Roger F. Gans

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A Unified Approach to Vibrations and Controls

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for Janet

Preface

This text presents a unified treatment of vibrations and control. Both of these topics fit under the broad title of mechanical systems, hence the main title of the book. I have taught a one-semester course dealing with both of these topics for several years. I have used two excellent books, but the notation is not consistent between the two, and asking students to buy two books is an imposition. This book provides one book with one notation for both topics.

One thing my teaching experience has taught me is that one semester is not enough, and I have included enough material for a 1-year course. Chapters 1–5 cover the topics in a standard vibrations course with some added material on electric motors and vibration measurement. Chapter 6 introduces state space and forms a bridge from the first half of the book to the second. If one were to use the book only for vibrations or only for controls, I would suggest including Chap. 6 in either course. Chapters 7–10 cover linear controls starting with PID control and moving quickly to state space control, where I address controllability, observability, and tracking. Chapter 11 introduces feedback linearization as an easily understandable venture in nonlinear control. I have taken advantage of the fact that kinematic chains with only revolute joints can be controlled using a variant of feedback linearization to introduce nonlinear control of simple robots with only revolute joints. This chapter is a bonus, not necessary for a first course in controls, but there for those who find it useful.

The text is aimed at juniors and seniors in mechanical and related engineering programs, such as applied mechanics, aeronautical engineering, and biomedical engineering, but I hope that it will be useful to working engineers as well. I expect the students to have had exposure to ordinary differential equations, linear algebra, and an elementary course in engineering dynamics. I use free body diagrams early in the text and expect that idea to be familiar to them. I use Euler-Lagrange methods from Chap. 3 onwards. These methods are introduced assuming no previous familiarity with them.

I have found that students don't much like abstract mathematics, so I have tried hard to include examples with a real-world flavor. Some of these examples (a servomotor, suspension of a steel ball by an electromagnet, and the overhead crane) appear early in the text and then reappear to illustrate the new material introduced in each chapter. I have also tried to refresh student memories and/or include basic material they may not have seen. In particular, I spend some time in Chap. 1, which provides an overview of the whole text, on complex notation and the connection between the complex exponential and trigonometric functions. As I work through the first few chapters, I use trigonometric functions and the complex exponential interchangeably to get students used to the connection. The complex exponential is essential once the book moves to state space.

Chapter 2 covers one degree of freedom problems using free body diagrams to develop the equations of motion. I also introduce the concepts of kinetic and potential energy here. Chapter 3 deals with systems with more than one degree of freedom. It includes a careful discussion of degrees of freedom and introduces the whole Euler-Lagrange process for deriving equations of motion. I also introduce a simple DC motor model in Chap. 3 so that I can use voltage as an input for the rest of the text instead of just saying "Here's a force" or "Here's a torque." Chapter 4 covers modal analysis. Chapter 5 discusses vibration measurement, and Fourier analysis, Fourier series, and Fourier transforms, primarily as tools for vibration analysis. I discuss the Gibbs phenomenon and Nyquist phenomena, including aliasing. I prove no theorems.

Most of the world cannot be well modeled by linear systems, so I have included the simulation of nonlinear systems using commercial packages for the integrations. Linearization is a skill that is not generally taught formally. It is very important, even in the age of computers, and I attack it twice, once in Chap. 3 and again in Chap. 6, more formally. The control designs in the text, except for Chap. 11, are all linear, based on choosing gains to drive errors to zero, but I assess their efficacy using numerical simulation of the nonlinear equations.

I cover control in the frequency domain in Chap. 7, introducing the Laplace transform (without theory) and transfer functions. I go through the classical PID control of a second-order (one degree of freedom) system and apply the idea of integral control to automotive cruise control.

I think that control in state space is much more useful. I introduce state space in Chap. 6 and state space control in the time domain in Chap. 8. The rest of the text is cast in state space and in the time domain. Chapter 8 includes the controllability theorem (the algebraic theorem without proof), the reduction of a controllable dynamical system to companion form, the placement of poles by choosing gains in the companion form, and the mapping of the gains back to the original space. Chapter 8 is limited to using full state feedback driving a dynamical system to a (vector) zero. I deal with both linear and nonlinear systems. I design controls for nonlinear systems by linearizing and applying a state space control algorithm, but then I test the controls by solving the full nonlinear systems numerically.

Chapter 9 introduces observers. I go through the examples in Chap. 8 using an observer. I also note that the observer doubles the size of the system and that one needs to take account of that in assessing the control that one designs. In particular, I note that the original system and the observed system may be independently stable, but that does not necessarily extend to the combined system, which can actually be unstable.

Chapter 10 explains how to track a time-dependent reference state, both assuming full state feedback and using an observer. Again I use linear control on nonlinear systems and verify the controls using numerical integration.

The basic control paradigm, executed in state space, is to linearize the system (if necessary), design a linear control, and then assess the linear control on the nonlinear system in simulation.

Chapter 11 addresses nonlinear control by feedback linearization. This is applicable to some simple robots, and I discuss three-dimensional motion briefly in order to deal with realistic robots.

Rochester, NY, USA Roger F. Gans

Acknowledgments

I owe thanks to several sets of students taking ME 213 at the University of Rochester. They showed what did and did not work. Even this year's class, whom I have only met five times at the time of writing, helped me clarify several points in Chap. 2. Thanks also to the TAs in this course, who gave me a separate feedback on what did and did not work. I also wish to thank several anonymous referees whose comments led me to further clarification.

Contents

Overview with Some Definitions and Mathematics

1

In which we introduce a number of concepts and techniques that will be important for the whole text.

1.1 General Introduction

Vibrations and controls are both key engineering topics. They have a great deal of mathematics and physics in common, which is why it makes sense to tackle them in the same text. Both deal with the response of mechanical systems to external forces. (I will include electromechanical systems in the term *mechanical systems*. This lets me use motors to control mechanical systems from time to time.) Mechanical systems can be as simple as a pendulum or as complicated as a jet aircraft. One of the important things an engineer has to do is to model complicated systems as simpler systems containing the important aspects of the real system. Our models will be mathematical—sets of differential equations. Real systems are generally nonlinear in a mathematical sense. Nonlinear differential equations are generally solvable only numerically, but it can be possible to *linearize* the differential equations, by which we mean: replace the nonlinear equations with linear equations that provide a good approximation to the nonlinear system over a limited range. Even the simple pendulum is a nonlinear system, one that can be approximated reasonably well by a linear system if the amplitude of the oscillation of the pendulum is not too large. (I'll discuss what I mean by too large later.) Linearization is an important topic that we will address in some detail later in the text.

We will restrict our mathematical analysis to linear systems, but we will not entirely neglect nonlinear systems, which we will deal with numerically. I will call a numerical model of a mechanical system a *simulation*, and I will use simulations to assess the utility of linear models. For example, we might design a control system based on a linear model of an actual system and test it on the nonlinear simulation. We can use simulations in general to assess the range of validity of a linearized model, the "too large" issue mentioned in the previous

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paragraph. The simulations will be sets of quasilinear first-order ordinary differential equations (*quasilinear* means that the derivatives enter linearly), which we will integrate using commercial software. These software packages typically implement something called the *Runge-Kutta method*. For more information, see Press et al. (1992).

Mechanical systems can be continuous, like a bending beam, or discrete, like a collection of masses connected by springs. Of course, at some level all systems are continuous (and at a still finer level, all systems are again discrete). Continuous systems can be modeled by discrete systems. This is the principle behind finite element analysis. This text addresses primarily discrete systems. I will look at three simple continuous systems in Chap. 4. Discrete vibrating systems can be modeled by finite collections of masses and springs and dampers. Mechanisms can also be so modeled, so the study of mechanisms and the study of vibrations are closely related. We want to be able to control mechanisms, so control also falls under this same broad umbrella. We will study mechanisms as mechanisms, as models for vibration, and as systems to be controlled.

A mechanism has some number of *degrees of freedom*. Each degree of freedom corresponds to an independent motion the mechanism can execute. A pendulum has but one degree of freedom; all it can do is swing back and forth in a fixed arc. A simplified model of a vehicle (a block lying on a plane) has three degrees of freedom: the location of its center of mass and the direction in which it is pointing. An object in space has six degrees of freedom, three translational and three rotational. An object confined to a plane has at most three degrees of freedom—two to define its location and one its orientation. This text deals almost exclusively with planar mechanisms (I will address some three-dimensional [robotic] control problems in Chap. 11), and I choose the Cartesian $x = 0$ plane as the plane in which they lie, so that the Cartesian translational variables will be y and z. These are coordinates in the world. I let z increase upward and y to the right. This gives me a right-handed coordinate system with unit vectors **j** and **k,** respectively. I will denote the orientation angle by θ. It will always increase in the counterclockwise direction, but I will choose different origins for different problems. Figure 1.1 shows a three-dimensional picture of a cylinder in the y, z ($x = 0$ plane). The arrow indicates the center of mass and the angle is the angle between the horizontal and the cylinder.

In general the more degrees of freedom a mechanism has, the more complicated it is. However, a mechanism can be quite complicated mechanically and still have a small number of degrees of freedom. For example, a twelve-cylinder internal combustion engine fixed to the ground has dozens of parts but only one degree of freedom. The location and orientation of every part in the engine is determined by the crank angle. (If the engine is installed in a vehicle, it has seven degrees of freedom, because it can rotate and translate as a whole. The rotation and translation will be limited by the motor mounts, but those degrees of freedom still exist.) I will discuss degrees of freedom in more detail in Chap. 3.

Mechanical systems have i*nputs* and *output*s. The inputs are whatever forces and torques that the world applies to the system. Sometimes these are fairly abstract. The inputs to a robot are the voltages applied to the motors driving the arms.

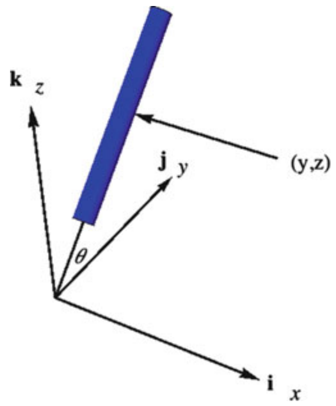


Fig. 1.1 Inertial space showing the $x = 0$ plane, to which our planar mechanisms will be confined

The outputs are whatever variables of the system that the world needs to know and perhaps control. The input to a cruise control is fuel flow and the output is speed. I will introduce a simple electric motor model in Chap. 3. Its input is a voltage, and its outputs are torque and rotation. For most of the models in this text the inputs will be forces or torques and the outputs positions and speeds. The control of linear systems with one input (single-input, SI systems) is a well-understood and well-studied topic, and it will be the focus of control in this text.

The mathematics involved is that of ordinary differential equations and linear algebra. The independent variable is the time, t. In order to address the mathematics, we need to be able to derive the equations, which we do by looking at models. Building models is actually the most difficult task before us. We need to be able to take a problem presented to us in words, possibly poorly, and devise a mechanical model, a mechanism made up of parts connected in some way. Mechanical systems have inertia, perhaps (usually) some way to store and release energy and ways of dissipating energy. The simpler the model, the simpler the mathematics. Our goal should be to find the simplest model that represents the important parts of the physics of the mechanism. Once we can identify the important parts of a mechanism and how they are attached, derivation of the relevant equations becomes simple (in principle). There are (at least) two ways to build equations: using free body diagrams and Newton's laws or considering the energies of the system and forming the *Euler-Lagrange equations*. We will start with the former, but most of our analysis will be based on the latter, which I will introduce in Chap. 3.

Once we have differential equations we need to solve them. This requires some skills and techniques in mathematics, most of which I will develop during the course of the text. All the sets of differential equations that we will tackle analytically will consist of linear, ordinary differential equations with constant coefficients. (As I noted above, we will have some systems of quasilinear differential equations, but we will tackle these numerically.) I expect the reader to have some familiarity with linear ordinary differential equations with constant coefficients. The independent variable will always be time. We are interested in the evolution of systems and their control as time passes. I will consider sets of first-order equations written in matrix-vector form, so some familiarity with linear algebra will be very helpful. The equations for mechanical systems occur "naturally" as sets of second-order equations, so we will need a method of converting these to pairs of first-order equations. This will lead us to the concept of *state space*. The solution of linear ordinary differential equations with constant coefficients is based on exponential and trigonometric functions (which are actually equivalent, as we will see when we learn a little about *complex numbers* at the end of this chapter). Trigonometric functions suggest the use of *Fourier series*, and we will look at those in Chap. 5. Exponentials and the *Laplace transform* are closely related, and we will learn how to use Laplace transforms. (The general theory of Laplace transforms is beyond the scope of this text.)

I will be using vectors and matrices throughout. The vectors are not physical vectors such as displacement, velocity, force, etc. They are abstract vectors in the sense of linear algebra. I will generally denote vectors by bold face lowercase roman letters, such as $\mathbf{u}$. I will denote the general component of a vector by a subscript, such as u_i. (I will generally denote scalars by roman italic, sometimes uppercase and sometimes lowercase.) I will denote matrices by bold face uppercase roman letters, such as $\mathbf{A}$. I will denote the general component of a matrix by a double subscript, the first referring to the row and the second to the column, such as A_{ij}. I will denote the transpose of a matrix by a superscript T, as $\mathbf{A}^{\mathrm{T}}$. I will denote its inverse by a superscript -1, as $\mathbf{A}^{-1}$. I will denote the identity matrix by $\mathbf{1}$ and the null matrix (and null vector, which one will be clear from context) by $\mathbf{0}$. Vectors are column vectors ($N \times 1$ matrices). I will write row vectors ($1 \times N$ matrices) as transposes, as $\mathbf{u}^{\mathrm{T}}$.

I will use a dot to indicate the total derivative with respect to time, as

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}$$

This means to differentiate each component of $\mathbf{x}$ with respect to time to form a new vector. It is frequently the case that $\mathbf{x}$ is not an explicit function of time, but that the time derivative exists because $\mathbf{x}$ depends on other variables that are themselves functions of time. For example

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}(q_1(t), q_2(t))}{dt} = \frac{\partial \mathbf{x}}{\partial q_1}\dot{q}_1 + \frac{\partial \mathbf{x}}{\partial q_2}\dot{q}_2$$

This is an example of the chain rule, and it will occur throughout the text.

I denote matrix multiplication by writing the matrix and vector next to each other with no symbol. For example, the vector differential equations governing the behavior of some dynamical system that has been written in state space form are

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{1.1a}$$

where $\mathbf{x}$ denotes an N-dimensional (column) vector, $\mathbf{A}$ and $\mathbf{B}$ are constant matrices, and $\mathbf{u}$ denotes an M-dimensional input vector. $\mathbf{A}$ is an $N \times N$ square matrix. Note that $\mathbf{B}$ is an $N \times M$ matrix. In the case of single-input systems, it becomes a vector $\mathbf{b}$ (an $N \times 1$ matrix), and Eq. (1.1a) becomes Eq. (1.1b)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{1.1b}$$

1.2 Vibrations

Everyone is familiar with vibration, from the annoying hum that can come from electronic devices (or a refrigerator) to the devastation that earthquakes can cause. Vibration is essentially oscillatory motion. Something moves up and down or back and forth (or torsionally). We speak of the frequency or frequencies of vibration. These can be clear, as in the vibration of a guitar string, or jumbled as in the roar of a subway train. We will learn that jumbled vibrations can be decomposed into a set of simple vibrations using *Fourier series*. Guitar strings are, of course, continuous, and I will discuss them in Chap. 4, together with two other simple continuous systems. For something to vibrate it needs to have inertia and some restoring force, and all real systems have mechanisms that dissipate energy, something that may be described in general as *friction*. Actual friction is complicated, and I will take that up in Chap. 2. We will generally limit ourselves to viscous friction, for which the frictional force is proportional to the speed of the motion. This is a good approximation for damping devices such as shock absorbers.

The pendulum can be viewed as a vibrating system. A simple pendulum has its mass concentrated in the bob. (See Fig. 1.2.) The mass of the rod can often be neglected, so the inertia component is simply the mass of the bob. The restoring force is gravity. If the pendulum moves from its straight-down equilibrium position, gravity pulls down on it. There may be some friction at the pivot, which will dissipate energy. The pendulum is also a nonlinear system, but we can analyze it in a linear limit. We can describe the position of the pendulum by an angle θ measured counterclockwise from the vertical. The linear analysis is valid when this angle is small. I will discuss the mathematical formulation and an ad hoc linearization of this simple problem in the next chapter.

Any one degree of freedom vibrating system has a single frequency, and the system can be reduced to an effective mass and an effective spring, so we will spend some time on mass-spring systems. The dashpot (shock absorber) is an ideal frictional element (its force proportional to velocity, as noted above), and we will learn how to introduce these when modeling mechanical systems. Figure 1.3 shows an abstract model of a one degree of freedom system. The object labeled c is our symbol for a dashpot, our ideal dissipation element. The system has one degree of freedom because the mass can only move back and forth. This is not explicitly shown in the diagram, but understood. We will discuss this model at length in the next chapter.

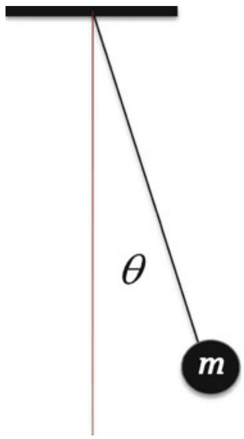


Fig. 1.2 A simple pendulum

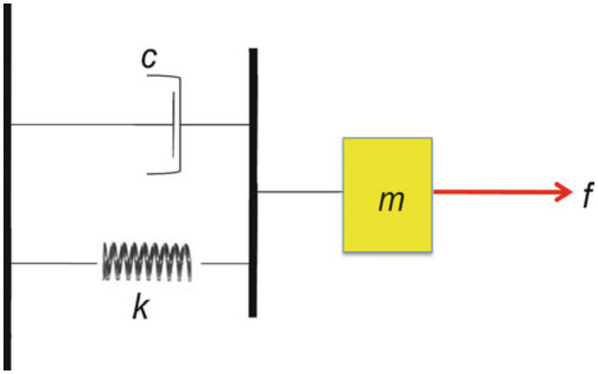


Fig. 1.3 A model of a one degree of freedom system: *m* denotes the mass (inertia term), *k* a spring constant (energy storage), and *c* a damping constant (energy dissipation). The input here is a force *f* and the output might be the position or velocity of the mass with respect to its equilibrium (undisturbed) position

1.3 Control

Everyone is also familiar with control, but not necessarily with the technical language used to describe it. Control is the way that we make mechanical (and thermal) systems do our bidding. A thermostat controls a furnace with the goal of keeping a living space at a constant temperature. A driver controls a vehicle by pushing on the accelerator and brake and turning the steering wheel. A cruise control system adjusts the fuel flow in a vehicle with the goal of maintaining a uniform speed. A batter controls a baseball by hitting it with a bat. One can divide controls and their associated systems in several ways: linear vs. nonlinear, open loop vs. closed loop, single input vs. multi-input, and single output vs. multi-output. *Linear* means that the associated mathematical model is linear. *Open loop* means that we work out the "best" force we can and apply it to the system to make it do what we want, but we do not check to see how well it works. *Closed loop* means that we monitor the system (sometimes just the output) and correct the

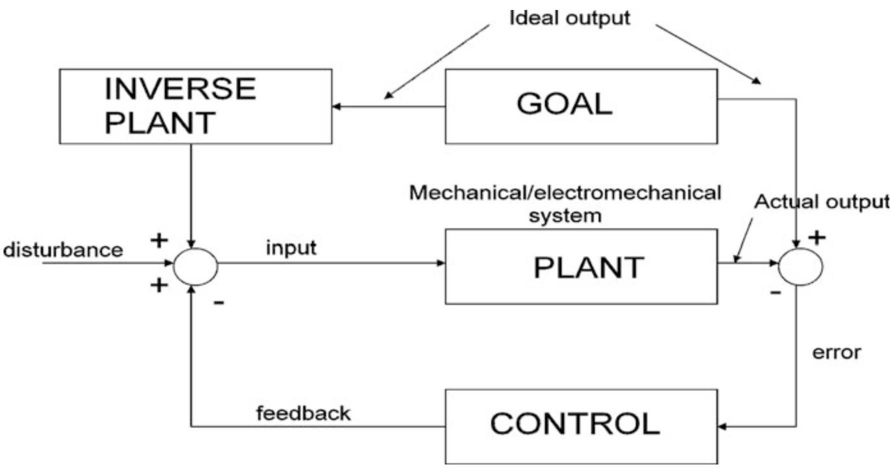


Fig. 1.4 Mechanisms, vibration, and control

input if the output is not behaving as we want. This is also called *feedback control*. This is the heart of the control analysis presented in this text. The thermostat is a closed-loop single-input single-output (SISO) nonlinear system. The car and driver is a closed-loop multi-input multi-output (MIMO) system that may be modeled as a linear or nonlinear system. The cruise control is a closed-loop SISO system that may be modeled as a linear or nonlinear system. The batter and the ball is an open-loop single-input nonlinear system. (The output is the position of the ball, but whether this is a single- or a multi-output system depends on what you want the ball to do. Think about it.) This text focuses on closed-loop single-input linear systems—feedback control. We will not focus all that much on the number of outputs. When we do we will generally look at single-output systems so that we can make use of the immense amount that is known about SISO systems. This sounds like an enormous restriction, but a surprisingly large number of practical engineering systems can be modeled in this manner, as we will see as we go along.

Figure 1.4 shows a schematic relation among the topics of the text: mechanical systems, vibrations, and control. The *plant* is the mechanical system to be controlled. It's a model of the actual system, and, as such, it covers both modeling of mechanical systems and their vibrations. The *goal* is what we want the plant to do. The *inverse plant* takes the goal and figures out what input(s) would lead to that goal. This is the open-loop component of the control. The error is the difference between the desired and actual behavior of the plant. This error is used to modify the input(s). This is the closed-loop part of the control. The second half of the book is devoted to *feedback control*, where we use the error to modify the input.

1.4 Energy

The concept of energy, kinetic and potential, is very useful in dynamics. These energies are fundamental to the Euler-Lagrange formulation. The kinetic energy of an infinitesimal particle is

$$dT = \frac{1}{2}dmv^2 = \frac{1}{2}\rho v^2 dV$$

The kinetic energy of a finite body can be obtained by integrating this over the body (see Gans 2013 for one approach). The result for a single object confined to a plane (as all our mechanisms will be [until Chap. 11]) is

$$T = \frac{1}{2}m\left(\dot{y}^2 + \dot{z}^2\right) + \frac{1}{2}I_x\dot{\theta}^2 \tag{1.2}$$

where y and z denote the Cartesian coordinates of the center of mass, and θ denotes the angle between some identifiable line on the object and a line defined in space. The parameters m and I_x denote the inertial terms the mass of the object and its moment of inertia about the x axis, supposed perpendicular to the plane containing the object. I generally evaluate the moment of inertia at the center of mass. I will drop the x subscript from now on. The kinetic energy of a mechanism is the sum of the kinetic energies of its parts. Of course, the coordinates of the different elements of a mechanism will be related by whatever constraints are applied to make the collection of elements into a mechanism. This is best understood in context.

I will often define the potential energy of an equilibrium condition to be zero. (The absolute value of the potential energy is not important. All we care about is the change in potential energy as the mechanism moves, so we are at liberty to choose coordinates to put the zero of potential energy wherever it is convenient for us.) The potential energy of any other configuration is equal to the work done in getting from the equilibrium configuration to the actual configuration. The two kinds of potential energy that will recur throughout this text are gravitational energy and spring (elastic) energy. If we suppose z to increase in the direction opposite to gravity, and define $z = 0$ to correspond to the equilibrium position of some mass, then the gravitational potential energy of the mass will be

$$V = \int_0^{s'} \mathbf{f} \cdot d\mathbf{s} = \int_0^z mgd\varsigma = mgz \tag{1.3}$$

The force required to move the object must be equal and opposite to the gravitational force trying to move the object down. A higher mass has more potential energy than the same mass at a lower height. We know that when we drop a mass from some height it accelerates at a rate $-mg$, so that its position as a function of time is

$$z = z_0 + w_0 t - \frac{1}{2} g t^2$$

assuming its initial position to be at z_0 and its initial speed to be w_0, typically zero. Its kinetic energy (for $w_0 = 0$) will be

$$T = \frac{1}{2} m (w_0 - gt)^2 \Rightarrow \frac{1}{2} m g^2 t^2$$

Its potential energy is

$$V = mgz = mg\left(z_0 + w_0 t - \frac{1}{2} g t^2\right) \Rightarrow mg\left(z_0 - \frac{1}{2} g t^2\right)$$

The initial potential energy is mgz_0 and the initial kinetic energy is zero. The potential energy vanishes at $t^2 = 2z_0/g$, at which time the kinetic energy is mgz_0, the same as the initial potential energy. Since z_0 is arbitrary, we see that energy is conserved by a falling object in the absence of any dissipative mechanism: $T + V = mgz_0$.

The force required to move a linear spring is ky, where k denotes the spring constant, and I generally choose the origin of y such that the spring is relaxed when $y = 0$. The potential energy can then be calculated as

$$V = \int_0^{s'} \mathbf{f} \cdot d\mathbf{s} = \int_0^{y} k y d\varsigma = \frac{1}{2} k y^2 \tag{1.4}$$

The work done compressing the spring is the same as that done stretching the spring. The fundamental equations for the energies are then

$$\begin{aligned} T &= \frac{1}{2} m\left(\dot{y}^2 + \dot{z}^2\right) + \frac{1}{2} I_x \dot{\theta}^2 \\ V_{\mathrm{g}} &= mgz \\ V_{\mathrm{k}} &= \frac{1}{2} k (y - y_0)^2 \end{aligned} \tag{1.5}$$

where I have reintroduced the specific relaxed length of the spring, which can be important in some contexts. I will usually choose coordinates such that $y_0 = 0$.

1.5 Scaling: Nondimensional Equations of Motion

We will be dealing with actual physical systems. There will be physical parameters, and all of our answers will have dimensions. It is often convenient to work in more generality to work problems that are independent of the specific physical problem being addressed. We can do this by writing the problem in terms of dimensionless quantities, solving the dimensionless problem, and then transforming the problem back to the physical domain where it was first posed. This way we can sometimes solve a whole class of problems once for all. Think of this as making a universal scale model of the problem. The details will vary with every problem, but this is an important technique, and so I am introducing the ideas here, using a simple problem.

We will learn in Chap. 2 that the equation of motion for the system shown in Fig. 1.3 is

$$m\ddot{y} + c\dot{y} + ky = f \tag{1.6}$$

We can make a scale model by supposing that the mass, length, and time scales can be chosen to be somehow representative of the problem. Here, for example, the mass is a good choice for the mass scale. This makes the dimensionless mass for this problem equal to unity. There may be a length scale and a time scale, but we cannot pick them out quite so obviously from the figure, so let us suppose that they exist and see what that does. Denote the length scale by L and the time scale by T. Write

$$y = Ly', \quad t = Tt' \tag{1.7}$$

You can think of this as the same as writing $y = y'$ meters and $t = t'$ seconds. Instead of meters and seconds, I have written L and T. We want the derivative with respect to time to be a derivative with respect to t' so that the entire system can be written in terms of the scaled variables. The following chain should be reasonably clear:

$$\frac{dy}{dt} = T\frac{dy}{dt'} \Rightarrow \frac{dy}{dt'} = \frac{1}{T}\frac{dy}{dt} \tag{1.8}$$

Every time derivative in the dimensional formulation corresponds to $1/T$ times the nondimensional time derivative. Substituting all of this into Eq. (1.6) gives

$$m\frac{L}{T^2}\ddot{y}' + c\frac{L}{T}\dot{y}' + kLy' = Ff' \tag{1.9}$$

where I have introduced a force scale F. The reader will note that I am using a dot to denote differentiation with respect to t' as well as differentiation with respect to t. It should be easy to figure out which is meant from context. I can make this equation dimensionless by dividing through by the dimensions. It is traditional to do this using the expression in the second derivative term, giving

$$\ddot{y}' + c\frac{T}{m}\dot{y}' + \frac{k}{m}T^2 y' = \frac{FT^2}{mL}f' \tag{1.10}$$

The parameters multiplying the primed variables are dimensionless, and we have the freedom to choose L, T, and F as we wish. One choice is to put $F = mL/T^2$, implying that the force is of the same size as the mass times the acceleration, an intuitively defensible choice. If we expect the acceleration to be controlled by the spring, then we can choose $T^2 = m/k$. (As we will learn in Chap. 2, the frequency of oscillation of a simple mass-spring system is $\sqrt{(k/m)}$, so this time scale is consistent with that observation.) Eq. (1.10) becomes

$$\ddot{y}' + \frac{c}{\sqrt{km}}\dot{y}' + y' = f' \tag{1.11}$$

The system has been reduced to a one parameter system, the behavior of which is controlled by a scaled dissipation parameter

$$2\zeta = \frac{c}{\sqrt{km}} \tag{1.12}$$

The factor of two is conventional. I will discuss this scaling for the dissipation in Chap. 2. Right now I want to leave you comfortable with the idea of choosing scales for mass, length, and time with a view to writing a set of equations in dimensionless form, with fewer parameters than the original dimensional set.

1.6 Complex Numbers and Trigonometric Functions

Complex numbers arise when one tries to find the roots of polynomials of degree higher than one. For example, what is the solution of $x^2 + 1 = 0$? We need a number which, when squared, is equal to -1. There is no such real number, so we define a symbol with that property. Mathematicians and physicists usually call it i, and engineers usually call it j. I will adopt the engineering usage:

$$j^2 = -1 = (-j)^2,$$

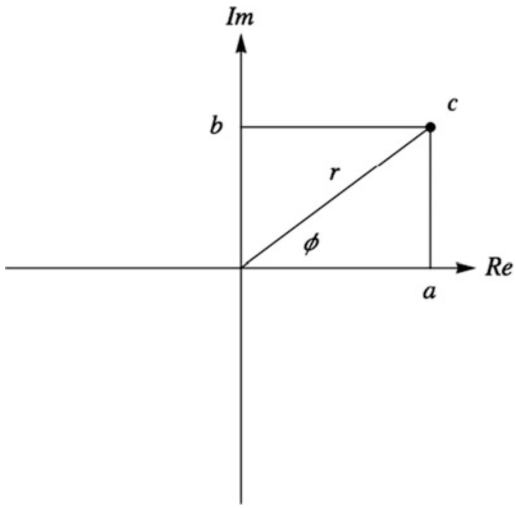
and the two roots of $x^2 + 1 = 0$ are $\pm j$. A general complex number $c = a + jb$. a is called the *real part* of c and b is called the *imaginary part* of c. Note that the imaginary part of c is a real number. Addition and subtraction of complex numbers is straightforward—one adds the real and imaginary parts separately. Multiplication requires an extra step and division two extra steps. In summary addition

$$c_1 = a_1 + jb_1, \quad c_2 = a_2 + jb_2$$
$$c_1 + c_2 = (a_1 + a_2) + j(b_1 + b_2)$$

and subtraction is obvious. Multiplication

$$c_1c_2 = (a_1 + jb_1)(a_2 + jb_2) = a_1a_2 - b_1b_2 + j(a_1b_2 + a_2b_1)$$

Fig. 1.5 The complex plane



and division

$$\frac{c_1}{c_2} = \frac{(a_1 + jb_1)}{(a_2 + jb_2)} = \frac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)} = \frac{a_1 a_2 + b_1 b_2 - j(a_1 b_2 - a_2 b_1)}{a_2^2 + b_2^2}$$

The quantity $a_2 - jb_2$ is called the *complex conjugate* of c_2 and often symbolized by c_2*. We find the complex conjugate of any complex number by changing the sign of the imaginary part. The product of a complex number and its complex conjugate is real and is the square of the *magnitude* of the complex number. The denominator in the expression for c_1/c_2 is the square of the magnitude of c_2. This language is suggested by the graphical representation of complex numbers in what is called the *complex plane*, a plane in which the horizontal axis represents the real part and the vertical axis the imaginary.

Figure 1.5 shows a complex number c plotted on the complex plane: r is called its magnitude and ϕ is its phase. A complex number can be written in terms of its magnitude and phase

$$c = r\cos\phi + jr\sin\phi$$

There is a connection between complex exponential and trigonometric functions that we will exploit later. We write

$$e^{j\phi} = \cos\phi + j\sin\phi$$

You can prove this statement by writing the Taylor series for the exponential, sine, and cosine and, using the properties of j, show that the left- and right-hand sides of this equation are identical. The series are absolutely convergent for all values of the argument, so the identity is established. We see by comparing the two equations that any complex number can be written in *polar notation*

$$c = r\cos\phi + jr\sin\phi = re^{j\phi}$$

We see that this allows us to write a complex exponential in terms of a real exponential and trigonometric functions

$$\exp(a + jb) = e^{a}(\cos b + j\sin b)$$

Polar notation makes it easy to calculate powers of complex numbers

$$c^{k} = \left(re^{j\phi}\right)^{k} = r^{k}e^{jk\phi}$$

This of course includes fractional powers. If $k = 1/n$, then we must have n roots, and we can obtain them by looking n equivalent expressions for c

$$c = re^{j\phi}, re^{j(\phi+2\pi)}, re^{j(\phi+4\pi)}, \ldots, re^{j(\phi+2(n-1)\pi)},$$

and the n roots of c will be

$$c^{\frac{1}{n}} = r^{\frac{1}{n}}e^{j\phi/n}, r^{\frac{1}{n}}e^{j(\phi+2\pi)/n}, r^{\frac{1}{n}}e^{j(\phi+4\pi)/n}, \ldots, r^{\frac{1}{n}}e^{j(\phi+2(n-1)\pi)/n}$$

The sequence terminates naturally, because the next term in the sequence would be equal to the first term ($\exp(2\pi j) = 1$). The five fifth roots of -1 are

$$\frac{1+\sqrt{5}}{4} + j\sqrt{\frac{5-\sqrt{5}}{8}}, \quad \frac{1-\sqrt{5}}{4} + j\sqrt{\frac{5+\sqrt{5}}{8}}, \quad -1, \quad \frac{1-\sqrt{5}}{4} - j\sqrt{\frac{5+\sqrt{5}}{8}}, \quad \frac{1+\sqrt{5}}{4} - j\sqrt{\frac{5-\sqrt{5}}{8}}$$

I plot them in Fig. 1.6. They come in conjugate pairs, and the angle between each root is $2\pi/5$.

We will use this later, when we will be able to write a displacement, say, as the real part of a complex displacement. We will be able to do the same with forces. The real part of a complex number is equal to half the sum of the number and its complex conjugate. I will apply this to write real forces in complex form to make solving real problems in complex space easier. It is not clear now why I might want to do that, but it will become obvious as we go forward.

We can represent real, physical quantities as the real parts of complex quantities. If we have an oscillatory displacement $y = d\cos(\omega t)$, we can represent it as the real part of $d\exp(j\omega t)$. The complex velocity will be the derivative of the complex displacement, $v = j\omega d\exp(j\omega t)$, and the complex acceleration will be $-d\omega^{2}\exp(j\omega t)$.

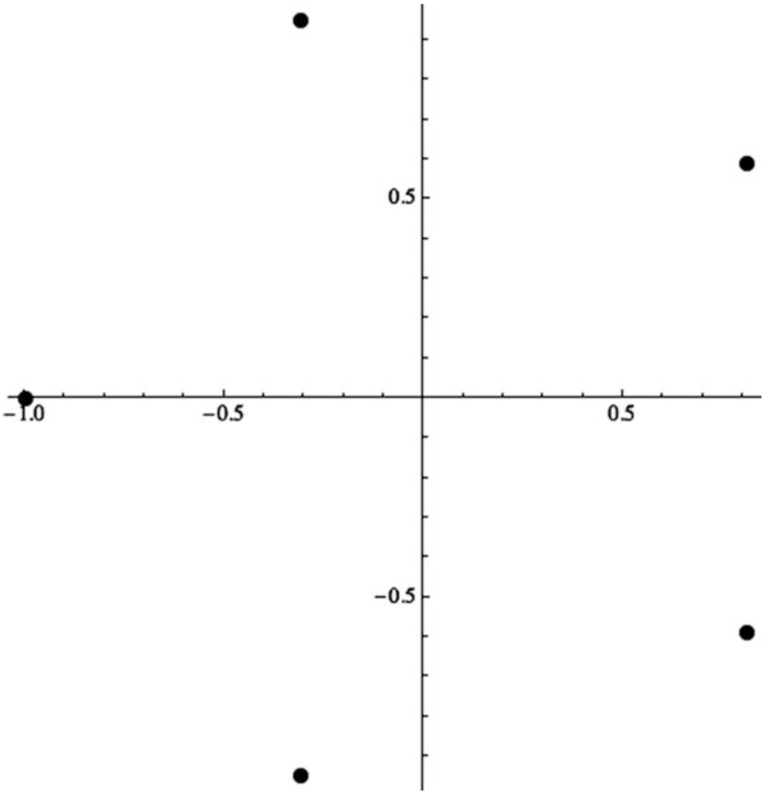


Fig. 1.6 The five fifth roots of −1

This is useful because the real part of the complex velocity is equal to the real, physical velocity, and the real part of the complex acceleration is equal to the real, physical acceleration. We can write this in equation form

$$\begin{aligned} y &= \mathrm{Re}\big(d\exp(j\omega t)\big) = d\cos\left(\omega t\right) \\ v &= \mathrm{Re}\big(j\omega d\exp(j\omega t)\big) = \mathrm{Re}\big(\omega d\big(j\cos\left(\omega t\right) - \sin\left(\omega t\right)\big)\big) = -\omega d\sin\left(\omega t\right) \\ a &= \mathrm{Re}\big(-\omega^2\exp(j\omega t)\big) = \mathrm{Re}\big(-\omega^2 d\big(\cos\left(\omega t\right) + j\sin\left(\omega t\right)\big)\big) = -\omega^2 d\cos\left(\omega t\right) \end{aligned} \tag{1.13}$$

This means that we can work entirely in the complex world and simply take the real part when we are done. We will find this very useful for analysis of both vibrations and controls. We have seen that we can write a complex number in the form $r\exp(j\theta)$. If $\theta = \omega t$, then differentiation of the form rotates the vector representing the function by $\pi/2$ in the counterclockwise direction and multiplies the length by (here) ω. Figure 1.7 shows Eq. (1.13) in graphical form for $\omega = 2$, illustrating the effect of differentiation.

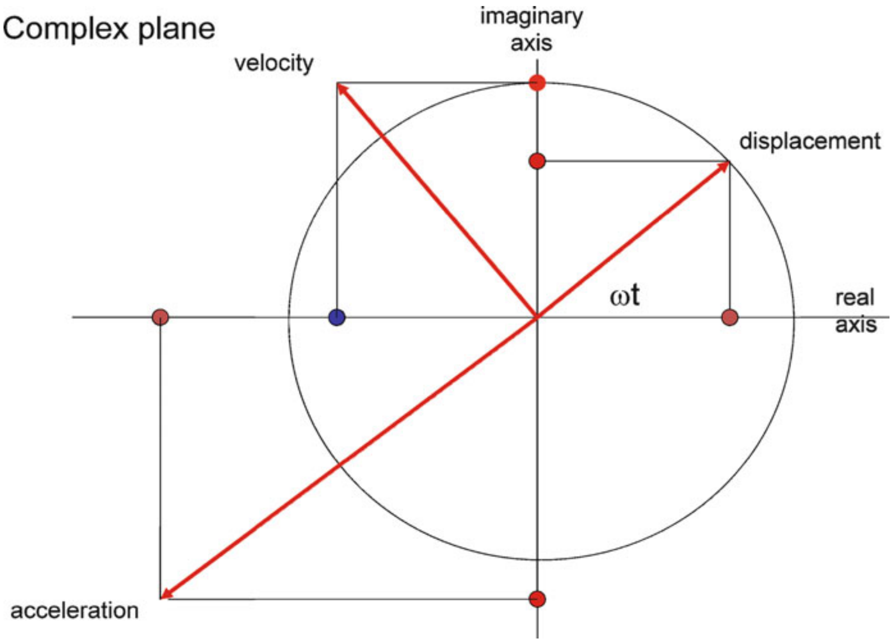


Fig. 1.7 Complex displacement, velocity, and acceleration on the complex plane

1.6.1 Harmonic Functions and Periodicity

Sines and cosines are harmonic by definition. We say that $\sin(\omega t + \phi)$ is a *harmonic function*. Here ω denotes the frequency and ϕ the phase. The connection between the trigonometric functions and the exponential functions means that $\exp(j\omega t + \phi)$ is also a harmonic function. Harmonic functions are periodic. Their period $T = 2\pi/\omega$. A periodic function duplicates itself exactly every period. All harmonic functions are periodic, but not all periodic functions are harmonic.

Linear combinations of harmonic functions can be periodic, but they are not harmonic. The period of a combination of harmonic functions is longer than the individual periods of the contributing harmonic functions. Consider a pair of harmonic functions and write

$$f(t) = a_1 \sin(\omega_1 t) + a_2 \sin(\omega_2 t) \tag{1.14}$$

The first function repeats itself every $T_1 = 2\pi/\omega_1$ time units and the second every $T_2 = 2\pi/\omega_2$ time units. The function will be periodic if there exist numbers m and n such that $nT_1 = mT_2$. We write

$$\frac{n}{m} = \frac{T_2}{T_1} = \frac{\omega_1}{\omega_2} \tag{1.15}$$

and seek the smallest values of n and m that satisfies the relation. From the point of view of a mathematician this equation can only be satisfied if the ratio on the right-hand side is a rational number. From an engineering perspective, we cannot distinguish between a rational and an irrational fraction by measurement, so we can suppose that the sum Eq. (1.14) is periodic. If n and m are large, then the function may have such a long period as not to be observable. Periods greater than the length of time we can observe a system are not accessible.

If the two frequencies in Eq. (1.14) are close together, then we can observe a phenomenon called beats. Let's rewrite Eq. (1.14) in terms of the mean and difference of the two frequencies

$$\omega_1 = \overline{\omega} + \delta\omega, \quad \omega_2 = \overline{\omega} - \delta\omega \tag{1.16}$$

where δ denotes the difference and the overbar the mean. The multiple angle formulas

$$\begin{aligned}\sin(\omega t + \phi) &= \sin(\omega t)\cos\phi + \cos(\omega t)\sin\phi \\ \cos(\omega t + \phi) &= \cos(\omega t)\cos\phi - \sin(\omega t)\sin\phi\end{aligned}$$

allow me to write the sum shown in Eq. (1.16) as

$$\sin(\omega_1 t) + \sin(\omega_2 t) = 2\sin(\overline{\omega} t)\cos(\delta\omega t) \tag{1.17}$$

This looks like a function that oscillates at the mean frequency while being modulated at the difference frequency. Figure 1.8 shows 100 periods (200π) of the high frequency of Eq. (1.17). One can see the beat frequency, and about two and one half of these appear in the figure.

The difference frequency is 0.025 and its period is $2\pi/0.025 = 80\pi$, so there will be two and one half long periods in 200π, as shown in the figure. The period of the entire function according to Eq. (1.15) comes from

$$\frac{n}{m} = \frac{0.975}{1.000} = \frac{975}{1,000} = \frac{39}{40},$$

and so it is 78π, a little less than the period defined by the difference, but significantly larger than either input period, 2π and 2.05π.

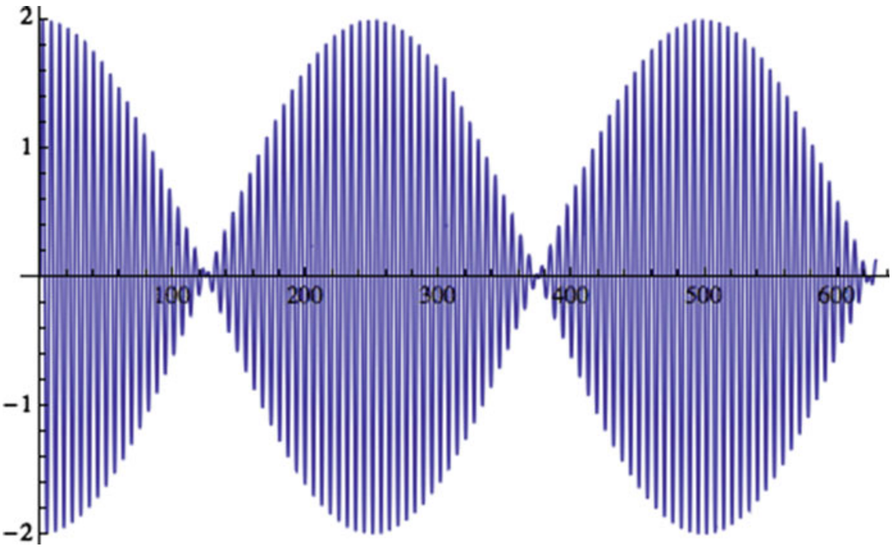


Fig. 1.8 Response of the function in Eq. (1.13) for $\omega_1 = 1$ and $\omega_2 = 0.975$

Exercises

1. Write the kinetic and potential energies of a body falling freely under gravity. Denote the position of the mass by z. Let the mass of the body be a mass scale. Choose a length scale L. Let the energy scale be mgL. Find a time scale such that

$$T' = \frac{1}{2}\dot{z}'^2$$

where the prime denotes the dimensionless quantity.
2. If the length scale chosen in exercise 1 is the diameter of the body, supposed spherical, what physical interpretation can you give to the time scale?
3. Newton's law of cooling says that the rate of change of the temperature of a body is proportional to the difference in the temperature of the body and its surroundings, supposed to be constant. This can be expressed in by a simple linear differential equation

$$\frac{dT}{dt} = -k(T - T_0)$$

where T denotes the temperature, T_0 the temperature of the surroundings, t the time, and k a physical constant with the dimensions of inverse time. Choose a temperature scale and a time scale that will reduce the problem to

$$\frac{dT'}{dt'} = -\left(T' - 1\right)$$

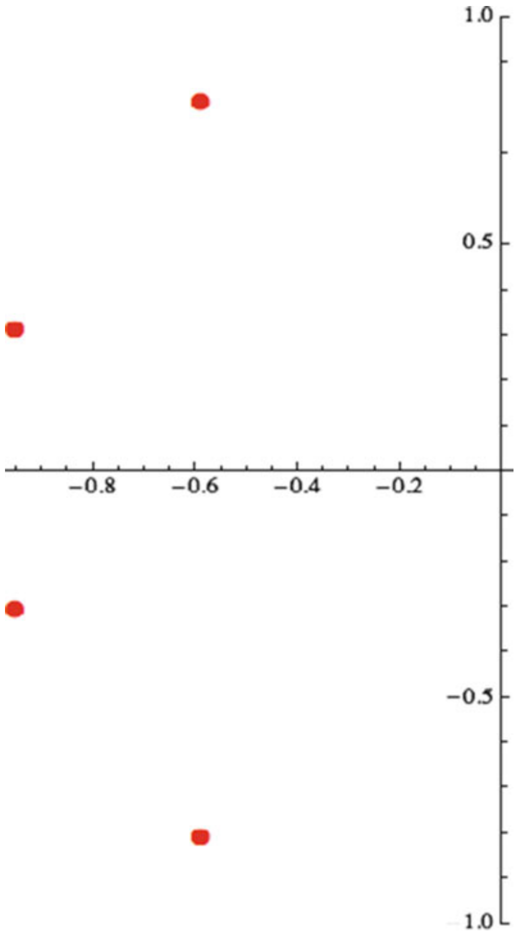
where again the prime denotes the dimensionless variables.

4. Show that the real part of a complex number is equal to the sum of the number and its complex conjugate.
5. Find the three cube roots of each of the following: -1, $1+j$, j, $-j$, $1-j$.
6. Suppose some displacement y can be written,

$$y = y_0 \sin(\omega t + \phi)$$

y_0 denotes the amplitude (peak to peak amplitude is $2y_0$), ω the phase, and ϕ the phase angle.
   (a) Find real numbers A and B such that $y = A\cos(\omega t) + B\sin(\omega t)$
   (b) Find a complex amplitude Y such that $y = \mathrm{Re}(Ye^{j\omega t})$
   (c) Find a complex amplitude Y such that $y = Ye^{j\omega t} + Y * \mathrm{e}^{-j\omega t}$, where the asterisk denotes complex conjugate.
   (d) Discuss the difference between the solutions to parts (b) and (c).
7. Find y if its complex representation is
   (a) $y = y_0 j\mathrm{e}^{j\omega t}$
   (b) $y = y_0(1+j)\mathrm{e}^{-j\omega t}$
   (c) $y = y_0\mathrm{e}^{j(\omega t+\phi)}$
   (d) $y = y_0\mathrm{e}^{-ky+j\omega t}$
   (e) $y = -y_0 j\mathrm{e}^{-(k+j\omega)t}$
   All the constants in these expressions are real.
8. Write the complex forms (proportional to $\exp(j\omega t)$) of the following:
   (a) $y = y_0 \cos(\omega t)$
   (b) $y = y_0 \cos(\omega t + \phi)$
   (c) $y = y_0 \mathrm{e}^{-kt} \sin(\omega t)$
   (d) $y = y_0 \sin(\omega t - \phi)$
9. Find the beat frequencies for the following pairs of harmonic functions:
   (a) $\sin(17t) + \sin(18t)$
   (b) $\sin(226t) + \sin(227t)$
   (c) $\cos(23t) + \sin(25t)$
   (d) $\sin(40t + \pi) + \cos(39t + \pi/4)$
10. Consider problem 6(b). Plot the displacement y and the velocity and the acceleration for the complex $y = Y\exp(j\omega t)$ at $\omega t = \pi/7$.
11. Find four complex numbers of unit amplitude arranged in the left half plane (imaginary parts less than zero) equally spaced in angle between $\pi/2$ and $3\pi/2$ as shown in the figure

(Exercise 11)



12. Repeat exercise 11 for ten such numbers.

Exercises 13–22: review some linear algebra.

Exercises 13–17: solve the matrix equation $\mathbf{Ax} = \mathbf{b}$ for the matrices and vectors given.

13.
$$\mathbf{A} = \begin{Bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{Bmatrix}$$

14.
$$\mathbf{A} = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{Bmatrix}$$

15.
$$\mathbf{A} = \begin{Bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

16.
$$\mathbf{A} = \begin{Bmatrix} 0 & 2 & 3 \\ 4 & 0 & 6 \\ 7 & 8 & 0 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 1 \\ 1 \\ 4 \end{Bmatrix}$$

17.
$$\mathbf{A} = \begin{Bmatrix} 0 & 2 & 3 \\ 4 & 0 & 4 \\ 7 & 2 & 0 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 2 \\ 1 \\ 0 \end{Bmatrix}$$

18–22. Show that the transpose of the product **Ab** is equal to $\mathbf{b}^T\mathbf{A}^T$ for the five sets of **A** and **b** in Exs. 13–17.

The following exercises are based on the chapter, but require additional input. Feel free to use the Internet and the library to expand your understanding of the questions and their answers.

23. Discuss the output(s) for the bat and ball system if the goal is to hit a home run. What does the bat have to impart to the ball?
24. The speed of a car on level ground is determined primarily by fuel flow and wind resistance. (Friction is small compared to wind resistance at highway speeds.) Identify the input(s) and output(s) for the speed of a car. Can you write a differential equation for this problem? (We'll come back to this later in the text.)
25. The Roomba™ is an autonomous vacuum cleaner. Discuss any control issues.
26. Discuss input and output issues (and sensors) for an autonomous vehicle operating in real traffic. These vehicles exist and being actively refined here in the early twenty-first century.
27. Discuss the control issues for a bicycle.
28. Discuss the control issues for an aircraft autopilot.
29. Discuss the control issues for a unicycle.
30. Discuss the control issues for the Segway™.
31. Identify control issues—sensors and feedback—inherent in driving a car (not an autonomous vehicle, but a regular car where the driver is input and sensors).

References

Gans RF (2013) Engineering dynamics: from the lagrangian to simulation. Springer, New York

Press WH, Teukolsky SA, Vettering WT, Flannery BP (1992) Numerical recipes in C. The art of scientific computing, 2nd edn. Cambridge University Press, New York

One Degree of Freedom Systems

2

In which we explore many facets of the analysis of mechanical systems in the context of a simple one degree of freedom system. . .

2.1 Development

The basic building blocks for models of mechanisms are masses, springs, and dampers (sometimes called dashpots). Automobile shock absorbers and the piston in screen and storm door closers are common examples of dampers. The motion of a mass is governed by the forces applied to it. Figure 1.2 shows a fundamental one degree of freedom system. The mass can only move in the horizontal (y) direction.

The spring and the damper are in parallel. This is the normal configuration. Rotated 90° counterclockwise this could represent an automobile suspension unit—coil spring and shock absorber (damper) in parallel. I will discuss other configurations later. The diagram shows a mass motion forced by an external force. The mass can also be made to move if the support moves. I will address this possibility below.

2.1.1 An Aside About Friction

Friction is a dissipative mechanism. If you rub your hands together briskly, the friction between the two will make them warmer. This is an example of *sliding friction*, also known as *dry friction*. Sliding friction is common, but difficult to deal with analytically. The simplest useful model (which apparently dates back to Leonardo da Vinci) supposes that the force of friction is proportional to the normal force W between the two sliding objects. The proportionality constant μ is called the *coefficient of friction*. (da Vinci apparently believed it to be a universal constant equal to 1/4.) The laws of dry friction:

R.F. Gans, *Mechanical Systems: A Unified Approach to Vibrations and Controls*,
DOI 10.1007/978-3-319-08371-1_2

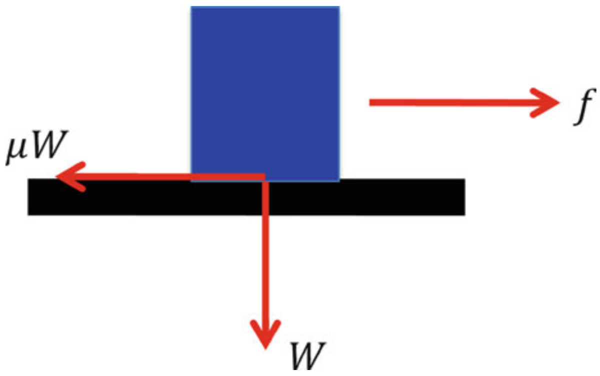


Fig. 2.1 Force balance for sliding friction

- The friction force is proportional to the applied load and independent of the contact area.
- The coefficient of sliding friction is independent of the speed of the motion.

are named for Amonton and Coulomb. Dry friction is often called *Coulomb friction*. Most models suppose that the coefficient of static friction μ_s is larger than the coefficient of sliding friction μ_d. The friction force always opposes the motion. Figure 2.1 shows a block and the forces involved.

The friction force will be less than μW if f is also less than μW. The friction force cannot induce motion; it can only impede motion. Figure 2.2 shows the friction force as a function of the applied force for this simple model. One can see the drop upon the commencement of motion and the subsequent constant friction force. The friction force adjusts to balance the applied force until it reaches its static limit $\mu_s W$, at which point the block starts to slide. The friction force drops because the coefficient of sliding friction is smaller than the coefficient of static friction, so the net force is positive and the block will accelerate.

It is relatively easy to measure two friction coefficients for this model using an inclined plane, as shown in Fig. 2.3. A simple static force balance gives the static friction coefficient

$$mg\sin\alpha = \mu mg\cos\alpha \rightarrow \mu_s = \tan\alpha \tag{2.1}$$

The angle α is often called the *friction angle*. The dynamic friction coefficient can be found by observing the fall of the block down the slope. I leave it to the exercises to show that

$$\mu_d = \mu_s - \frac{2s}{gT^2}\tan\alpha \tag{2.2}$$

where s denotes the distance slid in time T.

We cannot incorporate this model of friction into linear equations of motion. Dry friction does not depend on the motion linearly.

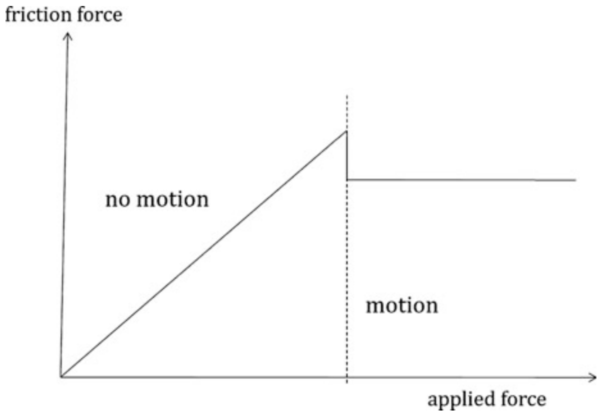


Fig. 2.2 Friction force as a function of applied force

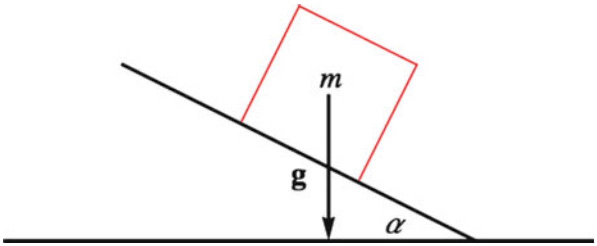


Fig. 2.3 An inclined plane for measuring friction coefficients

This does not mean that dry friction is unimportant. Dry friction is essential for stopping your car. Disc brakes work by forcing a pair of calipers to grip a disk rigidly attached to the wheel. The disk moves with respect to the calipers as long as the car is still moving, so the force on the disk depends on the coefficient of sliding friction. This friction force exerts a torque on the wheel, which causes it to slow down. The car slows because of the friction between the tire and the ground. Ultimately (as some tire commercials have pointed out) brakes don't stop your car: tires do. The patch of tire contacting the ground is stationary with respect to the ground (rolling without slipping). It can be approximated by a line contact for the present purposes. The appropriate coefficient of friction between the tire and the ground is the static coefficient. If you skid the appropriate friction coefficient becomes the coefficient of sliding friction, which is smaller, and your stopping power becomes less. Antilock brakes prevent this from happening, so you can stop more quickly.

The following discussion is a bit simplistic, but it will give a flavor of the design considerations involved in a braking system. The calipers are driven by a hydraulic cylinder. If the pad area is A, the pressure in the cylinder p and the appropriate coefficient of friction μ_1, then the force on the disk is

$$f = 2\mu_1 Ap$$

and the torque on the wheel is then r_D times this, where r_D denotes the effective radius of the contact points. The force on the ground will be this torque divided by the wheel radius r_W. This force cannot exceed the static friction force between the road and the tire, which can vary greatly depending on the condition of the road (and the tire). We can write this in equation form as

$$f = 2\mu_1 Ap \le \mu_2 N$$

where N denotes the normal force on the ground and μ_2 the coefficient of state friction between the tire and the road.

What happens if we interpose a layer of liquid between the block and the ground? The layer acts as a lubricant, and it provides no resistance to the initiation of motion. The resistance to motion parallel to the interface will be the integral of the shear stress in the liquid over the surface. (This leads to "aquaplaning" and accidents in the automotive application.) If the layer is thin, then the flow in the liquid will be laminar, and it can be approximated by plane Couette flow over most of the interface, the velocity varying linearly between the two surfaces. The stress is equal to the viscosity times the shear rate, which is constant for plane Couette flow. The shear rate is given by the speed divided by the thickness of the liquid layer. The resistance to motion parallel to the plane is thus proportional to the speed of motion and the contact area and inversely proportional to the thickness of the liquid layer. It is independent of the weight of the block. This is called *viscous friction*, friction proportional to the speed of motion. Any system where the resistance to motion is controlled by a liquid forced to pass through a narrow gap or opening where laminar flow is a good approximation will provide viscous friction to resist motion. Examples include lubricated bearings, shock absorbers, and screen door closers. The viscous friction approximation is convenient and a good approximation in many cases. It is amenable to linear analysis, which dry friction is not. I will use it more or less universally in this text; thus the model shown in Fig. 1.2 is an appropriate place to start our study of one degree of freedom problems.

2.1.2 The One Degree of Freedom Equation of Motion

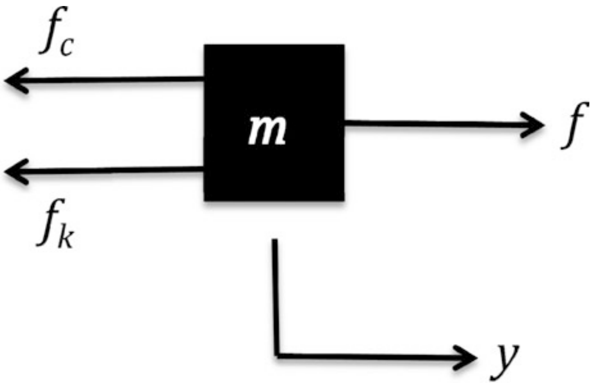
We can write a single differential equation governing the motion of the mass shown in Fig. 2.1 considering it to be a free body acted on by the force shown and the spring and damper forces. Figure 2.4 shows a free body diagram of the mass.

The rate of change of momentum is equal to the sum of the forces, giving an equation of motion

$$m\ddot{y} = f - f_k - f_c$$

where y denotes the departure from equilibrium, positive to the right in the figure, and f_k and f_c denote the spring force and the damper force, respectively. If the block

Fig. 2.4 Free body diagram corresponding to Fig. 1.2



moves to the right, the spring will be stretched, and it will exert a force on the mass to the left. I will suppose that the force in a linear spring is proportional to the displacement from equilibrium. The force it exerts on the mass is to the left, and the force it exerts on the wall is to the right. The equation for the mass becomes

$$m\ddot{y} = f - k(y - y_0) - f_c$$

where y_0 denotes the position of the mass when the spring is not stretched or compressed. If $y > y_0$ the force on the block is to the left and if $y < y_0$ the force is to the right. It is common to choose the origin for y such that $y_0 = 0$. I will do so here. There is no loss of generality.

The damper works the same way, except that the force is proportional to the speed of the mass. Thus we can write

$$m\ddot{y} = f - ky - c\dot{y} \Rightarrow \ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = \frac{f}{m} = a$$

The dimensions of k/m are $1/\text{time}^2$, and the dimensions of c/m are 1/time. We can introduce a *natural frequency*, ω_n, and *a damping ratio*, ζ, the latter dimensionless[1]

$$\frac{c}{m} = 2\zeta\omega_n, \quad \frac{k}{m} = \omega_n^2 \Leftrightarrow \omega_n^2 = \frac{k}{m}, \quad \zeta = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}} \tag{2.3}$$

and rewrite the one degree of freedom equation in standard form, where $a = f/m$ denotes the applied acceleration

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = a \tag{2.4}$$

[1] This is the parameter I introduced in Chap. 1 (Eq. 1.12).

Equation (2.4) contains all that one needs to understand the dynamics of linear one degree of freedom systems. The equation governing the motion of a mass driven by a moving support can be put in this form, as I will show below, as can the equation governing the small angle motion of a pendulum. Equation (2.4) is inhomogeneous, that is, $y = 0$ is not a solution. Specific problems are defined by adding initial conditions. A complete problem consists of a dynamical equation like Eq. (2.4) and a set of initial conditions. We'll have Eq. (2.4) and

$$y(0) = y_0, \quad \dot{y}(0) = v_0 \tag{2.5}$$

where y_0 and v_0 are constants, the position and speed of the mass at $t = 0$.

Let us attack the general problem, Eq. (2.4) subject to the conditions given in Eq. (2.5), in successively more complicated situations, starting with the unforced system ($a = 0$). Equation (2.4) becomes homogeneous. *Homogeneous equations with constant coefficients can always be solved in terms of exponential functions.* This is an important fact to remember. It applies to all systems of homogeneous differential equations with constant coefficients no matter the order of the individual equations or the number of equations. We learned in Chap. 1 that there is a connection between exponential and trigonometric functions. The solution to the simplest case, where there is no damping, can be found in terms of trigonometric functions alone.

2.2 Mathematical Analysis of the One Degree of Freedom Systems

2.2.1 Undamped Free Oscillations

Suppose $a = 0 = \zeta$. Equation (2.4) reduces to

$$\ddot{y} + \omega_{\mathrm{n}}^2 y = 0$$

There is no external forcing and no damping. The system is homogeneous. Since there is no damping, we expect any nontrivial ($y \neq 0$) solution to persist forever.

As noted above a differential equation by itself does not define a problem, but a class of problems. Its solution, the so-called general solution, has as many undetermined constants as the order of the equation. Side conditions, as many as the order of the differential equation, are needed to determine these constants. Here we have one second-order differential equation. It needs two side conditions, and these are usually taken to be the initial conditions—the value of y and its first derivative at the beginning of the motion. (If both are zero, there is no motion.) I suppose the problem to start from $t = 0$, so that Eq. (2.5) defines the initial conditions

$$y(0) = y_0, \quad \dot{y}(0) = v_0$$

The general solution of the differential equation can be written in terms of sines and cosines

$$y = A\cos(\omega_n t) + B\sin(\omega_n t) \tag{2.6}$$

Note that if we seek exponential solutions directly, $y = A\ \exp(st)$, the differential equation reduces to

$$s^2 A\exp(st) + \omega_n^2 A\exp(st) = 0 = \left(s^2 + \omega_n^2\right) A\exp(st)$$

so that we have a nontrivial solution if $s^2 = -{\omega_n}^2$ or $s = \pm j\omega_n$. We can use the connection between the exponential and the trigonometric functions given in Chap. 1 to convert the general exponential solution to the form of Eq. (2.6). This is only possible when s is purely imaginary. There is a similar transformation for complex values of s, which we will need when we have damping.

The solution given in Eq. (2.6) can also be expressed as

$$y = C\sin(\omega_n t + \phi) \tag{2.7}$$

where ϕ denotes a phase angle. You can verify either formula by direct substitution into the differential equation. To convert the form of Eq. (2.7) to that of Eq. (2.6), expand the sine using the usual multiple angle formulas

$$\begin{aligned} \sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{aligned}$$

to obtain

$$y = C\sin\phi\cos(\omega_n t) + C\cos\phi\sin(\omega_n t)$$

so that

$$A = C\sin\phi, \quad B = C\cos\phi \tag{2.8}$$

To convert from the form of Eq. (2.6) to that of Eq. (2.7), we can invert the process. We find directly from Eq. (2.8)

$$C^2 = A^2 + B^2, \quad \tan\phi = \frac{A}{B} \tag{2.9}$$

One has to be careful in calculating the phase. The inverse tangent is ambiguous. One can choose the appropriate quadrant by noting that

$$\sin\phi = \frac{A}{C}, \quad \cos\phi = \frac{B}{C}$$

The sine and cosine are both positive in the first quadrant ($0 < \phi < \pi/2$). The sine is positive and the cosine negative in the second quadrant ($\pi/2 < \phi < \pi$). The sine and cosine are both negative in the third quadrant ($\pi < \phi < 3\pi/2$). The sine is negative and the cosine positive in the fourth quadrant ($3\pi/2 < \phi < 2\pi$). The tangent is positive in the first and third quadrants and negative in the second and fourth quadrants.

The general solution becomes specific when the initial conditions are imposed, which is most easily done using the form in Eq. (2.6). We have

$$y(0) = A \Rightarrow A = y_0$$

$$\dot{y}(0) = \omega_n B \Rightarrow B = \frac{v_0}{\omega_n}$$

so that Eq. (2.10a)

$$y = y_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t) \tag{2.10a}$$

gives the general solution for unforced, undamped motion of a one degree of freedom system in terms of its initial conditions. This solution can also be written in terms of the amplitude and phase as in Eq. (2.7). It represents a sinusoidal response at the natural frequency, ω_n. Sinusoidal motion with a single frequency is called *harmonic motion*. Harmonic motion is periodic; not all periodic motion is harmonic. The response of an unforced, undamped single degree of freedom system is harmonic, representable in terms of sines and cosines of $\omega_n t$. The initial conditions determine A and B, hence C, and the amplitude and phase

$$\text{amplitude} = C = \sqrt{y_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}$$

$$\sin\phi = \frac{y_0}{\sqrt{y_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}}, \quad \cos\phi = \frac{v_0/\omega_n}{\sqrt{y_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}} \tag{2.11}$$

Equation (2.10a) can also be written in terms of exponential functions using the trigonometric-exponential correspondences given in Chap. 1.

$$\begin{aligned} y &= x_0 \frac{1}{2}\left(e^{j\omega_n t} + e^{-j\omega_n t}\right) + \frac{v_0}{\omega_n}\frac{1}{2j}\left(e^{j\omega_n t} - e^{-j\omega_n t}\right) \\ &= \frac{1}{2}\left(x_0 - j\frac{v_0}{\omega_n}\right)e^{j\omega_n t} + \frac{1}{2}\left(x_0 + j\frac{v_0}{\omega_n}\right)e^{-j\omega_n t} \end{aligned} \tag{2.10b}$$

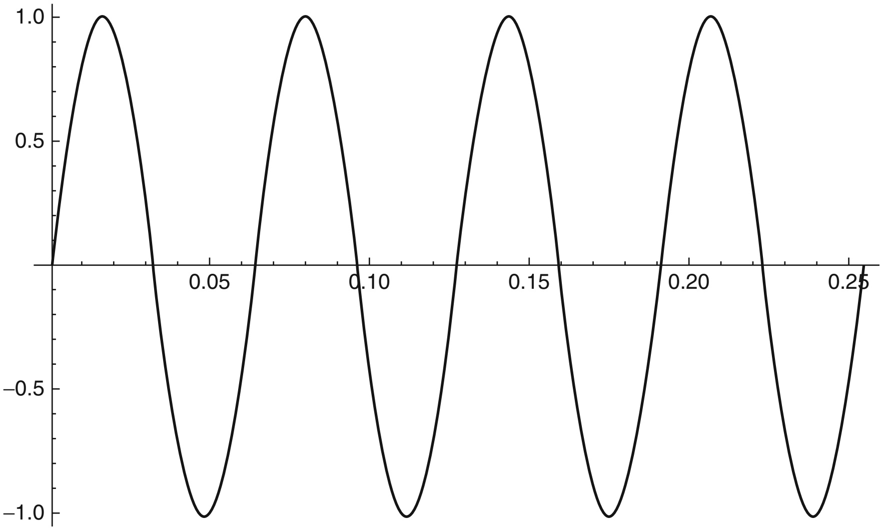


Fig. 2.5 Response of a one degree of freedom to an impulsive load (see text)

The two terms on the right are complex conjugates, so their sum is real, as it must be. We can solve differential equation in terms of complex exponentials and still arrive at physically meaningful real solutions. We'll see this shortly when we add dissipation to the homogeneous problem, but first let's look at a couple of examples.

Example 2.1 Response to an Impulse This simple one degree of freedom system seems a most artificial picture, but we can use it to examine the response of a simple system to an impulsive load. If we hit the system with a hammer some of its momentum will be transferred to the mass. In fact, if the hammer rebounds, then more than its momentum will be transferred. The hammer stops in a perfectly elastic collision, and all of the initial momentum of the hammer will be transferred to the mass. Suppose a 2 kg sledgehammer moving at 5 m/s collides elastically with a 10 kg mass attached to a 98,000 N/m spring. The momentum transferred will be $2 \times 5 = 10$ kg m/s. The velocity of the mass will jump "instantaneously" to 1 m/s, before the mass has a chance to move, so we can write our initial conditions as $v_0 = 1$ and $y_0 = 0$. The natural frequency of the system is $\sqrt{9{,}800} = 99$ rad/s (=15.8 Hz). The response of the system is then

$$y = \frac{1}{99}\sin(99t)\,\mathrm{m}$$

This is a harmonic response with an amplitude of about one cm. The frequency is a little below the threshold of human hearing.

Figure 2.5 shows a plot of the response over four periods. The amplitude is in cm and the horizontal scale in seconds. (In a real physical system there will be some damping, and the motion will decay. I will discuss this below.)

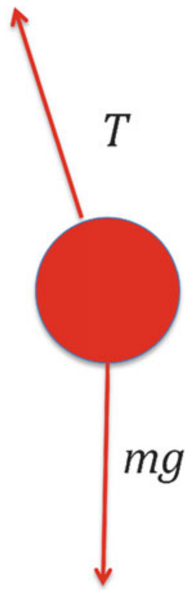


Fig. 2.6 Free body diagram of the bob of a simple pendulum

Example 2.2 The Simple Pendulum The simple pendulum is shown in Fig. 1.1. I denote the angle between the pendulum rod and the vertical by θ. This angle is zero when the pendulum hangs straight down and reckoned positive in the counter-clockwise direction ($\theta = \pi/2$ when the pendulum is extended horizontally to the right from its pivot point). The pendulum is confined to the plane. Let y denote the horizontal direction, positive to the right, and z denote the vertical, positive up following the convention I introduced in Chap. 1. Neglect the mass of the rod. The mass m, sometimes called the *bob*, is acted upon by two external forces: gravity in the $-z$ direction and a tension in the rod, directed parallel to the rod and in the upward direction, countering gravity. When the rod is vertical these two forces cancel and the pendulum is in static equilibrium. We can write two equations for the motion of the mass by drawing a free body diagram, Fig. 2.6.

We can resolve the forces in the y and z directions to give

$$\begin{aligned} m\ddot{y} &= -T\sin\theta \\ m\ddot{z} &= T\cos\theta - mg \end{aligned}$$

This is not the end of the story, because y and z are related. This is a one degree of freedom system; all the pendulum can do is swing back and forth along its circular arc. If we take the origin of the coordinate system to be at the base of the pendulum, where the rod attaches to the support, then we have

$$y^2 + z^2 = l^2$$

where l denotes the length of the rod. This expression can be parameterized in terms of the angle θ

$$y = l\sin\theta, \quad z = -l\cos\theta$$

The two differential equations become

$$ml\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = -T\sin\theta$$
$$ml\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) = T\cos\theta - mg$$

These can be combined into two equations, one of which determines θ as a function of time and the other the value of the tension as a function of time. Multiply the first by cosθ and the second by sinθ and add them to get the pendulum equation, Eq. (2.12a). Multiply the first by −sinθ and the second by cosθ and add them to get the tension equation, Eq. (2.12b).

We are primarily concerned with the first equation. It is unlikely that a simple pendulum will stretch or break the rod.

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{2.12a}$$

$$T = ml\dot{\theta}^2 + mg\cos\theta \tag{2.12b}$$

The equation for θ is nonlinear, but if θ remains small during the motion, then the sine can be replaced by θ. This is a common approximation that can be justified by the Taylor series for the sine, which is

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \cdots,$$

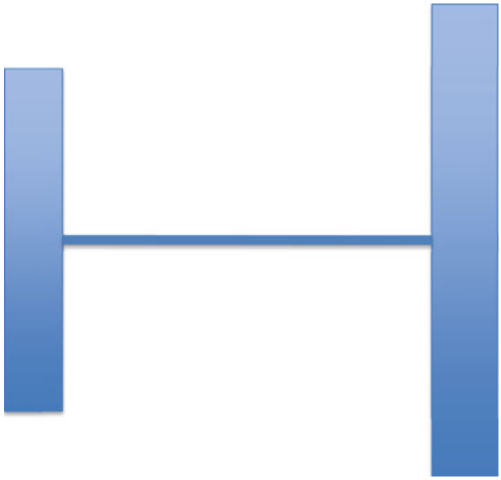
and if θ is small, θ^3 is much smaller than θ, and the higher order terms will be smaller still; thus sin$\theta \approx \theta$ will be a good approximation for small θ. (I will discuss a formal process of linearization in Chap. 3 and then again in Chap. 6.) Applying this approximation gives a linear equation exactly parallel to the mass-spring system, a realization of Eq. (2.4) with $\zeta = 0$ and $\omega_n{}^2 = g/l$

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

We find by analogy that the (radian) frequency of a simple pendulum is given by $\sqrt{(g/l)}$. Its circular frequency is this divided by 2π, and so the period of a simple pendulum is given by $2\pi\sqrt{(l/g)}$. The length of a pendulum with a one second period will be 248.5 mm.

The period of a simple pendulum is independent of the mass of the bob, a fact that Galileo observed in 1581 while he was a medical student in Pisa.

Fig. 2.7 A torsional system: moment-spring system



The mass-spring system and the pendulum are not the only systems that can be modeled by Eq. (2.4). The shafts in a gear train are under torsion. This is an elastic phenomenon, so the twist of a shaft away from equilibrium has an effective spring constant so long as the deformation remains in the elastic range. We can write the moment M associated with a given amount of twist as (see Crandall and Dahl 1959, or any equivalent strength of materials text)

$$M = \frac{GI_s}{l}\phi$$

where G denotes the shear modulus of the shaft material, I_s the polar moment of inertia of the shaft, l the length of the shaft, and ϕ the twist in radians. Consider the system shown in Fig. 2.7: two wheels connected by a shaft. Suppose the right-hand wheel to be fixed and consider the motion of the left-hand wheel. It may be subjected to an external torque, τ, and it is acted on by the twisting of the shaft. We can write its equation of motion, using the angle of twist as the dynamical variable, as

$$I_1\ddot{\phi} = \tau - \frac{GI_s}{l}\phi$$

where I_1 denotes the polar moment of the left-hand wheel, typically much greater than that of the shaft. We can rearrange this and deduce the natural frequency of the system to be

$$I_1\ddot{\phi} + \frac{GI_s}{l}\phi = \tau \Rightarrow \omega_n^2 = \frac{GI_s}{I_1 l}$$

Damping is usually pretty small in these systems, but one can certainly introduce damping, probably empirically. I will discuss measuring damping later in this chapter.

2.2.2 Damped Unforced Systems

The governing differential equation for unforced damped systems [from Eq. (2.4)] is

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0$$

It is still homogeneous, but it no longer has purely trigonometric solutions. Because it is homogeneous with constant coefficients, it does have exponential solutions. Let's see what they are by choosing an arbitrary exponential and substituting that into the differential equation

$$y = Ye^{st} \Rightarrow \ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0 = s^2 Y + 2\zeta\omega_n sY + \omega_n^2 Y$$

This is the first example of a technique we will use often. Y denotes an arbitrary (complex) constant. The constant parameter s is determined during the analysis. Y can be found from the initial conditions. This substitution converts the differential equation into an algebraic equation. Later we will generalize this method to convert systems of differential equations into systems of algebraic equations. (We will also find that the results look formally like the results of taking Laplace transform, but that connection must be deferred until Chap. 7.) We seek nontrivial solutions, solutions for which Y is not zero.

$$\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)Y = 0 \Rightarrow s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{2.13}$$

If Y is not to be zero, the quadratic equation in parentheses must vanish. This is an example of a *characteristic equation*, which I will treat more formally later on. This determines two values of s, the roots of the quadratic equation. Denote these by s_1 and s_2. The general solution is then

$$y = Y_1 e^{s_1 t} + Y_2 e^{s_2 t}, \tag{2.14}$$

and the values of Y_1 and Y_2 are determined by the initial conditions, just as they were for the trigonometric solution to the undamped problem. The initial conditions may be written

$$Y_1 + Y_2 = y_0, \quad s_1 Y_1 + s_2 Y_2 = v_0$$

from which

$$Y_1 = -\frac{s_2 y_0 - v_0}{s_1 - s_2}, \quad Y_2 = \frac{s_1 y_0 - v_0}{s_1 - s_2} \Rightarrow y = -\frac{s_2 y_0 - v_0}{s_1 - s_2} e^{s_1 t} + \frac{s_1 y_0 - v_0}{s_1 - s_2} e^{s_2 t} \tag{2.15}$$

Equation (2.15) is perfectly general. The nature of the solution depends on the values of the roots. We can apply the quadratic formula to obtain

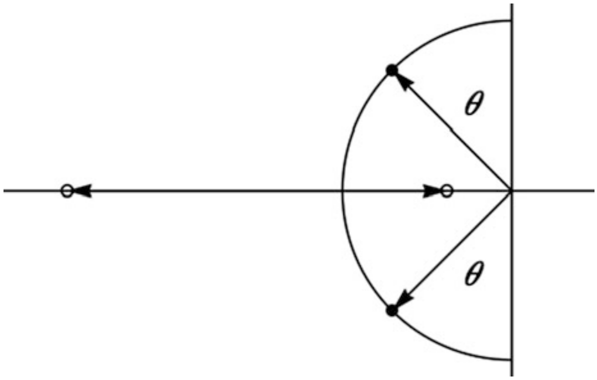


Fig. 2.8 Roots of Eq. (2.13b) (given by Eq. 2.16) in the complex plane

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \quad s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \tag{2.16}$$

The nature of the solution clearly depends on whether ζ is bigger or smaller than unity. The case $\zeta = 1$ is a special case. Engineering parameters are never determined to mathematical identity, and the difference in behavior between $\zeta = 0.99$ and 1.01 is generally indistinguishable. I will give an example below. The large ζ case is referred to as *overdamped*, and the small ζ case is referred to as *underdamped*. (The $\zeta = 1$ case is called *critically damped*, and we sometimes design for that.) The roots for the underdamped case are complex, having both real and imaginary parts. The roots are real (and negative) for the overdamped case. (The roots are purely imaginary for $\zeta = 0$, the undamped case we just looked at.)

We can visualize the behavior of the roots of Eq. (2.13b) as a function of ζ in the complex plane. Figure 2.8 shows the complex plane with the roots plotted for $\zeta = 1/2$ (underdamped, closed circles) and $\zeta = 3/2$ (overdamped, open circles). The roots lie on a circle of radius ω_n (unity in the figure) when the system is underdamped ($0 < \zeta < 1$). The roots are purely imaginary when $\zeta = 0$, the undamped case. When ζ reaches unity the two roots coincide and the system is critically damped. As ζ increases beyond unity, one root moves to the left and one to the right, as shown by the arrows, eventually approaching—∞ and zero (from below). The angle θ shown in the figure is defined by the ratio of the real part to the imaginary part of the root and so is directly related to the damping ratio:

$$\tan\theta = \frac{\zeta}{\sqrt{1 - \zeta^2}} \Rightarrow \zeta = \sin\theta \tag{2.17}$$

The general solution can be rewritten in a more useful form in the (common) underdamped case by making use of the relations between exponential and trigonometric functions

$$y = \exp(-\zeta\omega_n t)(A\cos(\omega_d t) + B\sin(\omega_d t)) \tag{2.18a}$$

where $\omega_d = \sqrt{1-\zeta^2}\omega_n$ is often called the *damped natural frequency*. This is the frequency you would measure. This form of the solution in terms of the initial conditions is

$$y = \exp(-\zeta\omega_n t)\left(y_0 \cos(\omega_d t) + \frac{v_0 - \zeta\omega_n y_0}{\omega_d} \sin(\omega_d t)\right) \tag{2.18b}$$

Equation (2.18b) is a much more useful form for underdamped systems than that given in Eq. (2.15).

We can learn about the behavior of the solutions by looking at how the impulse problem we have already studied changes when there is damping. I will make it simpler than before by setting $y_0 = 0$. I leave it to the reader to show that the solution to this problem [see Eq. (2.15)] for arbitrary ζ is

$$y = \frac{v_0}{2\omega_n\sqrt{\zeta^2-1}}\left(\exp\left(-\left(\omega_n\zeta + \sqrt{\zeta^2-1}\right)t\right) - \exp\left(-\left(\omega_n\zeta - \sqrt{\zeta^2-1}\right)t\right)\right) \tag{2.19}$$

If ζ is less than unity this can be written

$$\begin{aligned} y &= \frac{v_0}{2j\omega_n\sqrt{1-\zeta^2}}\left(\exp\left(-\left(\zeta + j\sqrt{1-\zeta^2}\right)\omega_n t\right) - \exp\left(-\left(\zeta - j\sqrt{1-\zeta^2}\right)\omega_n t\right)\right) \\ &= \frac{e^{-\omega_n\zeta t}}{\omega_n\sqrt{1-\zeta^2}} \frac{e^{\left(j\sqrt{1-\zeta^2}\omega_n t\right)} - e^{\left(-j\sqrt{1-\zeta^2}\omega_n t\right)}}{2j} \end{aligned} \tag{2.20}$$

The second quotient is recognizable as the sine, so we can write the underdamped solution as

$$y = \frac{\exp(-\zeta\omega_n t)}{\omega_n\sqrt{1-\zeta^2}} \sin\left(\sqrt{1-\zeta^2}\omega_n t\right) \tag{2.21}$$

Of course we could have found this result directly from Eqs. (2.18a) and (2.18b). If ζ is greater than unity the solution is as shown in Eq. (2.15)—both terms are real. Equation (2.21) gives the response of an underdamped system to a unit impulse.

Figure 2.9 shows the first two nominal periods ($4\pi/\omega_n$) of an underdamped case with $\zeta = 0.1$. One can see the effect of the damped frequency: the curve doesn't quite close because the actual period is longer than the ideal period. The damping also causes the solution to decay. There is significant decay even in the first quarter of the nominal period. The maximum value of the undamped response for the parameters in Fig. 2.9 is unity. The amplitude of the peak shown in Fig. 2.10 is 0.8626.

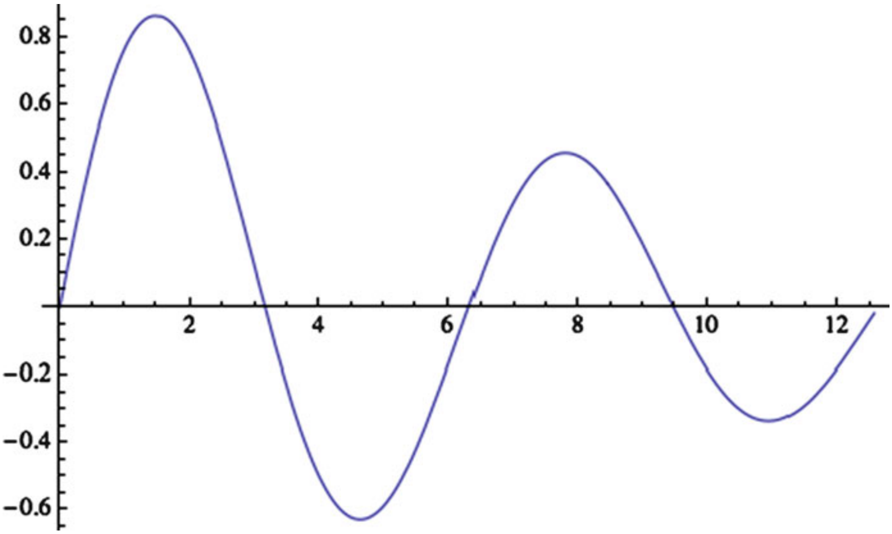


Fig. 2.9 The underdamped response to an impulsive load

Figure 2.10 shows the same time interval, but with a damping ratio of 10. The maximum amplitude is much reduced, but the decay time is much longer. This appears to be contradictory—more damping leads to a longer decay time. I leave it to you to think about why this is so. (Think about the implications of Fig. 2.8.) The most rapid decay is for a critically damped system, for which $\zeta = 1$.

I noted above that the difference between the response at a damping ratio of 0.99 and 1.01 was negligible. Figure 2.11 shows both responses to a unit impulse. The solid line is underdamped, and the dashed line is overdamped. There's little daylight between the two curves, although one can see the reduction in maximum response with increase in damping ratio. The difference is negligible (and probably unmeasurable) for engineering applications. Other approximations in modeling will overwhelm errors of this size.

We can summarize the behavior of an unforced one degree of freedom system as follows. If the system is in equilibrium and not disturbed, it will stay in equilibrium. If disturbed its behavior depends on the damping ratio ζ. If the damping ratio is zero there is no dissipation of energy and the system will oscillate at its natural frequency indefinitely—harmonic motion. If the damping ratio is not zero, the disturbed system will tend back to equilibrium, in an oscillatory fashion if the damping ratio is less than unity (underdamped) and without oscillations if the damping ratio is greater than unity (overdamped). The "frequency" of the oscillations in the underdamped case is less than the natural frequency. The decay time is a minimum for a damping ratio of unity (critically damped) and increases as the damping ratio increases from unity. Most of the systems with which we will be dealing will be underdamped. There will be some systems for which an undamped

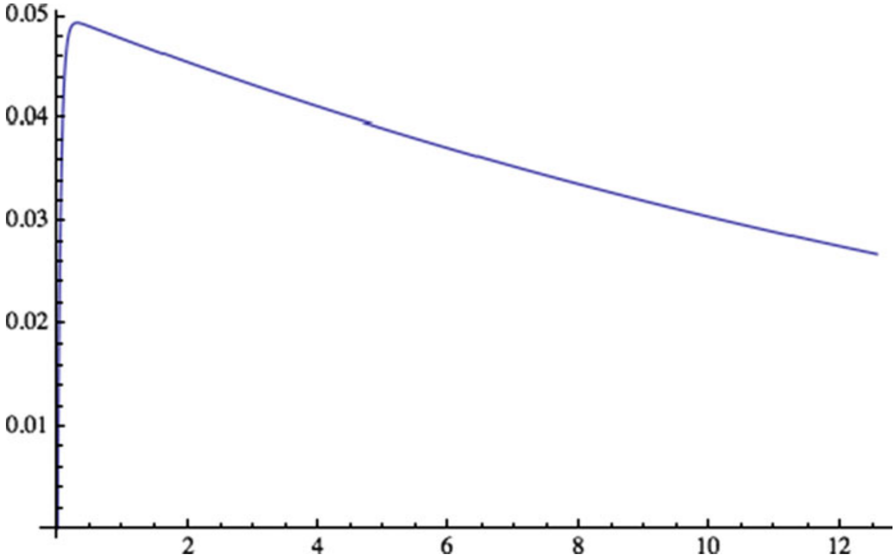


Fig. 2.10 The overdamped response to an impulsive load

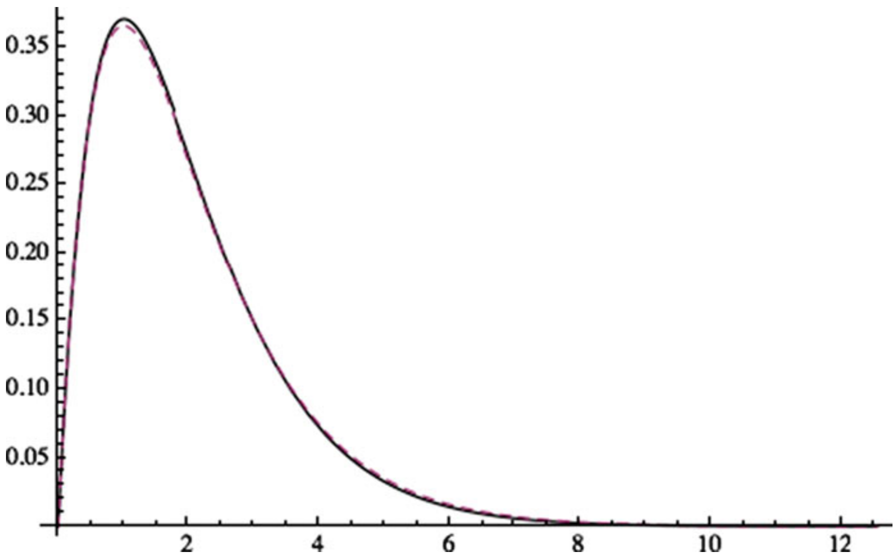


Fig. 2.11 The near critically damped response to a unit impulse

approximation makes sense, but all real systems have some damping. (Imagine the cacophony in the world were this not true!)

The e-folding time (the time it takes for the original value to decrease to $1/e$ times its initial value) for an underdamped system is $1/\zeta\omega_n$, so it will have disappeared for most practical purposes at about three times this number ($e^{-3} \approx 0.05$, 5 % of the initial amplitude). The 5 % time for the underdamped system just examined is about 30. For comparison, the critically damped version is down to 5 % in about 6 time units.

I can illustrate the techniques for dealing with forced system more clearly if I neglect damping to begin with—the various terms are simpler. But first let me say something about stability.

Equation (2.4) is our generalized one degree of freedom equation. The unforced system ($a = 0$) that we have looked at so far can oscillate, or oscillate while its amplitude decays, or just decay to zero, as I just noted. We can say this another way. Set $a = 0$ and multiply Eq. (2.4) by $\dot{y}$. We'll have

$$\dot{y}\ddot{y} + 2\zeta\omega_n\dot{y}^2 + \omega_n^2\dot{y}\ddot{y} = 0 \tag{2.22}$$

We can "integrate" Eq. (2.22) and rearrange it to get

$$\frac{1}{2}\frac{d}{dt}\left(\dot{y}^2 + \omega_n^2 y^2\right) = -2\zeta\omega_n\dot{y}^2 \tag{2.23}$$

The natural frequency is positive. The left-hand side of Eq. (2.23) is the derivative of the positive quantity $\dot{y}^2 + \omega_n^2 y^2$. The right-hand side is zero if $\zeta = 0$, negative if $\zeta > 0$, and positive if $\zeta < 0$. If the right-hand side is negative, then the quantity $\dot{y}^2 + \omega_n^2 y^2$ must decrease in amplitude until it goes (asymptotically) to zero. This is an example of absolute global stability.

2.2.3 Forced Motion

Forced systems obey Eq. (2.4) with its associated initial conditions. The classical way to deal with this is to divide the solution into two parts, a homogeneous solution y_H and a particular solution y_P. The homogeneous solution is simply the general unforced solution that we have been examining. The particular solution is *any* solution that satisfies the inhomogeneous equation, without regard to the initial conditions. I will give a general formula for such a solution later in this section. The actual solution is the sum of the homogeneous and particular solutions.

Why do we need a homogeneous solution? Because the particular solution may not satisfy the initial conditions. Indeed, it specifically ignores them. The homogeneous solution exists to cancel any incorrect initial values of the particular solution. Let's see how this goes. Suppose we have found the particular solution. We already know how to find the homogeneous solution. We find the initial conditions for the homogeneous solution by subtracting the initial values of the particular solution from the initial conditions specified in the problem:

$$y_{\mathrm{H}}(0) = y_0 - y_{\mathrm{P}}(0), \quad \dot{y}_{\mathrm{H}}(0) = v_0 - \dot{y}_{\mathrm{P}}(0),$$

and the analog of Eq. (2.15) will be

$$\begin{aligned} y = &-\frac{s_2(y_0 - y_{\mathrm{P}}(0)) - (v_0 - \dot{y}_{\mathrm{P}}(0))}{s_1 - s_2}\exp(s_1 t) \\ &+\frac{s_1(y_0 - y_{\mathrm{P}}(0)) - (v_0 - \dot{y}_{\mathrm{P}}(0))}{s_1 - s_2}\exp(s_2 t) \end{aligned}$$

If there is dissipation in the system, the homogeneous solution will decay away, and the long-term solution will be just the particular solution. In many situations we do not care about the initial conditions. In those cases we are said to *ignore the transients*, and then all we need to do is find the particular solution. This long-term solution is often referred to as the *steady solution*, even though it will be time dependent anytime the forcing a is time dependent. Deciding what to do about the transients (essentially the homogeneous solution) is a matter for engineering judgment. Sometimes it makes sense to ignore them, sometimes it does not. For now, let us assume that we need to take them into account and learn how to do this.

We know the homogeneous solution. It has two arbitrary constants, and we know that these can be determined from the initial condition once we have the particular solution. We need a particular solution. That is, we need to solve the inhomogeneous Eq. (2.4) (for now without damping). The solution clearly depends on the nature of the forcing acceleration a. If it is constant a_0, then it is clear by inspection that

$$y_{\mathrm{P}} = \frac{a_0}{\omega_{\mathrm{n}}^2}$$

If a is some power of time, then we can construct a polynomial for y_P of the same degree as the power. Let me illustrate this for t^3.

$$\ddot{y}_{\mathrm{P}} + 2\zeta\omega_{\mathrm{n}}\dot{y}_{\mathrm{P}} + \omega_{\mathrm{n}}^2 y_{\mathrm{P}} - t^3 = 0$$

Let

$$y_{\mathrm{P}} = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Substituting this into the differential equation leads to a polynomial of degree three that has to vanish for all time. This will only be true if each power of t vanishes separately, which gives four equations to determine the four coefficients in the expression for y_{P}. It is easy enough to show that these four equations are

$$\begin{gathered}\left(a_3\omega_n^2 - 1\right)t^3 = 0\\ \omega_n(6\zeta a_3 + \omega_n a_2)t^2 = 0\\ \left(6a_3 + \omega_n(4\zeta a_2 + \omega_n a_1)\right)t = 0\\ \left(2a_2 + \omega_n(2\zeta a_1 + \omega_n a_0)\right)t^0 = 0\end{gathered}$$

They can be solved successively for all four coefficients. I leave the details to the reader. If I can do it for any power, then I can do it for any function with a convergent Taylor series. Of course, this is of purely academic interest.

If a is a harmonic function, which is a much more important case, then y_P will also be harmonic at the forcing frequency, with a phase determined by the natural frequency and damping ratio

$$y_P = A\sin(\omega_f t + \phi)$$

Let's look at this case and use it to explore the combination of particular and homogeneous solutions.

Example 2.3 Response of an Undamped System to Harmonic Forcing Let $a = A_f\sin(\omega_f t)$. I can choose zero phase without loss of generality. It is important to remember that this forcing frequency is not the same as the natural frequency. (We'll see what happens when it is shortly.) The differential equation is

$$\ddot{y}_P + 2\zeta\omega_n\dot{y}_P + \omega_n^2 y_P - A_f\sin(\omega_f t) = 0$$

We see that there are two frequencies in the problem, the natural frequency and the forcing frequency. The particular solution depends on both. It oscillates at the forcing frequency, but its amplitude depends on both frequencies. We'll see shortly that the nature of the solution depends on the forcing frequency through its ratio to the natural frequency. I will denote this by $r = \omega_f/\omega_n$.

I will neglect damping for now, making the equation simpler

$$\ddot{y}_P + \omega_n^2 y_P - A_f\sin(\omega_f t) = 0$$

We are just looking for the particular solution right now, so we don't care about the initial conditions. Since the second derivative of the sine is proportional to the sine, we can find a particular solution by supposing it to be proportional to $\sin(\omega_f t)$: $y_P = Y_P\sin(\omega_f t)$. The differential equation becomes

$$\left(-\omega_f^2 Y_P + \omega_n^2 Y_P - A_f\right)\sin(\omega_f t) = 0 \Rightarrow Y_P = \frac{A_f}{\omega_n^2 - \omega_f^2} = \frac{1}{\omega_n^2(1 - r^2)}A_f$$

We see that the response is in phase with the excitation for low forcing frequencies (compared to the natural frequency, small r) and π radians out of phase for high forcing frequencies (large r). We see that the amplitude of the response is formally infinite if the forcing frequency equals the natural frequency ($r = 1$). This state of affairs is called *resonance*. Since the point of this example is how to connect the particular and homogeneous solutions, I will ignore the possibility of resonance for now and write

$$y_P = A_f \frac{\sin(\omega_f t)}{\omega_n^2 - \omega_f^2}, \quad \dot{y}_P = A_f \frac{\omega_f \cos(\omega_f t)}{\omega_n^2 - \omega_f^2} \Rightarrow y_P(0) = 0, \quad \dot{y}_P(0) = \frac{A_f \omega_f}{\omega_n^2 - \omega_f^2}$$

We know the homogeneous solution to the undamped system, so we have

$$y_H = A\cos(\omega_n t) + B\sin(\omega_n t) \Rightarrow y_H(0) = A, \quad \dot{y}_H(0) = \omega_n B$$

If $y = y_0$ and $\dot{y} = v_0$ at $t = 0$, then the initial conditions that determine A and B are

$$y(0) = A = y_0, \quad \dot{y}(0) = \omega_n B + \frac{A_f \omega_f}{\omega_n^2 - \omega_f^2} = v_0$$

from which

$$A = y_0, \quad B = \frac{1}{\omega_n}\left(v_0 - \frac{A_f \omega_f}{\omega_n^2 - \omega_f^2}\right) = \frac{v_0}{\omega_n} - \frac{r}{\omega_n^2(1 - r^2)} A_f$$

So we have the homogeneous solution

$$y_H = y_0 \cos(\omega_n t) + \left(\frac{v_0}{\omega_n} - \frac{r}{\omega_n^2(1 - r^2)} A_f\right) \sin(\omega_n t),$$

and the complete solution is the sum of the homogeneous and particular solutions

$$y = y_0 \cos(\omega_n t) + \left(\frac{v_0}{\omega_n} - \frac{r}{\omega_n^2(1 - r^2)} A_f\right) \sin(\omega_n t) + \frac{1}{\omega_n^2(1 - r^2)} A_f \sin(\omega_f t)$$

There are two harmonic terms, one at the forcing frequency and one at the natural frequency. The solution itself is not harmonic because it has more than one frequency.

2.2.4 The Particular Solution for a Harmonically Forced Damped System

The process is more complicated when the system is damped. Let us consider the particular solution to a harmonic forcing in the presence of damping. The homogeneous solution can be added at the end following the paradigm reviewed in Ex. 2.3. I can tackle this problem using trigonometric functions or complex exponentials.

This is a matter of taste, and I will explore both methods here. I suppose that I have the same forcing as the previous example. The particular solution must satisfy

$$\ddot{y}_\mathrm{P} + 2\zeta\omega_\mathrm{n}\dot{y}_\mathrm{P} + \omega_\mathrm{n}^2 y_\mathrm{P} - A_\mathrm{f}\sin(\omega_\mathrm{f}t) = 0$$

2.2.4.1 The Trigonometric Approach

The solution will be harmonic at the forcing frequency, but it cannot be proportional to the sine alone because the first derivative introduces a cosine term. Therefore we must write

$$y_\mathrm{P} = A_\mathrm{P}\cos(\omega_\mathrm{f}t) + B_\mathrm{P}\sin(\omega_\mathrm{f}t) \tag{2.24}$$

When this is substituted into the differential equation, there will be two terms, one proportional to the sine and one proportional to the cosine. The two terms must both vanish independently for the equation to be satisfied for all time. It is easy to verify that the coefficients of the cosine and sine in Eq. (2.24) are

$$2B_\mathrm{P}\zeta\omega_\mathrm{f}\omega_\mathrm{n} + \left(\omega_\mathrm{n}^2 - \omega_\mathrm{f}^2\right)A_\mathrm{P} = 0 = \left(\omega_\mathrm{n}^2 - \omega_\mathrm{f}^2\right)B_\mathrm{P} - 2A_\mathrm{P}\zeta\omega_\mathrm{f}\omega_\mathrm{n} - A_\mathrm{f}$$

This is a pair of inhomogeneous algebraic equations that can be solved for A and B. That result is

$$A_\mathrm{P} = -\frac{2\zeta\omega_\mathrm{f}\omega_\mathrm{n}A_\mathrm{f}}{\left(\omega_\mathrm{n}^2 - \omega_\mathrm{f}^2\right)^2 + (2\zeta\omega_\mathrm{f}\omega_\mathrm{n})^2}, \quad B_\mathrm{P} = \frac{\left(\omega_\mathrm{n}^2 - \omega_\mathrm{f}^2\right)A_\mathrm{f}}{\left(\omega_\mathrm{n}^2 - \omega_\mathrm{f}^2\right)^2 + (2\zeta\omega_\mathrm{f}\omega_\mathrm{n})^2}$$

or

$$A_\mathrm{P} = -\frac{2\zeta r A_\mathrm{f}}{\omega_\mathrm{n}^2\left(\left(1 - r^2\right)^2 + (2\zeta r)^2\right)}, \quad B_\mathrm{P} = \frac{\left(1 - r^2\right)A_\mathrm{f}}{\omega_\mathrm{n}^2\left(\left(1 - r^2\right)^2 + (2\zeta r)^2\right)} \tag{2.25a}$$

so that

$$y_\mathrm{P} = -\frac{2\zeta r A_\mathrm{f}}{\omega_\mathrm{n}^2\left(\left(1 - r^2\right)^2 + (2\zeta r)^2\right)}\cos(\omega_\mathrm{f}t) + \frac{\left(1 - r^2\right)A_\mathrm{f}}{\omega_\mathrm{n}^2\left(\left(1 - r^2\right)^2 + (2\zeta r)^2\right)}\sin(\omega_\mathrm{f}t) \tag{2.25b}$$

The amplitude of this response, Y_P, is given by $\sqrt{A_\mathrm{P}^2 + B_\mathrm{P}^2}$.

$$Y_{\mathrm{P}} = \frac{A_{\mathrm{f}}}{\omega_{\mathrm{n}}^2 \sqrt{\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2}} \tag{2.25c}$$

and the phase by

$$\tan\phi = -\frac{2\zeta r}{\left(1 - r^2\right)} \tag{2.25d}$$

The force applied to the system is m times A_{f}. The natural frequency is the square of the ratio of k to m, so that $A_{\mathrm{f}}/\omega_{\mathrm{n}}^{2} = F/k$, which is the amount that the spring would be compressed if F were constant, it provides a reference displacement; yk/F is the dimensionless response of the system. Equation (2.25c) tells us the amplitude of the response of a damped single degree of freedom system to harmonic excitation. We can plot the response as a function of r for various values of the damping ratio. Figure 2.12 shows the scaled magnitude, $y' = yk/F$, of the response for ζ ranging from 0.1 to 1.0 at intervals of 0.1. The dimensionless response is unity at zero excitation frequency and goes to zero as increases without bound. As we can see from Eq. (2.25c), the maximum amplitude is at the nominal resonance—the forcing frequency equal to the natural frequency ($r = 1$). We can use this value to find the damping ratio from the amplitude plot

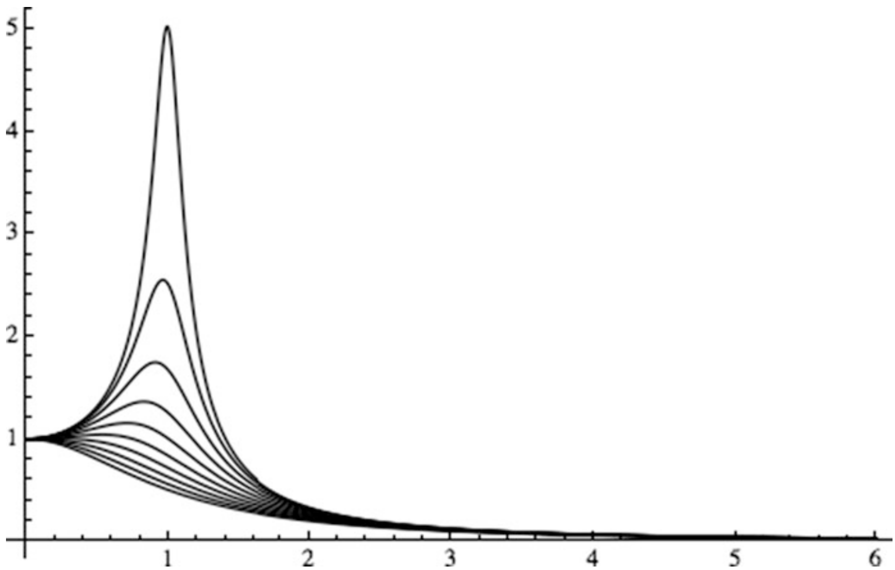


Fig. 2.12 The dimensionless amplitude of the response to a damped system to harmonic forcing. The vertical axis is relative displacement and the horizontal axis the ratio of forcing frequency to natural frequency

$$\zeta = \frac{1}{2y'_{max}} \tag{2.25e}$$

I'll discuss an alternate way of determining ζ when the system is underdamped later.

For high forcing frequencies (compared to the natural frequency, large r) the amplitude is small (asymptotically zero) and basically independent of the damping ratio. Physically the forcing is changing so rapidly that the inertia of the system gives it no time to respond. The response given by Eq. (2.15) is valid once the transients have decayed. If the transients are important, then a homogeneous solution—some realization of Eq. (2.9)—must be added to form a complete solution.

2.2.4.2 The Complex Variable Approach

We learned in Chap. 1 that the complex exponential is equivalent to the trigonometric functions. We can use this as an alternate way of finding the particular solution. We can rewrite the differential equation, replacing the sine by its complex equivalent

$$\ddot{y}_P + 2\zeta\omega_n\dot{y}_P + \omega_n^2 y_P = A_f\frac{1}{2j}\left(e^{j\omega_f t} - e^{-j\omega_f t}\right)$$

The two terms on the right-hand side are complex conjugates. The equation is linear, so the particular solution will have two parts, one forced by the first term on the right-hand side and one by its complex conjugate. These solutions will be complex conjugates of each other. We can find the solution by solving one of the equations and adding the complex conjugate of that solution to form the full solution. Symbolically, we can find y_{P1} as the solution to

$$\ddot{y}_{P1} + 2\zeta\omega_n\dot{y}_{P1} + \omega_n^2 y_{P1} = A_f\frac{e^{j\omega_f t}}{2j}$$

and write

$$y_P = y_{P1} + y_{P1}*$$

where the asterisk denotes complex conjugate. Now the sum of a function and its complex conjugate is twice the real part of the function

$$y_P = 2\mathrm{Re}(y_{P1})$$

I can incorporate the factor of two by multiplying the governing equation by 2 and define $y_Q = 2\ y_P$ to give

$$\ddot{y}_Q + 2\zeta\omega_n\dot{y}_Q + \omega_n^2 y_Q = -A_f j e^{j\omega_f t}$$

We solve this equation for y_Q and take its real part to give y_P. That is a relatively easy task. This is an important point, however, so I will spend a bit of time going through the details.

First multiply the original equation by 2 and move the j on the right-hand side to the numerator (multiply top and bottom by j)

$$2\ddot{y}_P + 4\zeta\omega_n\dot{y}_P + 2\omega_n^2 y_P = -jA_f\left(e^{j\omega_f t} - e^{-j\omega_f t}\right)$$

Replace y_P by its expression in terms of y_{P1} and its conjugate

$$2(\ddot{y}_{P1} + \ddot{y}_{P1}*) + 4\zeta\omega_n(\dot{y}_{P1} + \dot{y}_{P1}*) + 2\omega_n^2(y_{P1} + y_{P1}*) = -jA_f\left(e^{j\omega_f t} - e^{-j\omega_f t}\right)$$

Define y_Q as twice y_P, and write this as two equivalent equations

$$\begin{aligned}\ddot{y}_Q + 2\zeta\omega_n\dot{y}_Q + \omega_n^2 y_Q &= -jA_f e^{j\omega_f t}\\ \ddot{y}_Q * + 2\zeta\omega_n\dot{y}_Q * + \omega_n^2 y_Q * &= jA_f e^{-j\omega_f t}\end{aligned}$$

It is clear that the particular solution to either will be proportional to the exponential, and we find directly that the complex amplitude of the solution is

$$Y_Q = -\frac{jA_f}{\left(\omega_n^2 - \omega_f^2 + 2j\zeta\omega_n\omega_f\right)} = -\frac{jA_f(1 - r^2 - 2j\zeta r)}{\omega_n^2\left((1 - r^2)^2 + (2\zeta r)^2\right)}$$

Parts of this formula should look familiar.

The important thing to note is that one does not take the real part of Y_Q. One takes the real part of the actual solution, y_Q, which is given by

$$y_Q = Y_Q \exp(j\omega_f t) = -\frac{jA_f(1 - r^2 - 2j\zeta r)}{\omega_n^2\left((1 - r^2)^2 + (2\zeta r)^2\right)}\exp(j\omega_f t)$$

Expand the exponential and take the real part

$$\begin{aligned} y_P &= -\mathrm{Re}\left[\frac{jA_f(1 - r^2 - 2j\zeta r)}{\omega_n^2\left((1 - r^2)^2 + (2\zeta r)^2\right)}(\cos(\omega_f t) + j\sin(\omega_f t))\right]\\ &= \frac{-2\zeta r\cos(\omega_f t) + (1 - r^2)\sin(\omega_f t)}{\omega_n^2\left((1 - r^2)^2 + (2\zeta r)^2\right)}\end{aligned}$$

which one can see is identical to Eq. (2.23). The two methods are equivalent, as they must be. The choice of method is a matter of personal taste.

2.3 Special Topics

2.3.1 Natural Frequencies Using Energy

We found natural frequencies for undamped one degree of freedom systems by writing and solving the differential equations. This is how we will find natural frequencies for more complicated systems. However, it is interesting to note that we can find natural frequencies for one degree of freedom systems by consideration of energy alone. I want to address energy here because it is fundamental for the derivation of the equations of motion for the more complicated systems that we will see starting in Chap. 3.

An undamped, unforced system once set in motion will stay in motion perpetually (see Fig. 1.2 with the damper removed). Its energy is conserved: the sum of the maximum kinetic energy and the minimum potential energy must equal the sum of the minimum kinetic energy and the maximum potential energy

$$T_{\max} + V_{\min} = T_{\min} + V_{\max}$$

The minimum kinetic energy is zero, because the kinetic energy is proportional to the square of the speed, which is zero twice during any simple oscillation, so we have

$$T_{\max} = V_{\max} - V_{\min} \tag{2.26}$$

In one degree of freedom systems for which there is a restoring force (so that it will oscillate), the potential is an even function of the variable, which I will denote by y

$$V = V_0 + V_2 y^2 + \cdots$$

Thus

$$T_{\max} = V_{\max} - V_{\min} = V_2 y^2 + \cdots$$

where the constant term cancels. The idea of natural frequency makes sense only for small motions or for systems that are linear in their nature. In either case the +. . . terms are negligible, so that the maximum kinetic energy will be proportional to the square of the magnitude of the displacement. In most cases $V_{\min}$ will be zero. When it is not, it will be independent of the motion and will cancel from Eq. (2.26). This allows us to find the natural frequency. If we have harmonic motion, which we must have if we are to speak of natural frequencies, then we can write that $\ddot{y} = -\omega_n^2 y$. The left-hand side of Eq. (2.26) can be written in terms of the amplitude of the oscillation Y and the natural frequency.

$$T_{\max} \propto \omega_n^2 Y^2$$

The right-hand side can also be written in terms of the amplitude

$$V_{\max} - V_{\min} = V_2 y^2 + \cdots \approx V_2 Y^2$$

The proportionality constant for the kinetic energy and V_2 are different for each problem, but their ratio determines the square of the natural frequency. Let me go through some examples, starting with the simple mass-spring system.

Two points in this motion are special. When the (effective) spring is unstretched (the equilibrium position for a stationary mass) the mass is moving at its maximum speed. When the mass is stationary, the spring is at its maximum tension (compression). In the former case all the energy of the system is kinetic, in the latter, potential. We can write these two energies as

$$T = \frac{1}{2} m \dot{y}^2, \quad V = \frac{1}{2} k y^2$$

where we denote the kinetic energy by T and the potential energy by V. The potential energy supposes that I have chosen the origin for y such that the potential energy is zero when the spring is unstretched. We have already established that the motion is harmonic, so we can write (using the amplitude and phase notation for convenience)

$$y = Y \sin(\omega_n t + \varphi), \quad \dot{y} = Y \omega_n \cos(\omega_n t + \varphi)$$

The maximum displacement occurs when the sine is unity and the maximum speed when the cosine is unity. We can then find the maximum kinetic and potential energies and equate these.

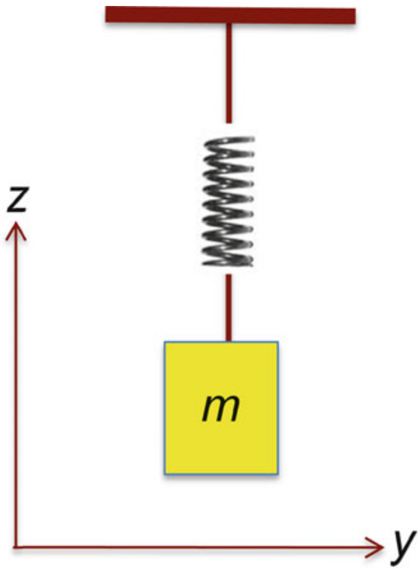
$$T_{\max} = \frac{1}{2} m Y^2 \omega_n^2 = V_{\max} = \frac{1}{2} k Y^2 \Rightarrow \omega_n^2 = \frac{k}{m}$$

The square of the natural frequency here is what we expect it to be, the ratio of the energy storage coefficient to the inertia coefficient.

The restoring force for the mass-spring example is the spring force. The restoring force for the pendulum is gravity. What happens when both gravity and a spring can act? It depends on the circumstances, but let's look now at the application of the energy argument for the mass-spring system in a vertical position. The potential energy here has both a spring and gravity component. Figure 2.13 shows the system.

This is a one degree of freedom system with motion only in the z direction. Denote the location of the mass where the spring is relaxed by z_0, and the equilibrium position of the spring, where the spring force just balances gravity, by z_1. A simple force balance—$mg = k(z_0 - z_1)$—shows that

Fig. 2.13 A vertical mass-spring system



$$z_1 = z_0 - \frac{mg}{k}$$

We can write the energies of the system

$$T = \frac{1}{2}m\dot{z}^2, \quad V = mgz + \frac{1}{2}(z - z_0)^2$$

We expect the system to oscillate harmonically about its equilibrium position. We can write

$$z = z_1 + Z\sin(\omega_n t + \phi)$$
$$\dot{z} = \omega_n Z\cos(\omega_n t + \phi)$$

where Z denotes the amplitude of the oscillation. The energies are

$$T = \frac{1}{2}m\omega_n^2 Z^2\cos^2(\omega_n t + \phi)$$

$$V = mg\left(z_1 + Z\sin(\omega_n t + \phi)\right) + \frac{1}{2}\left(z_1 + Z\sin\left(\omega_n t + \phi\right) - z_0\right)^2$$

Substituting for z_1 and expanding the potential energy gives

$$V = \frac{1}{2}kZ^2\sin^2(\omega_n t + \phi) + mgz_0 - \frac{m^2 g^2}{2k}$$

We can use Eq. (2.26) to get the natural frequency.

$$T_{\max} = V_{\max} - V_{\min}$$

The two minima for the horizontal mass-spring system were both zero. In the present case we have

$$\frac{1}{2} m\omega_{\mathrm{n}}^2 Z^2 = \frac{1}{2} kZ^2 + mgz_0 - \frac{m^2 g^2}{2k} - \left(mgz_0 - \frac{m^2 g^2}{2k} \right)$$

The constant terms cancel and we reproduce the previous formula.

The following example shows the power of the energy method.

Example 2.4 Find the Frequency of Sloshing in a U-Tube Filled with an Inviscid Liquid I first ran across this example in Den Hartog (1956). The approach below is different from his, but the result is, of course, the same.

Figure 2.14 shows a U-tube manometer. I assume the fluid to be inviscid, so there is no dissipation in the system. I also assume the liquid to be incompressible. We would like to find the natural frequency of this system as the liquid sloshes up and down. Denote the total length of the liquid by l, its density by ρ, and the cross-sectional area of the tube by A. The mass of the fluid is then $m = \rho A l$. The dynamical variable is h as shown in the figure. $h = 0$ corresponds to equilibrium. The kinetic energy is straightforward if we assume that the fluid moves as a unit:

$$T = \frac{1}{2} mv^2 = \frac{1}{2} \rho A l \dot{h}^2$$

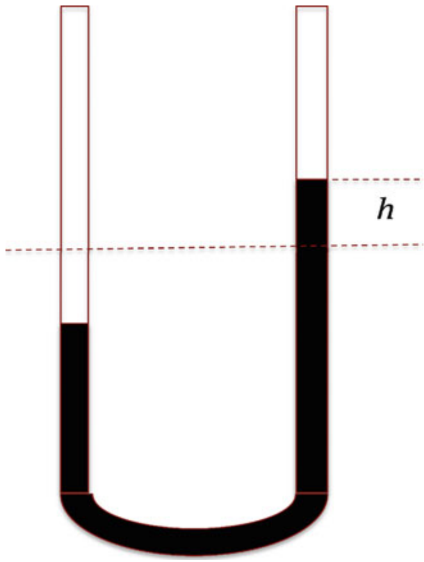


Fig. 2.14 A U-tube manometer. *h* denotes the distance the liquid has moved up and down from the equilibrium position, shown by the long *dashed line*

To get from the equilibrium configuration to the configuration shown in the figure, we moved a mass ρAh from the vacant spot below the dashed line on the left to the filled spot above the line. The center of mass moved up a distance h, so the change in potential energy was

$$V = \rho Ahg$$

(I will take the $h = 0$ equilibrium to define the reference state of the potential energy.)
Supposing the system to be harmonic and equating the two maxima gives

$$\omega_{\mathrm{n}}^2 = \frac{\rho Ag}{\frac{1}{2}\rho Al} = 2\frac{g}{l}$$

We can find the equation of motion from this, because in a nondissipative one degree of freedom system, the only parameter is the natural frequency. We'll have

$$\ddot{h} + 2\frac{g}{l}h = 0$$

2.3.2 A General Particular Solution to Eq. (2.4)

We can write an integral expression that purports to solve Eq. (2.4)

$$y = \int_0^t \phi(t-\tau)a(\tau)d\tau \tag{2.27}$$

Our task is to find the function ϕ such that Eq. (2.27) solves Eq. (2.4), which we can do by substituting Eq. (2.27) into Eq. (2.4). To so this we need to recall how to differentiate a definite integral. The derivative is equal to the integral of the derivative of the integrand, plus the derivative of the upper limit times the integrand evaluated at the upper limit, minus the derivative of the lower limit times the integrand evaluated at the lower limit. For y given by Eq. (2.26), we have

$$\dot{y} = \int_0^t \dot{\phi}(t-\tau)a(\tau)d\tau + \phi(0)a(t) - 0$$

Differentiating again gives

$$\ddot{y} = \int_0^t \ddot{\phi}(t-\tau)a(\tau)d\tau + \phi(0)\dot{a}(t) + \dot{\phi}(0)a(t)$$

Equation (2.4) becomes

$$\int_0^t \ddot{\phi}(t-\tau)a(\tau)d\tau + \phi(0)\dot{a}(t) + \dot{\phi}(0)a(t) + 2\zeta\omega_n\left(\int_0^t \dot{\phi}(t-\tau)a(\tau)d\tau + \phi(0)a(t)\right)$$
$$+\omega_n^2\int_0^t \phi(t-\tau)a(\tau)d\tau = a(t)$$

or combining all the integral terms under a single integral sign

$$\int_0^t \left(\ddot{\phi}(t-\tau) + 2\zeta\omega_n\dot{\phi}(t-\tau) + \omega_n^2\phi(t-\tau)\right)a(\tau)d\tau + \phi(0)\dot{a}(t) + \dot{\phi}(0)a(t) = a(t) \tag{2.28}$$

Equation (2.28) will be an identity if ϕ satisfies the homogeneous version of Eq. (2.4) with $\phi(0) = 0$ and its first derivative equal to unity. The integrand in Eq. (2.28) vanishes because the function satisfies the differential equation. The coefficient of $\dot{a}(t)$ vanishes, and the coefficient of $a(t)$ is unity. The function $\phi(t)$ is the solution for the unit impulse that we have already seen, and so we can write this for any value of ζ in terms of the two general exponents for the homogeneous equation.

$$\phi(t-\tau) = \frac{e^{s_1(t-\tau)} - e^{s_2(t-\tau)}}{s_1 - s_2} \tag{2.29a}$$

We can write this in terms of a general argument, replacing $t-\tau$ by ξ, to give Eq. (2.29b)

$$\phi(\xi) = \frac{e^{s_1\xi} - e^{s_2\xi}}{s_1 - s_2} \tag{2.29b}$$

This works because the two exponentials each satisfy the differential equation. The equation is linear, so their sum does as well. It is easy to establish that the initial conditions (here when the general argument $\xi = 0$) are as promised. It is worth noting that the underdamped version of Eq. (2.29b) is

$$\phi(\xi) = \exp(-\zeta\omega_n\xi)\frac{\sin(\omega_d\xi)}{\omega_d} \tag{2.29c}$$

the same as Eq. (2.21). The particular solution given by Eq. (2.27) and its first derivative both vanish at $t = 0$, the former by inspection, and the latter as follows:

$$\dot{y} = \phi(0)a(t) + \int_0^t \dot{\phi}(t-\tau)a(\tau)d\tau$$

The first term vanishes because $\phi(0)$ does and the second vanishes by inspection—the interval of integration goes to zero. This particular solution does not contribute

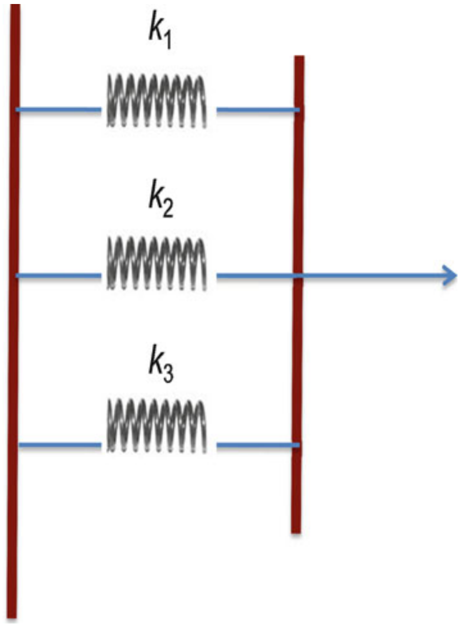


Fig. 2.15 Three springs in parallel

to the initial conditions. The initial conditions for the problem are taken care of entirely by the homogeneous solution. We can incorporate the initial conditions by adding an appropriate homogeneous solution, that given by Eq. (2.15).

This particular solution is restricted to one degree of freedom problems, but we will find similar integral expressions for problems of arbitrary complexity later.

2.3.3 Combining Springs and Dampers

Mechanical systems, even one degree of freedom systems, can have more than one spring or damper. We need to know how to combine springs and dampers to form effective springs and dampers so that we can use what we have learned about Eq. (2.4) for more complicated situations. Fortunately the rules are simple and straightforward and the same for dampers as for springs. I will work them out for springs.

The relation between force and displacement for a spring is linear, and the proportionality constant is the spring constant: $f = k\delta y$. We can find the spring constant by finding the relation between displacement and force. Figure 2.15 shows three springs in parallel.

We see that the force is the sum of the forces in the three springs so that the effective spring constant is simply the sum of the three individual constants. In equation form

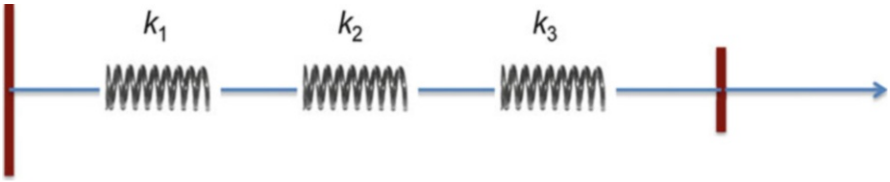


Fig. 2.16 Three springs in series

$$f = k_1\delta y + k_2\delta y + k_3\delta y = (k_1 + k_2 + k_3)\delta y$$

so that

$$k_{\text{eff}} = (k_1 + k_2 + k_3) \tag{2.30a}$$

Figure 2.16 shows three springs in series.

The system is stationary so the force between springs must be zero. Therefore, the force in each spring is the same, so each will deform in response to that force and the total displacement will be

$$\delta y = \frac{f}{k_1} + \frac{f}{k_2} + \frac{f}{k_3}$$

We can solve this for f in terms of the spring constants

$$f = \frac{\delta y}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}} = \frac{k_1k_2k_3}{k_1k_2 + k_1k_3 + k_2k_3}\delta y$$

so that the effective spring constant is given by

$$k_{\text{eff}} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}} = \frac{k_1k_2k_3}{k_1k_2 + k_1k_3 + k_2k_3} \tag{2.30b}$$

In summary the effective spring constant of springs in parallel is the sum of the individual spring constants; the effective spring constant of springs in series is the inverse of the sum of the inverses. If an electric analogy is helpful to you, you can say that springs in parallel add like resistors in series and springs in series add like resistors in parallel or that springs are the mechanical analog of capacitors.

The same combination rules apply to dampers. I leave it to you to verify this.

2.3.4 Measuring the Damping Ratio

So far this chapter has assumed that the parameters for Eq. (2.4) are given. Suppose we have some vibrating system that we think can be represented by a one degree of freedom model. We can hear or feel something that sounds like a vibration. If we want to analyze it further, we would need to provide values for the natural frequency and damping ratio. We know that the damping ratio is less than one or there would not be a detectable vibration. The model system must be underdamped. We might excite the system (e.g., by providing an impulse) and get a picture like Figure 2.9.

Can we deduce what we need to know from these data? We will learn methods for finding natural frequencies for systems of any complexity later. We can estimate the frequency for the sample data by simply counting zero crossings. We know that the system is underdamped because it is oscillating and decaying. How can we find the damping ratio from the data in Fig. 2.9? This is a traditional exercise. Extending it to more than one degree of freedom is not trivial, but the technique is intriguing and worth addressing.

We know the response to an underdamped system is of the form

$$y = \exp(-\zeta\omega_{\mathrm{n}}t)(A\cos(\omega_{\mathrm{d}}t) + B\sin(\omega_{\mathrm{d}}t))$$

If the response is to an impulse, $A = 0$, so we can write the response as [see Eq. (2.29c)]

$$y = Y\exp(-\zeta\omega_{\mathrm{n}}t)\sin(\omega_{\mathrm{d}}t)$$

The successive peaks are at $t_n = (2n+1)\pi/\omega_{\mathrm{d}}$ and so their values are

$$y_n = Y\exp\left(-\zeta\omega_{\mathrm{n}}\frac{(2n+1)}{\omega_{\mathrm{d}}}\pi\right) = Y\exp\left(-\frac{\zeta}{\sqrt{(1-\zeta^2)}}(2n+1)\pi\right)$$

This expression is independent of the natural frequency, and if we take ratios, we can make it independent of the amplitude Y. We have

$$\frac{y_n}{y_{n+1}} = \frac{\exp\left(-\frac{\zeta}{\sqrt{(1-\zeta^2)}}(2n+1)\pi\right)}{\exp\left(-\frac{\zeta}{\sqrt{(1-\zeta^2)}}(2n+3)\pi\right)} = \exp\left(\frac{\zeta}{\sqrt{(1-\zeta^2)}}2\pi\right)$$

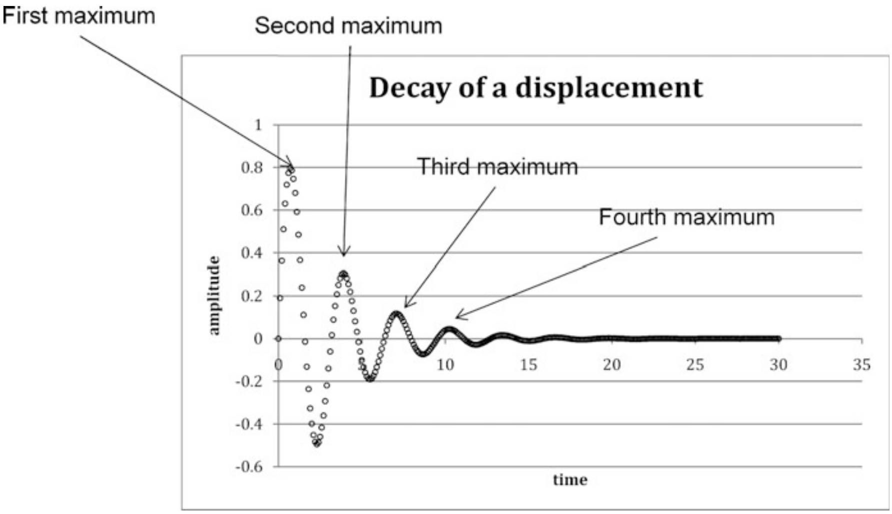


Fig. 2.17 A digital record of some artificial data

Take the logarithm of this

$$\delta = \ln\left(\frac{y_n}{y_{n+1}}\right) = 2\pi\frac{\zeta}{\sqrt{(1-\zeta^2)}} \tag{2.31}$$

The quantity δ is called the *log decrement*. We can measure the log decrement and solve the equation for the damping ratio.

$$\zeta = \frac{\delta}{\sqrt{(4\pi^2+\delta^2)}} \tag{2.32}$$

Note that the ratio is the same for each successive point, so one can measure several of these and arrive at an estimate based on several values of the ratio.

Example 2.5 Finding the Damping Ratio from Artificial Data Figure 2.17 shows a digital sampling of a decay curve for a system with a damping ratio of 0.15

$$y = \exp(-0.15\omega_{\mathrm{n}}t)\sin(0.9887\omega_{\mathrm{n}}t)$$

I eyeballed the data from the figure and built a table of successive maxima. I calculated ratios from these maxima, calculated the logarithms, then the damping ratios, and finally averaged the damping ratios, obtaining 0.1500, a very good estimate of the actual damping ratio. Table 2.1 shows the calculations.[2]

[2] Calculated using spreadsheet software.

