Matthew A. Davies · Tony L. Schmitz

System Dynamics for Mechanical Engineers

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To our Lord and Savior, Jesus Christ

Preface

In this textbook, we describe the fundamentals of system dynamics using Laplace transform techniques and frequency domain approaches as the primary analytical tools. It is aimed at the mechanical engineering student and, therefore, begins with a thorough discussion of the modeling of mechanical systems to provide the backdrop for the entire text. Once the fundamentals of mechanical system behavior are developed, the topic is broadened to include electrical, electromechanical, and thermal systems. Wherever possible, analogies between the less familiar systems and their mechanical counterparts are drawn upon to help clarify the subject matter. The topics in the book are concluded with a discussion of block diagrams, feedback control systems, and frequency response of dynamic systems including an introduction to vibrations. Example computational techniques using MATLAB® are incorporated throughout the text. The book is based upon undergraduate courses in system dynamics and mechanical vibrations that the authors currently teach. It is designed to be used in either a traditional 15-week semester or two quarters spanning 3–4 months. It is appropriate for undergraduate engineering students who have completed the basic courses in mathematics (through differential equations) and physics and the introductory mechanical engineering courses including statics and dynamics.

We organized the book into 11 chapters. The chapter topics are summarized here.

- Chapter 1—This chapter defines the concept of a dynamic system as it is commonly used in engineering. It gives examples of such systems and, in a broad sense, describes the importance of system dynamics in engineering. To prepare the reader for Chap. 2, it also links the idea of a system model to the mathematical concept of a differential equation.
- Chapter 2—This chapter describes the Laplace transform, the primary analysis and solution technique used in this book, and supporting topics.
- Chapter 3—This chapter introduces the fundamental lumped parameter elements used to model mechanical systems. These include translational, rotational, and transmission elements.
- Chapter 4—This chapter introduces modeling of a mechanical system with translation mechanical elements using the undamped and damped simple harmonic oscillator. The models are solved for common inputs. The concepts

of transfer function, characteristic equation, natural frequency, and damping ratio are introduced.
- Chapter 5—This chapter extends the concepts in Chap. 4 to include models with rotational degrees of freedom.
- Chapter 6—This chapter analyzes dynamic systems with transmission elements and includes the associated geometric and power constraints.
- Chapter 7—This chapter examines electrical circuits composed of resistors, capacitors, and inductors. The mathematical analogies between electrical and mechanical elements are discussed.
- Chapter 8—This chapter discusses electromechanical systems including electric motors and other electromagnetic actuators including voice coils. This discussion further emphasizes the mathematical analogies between mechanical and electrical elements.
- Chapter 9—This chapter describes bulk heat transfer showing the analogies between mechanical, electrical, and thermal elements. It also provides an introduction to proportional-integral-derivative feedback control in the context of a temperature control system.
- Chapter 10—This chapter condenses the book concepts into the formal language of block diagrams. Feedback and control systems are discussed in more detail.
- Chapter 11—This chapter describes the behavior of dynamic systems subjected to sinusoidal and other periodic inputs. It is a precursor to a mechanical vibrations course.

The text is written with the mechanical engineer in mind. This includes the organization, selection of examples, and range of topics. It will provide the engineering student not only with sound fundamentals, but also with the confidence to address new, multidisciplinary systems that are found in practice. It will equip the engineer with techniques to analyze the dynamics of modern systems.

We conclude by acknowledging the many contributors to this text. These naturally include our instructors, colleagues, collaborators, and students.

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Contents

1 Introduction

1.1 What Is a System?

The word *system* has a broad modern definition. The Merriam-Webster dictionary defines a system as "a regularly interacting or interdependent group of items forming a unified whole." For the engineer, a system consists of a combination of elements which, acting together, perform a specific task. An *input* to a system causes the system to exhibit a response which is observed as changes in the system *output*. All of the systems we discuss in this book are *causal*: the input, or cause, results in the output, or effect. Additionally, causality requires that the output depends only on current and previous input values. Future inputs do not affect the current output.

Systems are comprised of collections of elements that affect each other. Each element in a system has its own input/output relationship. For example, when an input force is applied to a linear spring, an output deflection that is proportional to the force is obtained. Similarly, if an input voltage is applied across a resistor, an output current flows through the resistor. System elements such as springs and resistors defined in this way are *static* because the output depends on the input only at the current time. A *dynamic* element produces an output that depends not only on current inputs, but also on the previous inputs (i.e., the input history). For example, consider a mass with a force input and a position output. The position of the mass depends not only on the current force value, but also on previous force values. In our analyses, this information is incorporated into the initial position and velocity of the mass. The input/output relationship for an entire system is developed by analyzing the interactions between all of the system elements. The output of a *static system* depends only on the inputs at the current time. The output of a *dynamic system* depends on the inputs and their history.

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1.2 System Boundaries

An example of a complex system is shown in Fig. 1.1a. This is a multi-axis ultra-precision machine tool used for manufacturing optics (lenses) and other components with an accuracy of better than 1 μm (one-millionth of a meter[1]). The machine adjusts the position of rotating tool using five angular and translational degrees of freedom as shown in Fig. 1.1b, and the rotating tool removes material (Fig. 1.1c) to produce an optical component with the commanded shape. The tool is positioned relative to the work material using five machine axes labeled X, Y, Z, B, and C. The X, Y, and Z axes produce linear motions, while the B and C axes produce rotary motions. With these five degrees of freedom, the machine can position the tool at an arbitrary location and an arbitrary angle within the limitation of its work volume.

The machine itself (Fig. 1.1a) is a system shown schematically in Fig. 1.2. At this high level, the input to the machine is a computer program that is uploaded into the machine through its computer-based user interface. This "part program" contains instructions that define the position of the cutting tool at each time during

