminus \{|x| = ct\}$, where $\mathbb{R}^2_+ := (-\infty,\ \infty) \times (0,\ \infty)$. Moreover, since the speed $c$ is finite, then $F(x,\ t) \equiv 0$, if $|x| > ct,\ t > 0$. By the same reasons the density

$$p(x,\ t) = \mathbb{P}\{X(t) \in \mathrm{d}x\}/\mathrm{d}x, \qquad x \in (-\infty, \infty), \quad t > 0, \tag{2.1.6}$$

has the support $[-ct,\ ct]$.

The term "density" is treated in the sense of generalised functions, $p \in \mathscr{D}'$. The distribution density $p = p(x,\ t)$ contains respective singular part. Furthermore,

$$\begin{aligned} p(x,\ t) = \frac{1}{2}\mathrm{e}^{-\lambda t}\left[\delta(x+ct) + \delta(x-ct)\right] \\ + P(x,t)\mathbb{1}_{\{|x|<ct\}}, \quad x \in (-\infty, \infty), \quad t > 0, \end{aligned} \tag{2.1.7}$$

where $\delta(x)$ is the Dirac's $\delta$-function, $P$ is the absolutely continuous part of the distribution with support $[-ct,\ ct]$, and $\mathbb{1}_A(x)$ is the indicator function

$$\mathbb{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases} \tag{2.1.8}$$

For any continuous function $\varphi\colon \mathbb{R} \to \mathbb{R}$ the expectation $\mathbb{E}\varphi(x + X(t))$ is

$$\mathbb{E}\varphi(x + X(t)) = \frac{1}{2}\mathrm{e}^{-\lambda t}\left[\varphi(x-ct) + \varphi(x+ct)\right] + \int_{-ct}^{ct} \varphi(x+y)P(y,\ t)\mathrm{d}y.$$

Fix the initial particle's direction $D(0) = \pm 1$.

Let $p_+(x,\ t)$ and $p_-(x,\ t)$ be the conditional distribution densities under fixed initial direction $D(0) = \pm 1$,

$$\begin{aligned} p_\pm(x,\ t) &= \mathbb{P}\{X^\pm(t) \in \mathrm{d}x\}/\mathrm{d}x \\ &= \mathbb{P}\{X(t) \in \mathrm{d}x \mid D(0) = \pm 1\}/\mathrm{d}x, \quad x \in (-\infty,\ \infty), \quad t > 0, \end{aligned} \tag{2.1.9}$$

such that $p(x,\ t) = \left(p_+(x,\ t) + p_-(x,\ t)\right)/2$. Similarly to (2.1.7) we conclude that

$$\begin{aligned} p_+(x,\ t) &= \mathrm{e}^{-\lambda t}\delta(x-ct) + P_+(x,\ t)\mathbb{1}_{\{|x|<ct\}}, \\ p_-(x,\ t) &= \mathrm{e}^{-\lambda t}\delta(x+ct) + P_-(x,\ t)\mathbb{1}_{\{|x|<ct\}}, \end{aligned} \tag{2.1.10}$$

where $P_+$ and $P_-$ are the absolutely continuous parts of these conditional distributions, $P(x,\ t) = \left(P_+(x,\ t) + P_-(x,\ t)\right)/2$.

The detailed description of functions $P(x, t)$, $P_{\pm}(x, t)$ defined in (2.1.7) and (2.1.10) is the main goal of forthcoming analysis.

## 2.2 Kolmogorov Equations

Let $\tau = \tau_1$ be the *first switching time* of the Markov process $\varepsilon = \varepsilon(t)$. Then for any $t > 0$ we have the following equality in conditional distribution (under fixed initial direction $D(0) = \sigma,\ \sigma = \pm 1$)

$$X(t) \overset{d}{=} \sigma ct \mathbb{1}_{\{\tau > t\}} + \left[\sigma c\tau + \tilde{X}(t-\tau)\right] \mathbb{1}_{\{\tau < t\}}. \tag{2.2.1}$$

Here the conditional distribution of $X(t)$ is considered under the fixed initial direction $D(0) = \sigma$. The telegraph process $\tilde{X} = \tilde{X}(t)$ starts at the opposite direction $-\sigma$ and is independent of $X$.

The distribution of $\tau$ is given by (1.4.3) (with $n = 1$), $\mathbb{P}\{\tau \in \mathrm{d}s\} = \lambda \mathrm{e}^{-\lambda s}\mathrm{d}s$. Hence equation (2.2.1) is equivalent to the following set of integral equations for conditional densities $p_{\pm}$ defined by (2.1.9),

$$\begin{aligned} p_+(x, t) &= \mathrm{e}^{-\lambda t}\delta(x - ct) + \int_0^t p_-(x - cs, t - s)\lambda \mathrm{e}^{-\lambda s}\mathrm{d}s, \\ p_-(x, t) &= \mathrm{e}^{-\lambda t}\delta(x + ct) + \int_0^t p_+(x + cs, t - s)\lambda \mathrm{e}^{-\lambda s}\mathrm{d}s, \end{aligned} \tag{2.2.2}$$

where $\delta = \delta(x)$ is the $\delta$-function. Here and thereafter we presume $\int_a^b \delta(t)\varphi(t)\mathrm{d}t = \varphi(0)$ for any continuous test-function $\varphi$ and $a < 0 < b$.

Conditioning on the *last switching time* we can get the integral equation for the joint distribution of the particle's position and its current direction, similar to Eq. (2.2.2). Let $\tau$ be the last switching instant till time $t$ (in the case of $N(t) > 0$), $\tau := \max\{\tau_n,\ n = 1, 2, \ldots \mid \tau_n < t,\ N(t) > 0\}$. Looking backwards, we deduce that

$$X(t) \overset{d}{=} D(t)ct\mathbb{1}_{\{N(t)=0\}} + [D(t)c(t-\tau) + X(\tau)]\,\mathbb{1}_{\{N(t)>0\}}. \tag{2.2.3}$$

It is known that for a Poisson process both forward and backward recurrence times are exponentially distributed, see e. g. [1]. Hence $t - \tau$ is exponentially distributed with parameter $\lambda$ as well.

Let $f(x, t)$ and $b(x, t)$ be the joint probability densities of the position and the direction of forward and backward moving particle, respectively,

$$\begin{aligned} f(x, t) &= \mathbb{P}\{X(t) \in \mathrm{d}x,\ D(t) = +1\}/\mathrm{d}x, \\ b(x, t) &= \mathbb{P}\{X(t) \in \mathrm{d}x,\ D(t) = -1\}/\mathrm{d}x, \end{aligned} \qquad x \in (-\infty, \infty), \quad t > 0. \tag{2.2.4}$$

The densities $f = f(x,\ t)$ and $b = b(x,\ t)$ satisfy the following set of integral equations similar to (2.2.2)

$$\begin{aligned} f(x,\ t) &= \frac{1}{2}\mathrm{e}^{-\lambda t}\delta(x-ct) + \int_0^t b(x-c(t-s),\ s)\lambda\mathrm{e}^{-\lambda(t-s)}\mathrm{d}s, \\ b(x,\ t) &= \frac{1}{2}\mathrm{e}^{-\lambda t}\delta(x+ct) + \int_0^t f(x+c(t-s),\ s)\lambda\mathrm{e}^{-\lambda(t-s)}\mathrm{d}s. \end{aligned} \tag{2.2.5}$$

It is easy to see, after the change of variables $s \to t-s$, that the integrals in (2.2.2) coincide with the respective integrals in (2.2.5).

Notice that $f(x,\ t),\ \ b(x,\ t)$ and $p_+(x,\ t),\ \ p_-(x,\ t)$ are equal to zero, if $|x| > ct$, since the speed $c$ is finite. Differentiating Eqs. (2.2.2) and (2.2.5) we obtain Kolmogorov equations in the differential form.

Define the matrix operator

$$\mathfrak{L} = \begin{pmatrix} \mathrm{L}_+^{x,t} + \lambda & -\lambda \\ -\lambda & \mathrm{L}_-^{x,t} + \lambda \end{pmatrix},$$

where the differential operators $\mathrm{L}_+^{x,t}$ and $\mathrm{L}_-^{x,t}$ are defined as

$$\mathrm{L}_\pm^{x,t} := \frac{\partial}{\partial t} \pm c\frac{\partial}{\partial x}. \tag{2.2.6}$$

In what follows, the superscript $T$ denotes the transposition of vectors.

**Theorem 2.1** [Kolmogorov equations] *Functions* $\mathbf{p} = (p_+,\ p_-)^T$ *and* $\mathbf{p} = (f,\ b)^T$ *satisfy the equation*

$$\mathfrak{L}\mathbf{p} = \mathbf{0}, \qquad |x| < ct. \tag{2.2.7}$$

*Moreover, if* $|x| > ct$ *then* $p_+(x,\ t) \equiv p_-(x,\ t) \equiv f(x,\ t) \equiv b(x,\ t) \equiv 0$*, and the initial conditions are*

$$p_+(x,\ +0) = p_-(x,\ +0) = \delta(x), \qquad f(x,\ +0) = b(x,\ +0) = \frac{1}{2}\delta(x). \tag{2.2.8}$$

*Proof* Initial conditions (2.2.8) follow immediately from Eqs. (2.2.2) and (2.2.5).

To get the equations we apply (in the sense of differentiation of generalised functions) the differential operators $\mathrm{L}_\pm^{x,t}$ (see 2.2.6) to Eqs. (2.2.2) and (2.2.5). Equation (2.2.7) can be obtained by exploiting the integration by parts in the resulting integral equations.

To prove that, first notice

$$\begin{aligned} \mathrm{L}_+^{x,t}\left[\mathrm{e}^{-\lambda t}\delta(x-ct)\right] &= -\lambda\mathrm{e}^{-\lambda t}\delta(x-ct), \\ \mathrm{L}_-^{x,t}\left[\mathrm{e}^{-\lambda t}\delta(x+ct)\right] &= -\lambda\mathrm{e}^{-\lambda t}\delta(x+ct), \end{aligned} \tag{2.2.9}$$

$$\mathrm{L}_{+}^{x,t}\left[p_-(x-cs,\ t-s)\right] = -\frac{\partial p_-(x-cs,\ t-s)}{\partial s},$$
$$\mathrm{L}_{-}^{x,t}\left[p_+(x+cs,\ t-s)\right] = -\frac{\partial p_+(x+cs,\ t-s)}{\partial s}.$$

Consequently, applying operator $\mathrm{L}_{+}^{x,t}$ to the first equation of (2.2.2) we get

$$\mathrm{L}_{+}^{x,t}\left[p_+(x,\ t)\right] = -\lambda \mathrm{e}^{-\lambda t}\delta(x-ct) + p_-(x-ct,\ 0)\lambda \mathrm{e}^{-\lambda t} - \int_0^t \frac{\partial p_-(x-cs,\ t-s)}{\partial s}\lambda \mathrm{e}^{-\lambda s}\mathrm{d}s.$$

Integrating by parts in the latter integral we have

$$\begin{aligned}\mathrm{L}_{+}^{x,t}\left[p_+(x,\ t)\right] &= -\lambda \mathrm{e}^{-\lambda t}\delta(x-ct) + p_-(x-ct,\ 0)\lambda \mathrm{e}^{-\lambda t}\\ &\quad - p_-(x-ct,\ 0)\lambda \mathrm{e}^{-\lambda t} + \lambda p_-(x,\ t)\\ &\quad - \lambda\int_0^t p_-(x-cs,\ t-s)\lambda \mathrm{e}^{-\lambda s}\mathrm{d}s\\ &= -\lambda p_+(x,\ t) + \lambda p_-(x,\ t),\end{aligned}$$

where in the last step we have used the first equation of (2.2.2). This equality coincides with the first component of (2.2.7) for $p_\pm(x,\ t)$. Proofs of the rest are left to the reader. □

**Corollary 2.1** *Densities $p_\pm$ and $f,\ b$ are related as*

$$f = \frac{1}{2}p_+, \qquad b = \frac{1}{2}p_-. \tag{2.2.10}$$

*Remark 2.1* System (2.2.7) is also named as *Cattaneo system*.

Equation (2.2.7) for function $\mathbf{p} = (f,\ b)^T$ has the sense of the forward Kolmogorov equation (or the Fokker-Planck equation) for the Goldstein-Kac telegraph process $X = X(t)$, see [2]. Here the matrix differential operator $\mathfrak{L}$ is the generator of process $X = X(t)$, while the matrix $\Lambda$,

$$\Lambda = \begin{pmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{pmatrix}, \tag{2.2.11}$$

is the infinitesimal matrix of the embedded two-state Markov chain $\varepsilon$ which controls the telegraph process $X = X(t)$.

*Remark 2.2* We introduce also the dual Kolmogorov equations (for expectations). Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a test-function (continuously differentiable). Consider the conditional expectations under the given initial direction $D(0) = \pm 1$,

$$u_\pm(x,\ t) = \mathbb{E}_\pm\{\varphi(x + X(t))\} = \int_{-\infty}^{\infty} \varphi(x+y)p_\pm(y,\ t)\mathrm{d}y,$$

and the expectations with respect to joint distribution of the position and the current direction

$$u^+(x,\ t) = \int_{-\infty}^{\infty} \varphi(x+y)f(y,\ t)\mathrm{d}y, \qquad u^-(x,\ t) = \int_{-\infty}^{\infty} \varphi(x+y)b(y,\ t)\mathrm{d}y.$$

Differentiating these equalities in $t$, applying equation (2.2.7) and then integrating by parts we conclude that functions $\mathbf{u} = (u_+,\ u_-)^T$ and $\mathbf{u} = (u^+,\ u^-)^T$ satisfy the equation

$$\mathfrak{L}'\mathbf{u} = \mathbf{0}, \tag{2.2.12}$$

where $\mathfrak{L}' = \begin{pmatrix} \mathrm{L}_-^{x,t} + \lambda & -\lambda \\ -\lambda & \mathrm{L}_+^{x,t} + \lambda \end{pmatrix}$ is the dual operator.

The initial conditions for Eq. (2.2.12) are

$$u_\pm(x, +0) = \varphi(x), \qquad u^\pm(x, +0) = \frac{1}{2}\varphi(x).$$

*Remark 2.3* It is easy to write down the backward Kolmogorov equation as well. Let

$$p_\pm(y,\ t;\ x,\ s) = \mathbb{P}\{X(t) \in \mathrm{d}y \mid X(s) = x,\ D(s) = \pm 1\}/\mathrm{d}y, \quad s < t$$

denote the densities related to the particle, which starts at time $s$ from the point $x$. Using an approach similar to (2.2.5) we obtain the integral equation

$$\begin{aligned} p_\pm(y,\ t;\ x,\ s) = {} & \mathrm{e}^{-\lambda(t-s)}\delta(y - x \mp c(t-s)) \\ & + \int_0^{t-s} p_\mp(y,\ t;\ x \pm c\tau,\ s+\tau)\lambda \mathrm{e}^{-\lambda\tau}\mathrm{d}\tau, \end{aligned}$$

since the medium is homogeneous. Differentiating similarly the proof of Theorem 2.1, we get the *differential* backward Kolmogorov equation,

$$\begin{aligned} -\mathrm{L}_+^{x,s}p_+ &\equiv -\frac{\partial p_+(y,\ t;\ x,\ s)}{\partial s} - c\frac{\partial p_+(y,\ t;\ x,\ s)}{\partial x} = -\lambda p_+(y,\ t;\ x,\ s) + \lambda p_-(y,\ t;\ x,\ s), \\ -\mathrm{L}_-^{x,s}p_- &\equiv -\frac{\partial p_-(y,\ t;\ x,\ s)}{\partial s} + c\frac{\partial p_-(y,\ t;\ x,\ s)}{\partial x} = -\lambda p_-(y,\ t;\ x,\ s) + \lambda p_+(y,\ t;\ x,\ s), \\ & x - c(t-s) < y < x + c(t-s), \qquad s < t. \end{aligned} \tag{2.2.13}$$

*Remark 2.4* There are several different interpretations of telegraph-type processes, cf. e. g. [3, 4]. Here we present somewhat exotic.

Let $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^2,\ \mathbf{e}_1 = (1,\ 0)^T$ and $\mathbf{e}_2 = (0,\ 1)^T$. Consider the Markov process $\boldsymbol{\xi} = \boldsymbol{\xi}_t = (1 - \varepsilon(t))\mathbf{e}_1 + \varepsilon(t)\mathbf{e}_2,\ t \ge 0$, where $\varepsilon = \varepsilon(t) \in \{0,\ 1\}$ is the two-state

Markov process defined by (2.1.1). Hence, the telegraph process $X = X(t),\ t \geq 0$ can be expressed as

$$X(t) = c\int_0^t \mathbf{D}\cdot\boldsymbol{\xi}_u \mathrm{d}u,$$

where $\mathbf{D} = \mathbf{e}_1 - \mathbf{e}_2 = (1,\ -1)^T$.

Notice that

$$\boldsymbol{\xi}_t - \int_0^t \Lambda\boldsymbol{\xi}_u \mathrm{d}u,$$

is a martingale, where $\Lambda$ is defined by (2.2.11). Hence, $\boldsymbol{\xi}_t$ can be represented as follows

$$\boldsymbol{\xi}_t = \boldsymbol{\xi}_0 + \int_0^t \Lambda\boldsymbol{\xi}_u \mathrm{d}u + \mathbf{M}_t, \tag{2.2.14}$$

where $\mathbf{M}_t$ is the martingale ($\mathbf{M}_0 = \mathbf{0}$). Equation (2.2.14) is equivalent to

$$\boldsymbol{\xi}_t = \mathrm{e}^{t\Lambda}\boldsymbol{\xi}_0 + \int_0^t \mathrm{e}^{(t-u)\Lambda}\mathrm{d}\mathbf{M}_u. \tag{2.2.15}$$

Representation (2.2.15) can be exploited as a source of various examples. If $X = X(t)$ is the basic telegraph process, defined by (2.1.3) (with $D(0) = +1$), then the state process $\boldsymbol{\xi}_t$ is defined by

$$\boldsymbol{\xi}_t = \left(\frac{1+(-1)^{N_t}}{2},\ \frac{1-(-1)^{N_t}}{2}\right)^T.$$

In this case the underlying martingale defined by (2.2.14) should be

$$\mathbf{M}_t = \left(\lambda\int_0^t (-1)^{N_u}\mathrm{d}u - \frac{1-(-1)^{N_t}}{2},\ -\lambda\int_0^t (-1)^{N_u}\mathrm{d}u + \frac{1-(-1)^{N_t}}{2}\right)^T. \tag{2.2.16}$$

Process $\mathbf{M}_t$ defined by (2.2.16) is actually the jump-telegraph martingale, see Theorem 4.1 (Chap. 4). The case of $D(0) = -1$ can be proceeded similarly.

Unfortunately, other examples of naturally defined martingales $\mathbf{M}_t$ produce rather exotic things. For instance, if the underlying martingale $\mathbf{M}_t$ has $M_t = (-1)^{N_t}\mathrm{e}^{2\lambda t}$ as a component (see e.g. [5]), then the resulting state process $\boldsymbol{\xi}_t$ can be expressed by means of

$$\xi_0 + \lambda\int_0^t (1 \pm \mathrm{e}^{-2\lambda(t-u)})\mathrm{e}^{2\lambda u}(-1)^{N_u}\mathrm{d}u + \sum_{k=1}^{N_t}(-1)^k\frac{1 \pm \mathrm{e}^{-2\lambda(t-\tau_k)}}{2}\mathrm{e}^{2\lambda\tau_k}$$

(see 2.2.15). This is the telegraph process with variable jumps and velocities of a very special comportment.

Nevertheless, representation (2.2.15) could be useful in various aspects. For $0 \leq s \leq t$

$$\mathbb{E}\{\boldsymbol{\xi}_t \mid \mathscr{F}_s\} = \mathbb{E}\{\boldsymbol{\xi}_t \mid \boldsymbol{\xi}_s\} = \mathrm{e}^{(t-s)\Lambda}\boldsymbol{\xi}_s.$$

Hence, the expectations of the telegraph process can be easily computed by

$$\mathbb{E}X(t) = c\int_0^t \mathbf{D}\cdot \mathrm{e}^{u\Lambda}\boldsymbol{\xi}_0 \mathrm{d}u,$$

etc.

## 2.3 Telegraph Equation

Hyperbolic systems (2.2.7) and (2.2.12) (and 2.2.13) of the first-order equations are equivalent to the hyperbolic differential equation of the second order (see 2.3.1).

**Theorem 2.2** *Functions $p_\pm = p_\pm(x,\ t)$ as well as functions $f = f(x,\ t),\ b = b(x,\ t)$ are solutions of the telegraph differential equation,*

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda\frac{\partial p}{\partial t} = c^2\frac{\partial^2 p}{\partial x^2}, \qquad t > 0, \quad x \in (-\infty, \infty). \tag{2.3.1}$$

*Proof* The proof is based on the so-called Kac's trick. We prove that Eq. (2.3.1) is valid for $p_\pm = p_\pm(x,\ t)$. The proof for $f$ and $b$ is left to the reader.

Let

$$p(x,\ t) = \frac{p_+(x,\ t) + p_-(x,\ t)}{2} \text{ and } w(x,\ t) = \frac{p_+(x,\ t) - p_-(x,\ t)}{2}.$$

In vectorial notations,

$$p = p(x,\ t) = (\mathbf{e}_+ \cdot \mathbf{p})/2, \qquad w = w(x,\ t) = (\mathbf{e}_- \cdot \mathbf{p})/2,$$

where $\mathbf{e}_+ = (1,\ 1),\ \mathbf{e}_- = (1,\ -1)$ and $\mathbf{p} = (p_+,\ p_-)^T$.

Notice that

$$\frac{1}{2}\mathbf{e}_+ \cdot \mathfrak{L}\mathbf{p} = \frac{\partial p}{\partial t} + c\frac{\partial w}{\partial x}, \qquad \frac{1}{2}\mathbf{e}_- \cdot \mathfrak{L}\mathbf{p} = \frac{\partial w}{\partial t} + c\frac{\partial p}{\partial x} + 2\lambda w.$$

Multiplying Eq. (2.2.7) by $\mathbf{e}_+/2$ and $\mathbf{e}_-/2$ we obtain the equivalent system

$$\begin{cases} \frac{\partial p}{\partial t} + c\frac{\partial w}{\partial x} = 0, \\ \frac{\partial w}{\partial t} + c\frac{\partial p}{\partial x} = -2\lambda w. \end{cases} \tag{2.3.2}$$

Then, we differentiate the first equation of system (2.3.2) in $t$ and the second one in $x$,

$$\frac{\partial^2 p}{\partial t^2} = -c\frac{\partial^2 w}{\partial x \partial t}, \qquad \frac{\partial^2 w}{\partial x \partial t} = -c\frac{\partial^2 p}{\partial x^2} - 2\lambda\frac{\partial w}{\partial x}$$

Eliminating the mixed derivative we obtain

$$\frac{\partial^2 p}{\partial t^2} = c^2\frac{\partial^2 p}{\partial x^2} + 2\lambda c\frac{\partial w}{\partial x}.$$

Applying again the first equation of system (2.3.2) we get (2.3.1) for $p = p(x, t)$.

In the similar way one can see that $w$ satisfies (2.3.1). It suffices to differentiate the first equation of (2.3.2) in $x$ and the second one in $t$, and then again to eliminate the mixed derivative.

Therefore the same equation is valid for $p_\pm = p_\pm(x, t)$. □

Equation (2.3.1) is called the *telegraph* or *damped wave* equation.

To specify the solution of this equation it is necessary to define the initial conditions. For the densities $p_\pm$ and $f, b$ (see 2.1.9 and 2.2.4 respectively) these conditions are obvious. For example, for conditional densities $p_\pm$ we have

$$p_\pm|_{t\downarrow 0} = \delta(x), \qquad \left.\frac{\partial p_\pm}{\partial t}\right|_{t\downarrow 0} = \mp c\delta'(x), \tag{2.3.3}$$

for $f, b$ the initial conditions are

$$f|_{t\downarrow 0} = b|_{t\downarrow 0} = \frac{1}{2}\delta(x), \qquad \left.\frac{\partial f}{\partial t}\right|_{t\downarrow 0} = -\left.\frac{\partial b}{\partial t}\right|_{t\downarrow 0} = -\frac{c}{2}\delta'(x),$$

and, hence, for the density $p = p(x, t) = \frac{1}{2}(p_+(x, t) + p_-(x, t)) = f(x, t) + b(x, t)$ (see 2.1.6)

$$p|_{t\downarrow 0} = \delta(x), \qquad \left.\frac{\partial p}{\partial t}\right|_{t\downarrow 0} = 0. \tag{2.3.4}$$

In each of these cases the first equality means that initially the particle is located at the origin, see initial conditions (2.2.8), and the second one follows directly from the first condition and from Eq. (2.2.7).

Let $\varphi\colon \mathbb{R} \to \mathbb{R}$, $\varphi \in C^2$ be any test-function. Consider the expectations

$$u_{\pm}(x,\ t) = \mathbb{E}_{\pm}\varphi(x + X(t)) = \int_{-\infty}^{\infty} \varphi(x + y)p_{\pm}(y,\ t)\mathrm{d}y \tag{2.3.5}$$

$$= \int_{-\infty}^{\infty} \varphi(y)p_{\pm}(y - x,\ t)\mathrm{d}y,$$

$$x \in (-\infty, \infty),\ t \geq 0.$$

Applying the differential operator $\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t} - c^2\frac{\partial^2}{\partial x^2}$ to (2.3.5) one can see that the differential equations for functions $p_{\pm}$ and for functions $u_{\pm}$ are the same.

**Theorem 2.3** *Functions $u_{\pm}$ as well as the function $u(x,\ t) = \mathbb{E}\varphi(x + X(t)) = \frac{1}{2}(u_{+}(x,\ t) + u_{-}(x,\ t))$ satisfy Eq. (2.3.1) with the following initial conditions*:

$$u_{+}(x,\ +0) = u_{-}(x,\ +0) = u(x,\ +0) = \varphi(x), \tag{2.3.6}$$

*and (see Eq. (2.2.12))*

$$\frac{\partial u_{\pm}}{\partial t}(x,\ +0) = \pm c\varphi'(x), \qquad \frac{\partial u}{\partial t}(x,\ +0) = 0. \tag{2.3.7}$$

*Remark 2.5* Applying the Kac's trick to system (2.2.13) we get another version of the telegraph equation, which has the sense of backward equation. Let $p = p_{\pm}(y,\ t;\ x,\ s)$, then

$$\frac{\partial^2 p}{\partial s^2} - 2\lambda\frac{\partial p}{\partial s} = c^2\frac{\partial^2 p}{\partial x^2}, \qquad s < t, \tag{2.3.8}$$

with the *terminal* conditions

$$p_{\pm}(y,\ t;\ x,\ s)|_{s\uparrow t} = \delta(y-x), \qquad \left.\frac{\partial p_{\pm}(y,\ t;\ x,\ s)}{\partial s}\right|_{s\uparrow t} = \pm c\delta'(y-x). \tag{2.3.9}$$

It is important to note that all equations and their transforms are treated in the generalised sense, that is, as the differentiation of generalised functions. More precisely, we assume that all the functions and their derivatives are considered in the space $\mathscr{D}'$.

Hyperbolic system (2.2.7) as well as hyperbolic Eq. (2.3.1) are well-posed and the solution is supported into the cone $\bar{S}$ between the bicharacteristics,

$$S = \{(x, t) \in \mathbb{R}^2_+ = \mathbb{R} \times (0, \infty)\ :\ |x| < ct\},$$

if the initial conditions are concentrated at the origin. The solution of (2.3.1), (2.3.4) is unique in the space $\mathscr{D}'$.

Solving the Cauchy problem for Eq. (2.3.1) with the initial conditions $(\delta(x), 0)$ is equivalent to solving the inhomogeneous telegraph equation

$$\frac{\partial^2 p(x,t)}{\partial t^2} + 2\lambda \frac{\partial p(x,t)}{\partial t} - c^2 \frac{\partial^2 p(x,t)}{\partial x^2} = \delta(x)\, \delta(t), \tag{2.3.10}$$

where the generalised function on the right-hand side of this equation represents the instant unit point-like source concentrated at the initial time $t = 0$ at the origin $x = 0$.

From (2.3.10) it follows that the transition density $p(x,t)$ of the telegraph process $X(t)$ is the fundamental solution (the Green's function) to the telegraph equation (2.3.1). This implies the fine analogy between the telegraph process $X(t)$ and the one-dimensional Brownian motion $w(t)$, whose transition density is the fundamental solution to the one-dimensional heat equation (1.2.2).

## 2.4 Characteristic Function

In this section we study the characteristic function (the Fourier transform of the transition density) of the telegraph process $X(t)$,

$$\widehat{p}(\xi,t) = \mathscr{F}_{x\to\xi}\,[p(x,\ t)] = \int_{-\infty}^{\infty} \mathrm{e}^{i\xi x}\ p(x,t)\, \mathrm{d}x, \tag{2.4.1}$$

where $p = p(x,t)$ is the transition density (see (2.1.6) and (2.1.7)).

**Theorem 2.4** *Let $d = d(\xi) = \lambda^2 - c^2\xi^2$, and $\mathbb{1}_A$ is the indicator function (see (2.1.8)).*

*For any $t > 0$ the characteristic function of the telegraph process $X(t)$ has the form :*

$$\widehat{p}(\xi,t) = \mathrm{e}^{-\lambda t}\left\{\left[\cosh\left(t\sqrt{d(\xi)}\right) + \frac{\lambda}{\sqrt{d(\xi)}}\ \sinh\left(t\sqrt{d(\xi)}\right)\right]\mathbb{1}_{\left\{|\xi|\le\frac{\lambda}{c}\right\}}\right. \tag{2.4.2}$$
$$\left. + \left[\cos\left(t\sqrt{-d(\xi)}\right) + \frac{\lambda}{\sqrt{-d(\xi)}}\ \sin\left(t\sqrt{-d(\xi)}\right)\right]\mathbb{1}_{\left\{|\xi|\,>\frac{\lambda}{c}\right\}}\right\}.$$

*Proof* According to (2.3.1) and (2.3.4), the characteristic function $\widehat{p}(\xi,t)$ is the solution of the initial-value problem: for any $\xi \in (-\infty, \infty)$ ($\xi$ is fixed)

$$\begin{aligned} &\frac{d^2\widehat{p}(\xi,t)}{dt^2} + 2\lambda\frac{d\widehat{p}(\xi,t)}{dt} + c^2\xi^2\widehat{p}(\xi,t) = 0, \\ &\widehat{p}(\xi,t)|_{t=0} = 1, \qquad \left.\frac{d\widehat{p}(\xi,t)}{dt}\right|_{t=0} = 0. \end{aligned} \tag{2.4.3}$$

The characteristic equation of the ordinary differential equation in (2.4.3) is

$$z^2 + 2\lambda z + c^2\xi^2 = 0$$

with the roots $z_1 = -\lambda - \sqrt{d(\xi)}, \quad z_2 = -\lambda + \sqrt{d(\xi)}$.

Therefore, the general solution of the ordinary differential equation in (2.4.3) has the form:

$$\widehat{p}(\xi,t) = C_1 e^{t\left(-\lambda-\sqrt{d(\xi)}\right)} + C_2 e^{t\left(-\lambda+\sqrt{d(\xi)}\right)}, \tag{2.4.4}$$

where $C_1, \ C_2$ are some constants which do not depend on $t$. The initial conditions of (2.4.3) give us the system

$$\begin{cases} C_1 + C_2 = 1, \\ C_1\left(-\lambda - \sqrt{d(\xi)}\right) + C_2\left(-\lambda + \sqrt{d(\xi)}\right) = 0. \end{cases}$$

The solution of this system is

$$C_1 = \frac{1}{2}\left(1 - \frac{\lambda}{\sqrt{d(\xi)}}\right), \qquad C_2 = \frac{1}{2}\left(1 + \frac{\lambda}{\sqrt{d(\xi)}}\right).$$

Substituting these values into (2.4.4) we obtain for $|\xi| \le \frac{\lambda}{c}$ (resp. for $d = d(\xi) = \lambda^2 - c^2\xi^2 > 0$):

$$\begin{aligned} \widehat{p}(\xi,t) &= \tfrac{1}{2}\left(1 - \tfrac{\lambda}{\sqrt{d}}\right) e^{t(-\lambda-\sqrt{d})} + \tfrac{1}{2}\left(1 + \tfrac{\lambda}{\sqrt{d}}\right) e^{t(-\lambda+\sqrt{d})} \\ &= e^{-\lambda t}\left[\tfrac{e^{t\sqrt{d}}+e^{-t\sqrt{d}}}{2} + \tfrac{\lambda}{\sqrt{d}}\ \tfrac{e^{t\sqrt{d}}-e^{-t\sqrt{d}}}{2}\right] \\ &= e^{-\lambda t}\left[\cosh\left(t\sqrt{d}\right) + \tfrac{\lambda}{\sqrt{d}}\ \sinh\left(t\sqrt{d}\right)\right]. \end{aligned} \tag{2.4.5}$$

Since $\cosh(iz) = \cos(z)$ and $\sinh(iz) = i\sin(z)$, we have for $|\xi| > \frac{\lambda}{c}$ (resp. for $d = d(\xi) = \lambda^2 - c^2\xi^2 < 0$):

$$\widehat{p}(\xi,t) = e^{-\lambda t}\left[\cos\left(t\sqrt{-d}\right) + \frac{\lambda}{\sqrt{-d}}\ \sin\left(t\sqrt{-d}\right)\right]. \tag{2.4.6}$$

Collecting (2.4.5) and (2.4.6) we obtain (2.4.2). Note that function (2.4.2) is continuous with respect to $\xi$ because

$$\lim_{\xi \uparrow \frac{\lambda}{c}} \widehat{p}(\xi, t) = \lim_{\xi \downarrow \frac{\lambda}{c}} \widehat{p}(\xi, t) = e^{-\lambda t}(1 + \lambda t).$$

The theorem is proved. □

## 2.5 Transition Density

The principal aim of this section is to obtain the transition density $p(x,\ t)$ (2.1.6) of the telegraph process $X(t)$. This is possible by computing the inverse Fourier transform of $\widehat{p}(\xi,\ t)$, but we prefer the direct calculation based on the Cauchy problem for telegraph equation (2.3.1) with initial condition (2.3.6) and (2.3.7) (see Theorem 2.3). This result is given by the following theorem. Another approach is presented in Sect. 4.1.1.

**Theorem 2.5** *For any $t > 0$ the transition density $p(x,\ t)$ of the telegraph process $X(t)$ has the form:*

$$\begin{aligned} p(x, t) = &\tfrac{e^{-\lambda t}}{2} \left[\delta(ct + x) + \delta(ct - x)\right] \\ &+ \tfrac{e^{-\lambda t}}{2c} \left[\lambda I_0\left(\tfrac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right) + \tfrac{\partial}{\partial t}\, I_0\left(\tfrac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right)\right] \mathbb{1}_{\{|x|<ct\}}, \\ &x \in (-\infty, \infty), \qquad t > 0, \end{aligned} \tag{2.5.1}$$

*where $I_0(x)$ is the modified Bessel function, $\delta = \delta(x)$ is Dirac's $\delta$-function.*

*Remark 2.6* Taking into account that $I_0'(z) = I_1(z)$ (see 1.5.3), we have

$$\frac{\partial}{\partial t}\, I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right) = \frac{\lambda ct}{\sqrt{c^2t^2 - x^2}}\, I_1\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right), \tag{2.5.2}$$

and, therefore, the transition density $p = p(x,\ t)$ given by (2.5.1) has the following alternative form:

$$\begin{aligned} p(x,\ t) = &\tfrac{e^{-\lambda t}}{2} \left[\delta(ct + x) + \delta(ct - x)\right] \\ &+ \tfrac{\lambda e^{-\lambda t}}{2c} \left[I_0\left(\tfrac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right) + \tfrac{ct}{\sqrt{c^2t^2 - x^2}}\, I_1\left(\tfrac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right)\right] \mathbb{1}_{\{|x|<ct\}}, \\ &x \in (-\infty, \infty), \qquad t > 0. \end{aligned} \tag{2.5.3}$$

*Proof* Due to Theorem 2.3 it is sufficient to prove that the expectation

$$\begin{aligned} u(x,\ t) = \mathbb{E}\varphi(x + X(t)) = &\frac{e^{-\lambda t}}{2} \left[\varphi(x - ct) + \varphi(x + ct)\right] \\ &+ \frac{e^{-\lambda t}}{2c} \int_{-ct}^{ct} \varphi(x + y) \left[\lambda I_0\left(\tfrac{\lambda}{c}\sqrt{c^2t^2 - y^2}\right) + \tfrac{\partial}{\partial t}\, I_0\left(\tfrac{\lambda}{c}\sqrt{c^2t^2 - y^2}\right)\right] dy \end{aligned} \tag{2.5.4}$$

satisfies telegraph equation (2.3.1) with the initial conditions

$$u(x,\ t)|_{t\downarrow 0} = \varphi(x), \qquad \left.\frac{\partial u(x,\ t)}{\partial t}\right|_{t\downarrow 0} = 0 \tag{2.5.5}$$

(see (2.3.6)–(2.3.7)) for any test-function $\varphi : \mathbb{R} \to \mathbb{R},\ \varphi \in C^2$.

After the change of variables $u = \mathrm{e}^{-\lambda t} v(x,\ t)$ it is easy to see that function $u = u(x,\ t)$ is the solution to the problem (2.3.1) and (2.5.5) if and only if the function $v = v(x,\ t)$ solves the equation

$$\frac{\partial^2 v}{\partial t^2}(x,\ t) - c^2 \frac{\partial^2 v}{\partial x^2}(x,\ t) = \lambda^2 v(x,\ t), \qquad x \in (-\infty, \infty), \quad t > 0. \tag{2.5.6}$$

Initial conditions (2.5.5) take the form

$$v(x,\ t)|_{t\downarrow 0} = \varphi(x), \qquad \left.\frac{\partial v(x,\ t)}{\partial t}\right|_{t\downarrow 0} = \lambda\varphi(x), \quad x \in (-\infty, \infty). \tag{2.5.7}$$

We find the solution of Cauchy problem (2.5.6)–(2.5.7) in the following terms. For any continuous function $\psi,\ \psi : \mathbb{R} \to \mathbb{R}$, we set

$$\begin{aligned} Z(x,\ t;\ \psi) &= \frac{1}{2} \int_0^t [\psi(x + cs) + \psi(x - cs)]\, I_0\left(\lambda\sqrt{t^2 - s^2}\right) \mathrm{d}s \\ &= \frac{1}{2} \int_{-t}^t \psi(x + cs) I_0\left(\lambda\sqrt{t^2 - s^2}\right) \mathrm{d}s. \end{aligned}$$

**Lemma 2.1** *The solution $v(x,\ t)$ to Eq. (2.5.6) supplied with the initial conditions*

$$v(x,\ t)|_{t\downarrow 0} = \varphi, \qquad \left.\frac{\partial v(x,\ t)}{\partial t}\right|_{t\downarrow 0} = \psi \tag{2.5.8}$$

*can be expressed in the form*

$$v(x,\ t) = Z(x,\ t;\ \psi) + \frac{\partial Z}{\partial t}(x,\ t;\ \varphi). \tag{2.5.9}$$

Theorem 2.5 follows from Lemma 2.1. Indeed, substituting $\lambda\varphi$ instead of $\psi$ in (2.5.9) and differentiating in $t$ we obtain

$$\begin{aligned} v(x,\ t) = &\frac{1}{2} [\varphi(x - ct) + \varphi(x + ct)] \\ &+ \frac{1}{2} \int_{-t}^t \varphi(x + cs) \left[\lambda I_0\left(\lambda\sqrt{t^2 - s^2}\right) + \frac{\partial}{\partial t}\, I_0\left(\lambda\sqrt{t^2 - s^2}\right)\right] \mathrm{d}s, \end{aligned}$$

which corresponds to (2.5.4) (after suitable change of variables).