Table 2.1 Estimating the damping ratio from artificial data

Eyeball			
Maxima	Ratios	ln(ratios)	Zetas
0.7965	2.592773438	0.952728128	0.149917725
0.3072	2.596787828	0.95427523	0.150155687
0.1183	2.605726872	0.957711666	0.150684156
0.0454	2.583949915	0.949319203	0.149393302
0.01757			0.150037717

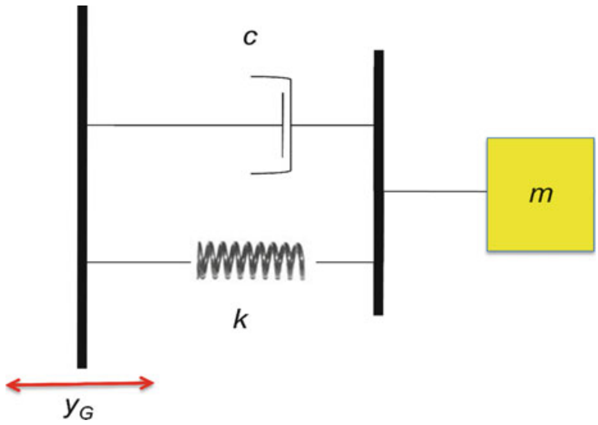


Fig. 2.18 A one degree of freedom system excited by motion of its support

2.3.5 Support Motion

I stated earlier that we can drive the mass by moving the support. We can redraw Fig. 1.2 with the external forcing replaced by an imposed displacement, shown in Fig. 2.18. The equation of motion associated with this system is

$$m\ddot{y} = -k(y - y_{\mathrm{G}}) - c(\dot{y} - \dot{y}_{\mathrm{G}}) \Rightarrow \ddot{y} + 2\zeta\omega_{\mathrm{n}}\dot{y} + \omega_{\mathrm{n}}^2 y = 2\zeta\omega_{\mathrm{n}}\dot{y}_{\mathrm{G}} + \omega_{\mathrm{n}}^2 y_{\mathrm{G}}$$

where y_{G} denotes the motion of the support. (I use the subscript G because the support will frequently be the ground.) The right-hand side plays the role of the acceleration a in Eq. (2.4). We can make the essential points more clearly if we neglect damping and suppose that the support motion is harmonic. Then we will have

$$\ddot{y} + \omega_{\mathrm{n}}^2 y = \omega_{\mathrm{n}}^2 Y_{\mathrm{G}} \sin(\omega_{\mathrm{G}} t)$$

In this simple case y will also be proportional to $\sin(\omega_{\mathrm{G}} t)$ and we find that

$$y_{\mathrm{P}} = \frac{\omega_{\mathrm{n}}^2 Y_{\mathrm{G}}}{\omega_{\mathrm{n}}^2 - \omega_{\mathrm{G}}^2} \sin(\omega_{\mathrm{G}} t) = \frac{Y_{\mathrm{G}}}{1 - r^2} \sin(\omega_{\mathrm{G}} t)$$

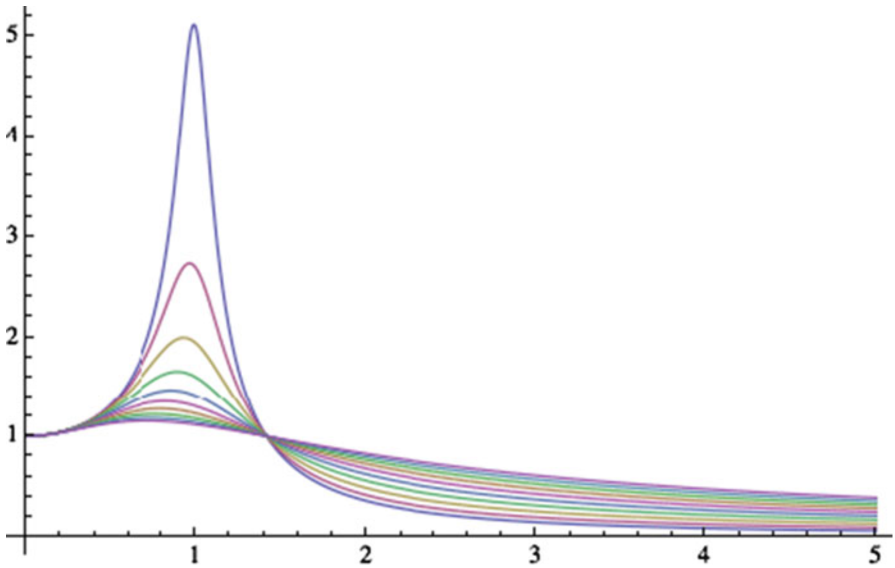


Fig. 2.19 Normalized response to harmonic ground motion

where r now denotes the ratio of the ground motion frequency to the natural frequency: $r = \omega_G/\omega_n$. Note that I have labeled this as a particular solution. It is fairly common to neglect the homogeneous solution in applications. All real systems have some damping, and the transients before damping can have its effect are often not interesting. When only the particular solution is used, it is called the *steady solution*, even though it is not actually steady, but a harmonic function of time at the same frequency as the driving input. The engineer has to decide when it is appropriate to use this steady (particular) solution. Generally the steady solution is fine if the transients are unimportant (or unknown).

There are two extreme limits. If ω_G is small compared to the natural frequency (small r), then the amplitude of the response is approximately equal to the input. If we add damping we get a somewhat more complicated expression, which I leave to the reader. Figure 2.19 shows the response normalized to the amplitude of the ground motion for the same range of damping ratios as in Fig. 2.14. Note the qualitative resemblance to Fig. 2.14. The quantitative differences stem from the appearance of the damping in the acceleration term.

The force on the mass also varies with forcing frequency, but not in the same way.
Figure 2.20 shows the normalized force for the same set of parameters:

$$\frac{f}{kY_G} = \frac{(y - y_G)}{Y_G} + 2\varsigma \frac{(\dot{y} - \dot{y}_G)}{\omega_n Y_G}$$

The force is zero at zero forcing frequency (nothing happens) and increases as the forcing frequency increases. The higher the damping ratio, the higher the force for large forcing frequency.

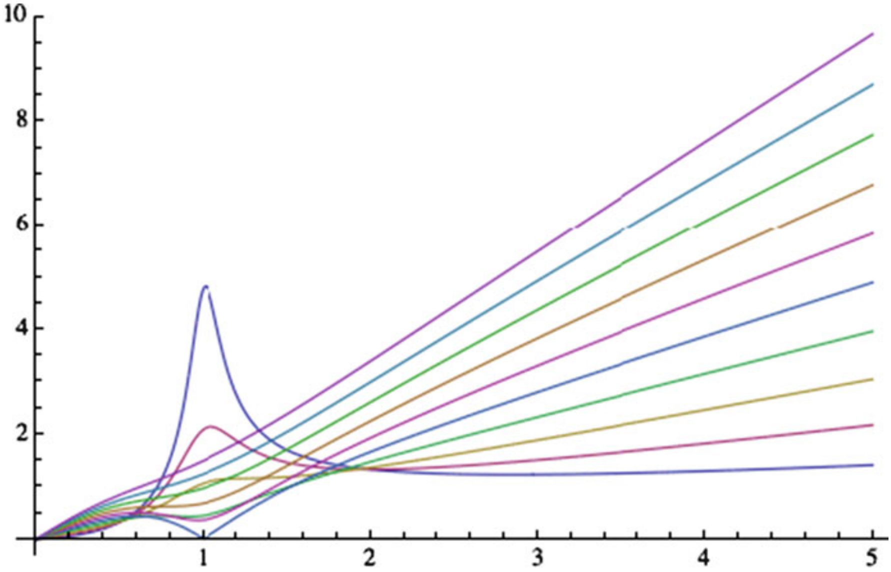


Fig. 2.20 Normalized force on the mass (see text)

2.4 Applications

2.4.1 Unbalanced Rotating Machinery

A piece of rotating machinery will react on its supports when rotating if it is not balanced. Examples include everything from an asymmetrically loaded washing machine to the turbine rotors in a jet engine. Automobile tires provide a homely example. We can look at this as a one degree of freedom if we suppose motion to be possible in only one direction.

Figure 2.21 shows a model of a system with a rotational imbalance. The small mass m rotates with the central shaft and is offset a distance d from the rotation axis. The shaft rotates at a rate ω as shown. We make this a one degree of freedom system by supposing the machine to be constrained to purely vertical motion. The unbalanced mass exerts a centripetal force on the axle, hence on the machine. If we denote the angle the unbalanced mass makes with the vertical by $\theta(=\omega t)$, then we can write the vertical component of the centripetal force as $md\omega^2 \cos\theta$, and we can write Eq. (2.4) as

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = \left(\frac{md}{M}\omega^2\right)\cos(\omega t) = \left(\frac{md}{M}\omega^2\right)\sin\left(\omega t - \frac{\pi}{2}\right)$$

In engineering practice we care about the long-term behavior of this system—will it shake itself apart or destroy its mount? Therefore we do not care about the

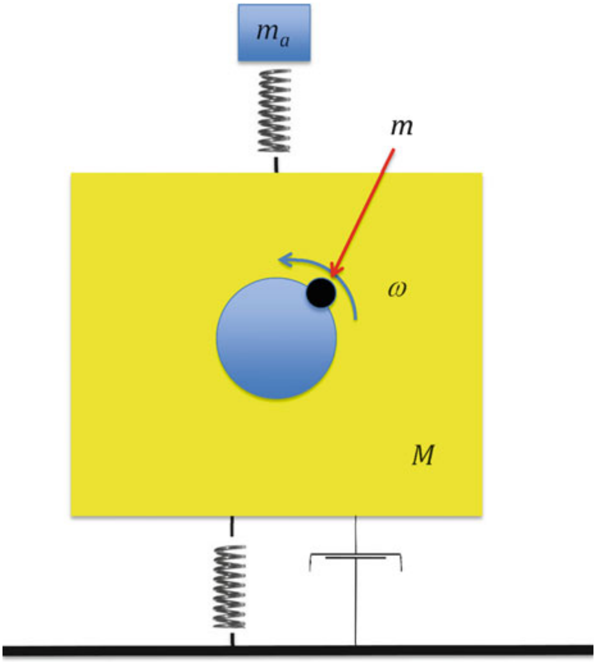


Fig. 2.21 Rotating machine with an imbalanced rotor

transient for this problem and can address the particular solution alone. This is a harmonically forced damped system, and we know the particular solution. We simply modify Eq. (2.13a) to obtain

$$z_P = -\frac{2\zeta\omega_f\omega_n}{\left(\omega_n^2 - \omega_f^2\right)^2 + \left(2\zeta\omega_f\omega_n\right)^2}\left(\frac{md}{M}\omega_f^2\right)\cos\left(\omega_f t - \frac{\pi}{2}\right) + \frac{\left(\omega_n^2 - \omega_f^2\right)}{\left(\omega_n^2 - \omega_f^2\right)^2 + \left(2\zeta\omega_f\omega_n\right)^2}\left(\frac{md}{M}\omega_f^2\right)\sin\left(\omega_f t - \frac{\pi}{2}\right)$$

where I have added a subscript f to the forcing frequency for clarity. We can write this in a more compact form

$$z_P = -\frac{2\zeta r}{\left(\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2\right)}\left(\frac{md}{M}r^2\right)\sin\left(\omega_f t\right) - \frac{\left(1 - r^2\right)}{\left(\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2\right)}\left(\frac{md}{M}r^2\right)\cos\left(\omega_f t\right) \tag{2.33}$$

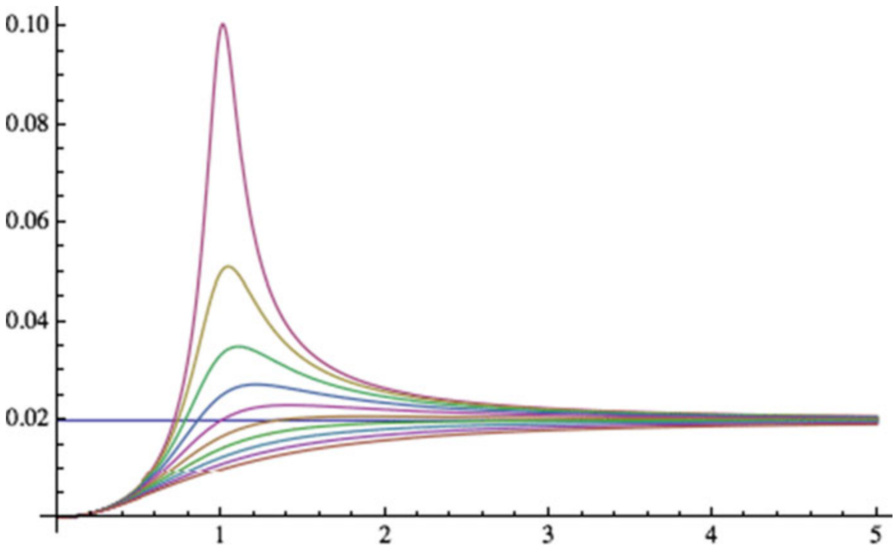


Fig. 2.22 Normalized displacement vs. exciting frequency

The amplitude of this can be gotten by substitution into Eq. (2.13b). That result is

$$\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\frac{m}{M}r^2 d \tag{2.34}$$

Now we see that the displacement is small at small forcing frequencies and is asymptotically equal to the mass ratio times the offset as the forcing frequency increases. (In most circumstances the product *md* is the only variable available. The actual location of the imbalance is not easily found. In the case of balancing a tire, weights are placed on the rim, so *d* and *m* can be determined independently for the compensatory weights.) Figure 2.22 shows the displacement divided by *d* for a mass ratio of 1/50. You can see the effects of resonance and the asymptotic result.

The force transmitted to the ground is often important. We can also obtain that from prior work. In this case the ground is not moving, so we have

$$f = kz + c\dot{z} = M\left(\omega_n^2 z + 2\zeta\omega_n\dot{z}\right)$$

We can plot the amplitude of *f*/*dk* as a function of the frequency ratio and the damping ratio, as we did for the displacement. That result is shown in Fig. 2.23.

Example 2.6 The Rotary Lawn Mower Consider the rotary lawn mower as a real-life example of a rotating imbalance with one degree of freedom (at least in a very simple model for which the wheels prevent sideways motion). Figure 2.24

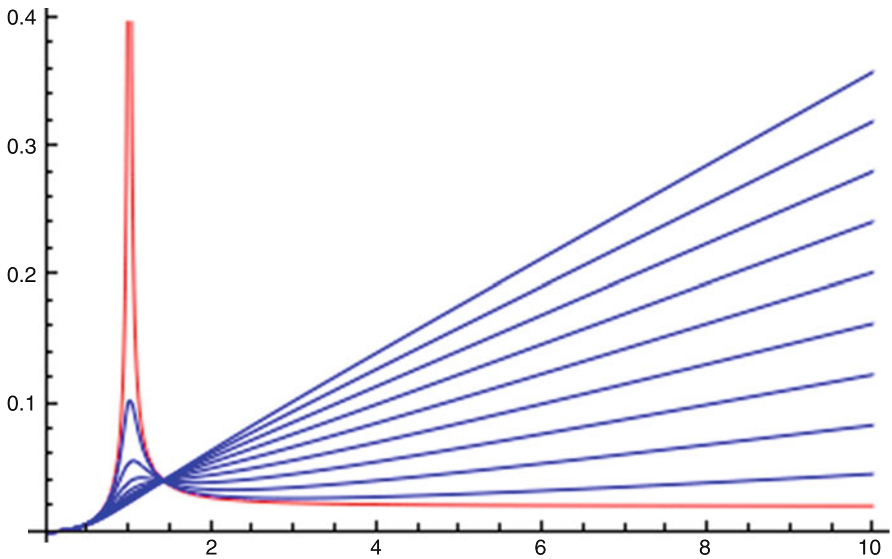


Fig. 2.23 Dimensionless ground force for the imbalanced rotor. The *red curve* is the undamped case. The *blue curves* show damping ratios from 0.1 to 0.9 in 0.1 increments. Damping increases the transmitted force for high frequency excitation

shows the bottom of a rotary lawn mower deck. It is reasonable to suppose that the wheels prevent transverse motion, so that the only motion that might be generated by an imbalance is in the nominal direction of travel. The force resisting this is complicated, requiring us to understand the interaction of the wheels with grass-covered ground. Since this is well beyond the scope of this text, let's take a very simple model and suppose there is no rolling resistance.

The blade shown is 533 mm (21″) long and its width is 54 mm (2 1/8″). Its mass is 0.70 kg (weight 1 lb 8.5 oz), and it is made of steel. The mower wheel base is 686 mm (27″) and its track is 508 mm (20″). We can model the blade as a uniform steel bar with the same mass and length and a reasonable width, a bar 533 mm long, 54 mm wide, and 3.10 mm thick. I take the mass of the entire lawn mower to be 15 kg.

How fast does the blade turn? Federal law limits the tip speed of the blade to 96.5 m/s (19,000 fpm). This speed is attained at approximately 3,460 rpm for this blade. Browsing the Internet suggests that rotary push lawn mower engines are set between 3,000 and 3,300 rpm. We can adopt some reasonable number, say 3,200 rpm, to explore the possible vibrations of this system caused by blade imbalance. The equation of motion for the system as a whole is

$$M\ddot{y} = md\omega^2 \cos(\omega t)$$

where M denotes the entire mass, md the unbalanced moment, and ω the rotation rate. The response is a simple 180° out of phase motion of amplitude md/M. This agrees with Eq. (2.31) in the limit that r goes to infinity—zero natural frequency. The worst case would have the location of the imbalance at the end of the blade,

Fig. 2.24 Lawn mower bottom (photo by the author)

so that we have an amplitude of 10.5 m/M inches. The unbalanced mass will be a small fraction of the blade mass, which is a small fraction of the system mass, so we do not expect much motion. However, there is a still a radial force on the motor bearings of $1.123 \times 10^5\, md$ N. A one gram unbalanced weight at the end of the blade makes this number approximately 30 N, not a negligible load on a bearing.

2.4.2 Simple Air Bag Sensor

The actual air bag sensor is a MEMS[3] device that we can look at as a small cantilever beam as shown in Fig. 2.25. We can fit this into our one degree of freedom model by finding the spring constant of the beam, either experimentally or by calculating it (beyond our capabilities at the moment). If the mass is much larger than the mass of the beam, then the calculation is simple, based on bending of a cantilever beam under end loading. I leave that to you. We are going to need to design an appropriate system eventually. For now, note that the system can be redrawn to fit into our model of a system driven by the motion of its support, as shown in Fig. 2.26. The sensor assembly is rigidly attached to the car so that the motion of the case of the assembly is the same as that of the car.

[3] Microelectromechanical systems.

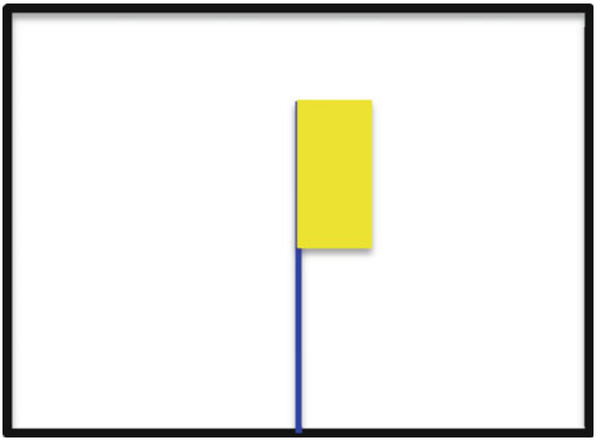


Fig. 2.25 A simple sketch of an air bag sensor

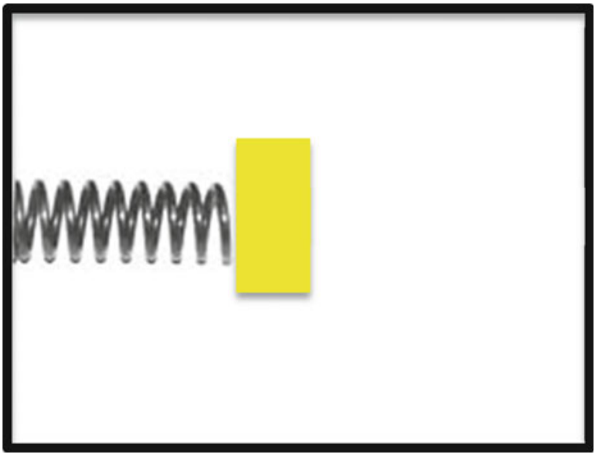


Fig. 2.26 Idealized model of an air bag sensor

Suppose the whole system to be moving to the right. When the vehicle hits a tree, the case stops, but the mass wants to keep moving. If the mass moves far enough, it will trigger the air bag, so we need to calculate how far it will travel for a given deceleration. Denote the position of the mass with respect to the case by y, and denote the motion of the case by y_W. The basic force balance is

$$m(\ddot{y} + \ddot{y}_{\mathrm{W}}) = -c\dot{y} - ky$$

which can be converted to an equivalent of Eq. (2.4)

$$\ddot{y} + 2\zeta\omega_{\mathrm{n}}\dot{y} + \omega_{\mathrm{n}}^2 y = -\ddot{y}_{\mathrm{W}}$$

We can get a general picture of how this works by neglecting damping (which will be small in any case) and assuming a constant deceleration. We need to solve

$$\ddot{y} + \omega_n^2 y = -a_0$$

subject to zero initial conditions. This is a problem for which the transient is all. Either the mass triggers the air bag during its first swing or it never will. This clearly depends on the deceleration (note that constant a_0 is a negative number) and the natural frequency. The particular solution is constant, and the homogeneous solution is the usual expression in terms of sine and cosine. The final result after applying initial conditions is

$$y = \frac{a_0}{\omega_n^2}(1 - \cos(\omega_n t))$$

I plot a scaled version as Fig. 2.27.

We see that the maximum displacement is twice the acceleration divided by the natural frequency and that it takes place at $\omega_n t = \pi$. We want the air bag to trigger early in the crash process, so we want a high natural frequency. The larger the design frequency, the smaller the response for a given deceleration, so we want the critical displacement to be small. Finally, we want the air bag to deploy only when the deceleration is extreme, more than one would expect from normal operation of the vehicle.

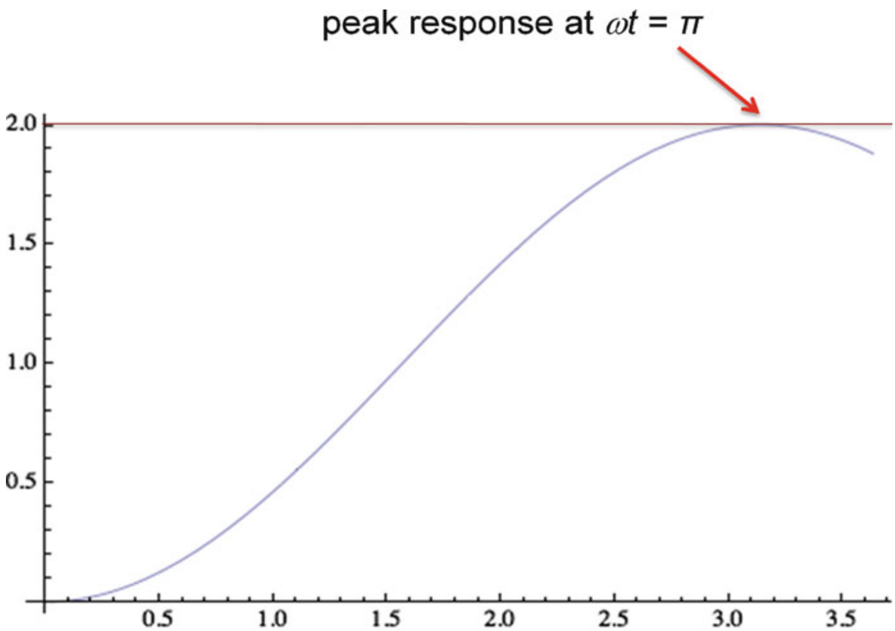


Fig. 2.27 Air bag sensor response

2.4.3 Seismometers and Accelerometers

Both seismometers and accelerometers work on the same principle as the air bag sensor. They have a proof mass connected to their world by a spring (and usually some sort of damper, whether deliberate or a consequence of natural dissipation) and a sensor that can detect motion between the proof mass and its world. The former measures displacement of the Earth and the latter the acceleration of whatever object it is attached to. How, then, are they different? We can address this question by considering again the simple one degree of freedom problem driven by ground motion.

Denote the motion of the proof mass by y and that of its world by y_G. The governing differential equation can be reduced to our standard form (recall how to do this)

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 2\zeta\omega_n\dot{y}_G + \omega_n^2 y_G$$

where ζ denotes the damping ratio and ω_n the natural frequency (here k/m). These systems require an unsteady input, and we can characterize such an input by some characteristic frequency. (In the case of a more complicated input, we can characterize that by a suite of frequencies.) The response will depend on the input frequency. Let

$$y_G = Y_G \sin(\omega_G t)$$

where Y_G is a constant and ω_G is the characteristic frequency. Substituting this into the differential equation leads to

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 2\zeta\omega_n\omega_G Y_G \cos(\omega_G t) + \omega_n^2 Y_G \sin(\omega_G t)$$

We care about the particular solution. (I'll say a little more about this when we've done the analysis.) The forcing is harmonic at ω_G and the particular solution must also be harmonic at ω_G. We can write

$$y = Y_1 \cos(\omega_G t) + Y_2 \cos(\omega_G t)$$

and substitute that into the differential equation. There will be sine and cosine terms on both sides of the equation, and they must satisfy the equation independently. The final result for the differential motion will be

$$y - y_G = \frac{2\zeta r^3}{(1-r^2)^2 + 4\zeta^2 r^2} Y_G \cos(\omega_G t) + \frac{r^2(1-r^2)}{(1-r^2)^2 + 4\zeta^2 r^2} Y_G \sin(\omega_G t)$$

where r denotes the ratio of the exciting frequency to the natural frequency: $r = \omega_G/\omega_n$. The damping will be small so that ζ is less than unity, probably considerably less than unity. The amplitude and phase of this signal are given by

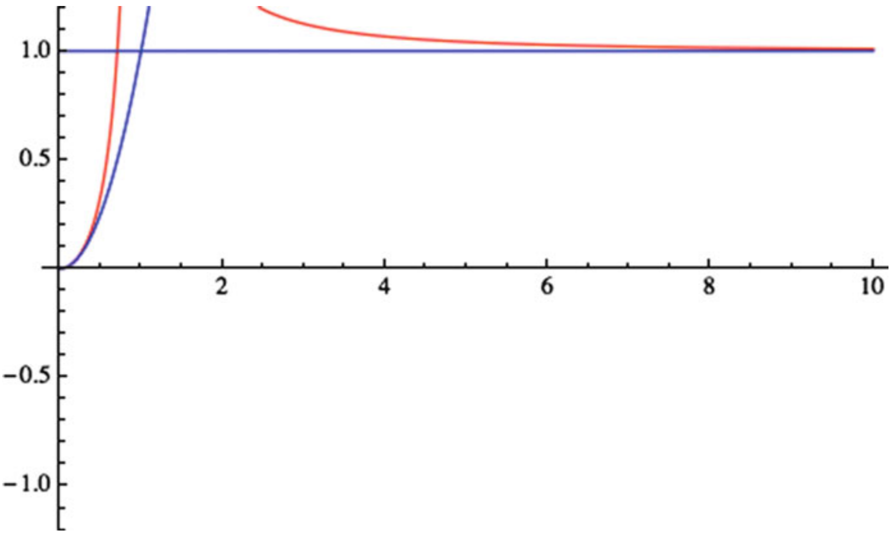


Fig. 2.28 The normalized amplitude of the response of an instrument driven by ground motion. The horizontal axis is the ratio of the exciting frequency to the natural frequency. The horizontal line is unity, and the curve is r^2. The damping ratio is zero, and the large values of the red curve cut off represent resonance at $r = 1$

$$A = \frac{r^2}{\sqrt{(1-r^2)^2 + 4\zeta^2 r^2}} Y_G, \quad \phi = \tan^{-1}\left(\frac{2r}{(1-r^2)}\right)$$

Figures 2.28, 2.29, and 2.30 show the amplitude divided by Y_G vs. r for r from zero to ten in red for damping ratios of 0, 1 (critically damped), and 0.707. The two blue lines are r^2 and unity, respectively. We see that the response for small r goes like the square of r, and for large r the response tends to a constant. Zero damping is clearly inadmissible, and critical damping reduces the range where the blue and red curves coincide. The intermediate damping seems to be a good choice. I encourage you to investigate this question further.

Figure 2.30 suggests that we can use this instrument to measure different things depending on whether r is large or small.

2.4.3.1 Seismometers

If r is large, then

$$y - y_G \approx \frac{2\zeta}{r} Y_G \cos(\omega_G t) - Y_G \sin(\omega_G t) \approx -Y_G \sin(\omega_G t)$$

The amplitude tends to Y_G and the phase to $-\pi$. The differential signal is proportional to the input displacement and 180° out of phase with it. The sensor will measure the displacement of the ground. Large r means that the natural frequency is

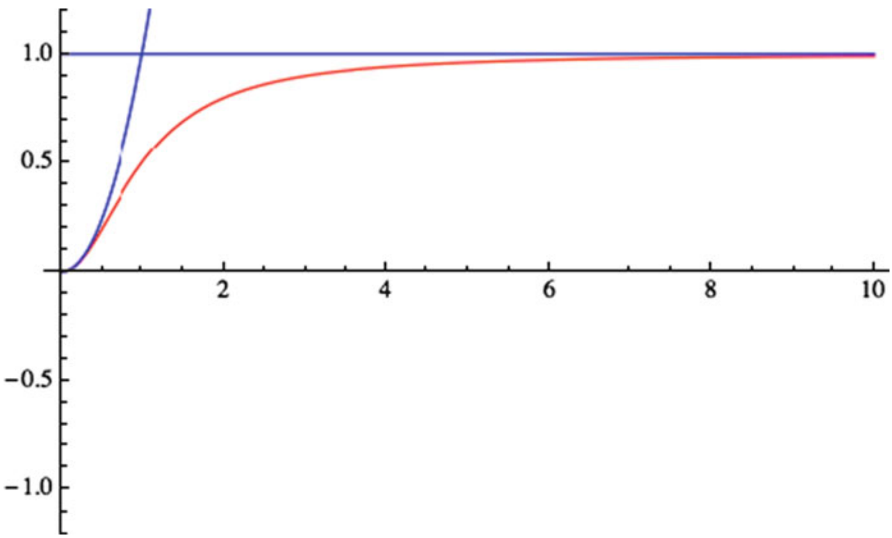


Fig. 2.29 The normalized amplitude of the response of an instrument driven by ground motion. The horizontal axis is the ratio of the exciting frequency to the natural frequency. The horizontal line is unity, and the curve is r^2. The damping ratio is unity

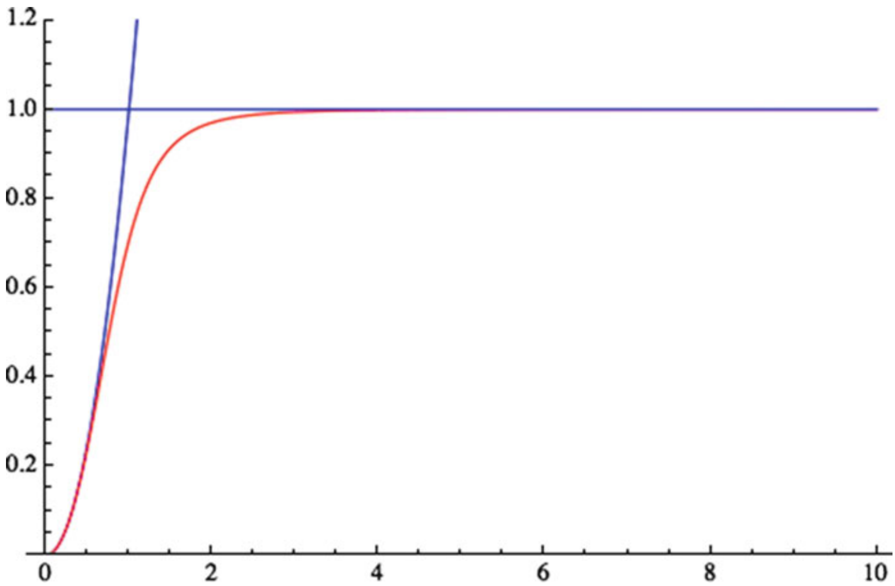


Fig. 2.30 The normalized amplitude of the response of an instrument driven by ground motion. The horizontal axis is the ratio of the exciting frequency to the natural frequency. The horizontal line is unity, and the curve is r^2. The damping ratio is 0.707

small compared to the displacement frequency—a large mass and a weak spring. Such a device can function as a seismometer, measuring ground motion at frequencies greater than its own natural frequency. Figure 2.30 suggests that ground motions at more than three times the natural frequency can be reliably measured. Maximum earthquake frequencies are about 20 Hz (with significant energies well below 1 Hz), so seismometers need very low natural frequencies. We can attain low frequencies (in mechanical seismometers) by using a horizontal pendulum. We know that the natural frequency of a pendulum is given by

$$\omega_{\mathrm{n}}^2 = \frac{g}{l}$$

If the pendulum is nearly horizontal the effective gravity is reduced and periods in excess of 30 s can be easily attained. It is also fairly easy to add damping to any such system using a damper at the pivot.

Let's take a look at how a pendulum responds to the motion of its pivot point. We don't need to worry about the horizontal aspect, just suppose that the effective gravitational constant is reduced. We can start with the same pendulum equations we had earlier in this chapter

$$m\ddot{y} = -T\sin\theta$$
$$m\ddot{z} = T\cos\theta - mg$$

The difference is that we must replace y by a term that takes account of the pivot motion:

$$y = y_{\mathrm{G}} + l\sin\theta$$

The expression for z remains the same. It is an easy matter to follow the pendulum argument to arrive at the modified equation for the pendulum [the equivalent of Eq. (2.12a)]

$$\ddot{\theta} + \frac{g}{l}\sin\theta = -\cos\theta\frac{\ddot{y}_{\mathrm{G}}}{l}$$

which linearizes to

$$\ddot{\theta} + \frac{g}{l}\theta = -\frac{\ddot{y}_{\mathrm{G}}}{l}$$

We can add damping proportional to the rotation rate of the pendulum, to give

$$\ddot{\theta} + \frac{c}{ml}\dot{\theta} + \frac{g}{l}\theta = -\frac{\ddot{y}_{\mathrm{G}}}{l}$$

We can use this to design our damping. If we want $\zeta = 0.707$, as in Fig. 2.28, we simply write

$$\frac{c}{ml} = 2\zeta\sqrt{\frac{g}{l}} \rightarrow c = 2\zeta m\sqrt{gl}$$

2.4.3.2 Accelerometers

On the other hand, if r is small, then we have

$$y - y_G \approx 2\zeta r^3 Y_G \cos(\omega_G t) + r^2 Y_G \sin(\omega_G t) \approx \frac{\omega^2}{\omega_n^2} Y_G \sin(\omega_G t)$$

The amplitude tends to

$$r^2 Y_G = \frac{\omega_G^2}{\omega_n^2} Y_G$$

and the phase to zero. The output provides a direct measurement of the acceleration ($\omega_G{}^2 y_G$) of the object to which the instrument is attached. This device can measure acceleration at frequencies below the natural frequency of the instrument, so a high natural frequency is desired. An accelerometer requires a tiny proof mass and a very stiff spring. I will discuss actual accelerometers in Chap. 5.

Note that neither instrument is all that much affected by the damping, so moderate damping (say $\zeta = 1/\sqrt{2}$) can be introduced to reduce any ringing from the sudden onset of the signal.

2.5 Preview of Things to Come

2.5.1 Introduction to Block Diagrams

We will have occasion to use block diagrams frequently in the course of this text. Block diagrams provide schematic diagrams equivalent to the differential equations governing any given system. Sometimes (for many people) this visual picture makes it easier to understand the dynamics. (Some people are content with the differential equations.) I will introduce block diagrams here in this simple setting, using the standard form of the one degree of freedom system as given by Eq. (2.4), reproduced here for convenience,

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 = a \tag{2.4}$$

I want to draw a picture of Eq. (2.4). The differential equation relates a function and its derivatives to the input. The mathematical operations are all differentiations. The mathematical operations in a block diagram are all integrations. The block diagram is, in some sense, the inverse of the differential equation.

Let's put together a block diagram of Eq. (2.4). This is a second-order equation involving two differentiations. Its block diagram equivalent will require two

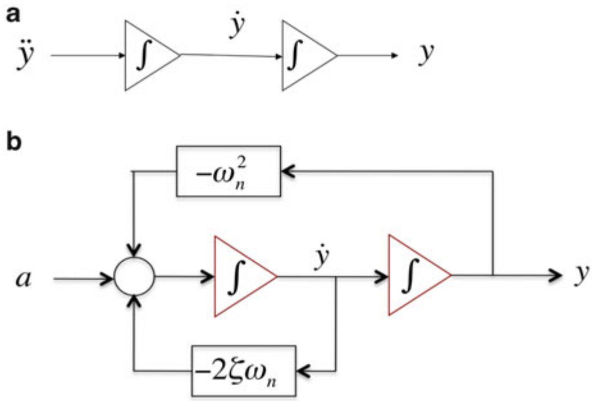


Fig. 2.31 (**a**) The spine of a second-order system. (**b**) Block diagram of the standard one degree of freedom system (Eq. 2.4)

integrations, which I will denote by triangles. There will be what I can call a second-order spine at the heart of the diagram, which I show as Fig. 2.31a.

One reads this diagram form left to right. We start with $\ddot{y}$, integrate to get $\dot{y}$, and then integrate a second time to get y. (It is conventional to ignore initial conditions when drawing block diagrams.) To complete the block diagram we need to have a picture of $\ddot{y}$, which we can obtain by solving Eq. (2.4) for $\ddot{y}$:

$$\ddot{y} = -2\zeta\omega_{\mathrm{n}}\dot{y} - \omega_{\mathrm{n}}^2 + a$$

There are three contributions to the second derivative—one from the input and one each from y and its first derivative. Once we have the second derivative, we integrate twice to get y. This is shown in Figure 2.31b, where I represent addition by a circle and multiplication by a box. The block diagram of the standard one degree of freedom system is shown as Fig. 2.31b.

The circle at the left gathers all three inputs to the second derivative: the actual input and the two "feedback" inputs. I put feedback in quotation marks here because I intend to use the word in a somewhat different sense later in the text. In both the present and future cases, the feedback is literally a feedback—already calculated variables are fed back to the beginning. The diagram is of what is called an *open-loop* system, open because there is no direct connection from the output y to the input a. Closing the loop, adding a connection from y back a, would make it a closed-loop system, and the connection is what we will generally mean by *feedback*. This process is the core of the second half of the text, beginning in Chap. 7.

Figure 2.31b shows a scalar block diagram, and it has two integrators. We can make a vector diagram of this and have a single (vector) integrator. This will be the way we will represent complicated systems later in the text, so it is a good idea to see how this goes. The easiest way to develop this picture is to go through the analysis. Instead of treating y and its derivative as connected scalars, we treat them as the components of a vector

$$\mathbf{x} = \begin{Bmatrix} y \\ \dot{y} \end{Bmatrix}, \quad y = \{1 \quad 0\}\mathbf{x}$$

and then write the differential equations for this vector

$$\frac{d}{dt}\begin{Bmatrix} y \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} \dot{y} \\ -\omega_{\mathrm{n}}^2 y - 2\zeta\omega_{\mathrm{n}}\dot{y} + a \end{Bmatrix}$$

Split out the external acceleration while maintaining the vector nature of the equations

$$\frac{d}{dt}\begin{Bmatrix} y \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} \dot{y} \\ -\omega_{\mathrm{n}}^2 y - 2\zeta\omega_{\mathrm{n}}\dot{y} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} a$$

This problem is linear so the homogeneous term on the right-hand side can be rewritten as a matrix times a vector

$$\frac{d}{dt}\begin{Bmatrix} y \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} 0 & 1 \\ -\omega_{\mathrm{n}}^2 & -2\zeta\omega_{\mathrm{n}} \end{Bmatrix}\begin{Bmatrix} y \\ \dot{y} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} a, \quad y = \{1 \quad 0\}\begin{Bmatrix} y \\ \dot{y} \end{Bmatrix}$$

We can write this system compactly in vector notation for any linear system, not just the simple one degree of freedom problem we are addressing, as Eq. (1.1b) augmented with a scalar output

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}a, \quad y = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

from which it is easy to draw the vector block diagram (Fig. 2.32)

In fact, this diagram and the accompanying vector equation can be generalized to systems that have more than one input and more than one output. In that case the vectors **b** and **c** become matrices. This is an example of a state space formulation, but I will defer further discussion of state space until Chap. 6.

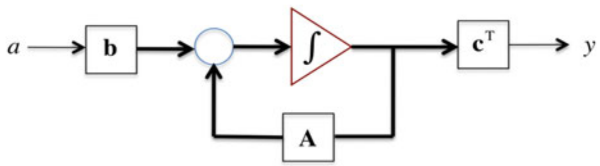


Fig. 2.32 Vector block diagram. The *thick lines* denote vector variables and the *thin lines* scalar variables

2.5.2 Introduction to Simulation: The Simple Pendulum

Most models of engineering problems are nonlinear. We can address linearized versions of these analytically, but frequently this is not sufficient. There is not much one can do analytically with nonlinear problems, and what there is is beyond the scope of this text (although this problem can be pursued further analytically, and I will do that). We can, however, address nonlinear problems numerically by integrating the governing equations. I will refer to this process as *simulation*, and to the results as a *simulation*. Numerical integration in time is usually based on finite differences, replacing the derivatives by differences. The most commonly used integration methods are the various Runge-Kutta schemes. Runge-Kutta schemes have different orders of accuracy. The more accurate the scheme, the longer the integration takes. Fourth-order accuracy is usually chosen, and it is also common to use an adaptive step size routine, taking larger steps where the solution is varying slowly. A thorough discussion of numerical integration is beyond the scope of the text. I refer the interested reader to Press et al. (1992), which not only contains the material but is one of the clearest mathematics books I know. Simulation is not only useful for modeling the dynamics of a mechanism, but it can be used to assess the validity of a linear solution: how well does the linear solution agree with the simulation? I will use commercial software to integrate the nonlinear equations, which can always be converted to a set of quasilinear first-order equations. I did the integrations in this book using Mathematica version 8.0.4.0, for which I believe the integration scheme to be a fourth-order adaptive step size Runge-Kutta scheme.

Example 2.7 Simulating the Simple Pendulum We found a solution for the simple pendulum that was valid for small angles. Let us assess how small these angles need to be by simulating the pendulum, integrating Eq. (2.12a), reproduced here for convenience.

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{2.12a}$$

We can convert this to a pair of first-order equations (state space form) for the state vector

$$\mathbf{x} = \begin{Bmatrix} \theta \\ \dot{\theta} \end{Bmatrix}$$

which gives a pair of coupled ordinary differential equations.

$$\dot{\theta} = \omega, \quad \dot{\omega} = -\frac{g}{l}\sin\theta$$

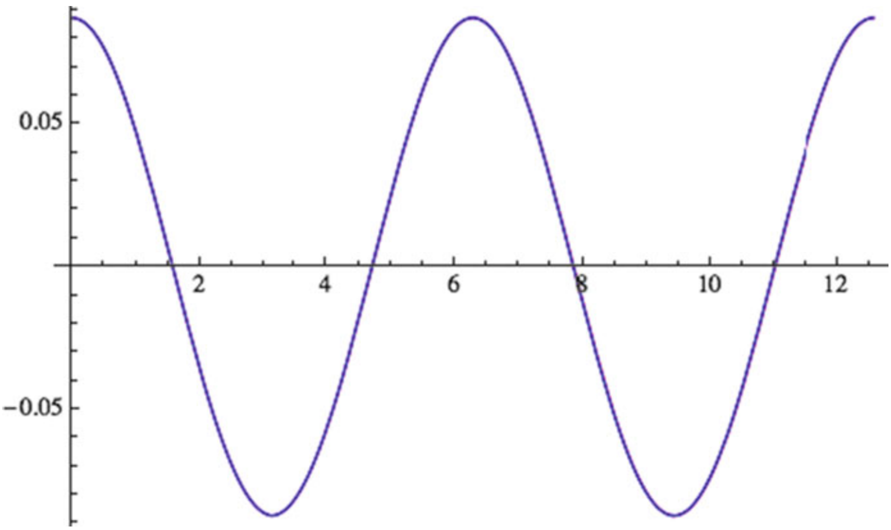


Fig. 2.33 Response of a pendulum started from rest at 5° from vertical

We can compare the linear and nonlinear response by comparing the behavior of the system starting from rest at $\theta = \theta_0$. The linear solution for this case is $\theta_0\cos(\surd(g/l)t)$. I can set $g = 1 = l$ without loss of generality. This makes the period of the linear pendulum equal 2π.

Figure 2.33 shows the linear and nonlinear response of the pendulum over two linear periods for an initial offset of $\pi/36$ (5°). The nonlinear solution (in red) is overlain almost perfectly by the linear solution, in blue. This suggests that 5° is certainly small enough, at least for a few periods.

There's not much visible difference in two periods for initial offsets of 10 and 15°. We can begin to detect a difference at 20°, as shown in Fig. 2.34. The nonlinear solution lags the linear solution, and, while it is periodic, it is no longer harmonic. (In fact, the nonlinear solution is never harmonic, as I will show eventually.)

If we go to 90° we see a sharp difference between the linear and nonlinear solutions, although the nonlinear solution still looks quite harmonic. This is shown in Fig. 2.35.

Finally, if I start the system in a nearly inverted position (0.99π) the entire character of the nonlinear solution changes, so much so that I have to plot four linear periods to give a good impression of the result, shown in Fig. 2.36.

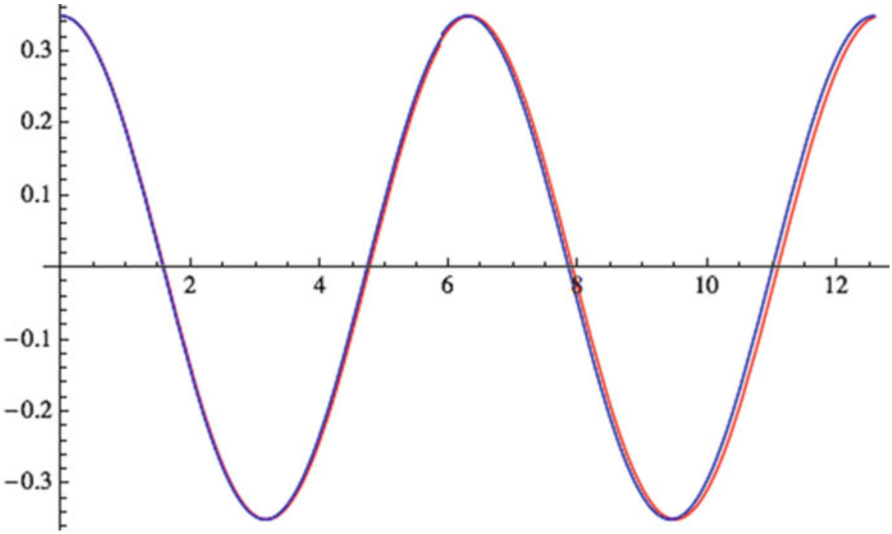


Fig. 2.34 Response of a pendulum started from rest at 20° from vertical

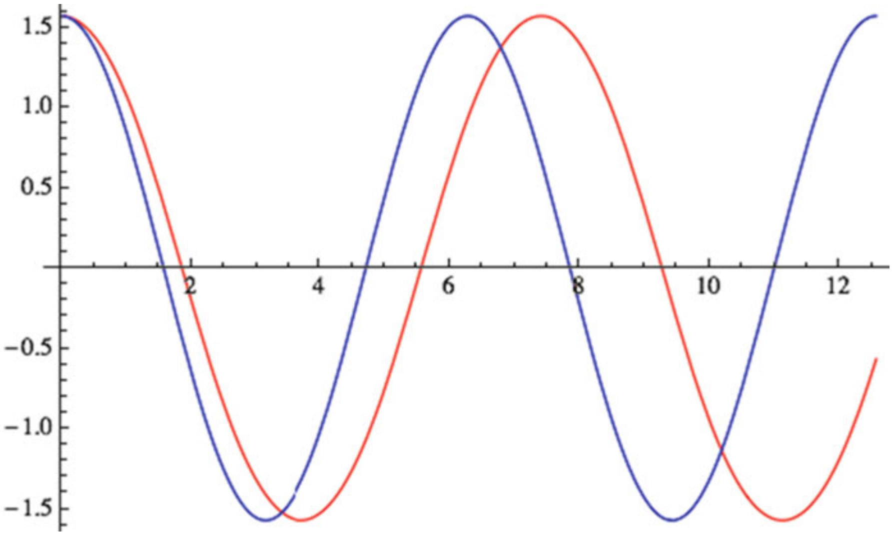


Fig. 2.35 Response of a pendulum started from rest at 90° from vertical

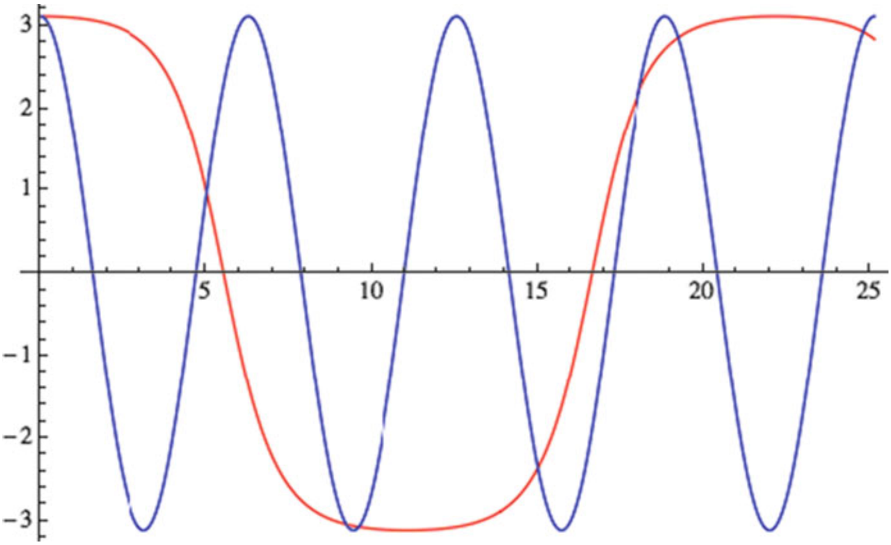


Fig. 2.36 Response of a pendulum started from rest in an almost vertical position

Exercises

1. Verify Eq. (2.2).
2. Solve the following differential equation with its initial conditions

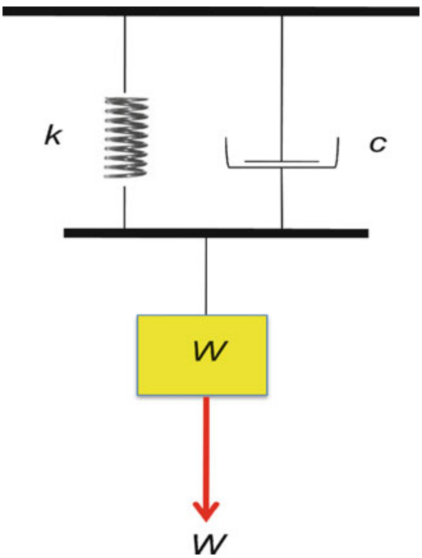
$$\frac{d^2y}{dt^2} + 2y = 0, \quad y(0) = 1, \quad \left.\frac{dy}{dt}\right|_{t=0} = 0$$

3. Solve the following differential equation with its initial conditions

$$\frac{d^2y}{dt^2} + 2y = \sin(t), \quad y(0) = 0, \quad \left.\frac{dy}{dt}\right|_{t=0} = 1$$

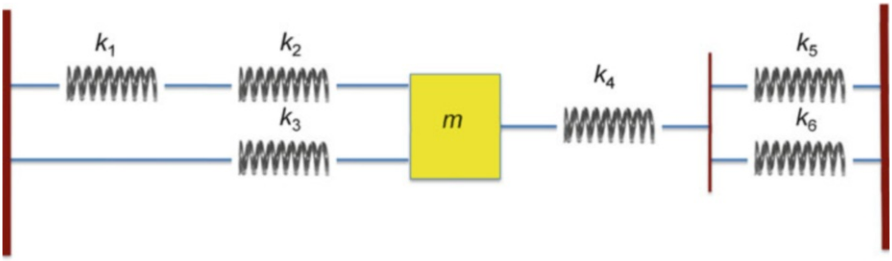
4. Find the natural frequency of a pendulum using the energy approach.
5. Find the amplitude and phase of the motion in terms of A and B in Eqs. (2.3) and (2.4).
6. Consider a simple pendulum initially in equilibrium (pointing straight down). What happens when the pendulum bob is struck impulsively in the horizontal direction? Find the maximum angle in terms of the momentum transferred using the linear approximation.
7. Find the equations of motion for a general pendulum for which the rod is not massless. Find the frequency of oscillation in the linear (small θ) limit.

8. Consider the system shown in Fig. 2.8. Write the differential equations from a free body diagram approach. How does gravity enter the problem? What is the difference between gravity and the spring?
9. Find the general solution to the homogeneous damped equation of motion when $\zeta = 1$.
10. Verify the damped impulse response Eq. (2.19).
11. Find the limit of the damped impulse response as $\zeta \to 1$ from below.
12. Complete the particular solution for an undamped system excited by $a = t^3$.
13. What is the damping ratio for a mass-spring-damper system in which every maximum amplitude is 2 % less than the prior maximum? Suppose that the mass weighs 1 lb and the spring constant is 10 lb/in, what is the value of c (include units)?
14. The deflection of the spring when the system is at rest is half an inch. The mass weighs 20 lbs. The amplitude of a free vibration decreases from 0.4 in to 0.1 in in 20 cycles. What is the damping constant in lb-sec/in?



15. A damped vibrating system consists of a spring of $k = 20$ lb/in and a weight of 10 lbs. It is damped so that each maximum amplitude is 99 % of the maximum one full cycle earlier. (a) Find ω_n. (b) Find the damping constant. (c) What amplitude of a force at ω_n will keep the amplitude of oscillation at 1 in?
16. Find an analytic expression for the force on a mass subject to support motion in the absence of damping.

17. Find the effective spring constant for the system shown in the diagram



mass-spring system for Ex. 17.

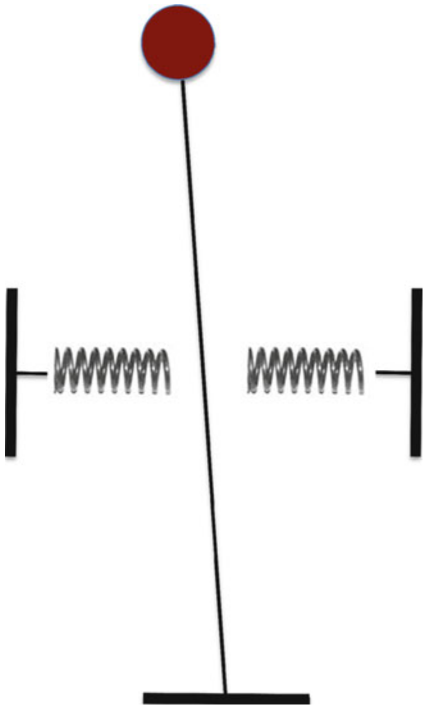
18. Show that dampers add in the same way as springs.
19. Verify Eqs. (2.18a), (2.18b), and (2.19).
20. Apply Eqs. (2.12a) and (2.12b) to the system shown in Fig 2.8.
21. A typical speed bump is about two feet wide and two inches high. Suppose it to have a sinusoidal shape and calculate the response of the simple vehicle examined above for various speeds. Do you think that speed bumps would be effective? Discuss.
22. Find the response of a damped mass-spring system to support motion, as shown in Figs. 2.18 and 2.19.
23. Design a pendulum seismometer with a period of 60 s and a damping ratio of 0.707.
24. Draw the block diagram for the linear pendulum.
25. Draw a block diagram for the two degree of freedom system

$$m_1\ddot{y}_1 + k(y_1 - y_2) = f_1, \quad m_2\ddot{y}_2 + k(y_2 - y_1) = f_2$$

26. Show that the system of Prob. 12 has a solution $y_1 = y_0 + v_0 t = y_2$ if $f_1 = 0 = f_2$. Can you construct a physical system that is represented by the two differential equations?
27. Draw a block diagram for the air bag sensor.
28. Draw a block diagram for the rotating imbalance.
29. Find the eigenvalues and eigenvectors for the system shown in Prob. 8.

The following exercises are based on the chapter, but require additional input. Feel free to use the Internet and the library to expand your understanding of the questions and their answers.

30. Find the natural frequency of a cantilever beam with a mass at its free end if the mass of the beam can be neglected.
31. Consider an inverted pendulum with two springs of constant k on either side, as shown in the figure. The pendulum can move 2° in either direction before coming into contact with the springs. Do not worry about the angles of the springs.



Exercise 31

Set up the equations of motion.

32. The metronome is based on an inverted pendulum with an adjustable position of the bob. How does it work?
33. Why does increasing the damping ratio beyond unity delay the decay of the system?
34. Design an air bag sensor. You will need to learn about how a car crashes.
35. Make a dissipation model for the U-tube of Ex. 2.3 based on fluid viscosity and calculate the behavior of the damped system. Calculate the behavior of the system if the height starts with a 10 % offset. (Choose the viscosity small enough to have an underdamped system.)

References

Crandall SH, Dahl NC (eds) (1959) An introduction to the mechanics of solids. McGraw-Hill, New York

Den Hartog JP (1956) Mechanical vibrations. Dover reprint 1985

Press WH, Teukolsky SA, Vettering WT, Flannery BP (1992) Numerical recipes in C. The art of scientific computing, 2nd edn. Cambridge University Press, New York

More Than One Degree of Freedom Systems and the Euler-Lagrange Process

3

In which we look at system with more than one degree of freedom, introduce the Euler-Lagrange process for deriving the equations of motion for systems of any complexity, introduce the idea of modes, introduce simple DC electric motor as drivers of mechanical systems, and take a look at linearization from a more formal perspective. . ..

3.1 Introduction: Degrees of Freedom

We saw in Chap. 1 that the number of degrees of freedom of a mechanical system corresponds to the number of independent motions the system is capable of. It can be easy to identify the degrees of freedom of simple systems simply using intuition or engineering judgment. I would like to formalize this so that we can have a ritual from which we can find the number of degrees of freedom of any mechanical system. Mechanical systems, mechanisms, are made up of parts, which I will refer to as *links*. All the mechanisms in this book (except for some robots in Chap. 11) are confined to two dimensions, the $x = 0$ plane. This means that an unconstrained link has three degrees of freedom: its center of mass can move in two directions, y and z, and it can rotate through an angle θ. Figure 3.1 shows a sketch of this statement.

If a link is constrained by being attached to the ground by a pin that allows rotation but not translation of the attachment point, the number of degrees of freedom is reduced by two. The angle θ completely determines the location of the center of mass, shown in Fig. 3.2.

This is a general principle. A pin removes two degrees of freedom. Figure 3.3 shows two links pinned together. This mechanism has four degrees of freedom ($2 \times 3 - 2$). If one of the links is pinned to the ground, the system loses two more degrees of freedom to become a two degree of freedom system. Most of this chapter concerns two degree of freedom systems.

A chain of N links pinned together has $3N - 2(N - 1) = N + 2$ degrees of freedom. If one of the links is pinned to the ground, then we have N degrees of freedom.

R.F. Gans, *Mechanical Systems: A Unified Approach to Vibrations and Controls*,
DOI 10.1007/978-3-319-08371-1_3

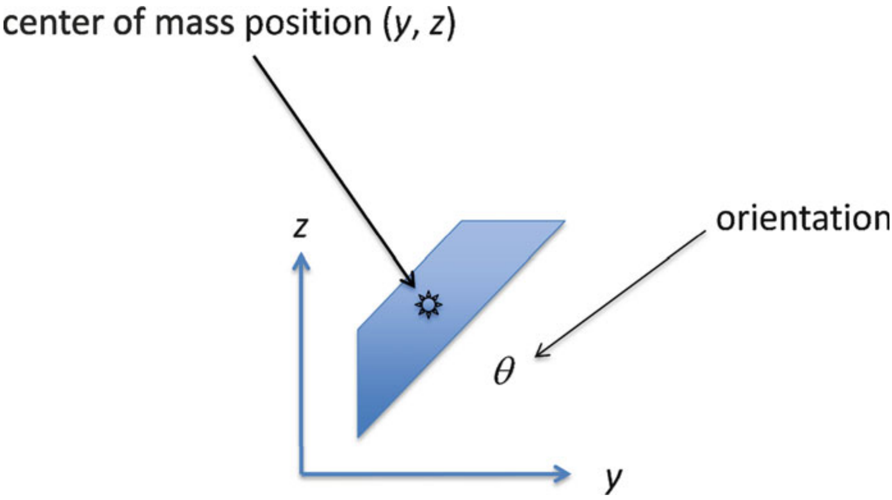


Fig. 3.1 A single unconstrained link

Fig. 3.2 One link constrained by being attached to the ground

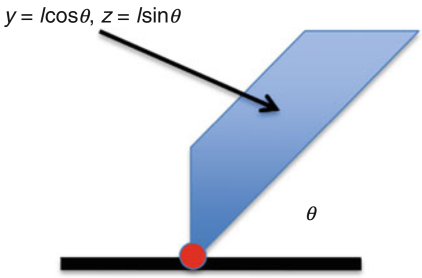
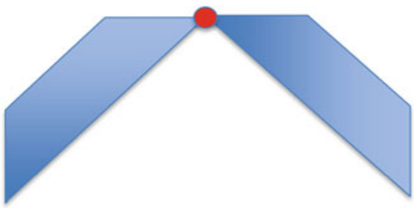


Fig. 3.3 A four degree of freedom mechanism



If two links are pinned to the ground we have a closed chain. I'm not going to pursue this except to note that the case of $N = 3$ gives one degree of freedom. This is the classical *four-bar linkage* that forms the simplest useful kinematic chain. It is called a four-bar linkage because the stretch of ground connecting the end links is considered to be a link in the mechanism.

Figure 3.4 shows a typical four-bar linkage. This one is a crank-rocker linkage. The crank rotates through a full circle and the follower rocks back and forth.

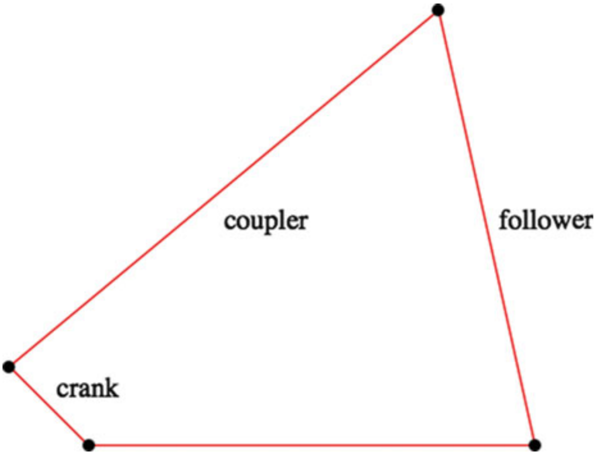


Fig. 3.4 A crank-rocker four-bar linkage

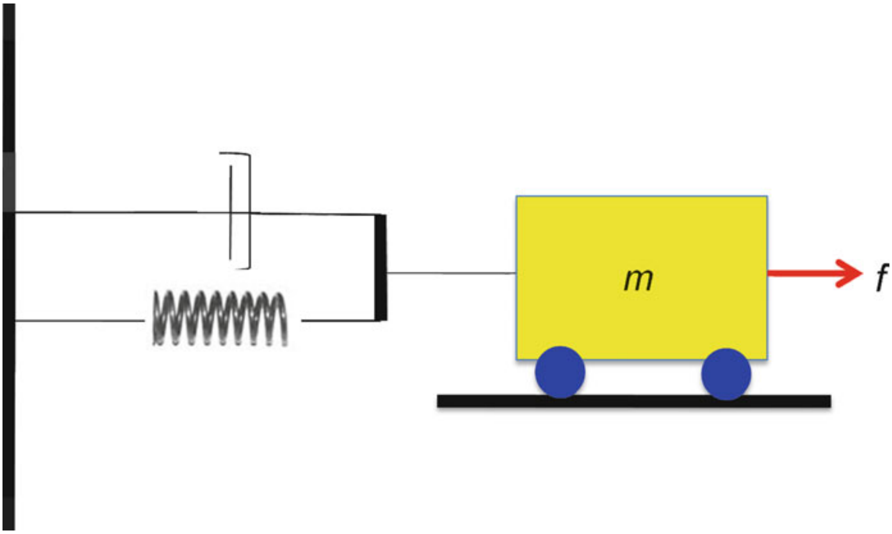


Fig. 3.5 A true one degree of freedom system

I will not discuss analysis of four-bar linkages here. One can see any book on kinematics (e.g., Gans 1991) for such a discussion.

Pins are not the only constraints. Most of the one degree of freedom mechanisms in Chap. 2 are constrained in a more ad hoc fashion. I constrained the simple mass-spring system by fiat, as in the air bag sensor, where I simply declared that the mass can only move in the *y* direction. We can look at it in a more realistic fashion by looking at a cart on a rail attached to a wall by a spring, as in Fig. 3.5, which is equivalent to Fig. 1.2. The rail imposes two constraints on the cart: it cannot move in the vertical direction and it cannot rotate. It is free to move back and forth in the *y* direction.

Fig. 3.6 A wheel inside a pipe

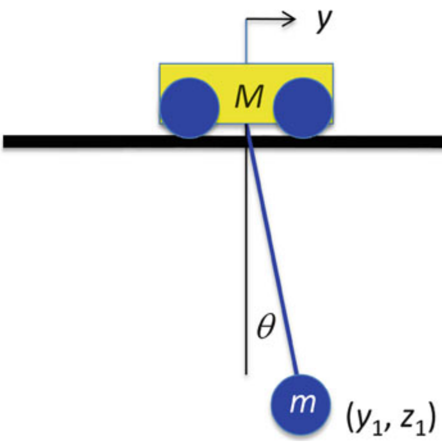


Fig. 3.7 A pendulum pinned to a cart constrained to roll along a rail: a model of the overhead crane

One might ask about the wheels. Technically they are two more links in the system. They are pinned to the cart, so they each have one degree of freedom remaining. If I assume that they roll without slipping on the rail, then their displacement in the y direction is completely determined by their angular position, or, to rephrase that, the angular position is determined by their y position, which is determined by the y position of the cart. Thus the wheels do not add any degrees of freedom (unless they slip), and the system shown in Fig. 3.5 has but one degree of freedom. Figure 3.5 also reminds us that springs and dampers neither add nor remove degrees of freedom. They do contribute forces to the dynamics.

Figure 3.6 shows another example of rolling. If the wheel is constrained to remain on the pipe and roll without slipping, then it has one degree of freedom, which can be written in terms of its angular position on the pipe. If it can slip, but is still constrained to remain on the pipe, then it has two degrees of freedom, which can be written in terms of its angular position on the pipe and the angular position of an imaginary line scribed on the wheel.

Figure 3.7 shows a system that I will discuss shortly (and often). We have a cart constrained to roll without slipping along a rail, which we now know has one degree of freedom, and we have a pendulum that is pinned to the cart. The pendulum adds

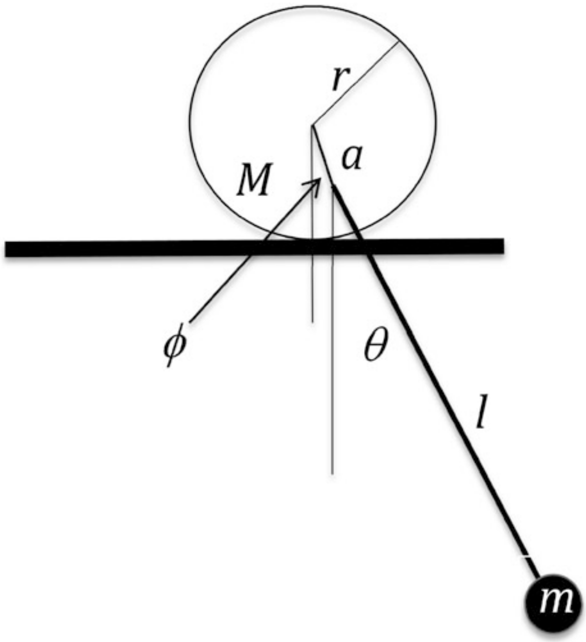


Fig. 3.8 A pendulum attached to a rotating disk

three degrees of freedom, and the pin subtracts two of them, so the system as a whole has two degrees of freedom, which we can choose to be the position of the cart, y, and the pendulum angle, θ.

We were able to derive equations of motion for one degree of freedom systems using free body diagrams. The two fundamental systems were the mass-spring-damper system and the pendulum, which can also be damped, although we did not look at that explicitly. The task of assembling free body diagrams becomes more complicated when there is more than one degree of freedom. There are internal forces of connection. For example, consider the case of a wheel that can roll without slipping on a straight track and suppose there is an off-center pendulum attached to the wheel as shown in Fig. 3.8—a problem in Den Hartog's (1956) text. M denotes the mass of the disk; m the mass of the bob, supposed to be a point mass (the rod is supposed massless); a the offset; r the radius of the disk; ϕ the angle between the line joining the center of the disk and the attachment point of the pendulum; and θ the angle the pendulum makes with the vertical. Both angles increase in the counterclockwise direction. What are the oscillation frequencies?

The system has two degrees of freedom. The wheel starts with three, but the rail prevents vertical motion, and the no-slip condition relates horizontal motion to rotation. The pendulum adds three degrees of freedom, but the pin removes two of them. I choose the two angles as the generalized coordinates.

Try to find the equations of motion using free body diagrams. You will find this difficult. The procedure is prone to error. We need a better way to find equations of motion, and the Euler-Lagrange procedure offers this. The Euler-Lagrange process leads to the Euler-Lagrange equations, which are perfectly compatible with Newton's laws. This is shown in many texts in physics and engineering (e.g., Gans 2013; Goldstein 1980; Meirovitch 1970). I will not derive the Euler-Lagrange equations, but refer the interested reader to any of the references just cited. I will address the problem shown in Fig. 3.8 (which is difficult) as soon as I have the simple Euler-Lagrange equations.

3.2 The Euler-Lagrange Equations

The Euler-Lagrange equations are based on energy, which we discussed in Chap. 1. Equation (1.1) gives the kinetic energy of a single object confined to two dimensions. There are contributions from each degree of freedom. The total kinetic energy is the sum of the kinetic energies of the individual objects making up the system. We can use Eq. (1.1) to find these, but we need to remember to correct the expressions to take account of the constraints. We can write the constraints mathematically by writing some of the physical coordinates in terms of other physical coordinates. How this works will become clear as we move forward. For example, the location of the bob of a simple pendulum (see Fig. 1.2, redrawn with additional notation as Fig. 3.9 below) can be expressed in terms of its y and z coordinates, which can in turn be expressed in terms of the length of the pendulum and the pendulum angle as

$$y = l \sin \theta, \quad z = -l \cos \theta$$

where I have chosen the pivot point as the origin of the y, z coordinate system.

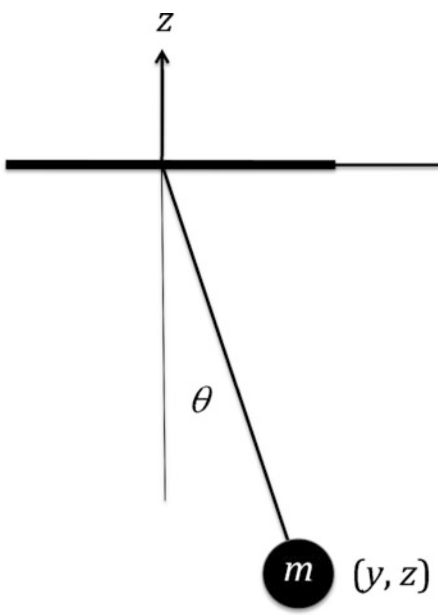


Fig. 3.9 The simple pendulum redrawn

The potential energy has contributions from gravity and whatever springs are in the system. It is best assessed for specific systems. For the simple pendulum

$$V = mgz = -mgl\cos\theta$$

We define the *Lagrangian*, equal to the difference between the potential and kinetic energies $L = T - V$. The Lagrangian is a function of the coordinates defining the system and their derivatives considered as separate variables.

We also need a set of *generalized coordinates*, one for each degree of freedom. These are usually denoted by q_i, and there will be as many of them as there are degrees of freedom. The generalized coordinates are defined in terms of the physical variables, and so the Lagrangian will be a function of the generalized coordinates and their derivatives—$L = L(q_i, \dot{q}_i)$. Mechanical systems can be subject to forces other than those that are contained in the potential, and we need a method to introduce these. We can distinguish between two kinds of nonpotential forces: external forces and forces associated with friction in the form of viscous friction. I will introduce external forces in the form of *generalized forces* and viscous friction through the *Rayleigh dissipation function*. (Other frictional models are nonlinear and are best included as explicit external forces.) We have the basic Euler-Lagrange equations in the absence of damping and external forces[1] and modified equations to include either or both forcing and damping. Let me consider these in turn.

3.2.1 The Basic Undamped, Force-Free Euler-Lagrange Equations

Suppose that we have the potential and kinetic energies in terms of a set of generalized coordinates and that we have formed the Lagrangian. The basic Euler-Lagrange equations are

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) - \frac{\partial L}{\partial q_m} = 0 \\ &= \frac{d}{dt}\left(\frac{1}{2}\left(\sum_{i,j=1}^{N} M_{mj}(q_k)\dot{q}_j\right) + \frac{1}{2}\left(\sum_{i,j=1}^{N} M_{im}(q_k)\dot{q}_i\right)\right) \\ &\quad - \frac{1}{2}\left(\sum_{i,j=1}^{N} \frac{\partial M_{ij}(q_k)}{\partial q_m}\dot{q}_i\dot{q}_j\right) + \frac{\partial V}{\partial q_m}\end{aligned}$$

The matrix $\mathbf{M}$ is symmetric, so the first two terms in the right-hand equation are the same and can be consolidated to give

[1] I will usually refer to external forces simply as forces.

$$\frac{d}{dt}\left(\sum_{i,j=1}^{N} M_{im}(q_k)\dot{q}_i\right) - \frac{1}{2}\left(\sum_{i,j=1}^{N} \frac{\partial M_{ij}(q_k)}{\partial q_m}\dot{q}_i\dot{q}_j\right) + \frac{\partial V}{\partial q_m} = 0 \tag{3.1}$$

There will be as many of these equations as there are generalized coordinates, and each of the equations will be second order. The second derivatives enter linearly, so Eq. (3.1) is quasilinear. The general expression is clumsy and not amenable to easy application. Let me construct an algorithm for constructing Eq. (3.1), and then illustrate the use of the algorithm, first with the offset pendulum introduced in Fig. 3.8, and then with the practical example of an overhead crane, using the simple mode shown in Fig. 3.7. I will use the latter to illustrate various techniques as we move through the text.

Take the Lagrangian in terms of the basic set of generalized coordinates. There will be one generalized coordinate for each degree of freedom. We must identify the degrees of freedom before we can assign generalized coordinates. We know how to find the number of degrees of freedom, and I will discuss sorting them out later in this chapter. For now let us assume that we know how to assign the generalized coordinates and move forward with the algorithm.

1. Take the partial derivative of the Lagrangian with respect to the derivative of the first generalized coordinate: $\frac{\partial L}{\partial \dot{q}_1}$.
2. Differentiate that expression with respect to time: $\frac{d}{dt}\left(\frac{\partial L}{\partial q_1}\right)$. Note that this is a total derivative, d not ∂, and that taking the derivative will generally involve the chain rule. $\frac{d}{dt} = \dot{q}_1\frac{\partial}{\partial q_1} + \ddot{q}_1\frac{\partial}{\partial \dot{q}_1} + \cdots$
3. Subtract from this the partial derivative of the Lagrangian with respect to the first generalized coordinate: $\frac{\partial L}{\partial q_1}$.

This gives the first Euler-Lagrange equation in the absence of damping or external forcing

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0 \tag{3.2}$$

Repeating these operations for all of the generalized coordinates will give a set of simultaneous, generally coupled, second-order ordinary differential equations. The independent variable is the time, t, and the dependent variables are the generalized coordinates. These equations apply when there is no external forcing and no damping. I will construct Eq. (3.2) for the offset pendulum now. There are no external forces and no damping.

Example 3.1 The Offset Pendulum Denote the horizontal position of the disk by y_D and the position of the bob by (y_B, z_B). Let the disk be uniform. The kinetic energy of the system is then

$$T = \frac{1}{2}M\dot{y}_D^2 + \frac{1}{4}Mr^2\dot{\phi}^2 + \frac{1}{2}m\left(\dot{y}_B^2 + \dot{z}_B^2\right),$$

and the only potential energy that can change, and thereby enter the dynamics, is that of the bob

$$V = mgz_B$$

The disk rolls without changing its height, so its potential does not change. These two equations contain four variables, but the system has only two degrees of freedom as noted above: the disk can roll with the pendulum fixed, and the pendulum can move with the disk fixed. We can reduce the number of variables by introducing constraints as I did for the simple pendulum. The location of the bob can be written in terms of the location of the center of the disk and the two angles. The disk rolls without slipping, so its speed is proportional to its rotation rate, and for this simple system that relation can be integrated. The constraints on the bob are

$$y_B = y_D + a\sin\phi + l\sin\theta, \quad z_B = r - a\cos\phi - l\cos\theta$$

where I have taken the origin of z to be the surface of the rail on which the disk rolls. The rolling constraint is just $y = -r\phi$, where y increases to the right. I choose the angles as the generalized coordinates: $q_1 = \phi$ and $q_2 = \theta$. The system is simple enough that I can use the physical coordinates directly, which makes the process easier to follow. These substitutions yield the Lagrangian in terms of ϕ and θ

$$L = \frac{1}{4}\left(2ma^2 + (2m + 3M)r^2 - 4mar\cos\phi\right)\dot{\phi}^2 + \frac{1}{2}ml^2\dot{\theta}^2 \\ + ml(a\cos(\theta - \phi) - r\cos\theta)\dot{\phi}\dot{\theta} - mg(r - a\cos\phi - l\cos\theta)$$

The nonlinear equations are pretty messy

$$\left(m\left(a^2 + r^2\right) - 2mar\cos\phi + \frac{3}{2}Mr^2\right)\ddot{\phi} + ml(a\cos(\theta - \phi) - r\cos\theta)\ddot{\theta} \\ + mar\sin\phi\dot{\phi}^2 + ml(r\sin\theta - a\sin(\theta - \phi))\dot{\theta}^2 + mga\sin\phi = 0$$

$$ml(a\cos(\theta - \phi) - r\cos\theta)\ddot{\phi} + ml^2\ddot{\theta} + mal\sin(\theta - \phi)\dot{\phi}^2 + mgl\sin\theta = 0$$

I will defer linearizing these equations until I have developed a more formal procedure for this. The offset is critical to making this an interesting problem. If $a = 0$ the only occurrence of the wheel angle is through its second derivative, so that $\phi = \phi_0 + \omega_{10}t$ is a solution. The wheel can roll independently of the pendulum if the pendulum pivot is at the axle of the wheel. I will say more about this when I consider the linearized version of this problem.

Fig. 3.10 An overhead crane at a sandstone quarry in the Indian state of Rajasthan (photo by the author)

I will need the potential in terms of the generalized coordinates later. It is

$$V = mgz_{\mathrm{B}} = mg(r - a\cos\phi - l\cos\theta)$$

Let's consider a "real" problem that can be reduced to a two degree of freedom model.

Example 3.2 The Overhead Crane One way of moving material from one place to another is by overhead crane or *gantry*. Figure 3.10 shows an overhead crane being used in a quarry. There is a motor-driven cart on rails that can travel back and forth and a set of cables that hold (in this case) a hook that can carry a load. In this case the whole crane can also be moved from in and out of the plane of the picture. I will ignore that degree of freedom. Let us build a model suitable for analysis.

I suppose the system to be fixed in a plane. I ignore the up and down motion in the cable. I suppose that whatever wheels the cart has roll without slipping. I neglect the flexibility of the cable, replacing it with a rigid rod, and do not consider the case where the length of the rod is changing. (A variable length rod is not an impossible problem, it's just more complicated than we are ready for.) I also restrict the motion to two dimensions. Figure 3.7 shows a cartoon of the simple model, an abstraction from the system shown in Fig. 3.10. I will suppose the crane to be driven in the y direction by some motor driving one or more of the wheels. This is effectively a force on the cart. I will incorporate this force once we have learned about generalized forces.

It should be reasonably clear that this reduced model is a two degree of freedom system, which I established more formally above. The cart can move back and forth and the pendulum can pivot. Formally, the cart is constrained to ride on the rail while its wheels roll without slipping, and the pendulum is pinned to the cart. We need one generalized coordinate for each degree of freedom, and the logical (to me) generalized coordinates are $q_1 = y$ and $q_2 = \theta$. I think the easiest way to construct the equations of motion is to use the Euler-Lagrange approach I just introduced. We can write the kinetic and potential energies most simply in terms of three of the variables in the figure (y, y_1, z_1) and then use constraint relations to rewrite the Lagrangian in terms of the two logical generalized coordinates. This approach minimizes mistakes. We have

$$T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\left(\dot{y}_1^2 + \dot{z}_1^2\right), \quad V = mgz_1$$

The constraints are

$$y_1 = y + l\sin\theta, \quad z_1 = -l\cos\theta$$

where l denotes the length of the (rigid) cable. I will need the potential later: $V = -mgl\cos\theta$. (The cart cannot move in the vertical, so it does not contribute to the potential energy.) The Lagrangian in terms of y and θ is

$$L = \frac{1}{2}(M + m)\dot{y}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\cos\theta\dot{y}\dot{\theta} + mgl\cos\theta$$

This is a sufficiently simple problem that I do not need to substitute the q notation. We can keep track of what is happening without that, and it is easier to maintain contact with the real world if we use the physical versions of the generalized coordinates.

It is essential to reduce the system to one in terms of just the generalized coordinates before moving on. One must eliminate all extraneous coordinates. We must apply the constraints before forming the Euler-Lagrange equations.

We obtain the Euler-Lagrange equations in the manner outlined above. Applying the algorithm given above to both generalized coordinates

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

leads to the Euler-Lagrange equations

$$(M + m)\ddot{y} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 = 0 \tag{3.3a}$$

$$ml\cos\theta\ddot{y} + ml^2\cos\theta\ddot{\theta} + mgl\sin\theta = 0 \tag{3.3b}$$

The first of these is a force balance and the second a torque balance. At this point we have neither external force nor external torque, so the right-hand sides are both zero. The equations are only quasilinear (the highest derivatives occur linearly, although other terms are nonlinear) and so not amenable to direct analytic solution. We will eventually want a numerical solution, but I will defer that until we have a more complete problem, one with forces.

If the angle is small we can construct a linear approximation of the Euler-Lagrange equations by replacing the sine by the angle and the cosine by unity and noting that the terms involving $\dot{\theta}^2$ are formally of third order. (I will discuss linearization in more depth later in this chapter and again in Chap. 6.) The resulting linear equations are

$$(M+m)\ddot{y} + ml\ddot{\theta} = 0 \tag{3.4a}$$

$$ml\ddot{y} + ml^2\ddot{\theta} + mgl\theta = 0 \tag{3.4b}$$

This formulation makes clear how the (linear) system behaves. We can use the first equation to eliminate $\ddot{y}$ from the second equation giving

$$\ddot{\theta} + \left(1 + \frac{m}{M}\right)\frac{g}{l}\theta = 0$$

This is the same as the ordinary linear pendulum equation [Eq. (2.12a)] with the frequency adjusted by the extra mass of the cart. As the mass of the cart increases, the frequency tends to that of the simple pendulum, and the acceleration of the cart goes to zero. Equation (3.4a) shows that the motion of the cart is entirely subordinate to that of the pendulum. If we know θ we can find y. Note that y enters only through its second derivative, so that we can add $Y_0 + V_0 t$ to any solution of the differential Eqs. (3.4a) and (3.4b).

We can treat the homogeneous problem as an initial value problem. All we need are initial conditions. Let's consider what the solution to this homogeneous problem is going to look like. We have a homogeneous set of constant coefficient ordinary differential equations, so we can seek exponential solutions, just as we did for the one degree of freedom problems.

Let

$$y = Y\mathrm{e}^{st}, \quad \theta = \Theta\mathrm{e}^{st}$$

substitute into the homogeneous differential equations and rearrange to obtain the algebraic equivalent of the problem

$$\begin{Bmatrix} s^2(M+m) & s^2ml \\ s^2ml & mgl + s^2ml^2 \end{Bmatrix}\begin{Bmatrix} Y \\ \Theta \end{Bmatrix} = 0$$

The determinant of the matrix must vanish for this to have a nontrivial solution. The determinant is

$$s^2(M+m)\left(mgl+s^2ml^2\right)-s^4m^2l^2=0=s^4+\frac{M+m}{M}\frac{g}{l}s^2$$

so we see that we have one pair of zero roots and one pair of purely imaginary roots. The frequency associated with the imaginary roots is a modified version of the usual pendulum frequency ($\sqrt{(g/l)}$)

$$\omega=\sqrt{\left(1+\frac{m}{M}\right)\frac{g}{l}}$$

In the limit that M goes to infinity the oscillatory motion of the cart stops, the nonzero frequency goes to the usual pendulum frequency, and the system looks like a simple pendulum.

We can shed a little more light on what is happening by looking at the constants that go with the two frequencies. Let $s=0$ and $s_2=\omega$. Then we can write the algebraic equations for the two frequencies as

$$\begin{Bmatrix}0 & 0\\ 0 & mgl^2\end{Bmatrix}\begin{Bmatrix}Y_1\\ \Theta_1\end{Bmatrix}=0,\quad \begin{Bmatrix}-\dfrac{(M+m)^2}{M}\dfrac{g}{l} & -\dfrac{(M+m)}{M}mg\\ -\dfrac{(M+m)}{M}mg & -\dfrac{m}{M}mgl\end{Bmatrix}\begin{Bmatrix}Y_2\\ \Theta_2\end{Bmatrix}=0$$

The constant vectors

$$\begin{Bmatrix}Y_1\\ \Theta_1\end{Bmatrix},\ \begin{Bmatrix}Y_2\\ \Theta_2\end{Bmatrix}$$

are called *modal vectors* and describe *mode shapes*. I will discuss modal analysis formally in Chap. 4.

The first case requires $\Theta_1=0$. We can interpret this to mean that the system is indifferent to the location of the cart, which is perfectly reasonable given our observation that $y=Y_0+V_0t$ is a solution to the homogeneous differential equations. The second case requires

$$Y_2=-\frac{ml}{M+m}\Theta_2$$

The cart moves in the opposite direction from the pendulum bob, and its motion decreases with increasing cart mass. All of this is in perfect accord with intuition, as well as with the brief discussion of the separated equations above.

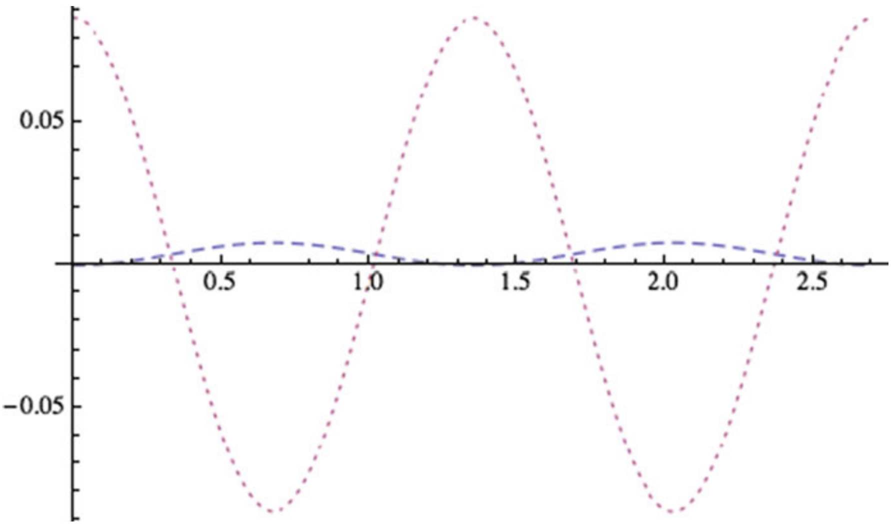


Fig. 3.11 Oscillation of the pendulum-cart model of an overhead crane. The *dashed line* denotes the motion of the cart (with respect to a pendulum length of ½) and the *dotted line* that of the pendulum (in radians)

It is more convenient and informative for this simple problem to write the complete homogeneous solution (including both values of s^2) in terms of trigonometric functions

$$y = Y_0 + V_0 t + Y_c \cos(\omega t) + Y_s \sin(\omega t), \quad \theta = \Theta_c \cos(\omega t) + \Theta_s \sin(\omega t)$$

where the frequency is that found above:

$$\omega = \sqrt{\left(1 + \frac{m}{M}\right)\frac{g}{l}}$$

and

$$Y_c = -\frac{ml}{M+m}\Theta_c, \quad Y_s = -\frac{ml}{M+m}\Theta_s$$

The initial conditions, $y(0) = y_0, \dot{y}(0) = v_0, \theta(0) = \theta_0, \dot{\theta}(0) = \omega_0$, determine the values of the four independent constants that we expect for a two degree of freedom system (a pair of second-order differential equations will require four initial conditions). I leave the determination of these constants to the reader. Figure 3.11 shows the motion of the cart and the pendulum for the linearized system (Eqs. 3.4a and 3.4b) starting from rest at an initial angle of 5°. The mass of the cart is ten times that of the bob and the pendulum length is 0.5 m.

3.2.2 External Forces (and Torques)

We can add external forces using the principle of virtual work. I will construct an algorithm shortly, but let me start with the simple idea of virtual work, generally covered at the end of most introductory statics courses. The idea is that one moves each generalized coordinate virtually without allowing the others to move and sees what force or torque does work. If a force (or torque) does virtual work when the *i*th generalized coordinate q_i moves, then it is entered on the right-hand side of the *i*th Euler-Lagrange equation as Q_i. In the case of the inverted pendulum on a cart, there will be a generalized force in the y equation if the cart is driven. Then Eq. (3.3a) becomes

$$(M + m)\ddot{y} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 = f \tag{3.3c}$$

where f denotes the force on the cart. Equation (3.3b) is unchanged because there is no direct torque on the pendulum.

We can construct a formal procedure that will allow the easy calculation of the generalized forces. We can write the virtual work in terms of virtual displacements and rotations and applied forces and torques. Note that only external forces and torques are to be included. The forces that constrain the mechanism, forces, and torques we might call *forces and torques of connectivity* are taken care of when we apply the constraints to the Lagrangian (and damping will be taken care of by the Rayleigh dissipation functions, introduced below). We can write the virtual work on a single element of a mechanism as

$$\delta W = \mathbf{f} \cdot \delta\mathbf{r} + \tau\delta\theta \tag{3.5}$$

where I use the symbol δ to denote a virtual quantity, W to denote work, and $\mathbf{r}$ to denote the position of the center of mass. The force acts on displacement and the torque on rotation. The total virtual work is the sum of the individual contributions. We generally cannot use Eq. (3.5) directly because it expresses the virtual work in terms of all three of the physical coordinates. These are generally connected by constraints. The virtual displacements and rotations must be consistent with the constraints on the mechanism. We must apply the constraints to Eq. (3.5) before we can proceed. This is perhaps more clearly understood through examples, which I will provide in due course. Let me continue in general a little bit longer. We can write Eq. (3.5) in terms of the generalized coordinates

$$\delta W = \mathbf{f} \cdot \delta\mathbf{r} + \tau\delta\theta = \mathbf{Q} \cdot \delta\mathbf{q} \tag{3.6}$$

We see that we could find the generalized forces by somehow differentiating the virtual work with respect to each component of the virtual displacement/rotation expressed in terms of the generalized coordinates. I can use this idea to construct an algorithm for finding the generalized forces.

Virtual displacements and rotations are fictional motions imposed by us as analysts. We can write them in any consistent manner. For example, I can write $\delta\mathbf{r} = \mathbf{v}\delta t$ because any possible virtual displacement must be parallel to the local instantaneous velocity. Thus I can rewrite Eq. (3.5) as

$$\delta W = \dot{W}\delta t = \mathbf{f} \cdot \mathbf{v}\delta t + \tau\dot{\theta}\delta t \tag{3.7}$$

We care about the sum of these terms over all the links in the mechanism, making

$$\delta W = \dot{W}\delta t = \sum_{i=1}^{N} \left(\mathbf{f}_i \cdot \mathbf{v}_i\delta t + \tau_i\dot{\theta}_i\delta t\right) = \mathbf{Q} \cdot \dot{\mathbf{q}}\delta t \tag{3.8}$$

We can factor out the δt and then note that

$$Q_m = \frac{\partial \dot{W}}{\partial \dot{q}_m} = \sum_{i=1}^{N} \left(\mathbf{f}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_m} + \tau_i \frac{\partial \dot{\theta}_i}{\partial \dot{q}_m}\right) \tag{3.9}$$

Equation (3.9) is a recipe for the generalized forces: write the rate of work in terms of the generalized coordinates and their derivatives and differentiate with respect to the derivatives.

Let's see how this works in a specific case, the pendulum on a cart shown in Fig. 3.7. Suppose that we have a force on the cart directed through the center of mass and that we have a motor attached to the cart that can provide a torque on the pendulum (and a reaction torque on the cart). We can write the rate of work done as

$$\dot{W} = f\dot{y} + \tau\dot{\theta}$$

The reaction torque does no work on the cart because the cart cannot rotate. The force is applied to the cart, not the pendulum. Internal forces of connectivity do not count. Equation (3.9) gives us $Q_1 = f$, and $Q_2 = \tau$.

We can do something similar for the pendulum on the disk, shown in Fig. 3.8. Let there be a motor that can provide torque at the pivot point of the pendulum. Suppose the torque on the pendulum to be τ. The reaction torque on the disk will be $-\tau$. In this case both the positive torque and the reaction torque do work. We have

$$\dot{W} = \tau\dot{\theta} - \tau\dot{\phi},$$

and the generalized forces given by Eq. (3.9) are $Q_1 = -\tau$, and $Q_2 = \tau$, where I treat the disk as the first link and the pendulum as the second link. It is important to remember reaction forces and torques, as sometimes they do contribute to the rate of work.

3.2.3 Dissipation and the Rayleigh Dissipation Function

In real systems there is friction. I discussed friction in Chap. 2, and I will limit myself to viscous friction at this point. The force associated with viscous friction is proportional to the derivative of the generalized coordinate. If we suppose that there is viscous friction at the joint between the cart and the pendulum in the example just worked, then the friction torque will be

$$\tau_{\mathrm{f}} = -c\dot{\theta}$$

where c denotes a damping coefficient and the minus sign comes from the fact that viscous friction opposes the motion. We can put this into the θ equation (because θ is the variable being displaced virtually) to obtain

$$ml^2\ddot{\theta} + ml\ddot{y} - mgl\theta = -\nu\theta\dot{\theta} \rightarrow ml^2\ddot{\theta} + ml\ddot{y} - mgl\theta - c_\theta\dot{\theta} = 0$$

We can do the same sort of thing if we suppose some friction in the bearings of the wheels. That will be proportional to the rotation rate of the wheels, which is directly proportional to the speed of the cart. This will enter the y equation to give

$$(M+m)\ddot{y} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 = -c_y\dot{y} \rightarrow (M+m)\ddot{y} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 - c_y\dot{y} = 0$$

I have introduced subscripts on the two damping constants to identify them.

The reader may note that the function

$$\mathsf{F} = \frac{1}{2}c_y\dot{y}^2 + \frac{1}{2}c_\theta\dot{\theta}^2$$

can be used to introduce these dissipative terms. We can generalize this to

$$\mathsf{F} = \frac{1}{2}\sum_{i,j=1}^{N} c_{ij}\dot{q}_i\dot{q}_j \tag{3.10}$$

which I will refer to as the *Rayleigh dissipation function*. There will be no cross terms in almost all cases (i.e., c_{ij} is usually zero if $i \neq j$). We can rewrite the Euler-Lagrange equations to include the gradient of F as well as the generalized force

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial \mathsf{F}}{\partial \dot{q}_i} = Q_i \tag{3.11}$$

I will take Eq. (3.11) to be the Euler-Lagrange equations. We can extend the algorithm to include the finding of the Rayleigh dissipation function. Note that the contribution to the equations of motion from the Rayleigh dissipation function is always linear in the generalized coordinates.

Let us see how this works by revisiting the pendulum on the cart model of the overhead crane shown in Figure 3.7. I add a force, f, on the cart; torque, τ, on the pendulum at the pivot point; viscous damping to the movement of the cart; and rotary damper at the base of the pendulum. The Rayleigh dissipation function has two terms and two dissipation constants

$$\mathsf{F} = \frac{1}{2}c_y\dot{y}^2 + \frac{1}{2}c_\theta\dot{\theta}^2$$

We've already found that the generalized force in the y equation is f and that in the θ equation is τ. The nonlinear governing equations become

$$(M+m)\ddot{y} + c_y\dot{y} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 = f \tag{3.3d}$$

$$ml^2\ddot{\theta} + c_\theta\dot{\theta} + ml\cos\theta\ddot{y} + mgl\sin\theta = \tau \tag{3.3e}$$

There are two new dissipation terms on the left-hand sides and two forcing terms on the right-hand side. I will repeat the linearization process for this complete problem after we have looked at linearization more formally.

3.3 Linearization and Stability I

Linearization is a powerful tool for attacking nonlinear problems. It can convert a very difficult (usually inaccessible analytically) problem into a tractable linear problem over a limited range of response. Linearization is often taught on an ad hoc basis. I linearized the overhead crane in an ad hoc fashion. I want to construct a formal procedure that can be applied to any problem. I will introduce the procedure here in the Euler-Lagrange context, and I will revisit the procedure for state space formulations in Chap. 6. While linearized problems are tractable, one should always check that the solution to a linear problem does not violate the assumptions on which the linearization was based. If it does violate those assumptions, it is highly unlikely to be correct. If I assume small angles to solve a problem, the angles in the solution to the problem must be small. (This is necessary but not sufficient. The validity of the linear solution can only be truly assessed by comparing it to a simulation based on the full nonlinear equations.) Linearization is most easily addressed in terms of state space, which I will do in Chap. 6. We have, however, already worked on the linear versions of several nonlinear problems. I will defer the state space linearization process until Chap. 6 and address linearization here in terms of the Euler-Lagrange equations.

Linearization means linearization with respect to some reference state, some *equilibrium*. Equilibria can be stable or unstable. We are said to linearize *about* the equilibrium reference state. The equilibrium state must satisfy the nonlinear equations of motion. All equilibria in this book will be time independent. Equilibria may or may not require external forces. If a small departure from the equilibrium grows without bound, the system is said to be *unstable*. If it decays it is said to be *stable*. If it neither grows nor decays, it is said to be *marginally stable*. The small departure from equilibrium satisfies the linearized equations of motion, which I am about to address. These linearized equations have exponential solutions, and the exponents determine the stability. The exponents are in general complex, and I can summarize the connection between the exponents and stability as follows:

- If the real parts of *all* the exponents are negative, then the equilibrium is stable.
- If the real part of *any* exponent is positive, then the equilibrium is unstable.
- If all the exponents have zero real parts and nonzero imaginary parts, then the equilibrium is marginally stable.
- Zero exponents are best discussed on a problem by problem basis. They generally imply an algebraic solution, as we saw for the overhead crane.

The equilibrium for an ordinary pendulum is with the pendulum vertical and no applied torque.[2] If the bob is hanging down ($\theta = 0$) it is a *stable* equilibrium. If it is pointing up ($\theta = \pi$), it is an *unstable* equilibrium. In either case the linear equations of motion govern the motion of the bob in the neighborhood of the equilibrium position. They give a good approximation in the stable case, but not in the unstable case, where the linear solution grows exponentially, quickly moving out of the range of the assumptions ($|\theta - \pi|$ small) that justified linearization.

Let's start by proceeding generally. Equations (3.1) can be modified to include dissipation and external forces.

$$\frac{d}{dt}\left(\sum_{i,j=1}^{N} M_{im}(q_k)\dot{q}_i\right) - \frac{1}{2}\left(\sum_{i,j=1}^{N} \frac{\partial M_{ij}(q_k)}{\partial q_m}\dot{q}_i\dot{q}_j\right) + \frac{\partial V}{\partial q_m} + \frac{\partial F}{\partial \dot{q}_m} = Q_m \tag{3.12a}$$

This is the mth Euler-Lagrange equation with dissipation and a generalized force. I consider only static equilibria (reference states), $\dot{\mathbf{q}}_0 = \mathbf{0} = \ddot{\mathbf{q}}_0$, so the reference state must satisfy N equations of motion of the form

$$\left.\frac{\partial V}{\partial q_m}\right|_{\mathbf{q}=\mathbf{q}_0} = Q_{m0} \tag{3.13}$$

where I have used the vector $\mathbf{q}$ to stand for the entire set of generalized coordinates and the zero subscript to denote the reference state. Equation (3.13) expresses a balance between potential forces and external forces.

Note that static equilibria do not involve the dissipation because viscous dissipation is proportional to the derivatives of the generalized coordinates. In an unforced equilibrium ($Q_{m0} = 0$) we simply seek zeroes of the gradient (in configuration space) of the potential. That is, we seek maxima and minima in configuration space.

[2] One can find an equilibrium that is not vertical for the proper imposed torque. This might be fun for an exercise, but it could be confusing in the present context.

Minima correspond to stable equilibria and maxima to unstable equilibria. We can see this in the simple example of an unforced simple pendulum. The potential is $mgz = -mgl\cos\theta$ if we take z positive up and gravity down. The gradient is proportional to $\sin\theta$, so the extrema are at $\theta = 0, \pi$. The former corresponds to a minimum and represents the stable bob-pointing-down position. The latter corresponds to a maximum, that of the (unstable) inverted pendulum.

We want to write differential equations that govern small departures from equilibrium. The easiest way to so this is to introduce an artificial small parameter ε and write each generalized coordinate as

$$q_i = q_i(\varepsilon) = q_{i0} + \varepsilon q_{i1} + \cdots \tag{3.14}$$

We do the same thing for the generalized forces. Then we can rewrite Eq. (3.12a) as

$$\sum_{i,j=1}^{N} M_{im}(q_k)\ddot{q}_i + \sum_{i,j=1}^{N} \frac{\partial M_{im}(q_k)}{q_k}\dot{q}_i\dot{q}_k - \frac{1}{2}\left(\sum_{i,j=1}^{N} \frac{\partial M_{ij}(q_k)}{\partial q_m}\dot{q}_i\dot{q}_j\right) + \frac{\partial V}{\partial q_m} + \sum_{i,=1}^{N} \nu_{im}\dot{q}_i = Q_m \tag{3.12b}$$

where I have used Eq. (3.10) to substitute for the Rayleigh dissipation term. This equation is a function of the small parameter ε. We know that when ε equals zero, it reduces to Eq. (3.13), which is satisfied. Equation (3.12b) has a Taylor series in ε. The linear term in ε is the linear equation governing the small departures from equilibrium. Let's take a look at this more formally.

Recall that the Taylor series of a function of ε around $\varepsilon = 0$ is given by

$$f(\varepsilon) = f(0) + \left.\frac{df}{d\varepsilon}\right|_{\varepsilon=0}\varepsilon + \left.\frac{1d^2f}{2d\varepsilon^2}\right|_{\varepsilon=0}\varepsilon^2 + \cdots,$$

so a linear approximation can be constructed by choosing just the first two terms, the equilibrium and the linear departure from equilibrium. How do we apply this to the Euler-Lagrange equations? We can move the forcing to the left-hand side of the equation, which makes Eq. (3.12b) a function of the generalized coordinates, their derivatives, and the forcing that is equal to zero.

$$\frac{d}{dt}\left(\sum_{i,j=1}^{N} M_{im}(q_k)\dot{q}_i\right) - \frac{1}{2}\left(\sum_{i,j=1}^{N} \frac{\partial M_{ij}(q_k)}{\partial q_m}\dot{q}_i\dot{q}_j\right) + \frac{\partial V}{\partial q_m} + \frac{\partial F}{\partial \dot{q}_m} - Q_m = 0$$
$$= \mathrm{E}_m(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \mathbf{Q}) \tag{3.15}$$

We see that each equation can be viewed as a function of $\mathbf{q}$, its derivatives, and $\mathbf{Q}$. These variables are functions of ε, so the function given by Eq. (3.15) is a function of ε. We must start with an equilibrium solution for which ε is equal to zero, which I can denote by $\mathbf{q}_0$ and $\mathbf{Q}_0$. Our equilibria will be stationary so the time derivatives of $\mathbf{q}_0$ are all zero. Linear equations can be obtained from the Taylor series representation of E_m, which is obtained by differentiation. We note that $\mathbf{q}$ and all its derivatives, as well as $\mathbf{Q}$, are functions of ε. We can differentiate with respect to ε by applying the chain rule.

$$\begin{aligned} E_m(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},Q_m) = E_m(\mathbf{q}_0,\dot{\mathbf{q}}_0,\ddot{\mathbf{q}}_0,Q_{m0}) &+ \left(\left.\frac{\partial E_m(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},\mathbf{Q})}{\partial Q_m}\right|_{\varepsilon=0}\frac{dQ_m}{d\varepsilon}\right)\varepsilon \\ &+ \left(\left.\frac{\partial E_m(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},\mathbf{Q})}{\partial q_1}\right|_{\varepsilon=0}\frac{dq_1}{d\varepsilon}+\left.\frac{\partial E_m(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},\mathbf{Q})}{\partial q_2}\right|_{\varepsilon=0}\frac{dq_2}{d\varepsilon}+\cdots\right)\varepsilon \\ &+ \left(\left.\frac{\partial E_m(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},\mathbf{Q})}{\partial \dot{q}_1}\right|_{\varepsilon=0}\frac{d\dot{q}_1}{d\varepsilon}+\left.\frac{\partial E_m(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},\mathbf{Q})}{\partial \dot{q}_2}\right|_{\varepsilon=0}\frac{d\dot{q}_2}{d\varepsilon}+\cdots\right)\varepsilon \\ &+ \left(\left.\frac{\partial E_m(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},\mathbf{Q})}{\partial \ddot{q}_1}\right|_{\varepsilon=0}\frac{d\ddot{q}_1}{d\varepsilon}+\left.\frac{\partial E_m(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},\mathbf{Q})}{\partial \ddot{q}_2}\right|_{\varepsilon=0}\frac{d\ddot{q}_2}{d\varepsilon}+\cdots\right)\varepsilon \\ &+\cdots = 0 \end{aligned}$$

For this to vanish the coefficient of each power of ε must vanish independently. The first term on the right-hand side is identically zero because the zeroth-order representation satisfies the nonlinear Euler-Lagrange equations by definition. We obtain the linear Euler-Lagrange equations by equating the terms multiplied by ε to zero. This is the linear term in the Taylor series, and it gives us the linear equations of motion.

The general picture looks very complicated. Do not let it intimidate you. I've put it in to justify what is actually a pretty simple procedure in practice. If we are confident we can do some of this "in our heads." The second and third terms of the linear part of the Taylor expansion are automatically zero because the reference state is constant, so the product of the derivatives is proportional to ε^2. The first and last terms on the left-hand side are proportional to ε, so the expansion of the inertia and dissipation matrices is not necessary. Thus, without doing any nasty mathematics we can see that the linear equations of motion can be derived from

$$\sum_{i=1}^{N} M_{im}(q_{k0})\varepsilon\ddot{q}_{i1} + \frac{\partial V}{\partial q_m} + \sum_{i=1}^{N} c_{im}\varepsilon\dot{q}_{i1} = Q_{m0} + \varepsilon Q_{m1}$$

The only thing we need to do is work out the expansion of the potential, and that is reasonably straightforward. (We'll see an example shortly.) We have, finally, the linear equations in general form

$$\sum_{i=1}^{N} M_{im}(q_{k0})\varepsilon\ddot{q}_{i1} + \sum_{i=1}^{N} \frac{\partial^2 V}{\partial q_i \partial q_m}\varepsilon q_{i1} + \sum_{i=1}^{N} c_{im}\varepsilon\dot{q}_{i1} = \varepsilon Q_{m1}$$

which I can now divide by ε to give me the linear equation set

$$\sum_{i=1}^{N} M_{im}(q_{k0})\ddot{q}_{i1} + \sum_{i=1}^{N} \frac{\partial^2 V}{\partial q_i \partial q_m} q_{i1} + \sum_{i=1}^{N} c_{im}\dot{q}_{i1} = Q_{m1} \tag{3.16}$$

This is not necessarily how one would actually approach the linearization problem. I will demonstrate a simpler way to linearize problems based only on the Euler-Lagrange equations. In a more general setting the idea of a Taylor expansion in an artificial small parameter will always carry one through to a successful linearization. It is our version of the mills of the gods—it takes a while, but it works.

It is essential to remember that linearization applies to the differential equations, NOT the Lagrangian. Linearizing the Lagrangian leads to disaster.[3] Find the general nonlinear equations (which you will often want for a simulation) and THEN linearize. On the other hand, if you can find the equilibrium without finding the equations of motion—as we can with the pendulum—then you can work directly from Eq. (3.11), which can be rewritten as

$$\mathbf{M}_0\ddot{\mathbf{q}}_1 + \mathbf{C}_0\dot{\mathbf{q}}_1 + \mathbf{K}_0\mathbf{q}_1 = \mathbf{Q}_1 \tag{3.17}$$

where the matrices can be extracted from Eq. (3.16). **M** and **C** are the *inertia matrix* and the matrix found in the Rayleigh dissipation function, which I will refer to as the *damping matrix*. They are symmetric. The matrix **K** comes from the potential function and is also symmetric, as I will establish. I will refer to the **K** matrix as the *stiffness matrix*. The phrase comes from structural engineering, for which the potential is usually elastic. There is no harm in adopting the name for any matrix derived from a restoring potential. The subscript 0 means to replace the variables in the matrices by their equilibrium values. The double derivative of V in Eq. (3.11) and the stiffness matrix **K** are not obviously connected, but it only takes a brief digression to connect them. We can write the gradient of the potential as a row vector

$$\nabla V = \frac{\partial V}{\partial q_m} = \left\{ \frac{\partial V}{\partial q_1} \quad \frac{\partial V}{\partial q_2} \quad \cdots \quad \frac{\partial V}{\partial q_N} \right\}$$

where N denotes the number of Euler-Lagrange equations and q_m on the left-hand side stands for all the variables at the same time. This is an indicial notation for vectors and matrices which I have generally avoided. The vector gradient of this, which is a matrix, may be written as

[3] One could "quadraticize" the Lagrangian, but that is likely to be confusing and more trouble than it is worth.

$$\frac{\partial^2 V}{\partial q_i \partial q_m} = \begin{Bmatrix} \frac{\partial}{\partial q_1} \\ \frac{\partial}{\partial q_2} \\ \vdots \\ \frac{\partial}{\partial q_N} \end{Bmatrix} \begin{Bmatrix} \frac{\partial V}{\partial q_1} & \frac{\partial V}{\partial q_2} & \cdots & \frac{\partial V}{\partial q_N} \end{Bmatrix}$$

$$= \begin{Bmatrix} \frac{\partial^2 V}{\partial q_1 \partial q_1} & \frac{\partial^2 V}{\partial q_1 \partial q_2} & \cdots & \frac{\partial^2 V}{\partial q_1 \partial q_N} \\ \frac{\partial^2 V}{\partial q_2 \partial q_1} & \frac{\partial^2 V}{\partial q_2 \partial q_2} & \cdots & \frac{\partial^2 V}{\partial q_2 \partial q_N} \\ \vdots & \cdots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \frac{\partial^2 V}{\partial q_N \partial q_N} \end{Bmatrix} = \mathbf{K}$$

The individual elements of **K** are the second derivatives. The matrix is obviously symmetric. (One must calculate the matrix directly from the potential before substituting the equilibrium condition.)

Note that we can tell whether a system is going to lead to linear equations directly from the energies. If the inertia matrix and the damping matrix are constant, and the potential energy is no more than quadratic in the generalized coordinates, then the differential equations derived from the Euler-Lagrange process are automatically linear and of the form of Eqs. (3.11) and (3.12a). We can actually avoid the Euler-Lagrange process and merely use its fruits if we limit ourselves to linear problems. We cannot dispose of the process entirely because we want to be able to simulate nonlinear systems, which we can only do using the full nonlinear equations of motion.

Example 3.3a $\boldsymbol{\varepsilon}$ Linearization Let me illustrate the general procedure for linearization based on ε using the equations from Example 3.1 (see Fig. 3.8) before looking at the compact method. I rewrite those nonlinear equations here for convenience

$$\left(m\left(a^2+r^2\right)-2mar\cos\phi+\frac{3}{2}Mr^2\right)\ddot{\phi}+ml(a\cos(\theta-\phi)-r\cos\theta)\ddot{\theta}$$
$$+mar\sin\phi\dot{\phi}^2+ml(r\sin\theta-a\sin(\theta-\phi))\dot{\theta}^2+mga\sin\phi=0$$

$$ml(a\cos(\theta-\phi)-r\cos\theta)\ddot{\phi}+ml^2\ddot{\theta}+mal\sin(\theta-\phi)\dot{\phi}^2+mgl\sin\theta=0$$

We can consider the equilibrium $\phi = 0 = \theta$, which is stable. We can do this in steps just as in the development above. First we drop the nonlinear derivative terms to obtain

$$\left(m\left(a^2 + r^2\right) - 2mar\cos\phi + \frac{3}{2}Mr^2\right)\ddot{\phi} + ml(a\cos(\theta - \phi) - r\cos\theta)\ddot{\theta}$$
$$+ mga\sin\phi = 0$$

$$ml(a\cos(\theta - \phi) - r\cos\theta)\ddot{\phi} + ml^2\ddot{\theta} + mgl\sin\theta = 0$$

Evaluate the coefficients of the second derivatives at equilibrium

$$\left(m\left(a^2 + r^2\right) - 2mar + \frac{3}{2}Mr^2\right)\ddot{\phi} + ml(a - r)\ddot{\theta} + mga\sin\phi = 0$$

$$ml(a - r)\ddot{\phi} + ml^2\ddot{\theta} + mgl\sin\theta = 0$$

Expanding the remaining trigonometric functions (which come from the potential, so we are expanding the derivative of the potential as in the general procedure) gives us the linear equations of motion, where ϕ and θ really represent $\varepsilon\phi_1$ and $\varepsilon\theta_1$.

$$\left(m(a - r)^2 + \frac{3}{2}Mr^2\right)\ddot{\phi} + ml(a - r)\ddot{\theta} + mga\phi = 0$$

$$ml(a - r)\ddot{\phi} + ml^2\ddot{\theta} + mgl\theta = 0$$

Example 3.3b Direct Matrix Linearization We can do this the short way as well. The kinetic energy after substitution is

$$T = \frac{1}{4}\left(2ma^2 + (2m + 3M)r^2 - 4mar\cos\phi\right)\dot{\phi}^2 + \frac{1}{2}ml^2\dot{\theta}^2$$
$$+ ml(a\cos(\theta - \phi) - r\cos\theta)\dot{\phi}\dot{\theta}$$

and the potential energy is

$$V = mg(r - a\cos\phi - l\cos\theta)$$

We can find the elements of the inertia matrix by differentiating the kinetic energy and then substituting the equilibrium values. The i, j element is

$$m_{ij} = \frac{\partial^2 T}{\partial\dot{q}_i\partial\dot{q}_j},$$

and the resulting matrix, after substituting the equilibrium values, is

$$\mathbf{M}_0 = \begin{Bmatrix} m(a-r)^2 + \frac{3}{2}Mr^2 & ml(a-r) \\ ml(a-r) & ml^2 \end{Bmatrix}$$

We obtain the elements of **K** in analogous fashion

$$k_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j},$$

and the resulting matrix, after substituting the equilibrium values, is

$$\mathbf{K}_0 = \begin{Bmatrix} mga & 0 \\ 0 & mgl \end{Bmatrix}$$

The differential equations become

$$\mathbf{M}_0\ddot{\mathbf{q}} + \mathbf{K}_0\mathbf{q} = 0,$$

and the reader can verify that these are identical to the equations found the long way (Ex. 3.3a)

Example 3.4 The Overhead Crane Revisited Now let's linearize the pendulum on a cart model of the overhead crane. We have the differential equations

$$(M+m)\ddot{y} + c_y\dot{y} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 = f$$

$$ml^2\ddot{\theta} + c_\theta\dot{\theta} + ml\cos\theta\ddot{y} + mgl\sin\theta = \tau$$

There is an equilibrium with the cart stationary (we can admit steady motion of the cart, but there is little to be gained from this) and the pendulum hanging straight down with the two generalized forces equal to zero. We must suppose that the generalized forces are also proportional to ε when we apply the ε method. Applying Eq. (3.10) (and keeping only the parts proportional to ε) to this problem and replacing the generalized forces gives

$$(M+m)\varepsilon\ddot{y}_1 + c_y\varepsilon\dot{y}_1 + ml\cos(\varepsilon\theta_1)\varepsilon\ddot{\theta}_1 - ml\sin(\varepsilon\theta_1)\varepsilon^2\dot{\theta}_1^2 = \varepsilon f_1$$

$$ml^2\varepsilon\ddot{\theta}_1 + c_\theta\varepsilon\dot{\theta}_1 + ml\cos(\varepsilon\theta_1)\varepsilon\ddot{y}_1 + mgl\sin(\varepsilon\theta_1) = \varepsilon\tau_1$$

We differentiate with respect to ε

$$(M+m)\ddot{y}_1 + c_y\dot{y}_1 + ml\cos(\varepsilon\theta_1)\ddot{\theta}_1 - ml\theta_1\sin(\varepsilon\theta_1)\varepsilon\ddot{\theta}_1 - 2ml\sin(\varepsilon\theta_1)\varepsilon\dot{\theta}_1^2$$
$$- ml\theta_1\cos(\varepsilon\theta_1)\varepsilon^2\dot{\theta}_1^2 = f_1$$

$$ml^2\ddot{\theta}_1 + c_\theta\dot{\theta}_1 + ml\cos(\varepsilon\theta_1)\ddot{y}_1 - ml\theta_1\sin(\varepsilon\theta_1)\varepsilon\ddot{y}_1 + mgl\theta_1\cos(\varepsilon\theta_1) = \tau_1$$

and then set ε equal to zero.

$$(M+m)\ddot{y}_1 + c_y\dot{y}_1 + ml\ddot{\theta}_1 = f_1$$

$$ml^2\ddot{\theta}_1 + c_\theta\dot{\theta}_1 + ml\ddot{y}_1 + mgl\theta_1 = \tau_1$$

These are the linear equations governing the motion of the overhead crane obtained by an explicit application of the ε method.

We can do the same thing by using the expansion of the E_m functions. Let me summarize that here for Eq. (3.3d). I will leave (3.3e) as an exercise for the reader. We have E_1 from Eq. (3.3d)

$$E_1 = (M+m)\ddot{y} + c_y\dot{y} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 - f$$

We note that the derivative terms in the expansion of E_1 are all of the form

$$\left.\frac{\partial X}{\partial \varepsilon}\right|_{\varepsilon\to 0} = X_1$$

where X stands for any of the variables appearing in the expansion. All the variables—generalized coordinates and their derivatives and f—are equal to zero at equilibrium. We form all the terms in the expansion by differentiation, followed by substitution of the equilibrium:

$$\frac{\partial E_1}{\partial Q_1} = -1 \to -1 \Rightarrow -f_1$$

$$\frac{\partial E_1}{\partial q_1} = 0 \to 0 \Rightarrow 0$$

$$\frac{\partial E_1}{\partial q_2} = ml\cos\theta\dot{\theta}^2 \to 0 \Rightarrow 0$$

$$\frac{\partial E_1}{\partial \dot{q}_1} = c_y \to c_y \Rightarrow c_y\dot{y}_1$$

$$\frac{\partial E_1}{\partial \dot{q}_2} = -2ml\sin\theta\dot{\theta} \to 0 \Rightarrow 0$$

$$\frac{\partial E_1}{\partial \ddot{q}_1} = M+m \to M+m \Rightarrow (M+m)\ddot{y}_1$$

$$\frac{\partial E_1}{\partial \ddot{q}_2} = ml\cos\theta \to ml \Rightarrow ml\ddot{\theta}_1$$

We add up the nonzero terms to obtain the linearized equation of motion, the same as that we obtained using the explicit ε method.

Let us look at Eq. (3.17) for this problem. The inertia matrix is

$$\mathbf{M} = \begin{Bmatrix} M+m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{Bmatrix} \Rightarrow \mathbf{M}_0 = \begin{Bmatrix} M+m & ml \\ ml & ml^2 \end{Bmatrix}$$

The dissipation matrix is

$$\mathbf{C} = \begin{Bmatrix} c_y & 0 \\ 0 & c_\theta \end{Bmatrix} = \mathbf{C}_0,$$

and the potential matrix is

$$\mathbf{K} = \begin{Bmatrix} 0 & 0 \\ 0 & mgl\cos\theta \end{Bmatrix} \Rightarrow \mathbf{K}_0 = \begin{Bmatrix} 0 & 0 \\ 0 & mgl \end{Bmatrix}$$

assuming that $q_1 = y$ and $q_2 = \theta$. I leave it to the reader to verify that these three matrices lead to the linear equations I derived by formal expansions.

3.4 Some Two Degree of Freedom Systems

I explore three two degree of freedom systems that I hope will illustrate the various techniques I have discussed in this chapter.

3.4.1 A Double Pendulum

We know the period of a simple pendulum. What can we say about the periods of a double pendulum? There are two degrees of freedom, so we expect two frequencies. Figure 3.12 shows a double pendulum, another two degree of freedom problem (we have two links and two pins: the angles shown in the figure are independent and can be chosen as generalized coordinates). I will treat the pendulums as simple pendulums, neglecting the masses of the rods. The kinetic and potential energies can be written in terms of the parameters in the figure. I start with the simple way of writing the energies using the positions of the bobs in y, z space;

$$T = \frac{1}{2}m_1\left(\dot{y}_1^2 + \dot{z}_1^2\right) + \frac{1}{2}m_2\left(\dot{y}_2^2 + \dot{z}_2^2\right) \tag{3.18}$$

$$V = m_1 g z_1 + m_2 g z_2 \tag{3.19}$$

We apply the pin constraints by expressing the Cartesian variables in terms of the angles

$$\begin{aligned} y_1 &= l_1\sin\theta_1, \quad z_1 = -l_1\cos\theta_1 \\ y_2 &= y_1 + l_2\sin\theta_2, \quad z_2 = z_1 - l_2\cos\theta_2 \end{aligned} \tag{3.20}$$

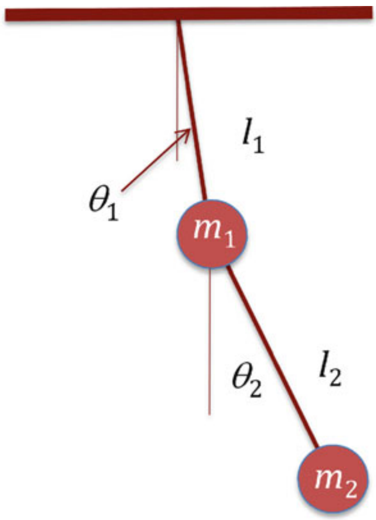


Fig. 3.12 A double pendulum

The Lagrangian in terms of the generalized coordinates is then

$$L = \frac{1}{2} l_1^2 (m_1 + m_2) \dot{\theta}_1^2 + \frac{1}{2} l_2^2 m_2 \dot{\theta}_2^2 + l_1 l_2 m_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 + g(l_1(m_1 + m_2) \cos\theta_1 + l_2 m_2 \cos\theta_2) \tag{3.21}$$

from which we can derive the nonlinear equations of motion for simulation. The pendulum is in (stable) equilibrium when it is hanging down motionless, and we can find the linear equations directly. We find the linear inertia and stiffness matrices by differentiating the energies and substituting the equilibrium values. The elements of the matrices are given by

$$M_{ij} = \left. \frac{\partial^2 T}{\partial \dot{\theta}_i \partial \dot{\theta}_j} \right|_{\mathbf{q} \to \mathbf{0}}, \quad K_{ij} = \left. \frac{\partial^2 V}{\partial \theta_i \partial \theta_j} \right|_{\mathbf{q} \to \mathbf{0}}, \tag{3.22}$$

and we can write out the two matrices defining the linear equations as

$$\mathbf{M}_0 = \begin{Bmatrix} m_1 l_1^2 + m_2 l_2^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{Bmatrix}, \quad \mathbf{K}_0 = \begin{Bmatrix} g(m_1 l_1 + m_2 l_2) & 0 \\ 0 & m_2 l_2 \end{Bmatrix} \tag{3.23}$$

The frequencies of this system can be obtained from

$$\det\left(s^2 \mathbf{M}_0 + \mathbf{K}_0\right) = 0 \Rightarrow s^4 + g \frac{(l_1 + l_2)(m_1 + m_2)}{m_1 l_1 l_2} s^2 + g^2 \frac{m_1 + m_2}{m_1 l_1 l_2} = 0 \tag{3.24}$$

One can see that all the frequencies will be proportional to the square root of gravity divided by some length. It will be helpful to choose actual numbers in order to follow what is going on. Let me consider two equal pendulums, according to which I can set gravity, both masses and both lengths equal to unity without further loss of generality. In that case the squares of the two distinct values of s are both negative:

$$s_1^2 = -\frac{1}{2}\left(4 - 2\sqrt{2}\right) \approx -0.5858, \quad s_2^2 = -\frac{1}{2}\left(4 + 2\sqrt{2}\right) \approx -3.4142$$

The corresponding value for a single pendulum under these values of the parameters is, of course, unity ($s^2 = -1$). One double pendulum frequency is higher than the simple pendulum frequency and one lower. The corresponding periods (2π divided by the frequency) are 11.6098 and 4.8089 for the double pendulum, bracketing the single pendulum period of 6.2832 (2π).

We can ask what the pendulum motion looks like for the two different periods. We can write

$$\theta_1 = A_{11}\exp(s_1 t) + A_{12}\exp(s_2 t), \quad \theta_2 = A_{21}\exp(s_1 t) + A_{22}\exp(s_2 t) \tag{3.25}$$

This is a homogeneous problem and there is a necessary relation between the pairs of coefficients. Substituting this form of the solution into the differential equation gives

$$\left(s_1^2\mathbf{M}_0 + \mathbf{K}_0\right)\begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix}\exp(s_1 t) + \left(s_2^2\mathbf{M}_0 + \mathbf{K}_0\right)\begin{Bmatrix} A_{12} \\ A_{22} \end{Bmatrix}\exp(s_2 t) \tag{3.26}$$

The vector coefficients multiplying the two exponentials must vanish independently, so we have

$$\left(s_1^2\mathbf{M}_0 + \mathbf{K}_0\right)\begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \left(s_2^2\mathbf{M}_0 + \mathbf{K}_0\right)\begin{Bmatrix} A_{12} \\ A_{22} \end{Bmatrix} \tag{3.27}$$

These define the two ratios. If we put in the numerical values we have been using we will arrive at

$$\begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ \sqrt{2} \end{Bmatrix}, \quad \begin{Bmatrix} A_{12} \\ A_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -\sqrt{2} \end{Bmatrix} \tag{3.28}$$

The two angles move in the same direction at the lower frequency and in opposite directions at the higher frequency. In both cases the lower pendulum moves through a larger angle than the upper pendulum. These two qualitatively different motions are called *modes*. The constant vectors in Eq. (3.28) are *modal vectors*. The nature of the modes is a universal result: the motion associated with higher frequencies are more complicated than the motion at lower frequencies. Chapter 4 is devoted to a discussion of modes. (Complicated is a subjective term. Here we see a more jagged appearance of the pendulum at its extreme position at the higher frequency. Both links of the pendulum move together in the simple mode; the links move in opposite senses for the complicated mode.)

3.4.2 The Pendulum on a Disk

This is the system shown in Fig. 3.7. It's a complicated system. What are its natural frequencies? Let me focus on the linear problem, from which we can find the natural frequencies. The energies are given by

$$T = \frac{1}{4}\left(2ma^2 + (2m + 3M)r^2 - 4mar\cos\phi\right)\dot{\phi}^2 + \frac{1}{2}ml^2\dot{\theta}^2$$
$$+ ml\left(a\cos(\theta - \phi) - r\cos\theta\right)\dot{\phi}\dot{\theta}$$
$$V = mg(r - a\cos\phi - l\cos\theta)$$

The two components of the gradient of the potential are the derivatives with respect to the two coordinates, and the equilibria are at $\phi = 0, \pi$ and $\theta = 0, \pi$. The minimum is at $\phi = 0 = \theta$, so this is the (marginally) stable equilibrium around which we wish to linearize. The general inertia matrix becomes

$$\mathbf{M} = \begin{Bmatrix} \frac{1}{2}\left(2ma^2 + (2m + 3M)r^2 - 4mar\cos\phi\right) & ml\left(a\cos(\theta - \phi) - r\cos\theta\right) \\ ml\left(a\cos(\theta - \phi) - r\cos\theta\right) & ml^2 \end{Bmatrix},$$

and the general **K** matrix is

$$\mathbf{K} = \begin{Bmatrix} mga\cos\phi & 0 \\ 0 & mgl\cos\theta \end{Bmatrix}$$

Their equilibrium values are

$$\mathbf{M}_0 = \begin{Bmatrix} m(a-r)^2 + \frac{3}{2}Mr^2 & ml(a-r) \\ ml(a-r) & ml^2 \end{Bmatrix}, \quad \mathbf{K}_0 = \begin{Bmatrix} mga & 0 \\ 0 & mgl \end{Bmatrix}$$

We have the usual set of ordinary differential equations with constant coefficients. We can write the exponential solution in the form of a modal vector times an exponential function of time.

$$\begin{Bmatrix} \phi \\ \theta \end{Bmatrix} = \begin{Bmatrix} \Phi \\ \Theta \end{Bmatrix}\exp(j\omega t)$$

There is no dissipation, so we can seek purely imaginary exponents, $s = j\omega$, as shown and replace the second derivative by $-\omega^2$, leading to the homogeneous algebraic problem

$$\left(\mathbf{K}_0 - \omega^2\mathbf{M}_0\right)\begin{Bmatrix} \Phi \\ \Theta \end{Bmatrix} = 0$$

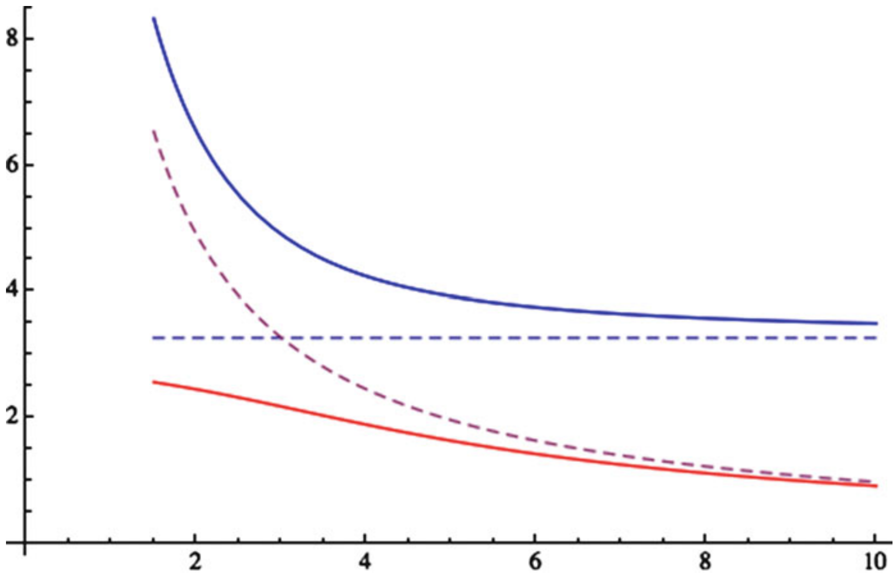


Fig. 3.13 Square of the frequency on the vertical axis vs. l on the horizontal axis. The horizontal dashed line is $g/3$, and the dashed curve is g/l. The lower frequency is clearly asymptotic to g/l

This has a nontrivial solution if and only if the determinant of the combined matrix vanishes. The determinant is a complicated quadratic equation determining ω^2. I can reduce the characteristic equation to a manageable size by scaling. Write

$$m = \mu M, \quad a = \alpha r, \quad l = \lambda r, \quad \omega^2 = \omega'^2 \frac{g}{l}$$

and divide out all the common factors. The resulting characteristic equation is

$$3\omega'^4 - \left(2\left(1 + \alpha^2 + \alpha(\lambda - 2)\right)\mu + 3\right)\omega'^2 + 2\alpha\lambda\mu = 0$$

There will be a zero frequency in certain special cases. If $\alpha = 0$ (corresponding to $a = 0$, the pendulum pivoting about the axle of the wheel), we have a zero frequency, and this is associated with the linear solution for ϕ that I noted at the beginning of the chapter, the wheel rolling along with the pendulum stationary. If $\mu = 0$ we not only have a zero frequency, but we also have the nonzero scaled frequency equal to unity. This is the limiting case where the mass of the wheel goes to infinity. The wheel does not move and the pendulum frequency is that of a simple pendulum. The remaining limit, $\lambda = 0$, has no physical meaning, because it gives us a pendulum of zero length. We can get an idea of the general problem by looking at the frequencies for a specific case. Set $M = 1 = m$, $r = 1$, and $a = 1/2$. Figure 3.13 shows the square of the frequencies plotted against l together with g/l, the limiting frequency for a pendulum. The low frequency tends to the pendulum frequency as the length of the pendulum increases. The square of the high frequency tends to the finite limit

$$\frac{2mga}{3Mr^2}$$

which equals $g/3$ for the example. The figure shows this limit as a horizontal dashed line.

The response of the two angles depends on the frequency. Both angles have the same sign for the lower frequency and opposite signs for the higher frequency. This is typical of multi-degree of freedom systems. The higher the frequency, the more "complicated" is the response, as I noted above. We will see this again and again as we go forward.

3.4.3 Vibration Absorption

Vibrating machinery can transmit oscillatory forces to their supports and eventually to the ground, as we have seen in Chap. 2. We can reduce the transmission (to zero in nondissipative systems) by adding a vibration absorber—an extra mass and spring. I will focus on the particular solution because in problems of this type, we care about the long-term behavior of the system, after damping has eaten up the homogeneous solution. I will start without damping to make clearer what happens in vibration absorption.

3.4.3.1 The Undamped Case

Consider the pair of masses shown in Figure 3.14. We apply a harmonic force to the first mass, and we can work out the motion of both masses. This is an abstract model, so I feel comfortable introducing a force without explaining where it comes from. We can look at a more specific example below. The system shown in Fig. 3.14 is simple enough that we can stick to a free body diagram approach and write the differential equations as

$$m_1\ddot{y}_1 = f - k_1 y_y + k_3(y_2 - y_1), \quad m_2\ddot{y}_2 = -k_3(y_2 - y_1)$$

or in more familiar form

$$\ddot{y}_1 + \frac{k_1 + k_3}{m_1}y_1 - \frac{k_3}{m_1}y_2 = a, \quad \ddot{y}_2 + \frac{k_3}{m_2}y_2 - \frac{k_3}{m_2}y_1 = 0$$

where $a = f/m_1$ is an acceleration. Suppose the acceleration to be proportional to sin (ωt). There is no damping, so all the terms of the particular solution will be proportional to sin(ωt). We write

$$y_1 = Y_1 \sin(\omega t), \quad y_2 = Y_2 \sin(\omega t),$$

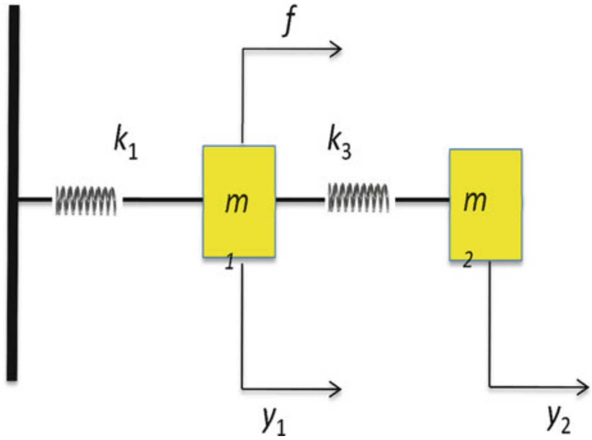


Fig. 3.14 A two-mass, two-spring system for vibration absorption

and the differential equations become algebraic. They are easily solved for the two coefficients

$$Y_1 = \frac{1}{\Delta}\left(\frac{k_3}{m_2} - \omega^2\right)A, \quad Y_2 = \frac{1}{\Delta}\frac{k_3}{m_2}A$$

where A denotes the amplitude of the acceleration a and

$$\Delta = \omega^4 - \left(\frac{k_1 + k_3}{m_1} + \frac{k_3}{m_2}\right)\omega^2 + \frac{k_1 k_3}{m_1 m_2}$$

The interesting thing about this result is that the response of the first mass can be made to vanish by tuning the second mass and spring to the exciting frequency. The vibration applied to the first mass is "absorbed" by the second mass. The motion of the second mass in this case is given by

$$Y_2 = -\frac{m_1}{m_2\omega^2}A$$

The smaller the second mass with respect to the first mass, the larger its excursions.