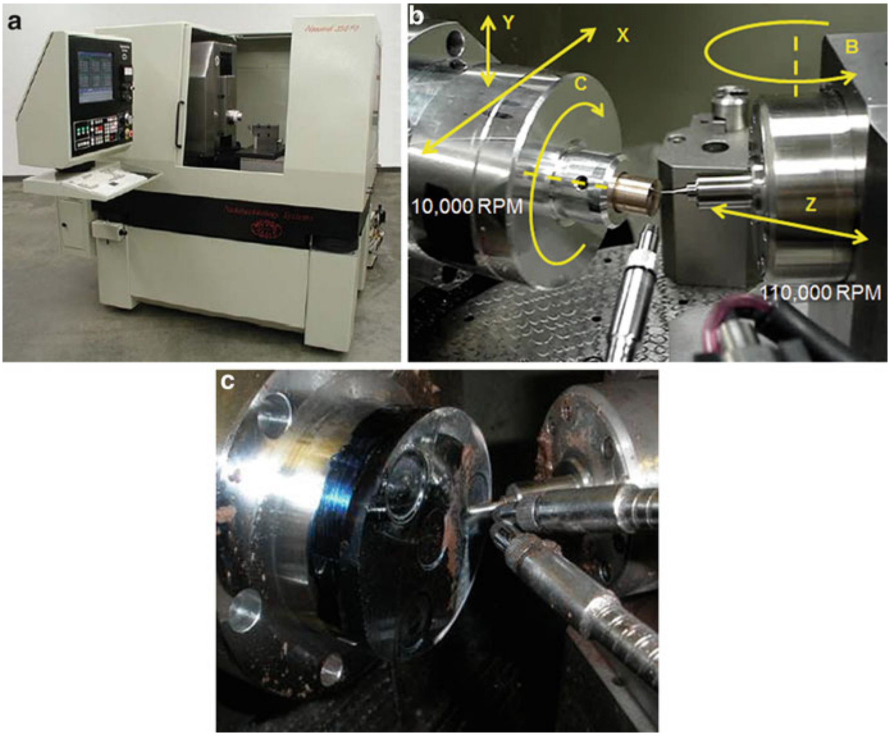


Fig. 1.1 Ultra-precision machine tool used for manufacturing optical components

[1] To provide a sense of scale, the diameter of a human hair is approximately 100 μm.

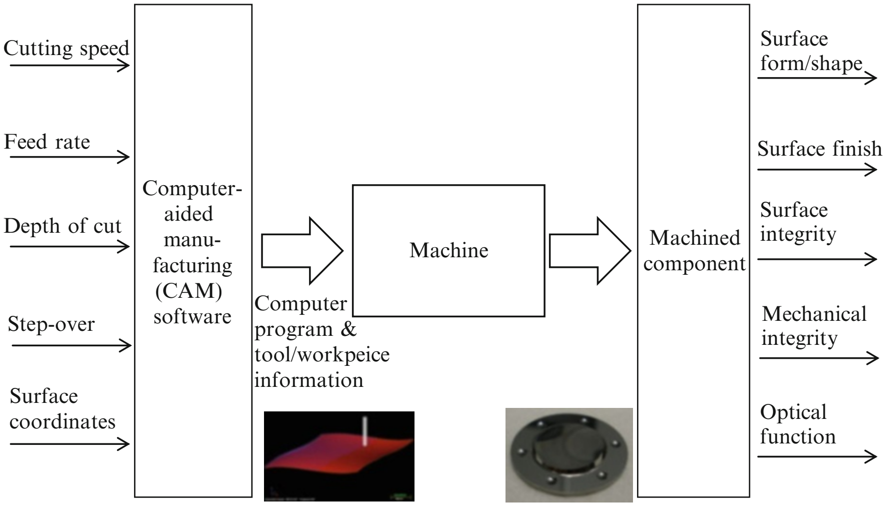


Fig. 1.2 Machine tool system schematic

the operation, the rate of rotation for the cutting tool, the tool velocity or feed rate, etc.; the part program is usually developed with the aid of a computer-aided manufacturing (CAM) software. Other inputs are the tool and work material, as well as information about the size and shape of the tool and the workpiece blank from which the final component is to be produced. The output of the machine is a component with (ideally) the commanded characteristics, such as form (shape), surface finish, and surface quality, which ultimately performs the desired optical function. The amount of information that is included in the part program for the selected machine is immense and includes prior knowledge of the tool-work material interaction, material specifications, tool specifications, etc. Also, the correct set-up of the tool and workpiece requires a great deal of knowledge in order to ensure a successful output. *Decisions about the system boundary therefore affect the inputs and outputs and, correspondingly, the system complexity.* An overly complex system might be difficult, or impossible, to model, while an overly simple model may not accurately represent the system behavior. A judicious choice of the system boundary is critical.

The complex machine shown in Fig. 1.1 consists of many individual subsystems. It is usually advantageous to model each subsystem separately before combining them to develop the model for the entire system. For example, the X-axis is a linear positioning system that uses a linear motor to drive a table (mass) along a set of hydrostatically supported guideways. The table position is measured with a glass-scale and this value is compared to the desired position in the machine's controller (i.e., the position feedback is used in a close-loop control system). Typically, the measured axis velocity, the position error, and the history of the position error are also used (this is proportional integral derivative, or PID, control). Errors in the position are corrected to the desired level, and any errors exceeding a preset limit cause a problem to be reported, which triggers the machine to stop.

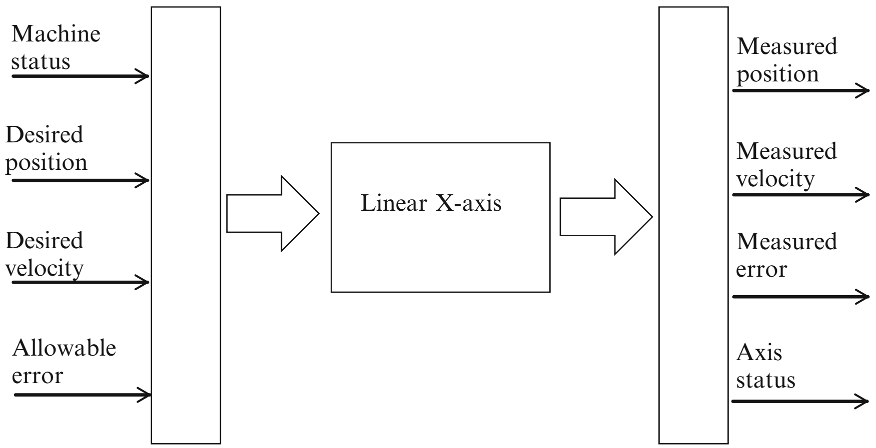


Fig. 1.3 Single linear positioning axis system

A schematic of this single axis system is shown in Fig. 1.3. The machine status is input to the axis. If the machine status is incorrect, which could be the result of a number of factors including errors in the other axes, low pressure in the hydrostatic bearings, or others, the axis is commanded to stop. If the machine status is correct, then the desired position, velocity, and allowable error tolerances are input to the axis. These inputs are compared to measured axis outputs and used to generate an input voltage for the linear motor. This input voltage causes a motor current, which produces a force on the axis table (mass). The imbalance between the motor force and any resistance forces or weight (depending on the axis orientation) causes the axis to accelerate. This acceleration leads to a changing velocity and position, which are measured by the machine scales and, subsequently, a new input voltage is generated. The axis status is reported back to the machine and the new machine status (which could include an error in the axis) is included in the new machine input. From this example, we see how a subsystem can be defined and how outputs from the system may also be used as inputs (i.e., feedback control).