*Proof of the Lemma* First we prove that $Z = Z(x,\ t;\ \psi)$ fits to Eq. (2.5.6).
Differentiating we obtain

$$\begin{aligned}\frac{\partial Z(x,\ t;\ \psi)}{\partial t} &= \frac{1}{2}\left(\psi(x+ct)+\psi(x-ct)\right) \\ &+ \frac{1}{2}\int_{-t}^{t}\psi(x+cs)\frac{\partial}{\partial t}I_0\left(\lambda\sqrt{t^2-s^2}\right)\mathrm{d}s \\ &= \frac{1}{2}\left(\psi(x+ct)+\psi(x-ct)\right) \\ &+ \frac{1}{2}\int_{-t}^{t}\psi(x+cs)I_1\left(\lambda\sqrt{t^2-s^2}\right)\frac{\lambda t}{\sqrt{t^2-s^2}}\mathrm{d}s.\end{aligned} \tag{2.5.10}$$

Then, using the limit relation $\lim_{z\to 0}\frac{I_1(z)}{z} = \frac{1}{2}$ we have

$$\begin{aligned}\frac{\partial^2 Z(x,\ t;\ \psi)}{\partial t^2} = \frac{c}{2}\left(\psi'(x+ct)-\psi'(x-ct)\right) + \frac{\lambda^2 t}{4}\left(\psi(x+ct)+\psi(x-ct)\right) \\ +\frac{1}{2}\int_{-t}^{t}\psi(x+cs)\frac{\partial^2}{\partial t^2}I_0\left(\lambda\sqrt{t^2-s^2}\right)\mathrm{d}s.\end{aligned} \tag{2.5.11}$$

Finally, notice

$$c^2\frac{\partial^2 Z(x,\ t;\ \psi)}{\partial x^2} = \frac{1}{2}\int_{-t}^{t}\frac{\partial^2\psi(x+cs)}{\partial s^2}I_0\left(\lambda\sqrt{t^2-s^2}\right)\mathrm{d}s.$$

Integrating by parts, similarly to (2.5.11), we get

$$\begin{aligned}c^2\frac{\partial^2 Z(x,\ t;\ \psi)}{\partial x^2} = \frac{c}{2}\left(\psi'(x+ct)-\psi'(x-ct)\right) + \frac{\lambda^2 t}{4}\left(\psi(x+ct)+\psi(x-ct)\right) \\ +\frac{1}{2}\int_{-t}^{t}\psi(x+cs)\frac{\partial^2}{\partial s^2}I_0\left(\lambda\sqrt{t^2-s^2}\right)\mathrm{d}s.\end{aligned} \tag{2.5.12}$$

Collecting (2.5.11) and (2.5.12) we have

$$\begin{aligned}&\frac{\partial^2 Z(x,\ t;\ \psi)}{\partial t^2} - c^2\frac{\partial^2 Z(x,\ t;\ \psi)}{\partial x^2} \\ &= \frac{1}{2}\int_{-t}^{t}\psi(x+cs)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial s^2}\right)I_0\left(\lambda\sqrt{t^2-s^2}\right)\mathrm{d}s.\end{aligned} \tag{2.5.13}$$

For any smooth function $\phi$

$$\frac{\partial^2\phi(t^2-s^2)}{\partial t^2} - \frac{\partial^2\phi(t^2-s^2)}{\partial s^2} = 4\phi'(t^2-s^2) + 4(t^2-s^2)\phi''(t^2-s^2).$$

In particular, if $\phi(z) := I_0(\lambda\sqrt{z})$, then

$$\phi'(z) = \frac{\lambda}{2\sqrt{z}} I_1(\lambda\sqrt{z}), \quad \phi''(z) = -\frac{\lambda}{4z^{3/2}} I_1(\lambda\sqrt{z}) + \frac{\lambda^2}{4z} I_2(\lambda\sqrt{z}),$$

and therefore

$$4\phi'(z) + 4z\phi''(z) = \frac{\lambda}{\sqrt{z}} I_1(\lambda\sqrt{z}) + \lambda^2 I_2(\lambda\sqrt{z}).$$

Finally, due to Bessel equation (see (1.5.1) and (1.5.3)), $I_2(z) + \frac{2}{z} I_1(z) = I_0(z)$, we have

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right) I_0\left(\lambda\sqrt{t^2 - s^2}\right) = \lambda^2 I_0\left(\lambda\sqrt{t^2 - s^2}\right).$$

This equality and (2.5.13) give us Eq. (2.5.6) for $Z$, and, therefore, the equation for $v$ defined by (2.5.9). From definition of $Z$ and equalities (2.5.10)–(2.5.11) we easily get

$$Z(x,\ +0;\ \psi) = 0, \quad \frac{\partial Z}{\partial t}(x,\ +0;\ \psi) = \psi(x), \quad \frac{\partial^2 Z}{\partial t^2}(x,\ +0;\ \psi) = 0$$

which is equivalent to (2.5.8). Lemma 2.1 and Theorem 2.5 are proved.

Notice that conditional densities $p_\pm(x,\ t)$ defined by (2.1.9) satisfy Eq. (2.3.1) with initial conditions (2.3.3). Hence functions $q_\pm(x,\ t) := \mathrm{e}^{\lambda t} p_\pm(x,\ t)$ fit for Eq. (2.5.6) with initial conditions

$$q_\pm(x,\ t)|_{t\downarrow 0} = \delta(x), \qquad \left.\frac{\partial q_\pm(x,\ t)}{\partial t}\right|_{t\downarrow 0} = \lambda\delta(x) \mp c\delta'(x).$$

Applying again Lemma 2.1 we obtain the explicit formulae for $p_\pm(x,\ t)$:

$$\begin{aligned} p_\pm(x,\ t) &= \mathrm{e}^{-\lambda t}\left[Z(x,\ t;\ \lambda\delta \mp c\delta') + \frac{\partial Z}{\partial t}(x,\ t;\ \delta)\right] \\ &= \mathrm{e}^{-\lambda t}\delta(ct \mp x) + P_\pm(x,\ t), \end{aligned} \tag{2.5.14}$$

where (see (2.1.10)) $P_\pm(x,\ t)$ are the absolutely continuous parts of the distributions,

$$\begin{aligned} P_\pm(x,\ t) &= P(x,\ t) \mp c\frac{\partial}{\partial x} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right) \mathbb{1}_{\{|x|<ct\}} \\ &= \frac{\lambda \mathrm{e}^{-\lambda t}}{2c}\left[I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right)\right. \\ &\quad \left. + \frac{ct \pm x}{\sqrt{c^2t^2 - x^2}}\ I_1\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right)\right] \mathbb{1}_{\{|x|<ct\}}, \end{aligned} \tag{2.5.15}$$

$$x \in (-\infty, \infty), \qquad t > 0.$$

For definition of $P(x, t)$ see (2.1.7).

Notice that $P_+(-x, t) \equiv P_-(x, t)$ and $p_+(-x, t) \equiv p_-(x, t)$.

Moreover, in addition to (2.2.10) we have

$$\frac{1}{2}p_-(x, t) = f(-x, t), \qquad \frac{1}{2}p_+(x, t) = b(-x, t). \tag{2.5.16}$$

Hence $f(x, t) \equiv b(-x, t)$.

The explicit formulae for $f(x, t)$ and $b(x, t)$ follow from (2.5.14) and (2.5.15).

*Remark 2.7* Equalities (2.5.16) have a probabilistic proof. Since the telegraph process is the renewal one, each sample path can be considered with inverted time direction. This means that if $X^+ = X^+(t)$ , $X^- = X^-(t)$ and $X = X(t)$ are the telegraph processes defined by (2.1.3) and (2.1.4) which are driven by the same Poisson process, then due to symmetry

$$\begin{aligned} X^{\pm}(t) &= \pm c \int_0^t (-1)^{N(s)} \mathrm{d}s = \pm c(-1)^{N(t)} \int_0^t (-1)^{N(t)-N(s)} \mathrm{d}s \\ &\overset{d}{=} \pm c(-1)^{N(t)} \int_0^t (-1)^{N(t-s)} \mathrm{d}s = \mp c(-1)^{N(t)} \int_0^t (-1)^{N(s)} \mathrm{d}s =: Y_{\mp}(t). \end{aligned}$$

Variable $Y_{\mp}(t)$ is distributed like a telegraph process with the *final direction* which is opposite to the initial direction of the original process $X^{\pm}$.

Identities (2.5.16) follow from these observations.

## 2.6 Convergence to Brownian Motion

In this section we establish the limiting behaviour of the telegraph process $X(t)$ as both the speed of the motion $c$ and the intensity of switchings $\lambda$ tend to infinity in such a way that the following condition holds

$$c \to \infty, \qquad \lambda \to \infty, \qquad \frac{c^2}{\lambda} \to \sigma^2. \tag{2.6.1}$$

Condition (2.6.1) is referred to as *Kac's condition.* The following theorem states that, under Kac's condition (2.6.1), the telegraph process $X(t)$ is asymptotically a Wiener process.

**Theorem 2.6** *For any fixed* $x \in (-\infty, \infty)$, $t > 0$, *under Kac's scaling condition (2.6.1) random variables* $X(x, t) := x + X(t)$ *converge in distribution to* $W_x(t)$, *where* $\{W_x(t)\}_{t>0}$ *is the diffusion process which starts from* $x \in (-\infty, \infty)$ *with zero drift and diffusion coefficient* $\sigma^2$,

$$x + X(t) \xrightarrow{d} W_x(t), \qquad x \in (-\infty, \infty), \quad t > 0. \tag{2.6.2}$$

*Proof* Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. It is sufficient to compute the limit of the function

$$\mathbb{E}\varphi(x + X(t)) = \frac{1}{2}\mathrm{e}^{-\lambda t}\left[\varphi(x + ct) + \varphi(x - ct)\right] + E_{c,\lambda}(x, t), \tag{2.6.3}$$

as $c, \ \lambda \to \infty, \ c^2/\lambda \to \sigma^2$. Here

$$E_{c,\lambda}(x, t) = \int_{-ct}^{ct} \varphi(x + y)P(y, \ t)\mathrm{d}y \tag{2.6.4}$$

and $P(y, \ t)$ is the absolutely continuous part of the distribution,

$$P(y, \ t) = \frac{\lambda \mathrm{e}^{-\lambda t}}{2c}\Big[I_0\Big(\frac{\lambda}{c}\sqrt{c^2t^2 - y^2}\Big) + \frac{ct}{\sqrt{c^2t^2 - y^2}} I_1\Big(\frac{\lambda}{c}\sqrt{c^2t^2 - y^2}\Big)\Big],$$

(see (2.1.7) and (2.5.3)).

The first summand in (2.6.3) (corresponding to the discrete component of the distribution) tends to zero, $\mathrm{e}^{-\lambda t}\left[\varphi(x + ct) + \varphi(x - ct)\right] \to 0$, because function $\varphi$ is bounded and $\mathrm{e}^{-\lambda t} \to 0$ as $\lambda \to \infty$.

We split the integral in (2.6.4) into two summands, $E_{c,\lambda}(x, \ t) = E^{(1)}(x, \ t) + E^{(2)}(x, \ t)$, where

$$E^{(1)}(x, \ t) = \int_{|y|<(ct)^{1/2}} \varphi(x + y)P(y, \ t)\mathrm{d}y,$$

$$E^{(2)}(x, \ t) = \int_{(ct)^{1/2}<|y|<ct} \varphi(x + y)P(y, \ t)\mathrm{d}y.$$

The first part reflects the integration on $[-(ct)^{1/2}, (ct)^{1/2}]$, such that under scaling condition (2.6.1)

$$\frac{\lambda}{c}\sqrt{c^2t^2 - y^2} \to \infty$$

for any $y, \ |y| < (ct)^{1/2}$. Applying asymptotics (1.5.8) of Bessel functions (with $\nu = 0, 1$) we obtain

$$\frac{\lambda e^{-\lambda t}}{2c}\int_{-(ct)^{1/2}}^{(ct)^{1/2}}\varphi(x+y)\left[I_0\left(\tfrac{\lambda}{c}\sqrt{c^2t^2-y^2}\right)+\tfrac{ct}{\sqrt{c^2t^2-y^2}}I_1\left(\tfrac{\lambda}{c}\sqrt{c^2t^2-y^2}\right)\right]dy$$
$$\sim\frac{\lambda}{c}\int_{-(ct)^{1/2}}^{(ct)^{1/2}}\varphi(x+y)e^{-\lambda t+\lambda\sqrt{t^2-y^2/c^2}}\frac{dy}{\sqrt{2\pi\lambda\sqrt{t^2-y^2/c^2}}}. \tag{2.6.5}$$

Notice that if $|y|<(ct)^{1/2}$, then under scaling (2.6.1) we have

$$-\lambda t+\lambda\sqrt{t^2-y^2/c^2}=\frac{-\lambda y^2/c^2}{t+\sqrt{t^2-y^2/c^2}}\to-\frac{y^2}{2\sigma^2 t}$$

and

$$\frac{\lambda/c}{\sqrt{2\pi\lambda\sqrt{t^2-y^2/c^2}}}=\frac{\sqrt{\lambda}/c}{\sqrt{2\pi\sqrt{t^2-y^2/c^2}}}\to\frac{1}{\sigma\sqrt{2\pi t}}.$$

Moreover, since function $\varphi$ is bounded, under the Kac's condition (2.6.1) we have

$$|\varphi(x+y)|\,e^{-\lambda t+\lambda\sqrt{t^2-y^2/c^2}}\frac{\lambda/c}{\sqrt{2\pi\lambda\sqrt{t^2-y^2/c^2}}}\mathbb{1}_{\{|y|<(ct)^{1/2}\}}$$
$$\le C\frac{\sqrt{\lambda}/c}{\sqrt{2\pi\sqrt{t^2-t/c}}}e^{\frac{-\lambda y^2/c^2}{t+\sqrt{t^2-t/c}}}\le Ae^{-ay^2}.$$

Here $A$ and $a$ depend on $t$ only, so the inequality is fulfilled uniformly in $y$.

Hence, by Lebesgue's dominated convergence theorem the latter integral in (2.6.5) converges to

$$\frac{1}{\sigma\sqrt{2\pi t}}\int_{-\infty}^{\infty}\varphi(x+y)e^{-\frac{y^2}{2\sigma^2 t}}dy.$$

To finish the proof it is sufficient to demonstrate that the second component $E^{(2)}(x,\ t)$ of $E_{c,\lambda}(x,\ t)$ vanishes under Kac's condition. Notice that (see (1.5.5))

$$I_0(z)\le e^z,\qquad \frac{I_1(z)}{z}\le\frac{1}{2}e^z,\qquad z\ge 0.$$

Applying these inequalities and then changing the variables, $y=\frac{c}{\sqrt{\lambda}}s$, we have the following estimates for $E^{(2)}(x,\ t)$,

$$\frac{\lambda e^{-\lambda t}}{2c}\int_{(ct)^{1/2}<|y|<ct}\varphi(x+y)\left(I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-y^2}\right)+\frac{ct}{\sqrt{c^2t^2-y^2}}I_1\left(\frac{\lambda}{c}\sqrt{c^2t^2-y^2}\right)\right)dy$$

$$\leq \frac{\lambda}{2c} \int_{(ct)^{1/2}<|y|<ct} \varphi(x+y)\left(1+\frac{\lambda t}{2}\right) \exp\left\{-\lambda t + \lambda\sqrt{t^2 - y^2/c^2}\right\} \mathrm{d}y$$

$$= \frac{\lambda}{2c} \int_{(ct)^{1/2}<|y|<ct} \varphi(x+y)\left(1+\frac{\lambda t}{2}\right) \exp\left\{\frac{-\lambda y^2/c^2}{t+\sqrt{t^2-y^2/c^2}}\right\} \mathrm{d}y = \left|y = \frac{c}{\sqrt{\lambda}} s\right|$$

$$= \frac{\sqrt{\lambda}}{2}\left(1+\frac{\lambda t}{2}\right) \int_{\sqrt{\frac{\lambda t}{c}}<|s|<\sqrt{\lambda} t} \varphi\left(x+\frac{c}{\sqrt{\lambda}} s\right) \exp\left\{-\frac{s^2}{t+\sqrt{t^2-s^2/\lambda}}\right\} \mathrm{d}s$$

$$\leq \frac{\sqrt{\lambda}}{2}\left(1+\frac{\lambda t}{2}\right) \int_{\sqrt{\frac{\lambda t}{c}}<|s|<\sqrt{\lambda} t} \varphi\left(x+\frac{c}{\sqrt{\lambda}} s\right) \exp\left\{-\frac{s^2}{2t}\right\} \mathrm{d}s$$

$$\leq C\lambda^{3/2} t^2 \exp\left\{-\frac{\lambda}{2c}\right\}$$

$$\sim C\lambda^{3/2} t^2 \exp\left\{-\frac{\sqrt{\lambda}}{2\sigma}\right\} \to 0$$

as $\lambda, c \to \infty$ and $c^2/\lambda \to \sigma^2$.

*Remark 2.8* Theorem 2.6 can be generalised up to the functional limit theorem (invariance principle). Under the Kac's scaling condition (2.6.1) the telegraph process $X(x,\ t) := x + X(t)$ converges in distribution in $C([0, T]; (-\infty, \infty))$ (equipped with the sup-norm and the $\sigma$-algebra generated by the open subsets) to the process $W_x(t),\ t > 0$, the diffusion process which starts from $x \in (-\infty, \infty)$ with zero drift and diffusion coefficient $\sigma^2$. To prove that, we need more detailed considerations comprising the compactness properties which are not included in the framework of this presentation. See [2].

Generalisations of the convergence (2.6.2) to the asymmetric case see in [6].

*Remark 2.9* We can also check that, under the Kac's condition (2.6.1), the characteristic function (2.4.2) of the telegraph process $X(t)$ converges to the characteristic function of the one-dimensional Brownian motion.

First we note that from Kac's condition (2.6.1) it follows that $(\lambda/c) \to \infty$ and, thus, $\mathbb{1}_{\left\{|\xi|\leq\frac{\lambda}{c}\right\}} \to 1,\quad \mathbb{1}_{\left\{|\xi|>\frac{\lambda}{c}\right\}} \to 0$. Therefore, for the characteristic function (2.4.2) we have the asymptotic relation (for $c \to \infty,\ \lambda \to \infty$):

$$\widehat{p}(\xi, t) \sim \mathrm{e}^{-\lambda t}\left[\cosh\left(t\sqrt{\lambda^2 - c^2\xi^2}\right) + \frac{\lambda}{\sqrt{\lambda^2 - c^2\xi^2}} \sinh\left(t\sqrt{\lambda^2 - c^2\xi^2}\right)\right]$$

$$= \mathrm{e}^{-\lambda t}\left[\cosh\left(\lambda t\sqrt{1-\frac{c^2}{\lambda^2}\,\xi^2}\right) + \frac{1}{\sqrt{1-\frac{c^2}{\lambda^2}\,\xi^2}} \sinh\left(\lambda t\sqrt{1-\frac{c^2}{\lambda^2}\,\xi^2}\right)\right]$$

$$\sim \; \mathrm{e}^{-\lambda t} \left[ \cosh\left(\lambda t \sqrt{1 - \frac{c^2}{\lambda^2}\,\xi^2}\right) + \sinh\left(\lambda t \sqrt{1 - \frac{c^2}{\lambda^2}\,\xi^2}\right) \right]$$

$$= \exp\left(-\lambda t + \lambda t \sqrt{1 - \frac{c^2}{\lambda^2}\,\xi^2}\right)$$

$$= \exp\left(-\frac{c^2\xi^2 t}{\lambda(1 + \sqrt{1 - c^2\xi^2/\lambda^2}}\right).$$

Due to Kac's condition (2.6.1) we have $\dfrac{c^2}{\lambda^2} \to 0$ and, therefore,

$$\lim_{\substack{c,\ \lambda \to \infty \\ (c^2/\lambda) \to \sigma^2}} \widehat{p}(\xi, t) = \exp\left(-\frac{1}{2}\, t\sigma^2\xi^2\right). \tag{2.6.6}$$

This is the characteristic function of the homogeneous one-dimensional Brownian motion with zero drift and diffusion coefficient $\sigma^2$ (see (1.2.3)).

## 2.7 Laplace Transforms

In this section we compute the Laplace transforms (in $t$) of the transition density of the telegraph process $X(t),\ t \geq 0$ and of its characteristic function.

Let $f : [0, \infty) \to \mathbb{R}$ be an integrable function. The Laplace transform $\widetilde{f}(s)$ is defined as

$$\mathscr{L}_{t\to s} f \equiv \widetilde{f}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t, \qquad s > 0, \tag{2.7.1}$$

see (1.6.6).

If $f \in \mathscr{D}'([0, \infty))$ is a distribution (generalised function) supported on $[0, \infty)$, then the Laplace transform is defined by means of

$$< \widetilde{f}, \varphi > = < f, \widetilde{\varphi} > \text{ for any test function } \varphi \in \mathscr{D}([0, \infty)).$$

**Theorem 2.7** *The Laplace transform (with respect to time t) of the transition density $p(x,\ t)$ of the telegraph process $X(t)$ is given by the formula: for $\forall x \in (-\infty, \infty)$*

$$\widetilde{p}(x,\ \cdot)(s) = \begin{cases} \frac{1}{2c}\sqrt{\frac{s+2\lambda}{s}}\ \mathrm{e}^{-\frac{|x|}{c}\sqrt{s(s+2\lambda)}}, & x \neq 0 \\ \frac{1}{2c}\left(\sqrt{\frac{s+2\lambda}{s}} + 1\right), & x = 0. \end{cases} \tag{2.7.2}$$

*Proof* First notice that due to Theorem 2.5 (see 2.5.1)

$$p(x,\ t) = \frac{e^{-\lambda t}}{2}\left[\delta(ct - x) + \delta(ct + x)\right] + P(x,\ t),$$

where

$$P(x,\ t) = \tfrac{e^{-\lambda t}}{2c}\left[\lambda I_0\left(\tfrac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right) + \tfrac{\partial}{\partial t}\ I_0\left(\tfrac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right)\right]\mathbb{1}_{\{t>|x|/c\}},\quad x \in (-\infty, \infty). \tag{2.7.3}$$

To calculate the Laplace transform of $P(x,\ \cdot)$, notice that

$$\int_{|x|/c}^{\infty} e^{-\lambda t - st}\frac{\partial}{\partial t}g(x,\ t)dt = -g(x,\ |x|/c + 0)e^{-(\lambda+s)\frac{|x|}{c}} + (\lambda + s)\widetilde{g}(x,\ \lambda + s). \tag{2.7.4}$$

The Laplace transform of the first term, $g_0(x,\ t) := \dfrac{\lambda e^{-\lambda t}}{2c} I_0\left(\tfrac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right)$ $\mathbb{1}_{\{t>|x|/c\}}$, is given by the formula (see e.g. [7], formula 29.3.93, p.1027)

$$\widetilde{g}_0(x, \cdot)(s) = \frac{\lambda/2c}{\sqrt{s(s + 2\lambda)}}e^{-\frac{|x|}{c}\sqrt{s(s+2\lambda)}}. \tag{2.7.5}$$

Applying now formula (2.7.4) to the second term of function $P(x,\ t)$ defined by (2.7.3) and collecting the result and (2.7.5), we get from (2.7.3)

$$\begin{aligned}\widetilde{P}(x,\ \cdot)(s) = &\tfrac{\lambda/2c}{\sqrt{s(s+2\lambda)}}e^{-\frac{|x|}{c}\sqrt{s(s+2\lambda)}}\\ &+\left(-\tfrac{1}{2c}e^{-(\lambda+s)\frac{|x|}{c}} + \tfrac{(\lambda+s)/2c}{\sqrt{s(s+2\lambda)}}e^{-\frac{|x|}{c}\sqrt{s(s+2\lambda)}}\right).\end{aligned} \tag{2.7.6}$$

The Laplace transform of the singular part $\frac{e^{-\lambda t}}{2}\left[\delta(ct - x) + \delta(ct + x)\right]$ is given by $\dfrac{1}{2c}e^{-(\lambda+s)\frac{|x|}{c}}$, if $x \neq 0$; and $\dfrac{1}{c}$, if $x = 0$.

Now, from (2.7.6) we obtain (2.7.2). □

*Remark 2.10* Due to continuity theorem (see e.g. [8], Chap. 13, Theorem 2a) and the convergence result (Theorem 2.6) we see that, under Kac's condition (2.6.1), the Laplace transform $\widetilde{p}(x, \cdot)$ converges to Laplace transform of the transition density of Brownian motion. This result can be obtained directly from formula (2.7.2). Indeed, from Kac's condition (2.6.1) it follows that

$$\frac{\sqrt{\lambda}}{c} \to \frac{1}{\sigma}, \qquad \frac{\lambda}{c} \to \infty,$$

and, therefore, function $\widetilde{p}(x,\ \cdot)$ defined by (2.7.2) turns into

$$\frac{1}{2c}\sqrt{\frac{s+2\lambda}{s}}\ \mathrm{e}^{-\frac{|x|}{c}\sqrt{s(s+2\lambda)}} = \frac{1}{2}\ \frac{\sqrt{\lambda}}{c}\ \sqrt{\frac{1}{s}\left(\frac{s}{\lambda}+2\right)}\ \mathrm{e}^{-|x|\ \frac{\sqrt{\lambda}}{c}\ \sqrt{s(\frac{s}{\lambda}+2)}} \\ \to \frac{1}{2}\ \frac{1}{\sigma}\ \sqrt{\frac{2}{s}}\ \mathrm{e}^{-|x|\ \frac{1}{\sigma}\ \sqrt{2s}} = \frac{1}{\sigma\sqrt{2s}}\ \mathrm{e}^{-|x|\sqrt{2s}/\sigma}, \tag{2.7.7}$$

and this is exactly the Laplace transform of the transition density of the homogeneous Brownian motion with zero drift and diffusion coefficient $\sigma^2$ (see (1.2.8)).

**Theorem 2.8** *The Laplace transform of the characteristic function $\widehat{p}(\xi,\ t)$ given by (2.4.2) of the telegraph process $X(t)$ has the form:*

$$\mathscr{L}_{t\to s}[\widehat{p}(\xi,\ t)] = \frac{s+2\lambda}{s^2+2\lambda s+c^2\xi^2}\ , \qquad s>0. \tag{2.7.8}$$

*Proof* Applying to (2.4.2) formulas (29.3.15)–(29.3.18) (p.1022, [7]) we have

$$\mathscr{L}_{t\to s}\left\{\mathrm{e}^{-\lambda t}\left[\cosh\left(t\sqrt{\lambda^2-c^2\xi^2}\right)+\frac{\lambda}{\sqrt{\lambda^2-c^2\xi^2}}\ \sinh\left(t\sqrt{\lambda^2-c^2\xi^2}\right)\right]\mathbb{1}_{\left\{|\xi|\le\frac{\lambda}{c}\right\}}\right\} \\ = \left[\frac{s+\lambda}{(s+\lambda)^2-(\lambda^2-c^2\xi^2)}+\frac{\lambda}{(s+\lambda)^2-(\lambda^2-c^2\xi^2)}\right]\mathbb{1}_{\left\{|\xi|\le\frac{\lambda}{c}\right\}}$$

and

$$\mathscr{L}_{t\to s}\left\{\mathrm{e}^{-\lambda t}\left[\cos\left(t\sqrt{c^2\xi^2-\lambda^2}\right)+\frac{\lambda}{\sqrt{\lambda^2-c^2\xi^2}}\ \sin\left(t\sqrt{c^2\xi^2-\lambda^2}\right)\right]\mathbb{1}_{\left\{|\xi|>\frac{\lambda}{c}\right\}}\right\} \\ = \left[\frac{s+\lambda}{(s+\lambda)^2+(c^2\xi^2-\lambda^2)}+\frac{\lambda}{(s+\lambda)^2+(c^2\xi^2-\lambda^2)}\right]\mathbb{1}_{\left\{|\xi|>\frac{\lambda}{c}\right\}}.$$

Summing up these equalities we obtain (2.7.8). □

## Notes

By the volume restrictions the content is rigourously selected. In particular, branching telegraph processes, travelling waves as well as applications in biology are not included in this chapter, see e.g. [9–12] and references therein.

Other references to this chapter are [4, 5, 13–30].

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# Chapter 3
# Functionals of Telegraph Process

**Abstract** In this chapter we study some important functionals of the telegraph processes. We derive the explicit formulae for the densities of the telegraph processes with reflecting and absorbing barriers, as well as first passage times and occupation time distributions. Our presentation corrects some inaccuracies and errors in this field.

**Keywords** Barriers · Occupation times · First passage time

## 3.1 Motions with Barriers

Let $X = X(t), t \geq 0$ be the telegraph process, $X(t) = c\int_0^t D(s)\mathrm{d}s$ and $D(t) = D(0)(-1)^{N(t)}$, as defined in (2.1.3). Here $N = N(t)$ is the counting Poisson process with parameter $\lambda$, $\lambda > 0$. Consider also two processes $X^{\pm} = X^{\pm}(t)$ with fixed initial direction (see (2.1.4)).

For arbitrary $a \in (-\infty, \infty)$ we define the first passage times for the processes $X^{\pm} = X^{\pm}(t)$ and $X = X(t)$,

$$T_a^{\pm} = \inf\{t \geq 0 :\ X^{\pm}(t) = a\}, \qquad T_a = \inf\{t \geq 0 :\ X(t) = a\}. \tag{3.1.1}$$

### *3.1.1 Telegraph Process with Reflecting Barrier*

Consider now the particle which moves in accordance with the same law, but with reverses of the direction at point $a$. Let $X^{ref}(t)$ be the current particle's position and $\tau^k = \tau^k(a), k = 1, 2, \ldots$ be the moment of $k$th reversal,

$$\tau^k = \inf\{t\ :\ t > \tau^{k-1}, X^{ref}(t) = a\}, \quad k \geq 1, \tau^0 = 0. \tag{3.1.2}$$

Let $a > 0$ (the case of $a < 0$ can be described similarly). Hence the particle's direction just after each reversal is $-1$ (in the case of $a < 0$ it is $+1$). The

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directions of $X(t)$ at collision points alternate, $D(\tau^k) = (-1)^{k-1}, k = 1, 2, \ldots$ Therefore, between collisions the direction of $X^{ref}(t)$ is $D^{ref}(t) = -D(\tau^k)D(t) = -(-1)^{N(\tau^k)}(-1)^{N(t)}$, if $\tau^k < t \le \tau^{k+1}, k = 1, 2, \ldots$

Hence the process $X^{ref}$ is defined as follows,

$$X^{ref}(t) = \begin{cases} X(t), & t \le \tau = \tau^1, \\ a - (-1)^{N(\tau^k)} c \int_{\tau^k}^t (-1)^{N(s)} \mathrm{d}s, & \tau^k < t \le \tau^{k+1}, k = 1, 2, \ldots \end{cases} \tag{3.1.3}$$

Moreover, at each collision with the point $x = a$ the process renews with initial direction $-1$. Indeed, at $t \in (\tau^k, \tau^{k+1}]$

$$\begin{aligned} X^{ref}(t) &= a - (-1)^{N(\tau^k)} c \int_{\tau^k}^t (-1)^{N(s)} \mathrm{d}s \\ &= a - c \int_0^{t-\tau^k} (-1)^{N(s+\tau^k)-N(\tau^k)} \mathrm{d}s = a - c \int_0^{t-\tau^k} (-1)^{N'(s)} \mathrm{d}s. \end{aligned} \tag{3.1.4}$$

Here $N'(s) = N(s + \tau^k) - N(\tau^k)$ is the number of Poisson events occurred after the moment $\tau^k$. It has the same distribution as $N(s)$.

Let

$$p^{ref}(x, t) = \mathbb{P}\{X^{ref}(t) \in \mathrm{d}x\}/\mathrm{d}x, \qquad p(x, t) = \mathbb{P}\{X(t) \in \mathrm{d}x\}/\mathrm{d}x$$

be the transition densities of $X^{ref}(t)$ and $X(t)$, respectively. Function $p = p(x, t)$ is defined by (2.5.1).

**Theorem 3.1** *If $t > a/c$, then*

$$p^{ref}(x, t) = p(x, t) + p(2a - x, t), \qquad x < a. \tag{3.1.5}$$

*Proof* Notice that, by definition (3.1.3), $X^{ref}(t) = X(t)$ if $t < \tau^1$. Further, it is easy to see that if $t \in [\tau^{2n-1}, \tau^{2n})$, then $X^{ref}(t)$ is distributed as $2a - X(t)$, and if $t \in [\tau^{2n}, \tau^{2n+1})$, then $X^{ref}(t)$ is distributed as the original process $X(t)$, $n = 1, 2, \ldots$

Let $\varphi = \varphi(x)$, $x < a$, be any smooth test-function. Then

$$\begin{aligned} \mathbb{E}\varphi\left(X^{ref}(t)\right) &= \mathbb{E}\left[\varphi\left(X(t)\right) \mathbb{1}_{\{t<\tau^1\}}\right] \\ &\quad + \sum_{n=1}^{\infty} \left\{\mathbb{E}\left[\varphi\left(X(t)\right) \mathbb{1}_{\{\tau^{2n} \le t < \tau^{2n+1}\}}\right]\right. \\ &\quad \left. + \mathbb{E}\left[\varphi\left(2a - X(t)\right) \mathbb{1}_{\{\tau^{2n-1} \le t < \tau^{2n}\}}\right]\right\} \\ &= \mathbb{E}\varphi(X(t))\mathbb{1}_{\{X(t)<a\}} + \mathbb{E}\varphi(2a - X(t))\mathbb{1}_{\{X(t)>a\}} \\ &= \int_{-\infty}^{a} \varphi(x)p(x, t)\mathrm{d}x + \int_{a}^{\infty} \varphi(2a - x)p(x, t)\mathrm{d}x \\ &= \int_{-\infty}^{a} \varphi(x)\left(p(x, t) + p(2a - x, t)\right)\mathrm{d}x. \end{aligned}$$

The theorem is proved. □

*Remark 3.1* If $t < a/c$, then $X^{ref}(t) = X(t)$, hence $p^{ref}(x,t) = p(x,t)$. Formula (3.1.5) is still valid because for $x < a$ and $t < a/c$ the density $p(2a - x, t)$ vanishes.

Notice that for $-ct < x < 2a - ct$ we also have $p(2a - x, t) = 0$ and hence again $p^{ref}(x,t) = p(x,t)$.

Similarly to (3.1.5) the conditional densities (with known initial direction)

$$p_{\pm}^{ref}(x,t) = \mathbb{P}\{X^{ref}(t) \in \mathrm{d}x \mid D(0) = \pm 1\}/\mathrm{d}x$$

take the following explicit representation (supported by $[-ct, a]$):

$$\begin{aligned} p_{-}^{ref}(x,t) &= p_{-}(x,t) + p_{-}(2a - x, t), \\ p_{+}^{ref}(x,t) &= p_{+}(x,t) + p_{+}(2a - x, t), \qquad x \le a. \end{aligned} \tag{3.1.6}$$

*Remark 3.2* It is interesting to note that densities (3.1.6) satisfy the following initial-boundary value problem. Functions $p_{\pm}^{ref}$ and $p^{ref}$ satisfy the telegraph equation

$$\frac{\partial^2 p(x,t)}{\partial t^2} + 2\lambda \frac{\partial p(x,t)}{\partial t} = c^2 \frac{\partial^2 p(x,t)}{\partial x^2}, \qquad x < a, \quad t > 0,$$

with the following initial and boundary conditions :

$$p_{\pm}^{ref}\Big|_{t=+0} = p^{ref}\Big|_{t=+0} = \delta(x),$$

$$\frac{\partial p_{\pm}^{ref}}{\partial t}\bigg|_{t=+0} = \mp c\delta'(x), \ \frac{\partial p^{ref}}{\partial t}\bigg|_{t=+0} = 0,$$

$$\frac{\partial p_{\pm}^{ref}}{\partial x}\bigg|_{x=a-} = \frac{\partial p^{ref}}{\partial x}\bigg|_{x=a-} = 0.$$

We get the proof by applying Theorem 2.2 to representation (3.1.6).

### *3.1.2 Telegraph Process with Absorbing Barrier*

Now we suppose that the particle is absorbed at the point $x = a > 0$. It means that the respective process denoted as $X^{abs}(t), t > 0$ coincides with $X(t)$ if $t < T_a$; if $t \ge T_a$ then $X^{abs}(t) = a$. Here $T_a$ is the first passage time defined by (3.1.1).

In contrast with the case of reflecting barrier it is more convenient to express the distribution of $X^{abs}(t), t > 0$ in terms of *last direction* densities. Let

$$\begin{aligned} f^{abs}(x,t) &= f^{abs}(x,t;a) = \mathbb{P}\{X^{abs}(t) \in \mathrm{d}x,\, D(t)=+1\}/\mathrm{d}x, \\ b^{abs}(x,t) &= b^{abs}(x,t;a) = \mathbb{P}\{X^{abs}(t) \in \mathrm{d}x,\, D(t)=-1\}/\mathrm{d}x, \end{aligned} \qquad x < a, \tag{3.1.7}$$

be the distribution densities for the particle which moves forwards and backwards, respectively. Similarly to Theorem 2.1 we can derive the Kolmogorov equations for $f^{abs}(x,t)$ and $b^{abs}(x,t)$.

**Theorem 3.2** *Densities $f^{abs}$ and $b^{abs}$ satisfy the following system,*

$$\begin{cases} \mathrm{L}_+^{x,t} f^{abs}(x,t) = -\lambda f^{abs}(x,t) + \lambda b^{abs}(x,t), \\ \mathrm{L}_-^{x,t} b^{abs}(x,t) = -\lambda b^{abs}(x,t) + \lambda f^{abs}(x,t), \end{cases} \quad \text{for } t > 0, x < a, \tag{3.1.8}$$

*with the initial and boundary conditions,*

$$\begin{aligned} & b^{abs}\big|_{t=0} = f^{abs}\big|_{t=0} = \frac{1}{2}\delta(x), \\ & b^{abs}\big|_{x=a-} = 0, \ \left(\mathrm{L}_+^{x,t} f^{abs} + \lambda f^{abs}\right)\big|_{x=a-} = 0. \end{aligned} \tag{3.1.9}$$

*Here the differential operators $\mathrm{L}_\pm^{x,t}$ are defined by (2.2.6).*

*Proof* Equations (3.1.8) and initial conditions of (3.1.9) follow from the system (2.2.7)–(2.2.8).

By conditioning on the last switching near the reflecting barrier $a$ we check the boundary conditions (3.1.9). Similarly to (2.2.5) we have for any $\varepsilon > 0$ and $t > a/c$

$$b^{abs}(a-\varepsilon, t) = \int_{t-\varepsilon/c}^{t} f^{abs}(a-\varepsilon + c(t-s), s)\lambda \mathrm{e}^{-\lambda(t-s)}\mathrm{d}s.$$

Passing to the limit, as $\varepsilon \to +0$, we obtain $b^{abs}(a-, t) = 0$. Notice that $b^{abs}(x,t) \equiv 0$, if $t \le a/c$, since the speed is finite. Boundary condition for $f^{abs}$ follows from the first equation of system (3.1.8). □

*Remark 3.3* Notice that for $t < a/c$ the motion cannot attain the barrier, so $f^{abs}(x,t) \equiv f(x,t)$ and $b^{abs}(x,t) \equiv b(x,t)$, $x \in [-ct, ct]$.

The (unique) solution of problem (3.1.8)–(3.1.9) is expressed in terms of transition densities $f$ and $b$ (see (2.2.4)).

**Theorem 3.3** *For $t > a/c$ the transition densities of the distribution of $X^{abs}(t)$ are*

$$\begin{aligned} b^{abs}(x,t) &= b(x,t) - b(2a-x,t), \\ f^{abs}(x,t) &= f(x,t) - f(2a-x,t) - \tfrac{2c}{\lambda} b'(2a-x,t) \end{aligned} \qquad \text{if } x < a, t \ge a/c. \tag{3.1.10}$$

*where* $b'(y,t) = \dfrac{\partial b(y,t)}{\partial y}$.