Tuning the system not only prevents the first mass from moving, but, because the force exerted on the support by the system is transmitted through the motion of the first mass, the entire system has been isolated from its support: no force is applied to the support.

Note that the analysis breaks down if the driving frequency ω is equal to one of the natural frequencies of the system, because Δ equals zero in that case. This cannot happen if we have tuned the system.

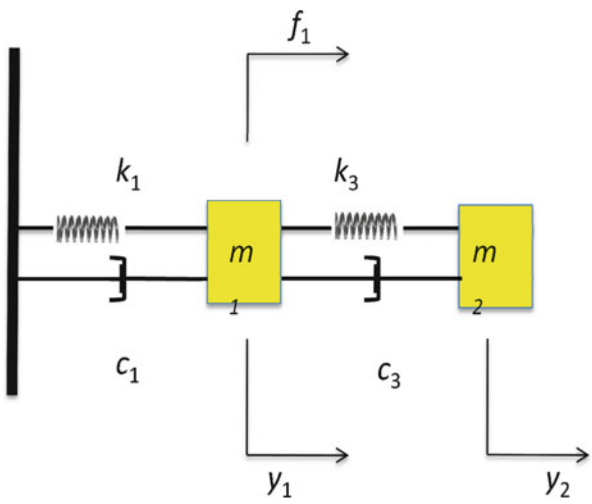


Fig. 3.15 A general two degree of freedom system with forcing and damping

3.4.3.2 The Damped Case

In the real world there will be damping, as shown in Figure 3.15, and that will spoil the broth. Let's take a look at the response of the dissipative system for the same harmonic forcing. I leave it to the reader to establish that the equations of motion are

$$\ddot{y}_1 + \frac{c_1 + c_3}{m_1}\dot{y}_1 + \frac{k_1 + k_3}{m_1}y_1 - \frac{c_3}{m_1}\dot{y}_2 - \frac{k_3}{m_1}y_2 = a,$$
$$\ddot{y}_2 + \frac{c_3}{m_2}\dot{y}_2 + \frac{k_3}{m_2}y_2 - \frac{c_3}{m_2}\dot{y}_1 - \frac{k_3}{m_2}y_1 = 0$$

Let me define some better notation, a partial scaling that allows me to make the dissipation dimensionless. Let

$$\mu = \frac{m_2}{m_1}, \quad \frac{k_1}{m_1} = \omega_1^2, \quad \frac{k_3}{m_2} = \omega_2^2, \quad \frac{c_1}{m_1} = 2\zeta_1\omega_1, \quad \frac{c_3}{m_2} = 2\zeta_2\omega_2$$

and the equations can be rewritten as

$$\ddot{y}_1 + 2(\zeta_1\omega_1 + \mu\zeta_2\omega_2)\dot{y}_1 + (\omega_1^2 + \mu\omega_2^2)y_1 - 2\mu\zeta_2\omega_2\dot{y}_2 - \mu\omega_2^2y_2 = a$$
$$\ddot{y}_2 + 2\zeta_2\omega_2\dot{y}_2 + \omega_2^2y_2 - 2\zeta_2\omega_2\dot{y}_1 - \omega_2^2y_1 = 0$$

The dissipation means that the response to a forcing at $A\sin\omega t$ will have sine and cosine terms. We can write

$$y_1 = Y_{1s}\sin(\omega t) + Y_{1c}\cos(\omega t), \quad y_2 = Y_{2s}\sin(\omega t) + Y_{2c}\cos(\omega t)$$

(We could also put this whole problem into complex notation. I pose that as a problem.) Plugging this into the differential equations and collecting terms in sine and cosine in each equation leads to four algebraic equations for the constants:

$$\left(\omega_1^2 + \mu\omega_2^2 - \omega^2\right)Y_{1c} + 2\omega(\zeta_1\omega_1 + \mu\zeta_2\omega_2)Y_{1s} - \mu\omega_2^2 Y_{2c} - 2\mu\zeta_2\omega\omega_2 Y_{2s} = 0$$

$$2\omega(\zeta_1\omega_1 + \mu\zeta_2\omega_2)Y_{1c} + \left(\omega_1^2 + \mu\omega_2^2 - \omega^2\right)Y_{1s} + 2\mu\zeta_2\omega\omega_2 Y_{2c} - \mu\omega_2^2 Y_{2s} = A$$

$$-\omega_2^2 Y_{1c} - 2\zeta_2\omega\omega_2 Y_{1s} + \left(\omega_2^2 - \omega^2\right)Y_{2c} + 2\zeta_2\omega\omega_2 Y_{2s} = 0$$

$$2\zeta_2\omega\omega_2 Y_{1c} - \omega_2^2 Y_{1s} - 2\zeta_2\omega\omega_2 Y_{2c} + \left(\omega_2^2 - \omega^2\right)Y_{2s} = 0$$

We can solve these. The results are quite complicated. The important result is that we cannot reduce the motion of the main mass to zero unless ζ_2 is zero. We can treat ζ_2 as a small number and expand the expression for the amplitude of the motion of the main mass $\sqrt{Y_{1c}^2 + Y_{1s}^2}$ in a Taylor series about $\zeta_2 = 0$. The result through the second order is

$$\frac{2}{\mu\omega^2}\left(\zeta_2 - 4\frac{\zeta_1\omega_1}{\mu\omega}\zeta_2^2 + \cdots\right)A$$

Example 3.5 An Unbalanced Machine on a Bench We looked at the vibrations associated with an unbalanced load back in Chap. 2. This is a system that one might well want to isolate from the floor. If such a machine is not isolated from the floor, then whoever is operating the machine will be subject to continuous vibrations, leading to fatigue and perhaps ill health. I will continue with the fiction that the motion is only in the vertical and complicate the picture by adding a dissipation-free vibration absorber. We can redraw Fig. 2.20 to add a vibration absorbing mass m_a (Fig. 3.16). I show both the connection to the floor and the connection to the absorber as springs, denoted by k and k_a, respectively. I leave it to the problems to examine the case with a more general connection. In this case the unbalanced machine will remain stationary. The amplitude of the motion of the absorber mass is given by

$$Y_2 = -\frac{M}{m_a\omega^2}A$$

where ω denotes the rotation rate and A the amplitude of the acceleration, which we can obtain from Chap. 2,

$$A = \frac{me}{M}\omega^2$$

from which

$$Y_2 = -\frac{me}{m_a}$$

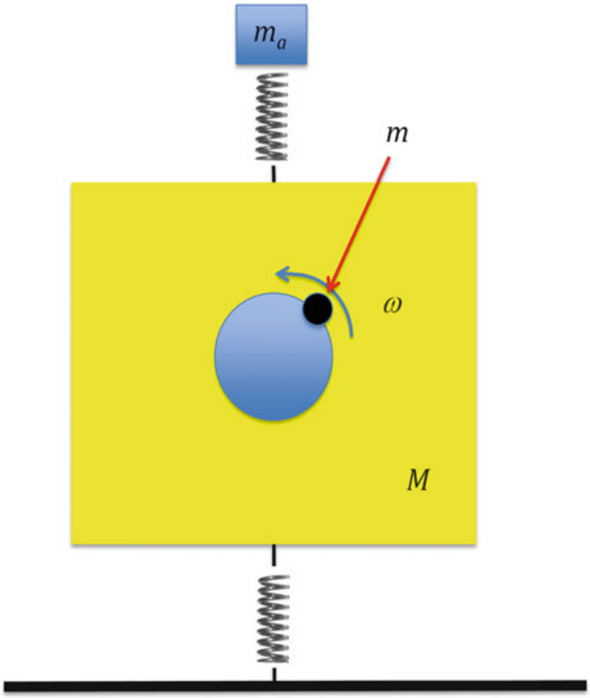


Fig. 3.16 Vibration absorber on a machine with an unbalanced rotor

the ratio of the unbalanced mass to the absorbing mass time the offset of the unbalanced mass.

3.5 Electromechanical Systems

3.5.1 Simple Motors

We need one more item to complete our dynamical picture. I would like to incorporate more realistic forces (and torques) in our examples. Rather than just saying "Apply a force," I would like to put a motor into the system and treat the voltage input to the motor as the input to the system rather than some arbitrary, unspecified, mysterious force. This is not the place for a long discussion of motors. I refer the reader who wants more to Chap. 4 in Hughes (2006). Let me just say that motors have an armature that rotates with respect to a stationary magnetic field. I will limit myself to direct current (DC) motors with the stationary field supplied by either permanent magnets or a separately excited field coil, so that all we need to deal with is the basics of the armature circuit. This is what I will mean when I use the word *motor*.

Consider a motor to be a system that has a voltage as an input and a rotation rate (which I will often call *speed*) and a torque as outputs. Figure 3.17 shows a

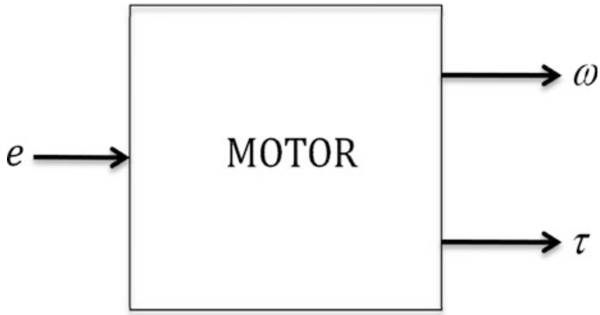


Fig. 3.17 Symbolic block diagram of a simple motor showing the input and the two outputs

symbolic block diagram. (I will construct a more detailed block diagram once we have a better understanding of how the motor works.) The two outputs are related, as we will see. The torque in a motor is proportional to the armature current, which is in turn depends on the armature voltage. The armature voltage is the difference between the input voltage and the *back emf*. The latter arises because the rotation of a conductor (the armature) in a magnetic field causes an induced voltage in the opposite sense. The back emf is proportional to the rotation rate of the motor. It is remarkable that the proportionality constant between the torque and armature current is equal to that between the rotation rate and the back emf for the simple motor models that will suffice for our purposes. I will denote this common constant by K and refer to it as the *motor constant*. Denote the input voltage by e, the armature current by i, the back emf by e_{b}, the rotation rate by ω, and the torque by τ. The relation between the current and voltage in the armature circuit is

$$e - e_{\mathrm{b}} = L\frac{di}{dt} + Ri \tag{3.29}$$

where L and R denote the armature inductance and resistance, respectively. In most motors the inductance is small enough so that if the current changes slowly[4] the inductive term is negligible and we can work with Ohm's law (obtained by setting L equal to zero). This makes the motor equations quite simple, and I will adopt that assumption unless I specifically state otherwise. The motor torque in terms of the input voltage and the rotation rate is

$$\tau = Ki = K\frac{e - v_{\mathrm{b}}}{R} = K\frac{e - K\omega}{R} \tag{3.30}$$

The maximum torque is developed at zero rotation rate. This is the so-called *starting torque* or *stall torque*. The maximum speed occurs when the torque is zero and is called the *no-load* speed. Figure 3.18 shows that relation between torque and rotation rate at some fixed input voltage. There is a family of parallel curves

[4] The meaning of *slowly* can be quantified, and I will discuss that at the end of this section.

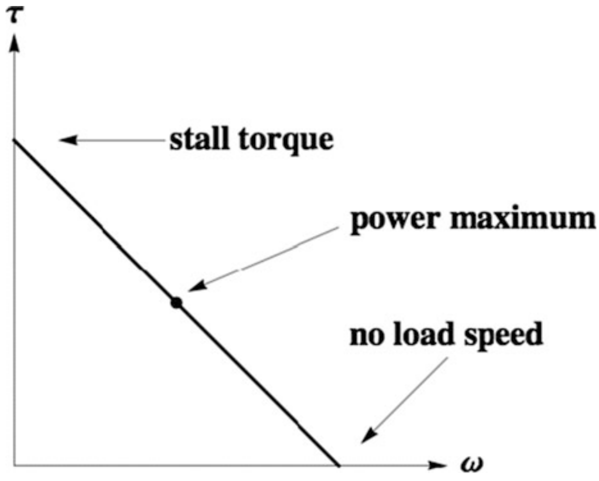


Fig. 3.18 Simple motor characteristics

parameterized by the input voltage. Any motor operates at some point on the curve. The power is the product of torque and speed, so we see that the maximum power is at half the no-load speed.

We can find K and R in terms of the stall torque and no-load speed per unit voltage.

$$K = \frac{e}{\omega_M}, \quad R = \frac{Ke}{\tau_M} = \frac{e^2}{\omega_M \tau_M}$$

These are not always given by the manufacturer. The nominal operating rotation rate is less than the no-load speed, so we know that the ratio of voltage to speed provides an upper bound for K, but we can do better. The typical motor is described by the rotation rate, input voltage, power, armature current, and nominal torque for its design operating condition, which is somewhere along the curve shown in Fig. 3.18. We can use these operating parameters to deduce an effective K and R. The power is the product of the current and the back emf, so we can find the back emf from the power and current. The back emf is the product of K and the speed, so we can find K at this point. The torque is the power divided by the speed, and we have an equation for the torque

$$\tau = K\frac{e - e_b}{R} \tag{3.31}$$

We know everything in this equation except the resistance, so we can use this to find the resistance. Let's go through the analysis for an actual motor [data from the Rotomag website (2012)]. Rotomag sells a class of 180 V, 1,500 rpm motors, and the estimate for K is therefore about 1.15. Let us consider the 1,500 W (2 hp) motor, for which the armature current is given as 11 A. We obtain a back emf of 136.4 V, and the correct value of K is 0.868. The nominal torque is 96 kg-cm (=9.42 N-m) and

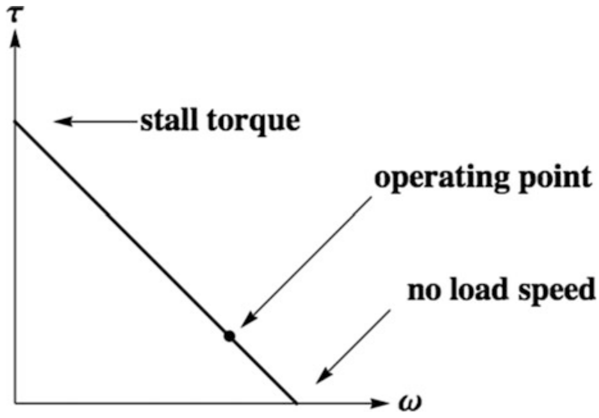


Fig. 3.19 Figure 3.18 redrawn for the specific motor under consideration

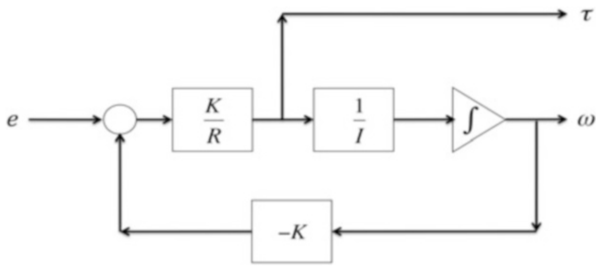


Fig. 3.20 Block diagram of a motor for which inductance is not important

we can deduce the resistance to be 3.97 Ω. We can find the no-load speed and the stall torque now that we have K and R. The no-load speed is 207 rad/s (1,980 rpm) and the stall torque is 39.6 Nm (404 kg-cm).

We can redraw Fig. 3.18 as Fig. 3.19 to show the operating point. If the motor is connected to an inertial load, the speed of the load is ω and the torque required to bring the load up to speed will be $I\dot{\omega}$, so we can write a dynamical equation

$$I\dot{\omega} = K\frac{e - K\omega}{R} \Rightarrow \dot{\omega} = -\frac{K^2}{IR}\omega + \frac{K}{IR}e \tag{3.32}$$

which is a first-order system in the speed. This assumes the absence of bearing friction or any other losses beyond the resistive losses. In particular I neglect the inductive effect, which implies that the rate of change of the current is not too fast. We can draw an actual block diagram corresponding to Fig. 3.17 based on Eqs. (3.31) and (3.32). I show that as Fig. 3.20.

Let's take the motor above and connect it to a flywheel with a moment of inertia of 1 kg m^2. Figure 3.21 shows the response of the system.

The no-load speed is 1,980 rpm. If we add friction equal to about 30 % of the resistive loss we obtain Fig. 3.22, showing a final speed of about 1,500 rpm.

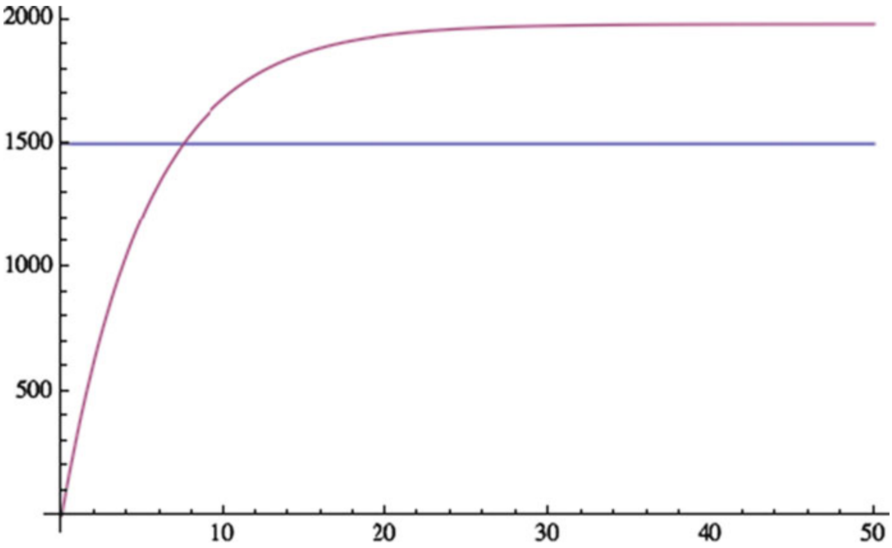


Fig. 3.21 Inertial load speed, rpm, vs. time, seconds. The *horizontal line* represents the nominal motor speed of 1,500 rpm

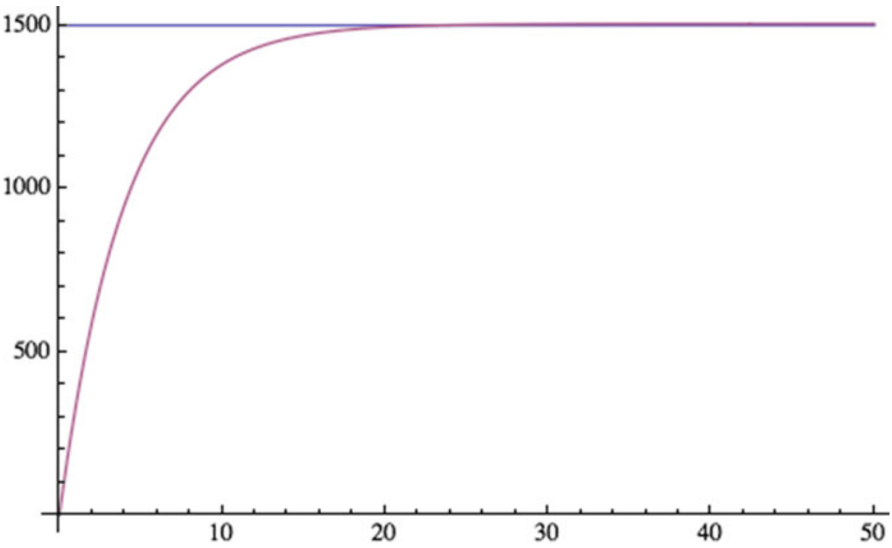


Fig. 3.22 Inertial load speed, rpm, vs. time, seconds for a system with friction equal to 32 % of the resistive loss coefficient

I will also want to be able to specify a force on a cart as well as torque on a shaft. I can do this by treating the cart as a wheeled vehicle that rolls without slipping and suppose the drive wheels to be driven by a motor. The relation between the wheel rotation rate and the speed on the cart will be $v = -\omega r$, where r denotes the wheel radius. The minus sign supposes positive rotation to be counterclockwise and positive velocity to be to the right. The force that the rail exerts on the cart is equal to the torque on the wheel divided by the radius of the wheel, so the force on the cart caused by the motor can be written

$$f = -\frac{\tau}{r} = -K\frac{e - K\omega}{Rr} = -K\frac{e + (K/r)v}{Rr} \tag{3.33}$$

from which we have

$$m\dot{v} = -K\frac{e + (K/r)v}{Rr} \Rightarrow m\dot{v} + \frac{K^2}{Rr^2}v = -\frac{K}{Rr}e \tag{3.34}$$

more or less the same first-order system. In applications where we are looking to drive a cart, it would be unusual to have the armature shaft directly connected to the drive wheel. There will be a gear, belt, or chain train so that the operating speed of the motor corresponds to a desired speed of the cart. We can incorporate this by replacing the wheel radius r by an *effective radius* such that $\omega r = v$, where w and v denote the design motor speed and design cart speed, respectively. I will use r in that sense when discussing motor-driven carts. We should, of course, also subtract potential losses from Eq. (3.33) to account for such things as bearing friction and losses in whatever gear or belt trains there are between the armature and the driven wheel. We can suppose these to be proportional to the speed, so that all that happens to Eq. (3.34) is an increase in the dissipative term.

Equations (3.31) and (3.32) give the torque and force outputs for a DC motor when the changes in input voltage are not too rapid. Let's take a quick look at how the system works when we cannot neglect the inductive terms, looking only at shaft speed for simplicity. We now have a second-order system

$$I\dot{\omega} = Ki$$

$$L\frac{di}{dt} = e - K\omega - Ri$$

It's linear, and we can make one second-order equation out of it

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{K^2}{IL}i = \frac{1}{L}\frac{de}{dt} \tag{3.35}$$

I show the block diagram of this second-order system as Fig. 3.23.

I do not know the inductance of this motor, but typical inductances are tens of mH (see Miller et al. 1999). I will take 50 mH for a value for simple thinking, so $R/L \approx 80$ and $K^2/IL \approx 15$. The "natural frequency" is then 3.88 rad/s and the damping ratio is about 10. The system is heavily overdamped so there are no free oscillations of this system.

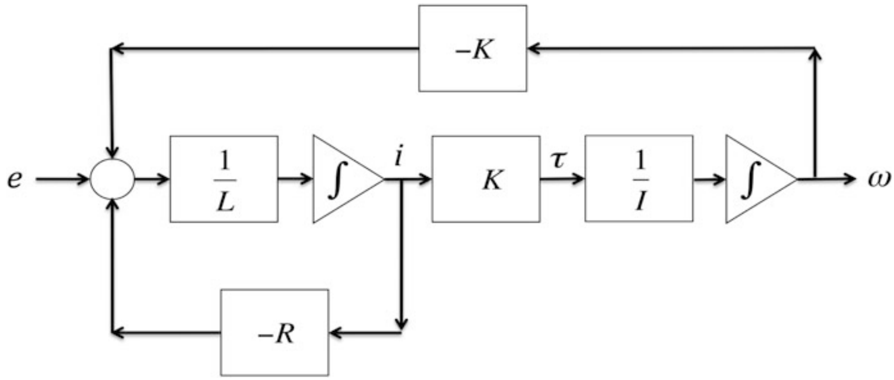


Fig. 3.23 Block diagram of a motor system showing the inductive effects

We can look at the forced behavior of the system by simply driving it with simple harmonic forcing. Let $e = E_0\sin(\omega t)$. I leave it to the reader to show (the formulation in terms of complex variables is probably the quickest way to do this) that the particular solution for the current is

$$i = \frac{E_0\omega}{\Delta \mathrm{L}}\left(\left(\frac{K^2}{IL} - \omega^2\right)\cos(\omega t) + \frac{\omega R}{L}\sin(\omega t)\right)$$

where

$$\Delta = \left(\frac{K^2}{IL} - \omega^2\right)^2 + \left(\frac{\omega R}{L}\right)^2,$$

and the torque is given by K times this. We see that there is a phase shift between the torque and the rate of change of the input voltage that vanishes when the resistance vanishes. This is perfectly reasonable, because whatever the value of ζ is, it is proportional to the resistance.

Example 3.6 Motor-Driven Overhead Crane Let's consider a "real" overhead crane, one for which the cart is driven by an electric motor. There will be no torque applied to the pendulum. Neglect the damping that appears in the Rayleigh dissipation function. (We'll find that we have damping from the motor.) I only care about the analytic solutions for now, so I will limit myself to the linear problem. The linear equations of motion are

$$(M+m)\ddot{y}_1 + ml\ddot{\theta}_1 = f_1 = -K\frac{e + (K/r)v}{Rr} = -\frac{K^2}{Rr^2}\dot{y} - \frac{K}{Rr}e$$

$$\Downarrow$$

$$(M+m)\ddot{y}_1 + \frac{K^2}{Rr^2}\dot{y} + ml\ddot{\theta}_1 = -\frac{K}{Rr}e$$

and

$$ml^2\ddot{\theta}_1 + ml\ddot{y}_1 + mgl\theta_1 = 0$$

What has happened to the frequencies? Suppose I seek exponential solutions and set the input $e = 0$? The characteristic polynomial is still a quartic, but now there is a linear term and a cubic term. There is still no constant term, so there is still a zero root, but now there is a nonzero real root. The characteristic polynomial is

$$s\left(s^3 + \frac{K^2}{MRr^2}s^2 + \frac{g(M+m)}{Ml}s + \frac{gK^2}{MlRr^2}\right) = 0$$

There are no changes of sign in the parenthetical segment,[5] so there are no real roots, so the real root must be negative. We can look at this more easily if we scale[6] s. Let s be proportional to $\surd(g/l)$, $s = \surd(g/l)s'$. We can then divide the cubic part of the characteristic polynomial by the cube of this and define a pair of dimensionless parameters

$$\mu = 1 + \frac{m}{M}, \quad \gamma = \frac{K^2}{MRr^2}\sqrt{\frac{l}{g}}$$

The parameter μ is greater than unity, but not much greater for a typical overhead crane. The characteristic polynomial in terms of the scaled variable is

$$s'^3 + \gamma s'^2 + \mu s' + \gamma = 0$$

The real root is proportional to $-\gamma$. The proportionality constant depends monotonically in a complicated way on the inverse of μ. It is clearly unity for μ equal to unity and decreases from there. It equals 0.759 at $\mu = 1.5$, which is as large as μ is likely to be. The other two roots are complex conjugates. Figures 3.24 and 3.25 show the real and imaginary parts for $\mu = 1.2$ as a function of γ. The real parts are always negative, and the imaginary parts are always near unity. This means that we have decaying exponentials with imaginary parts proportional to the usual pendulum frequency.

How does this work in "real life"? Let's put together a system using the motor we have already seen. Its operating speed is 1,500 rpm, and let us suppose we want the cart speed to be 2 fps at that speed. The effective wheel radius is then 3.88 mm.

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[5] Descartes' Rule of Signs states that the number of positive roots of a polynomial cannot exceed the number of changes in sign in the polynomial (here zero) and the number of negative roots cannot exceed the number of changes in sign of the polynomial with the variable replaced by its negative, which changes the sign of all the odd powers in the polynomial. Routh-Hurwitz procedures allow one to find the number of roots of a polynomial with positive real parts.

[6] I introduced scaling in Chap. 1. Scaling, also called nondimensionalization, replaces variables with units by variables without units. Among other things, this allows one to understand what is meant by small. If a variable has no units, then small means "small compared to unity."

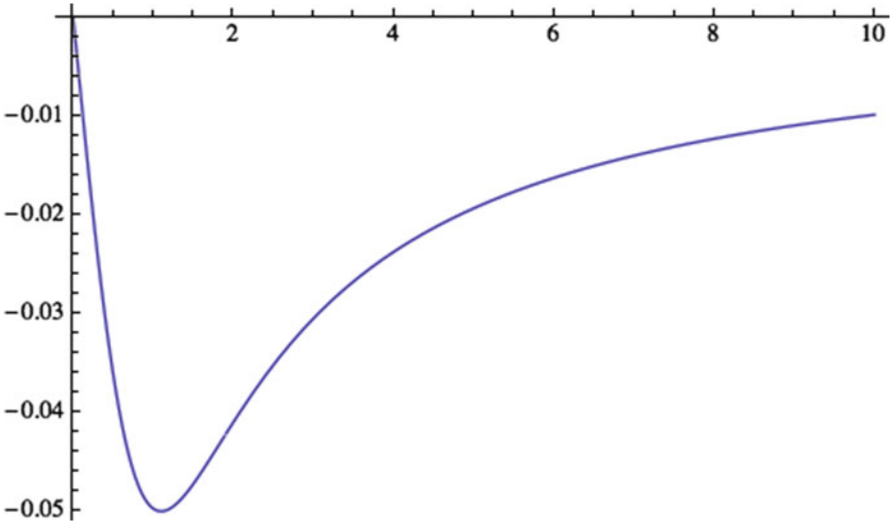


Fig. 3.24 The real part of the complex frequency for $\mu = 1.2$

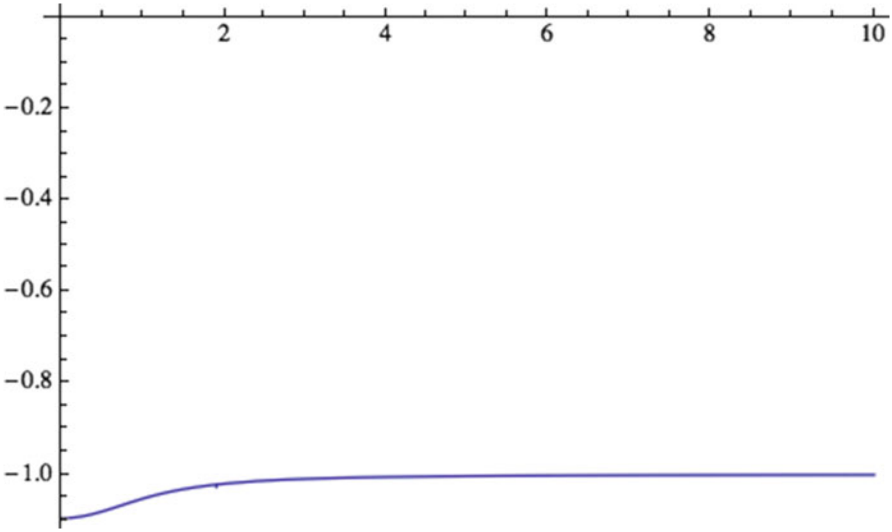


Fig. 3.25 The imaginary part of the complex frequency for $\mu = 1.2$

Table 3.1 shows a set of parameters for the motor in question and a 100 kg overhead crane carrying a 50 kg load at the end of a 2 m cable. The eigenvalues for these parameters are

$$-5.5 \times 10^{-6} \pm 2.7142j, \quad -22.0 \times 10^{-6}, \; 0$$

Table 3.1 Parameters for the overhead crane (SI units)

m	M	l	r	K	R	L	g
50	100	2	0.3	0.868	4	0.01	9.81

I will discuss this system in more detail in Chap. 6.

Motor-driven systems are not the only electromechanical systems of interest. We can consider systems with electromagnets.

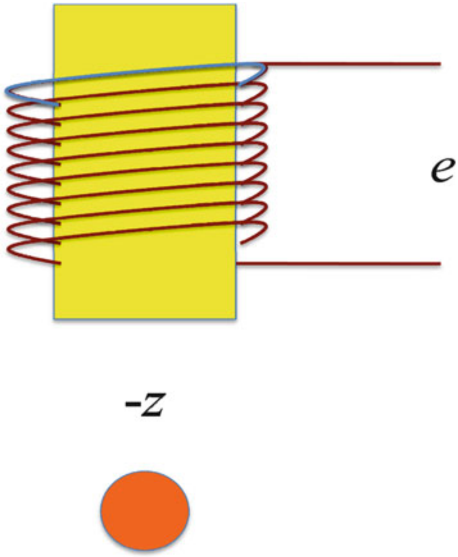


Fig. 3.26 The magnetic suspension system

3.5.2 Magnetic Suspension

A steel ball is attracted by a magnet, and it is possible to imagine one being suspended in midair below a magnet if the magnetic force and the gravitational force are equal (Fig. 3.26).

This is an equilibrium, but it is pretty easy to see that it is an unstable equilibrium. Displacing the ball up, closer to the magnet, increases the magnetic force while the gravitational force does not change. The system is unbalanced and the ball will continue to rise until it hits the magnet. Displacing the ball down, away from the magnet, decreases the magnetic force and the ball will continue to fall until it hits the floor. This is a good candidate for a control, which I will explore in Chap. 7. I just want to set up the equations and look at some aspects of the system here. (This example is modified from a similar example in Kuo (1991).) Suppose the magnet to be an electromagnet. The magnetic force is proportional to the square of the field, which is proportional to the current in the magnet. Choosing the rate at which the force decreases with distance is not simple. Kuo uses an inverse law. One is tempted to use an inverse square law, but "The only use, in fact, of the law of inverse squares, with respect to electromagnetism, is to enable you to write an answer when you want to pass an academical examination, set by some fossil examiner, who learned it years ago at the University, and never tried an experiment in his life to see if it was applicable to an electromagnet." (Thompson 1891 p. 110). Thompson reports some experiments by Dub that imply an inverse linear law, but it is hard to make sense of these experiments. The only nice experiments (Castañer et al. 2006) I've been able to find involve magnet-magnet interactions and suggest that an inverse fourth law might be appropriate. Fortunately the choice of any specific law is not important qualitatively for the purposes of control. I will suppose the force to obey an inverse *n*th power law in the distance between some reference point in the magnet and the center of the ball. The system varies only quantitatively with the choice of *n*

The force balance leads to

$$m\ddot{z} = C_n \frac{i^2}{z^n} - mg \tag{3.36}$$

where i denotes the current in the magnet; z the position of the ball relative to the reference position within the magnet, positive up; m the mass of the ball; g the acceleration of gravity; and C_n a constant depending on the power n, the parameters of the magnet and of the ball. The inductance is not negligible for the magnet, so the current evolves according to the extended Ohm's law

$$\frac{di}{dt} = -\frac{R}{L}i + \frac{1}{L}e \tag{3.37}$$

where R and L denote the resistance and the inductance of the magnet coil, respectively, and e the input voltage for the coil. Equation (3.36) is nonlinear, so if we are to look at analytic solutions, we will need to linearize it. We have an equilibrium defined by

$$C_n \frac{i_0^2}{z_0^n} = mg \tag{3.38}$$

with a corresponding equilibrium voltage $e_0 = Ri_0$. We can express all of this in term of the equilibrium location of the ball, z_0, which is negative. If n is odd, then C_n will be negative; if n is even, then C_n will be positive. The arguments of the square roots in Eq. (3.39) are thus positive and the current and voltage in Eq. (3.39) are real:

$$i_0 = \sqrt{\frac{mg}{C_n} z_0^n}, \quad e_0 = R\sqrt{\frac{mg}{C_n} z_0^n} \tag{3.39}$$

We can linearize Eq. (3.36) by setting $z = z_0 + \varepsilon z'$, $i = i_0 + \varepsilon i'$, $e = e_0 + \varepsilon e'$ and using the ε method. The nonlinear term in Eq. (3.36) becomes

$$C_n \frac{i^2}{z^n} = C_n \frac{\left(i_0 + \varepsilon i'\right)^2}{\left(z_0 + \varepsilon z'\right)^n} = C_n \frac{i_0^2 + 2\varepsilon i_0 i' + \cdots}{z_0^n + n\varepsilon z_0^{n-1} z' + \cdots}$$
$$= C_n \frac{i_0^2}{z_0^n} + \varepsilon C_n \left(\frac{2i_0}{z_0^n} i' - \frac{i_0^2}{z_0^n}\frac{n}{z_0} z'\right) + \cdots, \tag{3.40}$$

and the linear version of Eq. (3.36) is

$$\ddot{z}' = C_n \left(\frac{2i_0}{z_0^n} i' - \frac{i_0^2}{z_0^n}\frac{n}{z_0} z'\right) = mg\left(2\frac{i'}{i_0} - n\frac{z'}{z_0}\right) \tag{3.41}$$

where I have taken advantage of the equilibrium relation to simplify the appearance of the equation. The perturbed version of Eq. (3.37) becomes

$$\frac{di'}{dt} = -\frac{R}{L} i' + \frac{1}{L} e' \tag{3.42}$$

These are ordinary differential equations with constant coefficients, and so the homogeneous version admits exponential solutions. Substitute $z' = Z\exp(st)$, $i' = I\exp(st)$, which gives me a pair of algebraic equations

$$s^2 Z = mg\left(2\frac{I}{i_0} - n\frac{Z}{z_0}\right), \quad sI = -\frac{R}{L} I \tag{3.43}$$

This is an unusual problem. The current is uncoupled from the force balance in the sense that $i' = I\exp\left(-\frac{R}{L}t\right)$ is the only solution to the second of Eq. (3.43). There are two other modal values, but these have no reflection in the current. They are $s = \pm\sqrt{\frac{n}{-z_0}}$. These are real because z_0 is negative, so the mathematics agrees with our intuition about the instability of this system.

What happens for periodic forcing?

Let $e' = E_0 \sin(\omega t)$. It is easy to show that a particular solution to Eq. (3.43) is

$$i' = \frac{E_0}{R^2 + \omega^2 L^2}\left(\frac{R}{L}\sin(\omega t) - \omega\cos(\omega t)\right) \tag{3.44}$$

Equation (3.41) becomes

$$\ddot{z}' + \frac{mgn}{z_0} z' = 2mg\frac{i'}{i_0} = 2\frac{mg}{i_0}\frac{E_0}{R^2 + \omega^2 L^2}\left(\frac{R}{L}\sin(\omega t) - \omega\cos(\omega t)\right), \tag{3.45}$$

and I leave its solution to the problems.

I will revisit this problem as we go forward.

3.6 Summary

We have learned to build equations of motion for mechanical systems using the Euler-Lagrange process. We have learned how to figure out how many degrees of freedom such a system has and to assign generalized coordinates corresponding to the degrees of freedom. We have learned how to incorporate external forces and viscous dissipation in the formulation. This ability to form equations of motion is fundamental to everything that follows.

The governing equations are often nonlinear, and we have spent a little time discussing how to set them up for numerical integration. We have spent much more time learning how to construct linear models of the nonlinear systems, using an ε formalism, and a quicker method that draws on the long method. The former always works, but the latter is to be preferred in most cases. We will learn yet another method in Chap. 6.

We also spent a little time learning about how simple DC motors work and how to incorporate them into mechanisms.

The dynamics of mechanical systems can be found from four scalars: the kinetic energy T, the potential energy V, the Rayleigh dissipation function F, and the rate of doing work $\dot{W}$. This is true for both linear and nonlinear problems. We can only deal analytically with linear problems. Nonlinear problems require numerical methods. The linear (linearized) equations of motion can be written in matrix form as

$$\mathbf{M}_0\ddot{\mathbf{q}} + \mathbf{C}_0\dot{\mathbf{q}} + \mathbf{K}_0 = \mathbf{Q}_0 \tag{3.46}$$

where $\mathbf{q}$ denotes the vector made up of the generalized coordinates, and the other four terms can be derived from the four fundamental scalars. The 0 subscript denotes that the equation is valid in the neighborhood of an equilibrium for nonlinear problems. Equation (3.47) gives the elements of the system matrices and the generalized force vector:

$$M_{0ij} = \left.\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right|_{\mathbf{q}\to\mathbf{q}_0}, \quad C_{0ij} = \left.\frac{\partial^2 F}{\partial \dot{q}_i \partial \dot{q}_j}\right|_{\mathbf{q}\to\mathbf{q}_0}, \quad K_{0ij} = \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\mathbf{q}\to\mathbf{q}_0}, \quad Q_{0i} = \left.\frac{\partial \dot{W}}{\partial \dot{q}_i}\right|_{\mathbf{q}\to\mathbf{q}_0} \tag{3.47}$$

where $\mathbf{q}_0$ denotes the equilibrium state. If the system is linear to begin with, the substitution is not necessary.

The nonlinear equations of motion come from the Euler-Lagrange process. They are given by Eq. (3.12a) with the generalized forces given by

$$Q_i = \frac{\partial \dot{W}}{\partial \dot{q}_i} \tag{3.48}$$

These equations can be integrated numerically using any of a number of commercial software products. I use Mathematica throughout this book.

Exercises

1. Find the solution for the undamped overhead crane that results in Fig. 3.11.
2. Set up the damped vibration absorber problem using complex notation, verifying the formula given in the text.
3. Write the equations of motion for a pendulum with viscous damping at its pivot point.
4. Linearize Eq. (3.3e) using a formal linearization procedure.
5. Show that the $\mathbf{M}_0$, $\mathbf{C}_0$, $\mathbf{K}_0$ method applied to the overhead crane leads to the correct linear equations.
6. Consider the system shown in Fig. 3.6 and write the wheel angle in terms of the position angle, supposing the wheel to roll without slipping and supposing both to be zero when the wheel is at its lowest position.
7. Find the response of the unbalanced rotor plus absorber system if the two coupling springs are replaced by two spring-damper combinations. Suppose the damping to be small in the sense that any effective damping ratios are less than unity.
8. Show that choosing k_3 and m_3 to absorb the vibrations for the general system shown in Fig. 3.15 automatically avoids resonance.
9. Derive the equations of motion for a damped vibration absorber.
10. Find the damped natural frequencies for the overhead crane.
11. Calculate the force applied to the pivot point by a freely oscillating pendulum.
12. What is the vertical force exerted by the overhead crane pendulum on its cart?
13. If the overhead crane pendulum on the cart hits a building when it is at its lowest point, what impulse is imparted to the building? Assume a perfectly inelastic collision. Use the physical parameters in Table 3.1.
14. Discuss the phase relation between the cart motion and the pendulum motion for the linearized overhead crane. How does it depend on the motor resistance?
15. Find the equilibrium states for the following coupled equations and linearize about all of them

$$\dot{x} = \sigma(y - x), \quad \dot{y} = \rho x - y - xz, \quad \dot{z} = -\beta z + xy$$

16. Solve the magnetic suspension problem using the following parameters with $n = 4$. Suppose the ball to start at rest from its equilibrium position ($z_0 = -0.5$) and calculate how long it will take before the ball travels 0.25 m from its initial position. Use $g = 9.81$.

    A set of nominal parameters for the magnetic suspension model (SI units)

m	C_n	L	R
1	1	0.01	1

17. Set up the vibration absorption problem with damping using the Euler-Lagrange approach.
18. Set up the unbalanced machine on a bench using the Euler-Lagrange approach.

19. Suppose the voltage supplied to the magnet in the magnetic suspension problem to be $e_0 + 0.1\ \sin(5t)$. What is the behavior of the system if the ball starts from rest at its equilibrium position? Use the linear equations of motion.
20. Find the natural frequencies of a triple pendulum made up of identical steel cylinders one foot long and one inch in diameter.
21. Consider three simple pendulums hanging from the ceiling. Suppose them to be connected by springs. Find the natural frequencies of the system if the pendulums all have the same mass m, and length l, and both springs have the same spring constant k.

    The following exercises are based on the chapter, but require additional input. Feel free to use the Internet and the library to expand your understanding of the questions and their answers.
22. Can you set up the magnetic suspension problem using the Euler-Lagrange process? (There is more than one answer to this depending on how much research you are willing to do about magnetic energy.)
23. Set up the differential equations for the overhead crane if the bob of the pendulum can move up and down the rod (still supposed rigid and massless) in response to a force f.
24. Find the vibration frequencies of a system made up of ten identical rods confined to a plane if the rods are connected by torsional springs such that they exert a torque when the two rods they join are not parallel.

References

Castañer R, Medina JM, Cuesta-Bolao MJ (2006) The magnetic dipole interaction as measured by spring dynamometers. Am J Phys 74:510–513

Den Hartog JP (1956) Mechanical Vibrations. Dover reprint (1985)

Gans RF (1991) Analytical kinematics. Butterworth-Heynemann, Stoneham, MA

Gans RF (2013) Engineering dynamics: from the Lagrangian to simulation. Springer, New York

Goldstein H (1980) Classical mechanics. Addison Wesley, Reading, MA

Hughes A (2006) Chapter 4: Electric motors and drives. 3rd edn. Newnes (Elsevier) Burlington, MA, Amsterdam

Kuo BC (1991) Automatic control systems, 6th edn. Prentice Hall, Englewood Cliffs, NJ, pp 5–15

Meirovitch L (1970) Methods of analytical dynamics. McGraw-Hill, New York

Miller TJE, McGilp MI, Staton DA, Bremner JJ (1999) Calculation of inductance in permanent-magnet DC motors. IEE Proc-Electr Power Appl 146(2):129–137

Rotomag (2012) http://www.rotomag.com/pdf/industril_motors.pdf. Accessed 21 Aug 2012

Thompson SP (1891) Lectures on the electromagnet. WJ Johnson, New York, NY

Modal Analysis

4

In which we look at modal analysis of the vibration of discrete linear systems and the vibration of continuous systems, such as strings, beams, shafts. . ..

4.1 Discrete Systems

We have seen that a linear two degree of freedom system has two natural frequencies. Multi-degree of freedom systems have as many frequencies as degrees of freedom. The nature of the motion associated with each degree of freedom is unique: each frequency has its distinct mode shape. Generally the higher the frequency the more complicated the mode shape. We saw that physically for the undamped double pendulum. Damping changes things a bit (the exponents are no longer purely imaginary), but modes are still useful for lightly damped systems. To make the most of this we will need to modify our damping model. It should be clear from the pendulum example that modes say something important about a linear system. For this reason it would be nice to be able to work directly with the modes rather than the individual variables. I will outline how to do that in this chapter. (Inman (2001) provides an alternate derivation). Let me emphasize that modal analysis applies only to linear systems and is only useful for lightly damped systems (systems for which the modal damping ratio is less than unity for all the modes).

4.1.1 Undamped Systems

I will start with the standard undamped version of a linear system

$$\mathbf{M}_0\ddot{\mathbf{q}} + \mathbf{K}_0\mathbf{q} = 0 \tag{4.1}$$

to define the modes for that system, whether or not the actual system is damped. The zero subscript denotes that the matrices are constant, belonging to a linear system.

R.F. Gans, *Mechanical Systems: A Unified Approach to Vibrations and Controls*,
DOI 10.1007/978-3-319-08371-1_4

Equation (4.1) is a set of homogeneous linear ordinary differential equations, and so they admit exponential solutions. There are no first derivatives, and so we can replace the exponent s by $j\omega$ and write $\mathbf{q} = \exp(j\omega t)\mathbf{v}$, where $\mathbf{v}$ is a constant vector. This transforms Eq. (4.1) to

$$\left(-\omega^2\mathbf{M}_0 + \mathbf{K}_0\right)\mathbf{v} = 0 \tag{4.2}$$

Nontrivial solutions to Eq. (4.2) exist only if the determinant of the matrix in parentheses vanishes. This condition determines a set of frequencies, one for each degree of freedom. A vector $\mathbf{v}$ accompanies each frequency. Equation (4.2) then gives us a set of equations

$$\mathbf{K}_0\mathbf{v}_k = \omega_k^2\mathbf{M}_0\mathbf{v}_k \tag{4.3}$$

Each $\mathbf{v}$ vector represents a mode. I will call them *modal* vectors. I can write any $\mathbf{q}$ in terms of the modal vectors if they are independent, which they will be for most engineering problems. In that case we'll have

$$\mathbf{q} = u_1(t)\mathbf{v}_1 + u_2(t)\mathbf{v}_2 + \cdots \tag{4.4}$$

where the time-dependent coefficients u_k will be determined by the specific problem being addressed. They are the simple exponential functions we used to get Eq. (4.2) if the problem is homogeneous and undamped, but we can determine different coefficients for more complex problems, damped and/or forced. That will be the focus of this chapter.

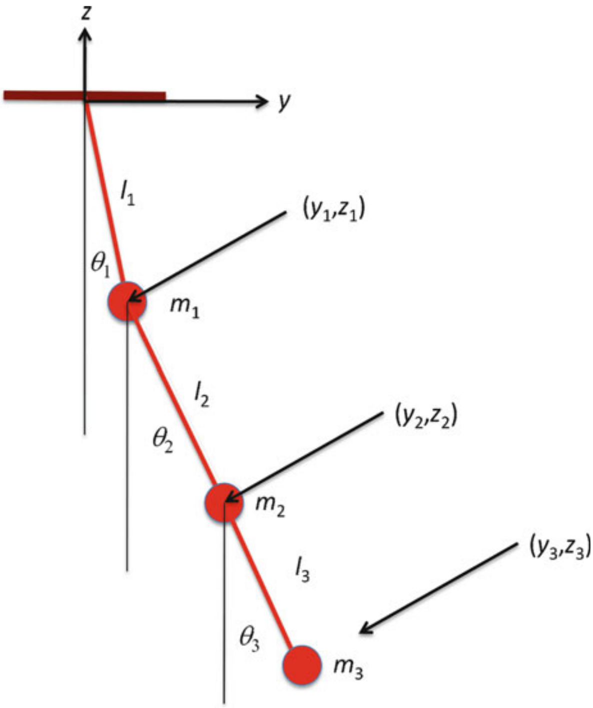
We can rewrite Eq. (4.4) in terms of a vector made up of the u functions and a matrix made up of the modal vectors. Let $\mathbf{V}$ denote a vector whose columns are the modal vectors $\mathbf{v}_k$, and let $\mathbf{u}$ denote a (column) vector whose components are the functions u_k. Then we can write

$$\mathbf{q} = \{\, \mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \}\begin{Bmatrix} u_1 \\ u_2 \\ \vdots \end{Bmatrix} = \mathbf{V}\mathbf{u} \tag{4.5}$$

Equation (4.5) will be invertible if the determinant of $\mathbf{V}$ is nonzero, which it will be if the modal vectors are independent. Equations (4.3) and (4.5) will be important as we go forward, but let's pause here and try to understand what the modal vectors represent. We can see what modes mean most easily by starting with an example where we can actually draw the modes. It happens to be an example derived from a nonlinear system, and I'll take the opportunity to review a little of what has gone before.

Example 4.1 A Triple Pendulum A triple simple pendulum provides a nice example of the Euler-Lagrange process with constraints as well as the new shortcut for linearizing equations. Figure 4.1 shows the system.

Fig. 4.1 A triple pendulum



These are simple pendulums, so the rods are massless and the bobs can be treated as point masses. I select the origin of the coordinate system to be at the pivot point of the upper pendulum. All the z coordinates are then negative for small (small here means much less than $\pi/2$) angles. The angles increase in the counterclockwise direction. In the general case, all the masses and lengths can be different, and I will set it up that way. The energies in terms of the six displacement coordinates are

$$T = \frac{1}{2} m_1 \left(\dot{y}_1^2 + \dot{z}_1^2\right) + \frac{1}{2} m_2 \left(\dot{y}_2^2 + \dot{z}_2^2\right) + \frac{1}{2} m_3 \left(\dot{y}_3^2 + \dot{z}_3^2\right)$$
$$V = m_1 g z_1 + m_2 g z_2 + m_3 g z_3$$

There are six variables in these expressions, but this is a three degree of freedom problem. We have three links and three pins. The easiest way to reduce the problem is to impose the following geometric (kinematic) constraints, which should be reasonably obvious from Fig. 4.1.

$$\begin{aligned} y_1 &= l_1 \sin\theta_1, & z_1 &= -l_1 \cos\theta_1 \\ y_2 &= y_1 + l_2 \sin\theta_2, & z_2 &= z_1 - l_2 \cos\theta_2 \\ y_3 &= y_2 + l_3 \sin\theta_3, & z_3 &= z_2 - l_3 \cos\theta_3 \end{aligned}$$

I can write the kinetic and potential energies in terms of the angles as

$$
\begin{aligned}
T =& \frac{1}{2} l_1^2 (m_1 + m_2 + m_3)\dot{\theta}_1^2 + \frac{1}{2} l_2^2 \left(m_2 + m_3\right)\dot{\theta}_2^2 + \frac{1}{2} l_3^2 m_3 \dot{\theta}_1^2 \\
&+ l_1 l_2 \left(m_2 + m_3\right) \cos\left(\theta_1 - \theta_2\right)\dot{\theta}_1\dot{\theta}_2 + l_1 l_3 m_3 \cos\left(\theta_1 - \theta_3\right)\dot{\theta}_1\dot{\theta}_3 \\
&+ l_2 l_3 m_3 \cos\left(\theta_2 - \theta_3\right)\dot{\theta}_2\dot{\theta}_3 \\
V =& g\left(l_1 (m_1 + m_2 + m_3) \cos\theta_1 + l_2 \left(m_2 + m_3\right) \cos\theta_2 + l_3 m_3 \cos\theta_3\right),
\end{aligned}
$$

and the resulting Lagrangian is

$$
\begin{aligned}
L =& \frac{1}{2} l_1^2 (m_1 + m_2 + m_3)\dot{\theta}_1^2 + \frac{1}{2} l_2^2 \left(m_2 + m_3\right)\dot{\theta}_2^2 + \frac{1}{2} l_3^2 m_3 \dot{\theta}_1^2 \\
&+ l_1 l_2 (m_2 + m_3) \cos\left(\theta_1 - \theta_2\right)\dot{\theta}_1\dot{\theta}_2 + l_1 l_3 m_3 \cos\left(\theta_1 - \theta_3\right)\dot{\theta}_1\dot{\theta}_3 \\
&+ l_2 l_3 m_3 \cos\left(\theta_2 - \theta_3\right)\dot{\theta}_2\dot{\theta}_3 g\left(l_1 (m_1 + m_2 + m_3) \cos\theta_1\right. \\
&\left.+ l_2 \left(m_2 + m_3\right) \cos\theta_2 + l_3 m_3 \cos\theta_3\right)
\end{aligned}
$$

The three angles are the logical choice for generalized coordinates, and the Euler-Lagrange process produces three second-order nonlinear equations, equations too lengthy to be written out here. This is not a linear problem: the inertia matrix is not constant and the potential energy is more than second order in the θs. The triple pendulum has a (marginally) stable equilibrium when all its angles are zero. (It has an unstable equilibrium when all the angles equal π.) Let us take a detour here and write the linear equations with respect to this equilibrium using the shortcut method. The inertia matrix evaluated at the equilibrium is

$$
\mathbf{M}_0 = \left\{ \begin{array}{ccc} (m_1 + m_2 + m_3) l_1^2 & (m_2 + m_3) l_1 l_2 & m_3 l_1 l_3 \\ (m_2 + m_3) l_1 l_2 & (m_2 + m_3) l_2^2 & m_3 l_2 l_3 \\ m_3 l_1 l_3 & m_3 l_2 l_3 & m_3 l_3^2 \end{array} \right\}
$$

and the stiffness matrix at equilibrium is diagonal

$$
\mathbf{K}_0 = \left\{ \begin{array}{ccc} (m_1 + m_2 + m_3) g l_1 & 0 & 0 \\ 0 & (m_2 + m_3) g l_2 & 0 \\ 0 & 0 & m_3 g l \end{array} \right\}
$$

I can write out Eq. (3.17) for this system as

$$
\begin{aligned}
&\left\{ \begin{array}{ccc} (m_1 + m_2 + m_3) l_1^2 & (m_2 + m_3) l_1 l_2 & m_3 l_1 l_3 \\ (m_2 + m_3) l_1 l_2 & (m_2 + m_3) l_2^2 & m_3 l_2 l_3 \\ m_3 l_1 l_3 & m_3 l_2 l_3 & m_3 l_3^2 \end{array} \right\} \left\{ \begin{array}{c} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{array} \right\} \\
&\quad + \left\{ \begin{array}{ccc} (m_1 + m_2 + m_3) g l_1 & 0 & 0 \\ 0 & (m_2 + m_3) g l_2 & 0 \\ 0 & 0 & m_3 g l \end{array} \right\} \left\{ \begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_3 \end{array} \right\} = 0
\end{aligned}
$$

This is once again a set of ordinary differential equations with constant coefficients, and it admits exponential solutions of the form

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} \exp(st)$$

where Θ_1, Θ_2, and Θ_3 are constants that can be arranged as a vector in configuration space as shown. We recognize that this is a dissipation-free problem so that we expect the exponents to be imaginary. We can set $s = j\omega$, where we can refer to ω as a frequency, and replace the second derivatives with $-\omega^2$ leading to the equivalent algebraic problem, Eq. (4.2) for this example

$$\left(\mathbf{K}_0 - \omega^2 \mathbf{M}_0\right) \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = 0 \tag{4.6}$$

Equation (4.6) will have a nontrivial solution if

$$\det\left(\mathbf{K}_0 - \omega^2 \mathbf{M}_0\right) = 0$$

This is a cubic equation for the square of the frequency. The solutions are distinct, giving *modal frequencies* for this problem, and there will be a distinct set of coefficients Θ, a distinct vector, for each value of the frequency. These are the modal vectors for this problem, defining the modes of vibration.

We can get a better feel for this problem by considering the case of identical pendulums. This will sweep away a lot of the fog of algebra and leave the physics more visible. I denote the common mass by m and the common length by l. The inertia and potential matrices become

$$\mathbf{M}_0 = \begin{Bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{Bmatrix} ml^2, \quad \mathbf{K}_0 = \begin{Bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{Bmatrix} mgl$$

It should be relatively clear that the square of the frequency will be proportional to g/l, and we can write $\omega = \sqrt{(g/l)}\,\omega'$, and the characteristic polynomial reduces to a simple equation for the nondimensional (scaled) frequency ω'

$$\omega'^6 - 9\omega'^4 + 18\omega'^2 - 6 = 0$$

The numerical values of ω' are 0.644806, 1.51469, and 2.50798. We can obtain the associated modes by finding the nontrivial solutions to Eq. (4.6)

$$\left(\mathbf{K}_0 - \omega^2 \mathbf{M}_0\right) \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = 0$$

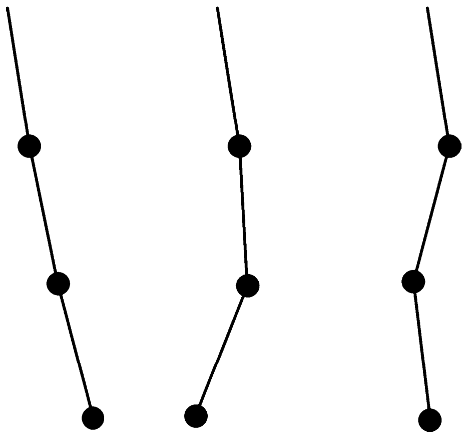


Fig. 4.2 The mode shapes for the triple pendulum

for each of the three values of the frequency. (One way to do this is to find the second and third components in terms of the first by setting the first two components of the equation equal to zero and then noting that the last component is automatically zero. The magnitude of the mode is undetermined because Eq. (4.6) is homogeneous, so one can choose an arbitrary value of the first component and write out the solution. If you want to be fancier you can normalize the modal vectors, but this is not necessary.) The following set of unnormalized modal vectors satisfies the homogeneous algebraic problem:

$$\mathbf{v}_1 = \begin{Bmatrix} 1 \\ 1.29211 \\ 1.63122 \end{Bmatrix}, \quad \mathbf{v}_2 = \begin{Bmatrix} 1 \\ 0.35286 \\ -2.39812 \end{Bmatrix}, \quad \mathbf{v}_3 = \begin{Bmatrix} 1 \\ -1.64497 \\ 0.766897 \end{Bmatrix}$$

The first modal vector, $\mathbf{v}_1$, corresponds to the lowest frequency; the second, $\mathbf{v}_2$, to the intermediate frequency; and the third, $\mathbf{v}_3$, to the highest frequency. The higher the frequency, the more complicated the mode shape. For the lowest frequency, all three pendulums rotate in the same direction. For the intermediate frequency, the first two rotate together, but the last one rotates in the opposite direction. For the highest frequency, the pendulums alternate their rotation directions. Figure 4.2 shows all three mode shapes when the first pendulum makes an angle of $\pi/20$ (which should to be small enough for us to have confidence in the linear theory) with the vertical.

This is another example of the general rule for multi-degree of freedom systems: the higher the frequency, the more complicated the mode shape. We can use the number of changes in sign in the modal vectors as a measure of complication (or complexity). Here we have zero, one, and two changes of signs, respectively.

Let's take a look at the limits of the linear approximation in Ex. 4.1 by simulating the triple pendulum before we move on. The easiest way to set this problem up for numerical integration is to convert the three Euler-Lagrange equations to six first-order equations. To do this solve the Euler-Lagrange equations for the second derivatives of the angles, and denote these expressions by α_1, α_2, and α_3, respectively. Denote the rates of change of the angles by ω_1, ω_2, and ω_3, respectively, and then the six ordinary differential equations can be written symbolically as

$$\begin{aligned}\dot{\theta}_1 &= \omega_1, \quad \dot{\theta}_2 = \omega_2, \quad \dot{\theta}_3 = \omega_3\\ \dot{\omega}_1 &= \alpha_2(\theta_1, \theta_2, \theta_3, \omega_1, \omega_2, \omega_3), \quad \dot{\omega}_1 = \alpha_2\left(\theta_1, \theta_2, \theta_3, \omega_1, \omega_2, \omega_3\right),\\ \dot{\omega}_3 &= \alpha_3\left(\theta_1, \theta_2, \theta_3, \omega_1, \omega_2, \omega_3\right)\end{aligned}$$

The expressions in the second line of the equations are too complicated to be written out, even in the simple case of identical pendulums. We saw in Chap. 2 that the linearized version of the simple pendulum behaved very like the nonlinear simple pendulum if the initial amplitude was not too large. The same is true here, although the meaning of "not too large" changes: the greater the number of pendulums, the more restrictive the constraint must be. I have chosen to assess the linear solution by substituting the modal configurations shown in Fig. 4.2 as initial conditions for the simulation, starting, of course, from rest. Figures 4.3, 4.4, and 4.5 show the three angles for the linear solution (blue) and for the simulation (red) for each mode. I plotted each figure over two periods of the linear solution. The more complicated the initial condition, the worse the linear solution agrees with the simulation.

We see that the first mode is very close to the simulation, the second mode less so, and the third mode still less so. This is another general rule of thumb: the more complicated an initial condition, the less well the linear solution represents the actual solution.

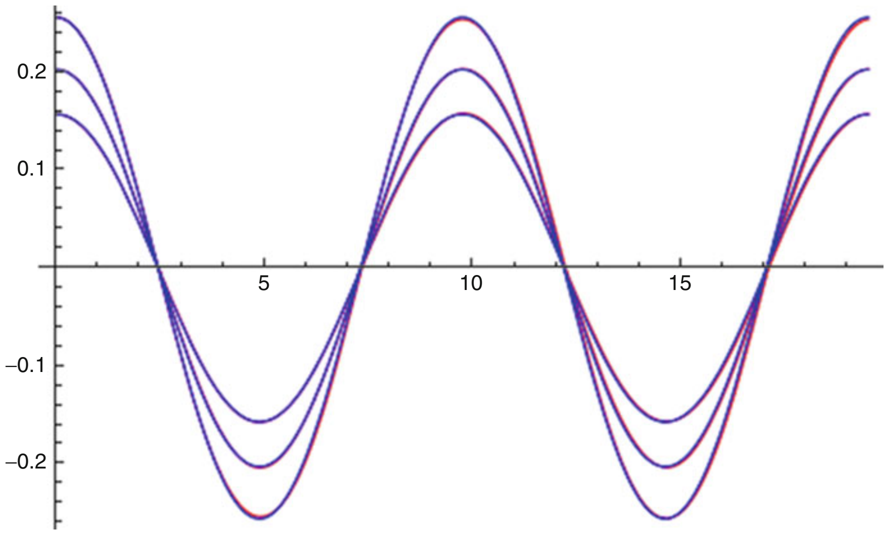


Fig. 4.3 The response of the first mode

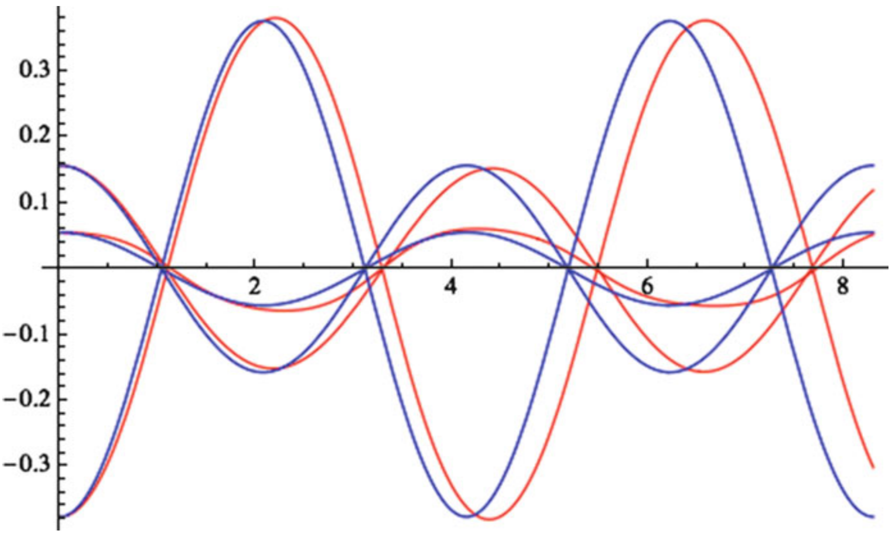


Fig. 4.4 The response of the second mode

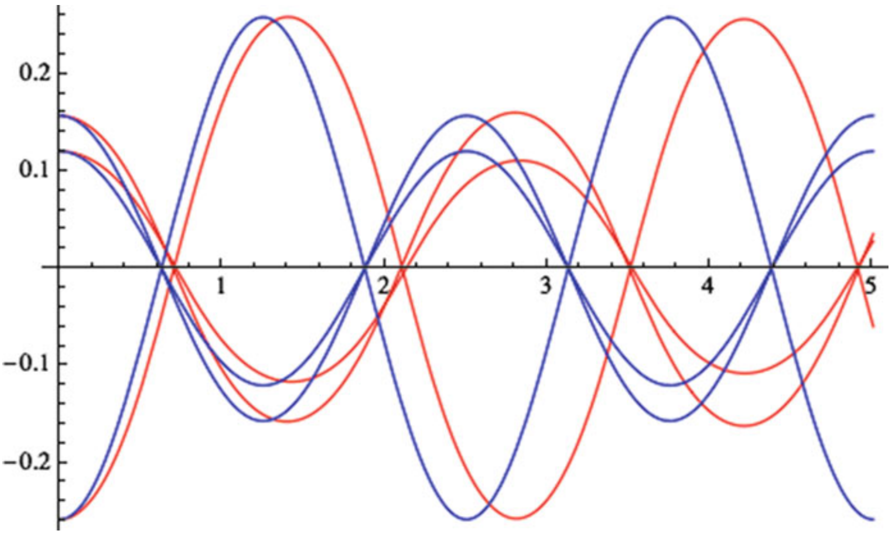


Fig. 4.5 The response of the third mode

4.1.2 Forced Motion

The modes and their modal vectors and frequencies come from the homogeneous solutions. How can we use them to solve problems of forced motion? Consider the undamped differential system Eq. (4.7).

$$\mathbf{M}_0\ddot{\mathbf{q}} + \mathbf{K}_0\mathbf{q} = \mathbf{f} \tag{4.7}$$

to begin with. I can substitute for $\mathbf{q}$ in terms of $\mathbf{u}$ and the modal vector basis matrix $\mathbf{V}$ [Eq. (4.5)] to give

$$\mathbf{M}_0\mathbf{V}\ddot{\mathbf{u}} + \mathbf{K}_0\mathbf{V}\mathbf{u} = \mathbf{f} \tag{4.8}$$

We can rewrite $\mathbf{K}_0\mathbf{V}\mathbf{u}$, making use of Eq. (4.3). The individual equations for each mode are

$$\mathbf{K}_0\mathbf{v}_1 = \mathbf{M}_0\omega_1^2\mathbf{v}_1, \quad \mathbf{K}_0\mathbf{v}_2 = \mathbf{M}_0\omega_2^2\mathbf{v}_2, \quad \mathbf{K}_0\mathbf{v}_3 = \mathbf{M}_0\omega_3^2\mathbf{v}_3, \ldots \tag{4.9}$$

from which we can deduce that

$$\mathbf{K}_0\mathbf{V} = \mathbf{M}_0\mathbf{V}\boldsymbol{\Omega}^2 \tag{4.10}$$

where $\boldsymbol{\Omega}^2$ denotes a diagonal matrix whose entries are the squares of the natural frequencies.

$$\Omega^2 = \text{diag}\left\{ \omega_1^2 \quad \omega_2^2 \quad \cdots \quad \omega_N^2 \right\} \tag{4.11}$$

Substitution into Eq. (4.8) gives the differential equations in modal form

$$\mathbf{M}_0\mathbf{V}\left(\ddot{\mathbf{u}} + \boldsymbol{\Omega}^2\mathbf{u}\right) = \mathbf{f} \tag{4.12}$$

If $\mathbf{f} = \mathbf{0}$, then the solutions of Eq. (4.12) are just the exponentials of the appropriate frequencies times the modal vectors—we are back to the beginning. Both $\mathbf{M}_0$ and $\mathbf{V}$ are invertible ($\mathbf{M}_0$ comes from the kinetic energy and so is positive definite and we have already supposed the modal vectors to be independent), so we can rewrite Eq. (4.12) as

$$\ddot{\mathbf{u}} + \boldsymbol{\Omega}^2\mathbf{u} = \mathbf{V}^{-1}\mathbf{M}_0^{-1}\mathbf{f} \tag{4.13}$$

The left-hand sides of Eq. (4.13) are now uncoupled (each component involves only one modal coefficient u_i), so that Eq. (4.13) can be rewritten as a set of equations of the form

$$\ddot{u}_i + \omega_i^2 u_i = \left(\mathbf{V}^{-1}\mathbf{M}_0^{-1}\mathbf{f}\right)_i \tag{4.14}$$

These are a set of equations that we know how to solve using the methods of Chap. 2. If the right-hand side is harmonic, proportional to $\sin(\omega_f t)$, say, then we can write

$$\ddot{u}_i + \omega_i^2 u_i = a_i \sin(\omega_f t)$$

where a_i denotes the acceleration of the ith mode. The particular solution for each mode is

$$u_i = \frac{a_i}{\omega_i^2 - \omega_f^2} \sin(\omega_f t)$$

I will assess homogeneous solutions as we move forward.

Example 4.2a Earthquake Response of an Undamped Multistory Building We can model a multistory building as a set of floors connected by beams. We can consider the floors to be rigid and the beams to be (comparatively) massless. Figure 4.6 shows such a model of a ten-story building. I will use the European system of labeling floors: ground floor, first floor, second floor, etc. The first rigid

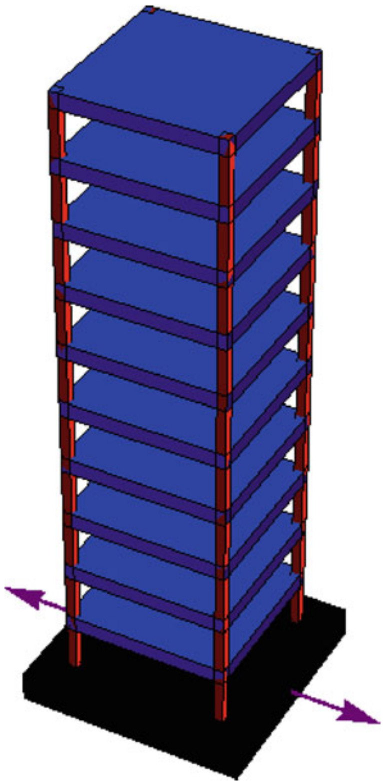


Fig. 4.6 A model of a ten-story building undergoing seismic shaking

floor in the model is then the floor of the (European) first floor and the ceiling of the ground floor. The motion of the ground floor is imposed by the earthquake. The motion of the other floors must be found. I consider building response to the horizontal ground motion shown by the arrows in Fig. 4.6.

The beams act as identical springs. I suppose the motion to be confined to the y direction, parallel to the arrows in Fig. 4.6. I label the position of the black ground element by y_0 and the positions of the floors by y_i, i running from one to ten, from the bottom to the top. I will specify the motion of the ground floor, y_0. I suppose the floors to be of equal mass m, equally spaced, and that the effective spring constant k between each floor is the same. The effective spring constant can be calculated by supposing that the floors remain parallel during the displacement. In that case the displacement of the neutral axis of each beam will be given by

$$\delta\left(3\left(\frac{y}{l}\right)^2 - 2\left(\frac{y}{l}\right)^3\right),$$

and the linear moment-curvature relation leads to an effective end force of $P = (6EI/l^3)\delta$, so that the spring constant per beam is $6EI/l^3$ [twice the stiffness of a simple cantilever beam], so the spring constant for each floor will be four times this.

The potential energy is given by

$$V = \frac{1}{2}k(y_1 - y_0)^2 + \frac{1}{2}k(y_2 - y_1)^2 + \cdots = \frac{1}{2}\sum_{i=1}^{10} k(y_i - y_{i-1})^2,$$

and the kinetic energy is given by

$$T = \frac{1}{2}m\dot{y}_1^2 + \frac{1}{2}m\dot{y}_1^2 + \cdots = \frac{1}{2}\sum_{i=1}^{10} m\dot{y}_i^2$$

The equilibrium is the state of no motion with no displacement of the floors. It is easy to see that the inertia matrix is simply the identity matrix multiplied by m. The stiffness matrix is a little more complicated, a tridiagonal matrix times the spring constant k.

$$\mathbf{K}_0 = k\left\{\begin{array}{cccccccccc} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{array}\right\}$$

All the frequencies will be proportional to the square root of k/m. The proportionality constants are

$$\{0.14946 \;\; 0.44504 \;\; 0.73068 \;\; 1.0000 \;\; 1.24698 \;\; 1.4661 \;\; 1.65248 \;\; 1.80194 \;\; 1.91115 \;\; 1.97766\},$$

so we know the natural frequencies. The proportionality constants can be viewed as dimensionless frequencies that we can use for analysis. We need the modal vectors, which define the shapes of the modes. The modal vectors satisfy the homogeneous equations

$$\begin{Bmatrix} 2-\omega_i'^2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2-\omega_i'^2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2-\omega_i'^2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2-\omega_i'^2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2-\omega_i'^2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2-\omega_i'^2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2-\omega_i'^2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2-\omega_i'^2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2-\omega_i'^2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1-\omega_i'^2 & 2 \end{Bmatrix}$$

$$\times \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ v_{10} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

where the primed frequencies denote proportionality constants between the frequency and $\sqrt{(k/m)}$. They denote dimensionless frequencies, the proportionality constants given above. This is actually straightforward. Choose the dimensionless frequency, set the first component equal to unity, and solve from the top down. The resulting unnormalized **V** matrix (rounded to three decimal places) is

Table 4.1 Nodes for the modes of the ten-story building

Mode	Node(s)
2	7
4	3, 6, 9
5	7
8	7

$$\mathbf{V} = \left\{ \begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1.978 & 1.802 & 1.466 & 1.000 & 0.445 & -0.1495 & -0.731 & -1.247 & -1.652 & -1.911 \\ 2.911 & 2.247 & 1.149 & 0 & -0.802 & -0.978 & -0.466 & 0.555 & 1.731 & 2.652 \\ 3.780 & 2.247 & 0.219 & -1.000 & -0.802 & 0.296 & 1.071 & 0.555 & -1.207 & -3.158 \\ 4.564 & 1.802 & -0.828 & -1.000 & 0.445 & 0.933 & -0.317 & -1.247 & 0.265 & 3.383 \\ 5.246 & 1.000 & -1.433 & 0 & 1.000 & -0.435 & -0.840 & 1.000 & 0.770 & -3.308 \\ 5.811 & 0 & -1.273 & 1.000 & 0 & -0.868 & 0.930 & 0 & -1.537 & 2.938 \\ 6.246 & -1.000 & -0.433 & 1.000 & -1.000 & 0.565 & 0.160 & -1.000 & 1.770 & -2.308 \\ 6.541 & -1.802 & 0.638 & 0 & -0.445 & 0.784 & -1.047 & 1.247 & -1.388 & 1.472 \\ 6.691 & -2.247 & 1.369 & -1.000 & 0.802 & -0.682 & 0.605 & -0.555 & 0.523 & -0.506 \end{array} \right\}$$

The columns of $\mathbf{V}$ contain the mode shapes, here reading from left to right. One can see how the mode shapes get progressively more complicated most easily by simply counting the changes in sign. There are no changes of sign in the first column, corresponding to the lowest frequency. Each additional column adds a change of sign, so that the mode shape of the highest frequency (in the tenth column) has nine changes in sign, the most complicated of the mode shapes. Four of the modal vectors have zero elements. These represent nodes. Table 4.1 shows the location of the nodes for those modes that have nodes.

The $\mathbf{u}$ equations are of the form

$$\ddot{u}_i + \omega_i^2 u_i = \left(\mathbf{V}^{-1}\mathbf{M}_0^{-1}\mathbf{F}\right)_i$$

We need to find the right-hand side, which can be calculated from the forcing term in the Euler-Lagrange equation, $\mathbf{f}$. I leave it to the problems to show that

$$\mathbf{f} = ky_0\{1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^{\mathrm{T}} = ky_0\mathbf{e}_1$$

$\mathbf{M}_0$ is diagonal, so its inverse is trivial, and we can write

$$\mathbf{M}_0^{-1}\mathbf{f} = \frac{ky_0}{m}\mathbf{e}_1$$

The inverse of $\mathbf{V}$ is nontrivial. We find the right-hand side of the modal equations to be

$$\frac{m}{k} \times 10^3 \times \mathbf{V}^{-1}\mathbf{M}_0^{-1}\mathbf{f}$$

$$= \{4.231 \;\; 35.858 \;\; 88.121 \;\; 142.85 \;\; 181.05 \;\; 189.41 \;\; 165.05 \;\; 116.43 \;\; 60.444 \;\; 16.549\}^{\mathrm{T}} y_0$$

We see that the ground motion contributes to all of the modes. The Euler-Lagrange equations are coupled but only one has an input. The modal equations are uncoupled but each has an input. The formulation in terms of modes is simpler because we can solve all the modal equations using the methods of Chap. 2. For this undamped case the forced building motion (the particular solution) will be in phase (or 180° out of phase) with the ground motion. I have left damping out of this example, but there will be some damping, and the particular solution will eventually dominate. (I'll look at the free modes and initial conditions later.) Consider the response of the building to simple harmonic forcing. If $y_0 = \delta\sin(\omega_f t)$, then the response for each mode will also be proportional to $\sin(\omega t)$. We'll have

$$u_i = \frac{1}{\omega_i^2 - \omega_f^2}\alpha_i\delta\sin(\omega t)$$

where α_i denotes the appropriate coefficient in the expression for the right-hand side. The solution for the third mode, for example, will be

$$u_3 = 88.121 \times 10^{-3}\frac{1}{\omega_3^2 - \omega_f^2}\frac{k}{m}\delta\sin(\omega t)$$

4.1.3 Damping

The uncoupling of the equations of motion that the modal transformation allows does not work for arbitrary damping. It does work for *proportional damping*, in which the damping matrix and the stiffness matrix are proportional, that is, $\mathbf{C}_0 = 2\eta\mathbf{K}_0$, where 2η denotes the proportionality constant (I introduced the factor of two for later convenience), and we can construct an analog of Eq. (4.10)

$$\mathbf{C}_0\mathbf{V} = 2\eta\mathbf{K}_0\mathbf{V} = 2\eta\mathbf{M}_0\mathbf{V}\Omega^2 \tag{4.15}$$

Remember that the matrix $\mathbf{V}$ is to be derived from the undamped homogeneous problem, Eq. (4.1). $\mathbf{C}_0$ and $\mathbf{K}_0$ are unlikely to be proportional in general, but we can still use modal equations if we are willing to estimate (maybe even measure) the damping rates of the modes independently. If we do that then we can impose a damping ratio for each mode. This method is called *modal damping*. We can do this by brute force, replacing the undamped modal equations by

$$\ddot{u}_i + 2\zeta_i\omega_i\dot{u}_i + \omega_i^2 u_i = \left(\mathbf{V}^{-1}\mathbf{M}_0^{-1}\mathbf{f}\right)_i \tag{4.16}$$

Equations (4.16) can be solved using the methods of Chap. 2, as I will demonstrate for the ten-story building shortly.

Note that proportional damping is a special case of modal damping. Consider the damped equations

$$\mathbf{M}_0\ddot{\mathbf{y}} + \mathbf{C}_0\dot{\mathbf{y}} + \mathbf{K}_0\mathbf{y} = \mathbf{f} \tag{4.17}$$

Expand the dependent variable in Eq. (4.17) in terms of the modal vectors as we did in the undamped case, and suppose that we have a case of proportional damping. Then Eq. (4.17) can be rewritten using Eq. (4.15) as

$$\mathbf{M}_0\mathbf{V}\ddot{\mathbf{u}} + \mathbf{C}_0\mathbf{V}\dot{\mathbf{u}} + \mathbf{K}_0\mathbf{V}\mathbf{u} = \mathbf{f} = \mathbf{M}_0\mathbf{V}\ddot{\mathbf{u}} + 2\eta\mathbf{K}_0\mathbf{V}\dot{\mathbf{u}} + \mathbf{K}_0\mathbf{V}\mathbf{u} \tag{4.18}$$

The proportionality constant 2η has the dimensions of time. We can make use of Eq. (4.10) to eliminate $\mathbf{K}_0$ and write Eq. (4.18) as

$$\mathbf{M}_0\mathbf{V}\ddot{\mathbf{u}} + 2\eta\mathbf{M}_0\mathbf{V}\boldsymbol{\Omega}^2\dot{\mathbf{u}} + \mathbf{M}_0\mathbf{V}\boldsymbol{\Omega}^2\mathbf{u} = \mathbf{f} \tag{4.19}$$

Multiply by the inverses of $\mathbf{M}_0$ and $\mathbf{V}_0$ and rearrange and we have the equivalent of Eq. (4.16), from which we can identify the damping ratios for the different modes

$$\ddot{u}_i + 2\eta\omega_i^2\dot{u}_i + \omega_i^2 u_i = \left(\mathbf{V}^{-1}\mathbf{M}_0^{-1}\mathbf{f}\right)_i \Rightarrow \zeta_i = \eta\omega_i \tag{4.20}$$

We see that the proportional damping damps the higher modes more than the lower modes, that is, the damping ratio for each mode is $\zeta_i = \eta\omega_i$ and so increases with the frequency. We can, of course, solve the individual equations in Eq. (4.17) using the methods of Chap. 2.

As long as ζ is less than unity we can solve the equations represented by Eq. (4.20) using the particular solution given by Eq. (2.27) with ϕ replaced by the expression given in Eq. (2.29c). If the input is harmonic, then the particular solution will be given by Eq. (2.25b) with a different r for each mode. If the system is excited impulsively, then the response will be made up of a set of free modes, all decaying at different rates.

Impulsive loading for the building would be sudden motion of the ground, which I will discuss in Ex. 4.2b. We have also seen impulsive loading in the form of a blow delivered to a system (see Chap. 2). I will discuss this in Ex. 4.3a.

4.1.3.1 A Scaling Example

We can reduce these problems to a very simple scaled problem when we have a system like the building for which all the masses, springs, and dampers are the same. We can rewrite Eq. (4.17) as

$$m\mathbf{M}_{00}\ddot{\mathbf{y}} + c\mathbf{C}_{00}\dot{\mathbf{y}} + k\mathbf{K}_{00}\mathbf{y} = \mathbf{f} \tag{4.21}$$

where m, c, and k denote the common mass, damping constant, and spring constant, respectively, and the doubly subscripted matrices are dimensionless. Divide by m and define $\omega_0 = \sqrt{(k/m)}$. Suppose that y can be written in terms of a common length, l, which can be the interfloor spacing or, more commonly, the expected displacement associated with disturbances, or perhaps the displacement of the ground floor, and that the time can be expressed in terms of a characteristic time τ. These assumptions allow me to rewrite Eq. (4.21) as

$$\frac{ml}{\tau^2}\mathbf{M}_{00}\ddot{\mathbf{y}}' + c\frac{l}{\tau}\mathbf{C}_{00}\dot{\mathbf{y}}' + kl\mathbf{K}_{00}\mathbf{y}' = \mathbf{f} \tag{4.22}$$

where the prime denotes a dimensionless quantity and the dot refers to differentiation with respect to the scaled time. I assume proportional damping such that $c = 2\eta k$. Multiply by τ^2/ml to get

$$\mathbf{M}_{00}\ddot{\mathbf{y}}' + 2\eta\frac{k}{m}\tau\mathbf{K}_{00}\dot{\mathbf{y}}' + \frac{k}{m}\tau^2\mathbf{K}_{00}\mathbf{y}' = \frac{\tau^2}{ml}\mathbf{f} \tag{4.23}$$

This suggests that choosing $\tau^2 = m/k$ will simplify this. This is a "natural" time scale, the inverse of a characteristic frequency. All the frequencies will be proportional to ω_0, the square root of k/m. We can scale $\eta = \tau\eta'$ and we can scale the force by F (which I will choose below), so that Eq. (4.23) becomes fully nondimensional.

$$\mathbf{M}_{00}\ddot{\mathbf{y}}' + 2\eta'\mathbf{K}_{00}\dot{\mathbf{y}}' + \mathbf{K}_{00}\mathbf{y}' = \frac{F}{kl}\mathbf{f}' \tag{4.24}$$

This suggests that the correct length scale is the force scale divided by the spring constant, which we have seen in another context in Chap. 2. It is just a static deflection under a constant force, a "natural" length scale. (Note that it does not correspond to any actual displacement, nor does the natural time scale correspond to any actual period of the system. These scales are typical or characteristic.) We apply the modal expansion and multiply by the inverse of $\mathbf{M}_{00}$ to give

$$\ddot{\mathbf{u}}' + 2\eta'\omega'^2\dot{\mathbf{u}}' + \omega'^2\mathbf{u}' = \mathbf{V}^{-1}\mathbf{M}_{00}^{-1}\mathbf{f}' \tag{4.25}$$

where $\mathbf{u}'$ denotes the dimensionless modal amplitudes. We can see that the scaled frequency plays the role of the natural frequency and that the damping ratio for each mode is given by $\zeta = \eta'\omega' = \eta\omega$ as noted above. (The scaled product is equal to the unscaled product because η and ω are scaled inversely by the same time scale.) Let's illustrate this by redoing the analysis for the building including proportional damping.

Example 4.2b The Ten-Story Building Revisited We are now in a position to add damping to the ten-story building and the look at two different excitations, a harmonic excitation and an impulsive excitation, the second half of Eq. (4.26)

$$(1)\ \ y_G = y_0 \sin(\omega_f t), \quad (2)\ \ y_G = \begin{cases} 0, & t < 0 \\ y_0, & t > 0 \end{cases} \tag{4.26}$$

I will suppose the damping to be small enough that all the nodes are underdamped ($\zeta_u < 1$), and I will assume proportional damping. For the first case I will seek only the particular solution. The second case will require me to find the homogeneous solution to cancel the particular solution at $t = 0$. I will use modal analysis, and I will use the mode structure I found for Ex. 4.2a. Thus I will write

$$\mathbf{y} = \mathbf{V}\mathbf{u} \Leftrightarrow \mathbf{u} = \mathbf{V}^{-1}\mathbf{y}$$

Equation (4.17) is the dimensional governing equation for this problem, with

$$\mathbf{f} = ky_G\{1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\} = ky_G\mathbf{e}_1 \tag{4.27}$$

Equation (4.27) defines $\mathbf{e}_1$, introduced informally in Ex. 4.2a. Let me further define $\mathbf{e}_i$ as a unit vector in configuration space with its ith component equal to unity and all the other components equal to zero. I will assume proportional damping and write $\mathbf{C}_0 = 2\eta\mathbf{K}_0$. Equation (4.17) can then be written

$$\mathbf{M}_0\mathbf{V}\ddot{\mathbf{u}} + 2\eta\mathbf{K}_0\mathbf{V}\dot{\mathbf{u}} + \mathbf{K}_0\mathbf{V}\mathbf{u} = ky_G\mathbf{e}_1 \tag{4.28}$$

The matrix $\mathbf{V}$ is the same as that I defined in Ex. 4.2a. We know from that example that $\mathbf{K}_0\mathbf{V}$ can be replaced, leading to

$$\mathbf{M}_0\mathbf{V}\ddot{\mathbf{u}} + 2\eta\mathbf{M}_0\mathbf{V}\Omega^2\dot{\mathbf{u}} + \mathbf{M}_0\mathbf{V}\Omega^2\mathbf{u} = ky_G\mathbf{e}_1 \tag{4.29}$$

which can be rewritten in modal equation form [see Eq. (4.20)]

$$\ddot{u}_i + 2\eta\omega_i^2\dot{u}_i + \omega_i^2 u = \frac{k}{m}y_G\left(\mathbf{V}^{-1}\mathbf{M}_{00}^{-1}\mathbf{f}'\right)_i = \alpha_i\frac{k}{m}y_G \tag{4.30}$$

where $\mathbf{M}_{00}$ denotes the mass (inertia) matrix divided by m, the dimensionless inertia matrix. In this case it is simply the identity matrix. The coefficients α_i are

$$\{4.231 \quad 35.858 \quad 88.121 \quad 142.85 \quad 181.05 \quad 189.41 \quad 165.05 \quad 116.43 \quad 60.444 \quad 16.549\}^T \times 10^{-3}$$

1. *Harmonic ground forcing*

We can find the particular solutions to the individual modal equations using the methods of Chap. 2. The trigonometric approach gives [see Eqs. (2.23) and (2.24a), resp]

$$u_i = A_i\cos(\omega_f t) + B_i\sin(\omega_f t) \tag{4.31}$$

where

$$\begin{aligned} A_i &= \alpha_i\frac{k}{m\omega_i^2}y_G\frac{\left(1-r_i^2\right)}{\left(\left(1-r_i^2\right)^2+\left(2\eta\omega_i r_i\right)^2\right)} \\ B_i &= \alpha_i\frac{k}{m\omega_i^2}y_G\frac{2\eta\omega_i r_i}{\left(\left(1-r_i^2\right)^2+\left(2\eta\omega_i r_i\right)^2\right)} \end{aligned} \tag{4.32}$$

Here r_i denotes the ratio of the forcing frequency to the modal frequency (different for each mode), and I have identified ζ_i with $\eta\omega_i$, different for each mode. The total $\mathbf{u}$ vector is then made up of the individual terms, and the $\mathbf{y}$ vector is obtained by multiplying by $\mathbf{V}$. We can write this compactly by supposing that A_i and B_i are the components of vectors $\mathbf{a}$ and $\mathbf{b}$, and then we'll have

Table 4.2 Peak amplitudes of the individual modes

Mode	Amplitude	Mode	Amplitude
1	63.36	6	3.005
2	20.34	7	1.829
3	11.29	8	0.9950
4	7.143	9	0.4330
5	4.669	10	0.1070

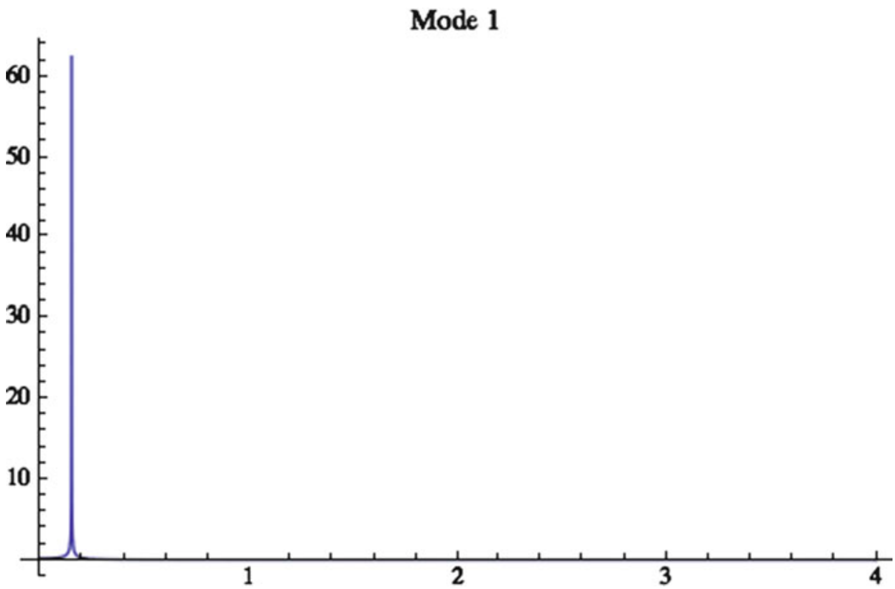


Fig. 4.7 Dimensionless amplitude of the first mode vs. r

$$\mathbf{u} = \mathbf{a}\cos(\omega_{\mathrm{f}}t) + \mathbf{b}\sin(\omega_{\mathrm{f}}t) \Rightarrow \mathbf{y} = \mathbf{Va}\cos(\omega_{\mathrm{f}}t) + \mathbf{Vb}\sin(\omega_{\mathrm{f}}t) \tag{4.33}$$

Each mode resonates at its own frequency, but the building resonates at all the frequencies, although with different levels of response at the various floors. I show the peak (dimensionless) amplitude of each mode for $\eta = 0.01$ in Table 4.2. The reduction in amplitude with increase in mode number happens because the effective damping ratio is proportional to the modal frequency, which increases. There is a corresponding broadening of each peak. Figures 4.7 and 4.8 show the first and tenth modal amplitudes, respectively. We see that we should not expect much response at the higher frequencies: the amplitude of the response of the first mode is 600 times that of the tenth.

The modal analysis reflects the intuitively obvious fact that each floor moves more than the floor below it. Imagine pushing the top floor—each floor below it will move less. The modal structure is more complicated, but the general principle holds for general forcing. The response to forcing at a modal frequency will, of course, be dominated by the modal vector associated with that frequency. I will discuss some implications of this shortly.

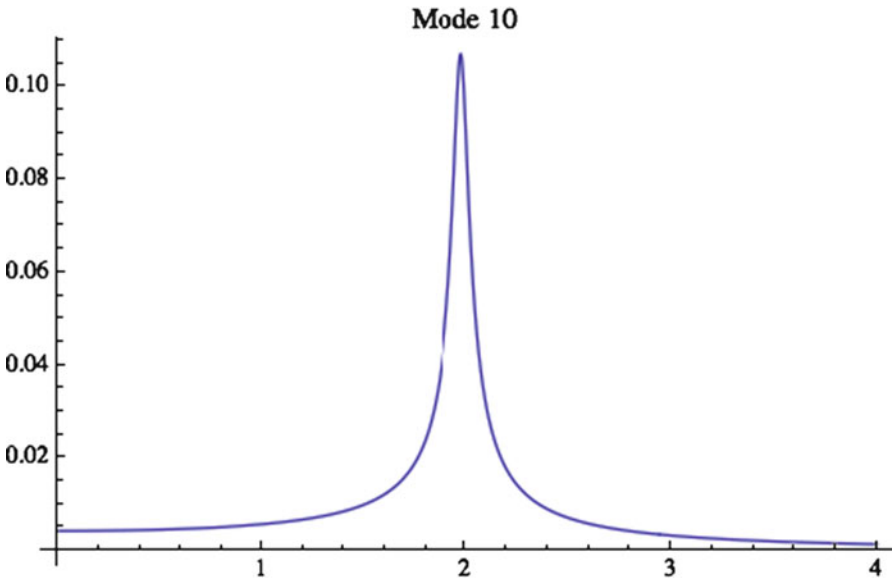


Fig. 4.8 Dimensionless amplitude of the tenth mode vs. r

Table 4.3 Dimensionless peak response at the lowest frequency

Mode	Amplitude	Mode	Amplitude
1	63.37	6	332.4
2	125.3	7	368.2
3	184.5	8	395.8
4	239.5	9	414.5
5	289.2	10	424.0

The amplitude of the response at any given floor can be deduced from Eq. (4.6). We recognize that the components of the generalized coordinate vector denote the motion of the floors, so the amplitude at the ith floor is the square root of $(\mathbf{Vu})_i^2$. Table 4.3 shows the amplitude of the response at each floor for the lowest modal frequency.

I show the amplitude of the top level in Fig. 4.9a. No modal vector has a node at this floor, so all contribute to the response, although the higher modes contribute very little. The vertical lines on the axis denote the modal frequencies. I clipped the plot at 50, which truncates the response at the first mode. The ninth and tenth modes are undetectable at this scale. Figure 4.9b shows a close-up of the high end. The ninth frequency may be detectable, and the tenth clearly is not!

Consider the nodes. The fourth mode has nodes at the third, sixth, and ninth floors, and the second, fifth, and eighth modes have nodes at the seventh floor (see Table 4.1). The modes will show up when the building is excited at these frequencies. The nodal floors will be stationary. Figure 4.10a shows the motion of the ninth floor, which has a node when excited at the fourth modal frequency. This is obvious in the figure. Figure 4.10b shows the motion of the seventh floor, which has nodes for the second, fifth, and eighth modal frequencies, and these nodes show up in the figure.

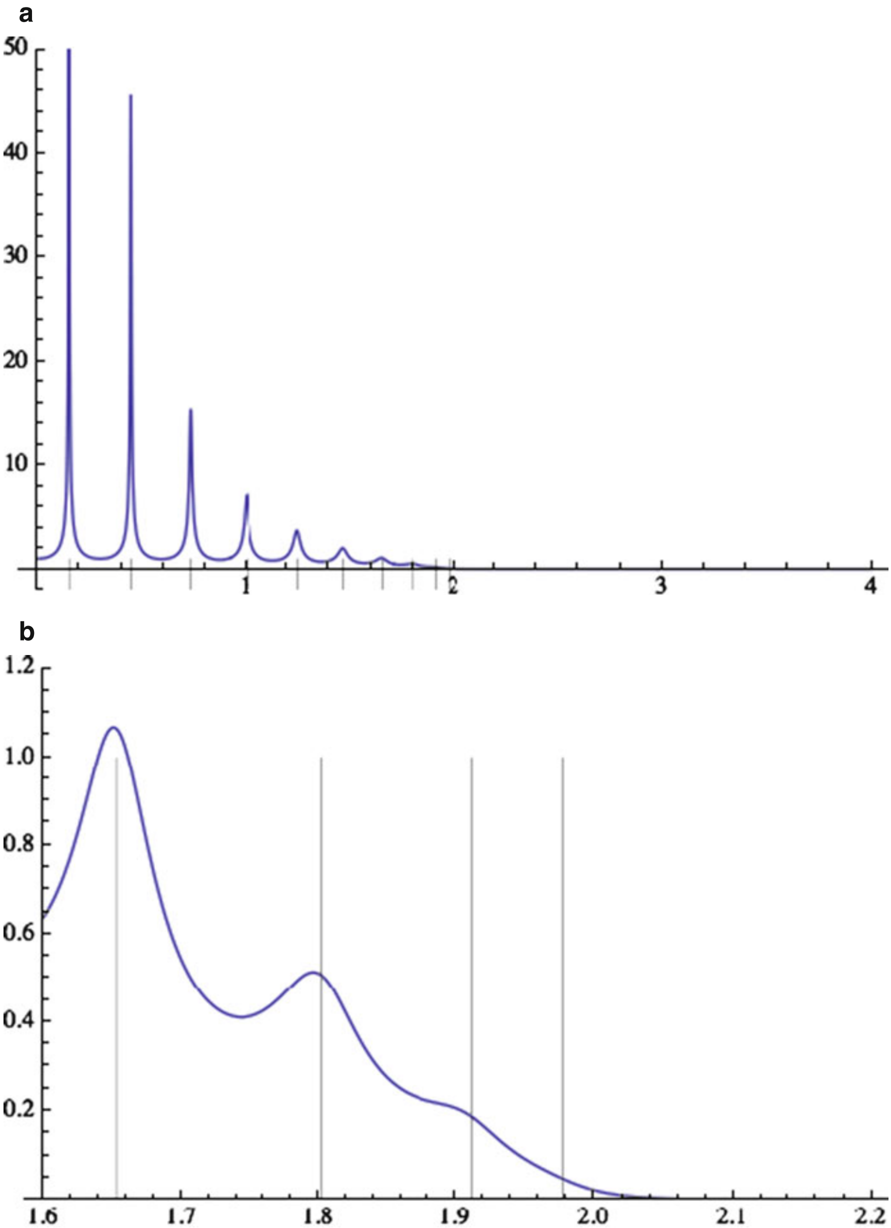


Fig. 4.9 (**a**) Amplitude of the tenth level vs. *r*. (**b**) The right-hand tail of Fig. 4.4a, showing the range covering the 7th through 10th modes, marked by *vertical lines*

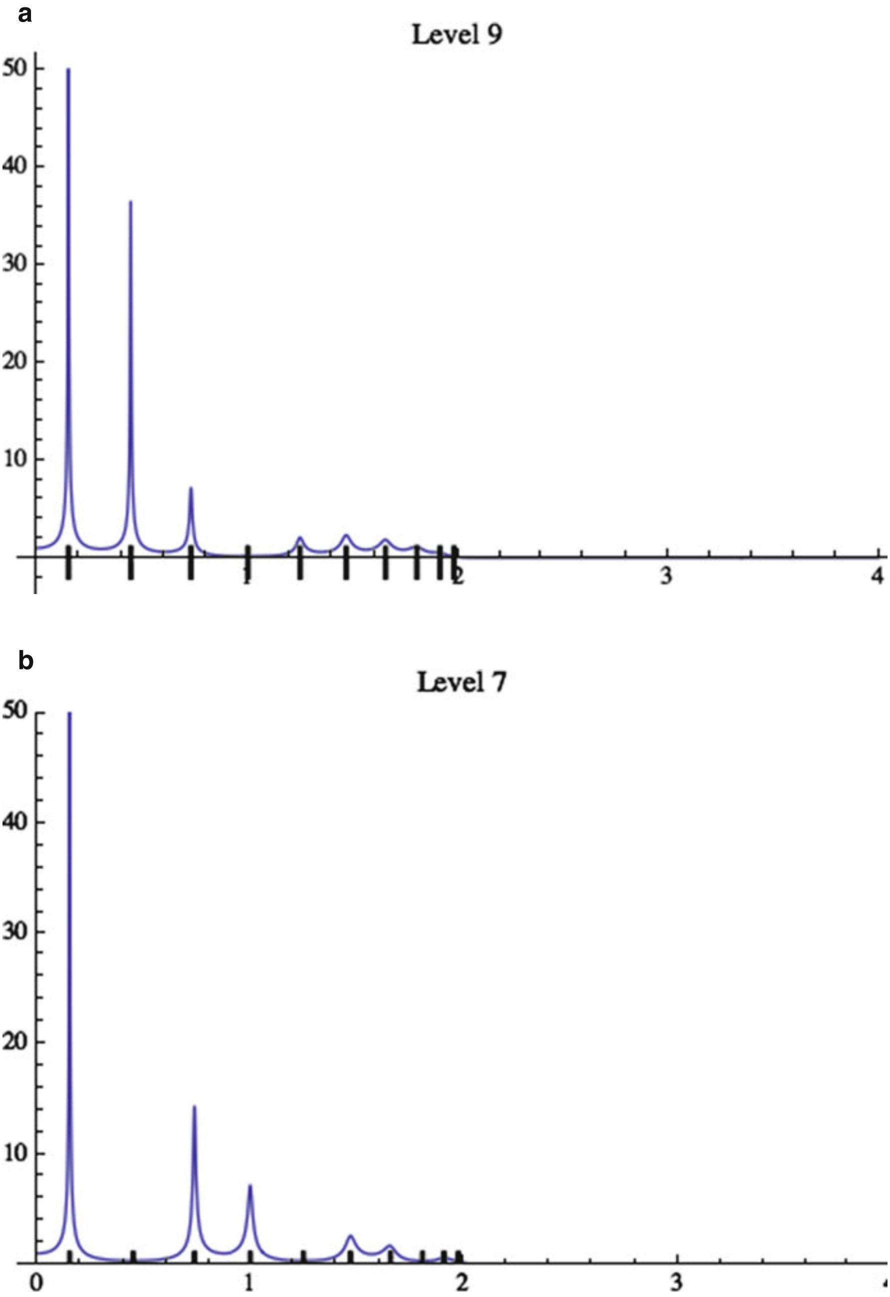


Fig. 4.10 (**a**) Response of the ninth floor vs. excitation frequency. Note the lack of motion at the fourth modal frequency. (**b**) Response of the seventh floor vs. excitation frequency. Note the absence of response at the second, fifth, and eighth modal frequencies

2. *Sudden ground motion*

I suppose that the ground moves to the right a unit amount at $t = 0$. The particular solution is simply $y_i = 1$ for all values of i. (Why?) The initial condition is that $y_i(0) = 0 = \dot{y}_i(0)$ for all i. I can formalize this by writing

$$\mathbf{y} = \mathbf{y}_{\mathrm{P}} + \mathbf{y}_{\mathrm{H}}, \quad \mathbf{y}(0) = 0 = \dot{\mathbf{y}}(0), \quad \mathbf{y}_{\mathrm{P}} = \mathbf{e}, \quad \mathbf{y}_{\mathrm{H}} = \mathbf{V}\mathbf{u}_{\mathrm{H}} \tag{4.34}$$

where I have introduced a vector, $\mathbf{e}$, every component of which is unity. We can address this problem using modal analysis. We'll have $\mathbf{u}_{\mathrm{P}} = \mathbf{V}^{-1}\mathbf{e}$, and the initial conditions on $\mathbf{u}$ are also zero. We can split $\mathbf{u}$ into a particular part and a homogeneous part. I've already given the particular part. The components of the homogeneous part satisfy the homogeneous equations

$$\ddot{u}_{\mathrm{H}i} + 2\eta\omega_i^2\dot{u}_{\mathrm{H}i} + \omega_i^2 u_{\mathrm{H}i} = 0$$

The derivative of each component is zero at $t = 0$, and the components themselves satisfy an initial condition derived from $\mathbf{u}_{\mathrm{H}}(0) = -\mathbf{V}^{-1}\mathbf{e}$.

$$u_{\mathrm{H}i}(0) = -\{\, 189 \quad 181 \quad 165 \quad 143 \quad 116 \quad 88.1 \quad 60.4 \quad 35.9 \quad 16.5 \quad 4.23 \,\} \times 10^{-3}$$

We can apply Eq. (2.17) directly with $y_0 = u_{\mathrm{H}i}(0)$ and $v_0 = 0$ to obtain

$$u_{\mathrm{H}i} = u_{\mathrm{H}i}(0)\exp\left(-\eta\omega_i^2 t\right)\left(\cos\left(\omega_{id}t\right) - \frac{\eta\omega_i^2}{\omega_{id}}\sin\left(\omega_{id}t\right)\right) \tag{4.35}$$

where the damped modal frequency is given by

$$\omega_{id} = \sqrt{1 - (\eta\omega_i)^2}\,\omega_i \tag{4.36}$$

How does this system behave? The higher modes decay much more rapidly than the lower modes. Figure 4.11a, b shows the first 100 time units for the first and tenth mode, respectively. Not only does the tenth mode decay much more rapidly than the first mode, it is also much smaller, even at the start, and Figure 4.12 shows the entire modal spectrum for 10,000 time units. It is dominated by the first mode. Even the second mode has essentially decayed by $t = 2{,}000$.

More interesting is the response of the building, which is given by $\mathbf{Vu}$. Figure 4.13a shows the response of the tenth floor (the roof) of the building for 500 time units, and Fig. 4.13b shows the response of all ten floors on the same graph. The maximum displacement of the roof is a little over twice the amplitude of the ground floor motion. We can see that the high frequencies decay rapidly, but that the low frequency persists. This is very clear in the smoothing of the roof response with

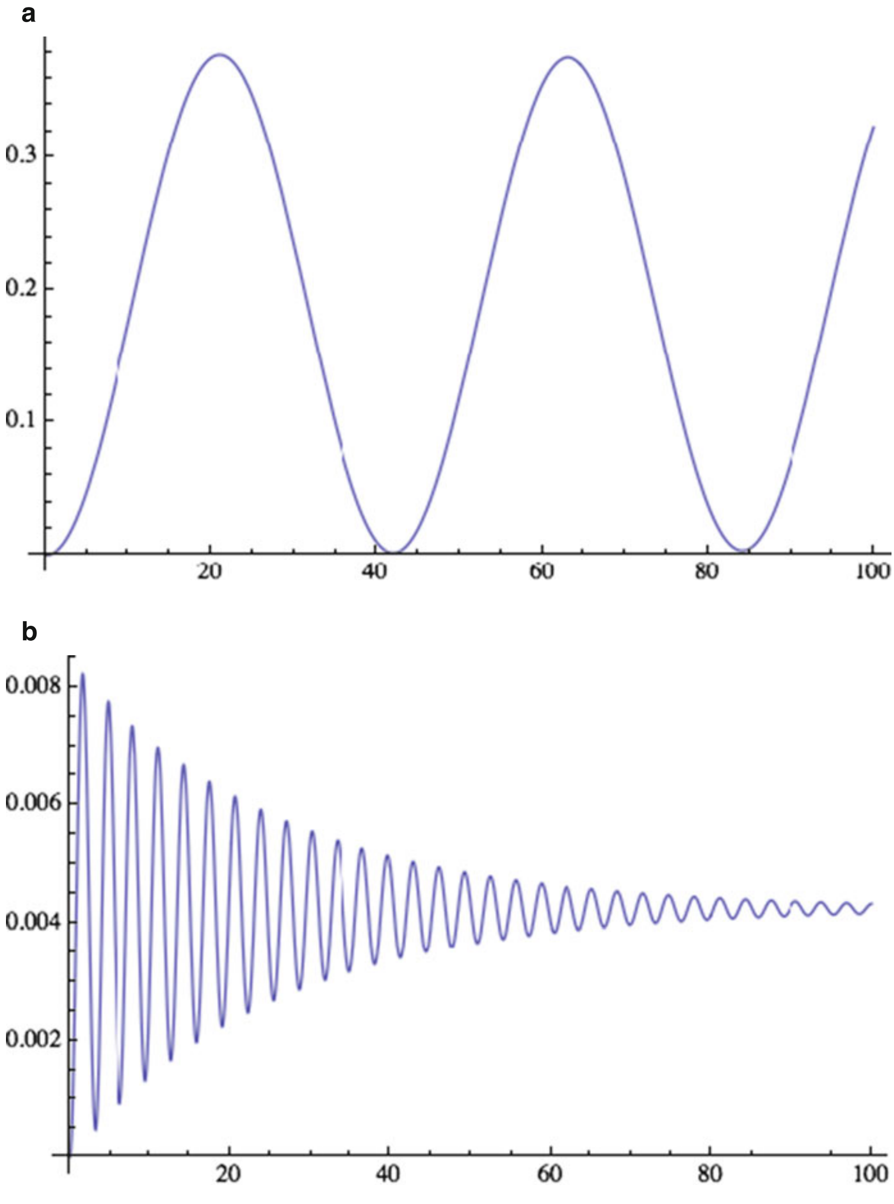


Fig. 4.11 (**a**) The amplitude of the first mode vs. time. (**b**) The amplitude of the tenth mode vs. time

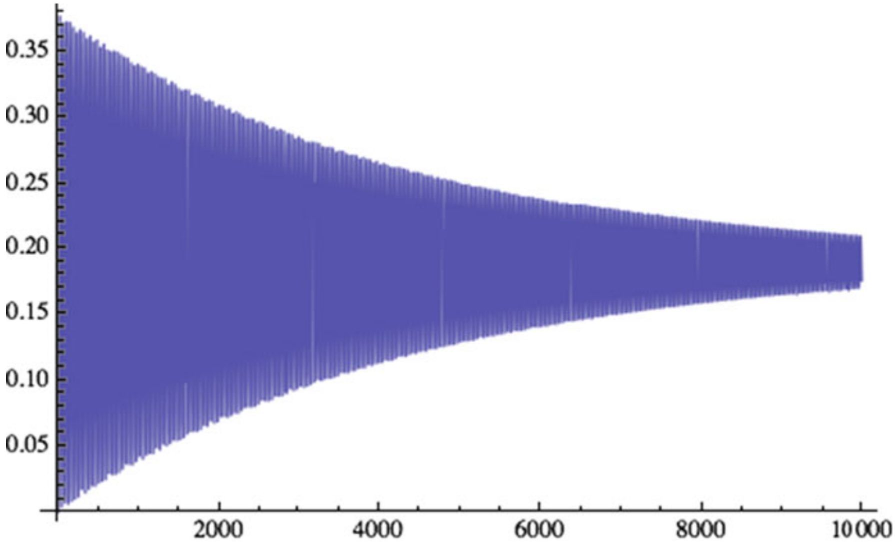


Fig. 4.12 All the modes vs. time

time in Fig. 4.13a. It takes a very long time for the lowest mode to decay. Figure 4.13b shows the response of all the floors vs. time over the same time interval.

Let's look at one more example to complete our exercises using modal analysis.

Example 4.3a A Four Degree of Freedom System Subjected to an Impulse Load Consider the behavior of the linear four degree of freedom system shown in Fig. 4.14 when it is struck from the right. Each spring represents a spring-damper combination. They are all identical, so this can be analyzed using modal analysis.

The problem is linear, and we can write the governing equations as

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f} \Rightarrow M\mathbf{M}_0 + c\mathbf{C}_0 + k\mathbf{K}_0 = \mathbf{f} \tag{4.37}$$

In this case the matrices $\mathbf{M}_0$, $\mathbf{C}_0$, and $\mathbf{K}_0$ are dimensionless. $\mathbf{C}_0$ is equal to $\mathbf{K}_0$, so this system has proportional damping. Let $c = \lambda k$, and note that λ has the dimensions of time. We can scale the problem

$$\mathbf{y} = \delta\mathbf{y}', \quad t = \sqrt{M/k}t', \quad \lambda = \sqrt{M/k}2\eta, \quad \mathbf{f} = k\delta\mathbf{f}' \tag{4.38}$$

The differential equations become

$$k\delta\mathbf{M}_0\ddot{\mathbf{y}}' + 2\eta k\delta\mathbf{C}_0\dot{\mathbf{y}}' + k\delta\mathbf{K}_0\mathbf{y}' = k\delta\mathbf{f}' \Rightarrow \mathbf{M}_0\ddot{\mathbf{y}}' + 2\eta\mathbf{K}_0\dot{\mathbf{y}}' + \mathbf{K}_0\mathbf{y}' = \mathbf{f}' \tag{4.39}$$

where the dot now refers to differentiation with respect to t'. The dimensionless matrices are

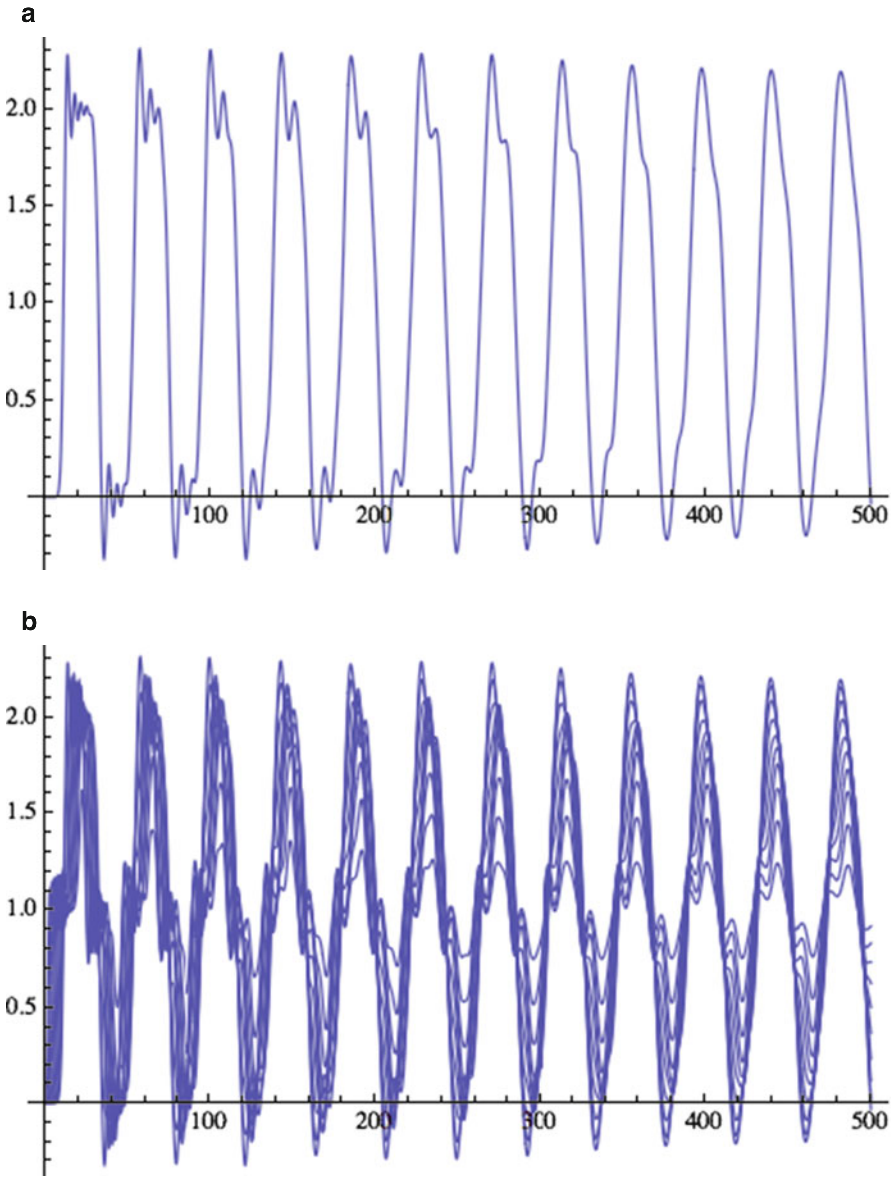


Fig. 4.13 (**a**) Response of the roof to the impulse. (**b**) Response of all the floors vs. time

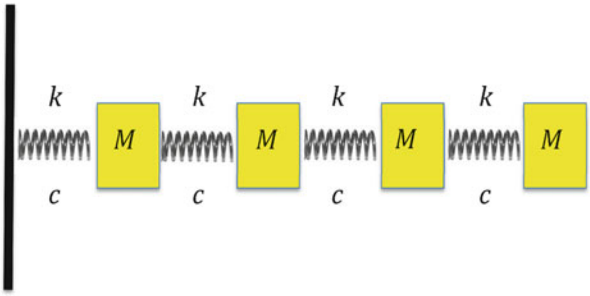


Fig. 4.14 A linear four degree of freedom problem

Table 4.4 Dimensionless modal parameters for the system in Fig. 4.14

Mode	ω_n	ζ
1	0.347296	0.347296η
2	1	η
3	1.53209	1.53209η
4	1.87939	1.87939η

$$\mathbf{M}_0 = \mathbf{1}, \quad \mathbf{K}_0 = \begin{Bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{Bmatrix}$$

Let's rehearse the modal damping procedure once again, this time with more focus on the homogeneous solution. To that end we put $\mathbf{y}' = \mathbf{V}\mathbf{u}$, where $\mathbf{V}$ denotes the matrix of the eigenvectors of the scaled version of Eq. (4.1) for this problem. The squares of the dimensionless frequencies are 0.120616, 1, 2.3473, and 3.53209. I denote these by ${\omega_n}^2$. Remember that these are dimensionless. We can find the true frequency squares by multiplying by k/M. The corresponding $\mathbf{V}$ matrix is

$$\mathbf{V} = \begin{Bmatrix} 0.347296 & -1 & 1.53209 & -1.87939 \\ 0.652704 & -1 & -0.532089 & 2.87939 \\ 0.879385 & 0 & -1.3473 & -2.53209 \\ 1 & 1 & 1 & 1 \end{Bmatrix} \tag{4.40}$$

We can combine all of this into a set of differential equations for the modal functions by setting $\mathbf{y}' = \mathbf{V}\mathbf{u}$. The differential equations become

$$\mathbf{M}_0\mathbf{V}\ddot{\mathbf{u}} + 2\eta\mathbf{K}_0\mathbf{V}\dot{\mathbf{u}} + \mathbf{K}_0\mathbf{V}\mathbf{u} = \mathbf{f}'$$

We can simplify this using Eq. (4.10) to eliminate $\mathbf{K}_0$, making the equation

$$\mathbf{M}_0\mathbf{V}\left(\ddot{\mathbf{u}} + 2\eta\boldsymbol{\Omega}^2\dot{\mathbf{u}} + \boldsymbol{\Omega}^2\mathbf{u}\right) = \mathbf{f}' \tag{4.41}$$

from which we obtain the damped modal equations, Eq. (4.20). Table 4.4 shows the natural frequency and the damping ratio for all four modes.

If all the modes are to be underdamped, then $\eta < 1/1.87939$. Let me explore how this system responds for an impulse applied to the right-hand mass. Let that mass be struck from the right with an impulse of magnitude P. This means that the initial value of the velocity of the end mass will be $-P/M$. All the position variables will be zero, as will the other derivatives. (See Ex. 2.1 for a refresher on how impulses work.) We need to work out the appropriate nondimensional initial condition. We can scale P, in a manner consistent with the rest of our scaling

$$P = M\delta\sqrt{\frac{k}{M}}v_0'$$

We can choose δ such that the dimensionless initial velocity is equal to unity: $\delta = P/\sqrt{Mk}$

The initial conditions on $\mathbf{u}$ come from those on $\mathbf{y}'$, so we have $\mathbf{u}(0) = \mathbf{0}$ and a somewhat more complicated picture for the derivative of $\mathbf{u}$ at zero: $\dot{\mathbf{u}}(0) = \mathbf{V}^{-1}\dot{\mathbf{y}}(0)$. The homogeneous solution for each mode can be obtained from Eq. (2.17)

$$u_i = \exp\left(-\zeta_i\omega_{ni}t'\right)\left(A_i\cos\left(\omega_{di}t'\right) + B_i\sin\left(\omega_{di}t'\right)\right) \tag{4.42}$$

All the A coefficients are equal to zero, because each mode must vanish at zero to satisfy the initial conditions. The derivative conditions come from $y_4(0) = -1$, projected onto $\mathbf{u}$.

$$\dot{\mathbf{u}}(0) = \mathbf{V}^{-1}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} -0.0519901 \\ -0.183634 \\ -0.333333 \\ -0.431043 \end{Bmatrix}$$

We can apply these conditions, find the B coefficients in Eq. (4.42), and plot the modal response and the response of the blocks ($\mathbf{y}$). Figure 4.15a shows the modal response for $\eta = 0.1$, and Fig. 4.15b shows the modal response for $\eta = 0.02$. Fig. 4.16a, b shows the response of the blocks for $\eta = 0.1$ and $\eta = 0.02$, respectively.

One can see the decay of each mode, the highest mode decaying most rapidly. The decay is less noticeable for the case of $\eta = 0.02$, as one would expect. In the more damped case (Fig. 4.16a) the block pictures look much like the lowest mode picture. The higher modes are more heavily damped and so quickly disappear from the figure. In the less damped case we can see the difference between the modes, nice underdamped functions, and their combination (Fig. 4.16b). We can plot the motion of any of the blocks for a long period of time in the zero damping case, and we see that the weighted sum of the modes is not particularly periodic or smooth. Figure 4.17 shows the motion of the fourth block, the one that was struck to start the system in motion. The various periods are apparent in Fig. 4.17, but since they are

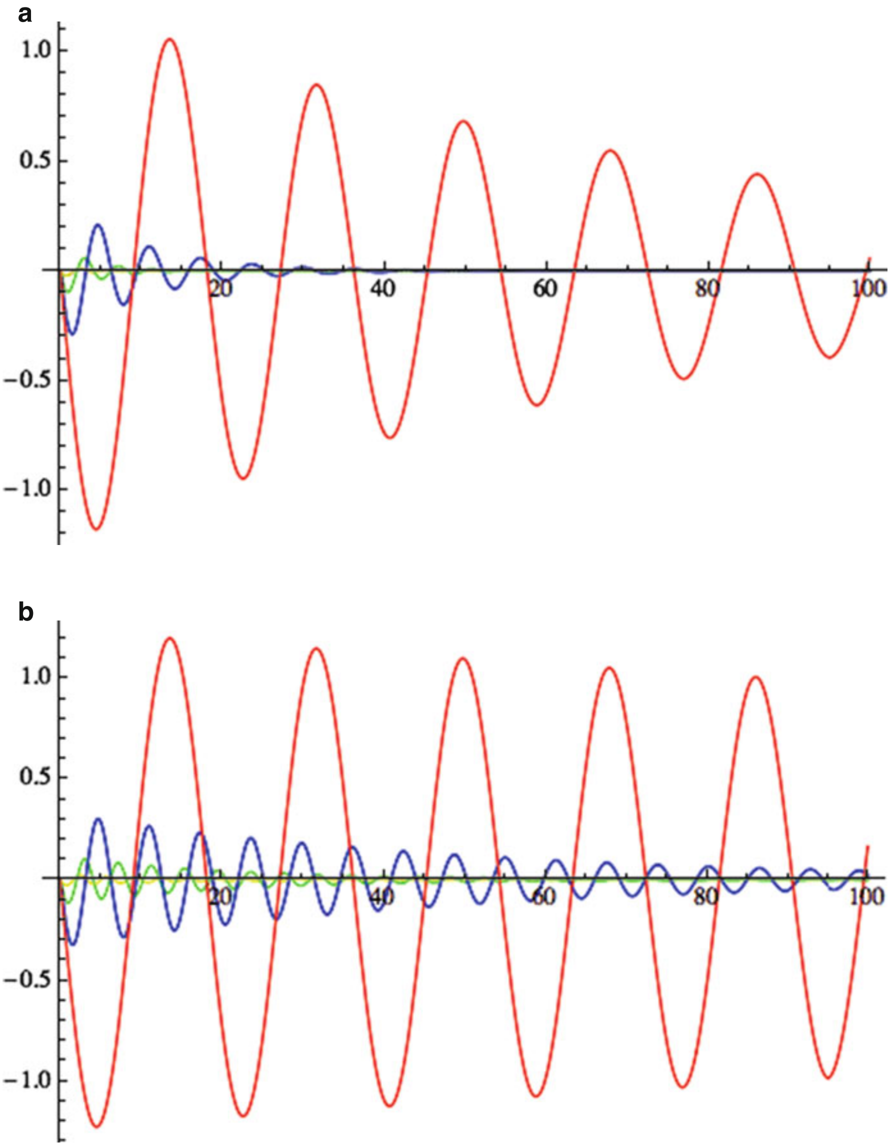


Fig. 4.15 (**a**) Modal behavior for $\eta = 0.1$: lowest mode in *red*, next in *blue*, next in *green*, and the highest mode in *yellow*. (**b**) Modal behavior for $\eta = 0.02$: lowest mode in *red*, next in *blue*, next in *green*, and the highest mode in *yellow*

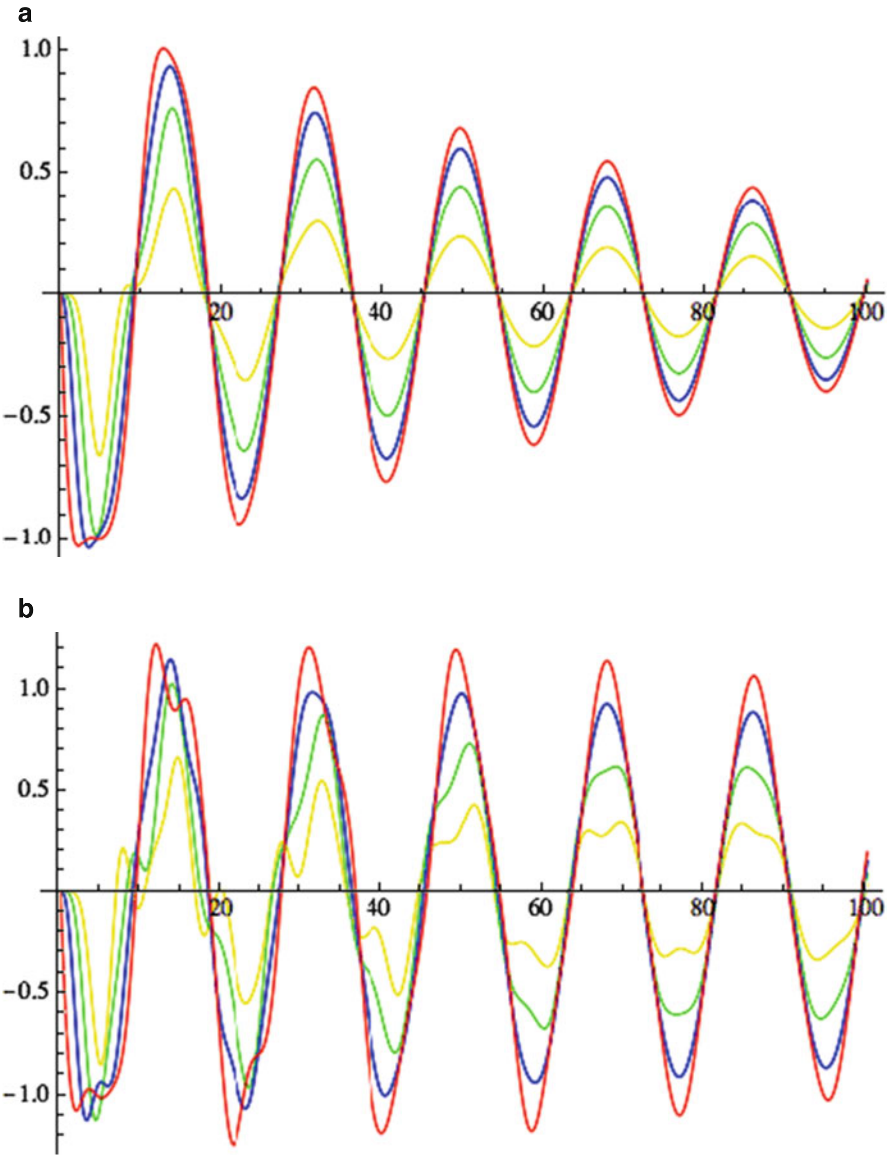


Fig. 4.16 (**a**) Motion of the blocks for $\eta = 0.1$. The blocks in numerical order from the left are depicted in *yellow*, *green*, *blue*, and *red*. (**b**) Motion of the blocks for $\eta = 0.02$. The blocks in numerical order from the left are depicted in *yellow*, *green*, *blue*, and *red*

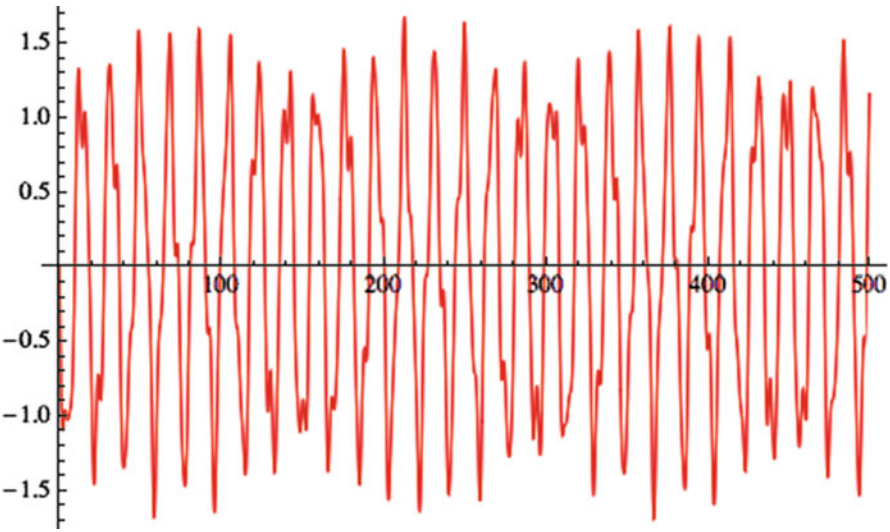


Fig. 4.17 Motion of the fourth block in the absence of damping

not practically commensurate (small integer multiples of each other) they do not form a periodic function.

It's often nice to do a sanity check: does the solution makes sense in terms of what we have said? A very simple check is to check the initial condition by plotting **y** for a small time interval starting from zero. Figure 4.18 is such a plot, and we see that all the components of the vector are zero at $t' = 0$, and that the derivatives of all but the fourth are zero, while the derivative of the fourth equals -1 as required. The solution passes this sanity check.