We can explore these concepts in more detail using a block diagram. Typically, axes are positioned using a DC servomotor. The term "servo" refers to the control, or feedback, inherent in the motor and implies a continuous measurement of velocity and its comparison to the desired velocity. The servomotors on the machine shown in Fig. 1.1 consist of brushless DC linear motors that produce a force through the electromagnetic interaction between coils, or wire windings, and permanent magnets. The motor produces a force that is proportional to the current flowing in the motor's windings and accelerates the machine table. A *block diagram* represents system components schematically; the simplified block diagram for a single linear axis is shown in Fig. 1.4. The input to the system is the commanded, or desired, position and is shown on the left side of the diagram.

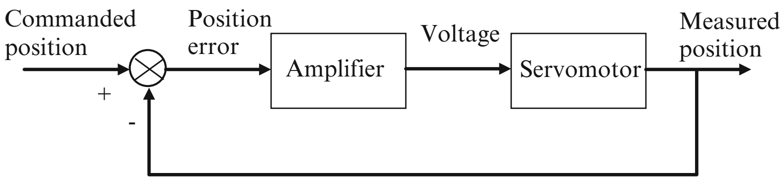


Fig. 1.4 Axis positioning system block diagram

The output is the measured, or actual, position and is shown on the right side of the diagram. The measured position is subtracted from the commanded position (see the summation circle) to determine the position error. This error is amplified and used to produce an input voltage to the motor.

1.3 Modeling and Analysis Tools

The modeling in this book is targeted at analytical closed form solutions for a system's response to input(s) that enable the designer to isolate and understand the effect of the most important system parameters while avoiding unnecessary complexity. Developing a meaningful model of a system as complicated as the machine shown in Fig. 1.1 requires a systematic approach and experience. A major modeling goal is to reduce or eliminate the need to build expensive prototypes and computer models.

The "art" of modeling is to include just enough detail to be meaningful, but not so much detail that the model cannot be solved analytically. While modern computer analysis tools such as finite element software enable detailed numerical models to be developed, these models often require significant development time/cost and are computationally expensive; they can be so complicated that they are essentially computer prototypes. The analytical approaches in this book typically come first in the design process and provide a quick assessment of feasibility and rapid estimate of the ranges for critical design parameters. Once this has been done, details can be checked with numerical models and then relatively advanced prototypes can be built and tested. The three design tools: (1) analytical modeling; (2) numerical modeling; and (3) physical prototyping are complementary and enable improved designs when used together.

To develop simplified models amenable to analytical solutions, *lumped parameter* and *linearized* approximations are adopted. To understand what is meant by this, consider an ideal linear spring. A linear spring has a stiffness, k. When a force, F, is applied across the spring, it deflects by an amount, $\Delta x = \frac{F}{k}$. This is a reasonably good model for small deflections of elastic elements. The deflection increases linearly with force as shown by the solid line in Fig. 1.5. However, the slope of

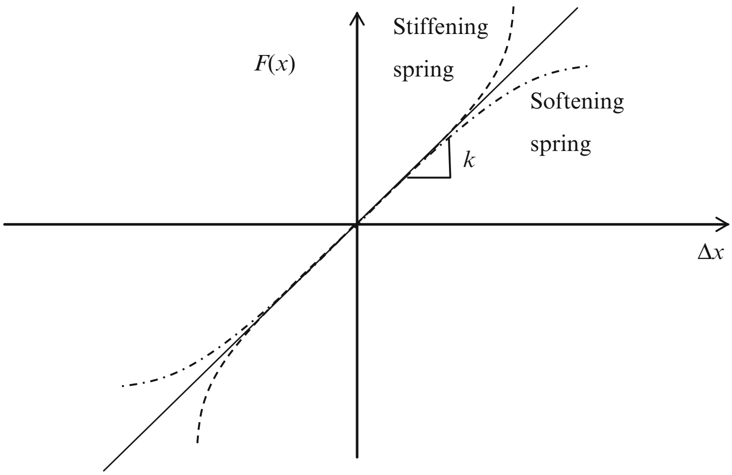


Fig. 1.5 Ideal and linearized spring behavior

the actual force deflection curve for an elastic element may increase (stiffening, dashed line) or decrease (softening, dot-dash line) with increased deflection. The linear approximation is appropriate if the deflection, Δx, is small. In this case we adopt a *linearized* model for the spring element, $F = k\Delta x$.

An elastic element that we model as a linear spring also has some mass. However, our model for a perfect linear spring ignores this mass. In many systems where a spring is used, such as an automotive suspension for example, the mass of the spring is small compared to the masses of the other system (chassis and body) and, therefore, it is justifiable to ignore the mass. Typically, this is a good first approximation if the mass of the spring is at least an order of magnitude less than the mass of the next least massive element in the system. Even in cases where the component mass is not negligible, it can be included (lumped) with the masses of other elements, while introducing relatively little error in the final model. When we develop models for dynamic systems, the input/output relationships for each element typically incorporate one lumped parameter behavior such as stiffness, mass, or energy loss, but not combined effects. The lumped parameter, linearized model for system elements may seem oversimplified, but it is surprising how many real systems can be adequately and, often, very accurately modeled in this manner.

A linear lumped parameter system model yields linear ordinary differential equations that describe the system's dynamic behavior. These differential equations can be solved by various techniques. The technique we adopt here is the *Laplace transform*. Because systems with apparently disparate physical elements give rise to models of the same form, analogous behavior occurs. Subsequently, an understanding of one system type (e.g., mechanical) leads to an understanding of other system types (such as electrical and thermal) as well.

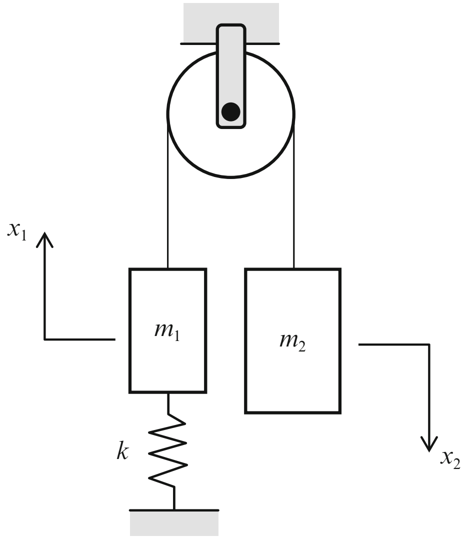


Fig. 1.6 Spring–mass–pulley system

1.4 Continuous Time Motions Versus Dynamic "Snapshots"

A first course in dynamics for mechanical engineers typically covers the motion of particles and rigid bodies using three related techniques: (1) direct application of Newton's second law; (2) the integral of Newton's second law over the time variable, which shows that the momentum of a system in two different configurations (states) is equal to the total impulse applied to the system; and (3) the integral of Newton's second law in the spatial variable, which shows that the kinetic energy of a system in two different configurations (states) is equal to the total work done on the system. Often the format of the questions in a first course is aimed at determining the dynamic behavior of the system at a certain instant in time or between two instants of time during which the system changes. In a system dynamics course, which we detail in this text, the intent is to predict the changes in a dynamic system as functions of time: we obtain continuous time solutions for the system dynamics and therefore determine how it changes between allowable states.