*Proof* We prove that the functions $f^{abs}$ and $b^{abs}$ defined by (3.1.10) solve system (3.1.8) with initial-boundary conditions (3.1.9).

First notice that functions $f = f(x,t)$ and $b = b(x,t)$ satisfy Eq. (2.2.7) with initial conditions given by (2.2.8). Applying operator $\mathrm{L}_-^{x,t}$ to the first equality of (3.1.10) and using (2.2.7) we have

$$\begin{aligned}
\mathrm{L}_-^{x,t} b^{abs}(x,t) &= \mathrm{L}_-^{x,t} b(x,t) - \mathrm{L}_+^{z,t} b(z,t)\big|_{z=2a-x} \\
&= \mathrm{L}_-^{x,t} b(x,t) - \mathrm{L}_-^{z,t} b(z,t)\big|_{z=2a-x} - 2cb'(2a-x,t) \\
&= -\lambda b(x,t) + \lambda f(x,t) + \lambda b(2a-x,t) \\
&\quad -\lambda f(2a-x,t) - 2cb'(2a-x,t) \\
&= -\lambda b^{abs}(x,t) + \lambda f^{abs}(x,t).
\end{aligned}$$

Next, using the similar technique we apply $\mathrm{L}_+^{x,t}$ to the second equality of (3.1.10). We have

$$\begin{aligned}
\mathrm{L}_+^{x,t} f^{abs}(x,t) &= \mathrm{L}_+^{x,t} f(x,t) - \mathrm{L}_-^{z,t} f(z,t)|_{z=2a-x} + \mathrm{L}_+^{x,t}\left[\frac{2c}{\lambda}\cdot\frac{\partial b(2a-x,t)}{\partial x}\right] \\
&= \mathrm{L}_+^{x,t} f(x,t) - \mathrm{L}_+^{z,t} f(z,t)|_{z=2a-x} + 2cf'(2a-x,t) \\
&\quad + \mathrm{L}_+^{x,t}\left[\frac{2c}{\lambda}\cdot\frac{\partial b(2a-x,t)}{\partial x}\right] \\
&= -\lambda f(x,t) + \lambda b(x,t) + \lambda f(2a-x,t) - \lambda b(2a-x,t) \\
&\quad - 2c\frac{\partial f(2a-x,t)}{\partial x} \\
&\quad + \frac{2c}{\lambda}\frac{\partial}{\partial x}\left[\frac{\partial b(2a-x,t)}{\partial t} + c\frac{\partial b(2a-x,t)}{\partial x}\right] \\
&= -\lambda f(x,t) + \lambda b(x,t) + \lambda f(2a-x,t) - \lambda b(2a-x,t) \\
&\quad + \frac{2c}{\lambda}\frac{\partial}{\partial x}\left[-\lambda f(2a-x,t) + \frac{\partial b(2a-x,t)}{\partial t} + c\frac{\partial b(2a-x,t)}{\partial x}\right] \\
&= -\lambda f(x,t) + \lambda b(x,t) + \lambda f(2a-x,t) - \lambda b(2a-x,t) \\
&\quad + \frac{2c}{\lambda}\frac{\partial}{\partial x}\left[-\lambda b(2a-x,t)\right] \\
&= -\lambda f^{abs}(x,t) + \lambda b^{abs}(x,t).
\end{aligned}$$

Therefore functions $f^{abs} = f^{abs}(x,t)$ and $b^{abs} = b^{abs}(x,t)$ defined by (3.1.10) satisfy Eq. (3.1.8).

The initial conditions in (3.1.9) follow from the initial conditions (2.2.8) for functions $f = f(x,t)$ and $b = b(x,t)$. Boundary conditions in (3.1.9) are fulfilled, since

$$b(x,t)|_{x=a} = b(2a-x,t)|_{x=a} = b(a,t)$$

and due to the first equation of (3.1.8). □

Denote by $p_{\pm}^{abs}$ the conditional transition densities of $X^{abs}$ when the initial direction is known:

$$\begin{aligned} p_{-}^{abs}(x,t) &= p_{-}^{abs}(x,t;a) = \mathbb{P}\{X^{abs}(t) \in \mathrm{d}x | D(0) = -1\}/\mathrm{d}x, \\ p_{+}^{abs}(x,t) &= p_{+}^{abs}(x,t;a) = \mathbb{P}\{X^{abs}(t) \in \mathrm{d}x | D(0) = +1\}/\mathrm{d}x, \end{aligned} \qquad x < a. \tag{3.1.11}$$

Notice that, due to symmetry and time-reversal trick (see Remark 2.7 and (2.5.16)), from Theorem 3.3 we easily obtain for $t > a/c$ and $x < a$

$$\begin{aligned} p_{-}^{abs}(x,t;a) &= 2f^{abs}(-x,t;a-x) = 2f(-x,t) \\ &\quad -2f(2a-x,t) - \tfrac{4c}{\lambda}b'(2a-x,t) \\ &= p_{-}(x,t) - p_{+}(2a-x,t) - \tfrac{2c}{\lambda}p'_{-}(2a-x,t), \\ p_{+}^{abs}(x,t;a) &= 2b^{abs}(-x,t;a-x) = b(-x,t) - b(2a-x,t) \\ &= p_{+}(x,t) - p_{-}(2a-x,t). \end{aligned} \tag{3.1.12}$$

*Remark 3.4* Notice that first formula of (3.1.12) differs from formula (3.3) of [1].

Let us remind notations $P(x,t)$ and $P_{\pm}(x,t)$ introduced by (2.5.1) and (2.5.15),

$$\begin{aligned} P(x,t) &= \tfrac{\lambda}{2c}\mathrm{e}^{-\lambda t}\left[I_0\left(\tfrac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) + \tfrac{ct}{\sqrt{c^2t^2-x^2}}I_1\left(\tfrac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right], \\ P_{\pm}(x,t) &= \tfrac{\lambda}{2c}\mathrm{e}^{-\lambda t}\left[I_0\left(\tfrac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) + \tfrac{ct \pm x}{\sqrt{c^2t^2-x^2}}I_1\left(\tfrac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right]. \end{aligned}$$

The survival probabilities of $X^{abs}(t)$ can be expressed as follows.

**Theorem 3.4** *If $t > a/c$, then*

$$\begin{aligned} \mathbb{P}\{X^{abs}(t) < a | D(0) = -1\} &= \int_{-a}^{a} P(x,t)\mathrm{d}x + \tfrac{2c}{\lambda}P_{-}(a,t), \\ \mathbb{P}\{X^{abs}(t) < a | D(0) = +1\} &= \int_{-a}^{a} P(x,t)\mathrm{d}x. \end{aligned} \tag{3.1.13}$$

*Proof* Applying the first formula of (3.1.12) we have

$$\mathbb{P}\{X^{abs}(t) < a | D(0) = -1\} = \int_{-\infty}^{a} p_{-}^{abs}(x,t;a)\mathrm{d}x = \mathrm{e}^{-\lambda t} + \int_{-ct}^{a} P_{-}^{abs}(x,t;a)\mathrm{d}x,$$

where $\mathrm{e}^{-\lambda t}$ is the probability that no one Poisson event occurs till time $t$ and $P_{-}^{abs}(x,t;a) = P_{-}(x,t) - P_{+}(2a-x,t) - \frac{2c}{\lambda}P'_{-}(2a-x,t)$ is the absolutely continuous part of the distribution, $P'_{-}(y,t) = \dfrac{\partial P_{-}(y,t)}{\partial y}$.

Hence

$$
\begin{aligned}
\mathbb{P}\{X^{abs}(t) < a | D(0) = -1\} \\
&= \mathrm{e}^{-\lambda t} + \int\limits_{-ct}^{a} P_{-}(x,t)\mathrm{d}x - \int\limits_{2a-ct}^{a} \left[ P_{+}(2a-x,t) + \frac{2c}{\lambda} P'_{-}(2a-x,t) \right] \mathrm{d}x \\
&= \mathrm{e}^{-\lambda t} + \int\limits_{-ct}^{a} P_{-}(x,t)\mathrm{d}x - \int\limits_{a}^{ct} P_{+}(x,t)\mathrm{d}x - \frac{2c}{\lambda} \int\limits_{a}^{ct} P'_{-}(x,t)\mathrm{d}x \\
&= \mathrm{e}^{-\lambda t} + \int\limits_{-ct}^{a} P_{-}(x,t)\mathrm{d}x - \int\limits_{-ct}^{-a} P_{-}(x,t)\mathrm{d}x - \frac{2c}{\lambda} \left[ P_{-}(ct,t) - P_{-}(a,t) \right] \\
&= \int\limits_{-a}^{a} P(x,t)\mathrm{d}x + \frac{2c}{\lambda} P_{-}(a,t),
\end{aligned}
$$

since $P_{+}(-x,t) \equiv P_{-}(x,t)$ (see (2.5.15)), $\int_{-a}^{a} P_{-}(x,t)\mathrm{d}x = \int_{-a}^{a} P_{+}(x,t)\mathrm{d}x = \int_{-a}^{a} P(x,t)$ $\mathrm{d}x$ and $P_{-}(ct,t) = \dfrac{\lambda}{2c}\mathrm{e}^{-\lambda t}$.

The proof of the second equality of (3.1.13) is similar. □

*Remark 3.5* Since

$$
\begin{aligned}
\mathbb{P}\{X^{abs}(t) < a\} &= \frac{1}{2} \left[ \mathbb{P}\{X^{abs}(t) < a | D(0) = -1\} + \mathbb{P}\{X^{abs}(t) < a | D(0) = +1\} \right] \\
&= \int\limits_{-a}^{a} P(x,t)\mathrm{d}x + \frac{c}{\lambda} P_{-}(a,t),
\end{aligned}
$$

we have the obvious inequalities

$$
\mathbb{P}\{X^{abs}(t) < a | D(0) = +1\} < \mathbb{P}\{X^{abs}(t) < a\} < \mathbb{P}\{X^{abs}(t) < a | D(0) = -1\}.
$$

## 3.2 Occupation Time Distributions

This section is devoted to the detailed description of the proportion of time spent by the telegraph process $X = X(t)$ on the positive semi-axis. This result generalises the famous and beautiful arcsine law by Paul Lévy which dates back about 70 years (see [2, Théorème 3, pp. 301–302]).

Let $w = w(t)$ be a standard Brownian motion on $\mathbb{R}$ starting from the origin ($w(0) = 0$), and consider the occupation time functional (see (1.2.12))

$$
\mathfrak{h}_T := \frac{1}{T} \int_0^T H(w(t))\mathrm{d}t, \qquad T > 0, \tag{3.2.1}
$$

where $H(x)$ is the Heaviside unit step function (i.e., $H(x) = 0$ for $x \le 0$ and $H(x) = 1$ for $x > 0$). The distribution of the random variable $\mathfrak{h}_T$ is given by the arcsine law, (1.2.13).

In this section we obtain the similar property of the telegraph process.

### 3.2.1 Feynmann-Kac Connection

The arcsine law for Brownian motion is usually derived using the Feynmann-Kac formula. Let us present the Feynmann-Kac formula for telegraph processes, which is of interest by itself.

Let $q = q(x),\ g = g(x),\ x \in (-\infty,\ \infty)$ be bounded functions, such that $g \in C^1(\mathbb{R})$ and $q$ is piecewise continuous, i. e., $q \in C(\mathbb{R} \setminus D_q)$, where $D_q$ is a finite set of discontinuities, and moreover, $q$ has finite left and right limits at the points of $D_q$. Consider the Cauchy problem

$$\begin{cases} \frac{\partial v_+}{\partial t} - c\frac{\partial v_+}{\partial x} & = \lambda(v_- - v_+) + qv_+, \\ \frac{\partial v_-}{\partial t} + c\frac{\partial v_-}{\partial x} & = \lambda(v_+ - v_-) + qv_-, \quad t > 0,\ x \in (-\infty, \infty), \\ \qquad v_\pm \mid_{t\downarrow 0} & = g(x), \qquad x \in (-\infty, \infty). \end{cases} \tag{3.2.2}$$

Let $X(x,\ t) = x + X(t)$ be the telegraph process which starts from point $x \in (-\infty, \infty)$.

**Theorem 3.5** *The functions*

$$v_+(x,\ t) = \mathbb{E}_+ \left\{ g(X(x,\ t)) \exp\left( \int_0^t q((X(x,\ s))\mathrm{d}s \right) \right\}, \tag{3.2.3}$$

$$v_-(x,\ t) = \mathbb{E}_- \left\{ g(X(x,\ t)) \exp\left( \int_0^t q((X(x,\ s))\mathrm{d}s \right) \right\} \tag{3.2.4}$$

*defined by the conditional distributions (given the initial direction $D(0) = \pm 1$) for all $(x, t) \in \mathbb{R} \times (0, \infty)$ such that $x \pm ct \notin D_q$ satisfy Cauchy problem (3.2.2).*

*Proof* The initial conditions of the Cauchy problem (3.2.2) follow immediately from (3.2.3)–(3.2.4). To prove that equations in (3.2.2) are valid for $v_\pm$ we first obtain the system of integral equations.

Denote by $f_+(y, t)$ and $f_-(y, t)$ the conditional probability densities for the particle that currently moves forward ($D(t) = +1$), given the initial direction $D(0) = +1$ and $D(0) = -1$, respectively,

$$f_\sigma(y,t) = \mathbb{P}\{X(t) \in \mathrm{d}y,\, D(t) = +1 \mid D(0) = \sigma\}/\mathrm{d}y, \qquad \sigma = \pm 1.$$

Conditional densities for the current backward direction are denoted by $b_+(y,t)$ and $b_-(y,t)$, namely

$$b_\sigma(y,t) = \mathbb{P}\{X(t) \in \mathrm{d}y,\, D(t) = -1 \mid D(0) = \sigma\}/\mathrm{d}y, \qquad \sigma = \pm 1.$$

Notice that functions $\mathbf{f} = (f_+(x,t), f_-(x,t))^T$ and $\mathbf{b} = (b_+(x,t), b_-(x,t))^T$ satisfy Eq. (2.2.7) and the initial conditions

$$\mathbf{f}|_{t=0} = (\delta(x), 0)^T \text{and } \mathbf{b}|_{t=0} = (0, \delta(x))^T. \tag{3.2.5}$$

The integral equations for functions $v_\pm$ are expressed in terms of conditional densities $f_\pm$ and $b_\pm$.

**Lemma 3.1** *Functions $v_\pm = v_\pm(x,t)$, which are defined by (3.2.3)–(3.2.4), satisfy the following system :*

$$\begin{aligned} v_+(x,t) = v_+^0(x,t) + \textstyle\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} q(x+y)\big[&v_+(x+y,t-s)f_+(y,s) \\ &+ v_-(x+y,t-s)b_+(y,s)\big]\,\mathrm{d}y, \end{aligned} \tag{3.2.6}$$

$$\begin{aligned} v_-(x,t) = v_-^0(x,t) + \textstyle\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} q(x+y)\big[&v_+(x+y,t-s)f_-(y,s) \\ &+ v_-(x+y,t-s)b_-(y,s)\big]\,\mathrm{d}y. \end{aligned} \tag{3.2.7}$$

*Here $v_\pm^0(x,t) = \mathbb{E}_\pm\{g(x+X(t))\}$.*

*Proof* (of the Lemma) First notice that for any locally integrable function $\Phi$ the following statement is valid: if

$$\varphi_1(t) := \exp\left(\int_0^t \Phi(s)\mathrm{d}s\right) \quad \text{and} \quad \varphi_2(t) := 1 + \int_0^t \Phi(s)\exp\left(\int_s^t \Phi(r)\mathrm{d}r\right)\mathrm{d}s,$$

then $\varphi_1 \equiv \varphi_2$. This is clear because both functions, $\varphi_1$ and $\varphi_2$, satisfy the equation

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \Phi(t)\varphi(t)$$

with the initial condition $\varphi(0) = 1$.

Since $\varphi_1 \equiv \varphi_2$, then

$$\exp\left(\int_0^t \Phi(s)\mathrm{d}s\right) = 1 + \int_0^t \Phi(s)\exp\left(\int_s^t \Phi(r)\mathrm{d}r\right)\mathrm{d}s,$$

with $\Phi(s) = q(x+X(s))$. Then multiplying by $g(x+X(t))$ and taking the expectations, we obtain

$$v_{\pm}(x,t) = v_{\pm}^{0}(x,t) + \mathbb{E}_{\pm}\left\{g(x+X(t))\int_0^t q(x+X(s)) \exp\left(\int_s^t q(x+X(r))\mathrm{d}r\right)\mathrm{d}s\right\}, \tag{3.2.8}$$

The process $(X(t), D(t))$ is renewable at any fixed time. We compute the expectations in (3.2.8) conditioning on the state at time $s$, $0 < s < t$. We get

$$\begin{aligned}
&\mathbb{E}_{\pm}q(x+X(s))g(x+X(t))\exp\left(\int_s^t q(x+X(r))\mathrm{d}r\right)\\
&=\int_{-\infty}^{\infty} q(x+y)\Big[f_{\pm}(y,s)\mathbb{E}_{+}\Big\{g(x+y+X(t-s))\\
&\qquad\qquad\exp\left(\int_0^{t-s} q(x+y+X(r))\mathrm{d}r\right)\Big\}\\
&\qquad+b_{\pm}(y,s)\mathbb{E}_{-}\left\{g(x+y+X(t-s))\exp\left(\int_0^{t-s} q(x+y+X(r))\mathrm{d}r\right)\right\}\Big]\mathrm{d}y\\
&=\int_{-\infty}^{\infty} q(x+y)\left[f_{\pm}(y,s)v_{+}(x+y,t-s)+b_{\pm}(y,s)v_{-}(x+y,t-s)\right]\mathrm{d}y.
\end{aligned}$$

Substituting this result into (3.2.8) we obtain (3.2.6)–(3.2.7). Lemma 3.1 is proved. □

Now we derive Eq. (3.2.2) from (3.2.6)–(3.2.7). We will check the first equation of (3.2.2). The proof of the second one is similar.

Differentiating the integral equation (3.2.6), and then integrating by parts in view of initial conditions (3.2.5), we have

$$\begin{aligned}
\frac{\partial v_{+}}{\partial t} &= \frac{\partial v_{+}^{0}}{\partial t} + \int_{-\infty}^{\infty} q(x+y)\left[v_{+}(x+y,0)f_{+}(y,t)+v_{-}(x+y,0)b_{+}(y,t)\right]\mathrm{d}y\\
&\qquad-\int_0^t \mathrm{d}s\int_{-\infty}^{\infty} q(x+y)\left[\frac{\partial v_{+}(x+y,t-s)}{\partial s}f_{+}(y,s)\right.\\
&\qquad\left.+\frac{\partial v_{-}(x+y,t-s)}{\partial s}b_{+}(y,s)\right]\mathrm{d}y\\
&=\frac{\partial v_{+}^{0}}{\partial t} + q(x)v_{+}(x,t)\\
&\qquad+\int_0^t \mathrm{d}s\int_{-\infty}^{\infty} q(x+y)\left[v_{+}(x+y,t-s)\frac{\partial f_{+}}{\partial s}(y,s)\right.\\
&\qquad\qquad\left.+v_{-}(x+y,t-s)\frac{\partial b_{+}}{\partial s}(y,s)\right]\mathrm{d}y,
\end{aligned}$$

and

$$c\frac{\partial v_+}{\partial x} = c\frac{\partial v_+^0}{\partial x} + c\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \frac{\partial}{\partial y}\left[q(x+y)v_+(x+y,t-s)\right] f_+(y,s)$$
$$+ \frac{\partial}{\partial y}\left[q(x+y)v_-(x+y,t-s)\right] b_+(y,s)$$
$$= c\frac{\partial v_+^0}{\partial x} - c\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} q(x+y)\left[v_+(x+y,t-s)\frac{\partial f_+}{\partial y}(y,s)\right.$$
$$\left. + v_-(x+y,t-s)\frac{\partial b_+}{\partial y}(y,s)\right]\mathrm{d}y.$$

The first equation of (3.2.2) follows from the latter two equations and (3.2.6). This can be demonstrated by using the Kolmogorov equations (2.2.7) for functions **f**, **b** and dual equations (2.2.12) for $v_\pm^0$. Theorem 3.5 is completely proved. □

### *3.2.2 Statement of the Main Result*

For $T > 0$ consider the following occupation time random variables

$$\eta_T := \frac{1}{T}\int_0^T H(X(t))\mathrm{d}t, \qquad \eta_T^\pm := \frac{1}{T}\int_0^T H(X^\pm(t))\mathrm{d}t, \tag{3.2.9}$$

where $H(x) = \mathbb{1}_{(0,\infty)}(x)$ is the Heaviside step function and $X(t)$, $X^\pm(t)$ are the telegraph processes introduced above (see (2.1.3) and (2.1.4)). Note that the total time spent by the processes $X^\pm(t)$, $0 \le t \le T$, at the origin (and at any fixed level $x$) almost surely (a.s.) is equal to zero, since by Fubini's theorem we have

$$\mathbb{E}\int_0^T \mathbb{1}_{\{x\}}(X^\pm(t))\mathrm{d}t = \int_0^T \mathbb{P}\{X^\pm(t) = x\}\mathrm{d}t = 0. \tag{3.2.10}$$

Hence, the complementary quantity $1 - \eta_T^\pm$ a.s. represents the proportion of time spent by the processes $X^\pm(t)$, $0 \le t \le T$ on the negative half of the axis,

$$1 - \eta_T^\pm = \frac{1}{T}\int_0^T \mathbb{1}_{(-\infty,0)}(X^\pm(t))\mathrm{d}t \qquad \text{(a.s.)},$$

and by symmetry it follows that

$$\eta_T^+ \stackrel{d}{=} 1 - \eta_T^-, \qquad \eta_T^- \stackrel{d}{=} 1 - \eta_T^+. \tag{3.2.11}$$

Let us consider the function $\varphi_T(t)$, $t \ge 0$ defined by

$$\varphi_T(t) := \frac{1}{4\pi\lambda T}\int_0^t \frac{1-\mathrm{e}^{-2\lambda Tu}}{u^{3/2}\sqrt{t-u}}\mathrm{d}u, \qquad t>0, \qquad \varphi_T(0) := \frac{1}{2}. \tag{3.2.12}$$

After the substitution $u = ty$, we have in the limit as $t \downarrow 0$,

$$\varphi_T(t) = \frac{1}{4\pi\lambda Tt}\int_0^1 \frac{1-\mathrm{e}^{-2\lambda Tty}}{y^{3/2}\sqrt{1-y}}\mathrm{d}y \to \frac{1}{2\pi}\int_0^1 \frac{1}{\sqrt{y(1-y)}}\mathrm{d}y = \frac{1}{2} \tag{3.2.13}$$

(cf (1.2.14)), and thus $\varphi_T(\cdot)$ is continuous at zero (and hence everywhere on $[0,\infty)$). Note the following useful scaling relation, which easily follows from the representation of $\varphi$ given by (3.2.13):

$$\varphi_{\alpha T}(t) = \varphi_T(\alpha t), \qquad t \ge 0, \quad \alpha > 0. \tag{3.2.14}$$

Let us also set

$$\psi_T(y) := 2\lambda T\varphi_T(y)\varphi_T(1-y), \qquad 0 \le y \le 1. \tag{3.2.15}$$

We are now ready to state our main result.

**Theorem 3.6** *The random variables $\eta_T^{\pm}$ defined in (3.2.9) have the distributions*

$$\mathbb{P}\{\eta_T^{\pm} \in \mathrm{d}y\} = 2\varphi_T(1)\delta_{x^{\pm}}(\mathrm{d}y) + \psi_T(y)\mathrm{d}y, \qquad 0 \le y \le 1, \tag{3.2.16}$$

*where $\delta_{x^{\pm}}$ is the Dirac measure (of unit mass) at point $x^{\pm}$, with $x^- = 0$, $x^+ = 1$. Furthermore, the distribution of $\eta_T$ (see (3.2.9)) is given by the formula*

$$\mathbb{P}\{\eta_T \in \mathrm{d}y\} = \varphi_T(1)\delta_0(\mathrm{d}y) + \varphi_T(1)\delta_1(\mathrm{d}y) + \psi_T(y)\mathrm{d}y, \qquad 0 \le y \le 1. \tag{3.2.17}$$

In other words, the distributions of $\eta_T^-$, $\eta_T^+$ has a discrete part with atom of mass $2\varphi_T(1)$ at point 0 or 1, respectively, and an absolutely continuous part with the density $\psi_T$ defined by (3.2.15). Similarly, the distribution of $\eta_T$ has atoms at points 0 and 1, both of mass $\varphi_T(1)$, and an absolutely continuous part with the density $\psi_T$ as above. Atoms correspond to the case of unreversed particle.

*Remark 3.6* The $\pm$-duality in (3.2.16) becomes clear from relation (3.2.11) and the symmetry property $\psi_T(y) \equiv \psi_T(1-y)$ (see (3.2.15)).

*Remark 3.7* Integrating in (3.2.16) over [0, 1], we arrive at the curious identity

$$2\varphi_T(1) + 2\lambda T\int_0^1 \varphi_T(y)\varphi_T(1-y)\mathrm{d}y = 1.$$

*Remark 3.8* Using an integral formula (see [3, 9.6.16, p. 376]) or (1.5.7) for the modified Bessel function $I_0$, it is easy to check that the function $\varphi_T$ defined by (3.2.12) admits another representation,

$$\varphi_T(t) = \frac{1}{2\lambda T t} \int_0^{\lambda T t} e^{-y} I_0(y) dy, \qquad t > 0. \tag{3.2.18}$$

Indeed, applying formula (1.5.7) we have

$$\begin{aligned}
\frac{1}{2\lambda T t} \int_0^{\lambda T t} e^{-y} I_0(y) dy &= \frac{1}{2\pi \lambda T t} \int_0^{\lambda T t} e^{-y} dy \int_{-1}^{1} \frac{e^{y\xi}}{\sqrt{1-\xi^2}} d\xi \\
&= \frac{1}{2\pi \lambda T t} \int_{-1}^{1} \frac{d\xi}{\sqrt{1-\xi^2}} \int_0^{\lambda T t} e^{-(1-\xi)y} dy \\
&= \frac{1}{2\pi \lambda T t} \int_{-1}^{1} \frac{1 - e^{-(1-\xi)\lambda T t}}{(1-\xi)^{3/2}(1+\xi)^{1/2}} d\xi \\
&= \frac{1}{4\pi \lambda T t} \int_0^{1} \frac{1 - e^{-2u\lambda T t}}{u^{3/2}(1-u)^{1/2}} du = \varphi_T(t),
\end{aligned}$$

proving (3.2.18).

Then formula (3.2.18) can be transformed (see [3][11.3.12, p. 483]) into

$$\varphi_T(t) = \frac{1}{2} e^{-\lambda T t} \big( I_0(\lambda T t) + I_1(\lambda T t) \big).$$

Thus, the distribution of $\eta_T^{\pm}(0)$ and $\eta_T(0)$ can be expressed in terms of the modified Bessel functions $I_0$ and $I_1$, like the distribution of the telegraph process (cf. (2.5.1), Theorem 2.5).

In view of the scaling properties of the telegraph process (Theorem 2.6), it is not surprising that the random variables $\eta_T, \eta_T^{\pm}$ converge in distribution to the arcsine law as $T \to \infty$.

**Theorem 3.7** *For* $0 \le \theta \le 1$,

$$\lim_{T\to\infty} \mathbb{P}\{\eta_T \le \theta\} = \lim_{T\to\infty} \mathbb{P}\{\eta_T^{\pm} \le \theta\} = \frac{2}{\pi} \arcsin \sqrt{\theta}. \tag{3.2.19}$$

### *3.2.3 Proof of Theorems 3.6 and 3.7*

For arbitrary $\beta \in \mathbb{R}$, set

$$v_T^{\pm}(\xi, t) = \mathbb{E}\left[ \exp\left\{ \frac{-\beta}{T} \int_0^{Tt} H(cT\xi + X^{\pm}(u)) du \right\} \right], \qquad \xi \in \mathbb{R}, \quad t \ge 0. \tag{3.2.20}$$

Since $H(\cdot)$ is a bounded function, the expectation in (3.2.20) is finite for all $\beta \in \mathbb{R}$.

Note that the value

$$v_T^{\pm}(0,1) = \mathbb{E}\left[\mathrm{e}^{-\beta\eta_T^{\pm}}\right] \tag{3.2.21}$$

represents the Laplace transform of the random variable defined in (3.2.9), and, hence, it can be used in order to characterise the distribution of $\eta_T^{\pm}$.

Let us record some simple properties of the function $v_T^{\pm}$.

**Lemma 3.2** *For each $\beta \in \mathbb{R}$ and any $T > 0$, the functions $v_T^{\pm}(\xi, t)$ are continuous in $\xi \in (-\infty, \infty)$ and in $t$, $t \geq 0$.*

*Moreover*

$$\lim_{\xi \to -\infty} v_T^{\pm}(\xi, t) = 1, \qquad \lim_{\xi \to +\infty} v_T^{\pm}(\xi, t) = \mathrm{e}^{-\beta t}. \tag{3.2.22}$$

*Proof* Continuity in $t \in [0, \infty)$ is obvious. As mentioned above (see (3.2.10)), for any $\xi_0 \in \mathbb{R}$ we have $cT\xi_0 + X_u^{\pm} \neq 0$ for all $u \in [0, Tt]$ except a (random) finite set. Since the function $H$ is continuous outside zero, this implies that, for such $u$, $H(cT\xi + X_u^{\pm}) \xrightarrow{\text{a.s.}} H(cT\xi_0 + X_u^{\pm})$ as $\xi \to \xi_0$ and hence, by Lebesgue's dominated convergence theorem, $\int_0^{Tt} H(cT\xi + X_u^{\pm})\mathrm{d}u \xrightarrow{\text{a.s.}} \int_0^{Tt} H(cT\xi_0 + X_u^{\pm})\mathrm{d}u$ as $\xi \to \xi_0$. The continuity of $v_T^{\pm}(\cdot, t)$ at point $\xi_0$ now follows by Lebesgue's dominated convergence theorem applied to the expectation (3.2.20), since

$$\exp\left\{\frac{-\beta}{T}\int_0^{Tt} H(cT\xi + X^{\pm}(u))\mathrm{d}u\right\}$$

is bounded (for any fixed $t$).

To prove (3.2.22), note that, for $T > 0$ and each $u \in [0, Tt]$, we have $cT\xi + X_u^{\pm} > 0$ (i. e. $v_T^{\pm}(\xi, t) = \mathrm{e}^{-\beta t}$) for $\xi > \frac{cTt}{cT} = t$, and $cT\xi + X_u^{\pm} < 0$ (i. e. $v_T^{\pm}(\xi, t) = 1$) for $\xi < -t$. Lemma 3.2 is proved. □

From definition (3.2.20) it follows that if $\beta \geq 0$ then, for each $\xi \in \mathbb{R}$, the functions $v_T^{\pm}(\xi, \cdot)$ are bounded on $[0, \infty)$, so the Laplace transform

$$w_T^{\pm}(\xi, s) := \int_0^{\infty} \mathrm{e}^{-st} v_T^{\pm}(\xi, t)\mathrm{d}t, \qquad s > 0 \tag{3.2.23}$$

is well defined.

**Lemma 3.3** *Set $\tilde{s} := s + \beta$. For any fixed $s > 0$ the functions $w_T^{\pm} = w_T^{\pm}(\xi, s)$ defined by (3.2.23) are continuous in $\xi \in \mathbb{R}$ and satisfy the following system*

$$\begin{cases} \frac{\mathrm{d}w_T^+}{\mathrm{d}\xi} = \lambda T\left(w_T^+ - w_T^-\right) + \left(s + \beta H(\xi)\right)w_T^+ - 1, \\ \frac{\mathrm{d}w_T^-}{\mathrm{d}\xi} = \lambda T\left(w_T^+ - w_T^-\right) - \left(s + \beta H(\xi)\right)w_T^+ + 1, \end{cases} \qquad \xi \neq 0, \tag{3.2.24}$$

*and the boundary conditions*

$$\lim_{\xi\to-\infty} w_T^{\pm}(\xi, s) = s^{-1}, \qquad \lim_{\xi\to+\infty} w_T^{\pm}(\xi, s) = \tilde{s}^{-1}. \tag{3.2.25}$$

*Proof* Functions $w_T^{\pm}(\xi, s)$ are continuous in $\xi$, since $v_T^{\pm}(\xi, t)$ are bounded and continuous (see Lemma 3.2 ). Further, applying Theorem 3.5 (with $g(x) \equiv 1$ and $q(x) = -\beta T^{-1} H(x)$), we see that the functions $v_T^{\pm} = v_T^{\pm}(\xi, t)$ defined by (3.2.20) satisfy the Cauchy problem

$$\begin{cases} -\frac{\partial v_T^+}{\partial t} + \frac{\partial v_T^+}{\partial \xi} = \lambda T\left(v_T^+ - v_T^-\right) + \beta H(cT\xi)v_T^+, \\ \frac{\partial v_T^-}{\partial t} + \frac{\partial v_T^-}{\partial \xi} = \lambda T\left(v_T^+ - v_T^-\right) - \beta H(cT\xi)v_T^-, \qquad t > 0, \quad T\xi \pm t \neq 0, \\ v_T^{\pm}(\xi, 0) = 1, \qquad \xi \in \mathbb{R}. \end{cases} \tag{3.2.26}$$

Integrating by parts and using the initial condition (3.2.26), we have

$$\int_0^\infty \mathrm{e}^{-st} \frac{\partial v_T^{\pm}(\xi, t)}{\partial t} \mathrm{d}t = -v_T^{\pm}(\xi, 0) + s \int_0^\infty \mathrm{e}^{-st} v_T^{\pm}(\xi, t)\mathrm{d}t = -1 + s w_T^{\pm}(\xi, s). \tag{3.2.27}$$

Applying the Laplace transformation (with respect to $t$) to Eq. (3.2.26) and taking into account (3.2.27), we immediately obtain the differential equation (3.2.24). Finally, the boundary conditions (3.2.25) readily follow from (3.2.29) by Lebesgue's dominated convergence theorem applied to (3.2.23). Lemma 3.3 is proved. □

**Lemma 3.4** *Suppose that for each $s > 0$, functions $w^{\pm_T}(\cdot, s)$ are continuous and satisfy Eq. (3.2.24) and boundary conditions (3.2.25). Then*

$$\begin{aligned} w_T^+(0, s) &= \tfrac{2}{\tilde{\kappa}+\tilde{s}} + \tfrac{2\lambda T}{(\kappa+s)(\tilde{\kappa}+\tilde{s})}, \\ w_T^-(0, s) &= \tfrac{2}{\kappa+s} + \tfrac{2\lambda T}{(\kappa+s)(\tilde{\kappa}+\tilde{s})}, \end{aligned} \tag{3.2.28}$$

*where $\kappa = \kappa(s) := \sqrt{s(s + 2\lambda T)}$ and $\tilde{\kappa} := \kappa(\tilde{s}) = \sqrt{\tilde{s}(\tilde{s} + 2\lambda T)}$.*

*Proof* Solving separately system (3.2.24) for $\xi < 0$ and for $\xi > 0$ with boundary conditions (3.2.25) at infinity, we find

$$w_T^{\pm}(\xi, s) = \begin{cases} K^{\pm}(s)\mathrm{e}^{\kappa\xi} + 1/s, & \xi < 0, \\ \tilde{K}^{\pm}(s)\mathrm{e}^{-\kappa\xi} + 1/\tilde{s}, & \xi > 0, \end{cases} \tag{3.2.29}$$

where the coefficients $K^{\pm} = K^{\pm}(s)$ and $\tilde{K}^{\pm} = \tilde{K}^{\pm}(s)$ satisfy the equations

$$\begin{cases} \kappa K^+ = \lambda T(K^+ - K^-) + sK^+, \\ \kappa K^- = \lambda T(K^+ - K^-) - sK^-, \\ \tilde{\kappa}\tilde{K}^+ = \lambda T(\tilde{K}^+ - \tilde{K}^-) + s\tilde{K}^+, \\ \tilde{\kappa}\tilde{K}^- = \lambda T(\tilde{K}^+ - \tilde{K}^-) - s\tilde{K}^-. \end{cases} \tag{3.2.30}$$

Taking into account the continuity of $w_T^{\pm}(\cdot, s)$ at zero, we also obtain

$$\begin{cases} K^+(s) + 1/s = \tilde{K}^+(s) + 1/\tilde{s}, \\ K^-(s) + 1/s = \tilde{K}^-(s) + 1/\tilde{s}. \end{cases} \tag{3.2.31}$$

The systems (3.2.30) and (3.2.31) are easily solved to yield

$$K^{\pm}(s) = \frac{-\beta}{s} \cdot \frac{\kappa \pm s}{s\tilde{\kappa} + \tilde{s}\kappa} < 0, \qquad \tilde{K}^{\pm}(s) = \frac{\beta}{\tilde{s}} \cdot \frac{\tilde{\kappa} \mp \tilde{s}}{s\tilde{\kappa} + \tilde{s}\kappa} > 0,$$

with $\kappa$ and $\tilde{\kappa}$ defined in the Lemma.

Hence, from Eq. (3.2.29) we obtain

$$\begin{aligned} w_T^{\pm}(0, s) &= K^{\pm}(s) + \frac{1}{s} = \frac{-\beta(\kappa \pm s)}{s(s\tilde{\kappa} + \tilde{s}\kappa)} + \frac{1}{s} \\ &= \frac{\kappa + \tilde{\kappa} \mp \beta}{s\tilde{\kappa} + \tilde{s}\kappa} = \frac{\kappa \pm s + \tilde{\kappa} \mp \tilde{s}}{s\tilde{\kappa} + \tilde{s}\kappa}. \end{aligned} \tag{3.2.32}$$

Note that

$$\kappa^2 - s^2 = 2\lambda T s, \qquad \tilde{\kappa}^2 - \tilde{s}^2 = 2\lambda T \tilde{s},$$

hence

$$w_T^{+}(0, s) = \frac{2\lambda T}{(\kappa - s)(\tilde{\kappa} + \tilde{s})}, \qquad w_T^{-}(0, s) = \frac{2\lambda T}{(\kappa + s)(\tilde{\kappa} - \tilde{s})},$$

which is equivalent to (3.2.28). Lemma 3.4 is proved. □

To finish the proof of Theorem 3.6, it is sufficient to calculate the inverse Laplace transformation on the right-hand side of Eq. (3.2.28).