4.2 Continuous Elastic Systems

4.2.1 Introduction

Finding the vibration frequencies and associated damping ratios of continuous elastic systems depends on an understanding of elasticity beyond the scope of this text, but we can deal with some simple systems. We can build a reasonable model of an elastic string. This is an important problem; the model is an excellent representation of the performance of stringed instruments. I will be able to lay out reasonable models for the longitudinal and transverse vibration of beams. I will tackle these in order, but first I'd like to outline how the general problem would appear using the linear theory of elasticity. Linear elasticity assumes that strains are small in the sense that squares and products of strain components can be

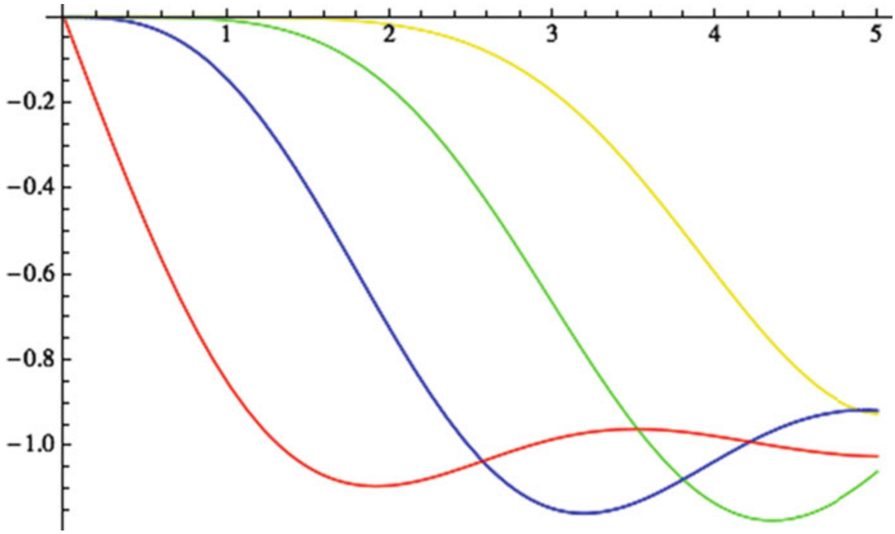


Fig. 4.18 Early time motion of the blocks. The blocks in numerical order from the left are depicted in yellow, green, blue, and red

neglected. We introduced an arbitrary small parameter ε when we did formal linearization in Chap. 3. A typical value of strain can serve as this small parameter. I will work in Cartesian coordinates, and when I get to examples I will consider motions in the y and z directions, which is consistent with the conventions I've used so far in this text. I need to abandon the planar limitation to outline the general situation. Strain and stress are tensors, each having six components, which I can write ε_{ij} and σ_{ij}, respectively. These are connected by the constitutive equation of linear elasticity

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)\delta_{ij} \tag{4.43}$$

Here δ_{ij} is equals unity when $i = j$ and zero otherwise, and λ and μ denote the Lamé constants of elasticity, which are related to the more familiar Young's modulus and Poisson's ratio by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \tag{4.44}$$

where E denotes Young's modulus and ν Poisson's ratio. Note that μ is the same as the usual shear modulus, generally symbolized by G, and that λ goes to infinity as ν goes to 1/2 (an incompressible material).

The strain can be expressed in terms of displacements. When a body B undergoes a deformation, the various points in the body move to new positions. The difference between the original positions and the final positions are called *displacements*. They form a physical vector, which I will denote by **u**. This is not the same as the modal vectors **u**. I will denote the components of the displacement by u_1, u_2, and u_3, which denote the displacement in the Cartesian x, y, and z directions, respectively. We can describe the location of any point in a material body by the vector **y**, which has the components x, y, and z, which can equally well be referred to as y_1, y_2, and y_3. The latter designation is very useful for the definition of the strain components

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i}\right) \tag{4.45}$$

The Navier equations of elasticity are a set of three second-order partial differential equations in the components of the displacement. I will not introduce them here. We don't need them to discuss the special vibration cases I've promised: transverse vibrations of the string and longitudinal and transverse vibrations of a slender beam.

4.2.2 The Vibrating String

We want to know what happens when we pluck a string, as a guitar string (or drag a bow across a violin string or hammer a piano string). The string is already under tension, stretched from some original length l_0 to an extended length which I will denote as l. This length is defined by the distance between two fixed points on the instrument. For a guitar, Fig. 4.19, the distance is from the nut to the bridge. The string is fixed at those two positions. Let the unplucked string lie along the y axis, and let the plucking take place in the z direction. I suppose that all points on the string move in the z direction, so that the displacement has one component, $u_3 = \mathrm{w}(y,t)$, which is much less than l, so we can neglect the change in length, which would cause a change in tension.

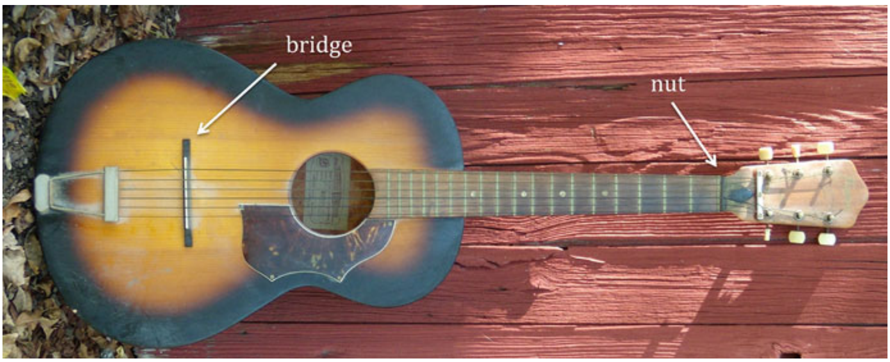


Fig. 4.19 A guitar (photo by the author)

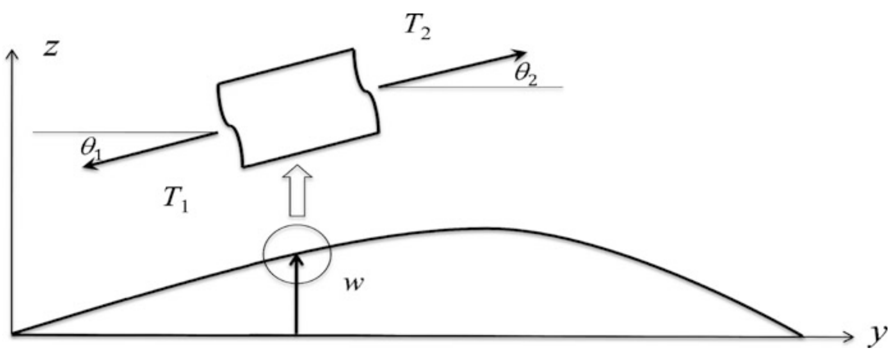


Fig. 4.20 A deflected string and its differential element

Figure 4.20 shows a sketch of the displaced string and an isolated infinitesimal element of the string.

The force balance on the infinitesimal element may be written in terms of the local tension and inclination of the string, supposing there to be no motion in the y direction

$$\begin{aligned} T_1 \cos\theta_1 &= T_2 \cos\theta_2 \\ \rho A dl \frac{\partial^2 w}{\partial t^2} &= T_2 \sin\theta_2 - T_1 \sin\theta_1 \end{aligned} \tag{4.46}$$

Here ρ denotes the density of the string, A its cross-sectional area, and dl the length of the differential element. The displacement of the string in a musical instrument will be small enough that we can suppose that the angles are small, and we can rewrite the governing equations as

$$\begin{aligned} T_1 &= T_2 \Rightarrow T_1 = T = T_2 \\ \rho A dl \frac{\partial^2 w}{\partial t^2} &= T\theta_2 - T\theta_1 \end{aligned} \tag{4.47}$$

The first equation establishes that the tension is approximately uniform throughout the string, and I have used that in writing the second equation. We can write the two angles in terms of the angle at the center of the element, which I will denote by θ, in terms of a Taylor series about the central angle

$$\theta_1 = \theta - \frac{1}{2}\frac{\partial\theta}{\partial l}dl + \cdots, \quad \theta_2 = \theta + \frac{1}{2}\frac{\partial\theta}{\partial l}dl + \cdots$$

I use a partial derivative because the angle is a function of both position and time. Substitution of the Taylor series into the second of Eq. (4.47) leads to

$$\rho A dl \frac{\partial^2 w}{\partial t^2} = T\frac{\partial\theta}{\partial l} + \cdots$$

We need to replace the angle by its expression in terms of the displacement w. The angle is the local slope of the curve represented by the string

$$\tan\theta = \frac{\partial w}{\partial l} \Rightarrow \theta = \frac{\partial w}{\partial l} + \cdots$$

Finally, we can replace the derivative with respect to l with the derivative with respect to y in this small angle limit. All these substitutions lead from the second of Eq. (4.47) to Eq. (4.48).

$$\rho A \frac{\partial^2 w}{\partial t^2} = T \frac{\partial^2 w}{\partial y^2} \tag{4.48}$$

This is the wave equation, and the wave speed for this equation is given by

$$c^2 = \frac{T}{\rho A} \tag{4.49}$$

There are different ways to address the wave equation. The best way for the string is through the Fourier series, but we haven't seen those yet, so we need to learn a little about Fourier series. I will discuss Fourier series at length in Chap. 5, but we need to look briefly at the Fourier sine series to interpret the vibration of the string.

You learned in calculus that functions can be represented by infinite series, and that infinite series can represent functions. You also learned that there are restrictions such as a limited range of the independent variable for which the series *converges*, that is, represents the function properly. The first infinite series that one usually encounters is the Taylor series, which is a series in which each term in the series is a power of the independent variable. We have used Taylor series in our discussion of linearization. *Fourier series* are series of sines and/or cosines. (It is actually more complicated than that, but I will defer a proper discussion to Chap. 5.) They are named for Jean Baptiste Joseph Fourier (1768–1830), their inventor.

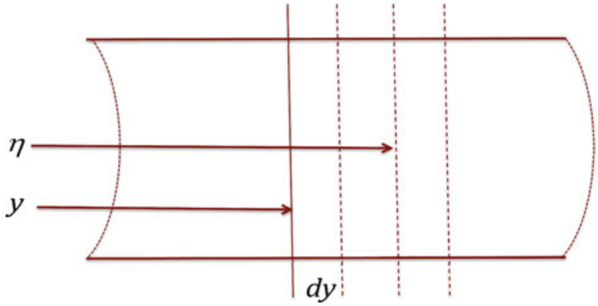
Fourier series are usually used to describe time-dependent functions, and I will use them in this way in Chap. 5. There is no need for this to be the case. We can also use Fourier series to describe spatial variations, and I will use them to describe the spatial variation of the string position. The vibrating string is continuous and of finite length and so suited to a Fourier representation.

There are four Fourier series, all closely related: the sine series, the cosine series, the complete Fourier series, and the exponential Fourier series. I will discuss these all in Chap. 5. The sine series is all I need to describe the vibrating string because the end points of the string are fixed, so that $w(0, t) = 0 = w(L,t)$. I can write

$$w(y, t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{\pi n}{L} y\right) \tag{4.50}$$

where L denotes the length of the string. Each component of the series satisfies the boundary conditions.

Fig. 4.21 Rod segment showing the slice being analyzed (after Den Hartog)



Substituting Eq. (4.50) into Eq. (4.48) leads to

$$\rho A \sum_{n=1}^{\infty} \ddot{b}_n(t) \sin\left(\frac{\pi n}{L} y\right) + T \sum_{n=1}^{\infty} \left(\frac{\pi n}{L}\right)^2 b_n(t) \sin\left(\frac{\pi n}{L} y\right) = 0$$

$$= \sum_{n=1}^{\infty} \left(\ddot{b}_n(t) + \frac{T}{\rho A}\left(\frac{\pi n}{L}\right)^2 b_n(t)\right) \sin\left(\frac{\pi n}{L} y\right) \tag{4.51}$$

The coefficient of each sine function must vanish independently, so we have an infinite set of ordinary differential equations

$$\ddot{b}_n(t) + \frac{T}{\rho A}\left(\frac{n\pi}{L}\right)^2 b_n(t) = 0 \tag{4.52}$$

These are homogeneous ordinary differential equations with constant coefficients and so they admit harmonic solutions. We see that the frequencies of oscillation will be

$$\omega_{\mathrm{n}} = \left(\frac{T}{\rho A L^2}\right)^{1/2} n\pi \tag{4.53}$$

We see that the frequency increases with the complexity of the shape of the displacement. The response at $n = 1$ is called the *fundamental*. The others are called *overtones*. The magnitude of the overtones depends on the initial excitation—how the string is plucked.

4.2.3 Longitudinal Vibrations of a Slender Beam

Consider a uniform rod of length l, cross-sectional area A, density ρ, and Young's modulus E. Let the position of any point on the rod be indicated by y. Figure 4.21 shows a segment of the rod and a slice of rod.

We want to derive the equations governing the motion of the slice. If the rod is deformed longitudinally, each point y moves to $y + \eta$, where η is small compared to l.

If this deformation is not uniform, it gives rise to a stress caused by the stretching of two adjacent points. We have

$$y \rightarrow y + \eta, \quad y + dy \rightarrow y + \eta + dy + d\eta$$

Their difference is a stretch, which we can write as

$$dy + d\eta = dy + \frac{\partial \eta}{\partial y} dy,$$

and the length that was dy is increased, and that increase is a local strain

$$\varepsilon = \frac{l' - l}{l} = \frac{\partial \eta}{\partial y},$$

and in our simple elastic model, this is accompanied by an elastic stress

$$\sigma = E \frac{\partial \eta}{\partial y}$$

The force on the slice is the stress on the right-hand side times the area minus that on the left-hand side times the area. I assumed a uniform rod, so the areas are the same. This difference comes from the usual Taylor series because the deformation is small. We then have a net force on the slice

$$F = AE \frac{\partial^2 \eta}{\partial y^2} dy$$

This is equal to the mass times the acceleration of the slice, leading to the partial differential equation

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 \eta}{\partial y^2} \tag{4.54}$$

Note that E/ρ has the dimensions of the square of a velocity. It defines the wave speed for this problem. Note also that Eq. (4.54) has exactly the same form as Eq. (4.48). It is another wave equation.

We can take a different approach to the wave equation for this problem. We chose spatial functions for the string. Let's try a time function here. Suppose that the oscillation is periodic in time at a frequency ω. Let $\eta = \mathrm{H}(y)\sin(\omega t)$. Substitution of this into Eq. (4.54) leads to an ordinary differential equation

$$\frac{d^2\mathrm{H}}{dy^2} + \frac{\rho \omega^2}{E} \mathrm{H} = 0 \tag{4.55}$$

We can solve this using the methods of Chap. 2.

$$H = A\cos\left(\sqrt{\frac{\rho\omega^2}{E}}y\right) + B\sin\left(\sqrt{\frac{\rho\omega^2}{E}}y\right) \tag{4.56}$$

The boundary conditions on H determine the frequencies. If both ends are fixed, then we find that $A = 0$ and the frequencies are determined by

$$\sin\left(\sqrt{\frac{\rho\omega^2}{E}}l\right) = 0 \Rightarrow \sqrt{\frac{\rho\omega^2}{E}}l = n\pi \Rightarrow \omega_n = \frac{n\pi}{l}\sqrt{\frac{E}{\rho}} \tag{4.57}$$

I leave it to the reader to find the frequencies for both ends free and for one end free.

4.2.3.1 The Organ Pipe

Den Hartog notes that the same general approach works for an open organ pipe. The frequency in that case is

$$\omega_{\mathrm{n}} = \frac{2n\pi c}{2l_{\mathrm{eff}}}, \quad n = 1, 2, \ldots$$

where c denotes the speed of sound in air and l_{eff} denotes an effective length, longer than the actual pipe. We can apply this to the tin whistle, comparing the lowest note on the whistle to the formula, and we find that the effective length, which accounts both for the inexactitude of the location of the zero overpressure at the open end and the nonideal closed end. Morse (1948) discusses this at some length. I measured an actual *D* whistle and also measured its lowest note. The length of the whistle is 265 mm, and the measured note was 596 Hz, 2 Hz sharp from the standard *D* major scale. The effective length is 289 mm.

4.2.4 Transverse Vibrations of a Slender Beam

The string, the longitudinal oscillations of a beam, and the organ pipe all satisfy the same partial differential equation with different constants. Transverse oscillations do not. We can find the natural frequencies of a cantilever beam (or any other configuration of a beam in simple bending) using the load-deflection equation for a beam, which can be found in any strength of material text, for example,

$$q = EI\frac{\partial^4 z}{\partial y^4}$$

where z denotes the deflection, y the length along the beam, E Young's modulus, and I the section moment. I will suppose a uniform beam, so EI is constant. Here q denotes a distributed load, force per unit length. There is no actual imposed load, but when a mass is in motion, it acts like an inertial load. One typically writes it as an inertial force $F_{\mathrm{I}} = -m\ddot{z}$. If we consider a slice as before, we can write the force on the slice as

$$F_{\mathrm{I}} = -\rho A dy \frac{\partial^2 z}{\partial t^2} \Rightarrow q = \frac{F_{\mathrm{I}}}{dy} = -\rho A \frac{\partial^2 z}{\partial t^2}$$

and write the force balance from the load-deflection relation

$$\rho A \frac{\partial^2 z}{\partial t^2} + EI \frac{\partial^4 z}{\partial y^4} = 0 \tag{4.58}$$

As before we can seek oscillatory solutions by supposing z to be proportional to $\sin(\omega t)$. Equation (4.58) becomes

$$-\omega^2 \rho A Z(y) + EI \frac{d^4 Z}{dy^4} = 0 \tag{4.59}$$

This is another homogeneous ordinary differential equation with constant coefficients, and so it admits exponential solutions, Z proportional to $\exp(ay)$. Equation (4.59) reduces to

$$-\omega^2 \rho A + a^4 EI = 0 \Rightarrow a = \left(\frac{\omega^2 \rho}{EI}\right)^{\frac{1}{4}} \tag{4.60}$$

There are four values of a: k, $-k$, jk, and $-jk$, where

$$k = \left| \left(\frac{\omega^2 \rho}{EI}\right)^{\frac{1}{4}} \right|$$

The imaginary roots lead to sines and cosines as they have before. The real roots can be rearranged into hyperbolic sines and cosines

$$\cos h(ky) = \frac{1}{2}\left(\mathrm{e}^{ky} + \mathrm{e}^{-ky}\right), \quad \sin h(ky) = \frac{1}{2}\left(\mathrm{e}^{ky} - \mathrm{e}^{-ky}\right)$$

to give the general solution for Z

$$Z = Z_1 \cos(ky) + Z_2 \sin(ky) + Z_3 \cos h(ky) + Z_4 \sin h(ky) \tag{4.61}$$

The boundary conditions determine the coefficients.

- A pinned boundary has neither deflection and nor moment: $Z = 0 = Z''$
- A free boundary has neither force nor moment: $Z'' = 0 = Z'''$
- A clamped boundary has neither displacement nor slope: $Z = 0 = Z'$

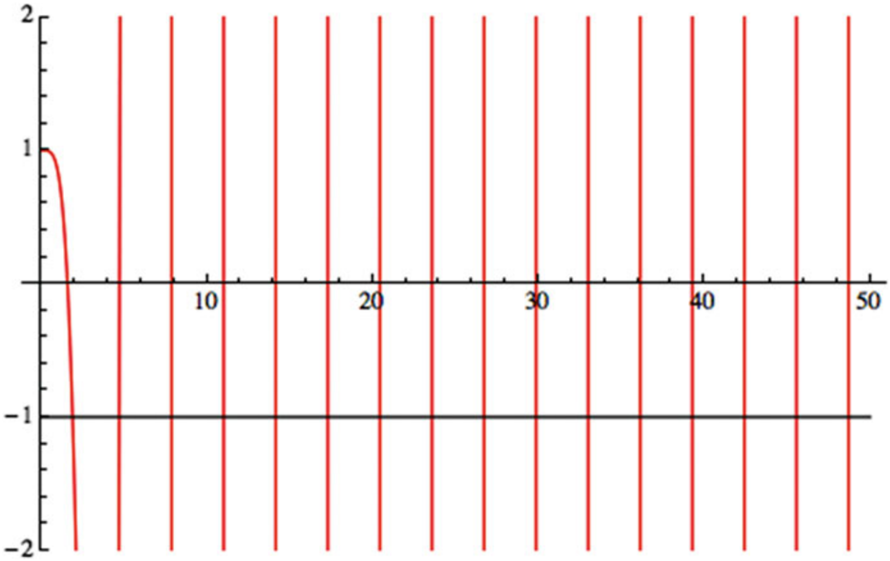


Fig. 4.22 cos(kl)cos h(kl) vs. kl. The *solid line* is −1, and the intersections give the roots for kl

No problem is as simple as the second-order problems. The conditions for a cantilever beam clamped at $y = 0$ and free at $y = l$ are

$$\begin{gathered} Z_1 + Z_3 = 0 \\ Z_2 + Z_4 = 0 \\ -Z_1 \cos(kl) - Z_1 \sin(kl) + Z_3 \cos h(kl) + Z_1 \sin h(kl) = 0 \\ Z_1 \sin(kl) - Z_1 \cos(kl) + Z_3 \sin h(kl) + Z_1 \cos h(kl) = 0 \end{gathered}$$

from which we can attain a determinant that must vanish to make it possible to find these constants. That determinant gives a transcendental relation that will determine, from which we can find ω.

$$0 = \det \begin{Bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\cos(kl) & -\sin(kl) & \cos h(kl) & \sin h(kl) \\ \sin(kl) & -\cos(kl) & \sin h(kl) & \cos h(kl) \end{Bmatrix}$$

The determinant is

$$\begin{aligned} &\sin^2(kl) + \cos^2(kl) - \sin h^2(kl) + \cos h^2(kl) + 2\cos(kl)\cos h(kl) = 0 \\ &\Rightarrow \cos(kl)\cos h(kl) = -1 \end{aligned}$$

Figure 4.22 shows cos(kl)cos h(kl) for kl from zero to 50. The roots tend to the zeroes of the cosine. Table 4.5 shows the first four actual roots, determined

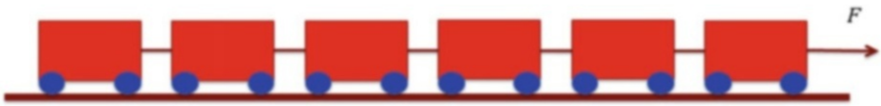
Table 4.5 The roots of kl for the free-clamped beam

Numerically determined root	Corresponding cosine zero
1.8751	1.5708
0.71239	4.71239
7.85476	7.85398
10.9955	10.9956

numerically, and the corresponding zeroes of the cosine. We see that the roots converge to their simple values very quickly.

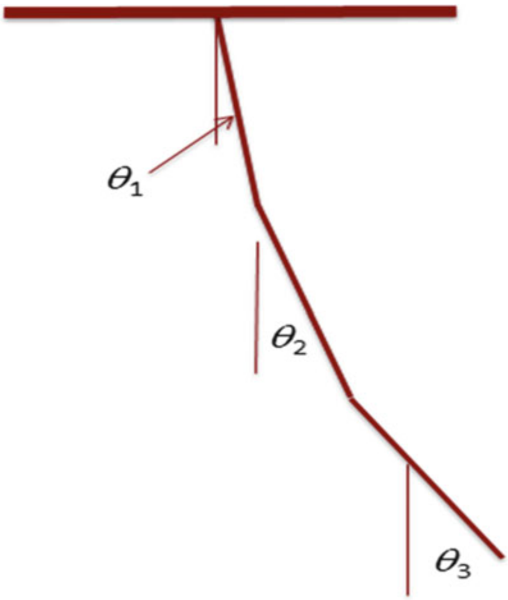
Exercises

Problems 1–5 refer to the six-car train shown below. The cars are identical and have mass M. The inertia of the wheels is negligible. The cars are coupled by identical damped springs with spring constant k and damping constant $2\eta\sqrt{(km)}$. The entire train is free to horizontally move along the track.



1. Find the number of degrees of freedom of this system (neglecting the wheels). Find the number of the degrees of freedom of the system if the wheels are included, and roll without slipping.
2. Find the governing equations and the modal frequencies and modal vectors.
3. What is the maximum value of η that insures that all the modal damping ratios are less than unity?
4. Find the steady state response of the system when $F = F_0\sin(\omega t)$ and when ω is half the lowest modal frequency and the maximum damping ratio equals 0.5.
5. Find the entire response of the system when F is an impulse equal to MV_0 and the maximum damping ratio equals 0.5.
6. Verify that Eqs. (4.4) and (4.6) are equivalent.
7. Follow the analysis of the triple pendulum with making the substitution $s = j\omega$. Do you get the same result?

Problems 8–11 refer to a triple pendulum made up of identical cylindrical steel rods one foot long and half an inch in diameter as in the sketch below.



8. Show that this is a three degree of freedom system.
9. Find the linearized equations of motion using the angles in the figure as generalized coordinates.
10. Find the response of the system starting from rest if all the initial angles are equal to $\pi/20$. What is the maximum angle in the response?
11. Find the response of the system if the far end of the third link is struck impulsively with an impulse P. Let the pendulum be at rest with all the links hanging straight down before it is struck. What is the maximum value of P for which all the angles remain less than $\pi/20$?
12. Repeat Prob. 11 assuming proportional damping with $\eta = 0.05$.
13. Derive the differential equations governing Ex. 4.3.
14. Find the natural frequencies and damping ratios for the data in table below.

Table 4.6 Data

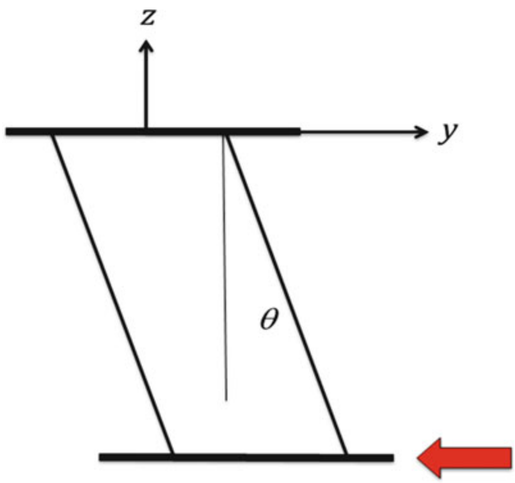
y	t	y	t	y	t
0.925533	0.152268	−0.423372	1.72503	−0.164076	3.61235
−0.790838	0.466821	0.360219	2.03958	0.140198	3.9269
0.675746	0.781373	−0.307795	2.35414	−0.119794	4.24145
−0.577403	1.09593	−0.224726	2.98324	0.10236	4.55601
0.493372	1.41048	0.192021	3.2969	−0.0874638	4.87056

15. Guitar strings are plucked and piano strings are hammered. Set up the differential equations and initial conditions for the two types of excitation. What difference do you expect in response?

16. What is the effective length of an organ pipe designed to produce a fundamental frequency of 20 Hz? What are its first three overtones?
17. What are the fundamental and first three overtones of an organ pipe ten feet long?
18. Find the natural frequency of longitudinal oscillations of an elastic beam if one end of the beam is free.
19. Find the natural frequency of longitudinal oscillations of an elastic beam if both ends of the beams are free.
20. What is the fundamental transverse vibration frequency of an aluminum rod five feet long and two inches in diameter if one end is clamped?

The following exercises are based on the chapter, but require additional input. Feel free to use the Internet and the library to expand your understanding of the questions and their answers.

21. What are the tensions in the six strings of a guitar in its normal tuning?
22. Find the lowest vibration frequency of a bow (as in bow and arrow). You may neglect damping and the initial curvature of the bow. (You may, of course, include the initial curvature if you wish a greater challenge.)
23. If you strike a hanging bar it will both move and vibrate. Consider the system shown in the figure. Find the motion and vibration of a 1″ diameter steel rod 3′ long. Suppose the impulse to be small enough that the motion of the entire rod is well represented as a pendulum.



System for Ex. 23

24. Calculate the oscillations of a diving board immediately after a 60n kg diver has left the end. Suppose the diver to have hit the rod at 6 m/s and to depart at 6 m/s upward.
25. How does the bow excite a violin string?

References

Inman DJ (2001) Engineering vibration. Prentice Hall, Upper Saddle River, NJ
Morse, PM (1948) Vibration and sound, 2nd edn. McGraw-Hill, New York, New York, pp. 220–223, 232–249, 284–287

Vibration Measurement 5

In which we look at the measurement of vibrations, and a very important tool both for measurement and for analysis, the Fourier series and Fourier transform. . ..

5.1 Vibration Measurement

It is difficult and impractical to find the natural frequencies and damping ratios of complicated machines and mechanisms analytically. We have found natural frequencies for some very simple continuous systems, but complicated systems are beyond us. We often do not know the details of most systems, so calculation will be beyond us. If we cannot calculate the frequencies, then we need to be able to measure them. We may also wish to measure vibration frequencies to trace where they may come from—to find the excitation. Your mechanism or building may be vibrating because of something going on elsewhere. If you can measure the frequencies of the vibrations you have, you may be able to unearth clues to the origin of the vibration. Let me give two quick examples.

Den Hartog (1956, p. 73) writes "A case is on record concerning a number of large single-phase generators installed in a basement in New York City. Complaints of a bad humming noise came from the occupants of an apartment house several blocks from where the generators were located, while the neighbors much closer to the course did *not* complain. The obvious explanation was that the complainers were unfortunate enough to have a floor or ceiling just tuned to 120 cycles per second. The cure for the trouble was found in mounting the generators on springs . . ."

The New York Times (Wakin 2009) reports a more recent example of a similar problem addressed during the renovation of Lincoln Center in New York City.

"Subway noise has been eliminated, Lincoln Center officials said. Ron Austin, the executive director of the development project, said the Metropolitan Transportation Authority welded down the train tracks, which are 2,000 ft away, and installed rubber pads to absorb vibrations."

R.F. Gans, *Mechanical Systems: A Unified Approach to Vibrations and Controls*,
DOI 10.1007/978-3-319-08371-1_5

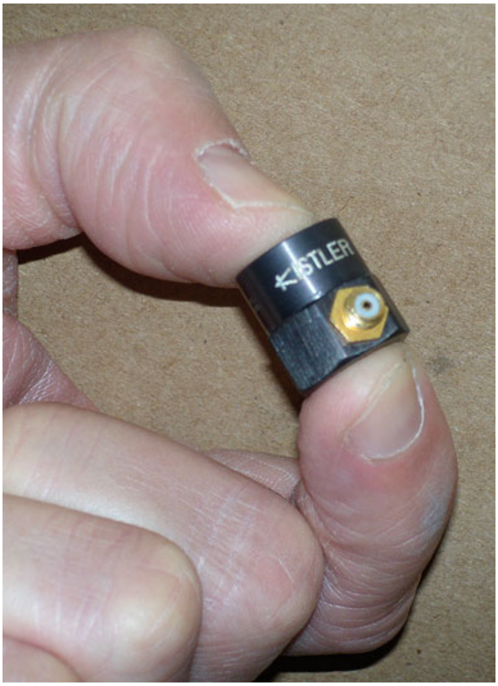


Fig. 5.1 A commercial accelerometer (Kistler 3-axis piezoelectric model). Note the diminutive size (photo by the author)

Both of these examples also bring up an important practical engineering consideration: the social and political aspects of a problem. The solutions given in the two examples are straightforward from a purely classroom engineering perspective, but in both cases the solution was remote from the problem. Someone needed to persuade the owner of the generators to remount them. Someone had to persuade the MTA to weld their tracks. Someone had to pay for these ameliorations.

The various parts of a system move when a system vibrates. We can measure the vibration by measuring this movement. We want to measure the movement without changing it, which means that we want our measurement system to be as noninterfering as possible. There are noncontact methods—optical and capacitance—but I will not treat them here. It is possible to use a microphone if the vibrations are in the acoustic range. I will give an example of this at the end of the chapter. I will focus on the use of accelerometers, which I have discussed briefly in Chap. 2. Accelerometers are typically small (see Fig. 5.1) and so have a negligible effect on the system inertia. (Not all parts move equally at all frequencies, as we saw in Ex. 4.2 a,b. The engineer may need to be clever when choosing where to mount sensors.)

It is worth noting that one can find frequencies using tiny cantilever beams, generally called *reeds* in this context. A cantilever beam with an attached mass, see Fig. 5.2a for a sketch, has a clean natural frequency and very little damping. One can adjust the location of the mass until the response is a maximum, which will determine one of the frequencies of the mechanism to which the beam was attached.

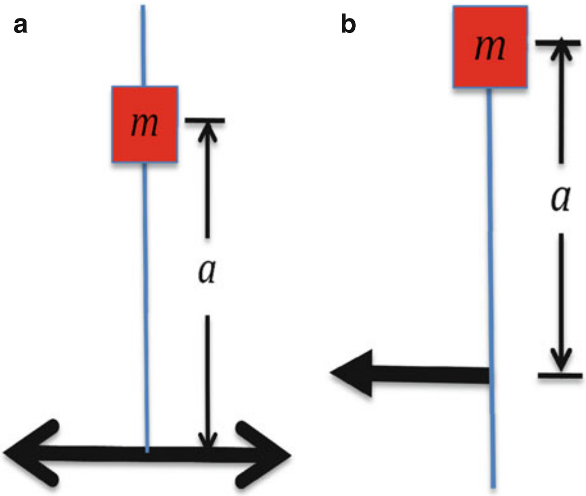


Fig. 5.2 (**a**) A cantilever beam (reed) as a vibration sensor. The distance *a* is adjustable, changing the natural frequency of the beam. The heavy black double arrow denotes the machine to which the beam is attached and its motion. (**b**) A single reed with an adjustable attachment point

(There are multiple-note tuning forks that work this way. See Fig. 5.3.) Alternatively one can use a fixed reed structure and vary the point of attachment, as shown in Fig. 5.2b.

The *Frahm tachometer* consists of several reeds, each tuned to a different frequency. This requires no adjustment to use. One merely needs to see which reed has the maximum response to identify the dominant frequency.

We looked at low-frequency systems that detect displacement at frequencies well above their natural frequency (seismometers) and high-frequency systems that detect accelerations at frequencies well below their natural frequency (accelerometers). Accelerometers are much smaller than seismometers and are ideally suited for measuring vibrations (sufficiently below their natural frequencies). We know that the response of the accelerometer is proportional to the acceleration of the object to which it is attached. We can integrate this signal twice to get the displacement. Of course, if all we want to know is the set of natural frequencies and damping ratios, we can get that directly from the acceleration. I will discuss how to do this shortly.

We need to be able to record the motion of the test mass in the accelerometer in order to measure its response. We can do this by adding a piezoelectric element to the basic instrument, as shown in Fig. 5.4.

The piezoelectric crystal is very stiff; the spring in Fig. 5.4 represents the effective spring constant of the sensor. Damping is generally very small, and my model of an accelerometer will be nondissipative. A piezoelectric material responds to stress by generating a voltage. Piezoelectricity was discovered in the nineteenth century in natural materials by the Curie brothers Jacques and Pierre (Curie and Curie 1880, cited by APC International 2002). A piezoelectric material has a principal axis of polarization, and it is pressure (deformation) in this axial direction that leads to the development of an electric field. We can think of this as the

Fig. 5.3 A tuning fork with adjustable weights to tune the frequency (photo by the author)

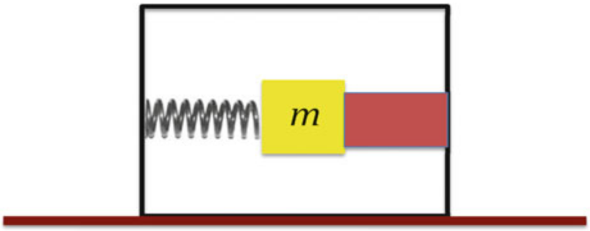


Fig. 5.4 An accelerometer with a proof mass *m*, a symbolic spring, and a piezoelectric crystal (in red)

generation of an electric field in response to a uniaxial compression or extension. This is called the *direct piezoelectric effect*. It works the other way as well—an applied voltage will deform a piezoelectric material—although that need not concern us.

Note that constant acceleration is problematical. It is necessary to draw some current from the sensor in order to measure the voltage signal. If the electric field is constant, then this current will slowly discharge the piezoelectric voltage. I will restrict our discussion to time-dependent vibrations, which is, of course, appropriate for a text dealing with vibrations. (One should really analyze the electric part of this as a system with the piezoelectric crystal and the measuring electronics as a system. This is beyond the scope of this text. An engineer wishing to use an accelerometer can generally rely on the manufacturer's information.)

An actual accelerometer, such as the one shown in Fig. 5.1, generally has three mutually perpendicular axes, so that it can measure acceleration in all directions.

The one-dimensional accelerometer modeled in Fig. 5.4 will suffice for the purposes of exploring how one analyzes the acceleration data.

We will analyze the data using computers, so we need to take the signal and convert it to a digital signal. We do this using some sort of analog to digital (A/D) converter. The converter introduces some issues. It takes a finite amount of time to convert a voltage to a digit, and the conversion has a certain level of precision. For this to give useful data, the conversion time must be short compared to time over which the input signal changes. This sets an upper bound to the frequencies that we can measure using a given A/D converter. I am not going to worry about that in the following development, but it is an issue that the engineer must consider. There is a second timing issue. The digital data are a set of discrete points separated by a fixed time interval Δt. This interval must be long compared to the conversion time. We can think of this interval as defining a *sampling frequency*, $f = 1/\Delta t$. The sampling frequency must be at least twice as high as the highest frequency in the signal in order to represent the frequencies accurately. I will discuss this at length in the next section.

Precision is expressed in terms of the number of binary digits in the digital signal. Of course, the more digits, the longer each conversion takes. If the input signal varies in the neighborhood of full scale, then 12–16 digits is enough. If the signal is always small compared to full scale, then more digits will be needed. You will generally want more than one accelerometer. Even one accelerometer generates three signals, so the A/D device must be capable of handling multiple channels. Such systems generally convert the channels sequentially, which introduces additional timing constraints. The more precision desired, the higher the sampling frequency, and the more channels needed, the more expensive the instrument. There are multichannel 24 bit MHz devices commercially available at the time of this writing.

How do we generate data? We must excite the system we wish to study. We can do this by hitting it (see the impulse response in Chap. 2 and Ex. 4.2b) or by shaking it, so that the input is a fixed harmonic function Ex. 4.2b. We can tune the harmonic input and observe the output magnitude as a function of the input magnitude. This is a more useful method. The idea is to increase the shaking frequency in steps and record the response amplitude as a function of the input frequency. We will get a plot that looks a lot like Fig. 4.3 if there is but one natural frequency. If there is more than one natural frequency, then we will get a more complicated picture like that shown in Fig. 4.9a.

Example 5.1 Measuring the Vibration of a Four Degree of Freedom System Let's look at an artificial example that will shed some light on measuring frequencies using an accelerometer. Consider the four degree of freedom system shown in Fig. 4.14. We add an accelerometer to the right-hand end of the system and then shake the system at different frequencies and observe the accelerometer signal. We expect intuitively its amplitude to be larger at the damped natural frequencies of the system than elsewhere. Figure 5.5 shows the augmented system. The output of the system will be the signal from the accelerometer, which we learned in Chap. 2 will be proportional to the difference between the motion of the accelerometer and the object to which it is attached. In the present case the signal will be $\ddot{y}_5 - \ddot{y}_4$.

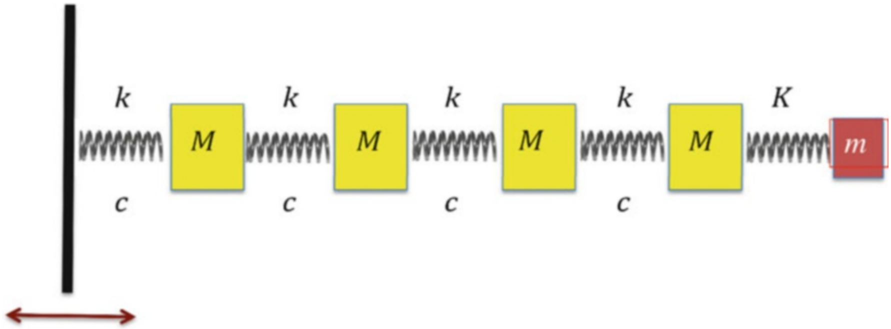


Fig. 5.5 A model four degree of freedom system with an accelerometer attached to its free end

If the system has one major natural frequency, then we can find that by simple observation. We can simply plot the amplitude of the response as a function of the exciting frequency, as we did back in Chap. 2, and pick off the natural frequency and work out the damping ratio (how?). In the case of more than one natural frequency, the procedure is not so simple.

First let us look at the effect of adding the extra mass and spring to the original four degree of freedom problem. I want the added mass to be small, and I want the natural frequency of the mass-spring system considered in isolation to be much higher than the frequencies to be observed. I write $m = \mu m$ and $K = \kappa k$. I suppose μ to be small and κ to be large. This should reduce the effect of the mass on the frequencies and insure that the accelerometer will behave as an accelerometer ought, its differential output proportional to the applied acceleration. I will also suppose there to be no damping associated with the K (accelerometer) spring for the sake of simplicity.

This system is not amenable to modal analysis, so we will have to use a more basic approach: actually solving the Euler-Lagrange equations, which we can write in the form of Eq. (3.45)

$$\mathbf{M}_0\ddot{\mathbf{y}} + \mathbf{C}_0\dot{\mathbf{y}} + \mathbf{K}_0 = \mathbf{Q}_0$$

The matrices in this equation are

$$\mathbf{M}_0 = M\left\{\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mu \end{matrix}\right\}, \quad \mathbf{C}_0 = c\left\{\begin{matrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}\right\},$$

$$\mathbf{K}_0 = k\left\{\begin{matrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1+\kappa & -\kappa \\ 0 & 0 & 0 & -\kappa & \kappa \end{matrix}\right\}$$

Table 5.1 Natural frequencies and damping ratios for the five degree of freedom system

Natural frequencies	Damping ratio for $\eta = 0.1$	Damping ratio for $\eta = 0.02$
0.345807	0.0345807	0.00691614
0.996704	0.0996701	0.019934
1.52933	0.152933	0.0305865
1.87844	0.187843	0.0375687
50.4977	38.89×10^{-6}	7.78×10^{-6}

The forcing appears only in the first equation and can be written $ky_G + c\dot{y}_G$.

The form of the matrices suggests scaling the problem as we did for Ex. 4.3a. We do not have proportional damping because of the addition of the accelerometer, but we can write $c = 2\eta\sqrt{(kM)}$, and η will play the same role as it did in Ex. 4.3a. The dimensionless oscillation frequencies can be obtained from the determinant of

$$s^2\frac{\mathbf{M}_0}{M} + 2\eta s\frac{\mathbf{C}_0}{c} + \frac{\mathbf{K}_0}{k}$$

The results depend on the ratios μ, κ, and η. I choose $\mu = 0.02$ and $\kappa = 50$ and show the resulting frequencies and damping ratios in Table 5.1 for $\eta = 0.1$ and 0.02, the same values I used in Ex. 4.3a. The nominal accelerometer frequency is then approximately 50 $\sqrt{(k/M)}$, a dimensionless value of 50, much higher than any of the other natural frequencies.

The first four frequencies are very close to their corresponding values in Table 4.4, and the natural frequency of the accelerometer mass is close to its nominal value of 50. We see that the additional small mass makes very little difference, as we had expected. The damping ratios are quite close to those given in Table 4.3 as well. Note that the motion of the accelerometer is slightly damped through the motion of the other links.

We obtain the dimensionless particular solution (using complex notation) in the form $\mathbf{y}' = \mathbf{Y}\exp(j\boldsymbol{\omega}t')$. The signal will be the amplitude of the difference between y_5 and y_4. I write it as $A(\omega) = \sqrt{(y_5 - y_4)(y_5 - y_4)*}$, where the asterisk denotes complex conjugate. I can plot this quantity as a function of frequency over and beyond the range of expected response frequencies. Figure 5.6 shows the result for low damping ($\eta = 0.02$), and Fig. 5.7 shows the results for higher damping ($\eta = 0.1$). The former shows all four frequencies; the latter is limited to the two lowest frequencies, with perhaps a hint at the third frequency.

What has happened to the fifth frequency? It can be seen with a sufficiently sensitive instrument. Figure 5.8 shows the amplitude of the response in the range (40, 60), and there is a clear spike in the neighborhood of 50, but the magnitude of the spike is very small, ten orders of magnitude less than the other peaks, and hence practically undetectable.

This is a rather ad hoc method. We can replace it with a method based on power spectra, but we need to explore the subject of Fourier series and transforms first.

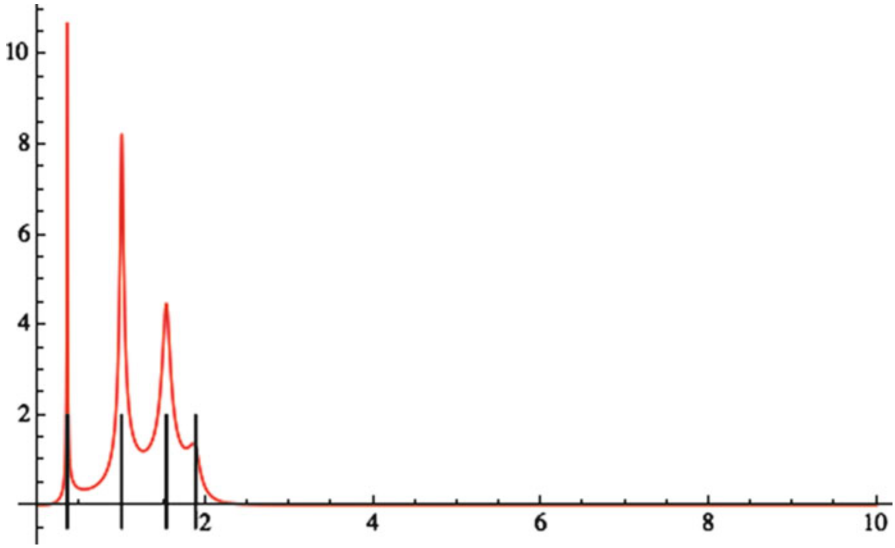


Fig. 5.6 The acceleration of the accelerometer mass vs. frequency for the low damping case. The *vertical lines* mark the positions of the damped natural frequencies

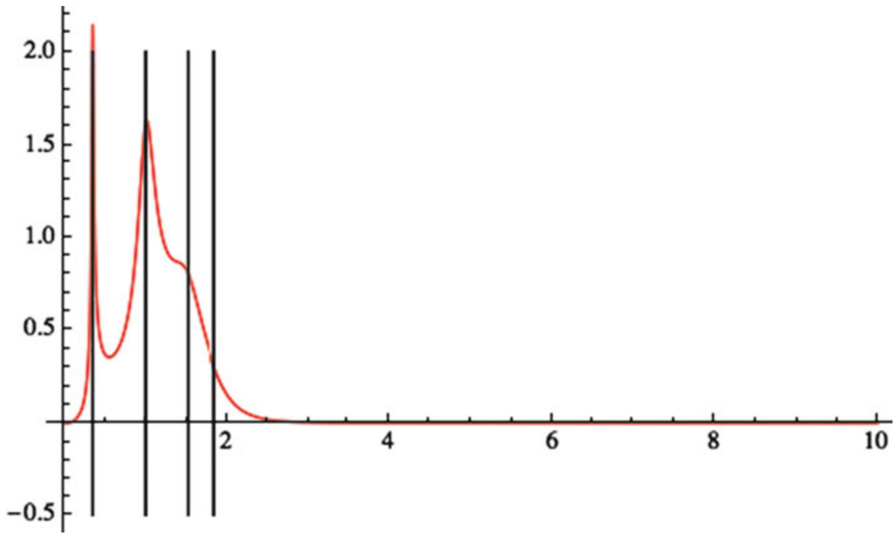


Fig. 5.7 Mark the positions of the damped natural frequencies

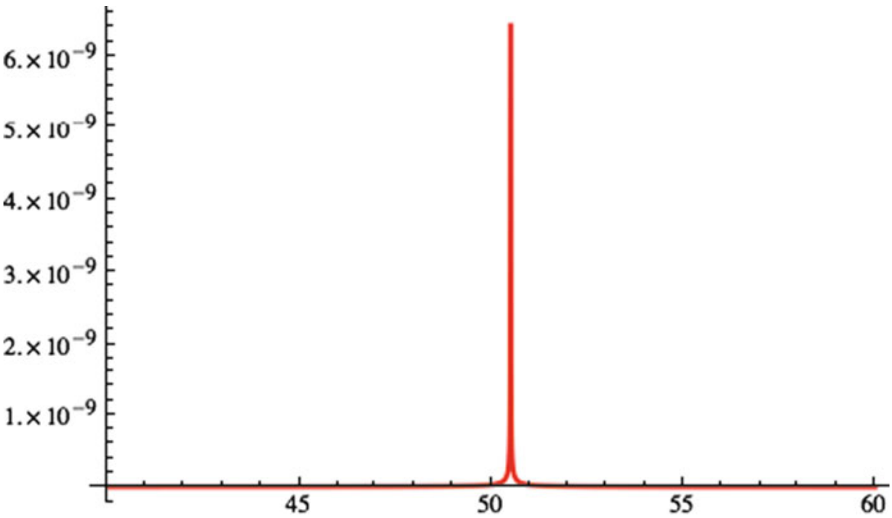


Fig. 5.8 The acceleration of the accelerometer mass vs. frequency for the low damping case in the range $40 < \omega < 60$. Note the sharp spike at 50

5.2 Fourier Series and Transforms and Power Spectra

Continuous systems have, in principle, an infinite number of vibration frequencies. As a practical matter only a few of them are likely to be important. The principle vibration frequencies need to be known so that one can avoid exciting them or, if excitation at those frequencies is unavoidable, so that one can redesign the system to eliminate them. We have looked at detecting frequencies informally. The standard method of measuring frequencies is to calculate power spectra. Power spectra are defined in terms of the Fourier transform. The Fourier transform requires a continuous signal over an infinite time interval. Data are not like this. Experimental data are typically a set of discrete data points taken over a finite interval of time. To use them to calculate a power spectrum requires a finite approximation to the Fourier transform. This *finite Fourier transform* is closely related to the Fourier series. I will start with the series, which came first chronologically, and then discuss the transform, which is really an extension of the series to an infinite interval. This is not a math book, so I will merely make the idea of Fourier series plausible without proving anything in a rigorous sense. Most books dealing with advanced calculus and partial differential equations cover the topic exhaustively. I will not.

I introduced the Fourier sine series informally at the end of Chap. 4. This section treats Fourier series in much more detail. I will repeat a little of what I said at the end of Chap. 4 so that this section is self-contained.

5.2.1 Fourier Series

You learned in calculus that functions can be represented by infinite series, and that infinite series can represent functions. You also learned that there are restrictions such as a limited range of the independent variable for which the series *converges*, that is, represents the function properly. The first infinite series that one usually encounters is the Taylor series, which is a series in which each term in the series is a power of the independent variable. One may recall that the Taylor series about some point t_0 is

$$f(t) \approx f(t_0) + \left.\frac{df}{dt}\right|_{t=t_0}(t - t_0) + \left.\frac{1 d^2 f}{2 dt^2}\right|_{t=t_0}(t - t_0)^2 + \cdots$$

The usual case is that $t_0 = 0$, so that the series is a simple series in powers of t. I will suppose data records to start at zero unless I specifically state otherwise. I use the time t as my independent variable, because this book deals with the behavior of mechanisms and systems as functions of time. The powers are said to be *complete*. Only complete sets can be used to represent a function in terms of an infinite series. Odd functions, functions for which $f(-t) = -f(t)$, can be represented by using only odd powers and even functions, functions for which $f(-t)$, $=f(t)$ can be represented by using only even powers.

The sine is an odd function and the cosine is an even function, and they possess power series representations of the appropriate forms (which you can find as Taylor series around the origin)

$$\sin(\omega t) = \omega t - \frac{1}{3!}(\omega t)^3 + \frac{1}{5!}(\omega t)^5 + \cdots,$$
$$\cos(\omega t) = 1 - \frac{1}{2!}(\omega t)^2 + \frac{1}{4!}(\omega t)^4 + \cdots$$

where ω denotes some frequency and the ! denotes the factorial, the product of the integers up to and including the number emphasized. For example, $4! = 1 \times 2 \times 3 \times 4 = 24$. It is not obvious, but nonetheless true that the sine and cosine of $n\omega t$ form complete odd and even sets as n runs from zero to infinity and can be used to represent odd and even functions. A combination of sines and cosines can represent any function, with a few caveats: the function being represented must be piecewise continuous (only a finite number of discontinuities), and the representation is valid only over a finite interval. These trigonometric representations are called *Fourier series* for Jean Baptiste Joseph Fourier (1768–1830), their inventor. (The Fourier transform extends the interval to infinity.)

The trigonometric functions are continuous, so they cannot represent even one discontinuity perfectly. Discontinuities lead to something called *Gibbs phenomenon*, which I will discuss shortly. The mathematical continuity condition is not an issue for engineering functions; mechanisms are continuous unless they break, and broken mechanisms are beyond the scope of this book. We all live in finite time, so that, too, is not, strictly speaking, a problem. However, the combination of finite intervals and discontinuities can lead to unexpected difficulties, which we can gather together under the name of Gibbs phenomena. You need to be aware of these things as we go forward. Starting and stopping data collection can introduce artificial discontinuities that can affect the interpretation of the measurements. We must be aware of this potential problem.

There are two important uses of Fourier series in the study of vibrations: analyzing observed vibrations and solving the (linear) differential equations governing forced vibrations. The latter is not much used anymore because we can solve these equations fruitfully using modern software.

We can think of vibrations in terms of frequencies. When the vibration frequencies are in the range of human hearing we can hear them as vaguely musical—think of the humming of a refrigerator or a furnace or an air conditioner. The frequencies contained in these vibrations can tell us about the processes that may be generating them and can also provide information about the best ways to minimize their impact on people. (We learned about vibration isolation back in Chap. 3.) Generally these frequencies are not immediately obvious, as are the frequencies produced by musical instruments (although, even there, there is more than one frequency—*overtones* of the *fundamental*). If we can convert the vibration to an electric signal, by attaching an accelerometer, say, we can analyze that signal using power spectra based on the finite Fourier transform, which we can relate to the Fourier series. This is a big step forward from the ad hoc method explored in Ex. 4.3b. Figure 5.9 shows a set of artificial vibration data collected over the time interval 12–47. The data are clearly oscillatory, but we can't tell very much about these data by looking at the figure. I'll discuss the application of Fourier series to these data once we have looked at how to calculate the series.

There are four Fourier series, all closely related: the sine series, the cosine series, the complete Fourier series, and the exponential Fourier series. The exponential Fourier series is closely related to the complete Fourier series. It is also the series that connects to the Fourier transform. The development below is loosely based on that in Churchill and Brown (1987), which is pretty much what one finds in most mathematical texts. Chapters 12 and 13 in Press et al. (1992) give practical information for all the Fourier methods I will discuss, focusing on the transform. The data are supposed to lie on the interval $(0, T)$ for the sine and cosine series and on $(-T, T)$ for the complete and exponential Fourier series.[1] (Churchill and Brown use $T = \pi$.) The data shown in the Fig. 4.27 lie on (12, 47), but the origin of time can easily be shifted to move our data into the appropriate range, so I will follow the standard development.

[1] Den Hartog (1956) uses a $(0, T)$ interval for the complete Fourier series, which is often more convenient. His T is, of course, twice that of Churchill and Brown.

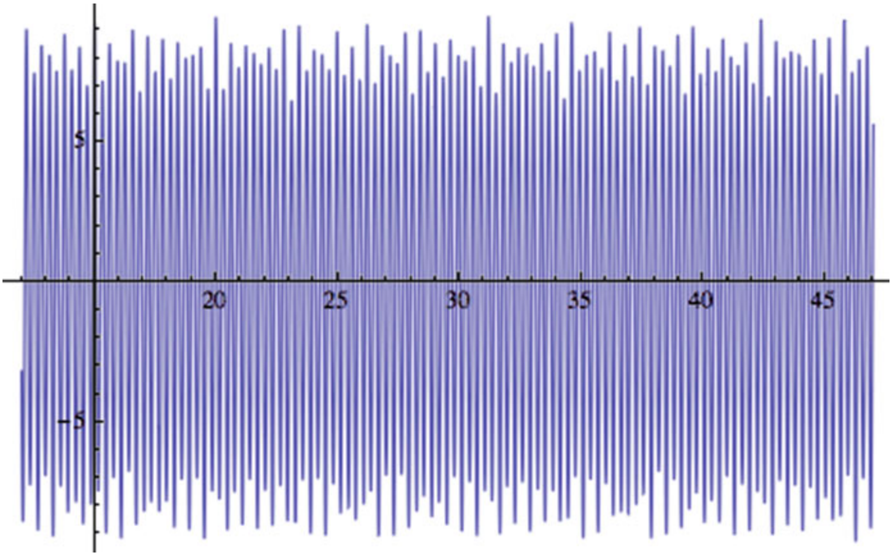


Fig. 5.9 Artificial data

5.2.1.1 Sine and Cosine Series

For the sine and cosine series we write $\omega = \pi/T$, so that $n\omega t = n\pi t/T$, and the sine and cosine series are given by

$$f(t) \approx \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{T}t\right), \quad f(t) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n}{T}t\right), \tag{5.1}$$

respectively. The data contributing to Eq. (5.1) lie on $0 < t < T$. If the data are collected on $t_1 < t < t_2$, as in Fig. 5.9, we can shift the origin and use the standard interval. I do not use an equals sign because the function and the series are not exactly equal. The series are representations. The coefficients are given by integrals over the interval

$$b_n = \frac{2}{T}\int_0^T \sin\left(\frac{\pi n}{T}t\right) f(t)dt, \quad a_n = \frac{2}{T}\int_0^T \cos\left(\frac{\pi n}{T}t\right) f(t)dt \tag{5.2}$$

The sine series is an odd function and the cosine series is an even function, and they are periodic with period $2T$, twice the interval during which the data were collected. The series represent the function on the interval (allowing for difficulties at isolated points), and their behavior outside the interval is governed by their periodicity and their oddness or evenness. There are some obvious caveats: if the function is not zero when t equals zero, the sine series is not appropriate. *The complete Fourier series*

The complete Fourier series is defined on $-T < t < T$. Equations (5.3) and (5.4) give the transform and its coefficients.

$$f(t) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n}{T} t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{T} t\right) \tag{5.3}$$

This looks like I have just added up the sine and cosine series, but I have not, as the formulas for the coefficients show.

$$a_n = \frac{1}{T}\int_{-T}^{T} \cos\left(\frac{\pi n}{T} t\right) f(t)dt, \quad b_n = \frac{1}{T}\int_{-T}^{T} \sin\left(\frac{\pi n}{T} t\right) f(t)dt \tag{5.4}$$

The interval has been doubled in size and the coefficients divided by 2. The complete Fourier series is also periodic with period $2T$, but it is neither odd nor even, and its period is equal to the length of the interval.

5.2.1.2 The Complex Fourier Series

The complex Fourier series is something of a simplification of the complete Fourier series, and it is most closely related to the finite Fourier transform. Consider the integral

$$\frac{1}{2T}\int_{-T}^{T} f(t) \exp\left(-j\frac{\pi n}{T} t\right) dt = a_n - jb_n = c_n \tag{5.5}$$

This defines *the Fourier coefficients*—the complex coefficients of the complex Fourier series. The complex Fourier series can be written in terms of the c_n as

$$f(t) \approx \sum_{n=-\infty}^{\infty} c_n \exp\left(j\frac{\pi n}{T} t\right) \tag{5.6}$$

This is a real function because the coefficients for negative n are the complex conjugates of those for positive n so the imaginary parts cancel. Note that the zero coefficient calculated according to Eq. (5.6) is equal to the mean value of f, so there is no need for a special calculation. Equation (5.6) with coefficients given by Eq. (5.5) is complete in itself.

5.2.1.3 Examples of Fourier Series

The complete Flourier series is to be preferred to either the sine or cosine series by itself. The complex Fourier series will be useful when we discuss the finite Fourier transform. The complete Fourier series and the complex Fourier series are equivalent. I think it is little easier to follow the complete series so that we do not have to worry about complex quantities. Note that T as used in the complete and complex Fourier series is half the length of the record. If one wishes to use the length of the record as the time base, then the expressions become

$$f(t) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T} t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T} t\right) \tag{5.7}$$

with coefficients given by

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \cos\left(\frac{2\pi n}{T} t\right) f(t)dt, \quad b_n = \frac{2}{T}\int_{-T/2}^{T/2} \sin\left(\frac{2\pi n}{T} t\right) f(t)dt$$

It is also possible to shift the interval of integration to (0, T), where T now denotes the length of the record, leaving the expression for the series unchanged and writing the coefficients as

$$a_n = \frac{2}{T}\int_{0}^{T} \cos\left(\frac{2\pi n}{T} t\right) f(t)dt, \quad b_n = \frac{2}{T}\int_{0}^{T} \sin\left(\frac{2\pi n}{T} t\right) f(t)dt \tag{5.8}$$

I will adopt Eqs. (5.7) and (5.8) as the definition of the complete Fourier series. This expression is clearly periodic with period T. This periodicity leads to two problems. First, the function being modeled is generally not periodic, so if t is outside the range over which the coefficients were calculated, the series no longer represents the function. The second problem is that if $f(0) \neq f(T)$, then the periodic Fourier series is discontinuous at the ends of the range. All the individual terms in the series are smooth, and they cannot represent a discontinuity exactly. The series will exhibit Gibbs phenomenon at each end. The Gibbs phenomenon also occurs if there are discontinuities in the function being represented within the range where the coefficients were calculated, but this is less of a problem for most engineering applications, particularly if we are using the series to analyze data. The time interval over which the Gibbs phenomenon extends decreases with the number of terms in the series, but the size of the jump never goes away. This is perhaps better illustrated by some examples.

Example 5.2 A Nonperiodic Function Exhibiting Gibbs Phenomenon at the End Points Let $f = 1 + t$ and let $T = 1$. The function is not periodic, and its end points are not equal. Figure 5.10 shows a 100-term Fourier representation on the interval (0, 1) in red with the function in blue. The Gibbs phenomena are clearly visible at the end points. The function is smooth, but its periodic extension is discontinuous.

Figure 5.11 shows the periodicity of the representation. Again the Fourier series is in red and the function in blue. The Gibbs phenomena are again clearly visible at the ends where the representation is periodic and the function is not.

The function f need only be C^0 continuous (the function is continuous but its first derivative is not) within its range to avoid Gibbs phenomenon if $f(0) = f(T)$, as can be seen in Ex. 5.3.

Example 5.3 A Sawtooth Function Let $f = t$ for $t < ½$ and $(1 - t)$ for $½ < t < 1$, and keep the time interval $T = 1$. Figure 5.12 shows the function and its representation. The full blue line completely covers the representation, and there is no sign of Gibbs phenomenon.

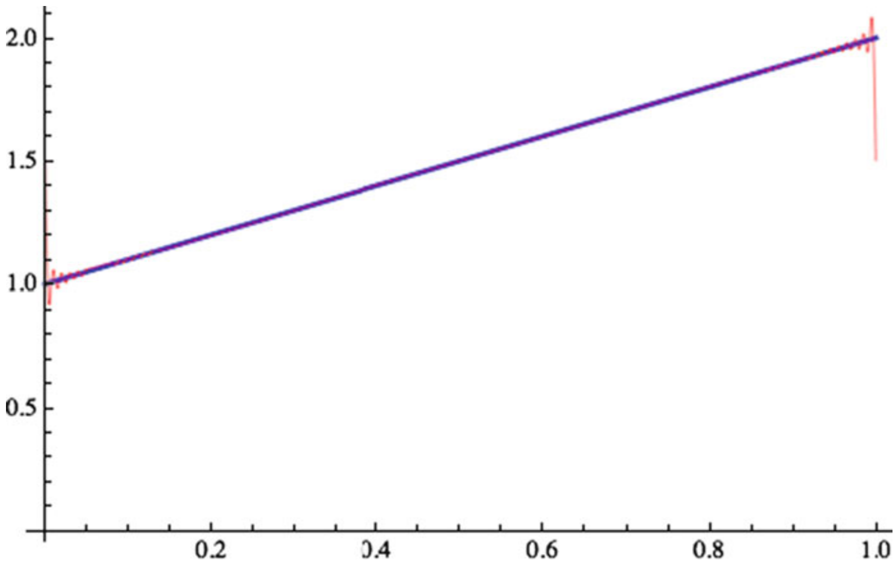


Fig. 5.10 Fourier representation of a linear function. Note the Gibbs phenomenon at both ends of the range

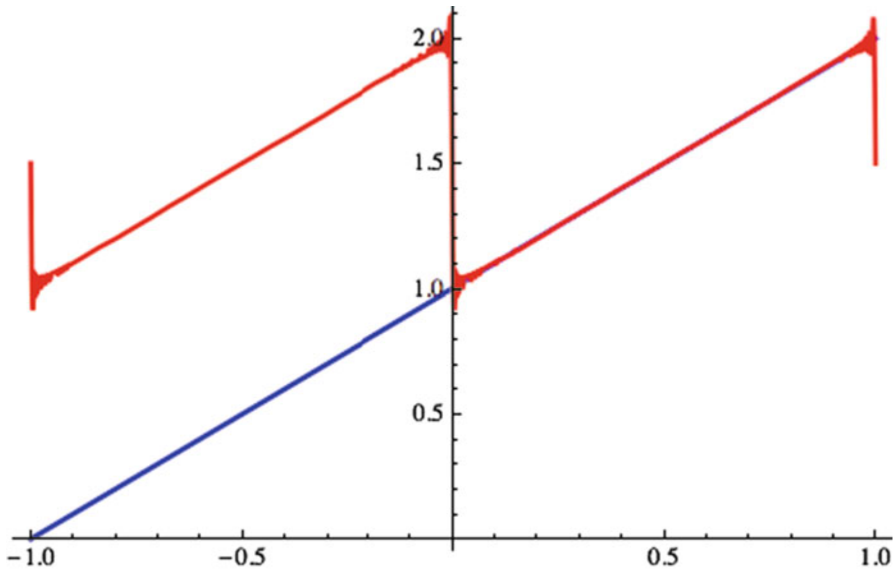


Fig. 5.11 The model of Fig. 5.10 extended outside the range of definition

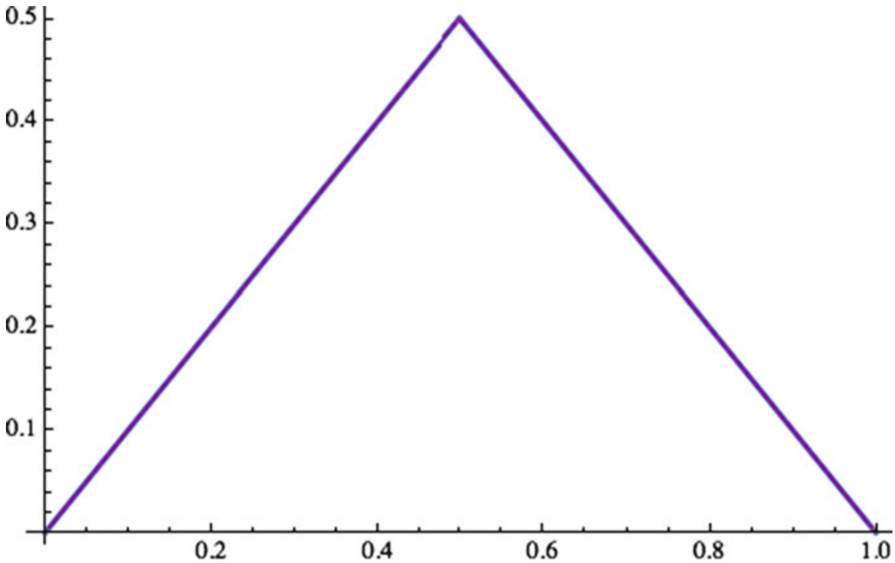


Fig. 5.12 Representation of a sharp hump

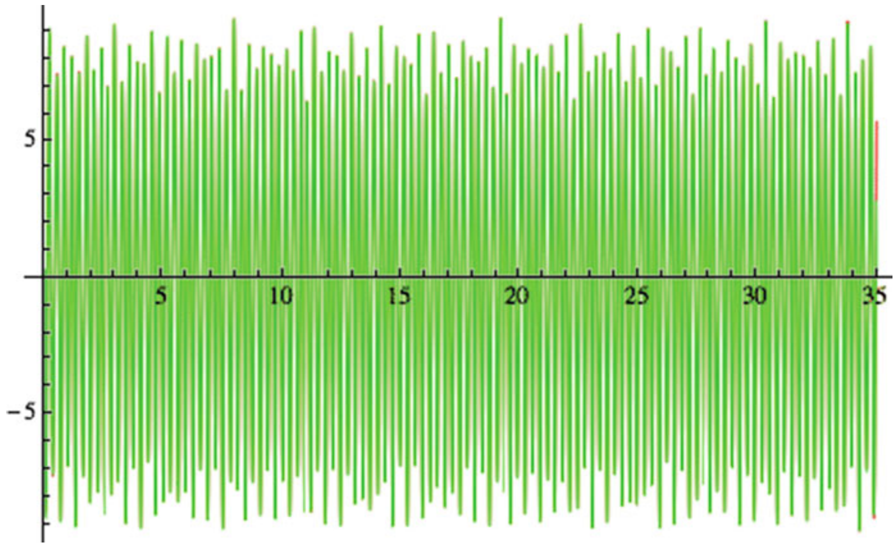


Fig. 5.13 The 500-term Fourier representation of the data shown in Fig. 4.27

I talked about identifying the frequencies in a signal by finding its Fourier series. Figure 5.13 shows the complete Fourier series representation of the data shown in Fig. 5.9, truncated after the first 500 terms. The series is in green and the data are in red. The series appears to overlay the data everywhere except at the very ends, where the data values are −3.18489 at the beginning and 5.61609 at the end. The representation

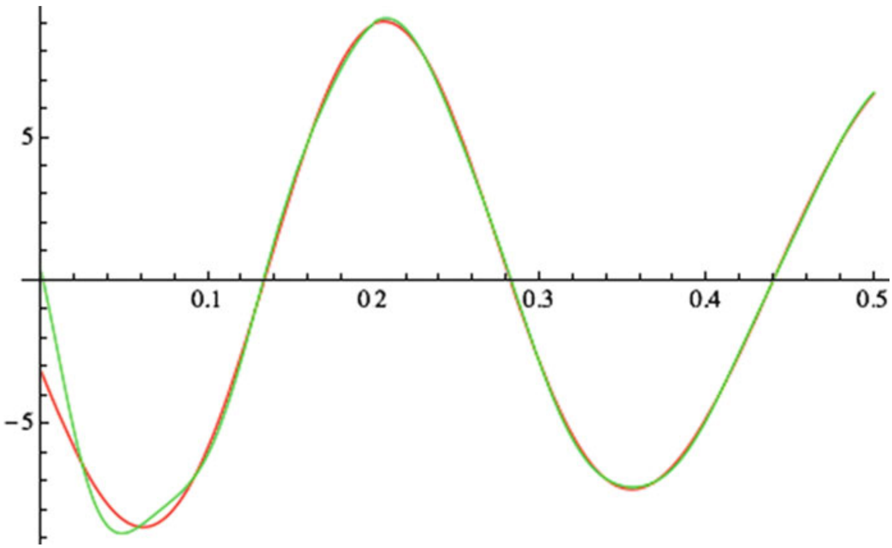


Fig. 5.14 The 500-term representation at the beginning

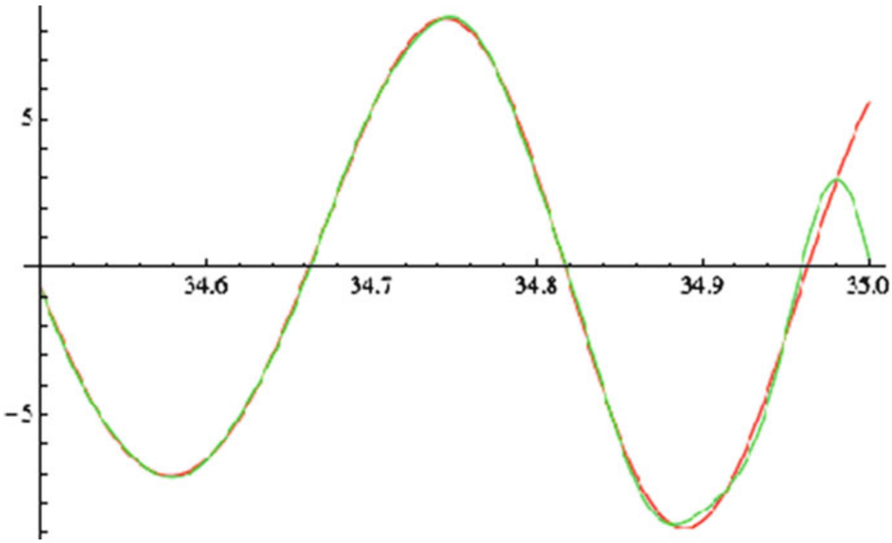


Fig. 5.15 The 500-term representation at the end

takes the value 0.2766 at both ends. The end values must be the same because the series is periodic. The end values equal the mean value of the signal.

Figures 5.14 and 5.15 show the details of the representation at the two ends. The discrepancy is obvious, but it does not look like a Gibbs phenomenon. The reason is that, for this function, 500 is actually a small number of terms. If we take 5,000 terms, we get very Gibbs-like behavior (Figures 5.16 and 5.17).

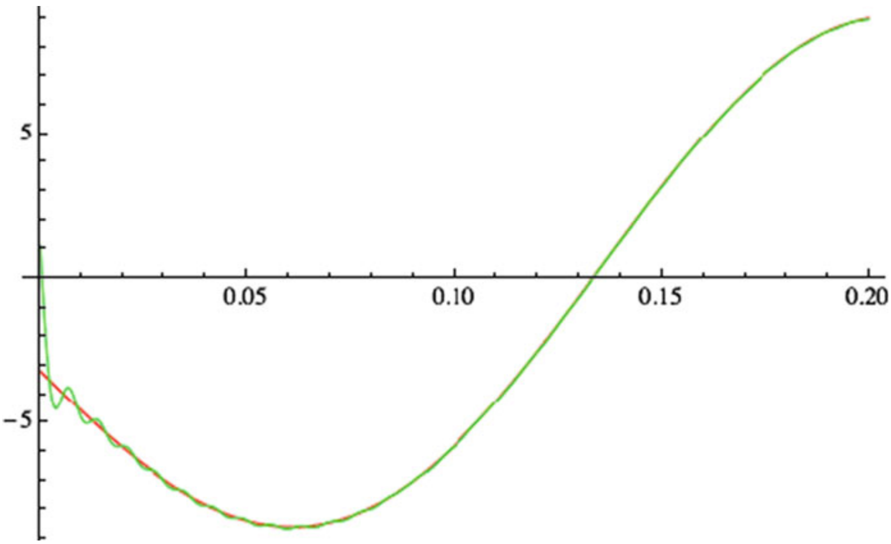


Fig. 5.16 The 5,000-term representation at the beginning, showing Gibbs phenomenon

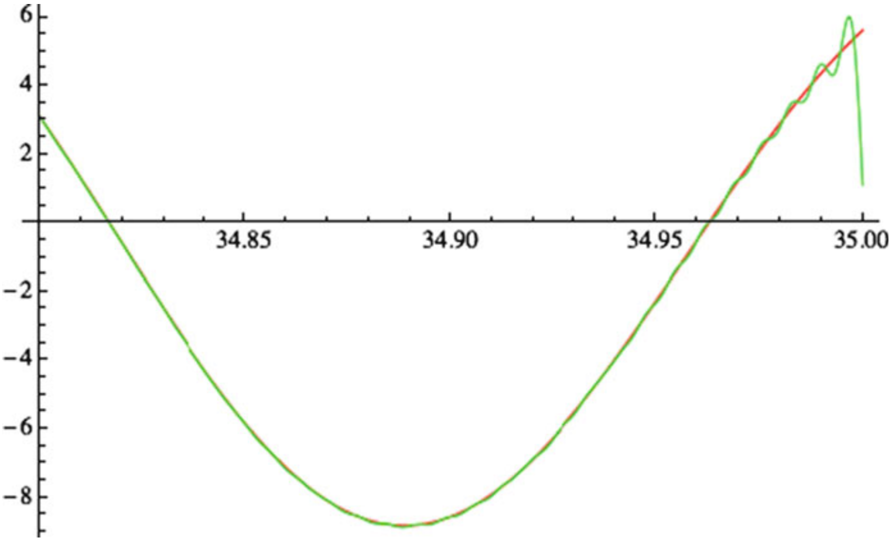


Fig. 5.17 The 5,000-term representation at the end, showing Gibbs phenomenon

5.2.2 Power Spectra

5.2.2.1 Fourier Transform

Power is defined in terms of the Fourier transform, so I will start by exploring that. The Fourier transform and its inverse are generally defined by

$$F(\omega) = \int_{-\infty,}^{\infty} f(t)\exp(-j\omega t)dt, \quad f(t) = \frac{1}{2\pi}\int_{-\infty,}^{\infty} F(\omega)\exp(j\omega t)d\omega \tag{5.9}$$

(Press et al. define them with the opposite sign. This makes no difference because the integrals run over all values of the variables, but it can make comparing the Fourier series and the Fourier transform a bit confusing.) The first thing we can note is that Eq. (5.5) is essentially the same as the first of Eq. (5.9). If $\omega = \omega_n = n\pi/T$, and we truncate the range of integration, then they are the same. The total power in a signal $f(t)$ is given by

$$P_{\mathrm{T}} = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \tag{5.10}$$

the square of the signal integrated for all time. The fact that the expression for power in terms of t is the same as that in terms of ω are equal is Parseval's theorem. We are interested in how much power there is for any frequency interval, how the power varies with frequency. We can define a (one-sided) power spectral density function

$$P(\omega) = |F(\omega)|^2 + |F(-\omega)|^2 = 2F(\omega)F(\omega)* \tag{5.11}$$

which expresses the power at the frequency ω. The total power [Eq. (5.10)] is the integral of this over all frequencies. There is no real distinction between positive and negative ω, so it makes sense to combine them. Of course we cannot find the Fourier transform of experimental data using Eq. (5.9) because the experiment ends after a finite interval. We have generally a discrete set of data collected at a discrete set of times. Denote the interval between data points by Δ, and suppose there are N such points, which I will number from zero to $N - 1$. We apply the transform by supposing that the function is zero outside the interval. Thus we have

$$F(\omega) = \int_{0,}^{(N-1)\Delta} f(t)\exp(-j\omega t)dt \approx \Delta\sum_{k=1}^{N-1} f_{\mathrm{k}}\exp(-j\omega k\Delta) \tag{5.12a}$$

where f_{k} denotes the value of $f(t)$ at $t = t_k = k\Delta$. The f_k are real, so the F are complex. When we apply Eq. (5.11) to calculate the power spectrum, we'll need the product of F and its complex conjugate.

We cannot expect to get more information from this than is contained in the data, so we can only expect to get the coefficients for N frequencies. We choose

$$\omega_{\mathrm{n}} = \frac{2\pi n}{N\Delta}, \quad n = -\frac{N}{2}, \cdots, \frac{N}{2} \tag{5.13}$$

There are, of course, $N + 1$ frequencies in Eq. (5.13), but the end frequencies are in effect one period apart, so there is no actual extra information here. That will become clear shortly. For now, we can rewrite Eq. (5.12a) as

$$F(\omega_{\mathrm{n}}) \approx \Delta \sum_{k=0}^{N-1} f_{\mathrm{k}} \exp\left(-j\frac{2\pi n}{N\Delta}k\Delta\right) = \Delta \sum_{k=1}^{N-1} f_{\mathrm{k}} \exp\left(-j\frac{2\pi nk}{N}\right) \tag{5.12b}$$

This is different from Eq. (12.1.6) in Press et al., because of the different initial definition. The *discrete Fourier transform* is the final sum (without the Δ).

Let's consider the inverse transform and go through a similar series of approximations.

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty,}^{\infty} F(\omega)\exp(j\omega t)d\omega \\ &\approx \frac{1}{2\pi}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} F(\omega_{\mathrm{n}})\exp(j\omega_{\mathrm{n}}t)\frac{2\pi}{\Delta N} = \frac{1}{\Delta N}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} F(\omega_{\mathrm{n}})\exp(j\omega_{\mathrm{n}}t) \end{aligned} \tag{5.14}$$

As before, we cannot expect more information than we have, so we can only find f at t_{k}. We also note that the values of F at the two ends of the range are identical, so we only keep one of them in doing the integration. The final result for the inverse is given by Eq. (5.15).

$$f(t_{\mathrm{k}}) \approx \frac{1}{\Delta N}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} -1F(\omega_{\mathrm{n}})\exp\left(j\frac{2\pi nk}{N}\right) \tag{5.15}$$

Press et al. make some changes to make the inverse resemble the transformation more closely.

We can look at a small example to establish that this works.

Example 5.4 The Discrete Fourier Transform for $N = 6$ Apply Eq. (5.13) to an arbitrary set of data. The result for the first six Fourier coefficients is

$$F = \Delta\left\{\begin{matrix} f_0 - f_1 + f_2 - f_3 + f_4 - f_5 \\ f_0 + \mathrm{e}^{2\pi j/3}f_1 + \mathrm{e}^{-2\pi j/3}f_2 + f_3 + \mathrm{e}^{2\pi j/3}f_4 + \mathrm{e}^{-2\pi j/3}f_5 \\ f_0 + \mathrm{e}^{\pi j/3}f_1 + \mathrm{e}^{2\pi j/3}f_2 - f_3 + \mathrm{e}^{-2\pi j/3}f_4 + \mathrm{e}^{-\pi j/3}f_5 \\ f_0 + f_1 + f_2 + f_3 + f_4 + f_5 \\ f_0 + \mathrm{e}^{-\pi j/3}f_1 + \mathrm{e}^{-2\pi j/3}f_2 - f_3 + \mathrm{e}^{2\pi j/3}f_4 + \mathrm{e}^{\pi j/3}f_5 \\ f_0 + \mathrm{e}^{-2\pi j/3}f_1 + \mathrm{e}^{2\pi j/3}f_2 + f_3 + \mathrm{e}^{-2\pi j/3}f_4 + \mathrm{e}^{2\pi j/3}f_5 \end{matrix}\right\}$$

The seventh Fourier coefficient is identical to the first. I leave it to the reader to establish this and to establish that the inverse defined by Eq. (5.15) succeeds in inverting this expression.

We can see from these arguments that the Fourier coefficients from the complex Fourier series contain all that we need to find power spectra.

$$P(\omega_{\mathrm{n}}) = c_n c_n * \tag{5.16}$$

5.2.2.2 Some Power Spectra

We can plot P_n as a function of n (the *power spectrum*) to identify the important frequencies in the input signal. Figure 5.18 shows the power spectrum for the data in Fig. 5.9.

I looked at all the values greater than 0.25. The major peak is at $n = 115$, corresponding to a frequency of 115 $(2\pi/35) = 20.64$ rad/s, and the secondary peak is at $n = 177$, corresponding to a frequency of 31.77 rad/s. I generated the artificial data using the function

$$f = \cos(31.43t) + 0.25\sin(5.1t) + 0.3\sin(7.3t) + 8\sin(20.2t),$$

so we see that the Fourier analysis does a pretty good job identifying the more important frequencies. If I reduce the threshold to 0.01 I pick up peaks at $n = 30$ and 43, corresponding to frequencies of 5.33 and 7.64 rad/s, near the frequencies of the other two sinusoidal components in the artificial signal. The precision of this "measurement" clearly depends on the number of terms in the series.

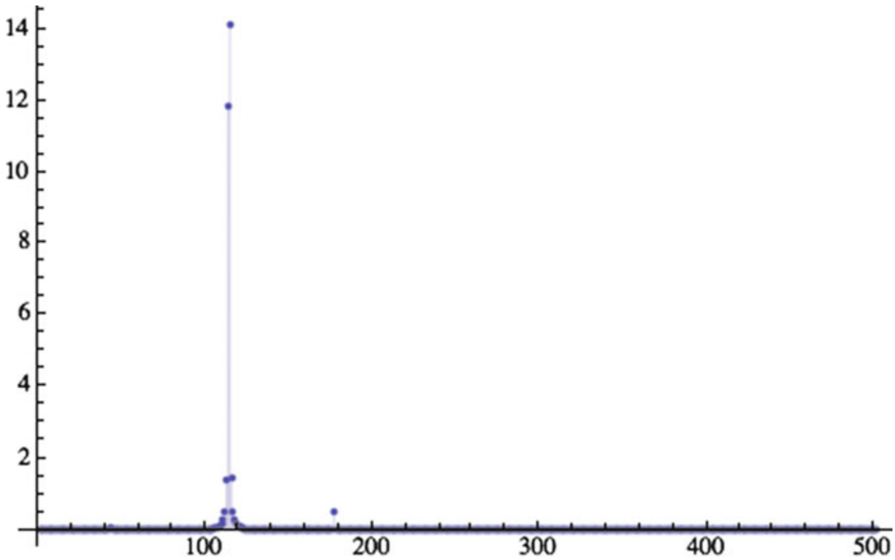


Fig. 5.18 The power spectrum of the data shown in Fig. 5.9

5.2.3 The Nyquist Phenomenon

The precision also depends, critically, on the sampling rate. A measurement system acquires signals digitally. It captures (samples) a signal and converts it to a digital number using a physical analog to digital converter. This converter has a fixed precision, expressed as the number of bits. It also has a fixed conversion time. The more bits, the higher the precision of the conversion and the longer it takes for each conversion. It is intuitively obvious that if the signal is changing more rapidly than it can be converted, this rapid variation cannot be captured. It is clearly impossible to sample more rapidly than the conversion time, and generally sampling is done at a rate slow compared to the conversion time. The sampling rate determines the frequencies that can be detected. There is a theorem to the effect that the maximum frequency that can be detected is half the sampling rate. This magic frequency is called the *Nyquist frequency*. It is a property of the sampling instrumentation.

The situation is even worse. Not only can we not detect signals at frequencies above the Nyquist frequencies, but those signals do not get lost. They reappear in the wrong place by a process called *aliasing*. This is best illustrated by an example. Consider a pure harmonic signal at a frequency of 40π (artificially generated in Excel), shown in Fig. 5.19.

If we sample slowly, we will see the apparent signal distort, and then, as we cross the Nyquist frequency, change dramatically. The actual frequency is 40π. The period of the signal is $2\pi/\omega = 1/20 = 0.05$. Our sampling interval must be no longer than half the period. We must sample at 80π or more or a sampling interval of less than 0.025. Figure 5.20 shows the curve sampled at intervals of 0.01 s, a frequency of 200π. The figure is qualitatively the same as Fig. 5.19, but the curve is no longer harmonic.

What happens if we sample at exactly the Nyquist frequency, a sampling interval of 0.025 s? We need to avoid beginning the sample at a zero, or all we'll get is

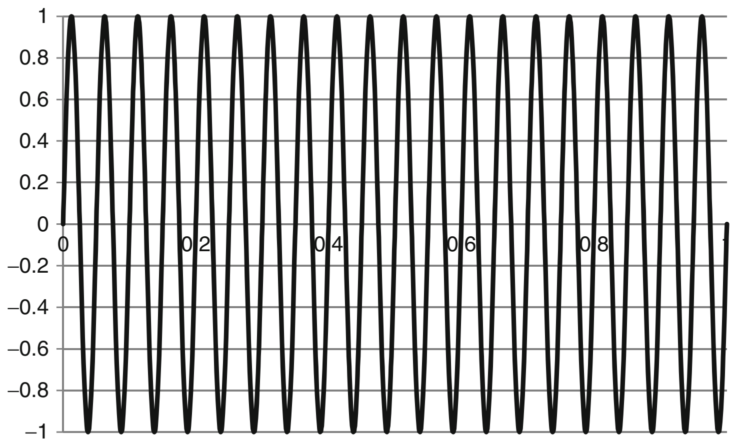


Fig. 5.19 A 40π sinusoidal signal of unit magnitude

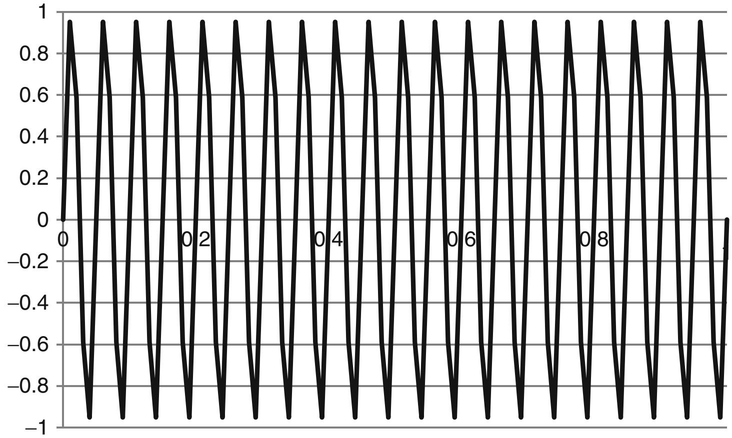


Fig. 5.20 The data of Fig. 5.19 sampled at intervals of 0.01 s

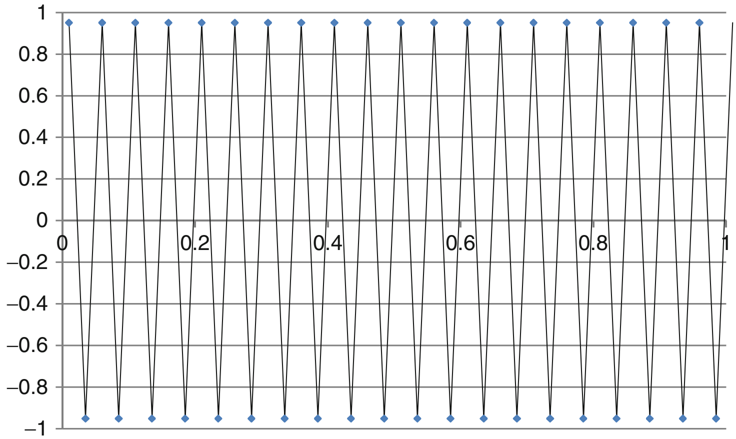


Fig. 5.21 The data from Fig. 5.19 sampled at the Nyquist frequency

zeroes. Figure 5.21 shows what happens if the sampling is begins at $t = 0.01$. The data points are at the peaks and we find the correct frequency. What happens if we sample too slowly?

Figure 5.22 shows the data sampled at an interval of 0.0255. There are approximately nineteen peaks. There should be twenty. The apparent frequency of these data is below the actual frequency. This is an example of *aliasing*. The 40π frequency did not disappear; it got aliased down and shows up as an apparently lower frequency. We can see this more clearly if we sample at a still lower rate.

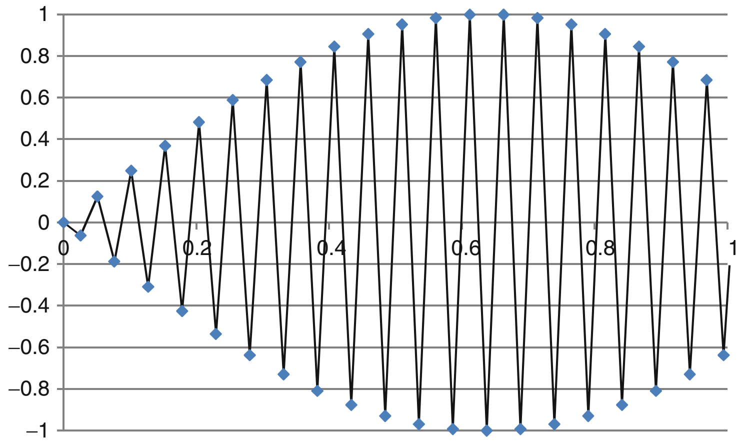


Fig. 5.22 The data from Fig. 5.19 sampled at an interval of 0.0255

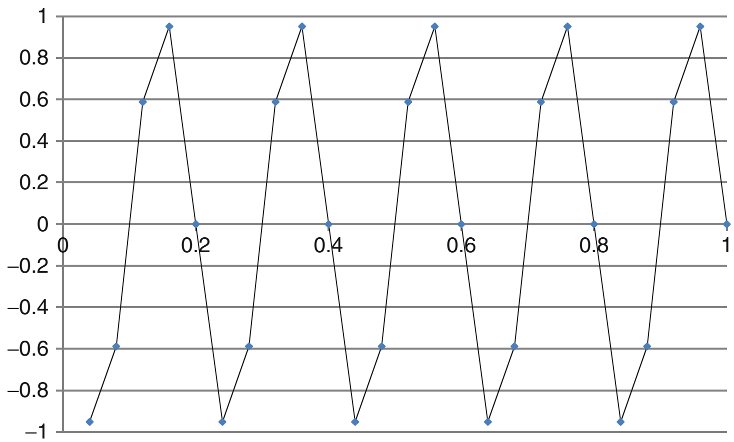


Fig. 5.23 The data from Fig. 5.19 sampled at an interval of 0.04

Figure 5.23 shows the result for a sampling interval of 0.04 s. It's a convincing signal, but it has nothing to do with the actual signal. The period is a little longer than 0.2, as opposed to the actual period of 0.05.

To correctly identify the frequency of an oscillatory signal, you must sample at a rate twice as high as the highest frequency you expect in the signal. This has nothing to do with instrumentation directly, merely that part of the system that samples the input signal. The instrumentation must, of course, be capable to sampling at the desired rate.

Example 5.5 An Experimental Spectrum It is reasonably well known that rubbing a damp finger along the rim of a glass partially filled with liquid can make a tone. Figure 5.24 shows a glass, and Fig. 5.25 shows the power spectrum of the tone.

The transducer here was a simple microphone. I analyzed the data using LabVIEW software.

Fig. 5.24 A cocktail glass (photo by the author)

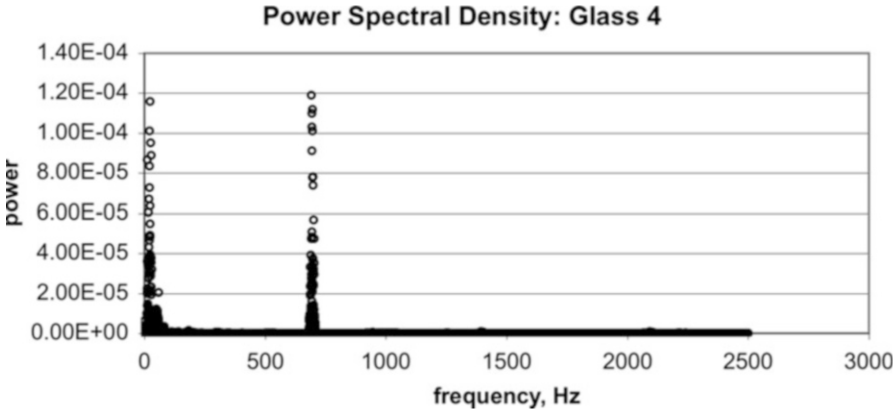


Fig. 5.25 Sample power spectral density plot for the glass shown in Fig. 5.24, filled to a level of 20.25 mm below the rim. The frequency peak is at 690.3 Hz and equals 1.19×10^{-4} (author's unpublished data)

Exercises

1. Calculate the response of a cantilever beam to a transverse ground motion. Consider the beam to be a simple mass-spring system.
2. Repeat exercise 1 supposing the beam to be elastic. Suppose the beam to be 400 mm long, 20 mm wide, and 3 mm thick and made of aluminum, and let the frequency of the ground motion be 100 Hz.
3. What are the first four natural frequencies of the beam in exercise 2?
4. Suppose you wish to measure the vibrations of the beam in exercise 2. One option would be to mount a strain gauge on the beam to measure the flexure. Where would be a good place to mount the strain gauge?
5. Calculate the response of a single reed designed for 100 Hz to a 200 Hz excitation.
6. Consider a one-axis accelerometer mounted on a pendulum perpendicular to the rod of the pendulum. What output signal do you expect?
7. Suppose the pendulum in the previous exercise is 200 mm long. What must the natural frequency of the accelerometer be to measure the pendulum motion effectively?
8. Consider an accelerometer with a natural frequency of 1,000 Hz. Suppose it to be attached to a falling mass (with its sensitive element aligned with the direction of fall). What signal do you expect from the accelerometer if the mass impacts the ground (assumed rigid) elastically after falling 20 ft?
9. What is the response of a seismometer with a 20 s period to a harmonic excitation at 1 Hz?
10. Design (choose bob mass, rod length, and tilt) a seismometer to measure ground motion from 0.1 Hz.
11. Apply the finite inverse Eq. (5.15) to the finite transform in Ex. 5.4.
12. Find the complete Fourier series for the linear function shown in Fig. 4.2.
13. Find the complete Fourier series for the function shown in Fig. 4.4.
14. Find the Fourier sine series on the interval (0, 2) for the discontinuous function

$$f = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

15. Find the complete Fourier series for $\tan(\pi t)$ valid on the interval $-3 < t < 5$.
16. Find the Fourier sine series for the function $t\,(t-1)$ on the interval (0, 1).
17. Find the complete Fourier series for the function of exercise 16.
18. Find the Fourier sine series for the function $t^2\,(t-1)\,(t-2)$ on the interval (0, 2).
19. Find the complete Fourier series for the function in exercise 18 on the interval (−2, 2).
20. Establish that the formula for power, Eq. (5.12b), is correct.

The following exercises are based on the chapter, but require additional input. Feel free to use the Internet and the library to expand your understanding of the questions and their answers.

21. Design a Frahm tachometer (find the masses and the spring constants) to cover two octaves starting at concert A (=440 Hz) and going upward. Go by whole notes and use fifteen reeds.
22. Suppose you wish to measure the acoustic spectrum of a symphony orchestra using a single microphone. Suppose the microphone to transduce the acoustic signal to an electric signal faithfully. What sampling rate and what sample size would you use to resolve the audible spectrum?
23. Consider how you would measure the acoustic spectrum of a New York City subway train.
24. How would you measure the vibrations of a baseball bat struck by a ball. What instrument would be appropriate? Where would you attach it? How would you model the bat?
25. How would you measure passenger comfort in an automobile?

References

APC International (2002) Piezoelectric ceramics: principles and applications. APC International, Mackeyville, PA

Churchill RV, Brown JW (1987) Fourier series and boundary value problems, 4th edn. McGraw-Hill, New York, NY

Curie J, Curie P (1880) "Développement par compression de l'électricité polaire dans les cristaux hémièdres à faces inclinées" (Development, via compression, of electric polarization in hemihedral crystals with inclined faces). Bulletin de la Société minérologique de France, vol 3. pp 90–93. Reprinted in: Jacques and Pierre Curie (1880) Développement, par pression, de l'électricité polaire dans les cristaux hémièdres à faces inclinées, "Comptes rendus . . .", vol 91. pp 294–295. See also: Jacques and Pierre Curie (1880) "Sur l'électricité polaire dans les cristaux hémièdres à faces inclinées" (On electric polarization in hemihedral crystals with inclined faces), Comptes rendus . . . , vol 91. pp 383–386

Den Hartog JP (1956) Mechanical vibrations. Dover reprint

Press WH, Teukolsky SA, Vettering WT, Flannery BP (1992) Numerical recipes in C. The art of scientific computing. Cambridge University Press, New York, NY

Wakin DJ (2009) "Musicians hear heaven in Tully Hall's new sound". New York Times. 29 Jan 2009. http://go.galegroup.com/ps/i.do?id=GALE%7CA192708483&v=2.1&u=nysl_ro_rochstru&it=r&p=ITOF&sw=w. 19 Aug 2013

State Space, Equilibrium, Linearization, and Stability

6

In which we convert the second-order systems of differential equations we have been using to first-order systems and use that to learn much more about mechanical systems . . . .

6.1 State Space

I have mentioned state space in passing in Chap. 1. I will be using state space formulations for the rest of the text, and so it is time to explore this a little more formally. We have seen that the Euler-Lagrange process applied to an N degree of freedom system yields coupled sets of N second-order ordinary differential equations. We can do pretty much everything we need to do for vibrations with this formulation,[1] but a reformulation in terms of first-order equations is more flexible and gives us a better base from which to work when we get to control problems. The basic idea is to define a set of variables to represent the first derivatives of the generalized coordinates. This leads to a set of N first-order equations of the form[2]

$$\dot{q}_i = p_i, \quad i = 1 \cdots N \tag{6.1}$$

The N second-order equations immediately become first-order equations because I can replace the derivatives of the generalized coordinates by the **p** vector, and the second derivatives of the generalized coordinates by the first derivatives of the **p** vector. Thus we have $2N$ quasilinear first-order equations.

[1] We can incorporate electric motors in the generalized forces.

[2] Some readers may be familiar with Hamilton's equations, in which **p** denotes the generalized momentum. This is *not* a Hamiltonian formulation!

R.F. Gans, *Mechanical Systems: A Unified Approach to Vibrations and Controls*,
DOI 10.1007/978-3-319-08371-1_6

Let me begin with the reformulation of the general one degree of freedom system from Chap. 2. Equation (2.4) is a second-order ordinary differential equation, the sort of equation that the Euler-Lagrange process will produce. We can convert it to a pair of first-order equations, and we can then do our analytic work in that context. [In fact, we can convert sets of second-order equations into sets of first-order equations using the transformation in Eq. (6.1).] There are a number of different ways to do this. I will choose the simplest by writing

$$\begin{aligned} \dot{y} &= v, \\ \dot{v} &= -2\zeta\omega_{\mathrm{n}}v - \omega_{\mathrm{n}}^2 y - a \end{aligned}$$

You can think of v as a velocity. I can define a *state vector* $\mathbf{x}$ here given by

$$\mathbf{x} = \begin{Bmatrix} y \\ v \end{Bmatrix}$$

The state equations can be written in matrix-vector form as

$$\dot{\mathbf{x}} = \begin{Bmatrix} x_2 \\ -2\zeta\omega_{\mathrm{n}}x_2 - \omega_{\mathrm{n}}^2 x_1 - a \end{Bmatrix} = \begin{Bmatrix} 0 & 1 \\ -\omega_{\mathrm{n}}^2 & -2\zeta\omega_{\mathrm{n}} \end{Bmatrix}\mathbf{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}a \tag{6.2}$$

This has the general form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{6.3}$$

where I have replaced the acceleration a by a general input u. This is the general form for a single-input system, no matter what its dimension. I will generally use u to denote the input to a single-input system, and $\mathbf{u}$ to denote the vector input to a multi-input system. Equation (6.3) changes slightly for a system with multiple inputs. The scalar u becomes the vector $\mathbf{u}$ and the vector $\mathbf{b}$ becomes a matrix $\mathbf{B}$.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{6.4}$$

[This is Eq. (1.1a) introduced with little comment at the end of Sect. 1.1. Equation (6.3) is the same as Eq. (1.1b).] This system can be solved in the same way as we solved the single second-order Eq. (2.4): by finding a particular solution and a homogeneous solution and combining them. Equations (6.3) or (6.4) will in general be accompanied by initial conditions, $\mathbf{x}(0) = \mathbf{x}_0$, but we can address the equations in isolation for now. We need more mathematics before we can solve Eq. (6.4) in general, but the homogeneous solution is simple enough that we can address that here, in general. Thus we can suppose that $\mathbf{u} = \mathbf{0}$ and write

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

This is simply a set of homogeneous first-order ordinary differential equations, and so it admits exponential solutions. We can suppose that $\mathbf{x}$ is proportional to e^{st}, so that we have

$$\mathbf{x} = \mathbf{v}e^{st} : \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \Rightarrow s\mathbf{v}e^{st} = \mathbf{A}\mathbf{v}e^{st} \Rightarrow (s\mathbf{1} - \mathbf{A})\mathbf{v}e^{st} = \mathbf{0}$$

where $\mathbf{v}$ denotes a constant vector, and $\mathbf{0}$ and $\mathbf{1}$ denote the empty and identity matrices, respectively. The exponential factor is never zero, so the only way we can satisfy the equation is to have

$$(s\mathbf{1} - \mathbf{A})\mathbf{v} = \mathbf{0} \tag{6.5}$$

which is the usual matrix eigenvalue problem. Let's take a moment to address that.

Equation (6.5) is a set of homogeneous algebraic equations. We know that these equations have a nontrivial solution if and only if the determinant of the coefficients is zero. That determinant is

$$\det(s\mathbf{1} - \mathbf{A}) = 0 \tag{6.6}$$

and it will be a polynomial in s of the same degree as the size of the $\mathbf{A}$ matrix, say N. The leading term will always be s^N. There will be N possible values of s. These are the *eigenvalues* of the matrix. (These do not have to be distinct, but usually will be for reasonable engineering problems.) There will be a nontrivial *eigenvector* $\mathbf{v}$ associated with each eigenvalue, so that

$$(s_i\mathbf{1} - \mathbf{A})\mathbf{v}_i = \mathbf{0} \tag{6.7}$$

The homogeneous solution will be a linear combination of the eigenvectors multiplied by the exponentials with the associated eigenvalues:

$$\mathbf{x}_H = a_1\mathbf{v}_1e^{s_1t} + a_2\mathbf{v}_2e^{s_2t} + \cdots + a_N\mathbf{v}_Ne^{s_Nt} \tag{6.8}$$

The initial conditions determine the coefficients

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_N\mathbf{v}_N\mathrm{e} = \mathbf{x}_H(0)$$

For this to succeed in general, the eigenvectors have to be independent. We can define a matrix

$$\mathbf{V} = \{\, \mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_N \,\} \tag{6.9}$$

whose columns are the eigenvectors of $\mathbf{A}$, and a vector $\mathbf{a}$ whose elements are the coefficients a_i, and then rewrite the initial condition in the compact vector notation

$$\mathbf{V}\mathbf{a} = \mathbf{x}(0) \rightarrow \mathbf{a} = \mathbf{V}^{-1}\mathbf{x}(0),$$

and one can see why the eigenvectors have to be independent, which they almost always are for practical engineering problems. We can write the solution in a very compact form if we are willing to define a diagonal matrix whose elements are the exponential terms in Eq. (6.8)

$$\boldsymbol{\Xi}(t) = \mathrm{diag}\{\, \mathrm{e}^{s_1 t} \quad \mathrm{e}^{s_2 t} \quad \cdots \quad \mathrm{e}^{s_N t} \,\} \tag{6.10}$$

Then the homogeneous solution Eq. (6.5) can be written

$$\mathbf{x}_{\mathrm{H}} = \mathbf{V}\boldsymbol{\Xi}(t)\mathbf{V}^{-1}\mathbf{x}(0) \tag{6.11}$$

Example 6.1 The Eigenvalues and Eigenvectors of Eq. (2.4) Let's see how this goes for Eq. (2.4). We want to look at the homogeneous version of Eq. (6.3). We find the eigenvalues by solving

$$\det\begin{Bmatrix} s & -1 \\ \omega_{\mathrm{n}}^2 & s + 2\zeta\omega_{\mathrm{n}} \end{Bmatrix} = 0 \rightarrow s^2 + 2\zeta\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2 = 0$$

the same characteristic polynomial we had before. We find the eigenvectors from

$$\begin{Bmatrix} s_i & -1 \\ \omega_{\mathrm{n}}^2 & s_i + 2\zeta\omega_{\mathrm{n}} \end{Bmatrix}\begin{Bmatrix} v_{i1} \\ v_{i2} \end{Bmatrix} = \mathbf{0}$$

Clearly the first row is satisfied if

$$\begin{Bmatrix} v_{i1} \\ v_{i2} \end{Bmatrix} = \begin{Bmatrix} 1 \\ s_i \end{Bmatrix}$$

Multiplying the second row out leads to the characteristic polynomial, which vanishes, so this is the expression for the eigenvectors of a two-dimensional state space. The matrix $\mathbf{V}$ is

$$\mathbf{V} = \begin{Bmatrix} 1 & 1 \\ s_1 & s_2 \end{Bmatrix},$$

and it is invertible as long as the two eigenvalues are distinct. (This omits the critically damped case. There are ways around this, but I have already noted that a system for which ζ is near unity is effectively critically damped without destroying the invertibility of $\mathbf{V}$.) Thus we can find the vector of coefficients and write the solution following Eq. (6.8). We have

$$\mathbf{V}^{-1} = \frac{1}{s_2 - s_2}\begin{Bmatrix} s_2 & -1 \\ -s_1 & 1 \end{Bmatrix}$$

so that we can collect all the pieces to obtain

$$\mathbf{x}_{\mathrm{H}} = \begin{Bmatrix} 1 & 1 \\ s_1 & s_2 \end{Bmatrix}\begin{Bmatrix} \exp(s_1 t) & 0 \\ 0 & \exp(s_2 t) \end{Bmatrix}\frac{1}{s_2 - s_2}\begin{Bmatrix} s_2 & -1 \\ -s_1 & 1 \end{Bmatrix}\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix}$$

Multiplying this out and simplifying gives the homogeneous solution in state space

$$\begin{aligned} x_1 &= \frac{1}{s_1 - s_2}\left(\left(x_2(0) - s_2 x_1(0)\right)\exp(s_1 t) + \left(s_1 x_1(0) - x_2(0)\right)\exp(s_2 t)\right) \\ x_2 &= \frac{1}{s_1 - s_2}\left(s_1\left(x_2(0) - s_2 x_1(0)\right)\exp(s_1 t) + s_2\left(s_1 x_1(0) - x_2(0)\right)\exp(s_2 t)\right) \end{aligned} \tag{6.12}$$

6.1.1 Some Comments on Eigenvalues and Eigenvectors

We have seen that the homogeneous solution to any equation of the form of Eq. (6.3) can be solved in terms of eigenvalues and eigenvectors. That is only useful if one can find the eigenvalues and the eigenvectors. As a practical matter, these are most easily found using one or another software package. According to Press et al. (1992) ". . . almost all canned routines in use nowadays trace their ancestry back to routines published in Wilinson and Reinnsch's *Handbook for Automatic Computation, Vil. II, Linear Algebra*" (which see). They are generally reliable for the matrices that arise for most mechanical (and electromechanical) systems. This is not the place to explore the numerical determination of eigenvalues and eigenvectors. Press et al. cover this admirably in their Chap. 11. The eigenvalues are generally complex and come in complex conjugate pairs. If the real parts of both eigenvalues are negative, then x_1 and x_2 will both go to zero, and we can say that the system is stable.

It is enough for us to note that the eigenvalues are determined by solving the characteristic polynomial given by Eq. (6.6). Each eigenvector can then be found by substituting the specific eigenvalue in Eq. (6.7) and finding the coefficients of the eigenvector $\mathbf{v}$. I demonstrated this for the two-dimensional case in Ex. 2.7, where I was able to find the eigenvectors almost by inspection. If the matrix $\mathbf{A}$ is moderately sparse (has a lot of zero entries), it is often possible to extend the process I used in the example. Define a general vector and write out the product given by Eq. (6.7). One can select one component of the general vector to be unity and solve the first $N-1$ equations simultaneously for the remaining coefficients. This gives a vector that satisfies all but the last component of the product, but because I have substituted the eigenvalue, the last equation is automatically satisfied. This is a very good check if you are finding eigenvectors by hand.

Let's see how the state space process works for a two degree of freedom problem.

Example 6.2 A Two-Mass, Two-Spring System Let me illustrate this using a very straightforward situation, a generic two degree of freedom problem, shown in Fig. 6.1 below

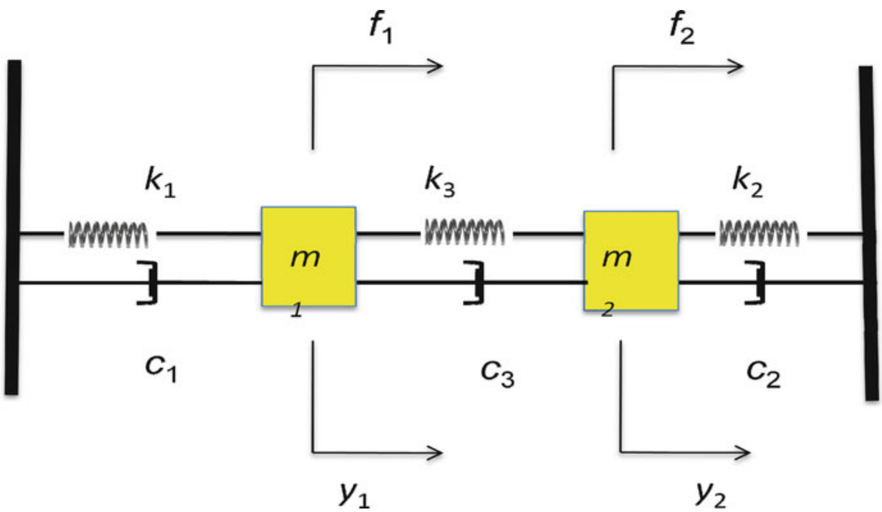


Fig. 6.1 A general two degree of freedom problem

We can choose y_1 and y_2 as generalized coordinates and work out the Euler-Lagrange equations fairly simply. The energies and Rayleigh dissipation function are given by

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2$$
$$V = \frac{1}{2} k_1 y_1^2 + \frac{1}{2} k_2 y_2^2 + \frac{1}{2} k_3 (y_1 - y_2)^2$$
$$F = \frac{1}{2} c_1 \dot{y}_1^2 + \frac{1}{2} c_2 \dot{y}_2^2 + \frac{1}{2} c_3 (\dot{y}_1 - \dot{y}_2)^2$$

I leave the derivation of the equations from this as an exercise for the reader. The result (after some algebra) is

$$\ddot{y}_1 = -\frac{c_1 + c_3}{m_1} \dot{y}_1 - \frac{k_1 + k_3}{m_1} y_1 + \frac{c_3}{m_1} \dot{y}_2 + \frac{k_3}{m_1} y_2 + \frac{f_1}{m_1}$$
$$\ddot{y}_2 = \frac{c_3}{m_2} \dot{y}_1 + \frac{k_3}{m_2} y_1 - \frac{c_2 + c_3}{m_2} \dot{y}_2 - \frac{k_2 + k_3}{m_2} y_2 + \frac{f_2}{m_2}$$

Now we apply Eq. (6.1) to this to obtain the first-order equations corresponding to the Euler-Lagrange equations

$$\dot{p}_1 = -\frac{c_1 + c_3}{m_1} p_1 - \frac{k_1 + k_3}{m_1} y_1 + \frac{c_3}{m_1} p_2 + \frac{k_3}{m_1} y_2 + \frac{f_1}{m_1}$$
$$\dot{p}_2 = \frac{c_3}{m_2} p_1 + \frac{k_3}{m_2} y_1 - \frac{c_2 + c_3}{m_2} p_2 - \frac{k_2 + k_3}{m_2} y_2 + \frac{f_2}{m_2}$$

These are to be combined with this instance of Eq. (6.1).

We define a *state vector* to represent **q** and **p** simultaneously

$$\mathbf{x} = \begin{Bmatrix} \mathbf{q} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} \tag{6.13}$$

The order of the components of the state vector is not unique. It is a matter of personal preference, and I prefer this order—all the variables followed by all their derivatives. The equations of motion here are linear

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -\frac{k_1 + k_3}{m_1} x_1 + \frac{k_3}{m_1} x_2 - \frac{c_1 + c_3}{m_1} x_3 + \frac{c_3}{m_1} x_4 + \frac{f_1}{m_1} \\ \dot{x}_4 &= \frac{k_3}{m_2} x_1 - \frac{k_2 + k_3}{m_2} x_2 + \frac{c_3}{m_2} x_3 - \frac{c_2 + c_3}{m_2} x_4 + \frac{f_2}{m_2} \end{aligned} \tag{6.14}$$

and so we can convert the state version of the equations directly to matrix form. We have a two-dimensional input vector **f** and a four-dimensional state vector **x** defined by Eq. (6.13). The equations are of the form

$$\dot{x}_i = \sum_{j=1}^{2N} A_{ij} x_j + \sum_{k=1}^{N} B_{ik} f_k \tag{6.15}$$

and the coefficients clearly form matrices. Here N denotes the number of degrees of freedom. I will shortly start using N to denote the number of dimensions in the state. I will try to be very clear about this, but the meaning of N ought in any case to be clear from context. I will write out the second term on the right-hand side for clarity.

$$\sum_{k=1}^{N} B_{ik} f_k = \begin{Bmatrix} B_{11} f_1 + B_{12} f_2 \\ B_{21} f_1 + B_{22} f_2 \\ B_{31} f_1 + B_{32} f_2 \\ B_{41} f_1 + B_{42} f_2 \end{Bmatrix}$$

In the present case $B_{1k} = 0 = B_{2k}$,

$$B_{31} = \frac{1}{m_1}, \quad B_{42} = \frac{1}{m_2},$$

and $B_{32} = 0 = B_{41}$, so that the sparse matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{Bmatrix}$$

B will always have as many rows as the dimension of the state and as many columns as there are inputs. States derived directly from an Euler-Lagrange process have at least twice as many dimensions as the number of degrees of freedom of the underlying system. If there are motors, then each motor may add a dimension, depending on whether you deem inductive effects to be important. The elements of the matrices can be found by inspection, or one can automate the process by defining the matrix elements by differentiating the governing equations:

$$A_{ij} = \frac{\partial \dot{x}_i}{\partial x_j}, \quad B_{ik} = \frac{\partial \dot{x}_i}{\partial f_k} \tag{6.16}$$

(The situation is similar when the equations are not linear. I will deal with that shortly.) The reader can easily verify that

$$\dot{\mathbf{x}} = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1 + k_3}{m_1} & -\frac{c_1 + c_3}{m_1} & \frac{k_3}{m_1} & \frac{c_3}{m_1} \\ \frac{k_3}{m_2} & \frac{c_3}{m_2} & -\frac{k_2 + k_3}{m_2} & -\frac{c_2 + c_3}{m_2} \end{Bmatrix} \mathbf{x} + \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{Bmatrix} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

is the equivalent of the state space equations, and also the equivalent of the Euler-Lagrange equations. I denote the first matrix by **A** and the second by **B**. I will denote the (here two-dimensional) force vector by **u**, and call it the *input*. (I will generally use **u** to denote input.) The matrix differential equations have the form given in Eq. (6.4). The vector **u** is called the *input*. Equation (6.16) defines it for this problem. In most of the cases we will deal with in control it will have but one component (a *single-input*, SI, system), and it will be convenient to call it a scalar u and replace the matrix **B** by a vector **b**. Equation (6.6) is the fundamental equation

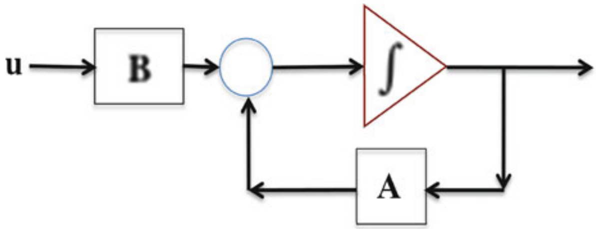


Fig. 6.2 Vector block diagram representing Eq. (6.4)

for linear dynamical systems. Figure 6.2 shows a vector block diagram for Eq. (6.4). Each specific instance of Eq. (6.4) will have its own scalar block diagram showing the relations among the components of the state vector. I leave it to the reader to construct a scalar block diagram reflecting the nature of **A** and **B**.

The general procedure is then to:

- Go through the Euler-Lagrange process as discussed in Sect. 3.2
- Define the vector **p** as the vector of the derivatives of **q**
- Select a state vector and find the state matrices **A** and **B** using Eq. (6.16)
- Combine the equations to get them into the form of Eq. (6.14) [or (6.3) for a single-input system]

Example 6.3 The Automobile Suspension System Figure 6.3 shows a model of an automobile suspension taken from Gillespie (1992) mapped into a mass-spring-dashpot model of the sort we are familiar with. Let us apply the procedure to this system. (This is a ground motion forced problem, but that is not an impediment to our endeavor.)

The basic functions we need for the Euler-Lagrange process for this problem are

$$T = \frac{1}{2}m_1\dot{z}_1^2 + \frac{1}{2}m_2\dot{z}_2^2$$
$$V = \frac{1}{2}k_1(z_1 - z_G)^2 + \frac{1}{2}k_3(z_1 - z_2)^2$$
$$F = \frac{1}{2}c_3(\dot{z}_1 - \dot{z}_2)^2$$

(In this case the forcing from the uneven road is incorporated in the Lagrangian, so we don't need a rate of work function.) The Euler-Lagrange equations are

$$\ddot{z}_1 = \frac{k_3}{m_1}(z_2 - z_1) + \frac{c_3}{m_1}(\dot{z}_2 - \dot{z}_1) - \frac{k_1}{m_1}z_1 + \frac{k_1}{m_1}z_G$$

$$\ddot{z}_2 = -\frac{k_3}{m_2}(z_2 - z_1) - \frac{c_3}{m_2}(\dot{z}_2 - \dot{z}_1)$$

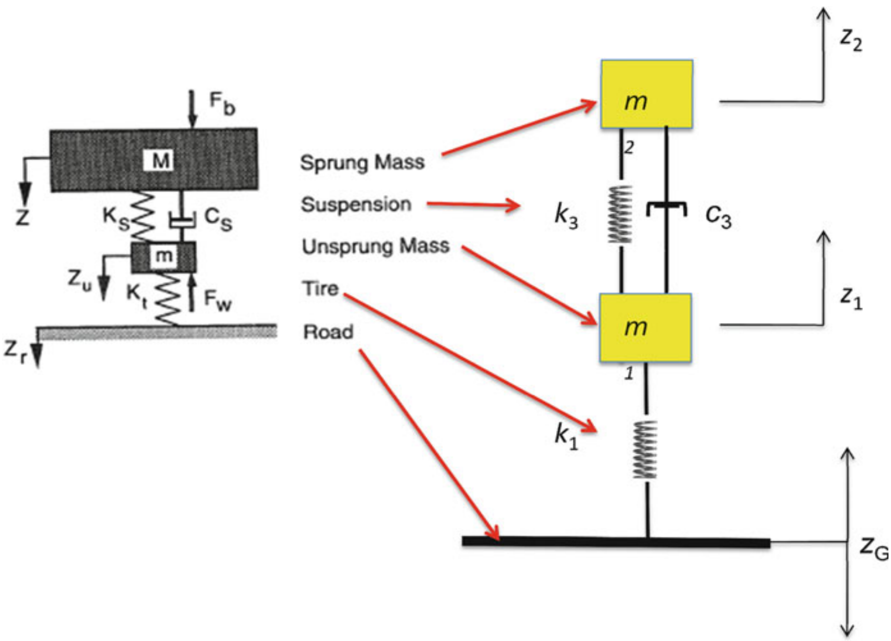


Fig. 6.3 A model of an automobile suspension (Gillespie 1992, p. 147)

We write $z_1 = q_1 = x_1, \quad z_2 = q_2 = x_2, \quad \dot{z}_1 = p_1 = x_3, \quad \dot{z}_2 = p_2 = x_4$ and form the obvious state vector

$$\mathbf{x} = \begin{Bmatrix} z_1 \\ z_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix}$$

This is another linear problem so we can write the equations for this problem directly in the form of Eq. (6.3).

$$\dot{\mathbf{x}} = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1 + k_3}{m_1} & \dfrac{k_3}{m_1} & -\dfrac{c_3}{m_1} & \dfrac{c_3}{m_1} \\ \dfrac{k_3}{m_2} & -\dfrac{k_3}{m_2} & \dfrac{c_3}{m_2} & -\dfrac{c_3}{m_2} \end{Bmatrix} \mathbf{x} + \begin{Bmatrix} 0 \\ 0 \\ \dfrac{k_1}{m_1} \\ 0 \end{Bmatrix} z_G$$

In this case we have but a single input, $u = z_G$, so the matrix $\mathbf{B}$ is actually a vector $\mathbf{b}$.

It's clear that we have a dynamical system, and equations in the form of Eq. (6.3) describe the dynamics. We need initial conditions to define a complete problem, and then, of course, we need to address how to solve the problem. As one can imagine, there will generally be a particular solution and a homogeneous solution, although I will not always call out these two parts of any given solution. The next section addresses solving problems in systems defined by Eq. (6.4).

6.2 Solving the General Inhomogeneous System of Linear Equations

The linear systems with which this text is concerned can all be reduced to sets of first-order linear differential equations that can be written in the form of Eq. (6.4)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

I denote the dimension of the state vector $\mathbf{x}$ by N (here N is equal to at least twice the number of degrees of freedom, the number of generalized coordinates in the Euler-Lagrange formulation), and the dimension of the input vector $\mathbf{u}$ by M. Thus $\mathbf{A}$ is an $N \times N$ matrix and $\mathbf{B}$ is an $N \times M$ matrix. In the common case that $M = 1$ $\mathbf{B}$ is an $N \times 1$ matrix, the vector $\mathbf{b}$. The second part of Eq. (6.16) becomes

$$b_i = \frac{\partial \dot{x}_i}{\partial u}$$

where u denotes the single input.

We will look at several ways of solving Eq. (6.4), beginning with a brute force symbolic method.

6.2.1 The State Transition Matrix

The solution to the one-dimensional system [scalar, Eq. (6.17)] is covered by any elementary differential equations book (see Boyce and DiPrima 1969, for example). One can show that the particular solution of a single linear first-order ordinary differential equation with constant coefficients,

$$\dot{x} = Ax + Bu, \tag{6.17}$$

is

$$x = \int_0^t \exp(A(t - \tau))Ba(\tau)d\tau \tag{6.18}$$

The homogeneous solution is proportional to exp(At), and so the system will be well behaved only if $A < 0$. The entire solution, satisfying an initial condition $x(0) = x_0$, is

$$x = \exp(At)x_0 + \int_0^t \exp(A(t-\tau))Ba(\tau)d\tau \tag{6.19}$$

The integral part of the solution vanishes at $t = 0$, and the other part vanishes as $t \to \infty$ (if $A < 0$). A solution to the vector problem is given by the direct vector analog of this expression

$$\mathbf{x} = \exp(\mathbf{A}t)\mathbf{x}_0 + \int_0^t \exp(\mathbf{A}(t-\tau))\mathbf{B}\mathbf{a}(\tau)d\tau \tag{6.20}$$

as one can easily verify by direct substitution, supposing that the derivative of exp($\mathbf{A}t$) with respect to time is $\mathbf{A}$ exp($\mathbf{A}t$). The question that arises is, of course, what does the exponential of a matrix represent? Does its derivative work as I just claimed? It clearly has to be an $N \times N$ matrix. This matrix is called the *state transition matrix*. We can define it in terms of the known infinite series that defines the exponential

$$\exp[\mathbf{A}t] = \mathbf{1} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}\cdot\mathbf{A}t^2 + \cdots$$

This is not the best way to find the state transition matrix [nor is the state transition matrix necessarily the best way to solve Eq. (6.4)]. One has to calculate a few terms and then be clever enough to recognize the functions that appear to be represented by the few terms of the series that appear in each element of the growing matrix. (We will learn how to find the state transition matrix using the Laplace transform in Chap. 7.) Let's look at three examples using the series technique.

Example 6.4 An Electric Motor with an Inertial Load (After Friedland 1986) Consider an electric motor attached to a flywheel (see Chap. 3 for motor details). The motor torque is proportional to the current, and the current is proportional to the difference between the input voltage and the back emf (assuming that the rate of change of the current is small enough that inductive effects are unimportant). The latter is proportional to the speed (rotation rate) of the motor. The rate of change of the rotation rate is proportional to the torque. The two proportionality constants are identical for the simple electric motor models appropriate to this text. If we suppose the voltage to be the input, and the state variables to be the motor angle θ and speed ω ($\dot{\theta}$), then we'll have a two-dimensional state $\left\{ \theta \quad \dot{\theta} \right\}^{\mathrm{T}}$, with state equations

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\frac{K^2}{IR}\omega + \frac{K}{IR}e$$

where I denotes the moment of inertia of the load, K the motor constant, and R the armature resistance, all positive. We can write this in the form of Eq. (6.3) using

$$\mathbf{x} = \begin{Bmatrix} \theta \\ \dot{\theta} \end{Bmatrix}, \quad \mathbf{A} = \begin{Bmatrix} 0 & 1 \\ 0 & -K\alpha \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ \alpha \end{Bmatrix}$$

where

$$\alpha = \frac{K}{IR} > 0$$

To apply the series for the exponential to find a state transition matrix, multiply **A** out through the fourth order to obtain an approximation to the state transition matrix (writing $a = K\alpha$ for convenience)

$$\exp[\mathbf{A}t] = \begin{Bmatrix} 1 & t - \frac{1}{2}at^2 + \frac{1}{3!}a^2t^3 - \frac{1}{4!}a^3t^4 + \cdots \\ 0 & 1 - at + \frac{1}{2}(at)^2 - \frac{1}{3!}(at)^3 + \frac{1}{4!}(at)^4 + \cdots \end{Bmatrix}$$

The (1,1) and (2,1) elements are exact in this expression (0 and 1, respectively). We can recognize the (2,2) element as $\exp(-at)$. The (1,2) element clearly has terms belonging to a negative exponential. If I multiply by $-a$ and add one, I get the negative exponential, so I claim that the state transition matrix here is

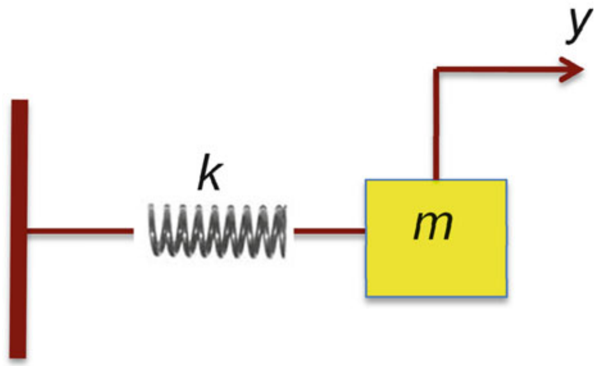
$$\exp[\mathbf{A}t] = \begin{Bmatrix} 1 & \frac{1}{a}\left(1 - \exp(-at)\right) \\ 0 & \exp(-at) \end{Bmatrix}$$

This was a carefully selected example for which the recognition process was fairly simple. The next example has a simple answer, but the functions are not so easily recognized.

Example 6.5 Undamped Mass-Spring System Figure 6.4 shows a simple mass-spring system. I leave it to you to verify that the **A** matrix for this problem is given by

$$\mathbf{A} = \begin{Bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{Bmatrix}$$

Fig. 6.4 An undamped mass-spring system



The fourth-order estimate for the state transition matrix is

$$\exp(At) = \begin{Bmatrix} 1 - \frac{1}{2}(\omega t)^2 + \frac{1}{4!}(\omega t)^4 + \cdots & t - \frac{1}{3!}\omega^2 t^3 + \cdots \\ -\omega^2 t + \frac{1}{3!}\omega^4 t^3 + \cdots & 1 - \frac{1}{2}(\omega t)^2 + \frac{1}{4!}(\omega t)^4 + \cdots \end{Bmatrix}$$

where, of course, $\omega^2 = k/m$. These look like relatives of the sine and cosine, and with a little thought we can recognize them. We can write

$$\exp(At) = \begin{Bmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega \sin(\omega t) & \cos(\omega t) \end{Bmatrix}$$

The **A** matrices in both examples were very simple: just 2×2 with only two nonzero elements. I will work one more example for which the result is sufficiently complicated to escape easy recognition, which will motivate us to find another way to find state transition matrices. (It's still a 2×2 matrix for a one degree of freedom problem, and we're going to need to be able to handle much bigger problems than this as we move forward.)

Example 6.6 Damped Mass-Spring System I add a damper, c, between the mass and the wall for the system shown in Fig. 6.3 (see also Fig. 1.3). The system can be characterized by a natural frequency ω and a damping ratio ζ. I will let you verify that the **A** matrix for this problem is given by

$$\mathbf{A} = \begin{Bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{Bmatrix}$$

The fourth-order estimate for the state transition matrix is pretty complicated and doesn't fit on the page. I show the third-order truncation instead

$$\exp(At) = \begin{Bmatrix} 1 - \frac{1}{2}(\omega t)^2 + \frac{1}{3}(\omega t)^3 + \cdots & t - \zeta\omega t^2 + \frac{1}{6}\omega^2 t^3 (4\zeta^2 - 1) + \cdots \\ -\omega^2 \left(t - \zeta\omega t^2 + \frac{1}{6}\omega^2 t^3 (4\zeta^2 - 1) \right) + \cdots & 1 - 2\zeta\omega t - \frac{1}{2}(\omega t)^2 (4\zeta^2 - 1) + \frac{2}{3}\zeta(\omega t)^3 (1 - 2\zeta^2) + \cdots \end{Bmatrix}$$

and this is not easy to recognize. (You probably think that you ought to know, but we'll find out shortly.)

I will give an alternate way to find the state transition matrix by using the Laplace transform in Chap. 7. In fact, one can solve the problem directly using the Laplace transform, but I will defer that until Chap. 7.

6.2.2 Diagonalization

The state transition matrix approach is frequently not the best way to attack these problems. The integrals are generally not accessible analytically, and numerical integration can only provide a result at a single moment in time. It makes sense to look at other methods for solving Eq. (6.4). All are based on understanding the matrix **A**. We can find the homogeneous solution in terms of the *eigenvalues* and *eigenvector*s of **A**. (See Eqs. (6.8), (6.9), (6.10), (6.11). The analysis here parallels the analysis there.) We can also build a particular solution using these eigenvalues and eigenvectors. (In degenerate cases we will need to take special measures.) I will start with the particular solution.

Suppose that **A** has distinct eigenvectors. *This is a necessary condition for this method to work*. Note that the eigenvalues need not be distinct, only the eigenvectors. The eigenvector **v** of a matrix **A** is a nontrivial vector satisfying

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

for some value of λ, called an eigenvalue. This is the matrix eigenvalue problem, which can be solved using many commercial software packages. I will not explore solution techniques here. Note that since the eigenvectors satisfy a homogeneous problem, they have no determined magnitudes. Analysts sometimes normalize them in the sense that $\mathbf{v}^T\mathbf{v} = 1$, but this is not necessary, and I generally will not bother to do this.

Take the distinct eigenvectors and let them be the columns of a matrix **V**. Because the eigenvectors are distinct **V** has an inverse $\mathbf{V}^{-1}$. It is known (see, for example, Strang 1988, Sect. 5.2) that

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda}$$

where $\mathbf{\Lambda}$ denotes a diagonal matrix the elements of which are the eigenvalues of **A**. I can map the vector **x** into another vector **z** by writing $\mathbf{x} = \mathbf{V}\mathbf{z}$. **V** is a constant matrix, so Eq. (6.4) can be rewritten

$$\mathbf{V}\dot{\mathbf{z}} = \mathbf{AVz} + \mathbf{Bu}$$

Multiplying this by the inverse of **V** gives me a simple set of equations

$$\mathbf{V}^{-1}\mathbf{V}\dot{\mathbf{z}} = \mathbf{V}^{-1}\mathbf{AVz} + \mathbf{V}^{-1}\mathbf{Bu} \Rightarrow \dot{\mathbf{z}} = \boldsymbol{\Lambda}\mathbf{z} + \mathbf{V}^{-1}\mathbf{Bu}$$

This uncouples the variables making up the vector **z**, making the **z** equations a set of simple uncoupled first-order equations, equations that we know how to solve using what we know from elementary differential equations (see Eq. 6.18). Note the strong resemblance of this method to the method of modal analysis discussed in Chap. 4. The modal equations explored in Chap. 4 were second order. These uncoupled equations are first order.

Let's see how this goes for a specific problem: a car model going over a speed bump.