As an example, consider the spring–mass–pulley system shown in Fig. 1.6. Two masses, m_1 and m_2, are suspended over a pulley by an inextensible cable and one mass is attached to a linear spring with stiffness, k. The friction in the system and pulley inertia are negligible. Suppose m_1 is 5 kg and m_2 is 30 kg and the spring stiffness is 300 N/m. A first course in dynamics typically poses a problem of the form: If the spring is unstretched and the system is released from rest, determine the velocity, v, of the system when m_2 has fallen by 0.5 m. The solution to this problem proceeds as follows. First, we declare coordinates which define the position of each mass, x_1 and x_2, as shown in Fig. 1.6. Next, we recognize that in the absence of frictional forces, the total kinetic and potential energy of the system is conserved between two states A and B. In state A, the system is at rest at a position such that the spring is not stretched. In state B, m_1 has an upward velocity of v_k, m_2 has a downward velocity of v_k, m_2 has dropped by a distance, h, and m_1 has risen by the same distance, h. If we define KE to be the system kinetic energy, PE_g to be the system gravitational potential energy, and PE_e to be the system elastic potential energy, then conservation of energy between states A (left) and B (right) requires that:

$$KE_A + PE_{g_A} + PE_{e_A} = KE_B + PE_{g_B} + PE_{e_B}. \tag{1.1}$$

If we declare the height reference for the potential energy of each mass to be the starting position, then the gravitational potential energy in state A is zero. The kinetic energy is also zero because the system is at rest, and the elastic potential energy is zero because the spring is unstretched. In state B, the masses have velocity, v, they have changed height by an amount, h, up or down, and the spring is stretched by an amount, h. This gives the energy balance in Eq. (1.2) (A on the left and B on the right).

$$0 = \frac{1}{2}m_1 v^2 + \frac{1}{2}m_2 v^2 + m_1 gh + m_2 g(-h) + \frac{1}{2}kh^2 \tag{1.2}$$

This can be solved for the unknown velocity.

$$v = \sqrt{\frac{2(m_2 - m_1)gh - kh^2}{m_1 + m_2}} \tag{1.3}$$

Substituting the numerical values into Eq. (1.3), we determine that the magnitude of the velocity to be 2.2 m/s.

If we examine this problem more closely, we recognize that in the absence of energy loss, the system will oscillate up and down indefinitely through the $h = 0.5$ m position. Each time it passes through this position, the velocity has a magnitude of 2.2 m/s, but the direction differs (upward/downward) depending on the point in the oscillation. A system dynamics course examines the entire oscillatory motion using Newton's second law to derive a *differential equation of motion* for the system.

To apply Newton's second law, we must first draw a free body diagram of each mass as shown in Fig. 1.7, where F_C is the force in the cable and F_k is the force in the spring. Newton's second law states that the sum of the forces is proportional to the product of the free body mass and its acceleration: $\sum F = m\ddot{x}$. Summing the forces, we obtain Eq. (1.4) for m_1 and Eq. (1.5) for m_2.

$$F_C - F_k - m_1 g = m_1 \ddot{x}_1 \tag{1.4}$$

$$-F_C + m_2 g = m_2 \ddot{x}_2 \tag{1.5}$$

The change in signs on the forces agrees with the change in the direction of the declared coordinates. Recognizing that the spring force on m_1 is kx_1, we rewrite Eq. (1.4).

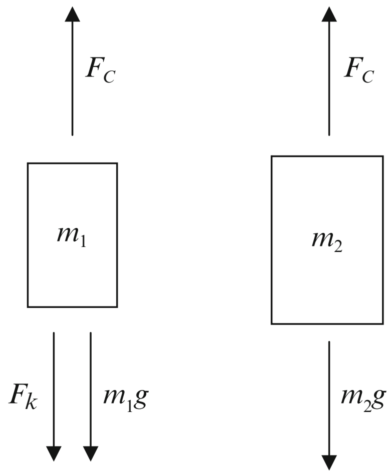


Fig. 1.7 Free body diagrams of each mass

$$F_C - kx_1 - m_1 g = m_1 \ddot{x}_1 \tag{1.6}$$

Solving Eq. (1.5) for F_C, substituting into Eq. (1.6), and recognizing that the cable constraint requires that $x_1 = x_2$, we obtain a combined differential equation describing the motions of the system with initial conditions: $x_2(0) = 0$ and $\dot{x}_2(0) = 0$.

$$(m_1 + m_2)\ddot{x}_2 + kx_2 = (m_2 - m_1)g \tag{1.7}$$

Substituting the numerical values, we obtain the equation for the system.

$$(35\,\mathrm{kg})\ddot{x}_2 + (300\,\mathrm{N/m})x_2 = 246\,\mathrm{N} \tag{1.8}$$

This linear ordinary differential equation can be solved by many methods. Removing the units and calculating the Laplace transform (see Chap. 2), we obtain Eq. (1.9).

$$X_2(s) = \frac{7.02}{s(s^2 + 8.57)} \tag{1.9}$$

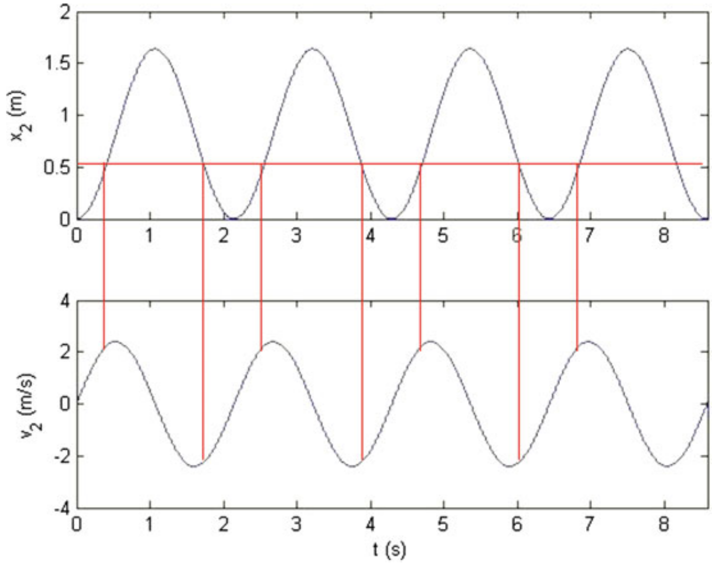
The inverse Laplace transform gives the time domain solution for the system motions.

$$x_2(t) = 0.819(1 - \cos(2.93t))\,\mathrm{m} \tag{1.10}$$

The derivative of Eq. (1.10) gives the mass velocity as a function of time.

$$v_2(t) = 2.4\sin(2.93t)\,\mathrm{m/s} \tag{1.11}$$

Plots of these functions are provided.