**Lemma 3.5** *Let $\varphi_T(t), t \geq 0$ be defined by (3.2.12). Then, for $s > 0$, we have*

$$\int_0^\infty e^{-st} \varphi_T(t) dt = \frac{1}{\kappa + s} \tag{3.2.33}$$

*and*

$$\int_0^\infty e^{-st} \left( \int_0^t e^{-\beta y} \varphi_T(y) \varphi_T(t - y) dy \right) dt = \frac{1}{(\kappa + s)(\tilde{\kappa} + \tilde{s})}, \tag{3.2.34}$$

*where $\kappa = \kappa(s)$ and $\tilde{\kappa} = \kappa(\tilde{s})$ are as defined in Lemma 3.4.*

*Proof* Note that

$$\begin{aligned}\int_0^\infty e^{-st}\varphi_T(t)dt &= \frac{1}{4\pi\lambda T}\int_0^\infty \frac{1-e^{-2\lambda Tu}}{u^{3/2}}\left(\int_u^\infty e^{-st}\frac{dt}{\sqrt{t-u}}\right)du\\ &= \frac{1}{4\pi\lambda T}\int_0^\infty \frac{e^{-su}(1-e^{-2\lambda Tu})}{u^{3/2}}\left(\int_0^\infty e^{-s\tau}\frac{d\tau}{\sqrt{\tau}}\right)du\\ &= \frac{1}{4\lambda T\sqrt{\pi s}}\int_0^\infty e^{-su}\left(1-e^{-2\lambda Tu}\right)u^{-3/2}du\\ &= \frac{1}{4\lambda T\sqrt{\pi s}}\cdot 2\sqrt{\pi}\left(\sqrt{s+2\lambda T}-\sqrt{s}\right)\\ &= \frac{1}{\sqrt{s(s+2\lambda T)}+s} = \frac{1}{\kappa+s},\end{aligned}$$

and (3.2.33) is proved. Furthermore, formula (3.2.33) readily implies (3.2.34) by the convolution property of the Laplace transformation. Indeed, recalling that $s+\beta=\tilde{s}$, the left-hand side of (3.2.34) is given by the product

$$\int_0^\infty e^{-st}\varphi_T(t)dt\int_0^\infty e^{-\tilde{s}t}\varphi_T(t)dt = \frac{1}{(\kappa+s)(\kappa(\tilde{s})+\tilde{s})},$$

and (3.2.34) follows. □

Thus, by Lemmas 3.4 and 3.5 and the relation (3.2.21) we get

$$\begin{aligned}\mathbb{E}\left[e^{-\beta\eta_T^-}\right] &= v_T^-(0,1) = 2\varphi_T(1) + 2\lambda T\int_0^1 e^{-\beta y}\varphi_T(y)\varphi_T(1-y)dy,\\ \mathbb{E}\left[e^{-\beta\eta_T^+}\right] &= v_T^+(0,1) = 2\varphi_T(1)e^{-\beta} + 2\lambda T\int_0^1 e^{-\beta y}\varphi_T(y)\varphi_T(1-y)dy.\end{aligned}\tag{3.2.35}$$

By inspection of the right-hand sides of (3.2.35), it is evident that the distribution of $\eta_T^\pm$ consists of an atom at point 0 (for $\eta_T^-$) or 1 (for $\eta_T^+$), and of an absolutely continuous part corresponding to the integral term in (3.2.35). More precisely, for each $\theta\in[0,1]$,

$$\mathbb{P}\{\eta_T^-\le\theta\} = 2\varphi_T(1) + 2\lambda T\int_0^\theta \varphi_T(y)\varphi_T(1-y)dy,$$

$$\mathbb{P}\{\eta_T^+\ge\theta\} = 2\varphi_T(1) + 2\lambda T\int_\theta^1 \varphi_T(y)\varphi_T(1-y)dy.$$

Theorem 3.6 is proved. □

Theorem 3.7 follows from Theorem 3.6 and from the following asymptotics of function $\varphi_T$. After change of variables we have

$$\sqrt{T}\varphi_T(t) = \frac{1}{4\pi\lambda\sqrt{T}}\int_0^t \frac{1-e^{-2\lambda T u}}{u^{3/2}\sqrt{t-u}}du = \frac{1}{4\pi}\int_0^{\lambda T t}\frac{1-e^{-2\tau}}{\tau^{3/2}\sqrt{\lambda t-\tau/T}}d\tau$$

$$= \frac{1}{4\pi}\int_0^{T^\alpha}\frac{1-e^{-2\tau}}{\tau^{3/2}\sqrt{\lambda t-\tau/T}}d\tau + \frac{1}{4\pi}\int_{T^\alpha}^{\lambda T t}\frac{1-e^{-2\tau}}{\tau^{3/2}\sqrt{\lambda t-\tau/T}}d\tau.$$

Let $\alpha \in (2/3, 1)$. Thus, the first summand converges to

$$\frac{1}{4\pi\sqrt{\lambda t}}\int_0^\infty \frac{1-e^{-2\tau}}{\tau^{3/2}}d\tau,$$

as $T \to \infty$. The second summand is

$$\frac{1}{4\pi}\int_{T^\alpha}^{\lambda T t}\frac{1-e^{-2\tau}}{\tau^{3/2}\sqrt{\lambda t-\tau/T}}d\tau = \frac{1}{4\pi\sqrt{T}}\int_{T^{\alpha-1}}^{\lambda t}\frac{1-e^{-2Tu}}{u^{3/2}\sqrt{\lambda t-u}}d\tau$$
$$\le \frac{1}{4\pi T^{3\alpha/2-1}}\int_{T^{\alpha-1}}^{\lambda t}\frac{du}{\sqrt{\lambda t-u}} \to 0.$$

Therefore, as $T \to \infty$,

$$\sqrt{T}\varphi_T(t) \to \frac{1}{4\pi\sqrt{\lambda t}}\int_0^\infty \frac{1-e^{-2\tau}}{\tau^{3/2}}d\tau = \frac{1}{\sqrt{2\pi\lambda t}}.$$

Thus, using Theorem 3.6, we have for $0 < a < b < 1$

$$\mathbb{P}\{a < \eta_T^\pm < b\} \to \int_a^b \frac{dy}{\pi\sqrt{y(1-y)}}.$$

Therefore, the distributions of $\eta_T^\pm$ converge, as $T \to \infty$, to the classical arcsine law for standard Brownian motion. Theorem 3.7 is proved. □

## 3.3 First Passage Time

In this section (as in Sect. 4.2) we consider the generalised telegraph process with alternating velocities $c_0 > 0 > c_1$, which is controlled by the Poisson process with alternating switching intensities $\lambda_0$ and $\lambda_1$.

The main objective is to compute the distribution of the first passage time

$$T(x) = \inf\{t \geq 0 \ : \ X(t) = x\}, \qquad x > 0,$$

assuming that $c_0 > 0 > c_1$.

It is easy to see that the conditional distributions (if the initial state is fixed) have the form

$$\mathbb{P}_0\{T(x) \in \mathrm{d}t\} = \mathrm{e}^{-\lambda_0 t}\delta_{x/c_0}(\mathrm{d}t) + Q_0(x,t)\mathrm{d}t, \quad \mathbb{P}_1\{T(x) \in \mathrm{d}t\} = Q_1(x,t)\mathrm{d}t, \tag{3.3.1}$$

where $\delta_{x/c_0}$ is the Dirac measure (of unit mass) at point $x/c_0$ corresponding to the motion which does not change its positive velocity $c_0$ and $Q_i(x,t)$, $i = 0, 1$ are the absolutely continuous parts of the distributions. If $t < x/c_0$, then the particle does not attain the level $x$ and $Q_i(x,t) = 0$, $i = 0, 1$.

**Theorem 3.8** *Conditional distribution densities $Q_0(x,t)$ and $Q_1(x,t)$ have the form :*

$$\begin{aligned} Q_0(x,t) &= \tfrac{\lambda_0\lambda_1 x \exp\{-\lambda_0\xi - \lambda_1(t-\xi)\}}{2c\sqrt{\lambda_0\lambda_1\xi(t-\xi)}} I_1\left(2\sqrt{\lambda_0\lambda_1\xi(t-\xi)}\right)\mathbb{1}_{\{t>x/c_0\}}, \\ Q_1(x,t) &= \tfrac{\lambda_1 \exp\{-\lambda_0\xi-\lambda_1(t-\xi)\}}{2c\xi}\Big[x I_0\left(2\sqrt{\lambda_0\lambda_1\xi(t-\xi)}\right) \\ &\qquad - \tfrac{c_1}{\sqrt{\lambda_0\lambda_1}}\sqrt{\tfrac{t-\xi}{\xi}} I_1\left(2\sqrt{\lambda_0\lambda_1\xi(t-\xi)}\right)\Big]\mathbb{1}_{\{t>x/c_0\}}, \end{aligned} \tag{3.3.2}$$

*where* $\xi = \xi(x,t) = \dfrac{x - c_1 t}{2c}$, $t - \xi = \dfrac{c_0 t - x}{2c}$, $2c = c_0 - c_1$.

*Proof* Consider the Laplace transform of these distributions,

$$\phi_i(x,s) = \mathbb{E}_i \mathrm{e}^{-sT(x)}, \qquad s > 0, i = 0, 1.$$

In view of (3.3.1), the Laplace transforms $\phi_i$ take the form

$$\phi_0(x,s) = \mathrm{e}^{-(\lambda_0+s)x/c_0} + \int_0^\infty \mathrm{e}^{-st}Q_0(x,t)\mathrm{d}t, \quad \phi_1(x,s) = \int_0^\infty \mathrm{e}^{-st}Q_1(x,t)\mathrm{d}t.$$

Let $t > x/c_0$. Conditioning on the first switching, we have

$$Q_0(x,t) = \int_0^{x/c_0} \lambda_0 \mathrm{e}^{-\lambda_0\tau}Q_1(x - c_0\tau, t-\tau)\mathrm{d}\tau, \tag{3.3.3}$$

$$\begin{aligned} Q_1(x,t) &= \textstyle\int_0^{+\infty} \lambda_1 \mathrm{e}^{-\lambda_1\tau}\mathrm{e}^{-\lambda_0(t-\tau)}\delta\left(t - \tau - (x - c_1\tau)/c_0\right)\mathrm{d}\tau \\ &\quad + \textstyle\int_0^{\frac{c_0 t - x}{2c}} \lambda_1 \mathrm{e}^{-\lambda_1\tau}Q_0(x - c_1\tau, t-\tau)\mathrm{d}\tau \\ &= \tfrac{\lambda_1 c_0}{2c}\mathrm{e}^{-\lambda_0\xi - \lambda_1(t-\xi)} + \textstyle\int_0^{\frac{c_0 t - x}{2c}} \lambda_1 \mathrm{e}^{-\lambda_1\tau}Q_0(x - c_1\tau, t-\tau)\mathrm{d}\tau. \end{aligned} \tag{3.3.4}$$

Hence,

$$\begin{aligned}\phi_0(x,s) &= \mathrm{e}^{-(\lambda_0+s)x/c_0} + \int_0^\infty \mathrm{e}^{-st}\mathrm{d}t \int_0^{x/c_0} \lambda_0 \mathrm{e}^{-\lambda_0\tau} Q_1(x-c_0\tau, t-\tau)\mathrm{d}\tau \\ &= \mathrm{e}^{-(\lambda_0+s)x/c_0} + \lambda_0 \int_0^{x/c_0} \mathrm{e}^{-\lambda_0\tau}\mathrm{d}\tau \int_\tau^\infty \mathrm{e}^{-st} Q_1(x-c_0\tau, t-\tau)\mathrm{d}t \\ &= \mathrm{e}^{-(\lambda_0+s)x/c_0} + \lambda_0 \int_0^{x/c_0} \mathrm{e}^{-(\lambda_0+s)\tau}\mathrm{d}\tau \int_0^\infty \mathrm{e}^{-st} Q_1(x-c_0\tau, t)\mathrm{d}t.\end{aligned}$$

Then

$$\phi_0(x,s) = \mathrm{e}^{-(\lambda_0+s)x/c_0} + \lambda_0 \int_0^{x/c_0} \mathrm{e}^{-(\lambda_0+s)\tau} \phi_1(x-c_0\tau, s)\mathrm{d}\tau. \qquad (3.3.5)$$

In the same manner we derive the integral equation for $\phi_1$. In view of (3.3.4), similarly to (3.3.5), we have

$$\begin{aligned}\phi_1(x,s) = \int_0^\infty \mathrm{e}^{-st} Q_1(x,t)\mathrm{d}t &= \frac{\lambda_1 c_0}{2c} \int_{x/c_0}^\infty \mathrm{e}^{-st}\mathrm{e}^{-\lambda_0\xi-\lambda_1(t-\xi)}\mathrm{d}t \\ &+ \lambda_1 \int_0^\infty \mathrm{e}^{-(\lambda_1+s)\tau}\left[\phi_0(x-c_1\tau, s) - \mathrm{e}^{-(\lambda_0+s)(x-c_1\tau)/c_0}\right]\mathrm{d}\tau.\end{aligned}$$

Thus,

$$\phi_1(x,s) = \lambda_1 \int_0^{+\infty} \mathrm{e}^{-(\lambda_1+s)\tau}\phi_0(x-c_1\tau, s)\mathrm{d}\tau. \qquad (3.3.6)$$

By differentiating Eq. (3.3.5)–(3.3.6) we obtain the system of ordinary differential equations

$$\begin{aligned} c_0\frac{\partial\phi_0}{\partial x}(x,s) &= -(\lambda_0+s)\phi_0(x,s) + \lambda_0\phi_1(x,s), \\ c_1\frac{\partial\phi_1}{\partial x}(x,s) &= \lambda_1\phi_0(x,s) - (\lambda_1+s)\phi_1(x,s), \end{aligned} \qquad (3.3.7)$$

The initial condition for $\phi_0$ follows from Eq. (3.3.5):

$$\phi_0(+0,s) = 1, \qquad s > 0, \qquad (3.3.8)$$

and, by definition of $T(x)$, we have at infinity

$$\phi_0(+\infty,s) = \phi_1(+\infty,s) = 0, \qquad s > 0. \qquad (3.3.9)$$

In the vector form system (3.3.7) reads

$$\frac{\partial\phi}{\partial x}(x,s) = \mathscr{A}\phi(x,s), \qquad x > 0. \qquad (3.3.10)$$

Here $\phi = (\phi_0, \phi_1)^T$ and matrix $\mathscr{A}$ is defined by

$$\mathscr{A} = \begin{pmatrix} -(\lambda_0+s)/c_0 & \lambda_0/c_0 \\ \lambda_1/c_1 & -(\lambda_1+s)/c_1 \end{pmatrix}.$$

Equation (3.3.10) with boundary conditions (3.3.8)–(3.3.9) can be solved to yield

$$\phi_0(x,s) = \mathrm{e}^{\alpha x}, \qquad \phi_1(x,s) = \left(1 + \frac{s+\alpha c_0}{\lambda_0}\right)\mathrm{e}^{\alpha x}, \tag{3.3.11}$$

where $\alpha = \alpha(s)$ is the negative eigenvalue of matrix $\mathscr{A}$, i. e. the negative root of the equation

$$\det(\mathscr{A} - \alpha I) \equiv c_0c_1\alpha^2 + 2(\tilde{a}+as)\alpha + s(2\lambda+s) = 0.$$

Thus,

$$\alpha = \alpha(s) = \frac{-\tilde{a} - as + d(s)}{c_0c_1} < 0,$$

where $d(s) = \sqrt{(\tilde{a}+as)^2 - c_0c_1s(2\lambda+s)}$. Here we exploit the notations of Sect. 4.2 and $\tilde{a} = (\lambda_1c_0 + \lambda_0c_1)/2$.

The Laplace transforms $\phi_0$ and $\phi_1$ can be inverted with help of [3] to yield the densities $Q_0$ and $Q_1$ in terms of modified Bessel functions.

Consider function $Q_0$ defined by (3.3.2),

$$\begin{aligned} Q_0(x,t) &= \tfrac{\lambda_0\lambda_1 x\exp\{-\lambda_0\xi-\lambda_1(t-\xi)\}}{2c\sqrt{\lambda_0\lambda_1\xi(t-\xi)}} I_1\left(2\sqrt{\lambda_0\lambda_1\xi(t-\xi)}\right)\mathbb{1}_{\{t\ge x/c_0\}} \\ &= \tfrac{x\sqrt{\lambda_0\lambda_1}\exp\{-(\tilde{c}t+\beta x)/c\}}{\sqrt{(c_0t-x)(x-c_1t)}} I_1\left(\sqrt{\lambda_0\lambda_1(c_0t-x)(x-c_1t)}/c\right)\mathbb{1}_{\{t\ge x/c_0\}}, \end{aligned} \tag{3.3.12}$$

where $\tilde{c} = (\lambda_1c_0 - \lambda_0c_1)/2$. The Laplace transform of $Q_0$ can be computed by using the table, see 29.3.96 and 29.2.14 of [3]:

$$\int_{x/c_0}^{\infty} \mathrm{e}^{-st}Q_0(x,t)\mathrm{d}t = \mathrm{e}^{\alpha x} - \mathrm{e}^{-(\lambda_0+s)x/c_0}.$$

Here the following identities are applied:

$$\lambda_0\xi + \lambda_1(t-\xi) = (\tilde{c}t+\beta x)/c, \qquad \xi(t-\xi) = \frac{(c_0t-x)(x-c_1t)}{4c^2}.$$

and

$$\left(\frac{t - ax/(c_0c_1)}{c}\right)^2 - \left(\frac{x}{c_0c_1}\right)^2 = (t - x/c_0)(t - x/c_1)/c^2.$$

To obtain the inverse Laplace transform of $Q_1$ we rewrite Eq. (3.3.12) as

$$Q_0(x-c_1\tau, t-\tau) = \frac{\lambda_0\lambda_1(x-c_1\tau)I_1\left(2\sqrt{\lambda_0\lambda_1\xi(t-\tau-\xi)}\right)\mathrm{e}^{-\lambda_0\xi-\lambda_1(t-\tau-\xi)}}{2c\sqrt{\lambda_0\lambda_1\xi(t-\tau-\xi)}}.$$

Therefore, formula (3.3.4) becomes

$$Q_1(x,t) = \frac{\lambda_1 c_0}{2c}\mathrm{e}^{-\lambda_0\xi-\lambda_1(t-\xi)} + \widetilde{Q}_1(x,t), \tag{3.3.13}$$

where

$$\widetilde{Q}_1(x,t) = \frac{\lambda_0\lambda_1^2\mathrm{e}^{-\lambda_0\xi-\lambda_1(t-\xi)}}{2c}\int\limits_0^{(c_0t-x)/(2c)}\frac{(x-c_1\tau)I_1\left(2\sqrt{\lambda_0\lambda_1\xi(t-\tau-\xi)}\right)}{\sqrt{\lambda_0\lambda_1\xi(t-\tau-\xi)}}\mathrm{d}\tau.$$

Changing the variables in this integral, $z = 2\sqrt{\lambda_0\lambda_1\xi(t-\tau-\xi)}$ (or, equivalently, $\tau = \tau(z) = t-\xi-z^2/(4\lambda_0\lambda_1\xi)$), and setting $z_0 = 2\sqrt{\lambda_0\lambda_1\xi(t-\xi)}$ we obtain

$$\begin{aligned}\widetilde{Q}_1(x,t) &= \frac{\lambda_1\mathrm{e}^{-\lambda_0\xi-\lambda_1(t-\xi)}}{x-c_1t}\int\limits_0^{z_0}(x-c_1\tau(z))I_1(z)\mathrm{d}z \\ &= \frac{\lambda_1\mathrm{e}^{-\lambda_0\xi-\lambda_1(t-\xi)}}{x-c_1t}\int\limits_0^{z_0}\left(\frac{c_0}{2c}(x-c_1t)+\frac{cc_1z^2}{2\lambda_0\lambda_1(x-c_1t)}\right)I_1(z)\mathrm{d}z \\ &= \frac{\lambda_1c_0\mathrm{e}^{-\lambda_0\xi-\lambda_1(t-\xi)}}{2c}\int\limits_0^{z_0}I_1(z)\mathrm{d}z + \frac{cc_1\mathrm{e}^{-\lambda_0\xi-\lambda_1(t-\xi)}}{2\lambda_0(x-c_1t)^2}\int\limits_0^{z_0}z^2I_1(z)\mathrm{d}z.\end{aligned}$$

The latter two integrals can be expressed as

$$\int_0^{z_0}I_1(z)\mathrm{d}z = I_0(z_0)-1, \qquad \int_0^{z_0}z^2I_1(z)\mathrm{d}z = z_0^2I_2(z_0)$$

(see, e.g. [3], 9.6.27 and 11.3.25).

Thus,

$$\widetilde{Q}_1(x,t) = \frac{\lambda_1}{2c}\left[c_0I_0(z_0)-c_0+c_1\frac{c_0t-x}{x-c_1t}I_2(z_0)\right]\mathrm{e}^{-\lambda_0\xi-\lambda_1(t-\xi)}. \tag{3.3.14}$$

By the first Eq. 9.6.26 of Ref. [3] we have the identity $I_2(z) = I_0(z)-2I_1(z)/z$. Hence,

$$\widetilde{Q}_1(x,t) = \frac{\lambda_1}{x - c_1 t}\left[x I_0(z_0) - c_0\xi - \frac{c_1}{\sqrt{\lambda_0\lambda_1}}\sqrt{\frac{c_0 t - x}{x - c_1 t}} I_1(z_0)\right] \mathrm{e}^{-\lambda_0\xi - \lambda_1(t-\xi)}. \tag{3.3.15}$$

The second equation of (3.3.2) follows from (3.3.13) and (3.3.15). □

*Remark 3.9* In the case of symmetric motion, $c_0 = -c_1 = c, \lambda_0 = \lambda_1 = \lambda$, formulae (3.3.12) and (3.3.15) take the form

$$Q_0(x,t) = \frac{\lambda x \mathrm{e}^{-\lambda t}}{\sqrt{c^2t^2 - x^2}} I_1\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right) \mathbb{1}_{\{t > x/c\}}, \tag{3.3.16}$$

$$Q_1(x,t) = \frac{\lambda \mathrm{e}^{-\lambda t}}{x + ct}\left[x I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right) + \frac{c}{\lambda}\sqrt{\frac{ct - x}{x + ct}} I_1 \times \left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right)\right] \mathbb{1}_{\{t > x/c\}}. \tag{3.3.17}$$

In the case of arbitrary $c_0, c_1, c_0 > 0 > c_1$, the conditional density of $T(x)$ given $\varepsilon(0) = 0$ is known (see [4] and [5] for symmetric motion , and [6] in general case). In [6] the equation similar to (3.3.4) is also derived (see Eq. (5.6) therein). To the best of our knowledge, if the initial velocity is negative (and $x > 0$), then the explicit form of first passage time distributions $Q_1$ is still unknown even in the symmetric case (see [6], Remark 5.1 ; cf. (3.3.17) and Theorem 4.1 of [1]).

## Notes

Telegraph processes with reflecting and absorbing barriers were studied by numerous physicists and mathematicians, see e.g. [4],[1, 5, 7–13]. Nevertheless, some erroneous results are still presented in this field. In this chapter we correct some inaccuracies, including the distributions of first-passage times.

Section 3.2 contains the recent results on occupation time distribution of telegraph processes, see detailed presentation in [14]. The distributions of the first passage times are analysed in [15].

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# Chapter 4
# Asymmetric Jump-Telegraph Processes

**Abstract** In this chapter we examine the more general jump-telegraph process with alternating velocities and alternating transition intensities in the presence of deterministic jumps at random time instants. The existence of the unique martingale measure is very important for financial modelling. Exploiting the analogue of Doob-Meyer decomposition (see e.g. [1]) we characterise the martingales based on the telegraph processes with jumps. A version of Girsanov's Theorem for jump-telegraph processes is obtained as well. The explicit formulae for the moments of the asymmetric telegraph process are also derived.

**Keywords** Jump-telegraph process · Expectations · Variances · Moments · Martingales · Girsanov's Theorem

## 4.1 Generalised Jump-Telegraph Processes

For the purposes of financial modelling we need some generalisation of the telegraph process defined in Chap. 2. We consider the telegraph process with alternating velocity values and alternating transition intensities. Moreover, we need a jump component to be added to the telegraph process.

The existence and the uniqueness of a martingale measure are very important for financial modelling (see Theorem 5.1 and 5.2). Exploiting the analogue of Doob-Meyer decomposition (see e.g. [1]) in this section we characterise the martingales based on the telegraph processes with jumps. The version of Girsanov's Theorem for jump-telegraph processes will be obtained as well.

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### 4.1.1 Transition Densities

On the filtered probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P})$ consider the Markov process $\varepsilon = \varepsilon(t) \in \{0, 1\},\ t \geq 0$, with alternating transition intensities $\lambda_0 > 0$ and $\lambda_1 > 0$ (cf. (2.1.1) of Chap. 2)

$$\mathbb{P}\{\varepsilon(t + \Delta t) \neq \varepsilon(t) \mid \varepsilon(t)\} = \lambda_{\varepsilon(t)}\Delta t + o(\Delta t), \qquad \Delta t \to +0. \tag{4.1.1}$$

Process $\varepsilon = \varepsilon(t)$ is assumed to be adapted to the filtration $\{\mathfrak{F}_t\}_{t\geq 0}$. All the paths of $\varepsilon$ are right-continuous. We will consider also the left-continuous version of $\varepsilon$,

$$\varepsilon(t-) := \begin{cases} \varepsilon(0) & \text{for } t = 0, \\ \lim\limits_{s\uparrow t} \varepsilon(s) & \text{for } t > 0. \end{cases}$$

Notice that the switching times $\tau_1 < \tau_2 < \ldots$ of the Markov process $\varepsilon = \varepsilon(t)$ have exponentially distributed and independent increments: $\mathbb{P}\{\tau_{n+1} - \tau_n > t \mid \mathfrak{F}_{\tau_n}\} = \exp\left(-\lambda_{\varepsilon(\tau_n)}t\right)$, $\tau_0 = 0$, and $\tau_{n+1} - \tau_n,\ n \geq 0$, are independent.

Let $c_1 < c_0$. We denote $V(t) = c_{\varepsilon(t)}$ and $X(t) = \int_0^t V(s)\mathrm{d}s$. The process $X$ is called the (inhomogeneous) telegraph process with alternating states $(c_0,\ \lambda_0)$ and $(c_1,\ \lambda_1)$.

*Remark 4.1* We have already changed the notations of previous chapters. The reasons are the following. We will consider the alternating pairs, $(c_0,\ \lambda_0)$ and $(c_1,\ \lambda_1)$, as the different market states. We say that $(c_0,\ \lambda_0)$ is the basic state, or 0-state. Staying on the optimistic "bearish" viewpoint, we use notation 0 for the basic state instead of + for the "growing" market (with velocity $c_0$), and 1 instead of − for the "falling" market (with velocity $c_1,\ c_1 < c_0$). When the market receives respective information (or the market signals) its current state changes from $(c_0,\ \lambda_0)$ to $(c_1,\ \lambda_1)$ or vice versa.

*Remark 4.2* In the homogeneous case of $\lambda_0 = \lambda_1$ and $-c_1 = c_0 = c$ properties of the telegraph process are well known (see Chap. 2). For the financial modelling we need the inhomogeneous process with alternating switching intensities defined above. The exact marginal distributions of inhomogeneous $X(t)$ are calculated in [2], [3]. Calibration for needs of financial market modelling has been suggested recently in [4], [5].

In this chapter we need also a pure jump (compound Poisson) process $J(t),\ t \geq 0$. Let $h_0,\ h_1$ be arbitrary fixed numbers that can be treated as the values of jumps, $h_0,\ h_1 \in (-\infty,\ \infty)$. The process $J = J(t)$ is specified as the (right-continuous) process $J = J(t) = \sum_{n=1}^{N(t)} h_{\varepsilon_n},\ t \geq 0$, with jumps occurring at the switching times $\tau_n,\ n = 1,\ 2,\ \ldots$ Here $\varepsilon_n = \varepsilon(\tau_n-)$ is the value of the Markov process $\varepsilon$ just before switching time $\tau_n$, and $N = N(t),\ t \geq 0$ is the counting Poisson process, $N(t) = \max\{n\ :\ \tau_n \leq t\}$.

We define a jump-telegraph process as the sum $X(t) + J(t),\ \ t \ge 0$. Fix the initial state $\varepsilon(0) = i \in \{0, 1\}$, so the process starts with the velocity $c_i$ and the first jump will be of value $h_i$ at the first switching time $\tau_1 = \tau_1^{(i)}$ which is exponentially distributed (with transition intensity $\lambda_i$). Then for any $t > 0$ we have the following equality in distribution

$$X(t) + J(t) \overset{d}{=} c_i t \mathbb{1}_{\{\tau_1 > t\}} + \left[c_i\tau_1 + h_i + \widetilde{X}(t-\tau_1) + \widetilde{J}(t-\tau_1)\right]\mathbb{1}_{\{\tau_1 < t\}}, \quad (4.1.2)$$

where both the telegraph process $\widetilde{X}$ and the jump process $\widetilde{J}$ start from the opposite state, $1 - i$, and they are independent of $X$ and $J$. The principle described by Eq. (4.1.2) is similar to (2.2.1) of Chap. 2.

Let

$$\mathbb{P}_i\{\cdot\} = \mathbb{P}\{\cdot \mid \varepsilon(0) = i\} \quad \text{and} \quad \mathbb{E}_i = \mathbb{E}\{\cdot \mid \varepsilon(0) = i\}, \quad i = 0, 1,$$

denote the conditional probabilities and conditional expectations under the initial state value $\varepsilon(0) = i$. The conditional transition densities of $X + J$ are defined by

$$p_i(x,\ t) := \mathbb{P}_i\left\{X(t) + J(t) \in \mathrm{d}x\right\}/\mathrm{d}x, \quad (4.1.3)$$

$$p_i(x,\ t;\ n) := \mathbb{P}_i\{X(t) + J(t) \in \mathrm{d}x,\ N(t) = n\}/\mathrm{d}x, \qquad n \ge 0,\ i = 0,\ 1.$$

It is clear that

$$p_i(x,\ t) = \sum_{n=0}^{\infty} p_i(x,\ t;\ n). \quad (4.1.4)$$

Using (4.1.2) we immediately get the following set of integral equations (cf. Chap. 2, Eq. (2.2.2))

$$\begin{aligned} p_0(x,\ t) &= \mathrm{e}^{-\lambda_0 t}\delta(x - c_0 t) + \int_0^t p_1(x - c_0 s - h_0,\ t - s)\lambda_0\mathrm{e}^{-\lambda_0 s}\mathrm{d}s, \\ p_1(x,\ t) &= \mathrm{e}^{-\lambda_1 t}\delta(x - c_1 t) + \int_0^t p_0(x - c_1 s - h_1,\ t - s)\lambda_1\mathrm{e}^{-\lambda_1 s}\mathrm{d}s \end{aligned} \quad (4.1.5)$$

and

$$\begin{aligned} p_i(x,\ t;\ 0) &= \mathrm{e}^{-\lambda_i t}\delta(x - c_i t), \quad n \ge 1, \qquad i = 0, 1, \\ p_i(x,\ t;\ n) &= \int_0^t p_{1-i}(x - c_i s - h_i,\ t - s;\ n - 1)\lambda_i\mathrm{e}^{-\lambda_i s}\mathrm{d}s. \end{aligned} \quad (4.1.6)$$

Here $\delta(\cdot)$ is Dirac's $\delta$-function, and for any test-function $\varphi$ we presume that (1.6.8) holds: $\int_a^b \delta(y - cs)\varphi(s)\mathrm{d}s = \varphi(y/c)/c, \quad -\infty \le a < b \le \infty,\ \ \text{if}\ y/c \in [a,\ b]$.

Notice that Eq. (4.1.5) can be obtained by summing up in (4.1.6).

Differentiating integral equations (4.1.5) and then integrating the result by parts (compare with the proof of Theorem 2.1) we easily derive the equivalent Cauchy problem

$$\begin{cases} \frac{\partial p_0}{\partial t}(x,\ t) + c_0\frac{\partial p_0}{\partial x}(x,\ t) = -\lambda_0 p_0(x,\ t) + \lambda_0 p_1(x - h_0,\ t), \\ \\ \frac{\partial p_1}{\partial t}(x,\ t) + c_1\frac{\partial p_1}{\partial x}(x,\ t) = -\lambda_1 p_1(x,\ t) + \lambda_1 p_0(x - h_1,\ t), \end{cases} \quad t > 0, \quad (4.1.7)$$

with the initial conditions $p_0(x,\ 0) = p_1(x,\ 0) = \delta(x)$.

In the same manner we show that integral equations (4.1.6) are equivalent to the set of equations, $n \geq 1$,

$$\begin{cases} \frac{\partial p_0}{\partial t}(x,\ t;\ n) + c_0\frac{\partial p_0}{\partial x}(x,\ t;\ n) = -\lambda_0 p_0(x,\ t;\ n) + \lambda_0 p_1(x - h_0,\ t;\ n-1), \\ \frac{\partial p_1}{\partial t}(x,\ t;\ n) + c_1\frac{\partial p_1}{\partial x}(x,\ t;\ n) = -\lambda_1 p_1(x,\ t;\ n) + \lambda_1 p_0(x - h_1,\ t;\ n-1), \end{cases} \quad (4.1.8)$$

$t > 0$, with the initial function $p_i(x,\ t;\ 0) = \mathrm{e}^{-\lambda_i t}\delta(x - c_i t),\ i = 0,\ 1$ and the initial conditions $p_i(x,\ 0;\ n) = 0,\ n \geq 1,\ i = 0,\ 1$.

We present the marginal distributions of the jump-telegraph process with alternating switching intensities as the solutions to (4.1.5) and (4.1.6) (or, equivalently, (4.1.7) and (4.1.8)).

We will use the notations of Theorem 3.8,

$$\xi = \xi(x, t) := \frac{x - c_1 t}{c_0 - c_1} \quad \text{and} \quad t - \xi = \frac{c_0 t - x}{c_0 - c_1}. \quad (4.1.9)$$

Notice that $0 < \xi(x, t) < t$, if $x \in (c_1 t,\ c_0 t)$. Using these notations we define functions $q_i(x, t; n)$, $i = 0, 1$: for $c_1 t < x < c_0 t$,

$$\begin{aligned} q_0(x, t; 2n) &= \frac{\lambda_0^n \lambda_1^n}{(n-1)!n!}\xi^n (t - \xi)^{n-1} \\ q_1(x, t; 2n) &= \frac{\lambda_0^n \lambda_1^n}{(n-1)!n!}\xi^{n-1} (t - \xi)^{n} \end{aligned} \quad n \geq 1, \quad (4.1.10)$$

and

$$\begin{aligned} q_0(x, t; 2n+1) &= \frac{\lambda_0^{n+1} \lambda_1^n}{(n!)^2}\xi^n (t - \xi)^{n} \\ q_1(x, t; 2n+1) &= \frac{\lambda_0^n \lambda_1^{n+1}}{(n!)^2}\xi^{n} (t - \xi)^{n} \end{aligned} \quad n \geq 0, \quad (4.1.11)$$

Denote $\theta(x, t) = \frac{1}{c_0 - c_1}\mathrm{e}^{-\lambda_0\xi - \lambda_1(t-\xi)}\mathbb{1}_{\{0<\xi<t\}}$.

**Proposition 4.1** *Equations* (4.1.6) *and* (4.1.8) *have the following solution*:

$$p_i(x,\ t;\ 0) = \mathrm{e}^{-\lambda_i t}\delta(x - c_i t), \quad (4.1.12)$$

$$p_i(x,\ t;\ n) = q_i(x - j_{in},\ t;\ n)\theta(x - j_{in},\ t), \quad n \geq 1, \quad i = 0,\ 1,$$

*where the displacements $j_{in}$ are defined as the sum of alternating jumps, $j_{in} = \sum_{k=1}^{n} h_{i_k}$, where $i_k = i$, if $k$ is odd, and $i_k = 1 - i$, if $k$ is even.*

*Proof* For $n \geq 2$ we directly substitute expressions (4.1.12) and (4.1.10)–(4.1.11) into system (4.1.6) of integral equations.

To make the necessary simplifications notice that for $0 \leq s \leq t$ and $c_1 t < x < c_0 t$

$$\xi(x - c_0 s, t - s) \equiv \xi(x, t) - s, \quad \xi(x - c_1 s, t - s) \equiv \xi(x, t), \tag{4.1.13}$$

and hence, for $i = 0, 1$

$$\lambda_i s + \lambda_0 \xi(x - c_i s, t - s) + \lambda_1 (t - s - \xi(x - c_i s, t - s)) \equiv \lambda_0 \xi(x, t) + \lambda_1 (t - \xi(x, t)).$$

The latter equality means that

$$\mathrm{e}^{-\lambda_0 s} \theta(x - c_0 s, t - s) \equiv \theta(x, t) \mathbb{1}_{\{s < \xi(x,t)\}}, \tag{4.1.14}$$

$$\mathrm{e}^{-\lambda_1 s} \theta(x - c_1 s, t - s) \equiv \theta(x, t) \mathbb{1}_{\{s < t - \xi(x,t)\}}.$$

To verify Eq. (4.1.6) first notice that, due to (4.1.13) and (4.1.14), the following identities are fulfilled:

$$\int_0^t \xi(x - c_0 s, t - s)^m (t - s - \xi(x - c_0 s, t - s))^k \cdot \mathrm{e}^{-\lambda_0 s} \theta(x - c_0 s, t - s) \mathrm{d}s$$

$$= \theta(x, t) \int_0^{\xi(x,t)} (\xi(x, t) - s)^m (t - \xi(x, t))^k \mathrm{d}s = \theta(x, t) \frac{\xi(x, t)^{m+1}}{m + 1} (t - \xi(x, t))^k$$

and

$$\int_0^t \xi(x - c_1 s, t - s)^m (t - s - \xi(x - c_1 s, t - s))^k \cdot \mathrm{e}^{-\lambda_1 s} \theta(x - c_1 s, t - s) \mathrm{d}s$$

$$= \theta(x, t) \int_0^{t - \xi(x,t)} \xi(x, t)^m (t - s - \xi(x, t))^k \mathrm{d}s = \theta(x, t) \xi(x, t)^m \frac{(t - \xi(x, t))^{k+1}}{k + 1}.$$

With these equalities in hand, it is easy to see that functions $p_i(x, t; n), i = 0, 1$ which are defined by (4.1.10)–(4.1.12), satisfy integral equations (4.1.6).

For $n = 1$ equations (4.1.6) can be solved by using the initial functions $p_i(x, t; 0) = \mathrm{e}^{-\lambda_i t} \delta(x - c_i t)$ and $p_i(x, t; 1)$ defined by (4.1.11)–(4.1.12), $p_i(x, t; 1) = \lambda_i \theta(x, t)$.

*Remark 4.3* In particular, in the homogeneous case $\lambda_0 = \lambda_1 = \lambda$ the multiplier $\theta$ becomes $\theta(x, t) = \mathrm{e}^{-\lambda t} \mathbb{1}_{\{c_1 t < x < c_0 t\}}$, cf. formula (2.5.1).

*Remark 4.4* Assume jump values to be symmetric, $h_0 + h_1 = 0$. Hence $j_{in} = 0$, if $n$ is even, and $j_{in} = h_i$, if $n$ is odd. Summing up in (4.1.4) and using explicit formulae (4.1.9)–(4.1.12) for $p_i(x, t; n)$ and series representations (1.5.3) for modified Bessel functions one can obtain the following expression for the distribution densities of the jump-telegraph process (cf. (2.5.3), (2.5.15))

$$\begin{aligned} p_i(x,\ t) = \mathrm{e}^{-\lambda_i t} \cdot \delta(x - c_i t) \\ + \frac{1}{c_0 - c_1}\left[\lambda_i \theta(x - h_i,\ t) I_0\left(2\frac{\sqrt{\lambda_0\lambda_1(c_0 t - x + h_i)(x - h_i - c_1 t)}}{c_0 - c_1}\right)\right. \\ \left. + \sqrt{\lambda_0\lambda_1}\theta(x, t)\left(\frac{x - c_1 t}{c_0 t - x}\right)^{\frac{1}{2} - i} I_1\left(2\frac{\sqrt{\lambda_0\lambda_1(c_0 t - x)(x - c_1 t)}}{c_0 - c_1}\right)\right], \end{aligned} \tag{4.1.15}$$

where $I_0(z)$ and $I_1(z)$ are the modified Bessel functions given by (1.5.3). Functions $p_0$ and $p_1$ defined by (4.1.15) solve PDE-system (4.1.7) as well. Formula (4.1.15) generalises formula (2.5.1) of Theorem 2.5.

Formulae similar to (4.1.15) have been presented in [2]. See also [6, 7]. The detailed explicit formulae of this type can be found in recent paper [8].

The plots of distribution densities $p_i(x,\ t)$ (continuous part) are presented in Fig. 4.1.