Example 6.7 The Automobile Suspension System Solved In order to work this problem we'll need specific values for the parameters. Symbolic manipulations of 4×4 systems lead to immense systems, too large for display. Let $m_1 = 0.194$ lb s^2/in., $m_2 = 2.479$ lb s^2/in., $k_1 = 1{,}198$ lb/in., $k_3 = 143$ lb/in., and $c_3 = 15.06$ lb s/in. (from Gillespie (1992), p. 154). We can substitute these into the matrix given in Example 6.1 to obtain a numerical matrix with which we can move forward:

$$\mathbf{A} = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6912.37 & 737.113 & -77.6289 & 77.6289 \\ 57.6846 & -57.6846 & 6.07603 & -6.07503 \end{Bmatrix}$$

The eigenvalues of this matrix (calculated using commercial software[3]) are

$$\begin{array}{c} -39.3025 \pm 70.4975j \\ -2.54942 \pm 6.94121j \end{array}$$

and the matrix of the eigenvectors (also calculated using commercial software) is

$$\mathbf{V} = \begin{Bmatrix} -0.00602 - 0.01079j & -0.00602 + 0.01079j & 0.00531 - 0.0142j & 0.00531 + 0.0142j \\ -0.00044 + 0.00080j & -0.00044 - 0.00080j & -0.0459 - 0.12499j & -0.0459 + 0.12499j \\ 0.99717 & 0.99717 & 0.0861 + 0.0731j & 0.0861 - 0.0731j \\ -0.03914 - 0.06292j & -0.03914 + 0.06292j & 0.9846 & 0.9846 \end{Bmatrix}$$

The eigenvalues are complex conjugate pairs (as they must be) and the eigenvectors also come in complex conjugate pairs. Thus $\mathbf{v}_1\exp(s_1t)$ is the complex conjugate of $\mathbf{v}_2\exp(s_2t)$. The solution will be real, as it must be. The eigenvectors are independent and so the matrix **V** is invertible.

[3] I used Mathematica 8.0.

The governing equations for $\mathbf{z}$ are uncoupled. They are first-order equations that we know how to solve. The solutions are

$$z_i = \left\{\mathbf{V}^{-1}\mathbf{b}\right\}_i \int_0^t \exp(\lambda_i(t-\tau)) z_G(\tau) d\tau$$

where the matrix $\mathbf{V}^{-1}\mathbf{b}$ is constant, so that it can be factored out of the integral. In this case it is given by

$$\mathbf{V}^{-1}\mathbf{b} = \left\{\begin{matrix} 3096.07 + j1682.66 \\ 3096.07 - j1682.66 \\ 15.5948 + 27.65613j \\ 15.5948 - 27.65613j \end{matrix}\right\}$$

(Note that these elements also come in complex conjugate pairs.) Suppose that displacement caused by the speed bump $z_G = 4\ \sin(\omega t)$ inches. (In principle we could define a speed bump using a step function to turn it on and off, but that turns out to be impractical for numerical work (unless you want to write your own integration routine).) We can calculate $\mathbf{z}$ once we have an explicit expression for the speed bump. The $\mathbf{z}$ response will be complex, but when we map it back to $\mathbf{x}$ all the imaginary parts will cancel.

The easiest way to solve this problem, for which the forcing is not uniformly continuous (the ends of the speed bump meet the road without any gap, but there is a discontinuity in the slope), is probably to define the response of the vehicle while it is on the speed bump using the integral expression, and to match that to a homogeneous solution after leaving the speed bump. Denote the former response by $\mathbf{z}_1$ and the latter by $\mathbf{z}_2$, which we can write as

$$\mathbf{z}_2 = \left\{\begin{matrix} c_1 \exp\left(\lambda_1\left(t - \dfrac{\pi}{\omega}\right)\right) \\ c_2 \exp\left(\lambda_2\left(t - \dfrac{\pi}{\omega}\right)\right) \\ c_3 \exp\left(\lambda_3\left(t - \dfrac{\pi}{\omega}\right)\right) \\ c_4 \exp\left(\lambda_4\left(t - \dfrac{\pi}{\omega}\right)\right) \end{matrix}\right\}$$

where the coefficients are determined by requiring $\mathbf{z}_1 = \mathbf{z}_2$ at $t = \pi/\omega$. The response must be continuous, and if the $\mathbf{z}$ response is continuous then the $\mathbf{x}$ response will also be continuous. The two time intervals for $\mathbf{z}$ must be mapped back to $\mathbf{x}$ individually, so we will have $\mathbf{x}_1 = \mathbf{V}\mathbf{z}_1$ and $\mathbf{x}_2 = \mathbf{V}\mathbf{z}_2$, each valid in its own time interval. In summary

$$\mathbf{x} = \begin{cases} \mathbf{x}_1, & 0 < t < \dfrac{\pi}{\omega} \\ \mathbf{x}_2, & t > \dfrac{\pi}{\omega} \end{cases}$$

The frequency is determined by the geometry of the speed bump and the speed of the vehicle. The y position of the vehicle is $y = Vt$, so I can write the following sequence to determine the frequency in the model problem:

$$z_G = h \sin\left(\pi \frac{y}{L}\right) \rightarrow h \sin\left(\pi \frac{Vt}{L}\right) \Rightarrow \omega = \frac{\pi V}{L}$$

Typically one might measure h in inches and L in feet. I will need the velocity in ft/s, so I have

$$\omega = \frac{88\,\pi}{60\,L} V_{\text{mph}}$$

with L in feet and V_{mph} the vehicle speed in miles per hour, the unit in which we accustomed to think. Let $L = 4$ ft, then $\omega = 1.15 V_{\text{mph}}$. Figures 6.5, 6.6, and 6.7 show the response of the body (in blue) and the tire (in red) for speeds of 1, 10, and 15 miles per hour for a total distance of six times the width of the speed bump. In all three cases the tire pretty much follows the bump. The faster the vehicle, the less motion the body has.

The moral of the story is *not* to drive over speed bumps as fast as you can to minimize body motion and maximize passenger comfort.

Figures 6.8, 6.9, and 6.10 show the accelerations of the body (in blue) and the tire (in red). The horizontal axis here is time. The point to these figures is that the acceleration is roughly proportional to the speed, and the acceleration is a measure of the force applied to the body and the tire, which is related to the force applied to the suspension. The faster one goes, the more the suspension parts are stressed.

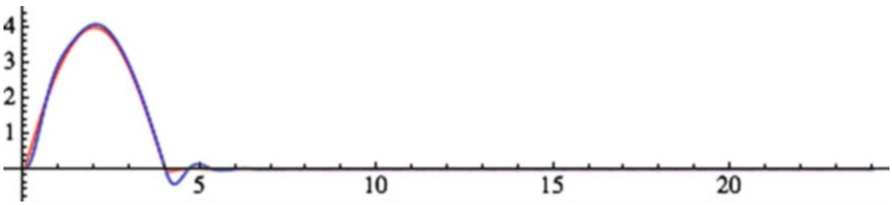


Fig. 6.5 Response at 1 mph. Vertical scale in inches; horizontal scale in feet

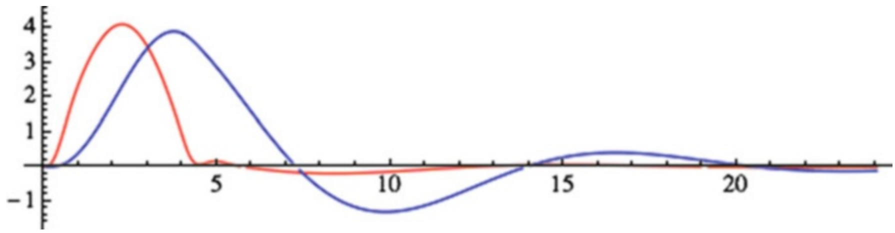


Fig. 6.6 Response at 10 mph. Vertical scale in inches; horizontal scale in feet

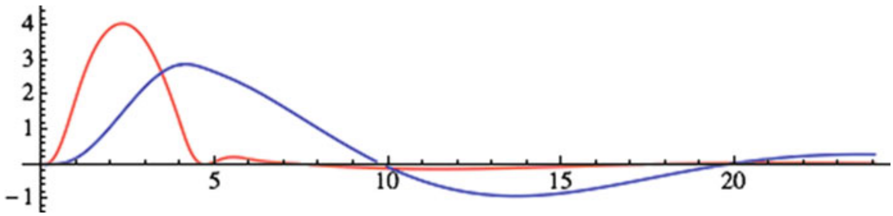


Fig. 6.7 Response at 15 mph. Vertical scale in inches; horizontal scale in feet

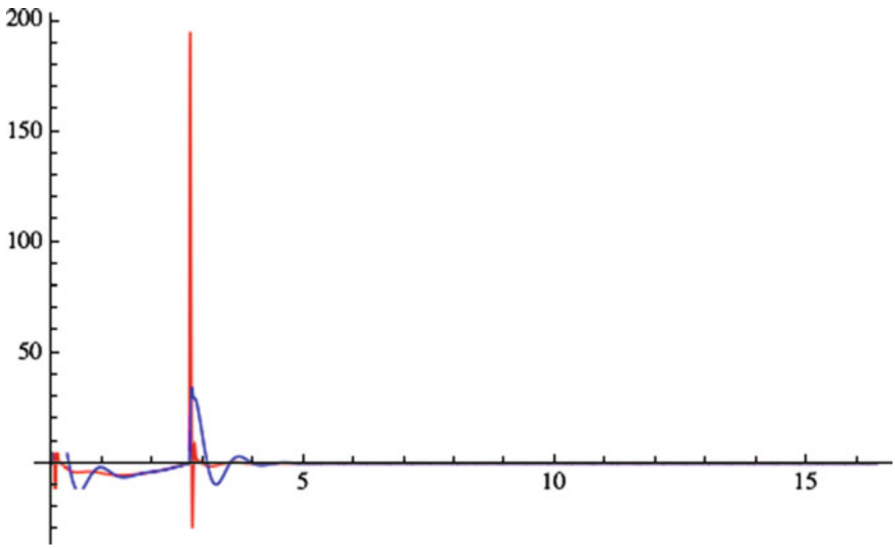


Fig. 6.8 Acceleration responses at 1 mph. Vertical scale (in./s^2); horizontal scale (s)

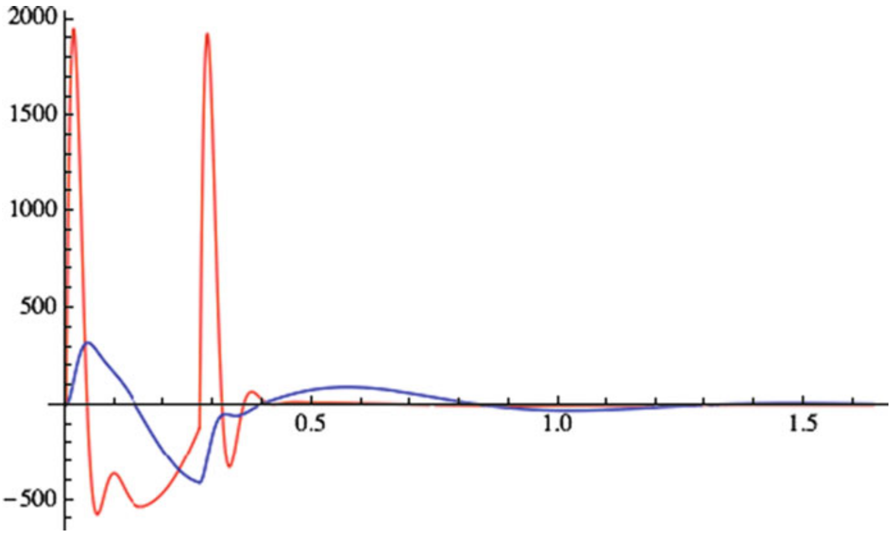


Fig. 6.9 Acceleration responses at 10 mph. Vertical scale (in./s^2); horizontal scale (s)

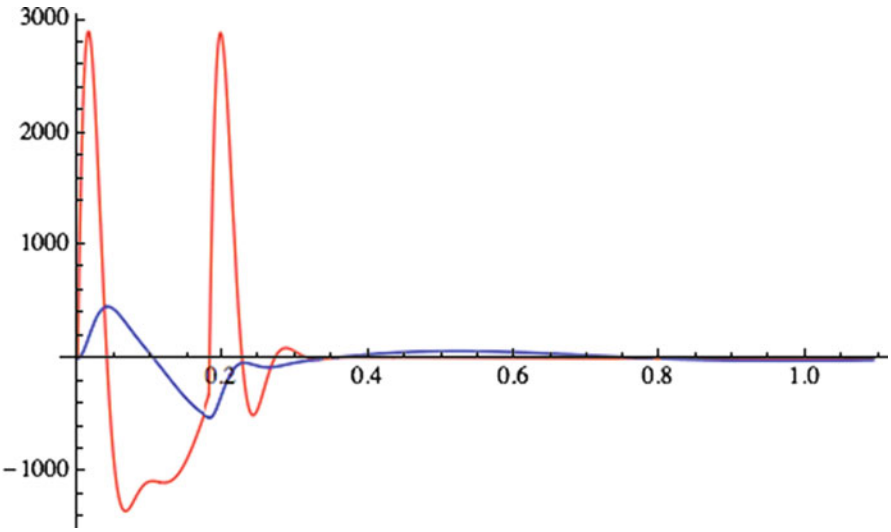


Fig. 6.10 Acceleration responses at 15 mph. Vertical scale (in./s^2); horizontal scale (s)

6.2.3 Companion Form

We learned one transformation from **x** to a new variable **z**—diagonalization. There is a second valuable transformation that leads to companion form, and it seems reasonable to introduce it here, although we will not be able to do much with it until Chap. 8. A system like our old friend Eq. (6.3)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

can sometimes be converted to companion form, which I will write as

$$\dot{\mathbf{z}} = \mathbf{A}_1\mathbf{z} + \mathbf{b}_1u \tag{6.21}$$

where $\mathbf{A}_1$ and $\mathbf{b}_1$ have the following special forms:

$$\begin{aligned} A_{1ij} &= 0, \quad i = 1\cdots N-1, \ j \neq i+1 \\ A_{1i,i+2} &= 1, \quad b_i = 0, \ i = 1\cdots N-1 \end{aligned} \tag{6.22}$$

with the last row of $\mathbf{A}_1$ arbitrary and the last element of $\mathbf{b}_1 = 1$. The following sixth-order $\mathbf{A}_1$ will give you an idea of the pattern

$$\mathbf{A}_1 = \left\{ \begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ A_{1,61} & A_{1,62} & A_{1,63} & A_{1,64} & A_{1,65} & A_{1,66} \end{array} \right\} \tag{6.23}$$

We effect the transformation using an invertible transformation matrix **T**, *not the same as the matrix that led to diagonalization!* We can write $\mathbf{z} = \mathbf{T}\mathbf{x}$, and then we have the sequence

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{T}^{-1}\dot{\mathbf{z}} = \mathbf{A}\mathbf{T}^{-1}\mathbf{z} + \mathbf{b}u \\ \mathbf{T}\mathbf{T}^{-1}\dot{\mathbf{z}} &= \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\mathbf{z} + \mathbf{T}\mathbf{b}u \\ \dot{\mathbf{z}} &= \mathbf{A}_1\mathbf{z} + \mathbf{b}_1u \end{aligned} \tag{6.24}$$

from which we deduce that

$$\mathbf{A}_1 = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}, \quad \mathbf{b}_1 = \mathbf{T}\mathbf{b} \tag{6.25}$$

We will learn how to find **T** in Chap. 8. I just want to introduce the companion form and display why it might be useful.

Denote the last row of $\mathbf{A}_1$, which is a row vector, by $\mathbf{a}_N^{\mathrm{T}}$, and then write out the last of Eq. (6.24) term by term. The first $N-1$ terms give

$$\begin{gathered}\dot{z}_1 = z_2 \Rightarrow z_2 = \dot{z} \\ \dot{z}_2 = \ddot{z}_1 = z_3 = \dot{z}_2 = \ddot{z}_1 \\ \vdots \\ z_N = z_1^{(N-1)}\end{gathered} \tag{6.26}$$

where the $(N - 1)$ in the exponent position means the N—first derivative. The Nth term of Eq. (6.24) is

$$\dot{z}_N = \mathbf{a}_N^{\mathrm{T}}\mathbf{z} \Leftrightarrow z_1^{(N)} = \mathbf{a}_N^{\mathrm{T}}\left\{ z_1 \quad \dot{z}_1 \quad \cdots \quad z_1^{(N-1)} \right\} + u \tag{6.27}$$

Equation (6.27) is an ordinary differential equation for z_1. This process has converted the N-dimensional vector equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$ to a single Nth-order differential equation for z_1. This is not always possible. We will learn when it is possible, and how to find the matrix $\mathbf{T}$ in Chap. 8.

6.3 Equilibrium, Linearization, and Stability

We have seen that systems containing pendulums are nonlinear because they involve trigonometric functions. Some systems are nonlinear for other reasons as well, such as nonlinear coupling. We know that the only method of dealing with most nonlinear problems in general is numerical integration, but it happens that in many instances we only care about "small" motions. I discussed linearization in some detail in Sect. 3.3. I want to extend that discussion to linearization in state space.

Problems derived by the Euler-Lagrange process are always quasilinear, that is to say, the highest derivatives enter linearly. The generalized forces also enter linearly, so the nonlinear equivalent of Eq. (6.4) is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u} \tag{6.28}$$

and we find it just as outlined above. The nonlinearity is confined to the vector function $\mathbf{f}$ and the matrix function $\mathbf{B}$. Equation (6.28) is general. It applies to all quasilinear dynamical systems for which the input enters linearly.

6.3.1 Equilibrium

We begin the linearization process by finding an equilibrium solution, such that

$$\dot{\mathbf{x}}_0 = \mathbf{f}(\mathbf{x}_0) + \mathbf{B}(\mathbf{x}_0)\mathbf{u}_0$$

Linearization always implies linearization with respect to some equilibrium state. Finding an equilibrium state is always necessary and often very simple. The equilibrium state is often simply $\mathbf{x} = \mathbf{0}$. A typical equilibrium solution is

$\mathbf{x}_0 = \mathbf{0} = \mathbf{u}_0$. For a double pendulum, for example, the equilibrium values of the two angles and their rates of change and the input vector are all zero. The two pendulums just hang down. This is a well-behaved equilibrium. Small perturbations will not grow without bound. All the equilibrium states in this text will be steady—$\dot{\mathbf{x}}_0 = 0$—and many will also have $u_0 = 0$.

6.3.2 Linearization in State Space

Once we have an equilibrium state, we can perturb that equilibrium and write

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}', \quad \mathbf{u} = \mathbf{u}_0 + \mathbf{u}' \tag{6.29}$$

where the primed quantities are supposed to be small departures from the equilibrium state. We can write, for example,

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}\left(\mathbf{x}_0 + \mathbf{x}'\right) = \mathbf{f}(\mathbf{x}_0) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}\to\mathbf{x}_0} \mathbf{x}' + \cdots \tag{6.30}$$

which is a shorthand notation for a multiple Taylor series around the equilibrium. We did this for scalar functions in Chap. 3. The vector case gives us an interesting object

$$\nabla \mathbf{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \tag{6.31}$$

which I will refer to as the *gradient of* **f**. For the moment note only that it must be a matrix because **f** and **x** are both vectors. In other words $\nabla \mathbf{f}\mathbf{x}'$ must be a vector [from Eq. (6.31)], and the most general proportionality between two vectors is a matrix of proportionality constants. The gradient of a scalar is a vector. The gradient of a vector is a matrix. The general connection between two vectors is a matrix of proportionality constants. Suppose **a** to be "proportional" to **b**. I mean by this that every component of **a** depends linearly on every component of **b**. For example, the first component of **a** would be written as

$$a_1 = A_{11}b_1 + A_{12}b_2 + \cdots$$

The proportionality constants A_{11}, A_{12}, etc., form the first row of a matrix. Each row of the matrix defines one component of **a**, so the complete "proportionality" will be

$$\mathbf{a} = \mathbf{A}\mathbf{b}$$

Substitution of the expansion given in Eq. (6.30) into Eq. (6.28) and truncating at the order indicated in Eq. (6.13) leads to the following set of ordinary differential equations:

$$\dot{\mathbf{x}}_0 + \dot{\mathbf{x}}' = \mathbf{f}(\mathbf{x}_0) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}\to\mathbf{x}_0} \mathbf{x}' + \mathbf{B}_0\mathbf{u}_0 + \mathbf{B}_0\mathbf{u}' + \left.\frac{\partial \mathbf{B}}{\partial \mathbf{x}}\right|_{\mathbf{x}\to\mathbf{x}_{\mathrm{eq}}} \mathbf{u}_0 \tag{6.32}$$

where $\mathbf{B}_0$ is shorthand for $\mathbf{B}$ evaluated at $\mathbf{x}_0$. The equilibrium parts cancel and we are left with a linear problem equivalent to Eq. (6.4)

$$\dot{\mathbf{x}}' = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}\to\mathbf{x}_0} \mathbf{x}' + \mathbf{B}_0\mathbf{u}' + \left(\left.\frac{\partial \mathbf{B}}{\partial \mathbf{x}}\right|_{\mathbf{x}\to\mathbf{x}_0} \mathbf{u}_0\right) \tag{6.33}$$

I put the last term in parentheses because it is almost always zero because the equilibrium is usually force-free, $u_0 = 0$. When the last term is zero we have a simple linear problem

$$\dot{\mathbf{x}}' = \left\{\left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}\to\mathbf{x}_0}\right\} \mathbf{x}' + \mathbf{B}_0\mathbf{u}' \tag{6.34}$$

and the gradient term in braces is the matrix $\mathbf{A}$. Note that this procedure is a generalization of the method we used to find the matrix $\mathbf{A}$ for the linear problem [see Eq. (6.16)]: differentiate the right-hand side of the differential equations with respect to the state variables, *and then set the state variables equal to their equilibrium values*. In the linear case the gradient of $\mathbf{f}$ is independent of the state variables, so there is no need for the substitution.

Let me discuss the gradient of $\mathbf{f}$ in more detail. We can find its components using the procedure I just outlined, but I'd like to look at it in the context of the linearization. Consider the first component of $\mathbf{f}$, which I can write as

$$f_1 = f_1(x_1, x_2, \ldots, x_N) = f_1\left(x_{1\mathrm{eq}} + x_1', x_{2\mathrm{eq}} + x_2', \ldots, x_{N\mathrm{eq}} + x_N'\right)$$

This is a function of all the state variables, and so it has a multiple Taylor series about the equilibrium position in each variable. We can write the Taylor series through their first terms as

$$\begin{aligned} f_1 = f_1\left(x_{1\mathrm{eq}}, x_{2\mathrm{eq}}, \ldots, x_{N\mathrm{eq}}\right) &+ \left.\frac{\partial f}{\partial x_1}\right|_{x_1\to x_{1\mathrm{eq}},\, x_2\to x_{2\mathrm{eq}},\cdots} x_1' + \cdots \\ &+ \left.\frac{\partial f}{\partial x_2}\right|_{x_1\to x_{1\mathrm{eq}},\, x_2\to x_{2\mathrm{eq}},\cdots} x_2' + \cdots \\ &\;\;\vdots \\ &+ \left.\frac{\partial f}{\partial x_N}\right|_{x_1\to x_{1\mathrm{eq}},\, x_2\to x_{2\mathrm{eq}},\cdots} x_N' + \cdots \end{aligned}$$

The first term beyond the equilibrium term is the gradient of f_1 dotted into the perturbation ($\mathbf{x}'$, the difference between $\mathbf{x}$ and $\mathbf{x}_{\mathrm{eq}}$)

$$\nabla f_1 \cdot \mathbf{x}'$$

Each of the other components of **f** will have a similar set of Taylor expansions. The general term in the *i*th row and *j*th column in the gradient of **f** will be the corresponding element of the matrix **A** entering the linearized equations of motion:

$$A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{x_1 \to x_{1\text{eq}}, x_2 \to x_{2\text{eq}}, \ldots}$$

We can write Eq. (6.26) in the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_{\text{eq}}\mathbf{u} \tag{6.35}$$

where I have dropped the prime for a cleaner-looking expression.

That was all pretty abstract. Let's take a look at this in a context we can understand.

Example 6.8 The Double Pendulum Figure 6.11 shows a double pendulum and the associated variables. I will address this problem under the assumption of zero damping. That makes it a bit simpler without obscuring the essentials of the linearization process. This is a two degree of freedom problem, and I choose the two angles as my generalized coordinates. It is easy enough to show that the energies are given by

$$T = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos\left(\theta_1 - \theta_2\right)$$
$$V = -m_1 g l_1 \cos\theta_1 + m_2 g(l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

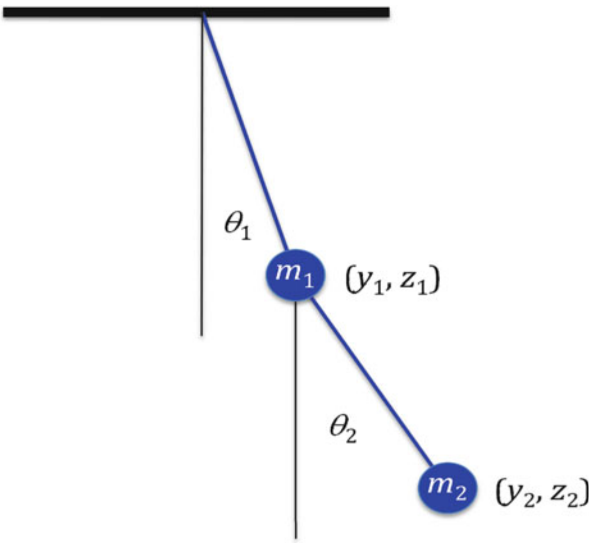


Fig. 6.11 The double pendulum

The problem has no damping as posed, so there is no Rayleigh dissipation function.

I will add a torque τ at the upper pivot so that we can see how generalized forces work. The rate of doing work (Eq. 3.9) is $\dot{W} = \tau\dot{\theta}_1$, so that $Q_1 = \tau$, and $Q_2 = 0$. The full nonlinear Euler-Lagrange equations become

$$(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2^2 + (m_1 + m_2)g l_1 \sin\theta_1 = \tau$$
$$m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + m_2 l_2^2\ddot{\theta}_2 - m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1^2 + m_2 g l_2 \sin\theta_2 = 0$$

Construct a state vector

$$\mathbf{x} = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix}$$

As usual I let the first N elements of the state be the generalized coordinates, and the second N elements be their derivatives. Solve the Euler-Lagrange equations for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ ($\dot{x}_3$ and $\dot{x}_4$) and write the state space equations in the form of Eq. (6.11). (This is a single-input system so that the matrix $\mathbf{B}$ is a vector $\mathbf{b}$, which we find by differentiating $\dot{\mathbf{x}}$ with respect to τ.) The elements of Eq. (6.11) are complicated and nonlinear, and we will have to linearize these to make progress analytically.

$$\mathbf{f} = \begin{Bmatrix} x_3 \\ x_4 \\ -\frac{1}{l_1^2\Delta}\left(g l_1\left((m_1+m_2)\sin x_1 - m_2\cos(x_1-x_2)\sin x_2\right) + m_2 l_1 \sin(x_1-x_2)\left(l_1\cos(x_1-x_2)x_3^2 + l_2\sin(x_1-x_2)x_4^2\right)\right) \\ \frac{1}{l_2\Delta}\left(g\left((m_1+m_2)\cos x_1 \sin(x_1-x_2)\right) + \sin(x_1-x_2)\left((m_1+m_2)l_1 x_3^2 + m_2 l_2\cos(x_1-x_2)x_4^2\right)\right) \end{Bmatrix}$$

$$\mathbf{b} = \frac{1}{\Delta}\begin{Bmatrix} 0 \\ 0 \\ \frac{1}{l_1^2} \\ -\frac{\cos(x_1-x_2)}{l_1 l_2} \end{Bmatrix}$$

where

$$\Delta = \left(m_1 + m_2\left(1 - \cos^2(x_1 - x_2)\right)\right)$$

This system has a simple equilibrium: no forcing ($\tau = 0$) and both angles and their derivatives equal to zero—just hanging there doing nothing. We can construct $\mathbf{A}$ and the equilibrium value of $\mathbf{b}$ by following the recipe given above. The details get messy. The equilibrium value of **b**, which I will denote by $\mathbf{b}_0$, is simple because τ is small (that is, $\tau_{eq} = 0$), and we only need **b** evaluated at equilibrium, which is

$$\mathbf{b}_0 = \frac{1}{m_1}\begin{Bmatrix} 0 \\ 0 \\ \frac{1}{l_1^2} \\ -\frac{1}{l_1 l_2} \end{Bmatrix}$$

We find **A** after some algebra

$$\mathbf{A} = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -g\frac{m_1+m_2}{m_1 l_1} & g\frac{m_2}{m_1 l_1} & 0 & 0 \\ g\frac{m_1+m_2}{m_1 l_2} & -g\frac{m_1+m_2}{m_1 l_2} & 0 & 0 \end{Bmatrix}$$

This matrix has distinct eigenvalues and eigenvectors. I will solve this linearized problem in the last section.

6.3.3 Stability

The meaning of stability in this book is *infinitesimal stability*—what happens if the state is displaced ever so slightly from its equilibrium. We can use the linearized equations of motion to assess the stability of an equilibrium. We say that an equilibrium is not unstable if small departures (which I will call *perturbations*) from that equilibrium remain close to the equilibrium. (We call it *stable* if the perturbations go to zero.) Any such perturbation will satisfy the homogeneous linearized equations. This set of homogeneous linearized equations is another example of a set of homogeneous differential equations with constant coefficients, having exponential solutions of the form exp(st). This little section introduces some nomenclature. I'll say more about stability later in the text. We know that exp(st) grows without bound if the real part of s is positive, and that it decays to zero if the real part of s is negative. If the real part of s is zero, then the exponential oscillates if the imaginary part of s is nonzero, and equals unity if both the real and imaginary parts of s equal zero. We have seen that homogeneous solutions to Eq. (6.4) involve exponential terms, the exponents of which are the eigenvalues of **A**. There will always be a homogeneous part of any solution to Eq. (6.4). Even if the particular solution formally satisfies the initial conditions, the actual initial conditions will always differ from the formal initial conditions by virtue of the imprecision of the world—nature's perturbations—so the homogeneous solution will always be present in the world if not in the formal text problem, and it will be important if it grows. If *any* of the eigenvalues of **A** have positive real parts, then that piece of the homogeneous solution will grow without bound. Solutions of this nature are said to be *unstable*. If *all* of the eigenvalues have negative real parts, the solution is said to be *stable* (technically, *asymptotically stable*, which I will simply call *stable*). If there are no eigenvalues with positive real parts, but some with zero real parts, the system is said to be *marginally stable*. Perturbations remain bounded, but do not decay to zero. (Zero eigenvalues are something of a special case that I'd like to address only in context.)

Suppose we have a nonlinear system. We want to assess its behavior. We can find an equilibrium. We can look at solutions near the equilibrium by linearizing to arrive at the problem defined by Eq. (6.27) with, generally, $\mathbf{u}'$ equal to zero, because we have no way of adjusting the equilibrium input. Since there is no input, the solution will be entirely the homogeneous solution, and we say that the equilibrium is stable, unstable, or marginally stable according to the eigenvalues of the matrix **A**.

Example 6.9 The Inverted Pendulum Equation (2.12a) governs the unforced (homogeneous) motion of a simple pendulum. It has two equilibrium positions, $\theta = 0$ and $\theta = \pi$. The linearized version of Eq. (2.11) is

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

in the neighborhood of the marginally stable equilibrium at $\theta = 0$. If we put $\theta = \pi + \theta'$ instead of supposing θ to be small we'll arrive at the linear problem

$$\ddot{\theta} - \frac{g}{l}\theta = 0$$

which is clearly unstable (what are its eigenvalues?). We see that we can convert the equations of motion for the marginally stable pendulum to those of the unstable pendulum by simply replacing g by $-g$. (We can do the same thing with the overhead crane to find the equations governing the motion of an inverted pendulum on a cart, which I will address in the second half of the text.)

We can add a torque at the pivot and convert to state space to write a set of linearized state space equations for the inverted pendulum

$$\mathbf{x} = \begin{Bmatrix} \theta \\ \dot{\theta} \end{Bmatrix} \Rightarrow \dot{\mathbf{x}} = \begin{Bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{1}{ml} \end{Bmatrix} \tau$$

We will learn how to stabilize an unstable equilibrium in the second half of this book using feedback. This is the essence of linear control theory.

6.4 Putting It All Together

We have seen how to convert a dynamical system found using the Euler-Lagrange method to a dynamical system in a suitably defined state space. We have seen how to solve a linear state space problem. We have seen how to make a nonlinear state space problem into a linear state space problem. We can combine all of this into a procedure to deal with any well-posed problem.

Example 6.10 The Double Pendulum Let's start by solving the double pendulum problem posed in Ex. 6.8. This is a mess in general terms, so let's choose a specific set of parameters. We'll lose generality but be able to follow what is happening. Let $m_1 = 1$, $m_2 = 2$, $l_1 = 2$, $l_2 = 1$, and choose a timescale such that $g = 1$. (One can always do the latter.) Figure 6.12 shows a scaled drawing of the example.

The eigenvalues for these numbers are

$$2.03407\,j, \quad -2.3407\,j, \quad 0.602114\,j, \quad -0.602114\,j$$

and the matrix of eigenvectors is

$$\mathbf{V} = \begin{Bmatrix} 0.186401\,j & -0.186401\,j & -1.46011\,j & 1.46011\,j \\ -0.491624\,j & 0.491624\,j & -1.66081\,j & 1.66081\,j \\ -0.379253 & -0.379253 & 0.879153 & 0.879153 \\ 1 & 1 & 1 & 1 \end{Bmatrix}$$

(Mathematica chose the scaling of the eigenvectors.) We see that the eigenvalues are complex conjugates, and so are the eigenvectors. The eigenvalues are purely imaginary, so the double pendulum is marginally stable (in the linear limit).

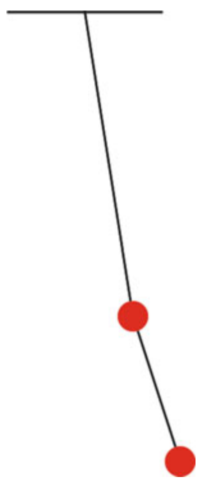


Fig. 6.12 The scaled pendulum with initial angles of $\pi/10$ and $\pi/20$

We can form a homogeneous solution from the eigenvalues and eigenvectors. Denote the columns of $\mathbf{V}$ by $\mathbf{v}_i$ and the eigenvalues by λ_i. The homogeneous solution is then

$$x_{\mathrm{H}} = \sum_{i=1}^{N_S} c_i \exp(\lambda_i t)\mathbf{v}_i$$

The initial conditions determine the values of the coefficients. Figure 6.13 shows the homogeneous solution for initial values of θ_1 and θ_2 of $\pm\,\pi/20$, respectively, for two nominal periods of the lower eigenfrequency (0.602114). The eigenvalues and eigenvectors are complex as are the c_i, but the sum is real. All the complex conjugates are such that the imaginary parts cancel. Note that the response is not at this frequency, because the eigenvectors associated with each eigenfrequency have components in both arms of the pendulum.

Suppose we excite the double pendulum with a sinusoidal torque with frequency ω applied at the upper pivot point as posed in Ex. 6.8. We can solve for the particular solution using diagonalization. Write $\mathbf{x} = \mathbf{Vz}$. Then we know that the components of $\mathbf{z}$ can be written

$$z_i = \left\{\mathbf{V}^{-1}\mathbf{b}\right\}_i \int_0^t \exp(\lambda_i(t-\tau))\tau(\xi)d\xi$$

The vector

$$\mathbf{V}^{-1}\mathbf{b} = \begin{Bmatrix} -0.27401 \\ -0.27401 \\ 0.02401 \\ 0.02401 \end{Bmatrix}$$

for this set of parameters. I will suppose that τ is proportional to $\sin(\omega t)$. The integrals are straightforward and we have

$$z_i = \frac{\omega}{\lambda_i^2 + \omega^2}\exp(\lambda_i t) - \frac{\omega\cos(\omega t) + \lambda_i \sin(\omega t)}{\lambda_i^2 + \omega^2}$$

Each component of $\mathbf{z}$ is complex (recall that the λ_ι are imaginary). We see that this has a homogeneous component at its eigenfrequency, and harmonic components at the forcing frequency. The appearance of a homogeneous component should not be upsetting. Recall that the particular solution is *any* solution that satisfies the differential equation, and this one certainly does. It satisfies zero initial conditions, as can be seen by direct substitution. Since $\mathbf{x}$ is proportional to $\mathbf{z}$, zero initial conditions on $\mathbf{z}$ correspond to zero initial conditions on $\mathbf{x}$ and vice versa. The components of $\mathbf{z}$ can be complex, because they do not represent the physical response of the system. That is represented by $\mathbf{x} = \mathbf{Vz}$. Both $\mathbf{V}$ and $\mathbf{z}$ are complex, but their product is real. The algebra to establish this is lengthy, and I will not

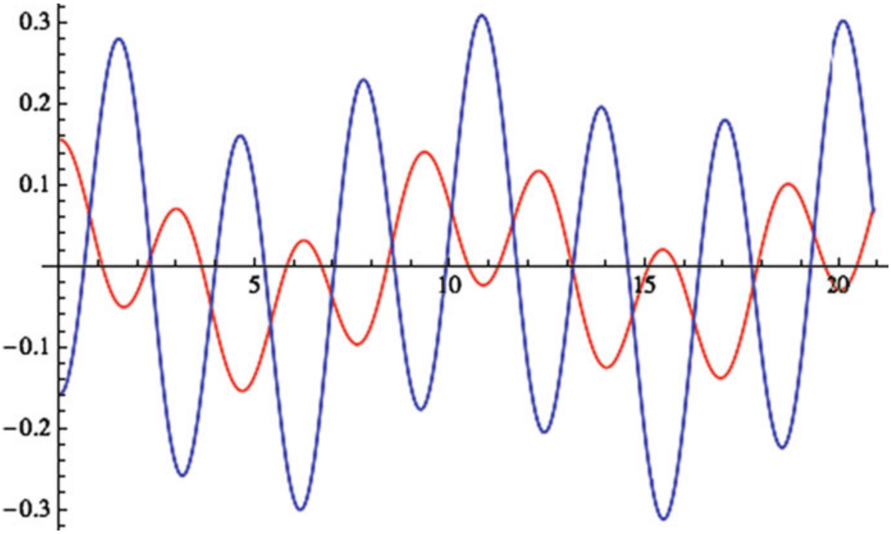


Fig. 6.13 Oscillation of the unforced double pendulum. The upper arm is in red and the lower arm in blue

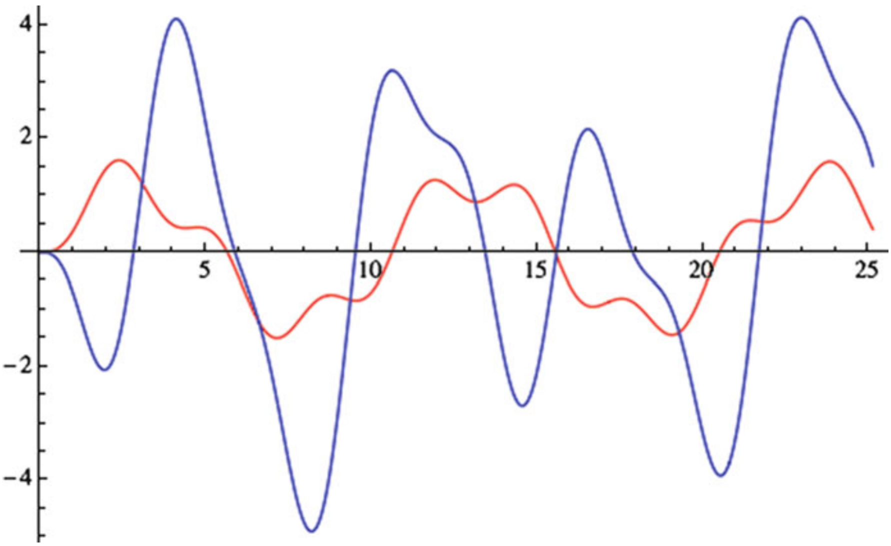


Fig. 6.14 Forced motion of the double pendulum. The forcing torque is 0.2 sin t Nm

display it here. Figure 6.14 shows the particular solution response to a forcing of 0.2 sin$t$ for four periods of the forcing.

This particular solution satisfies zero initial conditions. If there are nonzero initial conditions, the complete solution will be the sum of this particular solution and the appropriate homogeneous solution.

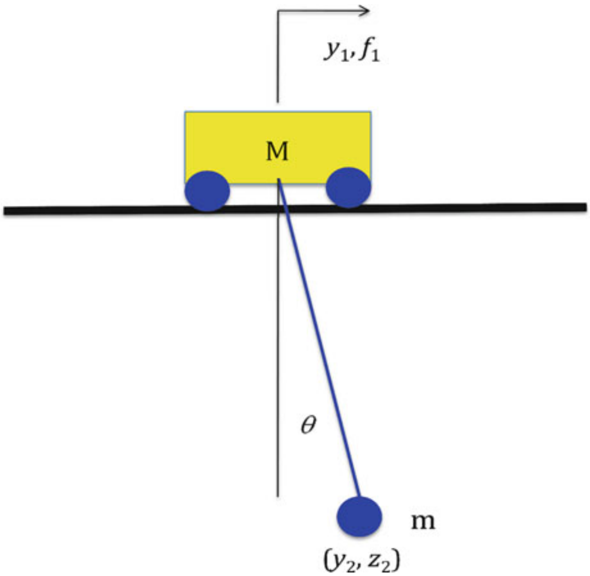


Fig. 6.15 The overhead crane

The system will never "settle down" to a nice forced harmonic motion because there is no damping. I encourage you to rework this problem with damping at one or the other (or both) of the pivots.

6.4.1 The Overhead Crane

We've established that the procedure works for the double pendulum. Let's take on what turns out to be a more difficult problem: the overhead crane first introduced in Chap. 3, a two degree of freedom problem that is nonlinear in its basic formulation. Figure 6.15 (redrawing of Fig. 3.9) shows the system

The Euler-Lagrange equations are a specialization of those governing Ex. 3.4:

$$(M + m)\ddot{y} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 = f$$

and

$$m\cos\theta\ddot{y} + ml\ddot{\theta} + c\dot{\theta} + mg\sin\theta = 0$$

where I have but one damping term and one forcing. Both of these are nonlinear because of the trigonometric functions, and the first is nonlinear because of cross terms as well. We can solve these for the second derivatives

$$\ddot{y} = -\frac{m\sin\theta}{\Delta}\left(l\dot{\theta}^2 + g\cos\theta\right) + \frac{c\cos\theta}{l\Delta}\dot{\theta} + \frac{1}{\Delta}f$$
$$\ddot{\theta} = -\frac{\sin\theta}{l\Delta}\left(ml\cos\theta\dot{\theta}^2 + (M+m)g\right) - \frac{c(m+M)}{ml^2\Delta}\dot{\theta} - \frac{\cos\theta}{l\Delta}f$$

where

$$\Delta = M + m\sin^2\theta$$

The force comes from an electric motor on the cart. If inductance is not important, then we can use Eq. (3.22) to represent the force in terms of the input voltage. If inductance is important, then we need to write $f = -\tau/r$, and write the torque in terms of the current as $\tau = Ki$. The current satisfies a first-order equation

$$L\frac{di}{dt} = e - K\omega - Ri$$

and the system becomes a fifth-order system. This is one of the advantages of the state space formulation: I can add variables without changing the structure of the problem because all the differential equations are first order. Let me look at the two cases in turn. Both are single-input systems with the voltage e as the input.

Example 6.11 The Overhead Crane in the Low-Inductance Limit (A Fourth-Order System) A logical state vector for the low-inductance case is

$$\mathbf{x} = \begin{Bmatrix} y \\ \theta \\ \dot{y} \\ \dot{\theta} \end{Bmatrix} \tag{6.36}$$

The force now comes from Eq. (3.22), and we can write the four nonlinear state equations as

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= \frac{m\sin x_2}{\Delta}\left(lx_4^2 + g\cos x_2\right) + \frac{c\cos x_2}{l\Delta}x_4 - \frac{1}{\Delta}\left(\frac{K}{rR}\left(e + \frac{K}{r}x_3\right)\right) \\
\dot{x}_4 &= -\frac{\sin x_2}{l\Delta}\left(ml\cos x_2 x_4^2 + (M+m)g\right) - \frac{c(m+M)}{ml^2\Delta}x_4 + \frac{\cos x_2}{l\Delta}\left(\frac{K}{rR}\left(e + \frac{K}{r}x_3\right)\right)
\end{aligned} \tag{6.37}$$

The nonlinear vector **b** is

$$\mathbf{b} = \frac{K}{lrR\Delta}\begin{Bmatrix} 0 \\ 0 \\ -l \\ \cos x_2 \end{Bmatrix}$$

This problem admits a somewhat more complicated equilibrium. You can verify that

$$\mathbf{x}_{\mathrm{eq}} = \begin{Bmatrix} Y_0 + V_0 t \\ 0 \\ V_0 \\ 0 \end{Bmatrix} \tag{6.38}$$

satisfies the nonlinear equations of motion for any values of Y_0 and V_0.

The (linear) matrix **A** derived from the state equations is

$$\mathbf{A} = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m}{M}g & -\frac{K^2}{Mr^2R} & \frac{c}{Ml} \\ 0 & -\frac{(M+m)}{Ml}g & \frac{K^2}{lMr^2R} & -\frac{(M+m)c}{mMl^2} \end{Bmatrix}, \quad \mathbf{b} = \frac{K}{MlrR}\begin{Bmatrix} 0 \\ 0 \\ -l \\ 1 \end{Bmatrix} \tag{6.39}$$

The eigenvalues of this matrix are not much affected by the value of c. The eigenvalues for $c = 0$ and the parameters shown in Table 6.1 (the motor parameters are those of the motor introduced in Chap. 3, for which the design voltage is 180) are:

$$-0.003488 \pm 2.7125j, \quad -0.01395, \quad 0.0$$

The system is marginally stable in the neighborhood of its static equilibrium.

(The characteristic polynomial of the matrix in Eq. (6.33) when c is equal to zero is that given in Ex. 3.5.) These remain qualitatively the same until c exceeds 677.23, when the complex eigenvalues become purely real. The zero eigenvalue persists for all values of c. I will discuss the system for $c = 0$ to make things a bit simpler.

Note that the decay rates for the oscillations are very low. If we decide to say that an initial condition has decayed when it has fallen by a factor of exp(−3), then the

Table 6.1 Parameters for the overhead crane (SI units)

m	M	l	r	K	R	L	g
50	100	2	0.3	0.868	4	0.01	9.81

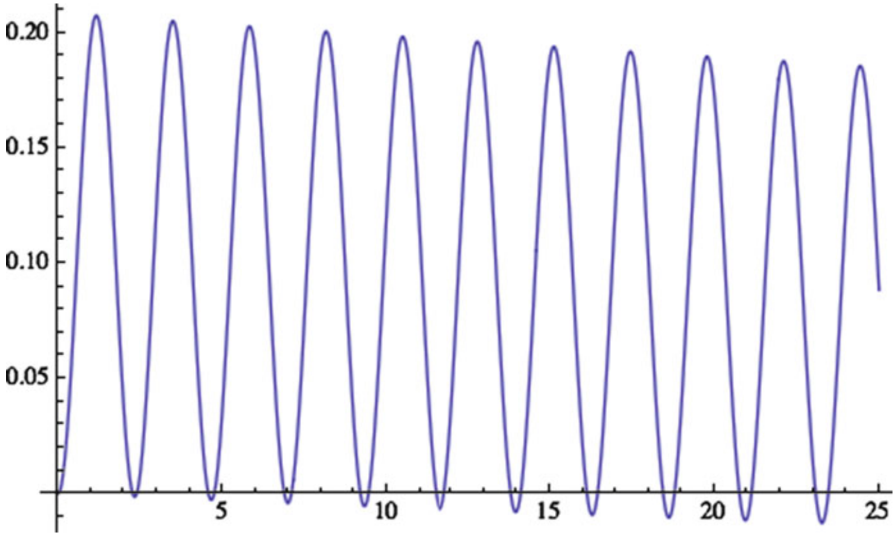


Fig. 6.16 Motion of the cart in meters

last of the transient will be over in about 2,000 s. The zero eigenvalue affects only y, as one can see by looking at the eigenvectors in Eq. (6.34).

The matrix $\mathbf{V}$ associated with this system (for $c = 0$, and rounded to the fourth decimal place) is

$$\mathbf{V} = \begin{Bmatrix} 0.0004 - 0.1919j & 0.0004 + 0.1919j & -1.0000 & 1 \\ 0.0002 + 0.2878j & 0.0002 - 0.2878j & 6 \times 10^{-6} & 0 \\ 0.5205 + 0.0015j & 0.5204 - 0.0015j & 0.0078 & 0 \\ -0.7807 & -0.7807 & -4 \times 10^{-8} & 0 \end{Bmatrix} \tag{6.40}$$

(The order of the vectors corresponds to the order in which I have listed the eigenvalues.) The zero eigenvalue corresponds to the position of the cart: its sole entry is in the y part of the state vector.

First let us note that the nonlinear problem has a solution for both the unforced and forced cases. Figures 6.16 and 6.17 show the simulated motion (using the nonlinear equations of motion) of the cart and the angle of the pendulum, respectively, for an unforced system starting from rest with the pendulum at an angle of $\pi/20$. Both execute what looks like nice underdamped motion. The "frequency" of the system should be about 2.71, corresponding to the imaginary part of the complex frequencies, giving a period of 2.34. The actual period is quite close to this. (Note that we can take the data shown in Fig. 6.15 or Fig. 6.16 and model it as an underdamped one degree of freedom from which we can calculate a natural frequency and a damping ratio. I will set this as an exercise.) The damping is small, so the decay is also slow, as I noted above. I show the time interval from 0 to 25 so that the oscillation can be seen.

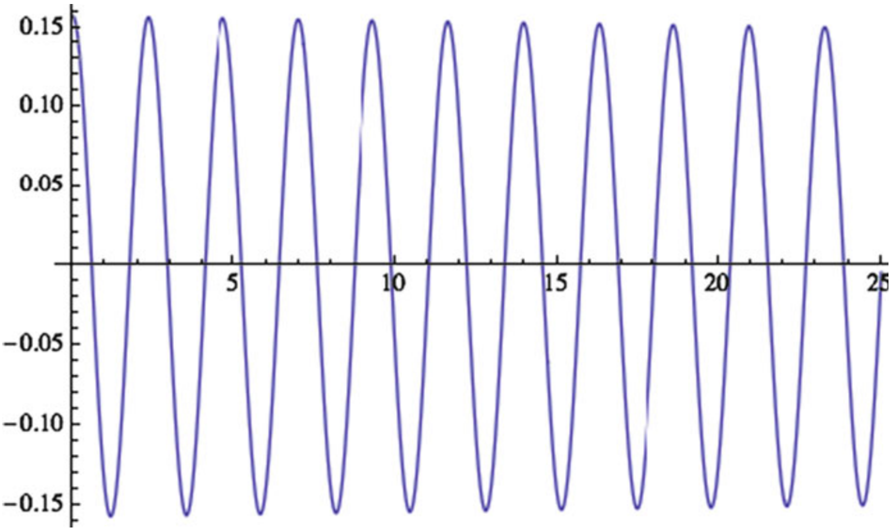


Fig. 6.17 Motion of the pendulum in radians

The behavior of the angle and position variables is qualitatively the same. We could look at displacement as a proxy for the entire system, but we ought to look at the pendulum motion because the use of a linear model relies on this angle being small. Figures 6.18, 6.19, and 6.20 show the response to an input voltage 180sin(ωt) for $\omega = 2.71$ (near resonance), 0.271, and 27.1. I start from equilibrium in each case. The damping parts are so small that it takes much more time than it is worth displaying to reach the final steady state (the particular solution)—about 1,000 s (quicker for the low-frequency excitation). I show the first 50 s to display the transient, and a time interval after the transient has mainly decayed, what we called steady motion in the earlier part of the book. The resonant case shows what looks like a beat frequency. It is not simply related to any of the frequencies in the problem. The excursions of the pendulum during the transient (Fig. 6.18a) exceed those of the steady solution (Fig. 6.18b).

The transient of the low-frequency response is but an overlay on the steady solution. The angle shown in Fig. 6.19a looks like a lumpy version of Fig. 6.19b with slightly larger excursions.

High-frequency forcing is quite different. The transient phase is dominated by the unforced response. One can barely see the forcing in Fig. 6.20a, where it appears as a bit of jitter on the transient. The amplitude of the response shown in Fig. 6.20b is an order of magnitude smaller than that in Fig. 6.20a.

Note that motion of the pendulum in the resonant case clearly exceeds what we have come to learn is a small angle, exceeding one radian (about 60°) in the early phase of the motion. The steady motion is an order of magnitude smaller, which is within the linear range.

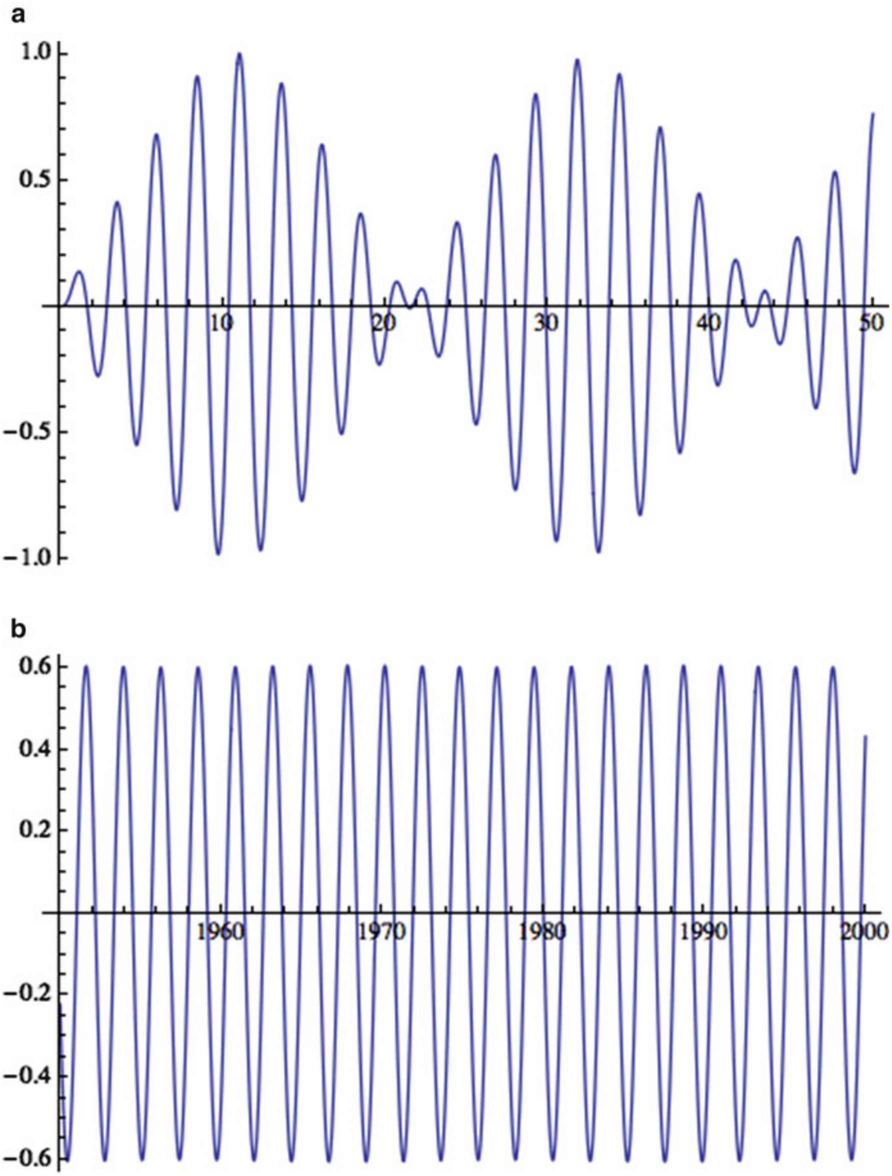


Fig. 6.18 (**a**) Initial motion of the pendulum at "resonance." (**b**) Steady motion of the pendulum at resonance

How does a linear solution compare to the simulation? Let me apply and expand the various techniques we have learned so far to find a linear solution supposing a sinusoidal forcing voltage. I can find a particular solution by writing $\sin(\omega t)$ as $-j\exp(j\omega t)$ and agree to take the real part of the solution as our particular solution.

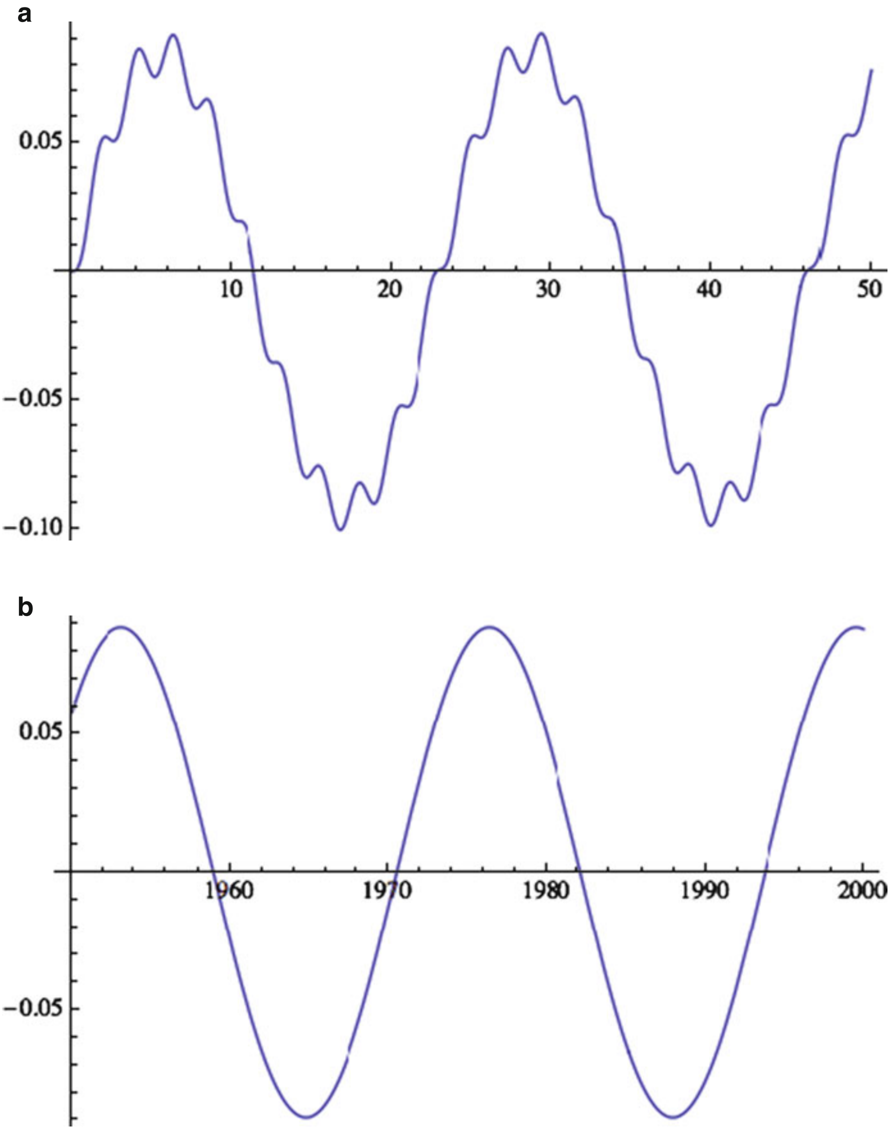


Fig. 6.19 (**a**) Initial motion of the pendulum for low-frequency forcing. (**b**) Steady motion of the pendulum for low-frequency forcing

I can put together the homogeneous solution using the eigenvalues and eigenvectors and then connect the two solutions by imposing initial conditions.

Let me start with the homogeneous solution, which I can write as

$$\mathbf{x}_{\mathrm{H}} = \sum_{i=1}^{4} c_i \mathbf{v}_i \exp(s_i t) \tag{6.41}$$

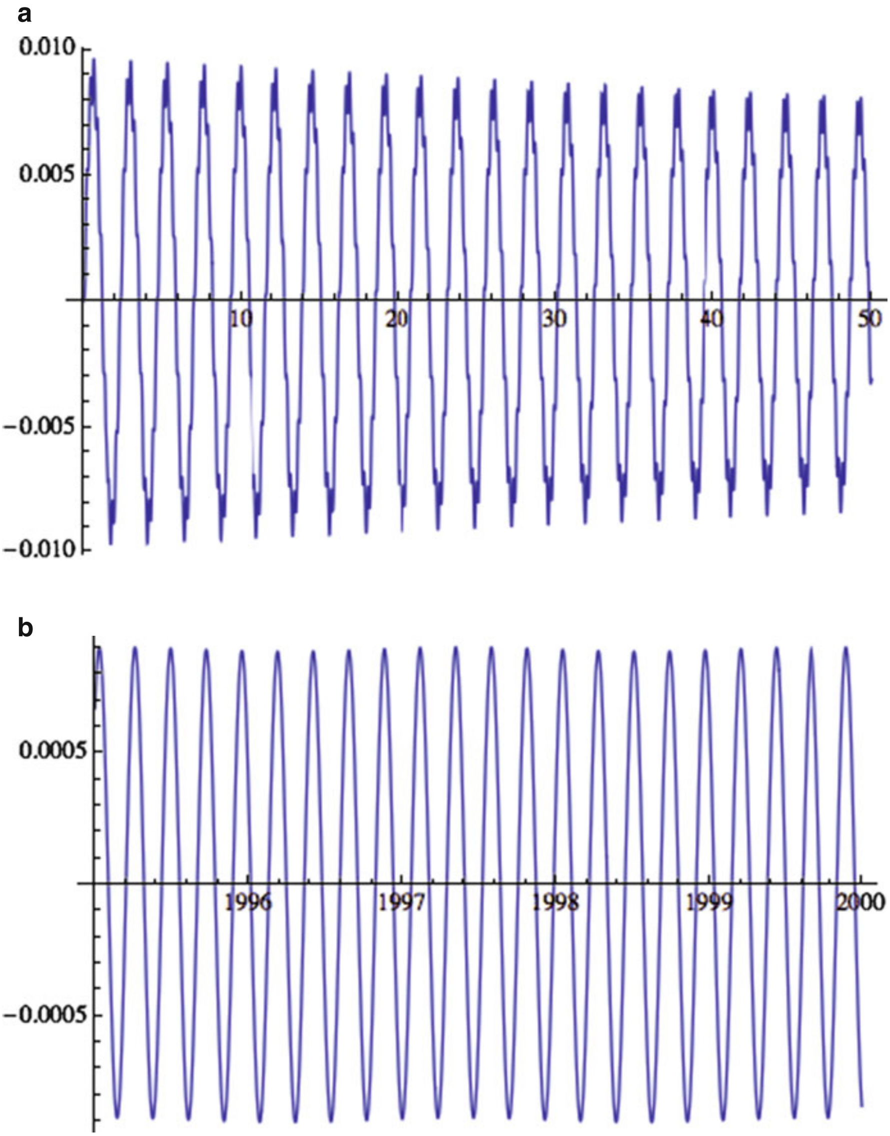


Fig. 6.20 (**a**) Initial motion of the pendulum for high-frequency forcing. (**b**) Steady motion of the pendulum for high-frequency forcing. I have taken a shorter time span because of the higher frequency

The initial conditions determine the coefficients c_i. For the initial conditions leading to Figs. 6.16 and 6.17 we have

$$c_1 = -18 \times 10^{-9} - 0.2729j, \quad c_2 = -18 \times 10^{-9} + 0.2729j,$$
$$c_3 = -0.1047, \quad c_4 = 0$$

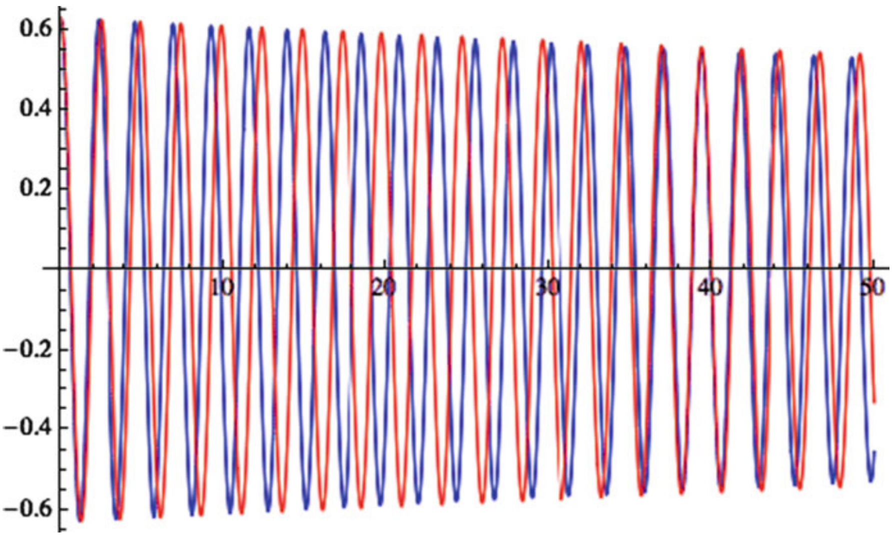


Fig. 6.21 Free oscillations starting from rest with an angle of $\pi/5$

Note that the two complex coefficients are mutual conjugates. (I suspect that the real parts come from roundoff errors in the eigenvalue calculation, but I do not know this.) We can show the response of any of the components. It is interesting to look at larger initial conditions so that we can compare the linear solution to the simulation. Figure 6.21 shows the angle for an initial angle of $\pi/5$. The coefficients for this initial condition are

$$c_1 = -74 \times 10^{-9} + 1.0915j, \quad c_2 = -74 \times 10^{-9} - 1.0915j,$$
$$c_3 = -0.4189, \quad c_4 = 0$$

The blue line is the linear solution and the red line the simulation. The results are different, as one might expect. The amplitude of the oscillation is essentially the same, but the frequency changes. We see the two start in phase and then drift in and out of phase. The beat period is about 40 s.

The forced problem requires a particular solution, and we can find that by analogy with the methods of Chap. 2. Recall that the particular solution is any solution that satisfies the inhomogeneous differential equations. We have $\mathbf{A}$ and $\mathbf{b}$ (and I neglect c as above), and I can write the inhomogeneous differential equations as

$$\dot{\mathbf{x}}_P = \mathbf{A}\mathbf{x}_P + \mathbf{b}(-j\exp(j\omega t)) \Rightarrow (j\omega\mathbf{1} - \mathbf{A})\mathbf{x}_P = e_0\mathbf{b}(-j\exp(j\omega t))$$

where e_0 denotes the amplitude of the input voltage. The actual particular solution will be the real part of

$$\mathbf{x}_P = e_0(j\omega\mathbf{1} - \mathbf{A})^{-1}\mathbf{b}(-j\exp(j\omega t))$$

The initial condition will be

$$\text{Re}\Big(e_0(j\omega\mathbf{1} - \mathbf{A})^{-1}\mathbf{b}(-j)\Big) + \sum_{i=1}^{4} c_i\mathbf{v}_i\exp(s_i t) = \mathbf{0}$$

The coefficients satisfying this initial condition are complicated functions of ω, and I won't write them out. They are complex. The final result for the state is

$$\mathbf{x} = \sum_{i=1}^{4} c_i\mathbf{v}_i\exp(s_i t) + \text{Re}\Big(e_0(j\omega\mathbf{1} - \mathbf{A})^{-1}\mathbf{b}(-j\exp(j\omega t))\Big) \tag{6.42}$$

I leave it to you to explore the nature of the difference between the linear and simulated solutions for the forced motion case.

Example 6.12 The Overhead Crane When Inductance Is Important (A Fifth-Order System) What happens if I put in the inductance? I need a new system, and it looks different. I have a new state vector

$$\mathbf{x} = \begin{Bmatrix} y \\ \theta \\ \dot{y} \\ \dot{\theta} \\ i \end{Bmatrix} \tag{6.43}$$

and a new set of state equations. (I added the new variable to the end of the previous state vector. I could have put it anywhere I wanted. It is a matter of personal choice.) I write the differential equations by adapting what we already have and adding one equation for the current,

$$\dot{x}_1 = x_3$$
$$\dot{x}_2 = x_4$$

$$\begin{aligned}
\dot{x}_3 &= \frac{m\sin x_2}{\Delta}\left(lx_4^2 + g\cos x_2\right) + \frac{K}{\Delta}x_5 \\
\dot{x}_4 &= -\frac{\sin x_2}{l\Delta}\left(ml\cos x_2 x_4^2 + (M+m)g\right) - \frac{K\cos x_2}{l\Delta}x_5 \\
\dot{x}_5 &= -\frac{K}{L}x_4 - \frac{R}{L}x_5 + \frac{e}{L}
\end{aligned} \tag{6.44}$$

Note that the input, which contributed to the third and fourth equations before, now appears only in the fifth equation. We can find **A** and **b** by differentiation. Denote the vector right-hand side of Eq. (6.44) by **f**. The elements of **A** and **b** are

$$A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{q}\to\mathbf{0}}, \quad b_i = \left.\frac{\partial f_i}{\partial e}\right|_{\mathbf{q}\to\mathbf{0}} \tag{6.45}$$

I have selected zero as my reference equilibrium state. The actual matrix and vector are given by

$$\mathbf{x} = \begin{Bmatrix} y \\ \theta \\ \dot{y} \\ \dot{\theta} \\ i \end{Bmatrix}, \quad \mathbf{A} = \begin{Bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \dfrac{mg}{M} & 0 & 0 & -\dfrac{K}{Mr} \\ 0 & -\dfrac{(m+M)g}{Ml} & 0 & 0 & \dfrac{K}{Mlr} \\ 0 & 0 & \dfrac{K}{L} & 0 & -\dfrac{R}{L} \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{1}{L} \end{Bmatrix} \tag{6.46}$$

The upper left-hand 4×4 partition of the matrix is the same as the 4×4 matrix for the previous problem.

The two systems are quite similar. The eigenvalues for the five-dimensional system (using the values from Table 6.1) are qualitatively the same as those of the four-dimensional system, with the addition of a rapidly decaying fifth eigenvalue (the first eigenvalue in the list)

$$\Lambda = \mathrm{diag}\{-399.979 \quad -0.0035 + j2.7125 \quad -0.0035 - j2.7125 \quad -0.01340 \quad 0.0\}$$

We should expect that the behavior should be more or less the same as long as the motion is not really rapid. We can see this by looking at a simulation with the same starting values as that shown in Fig. 6.16, which I show as Fig. 6.22. Comparing this to Fig. 6.16a we see there is very little difference.

There is little difference for the other two cases as well. When should we expect the difference to be important? The inductive term in the current law should become comparable to the resistive term: $\omega L \approx R$. In the present case (Table 6.1) we have $\omega \approx 400$ rad/s, well above anything we have looked at. If we excite the system at 400 rad/s and look at the steady behavior of the pendulum, we find a significant difference. Figure 6.22 shows the two curves plotted on the same graph. The low-inductance version has an extra, low-frequency, oscillation that is suppressed when the full system is considered (Fig. 6.23).

The high-inductance calculation takes significantly more computing time than the low-inductance case, so if there is a moral to be taken from this, it is not to use the more complicated method unless it is necessary.

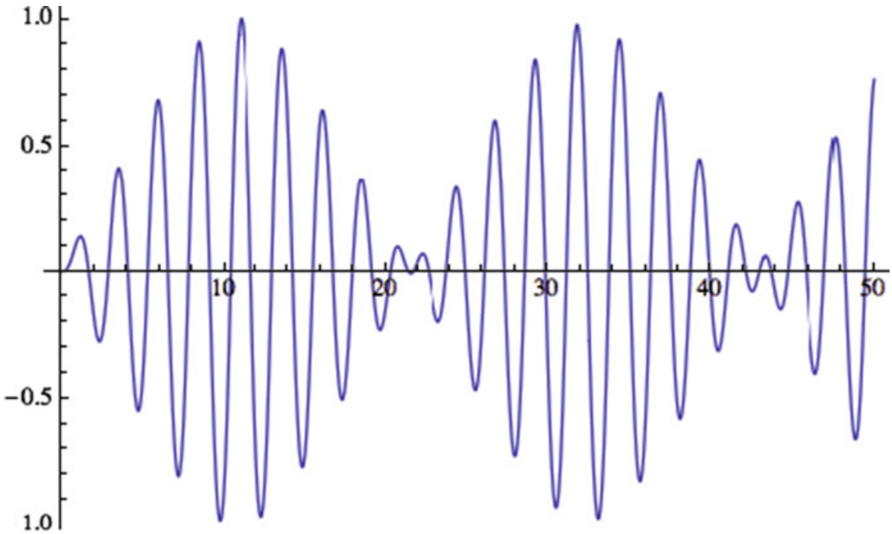


Fig. 6.22 The transient phase of the resonant case using the high-inductance equations of motion

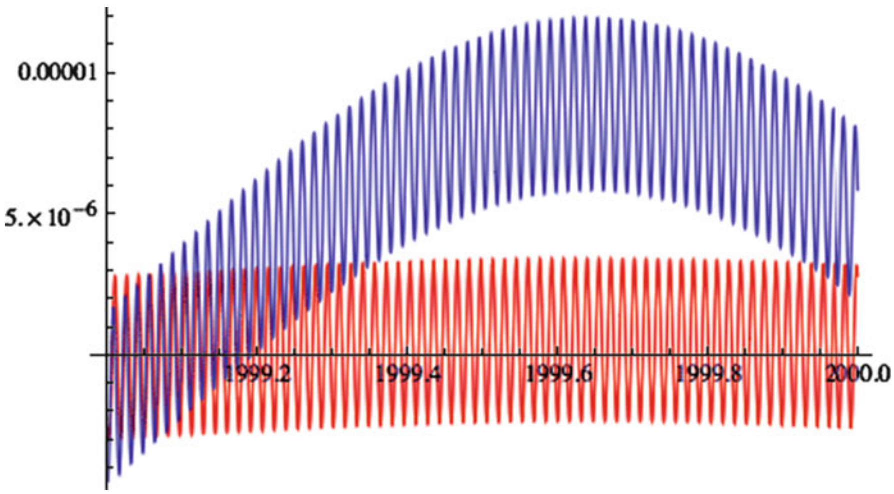


Fig. 6.23 The steady response of the pendulum at an excitation frequency of 400 rad/s. The low-inductance approximation is shown in *blue*, and the high-inductance response is shown in *red*

6.5 The Phase Plane

The phase plane is a pictorial representation of two-dimensional state space. It has a long history of its own. You should know about it, and this chapter is a logical place to introduce it. One degree of freedom systems have a two-dimensional state, which can be plotted on a piece of paper. This plot is called a phase diagram, drawn on the phase plane. W can integrate the differential equations of state given the state at $t = 0$. This means that specifying the function and its derivative, for example, is enough to specify the state at any time t. As time passes the points specified by the function and its derivative trace out a curve in the phase plane. We call this curve a *trajectory*. Plots in the phase plane deal with the basic system; it does not consider forcing. The phase plane gives us an interesting picture of a two-dimensional system, complementing the actual behavior of the system. It speaks to stability. The trajectory of a stable system converges to a point representing the equilibrium of the system. The trajectory of a marginally stable system converges to a closed orbit. The trajectory of an unstable system diverges. The phase diagram can be defined for nonlinear systems, and you can often deduce something about the stability of a system by considering the phase diagram. A thorough examination of the phase plane is beyond the scope of this text, but you should know a little about the phase plane, because you will run across it in your reading. This section is an introduction.

If the system has no damping or forcing its governing equation can be written as (from the Euler-Lagrange formulation)

$$m\ddot{y} + \frac{dV}{dy} = 0 \tag{6.47}$$

where V denotes a potential that is a function of the variable y. Equation (6.47) is not necessarily linear. We can write a similar equation for a rotational problem, θ taking the place of y. Equation 6.47 has a first integral. Multiply both sides by $\dot{y}$ and integrate

$$m\dot{y}\ddot{y} + \dot{y}\frac{\partial V}{\partial y} = \frac{d}{dt}\left(\frac{1}{2}m\dot{y}^2 + V\right) \Rightarrow \frac{1}{2}m\dot{y}^2 + V = E_0 \tag{6.48}$$

This is an expression of the conservation of energy, which is why I called the constant of integration E_0. It has the dimensions of energy. The energy is a function of y and $\dot{y}$. When $\dot{y} = 0$ the potential V is at its maximum (see the energy argument in Sect. 2.2) so $E'_0 = V_{max}$. (The argument in Sect. 2.2 supposes that $E'_0 > 0$, which is not necessarily the case. T_{max} need not equal V_{max}.) As time passes these two variables change, but the relationship between them is specified by Eq. (6.48). We can plot this trajectory on a phase plane with its horizontal axis y and its vertical axis $\dot{y}$.

We can sketch trajectories directly from Eq. (6.40). If $\dot{y}$ is positive then y must be increasing and vice versa. Thus trajectories in the upper part of the phase plane move to the right and those in the lower part of the phase plane move to the left. Closed trajectories, also called *orbits*, move in a clockwise sense. A stable trajectory will spiral to its equilibrium position in a clockwise sense.

6.5.1 The Simple Pendulum

The nonlinear simple pendulum gives us a good example of this. The differential equation is

$$ml^2\ddot{\theta} + mgl\sin\theta = 0 \tag{6.49}$$

where $\theta = 0$ when the pendulum points straight down. The integration to arrive at the energy equation is straightforward

$$ml^2\dot{\theta}\ddot{\theta} + mgl\dot{\theta}\sin\theta = 0 \Rightarrow \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = E_0 \tag{6.50}$$

We can scale Eqs. (6.49) and (6.50) to make all of this easier to illustrate. If we divide by ml^2, and choose a timescale such that $g/l = 1$, which we can always do without loss of generality, then Eq. (6.50) becomes

$$\frac{1}{2}\dot{\theta}^2 - \cos\theta = E_0' \tag{6.51}$$

and the differential equations for the scaled problem in state space form are

$$\mathbf{x} = \begin{Bmatrix} \theta \\ \dot{\theta} \end{Bmatrix}, \quad \dot{\mathbf{x}} = \begin{Bmatrix} x_2 \\ -\sin(x_1) \end{Bmatrix} \tag{6.52}$$

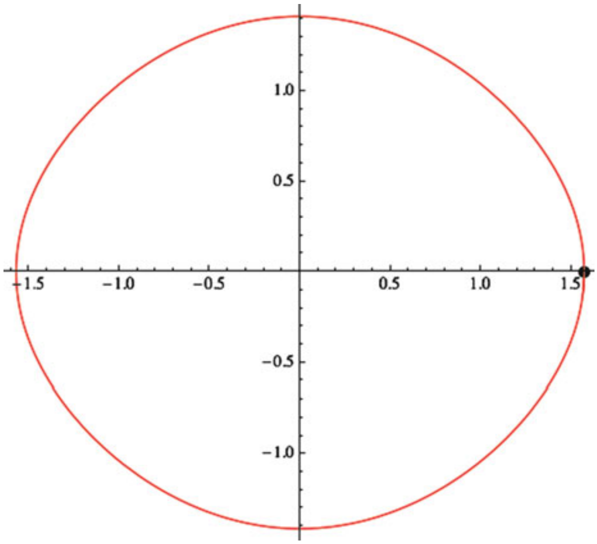
We can integrate Eq. (6.52) to give us the orbit. The initial conditions are related to the energy. If we start from $\cos\theta = 0$, then the initial value of $\dot{\theta}$ equals $\surd(2E'_0)$ (when $E'_0 > 0$). We can solve Eq. (6.51) for $\dot{\theta}$

$$\dot{\theta} = \pm\sqrt{2\left(E_0' + \cos\theta\right)} \tag{6.53}$$

Equation (6.51) shows that E'_0 cannot be less than -1. If $-1 < E'_0 < 1$, then θ_i is limited, and we can write those limits as $-\cos^{-1}(-E'_0) < \theta < \cos^{-1}(-E'_0)$. If $E'_0 > 1$, then θ is unlimited, but $\dot{\theta}$ cannot vanish. If it is positive at some point in the orbit it remains positive and vice versa.

Figure 6.24 shows the phase plane with a closed orbit between $-\pi/2$ and $\pi/2$ ($E'_0 = 0$). The blue dot represents the starting point for the calculation. The orbit moves in the clockwise direction, starting out down and to the left from its initial point. The pendulum oscillates back and forth between $\pi/2$ and $-\pi/2$, as we can see in Fig. 6.25. I show one period of the oscillation, which is approximately 2.4π. We saw that the period of the nonlinear pendulum is longer than the linear approximation in Chap. 2, so this is consistent with that observation.

Fig. 6.24 Phase plane for a pendulum starting from rest at $\theta = \pi/2$



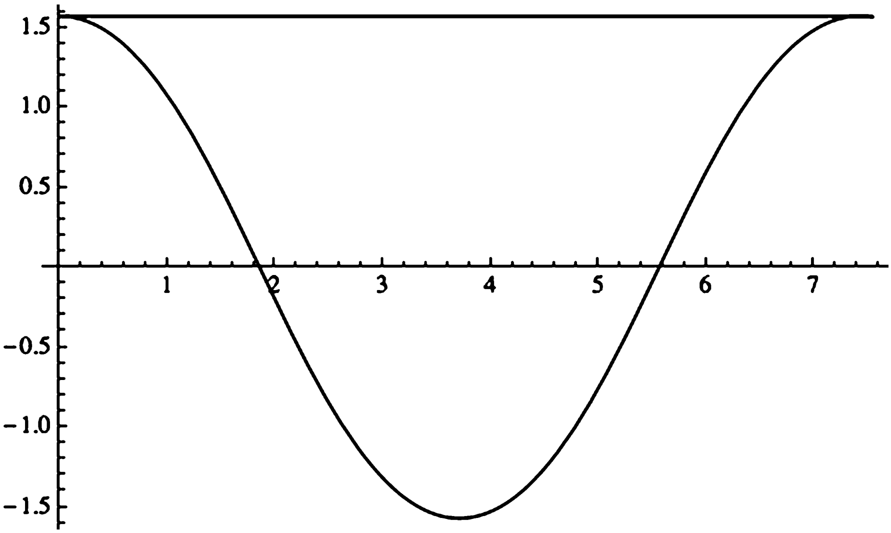


Fig. 6.25 Motion of the pendulum starting from rest at $\theta = \pi/2$. The *horizontal line* marks the initial value of $\pi/2$

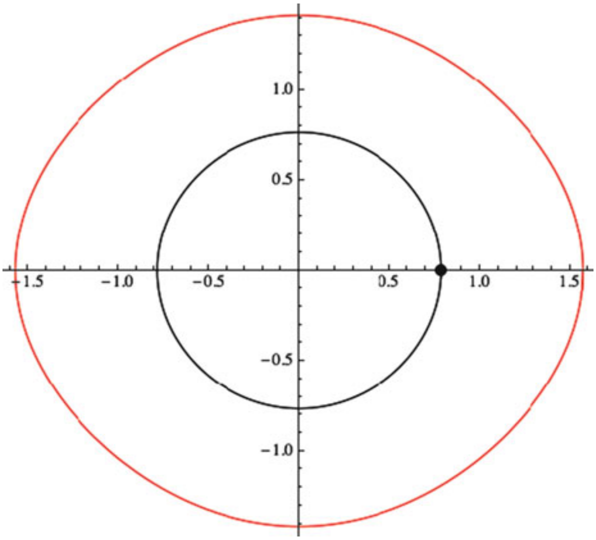


Fig. 6.26 Phase plane for the pendulum starting from rest at $\theta = \pi/4$. The *outer red orbit* is the reference orbit of Fig. 6.24

The motion of the pendulum is oscillatory. It's not harmonic, although it looks pretty harmonic. The linear pendulum is harmonic; the nonlinear pendulum is not harmonic, even though periodic. The period is greater than that of the linear approximation, as I noted above. I will refer to this state ($-\pi/2 < \theta < \pi/2$) as the *neutral orbit*, and we can use this result as a benchmark for other cases. If we start from rest at $\theta = \pi/4$, E'_0 is negative ($= -\cos(\pi/4) = -1/\sqrt{2}$), and we get a system that oscillates between $\pi/4$ and $-\pi/4$. It has a nice elliptical-looking phase plane, but the orbit is smaller than the orbit in Fig. 6.24. Figure 6.26 shows that. The period of this oscillation is approximately 2.1π.

If E'_0 is positive then the maximum excursion will be greater than $\pm\pi/2$. The pendulum will oscillate between $\pm\,\theta_0 < \pi$ if E'_0 is less than unity. If E'_0 is greater than unity, then the motion of the pendulum will never reverse. The pendulum will just spin round and round its pivot point, spending more time below the horizontal than above. We can model these by starting the system from $\theta = \pi/2$ with an initial speed less than and greater than $\sqrt{2}$. Figure 6.27 shows the phase plane for an initial $\dot{\theta} = 2 - 0.01$, which gives a closed orbit. Figure 6.28 shows the angle vs. time over period (approximately 4.93π). Figure 6.29 shows the phase plane for an initial $\dot{\theta} = 2 + 0.01$, for which the orbit cannot close because $\dot{\theta}$ cannot vanish.

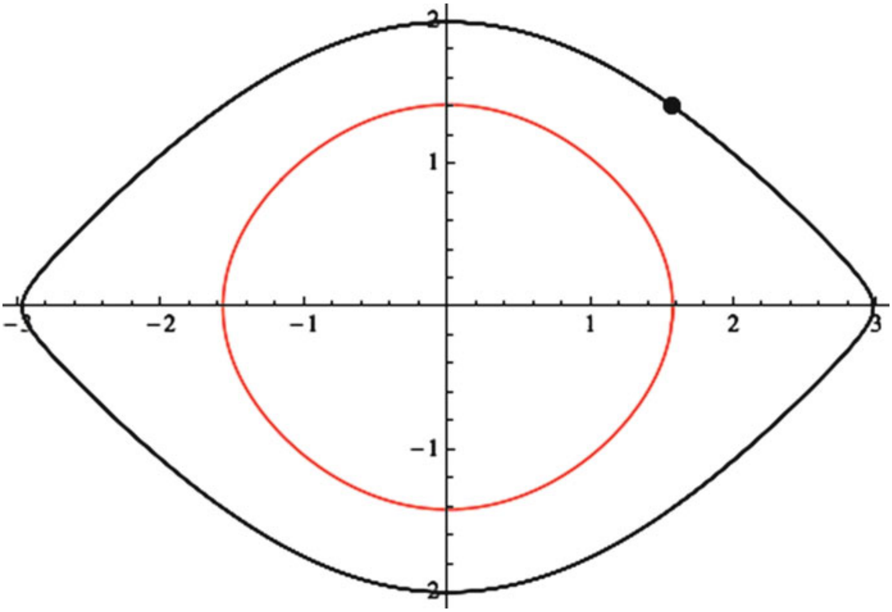


Fig. 6.27 Phase plane for the pendulum starting at $\theta = \pi/2$, with an initial rotation rate of $\sqrt{2} - 0.01$. The *red curve* denotes the neutral orbit

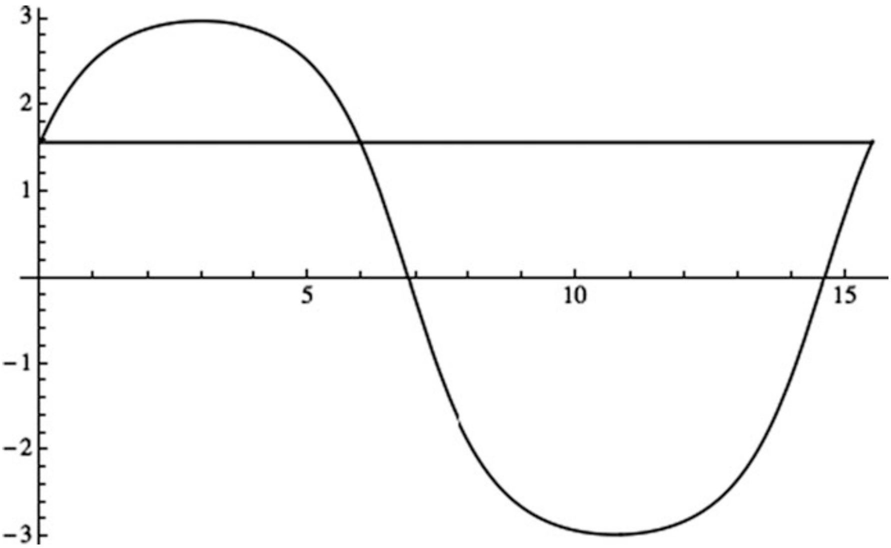


Fig. 6.28 Motion of the pendulum starting at $\theta = \pi/2$, with an initial rotation rate of $\sqrt{2} - 0.01$. It spends more time near the bottom position than it does near the top

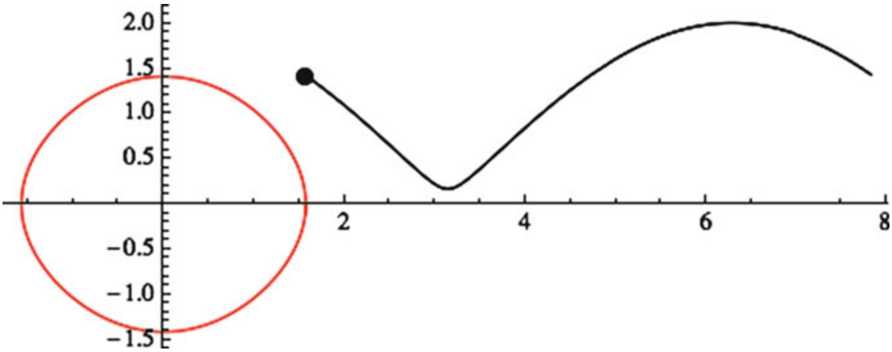


Fig. 6.29 Phase plane starting at $\theta = \pi/2$, with an initial rotation rate of $\sqrt{2} + 0.01$

We should note that the phase diagram in Fig. 6.29 is periodic in the rotation rate, but the mean is nonzero.

6.5.2 The van der Pol Oscillator

The classical illustration of the power of the phase plane picture is the van der Pol oscillator, from electrical engineering. It is a nonlinear second-order system that obeys the van der Pol equation

$$\ddot{y} - \mu\left(1 - y^2\right)\dot{y} + y = 0 \tag{6.54}$$

Figure 6.30 shows a trajectory of the van der Pol oscillator with $\mu = 1$ starting from rest at $y = 1$. I calculated the trajectory numerically using the state space version of the van der Pol equation

$$\begin{aligned} \dot{y} &= v \\ \dot{v} &= \mu(1 - v^2)v - y \end{aligned} \tag{6.55}$$

The vertical lines represent $y = \pm 1$. Outside these lines the system is stable with positive damping. Within the lines it is unstable with negative damping. The system goes back and forth between the two conditions and settles on a closed orbit, called a *limit cycle*. Note that unstable need not mean that y is growing. What grows is the distance in the phase plane from the origin. Figure 6.31 shows this variable plotted against y. You can see growth between $-1 < y < 1$ and shrinkage outside. We can look at this with a wee bit more rigor. Multiply Eq. (6.54) by $\dot{y}$ and rearrange to obtain Eq. (6.56)

$$\frac{1}{2}\frac{d}{dt}\left(\dot{y}^2 + y^2\right) = \mu\left(1 - y^2\right)\dot{y}^2 \tag{6.56}$$

which shows the growth for $y^2 < 1$ and the decay for $2 < 1$ and the decay for $y^2 > 1$.

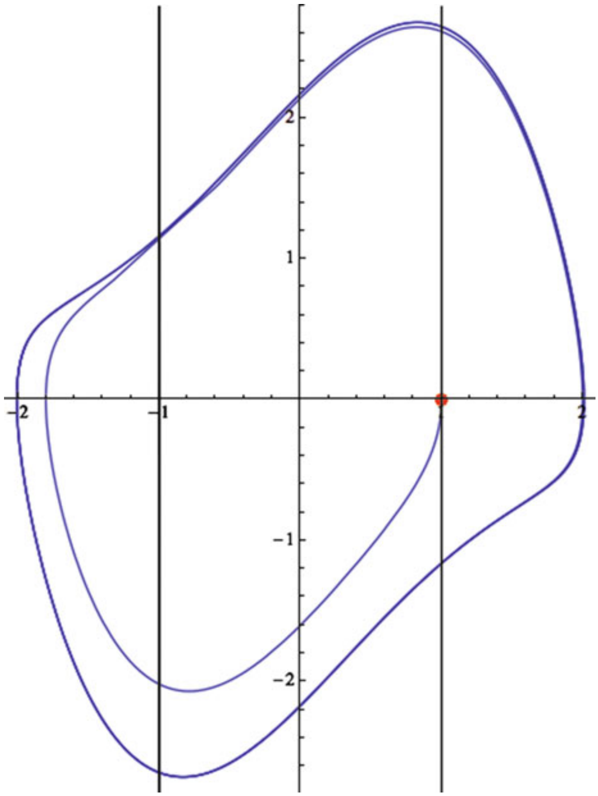


Fig. 6.30 The trajectory of a van der Pol oscillator. The *red dot* indicates the initial condition. The *vertical lines* indicate $y = \pm 1$, the stability limits. The motion on the trajectory is in the clockwise sense

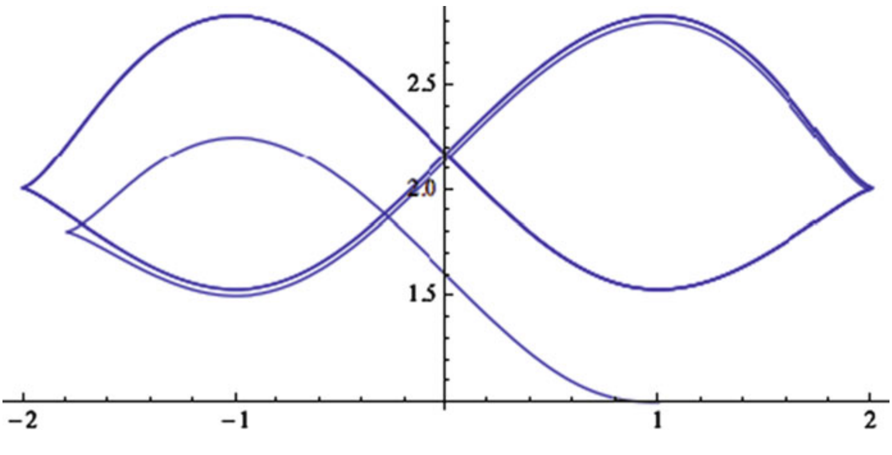


Fig. 6.31 Distance from the origin vs. y

6.6 Summary

We have looked at how to formulate electromechanical problems in state space. We can use the Euler-Lagrange method to find the dynamical equations and the generalized forces. We can convert these to state space form by defining the derivatives of the generalized coordinates as a new set of coordinates. This leads in general to a set of quasilinear first-order differential equations of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}$$

where $\mathbf{u}$ denotes a multidimensional input vector, $\mathbf{f}$ a vector function of the state, and $\mathbf{B}$ a matrix that may depend on the state variables. We will usually work with the single-input system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})u$$

for which $\mathbf{b}$ denotes a vector and u a scalar input. We looked at numerical solution of these equations. Unless the differential equations are linear, we need to linearize them to obtain analytic solutions. I discussed linearization in the state space context in Sect. 6.3. The linearized equations may be written

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where $\mathbf{A}$ and $\mathbf{B}$ are constant matrices. The vector $\mathbf{u}$ becomes a scalar u, and the matrix $\mathbf{B}$ becomes a vector $\mathbf{b}$ for a single-input system.

We looked at two methods of solving the linear state space problem: the state transition matrix (of which we will learn more in Chap. 7), and diagonalization. I introduced companion form, but we did not do much with it. It will reappear as an important form in Chap. 8. I also noted that the linear problems need not be stable. We learned that stability is determined by the eigenvalues of $\mathbf{A}$.

6.6.1 Four Linear Electromechanical Systems for Future Reference

I have provided several examples in the text so far. I'd like to collect four of them here for future reference: the simple inverted pendulum, the magnetic suspension, the low-inductance overhead crane, and the high-inductance overhead crane with gravity inverted to give the problem of the inverted pendulum on a cart. These are, respectively, two-, three-, four-, and five-dimensional systems. All but the first system is naturally nonlinear, so I had to linearize them to find linear state space problems for us to consider. The first and the third systems lead to marginally stable problems; the second and fourth systems lead to unstable problems. These are all single-input systems, so we are working with Eq. (1.1b) (Eq. 6.3). I give the state vector and $\mathbf{A}$ and $\mathbf{b}$ for all four systems.

6.6.1.1 S1. The Servomotor (from Ex. 6.4)

We can copy the servomotor directly from Ex. 6.4. That result is

$$\mathbf{x} = \begin{Bmatrix} \theta \\ \dot{\theta} \end{Bmatrix}, \quad \mathbf{A} = \begin{Bmatrix} 0 & 1 \\ 0 & -K\alpha \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ \alpha \end{Bmatrix} \tag{6.57}$$

where the input is the motor voltage, and I have neglected inductance effects. This is a linear problem as posed.

6.6.1.2 S2. Magnetic Suspension

I introduced the problem of suspending a ferromagnetic object beneath an electromagnet in Chap. 3, showing it to be an unstable configuration. I had not yet introduced state space as a regular tool, but it is easy to show that the perturbation Eqs. (3.34) and (3.35) can be converted to state space form

$$\mathbf{x} = \begin{Bmatrix} z' \\ \dot{z}' \\ i' \end{Bmatrix}, \quad \mathbf{A} = \begin{Bmatrix} 0 & 1 & 0 \\ -n\dfrac{g}{z_0} & 0 & 2\dfrac{g}{i_0} \\ 0 & 0 & -\dfrac{R}{L} \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{1}{L} \end{Bmatrix} \tag{6.58}$$

where i_0 is defined by Eq. (3.27) for any given value of z_0. Equation (6.28) is a linear equation for the departures from equilibrium, so it has a solution for $\mathbf{x} = \mathbf{0}$ with $u = 0$, u being the "extra" voltage—the difference between e and its equilibrium value.

6.6.1.3 S3. The Low-Inductance (Low Speed) Overhead Crane (Ex. 6.11)

I introduced both versions of the overhead crane in this chapter. Equations (6.38), (6.40), and (6.42) define the low-inductance version. The state space representation of the linearized equations of motion is then

$$\mathbf{x} = \begin{Bmatrix} y \\ \theta \\ \dot{y} \\ \dot{\theta} \end{Bmatrix}, \quad \mathbf{A} = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{m}{M}g & -\dfrac{K^2}{Mr^2R} & 0 \\ 0 & -\dfrac{(M+m)}{Ml}g & \dfrac{K^2}{lMr^2R} & 0 \end{Bmatrix}, \tag{6.59}$$

$$\mathbf{b} = \frac{K}{lrRM}\begin{Bmatrix} 0 \\ 0 \\ -l \\ 1 \end{Bmatrix}$$

where I have set the pivot damping $c = 0$, and the other parameters are as described in the example. The input here is the motor voltage. The equilibrium is $\mathbf{x} = \mathbf{0}$ with $e = 0$.

6.6.1.4 S4. An Inverted Pendulum on a Cart (from Ex. 6.12)

We must allow for rapid response for any control that we will eventually find to keep an inverted pendulum erect. Therefore we need to include the inductance if we want to stabilize an inverted pendulum on a motor-driven cart.

Figure 6.32 shows a model of the physical system. We know that we can get the equations for this system by simply changing the sign of gravity in the high-inductance version of the overhead crane, Ex. 6.12. Thus we can simply modify Eq. 6.46 to get the equations we need.

$$\mathbf{x} = \begin{Bmatrix} y \\ \theta \\ \dot{y} \\ \dot{\theta} \\ i \end{Bmatrix}, \quad \mathbf{A} = \begin{Bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\dfrac{mg}{M} & 0 & 0 & -\dfrac{K}{Mr} \\ 0 & \dfrac{(m+M)g}{Ml} & 0 & 0 & \dfrac{K}{Mlr} \\ 0 & 0 & \dfrac{K}{L} & 0 & -\dfrac{R}{L} \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{1}{L} \end{Bmatrix} \tag{6.60}$$

These equations are linearized about an unstable equilibrium at $\mathbf{x} = \mathbf{0}$, $e = 0$, where e denotes the motor voltage, which is the input to the system.

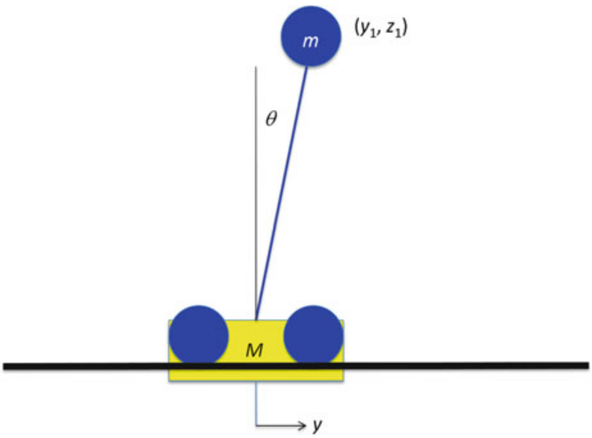


Fig. 6.32 The inverted pendulum on a cart

Exercises

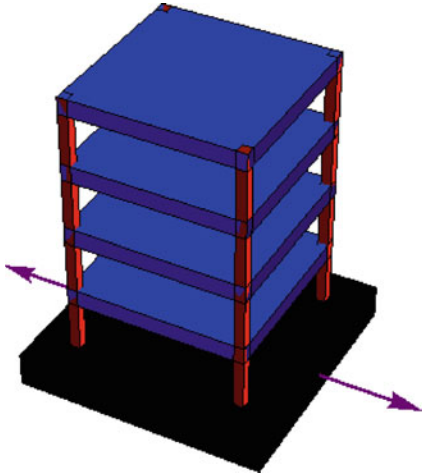
1. Use Eq. (6.5) to find the components of **B** for the problem defined by Eq. (6.3).
2. Let $A = -1$ and $B = 1$ in Eq. (6.7). Find the solution of x if $x(0) = 0$, and $u = \sin(\omega t)$ using Eq. (6.9).
3. Show that the series definition of $\exp(\mathbf{A}t)$ leads to the correct expression for its derivative.
4. Verify the differential equations given for Ex. 6.2.
5. Find the scalar block diagram for Ex. 6.2.
6. Use the state transition matrix in Ex. 6.2 to find the response if $e = \sin(\omega t)$.
7. Show that I have given the correct terms for the state transition matrix in Ex. 6.3.
8. Find the state transition matrix for Ex. 6.4.
9. Show that $\mathbf{x}_H$ given as the homogeneous solution to Ex. 6.8 is real.
10. Solve Ex. 6.8 using modal analysis and compare the modal frequencies and modal vectors to the eigenvalues and eigenvectors.
11. Add a damper between the first and second pendulums in Ex. 6.8 and find the steady response for an excitation at $w = 1$. What is the maximum input amplitude for which both pendulum angles remain less than 5°?
12. The table below shows the time and amplitude of the successive positive peaks shown in Fig. 6.15. Find an effective natural frequency and damping ratio for a one degree of freedom model of these data.

Table 6.2 Successive maxima vs. time

Time	Amplitude	Time	Amplitude
1.047	0.1482	10.481	0.02693
3.429	0.08224	12.825	0.019664
5.787	0.05340	15.169	0.01439
8.136	0.03735	17.512	0.01055

13. Show that the magnetic suspension system described in Chap. 3 can be rewritten as Eq. (6.31).
14. Draw a scalar block diagram for the linearized magnetic suspension problem.
15. Find the state space equations governing the motion of an inverted pendulum with a torque at its base if the torque is provided by an electric motor and the inductive contribution is important.
16. Find the eigenvalues and eigenvectors of the matrix **A** given by Eq. (6.33) using the constant values given in Table 6.1.

17. Consider a four-story building like the ten-story building of Ex. 4.2a and b.



Suppose the building to be dissipation-free. Find the eigenvalues and eigenvectors and compare them to the eigenvalues and eigenvectors of a linearized quadruple pendulum.

18. Write the linear state space equations for the triple pendulum. Find its eigenvalues and eigenvectors, and compare these to the modal frequencies and mode shapes given in Ex. 4.1. Draw a scalar block diagram for the linear system.
19. Write the linear state space equation for a quadruple pendulum about its stable equilibrium. Find the eigenvalues and eigenvectors. You may suppose the pendulums to be equal simple pendulums l units long with bobs of mass m.
20. Use the phase plane and the differential equations to discuss the van der Pol equation if $\mu < 0$. Which starting positions lead to stability and which lead to instability.
21. Redo Ex. 4.3a. in state space. Compare the eigenvalues and eigenvectors to those for Probs. 15 and 16. How do they compare to the modal frequencies and mode shapes?

The following exercises are based on the chapter, but require additional input. Feel free to use the Internet and the library to expand your understanding of the questions and their answers.

Exercises 22–25 require you to set up the ten-story building from Chap. 4 in state space.

22. Design a vibration absorber to reduce the response of the ten-story building to ground motion?
23. Suppose the ten-story building can move sideways with respect to the ground when excited. What is the response of the building to a harmonic disturbance with a sudden onset lasting only one period?

24. Find the response of the building in exercise 23 if the sideways motion is limited by linear springs.
25. Find the response of the building in exercise 24 if the sideways motion is limited by linear springs and dampers. Suppose the effective damping ratio to be $1/\sqrt{2}$.
26. Compare the eigenvectors for the free oscillation of the ten-story building to the modal vectors found in Chap. 4.

References

Boyce WE, DiPrima RC (1969) Elementary differential equations, 2nd edn. Wiley, New York, NY

Friedland B (1986) Control system design: an introduction to state space methods. Dover reprint 2007

Gillespie TD (1992) Fundamentals of vehicle dynamics. SAE, Warrendale, PA

Press WH, Teukolsky SA, Vetterling WT, Flannery BP (1992) Numerical recipes in C. The art of scientific computing 2nd edn. Cambridge U Press

Strang G (1988) Linear algebra and its applications, 3rd edn. Harcourt Brace Jovanovich, San Diego, CA

Classical Control

7

In which we look at the basics of the control of single-input–single-output systems from an intuitive time domain perspective, move on to the frequency domain, and introduce the idea of transfer functions . . .

7.1 Introduction

7.1.1 What Do We Mean by Control?

The idea of control is to find input(s) to a dynamical system such that the output(s) of the system do what you want them to do in the presence of unknown disturbances. This most general situation is shown in Fig. 7.1. The *plant* is the actual dynamical system, which I will suppose that we are able to model. It could be a robot or an autonomous vehicle, or any other electromechanical system that needs to be controlled. The input will include both the control that we wish to implement and any unavoidable disturbances that the world provides. This very general problem is beyond the scope of the text, but the general picture covers everything we will address.

The control part is made up of two parts: an *open loop* part and a *closed loop* part. The open loop part is simply a best guess. It is sufficient only in a perfect world, where the initial conditions can be perfectly imposed and there are no disturbances. The closed loop part compares the actual output to the desired output and uses the error, the difference between the two, as input to a control system. We call this feedback control. Figure 1.3 shows how this works in reality. The GOAL is what we want the system to do, the ideal output. The INVERSE PLANT calculates the input required to make this happen in the best of all possible worlds. One can see the comparison between the ideal output and the actual output. This difference is fed back to the input through the box labeled CONTROL. Finally one can see that the input to the plant is the sum of the ideal input (the open loop part), the

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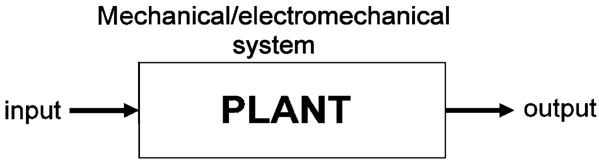


Fig. 7.1 A general situation calling for control

disturbance, and a feedback contribution depending on the error, coming from the box labeled CONTROL.

Dynamical models of real systems are often nonlinear. Direct nonlinear control of nonlinear systems is beyond the scope of this text, although I will address some special cases of direct nonlinear control in Chap. 11. This chapter will deal with the control of linear systems, or systems that can be linearized. We will need to be able to find nonlinear equilibrium results in order to be able to linearize. Remember that we must have equilibrium in order to linearize a system. The equilibrium corresponds to open loop control in some cases. We will also wish to preserve the nonlinear plant so that we can assess how the linear controls we design will work when applied to the nonlinear system, by simulating the nonlinear system using numerical integration. If a control does not work in simulation, it is very unlikely to work in real life and will need to be redesigned. (Effectiveness in simulation does not guarantee effectiveness in reality. The model of the plant may be defective.) This will be the limit of our analysis of nonlinear problems until Chap. 11: linearization to allow control design and simulation (numerical integration).

Linear systems can be analyzed in terms of a set of ordinary differential equations with time as the independent variable. This is called *time domain* analysis. This is what we have been doing so far. We can also analyze linear systems using the Laplace transform. This is called *frequency domain* analysis. Classical control is dominated by frequency domain analysis using the Laplace transform and transfer functions in the frequency domain. I prefer the time domain, but much of the development of control theory took place in the frequency domain, and the language of frequency domain analysis is common in the literature, so we need to look at both. I will start with simple linear systems with simple goals and proceed to more complicated problems as we go along. Most of the detailed analysis in this chapter will be concerned with single-input systems, and often with single-input–single-output (SISO) systems.

Constant goals are easier to deal with than time-dependent goals, and I will restrict this chapter to constant goals, generally a constant desired output. I will also restrict this chapter to linear systems with a single input and a single output. The output will then be given by $y = \mathbf{c}^T\mathbf{x}$, so a constant output generally implies a constant state. The state equation for a constant state reduces to

$$0 = \mathbf{A}\mathbf{x}_0 + \mathbf{b}u_0$$

where $\mathbf{x}_0$ and u_0 denote the equilibrium.[1] Both are often zero. The equilibrium can be stable or unstable. We can write $\mathbf{x} = \mathbf{x}_0 + \mathbf{e}$, $y = y_0 + y'$, and $u = u_0 + u'$, and form the equation for the error $\mathbf{e}$.

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{b}u', \quad y' = \mathbf{c}^{\mathrm{T}}\mathbf{e} \tag{7.1}$$

We want y' to go to zero, and I will do this by requiring $\mathbf{e}$ to go to zero (which is more than sufficient). Equation (7.1) is a state space model for error dynamics. There are other models for error dynamics, and this chapter will look at two: straightforward analysis in the time domain of the original differential equations and analysis in the frequency domain, which requires the use of the Laplace transform. I will start with the simplest cases, and when we have a better understanding of feedback control, I will go on to the frequency domain analysis. The alert reader will note strong similarities among all the approaches. I will point some out as we go along.

We will discover shortly that we almost always need to use the equivalent of the entire state to design an effective feedback control. If the entire state (or its equivalent) is not available (cannot be measured), then we must estimate the missing parts. The estimate of the state is called an *observer*, and I will discuss observers in Chap. 9 in the context of state space control.

Time-dependent goals fall under the general rubric of *tracking control*, which I will deal with in Chap. 10, again in the context of state space control.

The very simplest ideas in control can be conveyed in the time domain without particularly sophisticated analysis. I will begin with the well-known PID control of a second-order system, such as a one degree of freedom mechanical system.

7.1.2 PID Control of a Single-Input–Single-Output System

7.1.2.1 P Control

Mechanical systems are based on Newton's law (or its equivalent formulation using the Euler-Lagrange equations), and so the simplest basic system will be one second-order ordinary differential equation. The simplest such system is

$$\ddot{y} = u \tag{7.2}$$

where u denotes a single input and I will take y as the single output. The input is an acceleration and the output a displacement for a physical mechanical system. The right-hand side of Eq. (7.2), the input, comes from some force. Some effort is required to control the system. I will refer to the size of the control input force as the *control effort*. Any real system has a maximum possible control effort, limited by whatever the real source of the control is: how powerful a motor is available, say?

[1] This is also a special case of open loop control. I'll pick this up in Chap. 8.

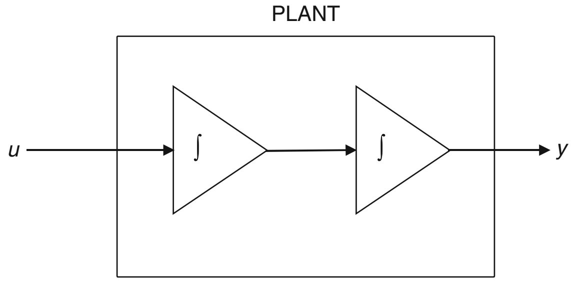


Fig. 7.2 The simplest second-order SISO system

This means that any control we design must take this into account. A control may work mathematically, but not practically.

We can work directly with Eq. (7.2) and take an intuitive approach for now. The plant is simply a double integration. Figure 7.2 shows its simple block diagram.

Suppose that y is initially equal to Y_0, and we want to make it go to zero automatically. Suppose for the nonce that there is no disturbing input. The equilibrium input is zero. We require a control input to move the output y from Y_0 to zero. If y is greater than zero we want to make it smaller, and if y is less than zero we want to make it bigger. This suggests that we ought to try $u = -g_P y$. The differential equation becomes

$$\ddot{y} + g_P y = 0 \tag{7.3}$$

The parameter g_P is called the *proportional gain* because it contributes a term to the feedback that is proportional to the output. The proportional gain obviously has the dimensions of the square of a frequency. If the system starts from rest ($\dot{y}(0) = 0$) at $y = Y_0$, then the solution to the differential equation is harmonic:

$$y = Y_0 \cos\left(\sqrt{g_P}t\right)$$

This control fails to reduce the initial error to zero. Its long-term average is zero, but the motion caused by the initial offset goes on forever.

7.1.2.2 PD Control

There is no dissipation in this problem, but we know from Chap. 2 that adding a term proportional to the derivative of y to the differential equation, Eq. (7.3), will introduce damping (if the proportionality constant is positive), so we add such a term to the feedback:

$$u = -g_P y - g_D \dot{y} \tag{7.4}$$

where g_D is called the *derivative gain*. It has the dimensions of frequency. Our differential equation becomes

$$\ddot{y} + g_D\dot{y} + g_P y = 0 \tag{7.5}$$

This is, of course, the same as Eq. (2.4) with a zero right-hand side, so we know the solution to this as well.

We know from Chap. 2 that the output y will go to zero most rapidly if the damping ratio is close to unity. (We can design for unity and hope to be close.) Comparing the terms in Eq. (7.5) to those in Eq. (2.4) we find

$$\omega_n^2 = g_P, \quad 2\zeta\omega_n = g_D \Rightarrow g_D = 2\zeta\sqrt{g_P} = 2\zeta\omega_n \tag{7.6}$$

We cannot choose the two gains independently. The derivative gain is limited by the proportional gain. If we suppose that ζ is close to but less than unity we can use Eq. (2.17) to express the solution. If we let $\zeta =$ unity, then we can use the critically damped solution, which is

$$y = Y_0(1 + \omega_n t)\exp(-\omega_n t), \quad \dot{y} = -\omega_n^2 t Y_0 \exp(-\omega_n t)$$

The bigger g_P (the higher the introduced natural frequency) the faster the output goes to zero. We cannot, however, simply set g_P as big as we'd like, because the bigger the gains, the bigger the control effort. I leave it to you to confirm that the control effort for the critically damped case is proportional to g_P $(= {\omega_n}^2)$:

$$u = -\omega_n^2 Y_i(1 - \omega_n t)\exp(-\omega_n t)$$

The maximum value of the control effort is $-{\omega_n}^2 Y_i$, occurring at $t = 0$.

We see that we need both proportional and derivative control for this simplest of all possible second-order problems, and we have seen how to choose the two parameters in Eq. (2.4) by choosing the gains. If we have sufficient control force available, we seem to be able to do anything we want.

There is another way of looking at this. We know that the solution of the problem under PD control is always in the form of exponential functions, and the exponents come from the characteristic equation, Eq. (2.12b), or, in the present notation:

$$s^2 + g_D s + g_P = 0$$

The coefficients of this equation are related to the roots of the equation (you may expand the product yourself)

$$s^2 + g_D s + g_P = 0 = s^2 - (s_1 + s_2)s + s_1 s_2 \tag{7.7}$$

so that we can write the roots in terms of the gains and vice versa. Thus we can choose the roots of the characteristic equation by choosing the gains. Choosing the roots of the characteristic polynomial is called *pole placement*. It will play a significant role in our control strategies from here onward. The idea can be extended to systems of any order, and I will do so. Remember that complex roots must be chosen in complex conjugate pairs.

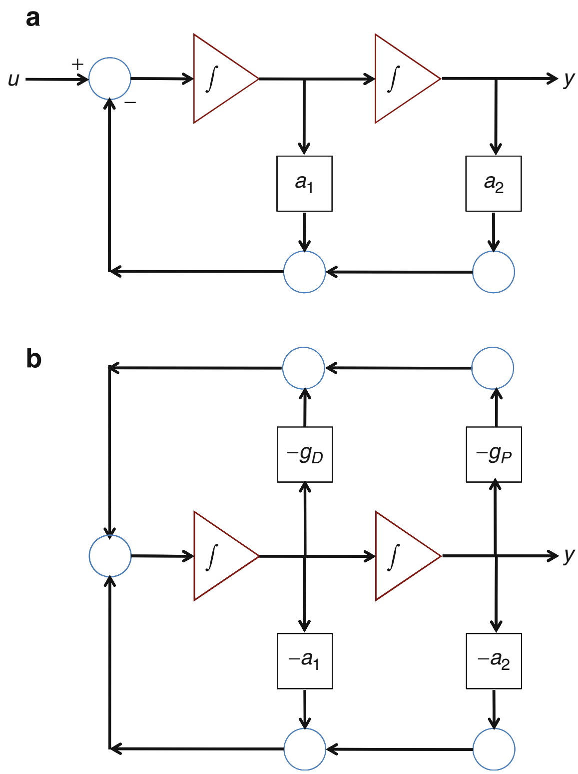


Fig. 7.3 (**a**) The open loop system of Eq. (7.8a). (**b**) The closed loop system of Eq. (7.8b)

We can extend PD to a general second-order system:

$$\ddot{y} + a_1\dot{y} + a_2 y = u \tag{7.8a}$$

We can still choose u to have negative feedback from both y and its derivative (as in Eq. 7.4), making the differential equation:

$$\ddot{y} + (a_1 + g_{\mathrm{D}})\dot{y} + (a_2 + g_{\mathrm{P}})y = 0 \tag{7.8b}$$

The extended version of Eq. (7.6) is then

$$g_{\mathrm{P}} = \omega_n^2 - a_2, \quad g_{\mathrm{D}} = 2\zeta\omega_n - a_1 \tag{7.9}$$

The gains correct for the constants in Eq. (7.8a) as well as giving the roots we want. If the homogeneous version of Eq. (7.8a) is unstable, then the control actually stabilizes the system. The control effort depends only on the gains. We can draw the block diagrams of the open loop (uncontrolled) and closed loop (controlled) versions of this extended second-order problem as Fig. 7.3a, b.

The addition of a_1 and a_2 makes no qualitative difference in the analysis. I will generally work with $a_1 = 0 = a_2$ in this section in the interest of clarity.

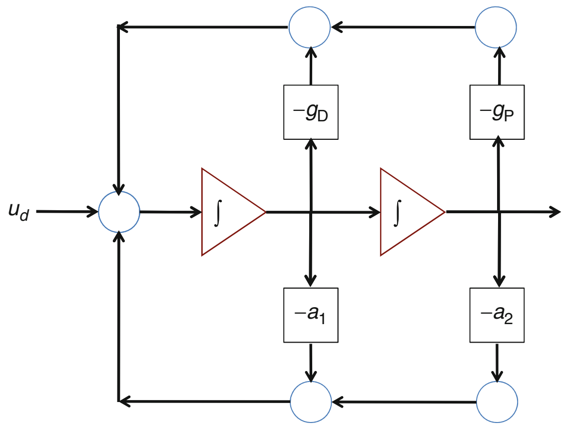


Fig. 7.4 The controlled system Eq. (7.8b) with an added disturbance

7.1.2.3 Disturbances

What happens when the system is disturbed by an unknown external force, which I will denote by u_d? Figure 7.4 shows this system, the controlled system of Fig. 7.3b with an added disturbance input. I will consider this in isolation, supposing that the system starts at its desired equilibrium, and I will drop a_1 and a_2 as unnecessary, working with Eq. (7.5) in the form of Eq. (2.4):

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = u_{\mathrm{d}}$$

where the proportional and derivative gains determine ω_n and ζ. We clearly need to retain the PD feedback so that when the disturbance ends y will go to zero. The solution can then be written in terms of the underdamped impulse solution (Eq. 2.24a), using Eq. (2.25) with the underdamped kernal function:

$$y = \int_0^t \exp(-\zeta\omega_n(t-\tau))\frac{\sin\left(\omega_{\mathrm{d}}(t-\tau)\right)}{\omega_{\mathrm{d}}}u(\tau)d\tau \tag{7.10}$$

We can get some idea of how this works by looking at harmonic disturbances. Let $u = \sin(r\omega_n t)$, where r denotes the ratio of the forcing frequency to the natural frequency as it did in Chap. 2. We only care about the long-term (particular) solution, which is

$$y_{\mathrm{P}} = \frac{U}{\omega_n^2}\left(\frac{1-r^2}{\Delta}\sin\left(r\omega_n t\right) - \frac{2\zeta r}{\Delta}\cos\left(r\omega_n t\right)\right)$$

where

$$\Delta^2 = \left(1-r^2\right)^2 + (2\zeta r)^2$$

(This expression should look familiar.)

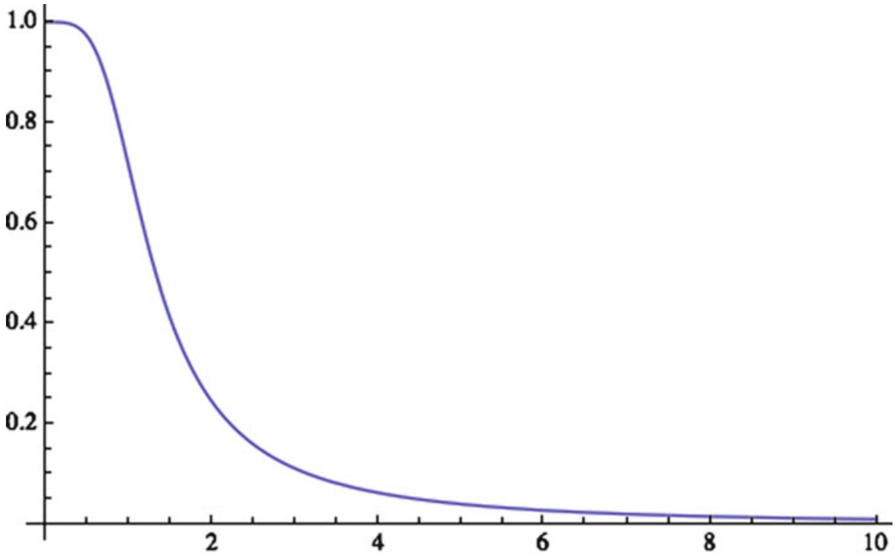


Fig. 7.5 Scaled amplitude of the output vs. r

For small r the control works poorly; low-frequency disturbances disturb the systems seriously. Figure 7.5 shows the scaled amplitude of y_P:

$$\frac{\omega_n^2 y_P}{U} = \frac{1}{\Delta}$$

vs. r for a damping ratio of 0.707. This is a large enough damping ratio to suppress the resonance effect at $r = 1$. The control works very well to suppress high-frequency disturbances, but fails to suppress low-frequency disturbances.

If low frequencies cannot be suppressed, what happens for a constant disturbance (zero frequency)? I will denote this constant disturbance by U. This question is actually quite easily answered in generality, independent of the values of the gains. Equation (7.5) becomes

$$\ddot{y} + g_D \dot{y} + g_P y = U$$

Its particular solution is $y = \frac{U}{g_P}$. There is a permanent offset. We know that the homogeneous solution will vanish as time increases (if the two gains are both positive), but this does not remove the permanent offset. The offset can be reduced by increasing the proportional gain, but we know that the gain cannot be increased without bound because increasing the gain increases the control effort. We appear to be stuck with a permanent error if the disturbance has a constant term.

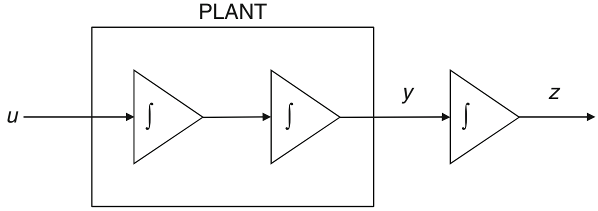


Fig. 7.6 The simple second-order system with an added integrator

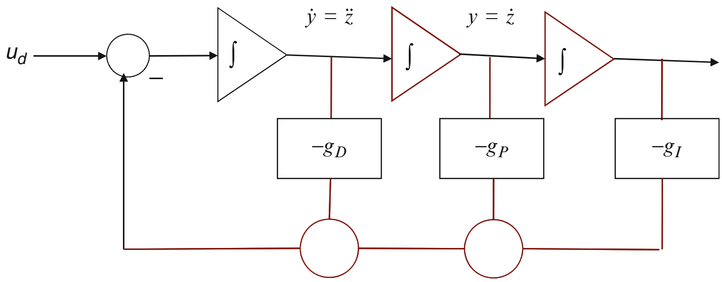


Fig. 7.7 The third-order version of the second-order system

7.1.2.4 PID Control

We can fix this by adding an integrator (which increases the order of the system). Let $y = \dot{z}$. (z is the integral of y.) The original second-order differential equation for y becomes a third-order differential equation for z: $\dddot{z} = u$. Figure 7.6 shows the block diagram of the new system.

We can add feedback from z. Figure 7.7 shows the system with all the feedbacks. The gain of the new feedback, g_I, is called the *integral gain*, which has the dimensions of frequency cubed. The combination of all three feedbacks gives the full PID controller.

We can add the integrated term to the feedback control: $u = -g_P\dot{z} - g_D\ddot{z} - g_I z$. The differential equation becomes

$$\dddot{z} + g_D\ddot{z} + g_P\dot{z} + g_I z = u_d \tag{7.11}$$

and the particular solution for a constant disturbance is $z = \frac{U}{g_I}$. We don't care about z, so we now have a system that drives the output y to zero at the expense of one integration. This final control is called *PID control*. Loosely speaking: the proportional part of the control drives the system in the opposite direction from the error, but overshoots, the derivative part drives the overshoot to zero, and the integral part removes any constant error. This is not a difficult control to implement. If we can measure y, then it is easy to integrate y to get z.

Equation (7.11) describes the behavior of a system with a PID control subject to a disturbance, and we are to design a set of gains to minimize the response to the disturbance. Recall that the response is y, corresponding to the derivative of z.

We have seen that introducing the integral gain g_I is able to suppress the effect of the forcing on y, transferring the constant error to z, its integral. (I will show how this idea can be applied to cruise control later in this chapter, and I will extend this idea to more complicated systems in Chap. 8). What can we say about this problem in more generality? How does the y response to a harmonic excitation depend on the gains? Is there a clever way to choose the gains to minimize this response?

Suppose u to be a unit harmonic function: $u = \sin(\omega t)$. The particular solution to Eq. (7.11) will also be harmonic:

$$z = A\cos(\omega t) + B\sin(\omega t) \tag{7.12}$$

This is a linear problem; the amplitude of the response z (not the output y) will be proportional to the amplitude of the forcing. The frequency of the response is, of course, the same as the forcing frequency and the amplitude of the response will be $Z = \sqrt{A^2 + B^2}$.

I can find A and B by simply substituting the general particular solution Eq. (7.12) into Eq. (7.11) and equating the sine and cosine terms to zero. I leave it to you to verify that the result is

$$A = -\omega \frac{g_P - \omega^2}{\Delta}, \quad B = \frac{g_I - g_D\omega^2}{\Delta} \tag{7.13}$$

where the denominator

$$\Delta = \omega^6 + \left(g_D^2 - 2g_P\right)\omega^4 + \left(g_P^2 - 2g_D g_I\right)\omega^2 + g_I^2 \tag{7.14}$$

The y response is the derivative of z:

$$y = -\omega A\sin(\omega t) + \omega B\cos(\omega t) \tag{7.15}$$

and its amplitude $Y = \omega Z$. This would be an easy problem were it possible to choose the numerators of A and B to be zero, apparently eliminating the response entirely. Unfortunately, if the numerators are both zero, so too is the denominator (try it and see), and the limits are infinite. We see that the y response tends to zero as ω goes to zero or infinity, but choosing the gains working from Eq. (7.7) is pretty obviously a hopeless cut and try procedure. Let us see what happens if we use the pole placement method (see Eq. 7.7) to assign the gains. We know that the gains can be written in terms of the poles by comparing the characteristic equations and equating the respective coefficients. The characteristic polynomial for the third-order system is

$$s^3 + g_D s^2 + g_P s + g_I = 0$$

from which we have

$$g_{\mathrm{P}} = s_1 s_2 + s_1 s_3 + s_2 s_3, \quad g_{\mathrm{I}} = -s_1 s_2 s_3, \quad g_{\mathrm{D}} = -(s_1 + s_2 + s_3)$$

How should we assign the poles? We know that they must have negative real parts, and that complex poles must occur in complex conjugate pairs. One common choice is to use Butterworth poles, proposed by Butterworth (1930). The Butterworth poles lie on a semicircle in the left half of the complex plane. I will refer to this as the *Butterworth circle* or the *pole circle*. There are as many poles as there are dimensions in the state, and they are given by

$$s_i = \rho\left(\cos\left(\frac{\pi}{2} + \frac{i\pi}{N+1}\right) + j\sin\left(\frac{\pi}{2} + \frac{i\pi}{N+1}\right)\right) \tag{7.16}$$

where ρ denotes the radius of the pole circle and N the dimension of the state, which is the same as the order of the governing equation. The poles lie in the second and third quadrant, where the cosine is negative. They occur in complex conjugate pairs, as they must. Poles have the dimensions of frequency, so it is often convenient to take account of the dimensions of the pole circle by writing $\rho = \omega_r$.

We can fit this third-order system to its set of Butterworth poles:

$$s_1 = -\omega_r, \quad s_2 = -\frac{\omega_r}{\sqrt{2}}(1 + j), \quad s_3 = -\frac{\omega_n}{\sqrt{2}} r(1 - j)$$

Figure 7.8 shows these poles for $\omega_r = 1$. The gains associated with these poles are

$$g_{\mathrm{D}} = \left(1 + \sqrt{2}\right)\omega_r, \quad g_{\mathrm{P}} = \left(1 + \sqrt{2}\right)\omega_r^2, \quad g_{\mathrm{I}} = \omega_r^3$$

The larger the reference frequency, the radius of the poles, the larger the gains and the larger the control effort.

In order to see better what happens, let us plot the amplitudes of z and y vs. frequency for fixed pole radius, here unity.

The scaled magnitudes of z and y can be written

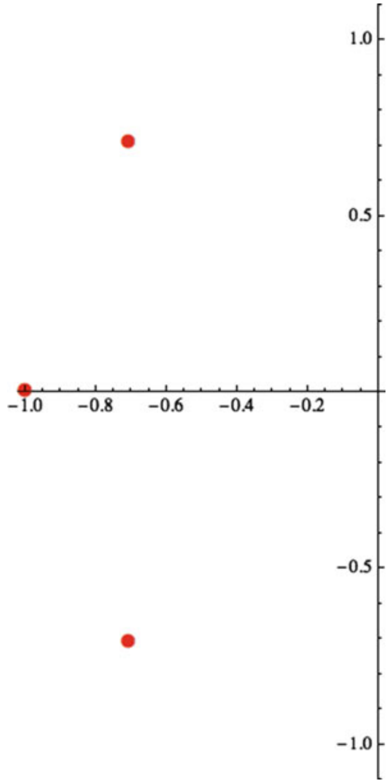
$$Y = \frac{\omega}{(1 + \omega^2)(1 + \omega^4)}, \quad Z = \frac{1}{(1 + \omega^2)(1 + \omega^4)} \tag{7.17}$$

and we can plot these as a function of r in Figs. 7.9 and 7.10.

We see that the addition of the integral gain has reduced the amplitude of y, moving some of the particular solution to z, and it preserves the elimination of the constant error in y.

Figure 7.11 shows the scaled amplitude of y as a function of ω and ω_r. We need a fairly large ω_r to suppress the response at intermediate frequencies. (There is a very large spike when both frequencies are small.) This means that the control effort will be high, but this is a topic for later. Suffice it to say now that we can use a PID controller to drive the response to a harmonic disturbance to a low level, particularly away from resonance.