We see that the velocity alternates between 2.2 and −2.2 m/s each time the height passes through 0.5 m. The maximum change in height is 1.64 m. For height changes greater than 1.64 m, Eq. (1.3) yields an imaginary solution which indicates that this system state cannot occur. This can be readily seen from the graphs of the motions. Equations (1.10) and (1.11) and the corresponding graphs give us a much more comprehensive view of the motions of the system than the energy analysis alone. They tell us not only what system states are occurring, but also how the system attained that state and which states will follow.

1.5 Summary

In this chapter, we introduced the following key concepts.

- A system is a group of interacting elements that performs a desired function.
- A system has inputs that lead to, or cause, outputs.
- For a static system, the output depends only on the input at the current time.
- For a dynamic system, the output depends on the input at the current and past times.
- The inputs and outputs of a system depend on the system boundaries.

- A lumped element is one that has been idealized to isolate one particular physical phenomenon.
- Elements are often linearized so that there is a linear relationship between input and output.
- System dynamics seeks to understand the entire functional form of a system's dynamic behavior.

Laplace Transform Techniques

2

2.1 Motivation

The Laplace transform is a powerful tool for analyzing linear differential equations that are used to model dynamic systems. These mathematical models are useful approximations of many dynamic systems including those that are:

- mechanical
- electrical
- electromechanical
- thermal
- hydraulic
- pneumatic

in nature. Because these systems can be described using linear, time-invariant differential equations,[1] the Laplace transform is a natural choice for their solution. For example, we will see how to use the Laplace transform to solve the differential equation of motion for the time-dependent mass displacement in a lumped parameter spring-mass-damper system.[2] Laplace transforms offer many useful advantages as a mathematical/modeling tool. In particular, the transformation from the time, t, to the Laplace, s, domain enables differential equations to be represented as algebraic equations. In this chapter, we will learn how to convert back and forth between the t and s domains. Although we may initially be more comfortable with the time domain, by the end of the chapter we will be able to interpret signals in the Laplace domain as well.

[1] Linear systems satisfy superposition and scaling. A time-invariant system is one where a shift in the input (a time shift, for example) results in the same shift in the output.

[2] In a lumped parameter spring-mass-damper system, we assume that all the mass is concentrated at the motion coordinate and the spring and damper are massless.

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2.2 Definition of the Laplace Transform

For the purposes of dynamic system analysis, the *Laplace transform* is applied to a dynamic function of time, t, denoted $f(t)$. This function describes the time domain behavior of a system of interest. The Laplace transform converts this well-behaved[3] time domain function into a new and related function in the Laplace domain-independent variable, s. The new function is denoted $F(s)$. As mentioned previously, the Laplace transform is particularly useful in solving linear differential equations because their solution in the Laplace domain is carried out through algebraic manipulation, rather than calculus as required in the time domain. Initially, the physical meaning of the Laplace transform and the variable s may not be apparent. Once we have introduced and defined the Laplace transform, we will provide examples to make the physical meaning more clear. However, a fully intuitive, physical understanding of the Laplace transform generally comes only through additional experience.

The Laplace transform is a definite integral that is defined in the following way, where $\mathcal{L}$ is the *Laplace operator*.

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)\mathrm{e}^{-st}dt \tag{2.1}$$

Here $f(t)$ is the time domain function to be transformed. The integral exists as long as $f(t)$ is locally integrable over the time interval from zero to infinity, $[0,\infty]$. From a dynamic systems perspective, the Laplace transform captures behavior from some initial condition, $f(0)$ (or the value of the function at $t=0$), to future times. For example, the function might describe the position of an oscillating mass starting from some initial position and velocity. While it is a mathematical conception to consider all future behavior of a system, it is often important to examine the dynamics of a system until the behavior attenuates to a negligible level and this is usually our practical interest. For the oscillating mass, we might want to observe its behavior until the motion magnitude is below a small threshold value.

Because Eq. (2.1) is a definite integral, the transform result, $F(s)$, is only a function of s. The Laplace variable s is a complex number whose real part is linked physically to energy dissipation and whose imaginary part is linked to the oscillating frequency. Since an integral is a linear operator, the Laplace transform exhibits the following important properties:

$$\begin{aligned} &\mathcal{L}(af(t)) = a\mathcal{L}(f(t)) = aF(s) \\ &\mathcal{L}(f(t)+g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t)) = F(s) + G(s) \end{aligned}, \tag{2.2}$$

[3] Although "well-behaved" depends on the application, for our purposes we will specify that the more times a function can be differentiated, the more well-behaved the function is considered to be.

where a is a constant and $g(t)$ is a function of time. These properties may be applied to determine the Laplace transform of many functions that we encounter in dynamic systems.

2.3 Complex Numbers

Because the Laplace variable s is complex with real and imaginary parts that have precise physical meanings, it is useful to review the algebra of *complex numbers* before proceeding with a discussion of Laplace transforms. The general form of a complex number, s, is given by:

$$s = \sigma + j\omega, \tag{2.3}$$

where we use $j = \sqrt{-1}$ as the imaginary variable instead of i to avoid confusion with the current flowing in an electrical circuit (see Chap. 7). The real part of s, denoted Re(s), is σ and the imaginary part of s, denoted Im(s), is ω. A complex number can be represented graphically as a vector quantity in the complex plane where the real part is plotted on the horizontal axis and the imaginary part is plotted on the vertical axis. Figure 2.1a shows this vector representation of a complex number; the similarity to vector mathematics is useful in further defining the properties of complex number mathematics.

The magnitude of a complex number can be determined from Fig. 2.1a:

$$|s| = \sqrt{(\mathrm{Re}(s))^2 + (\mathrm{Im}(s))^2} = \sqrt{\sigma^2 + \omega^2}. \tag{2.4}$$

and the phase angle between s and the real axis is:

$$\theta = \tan^{-1}\left(\frac{\mathrm{Im}(s)}{\mathrm{Re}(s)}\right) = \tan^{-1}\left(\frac{\omega}{\sigma}\right). \tag{2.5}$$

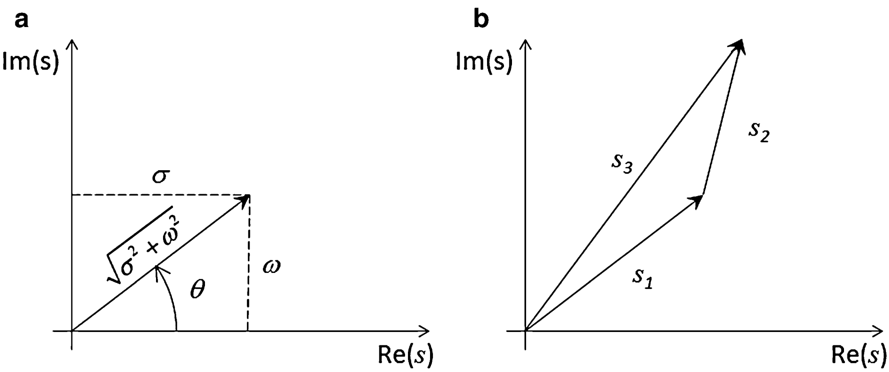


Fig. 2.1 (**a**) A complex number can be represented as a vector in the complex plane; and (**b**) the addition of complex numbers follows vector algebra