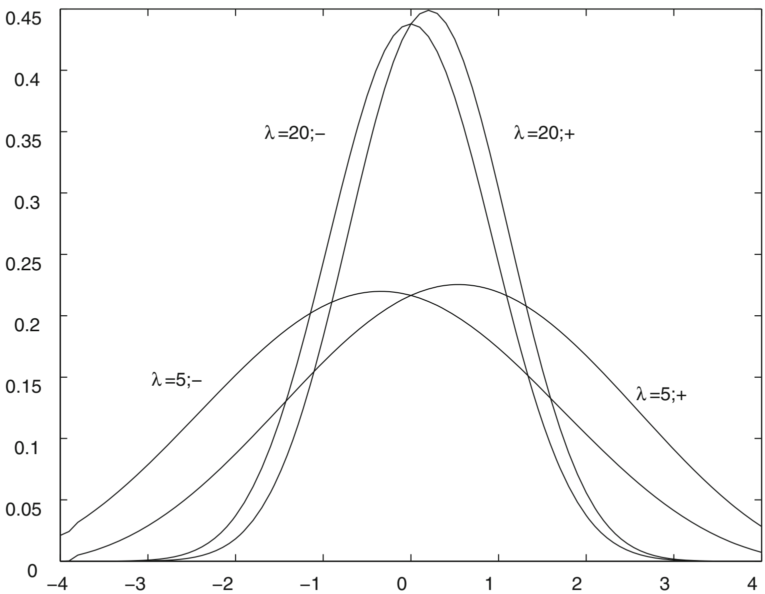


**Fig. 4.1** Conditional probability densities of jump telegraph process $X(t)$ (absolutely continuous part) with values $t = 1, c_0 = 4, c_1 = -4, h_0 = -0.2, h_1 = 0.2$ and with $\lambda_0 = \lambda_1 = 5$ or $\lambda_0 = \lambda_1 = 20$

### *4.1.2 Expectations and Variances Jump-Telegraph Martingales*

Applying again master Eq. (4.1.2) (or (4.1.5)) one can obtain the integral and differential equations for the expectations of jump-telegraph processes.

**Proposition 4.2** *The conditional expectations* $m_i(t) := \mathbb{E}_i\{X(t) + J(t)\},\ t > 0$, *satisfy the system*

$$m_i(t) = \frac{1 - \mathrm{e}^{-\lambda_i t}}{\lambda_i} d_i + \int_0^t \lambda_i \mathrm{e}^{-\lambda_i s} m_{1-i}(t-s)\mathrm{d}s, \quad i = 0, 1. \tag{4.1.16}$$

*Here numbers* $d_i := c_i + \lambda_i h_i,\ i = 0, 1$, *characterise the drift properties of the jump-telegraph process* $X(t) + J(t),\ t > 0$.

*Proof* To compute $m_i(t)$ we use integral equations (4.1.5). Indeed,

$$\begin{aligned} m_i(t) &= \int_{-\infty}^{\infty} x p_i(x,\ t)\mathrm{d}x \\ &= c_i t\mathrm{e}^{-\lambda_i t} + \int_{-\infty}^{\infty} x \left[\int_0^t p_{1-i}(x - c_i s - h_i,\ t-s)\lambda_i \mathrm{e}^{-\lambda_i s}\mathrm{d}s\right]\mathrm{d}x \\ &= c_i t\mathrm{e}^{-\lambda_i t} + \int_0^t \lambda_i \mathrm{e}^{-\lambda_i s}\mathrm{d}s \int_{-\infty}^{\infty} x p_{1-i}(x - c_i s - h_i,\ t-s)\mathrm{d}x. \end{aligned}$$

Changing the variables $x = x' + c_i s + h_i$ we have

$$m_i(t) = c_i t\mathrm{e}^{-\lambda_i t} + \int_0^t \lambda_i \mathrm{e}^{-\lambda_i s}(c_i s + h_i)\mathrm{d}s + \int_0^t \lambda_i \mathrm{e}^{-\lambda_i s} m_{1-i}(t-s)\mathrm{d}s.$$

Hence

$$m_i(t) = (c_i/\lambda_i + h_i)\left(1 - \mathrm{e}^{-\lambda_i t}\right) + \int_0^t \lambda_i \mathrm{e}^{-\lambda_i s} m_{1-i}(t-s)\mathrm{d}s,$$

proving (4.1.16). □

Integral Eq. (4.1.16) can be rewritten in the differential form.

**Corollary 4.1** *The set of Eq.* (4.1.16) *is equivalent to the Cauchy problem for the equation*

$$\frac{\mathrm{d}m_i}{\mathrm{d}t}(t) = d_i - \lambda_i m_i(t) + \lambda_i m_{1-i}(t), \qquad t > 0, \quad i = 0, 1, \tag{4.1.17}$$

*with the initial conditions* $m_0(0) = 0,\ m_1(0) = 0$.

*Proof* The initial conditions follow directly from Eq. (4.1.16). Differentiating in (4.1.16), integrating by parts and using the initial conditions $m_0(0) = m_1(0) = 0$, we have

$$\frac{\mathrm{d}m_i}{\mathrm{d}t}(t) = \mathrm{e}^{-\lambda_i t} d_i - \int_0^t \lambda_i \mathrm{e}^{-\lambda_i s} \frac{\partial m_{1-i}(t-s)}{\partial s} \mathrm{d}s$$

$$= \mathrm{e}^{-\lambda_i t} d_i + \lambda_i m_{1-i}(t) - \lambda_i \int_0^t \lambda_i \mathrm{e}^{-\lambda_i s} m_{1-i}(t-s) \mathrm{d}s$$

$$= d_i - \lambda_i m_i(t) + \lambda_i m_{1-i}(t).$$

To derive the final equality we again apply (4.1.16). □

The next theorem gives us the necessary and sufficient condition for a jump-telegraph process to be a martingale.

**Theorem 4.1** *Let $d_0 = c_0 + \lambda_0 h_0$ and $d_1 = c_1 + \lambda_1 h_1$ (as in Proposition 4.2). The process $X(t) + J(t),\ t \geq 0$, is a martingale if and only if*

$$d_0 = 0 \quad \textit{and} \quad d_1 = 0. \tag{4.1.18}$$

*Proof* First notice that, due to renewal character of $X(t) + J(t)$ for $s,\ t \in [0, T],\ s < t$, we have

$$\mathbb{E}\{X(t) + J(t) \mid \mathfrak{F}_s\} = \mathbb{E}\{X(t) + J(t) - X(s) - J(s) \mid \mathfrak{F}_s\} + \mathbb{E}\{X(s) + J(s) \mid \mathfrak{F}_s\}$$
$$= m_{\varepsilon(s)}(t-s) + X(s) + J(s).$$

Therefore, the process $X(t) + J(t),\ t \geq 0$, is a martingale if and only if $m_i(t) \equiv 0,\ i = 0,\ 1$. Meanwhile, the solution $m_0(t),\ m_1(t),\ t \geq 0$, of the Cauchy problem for Eq. (4.1.17) is equal to zero, $m_i(t) \equiv 0,\ i = 0,\ 1$, if and only if equalities (4.1.18) are fulfilled. □

System (4.1.17) can be generalised to describe the moments of the jump-telegraph process. The specific statement is given by the following.

**Proposition 4.3** *Let $f = f(x),\ x \in (-\infty, \infty)$ and $\alpha_0 = \alpha_0(t),\ \alpha_1 = \alpha_1(t),\ t \geq 0$, be arbitrary smooth deterministic functions. Let $X(t) + J(t),\ t \geq 0$, be the jump-telegraph process with the alternating sets of parameters, $(c_0,\ \lambda_0,\ h_0)$ and $(c_1,\ \lambda_1,\ h_1)$. Then the conditional expectations*

$$u_0(x,t) = \mathbb{E}_0 f(x - \alpha_0(t) + X(t) + J(t)), \qquad u_1(x,t) = \mathbb{E}_1 f(x - \alpha_1(t) + X(t) + J(t))$$

*satisfy the system*

$$\begin{cases} \frac{\partial u_0}{\partial t}(x,\ t) - \left(c_0 - \frac{\mathrm{d}\alpha_0}{\mathrm{d}t}(t)\right)\frac{\partial u_0}{\partial x}(x,\ t) = -\lambda_0\left[u_0(x,\ t) - u_1(x+\beta_0(t),\ t)\right], \\ \frac{\partial u_1}{\partial t}(x,\ t) - \left(c_1 - \frac{\mathrm{d}\alpha_1}{\mathrm{d}t}(t)\right)\frac{\partial u_1}{\partial x}(x,\ t) = -\lambda_1\left[u_1(x,\ t) - u_0(x+\beta_1(t),\ t)\right], \end{cases} \tag{4.1.19}$$

*where* $\beta_0(t) = h_0 - (\alpha_0(t) - \alpha_1(t))$, $\beta_1(t) = h_1 - (\alpha_1(t) - \alpha_0(t))$.

*Proof* First notice that the expectations $u_i = u_i(x,\ t)$ can be computed by the following integrals

$$u_i(x,t) = \int_{-\infty}^{\infty} f(x - \alpha_i(t) + y)p_i(y,\ t)\mathrm{d}y, \qquad i = 0,\ 1,$$

where the transition probability densities $p_0,\ p_1$ are defined by (4.1.3). Differentiating these equalities in $t$ we see that

$$\frac{\partial u_i}{\partial t} = -\frac{\mathrm{d}\alpha_i(t)}{\mathrm{d}t}\frac{\partial u_i(x,\ t)}{\partial x} + \int_{-\infty}^{\infty} f(x - \alpha_i(t) + y)\frac{\partial p_i(y,\ t)}{\partial t}\mathrm{d}y, \qquad i = 0,\ 1.$$

To finish the proof it is sufficient to apply Eq. (4.1.7) to the densities $p_0,\ p_1$ with subsequent integration by parts. □

The expectations $m_i(t) = \mathbb{E}_i(X(t) + J(t)),\ i = 0,\ 1,\ t > 0$, satisfy system (4.1.17), which can be rewritten in the following vector form:

$$\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}t} = \Lambda\mathbf{m} + \mathbf{v_1}.$$

Here

$$\Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}, \quad \mathbf{m} = (m_0(t),\ m_1(t))^T \text{ and } \mathbf{v}_1 = (d_0,\ d_1)^T.$$

We deduce a formula for the conditional variance under the given initial state $i = \varepsilon(0)$, $\mathbf{s} = (s_0(t),\ s_1(t))^T$, $s_i(t) = \mathrm{Var}_i(X(t) + J(t)),\ i = 0, 1$, by setting $f(x) = x^2,\ \alpha_i(t) = m_i(t),\ i = 0,\ 1$, in Proposition 4.3. In this case the set of Eq. (4.1.19) becomes

$$\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} = \Lambda\mathbf{s} + \mathbf{v_2}, \tag{4.1.20}$$

where $\mathbf{v}_2 = (\lambda_0(h_0 + m_1(t) - m_0(t))^2,\ \lambda_1(h_1 + m_0(t) - m_1(t))^2)^T$.

In general, such system

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \Lambda\mathbf{x}(t) + \mathbf{v}(t)$$

with zero initial conditions has the following solution

$$\mathbf{x}(t) = \int_0^t e^{(t-\tau)\Lambda}\mathbf{v}(\tau)d\tau. \tag{4.1.21}$$

The exponential $e^{t\Lambda}$ can easily be calculated and is found to be

$$e^{t\Lambda} = \frac{1}{2\lambda}\begin{pmatrix}\lambda_1 + \lambda_0 e^{-2\lambda t} & \lambda_0(1 - e^{-2\lambda t}) \\ \lambda_1(1 - e^{-2\lambda t}) & \lambda_0 + \lambda_1 e^{-2\lambda t}\end{pmatrix}, \tag{4.1.22}$$

where $2\lambda := \lambda_0 + \lambda_1$.

Substituting $e^{(t-\tau)\Lambda}$ given by (4.1.22) and $\mathbf{v}_1 = (d_0,\ d_1)^T$ instead of $\mathbf{v}(\tau)$ in (4.1.21) and then integrating, we obtain the explicit formula for the expectations:

$$\mathbf{m}(t) = \frac{t}{2\lambda}\begin{pmatrix}\lambda_1 + \lambda_0\Phi_\lambda(t) & \lambda_0 - \lambda_0\Phi_\lambda(t) \\ \lambda_1 - \lambda_1\Phi_\lambda(t) & \lambda_0 + \lambda_1\Phi_\lambda(t)\end{pmatrix}\begin{pmatrix}d_0 \\ d_1\end{pmatrix} \tag{4.1.23}$$

$$= \frac{t}{2\lambda}\left[(\lambda_1 d_0 + \lambda_0 d_1)\begin{pmatrix}1 \\ 1\end{pmatrix} + (d_0 - d_1)\Phi_\lambda(t)\begin{pmatrix}\lambda_0 \\ -\lambda_1\end{pmatrix}\right],$$

where $\Phi_\lambda(t) = \dfrac{1 - e^{-2\lambda t}}{2\lambda t}$.

To analyse the variances, notice that $m_0(t) - m_1(t) = t(d_0 - d_1)\Phi_\lambda(t)$. Hence

$$\mathbf{v_2}(\tau) = \begin{pmatrix}\lambda_0(h_0 + m_1(\tau) - m_0(\tau))^2 \\ \lambda_1(h_1 + m_0(\tau) - m_1(\tau))^2\end{pmatrix} = \begin{pmatrix}\lambda_0\left(h_0 - (d_0 - d_1)\tau\Phi_\lambda(\tau)\right)^2 \\ \lambda_1\left(h_1 + (d_0 - d_1)\tau\Phi_\lambda(\tau)\right)^2\end{pmatrix}$$
$$= \frac{1}{(2\lambda)^2}\begin{pmatrix}\lambda_0\left(2\lambda h_0 - (d_0 - d_1) + (d_0 - d_1)e^{-2\lambda\tau}\right)^2 \\ \lambda_1\left(2\lambda h_1 + (d_0 - d_1) - (d_0 - d_1)e^{-2\lambda\tau}\right)^2\end{pmatrix}.$$

With this formula in hand, the explicit expression for $\mathbf{s}(t)$ can be obtained from (4.1.21) and (4.1.22), but it is rather cumbersome. Nevertheless we can easily find the limits of $s_i(t)/t$ as $t \to 0$ and as $t \to \infty$:

$$\lim_{t\to 0}\frac{\mathbf{s}(t)}{t} = \lim_{t\to 0}\frac{1}{t}\int_0^t e^{(t-\tau)\Lambda}\mathbf{v_2}(\tau)d\tau = \mathbf{v_2}(0) = (\lambda_0 h_0^2,\ \lambda_1 h_1^2)^T. \tag{4.1.24}$$

To compute the limit at infinity, notice that the integrand in (4.1.21) can be written down by multiplying the matrix $e^{(t-\tau)\Lambda}$ defined by (4.1.22) and the vector $\mathbf{v}_2(\tau)$. The result has the following structure

$$\mathbf{A} + \mathbf{B}e^{-2\lambda\tau} + \mathbf{C}e^{-2\lambda(t-\tau)} + \mathbf{D}e^{-2\lambda(t+\tau)},$$

where $\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D}$ are constant vectors. The constant part $\mathbf{A}$ of this expression is

$$\mathbf{A} = \frac{1}{(2\lambda)^3}\begin{pmatrix}\lambda_1 & \lambda_0\\ \lambda_1 & \lambda_0\end{pmatrix}\cdot\begin{pmatrix}\lambda_0\,(2\lambda h_0-(d_0-d_1))^2\\ \lambda_1\,(2\lambda h_1+(d_0-d_1))^2\end{pmatrix}$$
$$= \frac{\lambda_0\lambda_1}{(\lambda_0+\lambda_1)^3}\left[(\lambda_1(h_0+h_1)+c_1-c_0)^2+(\lambda_0(h_0+h_1)+c_0-c_1)^2\right]\mathbf{e}_+,$$

where $\mathbf{e}_+ = (1,\ 1)^T$.

Therefore

$$\lim_{t\to\infty}\frac{s_0(t)}{t} = \lim_{t\to\infty}\frac{s_1(t)}{t}$$
$$= \frac{\lambda_0\lambda_1}{(\lambda_0+\lambda_1)^3}\left[(\lambda_1(h_0+h_1)+c_1-c_0)^2+(\lambda_0(h_0+h_1)+c_0-c_1)^2\right]. \tag{4.1.25}$$

*Remark 4.5* In the symmetric case when $\lambda_0 = \lambda_1 := \lambda$ one can simplify formulae for $\mathbf{m}$ and $\mathbf{s}$. We set $a = (c_0+c_1)/2, c = (c_0-c_1)/2, B = (h_0+h_1)/2, b = (h_0-h_1)/2,$ $\gamma_0 = -2c(c/\lambda+h_0)$, $\gamma_1 = -2c(c/\lambda-h_1)$. In these notations

$$m_i(t) = \left[a+\lambda B+(-1)^i(c+\lambda b)\Phi_\lambda(t)\right]t, \qquad i=0,\ 1, \tag{4.1.26}$$

$$s_i(t) = \left[\frac{c^2}{\lambda}+\lambda B^2+(c+\lambda b)^2\frac{\Phi_{2\lambda}(t)}{\lambda}+\gamma_i\Phi_\lambda(t)+(-1)^i 2B(c+\lambda b)\mathrm{e}^{-2\lambda t}\right]t. \tag{4.1.27}$$

### *4.1.3 Change of Measure for Jump-Telegraph Processes*

Fix the time horizon $T,\ T>0$. Let $\varepsilon = \varepsilon(t),\ t\in[0,\ T]$, be the underlying Markov process with alternating parameters $\lambda_0,\ \lambda_1 > 0$ defined on the filtered probability space $(\Omega, \mathfrak{F}, \{(\mathfrak{F})_t\}_{t\in[0,T]}, \mathbb{P})$ by Eq. (4.1.1). Let us present the measure change technique for this process.

Consider the telegraph process $X^*(t),\ t\geq 0$, with parameters $(c_0^*,\ \lambda_0)$ and $(c_1^*,\ \lambda_1)$, which is driven by the Markov process $\varepsilon$. Assume that $c_0^* < \lambda_0,\ c_1^* < \lambda_1$.

Consider the jump process

$$J^*(t) = -\sum_{n=1}^{N(t)} h^*_{\varepsilon_n}$$

with the jump values $h_0^* = -c_0^*/\lambda_0$ and $h_1^* = -c_1^*/\lambda_1,\ h_0^*,\ h_1^* > -1;\ \varepsilon_n := \varepsilon(\tau_n-)$ and the counting Poisson process $N(t),\ t\geq 0$. Due to Theorem 4.1 the sum $X^*(t)+J^*(t),\ t\geq 0$ is an $(\mathfrak{F}_t, \mathbb{P})$-martingale. The process

$$Z(t) = \mathscr{E}_t(X^*+J^*), \quad 0\leq t\leq T \tag{4.1.28}$$

is the $(\mathfrak{F}_t, \mathbb{P})$-martingale also. Here $\mathcal{E}_t(\cdot)$ denotes the (right-continuous) stochastic exponential.

Define a new measure $\mathbb{P}^*$ on $\mathfrak{F}_T$ by the distribution density

$$\left.\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}\right|_{\mathfrak{F}_T} = Z(T).$$

Integrating in (4.1.28) we see that ([9], Proposition 3.9.2)

$$Z(t) = \mathrm{e}^{X^*(t)}\kappa^*(t), \tag{4.1.29}$$

where

$$\kappa^*(t) = \prod_{s\le t}\left(1 + \Delta J^*(s)\right), \qquad \kappa^*(0) = 1.$$

Here $\Delta J^*(s) = J^*(s) - J^*(s-)$ is a jump value.

The jump component $\kappa^*(t)$ can be described as follows. Consider the recurrent sequence

$$\kappa_n^{*,i} = \kappa_{n-1}^{*,1-i}(1 + h_i^*), \quad n \ge 1, \quad i = 0,\ 1, \tag{4.1.30}$$

where $\kappa_0^{*,i} = 1$. Precisely speaking, if $n = 2k$, then $\kappa_n^{*,i} = (1 + h_i^*)^k(1 + h_{1-i}^*)^k$, and if $n = 2k + 1$, then $\kappa_n^{*,i} = (1 + h_i^*)^{k+1}(1 + h_{1-i}^*)^k$. Process $\kappa^*(t),\ t \ge 0$ can be expressed by means of the sequence $\kappa_n^{*,i},\ n \ge 0$, namely, $\kappa^*(t) = \kappa_{N(t)}^{*,i}$. Then (4.1.29) can be expressed as $Z(t) = \mathrm{e}^{X^*(t)}\kappa_{N(t)}^{*,i}$, where, remind, $i = \varepsilon(0) \in \{0,\ 1\}$ indicates the initial state of the system.

The following theorem is a counterpart of Girsanov's theorem in this setting.

**Theorem 4.2** *Under probability $\mathbb{P}^*$ with density $Z(T)$ relative to $\mathbb{P}$, the process $X = X(t),\ 0 \le t \le T$ is also the telegraph process with the parameters $(c_0,\ \lambda_0^*)$ and $(c_1,\ \lambda_1^*)$, where $\lambda_i^* = \lambda_i(1 + h_i^*) = \lambda_i - c_i^*, \quad i = 0,\ 1.$*

*Proof* Suppose that both the telegraph processes $X^*$ and $X$ are driven by the common Markov process $\varepsilon$, and have the velocities $c_i^*$ and $c_i,\ i = 0, 1$, respectively. Hence $X^*(t)$ and $X(t),\ t \ge 0$ are connected with each other as follows,

$$X^*(t) = \mu X(t) + at, \tag{4.1.31}$$

where the constants $\mu$ and $a$ solve the system

$$\begin{cases} \mu c_0 + a = c_0^*, \\ \mu c_1 + a = c_1^*. \end{cases}$$

Solving this system we have $\mu = \dfrac{\Delta c^*}{\Delta c} = \dfrac{c_0^* - c_1^*}{c_0 - c_1}$ and $a = \dfrac{c_1^* c_0 - c_0^* c_1}{c_0 - c_1}$.

Let $p_i(x,\ t;\ n)$ and $p_i^*(x,\ t;\ n)$ be the transition densities of $X(t)$ and $X^*(t)$ defined by (4.1.3). Functions $p_i(x,\ t;\ n)$ satisfy the set of Eq. (4.1.6) with $h_0 = h_1 = 0$.

To prove Theorem 4.2 it is sufficient to demonstrate that functions $p_i^*(x,\ t;\ n)$ satisfy Eq. (4.1.6) with $\lambda_0^*,\ \lambda_1^*$ instead of $\lambda_0,\ \lambda_1$ (and with $h_0 = h_1 = 0$).

By (4.1.29) and (4.1.31) the densities $p_i$ and $p_i^*$ are connected as follows:

$$\begin{aligned} p_i^*(x,\ t;\ n) &= \mathbb{E}\left\{Z(t)\mathbb{1}_{\{X(t)\in \mathrm{d}x,\ N(t)=n\}} \mid \varepsilon(0) = i\right\}/\mathrm{d}x \\ &= \kappa_n^{*,i}\mathrm{e}^{\mu x+at}p_i(x,\ t;\ n). \end{aligned} \tag{4.1.32}$$

Hence, due to Eq. (4.1.6) (with $h_0 = h_1 = 0$) we have

$$p_i^*(x,\ t;\ n) = \kappa_n^{*,i}\mathrm{e}^{\mu x+at}\int_0^t p_{1-i}(x - c_i s,\ t-s;\ n-1)\lambda_i \mathrm{e}^{-\lambda_i s}\mathrm{d}s.$$

Now we apply again identity (4.1.32):

$$p_i^*(x,\ t;\ n) = (1+h_i^*)\int_0^t \mathrm{e}^{(\mu c_i+a)s}p_{1-i}^*(x-c_i s, t-s; n-1)\lambda_i\mathrm{e}^{-\lambda_i s}\mathrm{d}s.$$

Application of the equalities

$$\mu c_i + a = c_i^*,\quad (1+h_i^*)\lambda_i = \lambda_i^* = \lambda_i - c_i^*,\quad i = 0, 1,$$

finishes the proof of the Theorem.

## 4.2 Moments

In this section we study moments of the asymmetric telegraph process $X(t),\ t \ge 0$. The approach presented is useful for the goals of financial applications and it is also interesting in general.

The explicit formulae for the moments can be obtained by differentiating the characteristic function $\widehat{p}(\xi, t)$, (2.4.1). Instead, we use an alternative approach based on the Kolmogorov equations. These equations look more natural with asymmetric velocities and switching rates. Thus we consider the telegraph process $X = X(t)$ which is driven by the Markov process $\varepsilon = \varepsilon(t)$ with arbitrary switching intensities $\lambda_i > 0,\ i = 0, 1$, such that the generator of $\varepsilon = \varepsilon(t)$ is defined by the matrix

$$\Lambda := \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}. \tag{4.2.1}$$

The respective velocities of $X$ are $c_0,\ c_1$, such that $X(t) := \int_0^t c_{\varepsilon(s)}\mathrm{d}s$. In this section we assume that $c_0 > 0 > c_1$.

Let $p_i = p_i(x,t)$ be the conditional densities of $X(t)$, if the initial state $i \in \{0,\ 1\}$ is fixed. Notice that in this setting the Kolmogorov Eq. (2.2.7) takes the form

$$\begin{aligned}
&\frac{\partial p_0}{\partial t}(x,t) + c_0\frac{\partial p_0}{\partial x}(x,t) = -\lambda_0 p_0(x,t) + \lambda_0 p_1(x,t),\\
& && t > 0 \qquad (4.2.2)\\
&\frac{\partial p_1}{\partial t}(x,t) + c_1\frac{\partial p_1}{\partial x}(x,t) = -\lambda_1 p_1(x,t) + \lambda_1 p_0(x,t),
\end{aligned}$$

with the initial conditions $p_0(x,0) = p_1(x,0) = \delta(x)$.

Moments of r. v. $X(t),\ t > 0$,

$$m_n^{(i)}(t) = \mathbb{E}_i\{X(t)^n\} = \int_{-\infty}^{\infty} x^n p_i(x,t)\mathrm{d}x, \qquad i = 0, 1, \quad n \ge 0, \qquad (4.2.3)$$

satisfy the following recursive chain of integral equations.

**Theorem 4.3** *For arbitrary $n \ge 1$ the expectations, defined by (4.2.3), satisfy the equations*

$$\begin{aligned}
m_n^{(0)}(t) &= \frac{n}{2\lambda}\left[\mathscr{I}\left(\lambda_1 c_0 m_{n-1}^{(0)} + \lambda_0 c_1 m_{n-1}^{(1)}\right)(t) + \lambda_0\mathscr{C}\left(c_0 m_{n-1}^{(0)} - c_1 m_{n-1}^{(1)}\right)(t)\right],\\
m_n^{(1)}(t) &= \frac{n}{2\lambda}\left[\mathscr{I}\left(\lambda_1 c_0 m_{n-1}^{(0)} + \lambda_0 c_1 m_{n-1}^{(1)}\right)(t) - \lambda_1\mathscr{C}\left(c_0 m_{n-1}^{(0)} - c_1 m_{n-1}^{(1)}\right)(t)\right],
\end{aligned} \qquad (4.2.4)$$

*where $2\lambda = \lambda_0 + \lambda_1$, and $m_0^{(i)}(t) \equiv 1, i = 0, 1$. Here the integral operators $\mathscr{I}$ and $\mathscr{C}$ are defined on $C[0,\infty)$ as follows,*

$$\mathscr{I}f = \mathscr{I}f(t) := \int_0^t f(s)\mathrm{d}s, \quad \mathscr{C}f = \mathscr{C}f(t) := \int_0^t \mathrm{e}^{-2\lambda(t-s)}f(s)\mathrm{d}s. \qquad (4.2.5)$$

*Proof* By differentiating (4.2.3) in $t$ we have

$$\frac{\mathrm{d}m_n^{(i)}(t)}{\mathrm{d}t} = \int_{-\infty}^{\infty} x^n\frac{\partial p_i}{\partial t}(x,t)\mathrm{d}x, \qquad i = 0, 1.$$

We apply Eq. (4.2.2) with subsequent integration by parts. For $n \ge 1,\ i = 0, 1$,

$$\frac{\mathrm{d}m_n^{(i)}(t)}{\mathrm{d}t} = nc_i\int_{-\infty}^{\infty} x^{n-1}p_i(x,t)\mathrm{d}x - \lambda_i\int_{-\infty}^{\infty} x^n p_i(x,t)\mathrm{d}x + \lambda_i\int_{-\infty}^{\infty} x^n p_{1-i}(x,t)\mathrm{d}x, \qquad (4.2.6)$$

which is equivalent to the system

$$\begin{cases} \dfrac{\mathrm{d}m_n^{(0)}(t)}{\mathrm{d}t} = -\lambda_0 m_n^{(0)}(t) + \lambda_0 m_n^{(1)}(t) + nc_0 m_{n-1}^{(0)}(t), \\ \dfrac{\mathrm{d}m_n^{(1)}(t)}{\mathrm{d}t} = -\lambda_1 m_n^{(1)}(t) + \lambda_1 m_n^{(0)}(t) + nc_1 m_{n-1}^{(1)}(t), \end{cases} \quad n \geq 1 \tag{4.2.7}$$

with initial conditions $m_n^{(0)}(0) = 0, m_n^{(1)}(0) = 0,\ n \geq 1$, and $m_0^{(0)}(t) \equiv m_0^{(1)}(t) \equiv 1$.

Introducing the vector notations

$$\mathbf{m}_n(t) = (m_n^{(0)}(t),\ m_n^{(1)}(t))^T, \quad \tilde{\mathbf{m}}_{n-1}(t) = (c_0 m_{n-1}^{(0)}(t),\ c_1 m_{n-1}^{(1)}(t))^T,$$

we can rewrite the latter system in the matrix form:

$$\frac{\mathrm{d}\tilde{\mathbf{m}}_n(t)}{\mathrm{d}t} = \Lambda \mathbf{m}_n(t) + n\tilde{\mathbf{m}}_{n-1}(t), \qquad n \geq 1, \tag{4.2.8}$$

where matrix $\Lambda$ is defined by (4.2.1).

Differential equation (4.2.8) with zero initial condition is equivalent to the set of integral equations

$$\mathbf{m}_n(t) = n \int_0^t \mathrm{e}^{(t-s)\Lambda} \tilde{\mathbf{m}}_{n-1}(s)\mathrm{d}s, \qquad n \geq 1. \tag{4.2.9}$$

The exponential of $t\Lambda$ can be easily calculated. It has the form

$$\mathrm{e}^{t\Lambda} = \frac{1}{2\lambda} \begin{pmatrix} \lambda_1 + \lambda_0 \mathrm{e}^{-2\lambda t} & \lambda_0(1 - \mathrm{e}^{-2\lambda t}) \\ \lambda_1(1 - \mathrm{e}^{-2\lambda t}) & \lambda_0 + \lambda_1 \mathrm{e}^{-2\lambda t} \end{pmatrix}, \qquad 2\lambda = \lambda_0 + \lambda_1. \tag{4.2.10}$$

Substituting this in (4.2.9) we verify recursive relations (4.2.4).

Let operators $\mathscr{I}$ and $\mathscr{C}$ be defined by (4.2.5). It is easy to see that operators $\mathscr{I}$ and $\mathscr{C}$ commute.

Let $U_{0,0} = U_{0,0}(t) \equiv 1$. We define the family of functions $U_{n,m} = U_{n,m}(t)$, $n, m \geq 0,\ t \geq 0$, by the following equalities:

$$U_{n,m}(t) := \mathscr{C}^n \mathscr{I}^m U_{0,0}(t). \tag{4.2.11}$$

First notice that $U_{0,m} = \mathscr{I}^m U_{0,0} = t^m/m!$ and by applying the $n$-fold convolution of $\mathscr{C}$ we have

$$U_{n,0}(t) = \mathscr{C}^n U_{00}(t) = (2\lambda)^{-n} \mathrm{e}^{-2\lambda t} \sum_{k=n}^{\infty} \frac{(2\lambda t)^k}{k!}. \tag{4.2.12}$$

Due to formula 6.5.13, [10]

$$U_{n,0}(t) = \frac{(2\lambda)^{-n}}{(n-1)!}\gamma(n, 2\lambda t), \tag{4.2.13}$$

where $\gamma(n,\cdot)$ is the incomplete gamma function, $\gamma(n,x) := \int_0^x \mathrm{e}^{-t}t^{n-1}\mathrm{d}t$.

Applying the series expansion of incomplete gamma function, 6.5.29 [10], we finally get

$$U_{n,0}(t) = \frac{(2\lambda)^{-n}}{(n-1)!}\sum_{k=0}^{\infty}\frac{(-1)^k(2\lambda t)^{n+k}}{k!(n+k)} = \frac{t^n}{(n-1)!}\sum_{k=0}^{\infty}\frac{(-2\lambda t)^k}{k!(n+k)}, \quad n \ge 1. \tag{4.2.14}$$

Notice, that functions $U_{n,m}$ can be expressed by means of the Kummer confluent hypergeometric functions $\Phi = \Phi(a, b; z)$. In particular, applying 6.5.12 [10] to (4.2.13), we have $U_{n,0}(t) = \dfrac{t^n}{n!}\Phi(n, n+1; -2\lambda t)$. Moreover, repeatedly integrating equality (4.2.14), we obtain for $n, m \ge 0$

$$\begin{aligned}
U_{n,m}(t) =& \mathscr{I}^m U_{n,0}(t) = \frac{1}{(n-1)!(2\lambda)^{n+m}}\sum_{k=0}^{\infty}\frac{(-2\lambda t)^{n+k+m}}{k!(n+k)(n+k+1)\ldots(n+k+m)} \\
&= \frac{1}{(n-1)!(2\lambda)^{n+m}}\sum_{k=0}^{\infty}\frac{(-2\lambda t)^{n+k+m}(n+k-1)!}{k!(n+k+m)!} \\
&= \frac{1}{(n+m)!(2\lambda)^{n+m}}\sum_{k=0}^{\infty}\frac{(-2\lambda t)^{n+k+m}(n)_k}{k!(n+m+1)_k} = \frac{t^{n+m}}{(n+m)!}\sum_{k=0}^{\infty}\frac{(-2\lambda t)^k(n)_k}{k!(n+m+1)_k} \\
&= \frac{t^{n+m}}{(n+m)!}\Phi(n, n+m+1; -2\lambda t).
\end{aligned} \tag{4.2.15}$$

It is easy to see that functions $U_{n,m} = U_{n,m}(t), \ \ n, m \ge 0$, are linearly independent. Solving the set of Eq. (4.2.4) we derive the formulae for $m_n^{(i)}(t)$ in the "almost-closed-form".

**Theorem 4.4** *Moments $m_n^{(i)}(t), \ i = 0, 1$ can be expressed by means of the linear combination of Kummer functions $\Phi(\cdot\,, n+1; -2\lambda t)$:*

$$m_n^{(i)}(t) = \frac{t^n}{n!}\sum_{k=0}^{n} a_{k,n-k}^{(i)}\ \Phi(k, n+1; -2\lambda t), \quad 2\lambda = \lambda_0 + \lambda_1. \tag{4.2.16}$$

*Here the coefficients $a_{n,m}^{(i)}$ are defined as follows*: $n, m \ge 1$,

$$\begin{aligned}
a_{0,0}^{(0)} &= a_{0,0}^{(1)} = 1,\\
a_{0,m}^{(0)} &= a_{0,m}^{(1)} = \frac{m}{2\lambda}\left[\lambda_1 c_0 a_{0,m-1}^{(0)} + \lambda_0 c_1 a_{0,m-1}^{(1)}\right],\\
a_{n,0}^{(0)} &= \frac{n\lambda_0}{2\lambda}\left[c_0 a_{n-1,0}^{(0)} - c_1 a_{n-1,0}^{(1)}\right],\ a_{n,0}^{(1)} = \frac{-n\lambda_1}{2\lambda}\left[c_0 a_{n-1,0}^{(0)} - c_1 a_{n-1,0}^{(1)}\right],
\end{aligned} \tag{4.2.17}$$

*and,* $n, m \geq 1$*,*

$$\begin{aligned}
a_{n,m}^{(0)} &= \frac{n+m}{2\lambda}\left[\left(\lambda_1 c_0 a_{n,m-1}^{(0)} + \lambda_0 c_1 a_{n,m-1}^{(1)}\right) + \lambda_0\left(c_0 a_{n-1,m}^{(0)} - c_1 a_{n-1,m}^{(1)}\right)\right],\\
a_{n,m}^{(1)} &= \frac{n+m}{2\lambda}\left[\left(\lambda_1 c_0 a_{n,m-1}^{(0)} + \lambda_0 c_1 a_{n,m-1}^{(1)}\right) - \lambda_1\left(c_0 a_{n-1,m}^{(0)} - c_1 a_{n-1,m}^{(1)}\right)\right].
\end{aligned} \tag{4.2.18}$$

*Proof* We write the solution of Eq. (4.2.4) in the form

$$m_n^{(i)}(t) = \sum_{k=0}^{n} a_{k,n-k}^{(i)} U_{k,n-k}(t), \tag{4.2.19}$$

where functions $U_{n,m}$ are defined by (4.2.11).

If $n = 0$, then (4.2.19) follows from $m_0^{(0)} = m_0^{(1)} = 1$ and $a_{0,0}^{(0)} = a_{0,0}^{(1)} = 1$, $U_{0,0} \equiv 1$.

Inserting (4.2.19) in Eq. (4.2.4) we obtain for $n \geq 1$:

$$\begin{aligned}
m_n^{(0)}(t) = \frac{n}{2\lambda}\Bigg[&\lambda_1 c_0 \sum_{k=0}^{n-1} a_{k,n-1-k}^{(0)} U_{k,n-k}(t) + \lambda_0 c_1 \sum_{k=0}^{n-1} a_{k,n-1-k}^{(1)} U_{k,n-k}(t)\\
&+ \lambda_0 c_0 \sum_{k=0}^{n-1} a_{k,n-1-k}^{(0)} U_{k+1,n-1-k}(t) - \lambda_0 c_1 \sum_{k=0}^{n-1} a_{k,n-1-k}^{(1)} U_{k+1,n-1-k}(t)\Bigg],\\
m_n^{(1)}(t) = \frac{n}{2\lambda}\Bigg[&\lambda_1 c_0 \sum_{k=0}^{n-1} a_{k,n-1-k}^{(0)} U_{k,n-k}(t) + \lambda_0 c_1 \sum_{k=0}^{n-1} a_{k,n-1-k}^{(1)} U_{k,n-k}(t)\\
&- \lambda_1 c_0 \sum_{k=0}^{n-1} a_{k,n-1-k}^{(0)} U_{k+1,n-1-k}(t) + \lambda_1 c_1 \sum_{k=0}^{n-1} a_{k,n-1-k}^{(1)} U_{k+1,n-1-k}(t)\Bigg].
\end{aligned}$$

Comparing these equations with (4.2.19) we get (4.2.17)–(4.2.18) by using linear independence of $U_{n,m}$.

Representation (4.2.16) now follows from (4.2.19) and (4.2.15).