Fig. 7.8 The third-order Butterworth poles



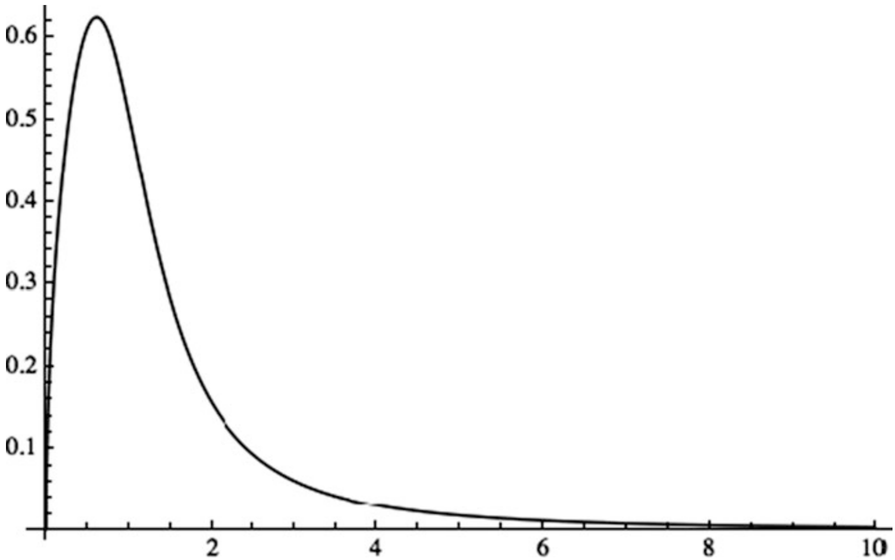


Fig. 7.9 Amplitude of y vs. r

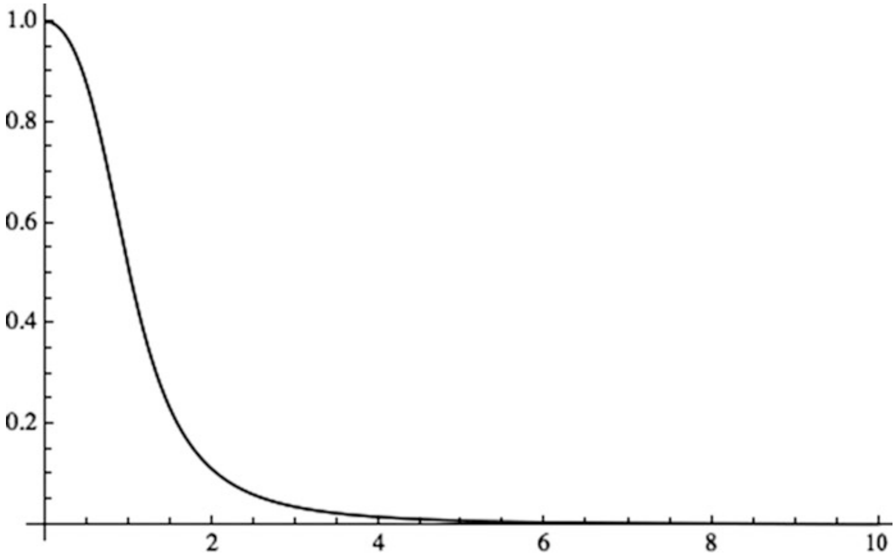


Fig. 7.10 Amplitude of z vs. r

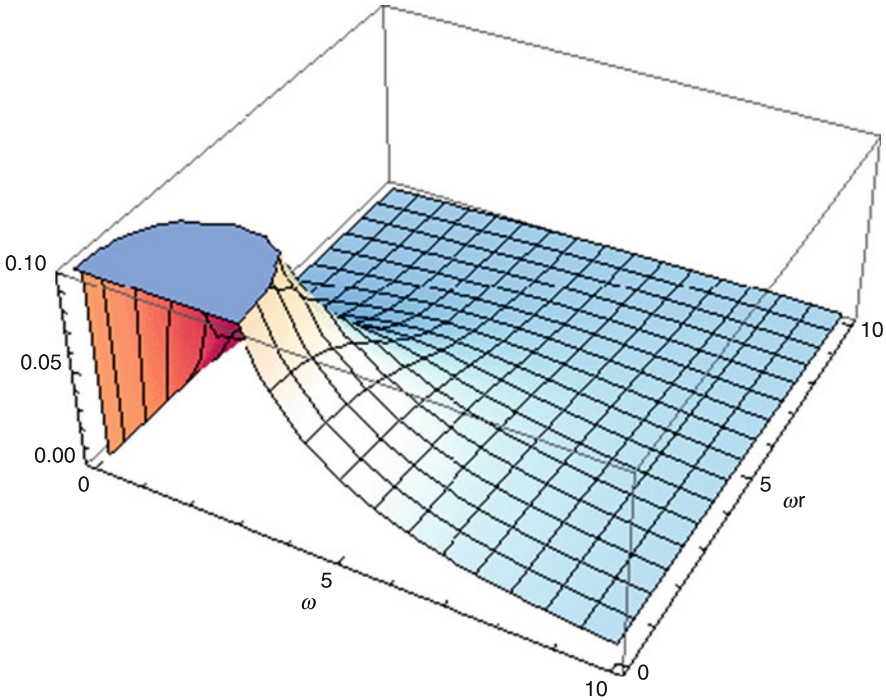


Fig. 7.11 The response (y) of a third-order PID-controlled system with Butterworth poles to a harmonic forcing vs. ω and ω_r

Let's consider a very simple example where augmenting the system from first order to second order makes it work better.

Example 7.1 Cruise Control The cruise control attempts to maintain the speed of a car in the presence of disturbances. I will look at a simple model that treats hills as the only disturbance. The car is driven by a force from the motor, retarded by air resistance and either slowed or accelerated by hills. I can write this as a force balance using the speed v as the sole variable—a one-dimensional state:

$$m\dot{v} = f - \frac{1}{2} C_D A \rho v^2 - mg\frac{dh}{dy}$$

Here m denotes the mass of the car, g the acceleration of gravity, C_D a drag coefficient, A the frontal area of the car, ρ the density of air, and f the force from the motor. I denote the elevation by h, and the direction of travel by y. This is a nonlinear problem, but its linearization is straightforward. I suppose $v = V + v'$, $h = h'$ to be small, and expand the force $f = F + f'$. Substitution of these into the governing equation leads to

$$m\dot{v}' = F + f' - \frac{1}{2} C_D A \rho \left(V + v'\right)^2 - mg\frac{dh}{dy}$$

This can be split into two equations: a reference equation and a linear equation for the departure of the speed from its desired value V:

$$0 = F - \frac{1}{2} C_D A \rho V^2, \quad m\dot{v}' = f' - C_D A \rho V v' - mg\frac{dh}{dy}$$

The reference equation defines open loop control for this problem. F is that force that would maintain the speed in the presence of air resistance in still air on a smooth, flat road. The second of these is our linear control problem. We want the error in speed, v', to go to zero. The last term is a disturbance caused by hills—the one disturbance I will consider in this exercise. Intuition suggests that the control force ought to be proportional to the error, giving us

$$m\dot{v}' + C_D A \rho V v' = -g_P v' - mg\frac{dh}{dy}$$

where I have introduced a proportional gain. This is where we started on our path to PID control. We see immediately that this is not going to do very much for us. We already have such a term in the air resistance. A constant hill will lead to a constant error, as we just saw in the more general development at the beginning of this section. Our cure will be the same. We can add an integral gain by introducing a position error y' such that $\dot{y}' = v$. This increases the order of the system from first to second. We can write the second-order system including the two gains as

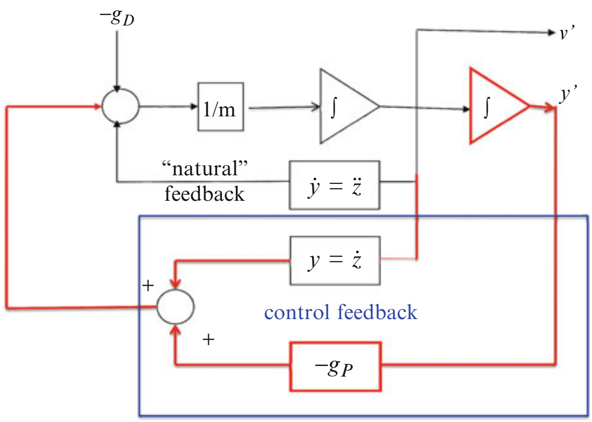


Fig. 7.12 Block diagram of the closed loop system

$$m\ddot{y}' + (C_D A\rho V + g_P)\dot{y}' + g_I y' = -mg\frac{dh}{dy}$$

This is now a closed loop system with a disturbance. We can see this more clearly if we draw block diagram of the system (Fig. 7.12).

The feedback part, the part that closes the loop, is outlined by a blue box in Fig. 7.11, and the feedback paths are shown in thick red lines. The crucial feedback for this system is the feedback from the displacement, which takes out any constant error in the speed, transferring it to the position, which is not a crucial variable for a cruise control.

Note that the introduction of an integral gain helps for any disturbance input. Recall that any time-dependent function can be written as a Fourier series. The integral gain will take out the constant term, the mean of the response to the disturbance. (You can also compare Figs. 7.9 and 7.10.)

7.2 The Laplace Transform

The Laplace transform provides an alternative way of solving dynamical problems and designing controls, and it will lead us to transfer functions. (We will find a new way to calculate state transition matrices along the way.)

7.2.1 The Transform

The Laplace transform is a method to transform a linear ordinary differential equation with constant coefficients into an algebraic equation. The algebraic equation can be solved, and the solution to the algebraic equation converted to the solution to the differential equation by taking the inverse Laplace transform. A full exposition of the Laplace transform and its inverse requires more mathematics than is appropriate for this text. I will state the formal definitions of each, but will not pursue them.

The Laplace transform of a function of time $f(t)$ is given by

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t) dt \tag{7.18}$$

where s is the transform variable. It is complex, and so is the transform. (It is a close relative of the exponential Fourier transform introduced in Chap. 5.) It is easy to show that the Laplace transform of a constant times a function is the constant times the transform, and that the Laplace transform of a sum is the sum of the Laplace transforms. (The Laplace transform of a product is NOT the product of the Laplace transforms, but their convolution, which I will address below.) We have

$$\mathcal{L}(af(t)) = \int_0^\infty e^{-st} af(t) dt = aF(s)$$

$$\mathcal{L}(af_1(t) + bf_2(t)) = \int_0^\infty e^{-st} af_1(t) dt + \int_0^\infty e^{-st} bf_2(t) dt = aF_1(s) + bF_2(s)$$

Finally we have

$$\mathcal{L}(f(at)) = \int_0^\infty e^{-st} f(at) dt = \frac{1}{a}\int_0^\infty e^{-\frac{sat}{a}} f(at) d(at) = \frac{1}{a} F\left(\frac{s}{a}\right)$$

The inverse transform is a contour integral in the complex s plane

$$\mathcal{L}^{-1}(F(s)) = f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^{st} F(s) ds \tag{7.19}$$

where c is a real constant large enough to ensure the existence of the integral. The integral must be evaluated using the techniques of contour integration, which are beyond the scope of the text. We will find inverse Laplace transforms using tables or convolution integrals. I will explain the latter shortly. First I will build up a very short table of Laplace transforms.

The Laplace transform of a constant C is simply C/s, which can be seen by inspection of Eq. (7.18). We can calculate the Laplace transform of $e^{\lambda t}$ directly from Eq. (7.19):

$$\mathcal{L}\left(e^{\lambda t}\right) = \int_0^\infty e^{(\lambda - s)t} f(t) dt = \left.\frac{e^{(\lambda - s)t}}{\lambda - s}\right|_0^\infty = \frac{1}{s - \lambda} \tag{7.20}$$

where the real part of s is supposed to be larger than the real part of λ so that the evaluation at the upper limit is zero. We can find the Laplace transform of the sine and the cosine by making use of the complex representation of each:

Table 7.1 A short table of Laplace transform pairs

$f(t)$	t^n	$e^{\lambda t}$	$\cos(\omega t)$	$\sin(\omega t)$
$F(s)$	$\frac{1}{s^{n+1}}$	$\frac{1}{s-\lambda}$	$\frac{s}{s^2-\omega^2}$	$\frac{\omega}{s^2-\omega^2}$

$$\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right), \quad \sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

Equation (7.20) gives the Laplace transform of the exponential terms in each expression, and these can be put over a common denominator to give

$$\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2}, \quad \mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2} \tag{7.21}$$

The transform of t can be found by an integration by parts. We write

$$\mathcal{L}(t) = \int_0^\infty t\exp(-st)dt = -\frac{1}{s}t\exp(-st)\big|_0^\infty - \frac{1}{s^2}\exp(-st)\big|_0^\infty = \frac{1}{s^2} \tag{7.22}$$

The Laplace transforms of the higher powers can be found in the same way. The result is that the Laplace transform of $t^n = 1/s^{n+1}$. These transform pairs will be adequate for most of our analyses. I collect them in Table 7.1.

7.2.2 Solving Single Linear Ordinary Differential Equations of Any Order

It is now time to see how the Laplace transform can be used to solve linear ordinary differential equations with constant coefficients. We need to know the Laplace transform of a derivative in order to apply the Laplace transform to differential equations. We can find this by integrating the definition (Eq. 7.18) by parts.

$$\mathcal{L}(\dot{f}) = \int_0^\infty e^{-xt}\dot{f}(t)ds = e^{-xt}f(t)\big|_0^\infty + s\int_0^\infty e^{-xt}f(t)ds = -f(0) + s\mathcal{L}(f) \tag{7.23}$$

This can clearly be extended to derivatives of any order, so, for example, the transform of the third derivative will be

$$\mathcal{L}(\dddot{f}) = -\ddot{f}(0) - s\dot{f}(0) - s^2 f(0) + s^3\mathcal{L}(f)$$

Example 7.2 Harmonically Forced One Degree of Freedom System Consider a typical one degree of freedom forced vibration problem with damping (our old friend Eq. 2.4). We are given initial conditions, the natural frequency, the damping ratio, and a forcing acceleration, which I suppose to be sinusoidal. Denote the displacement by y and its transform by an overbar. We can write the problem as an initial value problem in the time domain (Eq. 7.24):

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 = a\sin\left(\omega_f t\right), \quad y(0) = y_0, \quad \dot{y}(0) = v_0 \tag{7.24}$$

We know how to solve this problem by choosing a particular solution and a homogeneous solution. Let us see how to solve this using the Laplace transform. The problem can be rewritten in transform space by applying the Laplace transform to each term and rearranging the result:

$$s^2\bar{y} + 2\zeta\omega_n s\bar{y} + \omega_n^2\bar{y} = (s + 2\zeta\omega_n)y_0 + v_0 + \frac{a\omega_f}{s^2 + \omega_f^2} \tag{7.25}$$

This can be solved for the transform to give

$$\bar{y} = \frac{(s + 2\zeta\omega_n)}{s^2 + 2\zeta\omega_n s + \omega_n^2}y_0 + \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}v_0 + \frac{a\omega_f}{s^2 + \omega_f^2}\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{7.26}$$

The problem now is how to use what we know so far to invert the transform. We are not going to apply Eq. (7.19)!

The denominator of the first two terms has two roots, distinct unless $\zeta = 1$. These are the same exponents that we would find were we to find the homogeneous solution to the untransformed problem. Denote them by s_1 and s_2. They are in general complex conjugates. Rewrite Eq. (7.26) with the denominators formally factored:

$$\begin{aligned}\bar{x} = &\frac{(s + 2\zeta\omega_n)}{(s - s_1)(s - s_2)}x_0 + \frac{\nu_0}{(s - s_1)(s - s_2)} \\ &+ \frac{a\omega}{(s - s_1)(s - s_2)(s - j_\omega)(s + j_\omega)}\end{aligned} \tag{7.27}$$

The first two terms come from the initial conditions, and the third term comes from the forcing. It is the particular solution. We can use the method of *partial fractions* to rewrite the terms on the right-hand side. (Partial fractions is the name for undoing the operation of putting two or more fractions over a common denominator.) The second term is the simplest, so let's look at that in some detail as an illustration of the method of partial fractions. Replace the v_0 term with its quadratic denominator by two terms each with linear denominators, and then require that the two versions of the expression be the same by putting the latter term over the common quadratic denominator:

$$\frac{\nu_0}{(s - s_1)(s - s_2)} = \frac{a}{(s - s_1)} + \frac{b}{(s - s_2)} = \frac{(a + b)s - bs_1 - as_2}{(s - s_1)(s - s_2)}$$
$$\Rightarrow a + b = \nu_0, \quad bs_1 + as_2 = 0$$

from which we find

$$a = \frac{s_2 v_0}{s_2 - s_1}, \quad b = -\frac{s_1 v_0}{s_2 - s_1}$$

We recognize the two terms as Laplace transforms of exponentials, and the inverse transform of the second term in Eq. (7.26), one part of the homogeneous solution, is

$$\frac{s_2 v_0}{s_2 - s_1} e^{x_1 t} - \frac{s_1 v_0}{s_2 - s_1} e^{x_2 t}$$

The two roots are the roots we have seen before:

$$s_1 = -\omega_n\left(\zeta - \sqrt{\zeta^2 - 1}\right), \quad s_2 = -\omega_n\left(\zeta + \sqrt{\zeta^2 - 1}\right)$$

and we can partially simplify the answer to

$$\frac{\left(\zeta + \sqrt{\zeta^2 - 1}\right)\nu_0}{2\sqrt{\zeta^2 - 1}} e^{x_1 t} - \frac{\left(\zeta - \sqrt{\zeta^2 - 1}\right)\nu_0}{2\sqrt{\zeta^2 - 1}} e^{x_2 t}$$

I leave further simplification to you. The underdamped case for which $\zeta < 1$ is particularly interesting.

7.2.2.1 The Convolution Theorem and Its Use

This technique is clearly extendible as long as the forcing acceleration has a reasonable Laplace transform. There is, however, an even more general method. This depends on the convolution theorem, which I will state but not prove. Given the Laplace transforms $F(s)$ and $G(s)$ of two functions $f(t)$ and $g(t)$, the inverse Laplace transform of the product of F and G is the convolution of f and g, defined as follows:

$$f * g = \int_0^t f(\tau) g(t - \tau) d\tau = \int_0^t f(t - \tau) g(\tau) d\tau \tag{7.28}$$

If we know the individual inverse transforms, then we can find the inverse of the product by performing the integration in Eq. (7.28) (which may not be easy). In the present case we can tackle the forcing term by letting f be the homogeneous solution and g be the forcing. The homogeneous solution in question is

$$\frac{\left(\zeta + \sqrt{\zeta^2 - 1}\right)\nu_0}{2\sqrt{\zeta^2 - 1}} e^{x_1 t} - \frac{\left(\zeta - \sqrt{\zeta^2 - 1}\right)\nu_0}{2\sqrt{\zeta^2 - 1}} e^{x_2 t}$$

The particular solution is then given by

$$\begin{aligned} x_p &= \int_0^t \left(\frac{\left(\zeta + \sqrt{\zeta^2 - 1}\right)}{2\sqrt{\zeta^2 - 1}} e^{x_1 \tau} - \frac{\left(\zeta - \sqrt{\zeta^2 - 1}\right)}{2\sqrt{\zeta^2 - 1}} e^{x_2 \tau} \right) a \sin\left(\omega(t - \tau)\right) d\tau \\ &= \int_0^t \left(\frac{\left(\zeta + \sqrt{\zeta^2 - 1}\right)}{2\sqrt{\zeta^2 - 1}} e^{x_1 (t-\tau)} - \frac{\left(\zeta - \sqrt{\zeta^2 - 1}\right)}{2\sqrt{\zeta^2 - 1}} e^{x_2 (t-\tau)} \right) a \sin\left(\omega\tau\right) d\tau \end{aligned} \tag{7.29}$$

This is not nearly as bad as it looks: ζ is constant, and all that needs to be integrated is an exponential times a trigonometric function. The latter can be converted to exponential functions and then all the terms that need to be integrated are exponential functions, which are easy to integrate. This is primarily of academic interest. We already have a general particular solution of the same form that we obtained without using the Laplace transform.

7.2.3 Solving Systems of Linear Differential Equations

We can describe all of our linear systems in the form of Eq. (6.4). This chapter avoids state space generally, but this is a good place to look at the application of the Laplace transform to systems of first-order equations. We'll find an alternate method for calculating the state transition matrix. Consider the basic single-input–single-output (SISO) problem:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad y = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

The matrices **A**, **b**, and **c** are constant. (In the fully general system with more than one input, the vector **b** becomes a matrix **B**, but remains constant.) Because **A**, **b**, and **c** are constant, the Laplace transform of the vector problem is simply

$$s\overline{\mathbf{x}} - \mathbf{x}(0) = \mathbf{A}\overline{\mathbf{x}} + \mathbf{b}\overline{u}, \quad \overline{y} = \mathbf{c}^{\mathrm{T}}\overline{\mathbf{x}}$$

We can rearrange this:

$$(s\mathbf{1} - \mathbf{A})\overline{\mathbf{x}} = \mathbf{x}(0) + \mathbf{b}\overline{u}$$

so that

$$\overline{\mathbf{x}} = (s\mathbf{1} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{1} - \mathbf{A})^{-1}\mathbf{b}\overline{u}, \quad \overline{y} = \mathbf{c}^{\mathrm{T}}\overline{\mathbf{x}} \tag{7.30}$$

and

$$\mathbf{x} = \mathcal{L}^{-1}\left((s\mathbf{1} - \mathbf{A})^{-1}\right)\mathbf{x}(0) + \mathcal{L}^{-1}\left((s\mathbf{1} - \mathbf{A})^{-1}\mathbf{b}\overline{u}\right) \tag{7.31}$$

7.2.3.1 The State Transition Matrix

Compare Eq. (7.31) to Eq. (6.21). The first term of Eq. (7.30) will be the same as the first term of Eq. (6.20) if

$$\mathcal{L}^{-1}\left((s\mathbf{1}-\mathbf{A})^{-1}\right)=\exp(\mathbf{A}t)=\boldsymbol{\Phi}(t) \tag{7.32}$$

Is this an alternate way of calculating the state transition matrix? What does the second term of Eq. (7.31) look like? We apply the convolution theorem Eq. (7.28) to write it as

$$\int_0^t \mathcal{L}^{-1}\left((s\mathbf{1}-\mathbf{A})^{-1}\right)(t-\tau)\mathbf{b}u(\tau)d\tau$$

so that this too matches Eq. (6.20). Equation (7.32) provides a method of calculating the state transition matrix.

7.3 Control in the Frequency Domain

We have looked at first-order, second-order, and third-order control problems in the time domain, working directly with the differential equations. We can do the same thing in the frequency domain by taking the Laplace transform of the original problem. Let's start with the PID control of the simplest second-order system: $\ddot{y}=u$.

Take the Laplace transform:

$$s^2\overline{y}-\dot{y}(0)-sy(0)=\overline{u} \tag{7.33}$$

The initial conditions lead to homogeneous solutions, and we don't care much about these. Our goal is to get y to go to zero as t goes to infinity. The initial conditions are irrelevant to this goal, because they will decay for any functioning control. The particular solution will suffice and so all we need is

$$s^2\overline{y}=\overline{u}\Rightarrow\overline{y}=\frac{1}{s^2}\overline{u} \tag{7.34}$$

The second half of Eq. (7.34) gives a direct relation between the transform of the input and the transform of the output. The proportionality constant (a function of s) is called the *transfer function*, and I will follow common usage and write it as $H(s)$. Here $H(s)=1/s^2$.

We calculate the transfer function by taking the Laplace transform of an inhomogeneous linear ordinary differential equation with constant coefficients relating an input u and an output y:

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_N y = b_1 u^{(N-1)} + b_2 u^{(N-2)} + \cdots + b_N u \tag{7.35}$$

(We have yet to see a system in this chapter that involves derivatives of the input, but we will shortly. The right-hand side of Eq. (7.35) is always at least one degree lower than the left-hand side.) We find the transfer function following the recipe, and the result is

$$H(s) = \frac{b_1 s^{(N-1)} + b_2 s^{(N-2)} + \cdots + b_N}{s^{(N)} + a_1 s^{(N-1)} + \cdots + a_N} = \frac{P(s)}{Q(s)} \tag{7.36}$$

The transfer function is always a rational function (the ratio of two polynomials), and the polynomial in the numerator is always of lower degree than that in the denominator. The degree of the denominator is equal to the order of the system. The numerator is very often a constant. When it is not, we need to stop and think.

The system represented by Eq. (7.36) is not controlled, so the transfer function is that of an open loop system: the *open loop transfer function*. Note that the roots of the denominator, $Q(s)$, are the eigenvalues we would find if we let $y = Y\exp(st)$, so we can talk about stability in the context of transfer functions. We will refer to the zeroes of Q as the *pole*s of the transfer function and the zeroes of P as the *zeroes* of the transfer function.

We can see this in a little more generality by looking at Eq. (7.8a) and taking the Laplace transform (ignoring the initial conditions for the reasons I discussed above):

$$s^2\bar{y} + a_1 s\bar{y} + a_2\bar{y} = \bar{u} \tag{7.37}$$

from which we obtain the open loop transfer function:

$$H(s) = \frac{1}{s^2 + a_1 s + a_2} \tag{7.38}$$

We can control this system using feedback from y and its derivative (PD control) by setting

$$u = -g_P y - g_D \dot{y} \Rightarrow \bar{u} = -(g_P + s g_D)\bar{y} = -G(s)\bar{y} \tag{7.39}$$

This defines a *gain function* $G(s)$ in the frequency domain. We can find the closed loop system by combining Eqs. (7.37), (7.38), and (7.39):

$$\bar{y} = -H(s)G(s)\bar{y}$$

so that the closed loop system becomes

$$(1 + H(s)G(s))\bar{y} = 0 \tag{7.40}$$

This will converge to $y = 0$ if the poles of the closed loop transfer function $(1 + HG)$ all have negative real parts. Recall that the transfer function is a rational function, so we can rewrite Eq. (7.40) as

$$(Q(s) + P(s)G(s))\bar{y} = 0 \tag{7.41}$$

Recall that Q is an Nth degree polynomial in s. If P is constant, as it is here, and G is an $N - 1$ degree polynomial, then the gains can be used to place the poles for the closed loop system to ensure that y will converge to zero if disturbed from its equilibrium (desired) position. If the disturbance is ongoing, some u_{d}, then we will have

$$(Q(s) + P(s)G(s))\bar{y} = P(s)\bar{u}_{\mathrm{d}} \tag{7.42}$$

and there is a closed loop transfer function between the disturbance input and the output:

$$\bar{y} = H(s)\bar{u}_{\mathrm{d}} = \frac{P(s)}{(Q(s) + P(s)G(s))}\bar{u}_{\mathrm{d}} \tag{7.43}$$

We need to choose the poles of the closed loop system such that the system is stable.

We noted that a constant disturbance applied to the second-order system led to a constant error, and we also noted that this can be removed by adding an integral gain. That can be done very easily in the frequency domain. If multiplication by s is equivalent to differentiation, then division by s is equivalent to integration. We can use this idea to augment the gain function:

$$G(s) = \left(g_{\mathrm{P}} + sg_{\mathrm{D}} + \frac{1}{s}g_{\mathrm{I}}\right) \tag{7.44}$$

This increases the order of the denominator of the closed loop transfer function. Let's follow this in detail:

$$H(s) = \frac{P(s)}{(Q(s) + P(s)G(s))} = \frac{s}{(s(s^2 + a_1 s + a_2) + (sg_{\mathrm{P}} + s^2 g_{\mathrm{D}} + g_{\mathrm{I}}))} \tag{7.45}$$

The denominator is now third order, and there are three gains with which we can place poles. The output is proportional to the derivative of the input (if $P(s)$ is constant). We find its integral and differentiate to obtain the output.

$$\bar{z} = \frac{P_0 \bar{u}_{\mathrm{d}}}{(s(s^2 + a_1 s + a_2) + (sg_{\mathrm{P}} + s^2 g_{\mathrm{D}} + g_{\mathrm{I}}))}, \quad \bar{y} = s\bar{z} \tag{7.46}$$

where P_0 denotes the constant value of P. We can invert this using the convolution theorem to obtain either z or y. The denominator will become a simple factorable cubic equation once we have chosen the gains to place the poles, and we can write the $\bar{z}$ equation simply as

$$\bar{z} = \frac{P_0 \bar{u}_d}{(s - s_1)(s - s_2)(s - s_3)}$$
$$= \frac{P_0 u_d}{(s - s_1)(s_1 - s_2)(s_1 - s_3)} + \frac{P_0 u_d}{(s - s_2)(s_1 - s_2)(s_2 - s_3)}$$
$$+ \frac{P_0 u_d}{(s - s_3)(s_1 - s_3)(s_2 - s_3)} \tag{7.47}$$

The expansion in partial fractions means that I can find the inverse transform of H using the inverse transform of $1/(s - a)$. Of course if I know what the disturbance is, then I know the inverse transform for u_d. I can also work directly with y.

The transform of a constant disturbance is inversely proportional to the transform variable s, so the partial fraction expansion of Eq. (7.47) is of the form

$$\bar{z} = \frac{a_0}{s} + \frac{a_1}{(s - s_1)} + \frac{a_2}{(s - s_2)} + \frac{a_3}{(s - s_3)}$$

The first term in the expansion will give a constant contribution to z. The other three terms go to zero asymptotically with time supposing that I have chosen poles appropriately (stable poles). Since $\bar{y} = s\bar{z}$ there will be no such first term in the expansion for the transform of y. Thus we see that adding an integral control term eliminates the constant response to a constant disturbance, just as we found when working in the time domain.

Example 7.3 Undamped Mass-Spring System Consider the simple system shown in Fig. 7.13.
The governing equation of motion is

$$m\ddot{y} = f - ky$$

and the open loop transfer function between displacement and force is

$$H(s) = \frac{1/m}{s^2 + \omega_n^2}$$

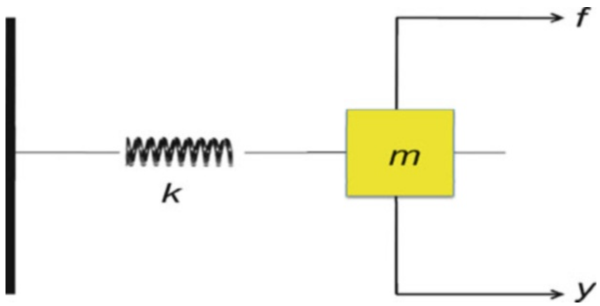


Fig. 7.13 Simple undamped one degree of freedom system

The system is marginally stable. It is a second-order system and we can place poles by introducing a gain function $G = g_P + g_D s$, leading to a closed form transfer function:

$$H(s) = \frac{1/m}{s^2 + \frac{g_D}{m}s + \frac{g_P}{m} + \omega_n^2}$$

We select the gains to place the poles. The reader can verify that

$$g_D = -m(s_1 + s_2), \quad g_P = m s_1 s_2$$

I leave it to the reader to discuss the addition of an integral gain.

Let's address a more interesting problem, a variation on the simple servomotor problem.

Example 7.4 Control of an Extended Servomotor Figure 7.14 shows a motor with a simple pendulum attached. The length of the pendulum is l, and the rod is massless. Gravity enters this problem. I denote the pendulum angle by θ, where θ increases in the counterclockwise direction and is zero when the rod points down. The torque balance equation is

$$ml^2\ddot{\theta} = \tau - mgl\sin\theta$$

and the torque is given by the motor equation. I assume that this system may react quickly so that I need to include inductance effects, making the torque and motor equations:

$$\tau = Ki, \quad \frac{di}{dt} = -\frac{R}{L}i + \frac{1}{L}e$$

where K denotes the motor constant, i the current, R the motor resistance, L the motor inductance, and e the input voltage.

This is a one degree freedom system that has a third-order representation because of the motor equations taking the inductance of the motor into account.

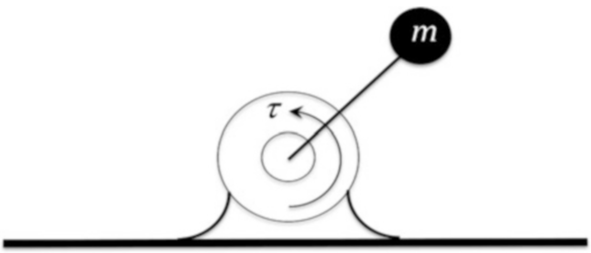


Fig. 7.14 A motor with an attached simple pendulum

I want to hold the pendulum at a fixed $\theta = \theta_0$, so I have an equilibrium, which requires an open loop control, an e_0. You can find e_0 by setting the time derivatives equal to zero in the governing equations:

$$\tau_0 = mgl\sin\theta_0 = Ki_0 = \frac{K}{R}e_0$$

The result for the equilibrium voltage is

$$e_0 = \frac{R}{K}mgl\sin\theta_0$$

Our control strategies can only deal with linear problems, so we need to linearize the governing equations about the equilibrium. The perturbation equations are

$$ml^2\ddot{\theta}' = Ki' - mgl\cos\theta_0\theta', \quad \frac{di'}{dt} = -\frac{R}{L}i' + \frac{1}{L}e'$$

where the prime denotes the departure from equilibrium. The basic solution does not enter explicitly, so there is no need to carry the prime further, so I will drop the primes for now.

This is not as simple as Ex. 7.3, but let's tackle it by taking the (zero initial conditions) Laplace transform. We get

$$ml^2s^2\bar{\theta} = K\vec{i} - mgl\cos\theta_0\bar{\theta}, \quad s\bar{i} = -\frac{R}{L}\bar{i} + \frac{1}{L}\bar{e}$$

from which we can derive a relation between the input e and the output θ.

$$\bar{i} = \frac{1/L}{s + R/L}\bar{e} \Rightarrow \left(s^2 + \frac{g}{l}\cos\theta_0\right)\bar{\theta} = \frac{K}{ml^2}\vec{i} = \frac{K/ml^2L}{s + R/L}\bar{e}$$

The transfer function here is

$$H(s) = \frac{K/ml^2L}{(s + R/L)\left(s^2 + \frac{g}{l}\cos\theta_0\right)}$$

and we see that this is a third-order system. We can use a third-order gain function:

$$\bar{e} = -\left(g_0 + g_1s + g_2s^2\right)$$

to place the poles to assure convergence of the perturbation angle to zero. We'll have

$$\bar{\theta} = -\frac{K/ml^2L}{(s + R/L)\left(s^2 + \frac{g}{l}\cos\theta_0\right)}\left(g_0 + g_1s + g_2s^2\right)\bar{\theta}$$

and the stability of the system is determined by the poles of

$$1 + \frac{K/ml^2L}{(s + R/L)\left(s^2 + \frac{g}{l}\cos\theta_0\right)}\left(g_0 + g_1 s + g_2 s^2\right)$$

which are the zeroes of

$$(s + R/L)\left(s^2 + \frac{g}{l}\cos\theta_0\right) + K/ml^2L\left(g_0 + g_1 s + g_2 s^2\right)$$

This is another pole placement problem. We must find gains such that

$$s^3 + \left(\frac{R}{L} + \frac{K}{ml^2L}g_2\right)s^2 + \left(\frac{g}{l}\cos\theta_0 + \frac{K}{ml^2L}g_1\right)s + \left(\frac{g}{l}\frac{R}{L}\cos\theta_0 + \frac{K}{ml^2L}g_2\right)$$
$$= (s - s_1)(s - s_2)(s - s_3)$$

The s^3 terms are identical. Each of the remaining terms has but one gain, so we can get all four coefficients to agree by choosing the gains.

The closed form transfer function can be written in the form of Eq. (7.42):

$$\frac{K/ml^2L}{(s + R/L)\left(s^2 + \frac{g}{l}\cos\theta_0\right) + \left(g_0 + g_1 s + g_2 s^2\right)} = \frac{K/ml^2L}{(s - s_1)(s - s_2)(s - s_3)}$$

We can use this closed from transfer function to find the response of this system to a disturbing torque. Suppose we add a (small) disturbing torque to the original dynamical equations:

$$ml^2\ddot{\theta}' = Ki' - mgl\cos\theta_0\theta' + \tau_d, \quad \frac{di'}{dt} = -\frac{R}{L}i' + \frac{1}{L}e'$$

We can find the transform of the controlled problem by following the steps in this example. The result of this is

$$(s - s_1)(s - s_2)(s - s_3)\overline{\theta} = \left(s + \frac{R}{L}\right)\frac{1}{ml^2}\overline{\tau}_d$$

Example 7.5 Magnetic Suspension Figure 7.15 shows the magnetic suspension system redrawn from Fig. 3.26.

The input is a voltage e and the output is the position of the ball, z. I chose z to be positive up and to equal zero at the face of the magnet. The governing equations are Eqs. (3.35) and (3.36). These are nonlinear, and so we need an equilibrium, which is provided by an open loop voltage, which we can find from Eq. (3.38):

$$i_0 = \sqrt{\frac{mg(-z_0)^n}{C_n}}, \quad e_0 = Ri_0$$

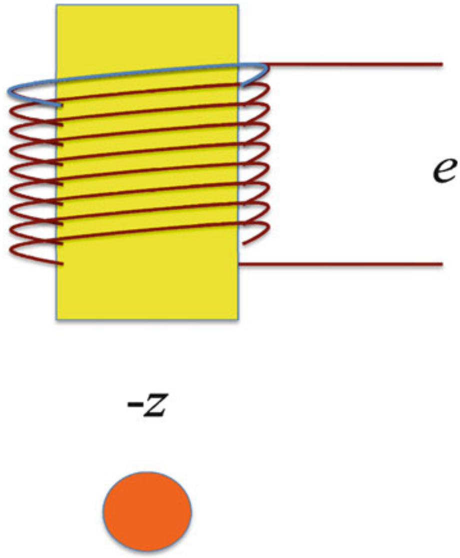


Fig. 7.15 The magnetic suspension system

Equations (3.40) and (3.41) give the linear equations for the system, which I reproduce here:

$$\ddot{z} = mg\left(2\frac{i}{i_0} + n\frac{z}{z_0}\right), \quad \frac{di}{dt} = -\frac{R}{L}i + \frac{1}{L}e$$

Here the dependent variables represent departures from the equilibrium, and z_0 is a positive number corresponding to the separation between the sphere and the magnet. This is another one degree of freedom problem represented as a third-order system. The Laplace transform of this system is

$$s^2\bar{z} = mg\left(2\frac{\bar{i}}{i_0} + n\frac{\bar{z}}{z_0}\right), \quad s\bar{i} = -\frac{R}{L}\bar{i} + \frac{1}{L}\bar{e}$$

and rearrange to get the single third-order system:

$$\bar{i} = \frac{1/L}{s + R/L}\bar{e} \Rightarrow \bar{z} = \frac{2mg/i_0L}{(s^2 - mgn/z_0)(s + R/L)}\bar{e}$$

from which the open loop transfer function can be found by inspection. It is of exactly the same form as the open loop transfer function found for Ex. 7.4. Further analysis follows the same path as Ex. 7.4, and I leave it to the exercises.

7.4 The Connection Between Transfer Functions and State Space

The transfer function relates the input and the output. So does the state space formulation. Suppose we have our familiar single-input–single-output system based on Eq. (6.3).

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad y = \mathbf{c}^{\mathrm{T}}\mathbf{x} \tag{7.48}$$

We've seen that we can solve this using the Laplace transform:

$$\overline{\mathbf{x}} = (s\mathbf{1} - \mathbf{A})^{-1}\mathbf{b}\overline{u}, \quad \overline{y} = \mathbf{c}^{\mathrm{T}}\overline{\mathbf{x}} \tag{7.49}$$

Combining these two equations gives an expression for $\overline{y}$ in terms of $\overline{u}$:

$$\overline{y} = \mathbf{c}^{\mathrm{T}}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{b}\overline{u} \Rightarrow H(s) = \mathbf{c}^{\mathrm{T}}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{b}\overline{u} \tag{7.50}$$

This is, of course, the open loop transfer function. The transfer function is a scalar; its components are vectors and matrices. It is therefore impossible to construct the original state based on the transfer function. Examples 7.2–7.4 gave us transfer functions with constant numerators, and we can use those transfer functions to construct a state space, but it is not necessarily the state that we would have gotten from the original differential equations. Let's look at Ex. 7.4. The open loop transfer function was

$$H(s) = \frac{K/ml^2L}{(s + R/L)\left(s^2 + \frac{g}{l}\cos\theta_0\right)}$$

and we can make a differential equation from this:

$$\dddot{\theta} + \frac{R}{L}\ddot{\theta} + \frac{g}{l}\cos\theta_0\dot{\theta} + \frac{R}{L}\theta = e \tag{7.51}$$

We can make a state space out of this:

$$\mathbf{x} = \begin{Bmatrix} \theta \\ \dot{\theta} \\ \ddot{\theta} \end{Bmatrix} \Rightarrow \dot{\mathbf{x}} = \begin{Bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{R}{L} & \frac{g}{l}\cos\theta_0 & -\frac{R}{L} \end{Bmatrix}\mathbf{x} + \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} e \tag{7.52}$$

Equation (7.52) is in companion form, but the state is not the obvious state. The obvious state in my opinion is $\mathbf{x} = \begin{Bmatrix} \theta & \dot{\theta} & i \end{Bmatrix}^{\mathrm{T}}$.

We can obtain both transfer functions for the low inductance overhead crane. The problem is defined by the state Eq. (5.30), which I reproduce here:

$$\mathbf{x} = \begin{Bmatrix} y \\ \theta \\ \dot{y} \\ \dot{\theta} \end{Bmatrix}, \quad \mathbf{A} = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{m}{M}g & -\dfrac{K^2}{Mr^2R} & 0 \\ 0 & -\dfrac{(M+m)}{Ml}g & \dfrac{K^2}{lMr^2R} & 0 \end{Bmatrix}, \quad \mathbf{b} = \frac{K}{lrRM}\begin{Bmatrix} 0 \\ 0 \\ -l \\ 1 \end{Bmatrix} \tag{5.30}$$

We have a choice of two $\mathbf{c}^T$ vectors: $\{1\ 0\ 0\ 0\}$ and $\{0\ 1\ 0\ 0\}$. These lead to the transfer functions:

$$H_y = \frac{gK/lMrR + s^2K/MrRs^2}{\Delta}, \quad H_\theta = \frac{s^2K/lMrR}{\Delta} \tag{7.53}$$

where

$$\Delta = s^4 + \frac{K^2}{Mr^2R}s^3 + (1+\mu)\frac{g}{l}s^2 + \frac{K^2}{Mr^2R}\frac{g}{l}s \tag{7.54}$$

These are the same transfer functions that I derived in Ex. 7.5. Using them for control still seems problematic.

Exercises

1. Write the nonlinear second-order cruise control in state space and linearize it about a constant speed V_0.
2. What gains are required for the cruise control if we want the poles to be at $-1+j$ and $-1-j$.
3. Solve the second-order cruise control problem for a harmonic disturbance (like rolling hills).
4. Integrate Eq. (7.27) for $\zeta = 0.1$ to obtain the particular solution to Ex. 7.2.
5. Establish the dimensions of the transfer function in Eq. (7.32).
6. Derive the relation between z and e that the transfer function gives for the magnetic suspension problem from Eq. (5.29).
7. Show that the solution to Eq. (5.10) for harmonic forcing is as shown.
8. Expand the last term in Eq. (7.25) using partial fractions and find its inverse transform.
9. Find the transfer function for problem *S4* assuming that the output is the angle of the pendulum. Discuss the nature of the feedback control required to control this system.

10. Find the closed form transfer function incorporating integral gain for Ex. 7.3.
11. Find the transfer function between e and θ for the linearized four-dimensional overhead crane.
12. Find the transfer function between e and y for the linearized four-dimensional overhead crane.
13. Place the poles for Ex. 7.4 as unit radius Butterworth poles and find the response of the system to a harmonic disturbance at $\omega = 0.1$ rad/s. Use the motor constants from Table 6.1, and let $m = 1 = l$.
14. Find the Laplace transform of t^2 by integrating by parts.
15. Find the inverse Laplace transform of $\frac{\omega}{s^3+s^2+\omega^2 s+\omega^2}$
16. Find the inverse Laplace transform of $\frac{1}{s^4-5s^2+4}$
17. Find the inverse Laplace transform of $\frac{1}{s^8+3s^7+2s^6}$
18. Construct a simulation for the magnetic suspension problem. Design a linear control based on Butterworth poles and find the radius of the poles required to control the system if the equilibrium $z_0 = 0.5$ and the initial position of the ball is $z = -0.45$. (Recall that $z_0 = 0.5$ means that the equilibrium value of $z = -0.5$.) Use the parameters in the table below

Magnetic suspension parameters (SI units)

m	n	C_n	R	L	g
1	2	1	1	0.01	9.81

19. Add an integral gain to the magnetic suspension system. Does this cancel a constant disturbance in the linear limit? Does it work for the nonlinear equations (in simulation).
20. Find the state transition matrix for the linearized magnetic suspension problem.
21. Consider the disturbed system from Ex. 7.4

$$(s - s_1)(s - s_2)(s - s_3)\bar{\theta} = \left(s + \frac{R}{L}\right)\frac{1}{ml^2}\bar{\tau}_d$$

Find the response of the angle to a constant disturbance. Can you eliminate this by adding an integral control component?

The following exercises are based on the chapter, but require additional input. Feel free to use the Internet and the library to expand your understanding of the questions and their answers.

22. Set up a model for cruise control to deal with variable headwinds.
23. Discuss control of the magnetic suspension system if the ball is disturbed by an impulse. How big an impulse will the control handle? Does it make a difference if the impulse is up or down?
24. What is the quickest way to drive a vehicle from point A to point B?

25. Repeat exercise 24 supposing that points A and B are 100 km apart, that the maximum speed of the vehicle is 120 mph, that the maximum acceleration of the vehicle is $0.25\,g$, and that the maximum deceleration is $0.9\,g$. How long does it take to get from point A to point B?

Reference

Butterworth S (1930) On the theory of filter amplifiers. Exp Wireless Wireless Eng October:536–542

The Basics of State Space Control

8

In which we look at feedback control of single-input linear systems from the point of view of state space. . ..

8.1 Introduction

We looked at control from a somewhat ad hoc position in Chap. 7. Our goal there was to choose an input that would drive an output to zero. We found that feedback based on the output and its derivatives could often accomplish this. We showed that

$$u = -g_0 y - g_1 \dot{y} - g_2 \ddot{y} - \cdots \tag{8.1}$$

could convert an open-loop dynamical system to a closed-loop dynamical system for which y went to zero as t went to infinity. The constants g_i were called gains. There were as many gains as there were dimensions in the state of which y is the output. (We also looked at adding integral gains, as in the PID controller and its extensions, which I will address further later in this chapter.) The feedback then depends on the output and its first $N - 1$ derivatives. If y and all its derivatives go to zero, then the state also goes to zero: $\mathbf{x} \to \mathbf{0}$. After some very simple problems that I could work directly in the time domain, I focused on the frequency domain, using the Laplace transform to find transfer functions that I could use to determine the gains in Eq. (8.1). Does this always work, or did I choose problems that work? We did run into a problem with the overhead crane. The difficulty stems from attempting to control one degree of freedom in a two degree of freedom system. Can I fix that? Can we deal directly with the state, avoiding the Laplace transform and the transfer functions? Can we find a feedback that depends on the components

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of the state even when they are not derivatives of the output? I will address these questions for linear SISO state space problems of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad \left(y = \mathbf{c}^{\mathrm{T}}\mathbf{x}\right), \quad \mathbf{x} \to \mathbf{0} \tag{8.2}$$

I put the output in parentheses because it is irrelevant to the basic problem of driving the state to zero, and if we can drive the state to zero, we automatically drive the output to zero.

8.1.1 Review of the Process: A Meta-algorithm

Start with an electromechanical system (some actual physical system out there in the world), and decide how many distinct links (and motors, perhaps) will be needed to construct a useful approximation to the mechanism. Then we can follow the usual path to a mathematical problem:

- Write the Lagrangian in as simple a form as you can.
- Write the constraints.
- Use the constraints to reduce the Lagrangian to an expression in the fewest number of variables required. These variables are the generalized coordinates.
- Write the Rayleigh dissipation function in terms of the generalized coordinates.
- Write the rate at which the external forces and torques do work in terms of the generalized coordinates using a rate of work function.
- Form the Euler-Lagrange equations using the Lagrangian, the Rayleigh dissipation function, and the rate of work function (to find the generalized forces). If electric motors provide the generalized forces and torques, the inputs become voltages instead of forces and torques, which you can find from the motor equations. (You can use Ohm's law if inductance can be neglected. If the system is going to have to react rapidly, then you need to replace Ohm's law by the first-order differential equation for the rate of change of the current (see Eq. 3.29), which you must add to the set of equations.)

At this point you will have a second-order, often nonlinear, ordinary differential Euler-Lagrange equation for each degree of freedom (or each generalized coordinate). These equations are generally coupled. You will also have a first-order (linear) ordinary differential equation for each motor for which the inductance is important.

The next step is to convert the set of equations to a state space representation. This leads to a set of first-order, nonlinear, ordinary differential equations. Each Euler-Lagrange equation must be rewritten in terms of the generalized coordinates and their derivatives, considered to be separate variables. The derivatives are defined by equations like Eq. (6.1). This set of differential equations can be integrated numerically to simulate the system. This simulation gives us a way to assess the validity and usefulness of the linear solutions that we obtain analytically.

We have seen how to drive some such systems (the one degree of freedom ones) to zero using feedback based on the output and its derivatives (and sometimes its integral). I will generalize that idea in this chapter, basing feedback on the elements of the state rather than the output and its derivatives. For sufficiently simple one degree of freedom systems these are the same thing. This is an important task, because we will look at more advanced problems from the point of view of finding equations for the vector error between what we want the state to be (the desired or reference state) and the actual state. We then want to devise a feedback control (meaning to find the gains—how much of each component of the state needs to be fed back) that will drive the vector error to zero. I will assume in this chapter that I have the entire state available (that I can measure every element of the state vector) from which to construct a feedback, and I will restrict this chapter to the simple problem for which the desired (reference) state is constant. I will discuss what to do when the full state is not available in Chap. 9. I will discuss unsteady desired (reference) states in Chap. 10. I will introduce some aspects of direct nonlinear control in Chap. 11. I will consider single-input systems for most of the examples, but the general topic of controllability applies to multi-input systems as well, so I will work in general when I can. I will be less concerned with the choice of output except when I consider how this formulation connects to the SISO approach through the transfer function.

The dynamical equation for a SISO system is Eq. (6.3) with an output y:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad y = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

The control problem for this chapter is to guide $\mathbf{x}$ to some constant value $\mathbf{x}_0$. If we think of this as controlling the output, and we want a desired output y_0, then we need to find $\mathbf{x}_0$ such that $y_0 = \mathbf{c}^{\mathrm{T}}\mathbf{x}_0$. This is generally a trivial task, particularly when the desired output is zero. We can rephrase the problem by letting $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}'$ and asking that the control drive $\mathbf{x}'$ to $\mathbf{0}$, a zero vector. The governing equation for $\mathbf{x}'$ is, of course,

$$\dot{\mathbf{x}}' = \mathbf{A}\mathbf{x}' + \mathbf{b}u' + \mathbf{A}\mathbf{x}_0 + \mathbf{b}u_0$$

The final two terms add up to zero, so the form of the equation for $\mathbf{x}'$ is the same as that for $\mathbf{x}$. Designing a control that makes $\mathbf{x}$ go to $\mathbf{0}$ is a perfectly general task when we are considering constant goals. Our problem is always reducible to Eq. (8.2). I will discuss nonconstant goals in Chap. 10.

Thus I will assume that $\mathbf{x}$ represents an error unless I specifically state otherwise. I want to make $\mathbf{x}$ go to zero by the proper choice of an input—feedback using the full state (one gain for each element of the state). We have not worried yet about whether or not we can do this for any problem. It turns out not to be always possible. The controllability theorem, which I will shortly state (without proof), tells us when it is possible.

8.1.2 Transfer Function vs. State Space

First let me review the connections among the state space formulation, the transfer function, and the conversion of a state back to a single differential equation in the single-input case. I gave the inverse transform of the transfer function relation Eq. (7.34), which I write here as Eq. (8.3).

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y = L(u) \tag{8.3}$$

where L denotes a differential operator

$$L(u) = b_0 u + b_1 \dot{u} + \cdots$$

If all but b_0 are zero, then we can use Eq. (8.3) directly to place poles and control the output. The Laplace transform of Eq. (8.3) provides a characteristic polynomial. If there are other nonzero terms, as for the overhead crane, then we have a more difficult problem.

We can write Eq. (8.3) as a set of N first-order differential equations using a state vector defined in terms of the output and its derivatives: $\mathbf{y} = \{ y \quad \dot{y} \quad \cdots \}^{\mathrm{T}}$. This state is not the same as the original state! The new state space equations for a four-dimensional system are

$$\dot{\mathbf{y}} = \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{Bmatrix} = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{Bmatrix} \mathbf{y} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} L(u) \tag{8.4}$$

and they are in companion form (see Chap. 6). Let's compare the equations we can derive from the transfer function to the state equations. We saw in Chap. 7 that they are not the same. When we build a system of equations from the output and its derivatives, we are actually constructing a new state. What is the relation between this state and the original state? I'd like to discuss this in the context of the fourth-order version of the overhead crane. We define the new state, call it $\mathbf{y}$, in terms of the derivatives by

$$\mathbf{y} = \begin{Bmatrix} y \\ \dot{y} \\ \ddot{y} \\ \dddot{y} \end{Bmatrix} \tag{8.5}$$

This is the logical state whenever $\mathbf{A}$ is in companion form. I can write this in terms of the original state and the input by simple substitution. I will write $\mathbf{A}$ and $\mathbf{b}$ (for the Eq. (6.17) representation) for this problem symbolically as Eq. (8.6)

$$\mathbf{A} = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & a_{42} & a_{42} & 0 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 0 \\ b_3 \\ b_4 \end{Bmatrix} \tag{8.6}$$

(Equation (6.41) gives the physical values of the elements of A and b.) I suppose the output to be y, so that $\mathbf{c}^{\mathrm{T}} = \{1\ 0\ 0\ 0\}$. The transfer function in this notation is

$$H(s) = \frac{a_{32}b_4 - a_{42}b_3 + b_3 s^2}{s^4 - a_{33}s^3 - a_{42}s^2 + (a_{33}a_{42} - a_{32}a_{43})s} \tag{8.7}$$

and the corresponding open-loop differential equation is

$$y^{(4)} - a_{33}\dddot{y} - a_{42}\ddot{y} + (a_{33}a_{42} - a_{32}a_{43})\dot{y} = (a_{32}b_4 - a_{42}b_3)e + b_3\ddot{e} \tag{8.8}$$

We can form a state space equation from Eq. (8.8) more or less by inspection

$$\dot{\mathbf{y}} = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_{32}a_{43} - a_{33}a_{42} & a_{42} & a_{33} \end{Bmatrix} \mathbf{y} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ (a_{32}b_4 - a_{42}b_3)\dot{e} + b_3\ddot{e} \end{Bmatrix} \tag{8.9}$$

The matrix **A** for the **y** problem is in companion form, but the input depends on e and its second derivative. There is also no clear relation between **x** and **y**, no linear transformation between the two. The input is inextricably involved in the transformation. We need to do better. It would be nice to have a linear transformation $\mathbf{z} = \mathbf{T}\mathbf{x}$ that leads to a set of ordinary differential equations in companion form without needlessly complicating the input. This is possible when the original system is controllable.

We can always write a single Nth-order differential equation in an N-dimensional state space form, the state being the generalization of Eq. (8.5) to N dimensions, components running from y to $y^{(N-1)}$. The state space equations in this case are always in companion form. We cannot, however, always write the single-input version of Eq. (5.6) as a single differential equation. The matrices **A** and **b** have to be in companion form for that to be possible. If a state space problem is in companion form, it can be rewritten as a single differential equation involving only the input—none of its derivatives or integrals. It is also controllable. If a system is controllable, then it can be converted to companion form. Controllability applies to the general multi-input problem, as well as the single-input system that leads to companion form.

The single-input system is, of course,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{8.10}$$

The development above (Eqs. 8.1, 8.2, and 8.3) is for a single-input system, and we can construct a method of transforming single-input system to companion form if it is controllable. If the system is controllable, then we can find an invertible linear transformation $\mathbf{T}$ between the original (physical) state $\mathbf{x}$ and a new state $\mathbf{z}$ ($\mathbf{z} = \mathbf{Tx}$) for which the equations are in companion form. I promised you this in Chap. 6. We typically prefer a transformation for which the output $y = z_1$, but this is not always possible. When possible we can write a version of Eq. (8.1) with y equal to the first element in $\mathbf{z}$. If u depends linearly on z and its derivatives, no matter the nature of z_1, then

$$u = -\sum_{i=0}^{N-1} g_i' z_1^i$$

where the superscript on z_1 denotes the number of differentiations and the primes here indicate that these gains are with respect to the $\mathbf{z}$ variables. I can write Eq. (8.11), a closed-loop version of Eq. (8.3)

$$z_1^{(N)} + \left(a_1 + g_{N-1}'\right) z_1^{(N-1)} + \cdots + \left(a_{N-1} + g_1'\right) z_1 = 0 \tag{8.11}$$

This is a homogeneous ordinary differential equation with constant coefficients, so it admits exponential solutions. If I write $z = Z_{\exp}(st)$ then Eq. (8.11) becomes

$$\left(s^N + \left(a_1 + g_{N-1}'\right) s^{N-1} + \cdots + \left(a_{N-1} + g_1'\right)\right) Z = 0 \tag{8.12}$$

and the exponents are determined by the coefficients of the algebraic equation in parentheses. That equation is the *characteristic equation* (or *characteristic polynomial*) for the closed-loop system, and it must vanish for Eq. (8.12) to have nontrivial solutions. The characteristic polynomial vanishes for the N distinct roots (poles) s_1, s_2,…, and the closed-loop system will converge to zero if $\mathrm{Re}(s_i) < 0$ for all i. Since each coefficient contains one and only one gain, we can easily put the poles anywhere we want and thus assure the required convergence. The roots of the characteristic equation are the poles of the corresponding closed-loop transfer functions. Because $\mathbf{z}$ is proportional to $\mathbf{x}$, the vanishing of $\mathbf{z}$ implies the vanishing of $\mathbf{x}$. The vanishing of $\mathbf{x}$ ensures the vanishing of y. The conclusion is that I can control the system Eq. (8.10) in the sense of driving the state vector $\mathbf{x}$ to zero if I can find a transformation to convert the original problem to companion form. We have found one such transformation using transfer functions. This works for some simple systems. These transformations are not always ideal, and we have no way of knowing in advance whether the method is going to work, that is, whether we can use it to find a feedback that will control the original system. The controllability theorem not only tells us whether or not we can transform $\mathbf{A}$ and $\mathbf{b}$ to companion form, it also gives an algorithm for the transformation when it is possible.

8.2 Controllability

8.2.1 The Controllability Theorem

Controllability is determined by **A** and **B**, and yes, it does apply to the general case, so I will state it for that case, even if most of the cases we will consider are single-input systems. The algorithm to find the companion form is for single-input systems only, but the theorem is general. Suppose the state to have N dimensions and the input M dimensions. The controllability matrix is the $N \times NM$ matrix formed from **B** and **A**

$$\mathbf{Q} = \left\{ \mathbf{B} \quad \mathbf{AB} \quad \cdots \quad \mathbf{A}^{(N-1)}\mathbf{B} \right\} \tag{8.13}$$

Equation (8.13) says that the first M columns of **Q** are given by **B**, the second M columns by **AB**, etc. The theorem says that if the rank of **Q** is N, then the problem is controllable. The matrix **Q** will be square for a single-input system. If an $N \times N$ square matrix has rank N its determinant will be nonzero and vice versa. We can state the controllability condition for a single-input system to be that **Q** has a nonzero determinant.

Example 8.1 The Simple (4D) Overhead Crane Let's check the controllability of the overhead crane. The matrix **A** and the vector **b** are given by Eq. (6.39). I will rewrite these in a reduced but equivalent form

$$\mathbf{A} = \left\{ \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \mu g & -\beta l & 0 \\ 0 & -(1+\mu)\dfrac{g}{l} & \beta & 0 \end{matrix} \right\}, \quad \mathbf{b} = \alpha \left\{ \begin{matrix} 0 \\ 0 \\ -1 \\ 1 \end{matrix} \right\}$$

The reader can verify that

$$\mu = \frac{m}{M}, \quad \alpha = \frac{K}{MlrR}, \quad \beta = \frac{K^2}{Mlr^2R}$$

The controllability matrix is given by

$$\mathbf{Q} = \alpha \left\{ \begin{matrix} 0 & -1 & \beta l^2 & \mu g - \beta^2 l^3 \\ 0 & 1 & -\beta l & \beta^2 l^2 + (1+\mu)\dfrac{g}{l} \\ -1 & \beta l^2 & \mu g - \beta^2 l^3 & \beta^3 l^4 - 2gl\beta\mu \\ 1 & -\beta l & \beta^2 l^2 + (1+\mu)\dfrac{g}{l} & -\beta\left(g - \beta^2 l^2\right) \end{matrix} \right\}$$

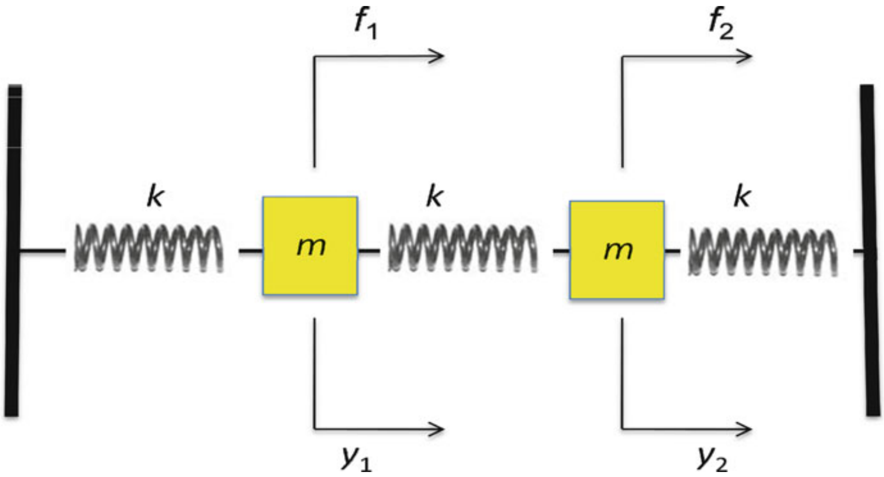


Fig. 8.1 A two-input system

and its determinant is $-\alpha^4 g^2(1+2\mu)^2$, which is never equal to zero, so this system is controllable. We failed to control it earlier, but we will control it before this chapter is done.

Controllability also applies to multi-input systems, so let's look at one of those.

Example 8.2 A Very Simple Multi-input Mass-Spring System Consider the system shown in Fig. 8.1, which is a very simple two-input, two degree of freedom system.

I leave it to the reader to show that the Euler-Lagrange equations for this system are

$$m\ddot{y}_1 = -k(2y_1 - y_2) + f_1, \quad m\ddot{y}_2 = -k(2y_2 - y_1) + f_2$$

We can define a state and find the state equations

$$\mathbf{x} = \begin{Bmatrix} y_1 \\ y_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix}, \quad \dot{\mathbf{x}} = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2\dfrac{k}{m} & \dfrac{k}{m} & 0 & 0 \\ \dfrac{k}{m} & -2\dfrac{k}{m} & 0 & 0 \end{Bmatrix} \mathbf{x} + \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

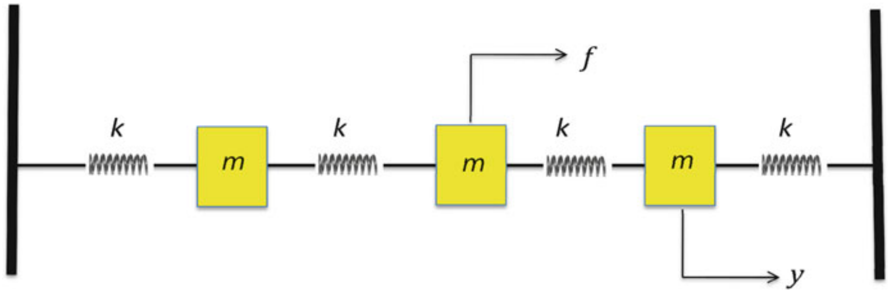


Fig. 8.2 An uncontrollable three degree of freedom system

from which we can define **A** and **B**, and then calculate **Q**, which will have four rows and eight columns. I leave it to the reader to do the simple matrix multiplication to obtain

$$\mathbf{Q} = \begin{Bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -2\frac{k}{m} & \frac{k}{m} \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{k}{m} & -2\frac{k}{m} \\ 1 & 0 & 0 & 0 & -2\frac{k}{m} & \frac{k}{m} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{k}{m} & -2\frac{k}{m} & 0 & 0 \end{Bmatrix}$$

The rank of **Q** is 4, which one can find by inspection, since the first four column vectors are independent. Thus the multi-input system shown in Fig. 8.1 is controllable.

Are there uncontrollable systems? The answer, unfortunately, is yes. The artificial system in the following example is uncontrollable.

Example 8.3 An Uncontrollable System Consider the system shown in Fig. 8.2. All the masses are equal, as are all the springs. There is no damping. We can write the equations of motion in terms of a single symbolic frequency $\omega = \sqrt{(k/m)}$. The input is applied to link two and the output will be the position of link three. I leave it to the reader to establish that this can be written as a sixth-order state equation where

$$
\mathbf{x} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{Bmatrix}, \quad \mathbf{A} = \begin{Bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2\omega^2 & \omega^2 & 0 & 0 & 0 & 0 \\ \omega^2 & -2\omega^2 & \omega^2 & 0 & 0 & 0 \\ 0 & \omega^2 & -2\omega^2 & 0 & 0 & 0 \end{Bmatrix},
$$

$$
\mathbf{b} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \mathbf{c}^T = \{0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0\}
$$

The transfer function supposing that y, the position of the third link in Fig. 8.2, is the output is

$$
H(s) = \frac{2\omega^4 + \omega^2 s^2}{s^6 + 6\omega^2 s^4 + 10\omega^4 s^2 + 4\omega^6}
$$

The numerator and denominator share a common factor, so the system is not truly a sixth-order system, but a fourth-order system. Will this be a problem? Is this problem controllable? We can form $\mathbf{Q}$ to answer this question. We obtain

$$
\mathbf{Q} = \begin{Bmatrix} 0 & 0 & 0 & \omega^2 & 0 & -4\omega^4 \\ 0 & 1 & 0 & -2\omega^2 & 0 & 6\omega^4 \\ 0 & 0 & 0 & \omega^2 & 0 & -4\omega^4 \\ 0 & 0 & \omega^2 & 0 & -4\omega^4 & 0 \\ 1 & 0 & -2\omega^2 & 0 & 6\omega^4 & 0 \\ 0 & 0 & \omega^2 & 0 & -4\omega^4 & 0 \end{Bmatrix}
$$

The first and third rows are identical, as are the fourth and sixth, so the determinant is zero and the system is not controllable.

8.2.2 Companion Form

It is nice to know that a system is controllable, but we still need to control it. We can convert a single-input system to companion form and then control the companion version of the problem, that is, find some u based on feedback from $\mathbf{z}$ that will drive

z to **0**. This same feedback will, of course, drive **x** to **0**. The controllability matrix gives us a method to find the (invertible) transformation matrix **T** for the single-input problem:

- Invert **Q**.
- The last row of the inverse is the first row of **T**.
- The successive rows of **T** can be obtained by multiplying **A** from the left by the previous row.

We can write this in equation form by defining a row vector $\mathbf{t}^{\mathrm{T}}$ with all but its last component equal to zero and the last component equal to unity. (In four dimensions, $\mathbf{t}^{\mathrm{T}} = \{0\ 0\ 0\ 1\}$.) Then we can follow the recipe

$$\mathbf{t}_1 = \mathbf{t}^{\mathrm{T}}\mathbf{Q}^{-1}, \quad \mathbf{t}_2 = \mathbf{t}_1\mathbf{A}, \quad \mathbf{t}_3 = \mathbf{t}_2\mathbf{A}, \cdots \Rightarrow \mathbf{T} = \left\{\begin{matrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \vdots \end{matrix}\right\} \tag{8.14}$$

This gives me the transformation between **x** and **z**, which is automatically invertible. We can substitute that into Eq. (8.4)

$$\mathbf{z} = \mathbf{T}\mathbf{x} \Leftrightarrow \mathbf{x} = \mathbf{T}^{-1}\mathbf{z} \Rightarrow \mathbf{T}^{-1}\dot{\mathbf{z}} = \mathbf{A}\mathbf{T}^{-1}\mathbf{z} + \mathbf{b}u \tag{8.15}$$

Multiply Eq. (8.15) by **T** (from the left)

$$\mathbf{T}\mathbf{T}^{-1}\dot{\mathbf{z}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\mathbf{z} + \mathbf{T}\mathbf{b}u \Rightarrow \dot{\mathbf{z}} = \mathbf{A}_1\mathbf{z} + \mathbf{b}_1 u \tag{8.16}$$

The matrix $\mathbf{A}_1$ and the vector $\mathbf{b}_1$ will be in companion form.

That was rather cryptic, so let's see how it works out in practice.

Example 8.4 Companion Form for the 4D Overhead Crane We found the controllability matrix in Ex. 8.1, and it is

$$\mathbf{Q} = \alpha\left\{\begin{matrix} 0 & -1 & \beta l^2 & \mu g - \beta^2 l^3 \\ 0 & 1 & -\beta l & \beta^2 l^2 + (1+\mu)\dfrac{g}{l} \\ -1 & \beta l^2 & \mu g - \beta^2 l^3 & \beta^3 l^4 - 2gl\beta\mu \\ 1 & -\beta l & \beta^2 l^2 + (1+\mu)\dfrac{g}{l} & -\beta\left(g - \beta^2 l^2\right) \end{matrix}\right\}$$

Its inverse is

$$\mathbf{Q}^{-1} = \frac{1}{\alpha}\begin{Bmatrix} -\beta & 0 & -\dfrac{1+\mu}{l(1+2\mu)} & \dfrac{\mu}{1+2\mu} \\ \dfrac{\mu}{1+2\mu} & \dfrac{\mu}{1+2\mu} & \dfrac{\beta l}{g(1+2\mu)} & \dfrac{\beta l^2}{g(1+2\mu)} \\ \dfrac{\beta l}{g(1+2\mu)} & \dfrac{\beta l^2}{g(1+2\mu)} & \dfrac{1}{g(1+2\mu)} & \dfrac{1}{g(1+2\mu)} \\ \dfrac{1}{g(1+2\mu)} & \dfrac{1}{g(1+2\mu)} & 0 & 0 \end{Bmatrix}$$

The last row of this is the first row of **T**, and following the algorithm leads to the entire transformation matrix

$$\mathbf{T} = \frac{1}{\alpha}\begin{Bmatrix} \dfrac{1}{g(1+2\mu)} & \dfrac{1}{g(1+2\mu)} & 0 & 0 \\ 0 & 0 & \dfrac{1}{g(1+2\mu)} & \dfrac{1}{g(1+2\mu)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}$$

I leave the calculation of its inverse and the calculation of $\mathbf{A}_1$ to the reader. The result is

$$\mathbf{A}_1 = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \beta g(1+2\mu) & (1+\mu)\dfrac{g}{l} & -\beta l \end{Bmatrix}$$

It is indeed in companion form. The zero in the last row does not change that. The entries in the last row need not be nonzero. I leave it to the reader to show that **b** is also converted to companion form.

8.3 Using the Companion Form to Control a System

We have learned how to figure out if a system is controllable. We have learned how to convert a controllable single-input system to companion form. It is now time to use this to find u to make $\mathbf{x}$ go to $\mathbf{0}$ when $\mathbf{x}$ satisfies Eq. (6.3).

We know that the behavior of the solutions of Eq. (6.3) depends on the eigenvalues of **A**. The transformation is a proper transformation, and it does not change the eigenvalues (see Strang (1988) or any other text on linear algebra for an explanation), so the eigenvalues of $\mathbf{A}_1$ are the same as those of **A**. Equation (6.3) is

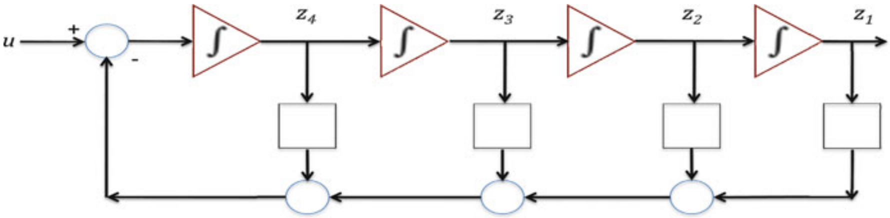


Fig. 8.3 Block diagram of a fourth-order system in companion form in open-loop form

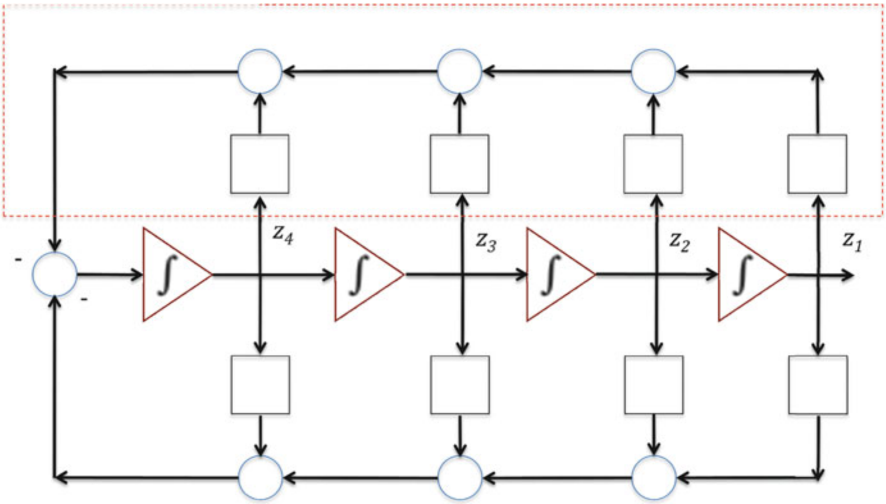


Fig. 8.4 The closed-loop version of Fig. 8.3

an open-loop system, as is its companion form. The block diagram for the companion form is simpler than the original block diagram. Figure 8.3 shows the block diagram corresponding to the companion form of the overhead crane (omitting the various constants to keep the main paths clearly in mind).

There is feedback from all the components of **z**. (There is no feedback from z_1 for the overhead crane, but that is a special case.) We can close the loop by making the input u proportional to the vector **z**, which provides additional feedback from the components of **z**, feedback that we can control. Figure 8.4 shows the closed-loop version. The dashed box contains the control feedback loop.

How do we do this mathematically? We let $u = -\mathbf{g}_z^{\mathrm{T}}\mathbf{z}$, where the subscript z emphasizes that I am working in the transformed **z** state space. The vector **g** is called the *gain vector*; this one is the gain vector in **z** space. The closed-loop system is then

$$\dot{\mathbf{z}} = \left(\mathbf{A}_1 - \mathbf{b}_1 \mathbf{g}_z^{\mathrm{T}}\right)\mathbf{z} \tag{8.17}$$

where $\mathbf{b}_1$ is a column vector with all but its last entry equal to zero and $\mathbf{g}^{\mathrm{T}}$ is a row vector. Their product is a square matrix, which I can represent symbolically

$$\mathbf{b}_1 \mathbf{g}_z^{\mathrm{T}} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{Bmatrix} \{ \bullet \quad \bullet \quad \cdots \quad \bullet \} = \begin{Bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & \cdots & \bullet \end{Bmatrix} \tag{8.18}$$

All but the last row of the matrix $\mathbf{b}_1 \mathbf{g}_z^{\mathrm{T}}$ is zero, and the last row is equal to $\mathbf{g}_z^{\mathrm{T}}$. The object in parentheses in Eq. (8.17) is therefore a square matrix and the behavior of the solutions to Eq. (8.17) depends on the eigenvalues of that matrix. We can determine these by our choice of $\mathbf{g_z}$. The second half of the matrix has nonzero entries only in its last row (because $\mathbf{b}_1$ has only one nonzero component, the last one), so the matrix (for the overhead crane, to be specific) is

$$\mathbf{A}_1 - \mathbf{b}_1 \mathbf{g}_z^{\mathrm{T}} = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -g_{z1} & -g_{z2} + \beta g(1 + 2\mu) & -g_{z3} + (1 + \mu)\dfrac{g}{l} & -g_{z4} - \beta l \end{Bmatrix} \tag{8.19}$$

The characteristic polynomial for any matrix in companion form can be deduced directly from the last row of the matrix, which provides the coefficients of the characteristic polynomial reading from left to right for a system of any size: the element in the lower left-hand corner is the coefficient of the constant term, the next is the coefficient of the linear term, and so on. The element in the lower right-hand corner is the coefficient of the $N - 1$ term. The coefficient of s^N is always unity. This applies to a system of any order in companion form (for a 4×4 system as in the example)

$$s^4 - A_{44}s^3 - A_{43}s^2 - A_{42}s - A_{41} = 0 \tag{8.20}$$

For the overhead crane matrix Eq. (8.19) we'll have

$$s^4 - (-g_{z4} - \beta l)s^3 - \left(-g_{z3} + (1 + \mu)\frac{g}{l}\right)s^2 - (-g_{z2} + \beta g(1 + 2\mu))s + g_{z1} = 0 \tag{8.21}$$

Each term in Eq. (8.21) contains one and only one of the gains, so we can find each coefficient easily. The coefficients of the characteristic polynomial are directly related to its roots; so if I choose the poles I want, I can calculate the coefficients that I want to be in Eq. (8.21) in order to drive **z** to 0. In this fourth-order case the desired version of Eq. (8.21) is

$$
\begin{gathered}
(s - s_1)(s - s_2)(s - s_3)(s - s_4) = 0 \\
\Downarrow \\
s^4 - (s_1 + s_2 + s_3 + s_4)s^3 + (s_1 s_2 + s_1 s_3 + s_1 s_4 + s_2 s_3 + s_2 s_4 + s_3 s_4)s^2 \\
-(s_1 s_2 s_3 + s_1 s_2 s_4 + s_1 s_3 s_4 + s_2 s_3 s_4)s + s_1 s_2 s_3 s_4 = 0
\end{gathered}
\tag{8.22}
$$

If I drive **z** to zero, then I will automatically drive the original state **x** to 0. We've seen this idea before. We equate the coefficients of the ideal and actual characteristic polynomials to obtain the gains (in **z** space).

8.3.1 Application to the Simple (4D) Overhead Crane

Let us design a control that will move the overhead crane 5 m to the right supposing that we can neglect inductive effects in the motor. Table 8.1 gives a set of parameters for this problem.

I show the **A** matrix and **b** vector in Eq. (8.23).

$$
\mathbf{A} = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{m}{M} g & -\dfrac{K^2}{Mr^2 R} & 0 \\ 0 & -\left(1 + \dfrac{m}{M}\right)\dfrac{g}{l} & \dfrac{K^2}{Mr^2 lR} & 0 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 0 \\ -\dfrac{K}{MrR} \\ \dfrac{K}{MrlR} \end{Bmatrix} \tag{8.23}
$$

I calculated the companion form of **A** in Ex. 8.4. I copy the result here as Eq. (8.24) for convenience.

Table 8.1 Parameters for the overhead crane (SI units)

M	m	l	r	R	K	g
100	50	2	0.4	4	0.868	9.81

$$\mathbf{A}_1 = \mathbf{TAT}^{-1} = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{gK^2}{Mr^2lR} & -\left(1+\dfrac{m}{M}\right)\dfrac{g}{l} & -\dfrac{K^2}{Mr^2R} \end{Bmatrix} \tag{8.24}$$

We can now let e be proportional to $\mathbf{z}$

$$e = -\mathbf{g}_z^{\mathrm{T}}\mathbf{z}$$

to close the loop. We have a closed-loop homogeneous system Eq. (8.17)

$$\left(\mathbf{A}_1 - \mathbf{b}_1\mathbf{g}_z^{\mathrm{T}}\right)\mathbf{z} = 0$$

The behavior of this system is controlled by the eigenvalues of $(\mathbf{A}_1 - \mathbf{b}_1\mathbf{g}_z^{\mathrm{T}})$. These can be adjusted by choosing the components of $\mathbf{g}_z$ to place poles. These are

$$\begin{aligned} g_{z1} &= s_1s_2s_3s_4 \\ g_{z2} &= -\frac{gK^2}{Mr^2lR} - (s_1s_2s_3 + s_1s_2s_4 + s_2s_3s_4 + s_1s_3s_4) \\ g_{z3} &= -\left(1+\frac{m}{M}\right)\frac{g}{l} + s_1s_2 + s_1s_3 + s_1s_4 + s_2s_3 + s_2s_4 + s_3s_4 \\ g_{z4} &= -\frac{gK^2}{Mr^2R} - (s_1 + s_2 + s_3 + s_4) \end{aligned} \tag{8.25}$$

where the s_j denotes the desired poles. We need to transform these back to $\mathbf{x}$ space in order to use them in the simulation $e = \mathbf{g}_z^{\mathrm{T}}\mathbf{z} = \mathbf{g}_z^{\mathrm{T}}\mathbf{Tx}$: $\mathbf{g}_x^{\mathrm{T}} = \mathbf{g}_z^{\mathrm{T}}\mathbf{T}$. The $\mathbf{x}$ gains are more complicated than the $\mathbf{z}$ gain, and I will not transcribe them here.

How well does this work? We can assess it using a simulation. Let me start from rest with $y = -5$ m, $\theta = 0$ using fourth-order Butterworth poles with unit radius:

$$-0.587785 \pm 0.809017j, \ -0.951057 \pm 0.309017j$$

I replace e in the nonlinear state equations by the control value, and then I integrate these nonlinear equations numerically to give me a simulation of the actual system. Figure 8.5 shows all four elements of the state vs. time. Figure 8.6 shows the pendulum angle in degrees. Note that it never exceeds 5°, so the small angle approximation holds during the entire motion. Figure 8.7 shows the voltage required for this control. The voltage exceeds the maximum design voltage for this motor (180 V) early in the motion. Let's modify the control to prevent this and see if the control still works.

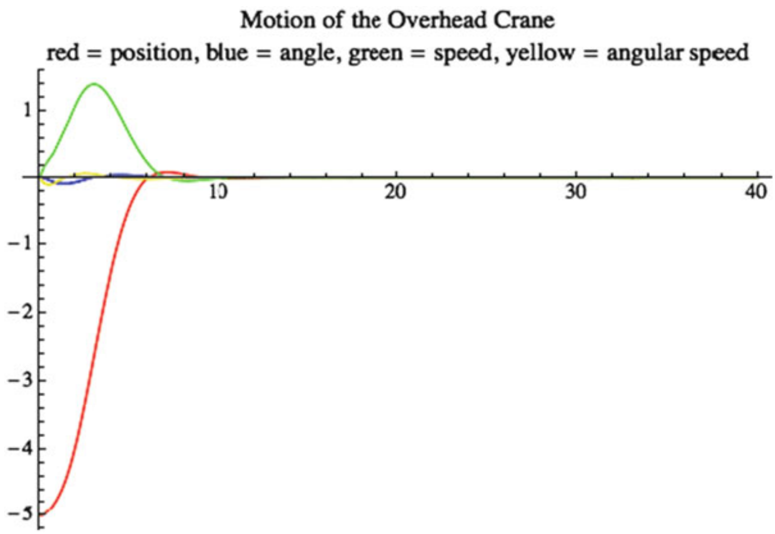


Fig. 8.5 The components of the state during the crane's motion

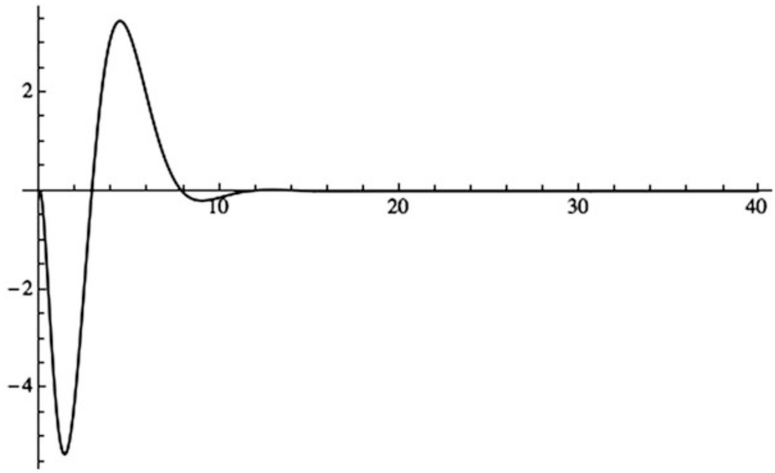


Fig. 8.6 The pendulum angle in degrees during the motion

Write

$$e = E_0 \tanh\left(\frac{\mathbf{g}^{\mathrm{T}}\mathbf{x}}{E_0}\right) \tag{8.26}$$

The hyperbolic tangent of x is approximately equal to x when x is small and tends to ± 1 as x tends to infinity. Thus Eq. (8.26) makes e very close to its design value $\mathbf{g}^{\mathrm{T}}\mathbf{x}$

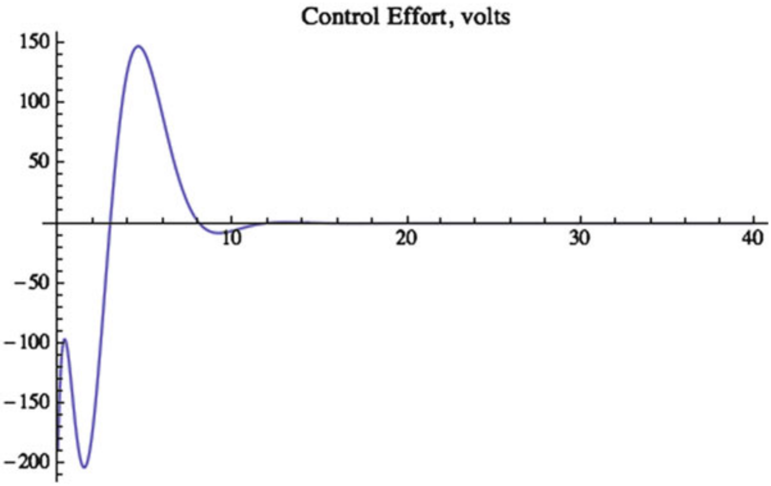


Fig. 8.7 Control voltage. There are two brief intervals with voltage greater than the design voltage of the motor

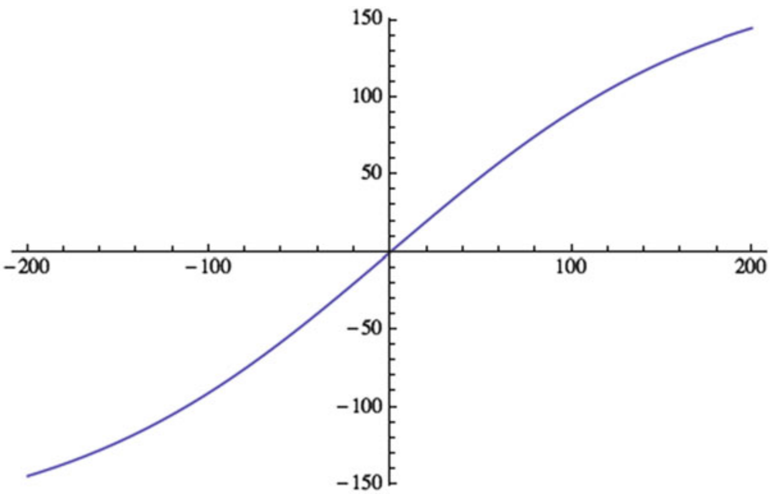


Fig. 8.8 Actual voltage vs. design voltage for the hyperbolic tangent model

when the design value is small but limits e to $\pm E_0$ as the design value of the control increases. Figure 8.8 shows a plot of e as given by Eq. (8.26) vs. the nominal design control voltage, $\mathbf{g}^T\mathbf{x}$ for an E_0 of 180 V.

Figures 8.9, 8.10, and 8.11 show the response corresponding to Figs. 8.5, 8.6, and 8.7 with the voltage limitation applied. There is not a lot of difference. The system takes a little longer to converge, the peak voltages are reduced, but the general state behavior is qualitatively the same.

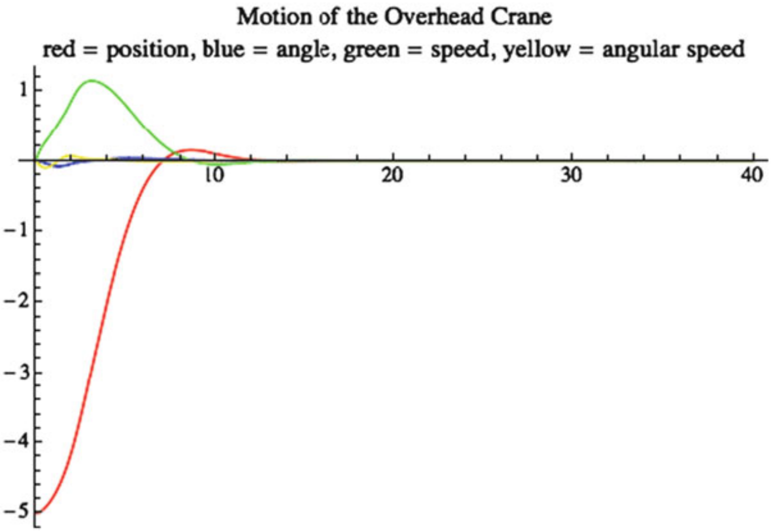


Fig. 8.9 Motion of the system with a limited voltage

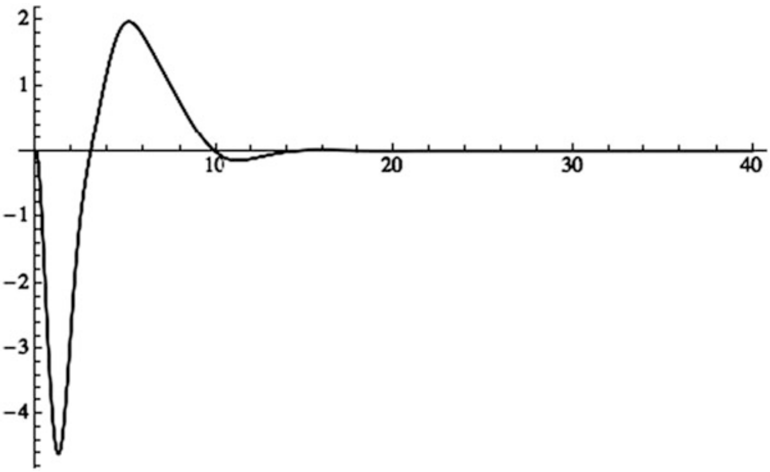


Fig. 8.10 Pendulum angle in degrees with a limited voltage

Let me summarize what I have done in this section. I started with the linearized equations of motion for the overhead crane in the low-inductance limit. I showed that the linear problem was controllable. I calculated the matrix **T** required to define **z** space following the algorithm. I let the voltage e be proportional to **z** and wrote a closed-loop system in **z** space (Eq. 8.20). I chose the eigenvalues for this system by choosing the components of $\mathbf{g}_z$. I mapped the gains back to **x** space, giving me a

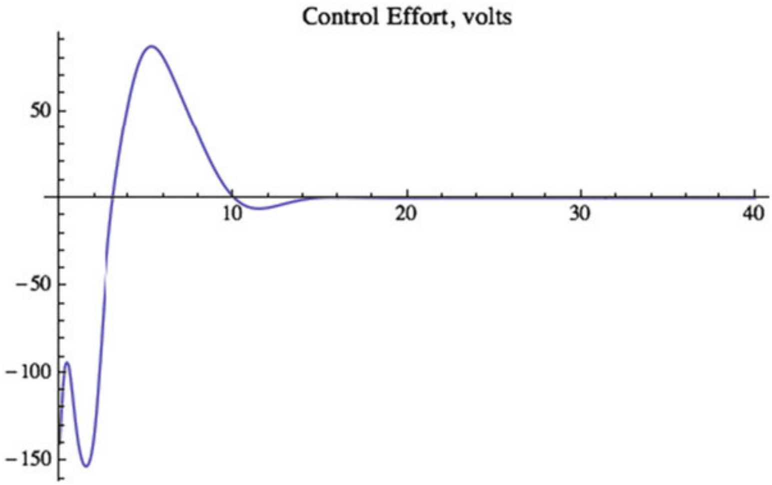


Fig. 8.11 Control voltage (limited to $-180 < e < 180$)

design input voltage. I then took the linear control and inserted it into the nonlinear equations of motion and integrated them numerically to give a simulation. Finally I introduced an additional level of reality by limiting the voltage to less than the control called for.

8.4 Three More Examples

I collected four systems at the end of Chap. 6. We've already explored the simple overhead crane, *S3*. Let's look at the remaining systems here.

Example 8.5 The Servo This system is given by Eq. (6.57)

$$\mathbf{x} = \begin{Bmatrix} \theta' \\ \dot{\theta}' \end{Bmatrix} \Rightarrow \dot{\mathbf{x}} = \begin{Bmatrix} 0 & 1 \\ 0 & -K\alpha \end{Bmatrix} + \alpha \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} e$$

This is a linear system so I can simulate it using these equations (or solve it directly). The controllability matrix is

$$\mathbf{Q} = \{ \mathbf{b} \quad \mathbf{Ab} \} = \begin{Bmatrix} 0 & \alpha \\ \alpha & \alpha^2 K \end{Bmatrix}$$

Its determinant is $-\alpha^2$, so the system is controllable. In this case the system is essentially in companion form from the beginning. The only difference is that the nonzero term in $\mathbf{b}$ is not unity, but α. The transformation matrix is simply the identity matrix divided by α:

$$\mathbf{T} = \frac{1}{\alpha}\mathbf{1}, \quad \mathbf{z} = \frac{1}{\alpha}\mathbf{x}, \quad \mathbf{x} = \alpha\mathbf{z}$$

The eigenvalues of $\mathbf{A}$ are real: 0 and $-K\alpha$. The characteristic polynomial for the combined matrix

$$\mathbf{A}_1 - \mathbf{b}_1\mathbf{g}_z^{\mathrm{T}} = \begin{Bmatrix} 0 & 1 \\ -g_{z1} & -(g_{z2} + K\alpha) \end{Bmatrix}$$

is

$$s^2 + (g_{z2} + K\alpha)s + g_{z1} = 0$$

If we want the eigenvalues to equal to s_1 and s_2, then we need to compare the characteristic polynomial to

$$s^2 - (s_1 + s_2)s + s_1 s_2 = 0$$

from which we get

$$g_{z1} = \frac{g}{l} + s_1 s_2, \quad g_{z2} = K\alpha - (s_1 + s_2)$$

We see that the proportional gain here eliminates the zero eigenvalue and the derivative gain moves negative eigenvalue (which means that if all we care about is stability, we could set it equal to zero, which means, in turn, that we have no need to measure the speed of the load, just its position).

We can convert the $\mathbf{z}$ gains to $\mathbf{x}$ gains so that our simulation will have an input of the form $e = -\mathbf{g}_x{}^{\mathrm{T}}\mathbf{x}$. That result is

$$\mathbf{g}_x^{\mathrm{T}} = \mathbf{g}_z^{\mathrm{T}}\mathbf{T} = \frac{1}{\alpha}\{\, s_1 s_2 \quad -(s_1 + s_2 + K\alpha)\,\}$$

Figure 8.12 shows the motion of a servo moving π/2 units. I took $K = 0.868$, $R = 4$, and $I = 10$, all in SI units, and I took the poles to be $-1 \pm j$. Figure 8.13 shows the input voltage, which is within the design voltage for the motor.

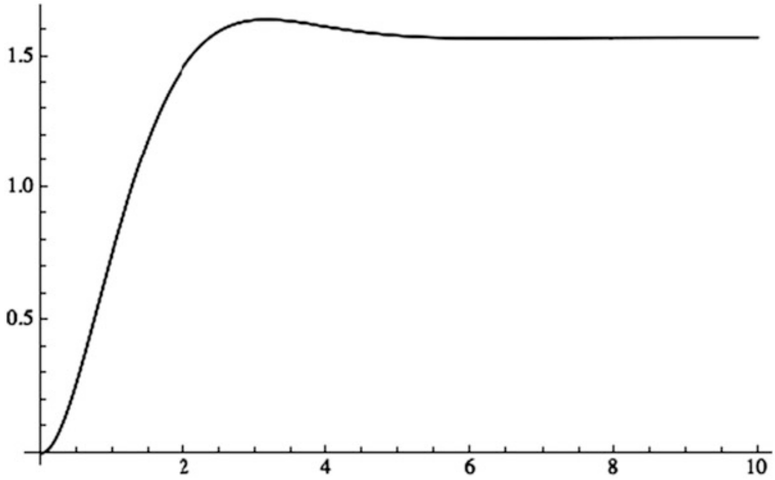


Fig. 8.12 Servo angle vs. time

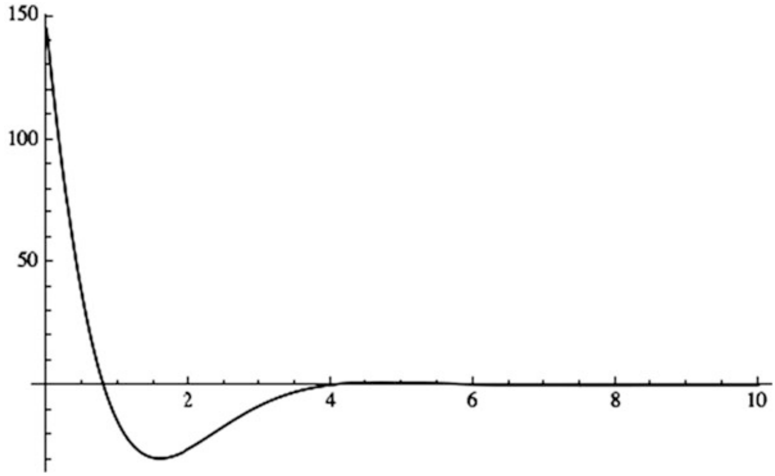


Fig. 8.13 Servo voltage vs. time

Example 8.6 Magnetic Suspension The linear equations for this problem are Eq. (6.59)

$$\mathbf{x} = \begin{Bmatrix} z' \\ \dot{z}' \\ i' \end{Bmatrix}, \quad \mathbf{A} = \begin{Bmatrix} 0 & 1 & 0 \\ -n\dfrac{g}{z_0} & 0 & 2\dfrac{g}{i_0} \\ 0 & 0 & -\dfrac{R}{L} \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{1}{L} \end{Bmatrix}$$

which I need to supplement with the nonlinear equations so that I can build a simulation to assess the control. These come from Eqs. (3.36) and (3.37), rewritten in state space form as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{C_n}{m}\frac{x_3^2}{x_1^n} - g$$
$$\dot{x}_3 = -\frac{R}{L}x_3 + \frac{1}{L}e$$

We design the control following the procedure I have outlined. The controllability matrix is

$$\mathbf{Q} = \begin{Bmatrix} 0 & 0 & 2\dfrac{g}{i_0 L} \\ 0 & 2\dfrac{g}{i_0 L} & -2\dfrac{gR}{i_0 L^2} \\ \dfrac{1}{L} & -\dfrac{R}{L^2} & \dfrac{R^2}{L^3} \end{Bmatrix}$$

which we can see is invertible by inspection (the three columns are independent, as are the three rows). Its inverse is

$$\mathbf{Q}^{-1} = \begin{Bmatrix} 0 & \dfrac{Ri_0}{2g} & L \\ \dfrac{Ri_0}{2g} & \dfrac{Li_0}{2g} & 0 \\ \dfrac{Li_0}{2g} & 0 & 0 \end{Bmatrix}$$

We can build the transformation matrix starting with the last row of this, and that result is

$$\mathbf{T} = \begin{Bmatrix} \dfrac{Li_0}{2g} & 0 & 0 \\ 0 & \dfrac{Li_0}{2g} & 0 \\ -n\dfrac{Li_0}{2z_0} & 0 & L \end{Bmatrix}$$

Table 8.2 Magnetic suspension parameters (SI units)

m	n	C_n	R	L	g
1	2	1	1	0.01	9.81

Applying the transformation to **A** leads to

$$\mathbf{A}_1 = \mathbf{TAT}^{-1} = \begin{Bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -n\dfrac{gR}{z_0 L} & -n\dfrac{g}{z_0} & -\dfrac{R}{L} \end{Bmatrix}$$

I let $u = -\mathbf{g}_z^{\mathrm{T}}\mathbf{z}$ and compare the closed-loop characteristic polynomial, the determinant of

$$s\mathbf{1} - \left(\mathbf{A}_1 - \mathbf{b}_1\mathbf{g}_z^{\mathrm{T}}\right),$$

to the characteristic polynomial with poles at s_1, s_2, and s_3

$$(s - s_1)(s - s_2)(s - s_3) = 0$$

As I noted above, each coefficient in this comparison contains one and only one component of $\mathbf{g}_z$, so we can solve for the **z** gains

$$g_{z1} = -s_1 s_2 s_3 - n\frac{gR}{z_0 L}, \quad g_{z2} = s_1 s_2 + s_1 s_3 + s_2 s_3 - n\frac{g}{z_0}, \quad \text{and}$$
$$g_{z3} = s_1 + s_2 + s_3 + \frac{R}{L}$$

This then needs to be mapped back to a set of **x** gains by multiplying by **T** from the right. The result is a bit scrambled, and I leave its calculation to the problems.

Once all of this is done, we can set $e = -\mathbf{g}_x^{\mathrm{T}}\mathbf{x}$ in the nonlinear differential equations and integrate those numerically to give a simulation of the magnetic suspension control. I use the parameters shown in Table 8.2.
I use the third-order Butterworth poles

$$\mathbf{p}^{\mathrm{T}} = \rho\begin{Bmatrix} -\dfrac{1}{\sqrt{2}}(1-j) & -\dfrac{1}{\sqrt{2}}(1+j) & -1 \end{Bmatrix}$$

which lie on a circle of radius ρ. Figure 8.14 shows the gains as a function of the radius of the pole circle. The bigger the circle, the more rapidly we would expect the control to work and the bigger the control effort needed to obtain that response.

This control works if the initial condition is not too far from the equilibrium. We cannot expect it to work everywhere because it is a linear control for a nonlinear system. Figure 8.15 shows the distance between the ball and the magnet when the

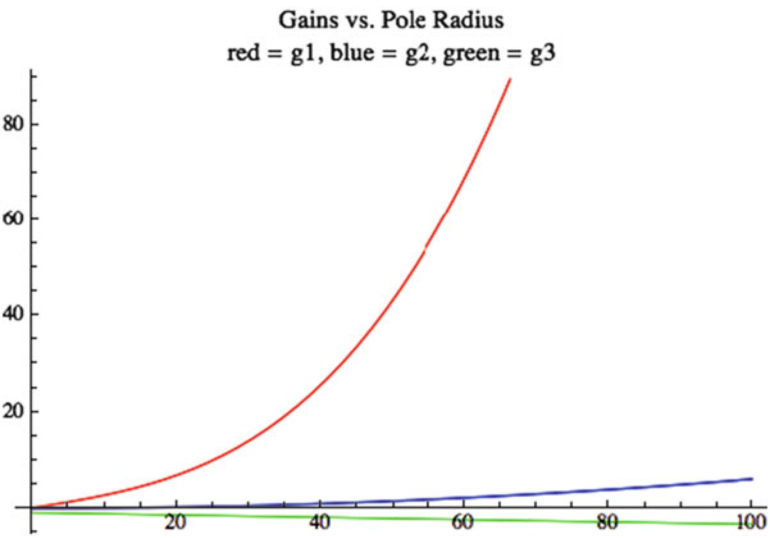


Fig. 8.14 Gains vs. pole-circle radius

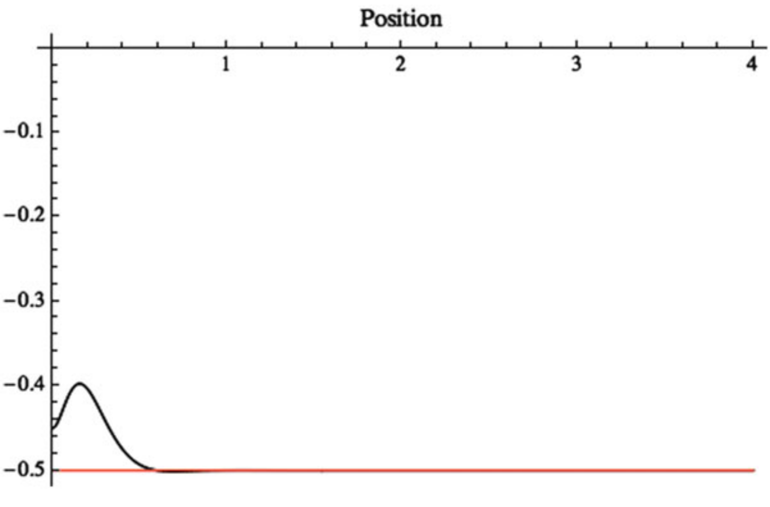


Fig. 8.15 Ball-magnet distance for the ball starting too close to the magnet

ball starts from rest at $z = -0.45$ when the equilibrium is $z_0 = -0.5$, and Fig. 8.16 shows the associated control voltage.

Figures 8.17 and 8.18 show the same things for an initial start at −0.55, too far from the magnet.

The red lines in Figs 8.15, 8.16, 8.17, and 8.18 denote the final equilibrium values. The poles for these examples are on a circle of radius 10. The reader may

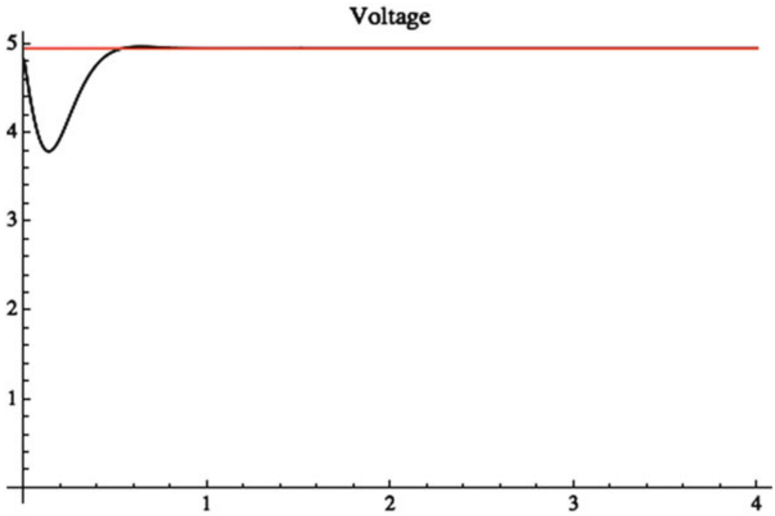


Fig. 8.16 Control voltage for the response shown in Fig. 8.14

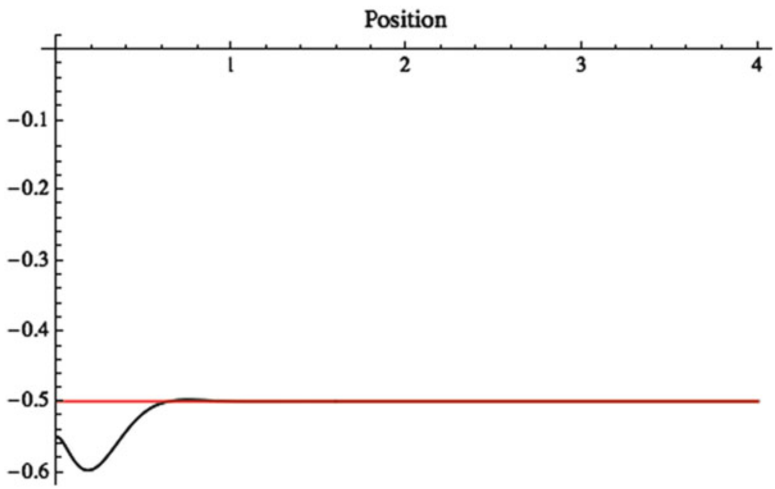


Fig. 8.17 Ball-magnet distance for the ball starting too far from the magnet

note that when the ball is too close to the magnet, its initial motion is toward the magnet, and when the ball is too far from the magnet, its initial motion is away from the magnet. This is consistent with the stability of this system.

It looks as if the linear control works for the nonlinear system, but, like the inverted pendulum, it will fail if the ball is initially too far from its design equilibrium. If the ball is initially closer than about 0.107, the control does not work. The control works for the ball initially very far from the magnet, but

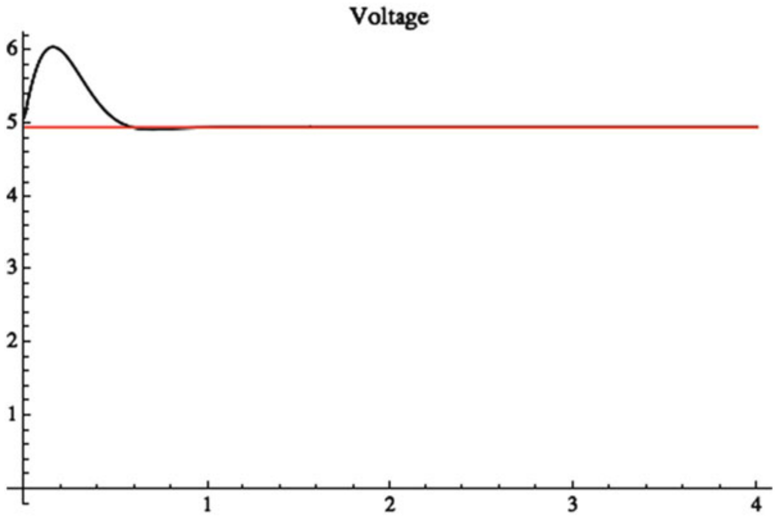


Fig. 8.18 Control voltage for the response shown in Fig. 8.16

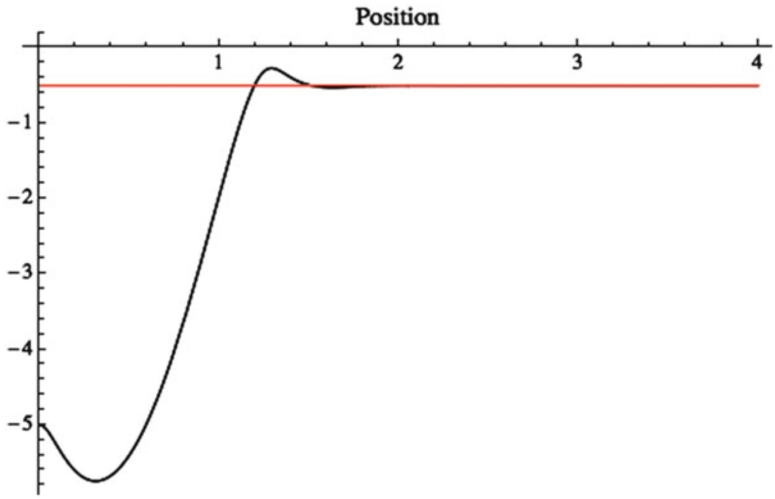


Fig. 8.19 Motion of the ball starting five units from the magnet

requires very large voltages. If the ball is initially five units from the magnet the voltage rises over 60 V, very large compared to the equilibrium voltage of just under five volts. I show the motion of the ball in Fig. 8.19 and the accompanying voltage in Fig. 8.20.

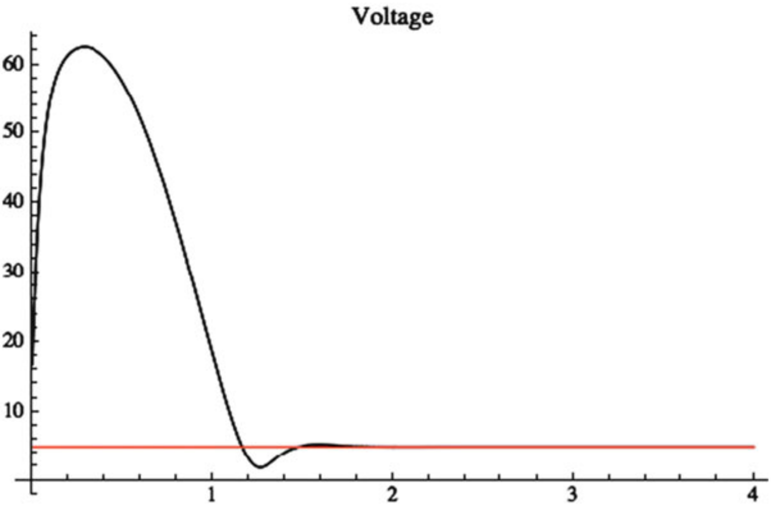


Fig. 8.20 Voltage for the motion shown in Fig 8.19

This is the last example that is small enough to show the intermediate stages in terms of symbols. I will work one more problem before leaving this chapter: the inverted pendulum on a cart. I will use this to illustrate some strategies for placing poles.

8.4.1 More on Pole Placement

We have learned how to choose poles to make a state go to zero. We have not said much about how to do this. We have accepted the idea that we can simply assign Butterworth poles and let it go at that. Are there other options? I haven't emphasized the fact, but some of the open-loop poles may be stable, that is, they may have negative real parts. Indeed, that has been the case for most of our examples so far. Should we move these stable poles or leave them where they are? These questions come under the topic of optimal control, which is beyond the scope of the text, but we ought to spend a little time thinking about these questions. We care about controlling a nonlinear system using a linear control. We'd like to do this with the smallest possible control effort. We need to measure all the elements of the state in order to use full state feedback, as we have been doing so far. This requires sensors. We can eliminate sensors if we can eliminate the need to use any of the elements of the state. I will discuss the estimation of missing state elements in Chap. 9. Example 8.7 allows us to eliminate one element of feedback, and I'll discuss that at the end of this section.

Table 8.3 Parameters for the inverted pendulum (SI units)

M	m	l	r	R	K	g
20	2	1	0.4	4	0.868	9.81

Example 8.7 The Inverted Pendulum on a Cart (System *S4*) I can copy **A** and **b** from Chap. 6

$$\mathbf{x} = \begin{Bmatrix} y \\ \theta \\ \dot{y} \\ \dot{\theta} \\ i \end{Bmatrix}, \quad \mathbf{A} = \begin{Bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\dfrac{mg}{M} & 0 & 0 & -\dfrac{K}{Mr} \\ 0 & \dfrac{(m+M)g}{Ml} & 0 & 0 & \dfrac{K}{Mlr} \\ 0 & 0 & \dfrac{K}{L} & 0 & -\dfrac{R}{L} \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{1}{L} \end{Bmatrix} \tag{6.60}$$

Here M and m denote the masses of the cart and the bob, respectively; l the length of the pendulum; r the effective radius of the cart wheels; g the acceleration of gravity; and K, R, and L denote the motor constant and the armature resistance and inductance. I will use the motor values from Table 8.1, but the pendulum needs to be smaller for that motor to be able to stabilize it in its inverted state. To that end I set $M = 20$, $m = 2$, and $l = 1$. Table 8.3 gives the parameters for this section.

The matrices **Q** and **T** are too complicated to be written out in terms of the parameters, although the determinant of **Q** can be written out in terms of the symbolic parameters

$$\det(\mathbf{Q}) = -\frac{g^2 K^4}{L(MlrL)^4}$$

This cannot be zero, so the system is controllable for any set of parameters. The system input is the armature voltage e.

The uncontrolled ($e = 0$) system is unstable. The eigenvalues of **A** for the values in Table 8.3 are

$$\{-399.895 \quad -3.28991 \quad -0.0951439 \quad 0 \quad 3.28038\}$$

I have ordered these from the smallest to the largest. The first eigenvalue is almost exactly equal to $-R/L$, showing that the motor damping is nearly independent of the rest of the system. Only the last two upset the apple cart and need to be changed. The question is how? We can certainly simply map all the poles to the fifth-order Butterworth poles at radius ρ or we could correct only the two bad poles. I will call these Strategy A and Strategy B, respectively. Either strategy will stabilize the linear problem, but we want to see if we can stabilize the nonlinear problem and keep the control effort low. I will design controls following each strategy and plug them into a simulation. As always the simulation is a numerical integration of the nonlinear equations, given by Eq. (6.44) with the sign of gravity reversed.

$$
\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= \frac{m \sin x_2}{\Delta}\left(l x_4^2 - g \cos x_2\right) + \frac{K}{\Delta} i \\
\dot{x}_4 &= -\frac{\sin x_2}{l\Delta}\left(ml \cos x_2 x_4^2 - (M+m) g\right) - \frac{K \cos x_2}{l\Delta} i \\
\dot{x}_5 &= -\frac{K}{L} x_4 - \frac{R}{L} x_5 + e
\end{aligned}
$$

I consider the behavior of the system starting from rest with $y_0 = 0$, $\theta_0 = \pi/20$, and $i_0 = 0$. Strategy A selects gains to move the five open-loop poles to the fifth-order Butterworth poles on a circle of radius ρ. Numerical experiments (which I urge the reader to duplicate) show that ρ must be large enough, but not too large. I find that Strategy A works in the simulated system in the range of radii $1.042 < \rho < 9.512$. Merely working is not enough. We care about the control effort, the voltage required to stabilize the pendulum. The maximum control voltage at $\rho = 1.042$ is about 142 V. The maximum control voltage at $\rho = 9.512$ is over 1,400. Intermediate radii give lower maximum control voltages. There's a local minimum (of the maximum control voltage) of about 84 V at $\rho = 1.75$, which I determined by numerical experiments. I invite you to set this problem up and see what you can learn about the Butterworth result. I show the response of the system and its control effort (voltage) for the best case $\rho = 1.75$ in Figs. 8.21 and 8.22.

I can implement Strategy B in different ways. The simplest is to leave the pole at -399.985 alone and consider it to be a modification of the $-\rho$ Butterworth pole. We can move the other four poles to a Butterworth circle of smaller radius. Convergence takes much longer. Figure 8.23 shows the response when I keep the large negative eigenvalue and move the other four poles to the complex Butterworth poles for $N = 5$ at a radius of 0.22. The maximum voltage (-47.6 V) occurs at the beginning. Figure 8.24 shows the control voltage.

We can compare the two strategies by looking at time to converge, maximum required voltage, and gains. Strategy A converges about ten times more quickly than Strategy B. Table 8.4 shows the gains and maximum voltage. Note that the maximum voltage does not necessarily correlate with the gains. (Strategy C is a modification of Strategy B that I will explain shortly.)

Note that Strategy B has a much reduced value of the current gain. This suggests a third strategy (Strategy C): set $g_5 = 0$, move four poles to the fourth-order

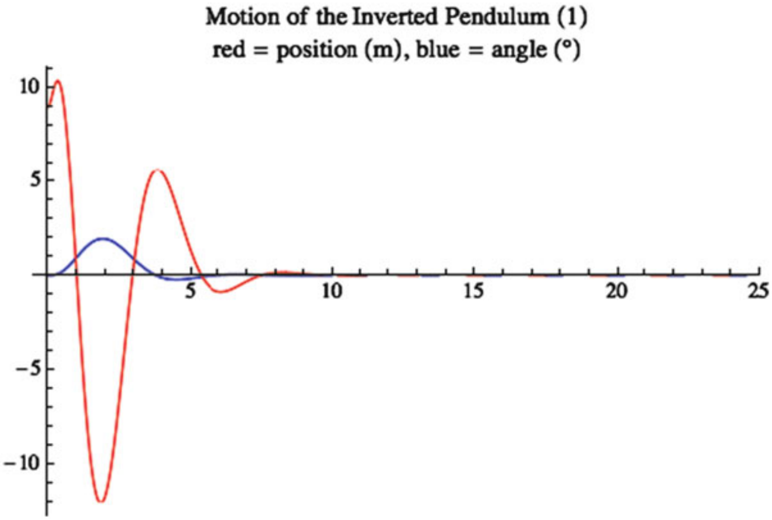


Fig. 8.21 Angle and position of the inverted pendulum under Butterworth poles with radius 1.4 (Strategy A)

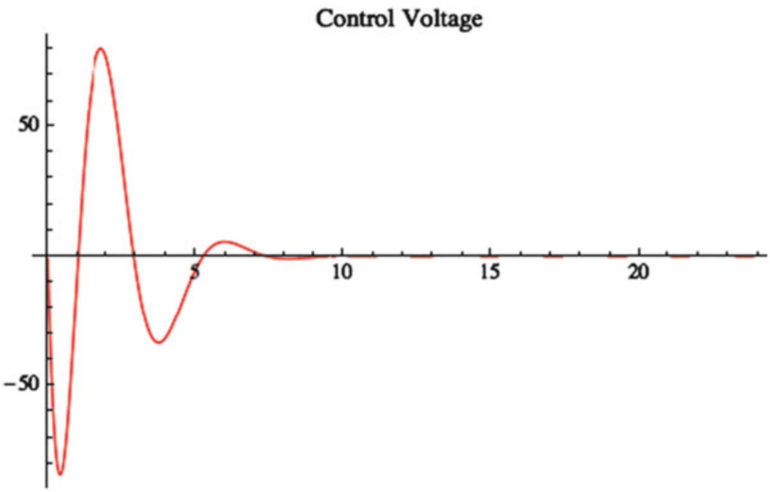


Fig. 8.22 Control voltage of the inverted pendulum under Butterworth poles with radius 1.4. The maximum effort is at the first positive peak (Strategy A)

Butterworth configuration by choosing the other four gains, and solve for the fifth pole. The fifth pole can be written in terms of the other four poles as

$$-\frac{R}{L} - (s_1 + s_2 + s_3 + s_4)$$

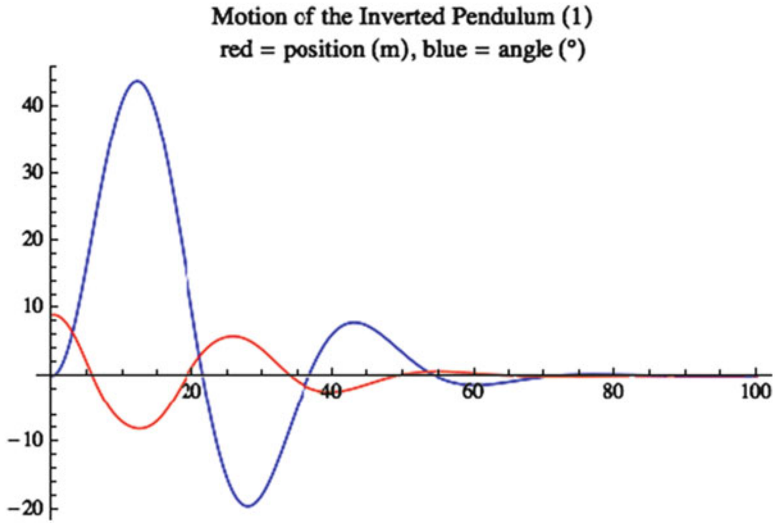


Fig. 8.23 Response for Strategy B

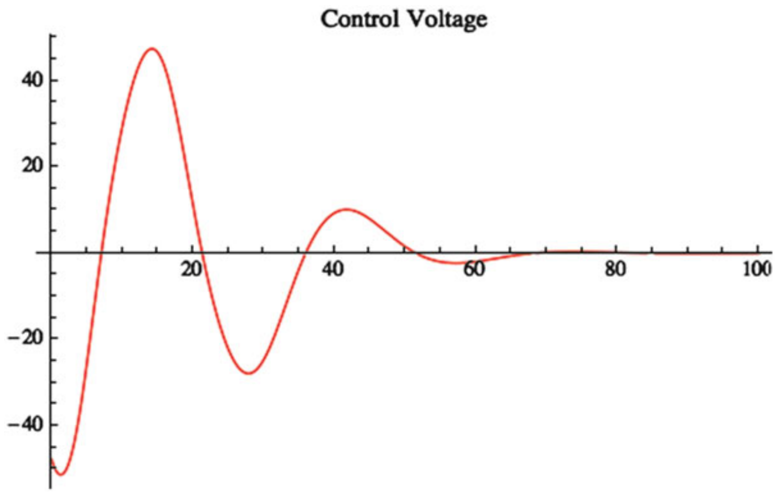


Fig. 8.24 Control voltage for Strategy B

Table 8.4 Comparison of the three control strategies for the inverted pendulum on a cart

	$g_1\ (y)$	$g_2\ (\theta)$	$g_3\ (\dot{y})$	$g_4\ (\dot{\theta})$	$g_5\ (i)$	$V_{\max}$
Strategy A	0.100048	7.0278	3.11297	2.25689	−3.93656	−84.1412
Strategy B	0.006609	303.741	2.97532	17.4554	0.0049638	−47.7115
Strategy C	0.0065914	304.037	2.98556	19.542	0	−47.758

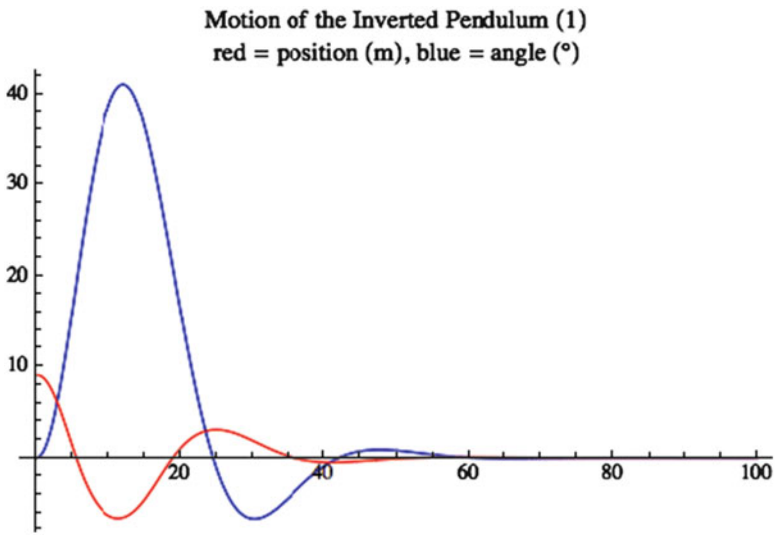


Fig. 8.25 Response of the system for Strategy C

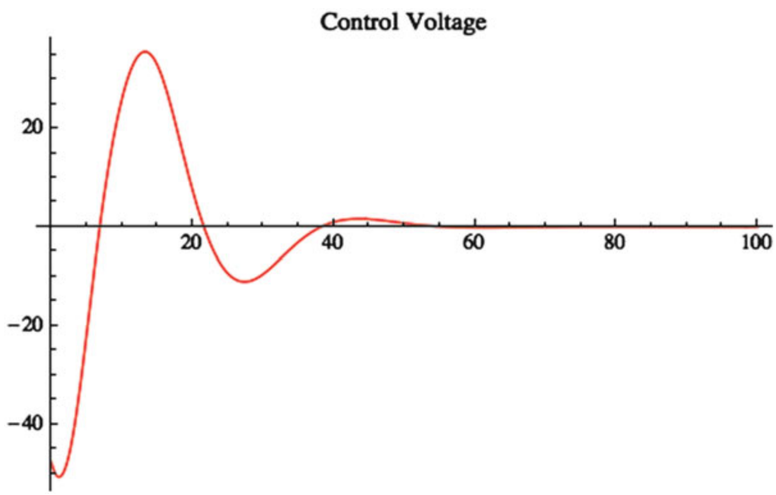


Fig. 8.26 Control voltage for Strategy C

Figures 8.25 and 8.26 show the system response and control voltage for Strategy C with $\rho = 0.22$. Table 8.4 shows that there is very little difference between Strategies B and C, but Strategy C is to be preferred because it eliminates one sensor: there is no need to measure the current.

8.4.2 Disturbances

We saw in Chap. 6 that the cruise control could be improved by adding an integral feedback, which moved the constant error caused by a uniform hill from the speed to the position, about which we didn't care all that much. Can we extend this to our more complicated nonlinear problems? The answer, at least in some cases, is yes. We start with the nonlinear equations of motion and whatever their equilibrium solution is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})u, \quad \mathbf{f}(\mathbf{x}_0) + \mathbf{b}(\mathbf{x}_0)u_0 = 0 \tag{8.27}$$

We add a variable, the integral of x_1, in such a way that the extended equilibrium is still steady—no explicit appearance of the time. The extra equation for the extra variable is

$$\dot{x}_0 = x_1 - x_{01} \tag{8.28}$$

The system is now an $N+1$-dimensional system. It can be linearized just as the original system was linearized. If the new system is controllable, then we can design a control in exactly the same way, with a new set of poles (having one more pole than the original set). There will be feedback from x_0 as well as the components of the original vector $\mathbf{x}$. The error in response to a constant disturbance will move from x_1 to x_0. The systems I've been addressing in this chapter are much more complicated than the cruise control, so there may be other consequences as well. Let's look at this in terms of the magnetic suspension problem from Ex. 8.6.

Example 8.8 Disturbing the Magnetic Suspension System We consider the same system as for Ex. 8.6, but we allow a possible disturbance. Figure 8.27 shows the position of the ball if the voltage is disturbed by a constant 0.1 V. The system remains in equilibrium, but the equilibrium position of the ball is much closer to the surface of the magnet, and the final voltage, shown in Fig. 8.28, is much less than the voltage required to hold it at the original equilibrium of −0.5.

Figure 8.29 shows the response for a harmonic disturbance of 0.1 V at a frequency of 5, and Fig. 8.30 shows the corresponding voltage. This is a serious disturbance. Not only does it show large fluctuations, but the mean error is −0.0436, close to 10 % of the equilibrium position. (It is worth noting that higher frequency disturbances are unimportant. The response at a frequency of 50 is just a tiny ripple, shown in Fig. 8.31.)

Now let's look at the four-dimensional version of this problem. It is trivial to add the new equation to the set. We'll have

$$\dot{x}_0 = x_1 - z_0 \tag{8.29}$$

where, in our case, $z_0 = -0.5$. The rest of the adjustments are nontrivial. We can linearize, obtaining

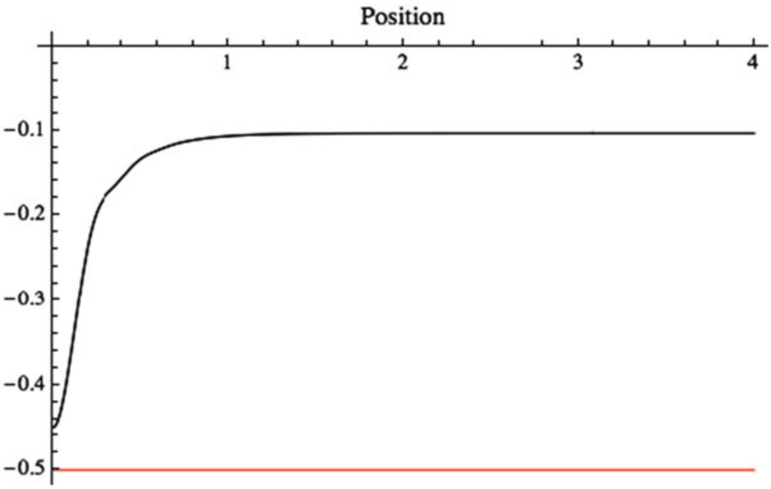


Fig. 8.27 Position of the ball when the voltage is disturbed by a constant 0.1 V. The *red line* denotes the equilibrium voltage for the undisturbed system. The pole radius was 10

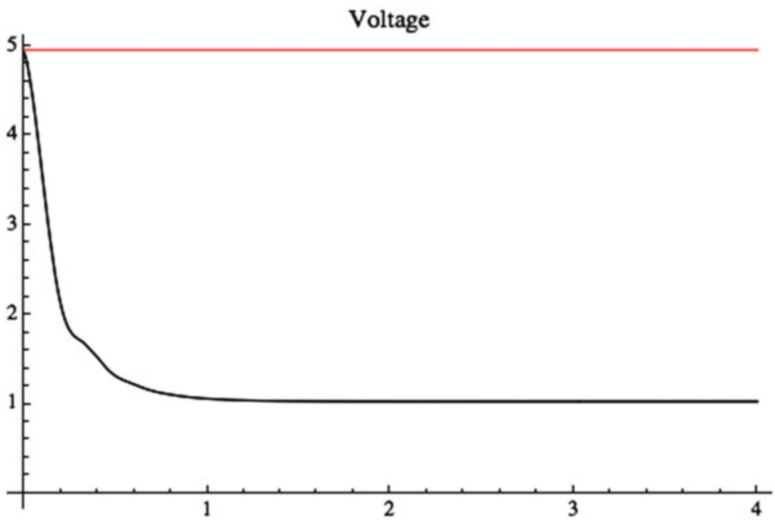


Fig. 8.28 The control voltage for the position shown in Fig. 8.27. The *red line* is the equilibrium voltage for the undisturbed case

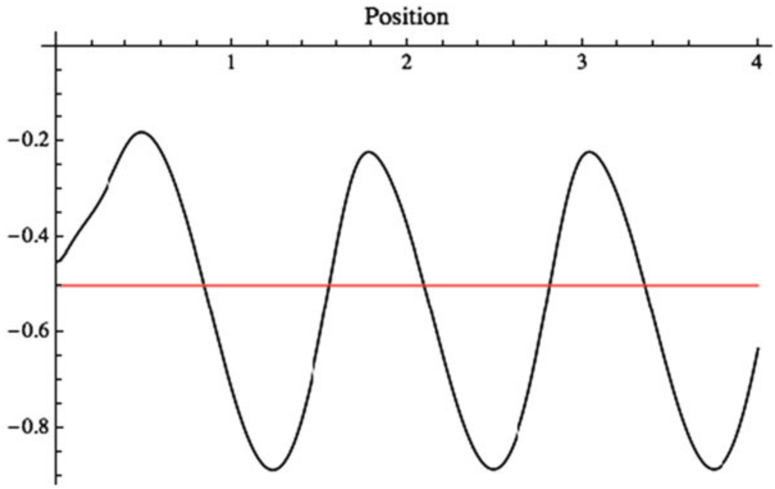


Fig. 8.29 Response at a frequency of 5

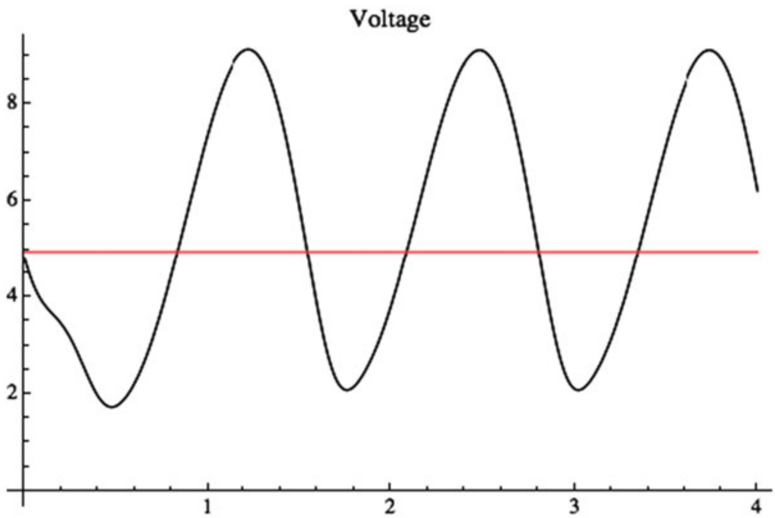


Fig. 8.30 Voltage for the response shown in Fig. 8.29

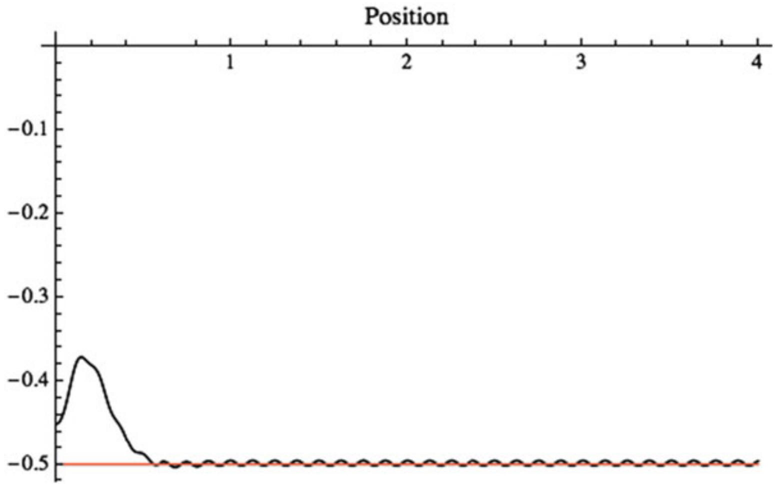


Fig. 8.31 Response at a frequency of 50

$$\mathbf{A} = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\dfrac{2g}{z_0} & 0 & -\dfrac{2}{z_0}\sqrt{\dfrac{cg}{m}} \\ 0 & 0 & 0 & -\dfrac{R}{L} \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{L} \end{Bmatrix} \tag{8.30}$$

The eigenvalues of **A** include one unstable eigenvalue and a zero eigenvalue that comes from the added variable. The system is controllable. I will spare you the manipulations required to find the gains. If I choose fourth-order Butterworth poles on a circle of radius ρ, these gains are 10^{-3} times

$$0.252409\rho^4, 0.155367(1972 + 5\rho^2), 0.0504819(1962 + 21.1803\rho^2), -1000 + 30.776\rho^2$$

(I give them only in case you want to find them yourself and want to check the result.) The input is the usual, expressed in terms of the four-dimensional state vector

$$e = e_0 - \mathbf{g}^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_0)$$

Figure 8.32 shows the position of the ball subject to a constant disturbance of 0.1 V.