Further, the sum of two complex numbers is calculated using the sums of their real and imaginary parts, respectively.

$$s_1 + s_2 = (\sigma_1 + \sigma_2) + j(\omega_1 + \omega_2). \tag{2.6}$$

Equation (2.6), together with Fig. 2.1b, illustrates the analogy between complex numbers and vectors, where the real and imaginary components specify the coordinate system (or basis set). The vector representation of complex numbers is powerful and forms the foundation for the analysis of dynamic systems in terms of phasors. We will not use that approach in this text, but the interested student can reference the classic vibrations text by Den Hartog [1], for example.

Complex conjugates, denoted s and $\bar{s}$, are complex numbers having the same real part, but imaginary parts of equal magnitude with opposite signs. Thus, the complex conjugate of s is calculated by reversing the sign on the imaginary part.

$$\bar{s} = \sigma - j\omega \tag{2.7}$$

The product of a complex number and its conjugate is a real number. Further, we can see that the magnitude of the complex number s can be calculated from the product of the number and its conjugate. We will demonstrate the application of Eq. (2.8) using Example 2.1.

$$|s| = \sqrt{s\bar{s}} = \sqrt{\sigma^2 + j\sigma\omega - j\sigma\omega - j^2\omega^2} = \sqrt{\sigma^2 + \omega^2}. \tag{2.8}$$

Example 2.1 Plot each of the following complex numbers and their complex conjugates in the complex plane. Find the magnitude and phase angle of each.

(a) $s = j3$
(b) $s = -4 + j3$
(c) $s = -4$

Each of the numbers and their conjugates are shown in the complex plane in Fig. 2.2. The complex conjugates are identified by the gray arrows.

Solution We directly apply the definition of the complex conjugate to complete this example.

(a) The complex conjugate of a purely imaginary number is found by changing the sign.

$$\bar{s} = -j3$$

The magnitude is then determined by applying Eq. (2.8).

$$|s| = \sqrt{s\bar{s}} = \sqrt{(j3)(-j3)} = \sqrt{-(j^2)(9)} = \sqrt{-(-1)(9)} = \sqrt{9} = 3$$

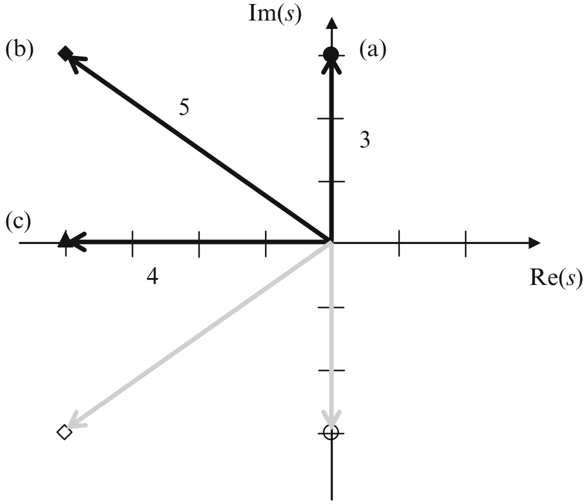


Fig. 2.2 The complex numbers in Example 2.1 and their conjugates

The phase angle is calculated using Eq. (2.5).

$$\theta = \tan^{-1}\left(\frac{3}{0}\right) \rightarrow \frac{\pi}{2}\text{rad (or } 90^\circ\text{)}$$

(b) The complex conjugate is found by changing the sign on the imaginary part of the number.

$$\bar{s} = -4 - 3j$$

Equation (2.8) is applied to calculate the magnitude.

$$|s| = \sqrt{s\bar{s}} = \sqrt{(-4+j3)(-4-j3)} = \sqrt{16 - j^2 9} = \sqrt{16+9} = 5$$

Equation (2.5) is used to determine the phase angle.

$$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 2.498\,\text{rad}\,(143.1^\circ)$$

(c) The complex conjugate of a real number is simply equal to the real number.

$$\bar{s} = -4$$

The magnitude of a real number is the absolute value of the number, which agrees with Eq. (2.8).

$$|s| = \sqrt{s\bar{s}} = \sqrt{(-4)(-4)} = \sqrt{16} = 4$$

Visual inspection yields a phase angle of 180° which also follows Eq. (2.5).

$$\theta = \tan^{-1}\left(\frac{0}{-4}\right) = \pi\,\mathrm{rad}\,(180^\circ)$$

2.4 Phasors

The treatment of complex numbers and dynamic systems using *phasor* mathematics is rooted in *Euler's formula*, named after the mathematician Leonhard Euler. This formula establishes a relationship between the complex exponential function and the trigonometric sine and cosine functions:

$$e^{j\theta} = \cos\theta + j\sin\theta, \tag{2.9}$$

where $\cos\theta$ represents the real part of the complex number and $\sin\theta$ is the imaginary part. Equation (2.9) can be proven using a Taylor series and is generally considered one of the most profound and useful results in all of mathematics. From Euler's formula, it can be seen that any complex number with a magnitude A may be represented as a single complex exponential as shown in Fig. 2.3. This can readily be verified by multiplying both sides of Eq. (2.9) by a constant A.

$$Ae^{j\theta} = A\cos\theta + jA\sin\theta \tag{2.10}$$

The magnitude of a complex number is $\sqrt{\mathrm{Re}^2 + \mathrm{Im}^2}$ and, using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, we see from the right-hand side of Eq. (2.10) that the magnitude of the complex number is A. Additionally, the phase angle θ is given by the inverse tangent of the imaginary part of the complex number divided by the real part, $\theta = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right)$

Not only does Euler's formula provides a compact method for representing complex numbers, it also provides insight into the physical meaning of the Laplace variable, s, as demonstrated in Examples 2.2 and 2.3.