*Remark 4.6* Formulae (4.2.17) in closed form become:

$$a_{0,m}^{(0)} = a_{0,m}^{(1)} = \frac{m!}{(2\lambda)^m}(\lambda_1 c_0 + \lambda_0 c_1)^m, \quad m \geq 0, \tag{4.2.20}$$

$$\begin{aligned} a_{n,0}^{(0)} &= \frac{\lambda_0(c_0-c_1)n!}{(2\lambda)^n}(\lambda_0c_0+\lambda_1c_1)^{n-1}, \\ a_{n,0}^{(1)} &= -\frac{\lambda_1(c_0-c_1)n!}{(2\lambda)^n}(\lambda_0c_0+\lambda_1c_1)^{n-1} \end{aligned} \quad n \geq 1. \tag{4.2.21}$$

In particular, (4.2.20) and (4.2.21) give

$$a_{0,1}^{(0)} = a_{0,1}^{(1)} = \frac{1}{2\lambda}(\lambda_1c_0+\lambda_0c_1),\, a_{1,0}^{(0)} = \frac{\lambda_0}{2\lambda}(c_0-c_1),\, a_{1,0}^{(1)} = -\frac{\lambda_1}{2\lambda}(c_0-c_1).$$

By definition of Kummer function, $\Phi(0, 2; z) \equiv 1$, $\Phi(1, 2; z) = \dfrac{e^z-1}{z}$. Applying (4.2.16) with $n = 1$ we have the following expressions for the first moments,

$$\begin{aligned} m_1^{(i)}(t) &= t\left[a_{0,1}^{(i)}\Phi(0,2;-2\lambda t) + a_{1,0}^{(i)}\Phi(1,2;-2\lambda t)\right] \\ &= \frac{1}{2\lambda}\left[(\lambda_1c_0+\lambda_0c_1)t + (-1)^i\lambda_i(c_0-c_1)\left(\frac{1-e^{-2\lambda t}}{2\lambda}\right)\right], \quad i = 0, 1. \end{aligned}$$

Similarly, the second moments are

$$\begin{aligned} m_2^{(i)}(t) &= \frac{t^2}{2}\left[a_{0,2}^{(i)}\Phi(0,3;-2\lambda t) + a_{1,1}^{(i)}\Phi(1,3;-2\lambda t) + a_{2,0}^{(i)}\Phi(2,3;-2\lambda t)\right] \\ &= \frac{1}{4\lambda^2}\left\{(\lambda_1c_0+\lambda_0c_1)^2t^2 + \frac{2\lambda_0\lambda_1c^2}{\lambda^2}(e^{-2\lambda t}-1+2\lambda t)\right. \\ &\left. + (-1)^i\frac{2\lambda_ic}{\lambda}\left[\frac{2\beta c}{\lambda}(1-e^{-2\lambda t}) + (\lambda_1c_0+\lambda_0c_1)t - (\lambda_0c_0+\lambda_1c_1)te^{-2\lambda t}\right]\right\}. \end{aligned}$$

Here the following equalities have been used (see [10]):

$$\Phi(0,3;z) \equiv 1, \quad \Phi(1,3;z) = 2(e^z-1-z)/z^2, \quad \Phi(2,3;z) = 2(ze^z-e^z+1)/z^2,$$

$$\begin{aligned} a_{0,2}^{(0)} &= a_{0,2}^{(1)} = \frac{(\lambda_1c_0+\lambda_0c_1)^2}{2\lambda^2}, \qquad \text{(see (4.2.20))}, \\ a_{1,1}^{(i)} &= \frac{\lambda_i(c_0-c_1)}{2\lambda^2}\left[\lambda_{1-i}(c_0-c_1) + (-1)^i(\lambda_1c_0+\lambda_0c_1)\right], \quad \text{(see (4.2.18))}, \\ a_{2,0}^{(i)} &= (-1)^i\frac{\lambda_i(c_0-c_1)}{2\lambda^2}(\lambda_0c_0+\lambda_1c_1), \quad \text{(see (4.2.21))}, \quad i = 0, 1. \end{aligned}$$

Equations (4.2.18) have the following equivalent form: for $n, m \geq 1$

$$\begin{aligned} a_{n,m}^{(0)} - a_{n,m}^{(1)} &= (n+m)(c_0a_{n-1,m}^{(0)} - c_1a_{n-1,m}^{(1)}), \\ \lambda_1a_{n,m}^{(0)} + \lambda_0a_{n,m}^{(1)} &= (n+m)(\lambda_1c_0a_{n,m-1}^{(0)} + \lambda_0c_1a_{n,m-1}^{(1)}). \end{aligned} \tag{4.2.22}$$

*Remark 4.7* For the *symmetric* telegraph process formulae (4.2.16) have a closed form. If $\lambda_0 = \lambda_1 =: \lambda,\ c_0 = -c_1 =: c$, then formulae (4.2.22) become:

$$\begin{aligned} a_{n,m}^{(0)} - a_{n,m}^{(1)} &= c(n+m)(a_{n-1,m}^{(0)} + a_{n-1,m}^{(1)}), \\ a_{n,m}^{(0)} + a_{n,m}^{(1)} &= c(n+m)(a_{n,m-1}^{(0)} - a_{n,m-1}^{(1)}), \end{aligned}$$

which is equivalent to

$$\begin{aligned} a_{n,m}^{(0)} - a_{n,m}^{(1)} &= c^2(n+m)(n+m-1)(a_{n-1,m-1}^{(0)} - a_{n-1,m-1}^{(1)}), \\ a_{n,m}^{(0)} + a_{n,m}^{(1)} &= c^2(n+m)(n+m-1)(a_{n-1,m-1}^{(0)} + a_{n-1,m-1}^{(1)}). \end{aligned}$$

Therefore,

$$a_{n,m}^{(i)} = c^2(n+m)(n+m-1)a_{n-1,m-1}^{(i)}, \qquad n, m \geq 1,\ i = 0, 1. \tag{4.2.23}$$

The initial coefficients are $a_{0,0}^{(i)} = 1,\ a_{1,0}^{(0)} = c,\ a_{1,0}^{(1)} = -c$ (see (4.2.20) with $m = 0$ and (4.2.21) with $n = 1$). The "boundary" values in the symmetric case become (see (4.2.20) and (4.2.21)) $a_{0,m}^{(i)} = a_{n,0}^{(i)} = 0$ for any $m \geq 1,\ n \geq 2$.

Thus, (4.2.23) gives $a_{n,n}^{(i)} = (2n)!c^{2n},\quad a_{n+1,n}^{(i)} = (-1)^i(2n+1)!c^{2n+1},\quad n \geq 0$ and $a_{n,m}^{(i)} = 0$ for other $n, m,\ i = 0, 1$.

Hence, formula (4.2.16) can be easily simplified to

$$\begin{aligned} m_{2n}^{(i)}(t) &= (ct)^{2n}\Phi(n, 2n+1; -2\lambda t), \\ m_{2n+1}^{(i)}(t) &= (-1)^i(ct)^{2n+1}\Phi(n+1, 2n+2; -2\lambda t). \end{aligned} \tag{4.2.24}$$

From (4.2.24) the unconditional moments follow:

$$m_{2n}(t) = (ct)^{2n}\Phi(n, 2n+1; -2\lambda t), \quad m_{2n+1}(t) \equiv 0.$$

The moments of symmetric telegraph processes have been presented in [4] and [11] in terms of modified Bessel functions, but formulae (4.2.24) are more succinct.

Applying the asymptotic formula, (see [10], 13.1.5):

$$\Phi(a, b; z) = \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a}[1 + O(|z|^{-1})], \quad |z| \to \infty,\ \mathrm{Re}\, z < 0, \tag{4.2.25}$$

to formulae (4.2.24), we obtain the limits under scaling condition (2.6.1):

$$m_{2n}^{(i)}(t) \to (\sigma^2 t)^n (2n-1)!!, \qquad m_{2n+1}^{(i)}(t) \to 0, \quad i = 0, 1 \tag{4.2.26}$$

and this coincides with moments of Brownian motion with diffusion coefficient $\sigma^2$. This accords with the classic result by M. Kac [12].

## Notes

This chapter is based preferably on papers by the second author, see [3, 7, 13]. Section 4.2 follows paper [14].

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# Chapter 5
# Financial Modelling and Option Pricing

**Abstract** In Chapter 5 we apply the results of previous chapters for option pricing. The fundamental building block of all financial modelling is the concept of arbitrage-free and complete market. For the time- and space-continuous stochastic models the unique underlying process satisfying this concept is the geometric Brownian motion. In contrast, we suggest another approach to the continuous-time stochastic modelling of financial markets based on the telegraph processes. We construct a simple model, which is *free of arbitrage and complete*.

**Keywords** Option pricing · Hedging strategies · Martingales · Jump-telegraph processes · Rescaling · Implied volatility

Applications of stochastic processes in financial modelling date back to the beginning of XX century. Louis Bachelier in his thesis (1900) [1] discovered the concept of Brownian motion with a view to study price movements on the Paris exchange. Unfortunately, his discovery has been lost for a long time. The name of Bachelier was returned in scientific circulation more than a half century later, see this story in [2]. From 1973, beginning with the papers by Merton, [3], Black and Scholes, [4], the probabilistic exploration in the field becomes extremely fruitful. More and more sophisticated models have been constructed in attempts to describe financial markets. Now the shelf of textbooks on mathematical finance is almost infinite. The reader is referred to [5] or [6]. We can also mention such textbooks as [7–14], to name a few.

The fundamental building block of all financial modelling is the concept of arbitrage-free and complete market. If we proceed with continuous-time (and space) stochastic models, the unique underlying process which satisfies this concept is the Brownian motion. In this chapter we suggest another approach for continuous-time stochastic modelling of financial markets based on the telegraph processes. We construct a simple model, which is *free of arbitrage and complete*.

The majority of these results has been recently published (see [15–17]). The main objective of the next section is to give a brief introduction to the methods and concepts required in this field.

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## 5.1 Hedging Strategies and Option Pricing

First, we remind briefly the Black-Scholes settings and some general principles of financial modelling, referring the reader to more detailed presentations (see, for example, [5] or [6] and references therein).

### 5.1.1 Option Pricing, Hedging and Martingales

Consider the financial market of two assets, namely, a risky asset which is driven by a stochastic process, and a deterministic bond (bank account).

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space. Fix the trading horizon $T, T > 0$. Let $\{\mathfrak{F}_t\}_{t\in[0,T]}$ be a filtration and $\mathfrak{F}_t$ can be interpreted as an information available for investors at time $t$. We assume that $\mathfrak{F}_0$ consists of all $\mathbb{P}$-null sets and their complements, $\mathfrak{F}_T = \mathfrak{F}$.

Let stochastic process $S = S(t),\ t \in [0, T]$ represents the price of risky asset, $B = B(t),\ t \in [0, T]$ is the (non-random) bond price. The process $S = S(t)$ is assumed to be adapted to the filtration $\{\mathfrak{F}_t\}_{t\in[0,T]}$.

The underlying assets $S$ and $B$ are traded continuously at time instants $t \in [0, T]$, a portfolio (or a strategy) $\pi_t,\ t \in [0, T]$ is given by (two-dimensional) random process $(\varphi_t,\ \psi_t)$ with the wealth process $V_t = \varphi_t S(t) + \psi_t B(t)$. So, $\varphi_t$ and $\psi_t$ are the numbers of units of the risky asset and of the bond, respectively, in the portfolio which is formed at time $t,\ t \in [0, T]$. The processes $\varphi_t,\ \psi_t,\ t \in [0, T]$ are assumed to be predictable (i. e. their trajectories are left-continuous a.s.).

The strategy is said to be *admissible*, if $V_t \geq 0$ a.s. for all $t \in [0, T]$. We call the strategy $\pi_t$ *self-financing*, if any changes in the value $V_t$ result entirely from the changes in prices of the basic assets:

$$\mathrm{d}V_t = \varphi_t \mathrm{d}S(t) + \psi_t \mathrm{d}B(t), \qquad t \in [0, T]. \tag{5.1.1}$$

We say that the market model possesses *arbitrage opportunities* if there exists an admissible market strategy $\pi_t,\ t \in [0, T]$ such that the value process $V_t,\ t \in [0, T]$ satisfies the following conditions

$$V_0 = 0, \qquad \mathbb{E}\{V_T\} > 0. \tag{5.1.2}$$

We say that the market model is *viable* or it is *arbitrage-free* if it contains no arbitrage opportunities.

Consider a non-negative random variable $\mathcal{H}$ on the probability space $(\Omega, \mathfrak{F}_T, \mathbb{P})$ as a contingent claim with maturity $T$. The claim $\mathcal{H}$ is *replicable*, if there exists an admissible self-financing strategy, such that the final strategy value coincides with $\mathcal{H}:\ V_T = \mathcal{H}$ a.s. This strategy is named the *hedging strategy* for the claim $\mathcal{H}$.

If the model is arbitrage-free, the value process which is associated with replicable strategy is unique (so called the *law of one price*).

The *law of one price* can be proved by constructing an arbitrage strategy.

To describe the model without arbitrage we need the notion of martingale.

**Definition 5.1** Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in[0,T]}, \mathbb{P})$ be the filtered probability space. An $\mathfrak{F}_t$-adapted process $M = (M_t)_{t\in[0,T]}$ is $\mathbb{P}$-martingale, if $\mathbb{E}|M_t| < \infty$ for all $t \in [0, T]$ and

$$\mathbb{E}_{\mathbb{P}}\{M_t \mid \mathfrak{F}_s\} = M_s \text{ for all } s, t \in [0, T], \; s < t. \tag{5.1.3}$$

Here $\mathbb{E} = \mathbb{E}_{\mathbb{P}}\{\cdot\}$ and $\mathbb{E}_{\mathbb{P}}\{\cdot \mid \mathfrak{F}_s\}$ denote expectations with respect to probability measure $\mathbb{P}$.

We say that probability measure $\mathbb{P}^*$ is equivalent to measure $\mathbb{P}$ if $\mathbb{P}^*$ has the same null sets as $\mathbb{P}$. We use the notation $\mathbb{P}^* \sim \mathbb{P}$.

The probability measure $\mathbb{P}^* \sim \mathbb{P}$ is the equivalent martingale measure (EMM) for the market model $(S, \; B)$ if the discounted price $\bar{S}(t) = B(t)^{-1}S(t)$ is a martingale under measure $\mathbb{P}^*$ and filtration $\{\mathfrak{F}_t\}_{t\in[0,T]}$.

**Theorem 5.1** **(First fundamental theorem)** *The following statements are equivalent*:

- *There exists an equivalent measure $\mathbb{P}^* \sim \mathbb{P}$ such that the discounted price $B(t)^{-1}S(t)$ is a $\mathbb{P}^*$-martingale.*
- *The market model is arbitrage-free.*

**Definition 5.2** Market model $(S, \; B)$ is said to be complete if any $\mathfrak{F}_T$-measurable claim $\mathcal{H}$ can be hedged by an admissible self-financing strategy.

**Theorem 5.2** **(Second fundamental theorem)** *Let the market model $(S, \; B)$ possesses an EMM. The following statements are equivalent*:

- *The equivalent martingale measure is unique.*
- *The market model is complete.*

See the extensive literature on fundamental theorems beginning with two works by Harrison and Pliska [18, 19]. Details can also be found in [20].

It is known (see, e. g. [6]) that if the EMM exists and is unique, then for any replicable contingent claim $\mathcal{H}$ the hedging strategy value $V_t$ can be calculated as

$$V_t = B(t)\mathbb{E}_{\mathbb{P}^*}\{B(T)^{-1}\mathcal{H} \mid \mathfrak{F}_t\}. \tag{5.1.4}$$

In particular, the initial strategy value (or the arbitrage price of derivative) is $\mathfrak{c} = V_0 = \mathbb{E}_{\mathbb{P}^*}\{B(T)^{-1}\mathcal{H}\}$.

To construct the equivalent martingale measure we use the standard measure change technique.

Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in[0,T]}, \mathbb{P})$ be the filtered probability space. Suppose $\mathbb{P}^*$ is the probability measure on the space $(\Omega, \mathfrak{F}_T)$ which is absolutely continuous with respect to $\mathbb{P}$:

$$Z_T := \frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}.$$

Consider also the $\mathbb{P}$-martingale $Z_t = \mathbb{E}_{\mathbb{P}}\{Z_T \mid \mathfrak{F}_t\},\ t \in [0,\ T]$.

Let $M_t,\ t \in [0, T]$ be a measurable process on $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in[0,T]}, \mathbb{P})$. It is well-known that $M_t Z_t$ is a martingale under $\mathbb{P}$ if and only if $M_t$ is a martingale under $\mathbb{P}^*$ (see e.g. [5], Lemma 7.2.2).

### *5.1.2 Black-Scholes Model and Girsanov's Theorem*

Consider the market model of two assets, the bond

$$B(t) = \mathrm{e}^{rt},\ t \in [0, T], \tag{5.1.5}$$

and the risky asset

$$S(t) = S_0 \mathrm{e}^{\sigma w(t)+\mu t},\ t \in [0, T], \tag{5.1.6}$$

where $r > 0,\ \sigma > 0$ and $\mu \in (-\infty, \infty)$ are constants and $w = w(t)$ is the standard Brownian motion (see Definition 1.1). Hence, by Itô's Theorem (Theorem 1.2)

$$\mathrm{d}S(t) = S(t)\left(\sigma \mathrm{d}w(t) + \left(\mu + \sigma^2/2\right)\mathrm{d}t\right).$$

This market model is named the Black-Scholes model. The discounted risky asset price $\bar{S}(t) := B(t)^{-1}S(t),\ t > 0$ satisfies the stochastic equation

$$\mathrm{d}\bar{S}(t) = \bar{S}(t)\left(\sigma \mathrm{d}w(t) + (\mu - r + \sigma^2/2)\mathrm{d}t\right). \tag{5.1.7}$$

To eliminate the drift component we use the following classic result.

**Theorem 5.3 (Girsanov's Theorem)** *Let $\gamma_t,\ t \in [0, T]$ be an adapted measurable process such that $\int_0^T \gamma_s^2 \mathrm{d}s < \infty$ a.s. Consider the $\mathbb{P}$-martingale*

$$Z_t = \exp\left(-\int_0^t \gamma_s \mathrm{d}w(s) - \frac{1}{2}\int_0^t \gamma_s^2 \mathrm{d}s\right).$$

*If the new measure $\mathbb{P}^*$ is defined by the density $Z_T$,*

$$\left.\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}\right|_{\mathfrak{F}_T} = Z_T,$$

*then the process*

$$\tilde{w}(t) = w(t) + \int_0^t \gamma_s \mathrm{d}s$$

*is the standard Brownian motion under new measure* $\mathbb{P}^*$.

For the proof see e.g. [5].

Applying Girsanov's Theorem with $\gamma_t = \gamma = \frac{\mu - r + \sigma^2/2}{\sigma}$ we transform Eq. (5.1.7) into

$$\mathrm{d}\bar{S}(t) = \sigma \bar{S}(t)\mathrm{d}\tilde{w}(t),$$

where $\tilde{w}(t) := w(t) + \gamma t,\ \ t > 0$, is the Brownian motion with respect to measure $\mathbb{P}^*$ and filtration $\{\mathfrak{F}_t\}_{t\in[0,\ T]}$. Hence $\bar{S}(t) = B(t)^{-1}S(t)$ is a $\mathbb{P}^*$-martingale and

$$\mathrm{d}S(t) = S(t)\left(\sigma \mathrm{d}\tilde{w}_t + r\mathrm{d}t\right), \tag{5.1.8}$$

or, by integrating,

$$S(t) = S_0 \mathrm{e}^{\sigma \tilde{w}(t) + (r - \sigma^2/2)t}, \qquad t > 0.$$

Let $\mathcal{H} = f(S(T))$ be the contingent claim. To describe the hedging strategy $\pi_t = (\varphi_t,\ \psi_t)$, let us consider the function

$$F(x,\ t) = \mathrm{e}^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}\left\{f(S(T)) \mid S(t) = x\right\}. \tag{5.1.9}$$

This function can be interpreted as the option price at time $t \in [0,\ T]$ with maturity time $T$, if at time $t$ the risky asset price is equal to $x$. In other words, the strategy value $V_t = \varphi_t S(t) + \psi_t B(t)$ and function $F(x,\ t)$ are connected as follows: $V_t = F(S(t),\ t),\ t \in [0, T]$ (see (5.1.4)).

Applying the Black-Scholes analysis [4] we derive the fundamental equation. Differentiating $F(S(t),\ t)$ and exploiting Itô's formula (1.3.3), we have

$$\begin{aligned}
\mathrm{d}V_t &= \mathrm{d}\left(F(S(t),\ t)\right)\\
&= \frac{\partial F}{\partial x}(S(t),\ t)\,\mathrm{d}S(t) + \frac{\partial F}{\partial t}(S(t),\ t)\,\mathrm{d}t + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 F}{\partial x^2}(S(t),\ t)\,\mathrm{d}t.
\end{aligned}$$

Substituting (5.1.8) we get

$$\mathrm{d}V_t = \sigma S(t)\frac{\partial F}{\partial x}\mathrm{d}\tilde{w}_t + \left(rS(t)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 F}{\partial x^2} + \frac{\partial F}{\partial t}\right)\mathrm{d}t. \tag{5.1.10}$$

On the other hand, due to self-financing condition (5.1.1), $\mathrm{d}V_t = \varphi_t \mathrm{d}S(t) + \psi_t \mathrm{d}B(t) = \varphi_t \mathrm{d}S(t) + r\psi_t B(t)\mathrm{d}t$. Hence, by (5.1.8)

$$dV_t = \sigma S(t)\varphi_t \mathrm{d}\tilde{w}_t + r(S(t)\varphi_t + \psi_t B(t))\mathrm{d}t = \sigma S(t)\varphi_t \mathrm{d}\tilde{w}_t + rV_t\mathrm{d}t. \tag{5.1.11}$$

According to the uniqueness property of diffusion processes [21] we compare the coefficients of the differentials $\mathrm{d}\tilde{w}(t)$ and $\mathrm{d}t$ in (5.1.10)–(5.1.11). Hence

$$\varphi_t = \frac{\partial F}{\partial x}(S(t),\ t), \tag{5.1.12}$$

$$\frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 F}{\partial x^2}(S(t),\ t) + rS(t)\frac{\partial F}{\partial x}(S(t),\ t) + \frac{\partial F}{\partial t}(S(t),\ t) = rF(S(t),\ t). \tag{5.1.13}$$

Formula (5.1.12) gives the number of risky assets which should be held into the hedging portfolio. The number of bonds can be defined as $\psi_t = B(t)^{-1}V_t - \varphi_t \bar{S}(t)$.

We rewrite the fundamental equation (5.1.13) as follows,

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(x,\ t) + rx\frac{\partial F}{\partial x}(x,\ t) + \frac{\partial F}{\partial t}(x,\ t) = rF(x,\ t), \quad 0 < t < T. \tag{5.1.14}$$

Due to definition (5.1.9), equation (5.1.14) is supplied with the terminal condition

$$F(x,\ T) = f(x). \tag{5.1.15}$$

The solution of this terminal value problem gives us another way of finding the option price.

Using the probabilistic reasoning (5.1.4) or, alternatively, taking the solution of the terminal value PDE-problem (5.1.14)–(5.1.15) one can obtain the famous Black-Scholes formula for the call option price: if asset prices are presented in (5.1.5)–(5.1.6) and the claim is $\mathcal{H} = (S_T - k)^+$ with fixed level $k$ of negotiated price, then the derivative value at time $t$ is equal to

$$V_t = S(t)\Phi(z_+) - k\mathrm{e}^{-r(T-t)}\Phi(z_-), \quad 0 \le t < T, \tag{5.1.16}$$

where $z_\pm := \dfrac{\ln(S(t)/k) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$ and $\Phi(z) = \dfrac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{z} \mathrm{e}^{-x^2/2}\,\mathrm{d}x$.

We have mentioned not all mathematical principles and aspects which are applied in finance. In particular, the optimisation theory under a random environment is omitted (see a review of these methods in [22]).

## 5.2 Market Model Based on Jump-Telegraph Processes

Assume that the price of a risky asset $S(t)$ is driven by the stochastic equation

$$\mathrm{d}S(t) = S(t-)\mathrm{d}\left(X(t) + J(t)\right), \quad t > 0. \tag{5.2.1}$$

Here $X = X(t),\ t \geq 0$, is the telegraph process with the parameters $(c_0,\ \lambda_0)$, $(c_1,\ \lambda_1)$, where $c_0 \geq c_1$, and $J = J(t) = \sum_{n=1}^{N(t)} h_{\varepsilon(\tau_n -)}$ is the jump process with $h_0,\ h_1 > -1$. Suppose that $S(t),\ t > 0$, is right-continuous.

The price of a non-risky asset (bond or bank account) has the form

$$B(t) = \mathrm{e}^{Y(t)}, \qquad Y(t) = \int_0^t r_{\varepsilon(t')} \mathrm{d}t', \tag{5.2.2}$$

where $r_0,\ r_1 > 0$ are the interest rates of the respective market states. Notice that in this framework we assume the bond price to be a stochastic telegraph process.

Integrating (5.2.1) we have

$$S(t) = S_0 \mathscr{E}_t\left(X + J\right), \tag{5.2.3}$$

where $S_0 = S(0)$. Here the stochastic exponential $\mathscr{E}(\cdot)$ is defined by (4.1.29) with $c_i$ and $h_i$ instead of $c_i^*$ and $h_i^*,\ i = 0,\ 1$. Precisely speaking, $S(t) = S_0 \mathrm{e}^{X(t)} \kappa(t)$, where $\kappa(t) = \kappa^i_{N(t)}$. The sequence $\kappa^i_n,\ n \geq 0$ is defined by (4.1.30), but now we use $h_0$ and $h_1$ instead of $h_0^*$ and $h_1^*$.

*Remark 5.1* Let us give some historical and substantial remarks. Telegraph processes have been exploited first for stochastic volatility modelling (see [23]). Then, the model based on a pure jump process have been studied in [24].

In general, the telegraph market model possesses arbitrage opportunities, if the jump component vanishes, i. e. $h_0 = h_1 = 0$ (cf. the models considered in [25] and in [26]). It reflects the widely accepted opinion that the telegraph process has a persistent character. The corresponding arbitrage strategy can be described as follows.

Assume $r_0 = r_1 = 0$ for simplicity. Take levels $A,\ B$ such that $S_0 < A < B < S_0 \mathrm{e}^{c_0 T}$. Consider the following strategy: buy the risky asset at time $\mathfrak{t}_1 = \min\{t \in [0,\ T] :\ \ S(t) = A\}$, and then sell it at time $\mathfrak{t}_2 = \min\{t \in (\mathfrak{t}_1,\ T] :\ \ S(t) = A \text{ or } S(t) = B\}$. This strategy has no losses at time $\mathfrak{t}_1$, because $\mathfrak{t}_1$ coincides with the switching time of $X$ with zero probability. Hence the strategy yields a positive profit with positive probability $\mathbb{P}\{S(\mathfrak{t}_2) = B\}$.

In the sequel, we assume that $h_0,\ h_1 \neq 0$. Moreover, let the parameters of model (5.2.1)–(5.2.2) satisfy the inequalities

$$\frac{r_0 - c_0}{h_0} > 0, \qquad \frac{r_1 - c_1}{h_1} > 0. \tag{5.2.4}$$

Under such conditions, we can find a unique martingale measure in the framework of market model (5.2.1)–(5.2.2). Recall that measure $\mathbb{P}^*$ is a martingale measure for model (5.2.1)–(5.2.2) if the process $B(t)^{-1} S(t),\ \ t \geq 0$ is a $\mathbb{P}^*$-martingale. We define this measure by the density $Z(t), t \geq 0$, see (4.1.29). Notice that measure $\mathbb{P}^*$ is defined by the pair of real numbers $c_0^*,\ c_1^*$ (and by $h_i^* = -c_i^*/\lambda_i > -1,\ i = 0,\ 1$).

**Theorem 5.4** *Measure $\mathbb{P}^*$, defined by (4.1.29), is the equivalent martingale measure for model (5.2.1)–(5.2.2) if and only if*

$$c_0^* = \lambda_0 + \frac{c_0 - r_0}{h_0}, \qquad c_1^* = \lambda_1 + \frac{c_1 - r_1}{h_1}. \tag{5.2.5}$$

*Moreover, under measure $\mathbb{P}^*$ the process $N = N(t)$ is again the Poisson process with alternating intensities*

$$\lambda_0^* = \frac{r_0 - c_0}{h_0} > 0, \qquad \lambda_1^* = \frac{r_1 - c_1}{h_1} > 0. \tag{5.2.6}$$

*Proof* By Theorem 4.2 the switching intensities under the changed measure $\mathbb{P}^*$ are $\lambda_i^* = \lambda_i - c_i^*,\ i = 0,\ 1$. Then, notice that conditions (5.2.4) and (5.2.5) provide the following inequalities: for $i = 0,\ 1$

$$h_i^* = -c_i^*/\lambda_i = -1 + (r_i - c_i)/(\lambda_i h_i) > -1, \qquad \lambda_i^* = \lambda_i - c_i^* = (r_i - c_i)/h_i > 0.$$

Therefore, the process $Z = Z(t) = \mathcal{E}_t(X^* + J^*)$ with $c_i^* = \lambda_i + \frac{c_i - r_i}{h_i}$ and $h_i^* = -c_i^*/\lambda_i,\ i = 0,\ 1$ defines the density of new probability measure $\mathbb{P}^*$ correctly.

According to Theorem 4.2, the process $X - Y$ is the telegraph process (with respect to measure $\mathbb{P}^*$) with the parameters $(c_i - r_i,\ \lambda_i^*),\ \lambda_i^* = \lambda_i - c_i^*,\ i = 0,\ 1$. By Theorem 4.2 the jump-telegraph process $X(t) - Y(t) + J(t),\ t \geq 0$ is a $\mathbb{P}^*$-martingale if and only if $(\lambda_i - c_i^*)h_i = -(c_i - r_i),\quad i = 0,\ 1$. Hence $c_i^* = \lambda_i + (c_i - r_i)/h_i$. Moreover, $h_i^* = -c_i^*/\lambda_i = -1 + (r_i - c_i)/\lambda_i h_i$ and $\lambda_i^* = \lambda_i - c_i^* = (r_i - c_i)/h_i$. The theorem is proved. □

Due to Theorem 5.4, the market model (5.2.1)–(5.2.2) is arbitrage-free and complete if its parameters satisfy conditions (5.2.4).

## 5.3 Diffusion Rescaling and Natural Volatility

In this section we discuss the convergence of this model to the model based on the geometric Brownian motion.

We begin with some generalisation of the limit theorem of Chap. 2 (see Theorem 2.6). We assume that switching intensities $\lambda_0,\ \lambda_1$ are different, but $\lambda_0 - \lambda_1 = o(\lambda_0 + \lambda_1),\ $ as $\lambda_0,\ \lambda_1 \to \infty$. Precisely speaking, let $X = X(t),\ t \geq 0$ be a telegraph process with velocities $c_0 = -c_1 =: c,\ c > 0$, and with alternating switching intensities $\lambda_0,\ \lambda_1 > 0$. Let $c \to \infty,\ \lambda_0,\ \lambda_1 \to \infty$ such that

$$c^2/\lambda \to \sigma^2, \quad c\mu/\lambda \to \beta, \qquad \sigma > 0,\ \beta \in (-\infty, \infty), \tag{5.3.1}$$

where $\lambda := (\lambda_0 + \lambda_1)/2$ and $\mu := (\lambda_0 - \lambda_1)/2$. Notice that from (5.3.1) it follows that $\mu/\sqrt{\lambda} \to \beta/\sigma$.

**Theorem 5.5** *Under scaling conditions (5.3.1) the following weak convergence holds*: $\forall t > 0$

$$X(t) \xrightarrow{d} \sigma w(t) - \beta t, \tag{5.3.2}$$

*where $w = w(t),\ t > 0$ is the standard Brownian motion.*

*Proof* Consider the conditional characteristic functions

$$\widehat{p}_j(\xi,\ t) := \mathbb{E}\left\{ \mathrm{e}^{i\xi X(t)} \;\middle|\; \varepsilon(0) = j \right\} = \int_{-\infty}^{\infty} \mathrm{e}^{i\xi x} p_j(x,\ t)\mathrm{d}x, \quad j = 0,\ 1.$$

Due to Prokhorov's Theorem [27] it is sufficient to prove the convergence of $\widehat{p}_j(\xi,\ t)$ to $\exp\{-\sigma^2\xi^2 t/2 - i\xi\beta t\}$.

Similarly to Eq. (2.4.3) one can see that function $\widehat{\mathbf{p}} := (\widehat{p}_0,\ \widehat{p}_1)^T$ satisfies the following initial value problem

$$\frac{\mathrm{d}\widehat{\mathbf{p}}}{\mathrm{d}t}(\xi,\ t) = \mathscr{A}\widehat{\mathbf{p}}(\xi,\ t), \qquad t > 0 \tag{5.3.3}$$

with the initial conditions $\widehat{p}_0(\xi, 0) = \widehat{p}_1(\xi, 0) = 1$, where the matrix $\mathcal{A}$ is defined by

$$\mathcal{A} := \begin{pmatrix} -\lambda_0 + ic\xi & \lambda_0 \\ \lambda_1 & -\lambda_1 - ic\xi \end{pmatrix}.$$

The solution of the initial value problem for (5.3.3) can be expressed as

$$\widehat{\mathbf{p}}(\xi,\ t) = \mathrm{e}^{tz_1}\mathbf{e}_1 + \mathrm{e}^{tz_2}\mathbf{e}_2. \tag{5.3.4}$$

Here $z_1$ and $z_2$ are the eigenvalues of matrix $\mathscr{A}$, $\mathbf{e}_1$ and $\mathbf{e}_2$ are the respective eigenvectors. Eigenvalues $z_1, z_2$ are the roots of the equation $\det(\mathscr{A} - zI) = 0$, where

$$\det(\mathscr{A} - zI) = z^2 - \mathrm{Tr}(\mathscr{A})z + \det(\mathscr{A}) = z^2 + 2\lambda z + c^2\xi^2 + 2ic\mu\xi.$$

Hence the eigenvalues are $z_1 = -\lambda - \sqrt{D},\ z_2 = -\lambda + \sqrt{D}$ with $D = \lambda^2 - c^2\xi^2 - 2ic\mu\xi$.

Notice that

$$\mathscr{A} - z_{1,2}I = \begin{pmatrix} -\mu + ic\xi \pm \sqrt{D} & \lambda_0 \\ \lambda_1 & \mu - ic\xi \pm \sqrt{D} \end{pmatrix}.$$

From the initial conditions $\widehat{p}_0(\xi, 0) = \widehat{p}_1(\xi, 0) = 1$ and (5.3.4) it follows that $\mathbf{e}_1 + \mathbf{e}_2 = (1,\ 1)^T$.

Let $\mathbf{e}_i = (x_i,\ y_i),\ i = 1,\ 2$. To compute eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ we have the following system, $\mathscr{A}\mathbf{e}_i = z_i\mathbf{e}_i,\ i = 1, 2$, and $\mathbf{e}_1 + \mathbf{e}_2 = (1,\ 1)^T$. This is equivalent to

$$\begin{cases} (-\mu + ic\xi + \sqrt{D})x_1 + \lambda_0 y_1 = 0, \\ \lambda_1 x_1 + (\mu - ic\xi + \sqrt{D})y_1 = 0, \\ (-\mu + ic\xi - \sqrt{D})x_2 + \lambda_0 y_2 = 0, \\ \lambda_1 x_2 + \mu - ic\xi - \sqrt{D}y_2 = 0, \\ x_1 + x_2 = 1, \\ y_1 + y_2 = 1. \end{cases}$$

Solving this system we can easily obtain

$$\mathbf{e}_1 = \frac{1}{2}\left(1 - \lambda/\sqrt{D} - ic\xi/\sqrt{D},\ 1 - \lambda/\sqrt{D} + ic\xi/\sqrt{D}\right)^T,$$
$$\mathbf{e}_2 = \frac{1}{2}\left(1 + \lambda/\sqrt{D} + ic\xi/\sqrt{D},\ 1 + \lambda/\sqrt{D} - ic\xi/\sqrt{D}\right)^T.$$

Then, notice that under conditions (5.3.1), $c^2/\lambda^2 \to 0,\ c\mu/\lambda^2 \to 0$. Hence $\lambda/\sqrt{D} \to 1$, $c/\sqrt{D} \to 0$ and $\mathbf{e}_1 \to (0, 0)^T$, $\mathbf{e}_2 \to (1, 1)^T$. Moreover, Re $z_1 < 0$ and

$$z_2 = -\lambda + \sqrt{D} = \frac{D - \lambda^2}{\sqrt{D} + \lambda} = -\frac{c^2\xi^2 + 2ic\mu\xi}{\sqrt{D} + \lambda} = -\frac{c^2\xi^2/\lambda + 2i\xi c\mu/\lambda}{\sqrt{D}/\lambda + 1}$$
$$\to -\frac{1}{2}\left(\sigma^2\xi^2 + 2i\xi\beta\right).$$

Passing to the limit in (5.3.4) we obtain the convergence of $\widehat{p}_0(\xi,\ t)$ and $\widehat{p}_1(\xi,\ t)$ to $\mathrm{e}^{-(\sigma^2\xi^2 + 2i\xi\beta)t/2}$, which coincides with the characteristic function of $\sigma w(t) - \beta t$. The theorem is proved. □

The following theorem provides the necessary connection between stock prices driven by a geometric jump-telegraph process and a geometric Brownian motion.

Let the underlying telegraph process $X = X(t)$ with velocities $c_0,\ c_1$, and jump process $J = J(t)$ with jump values $h_0,\ h_1 > -1$ be driven by the Markov process $\varepsilon = \varepsilon(t)$ with alternating switching intensities $\lambda_0,\ \lambda_1$. Let the price of risky asset be defined by (5.2.3). Denote $c = (c_0 - c_1)/2,\ a = (c_0 + c_1)/2$.

We assume that $c,\ \lambda \to \infty$, such that (5.3.1) is fulfilled. Moreover $h_0,\ h_1 \to 0$ and

$$\frac{\lambda_0 h_0 - \lambda_1 h_1}{\sqrt{\lambda}} \to \gamma, \qquad \sqrt{\lambda_i}h_i \to \alpha_i, \qquad i = 0,\ 1. \tag{5.3.5}$$

**Theorem 5.6** *Under scaling conditions (5.3.1) and (5.3.5) we additionally assume that the following limit exists*:

$$a + \frac{1}{2}\left[\lambda_0 \ln(1+h_0) + \lambda_1 \ln(1+h_1)\right] \to \delta. \tag{5.3.6}$$

*Then model (5.2.3) converges in distribution to the Black-Scholes model*:

$$S(t) \xrightarrow{d} S_0 \mathrm{e}^{vw(t)+dt}, \qquad t \in [0,\ T], \tag{5.3.7}$$

*where*

$$v^2 = \sigma^2 + \frac{\alpha_0^2+\alpha_1^2}{2} - \frac{\gamma^2}{4}, \qquad d = \delta - \beta\left(1 + \frac{\gamma}{2\sigma}\right). \tag{5.3.8}$$

*Remark 5.2* We notice that $\frac{\alpha_0^2+\alpha_1^2}{2} - \frac{\gamma^2}{4} \ge 0$, so $v^2 \ge \sigma^2$. To prove this inequality notice that

$$\frac{\lambda_0 h_0^2 + \lambda_1 h_1^2}{2} - \frac{(\lambda_0 h_0 - \lambda_1 h_1)^2}{4\lambda} \equiv \frac{\lambda_0\lambda_1(h_0+h_1)^2}{4\lambda} \ge 0 \qquad \forall \lambda_0,\ \lambda_1 > 0,$$

and apply (5.3.5).

*Proof* For arbitrary $z \in \mathbb{R}$ consider the moment generating function of $\ln(S(t)/S_0)$,

$$f(z,\ t) = \mathbb{E}\mathrm{e}^{z\ln(S(t)/S_0)} = \mathbb{E}\mathrm{e}^{z(X(t)+\ln\kappa(t))} = \mathbb{E}\mathrm{e}^{zX(t)}\kappa(t)^z.$$

To prove the convergence (5.3.7) it is sufficient to verify that under scaling conditions (5.3.1), (5.3.5) and (5.3.6)

$$f(z,\ t) \to \mathrm{e}^{v^2z^2t/2+dzt}. \tag{5.3.9}$$

First notice that $X(t) = at + X^0(t)$, where $X^0 = X^0(t)$ is the telegraph process with symmetric velocities $\pm c$, which is driven by the same Markov switching process $\varepsilon = \varepsilon(t)$. Fix the initial market state $i = \varepsilon(0) \in \{0,\ 1\}$. The conditional generating function $f_i(z,\ t)$ can be expressed as

$$\begin{aligned} f_i(z,t) &= \mathbb{E}_i\left\{\mathrm{e}^{z(at+X^0(t))}\kappa(t)^z\right\} \\ &= \mathrm{e}^{(c_iz-\lambda_i)t} + \mathrm{e}^{(az-\lambda)t}\sum_{n=1}^{\infty}(\kappa_n^i)^z \int_{-ct}^{ct} \mathrm{e}^{zx-\mu x/c} q_i^0(x,t;n)\mathrm{d}x. \end{aligned}$$

Here $q_i^0(x,t;n)$ are defined by (4.1.10)–(4.1.11) with symmetric speed $c_0{=}-c_1{=}c$.