You can see that the ball is stabilized at its desired equilibrium. The error has been moved to the integral, shown in Fig. 8.33. The integral goes to zero in the absence of a disturbance, although I do not show that here.

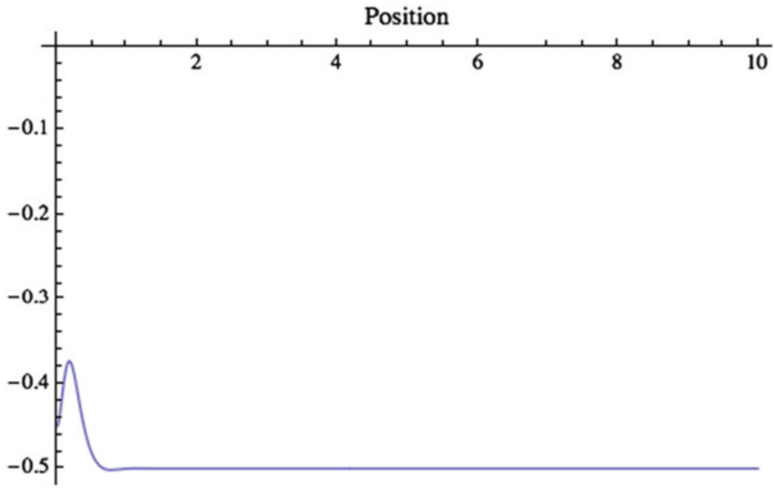


Fig. 8.32 Position of the ball under a constant disturbance of 0.1 V

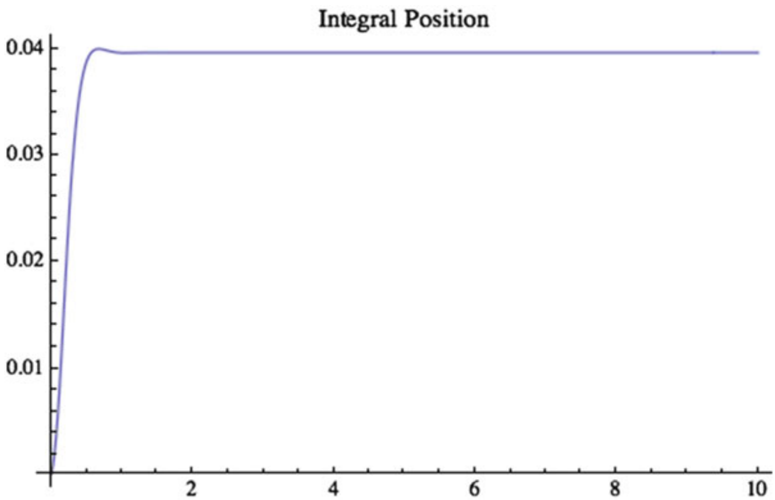


Fig. 8.33 The integral of the position, showing the transfer of the error from the position to the integral

We saw in an earlier work that the integral gain cannot entirely suppress harmonic disturbances, but Fig. 8.34 shows the response at a frequency of 5, and you can see that the oscillations are less than half those without the added integral gain, and the mean error has been reduced to −0.003937, more than a factor of ten smaller.

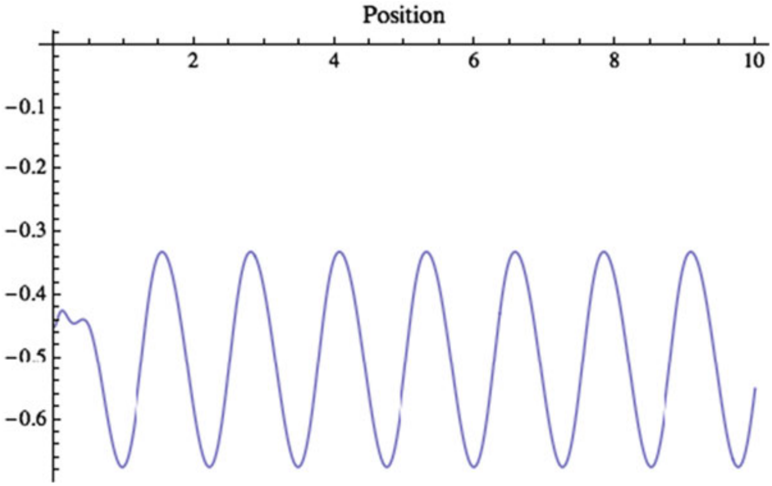


Fig. 8.34 Response to a disturbance at a frequency of 5

8.5 Summary

We considered electromechanical systems (which can include purely mechanical systems). We found equilibrium positions for these systems, whether stable, marginally stable, or unstable, that can be used as reference configurations. These can be written in state space as $\mathbf{x}_0$, a constant. All that is necessary for such a reference position is that we can write the equilibrium as

$$0 = \mathbf{f}(\mathbf{x}_0) + \mathbf{b}(\mathbf{x}_0)u_0$$

That is, it must satisfy the nonlinear equations exactly. We linearize the system about this equilibrium and write a set of equations governing the departure from equilibrium, which I will also symbolize by $\mathbf{x}$. These equations are in our usual standard form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

We determine controllability by looking at the controllability matrix

$$\mathbf{Q} = \left\{ \mathbf{b} \quad \mathbf{Ab} \quad \mathbf{AAb} \quad \cdots \quad \mathbf{A}^{(N-1)}\mathbf{b} \right\}$$

The system is controllable if the determinant of $\mathbf{Q}$ is nonzero. In that case we build a transformation matrix $\mathbf{T}$ starting with the last row of the inverse of $\mathbf{Q}$. (Don't forget to take the inverse!) We have

$$\mathbf{T}_1^{\mathrm{T}} = \{0 \quad 0 \quad 0 \quad \cdots \quad 1\}\mathbf{Q}^{-1}, \quad \mathbf{T}_2^{\mathrm{T}} = \mathbf{T}_1^{\mathrm{T}}\mathbf{A}, \quad \mathbf{T}_3^{\mathrm{T}} = \mathbf{T}_2^{\mathrm{T}}\mathbf{A}, \cdots$$

$$\mathbf{T} = \begin{Bmatrix} \mathbf{T}_1^{\mathrm{T}} \\ \mathbf{T}_2^{\mathrm{T}} \\ \mathbf{T}_3^{\mathrm{T}} \\ \vdots \\ \mathbf{T}_N^{\mathrm{T}} \end{Bmatrix}$$

We can then use **T** and its inverse to put **A** and **b** into companion form

$$\mathbf{A}_1 = \mathbf{TAT}^{-1}, \quad \mathbf{b}_1 = \mathbf{Tb}$$

(Of course the form of $\mathbf{b}_1$ is known, but it's helpful to calculate it as a sanity check.) The original linear equation becomes

$$\mathbf{z} = \mathbf{Tx}, \quad \dot{\mathbf{z}} = \left(\mathbf{A}_1 - \mathbf{b}_1\mathbf{g}_z^{\mathrm{T}}\right)\mathbf{z}$$

where I have introduced gains in **z** space, writing

$$u = -\mathbf{g}_z^{\mathrm{T}}\mathbf{z}$$

I choose the gains to place the poles of the combined matrix

$$\left(\mathbf{A}_1 - \mathbf{b}_1\mathbf{g}_z^{\mathrm{T}}\right)$$

in the left half plane to make the **z** system converge to $\mathbf{z} = 0$, usually by assigning Butterworth poles of the appropriate order, but any choice of poles with negative real parts that occur in complex conjugate pairs will stabilize the linear problem. These poles will be the same as the poles of the **x** space version

$$\left(\mathbf{A} - \mathbf{b}\mathbf{g}_x^{\mathrm{T}}\right)$$

The **z** gains need to be mapped back to **x** space

$$\mathbf{g}_x^{\mathrm{T}} = \mathbf{g}_z^{\mathrm{T}}\mathbf{T}$$

and the control inserted into the nonlinear equations to simulate the system to assess the utility of the control.

Finally I introduced an analog of integral control as originally given as part of a linear PID control and showed that it worked the same way for the magnetic suspension problem, moving a constant error from the position to its integral.

Exercises

1. Write the transfer function between f and y for Ex. 8.3. What is the corresponding differential equation?
2. What is the transfer function between e and i for the magnetic suspension problem?
3. Does the system in Ex. 8.3 become controllable if the one of the masses is different from the other two? If so, design a control to drive y to zero from a nonzero position. Does it matter which mass is different?
4. Determine the controllability of the system in Ex. 8.3 if proportional damping is added.
5. Verify that the characteristic polynomial for a matrix in companion form has the form stated in the text.
6. Compare the magnetic suspension control derived in Chap. 6 using a transfer function to that in Ex. 8.6. Can you write simulations for both? Is one better than the other (more rapid convergence for the same control effort, say)?
7. Consider the simple inverted pendulum stabilized by a torque at its base. Suppose there is a constant disturbance in the form of an added torque. Find a control that will compensate for this disturbance, and investigate how it works in simulation.
8. A position error in moving the overhead crane would be serious. The crane could miss its target or crash into the stop. Expand the overhead crane system by adding an integral term, find the resulting linear control, and verify that it works in simulation.
9. Set up a double inverted pendulum to be controlled by a torque at its base. (Assume the pendulums are simple and don't worry about the source of the torque.) Is it controllable? If so, design a control and test it by writing a simulation. How small must the initial angles be to ensure convergence to the inverted position? (This can be answered by numerical experiment.)
10. Repeat Ex. 8.5 with the torque supplied by a motor. Size the pendulums such that the motor we have been using in the text can provide the torque necessary and include the effects of inductance.
11. Consider another strategy for the inverse pendulum on a cart. Set $g_5 = 0$, leave the two good poles alone, and move the two bad poles to a second-order Butterworth configuration. Will this work? If so, implement it and verify that it works.
12. Consider the inverse pendulum on a cart and replace the motor driving the cart by a motor mounted on the cart that can apply a torque at the base of the pendulum. Is this system controllable? If so, write a stabilizing control. If not, can you at least keep the pendulum erect?
13. Is the overhead crane controllable if the single pendulum is replaced by a double pendulum?

14. Write a control that will hold the pendulum in Ex. 6.4 45° from the vertical, as shown in Fig. 6.14.
15. Consider a three-car train where the cars are coupled by identical spring-damper couplings. Is the position of the third car controllable if the front car is driven by a DC motor? If so, write a control to move the third car a fixed distance Y.
16. Consider a unicycle on the top of a hill. Ignore balancing problems and write a control for the torque on the wheel that will hold the unicycle stationary on the top of the hill. Use a parabolic hill $\frac{h}{h_0} = 1 - \left(\frac{y}{h_0}\right)^2$.
17. Turn the magnetic suspension problem upside down and suppose the ball to be magnetized so that the force between the ball and the magnet is repulsive. Find the equilibrium (suppose that the sphere cannot rotate). Assume an inverse fourth law. Is the equilibrium stable? Design a control to move the ball from $z = z_1$ to $z = z_2 > z_1$.
18. Consider a vertical massless rod with a sphere on top (motion is confined to the plane). Can this configuration be stabilized by a horizontal force at the bottom of the rod? If so, write the control. (This is not just the inverted pendulum; the sphere is free to move with respect to the rod.)
19. Can you balance one cylinder on top of another by moving the base horizontally? Assume everything rolls without slipping.
20. Repeat problem 19 with three cylinders stacked atop one another.

The following exercises are based on the chapter, but require additional input. Feel free to use the Internet and the library to expand your understanding of the questions and their answers.

21. Find a control in the form of a torque at the base of a cantilever beam that will return the beam to a horizontal equilibrium position if it is disturbed by some impulse. Treat the beam as an elastic beam and use what you know about transverse oscillations of a beam from Chap. 4.
22. Consider there to be a transverse force at the base of a ten-story building. Design a control for that force to return the building to stationary equilibrium after a transverse impulse.
23. Design a motor that will serve to control an inverted pendulum 10 feet long with a 100 pound bob. Suppose that the pendulum is never to be displaced from its equilibrium by more than 5°.
24. Design a control in the form of a base torque to stabilize an inverted triple pendulum. How large an initial disturbance can this control handle?
25. Design a control for a double inverted pendulum on a cart. Take the inductance of the cart motor into account. Pick a motor that will work. How large a disturbance can the control stabilize?

26. What happens to the inverted pendulum on a cart if a bird lands on the bob? Suppose the mass of the bird to be the same as that of the bob. Choose initial conditions you find interesting and explain what happens. Under what circumstances will the control continue to function?

Reference

Strang G (1988) Linear algebra and its applications, 3rd edn. Saunders HBJ, Philadelphia, PA, §5.6

Observers

9

In which we learn how to implement a full state feedback control when we don't know (can't measure) the full state. . ..

9.1 Introduction

We have learned how to design controls to stabilize systems, that is, drive them to a constant reference state. Almost all of these controllers required knowledge of the entire state at all times. They were full state feedback systems, in which the error from each element of the state entered the feedback loop. It is seldom possible (or practical) to instrument a physical system completely. We generally cannot measure every element of the state. Perhaps we can only measure a lower dimensional output—often a one-dimensional output. We must devise estimates of the missing measurements if we are to use a full state feedback control. We do this by constructing an artificial auxiliary dynamical system, the output of which is an estimate of the state. This system generally has the same dimensions as the actual state. It is called an *observer*.

The bulk of this section is limited to SISO systems, and most of the general procedures apply only to such systems. The single-output measurement is, of course, the most restrictive. I also suppose that the goal of the control is to drive a state variable to zero. Any control driving a state vector to a fixed point can be reduced to this as we saw in Chap. 8. I will address tracking control in Chap. 10, including tracking control using an observer.

R.F. Gans, *Mechanical Systems: A Unified Approach to Vibrations and Controls*,
DOI 10.1007/978-3-319-08371-1_9

9.2 General Analysis

9.2.1 Linear Systems

I will start by deriving the observer for a simple linear single-input-single-output (SISO) system that is to be controlled to zero. The system may have been derived from a nonlinear system by linearization. If so it must be assessed using a simulation. I will discuss nonlinear systems later in this section. The general linear SISO system is the familiar system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad y = \mathbf{c}^{\mathrm{T}}\mathbf{x} \tag{9.1}$$

where $\mathbf{x}$ denotes the state; $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{c}^{\mathrm{T}}$ have their usual meanings; and y denotes the scalar output, which I will suppose to be the only measurable element of the state. Figure 9.1 shows a vector block diagram of this system for future reference. The thick lines denote vectors and the thin lines denote scalars. You should be able to construct Fig. 9.1 from Eq. (9.1) and vice versa.

We always need to assess controllability when tackling a control problem. I assume the system to be controllable, so that we can find a set of gains in terms of its desired poles, and we can write a control as we learned in Chap. 8.

$$u = -\mathbf{g}^{\mathrm{T}}\mathbf{x}$$

Now suppose that $\mathbf{x}$ is not completely known. Suppose that only the single output

$$y = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

is known. We will have to write the input in terms of an estimate for $\mathbf{x}$

$$u = -\mathbf{g}^{\mathrm{T}}\hat{\mathbf{x}}$$

where $\hat{\mathbf{x}}$ denotes the estimate. The gains will be those we calculated from the original problem, Eq. (9.1).

I suppose that the estimate will satisfy a dynamical system closely related to that of the actual state. After all, we want the estimate to be as close to the actual state as possible. The dynamical system for the estimated state will be driven by the input and also by the output of the actual state. Such a system looks like

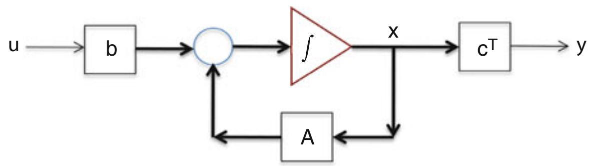


Fig. 9.1 The basic SISO vector system showing the input and output

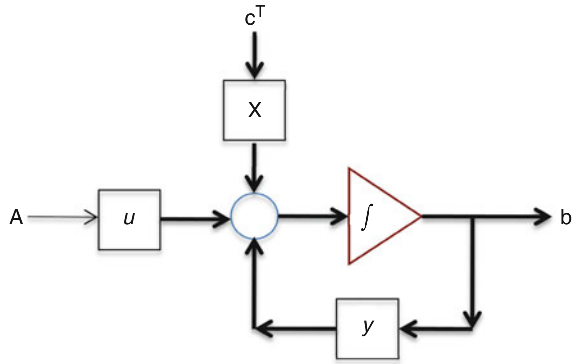


Fig. 9.2 Block diagram of the observer. Note the two inputs, one from u and one from y

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{b}}u + \mathbf{k}y \tag{9.2}$$

where the matrix and the two vectors on the right-hand side are to be determined by requiring that the difference between the state and its estimate go to zero as time goes to infinity. $\hat{\mathbf{A}}$ has the same dimensions as $\mathbf{A}$, and $\hat{\mathbf{b}}$ and $\mathbf{k}$ denote column vectors with the same dimensions as $\mathbf{b}$. We need a closed-loop (homogeneous) system for the error, and its poles must lie in the negative half plane. Figure 9.2 shows a block diagram of the observer—the system represented by Eq. (9.2). It has two scalar inputs—u and y—the latter being the connection between the observer and the thing being observed. The output of the observer is the entire estimated state.

The observer has two scalar inputs and a vector output of the same dimension as the state. Figure 9.3 shows the two systems coupled together. The output of this combined system is $\hat{\mathbf{x}}$ and its input is u.

The system shown in Fig. 9.3 is still open loop because the input is not connected to the output. The input u drives both systems. I will discuss the combined system in some detail below.

We can obtain the equation for the vector error $\mathbf{e}$ by taking the difference of the two systems.

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{A}\mathbf{x} + \mathbf{b}u - \hat{\mathbf{A}}\hat{\mathbf{x}} - \hat{\mathbf{b}}u - \mathbf{k}y$$

I can write the output, y, and the estimate, $\hat{\mathbf{x}}$, in terms of $\mathbf{x}$ and $\mathbf{e}$. This allows me to write an error equation in terms of $\mathbf{x}$, $\mathbf{e}$, and u:

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e} + \left(\mathbf{A} - \hat{\mathbf{A}} - \mathbf{k}\mathbf{c}^{\mathrm{T}}\right)\mathbf{x} + \left(\mathbf{b} - \hat{\mathbf{b}}\right)u \tag{9.3}$$

This will be a nice homogeneous equation for the evolution of the error if the second and third terms on the right-hand side were to vanish, and we can choose $\hat{\mathbf{A}}$ and $\hat{\mathbf{b}}$ to ensure this:

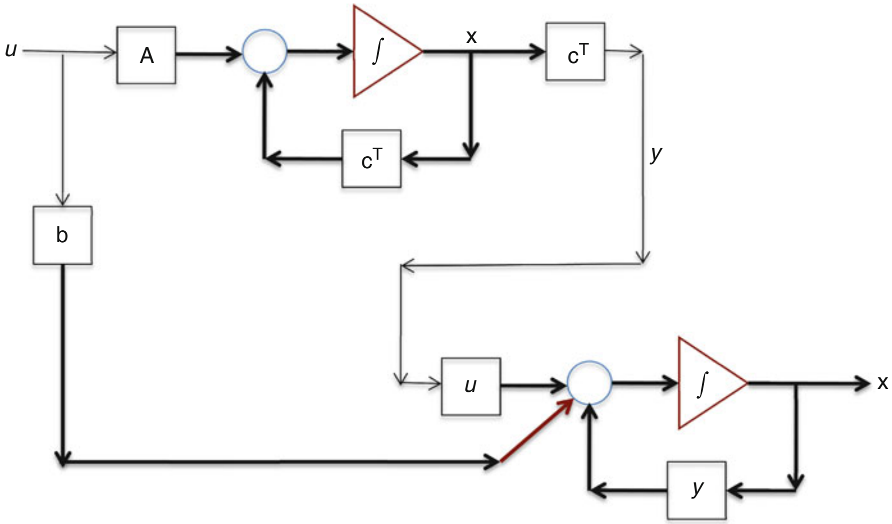


Fig. 9.3 The system and its observer

$$\hat{\mathbf{A}} = \mathbf{A} - \mathbf{k}\mathbf{c}^{\mathrm{T}}, \quad \hat{\mathbf{b}} = \mathbf{b} \tag{9.4}$$

leaving us an evolution equation for the error (Eq. 9.5) in terms of the original **A** matrix, which we know, and the **k** vector, which it is our job to determine in such a way that the error converges to zero.

$$\dot{\mathbf{e}} = \left(\mathbf{A} - \mathbf{k}\mathbf{c}^{\mathrm{T}}\right)\mathbf{e} \tag{9.5}$$

We can make the error disappear by the proper selection of poles of the matrix in Eq. (9.5). If those poles have negative real parts, then $\mathbf{e} \to 0$. The question arises as to whether this can be done and if so, how. The vector **k** has as many elements as the poles to be placed, so it is not impossible on its face. Let's consider this in more detail. Write the matrix on the right-hand side of Eq. (9.5) to show the shapes of the various terms as Eq. (9.6a).

$$\hat{\mathbf{A}} = \left(\mathbf{A} - \mathbf{k}\mathbf{c}^{\mathrm{T}}\right) \Leftrightarrow \left\{ \quad \mathbf{A} \quad \right\} - \left\{ \mathbf{k} \right\} \left\{ \quad \mathbf{c}^{\mathrm{T}} \quad \right\} \tag{9.6a}$$

This looks just like the equation that we use to find gains, with **k** playing the role of **b** and $\mathbf{c}^{\mathrm{T}}$ playing the role of $\mathbf{g}^{\mathrm{T}}$. Of course, what we want to do is to select **k**; we cannot select $\mathbf{c}^{\mathrm{T}}$, which comes from the original system, Eq. (9.1). Fortunately the eigenvalues of the transpose of a matrix are equal to those of the matrix itself. The transpose of Eq. (9.6a) is Eq. (9.6b)

$$\hat{\mathbf{A}}^{\mathrm{T}} = \left(\mathbf{A} - \mathbf{k}\mathbf{c}^{\mathrm{T}}\right)^{\mathrm{T}} \Leftrightarrow \left\{ \quad \mathbf{A}^{\mathrm{T}} \quad \right\} - \left\{ \mathbf{c} \right\} \left\{ \quad \mathbf{k}^{\mathrm{T}} \quad \right\} \tag{9.6b}$$

and now we have the exact analog of the equation we used to determine the gains in the first half of this process, and we can write the analog of the controllability matrix

$$\mathbf{N}^{\mathrm{T}} = \left\{ \mathbf{c} \quad \mathbf{A}^{\mathrm{T}}\mathbf{c} \quad \mathbf{A}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{c} \quad \cdots \right\} \tag{9.7}$$

We can place the poles for the observer if and only if this matrix is of full rank. The common definition of the *observability matrix* is the transpose of this

$$\mathbf{N} = \left\{ \mathbf{c}^{\mathrm{T}} \quad \mathbf{c}^{\mathrm{T}}\mathbf{A} \quad \mathbf{c}^{\mathrm{T}}\mathbf{A}\mathbf{A} \quad \cdots \right\} \tag{9.8}$$

which is why I chose the notation I did for the first form of the observability matrix. The observability matrix **N** can be calculated directly from the terms in Eq. (9.1) without the need for the transpose of **A** or **c**. We can determine controllability and observability directly from the state space version of the original (linearized if necessary) problem. We can construct an observer for a system if and only if it is observable, and it is observable if and only if the observability matrix **N** given by Eq. (9.8) is of full rank.[1] In that case we can put an effective equivalent of the error equation, Eq. (9.5), into companion form or, rather, we can put the matrix Eq. (9.6b) into companion form by a transformation akin to that we used for gain calculation. Let me review that process briefly.

We started with the standard single-input system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

We used **A** and **b** to determine controllability, and if the system was controllable, we found the transformation **T** that converted the problem to companion form

$$\mathbf{z} = \mathbf{T}\mathbf{x} \Rightarrow \dot{\mathbf{z}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\mathbf{z} + \mathbf{T}\mathbf{b}u = \mathbf{A}_1\mathbf{z} + \mathbf{b}_1 u$$

where $\mathbf{A}_1$ and $\mathbf{b}_1$ denote the companion forms of **A** and **b**. We then let

$$u = -\mathbf{g}_z^{\mathrm{T}}\mathbf{z}$$

and compared the coefficients of the characteristic polynomial of

$$\mathbf{A}_1 - \mathbf{b}_1\mathbf{g}_z^{\mathrm{T}}$$

[1] I use the rank condition because this is extensible to systems with more than one input.

to those of the desired polynomial

$$(s - s_1)(s - s_2)\cdots(s - s_N) = 0$$

to determine the gains in **z** space in terms of the desired poles. These gains then need to be transformed back to **x** space by multiplying from the right by **T**.

$$\mathbf{g}^{\mathrm{T}} = \mathbf{g}_x^{\mathrm{T}} = \mathbf{g}_z^{\mathrm{T}}\mathbf{T}$$

Remember that the gains that we find from $\mathbf{A}_1$ and $\mathbf{b}_1$ need to be transformed back to the original **x** space.

Let's apply this to the observer problem. Consider the matrix in Eq. (9.6b), which we have seen is analogous to the closed-loop matrix for the gain problem. We can write it compactly as

$$\mathbf{A}^{\mathrm{T}} - \mathbf{c}\mathbf{k}^{\mathrm{T}}$$

I want to transform this to its companion form, which I can do if it is observable. We can form the transformation matrix in the same way that we formed the transformation matrix for the usual gain calculation. Construct $\mathbf{N}^{\mathrm{T}}$ according to Eq. (9.7). Take the inverse of $\mathbf{N}^{\mathrm{T}}$, which exists if the system is observable. Let the first row of the transformation matrix be equal to the last row of the inverse of $\mathbf{N}^{\mathrm{T}}$. We form the successive rows of the transformation matrix, which I will denote by $\hat{\mathbf{T}}$, by repeated multiplications from the right by $\mathbf{A}^{\mathrm{T}}$ (see Eq. (9.9)). The vector $\mathbf{k}^{\mathrm{T}}$ plays the role of $\mathbf{g}^{\mathrm{T}}$, and so it is reasonable to call its elements gains. I'll call them *observer gains* to distinguish them from the elements of $\mathbf{g}^{\mathrm{T}}$, which we can just call gains.

$$\begin{gathered}\hat{\mathbf{T}}_1 = \{0 \quad 0 \quad \cdots \quad 1\}\left(\mathbf{N}^{\mathrm{T}}\right)^{-1}\\ \hat{\mathbf{T}}_2 = \hat{\mathbf{T}}_1\mathbf{A}^{\mathrm{T}}\\ \vdots\\ \hat{\mathbf{T}}_N = \hat{\mathbf{T}}_{(N-1)}\mathbf{A}^{\mathrm{T}}\end{gathered} \tag{9.9}$$

Once we have found $\hat{\mathbf{T}}$ the analysis proceeds as usual: find the observer gains in the transformed space, map them back to the original observer space ($\mathbf{k}^{\mathrm{T}} \Rightarrow \mathbf{k}^{\mathrm{T}}\hat{\mathbf{T}}$), and solve the two systems simultaneously. To define the observer gains, multiply from the left by the transformation matrix $\hat{\mathbf{T}}$ and from the right by the inverse of the transformation matrix

$$\hat{\mathbf{T}}\mathbf{A}^{\mathrm{T}}\hat{\mathbf{T}}^{-1} - \left(\hat{\mathbf{T}}\mathbf{c}\right)\left(\mathbf{k}^{\mathrm{T}}\hat{\mathbf{T}}^{-1}\right) = \left(\mathbf{A}^{\mathrm{T}}\right)_1 - \mathbf{c}_1\mathbf{k}'^{\mathrm{T}} \tag{9.10}$$

Here $\mathbf{c}_1$ is the companion form of **c**: all but the last component is zero, and the last component is unity. $(\mathbf{A}^{\mathrm{T}})_1$ is the companion form of $\mathbf{A}^{\mathrm{T}}$, and the parentheses are to emphasize that this is to be read *A transpose one*, not *A one transpose*. Finally, $\mathbf{k}'^{\mathrm{T}}$ denotes the observer gains in the transformed observer space, which we can calculate by comparing the actual characteristic polynomial to the desired

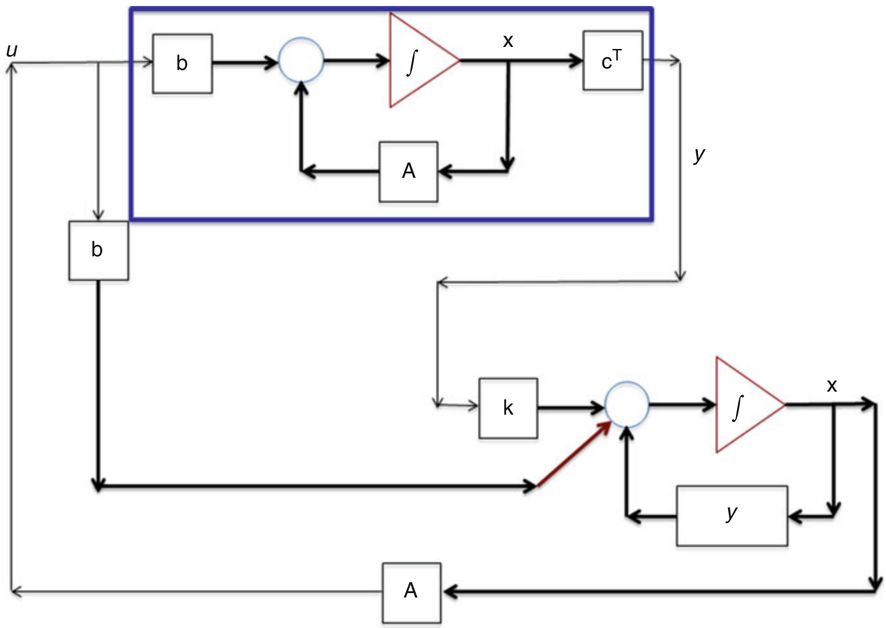


Fig. 9.4 The complete control system for a general linear SISO system with an observer. The original system lies in the *blue box*

characteristic polynomial as we did for the calculation of gains. As the first version of Eq. (9.10) shows, this must be mapped back to the error space by multiplying from the right by $\hat{\mathbf{T}}$.

We have already found the gains ($\mathbf{g}^T$) for the full state feedback. The final picture including the control is shown in Fig. 9.4. This is a closed-loop system, the closed-loop version of Fig. 9.3. The input feeds both systems, the output of the actual system feeds the observer system, and the loop is closed by the output of the observer system. Tracing the flow of information starting in the upper left-hand corner of the figure, we can see that the input u from the world drives both systems. The output from the actual system, at the upper right-hand corner, provides an additional driver for the observer system, and the output of the observer closes the loop.

Let's summarize once more what we just did, and then apply it to a simple familiar system. This summary is perhaps a bit telegraphic, but....

We start with an actual dynamical system with the goal of driving the state to zero

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad y = \mathbf{c}^T\mathbf{x} \tag{9.1}$$

We want to let $u = -\mathbf{g}^T\mathbf{x}$, but we cannot measure all of $\mathbf{x}$, so we must build an estimate for $\mathbf{x}$, and this leads us to a second, artificial dynamical system

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{k}\mathbf{c}^{\mathrm{T}}\mathbf{x} \tag{9.2}$$

where I have substituted for y in terms of $\mathbf{x}$. We use the estimated $\mathbf{x}$ to control the system, leading to the coupled pair (where I have replaced $\mathbf{c}^{\mathrm{T}}\mathbf{x}$ by y in the second of Eq. (9.11))

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} - \mathbf{b}\mathbf{g}^{\mathrm{T}}\hat{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\mathbf{g}^{\mathrm{T}}\hat{\mathbf{x}} + \mathbf{k}\mathbf{c}^{\mathrm{T}}\mathbf{x} \end{aligned} \tag{9.11}$$

Note that these are coupled in both directions: $\hat{\mathbf{x}}$ appears in the first equation and $\mathbf{x}$ appears in the second equation. We have supposed that the two vectors $\mathbf{g}$ and $\mathbf{k}$ are independent, and we have seen how to derive them under that assumption. Equation (9.10) is the differential equations represented by Fig. 9.4. They are linear equations and so are solvable analytically, but it is easier to solve them numerically to generate the response of our examples. (See Friedland (1986) for a somewhat different approach to the observer problem.)

9.2.1.1 The Coupled Problem

Equations (9.1) and (9.2) must be solved simultaneously, so they can be viewed as a single $2N$-order system in the state variable $\mathbf{x}_{\mathrm{C}}$ made up of the two individual state variables. We can write the open-loop system in block matrix form

$$\mathbf{x}_{\mathrm{C}} = \begin{Bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{Bmatrix}, \quad \dot{\mathbf{x}}_{\mathrm{C}} = \begin{Bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{k}\mathbf{c}^{\mathrm{T}} & \hat{\mathbf{A}} \end{Bmatrix} + \begin{Bmatrix} \mathbf{b} \\ \hat{\mathbf{b}} \end{Bmatrix} u \tag{9.12}$$

where the C subscript denotes the combined problem. We can look at the combined version of Eqs. (9.1) and (9.2)

$$\dot{\mathbf{x}}_{\mathrm{C}} = \mathbf{A}_{\mathrm{C}}\mathbf{x}_{\mathrm{C}} + \mathbf{b}_{C}u \tag{9.13}$$

where

$$\mathbf{A} = \mathbf{A}_{\mathrm{C}} = \begin{Bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{k}\mathbf{c}^{\mathrm{T}} & \hat{\mathbf{A}} \end{Bmatrix}, \quad \mathbf{b} = \mathbf{b}_{\mathrm{C}} = \begin{Bmatrix} \mathbf{b} \\ \hat{\mathbf{b}} \end{Bmatrix} u \tag{9.14}$$

This differs from the problems we have solved before because the vector $\mathbf{k}$, which is unknown, appears in the combined matrix $\mathbf{A}_{\mathrm{C}}$. We cannot address this using our usual controllability algorithm, because, even if it turns out to be formally controllable (and generally it does not), the feedback depends only on the estimate, so we only have half as many gains as the dimension of the combined system.

We can look at the closed-loop system (see Fig. 9.4) to see how the eigenvalues compare to the eigenvalues we chose independently. (We won't be able to do this symbolically, but we can do it for the example problems.) We construct the closed-loop problem by substituting for $u = -\mathbf{g}^{\mathrm{T}}\hat{\mathbf{x}}$, which gives Eq. (9.15)

$$\frac{d}{dt}\begin{Bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{A} & -\mathbf{b}\mathbf{g}^{\mathrm{T}} \\ \mathbf{k}\mathbf{c}^{\mathrm{T}} & \hat{\mathbf{A}} - \mathbf{b}\mathbf{g}^{\mathrm{T}} \end{Bmatrix}\begin{Bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{Bmatrix} \tag{9.15}$$

Now let's look at this for a simple and familiar system.

Example 9.1 The Servomotor We first looked at the one degree of freedom servomotor connected to an inertial load back in Chap. 3. We looked at controlling it in Ex. 8.5. Here I want to look at the problem as a second-order system for the actual angle, θ. The problem is simplest when we suppose that inductive effects are negligible and we can write the system as a second-order system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \rightarrow \frac{d}{dt}\begin{Bmatrix} \theta \\ \omega \end{Bmatrix} = \begin{Bmatrix} 0 & 1 \\ 0 & -K\alpha \end{Bmatrix}\begin{Bmatrix} \theta \\ \omega \end{Bmatrix} + \begin{Bmatrix} 0 \\ \alpha \end{Bmatrix}u$$

where K denotes the motor constant and $\alpha = K/IR$, where I denotes the moment of inertia of the load and R the armature resistance of the motor. The input is the voltage supplied to the motor. I will suppose that all that we can observe is the position of the motor, so that $\mathbf{c}^{\mathrm{T}} = \{1\ 0\}$. The goal is to move the motor ϕ radians from its initial position. This fits into the $\mathbf{x} \rightarrow 0$ category because we can always define the origin such that the initial condition is $\theta(0) = -\phi$. We found the gains ($\mathbf{g}^{\mathrm{T}}$) for this problem in Ex. 8.5. That result is

$$\mathbf{g}^{\mathrm{T}} = \begin{Bmatrix} \dfrac{s_1 s_2}{\alpha} & -\dfrac{K\alpha + s_1 + s_2}{\alpha} \end{Bmatrix}$$

where s_1 and s_2 denote the poles chosen for the system, perhaps the second-order Butterworth poles.

$\mathbf{A}^{\mathrm{T}}$ is the analog of $\mathbf{A}$, $\mathbf{c}$ is the analog of $\mathbf{b}$, and $\mathbf{k}^{\mathrm{T}}$ is the analog of $\mathbf{g}^{\mathrm{T}}$. We can form the analog of the controllability matrix, the observability matrix, in the form of Eq. (9.7) or Eq. (9.8). The transformation matrix is most easily found from the Eq. (9.7) formulation, so I will use that.

The matrix $\mathbf{N}^{\mathrm{T}}$ turns out to be the identity matrix

$$\mathbf{N}^{\mathrm{T}} = \{\mathbf{c} \quad \mathbf{A}^{\mathrm{T}}\mathbf{c}\},\quad \mathbf{c} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix},\quad \mathbf{A}^{\mathrm{T}}\mathbf{c} = \begin{Bmatrix} 0 & 0 \\ 1 & -K\alpha \end{Bmatrix}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \Rightarrow \mathbf{N}^{\mathrm{T}} = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}$$

It is then easy to show that the transformation matrix is

$$\hat{\mathbf{T}}_1 = \{0 \quad 1\},\quad \hat{\mathbf{T}}_2 = \hat{\mathbf{T}}_1\mathbf{A}^{\mathrm{T}} = \{1 \quad -K\alpha\} \Rightarrow \hat{\mathbf{T}} = \begin{Bmatrix} 0 & 1 \\ 1 & -K\alpha \end{Bmatrix},$$

$$\hat{\mathbf{T}}^{-1} = \begin{Bmatrix} -K\alpha & -1 \\ -1 & 0 \end{Bmatrix}$$

After a little more algebra we have

$$\left(\mathbf{A}^{\mathrm{T}}\right)_1 - \mathbf{c}_1\mathbf{k}'^{\mathrm{T}} = \begin{Bmatrix} 0 & 1 \\ -k_1' & -K\alpha - k_2' \end{Bmatrix}$$

where $\mathbf{k}'$ denotes the $\mathbf{k}$ vector with respect to the transformed space. We'll have $\mathbf{k}^{\mathrm{T}} = \mathbf{k}'^{\mathrm{T}}\mathbf{T}$.

The characteristic polynomial is

$$s^2 + \left(K\alpha - k_2'\right)s + k_1' = 0$$

where the primes on the components of $\mathbf{k}$ serve to indicate that these are the values in the transformed space, which are $k_1{}' = s_3 s_4, k_2{}' = K\alpha - s_3 - s_4$. We need to map them back to the original space by multiplying from the right by $\hat{\mathbf{T}}$. That result gives us the $\mathbf{k}$ vector in terms of the $\mathbf{k}$ poles:

$$\mathbf{k}^{\mathrm{T}} = \{ -s_3 - s_4 - K\alpha \quad (s_3 - K\alpha)(s_4 - K\alpha) \}$$

How does this system behave? Figures 9.5 and 9.6 show the response of the system for $K = 1 = \alpha$ when the four poles are all different: the poles associated with the gains are the upper and lower of the fourth-order Butterworth poles $(-0.588 \pm 0.809j)$ and those associated with $\mathbf{k}$ are the other two fourth-order Butterworth poles $(-0.951 \pm 0.309j)$—all with unit radius. (I calculated the response shown numerically.) The system started from rest with an initial angle of $\pi/20$ and an initial estimated angle of zero. We see a very nice convergence to zero, and we can also see that the estimate and the actual state elements come together on their way to zero. The red curves denote the actual solution and the blue curves the observer in both figures.

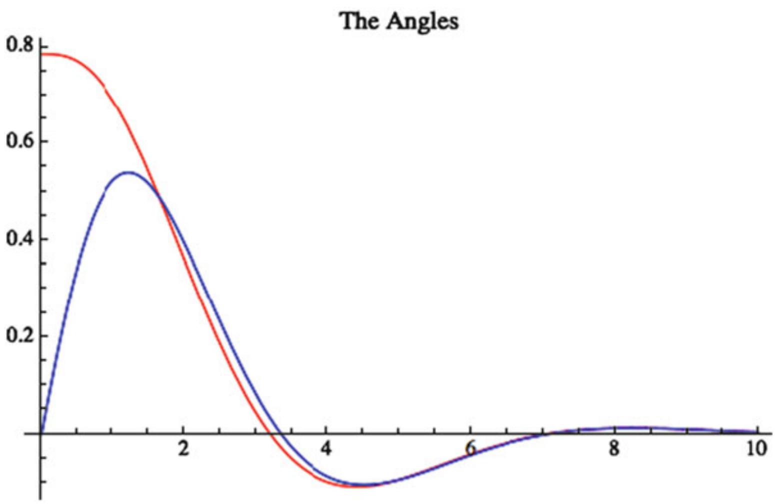


Fig. 9.5 Response of the servomotor for Butterworth poles (see text)

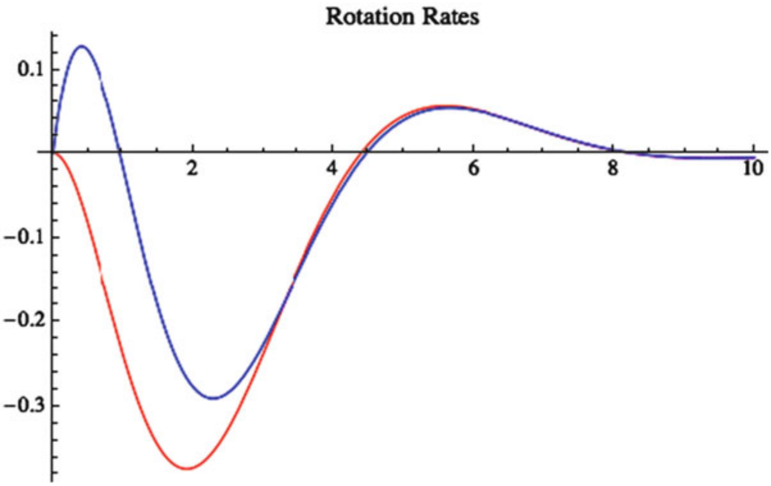


Fig. 9.6 Response of the servomotor for Butterworth poles (see text)

Table 9.1 Comparison of the eigenvalues (see text)

Poles	$-0.588+0.809j$	$-0.588-0.809j$	$-0.951+0.309j$	$-0.951-0.309j$
$\lambda(\mathbf{A}_4)$	-1.0425	$-0.892+1.269j$	$-0.892-1.269j$	-0.348

We expect that the system will respond more rapidly the farther the poles are from the origin, and that proves to be the case. I invite you to try for yourselves.

Let's take a look at the fourth-order closed-loop problem. What are its eigenvalues? How can we solve it analytically? The reader can verify that the matrix **A** given in block form in Eq. (9.14) for the present case is

$$\mathbf{A}_4 = \left\{ \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & -K\alpha & -s_1 s_2 & s_1 + s_2 + K\alpha \\ -s_3 - s_4 - K\alpha & 0 & -s_3 s_4 & 1 \\ s_3 s_4 + K\alpha(s_3 + s_4 + K\alpha) & 0 & -s_1 s_2 + s_3 + s_4 + K\alpha & s_1 + s_2 \end{matrix} \right\}$$

where s_1 and s_2 denote the poles used to determine the gains and s_3 and s_4 denote the poles used to determine the observer gains. Table 9.1 shows the set of poles I used to determine the gains and the observer gains and the eigenvalues of $\mathbf{A}_4$ for the parameters used in Figs. 9.5 and 9.6.

The first two columns show the poles used to determine the gains and the second two columns those used to determine the observer gains. There is no clear relation between the poles in the first row and the eigenvalues in the second row. All the eigenvalues of $\mathbf{A}_4$ have negative real parts. Were this not so, the system would not behave as shown in Figs. 9.5 and 9.6.

The eigenvector matrix associated with these eigenvalues is

$$\mathbf{V} = \begin{Bmatrix} -1.103 & -0.166 + 0.303j & -0.166 - 0.303j & -1.478 \\ 1.149 & 0.533 - 0.0595j & 0.533 + 0.0595j & 0.515 \\ -0.127 & -0.157 + 0.683j & -0.157 - 0.683j & -0.511 \\ 1 & 1 & 1 & 1 \end{Bmatrix}$$

The complex eigenvectors go with the complex eigenvalues. I invite you to construct an analytic solution to the problem given the eigenvectors and eigenvalues. I also leave it to the interested reader to determine whether there are poles for the gains and the observer gains that lead to an unstable coupled system.

9.2.2 Nonlinear Systems

Most mechanical and electromechanical systems are nonlinear. This nonlinearity dwells in the dynamics; the input enters linearly, and we can write such a system as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})u, \quad y = \mathbf{c}^{\mathrm{T}}\mathbf{x} \tag{9.16}$$

The block diagram looks just like that for the linear problem, except that $\mathbf{A}$ is replaced by a functional relationship. The state is fed back through a nonlinear function. The single output is related to the state in the same way as before. Figure 9.7 shows the block diagram.

We try to control these nonlinear systems by linearizing, following the control rituals, and verifying the control using the nonlinear system. Let's see how that works when an observer is necessary.

We find the linear system following the method outline in Sect. 6.3

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u = \left.\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right|_{\mathbf{x}\to\mathbf{0}} \mathbf{x} + \mathbf{b}u$$

The observer can be supposed to obey the same linear model used above. All the analysis for the observer-linear system combination is unchanged. The only difference is that now we need to use the nonlinear model when assessing the result.

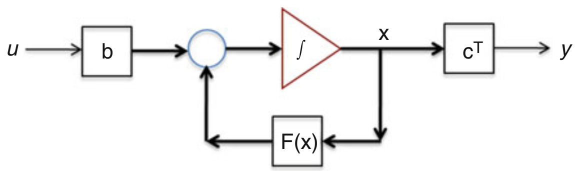


Fig. 9.7 Block diagram of a SISO nonlinear system

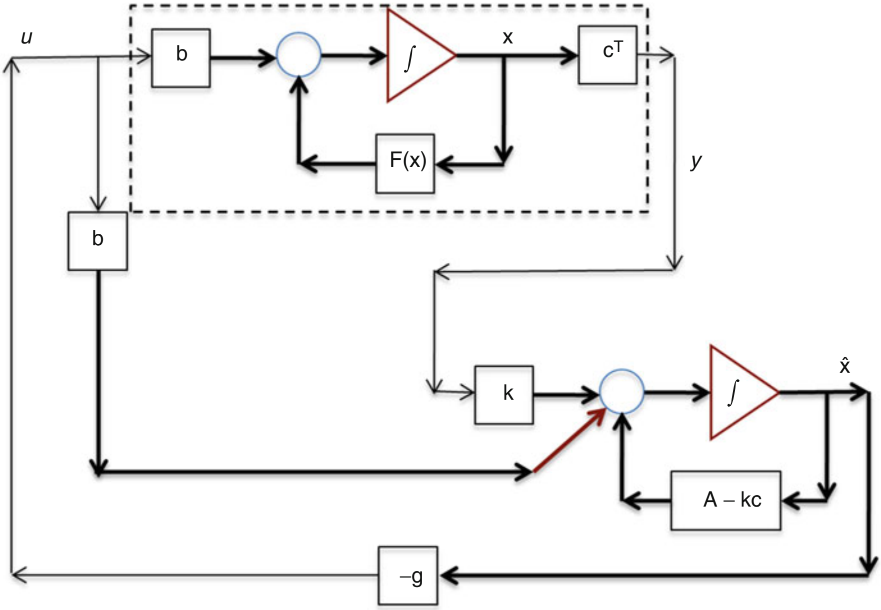


Fig. 9.8 A nonlinear system with an observer. The segment enclosed by the *dashed box* is the nonlinear system

The final picture is shown in Fig. 9.8. The difference between Figs. 9.8 and 9.4 is the nonlinear function inside the dashed box. The nonlinearity is formally invisible to the controller.

We know how to design a linear control for this, and that linear control comes from the linearized version of Eq. (9.14), which is of the form of Eq. (9.1). If we cannot measure the entire state, then we will still need an observer. The observer satisfies the linear observer equation (Eq. 9.2). We can go through the entire exercise we just rehearsed, but when it comes to test the system, we must combine Eq. (9.15) with Eq. (9.2), and that requires just a bit of thought. Equation (9.16) replaces Eq. (9.1), and that redefines the state vector. This leads to two changes from the linear problem. These stem from coupling a full nonlinear state equation to an observer based on the deviation from the full nonlinear state. Let me deal with them in turn.

The state vector in Eq. (9.1) is the deviation from the reference state; the state vector in Eq. (9.16) includes the reference state. The feedback to u is based on the deviation from the references state, and Eq. (9.2) is for the deviation observer. The observer does not care about the full state. Linearization supposes that there is an equilibrium state $\mathbf{x}_0$, u_0, about which we linearized

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}', \quad u = u_0 + u'$$

We can denote deviations from the equilibrium by $\mathbf{x}'$ and u', respectively. The u that enters Eq. (9.16) is the sum of the equilibrium input and the deviation from the equilibrium state. The u that we derive from the observer u' is based on the deviations, so the input to Eq. (9.16) must be

$$u = u_0 - \mathbf{g}^{\mathrm{T}}\hat{\mathbf{x}}$$

The u that enters Eq. (9.2) remains u' $(= -\mathbf{g}^{\mathrm{T}}\hat{\mathbf{x}})$.

The second change is in the feed forward from the state to the observer. The observer is expecting $\mathbf{kc}^{\mathrm{T}}\mathbf{x}'$, and we must be sure to provide that. After these two effects are accounted for we have the state and observer equations in the form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})(u_0 - \mathbf{g}^{\mathrm{T}}\hat{\mathbf{x}}), \quad y = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \dot{\hat{\mathbf{x}}} &= \hat{\mathbf{A}}\,\hat{\mathbf{x}} - \hat{\mathbf{b}}\,\mathbf{g}^{\mathrm{T}}\hat{\mathbf{x}} + \mathbf{kc}^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_0) \end{aligned} \tag{9.17}$$

Equation (9.15) replaces Eq. (9.11). The vector $\mathbf{x}$ is the full nonlinear state vector, while $\hat{\mathbf{x}}$ represents the observer's estimate of the error.

Let me look at this for the magnetic suspension problem. We know that the linear problem is controllable with full state feedback and that the linear control can control the nonlinear system over a certain range of poles. Let's now look at it using an observer, supposing that the only measureable variable is the position z.

Example 9.2 Magnetic Suspension I introduced this problem in Chap. 3. The nonlinear equations governing its behavior are Eqs. (3.36) and (3.37) (which is linear). I can convert Eq. (3.36) to state space form and write the three state equations as

$$\dot{z} = w, \quad \dot{w} = C_n \frac{i^2}{mz^n} - g, \quad \frac{di}{dt} = -\frac{R}{L}i + \frac{1}{L}e$$

There is an equilibrium for any specific value of z_0 (less than zero) given by Eq. (3.39):

$$i_0 = \sqrt{\frac{mg}{C_n}z_0^n}, \quad e_0 = R\sqrt{\frac{mg}{C_n}z_0^n} \tag{3.39}$$

I gave the linearized equations in state space form at the end of Chap. 6. They are

$$\mathbf{x} = \begin{Bmatrix} z' \\ \dot{z}' \\ i' \end{Bmatrix}, \quad \mathbf{A} = \begin{Bmatrix} 0 & 1 & 0 \\ -n\dfrac{g}{z_0} & 0 & 2\dfrac{g}{i_0} \\ 0 & 0 & -\dfrac{R}{L} \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{1}{L} \end{Bmatrix} \tag{6.58}$$

I found the gains in $\mathbf{z}$ space for this problem in Ex. 8.6. I left the mapping into $\mathbf{x}$ space to the problems. That result was

$$g_1 = \sqrt{\frac{mL^2 z_0^{n-2}}{4C_n g}}(ng(s_1 + s_2 + s_3) - z_0 s_1 s_2 s_3)$$

$$g_2 = \sqrt{\frac{mL^2 z_0^n}{4C_n g}} s_2 \left(s_1 s_2 + s_1 s_3 + s_2 s_3 - \frac{ng}{z_0} \right)$$

$$g_3 = -R - L(s_1 + s_2 + s_3)$$

where s_i denotes the desired gain poles.

The next step is to find the observer poles **k**. I find

$$\hat{\mathbf{A}} = \mathbf{A} - \mathbf{k}\mathbf{c}^{\mathrm{T}} = \begin{Bmatrix} -k_1 & 1 & 0 \\ -k_2 - \dfrac{ng}{z_0} & 0 & \sqrt{\dfrac{4C_n g}{m z_0^n}} \\ -k_3 & 0 & -\dfrac{R}{L} \end{Bmatrix}$$

The observability matrix

$$\mathbf{N} = \begin{Bmatrix} \mathbf{c}^{\mathrm{T}} \\ \mathbf{c}^{\mathrm{T}}\mathbf{A} \\ \mathbf{c}^{\mathrm{T}}\mathbf{A}\mathbf{A} \end{Bmatrix} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\dfrac{ng}{z_0} & 0 & \sqrt{\dfrac{4C_n g}{m z_0^n}} \end{Bmatrix}$$

Its determinant is nonzero by inspection (just the product of the main diagonal elements), so the system is observable. The last row of the inverse of the transpose of this provides the first row of the $\hat{\mathbf{T}}$ matrix, and successive multiplications lead to the transformation matrix

$$\hat{\mathbf{T}} = \begin{Bmatrix} 0 & 0 & \dfrac{i_0}{2g} \\ 0 & 1 & -\dfrac{R i_0}{2gL} \\ 1 & -\dfrac{R}{L} & \dfrac{R^2 i_0}{2gL^2} \end{Bmatrix}$$

Its inverse is straightforward, and the companion form of $\mathbf{A}^{\mathrm{T}}$ is

$$\left(\mathbf{A}^{\mathrm{T}}\right)_1 = \hat{\mathbf{T}}\,\mathbf{A}^{\mathrm{T}}\hat{\mathbf{T}}^{-1} = \begin{Bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\dfrac{ngR}{z_0 L} & -\dfrac{ng}{z_0} & -\dfrac{R}{L} \end{Bmatrix}$$

We can deduce the components of **k** from this, and that result, after mapping back to the correct space, is

$$k_1 = -\frac{R}{L} - (s_4 + s_5 + s_6)$$
$$k_2 = \frac{R^2}{L^2} + \frac{R}{L}(s_4 + s_5 + s_6) + s_4 s_5 + s_4 s_6 + s_5 s_6$$
$$k_3 = \left(\frac{R}{L} + s_4\right)\left(\frac{R}{L} + s_5\right)\left(\frac{R}{L} + s_6\right)\sqrt{\frac{m z_0^n}{4 C_n g}}$$

where s_4, s_5, and s_6 denote the poles used for assigning observer gains.

Before testing this control, let's take a look at the eigenvalues of the combined problem, the eigenvalues of the matrix given by Eq. (9.13). It's a six-by-six matrix and rather too complicated to be written out here, but we can look at its eigenvalues in specific cases. Let's adopt the parameters from Table 8.2. I would like to assign the poles using the Butterworth poles, but I have to use the third-order Butterworth poles twice rather than the sixth-order Butterworth poles because each set of poles is odd and requires at least one real pole. I assign the poles as

$$s_1 = -\frac{\rho}{\sqrt{2}}(1 - j) = s_4, \quad s_2 = -\rho = s_5, \quad s_3 = -\frac{\rho}{\sqrt{2}}(1 - j) = s_6$$

where ρ denotes the radius of the pole circle. (The gain poles and the observer gain poles need not lie on the same pole circle. I will use different circles in Chap. 10.) Figure 9.9 shows the real parts of the eigenvalues of $\mathbf{A}_6$ as a function of ρ. We see that while the individual problems are stable, the joint problem is not stable unless ρ exceeds a critical value, here 0.155849. Note that this has nothing to do with the fact

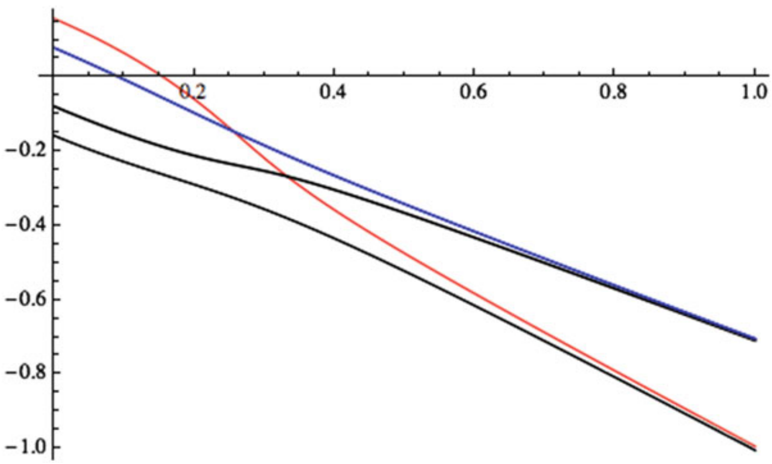


Fig. 9.9 The real parts of the eigenvalues of $\mathbf{A}_6$. The *blue curve* is a pair of complex roots and the *red curve* is a real root

that the original problem was nonlinear. It is simply a stark reminder that the process of determining control gains and observer gains independently can lead to trouble.

All that remains is to test the control in simulation. We need to integrate the system Eq. (9.17) for the problem at hand. I use the third-order Butterworth poles and the physical parameters given in Table 8.2. I have done some numerical experimentation (and I urge you to do the same) and discovered that $\rho = 29$ is about the smallest Butterworth radius for which the control works. Figure 9.10 shows the motion of the ball (red) and the observer ball (blue). (The observer trace is the output of the observer plus the desired final state so that curves represent the full state, not the perturbation state.) Both converge to -0.5 and they converge to each other on the way. Figure 9.11 shows the actual voltage. The reference voltage for this set of parameters is 4.95227, shown in blue on the figure.

Example 9.2 was pretty straightforward. I followed the procedures outlined in the first half of Sect. 9.2. The choice of variable to measure was obvious. We wanted to control the location of the ball, so that seemed to be the thing to measure. This would certainly be our choice were we to look at this as an SISO system. The uncontrolled problem was unstable. The gains and the observer gains must be chosen separately. Magnetic suspension is a third-order problem, so one of the poles must be real. I chose the third-order Butterworth poles to determine both sets of gains. The control worked in the nonlinear system with moderate input voltages.

This problem does not settle all possible issues. I'd like to consider a more practical example, the high-inductance (fifth-order) overhead crane, in more detail. It is marginally stable rather than unstable, but it is challenging, and I will devote the entire next section to this problem.

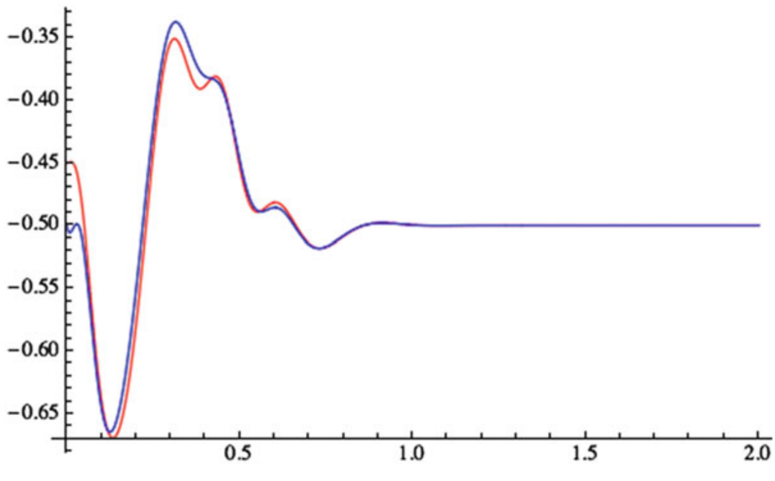


Fig. 9.10 Control of the magnetic suspension using an observer (see text)

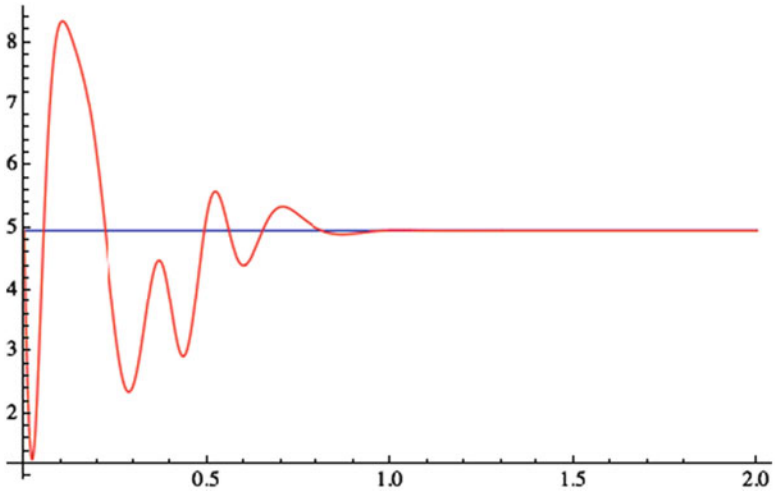


Fig. 9.11 Total voltage for the suspension behavior shown in Fig. 9.8

9.3 The High-Inductance Overhead Crane

This problem (see Ex. 6.12) is a fifth-order nonlinear problem, but it is only marginally stable, not unstable. Its equilibrium can be taken to be $\mathbf{x}_0 = 0$ and $u_0 = 0$, which makes some of the equations simpler. Example 6.12 gives us most of what we need to get started. Equation (6.44) gives the nonlinear equations

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= \frac{m \sin x_2}{\Delta}\left(l x_4^2 + g \cos x_2\right) + \frac{K}{\Delta} x_5 \\
\dot{x}_4 &= -\frac{\sin x_2}{l\Delta}\left(ml \cos x_2 x_4^2 + (M+m)g\right) - \frac{K \cos x_2}{l\Delta} x_5 \\
\dot{x}_5 &= -\frac{K}{L} x_4 - \frac{R}{L} x_5 + \frac{e}{L}
\end{aligned} \tag{6.44}$$

and Eq. (6.41) gives the linear approximation.

$$\mathbf{x} = \begin{Bmatrix} y \\ \theta \\ \dot{y} \\ \dot{\theta} \\ i \end{Bmatrix}, \quad \mathbf{A} = \begin{Bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \dfrac{mg}{M} & 0 & 0 & -\dfrac{K}{Mr} \\ 0 & -\dfrac{(m+M)g}{Ml} & 0 & 0 & \dfrac{K}{Mlr} \\ 0 & 0 & \dfrac{K}{L} & 0 & -\dfrac{R}{L} \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{1}{L} \end{Bmatrix} \tag{6.46}$$

The numerical parameters for the eventual simulation come from Table 6.1, (p. 234).

The eigenvalues of $\mathbf{A}$ for the parameters in Table 6.1 are -399.988, $-0.00196203 \pm 2.71248j$, -0.00784835, and 0. These are similar to the eigenvalues for the inverted pendulum in Chap. 8. One eigenvalue is an outlier, very far from the other four on the complex plane. The system is marginally stable (and we know from previous analysis that the marginally stable part comes from the indifference of the problem to the location of the cart).

9.3.1 The Gains

The controllability matrix is too complicated to be written out symbolically, but its determinant is

$$-\frac{g^2K^4}{(MlrL)^4L} < 0$$

so the system is controllable. The transformation matrix $\mathbf{T}$ is also too complicated to be written out symbolically. The companion form of $\mathbf{A}$ is

$$\mathbf{TAT}^{-1} = \mathbf{A}_1 = \begin{Bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -\dfrac{gK^2}{Mlr^2L} & -\dfrac{g(m+M)R}{MlL} & -\dfrac{g(m+M)r^2L + lK^2}{Mlr^2L} & -\dfrac{R}{L} \end{Bmatrix}$$

We can find the gains by forming the determinant of $s\mathbf{1} - (\mathbf{A}_1 - \mathbf{b}_1\mathbf{g}\mathrm{T})$ and equating the coefficients of s to those of the coefficients of the desired characteristic polynomial. Again the abstract expressions are too complicated to display. I chose fifth-order Butterworth poles on a radius ρ, and the gains (in $\mathbf{x}$ space) for the parameters in Table 6.1 are

$$\mathbf{g} = \left\{ \begin{array}{c} -0.093951\rho^5 \\ -25.3074\rho + 5.9577\rho^3 - 0.187902\rho^5 \\ 2.17 - 0.35083\rho^4 \\ -6.78111 + 5.9577\rho^2 - 0.70126\rho^4 \\ -4 + 0.0373205\rho \end{array} \right\}$$

You should always substitute these back into the original problem, here Eq. (6.46), to make sure the eigenvalues survived. I checked it for $\rho = 1$ and it was fine.

Do the gains actually control the system as promised given full state feedback? Figures 9.12, 9.13, and 9.14 show the response of the cart and the pendulum and the input voltage for $\rho = 0.5$.

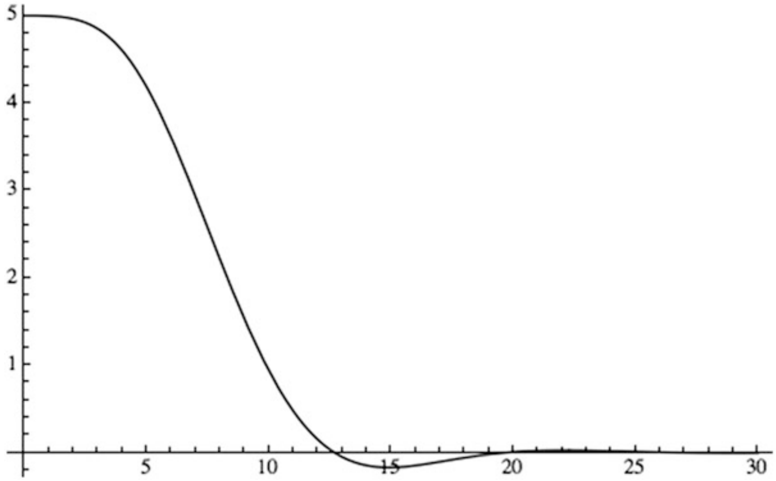


Fig. 9.12 Position of the cart originally at $y = 5$ m

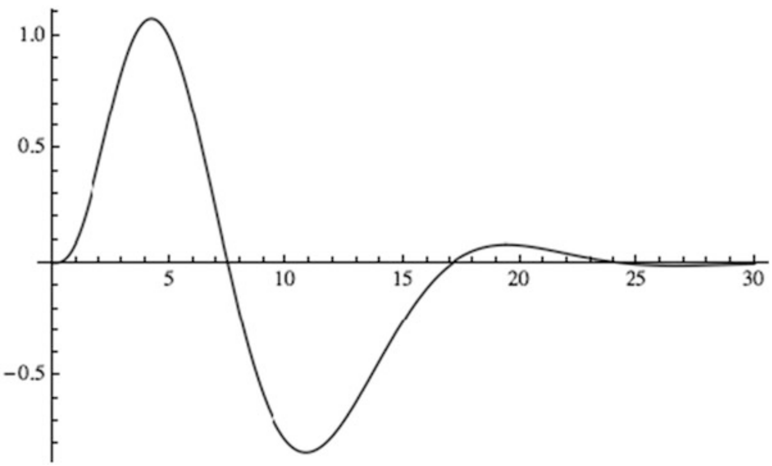


Fig. 9.13 Pendulum angle in degrees

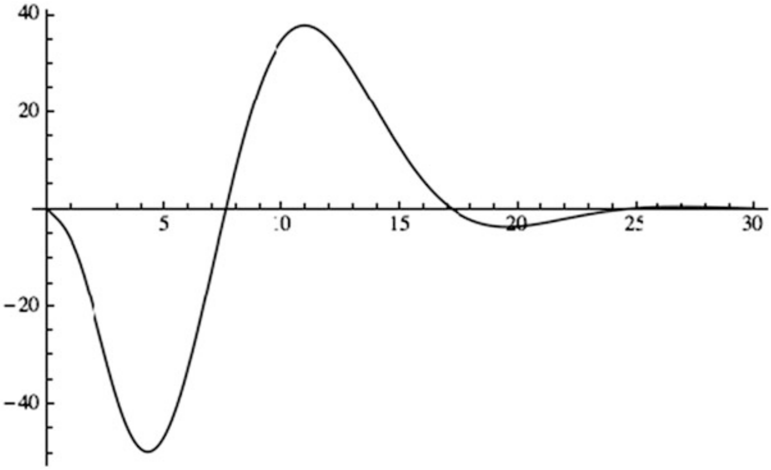


Fig. 9.14 Input voltage

The cart overshoots its end position a bit. This may be unacceptable. If it is, the remedy is to choose all real poles. The pendulum angle hardly changes from zero, which is very good. The whole system travels 5 m in 30 s. Whether this is acceptable performance depends on whether or not your client is happy with it.

The input voltage is small compared to the limits of the motor we are using. This suggests that we can work with a smaller motor or speed up the response by increasing the radius of the Butterworth circle (or otherwise shifting the poles to insure faster convergence).

9.3.2 The Observer Gains

That was fine, but we need to proceed on the assumption that we cannot measure the full state and therefore must design an observer. The fact that the first column of **A** is all zeroes limits our choices of outputs to observe. If the first column of **A** is zero, then the first row of $\mathbf{A}^{\mathrm{T}}$ will also be zero, and $\mathbf{N}^{\mathrm{T}}$ in the form of Eq. (9.7) will have a zero first row, and hence the system will be unobservable unless **c** has a nonzero entry in its first position. The only candidate for an output is then y $(=x_1)$. We cannot choose the angle. We must choose the cart position, for which

$$\mathbf{c}^{\mathrm{T}} = \{1 \quad 0 \quad 0 \quad 0 \quad 0\}$$

The observability matrix is too complicated to write out fully. Its pattern is

$$\mathbf{N} = \begin{Bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & g\dfrac{m}{M} & 0 & 0 & -\dfrac{K}{MR} \\ 0 & 0 & -\dfrac{K^2}{Mr^2L} & -g\dfrac{m}{M} & \dfrac{KR}{MrL} \\ 0 & \bullet & \dfrac{K^2R}{Mr^2L^2} & 0 & \bullet \end{Bmatrix} \tag{9.18}$$

where the bullets stand for two different complicated terms. Its determinant is

$$\det(\mathbf{N}) = \frac{g^2Km^2}{m^3rl}\left(g + l\frac{R^2}{L^2}\right)$$

which is always nonzero, so the system is observable. We can form the transformation matrix $\hat{\mathbf{T}}$ from the inverse of the transpose of the observability matrix. That result is remarkably complicated. It is naturally invertible, and we can form the companion form

$$\mathbf{A}_1^{\mathrm{T}} = \begin{Bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -\dfrac{gK^2}{Mlr^2L} & -\dfrac{g}{l}\dfrac{m+M}{m}\dfrac{R}{L} & \bullet & -\dfrac{R}{L} \end{Bmatrix} \tag{9.19}$$

where again I use a bullet to stand for a complicated term.

We find the observer gains in terms of the observer poles by finding the characteristic polynomial of $\mathbf{A}_1 - \mathbf{c}_1\mathbf{k}^{\mathrm{T}}$ and equating its coefficients to those of the desired characteristic polynomial. This much is straightforward. The next step calls for judgment: we must pick the desired observer poles.

Your first thought is likely to choose the fifth-order Butterworth poles. This was a successful strategy for magnetic suspension, but it fails dismally here. The observer gains for the Butterworth poles on a unit Butterworth circle are

$$\mathbf{k}^{\mathrm{T}} = \{-399.732 \quad 199.88 \quad 159880 \quad -79937.7 \quad 2.94706\times10^9\} \tag{9.20}$$

These huge values lead to a huge input to the observer equations whenever y is not zero. The simulation fails after a few tens of milliseconds. We clearly need to do something else.

I discussed other pole selection options in Chap. 8. I pointed out there that it takes effort to move the uncontrolled poles, and I worked with the inverted

Table 9.2 Poles for the combined problem

Gain poles	$-1/2+\sqrt{3}/2j$	$-1/2+\sqrt{3}/2j$	-1	$-\sqrt{3}/2+1/2j$	$-\sqrt{3}/2-1/2j$
Obs poles	-399.988	$-10+j$	$-10-j$	-2	-3
Comb poles	$-1/2+\sqrt{3}/2j$	$-1/2+\sqrt{3}/2j$	-1	$-\sqrt{3}/2+1/2j$	$-\sqrt{3}/2-1/2j$
Comb poles	-399.988	$-10+j$	$-10-j$	-2	-3

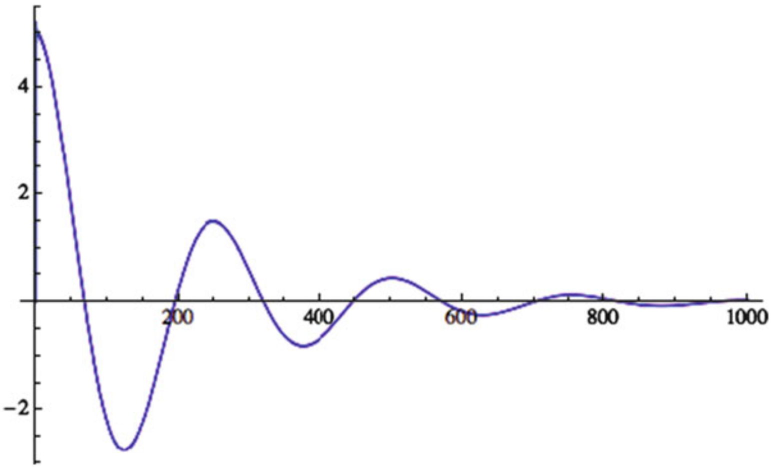


Fig. 9.15 Position of the cart vs. time

pendulum leaving the large negative pole alone. We can take a similar approach here. I will leave the large negative pole alone. I will move the two real poles, 0 and -0.00784843, further into the left half, to -2 and -3, respectively. Finally I move the complex conjugate pair, which have very small real parts, to $-10 \pm j$ increasing the real parts significantly. The observer gains for this cleverer choice of observer poles are much reduced from the values in Eq. (9.19)

$$\mathbf{k}^{\mathrm{T}} = \{\, 24.9882 \quad 89.9239 \quad 199.348 \quad -175.767 \quad 107.548 \,\} \tag{9.21}$$

The next question is whether the eigenvalues of the combined problem have negative real parts for this choice of poles. Table 9.2 shows the eigenvalues of the combined problem and the poles for the gains (with $\rho = 1$) and the observer gains. There are ten combined poles because the combined system is a 10×10 system. I show the poles for the combined system in the last two rows of Table 9.2. In this case they are the same. We saw in Ex. 9.2 that this is not a universal result.

The final test is whether this control works in simulation. It does work, but not nearly as well as true full state feedback. I invite you to experiment with this problem to see if you can design a better set of poles. Figures 9.15, 9.16, and 9.17 show the response of the system and the input voltage for $\rho = 1.3$. (The system

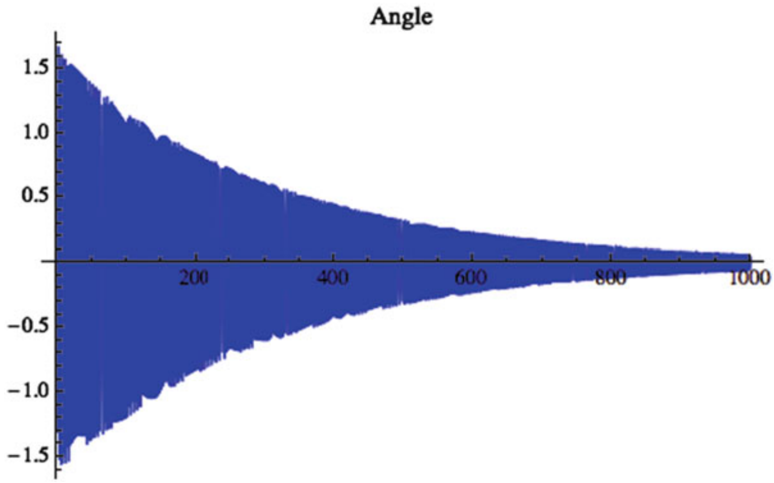


Fig. 9.16 Angle in degrees vs. time

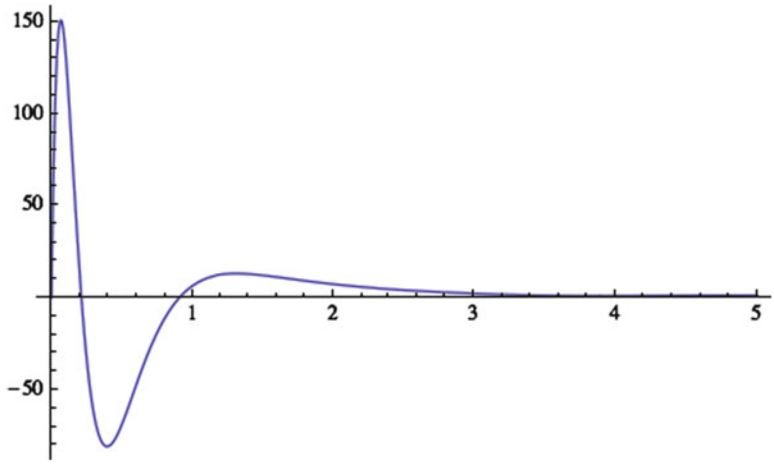


Fig. 9.17 Control voltage over the first 5 s

responds more rapidly for larger values of ρ, but the voltage exceeds the 180 V limit for the motor in question.)

It takes about 1,000 s to move the cart to its final position and for the entire system to converge. Figure 9.16 shows that the angle never gets very big. The observer angle gets over 600°, but this is not a problem for the system, because it is fictional. The control effort converges to zero very quickly. Figure 9.17 shows the input voltage for the first 5 s. The maximum is 150.825 V.

This control is probably not acceptable—it takes too long. It may be possible to redesign this control, or it may be necessary to add a sensor or two. Unfortunately multi-output observers are beyond the scope of this text.

9.4 Summary

This chapter has added another tool to our tool box. We now know how to set about designing a linear control for a system when we can measure but one element of the state. It is not foolproof, but we have found a procedure that has a good chance of working. We start the same way we start any problem—by finding a model and deriving appropriate equations of motion, generally using the Euler-Lagrange approach. These equations will be nonlinear for most practical problems.

We then linearize the equations of motion and design a full state feedback control for the linearized equations. Do not forget to construct the controllability matrix to be sure that the problem is controllable. If it isn't, you can't control it. The main issue here is the choice of poles from which to calculate the gains. An appropriate set of Butterworth poles often works, but you should be alert to the possibility of other choices. In particular, if one or more of the original poles are very stable (large negative real parts) you might want to retain those. Remember when using something other than the Butterworth poles you must be sure to choose complex poles in complex conjugate pairs. (The Butterworth poles do this automatically.) You should build a simulation and test whether the full state feedback control works for the nonlinear system before moving onward. If it does not, you should go back and adjust the gain poles until it does.

The full state feedback you just designed requires you to measure all the elements of the state, which is generally impossible or too expensive to be practical. The method I have outlined in this chapter requires only one measurement or combination of measurements of what we select as the output. The choice of output may be obvious, as it was for the magnetic suspension problem, or it may be forced on you by the observability condition, as it was for the overhead crane. In any case, the system must be observable in order to construct an observer. If it is observable, then you can follow the procedure in this chapter. Again the main issue is the choice of poles from which to calculate the observer gains. We saw this in great detail in Sect. 9.3. It is worth looking at the observer gains when you have found them. If they are too large (not a well-defined number) you will need to choose new poles. This may turn out to be a tedious chore. Once you have found these gains, you should build the joint linear problem and check its eigenvalues. These must all have negative real parts or the joint linear system will not be stable.

Finally you need to build a simulation and verify that the control works in the coupled system (see Eq. 9.17). If it doesn't work, then you have to go back and start over again.

Exercises

1. Suppose that **g** and **k** for Ex. 9.1 are distributed according to the fourth-order Butterworth poles. Remember that the **g** poles and the **k** poles must be complex conjugates independently and choose the pole assignments accordingly. Find the behavior of the system for $\rho = 1$ and 2.
2. Let $K = 1 = a$ in Example 9.1. Suppose

$$s_1 = -\frac{1}{2}\rho\left(\sqrt{3}+j\right) = s_3, \quad s_2 = -\frac{1}{2}\rho\left(\sqrt{3}-j\right) = s_4$$

   Find the limits (if any) of ρ for which the eigenvalues of $\mathbf{A}_4$ have negative real parts.
3. Consider the general one degree of freedom problem for a variable y from an analysis using the Euler-Lagrange process (Eq. (2.4) converted to state space). The matrix **A** and the vector **b** are automatically in companion form (for properly scaled input). Suppose the output to be y. Find $\mathbf{A}_4$. Show that it can be put in companion form, and show that the elements of **g** and **k** enter nonlinearly.
4. Consider the cruise control as a full nonlinear system. Construct an observer for the PI control model supposing that all you can measure is the speed. Let the drag coefficient be 0.35 and the frontal area to be 2 m^2. Linearize around 60 mph. Choose poles for the gain and the observer gain and assess the matrix $\mathbf{A}_4$.
5. Write an observer for Ex. 6.4 and assess it by controlling the inverted pendulum.
6. Write an observer for exercise 7.16 supposing the wheel angle to be measurable. Does the control work? Assess $\mathbf{A}_4$.
7. Can you write an observer for exercise 7.15 supposing the position of the lead car to be measurable? If so, do it, control the system and assess the combined matrix, here $\mathbf{A}_{12}$.
8. If you can measure more than one output it is possible to use them in an observer equation. Modify Eq. (9.2) to do this. What happens to the error equation?
9. Reconsider the train problem (exercise 7.15) assuming you know the positions of all three cars.
10. What do you need to know to make the torque-driven double inverted pendulum observable?
11. Repeat problem 10 for an inverted triple pendulum.
12. Draw a block diagram for an observer-based system subject to disturbances.
13. What happens to the system discussed in Sect. 9.3 if it is disturbed by a torque at the base of the pendulum?
14. Implement PI control for the cruise control using an observer and supposing that all you can measure is the speed. Use the parameters from exercise 4 and simulate the result.

15. Repeat exercise 14 assuming that all you can measure is distance.
16. Repeat exercise 14 supposing that there are disturbances in the form of headwinds (and tail winds).
17. Draw the block diagram for the PI cruise control with speed as the input to an observer.
18. What are the possible outputs for the magnetic suspension problem that allow the successful construction of an observer?
19. Show that a system in companion form will be observable if $\mathbf{c}^T = \{1\ 0\ ...\ 0\}$.
20. Control the inverted pendulum on a cart using cart position to form an observer. What is the limit on the initial departure from vertical of the pendulum? You will need a simulation to answer that question.

The following exercises are based on the chapter but require additional input. Feel free to use the Internet and the library to expand your understanding of the questions and their answers.

21. Is a system for which **A** is in companion form always observable? Explain.
22. Can you control an inverted double pendulum if all you can observe is the angle of the lower pendulum? If so, design the control and verify it in simulation. If not, what input is necessary to control the inverted double pendulum using an observer?

Reference

Friedland B (1986) Control system design. An Introduction to State Space Methods (Chapter 7). Dover reprint 2005

Tracking Control 10

In which we learn how to make a mechanical system track a time-dependent goal. . ..

10.1 Introduction

We have learned how to drive a (linear) mechanical system in state space form to a zero state, and we have seen that this can be quite useful. However it is also necessary to be able to track a time-dependent goal. For example, one might want a vehicle to follow some path in time or a robot to execute a specific task. This section will deal with problems of this nature, starting with our usual linear single-input system rewritten as Eq. (10.1)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{10.1}$$

The goal here is that $\mathbf{x}$ should follow a reference state, $\mathbf{x}_r$, which can be a function of time. The reference state must be consistent with the basic physics of the problem. We can't expect to be able to track impossible states. When we wanted to track a stationary state, we asked that some error be reduced to zero, and we can take the same approach here, but it will prove to be more complicated. Let's define the error $\mathbf{e}$ to be the difference between the state and its reference state, then we can write $\mathbf{x} = \mathbf{x}_r + \mathbf{e}$. We can write $\mathbf{x}$ in Eq. (10.1) in terms of the reference state and the error and rearrange it to give an equation for the error in terms of the reference state and the constants in Eq. (10.1)

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{b}u + \mathbf{A}\mathbf{x}_r - \dot{\mathbf{x}}_r \tag{10.2}$$

The task is to find u such that the error $\mathbf{e} \to 0$. We can deal with Eq. (10.2) in two different ways, the choice depending on $\mathbf{x}_r$.

R.F. Gans, *Mechanical Systems: A Unified Approach to Vibrations and Controls*,
DOI 10.1007/978-3-319-08371-1_10

In the first method, which I will call *reference dynamics*, we find a matrix $\mathbf{A}_r$ such that

$$\dot{\mathbf{x}}_r = \mathbf{A}_r \mathbf{x}_r \tag{10.3}$$

which eliminates $\dot{\mathbf{x}}_r$ from Eq. (10.2). This cannot be done in general, as you can establish by attempting the exercise. The reference dynamics described by Eq. (10.3) are by definition homogeneous. We can construct a set of reference dynamics for reference states that can be expressed in terms of exponentials, real or imaginary.

In the second method, which I will call *reference input*, we must find an open-loop control corresponding to the reference state. (We did that for the cruise control.) We can write

$$u = u_r + u' \tag{10.4}$$

where u_r is such that

$$\dot{\mathbf{x}}_r = \mathbf{A}\mathbf{x}_r + \mathbf{b}u_r \tag{10.5}$$

Equation (10.2) can then be rewritten

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{b}u' \Rightarrow \dot{\mathbf{e}} = \mathbf{A}\mathbf{e} - \mathbf{b}\mathbf{g}^T\mathbf{e} \tag{10.6}$$

This completely eliminates all reference to the reference state from the error equation, Eq. (10.2). The reference state is still included in the control of the basic system, of course. Equation (10.1) becomes

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{b}\mathbf{g}^T(\mathbf{x} - \mathbf{x}_r) \tag{10.7}$$

It should be clear that constructing a reference input is also not always possible.

10.2 Tracking with Full State Feedback

I will start our discussion of tracking by supposing that we can measure the full state. I will deal with tracking using an observer in Sect. 10.4. The goal of tracking is to find a control input that will cause a state vector to track a reference state vector, that is, find u such that the solution $\mathbf{e}$ to Eq. (10.2) goes to zero. This is a difficult problem, and it does not always have an exact solution. One can, however, often track a desired output. I will develop the mathematics for tracking and then take a look at what can work and what cannot.

Consider an N-dimensional linear single-input system in its usual form Eq. (10.1)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

Denote the reference state by $\mathbf{x}_r$. The reference state cannot be inconsistent with the overall physics of the system. There are other constraints that I will discuss in due course. For now suppose that we have an acceptable reference state.

We start with Eq. (10.2)

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{A}\mathbf{x}_r - \dot{\mathbf{x}}_r + \mathbf{b}u$$

The difficulty in dealing with Eq. (10.2) lies in the appearance of the reference state. There are two approaches to dealing with this, as I noted above, and they give two different error equations.

10.2.1 Reference Dynamics

We seek a matrix $\mathbf{A}_r$ such that

$$\dot{\mathbf{x}}_r = \mathbf{A}_r\mathbf{x}_r \tag{10.8}$$

This is not possible for all choices of $\mathbf{x}_r$. In general we can do this with exponential and trigonometric reference states. I'll make this clear in terms of examples as we move through this section. Substitution of Eq. (10.8) into Eq. (10.2) gives the reference dynamics error equation Eq. (10.9)

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + (\mathbf{A} - \mathbf{A}_r)\mathbf{x}_r + \mathbf{b}u \tag{10.9}$$

Figure 10.1 shows a block diagram of Eq. (10.9).

We need to choose some u to drive $\mathbf{e}$ to zero. We can reason by analogy with what has come before that u must have feedback from $\mathbf{e}$ and from $\mathbf{x}_r$ in order to deal with two possible destabilizing terms.

$$u = -\mathbf{g}^T\mathbf{e} - \mathbf{g}_r^T\mathbf{x}_r \tag{10.10}$$

These two feedbacks convert the error equation Eq. (10.2) to Eq. (10.11)

$$\begin{aligned}\dot{\mathbf{e}} &= \mathbf{A}\mathbf{e} + (\mathbf{A} - \mathbf{A}_r)\mathbf{x}_r - \mathbf{b}\left(\mathbf{g}^T\mathbf{e} + \mathbf{g}_r^T\mathbf{x}_r\right) \\ &= \left(\mathbf{A} - \mathbf{b}\mathbf{g}^T\right)\mathbf{e} + \left(\mathbf{A} - \mathbf{A}_r - \mathbf{b}\mathbf{g}_r^T\right)\mathbf{x}_r\end{aligned} \tag{10.11}$$

Figure 10.2 shows the block diagram corresponding to Eq. (10.11).

As with any control package, $\mathbf{g}^T$ must be chosen to make the matrix $(\mathbf{A} - \mathbf{b}\mathbf{g}^T)$ stable. The system defined by Eq. (10.1) must be controllable for this to be possible. If it is controllable, then we can go through the control procedure introduced in Chap. 8 to find the gains making up $\mathbf{g}^T$ in terms of a set of gain poles. I will go through this part in the examples that follow. We can make the system described by

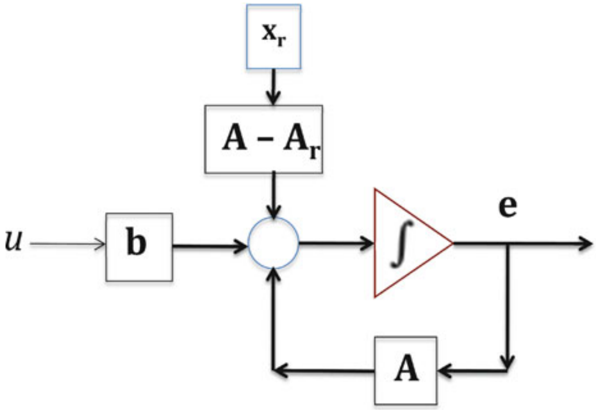


Fig. 10.1 Block diagram of the error equation Eq. (10.9)

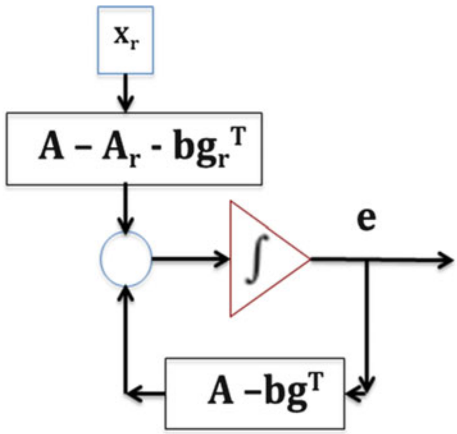


Fig. 10.2 Block diagram of Eq. (10.11)

Fig. 10.2 look more like a closed-loop system by adding the equation governing the state to be tracked to the diagram, combining Eqs. (10.8) and (10.11). I show the result as Fig. 10.3.

Now let us move on to the choice of $\mathbf{g}_r^T$ knowing that we can find $\mathbf{g}^T$. If we could find a $\mathbf{g}_r$ such that

$$\left(\mathbf{A} - \mathbf{A}_r - \mathbf{b}\mathbf{g}_r^T\right) = 0 \tag{10.12}$$

we'd be done, because we would have eliminated $\mathbf{x}_r$ from Eq. (10.11), severing the error dynamics from the reference dynamics (replace $\mathbf{A} - \mathbf{A}_r - \mathbf{b}\mathbf{g}_r^T$ by 0 in Fig. 10.3). We generally cannot do this. What we can often do is choose $\mathbf{g}_r$ such that the error is constant. A zero error will certainly be constant, although a constant error need not be zero. Let's set $\dot{\mathbf{e}}$ equal to zero and see if we can find a consistent set of equations. This seems like a bootstrap operation, and it is. We must verify the results when we are done. If $\dot{\mathbf{e}} = 0$ then Eq. (10.11) becomes

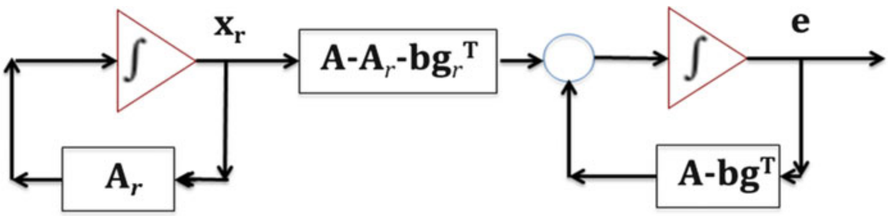


Fig. 10.3 The block diagram of Fig. 10.2 redrawn to show the dynamics of the reference state: Eqs. (10.8) and (10.11) combined

$$\left(\mathbf{A} - \mathbf{b}\mathbf{g}^{\mathrm{T}}\right)\mathbf{e}_{\mathrm{c}} + \left(\mathbf{A} - \mathbf{A}_{\mathrm{r}} - \mathbf{b}\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}\right)\mathbf{x}_{\mathrm{r}} = 0 \tag{10.13}$$

The matrix $(\mathbf{A} - \mathbf{b}\mathbf{g}^{\mathrm{T}})$ is invertible for a controllable system, so we can solve Eq. (10.13) for the constant error $\mathbf{e}_{\mathrm{c}}$, which is proportional to the reference state.

$$\mathbf{e}_{\mathrm{c}} = -\left(\mathbf{A} - \mathbf{b}\mathbf{g}^{\mathrm{T}}\right)^{-1}\left(\mathbf{A} - \mathbf{A}_{\mathrm{r}} - \mathbf{b}\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}\right)\mathbf{x}_{\mathrm{r}} = \mathbf{E}\mathbf{x}_{\mathrm{r}} \tag{10.14}$$

The error vector has the same number of dimensions as the states, so the matrix $\mathbf{E}$ is an $N \times N$ matrix with N^2 components. There are only N components of $\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}$, so we cannot make $\mathbf{E}$ vanish in the general case. We can, however, usually make a subset of $\mathbf{E}$ vanish. Often we really only care about part of the error, the error in the (single) output. That is, if the output is defined by $y = \mathbf{c}^{\mathrm{T}}\mathbf{x}$, we can make $\mathbf{c}^{\mathrm{T}}\mathbf{E}$, which has only N components, vanish. We can write

$$y = \mathbf{c}^{\mathrm{T}}\mathbf{x} = \mathbf{c}^{\mathrm{T}}\mathbf{x}_{\mathrm{r}} + \mathbf{c}^{\mathrm{T}}\mathbf{e} \Rightarrow \mathbf{c}^{\mathrm{T}}\mathbf{x}_{\mathrm{r}} + \mathbf{c}^{\mathrm{T}}\mathbf{E}\mathbf{x}_{\mathrm{r}}$$

and sometimes we can find $\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}$ such that $\mathbf{c}^{\mathrm{T}}\mathbf{E} = 0$ independent of the details of $\mathbf{x}_{\mathrm{r}}$. We can look at this analytically.

Multiply Eq. (10.14) by $\mathbf{c}^{\mathrm{T}}$. If we drop $\mathbf{x}_{\mathrm{r}}$ we can rearrange Eq. (10.14) to give

$$\mathbf{c}^{\mathrm{T}}\left(\mathbf{A} - \mathbf{b}\mathbf{g}^{\mathrm{T}}\right)^{-1}(\mathbf{A} - \mathbf{A}_{\mathrm{r}}) = \mathbf{c}^{\mathrm{T}}\left(\mathbf{A} - \mathbf{b}\mathbf{g}^{\mathrm{T}}\right)^{-1}\mathbf{b}\mathbf{g}_{\mathrm{r}}^{\mathrm{T}} \tag{10.15}$$

which we can solve formally for $\mathbf{g}_{\mathrm{r}}$. The matrix multiplying $\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}$ is square and invertible if the input and the output have the same dimensions. In the single-input-single-output case $\mathbf{c}^{\mathrm{T}}$ is a $1 \times N$ matrix and $\mathbf{b}$ is a $N \times 1$ matrix and the matrix multiplying $\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}$ is a scalar. We can rewrite the final form of Eq. (10.15) to make the dimensions clearer

$$\mathbf{c}_{1\times N}{}^{\mathrm{T}}\left(\mathbf{A} - \mathbf{b}\mathbf{g}^{\mathrm{T}}\right)_{N\times N}{}^{-1}(\mathbf{A} - \mathbf{A}_{\mathrm{r}})_{N\times N} = \left(\mathbf{c}_{1\times N}{}^{\mathrm{T}}\left(\mathbf{A} - \mathbf{b}\mathbf{g}^{\mathrm{T}}\right)_{N\times N}{}^{-1}\mathbf{b}_{N\times 1}\right)\left(\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}\right)_{1\times N} \tag{10.16}$$

The left-hand side of the equation is a $1 \times N$ matrix, and the matrix coefficient multiplying $\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}$ on the right-hand side is a scalar (a 1×1 matrix), so Eq. (10.17) gives an explicit expression for $\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}$ for the SISO case:

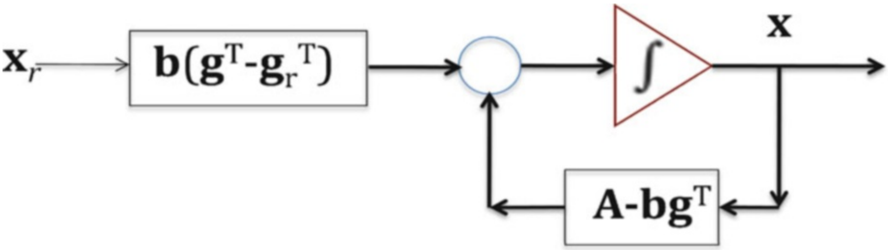


Fig. 10.4 The general tracking control, Eq. (10.19)

$$\mathbf{g}_r^T = \frac{\mathbf{c}^T(\mathbf{A}-\mathbf{bg}^T)^{-1}(\mathbf{A}-\mathbf{A}_r)}{\left(\mathbf{c}^T(\mathbf{A}-\mathbf{bg}^T)^{-1}\mathbf{b}\right)} \tag{10.17}$$

This choice of $\mathbf{g}_r$ insures that the constant error in the output will be zero so the output will track its desired output $\mathbf{c}^T\mathbf{x}_r$. This does not guarantee that $\mathbf{x}$ will track all of $\mathbf{x}_r$.

How do we apply this to the original system? We can rewrite Eq. (10.10) in terms of $\mathbf{x}$ and $\mathbf{x}_r$ by direct substitution for $\mathbf{e}$. Replace $\mathbf{e}$ by $\mathbf{x}-\mathbf{x}_r$, so that

$$u = -\mathbf{g}^T(\mathbf{x}-\mathbf{x}_r) - \mathbf{g}_r^T\mathbf{x}_r \tag{10.18}$$

and, after some cancellations, we have a general expression for the evolution of the state in terms of the gain vectors and the desired state. In general we have

$$\dot{\mathbf{x}} = \left(\mathbf{A}-\mathbf{bg}^T\right)\mathbf{x} + \mathbf{b}\left(\mathbf{g}^T - \mathbf{g}_r^T\right)\mathbf{x}_r \tag{10.19}$$

We can draw a block diagram of this system (Fig. 10.4), which looks like a simple stable system forced by $\mathbf{x}_r$.

10.2.2 The Reference State and the Matrix $\mathbf{A}_r$

It is difficult and, I believe, ultimately unproductive to discuss this topic entirely in the abstract. I can say a few general things. If the desired output is constant or can be written in terms of exponentials, real or imaginary, and the dimension of the state is small, you can frequently choose a reference state and the matrix $\mathbf{A}_r$ by inspection or intuition. We'll see this in Exs. 10.1 and 10.2. The state will not in general track the entire reference state, so the important thing is that the output track the reference output. This makes it easier to choose a reference state, since all that is essential is that the element of the reference state that represents the output be equal to the reference output: $\mathbf{c}^T\mathbf{x} \to \mathbf{c}^T\mathbf{x}_r$. I will work through a series of examples in the hope that principles and procedures will well up from these examples.

Example 10.1 The Servomotor I will start with the simple second-order DC servomotor that we have seen before. The differential equations are

$$\dot{\mathbf{x}} = \frac{d}{dt}\begin{Bmatrix} \theta \\ \omega \end{Bmatrix} = \begin{Bmatrix} 0 & 1 \\ 0 & -K\Gamma \end{Bmatrix}\begin{Bmatrix} \theta \\ \omega \end{Bmatrix} + \begin{Bmatrix} 0 \\ \Gamma \end{Bmatrix} e$$

where K and Γ denote motor and load parameters. We will want two gain matrices: $\mathbf{g}^{\mathrm{T}}$ and $\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}$. The former can be found without reference to the reference state. Let's do that.

Note that the system is close enough to companion form that there is no real reason to go to the effort of finding the companion form. Let $e = -\mathbf{g}^{\mathrm{T}}\mathbf{x}$ and find $\mathbf{g}^{\mathrm{T}}$ in terms of the poles. We'll have

$$\mathbf{A} - \mathbf{b}\mathbf{g}^{\mathrm{T}} = \begin{Bmatrix} 0 & 1 \\ -\dfrac{K}{IR}g_1 & -\dfrac{K^2}{IR} - \dfrac{K}{IR}g_2 \end{Bmatrix}$$

for which the characteristic polynomial is

$$s^2 + \frac{K}{IR}(g_2 + K)_1 s + \frac{K}{IR}g$$

and the gains in terms of the two poles are

$$g_1 = \frac{IR}{K}s_1 s_2, \quad g_2 = -\left(K + \frac{IR}{K}(s_1 + s_2)\right)$$

Any desired reference state must be compatible with the physics: the second element must be the derivative of the first. Suppose we want to make the servo oscillate at a frequency ω. Then we'll have

$$\mathbf{x}_{\mathrm{r}} = \begin{Bmatrix} \alpha \sin(\omega t) \\ \alpha\omega \cos(\omega t) \end{Bmatrix}$$

clearly compatible with the physics of the system. We can find $\mathbf{A}_{\mathrm{r}}$ by inspection

$$\mathbf{A}_{\mathrm{r}} = \begin{Bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{Bmatrix}$$

Now let us construct the error matrix in Eq. (10.14). It's a simple matter of matrix algebra, and I will leave it to the reader to verify that the result is

$$\mathbf{E} = \frac{1}{s_1 s_2}\begin{Bmatrix} \dfrac{K}{IR}g_{\mathrm{r}1} - \omega^2 & \dfrac{K}{IR}(g_{\mathrm{r}2} + K) \\ 0 & 0 \end{Bmatrix}$$

and so the entire error can be driven to zero for any specific input frequency. If all we care about is the angle, not its rate of change, then all we need to select is g_{r1}. Note that $\mathbf{g}_r^T$ is determined entirely by the problem parameters. There are no poles to be chosen (s_1 and s_2 are the poles determining $\mathbf{g}$).

To summarize the results of the control analysis in this example we have

$$g_1 = \frac{IR}{K} s_1 s_2, \quad g_2 = -\left(K + \frac{IR}{K}(s_1 + s_2)\right) \text{ and } g_{r1} = \frac{IR}{K}\omega^2, \quad g_{r2} = -K$$

The two matrices in Fig. 10.4 for this example are

$$\mathbf{A} - \mathbf{b}\mathbf{g}^T = \begin{Bmatrix} 0 & 0 \\ -s_1 s_2 & s_1 + s_2 \end{Bmatrix} \text{ and } \mathbf{b}\left(\mathbf{g}^T - \mathbf{g}_r^T\right) = \begin{Bmatrix} 0 & 0 \\ s_1 s_2 - 4\pi^2 & -s_1 - s_2 \end{Bmatrix}$$

If I choose $s_1 = -1 + j$ and $s_2 = -1 - j$ (Butterworth poles with radius √2) the matrices become

$$\mathbf{A} - \mathbf{b}\mathbf{g}^T = \begin{Bmatrix} 0 & 0 \\ -2 & -2 \end{Bmatrix} \text{ and } \mathbf{b}\left(\mathbf{g}^T - \mathbf{g}_r^T\right) = \begin{Bmatrix} 0 & 0 \\ 2 - 4\pi^2 & 2 \end{Bmatrix}$$

Figure 10.5 shows the tracking response for $\omega = 2\pi$. The tracking algorithm works very nicely. It takes about two periods for the servo to catch up to its desired path.

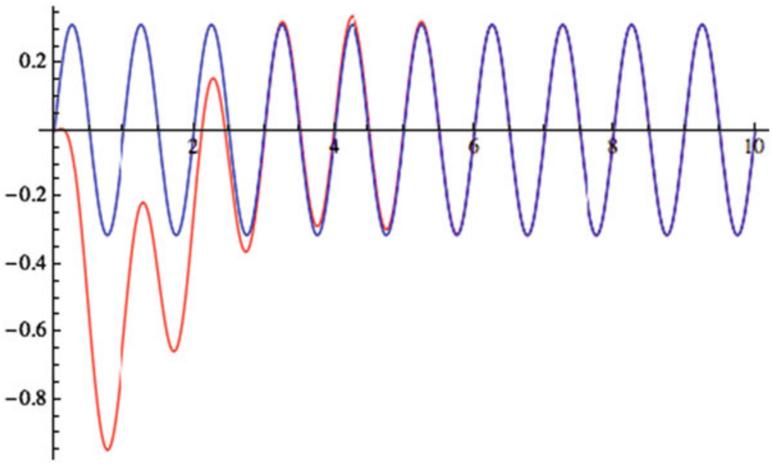


Fig. 10.5 Tracking behavior for the servo. The desired angle is in *blue* and the actual angle in *red*

10.2.2.1 Nonlinear Problems

We need to adapt this procedure a little to deal with nonlinear problems, as we did in Chap. 9. The actual dynamics satisfies the usual (single-input) nonlinear system.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})u, \quad y = \mathbf{c}^{\mathrm{T}}\mathbf{x} \tag{10.20}$$

Linearization supposes that there is an equilibrium state $\mathbf{x}_0$ (not necessarily stable) about which we linearized

$$\mathbf{x} = \mathbf{x}_0, \quad u = u_0$$

The u that enters Eq. (10.20) is the sum of the reference input u_0 and the control u from Eq. (10.10). The error in Eq. (10.10) is based on the difference between the actual departure from equilibrium and the desired departure from equilibrium. We cannot write $\mathbf{e} = \mathbf{x} - \mathbf{x}_r$, we must write $\mathbf{e} = \mathbf{x} - \mathbf{x}_0 - \mathbf{x}_r$ and the input to Eq. (10.20) will be

$$u = u_0 - \mathbf{g}^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_0 - \mathbf{x}_r) - \mathbf{g}_r^{\mathrm{T}}\mathbf{x}_r$$

giving the problem nonlinear problem controlled with the linear control:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})\left(u_0 - \mathbf{g}^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_0 - \mathbf{x}_r) - \mathbf{g}_r^{\mathrm{T}}\mathbf{x}_r\right), \quad y = \mathbf{c}^{\mathrm{T}}\mathbf{x} \tag{10.21}$$

Equation (10.21) is the nonlinear equivalent of Eq. (10.20). The state variable $\mathbf{x}$ refers to the entire state, as does the equilibrium state $\mathbf{x}_0$, while the reference state $\mathbf{x}_r$ refers to the desired departure from equilibrium, that is, our task is to make $\mathbf{x}$ converge to $\mathbf{x}_0 + \mathbf{x}_r$. We can look at how this goes in the context of the magnetic suspension system, a nonlinear problem with a nontrivial and unstable equilibrium state.

Example 10.2 Magnetic Suspension Revisited I introduced this problem in Chap. 3, where we found its equilibrium. We learned how to control it to equilibrium in Chap. 8, and how to control it to equilibrium using an observer in Chap. 9. Table 8.2 gives us a set of physical parameters. Let's see how we can make the ball oscillate, that is, track an oscillatory reference state. The state equations are

$$\dot{\mathbf{x}} = \begin{Bmatrix} x_2 \\ g - C\dfrac{x_3^2}{mx_1^n} \\ -\dfrac{R}{L}x_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dfrac{1}{L} \end{Bmatrix} e$$

and the linearized state equations are

$$\dot{\mathbf{x}}' = \mathbf{A}\mathbf{x}' + \mathbf{b}u' = \begin{Bmatrix} 0 & 1 & 0 \\ n\dfrac{g}{z_0} & 0 & -2\dfrac{g}{i_0} \\ 0 & 0 & -\dfrac{R}{L} \end{Bmatrix}\mathbf{x}' + \begin{Bmatrix} 0 \\ 0 \\ \dfrac{1}{L} \end{Bmatrix}u'$$

where we need to remember that $\mathbf{x}'$ here refers to the departure from equilibrium. The gains (mapped back into $\mathbf{x}$ space) required to maintain equilibrium for the parameters in Table 8.2 using the third-order Butterworth poles on a pole circle of radius 3 (which is the smallest circle on which the basic control works) are

$$\mathbf{g}^{\mathrm{T}} = \{-0.724166 \quad -0.10453 \quad -0.927574\}$$

We want to make the ball follow a harmonic oscillation about its equilibrium. This is the problem of making the departure variable $\mathbf{x}'$ track a harmonic system. You can verify that

$$\mathbf{x}_{\mathrm{r}} = \begin{Bmatrix} \delta \sin(\omega t) \\ \delta\omega \cos(\omega t) \\ i_{\mathrm{r}} \sin(\omega t) \end{Bmatrix}, \quad \mathbf{A}_{\mathrm{r}} = \begin{Bmatrix} 0 & 1 & 0 \\ \omega^2 & 0 & 0 \\ 0 & \dfrac{i_{\mathrm{r}}}{\delta} & \end{Bmatrix}$$

is a physically possible state that includes an oscillation about the equilibrium. The parameter δ is the magnitude of the oscillation and ω its frequency. The current parameter i_{r} has no specific meaning. We will discover that we can use it to refine the analysis. We can use this reference state and matrix $\mathbf{A}_{\mathrm{r}}$ to design our tracking control.

We need to form the error matrix $\mathbf{E}$ I introduced in Eq. (10.14)

$$\mathbf{E} = \left(\mathbf{A} - \mathbf{b}\mathbf{g}^{\mathrm{T}}\right)^{-1}\left(\mathbf{A} - \mathbf{A}_{\mathrm{r}} - \mathbf{b}\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}\right)$$

The second row of $\mathbf{E}$ is all zeroes. The first and third rows are nonzero, and either row can be used to find $\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}$. The first row leads to the error in position, so that is the one we want to use. (We can write this as $\mathbf{c}^{\mathrm{T}}\mathbf{E} = 0$.) That result, with the parameters from Table 8.2 substituted, is

$$g_{\mathrm{r}1} = \frac{(1+\sqrt{2})\rho(1962 + 5\omega^2)}{600\sqrt{1090}}, \quad g_{\mathrm{r}2} = -\frac{i_{\mathrm{r}}}{100\delta}, \quad g_{\mathrm{r}3} = \frac{(1+\sqrt{2})\rho - 100}{100}$$

where ρ denotes the radius of the Butterworth circle for the gains $\mathbf{g}^{\mathrm{T}}$. The entire error for this problem can be reduced to zero. We have one additional arbitrary parameter, i_{r}. At this point in the analysis the first two components of $\mathbf{E}\mathbf{x}_{\mathrm{r}}$ are equal to zero, and the third component is

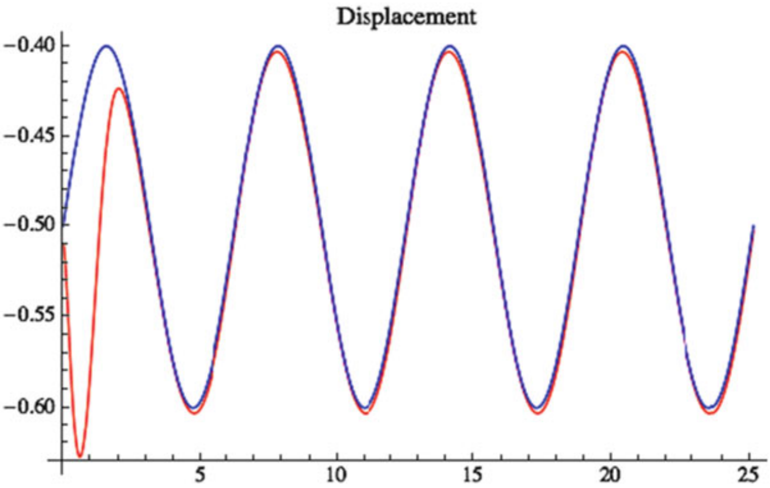


Fig. 10.6 Displacement tracking at $\omega = 1$, $\rho = 3$. The reference motion is in *blue* and the actual motion in *red*

$$\left(i_r - \frac{\delta(1962 + 5\omega^2)}{6\sqrt{1090}}\right) \sin(\omega t)$$

so we can eliminate this by the choice of i_r. The desired tracking state is now

$$\mathbf{x}_r = \delta \begin{Bmatrix} \sin(\omega t) \\ \omega \cos(\omega t) \\ \dfrac{1962 + \omega^2}{6\sqrt{1090}} \sin(\omega t) \end{Bmatrix}$$

We now have everything we need to use Eq. (10.19).

We can drive the system to equilibrium with ρ as small as 3. The tracking system will work at this level if the frequency to be tracked is low. Figure 10.6 shows the displacement of the system tracking at $\omega = 1$, and Fig. 10.7 shows the associated input voltage. I started the system from rest at $z = -0.51$. (The equilibrium position is -0.50.)

Increasing the frequency to be tracked increases the minimum radius of the pole circle required to track successfully. Figures 10.8 and 10.9 show the tracking result and the control voltages for a tracking frequency $\omega = 30$. The pole radius $\rho = 43$. The control effort is more than twice as high.

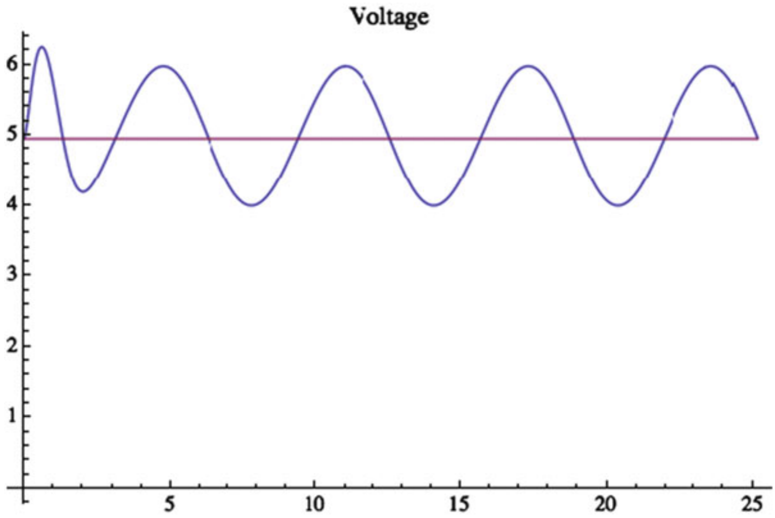


Fig. 10.7 The control voltage for the tracking shown in Fig. 10.6. The horizontal line indicates the equilibrium voltage

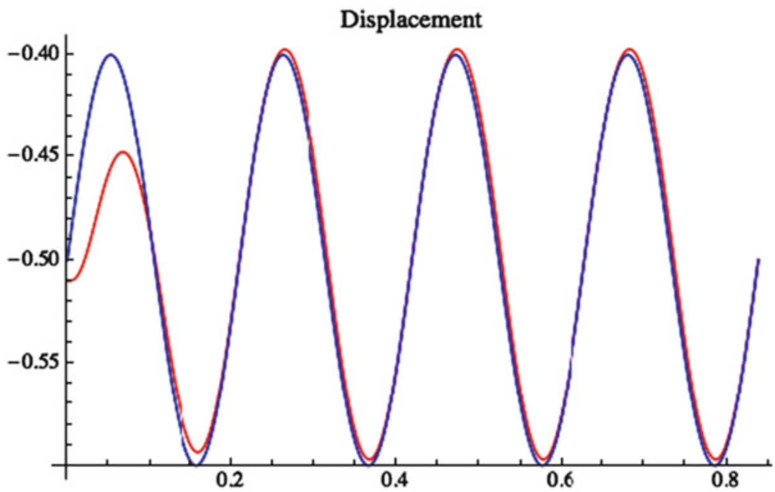


Fig. 10.8 Displacement tracking at $\omega = 30$, $\rho = 43$. The reference motion is in *blue* and the actual motion is in *red*

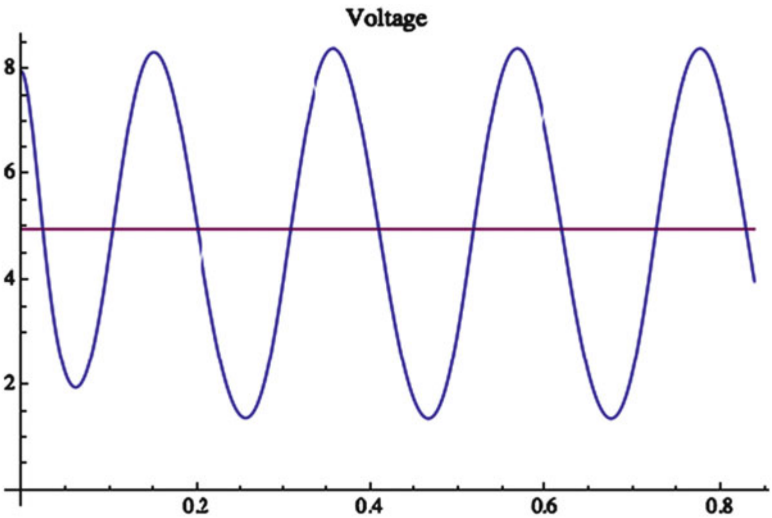


Fig. 10.9 Control voltage for the motion shown in Fig. 10.8

10.3 The Overhead Crane as an Extended Example of Tracking Control

10.3.1 General Comments

I discussed two methods of tracking a reference state: reference dynamics and reference input. There are two steps here. One is to choose the reference state and the other is to find either the matrix $\mathbf{A}_r$ or the reference input u_r. Note that the reference states need not be identical, although they must share the same reference output. I will compare the two methods for the overhead crane following a sinusoidal displacement, and I will exercise the second method for a reference path that does not lend itself to the first method. We can select an initial marginally stable equilibrium state $\mathbf{x}_0 = 0$ in both cases.

10.3.2 Tracking a Sinusoidal Path

I want the position of the cart to follow $y_r = d\sin(\omega_f t)$. I need to devise a state that contains this and lends itself to the construction of $\mathbf{A}_r$. The state will be the usual five-dimensional state

$$\mathbf{x} = \begin{Bmatrix} y \\ \theta \\ \dot{y} \\ \dot{\theta} \\ i \end{Bmatrix} \tag{10.22}$$

Suppose the first element of the state goes like the sine. I will choose a reference state for which y, θ, and i are in phase, given by Eq. (10.23)

$$\mathbf{x}_{\mathrm{r}} = \begin{Bmatrix} d \sin\left(\omega_f t\right) \\ \Theta \sin\left(\omega_f t\right) \\ \omega_f d \cos\left(\omega_f t\right) \\ \omega_f \Theta \cos\left(\omega_f t\right) \\ I \sin\left(\omega_f t\right) \end{Bmatrix} \tag{10.23}$$

Θ and I are constants that I will determine as part of the analysis.

10.3.2.1 Reference Dynamics

You can verify that

$$\mathbf{A}_{\mathrm{r}} = \begin{Bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\omega_f^2 & 0 & 0 & 0 & 0 \\ 0 & -\omega_f^2 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{I}{d} & 0 & 0 \end{Bmatrix} \tag{10.24}$$

satisfies the required conditions. This is not the only $\mathbf{A}_{\mathrm{r}}$ that satisfies these conditions. I have chosen it because I think I ought to link the current to the position because the motor acts directly on the cart. This is simply an intuitive thought; I cannot prove this to be the best choice.

The input to the system is given by Eq. (10.10), which we can rearrange to give Eq. (10.25)

$$u = e = -\mathbf{g}^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_{\mathrm{r}}) - \mathbf{g}_{\mathrm{r}}^{\mathrm{T}} \tag{10.25}$$

We need to find the two gain matrices. We can find $\mathbf{g}^{\mathrm{T}}$ in the usual fashion: test for controllability, find the transformation matrix to put the basic system into companion form, pick poles and find the gains in $\mathbf{z}$ space, and finally transform the gains back to $\mathbf{x}$ space. In this case we find the determinant of the controllability matrix to be

$$-\frac{g^2K^4}{M^4l^4r^4L}$$

The transformation matrix is

$$\mathbf{T} = \begin{Bmatrix} -\frac{MlrL}{gK} & -\frac{Ml^2rL}{gK} & 0 & 0 & 0 \\ 0 & 0 & -\frac{MlrL}{gK} & -\frac{Ml^2rL}{gK} & 0 \\ 0 & \frac{MlrL}{K} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{MlrL}{K} & 0 \\ 0 & -\frac{g(M+m)rL}{K} & 0 & 0 & L \end{Bmatrix}$$

The gains for the Butterworth poles on a Butterworth circle of radius ρ are

$$\mathbf{g}^{\mathrm{T}} = \left\{ -\frac{MlL}{gK}\rho^5 \bullet\bullet\bullet \left(2+\sqrt{3}\right)\rho - R \right\}$$

where the bulleted terms are too long to be written out here.

This is a SISO system so we can find $\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}$ from Eq. (10.18). Equation (10.15) gives the error. I choose to eliminate the error in the output y, so

$$\mathbf{c}^{\mathrm{T}} = \{ 1 \quad 0 \quad 0 \quad 0 \quad 0 \}$$

and the error from which we can calculate the gains, $\mathbf{c}^{\mathrm{T}}\mathbf{E}$, is complicated, so I cannot write it out here. It is a five-dimensional row vector, and each of the five terms contains one of the reference gains, so they are easy to find. They are, regrettably, fairly complicated. Do note that they depend on $\mathbf{g}^{\mathrm{T}}$ as well as the parameters of the reference state. The reference gains cannot be found in a vacuum.

In order to determine whether the tracking control works we need to construct a simulation. It is a good idea to test the simulation before applying the tracking control. We can look at what happens when there is no input. If we start from rest with the pendulum a little off-center we expect the pendulum to continue to oscillate with a decreasing amplitude as the armature resistance gradually eats up the energy. (The eigenvalues of $\mathbf{A}$ for the parameters in Table 6.1 are $-3{,}109.988$, $-0.001962 \pm 2.7125j$, and -0.007848, 0.) Figure 10.10 shows that this is what happens.

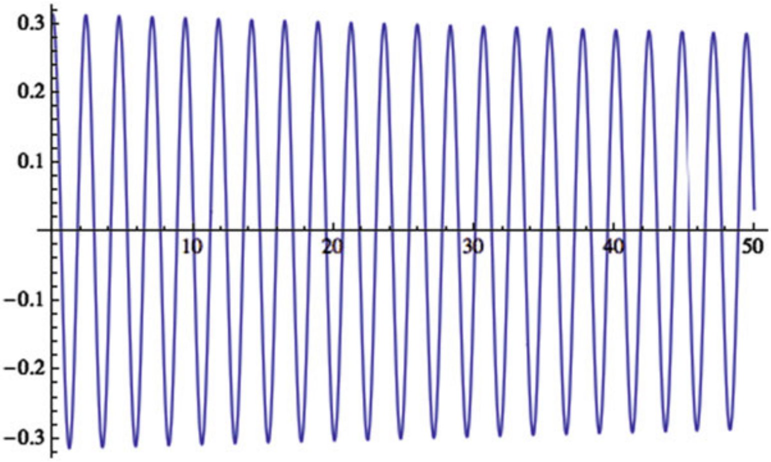


Fig. 10.10 Pendulum oscillation of the overhead crane starting from rest with an initial angle of $\pi/10$

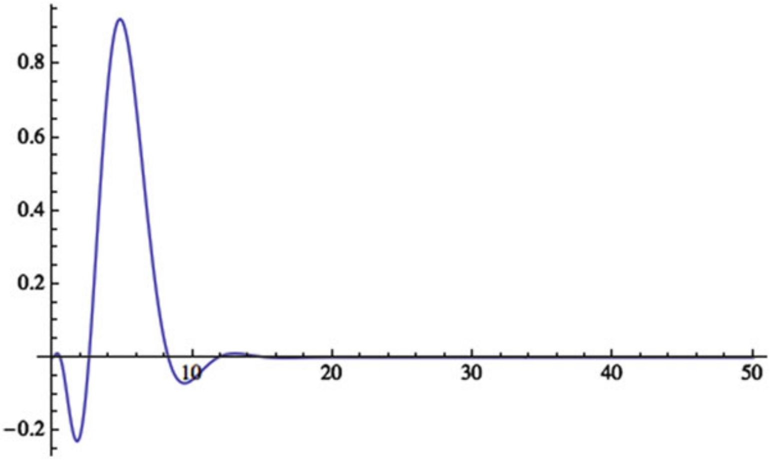


Fig. 10.11 Controlled motion of the cart for $\mathbf{x}_r = 0$

The decay is very slow. These data look like a decaying harmonic motion from which one ought to be able to measure a frequency and a damping ratio. I have set this as a review exercise.

The simulation works for no applied voltage. If we try to track a constant, we can use the input with $\mathbf{g}_r = 0$. Figures 10.11 and 10.12 show the control when the constant $\mathbf{x}_r = 0$.

Now we are ready to try the tracking control. First, note that it works in so much as the cart will track the reference position if you adjust the size of the Butterworth

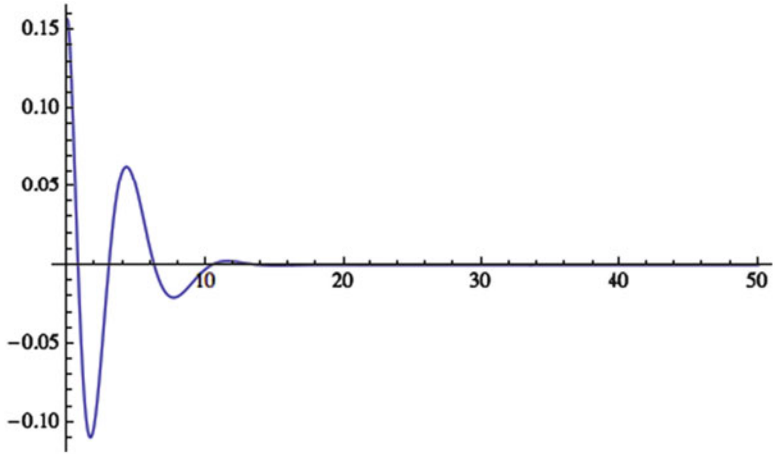


Fig. 10.12 Controlled motion of the pendulum for $\mathbf{x}_r = 0$

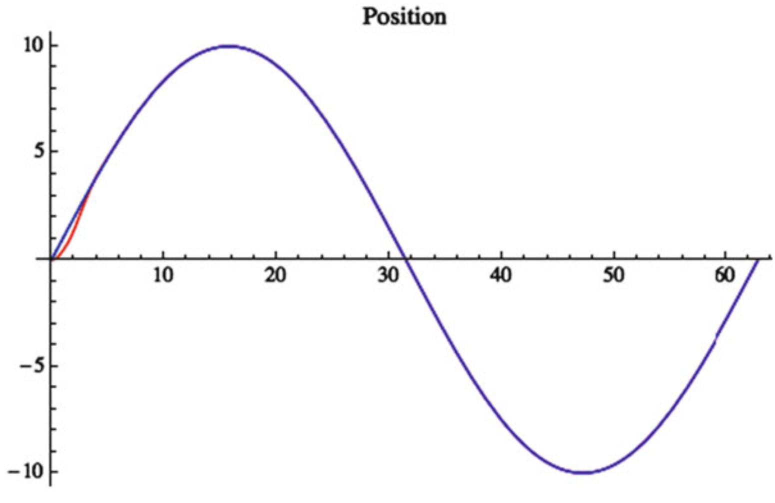


Fig. 10.13 Position of the cart vs. time for $\omega_f = 0.1$. The *blue curve* is the reference position and the *red* the actual position

radius. I chose a peak-to-peak amplitude of 20 m, and I varied ω_f. The larger ω_f, the larger the swing of the pendulum. This is in accord with intuition, because the acceleration of the cart is proportional to the square of the frequency, and the pendulum will swing more the more its attachment point is jerked. The angle also does not go smoothly to zero at the end of the motion. The behavior of the system is acceptable if the frequency is 0.1 rad/s, which corresponds to a maximum speed of 1 m/s, which is probably fast enough for a practical overhead crane. Figures 10.13, 10.14, and 10.15 show the position over one period of the reverence

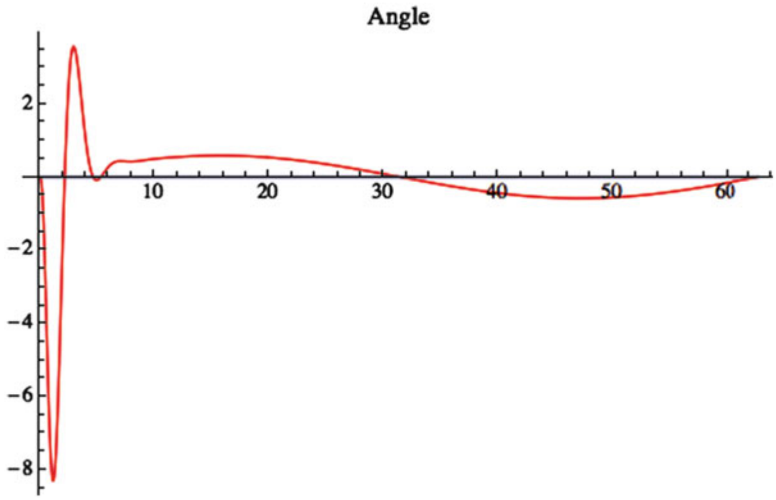


Fig. 10.14 The angle in degrees during the motion shown in Fig. 10.13

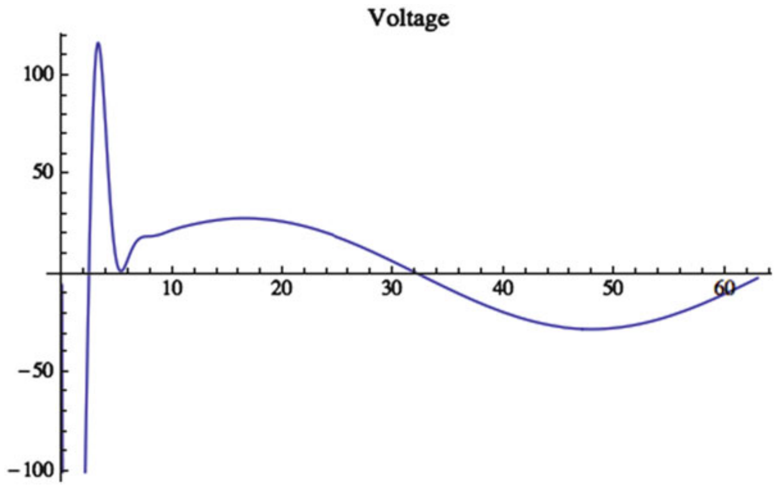


Fig. 10.15 The control voltage for the motion shown in Fig. 10.13

state, $2\pi/\omega_f$, the angle over the same period, and the input voltage required to obtain the results I show. The angle for this control remains quite small, and the voltage is generally within the operating range of the motor. I used a Butterworth radius of 2.

The position tracking works in simulation for higher frequencies. Figure 10.16 shows four periods of the motion for unit frequency. The Butterworth radius was 2.9. Figures 10.17 and 10.18 show the angle and the voltage for this case. Both are unacceptably large.