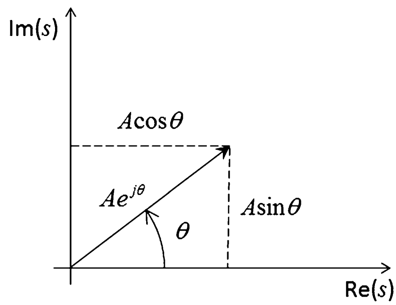


Fig. 2.3 Euler's formula shows that a vector in the complex plane can also be represented by a complex exponential function

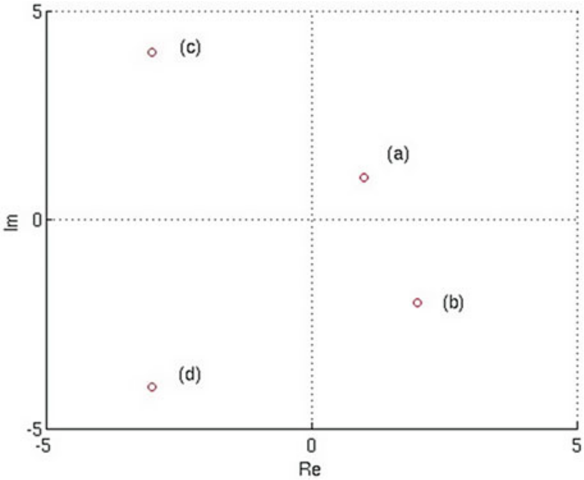


Fig. 2.4 Complex numbers plotted in the complex plane using MATLAB®

Example 2.2 Find the magnitude of the following complex numbers and then plot each in the complex plane using MATLAB®.

(a) $s = 1 + j$
(b) $s = 2 - j2$
(c) $s = -3 + j4$
(d) $s = -3 - j4$

Solution The following commands, entered at the MATLAB® command prompt, will produce the desired plot see Fig. 2.4. We define the variable j to match our notation because MATLAB® uses i as the imaginary variable by default.

```
>> j = sqrt(-1);
>> sa = 1+j;
>> sb = 2-2*j;
>> sc = -3+4*j;
>> sd = -3-4*j;
>> figure(1)
>> hold on
>> plot(real(sa), imag(sa), 'ro');
>> plot(real(sb), imag(sb), 'ro');
>> plot(real(sc), imag(sc), 'ro');
>> plot(real(sd), imag(sd), 'ro');
>> axis([-5 5 -5 5])
>> xlabel('Re')
>> ylabel('Im')
```

Example 2.3 Using Euler's formula, determine the exponential representation of the complex numbers plotted in Example 2.2. Evaluate the complex exponentials using MATLAB® and show that they are equivalent.

(a) $s = 1 + j$
(b) $s = 2 - j2$
(c) $s = -3 + j4$
(d) $s = -3 - j4$

Solution From Fig. 2.3, we observe that A is the magnitude and that the phase angle, θ, is the inverse tangent of the quotient of the imaginary and real parts of the complex number.

(a) $s = 1 + j$

$$A = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}\,\mathrm{rad}$$

Substituting A and θ into Eq. (2.10), we write the complex exponential form of the number.

$$s = \sqrt{2}e^{j\frac{\pi}{4}}$$

This representation of s can also be interpreted graphically as a vector of length $\sqrt{2}$ rotated by $\frac{\pi}{4}$ rad (or 45°) from the positive real axis in the counterclockwise direction.

(b) $s = 2 - j2$

The magnitude and phase angle of the complex number are:

$$A = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2} \text{ and } \theta = \tan^{-1}\left(\frac{-2}{2}\right) = -\frac{\pi}{4}\mathrm{rad}.$$

Inserting these values into the exponential representation yields:

$$s = 2\sqrt{2}e^{-j\frac{\pi}{4}}.$$

We must be careful about the quadrant[4] of the complex number when calculating θ. In MATLAB® the `atan2` function can be used to identify the correct phase angle. It accepts two arguments: the first is the imaginary part and the second is the real part of the complex variable. It reports the angle in radians.

[4] The complex plane can be divided into four quadrants: I (Re > 0, Im > 0), II (Re < 0, Im > 0), III (Re < 0, Im < 0), and IV (Re > 0, Im < 0).

```
>> atan2(-2, 2)
ans =
 -0.7854
```

Alternately, the correct angle can be obtained graphically from Fig. 2.3.

The remaining two complex numbers can be converted to exponential form in the same manner.

(c) $s = -3 + j4 \quad s = 5e^{j2.2143}$
(d) $s = -3 - j4 \quad s = 5e^{-j2.2143}$

We see that the complex conjugates have the same magnitude but equal and opposite phase angles (i.e., the complex conjugates are mirrored about the horizontal real axis).

MATLAB® is a powerful program for completing complex mathematics. It recognizes and evaluates complex numbers readily. To verify our answers in MATLAB®, we can simply type the complex exponentials directly at the command prompt and examine the results.

```
>> j = sqrt(-1);
>> sa = sqrt(2)*exp(pi/4*j)
sa =
 1.0000 + 1.0000i
>> sb = 2*sqrt(2)*exp(-pi/4*j)
sb =
 2.0000 - 2.0000i
>> sc = 5*exp(2.2143*j)
sc =
-3.0000 + 4.0000i
>> sb = 5*exp(-2.2143*j)
sb =
-3.0000 - 4.0000i
```

Example 2.4 Determine the real and imaginary parts of the complex fractional number, $s = \frac{2+j6}{1+j}$.

Solution One approach is to multiply both the top and the bottom of the fraction by the complex conjugate of the denominator.

$$s = \left(\frac{2+j6}{1+j}\right)\left(\frac{1-j}{1-j}\right) = \frac{2-j2+j6+6}{2} = 4 + j2$$

An alternate approach is to use the complex exponential representation of the numerator and denominator. We then divide the magnitudes and difference the phase angles before applying Euler's identity to rewrite the resulting exponential.

$$
\begin{aligned}
s &= \frac{2\sqrt{10}e^{j1.249}}{\sqrt{2}e^{j\frac{\pi}{4}}} = \frac{2\sqrt{2\cdot 5}}{\sqrt{2}}e^{j\left(1.249-\frac{\pi}{4}\right)} = 2\sqrt{5}e^{j0.4636} \\
&= 2\sqrt{5}\cos 0.4636 + 2\sqrt{5}\sin 0.4636 = 4 + j2
\end{aligned}
$$

The results are the same.

2.5 Laplace Transforms of Common Functions

What do we mean by common functions? After making approximations to linearize the behavior of various elements, the majority of systems encountered by mechanical engineers can be modeled, at least approximately, by linear differential equations with constant coefficients that have well-behaved input functions. As you may have encountered in a previous differential equations course, these equations have solutions in the form of complex exponentials or, perhaps more familiarly, products of real exponentials and sine or cosine functions. Thus, when we refer to common functions in the title of this section, we refer to the types of functions most likely used to describe the systems commonly encountered by the mechanical engineer. Furthermore, using Fourier series[5] techniques, it is often possible to develop a solution to a more complex system model by combinations of exponential and trigonometric functions.