Using equalities (4.1.12) and (4.1.10)–(4.1.11) the factors $(\kappa_n^i)^z$ can be absorbed as follows

$$f_i(z,t) = \mathrm{e}^{(c_i z-\lambda_i)t} + \mathrm{e}^{(az-\lambda)t}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{(z-\mu/c)x}\hat{q}_i^0(x,t;n)\mathrm{d}x,$$

where functions $\hat{q}_i^0(x,t;n)$ are defined as above by (4.1.10)–(4.1.11) with $c_0 = -c_1 = c$ and $\hat{\lambda}_0 = \lambda_0(1+h_0)^z,\ \hat{\lambda}_1 = \lambda_1(1+h_1)^z$ instead of $\lambda_0$ and $\lambda_1$, respectively.

Functions $\hat{q}_i^0(x,\ t;\ n),\ i=0,\ 1,\ n\geq 1$, are related to the transition densities $\hat{p}_i^0(x,\ t;\ n)$ of $\hat{X}^0(t)$, the telegraph process with velocities $\pm c$, which is driven by the Markov process $\hat{\varepsilon}$ with alternating intensities $\hat{\lambda}_i,\ i=0,\ 1$:

$$\hat{p}_i^0(x,\ t;\ n) = \mathrm{e}^{-\hat{\lambda}t-\hat{\mu}x/c}\hat{q}_i^0(x,\ t;\ n),$$

where $\hat{\lambda} = (\hat{\lambda}_0+\hat{\lambda}_1)/2$ and $\hat{\mu} = (\hat{\lambda}_0-\hat{\lambda}_1)/2$ (see Proposition 4.1). Hence

$$f_i(z,\ t) = \mathrm{e}^{(c_i z-\lambda_i)t} + \mathrm{e}^{(az-\lambda+\hat{\lambda})t}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{(z-\mu/c+\hat{\mu}/c)x}\hat{p}_i^0(x,\ t;\ n)\mathrm{d}x \quad (5.3.10)$$

$$= \mathrm{e}^{(c_i z-\lambda_i)t} + \mathrm{e}^{(\hat{\lambda}-\lambda+az)t}\int_{-\infty}^{\infty}\mathrm{e}^{((\hat{\mu}-\mu)/c+z)x}\hat{p}_i^0(x,\ t)\mathrm{d}x.$$

Here $\hat{p}_i^0(x,\ t) = \sum_{n=1}^{\infty}\hat{p}_i^0(x,\ t;\ n)$ is the absolutely continuous part of the distribution of the telegraph process $\hat{X}^0(t)$ with symmetric velocities $\pm c$ and with the alternating switching intensities $\hat{\lambda}_0 = \lambda_0(1+h_0)^z,\ \hat{\lambda}_1 = \lambda_1(1+h_1)^z$.

We will apply Theorem 5.5 to process $\hat{X}^0(t)$. To verify assumptions (5.3.1), we need to compute the limits of $c^2/\hat{\lambda}$ and $c\hat{\mu}/\hat{\lambda}$ under scaling conditions (5.3.1), (5.3.5), where $\hat{\lambda} = (\hat{\lambda}_0+\hat{\lambda}_1)/2,\ \hat{\mu} = (\hat{\lambda}_0-\hat{\lambda}_1)/2$.

It is easy to see that

$$\frac{c^2}{\hat{\lambda}} = \frac{2c^2}{\lambda_0(1+h_0)^z+\lambda_1(1+h_1)^z} = \frac{c^2}{\lambda}\cdot\frac{\lambda_0+\lambda_1}{\lambda_0(1+h_0)^z+\lambda_1(1+h_1)^z}$$
$$= \frac{c^2}{\lambda}\cdot\frac{\lambda_0+\lambda_1}{\lambda_0+\lambda_1+o(\lambda_0+\lambda_1)} \to \sigma^2,$$

and

$$\frac{c\hat{\mu}}{\hat{\lambda}} = c\cdot\frac{\hat{\lambda}_0-\hat{\lambda}_1}{\hat{\lambda}_0+\hat{\lambda}_1} = c\cdot\frac{\lambda_0(1+h_0)^z-\lambda_1(1+h_1)^z}{\lambda_0(1+h_0)^z+\lambda_1(1+h_1)^z}$$
$$= c\cdot\frac{\lambda_0-\lambda_1+z(\lambda_0 h_0-\lambda_1 h_1)+o(\lambda_0 h_0-\lambda_1 h_1)}{\lambda_0+\lambda_1+o(\lambda_0+\lambda_1)}$$

$$= \frac{c \cdot \dfrac{\lambda_0 - \lambda_1}{2\lambda} + z\dfrac{c}{\sqrt{2\lambda}} \cdot \dfrac{\lambda_0 h_0 - \lambda_1 h_1}{\sqrt{2\lambda}} + o(1)}{1 + o(1)} \to \beta + \frac{z\sigma\gamma}{2}.$$

With this in hand, applying Theorem 5.5 we have the following convergence to the normal distribution

$$\hat{X}^0(t) \xrightarrow{d} \sigma w(t) - (\beta + z\sigma\gamma/2)t \sim \mathcal{N}(-(\beta + z\sigma\gamma/2)t,\ \sigma^2 t),$$

and hence,

$$\hat{p}_i^0(x,\ t) \to \frac{1}{\sigma\sqrt{2\pi t}} \mathrm{e}^{-(x+\hat{\beta}t)^2/(2t\sigma^2)}, \quad i = 0,\ 1, \tag{5.3.11}$$

where $\hat{\beta} = \beta + z\sigma\gamma/2$.

The limiting normal distribution in (5.3.11) has the following moment generating function,

$$\int_{-\infty}^{\infty} \mathrm{e}^{zx} \frac{1}{\sigma\sqrt{2\pi t}} \mathrm{e}^{-(x+\hat{\beta}t)^2/(2t\sigma^2)} \mathrm{d}x = \exp\left\{\frac{z^2\sigma^2 t}{2} - z\hat{\beta}t\right\}. \tag{5.3.12}$$

We finish the proof of (5.3.9) by applying scaling conditions (5.3.1), (5.3.5) to (5.3.10). First, notice

$$\frac{\hat{\mu} - \mu}{c} = \frac{(\hat{\lambda}_0 - \lambda_0) - (\hat{\lambda}_1 - \lambda_1)}{2c} = \frac{\lambda_0\left((1+h_0)^z - 1\right) - \lambda_1\left((1+h_1)^z - 1\right)}{2c}$$

$$= \frac{\sqrt{\lambda}}{c} \cdot \frac{\lambda_0\left((1+h_0)^z - 1\right) - \lambda_1\left((1+h_1)^z - 1\right)}{2\sqrt{\lambda}} \to \frac{\gamma z}{2\sigma}. \tag{5.3.13}$$

Then, using (5.3.6) we obtain

$$\begin{aligned} az + \hat{\lambda} - \lambda &= az + \frac{1}{2}\left[\lambda_0 \mathrm{e}^{z\ln(1+h_0)} + \lambda_1 \mathrm{e}^{z\ln(1+h_1)}\right] - \frac{\lambda_0 + \lambda_1}{2} \\ &= az + \frac{z}{2}\left[\lambda_0 \ln(1+h_0) + \lambda_1 \ln(1+h_1)\right] \\ &\quad + \frac{z^2}{4}\left[\lambda_0 \ln^2(1+h_0) + \lambda_1 \ln^2(1+h_1)\right] + o(1) \\ &\to \delta z + \frac{\alpha_0^2 + \alpha_1^2}{4} z^2. \end{aligned} \tag{5.3.14}$$

Applying (5.3.11)–(5.3.14) to (5.3.10) and using continuity of moment generating functions, convergence (5.3.9) emerges. Indeed,

$$f_i(z,\ t) \to$$
$$\exp\left\{\delta z t + \frac{\alpha_0^2+\alpha_1^2}{4}z^2 t + \frac{z^2\sigma^2 t}{2}\left(1+\frac{\gamma}{2\sigma}\right)^2 - z\left(1+\frac{\gamma}{2\sigma}\right)t\cdot\left(\beta+\frac{z\sigma\gamma}{2}\right)\right\}$$
$$= \exp\left\{zt\left(\delta - \beta\left(1+\frac{\gamma}{2\sigma}\right)\right) + \frac{z^2 t}{2}\left(\frac{\alpha_0^2+\alpha_1^2}{2} + \left(\sigma+\frac{\gamma}{2}\right)^2 - \sigma\gamma\left(1+\frac{\gamma}{2\sigma}\right)\right)\right\}$$

$$= \exp\left\{dzt + \frac{z^2 t}{2}\left[\frac{\alpha_0^2+\alpha_1^2}{2} + \sigma^2 - \frac{\gamma^2}{4}\right]\right\} = \exp\left\{dzt + v^2 z^2 t/2\right\}.$$

The theorem is proved. □

*Remark 5.3* Due to the structure of limiting volatility (see (5.3.8)) the value $\sigma^2 = \lim(c^2/\lambda)$ can be interpreted as the "telegraph" component of volatility, and the terms

$$\frac{\alpha_0^2+\alpha_1^2}{2} - \frac{\gamma^2}{4} = \lim\left\{\frac{\lambda_0 h_0^2 + \lambda_1 h_1^2}{2} - \frac{(\lambda_0 h_0 - \lambda_1 h_1)^2}{2(\lambda_0+\lambda_1)}\right\}$$

are the components of volatility engendered by jumps. Here lim denotes the limit under conditions (5.3.5).

It is reasonable to define the volatility *vol* of the model (5.2.1)–(5.2.2) as pre-limit value, see (5.3.8) and Remark 5.2,

$$vol^2 = \frac{c^2}{\lambda} + \frac{\lambda_0\lambda_1(h_0+h_1)^2}{4\lambda}. \tag{5.3.15}$$

Notice that the volatility increases if the model is asymmetric in jump values, i. e. $h_0 + h_1 \neq 0$.

*Remark 5.4* Condition (5.3.6) can be changed to

$$a + \frac{\lambda}{2}(h_0+h_1) \to \delta',\ \text{ where } \delta' := \delta + \frac{1}{4}(\alpha_0^2+\alpha_1^2).$$

This is obvious due to the following limit relation

$$\frac{\lambda}{2}\left[\ln(1+h_0) + \ln(1+h_1) - (h_0+h_1)\right] = -\frac{\lambda}{2}\frac{h_0^2+h_1^2}{2} + o(1) \to -\frac{\alpha_0^2+\alpha_1^2}{4}.$$

In these terms the drift coefficient in Theorem 5.6 takes the form

$$d = \delta' - \frac{1}{4}(\alpha_0^2+\alpha_1^2) - \beta\left(1+\gamma/2\sigma\right).$$

## 5.4 Fundamental Equation and Perfect Hedging

Consider European option with the maturity time $T$ and the payoff function $\mathcal{H} = f(S(T))$. We assume that $f$ is a continuous function. To price the option, we need to study the master-functions

$$F_i(t,\ x) = F(t,\ x;\ i) = \mathbb{E}_i^* \left\{ \mathrm{e}^{-Y(T-t)} f(x\mathrm{e}^{X(T-t)}\kappa(T-t)) \right\}, \tag{5.4.1}$$

$$i = 0,\ 1, \quad 0 \le t \le T,$$

where $\mathbb{E}_i^*$ denotes the (conditional) expectation with respect to the martingale measure $\mathbb{P}^*$ under the initial state $\varepsilon(0) = i$.

Similarly to Eq. (4.1.5) and using (4.1.2) we obtain the system of integral equations, $i = 0,\ 1,\ t < T$,

$$F_i(t,\ x) = \mathrm{e}^{-(\lambda_i^*+r_i)(T-t)} f(x\mathrm{e}^{c_i(T-t)}) \tag{5.4.2}$$
$$+ \lambda_i^* \int_t^T F_{1-i}(s,\ x(1+h_i)\mathrm{e}^{c_i(s-t)})\mathrm{e}^{-(\lambda_i^*+r_i)(s-t)}\mathrm{d}s,$$

with the terminal conditions $F_i(t,\ x)\ |_{t\uparrow T} = f(x)$. Here risk-free intensities $\lambda_i^*$ are defined by $\lambda_i^* = (r_i - c_i)/h_i,\ i = 0,\ 1$ (see Theorem 5.4, formula (5.2.6)). Differentiating in $t$ and integrating by parts in the result we obtain the PDE-form of (5.4.2):

$$\frac{\partial F_i}{\partial t}(t,\ x) + c_i x \frac{\partial F_i}{\partial x}(t,\ x) = (r_i + \lambda_i^*) F_i(t,\ x) - \lambda_i^* F_{1-i}(t,\ x(1+h_i)), \tag{5.4.3}$$

$$0 < t < T, \quad i = 0,\ 1.$$

*Remark 5.5* Systems (5.4.2) and (5.4.3) play the same role for our model as the fundamental Black-Scholes Eqs. (5.1.14)–(5.1.15). In contrast with the classic theory, system (5.4.3) is hyperbolic. In particular, it implies the finite propagation velocity, which corresponds better to the intuitive understanding of financial markets. Note that Eqs. (5.4.2) and (5.4.3) do not depend on $\lambda_0$ and $\lambda_1$, just like the respective Eq. (5.1.14) in the Black-Scholes model (5.1.5)–(5.1.6) does not depend on the drift parameter.

Equation (5.4.3) can be used for the explicit description of the hedging strategy. Remind that the admissible self-financing strategy $\pi = (\varphi_t,\ \psi_t),\ 0 \le t \le T$ is called a hedge (perfect hedge, replicating strategy) of the option with maturity at time $T$ and payoff function $\mathcal{H}$, if its terminal value is equal to the payoff of the option: $V_T^\pi = \mathcal{H}$ a.s.

For the wealth process

$$V_t = V_t^{\pi} = \varphi_t S(t) + \psi_t B(t), \qquad 0 \le t \le T, \tag{5.4.4}$$

we require the self-financing property to be fulfilled,

$$\mathrm{d}V_t = \varphi_t \mathrm{d}S(t) + \psi_t \mathrm{d}B(t). \tag{5.4.5}$$

Let $S,\ B$ follow the model (5.2.1)–(5.2.2). Equation (5.4.5) can be written in the integral form

$$V_t = V_0 + \int_0^t \varphi_s S(s)\mathrm{d}X(s) + \int_0^t \psi_s \mathrm{d}B(s) + \sum_{n=1}^{N(t)} \varphi_{\tau_n} h_{\varepsilon(\tau_n-)} S(\tau_n-).$$

Using the identity $\psi_t \equiv B(t)^{-1}(V_t - \varphi_t S(t))$ (see the balance in (5.4.4)), we obtain

$$V_t = V_0 + \int_0^t r_{\varepsilon(s)} V_s \mathrm{d}s + \int_0^t \varphi_s S(s)(c_{\varepsilon(s)} - r_{\varepsilon(s)})\mathrm{d}s + \sum_{n=1}^{N(t)} \varphi_{\tau_n} h_{\varepsilon(\tau_n-)} S(\tau_n-). \tag{5.4.6}$$

To identify such a strategy in the case $\mathcal{H} = f(S(T))$, note that (5.1.4) becomes

$$V_t = V_t^{\pi} = B(t)\mathbb{E}^*[B(T)^{-1}\mathcal{H} \big| \mathfrak{F}_t] = F(t,\ S(t);\ \varepsilon(t)), \tag{5.4.7}$$

where the functions $F(t,\ x;\ i) = F_i(t,\ x),\ i = 0, 1$ defined by (5.4.1) satisfy the fundamental equation (5.4.3). Notice that the strategy value $V_t$ depends on the current market's state $\varepsilon(t)$(or in other words, it depends on the *direction of market's movement*). Exploiting Itô's formula (1.3.3), we get

$$V_t = V_0 + \int_0^t \frac{\partial F}{\partial s}(s,\ S(s);\ \varepsilon(s))\mathrm{d}s + \int_0^t \frac{\partial F}{\partial x}(s,\ S(s);\ \varepsilon(s)) S(s) c_{\varepsilon(s)} \mathrm{d}s \tag{5.4.8}$$

$$+ \sum_{n=1}^{N(t)} (V_{\tau_n} - V_{\tau_n-}).$$

Comparing Eqs. (5.4.6) and (5.4.8), using the fundamental equation (5.4.3) and the definition of $\lambda_i^*,\ \lambda_i^* = (r_i - c_i)/h_i$, we have (between jumps): for $t \in (\tau_{n-1},\ \tau_n),\ n \in \mathbb{N}$,

$$\varphi_t = \frac{S(t) c_{\varepsilon(t)} \dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial t} - r_{\varepsilon(t)} V_t}{S(t)(c_{\varepsilon(t)} - r_{\varepsilon(t)})} \tag{5.4.9}$$

$$= \frac{F(t,\ S(t)(1+h_{\varepsilon(t)}),\ 1-\varepsilon(t)) - F(t,\ S(t),\ \varepsilon(t))}{S(t)h_{\varepsilon(t)}}.$$

Moreover, from (5.4.6) and (5.4.8), we obtain the values of $\varphi_{\tau_n}$:

$$\varphi_{\tau_n} = \frac{F_{\tau_n} - F_{\tau_n-}}{S(\tau_n-)h_{\varepsilon(\tau_n-)}} = \frac{F(\tau_n, S(\tau_n), \varepsilon(\tau_n)) - F(\tau_n, S(\tau_n-), 1-\varepsilon(\tau_n))}{S(\tau_n-)h_{\varepsilon(\tau_n-)}}. \tag{5.4.10}$$

It turns out that the process $\varphi_t$ is left-continuous. To prove this, note that from (5.2.3) it follows that

$$S(\tau_n-)(1+h_{\varepsilon(\tau_n-)}) = S(\tau_n). \tag{5.4.11}$$

The left-continuity of $\varphi_t$ can be proved by applying (5.4.11) to (5.4.9)–(5.4.10).

## 5.5 Pricing Call Options

The main goal of this section is to derive an explicit formula for the initial price $\mathfrak{c}$ of a call option with payoff $\mathcal{H} = (S(T) - K)^+,\ K > 0$, in the framework of the market model (5.2.1)–(5.2.2). According to the theory of option pricing we have

$$\mathfrak{c} = \mathbb{E}^*\left\{ B(T)^{-1}(S(T)-K)^+ \ \middle|\ \varepsilon(0)\right\} = \mathbb{E}^*_{\varepsilon(0)}\left\{B(T)^{-1}(S(T)-K)^+\right\}, \tag{5.5.1}$$

see (5.1.4). Here $K$ is the strike price and $\mathbb{E}^*\{\cdot\}$ denotes the expectation with respect to the martingale measure $\mathbb{P}^*$ which was constructed in Sect. 5.2.

Fix the initial state $i = \varepsilon(0)$. Formula (5.5.1) has the standard form,

$$\mathfrak{c} = \mathfrak{c}_i = S_0 U^{(i)}(y,\ T) - K u^{(i)}(y,\ T), \tag{5.5.2}$$

where $y = \ln K/S_0$ and

$$u^{(i)}(y,\ T) = \mathbb{E}^*_i\left\{ \mathrm{e}^{-Y(T)} \mathbb{1}_{\{X(T)+\ln\kappa(T)>y\}}\right\}, \tag{5.5.3}$$

$$U^{(i)}(y,\ T) = \mathbb{E}^*_i\left\{ \mathrm{e}^{-Y(T)+X(T)}\kappa(T) \mathbb{1}_{\{X(T)+\ln\kappa(T)>y\}}\right\}. \tag{5.5.4}$$

To compute the functions $u^{(i)}$ and $U^{(i)}$ defined by (5.5.2)–(5.5.4) first notice that

$$Y(t) \overset{d}{=} \mu_r X(t) + a_r t, \tag{5.5.5}$$

where $\mu_r = \dfrac{r_0 - r_1}{c_0 - c_1}$ and $a_r = \dfrac{c_0 r_1 - c_1 r_0}{c_0 - c_1}$. Formula (5.5.3) becomes

$$u^{(i)}(y,\ T) = \sum_{n=0}^{\infty} u_n^{(i)}(y - b_n,\ T),$$

where $b_n = \ln \kappa_n^{(i)}$ and

$$u_n^{(i)}(y,\ t) = \mathrm{e}^{-a_r t} \int_y^{\infty} \mathrm{e}^{-\mu_r x} p_i^*(x,\ t;\ n)\mathrm{d}x. \tag{5.5.6}$$

Numbers $\kappa_n^{(i)}$ are defined by (4.1.30) with $h_i$ instead of $h_i^*$, and the transition densities $p_i^*(x,\ t;\ n)$ are defined by (4.1.3) with $\mathbb{P}_i^*$ instead of $\mathbb{P}_i$.

Conditioning again on the number of switchings we have

$$U^{(i)}(y,\ T) = \sum_{n=0}^{\infty} U_n^{(i)}(y - b_n,\ T),$$

where (see (5.5.4) and (5.5.5))

$$U_n^{(i)}(y,\ t) = \kappa_n^i \mathrm{e}^{-a_r t} \int_y^{\infty} \mathrm{e}^{-\mu_r x + x} p_i^*(x,\ t;\ n)\mathrm{d}x. \tag{5.5.7}$$

To obtain the explicit pricing formula from (5.5.2) we should integrate in (5.5.6)–(5.5.7). It is easy to see that functions $u_n^{(i)}$ and $U_n^{(i)},\ i = 0,\ 1,\ n \geq 1$, are continuous and piecewise continuously differentiable, whereas $u_0^{(i)}(y,\ t) = \mathrm{e}^{-(\lambda_i^* + r_i)t} \mathbb{1}_{\{c_i t > y\}}$, $U_0^{(i)}(y,\ t) = \mathrm{e}^{-(\lambda_i^* + r_i - c_i)t} \mathbb{1}_{\{c_i t > y\}}$.

We derive now the PDEs for functions $u_n^{(i)}$ and $U_n^{(i)},\ i = 0,\ 1,\ n \geq 0$, using the notation $\mathcal{L}_i u := \dfrac{\partial u}{\partial t} + c_i \dfrac{\partial u}{\partial y}$.

**Proposition 5.1** Functions $u_n^{(i)}$ and $U_n^{(i)},\ i = 0,\ 1,\ n \geq 0$, defined by (5.5.6) and (5.5.7) satisfy the following systems, $i = 0,\ 1,\ n \geq 1,\ \ t > 0$

$$\mathscr{L}_i u_n^{(i)}(y,\ t) = -(\lambda_i^* + r_i) u_n^{(i)}(y,\ t) + \lambda_i^* u_{n-1}^{(1-i)}(y,\ t), \tag{5.5.8}$$

$$\mathscr{L}_i U_n^{(i)}(y,\ t) = -(\lambda_i^* + r_i - c_i) U_n^{(i)}(y,\ t) + \lambda_i^*(1 + h_i) U_{n-1}^{(1-i)}(y,\ t) \tag{5.5.9}$$

with the initial conditions $u_n^{(i)}\Big|_{t=0} = 0,\ \ U_n^{(i)}\Big|_{t=0} = 0.$

*Proof* Equation (5.5.8) follows, from Eq. (4.1.8) for distribution densities $p_n^{*,i}$. Indeed, differentiating in formula (5.5.6) we have

$$\frac{\partial u_n^{(i)}(y,\ t)}{\partial t} = -a_r u_n^{(i)}(y,\ t) + \mathrm{e}^{-a_r t}\int_y^\infty \mathrm{e}^{-\mu_r x}\frac{\partial p_i^*}{\partial t}(x,\ t;\ n)\mathrm{d}x.$$

Applying Eq. (4.1.8) (with $h_0 = h_1 = 0$) and integrating by parts we have

$$\frac{\partial u_n^{(i)}}{\partial t}(y,\ t) = -(a_r + \mu_r c_i + \lambda_i^*)u_n^{(i)}(y,\ t) + \lambda_i^* u_{n-1}^{(1-i)}(y,\ t) + c_i \mathrm{e}^{-\mu_r y - a_r t} p_i^*(y,\ t;\ n). \tag{5.5.10}$$

We have used here the identity

$$\mathrm{e}^{-a_r t}\int_y^\infty \mathrm{e}^{-\mu_r x} p_{1-i}^*(x,\ t;\ n-1)\mathrm{d}x = u_{n-1}^{(1-i)}(y,\ t), \tag{5.5.11}$$

which follows from Eq. (5.5.6).

On the other hand, differentiating (5.5.6) in $y$ we have

$$c_i \frac{\partial u_n^{(i)}(y,\ t)}{\partial y} = -c_i \mathrm{e}^{-\mu_r y - a_r t} p_i^*(y,\ t;\ n). \tag{5.5.12}$$

From Eqs. (5.5.10) and (5.5.12) using the equalities $a_r + \mu_r c_i = r_i,\ i = 0,\ 1$, see (5.5.5), we get Eq. (5.5.8).

The proof of (5.5.9) is similar. The only difference is that instead of (5.5.11) we should use the identity (see (5.5.7))

$$\kappa_n^i \mathrm{e}^{-a_r t}\int_y^\infty \mathrm{e}^{-\mu_r x + x} p_{1-i}^*(x,\ t;\ n-1)\mathrm{d}x = (1 + h_i)U_{n-1}^{(1-i)}(y,\ t).$$

The proposition is proved. □

To underline the dependence on the parameters we will use the notations

$$u_n^{(i)} = u_n^{(i)}(y,\ t;\ \lambda_0,\ \lambda_1,\ c_0,\ c_1,\ r_0,\ r_1),$$
$$U_n^{(i)} = U_n^{(i)}(y,\ t;\ \lambda_0,\ \lambda_1,\ c_0,\ c_1,\ r_0,\ r_1)$$

for functions $u_n^{(i)}$ and $U_n^{(i)}$ defined by (5.5.6)–(5.5.7). Here $\lambda_i$ are the switching intensities, $c_i$ are the velocities of the main telegraph process $X = X(t)$ and $r_i$ are the interest rates of the bond $B = B(t)$. Comparing Eqs. (5.5.8) and (5.5.9), we note that

$$U_n^{(i)}(y,\ t;\ \lambda_0^*,\ \lambda_1^*,\ c_0,\ c_1,\ r_0,\ r_1) = u_n^{(i)}(y,\ t;\ \hat{\lambda}_0,\ \hat{\lambda}_1,\ c_0,\ c_1,\ 0,\ 0), \tag{5.5.13}$$

where, remind, $\hat{\lambda}_i := \lambda_i^*(1+h_i) = \lambda_i^* + r_i - c_i,\ i = 0,\ 1$ (see (5.2.6)). Hence, to obtain the exact pricing formulae it is sufficient to get the exact formulae for $u_n^{(i)}$, and then use (5.5.13) and (5.5.2).

Notice that $\mathbb{P}$-a.s. $c_1 t \le X(t) \le c_0 t$. Due to Eq. (5.5.6), for $n \ge 0,\ i = 0,\ 1$,

$$u_n^{(i)}(y,\ t) \equiv 0, \qquad \text{if } y > c_0 t, \tag{5.5.14}$$

$$u_n^{(i)}(y,\ t) \equiv \rho_n^{(i)}(t) := \mathrm{e}^{-a_r t} \int_{-\infty}^{\infty} \mathrm{e}^{-\mu_r x} p_i^*(x,\ t;\ n)\mathrm{d}x, \qquad \text{if } y < c_1 t. \tag{5.5.15}$$

In the latter case, system (5.5.8) takes the form

$$\frac{\mathrm{d}\rho_n^{(i)}}{\mathrm{d}t} = -(\lambda_i^* + r_i)\rho_n^{(i)} + \lambda_i^* \rho_{n-1}^{(1-i)}, \quad t > 0,\ n \ge 1,\ i = 0,\ 1. \tag{5.5.16}$$

Here, by Eq. (5.5.6),

$$\rho_0^{(i)}(t) = \mathbb{E}_i^* \left\{ B(t)^{-1} \mathbb{1}_{\{N(t)=0\}} \right\} = \mathrm{e}^{-(\lambda_i^* + r_i)t}, \qquad t \ge 0,\ i = 0,\ 1. \tag{5.5.17}$$

System (5.5.16) is supplied with the initial conditions

$$\rho_n^{(i)}(0) = 0, \quad n \ge 1,\ i = 0,\ 1. \tag{5.5.18}$$

**Lemma 5.1** *The solution of system (5.5.16) can be represented in the form*

$$\rho_n^{(i)}(t) = \mathrm{e}^{-(\lambda_1^* + r_1)t} \Lambda_n^{(i)} P_n^{(i)}(t), \qquad i = 0, 1,\ n \ge 0,$$

*where* $\Lambda_n^{(i)} = (\lambda_i^*)^{[(n+1)/2]} (\lambda_{1-i}^*)^{[n/2]},\ n \ge 0,$ *and functions* $P_n^{(i)}$ *are defined as follows*:

$$P_0^{(0)}(t) = \mathrm{e}^{-at}, \qquad P_0^{(1)} \equiv 1, \tag{5.5.19}$$

$$P_n^{(i)} = P_n^{(i)}(t) = \frac{t^n}{n!} \left( 1 + \sum_{k=1}^{\infty} \frac{(m_n^{(i)} + 1)_k}{(n+1)_k} \cdot \frac{(-at)^k}{k!} \right), \quad i = 0, 1,\ n \ge 1.$$

*Here*

$$m_n^{(0)} = [n/2]\,, \quad m_n^{(1)} = [(n-1)/2]\,,$$
$$(m)_k = m(m+1)\ \ldots\ (m+k-1), \quad a = \lambda_0^* - \lambda_1^* + r_0 - r_1,$$

$[\cdot]$ *is the integer part of a number.*

*Proof* Notice that in the particular case $\lambda_0^* = \lambda_1^* = \lambda$ and $r_0 = r_1 = 0$, the solution of system (5.5.16) is well known: $\rho_n^{(i)}(t) = \pi_n(t) = \mathbb{P}\{N(t) = n\} = \dfrac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t}$, which coincides with the result of Lemma 5.1.

In the general case, we apply the following change of variables

$$\rho_n^{(i)}(t) = \mathrm{e}^{-(\lambda_1^*+r_1)t}\Lambda_n^{(i)}P_n^{(i)}(t).$$

Due to (5.5.17), in these notations, for $n = 0$, we have $P_0^{(1)}(t) \equiv 1,\ \ P_0^{(0)}(t) = \mathrm{e}^{-at}$, where $a = (\lambda_0^* + r_0) - (\lambda_1^* + r_1)$. Initial conditions (5.5.18) lead to the initial conditions $P_n^{(i)}\Big|_{t=0} = 0,\ n \ge 1,\ i = 0, 1$, for the system

$$\begin{cases} \dfrac{\mathrm{d}P_n^{(0)}}{\mathrm{d}t} + aP_n^{(0)} = P_{n-1}^{(1)} \\ \dfrac{\mathrm{d}P_n^{(1)}}{\mathrm{d}t} = P_{n-1}^{(0)} \end{cases}, \quad n \ge 1. \tag{5.5.20}$$

It is easy to see that functions $P_n^{(i)}$ defined by (5.5.19) satisfy system (5.5.20). □

*Remark 5.6* Formulae (5.5.19) can be expressed by means of hypergeometric functions, see [28], Chap. 13:

$$P_n^{(i)}(t) = \frac{t^n}{n!}\,{}_1F_1(m_n^{(i)} + 1;\ n + 1;\ -at), \quad i = 0, 1.$$

Formulae (5.5.19) can be rewritten in detail:

$$P_{2n+1}^{(0)} = P_{2n+1}^{(1)} = P_{2n+1} = \frac{t^{2n+1}}{(2n+1)!}\left[1 + \sum_{k=1}^{\infty}\frac{(n+1)\dots(n+k)}{(2n+2)\dots(2n+k+1)}\cdot\frac{(-at)^k}{k!}\right],$$

$$P_{2n}^{(1)} = \frac{t^{2n}}{(2n)!}\left[1 + \sum_{k=1}^{\infty}\frac{n(n+1)\dots(n+k-1)}{(2n+1)\dots(2n+k)}\cdot\frac{(-at)^k}{k!}\right],$$

$$P_{2n}^{(0)} = \frac{t^{2n}}{(2n)!}\left[1 + \sum_{k=1}^{\infty}\frac{(n+1)\dots(n+k)}{(2n+1)\dots(2n+k)}\cdot\frac{(-at)^k}{k!}\right], \tag{5.5.21}$$

Further, using (5.5.21) it is easy to check the useful identity:

$$P_{2n}^{(1)}(t) - P_{2n}^{(0)}(t) \equiv aP_{2n+1}(t), \quad t \ge 0,\ n \ge 0.$$

In the similar manner we can express the general solution of Eq. (5.5.8).

Define the coefficients $\beta_{k,j},\ \ j < k$:

$$\beta_{k,0} = \beta_{k,1} = \beta_{k,k-2} = \beta_{k,k-1} = 1, \tag{5.5.22}$$

$$\beta_{k,j} = \frac{(k-j)_{[j/2]}}{[j/2]!}, \qquad k \geq 1, \ j < k.$$

We use coefficients $\beta_{k,j}$ to define the following functions $\varphi_{k,n} = \varphi_{k,n}(t), \ \ k \leq n$:

$$\varphi_{0,n} = P_{2n+1}(t), \tag{5.5.23}$$

$$\varphi_{k,n} = \sum_{j=0}^{k-1} a^{k-j-1} \beta_{k,j} P^{(1)}_{2n-j}(t), \qquad t \geq 0, \ 1 \leq k \leq n.$$

Finally, we define functions $v^{(i)}_n = v^{(i)}_n(p, \ q)$ (for positive $p, \ q$) as follows. For $n = 0$ set $v^{(1)}_0 \equiv 0, \ v^{(0)}_0 = \mathrm{e}^{-ap}$, and for $n \geq 1$

$$v^{(0)}_{2n+1}(p, \ q) = v^{(1)}_{2n+1}(p, \ q) = P_{2n+1}(p) + \sum_{k=1}^{n} \frac{q^k}{k!} \varphi_{k,n}(p), n \geq 0,$$

$$v^{(1)}_{2n}(p, \ q) = P^{(1)}_{2n}(p) + \sum_{k=1}^{n-1} \frac{q^k}{k!} \varphi_{k+1,n}(p), n \geq 1, \tag{5.5.24}$$

$$v^{(0)}_{2n}(p, \ q) = P^{(0)}_{2n}(p) + \sum_{k=1}^{n} \frac{q^k}{k!} \varphi_{k-1,n-1}(p), n \geq 1.$$

Now we can derive the expressions for $u^{(i)}_n = u^{(i)}_n(y, \ t), \ i = 0, \ 1$, in the interval $c_1 t < y < c_0 t$.

**Theorem 5.7** *System (5.5.8) admits the unique solution of the form*

$$u^{(i)}_n = \begin{cases} 0, & y > c_0 t, \\ w^{(i)}_n(p, \ q), & c_1 t \leq y \leq c_0 t, \\ \rho^{(i)}_n(t), & y < c_1 t, \end{cases} \qquad i = 0, 1. \tag{5.5.25}$$

*Here*

$$w^{(i)}_n = \mathrm{e}^{-(\lambda^*_0 + r_0)q - (\lambda^*_1 + r_1)p} \Lambda^{(i)}_n v^{(i)}_n(p, \ q), \quad i = 0, \ 1, \qquad n \geq 0, \tag{5.5.26}$$

$$p = \frac{c_0 t - y}{c_0 - c_1}, \quad q = \frac{y - c_1 t}{c_0 - c_1}.$$

*Coefficients* $\Lambda_n^{(i)}$ *and functions* $\rho_n^{(i)}$ *are defined in Lemma 5.1. Functions* $v_n^{(i)} = v_n^{(i)}(p,\ q)$ *are defined in (5.5.24).*

*Proof* By Eq. (5.5.6), $u_n^{(i)}(y,\ t) \equiv 0$, if $p < 0$, and $u_n^{(i)}(y,\ t) \equiv \rho_n^{(i)}(t)$, if $q < 0$.

We solve now system (5.5.8) for $c_1 t \le y \le c_0 t$, i. e. for $p,\ q > 0$. First notice that for any smooth function $w = w(p,\ q)$

$$\begin{aligned}\mathscr{L}_0 w(p,\ q) &= \frac{\partial w}{\partial t} + c_0 \frac{\partial w}{\partial y}\\ &= \frac{1}{c_0 - c_1}\left(c_0 \frac{\partial w}{\partial p} - c_1 \frac{\partial w}{\partial q}\right) + c_0 \frac{1}{c_0 - c_1}\left(-\frac{\partial w}{\partial p} + \frac{\partial w}{\partial q}\right) = \frac{\partial w}{\partial q}\end{aligned}$$

and

$$\begin{aligned}\mathscr{L}_1 w(p,\ q) &= \frac{\partial w}{\partial t} + c_1 \frac{\partial w}{\partial y}\\ &= \frac{1}{c_0 - c_1}\left(c_0 \frac{\partial w}{\partial p} - c_1 \frac{\partial w}{\partial q}\right) + c_1 \frac{1}{c_0 - c_1}\left(-\frac{\partial w}{\partial p} + \frac{\partial w}{\partial q}\right) = \frac{\partial w}{\partial p}.\end{aligned}$$

Therefore, in the notations of (5.5.25)–(5.5.26) for $p,\ q > 0$ (i. e. for $c_1 t < y < c_0 t$) Eq. (5.5.8) are equivalent to the system

$$\begin{cases} \frac{\partial v_n^{(0)}}{\partial q} = v_{n-1}^{(1)} \\ \frac{\partial v_n^{(1)}}{\partial p} = v_{n-1}^{(0)} \end{cases}, \quad n \ge 1. \tag{5.5.27}$$

To define the initial ($n = 0$) functions $v_0^{(i)},\ i = 0,\ 1$ we apply definition (5.5.6) of functions $u_n^{(i)}$:

$$u_0^{(i)} = \mathrm{e}^{-(\lambda_i^* + r_i)t} \mathbb{1}_{\{y < c_i t\}} \equiv \mathrm{e}^{-(\lambda_i^* + r_i)(p+q)} \mathbb{1}_{\{y < c_i t\}},$$

and (5.5.26):

$$u_0^{(i)} = \mathrm{e}^{-(\lambda_0^* + r_0)q - (\lambda_1^* + r_1)p}\ \ v_0^{(i)}(p,\ q).$$

Comparing these two expressions we have

$$v_0^{(0)} = \mathrm{e}^{-ap} \mathbb{1}_{\{p > 0\}}, \quad v_0^{(1)} = \mathrm{e}^{aq} \mathbb{1}_{\{q < 0\}}, \tag{5.5.28}$$

where $a = (\lambda_0^* + r_0) - (\lambda_1^* + r_1)$.

Applying again definitions (5.5.6), and (5.5.26) and Lemma 5.1 we have the following boundary conditions for (5.5.27):

$$v_n^{(i)}\Big|_{p<0} \equiv 0, \qquad v_n^{(i)}\Big|_{q<0} = \mathrm{e}^{aq} P_n^{(i)}(p+q), \tag{5.5.29}$$

where $P_n^{(i)} = P_n^{(i)}(t),\ n \geq 0,\ i = 0, 1$, are defined by (5.5.19).

We will find the solution of Eq. (5.5.27) in the form (5.5.24) with indefinite $\varphi_{k,\,n}$.

First, if the number of equation is odd (i. e. for $2n+1$ instead of $n$), then substituting $v_{2n+1}^{(0)}$ and $v_{2n+1}^{(1)},\ v_{2n+1}^{(0)} = v_{2n+1}^{(1)}$ defined by (5.5.24) into system (5.5.27) we see that Eq. (5.5.27) are fulfilled if

$$\sum_{k=1}^{n} \frac{q^{k-1}}{(k-1)!}\varphi_{k,\,n}(p) = P_{2n}^{(1)}(p) + \sum_{k=1}^{n-1} \frac{q^k}{k!}\varphi_{k+1,\,n}(p), \qquad \forall p,\ q > 0$$

and

$$P_{2n+1}'(p) + \sum_{k=1}^{n} \frac{q^k}{k!}\varphi_{k,\,n}'(p) = P_{2n}^{(0)}(p) + \sum_{k=1}^{n} \frac{q^k}{k!}\varphi_{k-1,\,n-1}(p), \qquad \forall p,\ q > 0.$$

Here the prime denotes the derivative.