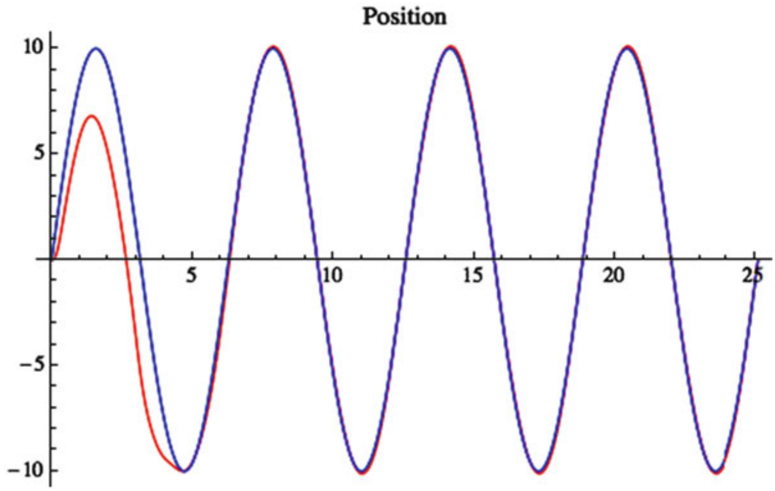


Fig. 10.16 Position of the cart vs. time for $\omega_f = 1$. The *blue curve* is the reference position and the *red* the actual position

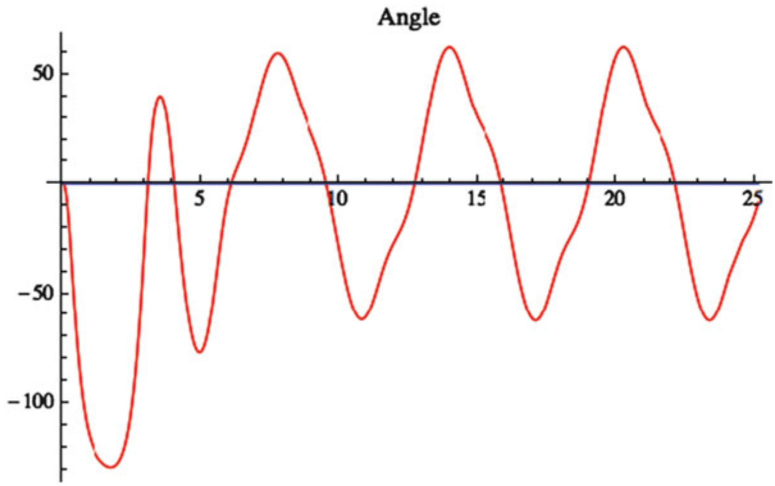


Fig. 10.17 Angle of the pendulum in degrees vs. time for $\omega_f = 1$

10.3.2.2 Reference Input

Much of what we did for the first method carries over. We can adapt the reference state I just used. The basic gains $\mathbf{g}^T$ are the same. We do not need $\mathbf{g}_r^T$. We do need to reduce $\dot{\mathbf{x}}_r - \mathbf{A}\mathbf{x}_r$ to a single nonzero component, the fifth one, so that we can choose a reference input. The first and second components are automatically zero because

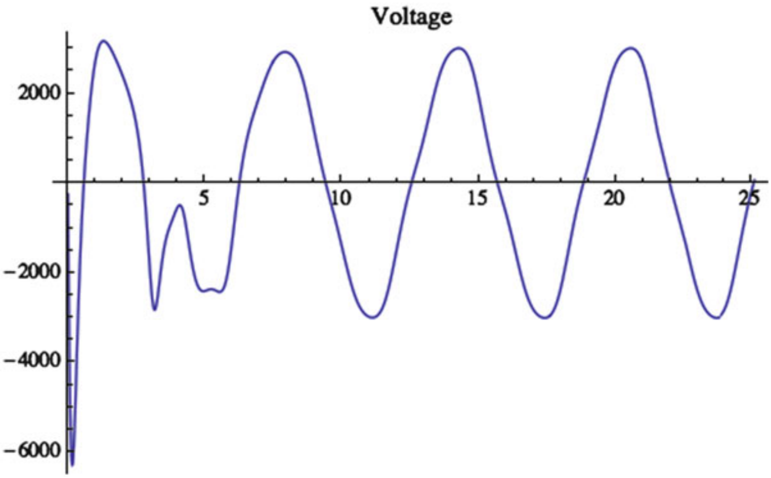


Fig. 10.18 The voltage required for the motion shown in Fig. 10.16

of the way we have chosen the position and angle functions. We can make the third and fourth components vanish by choosing the coefficients of the angle and current functions in terms of d. We find

$$\Theta = \frac{\omega_f^2}{g - l\omega_f^2} d, \quad I = \frac{r\omega_f^2 \left(g(M+m) - Ml\omega_f^2 \right)}{\left(g - l\omega_f^2 \right) K} d \tag{10.26}$$

There is a possibility of resonance here. This is in accord with intuition. We expect a large response from a pendulum when its pivot point is oscillated at its natural frequency. We calculate the reference input from the last term in $\dot{\mathbf{x}}_r - \mathbf{A}\mathbf{x}_r$, which must be equal to $\mathbf{b}u_r$. In this case we obtain

$$\begin{aligned} u_r = {} & \left(K^2 l\omega_f^2 - lLMr^2\omega_f^4 - g\left(K^2 - (M+m)r^2\omega_f^2 \right) \right) \frac{\omega_f d \cos(\omega_f t)}{rK\left(g - l\omega_f^2 \right)} \\ & + \left(Ml\omega_f^2 - (M+m)g \right) \frac{\omega_f^2 r^2 R d \sin(\omega_f t)}{rK\left(g - l\omega_f^2 \right)} \end{aligned} \tag{10.27}$$

The control input is now the sum of u_r and the usual error feedback

$$u = u_r - \mathbf{g}^T(\mathbf{x} - \mathbf{x}_r) \tag{10.28}$$

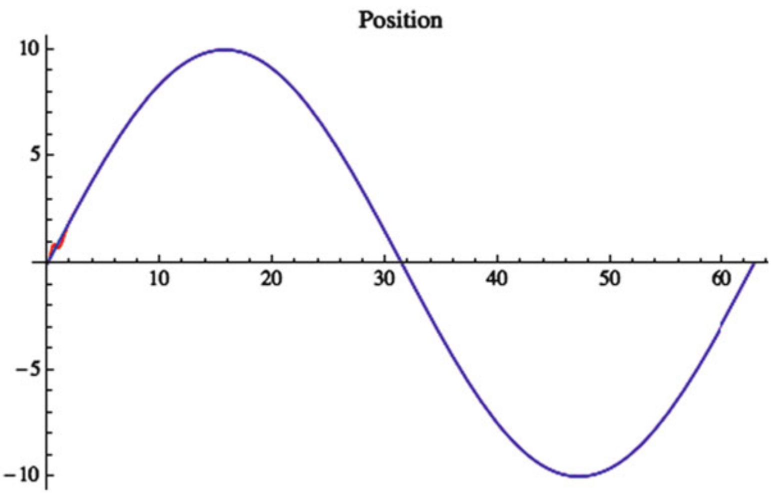


Fig. 10.19 Motion of the cart for $\omega_f = 0.1$ under the second method

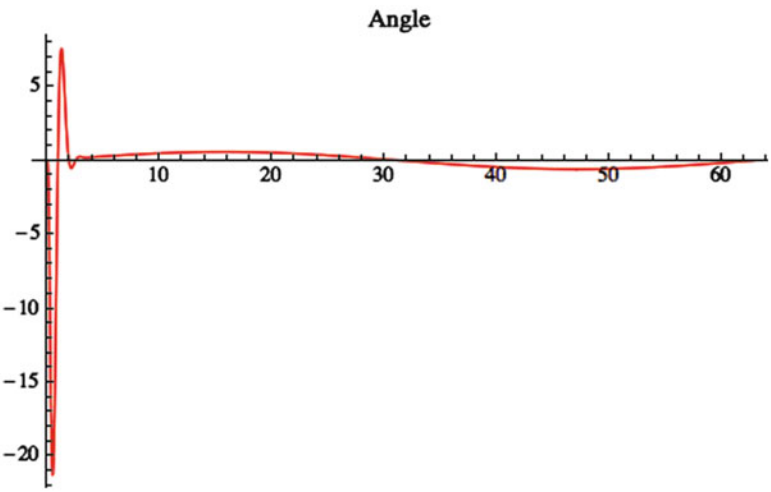


Fig. 10.20 Angle vs. time for the motion shown in Fig. 10.19. The negative peak of about $-21°$ is probably acceptable

So, how does this work? Consider the case represented by Figs. 10.19, 10.20, and 10.21. Figure 10.19 shows one period of the cart motion.

The position tracking is so good that we can barely see the distinction between the reference position and the actual position. As Fig. 10.20 shows, the angle is small after some largish excursions at the beginning. The beginning is the fly in the ointment. Figure 10.21 shows the voltage, which has a very large spike at the

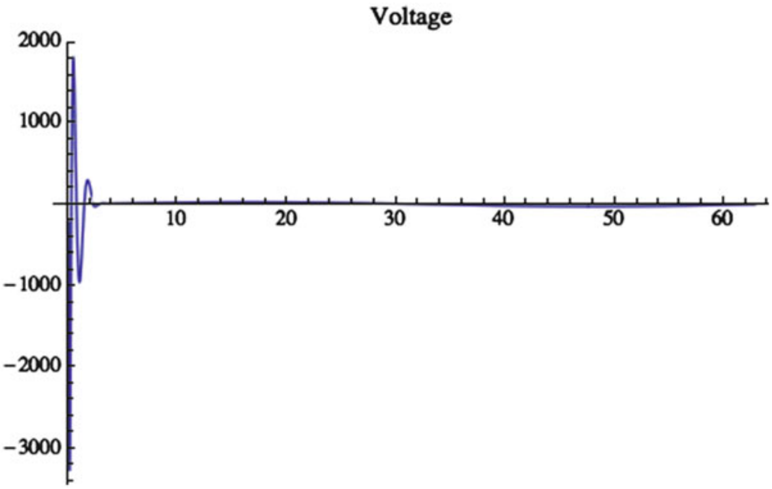


Fig. 10.21 Control voltage for the motion in Fig. 10.19

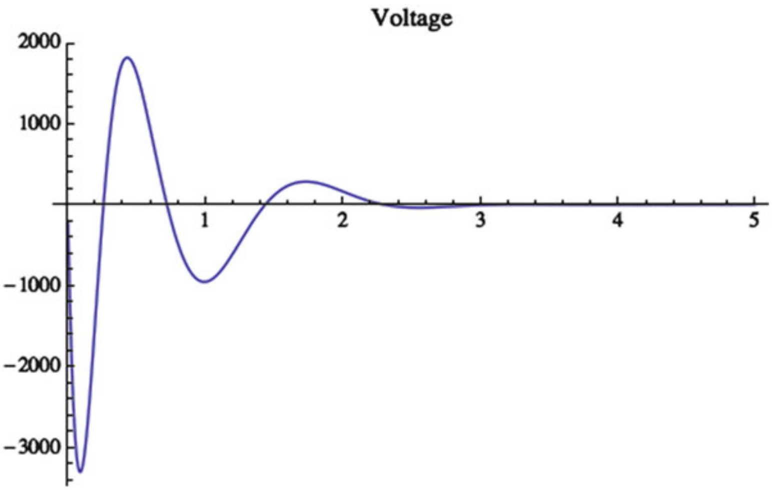


Fig. 10.22 A close-up of the voltage shown in Fig. 10.21

beginning. Figure 10.22 shows the first 5 s of the input voltage, showing a negative voltage spike of over 3,000 V. I tried to add a voltage limitation scheme like that I used in Chap. 8, but it did not work. The voltage spikes are apparently necessary for the control to work.

Reference dynamics works better than reference input for this case.

10.3.3 Tracking a More Useful Position

What we really want to do with an overhead crane is to move it from one place to another, where it will stop, all this without extreme movement of the pendulum. We can do this using a simple scheme to drive the state to zero in the same way that we stabilized the inverted pendulum on a cart in Chap. 8. The equations differ only in the sign of g. We could specify that the final state was an equilibrium (which it is), and simply start from a nonequilibrium position, using $\mathbf{g}^{\mathrm{T}}$ to control it. That may well be the best control strategy, but I want to look at something a little more difficult as an exercise.

I will choose a reference position such that the initial position is zero and the final position is d. I will also require that the first and second derivatives of the position vanish at the beginning and end of the motion. You can verify that

$$y_{\mathrm{r}} = \left(10\left(\frac{t}{t_{\mathrm{f}}}\right)^3 - 15\left(\frac{t}{t_{\mathrm{f}}}\right)^4 + 6\left(\frac{t}{t_{\mathrm{f}}}\right)^5\right)d \tag{10.29}$$

where t_{f} denotes the final time satisfies the conditions I stated. I have been unable to find a complete state and an accompanying $\mathbf{A}_{\mathrm{r}}$ to use to apply the standard method, so my only recourse is to try the reference input method. To use this method we must find a state such that

$$\dot{\mathbf{x}}_{\mathrm{r}} - \mathbf{A}\mathbf{x}_{\mathrm{r}} = \mathbf{b}u_{\mathrm{r}} \tag{10.30}$$

The only nonzero component of $\mathbf{b}$ for the overhead crane is the fifth component, so we need a state for which the first component is given by Eq. (10.27) and which satisfies Eq. (10.28).

I start with a state with two unknown functions

$$\mathbf{x}_{\mathrm{r}} = \begin{Bmatrix} \left(10\left(\frac{t}{t_{\mathrm{f}}}\right)^3 - 15\left(\frac{t}{t_{\mathrm{f}}}\right)^4 + 6\left(\frac{t}{t_{\mathrm{f}}}\right)^5\right)d \\ f_{\mathrm{r}2} \\ \left(30\left(\frac{t}{t_{\mathrm{f}}}\right)^2 - 60\left(\frac{t}{t_{\mathrm{f}}}\right)^3 + 30\left(\frac{t}{t_{\mathrm{f}}}\right)^4\right)\frac{d}{t_{\mathrm{f}}} \\ \dot{f}_{\mathrm{r}2} \\ f_{\mathrm{r}5} \end{Bmatrix} \tag{10.31}$$

The first and second components of Eq. (10.28) equal zero automatically. We can solve the third component for $f_{\mathrm{r}5}$

$$f_{\mathrm{r}5} = \frac{r}{Kt_{\mathrm{f}}^5}\left(mgt_{\mathrm{f}}^5 f_{\mathrm{r}2} - 60Mt\left(2t^2 - 3tt_{\mathrm{f}} + t_{\mathrm{f}}^2\right)\right) \tag{10.32}$$

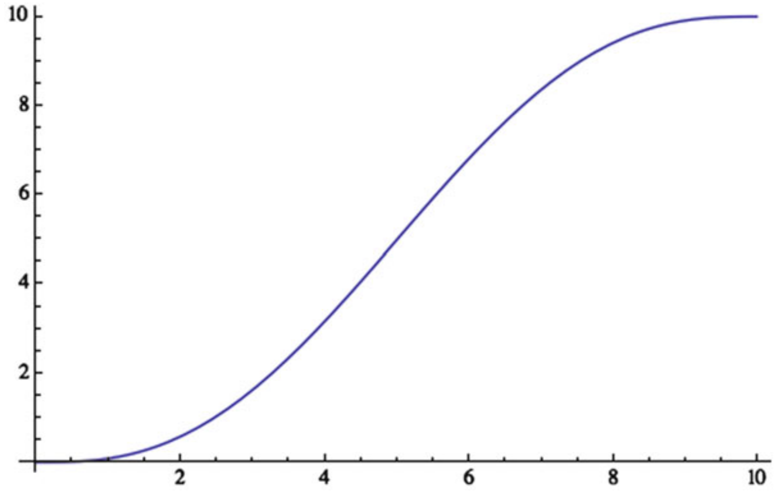


Fig. 10.23 Reference motion of the cart

The fourth component of Eq. (10.28) is now an inhomogeneous ordinary differential equation for f_{r2}

$$\ddot{f}_{r2} + \frac{g}{l} f_{r2} = -60 \frac{dt}{l t_f^3} \left(2 \left(\frac{t}{t_f} \right)^2 - 3 \left(\frac{t}{t_f} \right) + 1 \right) \tag{10.33}$$

This second-order equation requires two side conditions. Asking it to vanish at each end of the time interval seems to be the best choice. With that choice we have

$$\begin{aligned} f_{r2} = & -\frac{60d}{g^2 t_f^5} \left(2gt^3 - 3g t_f t^2 + \left(g t_f^2 - 12l \right) t + 6 l t_f \right) \\ & + \frac{360 l d}{g^2 t_f^4} \left(\cos(\omega t) - \cot\left(\frac{1}{2} \omega t_f \right) \sin(\omega t) \right) \end{aligned} \tag{10.34}$$

where $\omega^2 = g/l$.

What does this look like? Figure 10.23 shows the reference position for $d = 10$ and $t_f = 10$. It looks like what we planned, which is unsurprising.

Figure 10.24 shows the angle, the function f_{r2}. The angle is well within acceptable limits. The current gets pretty large, but the input voltage is within the specifications for the motor, and we will look at it as part of assessing the accuracy of the tracking.

We can solve the fifth component of Eq. (10.26) for u_r. The result is too complicated to display here. The input to the original nonlinear system is $u = u_r - \mathbf{g}^T(\mathbf{x} - \mathbf{x}_r)$ so we obtain the simulation by integrating

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b} u_r - \mathbf{b}\mathbf{g}^T(\mathbf{x} - \mathbf{x}_r) \tag{10.35}$$

where the vector **f** is given by the right-hand side of Eq. (6.39), and $\mathbf{g}^T$ is the same as that found earlier in this section.

Figure 10.25 shows the tracking behavior of the position for the Butterworth radius equal to unity. The tracking is excellent. Figure 10.26 shows the angle during the motion, less than 3° throughout. Figure 10.27 shows the input voltage, well within the capacity of the motor.

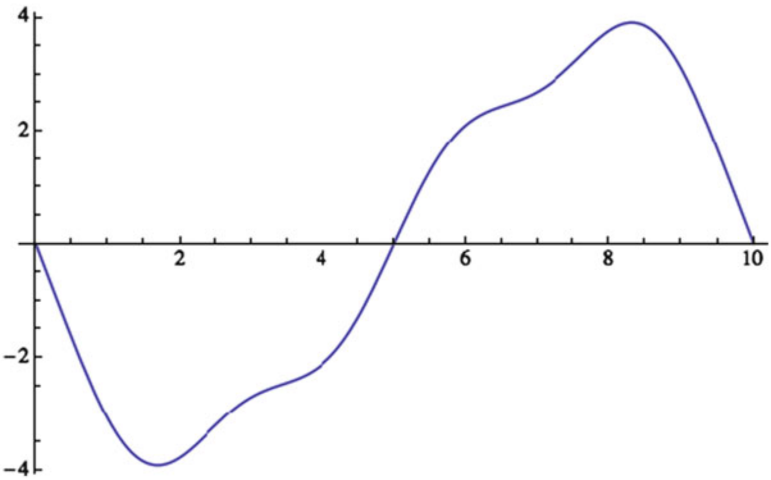


Fig. 10.24 The reference angle in degrees corresponding to the reference displacement of Fig. 10.23

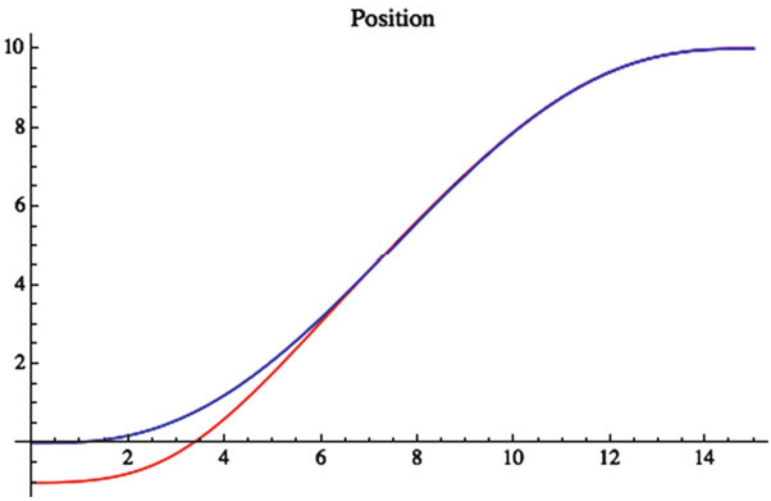


Fig. 10.25 Position tracking. The *blue curve* is the desired response and the *red curve* the actual response. The tracking is so good that I had to start the cart off the path to display a difference

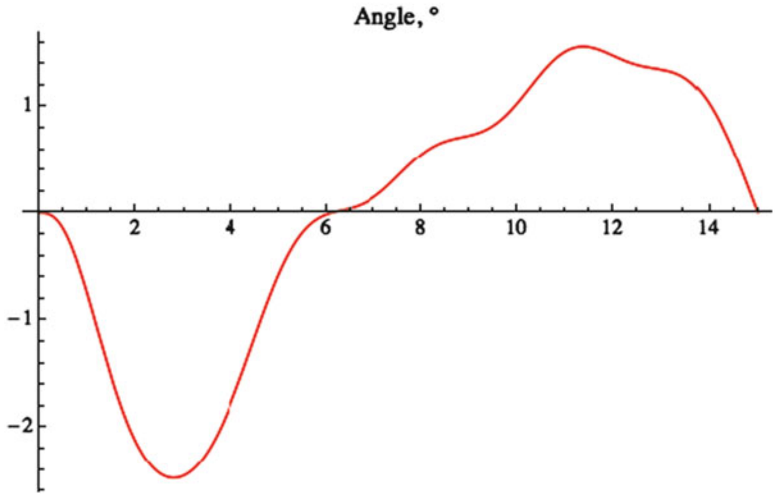


Fig. 10.26 The actual angle (in degrees) for the cart motion shown in Fig. 10.25

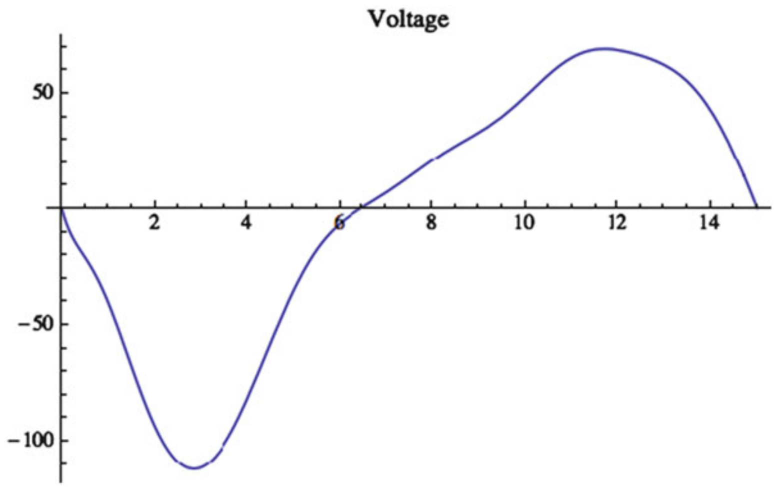


Fig. 10.27 The control voltage required for the tracking exercise

10.4 Tracking with an Observer

Suppose we want to design a tracking control but we cannot measure the entire state. Control using an observer requires us to combine the material in this chapter with that in Chap. 9. The tracking equation, a rewritten version of Eq. (10.15), is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}u_0 - \mathbf{b}\mathbf{g}^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_0 - \mathbf{x}_{\mathrm{r}}) - \mathbf{b}\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}\mathbf{x}_{\mathrm{r}} \tag{10.36}$$

We know how to find the two gain vectors: $\mathbf{g}^{\mathrm{T}}$ comes from placing poles for the closed-loop matrix $\mathbf{A} - \mathbf{b}\mathbf{g}^{\mathrm{T}}$ and $\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}$ comes from setting the error in the output to zero. $\mathbf{A}$ denotes the matrix obtained by linearizing $\mathbf{f}(\mathbf{x})$. We have gone through some examples of this much of the procedure. The procedure relies on knowing the full state, and this is not always the case. When we don't know the full state, we form an observer. The observer satisfies Eq. (9.2)

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\,\hat{\mathbf{x}} + \hat{\mathbf{b}}\,u + \mathbf{k}y \tag{9.2}$$

and Eq. (10.36) must be modified to replace $\mathbf{x} - \mathbf{x}_0$ by its estimate in the input. (The $\mathbf{x}$ that enters $\mathbf{f}$ is the actual $\mathbf{x}$, part of the system. It is automatically there as part of the analysis. We do not need to measure it.) The modified equation is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}u_0 - \mathbf{b}\mathbf{g}^{\mathrm{T}}(\hat{\mathbf{x}} - \mathbf{x}_{\mathrm{r}}) - \mathbf{b}\mathbf{g}_{\mathrm{r}}^{\mathrm{T}}\mathbf{x}_{\mathrm{r}} \tag{10.37}$$

There is no new analysis required here. We know both gain matrices from working the tracking problem, and we know $\mathbf{k}$ from working the observer problem. All that we need to do is solve (numerically) Eqs. (9.2) and (10.37) simultaneously. I replaced $\mathbf{x} - \mathbf{x}_0$ by $\hat{\mathbf{x}}$ in going from Eq. (10.36) to Eq. (10.37). This is because the output of the observer is an estimate of the perturbation state $\mathbf{x}' = \mathbf{x} - \mathbf{x}_0$.

Now let's look at an example: the magnetic suspension problem. This is only a third-order system, but it is nonlinear with an unstable equilibrium, and so it has the potential to be a challenge. We'll find a number of interesting subtleties as we work through the example.

Example 10.3 Tracking Control of the Magnetic Suspension System Using an Observer Example 8.6 gives the basic nonlinear equations governing the motion of this system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{C_n}{m}\frac{x_3^2}{x_1^n} - g$$
$$\dot{x}_3 = -\frac{R}{L}x_3 + \frac{1}{L}e$$

as well as the gain vector $\mathbf{g}^T$ in $\mathbf{z}$ space. I gave the expression in $\mathbf{x}$ space in Ex. 9.2.

$$g_1 = \sqrt{\frac{mL^2 z_0^{n-2}}{4C_n g}}(ng(s_1 + s_2 + s_3) - z_0 s_1 s_2 s_3)$$
$$g_2 = \sqrt{\frac{mL^2 z_0^n}{4C_n g}} s_2 \left(s_1 s_2 + s_1 s_3 + s_2 s_3 - \frac{ng}{z_0} \right)$$
$$g_3 = -R - L(s_1 + s_2 + s_3)$$

Equation (3.38) (reproduced in Ex. 8.2) gives the equilibrium values of the voltage and current in terms of the equilibrium displacement, supposed negative.

$$i_0 = \sqrt{\frac{mg}{C_n}} z_0^n, \quad e_0 = R\sqrt{\frac{mg}{C_n}} z_0^n \tag{3.38}$$

Example 9.2 gives the observer for this system. The observer gains are

$$k_1 = -\frac{R}{L} - (s_4 + s_5 + s_6)$$
$$k_2 = \frac{R^2}{L^2} + \frac{R}{L}(s_4 + s_5 + s_6) + s_4 s_5 + s_4 s_6 + s_5 s_6$$
$$k_3 = \left(\frac{R}{L} + s_4\right)\left(\frac{R}{L} + s_5\right)\left(\frac{R}{L} + s_6\right)\sqrt{\frac{m z_0^n}{4C_n g}}$$

and Ex. 10.2 gives the tracking control. The tracking gains are

$$g_{r1} = \frac{(1 + \sqrt{2})\rho(1962 + 5\omega^2)}{600\sqrt{1090}}, \quad g_{r2} = -\frac{i_r}{100\delta}, \quad g_{r3} = \frac{(1 + \sqrt{2})\rho - 100}{100}$$

where ω denotes the frequency of the oscillatory reference displacement from Ex. 10.2 and the current coefficient

$$i_r = \frac{\delta(1962 + 5\omega^2)}{6\sqrt{1090}}$$

for the parameters in Table 8.2. All that we need to do now is connect these into one coherent whole, and verify that this works in simulation. We need the specific realization of Eq. (10.27).

The efficacy of the system depends on the choice of poles. I chose third-order Butterworth poles on radii of ρ_1 and ρ_2 for the gain poles and the observer gains poles, respectively. I spent a little time playing with the two radii. I obtained the best result using $\rho_1 = 27$ and $\rho_2 = 100$. Figure 10.28 shows the displacement of the ball (red), the reference displacement (blue), and the observer displacement (green) for this choice of radii. The latter soon tracks the actual displacement so closely that

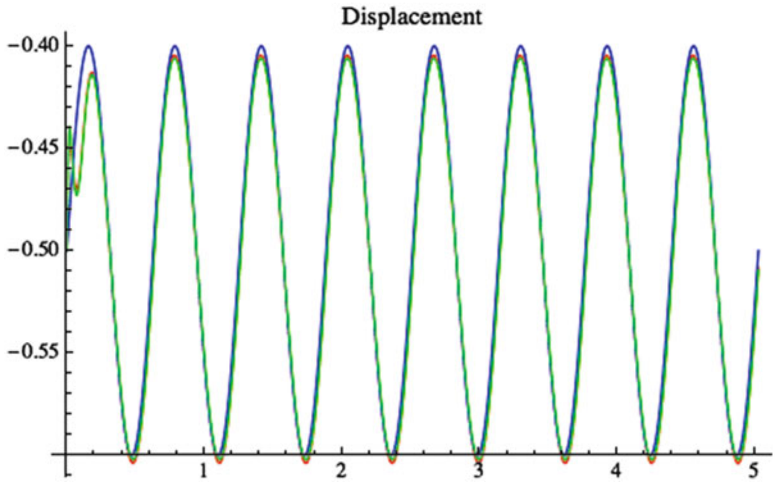


Fig. 10.28 Displacement vs. time: *blue* = reference displacement, *red* = actual displacement, and *green* = observer displacement. The actual and observer displacements quickly become equal and the *green* overlays the *red*, hiding it

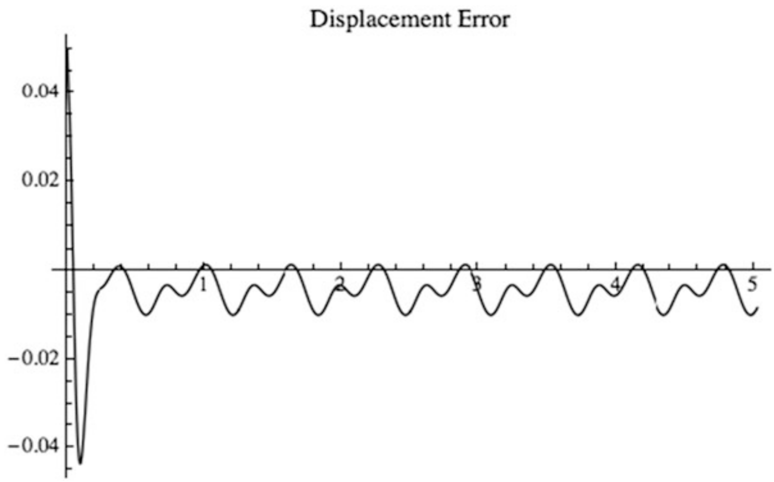


Fig. 10.29 Displacement error, actual displacement minus the reference displacement

the green curve overlies the red curve, which cannot be seen. Figure 10.29 isolates the error, the difference between the actual displacement and the reference displacement. The reference oscillation has a peak-to-peak amplitude of 0.2. The error has a peak-to-peak amplitude of about 0.01, some 5 % of the solution to be tracked.

Figure 10.30 shows the voltage. The red curve is the actual voltage, the control effort for this problem. The horizontal black line denotes the equilibrium voltage. We see that the control effort is reasonable. The maximum voltage is about 6.5 compared to the steady voltage of about 5 V (4.95227 V).

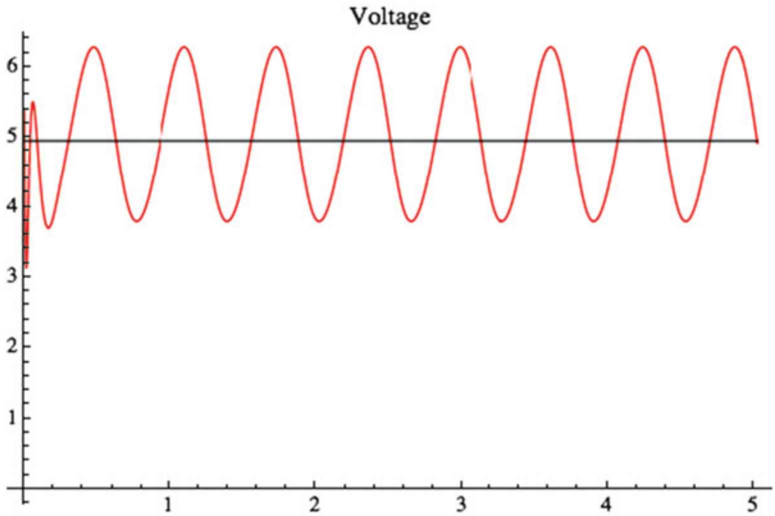


Fig. 10.30 The voltage to control the ball tracking

The system is moderately robust to disturbances. A constant error voltage moves the mean error up or down. High-frequency disturbances make very little difference. Low-frequency disturbances are more disturbing.

10.5 Summary of Linear Control

Chapters 8 and 9 and this chapter cover most of what can be done using linear control in state space to control nonlinear systems. They build on what has come before. We should not lose sight of the need to convert problems stated in words into models suitable for mathematical analysis. Our primary tool for this is the Euler-Lagrange process (Chap. 3), which requires us to understand energies, dissipation, and the devising of generalized forces through a rate of work function. We need to recall how motors work if motors provide the generalized forces (Chap. 3). Once we have a model, we need to convert that to state space form: a set of first-order ordinary differential equations (Chap. 6). There will generally be two equations for each degree of freedom, with an additional equation for each motor. The state equations will not all be linear. We do not know how to address control for nonlinear systems without linearizing, so we need to linearize (Chaps. 3 and 6). To do this we need to find an equilibrium state that we can linearize about.

Linear control is based on linear feedback from the linear error, the difference between the actual state and some desired reference state. We calculate the error, multiply each component of the error by an appropriate gain, and add this composite input to the equilibrium input (if any). Chapter 8 shows how to do this if the reference state is **0**, and the techniques there require us to know (be able to measure) the entire state, because we need (in general) all the components of the error to construct the necessary feedback. The gain calculation methods in Chap. 8 extend to the more complicated problems addressed in Chap. 9 and this chapter. Chapter 8 also shows that adding an integral variable to a state can suppress constant disturbances, picking up on the PID control introduced in Chap. 7.

The controls designed in Chap. 8 require us to be able to measure the entire state. This is often impractical or even impossible. Every component of the state may require an additional sensor. The more sensors required, the more complicated and expensive the control solution will be to implement. Chapter 9 introduces the idea of an observer, an estimate of the state, so that we can use the full state feedback controls designed using the techniques of Chap. 8, replacing the actual measurements with the estimates so we don't have to measure the entire state. All we need to do is measure one element of the state, typically the output for a single-output system.

This chapter expands the reference state from **0** to arbitrary, physically possible, time-dependent reference states. I started with full state feedback and then added an observer, so it is possible to track arbitrary, physically possible with feedback limited to the output. This is pretty good.

Chapter 11 introduces some techniques of nonlinear control, avoiding the need to linearize. It is qualitatively different from the rest of the book.

Exercises

1. Solve the tracking equations for the servo (Ex. 10.1) and show that the result is as depicted in Fig. 10.5.
2. Find the tracking control for the servo if you want it to track $\theta = \pi/2(1 - \exp(-3t))$, and demonstrate that it works, either analytically or in simulation.
3. The table shows below the peaks of the data in Fig. 10.10. Use these data to estimate a natural frequency and damping ratio for these data. Compare the result to the eigenvalues for the problem.

t	2.35842	4.71647	7.07416	9.43149	11.7885	14.1451	16.5013	18.8572
θ	0.312761	0.311369	0.309984	0.308604	0.307229	0.305861	0.304499	0.303143

4. Find the eigenvectors for the overhead crane using the values in Table 5.1. Can you identify the nature of the modes associated with these eigenvectors?
5. Design a control to move the ball in the magnetic suspension system from $z_0 = -0.5$ to $z_1 = -0.75$. Use the physical parameters in Table 8.2 and let the movement be essentially complete in 30 s.
6. Do the observer analysis by putting the system into phase canonical form and verify that the **k** vector found this way agrees with the **k** vector given in the text.
7. Modify the analysis in Sect. 10.3 to track the reference state shown in Fig. 10.25 using a position observer. Simulate the system.

The following design problems are somewhat open ended and require you to invent plausible values for physical parameters. Some will require considerable thought as well.

8. Sinusoidal reference states are comparatively easy to track because constructing $\mathbf{A}_r$ is easy. Design a control to move the magnetically suspended ball from $-z_0$ to $-2z_0$ and back in 5 s using the standard parameters.
9. Repeat exercise 8 using a position observer.
10. Design a control for Ex. 6.4 that will move the arm from $\theta = \pi/2$ to $\theta = \pi$ with a constant acceleration.
11. Airport shuttle trains are automatic. Design a tracking control that will move a single-car train through a four-station loop. Suppose the stations to be 0.5 km apart and that the train must stop at each station for 45 s. The maximum acceleration and deceleration cannot exceed 0.1 g.
12. Repeat exercise 11 with a three-car train like that in exercise 8.15.
13. Design a control to move a car from point A to point B under the following constraints: maximum speed 60 mph and maximum acceleration and deceleration 0.5 g (this is not slow—work out the 0–60 time). Take wind resistance into account. Suppose the drag coefficient to be 0.35 and the frontal area to be 2 m^2. The mass of the car is 1.5 tonnes. How long does the trip take?
14. You can model an elevator as a mass at the end of a stiff spring. Design a control to move an elevator with a gross weight of 1,000 lbs smoothly from the first floor to the sixth floor. The maximum allowable acceleration and deceleration is 0.05 g.
15. Consider the 4D overhead crane and modify it such that the bob can move up and down the rod in response to a force independent of the motor driving the cart. This is a two-input system. Is it controllable? If so, design a control that will move the cart 10 m to the right while the bob moves from the bottom of a two-meter rod to the middle of the rod and then back down.
16. A wrecking ball is essentially a simple pendulum hanging from the end of a crane. It will have its maximum destructive power if it hits the wall at the bottom of its arc. Design the motion of the end of the crane (which you may assume is in a straight line) to assure this. You may suppose the crane to be taller than the building so that you need not concern yourself with the possibility that the crane will hit the building.
17. Design a control to pick up a steel ball from a surface and suspend it 1 m from that surface using an electromagnet at a height of 1.5 m above the surface. Let the ball come smoothly (velocity and acceleration go to zero at the end point) to its final position. You may assume full state feedback.
18. Repeat exercise 17 using a position observer.
19. Design a control to move a car from point A to point B smoothly, such that the initial and final values of velocity and acceleration are zero. Suppose the maximum acceleration and deceleration to be 0.5 g. Assume full state feedback.
20. Repeat exercise 19 using a position observer.

The following exercises are based on the chapter but require additional input. Feel free to use the Internet and the library to expand your understanding of the questions and their answers.

21. Automated airport shuttle trains stop at several stations while running their routes. Consider a single-car train and ask it to stop at four stations 500 m apart. The train is to stay at each stop for 45 s then move to the next station. The maximum allowable acceleration is limited to 0.1 g and the top speed to 15 m/s. Design a tracking control to perform this. (The major problem here is that the mass of the train plus passengers is not always the same.) Verify the control in simulation for a range of masses. You may use a NYC subway train to help define the properties of the train.
22. Repeat exercise 21 supposing that all you can observe is the position of the train.

11 Introduction to Nonlinear Control

In which we learn about feedback linearization and the nonlinear control of some special systems, which systems include some practical robots. . ..

11.1 Feedback Linearization

I am starting something new, so I want to make it as simple as possible: find a control to drive a system to a constant equilibrium. I consider a quasilinear single-input system in state space of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})u \tag{11.1}$$

A constant equilibrium must satisfy Eq. (11.2)

$$0 = \mathbf{f}(\mathbf{x}_0) + \mathbf{b}(\mathbf{x}_0)u_0 \tag{11.2}$$

We need to convert the nonlinear problem in Eq. (11.1) to one for which $\mathbf{x}$ represents the departure from equilibrium. The new equation will be of the same form as Eq. (11.1), so there's no need to make a change. We can make the transformation as follows. Split the state variable

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}', \quad u = u_0 + u'$$

Substitute into the equations

$$\begin{aligned} \dot{\mathbf{x}}' &= \mathbf{f}\left(\mathbf{x}_0 + \mathbf{x}'\right) + \mathbf{b}\left(\mathbf{x}_0 + \mathbf{x}'\right)\left(u_0 + u'\right) \\ &= \mathbf{f}\left(\mathbf{x}_0 + \mathbf{x}'\right) + \mathbf{b}\left(\mathbf{x}_0 + \mathbf{x}'\right)u_0 + \mathbf{b}\left(\mathbf{x}_0 + \mathbf{x}'\right)u' \end{aligned}$$

Since $\mathbf{x}_0$ is just a constant vector, we can write this in the same form as Eq. (11.1)

R.F. Gans, *Mechanical Systems: A Unified Approach to Vibrations and Controls*,
DOI 10.1007/978-3-319-08371-1_11

$$\dot{\mathbf{x}}' = \mathbf{f}'\left(\mathbf{x}'\right) + \mathbf{b}'\left(\mathbf{x}'\right)u'$$

where the primes on **f** and **b** indicate that these are different functions from **f** and **b** in Eq. (11.1). There is no need to carry the primes further in this general development, so I will drop them. It is necessary to remember how I developed these equations when we attempt to apply them. This is best addressed in the context of individual problems, so let's suppose that we have done that conversion.

We know how to address the control of the linear version of this system. We discover whether it is controllable or not. If it is then we convert to companion form and design a linear control. At this point we know how to design linear tracking controls (which of course includes tracking to the null state (**0**) and how to proceed if we do not have access to the full state. Once we have designed the linear control, we can test it by solving Eq. (11.1) numerically and seeing if **x** did actually converge to the reference state, here **0**.

There is an analogous nonlinear procedure for choosing u that we can apply directly to this quasilinear problem. It makes use of similar feedback and a trick to get rid of the nonlinear term. The controllability theorem has an analog, which is not all that useful in practical situations. (If you are interested, see Slotine and Li (1991); Sects. 6.2–6.4.) It makes more sense to try to find an appropriate transformation without testing the system. This is not a good strategy for systems of high dimension (and running them through the theorem is outrageous), so cut and try is not so bad for the systems we can address. I will give some examples below.

The basic idea is to put Eq. (11.1) into something that looks like companion form by applying a nonlinear transformation. The linear transformation is $\mathbf{z} = \mathbf{T}\mathbf{x}$, $\mathbf{x} = \mathbf{T}^{-1}\mathrm{z}$. We write a general nonlinear functional transformation

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}) \tag{11.3}$$

which must be invertible. That is, there must exist an inverse transform

$$\mathbf{x} = \mathbf{X}(\mathbf{z}), \quad \mathbf{Z}(\mathbf{X}(\mathbf{z})) = \mathbf{z} \tag{11.4}$$

This is essential, because we need to control **x**, not **z**, so we need to be able to recover **x** from **z**, which requires the inverse transformation, Eq. (11.4). The forward transformation in Eq. (11.3) converts Eq. (11.1) to a pseudocompanion form. The first $N - 1$ equations of the pseudocompanion form are the same as those of the usual linear companion form. The Nth equation is nonlinear, but it contains u, the input we are trying to find, and we can use u shrewdly to linearize the last line. This is called *feedback linearization*. Equation (11.5) summarizes this

$$\dot{z}_i = z_{i+1}, \quad i = 1 \cdots N_S - 1, \quad \dot{z}_N = f'_{N_S} + b'_N u \tag{11.5}$$

where I have introduced the prime to denote that the components are not the components of the original **f** and **b** vectors. Of course, feedback linearization is not always possible. I will show some examples of this method, but solely on an ad hoc basis.

The way that this usually works is that we select one element of the state **x** and call that z_1. The output ($y = \mathbf{c}^T\mathbf{x}$) of a single-output system is an obvious choice, and it frequently works. Once we've chosen z_1, Eq. (11.1) gives us z_2, since it is just the derivative of z_1. The first step does not work unless $b_1 = 0$. We can then find z_3, and we can continue the process until we have defined the entire transformation. We need to check that the input does not intrude during the process, and that the transformation is invertible. This is not automatic. Note that when $z_1 = y$, the output, each succeeding component of **z** is a derivative of y:

$$z_1 = y, z_2 = \dot{y}, \cdots z_j = y^{(j-1)} \cdots$$

Once a system is in companion form, we know how to choose the coefficients of the last row to give us the poles that we want, the poles required to make $\mathbf{z} \to 0$. We can replace the last line of Eq. (11.5) by a linear system that would control the system by the proper choice of u.

$$\dot{z}_N = f_N' + b_N' u = \sum_{i=1}^{N-1} g_i z_i \tag{11.6}$$

where the g_i terms can be chosen to place a set of poles. These are just a set of gains in **z** space, the same thing that we learned how to do in Chap. 8. The key difference is that the input defined by Eq. (11.6) can remove the nonlinear terms from the **z** equations without approximation in the **x** equations. We solve Eq. (11.6) to find the required input in terms of the nonlinear forcing terms and the gains (which we are to find)

$$u = \frac{1}{b_{N_S}'} \left(\sum_{i=1}^{N_S - 1} k_i z_i - f_{N_S}' \right) \tag{11.7}$$

This will work as long as u given by Eq. (11.7) is not singular.

This looks quite awkward, and I'd like to run through a symbolic example here before moving on to specific examples.

11.1.1 A Symbolic Third-Order System

Consider a specific third-order realization of Eq. (11.1)

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2, x_3)\\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + u\end{aligned} \tag{11.8}$$

This is special because f_1 does not depend on x_3 and because the force occurs only in the third equation. I suppose the output to be $y = x_1$. If we let $z_1 = y$, then we have the following chain of calculations:

$$\begin{aligned} z_1 &= x_1\\ z_2 &= \dot{z}_1 = \dot{x}_1 = \frac{\partial f_1}{\partial x_1}\dot{x}_1 + \frac{\partial f_1}{\partial x_2}\dot{x}_2 = \frac{\partial f_1}{\partial x_1}f_1 + \frac{\partial f_1}{\partial x_2}f_2\\ z_3 &= \dot{z}_2 = \frac{\partial}{\partial x_1}\left(\frac{\partial f_1}{\partial x_1}f_1 + \frac{\partial f_1}{\partial x_2}f_2\right)f_1 + \frac{\partial}{\partial x_2}\left(\frac{\partial f_1}{\partial x_1}f_1 + \frac{\partial f_1}{\partial x_2}f_2\right)f_2\\ &\quad + \frac{\partial}{\partial x_3}\left(\frac{\partial f_1}{\partial x_1}f_1 + \frac{\partial f_1}{\partial x_2}f_2\right)f_3 \end{aligned} \tag{11.9}$$

It is not clear from the symbolic form that this transformation is invertible. It depends on the nature of the three functions in Eq. (11.8). If it is we can continue and write the differential equations governing the evolution of z as Eq. (11.10).

$$\begin{aligned} \dot{z}_1 &= z_2\\ \dot{z}_2 &= z_3\\ \dot{z}_3 &= \frac{\partial}{\partial x_1}\left(\begin{array}{c}\frac{\partial}{\partial x_1}\left(\frac{\partial f_1}{\partial x_1}f_1 + \frac{\partial f_1}{\partial x_2}f_2\right)f_1 + \frac{\partial}{\partial x_2}\left(\frac{\partial f_1}{\partial x_1}f_1 + \frac{\partial f_1}{\partial x_2}f_2\right)f_2\\ + \frac{\partial}{\partial x_3}\left(\frac{\partial f_1}{\partial x_1}f_1 + \frac{\partial f_1}{\partial x_2}f_2\right)f_3\end{array}\right)f_1\\ &\quad + \frac{\partial}{\partial x_2}\left(\begin{array}{c}\frac{\partial}{\partial x_1}\left(\frac{\partial f_1}{\partial x_1}f_1 + \frac{\partial f_1}{\partial x_2}f_2\right)f_1 + \frac{\partial}{\partial x_2}\left(\frac{\partial f_1}{\partial x_1}f_1 + \frac{\partial f_1}{\partial x_2}f_2\right)f_2\\ + \frac{\partial}{\partial x_3}\left(\frac{\partial f_1}{\partial x_1}f_1 + \frac{\partial f_1}{\partial x_2}f_2\right)f_3\end{array}\right)f_2\\ &\quad + \frac{\partial}{\partial x_3}\left(\begin{array}{c}\frac{\partial}{\partial x_1}\left(\frac{\partial f_1}{\partial x_1}f_1 + \frac{\partial f_1}{\partial x_2}f_2\right)f_1 + \frac{\partial}{\partial x_2}\left(\frac{\partial f_1}{\partial x_1}f_1 + \frac{\partial f_1}{\partial x_2}f_2\right)f_2\\ + \frac{\partial}{\partial x_3}\left(\frac{\partial f_1}{\partial x_1}f_1 + \frac{\partial f_1}{\partial x_2}f_2\right)f_3\end{array}\right)f_3 + u \end{aligned} \tag{11.10}$$

This looks terrible, but you need to think of it as an algorithm that you can program in whatever symbolic language you choose, or even by hand if the functions are simple enough. I cannot write the inverse transformation without knowing the functions f_1, f_2, and f_3. The inverse (if it exists) is system specific. The point of this bit of analysis is to lay out the forward transformation algorithm in some detail. You should also note that this succeeded because the only place that u appeared was in the derivative of x_3, and that the derivative of x_3 did not appear until I differentiated z_3. The problem has to be such that the appearance of the forcing can be delayed until the last **z** equation. This can require clever manipulation of the original problem and/or a clever choice of z_1. *It is not always possible*!

The final step is then to determine u, which we get by equating the right-hand side of the third line of Eq. (11.10) to the desired linear feedback.

$$
\begin{aligned}
u = &-\frac{\partial}{\partial x_1}\begin{pmatrix}\frac{\partial}{\partial x_1}\left(\frac{\partial f_1}{\partial x_1}f_1+\frac{\partial f_1}{\partial x_2}f_2\right)f_1+\frac{\partial}{\partial x_2}\left(\frac{\partial f_1}{\partial x_1}f_1+\frac{\partial f_1}{\partial x_2}f_2\right)f_2\\ +\frac{\partial}{\partial x_3}\left(\frac{\partial f_1}{\partial x_1}f_1+\frac{\partial f_1}{\partial x_2}f_2\right)f_3\end{pmatrix}f_1\\
&-\frac{\partial}{\partial x_2}\begin{pmatrix}\frac{\partial}{\partial x_1}\left(\frac{\partial f_1}{\partial x_1}f_1+\frac{\partial f_1}{\partial x_2}f_2\right)f_1+\frac{\partial}{\partial x_2}\left(\frac{\partial f_1}{\partial x_1}f_1+\frac{\partial f_1}{\partial x_2}f_2\right)f_2\\ +\frac{\partial}{\partial x_3}\left(\frac{\partial f_1}{\partial x_1}f_1+\frac{\partial f_1}{\partial x_2}f_2\right)f_3\end{pmatrix}f_2\\
&-\frac{\partial}{\partial x_3}\begin{pmatrix}\frac{\partial}{\partial x_1}\left(\frac{\partial f_1}{\partial x_1}f_1+\frac{\partial f_1}{\partial x_2}f_2\right)f_1+\frac{\partial}{\partial x_2}\left(\frac{\partial f_1}{\partial x_1}f_1+\frac{\partial f_1}{\partial x_2}f_2\right)f_2\\ +\frac{\partial}{\partial x_3}\left(\frac{\partial f_1}{\partial x_1}f_1+\frac{\partial f_1}{\partial x_2}f_2\right)f_3\end{pmatrix}f_3\\
&-g_{z1}z_1-g_{z2}z_2-g_{z3}z_3
\end{aligned}
\tag{11.11}
$$

This converts the **z** equations Eq. (11.10) to companion form

$$
\dot{\mathbf{z}} = \begin{Bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -g_{z1} & -g_{z2} & -g_{z3} \end{Bmatrix}\mathbf{z} \tag{11.12}
$$

which can be controlled by assigning the **z** gains.

If the transformation is invertible, you can either solve the simple **z** equations, Eq. (11.12) or you can convert u back to **x** space by applying the inverse transformation directly to the input as given by Eq. (11.11) and apply that directly to Eq. (11.8). (The former seems to work better numerically.)

Let me go on to some examples of increasing complexity.

Example 11.1 Inverting a Pendulum Consider a pendulum mounted on a motor, as in Fig. 6.14.

We know that the pendulum equations for a simple pendulum are

$$ml^2\ddot{\theta} + mgl\sin\theta = \tau \tag{11.13}$$

where m denotes the mass of the bob, l the length of the pendulum, g the acceleration of gravity, and τ the torque applied at the pivot of the pendulum. Equation (11.13) assumes that the pendulum is pointing straight down when $\theta = 0$. We know that this system has two equilibria: a stable equilibrium about $\theta = 0$ and an unstable equilibrium about $\theta = \pi$. The unstable equilibrium is stabilizable using linear control, but we have seen that when the initial departure from equilibrium is too large, the linear control fails. Can we control this using nonlinear control? The answer is yes, and I will now explore how that works. This problem is sufficiently simple that I can work it without transforming to state space, and it is perhaps easier to understand the process that way. I'll do the state space picture afterwards.

My goal is to get θ to go to π and stay there. I can write $\theta = \pi + e$, where e denotes the error. I need to do this because I am not yet tracking, merely using a nonlinear approach to drive a system to zero. I can form the error equation from Eq. (11.12).

$$ml^2\ddot{e} + mgl\sin(\pi + e) = \tau \Rightarrow \ddot{e} = \frac{g}{l}\sin e + \frac{\tau}{ml^2} \tag{11.14}$$

We know the error would go to zero if

$$\ddot{e} = -2\zeta\omega\dot{e} - \omega^2 e$$

Suppose that to be the case and find τ.

$$\frac{\tau}{ml^2} = -\frac{g}{l}\sin e - 2\zeta\omega\dot{e} - \omega^2 e \tag{11.15}$$

This is a nonlinear function of e and its derivative, and this choice of τ will drive the error to zero, and it will do so for any initial condition. The second-order nonlinear differential equation becomes

$$\ddot{\theta} = -2\zeta\omega\dot{e} - \omega^2 e = -2\zeta\omega\dot{\theta} - \omega^2(\theta - \pi) \tag{11.16}$$

Let's look at the state space formulation by converting Eq. (11.13) to state space form. We have

$$\dot{\mathbf{x}} = \frac{d}{dt}\begin{Bmatrix} \theta \\ \omega \end{Bmatrix} = \begin{Bmatrix} \omega \\ \dfrac{\tau}{ml^2} - \dfrac{g}{l}\sin\theta \end{Bmatrix}$$

Figure 11.1 shows a block diagram of this system.

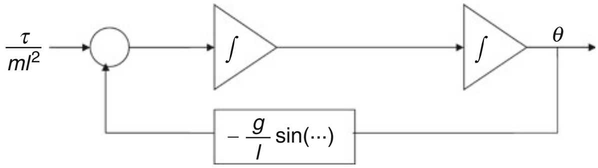


Fig. 11.1 Block diagram of the nonlinear open-loop system

We want to drive θ to π, and one way to attack this is to move the origin by setting $\theta = \pi + \phi$, as I did above and solving a new set of equations (Eq. 11.17) for ϕ, for which the goal is $\phi \to 0$. I have simply specified a new $\mathbf{x}_0$. It's unstable, but that is not an issue.

$$\dot{\mathbf{x}} = \frac{d}{dt}\begin{Bmatrix} \phi \\ \omega \end{Bmatrix} = \begin{Bmatrix} \omega \\ \dfrac{\tau}{ml^2} + \dfrac{g}{l}\sin\phi \end{Bmatrix} \tag{11.17}$$

There is no real need to transform from $\mathbf{x}$ to $\mathbf{z}$, because Eq. (11.17) is already in pseudocompanion form, but let us do that anyway. Let $\phi = z_1$ and then $\omega = z_2$. Then we have a pair of equations governing the evolution of $\mathbf{z}$.

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = \frac{\tau}{ml^2} + \frac{g}{l}\sin z_1 \tag{11.18}$$

This will be well behaved (drive ϕ to zero) if I follow the same path as above.

$$\dot{z}_2 = -\omega_n^2 z_1 - 2\zeta\omega_n z_2 \Rightarrow \frac{\tau}{ml^2} - \frac{g}{l}\sin z_1 = -\omega_n^2 z_1 - 2\zeta\omega_n z_2 \tag{11.19}$$

This determines the control torque as a nonlinear function of $\mathbf{z}$:

$$\dot{z}_2 = -\omega_n^2 z_1 - 2\zeta\omega_n z_2 \Rightarrow \frac{\tau}{ml^2} + \frac{g}{l}\sin z_1 = -\omega_n^2 z_1 - 2\zeta\omega_n z_2$$

or, in terms of θ,

$$\tau = -ml^2\left(\omega_n^2(\theta - \pi) + 2\zeta\omega_n\omega - \frac{g}{l}\sin\theta\right) \tag{11.20}$$

I will call the $-mgl\sin\theta$ term the *feed forward* term. Figure 11.2 shows the closed-loop block diagram where I have identified the feed forward term, which cancels the nonlinear term. I can plug the torque from Eq. (11.20) into the original set of equations, Eq. (11.16), and integrate in time starting with the pendulum pointing straight down. This nonlinear control should be able to drive the pendulum from its stable equilibrium to its unstable equilibrium and maintain it there. I need to put in numbers to verify this. I let $m = 1 = l$, and I choose $\zeta = 0.25$ and $\omega_n = 1$. The former choice will lead to some oscillations on the way to its final position, and

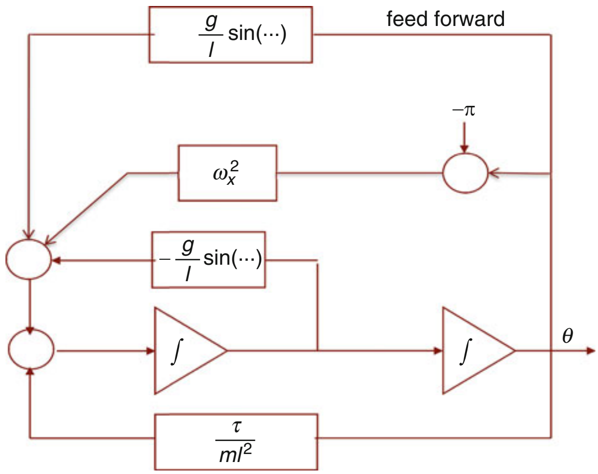


Fig. 11.2 Closed-loop block diagram

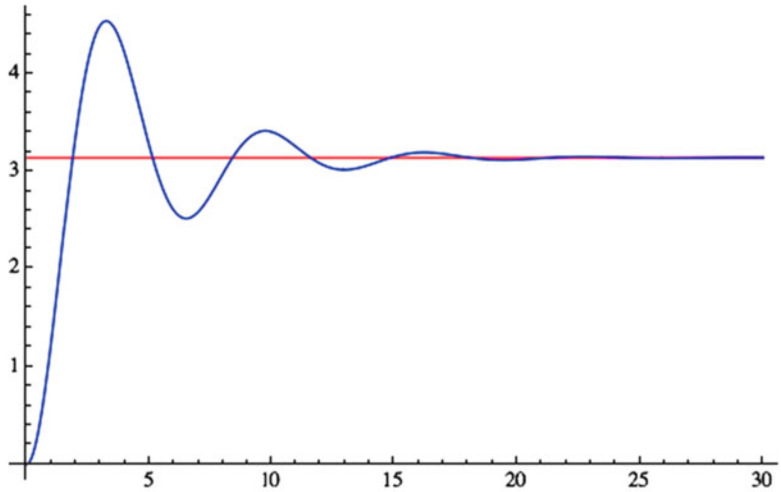


Fig. 11.3 Erecting a pendulum (see text)

the latter choice determines the time scale. Figure 11.3 shows the angle vs. time, and Fig. 11.4 shows the torque required to erect the pendulum.

Let's look at a second, more challenging example, the magnetic suspension. We have seen that the linear model fails if the ball is too close to the magnet. Can we fix that?

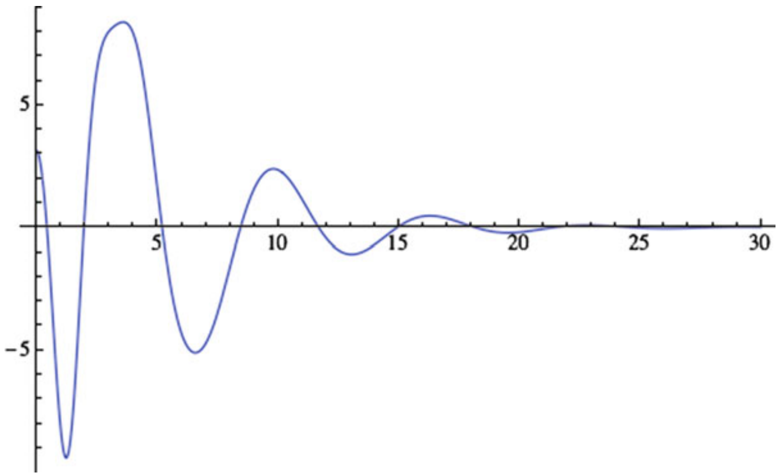


Fig. 11.4 The torque required to erect the pendulum

Example 11.2 Magnetic Suspension We start with the nonlinear state space differential equations that we can deduce from Eqs. (3.35) and (3.36)

$$\mathbf{x} = \begin{Bmatrix} z \\ \dot{z} \\ i \end{Bmatrix}, \quad \dot{\mathbf{x}} = \begin{Bmatrix} x_2 \\ \dfrac{C_n}{m}\dfrac{x_3^2}{x_1^n} - g \\ -\dfrac{R}{L}x_3 + \dfrac{1}{L}e \end{Bmatrix}$$

We know that there is an equilibrium for any fixed value of the input voltage. I find it convenient to describe the equilibrium in terms of the equilibrium position of the ball. I take the origin of the z coordinate to be the face of the magnet, so that the equilibrium value of z, z_0 will be a negative number. We know that the system is unstable, so that an input voltage that will balance gravity at the equilibrium point will not hold the ball, because any perturbation will cause it to move away from its equilibrium position. I want to design a nonlinear control using feedback linearization to hold the ball at its equilibrium position. So far we have learned how to make a state go to zero, so if we are to use that knowledge, we need to convert the initial problem to a zero problem. To that end we write

$$z \to z_0 + z', \quad i \to i_0 + i', \quad e \to e_0 + e'$$

where the zero current and voltage can be expressed in terms of the equilibrium position by

$$i_0 = \sqrt{\frac{mgz_0^n}{C_n}}, \quad e_0 = R\sqrt{\frac{mgz_0^n}{C_n}}$$

(The inductance plays no role in the steady solution.)

In previous chapters we have linearized about the equilibrium position. Here we want to find a nonlinear control, so linearization is not appropriate. We want a problem for $\mathbf{x}'$, the deviation from the (unstable) equilibrium, which problem is

$$\dot{x}_1' = x_2', \quad \dot{x}_2' = \frac{C_n}{m}\frac{\left(\sqrt{mgz_0^n/C_n} + x_3'\right)^2}{(z_0 + x_1)^n} - g, \quad \dot{x}_3' = -\frac{R}{L}x_3' + \frac{1}{L}e'$$

You can verify that these equations are satisfied by $\mathbf{x}' = 0$.

In order to apply feedback linearization I need to find a transformation $\mathbf{x}' \to \mathbf{z}$. I choose z_1 to be equal to x_1, and then I can go through the chain of derivatives to find

$$z_1 = x_1', \quad z_2 = x_2', \quad z_3 = \dot{x}_2' = \frac{C_n}{m}\frac{\left(\sqrt{mgz_0^n/C_n} + x_3'\right)^2}{(z_0 + x_1)^n} - g$$

The transform has to be invertible, and you can verify that it is and that its inverse transformation is

$$x_1 = z_1, \quad x_2 = z_2, \quad x_3 = \sqrt{\frac{m}{C_n}(z_0 + z_1)^n(g + z_3)} - \sqrt{\frac{gm}{C_n}z_0^n}$$

My plan is to solve the problem in $\mathbf{z}$ space and then project the answers back to $\mathbf{x}'$ space. An alternative approach might be to find the control voltage in $\mathbf{z}$ space and apply that voltage in terms of $\mathbf{x}'$ in the $\mathbf{x}$ space equations. As it happens this leads to numerical problems. It is more stable computationally to do the simulation in $\mathbf{z}$ space and then use the inverse transformation just given to find the behavior of the actual system.

We are on the way to getting the transformed problem into companion form. All we need to do is to choose e', the control voltage. We want

$$\dot{z}_3 = A_{31}z_1 + A_{32}z_2 + A_{33}z_3$$

We get the derivative of z_3 by differentiating it with respect to time

$$\dot{z}_3 = \frac{\partial}{\partial x_1'}\left(\frac{C_n}{m}\frac{\left(\sqrt{mgz_0^n/C_n} + x_3'\right)^2}{(z_0 + x_1)^n}\right)\dot{x}_1' + \frac{\partial}{\partial x_3'}\left(\frac{C_n}{m}\frac{\left(\sqrt{mgz_0^n/C_n} + x_3'\right)^2}{(z_0 + x_1)^n}\right)\dot{x}_3'$$

and then substituting for $\mathbf{x}$ in terms of $\mathbf{z}$. The result is rather messy, but the derivative of x'_3 contributes a term involving e' so it is possible to find an expression for e' to satisfy the condition. It's a lengthy expression, and I will not reproduce it here. I will point out that it has two parts. One cancels the complicated nonlinear

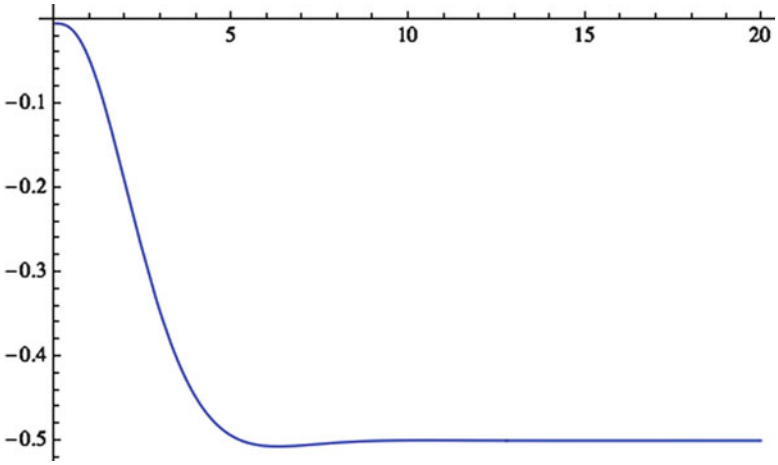


Fig. 11.5 Ball position for an initial position of −0.01

terms (the feed forward term) and the other inserts the coefficients that will eventually give me my stabilized **z** system (the feedback term). The coefficients in the expression for the derivative of z_3 can then be assigned to give the poles that we want for the pseudocompanion form

$$A_{31} = s_1 s_2 s_3, \quad A_{32} = -(s_1 s_2 + s_1 s_3 + s_2 s_3), \quad A_{33} = s_2 + s_2 + s_3$$

and the governing equations for **z** are the simple closed-loop linear equations

$$\dot{\mathbf{z}} = \begin{Bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ s_1 s_2 s_3 & -(s_1 s_2 + s_1 s_3 + s_2 s_3) & s_2 + s_2 + s \end{Bmatrix} \mathbf{z}$$

We saw in Chap. 8 that the linear control failed when the ball started too close to the magnet. Let's look at a system where the ball starts at rest only 0.01 units from the face of the magnet. Figure 11.5 shows the path of the ball, and Fig. 11.6 shows the control voltage. The physical parameters are those given in Table 11.1, and I used Butterworth poles at a radius of unity to generate these plots.

I wrote the general procedure in terms of tracking. Example 11.1 tracked to 0, as did Ex. 11.2 once I reset the origin. Let me work the problem of Ex. 11.1 as a "real" tracking problem. Instead of the reference state (the goal) θ_r, being π, we can choose it as an arbitrary function of time. The analysis parallels that above. I will also let inductance be important, so we have a third-order problem, which encourages us to use state space and going through the formal transformation process.

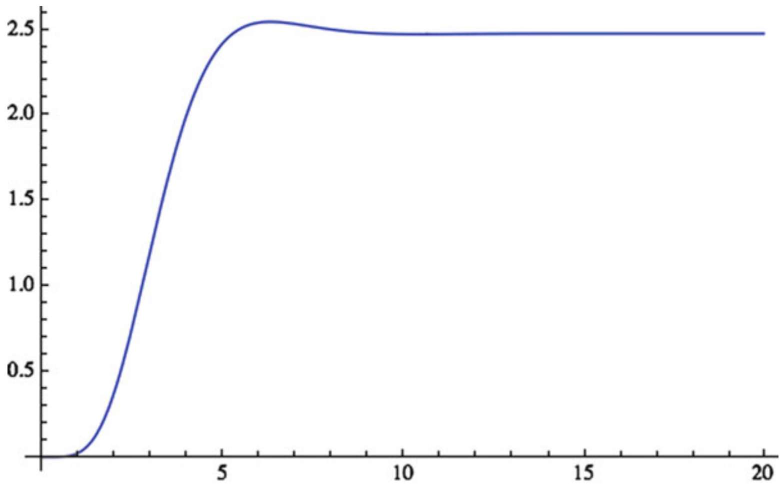


Fig. 11.6 Control voltage for the response shown in Fig. 11.5

Table 11.1 Parameters for the magnetic suspension problem

m	C_n	n	L	R	g
1	1	4	0.01	1	9.81

Example 11.3 A Third-Order Tracking Problem for the Pendulum What happens to this if I replace the arbitrary torque with a motor in the high-inductance limit, so that the system goes to third order? The two-dimensional state equations become

$$\dot{\mathbf{x}} = \frac{d}{dt}\begin{Bmatrix} \theta \\ \omega \end{Bmatrix} = \begin{Bmatrix} \omega \\ \dfrac{Ki}{ml^2} - \dfrac{g}{l}\sin\theta \end{Bmatrix}$$

where K denotes the motor constant and i the current. These must be supplemented by a current equation

$$\dot{i} = \frac{1}{L}\left(e - iR - K\dot{\theta}\right) \tag{11.21}$$

where L and R denote the inductance and resistance, respectively, and the state must become a three-dimensional state. I'll add the current as the third component of the state. There is a reference current that we can obtain from the old state equations

$$i_r = \frac{ml^2}{K}\left(\frac{g}{l}\sin\theta_r + \ddot{\theta}_r\right) \tag{11.22}$$

We can find a transformation based on the full state space representation of this system

$$\mathbf{x} = \begin{Bmatrix} \theta \\ \dot{\theta} \\ i \end{Bmatrix}, \quad \dot{\mathbf{x}} = \begin{Bmatrix} x_2 \\ -\frac{g}{l}\cos x_1 + \frac{K}{ml^2}x_3 \\ -\frac{1}{L}(Rx_3 + Kx_2) \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{Bmatrix} e \tag{11.23}$$

Since $\mathbf{x} = 0$ satisfies Eq. (11.23) (it is an equilibrium) I do not need to go through the ritual of finding an equilibrium as I did in Ex. 11.2. We want to find a vector $\mathbf{z}$ such that

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = z_3$$

where none of the components of $\mathbf{z}$ involve the input e. We want θ to track a reference, so θ is the output for this problem, and I will choose z_1 to be θ (x_1), then I have

$$\dot{z}_1 = x_2 = z_2, \quad \dot{z}_2 = -\frac{g}{l}\cos x_1 + Kx_3 = z_3$$

We can solve the last of these equations for x_3, the current

$$z_3 = -\frac{g}{l}\cos x_1 + \frac{K}{ml^2}x_3 \Leftrightarrow x_3 = \frac{ml^2}{K}\left(z_3 + \frac{g}{l}\cos z_1\right) \tag{11.24}$$

or

$$z_1 = \theta, \quad z_2 = \dot{\theta}, \quad z_3 = \ddot{\theta}$$

The inverse transformation is nonsingular and may be written as

$$x_1 = z_1, \quad x_2 = z_2, \quad x_3 = \frac{ml^2}{L}\left(\frac{g}{l}\cos z_1 + z_3\right) \tag{11.25}$$

The transformation does not involve the input, and it is invertible. The system in z space is

$$\dot{z}_1 = z_2, \;\; \dot{z}_2 = z_3, \;\; \dot{z}_3 = \frac{gR}{lL}\cos z_1 - \left(\frac{K^2}{ml^2L} + \frac{g}{l}\sin z_1\right)z_2 - \frac{R}{L}z_3 + \frac{K}{ml^2L}e \tag{11.26}$$

Equation (11.26) presents the original problem. We can write it as a third-order equation for θ by substituting for the components of $\mathbf{z}$ in the third of Eq. (11.26).

$$\dddot{\theta} + \frac{gR}{lL}\cos\theta + \left(\frac{K^2}{ml^2L} - \frac{g}{l}\sin\theta\right)\dot{\theta} + \frac{R}{L}\ddot{\theta} - \frac{K}{ml^2L}e = 0 \tag{11.27}$$

We can convert this to an error equation by writing $\theta = \theta_r + \theta'$

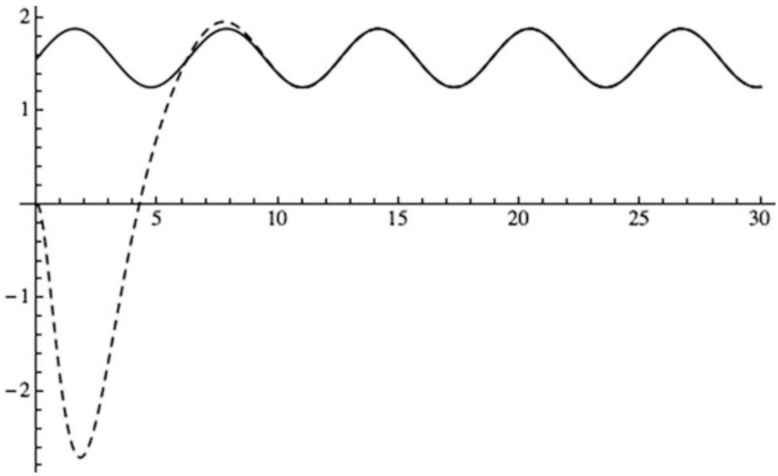


Fig. 11.7 Motor control of the pendulum. The *solid line* denotes the reference state and the dashed line the angle response

$$\dddot{\theta}_r' + \dddot{\theta}' + \frac{gR}{lL}\cos\left(\theta_r + \theta'\right) + \left(\frac{K^2}{ml^2L} - \frac{g}{l}\sin\left(\theta_r + \theta'\right)\right)\left(\dot{\theta}_r + \dot{\theta}'\right)$$
$$+\frac{R}{L}\left(\ddot{\theta}_r + \ddot{\theta}'\right) - \frac{K}{ml^2L}e = 0 \tag{11.28}$$

We want the error to vanish, and we can do this using pole placement. We want Eq. (11.28) to equal Eq. (11.29) (Why? Where does this come from?)

$$\dddot{\theta}' - (s_1 + s_2 + s_3)\ddot{\theta}' + (s_1s_2 + s_1s_3 + s_2s_3)\dot{\theta}' - s_1s_2s_3\theta' = 0 \tag{11.29}$$

We can do this by equating them and choosing the input voltage, which will come out in terms of the poles, the reference angle and its derivatives, and the error and its derivatives. We want it in terms of the actual angle and its derivatives and the reference and its derivatives, so we need to substitute for the error, writing it as the difference between θ and θ_r. We arrive at a fairly complicated expression for the voltage that I do not want to write out.

Figures 11.7 and 11.8 show the angle response and the control voltage, respectively, for poles at -1, $-(1 \pm j)/\sqrt{2}$, the third-order Butterworth poles on a unit circle.

The initial response shown in Figs. 11.6 and 11.7 differs because the two systems are of different order. The first system is second order in the angle; the second system is third order in the angle. Both track perfectly after the transient has gone by. The initial response always depends on the order. The input acts on the second derivative of the angle in the second-order problem—directly on the torque. It acts

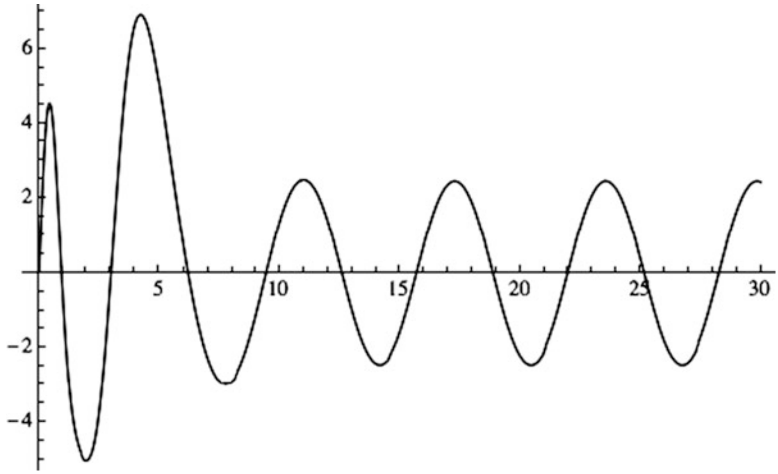


Fig. 11.8 Control voltage for the response shown in Fig. 11.7

on the third derivative of the angle in the third-order problem, so there is one integration between the input and the torque, which results in the delay that one can see when comparing the two figures.

11.2 Nonlinear Control of a Kinematic Chain

The control that we designed to control a pendulum in the previous section can be extended to kinematic chains that contain only revolute joints. A kinematic chain is a set of links joined together, and revolute joints are rotational joints. This idea forms the basis for some robotic controls, which I will discuss in Sect. 11.3. There is little point in trying to do the general problem symbolically, so I will select a case that is complicated enough to give an idea of how this proceeds, but simple enough that we can follow what is happening.

Figure 11.9 shows a three-link kinematic chain. It's a three degree of freedom system and I expect a six-dimensional state space.[1] I suppose there is a torque at each joint. I will number the three links 1, 2, and 3 distally from the ground. I will call the ground (the thick black line) link 0. I will denote each torque by τ_{ij}, where i denotes the link providing the torque and link j the link to which the torque is applied. I will let all the links be the same for convenience, and suppose them to be symmetric so that the center of mass of each link is at its geometric center. I define the orientation of each link by the angle it makes from the horizontal, which I denote by θ_i, increasing in the counterclockwise direction. Each link has the same

[1] Assuming that I either specify the torques or use a motor at the low-inductance limit, which is reasonable if the reference state does not vary too rapidly.

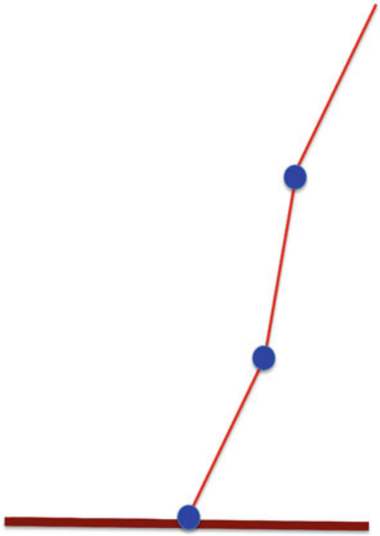


Fig. 11.9 A three-link kinematic chain

mass m and the same length l. They are uniform rods, so that their moments of inertia about their centers of mass equal $ml^2/12$.

If we can specify the torques independently of the dynamics then we can design the entire control using the Euler-Lagrange equations with no need to introduce a state space. I will restrict the discussion here to the torque-driven case, which is equivalent to the low-inductance motor-driven case.

We can find the equations of motion using the Euler-Lagrange process. They are complicated, and I will simply outline the procedure, starting with the simple energy expressions

$$T = \frac{1}{2}m\left(\dot{y}_1^2 + \dot{z}_1^2 + \dot{y}_2^2 + \dot{z}_2^2 + \dot{y}_3^2 + \dot{z}_3^2\right) + \frac{1}{24}ml^2\left\{\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right\}$$
$$V = mg(z_1 + z_2 + z_3) \tag{11.30}$$

The positions of the centers of mass can be written in terms of the angles; this is a three degree of freedom system as I noted earlier. These geometric (kinematic) constraints are

$$y_1 = \frac{1}{2}l\cos\theta_1, \quad z_1 = \frac{1}{2}l\sin\theta,$$
$$y_2 = l\cos\theta_1 + \frac{1}{2}l\cos\theta_2, \quad z_2 = l\sin\theta_1 + \frac{1}{2}l\sin\theta_2 \tag{11.31}$$
$$y_2 = l\cos\theta_1 + l\cos\theta_2 + \frac{1}{2}l\cos\theta_3, \quad z_2 = l\sin\theta_1 + l\sin\theta_2 + \frac{1}{2}l\sin\theta_3$$

We can substitute the constraints and their derivatives into the energies, find the Lagrangian, and choose the angles as the three generalized coordinates. These involve generalized forces, which we can find from the rate of work. We have

$$\begin{aligned} \dot{W} &= (\tau_{01} - \tau_{12})\dot{\theta}_1 + (\tau_{12} - \tau_{23})\dot{\theta}_2 + \tau_{23}\dot{\theta}_3 \\ Q_1 = (\tau_{01} - \tau_{12}),& \quad Q_2 = (\tau_{12} - \tau_{23}), \quad Q_3 = \tau_{23} \end{aligned} \tag{11.32}$$

The equations are too complicated to write out here, but I can write out their forms.

$$\begin{aligned} &\frac{5}{2}mgl\cos\theta_1 + f_{12}\dot{\theta}_2^2 + f_{13}\dot{\theta}_3^2 + a_{11}\ddot{\theta}_1 + a_{12}\ddot{\theta}_2 + a_{13}\ddot{\theta}_3 = \tau_{01} - \tau_{12} \\ &\frac{3}{2}mgl\cos\theta_2 + f_2\dot{\theta}_1^2 + f_{23}\dot{\theta}_3^2 + a_{21}\ddot{\theta}_1 + a_{22}\ddot{\theta}_2 + a_{23}\ddot{\theta}_3 = \tau_{12} - \tau_{23} \\ &\frac{1}{2}mgl\cos\theta_3 + f_{31}\dot{\theta}_1^2 + f_{32}\dot{\theta}_2^2 + a_{31}\ddot{\theta}_1 + a_{32}\ddot{\theta}_2 + a_{33}\ddot{\theta}_3 = \tau_{23} \end{aligned} \tag{11.33}$$

The various coefficients are functions of the angles.

I will adopt and extend the tracking procedure from Sect. 11.1, setting

$$\theta_i = \theta_{ri} + e_i$$

We can substitute this into the equations of motion, which are then second-order differential equations for e_i, and we will want to make each e go to zero. We know that if each e_i satisfies the differential equation of the form

$$\ddot{e}_i + 2\zeta_i\dot{e}_i + \omega_{ni}^2\theta = 0$$

then each component of the error will go to zero with time and we can expect the chain to track its reference state. We can solve Eq. (11.38) for torques that will assure this. These torques come out in terms of the errors, their first derivatives and the reference angles, and their first and second derivatives. We can write the errors and their derivatives in terms of the angles and their reference values and substitute those back into Eq. (11.38) to give a set of second-order ordinary differential equations in terms of the angles and the reference angles with torques that ought to drive the error to zero.

In summary:

- Find the Euler-Lagrange (EL) equations.
- Find the generalized forces in terms of the physical inputs.
- Write the EL equations in terms of the reference state and deviations from the reference state (the errors).
- Solve the modified EL equations for the physical inputs to make the errors go to zero.

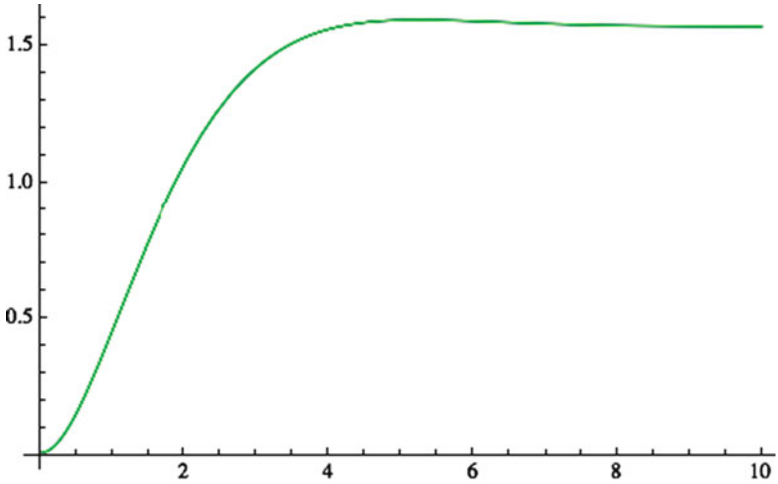


Fig. 11.10 Raising the three-link chain specifying just the initial and final positions

- Replace the errors in the physical inputs by their definitions in terms of the state and the reference state, and substitute this into the original EL equations.
- Integrate these numerically to verify that the control designed works

Let me show some results.

Example 11.4 The Reference State a Constant Suppose that we want the three angles to have specified values. Suppose that the linkage in Fig. 11.7 is horizontal ($\theta_1 = \theta_2 = \theta_3 = 0$), and we want it to be erect. We select an initial condition starting from rest with all the angles equal to zero.[2] All three angles in the reference state equal $\pi/2$, and the derivatives of the reference state are all zero. I select $m = 1 = l$ and choose time scale such that g is also equal to unity. There is no loss of generality in this choice. Figure 11.10 shows the simulated angles as a function of time for $\zeta = 0.8$ and $\omega_n = 1$. I can remove the overshoot by selecting $\zeta = 1$, and the overshoot will be greater if ζ is smaller. The reader is invited to experiment with this system.

Example 11.5 Tracking Harmonic Functions We can also track time-dependent reference states. Figure 11.11 shows the angles of a three-link chain for which the reference state is

[2] I actually needed to choose very small initial angles because the numerical software I am using thinks the equations for zero angles are singular.

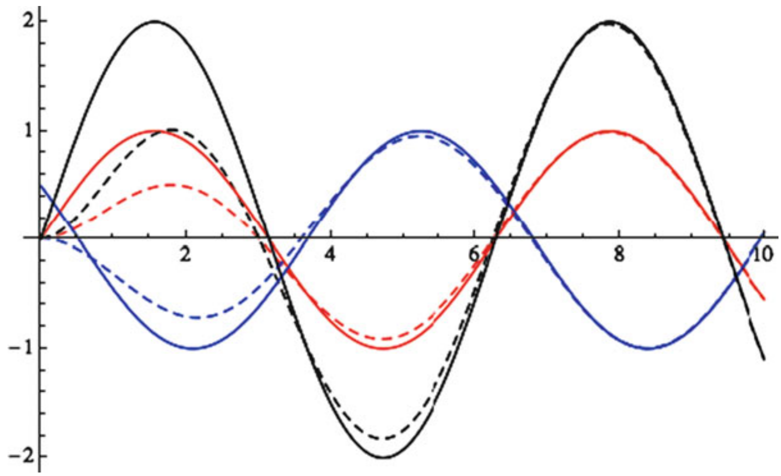


Fig. 11.11 Tracking three harmonic functions. The first angle is in *red*, the second in *blue*, and the third in *black*. The *solid lines* show the reference state and the *dashed line* the actual state

$$\theta_{r1} = \sin t, \quad \theta_{r2} = \cos\left(t + \frac{\pi}{3}\right), \quad \theta_{r3} = 2\sin t \tag{11.34}$$

The initial condition is the same as the previous example. I have reduced ζ to 1/2 to make the transient while the system moves to track the reference state easier to see.

We looked at a simple motion taking the angles from one set to another in Ex. 11.4. Suppose we want to specify the smoothness of this process and the path along the way. The Euler-Lagrange equations are second order, and so smoothness implies the vanishing of the function and its first and second derivatives at the end of its motion. The reader can verify that

$$\theta = \theta_i + \left(\theta_f - \theta_i\right)\left(10\left(\frac{t}{t_\mathrm{f}}\right)^3 - 15\left(\frac{t}{t_\mathrm{f}}\right)^4 + 6\left(\frac{t}{t_\mathrm{f}}\right)^5\right) \tag{11.35}$$

takes on the value θ_i at $t = 0$ and θ_f at $t = t_\mathrm{f}$ and has zero first and second derivatives at the two time end points.

Example 11.6 Smooth Change of Position Let me move the chain from a folded position with $\theta_1 = \pi/2$, $\theta_2 = -\pi/2$, and $\theta_3 = \pi/2$[3] to an outstretched position with all three angles equal to $\pi/4$ in 11 time units with $\zeta = 1/2$ and $\omega_n = 1$. Figure 11.12 shows the smooth response of all three angles, and Fig. 11.13 shows the input torques, the control effort.

[3] Link 1 points up, link 2 points down, and link 3 points up.

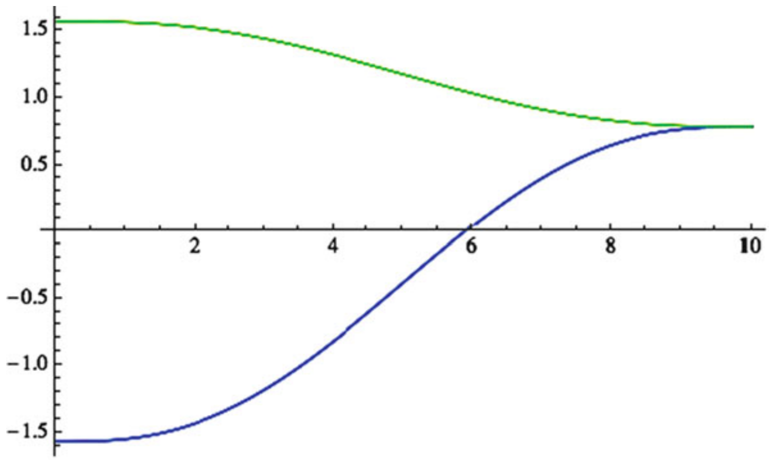


Fig. 11.12 Smooth change of position (see text)

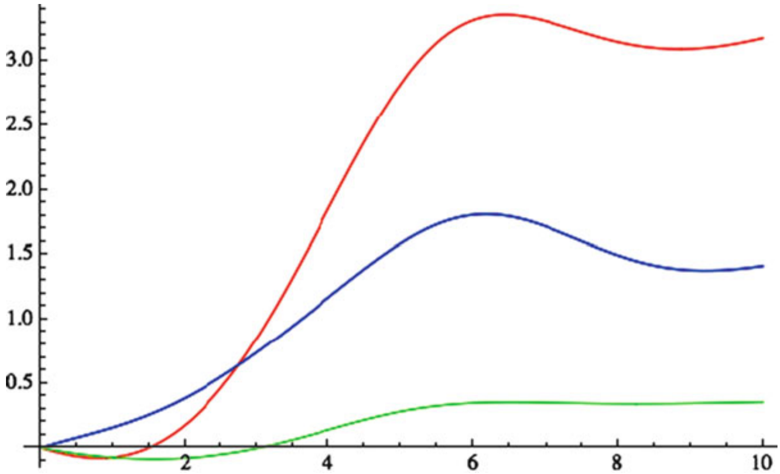


Fig. 11.13 Control effort for the motion shown in Fig. 11.10. τ_{01} is in *red*, τ_{12} in *blue*, and τ_{23} in *green*

All three start from zero, because no control effort is required when the links are vertical. The different final results are to be expected. The first torque supports all three links, the second the outer two links and the third only the outer link.

Note that this method of control ensures a smooth approach to the final position.

11.3 Elementary Robotics

We can extend the development in the previous section to include elementary robots, those with only simple revolute joints. I will consider only torque-driven cases as in Sect. 11.2. I will focus on the three-link robot consisting of a vertical link that rotates about the vertical axis and two links that rotate about horizontal axes. Figure 11.14 shows a simple schematic for which the links are identical cylinders. I denote the masses of the cylinders by m, their radii by r, and their lengths by l. The center of mass of each is at the center of the link, $l/2$ from the end. The moments of inertia with respect to their centers of mass are then

$$A = \frac{1}{12}m\left(3r^2 + l^2\right) = B, \quad C = \frac{1}{2}mr^2 \tag{11.36}$$

I suppress the details of the hinges. The unit vectors shown are fixed in space (and I'll be more precise about that shortly). The end of the robot can cover a volume of space, allowing it to put an end effector, which does the actual robotic task, anywhere in that volume. Our control problem is to move the end effector along some predetermined path using the same technique I used for the kinematic chain. The red base can move only about the vertical. The blue and green links rotate about the vertical with the red link, and in addition they can rotate about an axis fixed in the robot (not in space).

The work space available to the robot is most easily defined if I put the origin at the upper end of link 1 (red-blue junction in Fig. 11.12). Then we have

$$x^2 + y^2 + z^2 \leq (l_2 + l_3)^2 \text{ and } z \geq -l_1 \tag{11.37}$$

Cylindrical coordinates are better for defining the position of the end effector. Write the usual $x = \varpi \cos\varphi$, $y = \varpi \sin\varphi$. The coordinates of ϖ and z are determined

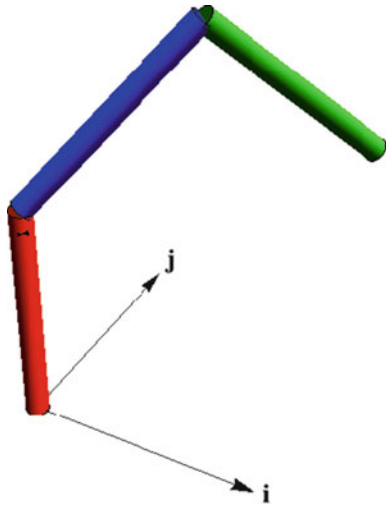


Fig. 11.14 Schematic of a three-link robot

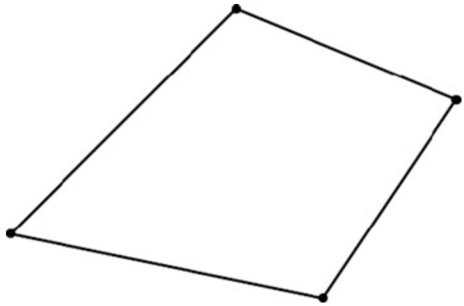


Fig. 11.15 A pair of compatible configurations

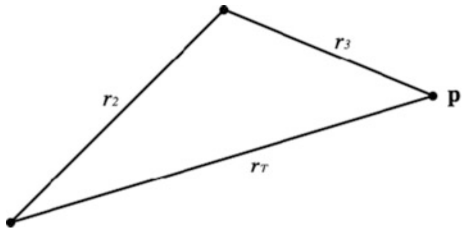


Fig. 11.16 A sketch of the upper configuration shown in Fig. 11.13, showing the definitions of the lengths in the formulas for the angles: r_2 denotes the second arm of the robot and r_3 the third arm

by the angles θ_2 and θ_3. The plane in which they are located is determined by ψ_1. The inverse problem, determining the angles from the position, is not trivial, but it is addressed in any kinematics text. One important point is that generally each pair ϖ and z can be realized by two different pairs of θ_2 and θ_3. The two configurations form the sides of an irregular quadrilateral. We have

$$\varpi = l_2 \cos\theta_2 + l_3 \cos\theta_3, \quad z = l_2 \sin\theta_2 + l_3 \sin\theta_3$$

where I measure the angle counterclockwise from the horizontal. Figure 11.15 shows the two configurations for a single point: {2.800, 0.840}. Here $l_2 = 2$ and $l_3 = 1.5$. I have not calculated the two pairs, so the picture is not quite right. The angles for the upper pair are $\pi/4$ and $15\pi/8$, and for the lower pair they are approximately 6.08 and 0.98 radians. I found them by trial and error, but there are analytic techniques to find them. These are outlined in any kinematics book. The following comes from mine (Gans 1991) as transcribed in Gans (2013). We have

$$\theta_2 = \theta_T \pm \cos^{-1}\left(\frac{r_3^2 - r_T^2 - r_2^2}{2r_2 r_T}\right), \quad \theta_3 = \theta_T \mathrm{m} \cos^{-1}\left(\frac{r_2^2 - r_T^2 - r_3^2}{2r_3 r_T}\right) \tag{11.38}$$

where r_T is shown in Fig. 11.16 and θ_T is the angle this resultant makes with the horizontal. Note that r_T is the same for either the upper or lower configuration. The angle θ_T can be found from $\tan\theta_T = p_z/p_\rho$. It is to be measured in the counterclockwise direction from **p**, so in this case it lies in the third quadrant. The length r_T can be calculated from the law of cosines. We get the two configurations shown in Fig. 11.15 by choosing either the upper pair of signs or the lower pair of signs.

In the present case $p_\rho = 2.80$ and $p_z = 0.84$, so we have $\tan\theta_T = 0.3$, and the third quadrant angle corresponding to this is 3.433 radians. I can find θ_2 in the same way, nothing that it lies in the first quadrant for the configuration shown in Fig. 11.16. The result is $\pi/4$ (which is what I set it up to be in the first place). The angle opposite r_3 is the difference between that and 0.29, the first quadrant angle corresponding to r_T. The law of cosines gives

$$r_3^2 = r_2^2 + r_T^2 - 2r_2 r_T \cos(0.495)$$

which I can solve for r_T, obtaining 2.9207 radians. The two expressions for the angles Eq. (11.43) are $\theta_2 = \pi/4$, 6.080 and $\theta_3 = 15\pi/8$, 0.977, and these give the results shown in Fig. 11.14.

11.3.1 Some Comments on Three-Dimensional Motion

We need to spend some time learning about three-dimensional motion and its representation in order to reduce the paragraph above to mathematics. Up until now we have limited ourselves to planar motion in the $x = 0$ plane. We must now consider motion in three dimensions.

A body in three dimensions, in our case, a robot link, has six degrees of freedom, three for the position of its center of mass and three for its orientation. I will use Euler angles to describe orientation. There are several Euler angle conventions. The most common is the one that I will call the *z*–*x*–*z* set. This convention is described in Gans (2013), Goldstein (1980), and Meirovitch (1970), among many other sources. The Euler angles describe the relations between a set of unit vectors in inertial space **i**, **j**, and **k**, the spatial axes (those shown in Fig. 11.12), and a set of unit vectors attached to the body, **I**, **J**, and **K** (with associated coordinates X, Y, and Z), the body axes. The Euler angles are defined operationally as follows:

Begin with the body and spatial axes aligned.

- Rotate about the body axis **K**, concurrent with the spatial axis **k** during this rotation. This defines the first Euler angle, ϕ.
- Rotate the body about the body axis **I**, which is no longer in its original position because of the first rotation. This defines the second Euler angle, θ.
- Rotate the body about the body axis **K**, which has moved under the second rotation. This defines the third Euler angle ψ.

In our two-dimensional world the body axis **I** never moved, and so the angle θ I used to describe rotations was the second Euler angle. One way to describe a two-dimensional world is to set $\phi = 0 = \psi$. Note that if there is no rotation about the body axis **I**, then the angles ϕ and ψ are indistinguishable. In that case I adopt the convention that I use the angle ψ to describe that rotation. I can summarize the Euler angle rotations by writing the orientation of the body axes in the spatial system as

$$
\begin{aligned}
\mathbf{I} &= (\cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi)\mathbf{i} + (\sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi)\mathbf{j} \\
&\quad + \sin\theta\sin\psi\mathbf{k} \\
\mathbf{J} &= -(\cos\phi\sin\psi + \sin\phi\cos\theta\cos\psi)\mathbf{i} - (\sin\phi\sin\psi - \cos\phi\cos\theta\cos\psi)\mathbf{j} \\
&\quad + \sin\theta\cos\psi\mathbf{k} \\
\mathbf{K} &= \sin\phi\sin\theta\mathbf{i} - \cos\phi\sin\theta\mathbf{j} + \cos\theta\mathbf{k}
\end{aligned} \tag{11.39}
$$

The rotational kinetic energy can be written in terms of the Euler angles as

$$
\begin{aligned}
T_{\text{rot}} &= \frac{1}{2}A\left(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi\right)^2 + \frac{1}{2}B\left(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi\right)^2 \\
&\quad + \frac{1}{2}C\left(\dot\phi\cos\theta + \dot\psi\right)^2
\end{aligned} \tag{11.40}
$$

where A, B, and C denote the principal moments of inertia about the **I**, **J**, and **K** axes, respectively. These axes are attached to the body, so these moments of inertia do not change. They are a property of the body no matter where it goes. In our planar case this reduces to

$$
T_{\text{rot}} = \frac{1}{2}A\dot\theta^2 \tag{11.41}
$$

and we see that A corresponds to the single moment of inertia I we used up until now.

We will need the angular velocity in order to find the rate of work, which we need to find generalized forces. The forces and torques will be fixed in the body system when we are dealing with robots, so we want the angular velocity in body coordinates. This is

$$
\begin{aligned}
\Omega &= \left(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi\right)\mathbf{I} + \left(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi\right)\mathbf{J} \\
&\quad + \left(\dot\phi\cos\theta + \dot\psi\right)\mathbf{K}
\end{aligned} \tag{11.42}
$$

How does all these apply to the simple robot skeleton of Fig. 11.14? Suppose the robot to begin with all the links in the vertical, with all their body axes aligned with the space axes. I will align the **K** body axes with the long axes of the links. Link 1 rotates about the vertical, carrying links 2 and 3 with it. Link 1 is not going to rotate again, so there is no distinction between ϕ and ψ. Our convention says that we are to call this angle ψ. The other two angles remain zero. The common **I** axes, about which links 2 and 3 will rotate, become

$$
\mathbf{I} = \cos\psi_1\mathbf{i} + \sin\psi_1\mathbf{j}
$$

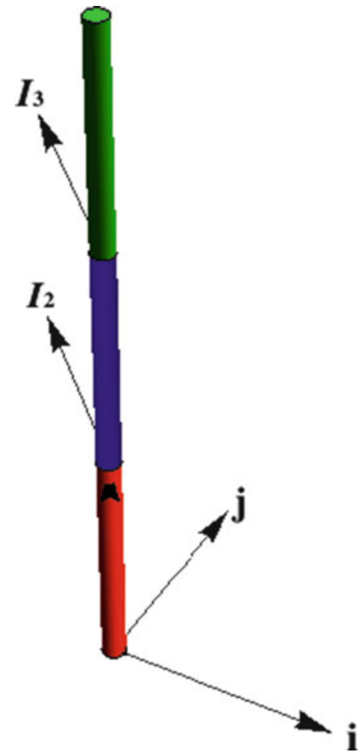


Fig. 11.17 First physical rotation: $\psi_1 = \phi_2 = \phi_3 = 3\pi/4$

The physical appearance of the system does not change until the two distal links rotate about their **I** axes by θ_2 and θ_3, respectively. The sketch shown in Fig. 11.14 has $\psi_1 = 3\pi/4$, $\theta_1 = \pi/4$, and $\theta_3 = 3\pi/4$. Let's take a look at the intermediate steps and compare the physical motions with their expression in terms of the Euler angles. I will use physical rotations, in which the distal links move with the proximal links, as opposed to introducing each rotation separately because I think it is easier to follow the rotations this way. Figure 11.17 shows the system after the rotation of the entire system about the vertical: $\psi_1 = \phi_2 = \phi_3 = 3\pi/4$. I show the **I** axes for the second and third links at their hinge locations.

Figure 11.18 shows the second physical rotation, about the axis $\mathbf{I}_2$: $\theta_2 = \pi/4 = \theta_3$. Link 3 is distal to link 2 and so I rotate it as well. Link 1 does not move under this rotation.

Finally we get to the final position by completing the rotation about $\mathbf{I}_3$: $\theta_3 = 3\pi/4$. This is the final value of θ_3; the additional rotation from Fig. 11.18 to Fig. 11.19 is only $\pi/2$. Figure 11.19 reproduces Fig. 11.14 showing the two interesting **I** vectors, those about which links 2 and 3 rotate.

We are now in a position to set up the Euler-Lagrange equations for this simple robot. The kinetic energy of each link is given by the translational kinetic energy

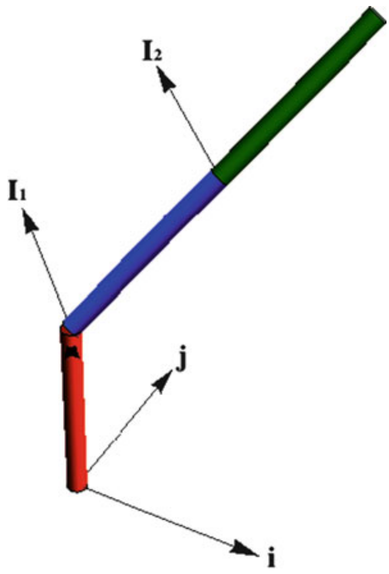


Fig. 11.18 Second physical rotation: $\psi_1 = \phi_2 = \phi_3 = 3\pi/4$, $\theta_2 = \pi/4 = \theta_3$

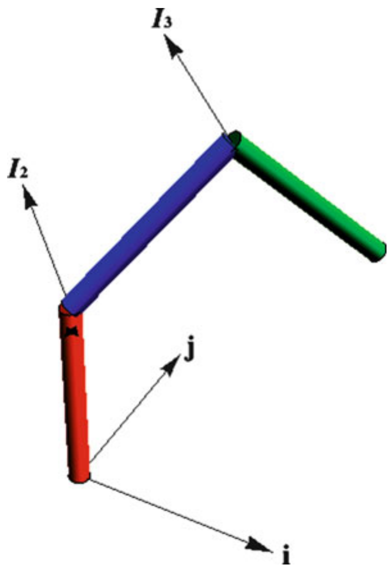


Fig. 11.19 The final position of the robot: $\psi_1 = \phi_2 = \phi_3 = 3\pi/4$, $\theta_2 = \pi/4$, $\theta_3 = 3\pi/4$

$$T_i = \frac{1}{2} m_1 \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \right)$$

plus the rotational energy as given by Eq. (11.40). The rotational energies simplify both because of symmetry and because of the rotational constraints.

$$\phi_1 = 0 = \theta_1, \quad \phi_2 = \psi_1 = \phi_3, \quad \psi_2 = 0 = \psi_3 \tag{11.43}$$

We can substitute Eq. (11.43) into Eq. (11.40) to get the rotational kinetic energies. The kinetic energy of the first link is

$$T_1 = \frac{1}{2}C\dot{\psi}_1^2$$

The rotational kinetic energies of the other two links follow the same form

$$T_{\text{rot}i} = \frac{1}{2}A_i\left(\dot{\theta}_i^2 + \dot{\psi}_1{}^2 \sin^2\theta_i\right) + \frac{1}{2}C_i\dot{\psi}_1^2 \cos^2\theta_i, \quad i = 2, 3$$

where I have used the constraints Eq. (11.43) to replace $\dot{\phi}_2$ and $\dot{\phi}_3$. We'll need to constrain the center of mass coordinates because they are determined by the angles, which I will adopt as the generalized coordinates. This was fairly easy in the planar case. We now need to use the corresponding vector relations.

Denote the locations of the centers of mass by the vector $\mathbf{r}$. The centers of mass are at the geometric centers of the links and are connected vectorially.

$$\mathbf{r}_1 = \frac{1}{2}l_1\mathbf{K}_1, \;\; \mathbf{r}_2 = l_1\mathbf{K}_1 + \frac{1}{2}l_2\mathbf{K}_2, \;\; \mathbf{r}_3 = l_1\mathbf{K}_1 + l_2\mathbf{K}_2 + \frac{1}{2}l_3\mathbf{K}_3 \tag{11.44}$$

We now have all the pieces needed to write the Lagrangian. It is complicated, but if I let the three links be identical, it just fits on the page

$$\begin{aligned} L = &\begin{pmatrix} \dfrac{r^2}{16}(10 + \cos(2\theta_2) + \cos(2\theta_3)) \\ + \dfrac{l^2}{12}(5 - 4\cos(2\theta_2) + 6\sin(\theta_2)\sin(\theta_3) - \cos(2\theta_3)) \end{pmatrix}\dot{\psi}_1^2 \\ &+ \frac{m}{24}\left(16l^2 + 3r^2\right)\dot{\theta}_2^2 + \frac{m}{24}\left(4l^2 + 3r^2\right)\dot{\theta}_3^2 + \frac{1}{2}ml^2\cos(\theta_2 - \theta_3)\dot{\theta}_2\dot{\theta}_3 \\ &- mgl(5 + 3\cos\theta_2 + \cos\theta_3) \end{aligned} \tag{11.45}$$

We can find the three second-order governing equations from Eq. (11.45) using the Euler-Lagrange process. They are too long to display here. The generalized forces come from the rate of work, which I can write as

$$\dot{W} = \boldsymbol{\omega}_1 \cdot (\boldsymbol{\tau}_{01} - \boldsymbol{\tau}_{12}) + \boldsymbol{\omega}_2 \cdot (\boldsymbol{\tau}_{12} - \boldsymbol{\tau}_{23}) + \boldsymbol{\omega}_3 \cdot \boldsymbol{\tau}_{23} \tag{11.46}$$

The three torques are aligned with the $\mathbf{k}$, $\mathbf{I}_1$, and $\mathbf{I}_2$ axes, respectively. We can write out the scalar rate of work as

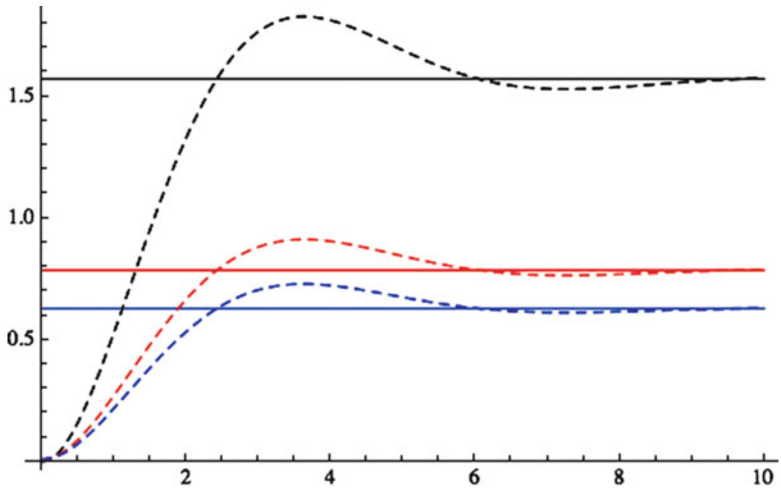


Fig. 11.20 Motion of the robot to a constant state (see text)

$$\dot{W} = \tau_{01}\dot{\psi}_1 + (\tau_{12} - \tau_{23})\dot{\theta}_2 + \tau_{23}\dot{\theta}_3$$

where the scalar torques denote the magnitudes of the vector torques given in Eq. (11.46).

We construct the error equations by writing

$$\psi_1 = \psi_{1r} + e_1, \quad \theta_2 = \theta_{2r} + e_2, \quad \theta_3 = \theta_{3r} + e_3$$

This leads to three equations for the three errors, and if we require the errors to satisfy the equation introduced earlier

$$\ddot{e}_i + 2\zeta_i\dot{e}_i + \omega_{ni}^2\theta = 0$$

We can find torques to make this the case. Solve the desired error equation for the second derivatives of the errors. Substitute these into the error equations, and solve those for the torques. These torques will be functions of the reference state and all its derivatives and the errors and their first derivatives. We removed the second derivatives of the error. We can then substitute these torques into the original Euler-Lagrange equations. This follows the algorithm outlined in the previous section.

I can exercise this algorithm by specifying the parameters of the robot shown in Fig. 11.14 and choosing a reference state. I will choose $m = 1 = l$, $r = 0.05$, and I will use $g = 9.81$, all in SI units. I will start the robot from rest with all the angles equal to zero[4] for all the examples I show below. Figure 11.20 shows the angle

[4] The θ angles start from 0.01 to avoid a numerical singularity.

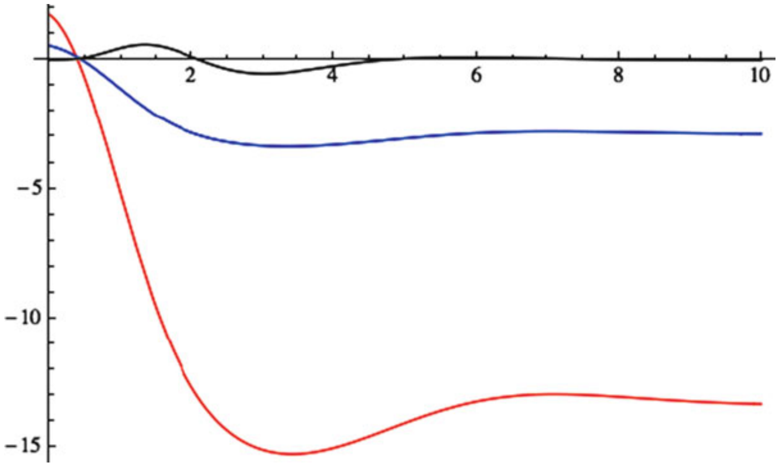


Fig. 11.21 The torques required for the motion shown in Fig. 11.18

response for a reference state of $\psi_1 = \pi/2$, $\theta_2 = \pi/4$, and $\theta_3 = \pi/5$. I use $\zeta = 0.5$ and $\omega_n = 1$. The solid lines in Fig. 11.20 show the reference state and the dashed lines the actual state. Figure 11.21 shows the required torques. Black denotes ψ_1, red θ_2, and θ_3 in both figures. Note that the torque on the first link goes to zero when the reference state is attained—gravity does not act on any possible motion of that link.

Let me look at a time-dependent tracking problem. Let

$$\psi_{1r} = \frac{\pi}{2}, \quad \theta_{2r} = \frac{\pi}{4} + \frac{\pi}{10}\sin t, \quad \theta_{3r} = \frac{\pi}{5} + \frac{\pi}{30}\sin t \tag{11.47}$$

Figure 11.22 shows the response of the angles for $\zeta = 0.25$, and Fig. 11.23 shows the response for a "critically damped" control, $\zeta = 1$, both for unit natural frequency. The conventions defining the curves remain the same.

Note that the critically damped choice prevents overshoot for the constant angle (in this case), but not for the time-dependent reference angles.

Increasing the natural frequency makes the system converge more rapidly and also increases the required torques. Figure 11.24 shows the response for a natural frequency of 2.

Finally, Figs. 11.25 and 11.26 show the torques required for the responses shown in Figs. 11.23 and 11.24, respectively.

11.3.2 Path Following

I have shown how to control the angles of a three-link robot, but I have not explained how to choose which angles to follow. We don't really want to follow angles; we want to follow some path in space. The angles are secondary. We want

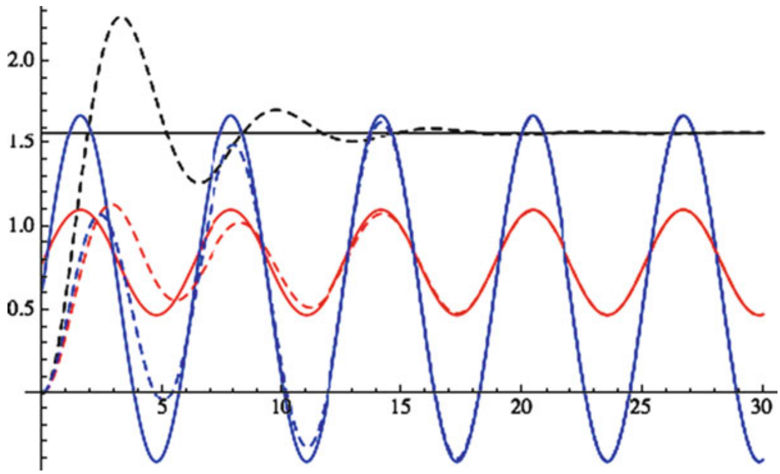


Fig. 11.22 Tracking control for $\zeta = 0.25$ (see text)

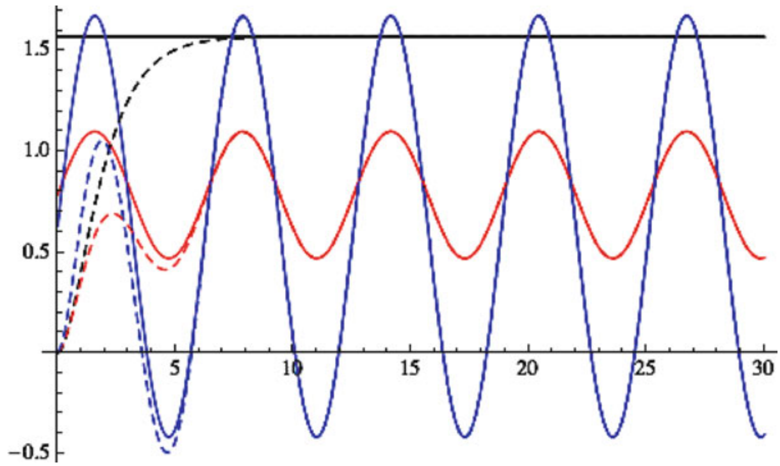


Fig. 11.23 Tracking control for $\zeta = 1.0$ (see text)

the end of the robot arm to follow some path in space that we specify. It is not immediately obvious how to translate the path to a set of angles. The position of the end effector is at the end of the third arm of our primitive robot. For the sake of simplicity I will continue the convention that each arm is of unit length. The vector position of the end of the arm is then

$$\mathbf{p} = \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3 \tag{11.48}$$

We would like that to follow some reference path in space $\mathbf{p}_r$. Equating Eq. (11.48) to the reference path gives three transcendental equations, which are

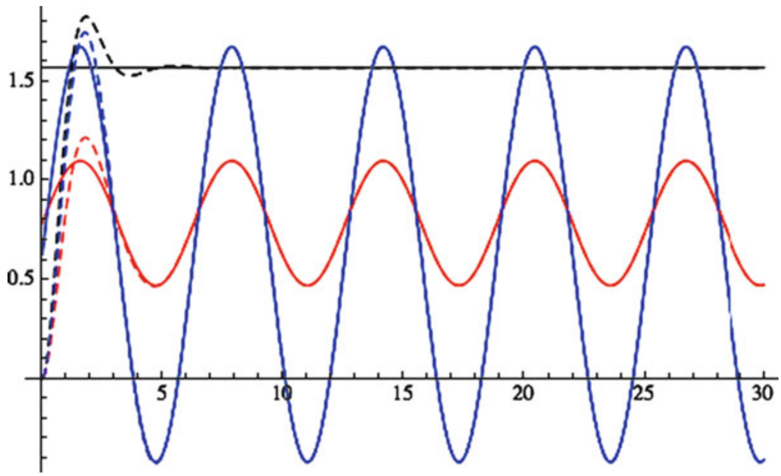


Fig. 11.24 Critically damped case for $\omega_n = 2$ (see text)

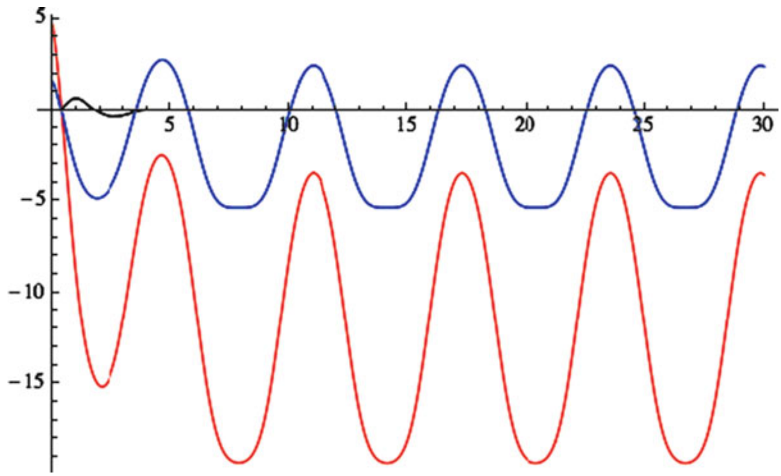


Fig. 11.25 Torques for the response shown in Fig. 11.19

not amenable to analytic solution. Fortunately there is an alternative. We can differentiate the path and the reference path and equate the velocities. This will give us a set of quasilinear first-order ordinary differential equations that we can solve to give the path angles in numerical form. These can then be used in the nonlinear tracking controls I have already presented. The path vector **p** can be written as

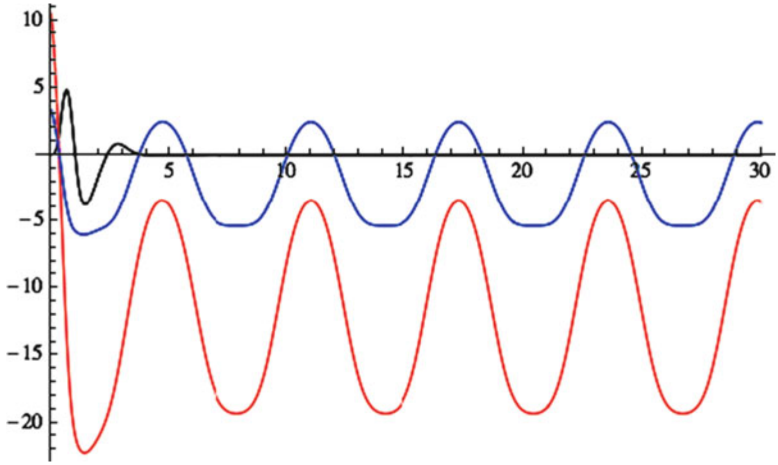
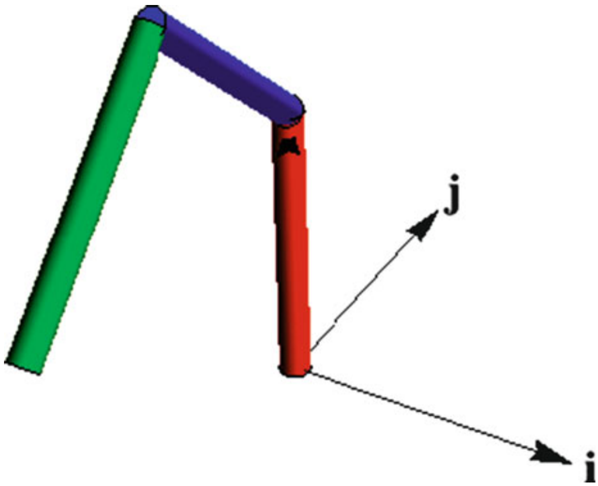


Fig. 11.26 Torques required for the response shown in Fig. 11.20

Fig. 11.27 Initial posture of the robot



$$\mathbf{p} = (\sin\theta_2 \sin\psi_1 + \sin\theta_2 \sin\psi_1)\mathbf{i} - (\sin\theta_2 \cos\psi_1 + \sin\theta_2 \cos\psi_1)\mathbf{j} + (1 + \cos\theta_2 + \cos\theta_3)\mathbf{k}$$

and its derivative is messy but straightforward. If we specify the path by its starting location and velocity, then we can integrate these three equations from the starting angles to the end of the motion to find the path in terms of the angles.

For example, suppose we want the end effector to start at unit height at $x = 0$ and $y = -\sqrt{2}$ and move in the $\mathbf{j}$ direction to the end of its range. The initial angles will be $\theta_2 = \pi/4$ and $\theta_3 = 3\pi/4$. Figure 11.27 shows the initial posture of the robot.

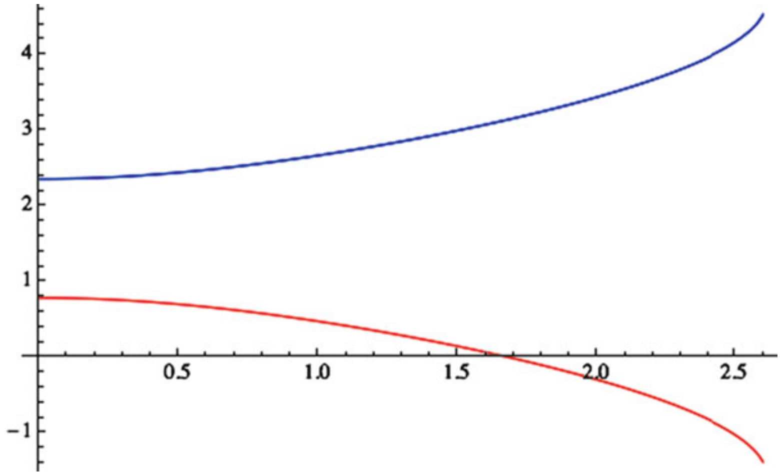


Fig. 11.28 The reference angles for a straight line path in the **j** direction. θ_2 is in *red* and θ_3 in *blue*

We have

$$\dot{\mathbf{p}} \cdot \mathbf{i} = 0 = \dot{\mathbf{p}} \cdot \mathbf{k}, \quad \dot{\mathbf{p}} \cdot \mathbf{j} = Vt \tag{11.49}$$

We can solve these three equations for the derivatives of the angles

$$\dot{\psi}_1 = \frac{Vt \sin \psi_1}{\sin \theta_2 + \sin \theta_3}, \quad \dot{\theta}_2 = \frac{Vt \cos \psi_1 \sin \theta_3}{\sin(\theta_2 - \theta_3)}, \quad \text{and} \quad \dot{\theta}_3 = -\frac{Vt \cos \psi_1 \sin \theta_2}{\sin(\theta_2 - \theta_3)} \tag{11.50}$$

The differential equations in Eq. (11.50) need to be added to the simulation equations and integrated simultaneously to give the reference path.

I specified an initial condition where $\psi_1 = 0$, so its derivative is zero and y_1 never changes. Figure 11.28 shows the other two angles, and Fig. 11.29 shows the reference path of the end effector.

Suppose we now try to use this path to control the robot. Figure 11.30 shows the reference angles (solid curves) and the actual angles (dashed curves) if the robot starts at $\theta_2 = \pi/5$ and $\theta_3 = 3\pi/5$, quite far from the reference curve's starting position. The tracking works quite well. Figure 11.31 shows the reference path as a thin black line and the actual path as a thick, dashed black line. Finally, Fig. 11.32 shows the initial and final configuration of the robot simultaneously.

The alert reader will note that the transformation hinted at in Fig. 11.32 is on the face of it impossible because link 3 would have to pass through link 2. If the links

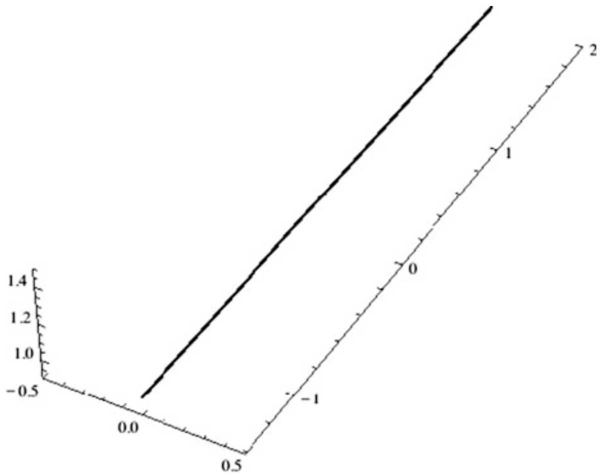


Fig. 11.29 The reference path, which is directed away from the observer

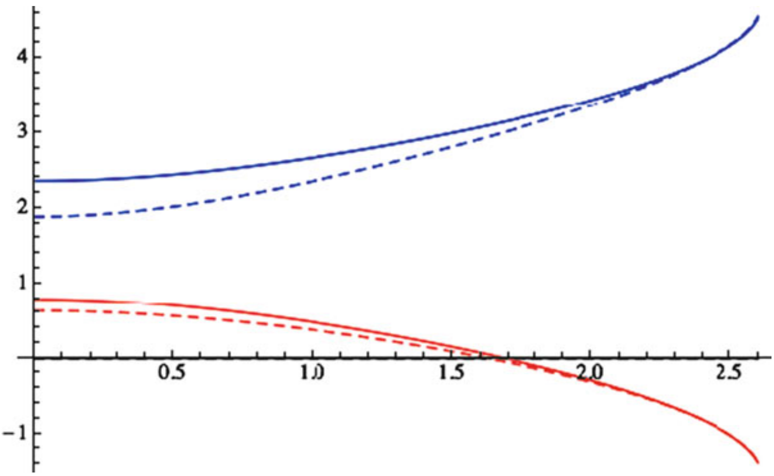


Fig. 11.30 The reference angles (*solid*) and the actual angles (*dashed*) under the nonlinear control

were indeed actually as the drawing shows, the motion specified would be impossible. In an actual robot the links are offset so that link 3 can pass by link 2, not go through it. If the end point is to stay at a height of unity, then θ_3 must increase as θ_2 decreases. Both motions move the end point in the **j** direction.

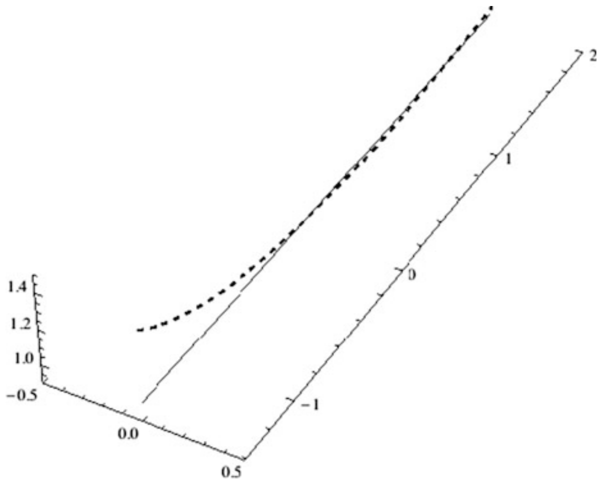


Fig. 11.31 The reference path (*thin*, *solid line*) and the actual path (*thick*, *dashed*)

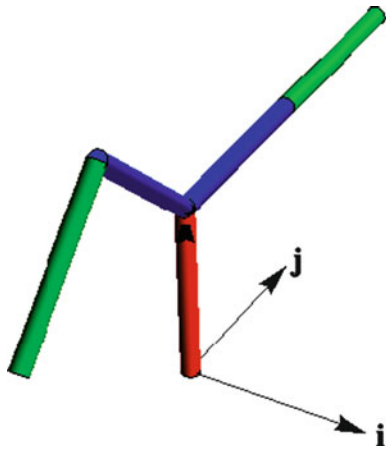


Fig. 11.32 The initial and final robot configurations

Exercises

1. Draw a block diagram for Ex. 11.2. Identify the feed forward and feedback lines.
2. Verify the inverse transformation in Ex. 11.2.
3. Repeat Ex. 11.4 for a reference state

$$\theta_{r1} = \frac{\pi}{2} + \frac{\pi}{10}\sin t, \quad \theta_{r2} = \frac{\pi}{2} + \frac{\pi}{5}\sin t, \quad \theta_{r3} = \frac{\pi}{2} + \frac{\pi}{4}\sin t$$

4. Redo the three-link kinematic chain using three high-inductance motors—a nine-dimensional state space.
5. Discuss the control of a three-link kinematic chain falling freely through the air. This can be a very simple model of a diver or a gymnast.

6. Design a control to move a six-link kinematic chain from lying flat on the ground to a closed hexagon.
7. Draw a block diagram of the three-link robot.

8–11. Set up a simulation of the three-link robot and track the following paths:

8. A straight line from {0, 0, 0} to {1, 1, 1}
9. A circle from {0, 1, 1} to {1, 0, 1}
10. A straight line from {0, 0, 1} to {1, 1, 1}
11. Consider a five-link chain (identical links) lying on a frictionless horizontal surface. Suppose one end of one of the links to be welded to the surface. Design a set of joint torques that will move the links to form a pentagram.
12. Consider a three-link chain (identical links) attached to the ground and confined to a vertical plane. Design a set of join torques to move it from the ground to a zigzag configuration with the distal end of the third link at a height of twice the length of an individual link.
13. Consider a three-link chain made of uniform cylindrical rods. Let the two end links be 1 unit long and the center link 1/2 unit long. Let them all have the same mass, which we can take as unity. Design a control to take the system from an extended position to a position in which the distal ends of the end links touch, and then straightens back out, the entire cycle to be completed in 10 s. (You can think of this as a very simple model of a jackknife dive.)
14. Repeat exercise 13 supposing the whole system to start with all three links moving up at 1 m/s and rotating counterclockwise at 1 rad/s. Let the initial angle with respect to the vertical of all three angles be $\pi/20$. Let gravity act on the system. What is the position and orientation of the system after 20 s?
15. Consider a three-link chain. Let the middle link have a mass of 2 and a length of ½. Let the two outer links have mass ½ and length 2. Design a control to flap the outer links through a 60° arc centered on $\pi/2$, while the center link starts at an angle of $\pi/2$. What happens to the middle link if (a) the flapping is in phase and (b) the flapping is π out of phase?

Exercises 16–20 make use of the robot introduced in Sect. 11.3.

16. Design a control to make the end effector move in a horizontal circle in the x-y plane of radius 0.50 with its center (at $x = 0 = y$) 2.9365 units above the ground in 10 s.
17. Design a control to make the end effector move in a horizontal circle in the x-y plane of radius 0.50 with its center (at $x = 0 = y$) 1.5 units above the ground in 10 s.
18. Design a control to make the end effector move in a vertical circle in the y-z plane of radius 0.20 with its center 1/2 unit above the ground in 10 s.
19. Design a control to make the end effector move in a straight line from the ground to a point 2 units above the ground over the point $x = 0.2 = y$ in 10 s.
20. Design a control to make the end effector move from $(x, y, z) = (0, 0, 0)$ to (1, 1, 1) in 10 s. Choose the origin for the z coordinate to be on the ground for this exercise.

The following exercises are based on the chapter but require additional input. Feel free to use the Internet and the library to expand your understanding of the questions and their answers.

21. Do some research on the sidewinder rattlesnake and build a six-link model of the snake and its motion.
22. Repeat exercise 22 for an eel.
23. Design a kinematic chain and its control capable of moving to represent any of the seven segment digits used in LED displays.
24. Consider exercise 14 as a model of platform diving. Build a realistic (but simple!) model of a diver and work out the control and initial conditions for a simple jackknife dive from a 3 m platform.
25. Expand on exercise 24 by considering other Olympic dives. (Feel free to restrict yourself to planar dives. If you want to be adventurous, think about how you would formulate dives with motion in three dimensions, which I think is a research-level exercise.)

References

Gans RF (1991) Analytical kinematics. Butterworth-Heinemann, Stoneham, MA

Gans RF (2013) Engineering dynamics: from the Lagrangian to simulation. Springer, New York, NY

Goldstein H (1980) Classical mechanics 2nd edn. Addison Wesley Reading MA

Meirovitch LM (1970) Methods of analytical dynamics McGraw-Hill Upper Saddle River NJ New York, NY

Slotine J-JE, Li W (1991) Applied nonlinear control. Prentice-Hall

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