These relations are equivalent to

$$\varphi_{1,\,n}(p) = P_{2n}^{(1)}(p), \tag{5.5.30}$$

$$P_{2n+1}'(p) = P_{2n}^{(0)}(p) \tag{5.5.31}$$

and

$$\varphi_{k,n}'(p) = \varphi_{k-1,n-1}(p), \qquad p > 0, \quad 1 \leq k \leq n, \quad n \geq 1. \tag{5.5.32}$$

Second, if the number of equation is even (i. e. for $2n$ instead of $n$), then substituting $v_{2n}^{(0)}$ and $v_{2n}^{(1)}$ defined by (5.5.24) into system (5.5.27) we see that Eq. (5.5.27) are fulfilled if

$$\sum_{k=1}^{n} \frac{q^{k-1}}{(k-1)!}\varphi_{k-1,\,n-1}(p) = P_{2n-1}(p) + \sum_{k=1}^{n-1} \frac{q^k}{k!}\varphi_{k,\,n-1}(p)$$

and

$$P_{2n}^{(1)'}(p) + \sum_{k=1}^{n-1} \frac{q^k}{k!}\varphi_{k+1,\,n}'(p) = P_{2n+1}(p) + \sum_{k=1}^{n-1} \frac{q^k}{k!}\varphi_{k,\,n-1}(p).$$

These equations are equivalent to

$$\varphi_{0,\,n-1}(p) = P_{2n-1}(p), \tag{5.5.33}$$

$$P_{2n}^{(1)'}(p) = P_{2n-1}(p) \tag{5.5.34}$$

and

$$\varphi'_{k+1,\,n}(p) = \varphi_{k,\,n-1}(p), \qquad p > 0, \quad 1 \le k \le n-1, \quad n \ge 1. \tag{5.5.35}$$

Notice that equalities (5.5.31) and (5.5.34) follow from the second equation of (5.5.20). Equations (5.5.32) and (5.5.35) are equivalent.

We define the functions

$$\varphi_{0,n}(p) := P_{2n+1}(p), \tag{5.5.36}$$

$$\varphi_{k,n}(p) := \sum_{j=0}^{k-1} a^{k-j-1}\beta_{k,j} P^{(1)}_{2n-j}(p), \qquad 1 \le k \le n, \quad n \ge 1.$$

In particular, due to the definition of $\beta_{k,j}$ (see (5.5.22)), coefficient $\beta_{1,0} = 1$ and hence $\varphi_{1,n}(p) = P^{(1)}_{2n}(p)$, which coincides with (5.5.30). The first equality of (5.5.36) yields (5.5.33).

It remains to check that functions $\varphi_{k,n}$ (5.5.36) solve system (5.5.32). Differentiating (5.5.36) and using (5.5.20) we get

$$\varphi'_{k,n}(p) = \sum_{j=0}^{k-1} a^{k-j-1}\beta_{k,j} P^{(0)}_{2n-j-1}(p).$$

By the identities $P^{(0)}_{2n+1} \equiv P^{(1)}_{2n+1}$ and $P^{(1)}_{2n} - P^{(0)}_{2n} \equiv aP_{2n+1}$, $n \ge 0$ (see Remark 5.6), we have

$$\begin{aligned}\varphi'_{k,n}(p) = &\sum_{\substack{j\ge 0,\\ j \text{ is even}}} a^{k-j-1}\beta_{k,j} P_{2n-j-1}(p) \\ &+ \sum_{\substack{j\ge 0,\\ j \text{ is odd}}} a^{k-j-1}\beta_{k,j} P^{(1)}_{2n-j-1}(p) - \sum_{\substack{j\ge 0,\\ j \text{ is odd}}} a^{k-j}\beta_{k,j} P_{2n-j}(p).\end{aligned}$$

To complete the proof, it is sufficient to apply the identities $\beta_{k,2m+1} = \beta_{k-1,2m}$ and $\beta_{k,2m} - \beta_{k,2m+1} = \beta_{k-1,2m-1}$, which evidently follow from the definition of $\beta_{k,n}$ (see (5.5.22)). Theorem 5.7 is completely proved. □

*Remark 5.7* If $\lambda_0^* = \lambda_1^* = \lambda$, $r_0 = r_1 = r$, then $a = \lambda_0^* - \lambda_1^* + r_0 - r_1 = 0$ and formula (5.5.25) looks more simple. First, by definition (5.5.21) we simplify $P_n^{(i)}$ to $P_n^{(i)}(t) = \dfrac{t^n}{n!}$. Moreover, $\pi_n^{(i)} \equiv \pi_n = \dfrac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t}$ and $\rho_n^{(i)} = \mathrm{e}^{-rt}\pi_n(t)$. The sum (5.5.36) for $\varphi_{k,\,n}$ contains the only term $\varphi_{k,\,n}(t) = P^{(i)}_{2n-k+1}(t) = \dfrac{t^{2n-k+1}}{(2n-k+1)!}$. Therefore,

$$v_n^{(i)}(p,\ q) = \frac{1}{n!}\sum_{k=0}^{m_n^{(i)}} \binom{n}{k} q^k p^{n-k}$$

and

$$u_n^{(i)} = \lambda^n \mathrm{e}^{-(\lambda+r)t} \begin{cases} 0, & y > c_0 t, \\ v_n^{(i)}(p,\ q), & c_1 t \le y \le c_0 t, \\ \dfrac{t^n}{n!}, & y < c_1 t. \end{cases}$$

*Remark 5.8* It follows from (5.5.25) that functions $u_0^{(1)}$ and $u_0^{(0)}$ are discontinuous at $q = 0$ and $p = 0$, respectively. All subsequent functions $u_n^{(i)},\ n \ge 1$, defined by (5.5.25), are continuous. The points of possible discontinuity of the first derivatives are concentrated on the lines $p = 0$ and $q = 0$. For example, for $u_1^{(0)}$ and $u_1^{(1)}$ we have

$$\left.\frac{\partial u_1^{(i)}}{\partial q}\right|_{q=+0} - \left.\frac{\partial u_1^{(i)}}{\partial q}\right|_{q=-0} = \lambda_i^* \mathrm{e}^{-(\lambda_0^*+r_0)p},$$
$$\left.\frac{\partial u_1^{(i)}}{\partial p}\right|_{p=+0} - \left.\frac{\partial u_1^{(i)}}{\partial p}\right|_{p=-0} = \lambda_i^* \mathrm{e}^{-(\lambda_0^*+r_0)q}, \quad i = 0, 1.$$

In general, using (5.5.25) one can prove that functions $u_n^{(i)}$ and their derivatives up to $(n-1)$-th order are continuous.

*Remark 5.9* The formulae in (5.5.2)–(5.5.7) have a different structure, which depends on the sign of $\ln(1+h_1)(1+h_0)$.

1. If $(1+h_1)(1+h_0) < 1$, then $\ln(1+h_1) + \ln(1+h_0) < 0$, hence $b_n \to -\infty$. The price of a call option is given by formula (5.5.2) with

$$u = u^{(i)}(y,\ T) = \sum_{k=0}^{n_1^{(i)}} \rho_k^{(i)}(T) + \sum_{k=n_1^{(i)}+1}^{n_0^{(i)}} u_k^{(i)}(y - b_k,\ T;\ \lambda_0^*, \lambda_1^*, c_0, c_1, r_0, r_1),$$

and

$$U = U^{(i)}(y,\ T) = u^{(i)}(y,\ T;\ \hat{\lambda}_0, \hat{\lambda}_1, c_0, c_1, 0, 0), \tag{5.5.37}$$

where $y = \ln K/S_0$ and

$$n_1^{(i)} = \min\{n\ :\ y - b_n > c_1 T\}, \quad n_0^{(i)} = \min\{n\ :\ y - b_n > c_0 T\}, \ i = 0, 1.$$

2. If $(1+h_1)(1+h_0) > 1$, then $\ln(1+h_1) + \ln(1+h_0) > 0$, hence $b_n \to +\infty$. Denoting

$$m_1^{(i)} = \max\{n\ :\ y - b_n > c_1 T\},\ m_0^{(i)} = \max\{n\ :\ y - b_n > c_0 T\},\ i = 0, 1,$$

we obtain the call option price formula of the form (5.5.2) with

$$u^{(i)}(y,\ T) = \sum_{k=m_0^{(i)}}^{m_1^{(i)}} u_k^{(i)}(y - b_k,\ T;\ \lambda_0^*, \lambda_1^*, c_0, c_1, r_0, r_1) + \sum_{k=m_1^{(i)}+1}^{\infty} \rho_k^{(i)}(T),$$

and $U^{(i)}(y,\ T)$ is defined in (5.5.37). Consider the following examples.

*Example 5.1* The Merton model.

If $r_0 = r_1 = r,\ \ c_0 = c_1 = c,\ \ h_0 = h_1 = -h,\ \ h < 1,\ \ \lambda_0 = \lambda_1 = \lambda$, Eq. (5.2.1) takes the form

$$\mathrm{d}S(t) = S(t-)(c\mathrm{d}t - h\mathrm{d}N(t)),$$

where $N = N(t),\ t \geq 0$, is a (homogeneous) Poisson process with parameter $\lambda > 0$. In this case functions $U^{(i)} \equiv U$ and $u^{(i)} \equiv u$ in (5.5.2) do not depend on $i$. Functions $u$ and $U$ are defined as follows.

If $0 < h < 1$ and $c > r$, then $b_n = n \ln(1 - h) \to -\infty$ and

$$u = u(\ln K/S_0,\ T) = \mathrm{e}^{-rT} \sum_{n=0}^{n_0} u_n^{(i)}(\ln(K/S_0) - b_n,\ T)$$

$$= \mathrm{e}^{-rT}\mathbb{P}(N(T) \leq n_0) = \mathrm{e}^{-rT}\Psi_{n_0}(\lambda^* T),$$

where $\lambda^* = (c - r)/h > 0$ and $\Psi_{n_0}(z) = \mathrm{e}^{-z} \sum_{n=0}^{n_0} \frac{z^n}{n!}$. In this case, function $U$ has the form

$$U(y,\ T) = \Psi_{n_0}(\lambda^*(1 - h)T).$$

For $h < 0$ and $c < r$, i. e. $b_n = n \ln(1 - h) \to +\infty$, we have

$$u(y,\ T) = \mathrm{e}^{-rT}\left(1 - \Psi_{n_0}(\lambda^* T)\right),$$

$$U(y,\ T) = 1 - \Psi_{n_0}(\lambda^*(1 - h)T).$$

In both cases,

$$n_0 = \inf\{n : S_0 \mathrm{e}^{n \ln(1-h)+(c-r)T} > B(T)^{-1}K\} = \left[\frac{\ln(K/S_0) - cT}{\ln(1 - h)}\right].$$

*Example 5.2* Let us consider another symmetric case $\lambda_0 = \lambda_1 = \lambda,\ \ r_0 = r_1 = r,$ $c_0 = r + c,\ \ c_1 = r - c$ and $h_0 = -h,\ \ h_1 = h;\ \ c > 0,\ \ 0 < h < 1$. These assumptions simplify the form of $u^{(i)}$. In this case we have $\lambda_0^* = \lambda_1^* = c/h$ and $b_n \to -\infty$. Here $b_{2n} = n \ln(1 - h^2)$ and $b_{2n+1} = n \ln(1 - h^2) + \ln(1 - (-1)^i h)$. We denote

$$n_i = \left[\frac{\ln(K/S_0) - (c + (-1)^i r)T}{\ln(1 - h^2)}\right], \quad i = 0, 1. \tag{5.5.38}$$

Function $u^{(0)}$ has the form (see Remark 5.7)

$$u^{(0)}(y,\ T) = \mathrm{e}^{-(c/h+r)T}\left\{\sum_{n=0}^{2n_1}\frac{(cT/h)^n}{n!} + \sum_{n=2n_1+1}^{2n_0}\frac{(c/h)^n}{n!}\sum_{k=0}^{[n/2]}\binom{n}{k}q_n^k p_n^{n-k}\right\},$$

where $p_n = \dfrac{(r+c)T - y + b_n}{2c} \geq 0,\ \ q_n = \dfrac{y-(r-c)T-b_n}{2c} \geq 0$ and $y = \ln(K/S_0)$.
Function $u^{(1)}$ has the similar form.

## 5.6 Historical and Implied Volatilities in the Jump Telegraph Model

### *5.6.1 Historical Volatility*

Historical volatility is defined as

$$\mathrm{HV}(t) = \sqrt{\frac{\mathrm{Var}\{\log S(t+\tau)/S(\tau)\}}{t}}. \tag{5.6.1}$$

For the classic Black-Scholes model $\log S(t+\tau)/S(\tau) \overset{d}{=} at + \sigma w(t)$, where $w = w(t),\ t \geq 0$, is a standard Brownian motion, the historical volatility is constant: $\mathrm{HV}_{\mathrm{BS}}(t) \equiv \sigma$.

Consider a moving-average type model (see [29]), which is described by

$$\log S(t)/S(0) = at + \sigma w(t) - \sigma\int_0^t \mathrm{d}\tau \int_{-\infty}^{\tau}\lambda_0 \mathrm{e}^{-(\lambda_0+\lambda_1)(\tau-u)}\mathrm{d}w(u),$$
$$\sigma,\ \lambda_1,\ \lambda_0+\lambda_1 > 0.$$

Usually these models are applied to capture memory effects of the market [24, 30].

The historical volatility (5.6.1) for this model can be exactly described [29],

$$\mathrm{HV}(t) = \frac{\sigma}{2\lambda}\sqrt{\lambda_1^2 + \lambda_0(2\lambda_1+\lambda_0)\Phi_\lambda(t)} \tag{5.6.2}$$

with $2\lambda = \lambda_0 + \lambda_1$ and $\Phi_\lambda(t) = (1-\mathrm{e}^{-2\lambda t})/(2\lambda t)$.

The historical volatility of jump telegraph model (5.2.1) is defined by

$$\mathrm{HV}_{\varepsilon(\tau)}(t) = \sqrt{\frac{s_{\varepsilon(\tau)}(t)}{t}}, \tag{5.6.3}$$

where $s_i(t) = \mathrm{Var}_i\,[X(t) + \ln\kappa(t)]$ and $i = \varepsilon(0)$ is the initial state. Due to (4.1.20), historical volatility $\mathbf{HV} = (\mathrm{HV}_0(t), \mathrm{HV}_1(t))^T$ can be computed by the formula

$$\mathbf{HV}(t) = \sqrt{\frac{1}{t}\int_0^t \mathrm{e}^{(t-\tau)\Lambda}\ \mathbf{v}(\tau)\mathrm{d}\tau}, \tag{5.6.4}$$

where $\mathbf{v} = (v_0(\tau),\ v_1(\tau))^T$ is defined as in (4.1.20), but with $\ln(1+h_i)$ instead of $h_i,\ i = 0, 1$:

$$v_0(\tau) = \lambda_0\,[\ln(1+h_0) - C\tau\Phi_\lambda(\tau)]^2\,, \quad v_1(\tau) = \lambda_1\,[\ln(1+h_1) + C\tau\Phi_\lambda(\tau)]^2\,.$$

Here $C = c_0 + \lambda_0\ln(1+h_0) - c_1 - \lambda_1\ln(1+h_1)$ and $\Phi_\lambda(\tau) = (1 - \mathrm{e}^{-2\lambda\tau})/(2\lambda\tau)$, $2\lambda = \lambda_0 + \lambda_1$.

Historical volatility in jump telegraph model has the following very natural limiting behaviour (see (4.1.24)–(4.1.25)):

$$\lim_{t\to 0}\mathrm{HV}_i(t) = \sqrt{\lambda_i}\,|\ln(1+h_i)|,$$
$$\lim_{t\to\infty}\mathrm{HV}_i(t) = \sqrt{\frac{\lambda_0\lambda_1}{2\lambda^3}\left[(\lambda_0 B + c)^2 + (\lambda_1 B - c)^2\right]}, \quad i = 0, 1,$$

($2B = \ln(1+h_0)(1+h_1)$, $2c = c_0 - c_1$; see (4.1.25)). These limits look quite reasonable: the limit at 0 is engendered by jumps only, the limit at $\infty$ contains both the drift component and a long-term influence of jumps.

Using (4.1.27) and (5.6.1), in the symmetric case $\lambda_0 = \lambda_1 = \lambda$ formula (5.6.4) takes the form similar to (5.6.2),

$$\mathrm{HV}_i(t) = \sqrt{\frac{c^2}{\lambda} + \lambda B^2 + (c+\lambda b)^2\frac{\Phi_{2\lambda}(t)}{\lambda} + \gamma_i\Phi_\lambda(t) + (-1)^i 2B(c+\lambda b)\mathrm{e}^{-2\lambda t}}, \tag{5.6.5}$$
$$i = 0,\ 1.$$

The limits of historical volatility under a standard diffusion scaling (see Theorem 5.6) look more complicated. Nevertheless, in the symmetric case $\lambda_0 = \lambda_1 = \lambda$, we have under the scaling conditions of Theorem 5.6, i. e. $\lambda,\ c \to \infty,\ h_i \to 0,\ c^2/\lambda \to \sigma^2,\ \sqrt{\lambda}h_i \to \alpha_i$ that the historical volatility $\mathrm{HV}_i(t),\ i = 0,\ 1$ defined by (5.6.5) converges to $v = \sqrt{\sigma^2 + \left(\frac{\alpha_0+\alpha_1}{2}\right)^2}$. Indeed, according to scaling conditions, we have the convergence of the first two terms $\frac{c^2}{\lambda} + \lambda B^2$ to $v^2 = \sigma^2 + \left(\frac{\alpha_0+\alpha_1}{2}\right)^2$.

The remaining part vanishes, because $\Phi_\lambda(t) \to 0,\ \mathrm{e}^{-2\lambda t} \to 0,\ \forall t > 0,$ as $\lambda \to \infty$, and the limits of $\dfrac{(c+\lambda b)^2}{\lambda},\ \gamma_i,\ B(c+\lambda b)$ are finite.

## 5.6.2 Implied Volatility and Numerical Results

Define the Black-Scholes call price function $f(\mu, v),\ \mu = \log K$, by

$$f(\mu, v) = \begin{cases} F\left(\frac{-\mu}{\sqrt{v}} + \frac{\sqrt{v}}{2}\right) - \mathrm{e}^{\mu} F\left(\frac{-\mu}{\sqrt{v}} - \frac{\sqrt{v}}{2}\right), & \text{if } v > 0, \\ (1-\mathrm{e}^{\mu})^+, & \text{if } v = 0, \end{cases}$$

where $F(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{z} \mathrm{e}^{-x^2/2} \mathrm{d}x$ is the distribution function of the standard normal law. The processes $V_i(\mu, t),\ t \ge 0,\ \mu \in \mathbb{R},\ i = 0,\ 1$, defined by the equation

$$\mathbb{E}\left[(S(t+\tau)/S(\tau) - \mathrm{e}^{\mu})^+ | \mathfrak{F}_\tau\right] = f(\mu,\ V_{\varepsilon(\tau)}(\mu, t)), \tag{5.6.6}$$

are referred to as the implied variance processes.

The implied volatilities $\mathrm{IV}_i(\mu, t)$ are

$$\mathrm{IV}_i(\mu, t) = \sqrt{\frac{V_i(\mu, t)}{t}}. \tag{5.6.7}$$

Below, we perform the numerical valuation of the jump telegraph volatility (5.3.15) and the historical volatility (5.6.4) which are compared with the implied volatilities (5.6.7) with respect to different moneyness and to the initial market states. The implied volatilities are calculated by the explicit formulae (5.5.1)–(5.5.2), (5.5.13), (5.5.25), and (5.6.7). First, we consider the symmetric case: $\lambda_0 = \lambda_1 = 10$, $c_0 = +1,\ c_1 = -1$ and $h_0 = -0.1,\ h_1 = +0.1$. In Fig. 5.1 we plot implied volatilities in this simple case. Table 5.1 lists call prices and implied volatilities of this volatility smile numerically. Notice that these frowned smiles of implied volatilities $\mathrm{IV}_0$ and $\mathrm{IV}_1$ intersect at $K/S_0 \approx 1.17$.

Table 5.2 and Fig. 5.2 show the implied volatility picture for skewed movement, when the market prices have a drift: both velocities are positive, and to avoid an arbitrage, we suppose jump values to be negative. This figure has unstable oscillations for deep-out-of-the-money options. Moreover, only in this case historical and jump telegraph volatilities are less then implied volatilities values for at-the-money options.

Finally, we present the case taken asymmetrically with $\lambda_1 = 48.53,\ \lambda_0 = 34.61,\ h_1 = -0.0126,\ h_0 = -0.0358,\ c_1 = 0.61,\ c_0 = 1.24$. These parameters correspond to the simulations of a preferably bullish market with small jump corrections. The main feature of this market consists in the redundancy of small

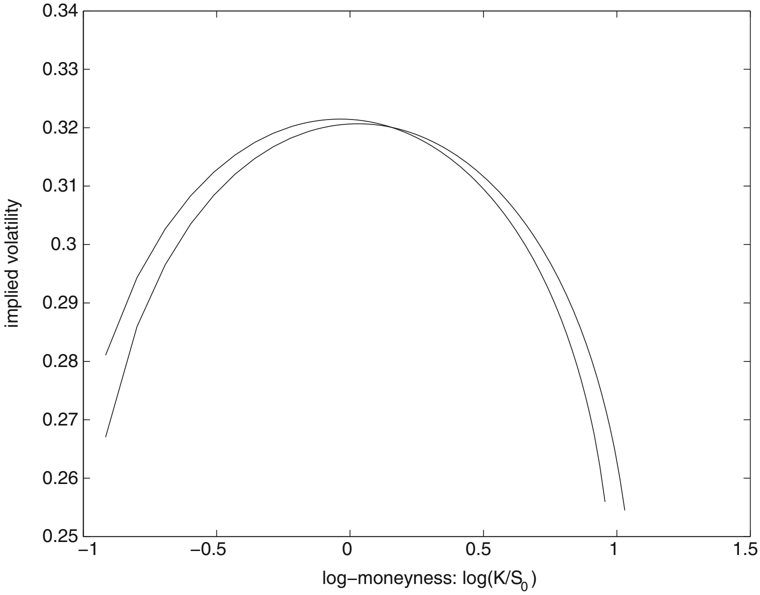


**Fig. 5.1** Symmetric smile, $t = 1$, $S_0 = 100$, $\lambda_0 = \lambda_1 = 10$, $h_0 = -0.1$, $h_1 = 0.1$, $c_0 = 1$, $c_1 = -1$, $\mathrm{HV}_{\pm} = 0.3162$

**Table 5.1** Symmetric smile, $t = 1$, $S_0 = 100$, $\lambda_0 = \lambda_1 = 10$, $h_0 = -0.1$, $h_1 = +0.1$, $c_0 = 1$, $c_1 = -1$

| $K$ | 40 | 70 | 100 | 117 | 130 | 160 | 190 | 220 | 250 | 280 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathfrak{c}_1$ | 60.0013 | 31.6774 | 12.7370 | 6.9036 | 4.1565 | 1.1433 | 0.2632 | 0.0478 | 0.0058 | 0.0002 |
| $\mathfrak{c}_0$ | 60.0026 | 31.7257 | 12.7680 | 6.9039 | 4.1382 | 1.1128 | 0.2430 | 0.0390 | 0.0032 | 0.0 |
| $\mathrm{IV}_1$ | 0.2670 | 0.3147 | 0.3206 | 0.3200 | 0.3186 | 0.3128 | 0.3045 | 0.2935 | 0.2787 | 0.2545 |
| $\mathrm{IV}_0$ | 0.2811 | 0.3175 | 0.3214 | 0.3200 | 0.3180 | 0.3109 | 0.3010 | 0.2875 | 0.2671 | 0.0 |

**Table 5.2** Skewed smile, $t = 1$, $S_0 = 100$, $\lambda_0 = \lambda_1 = 10$, $h_1 = -0.03$, $h_0 = -0.19$, $c_1 = 0.3$, $c_0 = 1.9$

| $K$ | 50 | 100 | 150 | 200 | 250 | 300 | 350 | 400 | 450 | 500 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathfrak{c}_1$ | 50.8133 | 17.6956 | 5.5624 | 1.8243 | 0.6350 | 0.2325 | 0.0882 | 0.0347 | 0.0127 | 0.0053 |
| $\mathfrak{c}_0$ | 50.9762 | 18.5944 | 6.3367 | 2.2640 | 0.8586 | 0.3454 | 0.1413 | 0.0621 | 0.0279 | 0.0099 |
| $\mathrm{IV}_1$ | 0.4475 | 0.4473 | 0.4539 | 0.4590 | 0.4620 | 0.4632 | 0.4630 | 0.4624 | 0.4577 | 0.4565 |
| $\mathrm{IV}_0$ | 0.4662 | 0.4704 | 0.4776 | 0.4827 | 0.4856 | 0.4875 | 0.4868 | 0.4873 | 0.4867 | 0.4766 |

jumps. The calibrated martingale distribution is strongly asymmetric (Table 5.3 and Fig. 5.3).

The behaviour of implied volatility in the jump-telegraph model for these data surprisingly resembles the calibration results for stochastic volatility models of

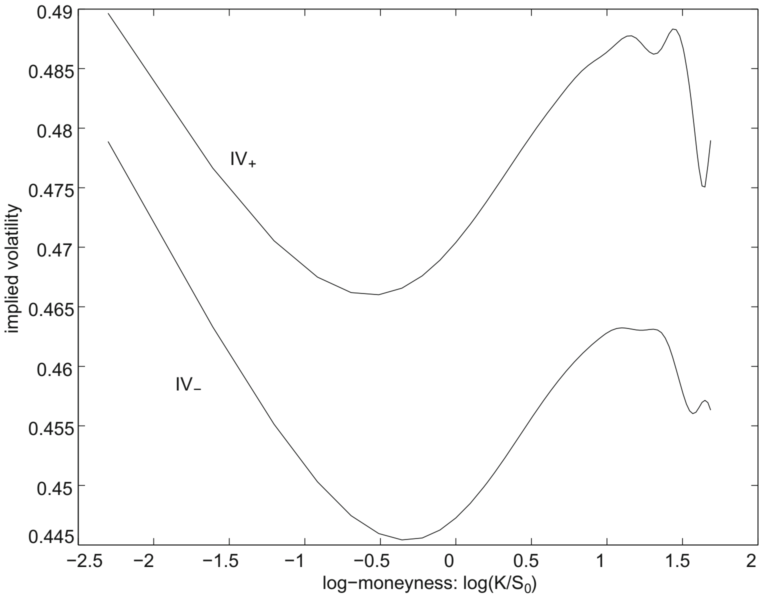


**Fig. 5.2** Skewed smile, $t = 1$, $S_0 = 100$, $\lambda_0 = \lambda_1 = 10$, $h_1 = -0.03$, $h_0 = -0.19$, $c_1 = 0.3$, $c_0 = 1.9$ , $\text{HV}_1 = 0.4198$, $\text{HV}_0 = 0.4402$

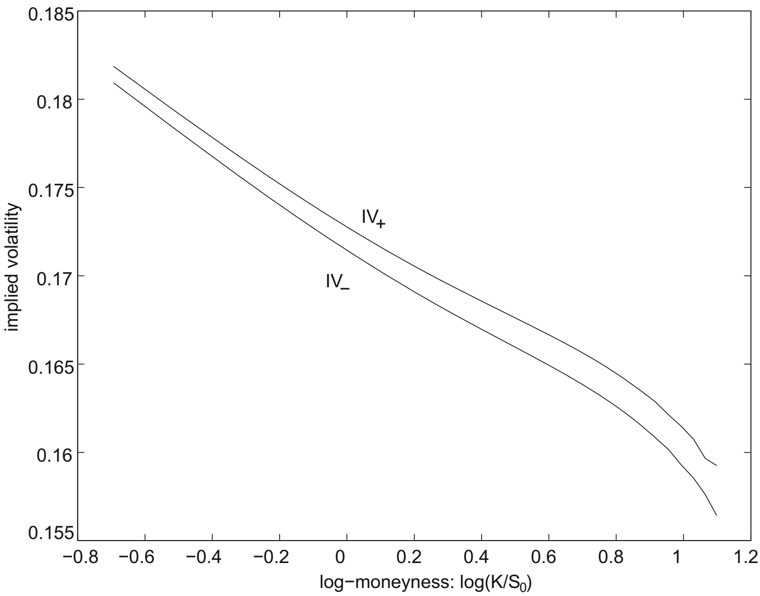


**Fig. 5.3** Asymmetric smile, $t = 1$, $S_0 = 100$, $\lambda_1 = 48.53$, $\lambda_0 = 34.61$, $h_1 = -0.0126$, $h_0 = -0.0358$, $c_1 = 0.61$, $c_0 = 1.24$ , $\text{HV}_1 = 0.1630$, $\text{HV}_0 = 0.1642$

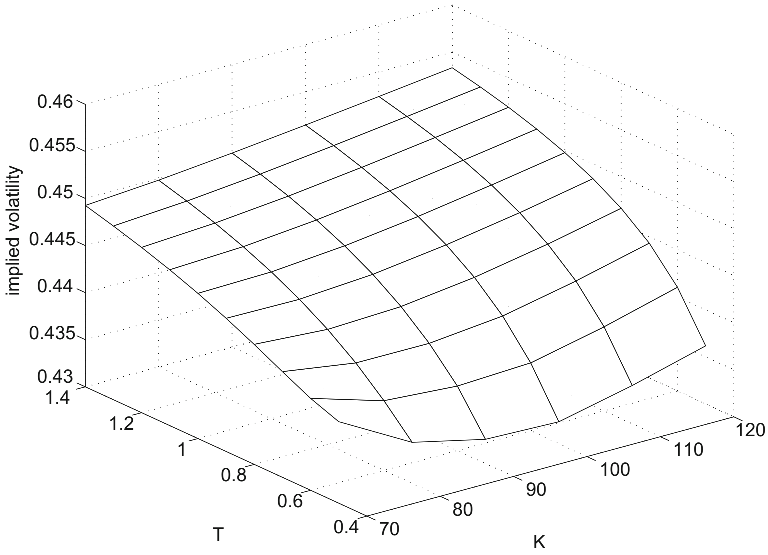


**Fig. 5.4** Skewed smile, $S_0 = 100$, $\lambda_0 = \lambda_1 = 10$, $h_1 = -0.03$, $h_0 = -0.19$, $c_1 = 0.3$, $c_0 = 1.9$

**Table 5.3** Asymmetric smile, $t = 1$, $S_0 = 100$, $\lambda_1 = 48.53$, $\lambda_0 = 34.61$, $h_1 = -0.0126$, $h_0 = -0.0358$, $c_1 = 0.61$, $c_0 = 1.24$

| $K$ | 50 | 70 | 100 | 130 | 160 | 190 | 220 | 250 |
|---|---|---|---|---|---|---|---|---|
| $\mathfrak{c}_1$ | 50.0002 | 30.1167 | 6.8313 | 0.4913 | 0.0146 | 0.0002 | 0.0000 | 0.0000 |
| $\mathfrak{c}_0$ | 50.0002 | 30.1215 | 6.8838 | 0.5117 | 0.0162 | 0.0003 | 0.0000 | 0.0000 |
| $\mathbb{IV}_1$ | 0.1809 | 0.1762 | 0.1714 | 0.1684 | 0.1663 | 0.1645 | 0.1628 | 0.1608 |
| $\mathbb{IV}_0$ | 0.1819 | 0.1773 | 0.1728 | 0.1699 | 0.1679 | 0.1662 | 0.1646 | 0.1629 |

the Ornstein-Uhlenbeck type (see [31], Fig. 5.1, where implied volatilities of OU-stochastic volatility model was depicted). Similar comportment is observed in jump-diffusion models (see Table 2 of [32]). All calculations were prepared by considering a data set of European call options on S&P 500 index).

Figure 5.4 depicts an implied volatility surface with respect to strike prices and maturity times.

## 5.7 Pricing Exotic Options

In the framework of model (5.2.3) it is possible to price more exotic than the standard call options. Here we give some hints to *up-and-out* and $\alpha$*-quantile* call options.

Consider the market with the underlying assets defined by (5.2.1)–(5.2.2). Up-and-out call option is a European call, expiring at time $T$, with strike $K$ and up-and-out barrier $B$. We assume $K < B$. Denote $M(t) = \max_{0\le s\le t} X(s)$. The payoff function of this option is $(S(T) - K)^+ \mathbb{1}_{\{M(T)<B\}}$.

The price of this option at time $t$ is represented by

$$F_i(t,\ x) = \mathbb{E}_i^*\left[\mathrm{e}^{-Y(T-t)}(S(T-t) - K)^+ \mathbb{1}_{\{M(T-t)<B\}}\right], \qquad x = S_0 = S(0).$$

In the rectangle $\{(t, x) :\ 0 \le t \le T,\ 0 \le x \le B\}$ functions $F_i(t,\ x),\ i = 0, 1$ satisfy integral and differential Eqs. (5.4.2), (5.4.3) with the terminal condition

$$F_i(T,\ x) = (x - K)^+, \qquad 0 \le x \le B,$$

and the boundary conditions (cf. [13], Theorem 7.3.1):

$$F_i(t,\ 0) = 0, \qquad 0 \le t \le T,$$

$$F_i(t,\ B) = 0, \qquad 0 \le t < T.$$

The option price formulae can be obtained by solving this boundary value problem similarly to formulae (5.5.2)–(5.5.4).

Another example of path-dependent options based on telegraph processes which can be easily proceeded is so-called the $\alpha$-quantile option. We define the payoff function using the following notation. Let

$$l(x,\ y,\ t) = \int_0^t \mathbb{1}_{\{S(x,s)\le y\}} \mathrm{d}s$$

be the occupation time with respect to $t$ of the process $S = S(x, t) = x\mathrm{e}^{X(t)}\kappa(t)$ lying below the level $y$. The $\alpha$-quantile $(0 < \alpha < 1)$ of stochastic process $S$ is defined by $M_T^\alpha = \inf\{y\ :\ l(x,\ y,\ T) > \alpha T\}$. The terminal payoff function of the fixed strike $\alpha$-quantile call option is defined by $(M_T^\alpha - K)^+$.

To price this option, we operate similarly to Sect. 3.2. Consider the double Laplace transform

$$w_i(x) = \int_0^\infty \mathrm{e}^{-bt} v_i(x,\ t)dt,$$

where $v_i(x,\ t) = \mathbb{E}_i \exp(-\beta l(x,\ y,\ t)),\ \beta > 0$. Notice that functions $v_i$ satisfy the Feynman-Kac system

$$\begin{cases} -\frac{\partial v_0}{\partial t} + c_0 x \frac{\partial v_0}{\partial x} = \lambda_0 \left[v_0(x,\ t) - v_1(x(1+h_0),\ t)\right] + k(x)v_0(x,\ t), \\ -\frac{\partial v_1}{\partial t} + c_1 x \frac{\partial v_1}{\partial x} = \lambda_1 \left[v_1(x,\ t) - v_0(x(1+h_1),\ t)\right] - k(x)v_1(x,\ t), \end{cases} \tag{5.7.1}$$

with

$$k(x) = \begin{cases} 0, & x > y, \\ \beta, & x \le y. \end{cases}$$

To prove that, we use again the conditioning trick. Let

$$Y_t^{(i)}(x) = \exp\left\{-\beta \int_0^t \mathbb{1}_{\{S_i(x,s)\le y\}} \mathrm{d}s\right\}, \qquad i = 0, 1.$$

Notice that, for any $x,\ y > 0$, the root $s = s_i^*$ of the equation $x\mathrm{e}^{c_i s} = y$ is unique. Set

$$\tau_i^*(x,t) = s_i^* = \frac{\ln(y/x)}{c_i}, \quad \text{if } s_i^* \in [0, t];$$
$$\tau_i^*(x,t) = t, \quad \text{if } s_i^* > t;$$
$$\tau_i^*(x,t) = 0, \quad \text{if } s_i^* < 0.$$

Conditioning on the first switching, we obtain the following equality in distribution

$$Y_t^{(0)}(x) \stackrel{d}{=} \mathrm{e}^{-\beta\tau_0^*(x,t)} \cdot \mathbb{1}_{\{\tau>t\}} + \int_0^t \lambda_0 \mathrm{e}^{-\lambda_0 s} Y_{t-s}^{(1)}(x\mathrm{e}^{c_0 s}(1+h_0))\mathrm{e}^{-\beta\tau_0^*(x,s)}\mathrm{d}s,$$
$$Y_t^{(1)}(x) \stackrel{d}{=} \mathrm{e}^{-\beta(t-\tau_1^*(x,t))} \cdot \mathbb{1}_{\{\tau>t\}} + \int_0^t \lambda_1 \mathrm{e}^{-\lambda_1 s} Y_{t-s}^{(0)}(x\mathrm{e}^{c_1 s}(1+h_1))\mathrm{e}^{-\beta(t-\tau_1^*(x,s))}\mathrm{d}s.$$

Here $\tau$ is the first switching time.

Therefore,

$$v_0(x,t) = \mathrm{e}^{-\beta\tau_0^*(x,t)-\lambda_0 t} + \int_0^t \lambda_0 \mathrm{e}^{-\lambda_0 s} v_1(x\mathrm{e}^{c_0 s}(1+h_0), t-s)\mathrm{e}^{-\beta\tau_0^*(x,s)}\mathrm{d}s, \quad (5.7.2)$$

$$v_1(x,t) = \mathrm{e}^{-\beta(t-\tau_1^*(x,t))-\lambda_1 t} + \int_0^t \lambda_1 \mathrm{e}^{-\lambda_1 s} v_1(x\mathrm{e}^{c_1 s}(1+h_1), t-s)\mathrm{e}^{-\beta(t-\tau_1^*(x,s))}\mathrm{d}s. \quad (5.7.3)$$

Differentiating Eqs. (5.7.2)–(5.7.3) we obtain system (5.7.1) by applying the following identity

$$\left[-\frac{\partial}{\partial s} + c_i x \frac{\partial}{\partial x}\right] \tau_i^*(x,s) = -1, \qquad i = 0, 1.$$

System (5.7.1) leads to the set of ordinary differential equations for $(w_0,\ w_1)$:

$$\begin{cases} c_0 x w_0'(x) = \lambda_0(w_0(x) - w_1(x(1+h_0))) + (b + k(x))w_0(x) - 1 \\ c_1 x w_1'(x) = \lambda_1(w_1(x) - w_0(x(1+h_1))) + (b - k(x))w_1(x) - 1. \end{cases}$$

Let $\mathcal{L}_b^{-1}$ and $\mathcal{L}_\beta^{-1}$ denote the Laplace inversion with respect to Laplace variables $b$ and $\beta$, and let $t$ and $\tau$ be the respective variables after Laplace inversion. The price of the $\alpha$-quantile option is

$$c^i_{K,\ \alpha} = \mathrm{e}^{-rT} \int_K^\infty \mathbb{P}_i^*(M_T^\alpha \geq x)\mathrm{d}x = \mathrm{e}^{-rT} \int_K^\infty \mathcal{L}_b^{-1}\mathcal{L}_\beta^{-1}\left[\frac{w_i(x)}{b\beta}\right]\Big|_{\{t=T,\ \tau=\alpha T\}}\,\mathrm{d}x \tag{5.7.4}$$

(cf. [33], formula (3.6)). By performing the integration one can obtain the integral representation of (5.7.4) as well as the distribution of $l(x,\ y,\ t)$. In the presence of jumps, the closed formulae for distribution of $l(x,\ y,\ t)$, similar to (3.2.16)–(3.2.17), are yet unknown.

## Notes

Option pricing model based on the generalised jump-telegraph processes was first introduced by the second author [34]. Later this model has been substantially developed in [15–17], [35].

This book does not contain extensively developed theory of continuous time random walks (CTRW) with its multiple applications to finance [36]. Market models based on diffusion-telegraph processes, see [37], are also omitted.

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