Given this introduction, there are two functions that are commonly used as approximations of traditional system inputs. These are the *step function*, $u(t-a)$, and the Dirac delta function, $\delta(t-a)$, or simply the *delta function*. Physically, the step function occurs when a steady load is suddenly applied to a system. For example, a step function could be used to represent the step force obtained when placing a sandbag in the trunk of an automobile. An electrical analogy is turning on a switch, which applies a discrete change in voltage to the system when the switch is closed. A delta function can be thought of as a sudden impact. An approximation of a delta function is often used in vibration testing, where the system is excited, or "hit," by an instrumented hammer which also measures the input force profile. The functions are shown graphically in Fig. 2.5. The constant a provides a mechanism for shifting the functions in time (i.e., they do not necessarily have to be applied at $t=0$).

The mathematical definition of a delta function is:

$$
\delta(t-a) = \begin{cases} 0 & t < a \\ \infty & t = a \\ 0 & t > a \end{cases} \tag{2.11}
$$

[5] A Fourier series expresses a periodic function as a sum of sine and cosine functions.

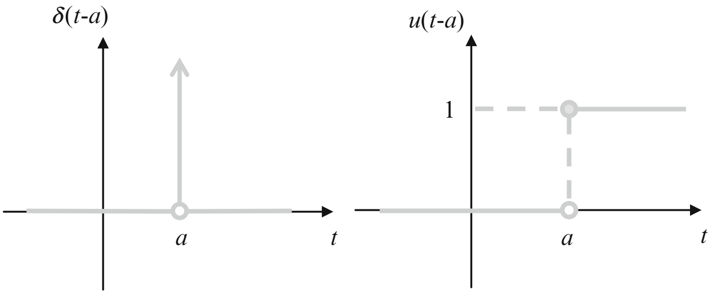


Fig. 2.5 (**a**) Delta function; (**b**) unit step function

which is a mathematical idealization describing a function that has an infinitely large height and infinitesimally narrow width at $t = a$ and is zero for all other times. Further, the delta function is defined as having a total area of unity.

$$\int_{-\infty}^{\infty} \delta(t-a)dt = 1 \tag{2.12}$$

At present, this may seem abstract, but we will see that the delta function is useful for approximating an actual impulsive input such as an impact.[6]

The mathematical definition of the step function is:

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases} \tag{2.13}$$

which is a mathematical idealization describing a function that is zero for $t < a$ and unity for $t \geq a$. The step and impulse functions are related by a time derivative.

$$\delta(t-a) = \frac{d}{dt}(u(t-a)) \tag{2.14}$$

Equation (2.14) can be described in a graphical sense. The slope of the step function (i.e., its derivative) is infinite at time $t = a$, where the delta function is infinite, and zero everywhere else, where the delta function is zero.

Example 2.5 Determine the Laplace transform of the function $C\delta(t)$, where C is a constant, using the definition of the Laplace transform provided in Eq. (2.1).

[6] A hammer tap can be used to excite a structure so that its vibration response can be measured. This is a common strategy in modal testing.

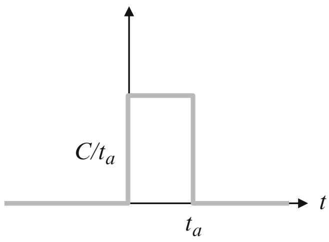


Fig. 2.6 Plot of a pulse function beginning at $t = 0$, terminating at $t = t_a$ and having a duration-dependent amplitude

Solution The Laplace transform of a delta function comes from the limit of a function that has finite width and finite height. Consider the pulse function, shown in Fig. 2.6, described as $f(t) = \begin{cases} \dfrac{C}{t_a} & 0 < t < t_a \\ 0 & t < 0, t > t_a \end{cases}$, where C is a finite constant and the function has a variable duration t_a. Therefore, the height of the function increases as the duration decreases. In the limit of a short duration, this function approximates $C\delta(t)$. We begin by determining the Laplace transform of this function and then consider the limiting case as the time interval goes to zero.

Applying the Laplace definition to the function, we obtain $\mathcal{L}(f(t)) = \int_0^\infty \frac{C}{t_a} \mathrm{e}^{-st_a} dt = \frac{C}{t_a s}(1 - \mathrm{e}^{-st_a})$. The delta function is the limiting case of this pulse function as t_a goes to zero. Applying the limit and using L'Hopital's rule, we obtain the following for the Laplace transform of the delta function.

$$\mathcal{L}[C\delta(t)] = \lim_{t_a \to 0} \left(\frac{C}{t_a s}(1 - \mathrm{e}^{-st_a}) \right) = \frac{C}{s} \lim_{t_a \to 0} \left(\frac{1 - \mathrm{e}^{-st_a}}{t_a} \right) = \frac{C}{s} \lim_{t_a \to 0} \left(\frac{s\mathrm{e}^{-st_a}}{1} \right) = C$$

For a unit impulse, $C = 1$ and, therefore, the Laplace transform of $\delta(t)$ is 1.

Example 2.6 Determine the Laplace transform of the unit step function $u(t - a)$ using the definition of the Laplace transform (Eq. 2.1).

Solution Applying the definition we obtain $\mathcal{L}[u(t-a)] = \int_0^\infty u(t-a)\mathrm{e}^{-st} dt$. Next, we apply a change of variables where we let $\rho = t - a$. Rewriting gives $t = \rho + a$. Taking the derivative, we obtain $dt = d\rho$. After substitution, we complete the integral.

$$\begin{aligned}
\mathcal{L}[u(t-a)] &= \int_{-a}^{\infty} u(\rho)\mathrm{e}^{-s(\rho+a)}d\rho \\
&= \mathrm{e}^{-sa}\int_{-a}^{\infty} u(\rho)\mathrm{e}^{-s\rho}d\rho = \mathrm{e}^{-sa}\left(\int_{-a}^{0} u(\rho)\mathrm{e}^{-s\rho}d\rho + \int_{0}^{\infty} u(\rho)\mathrm{e}^{-s\rho}d\rho\right) \\
&= \mathrm{e}^{-sa}\left(0 + \int_{0}^{\infty} \mathrm{e}^{-s\rho}d\rho\right) \\
&= \mathrm{e}^{-sa}\left(\frac{-1}{s}\right)\left(\mathrm{e}^{-s\rho}\big|_0^{\infty}\right) \\
&= \mathrm{e}^{-sa}\left(\frac{-1}{s}\right)(0-1) \\
&= \frac{\mathrm{e}^{-sa}}{s}
\end{aligned}$$

If the starting time of the step function is zero (i.e., $a=0$), the exponential in the numerator becomes 1 ($\mathrm{e}^{-sa}=\mathrm{e}^{0}=1$) and the expression is simplified to $\mathcal{L}[u(t)] = \frac{1}{s}$.

Example 2.7 Find the Laplace transform of the exponential function, $f(t)=\mathrm{e}^{-at}$, using Eq. (2.1).

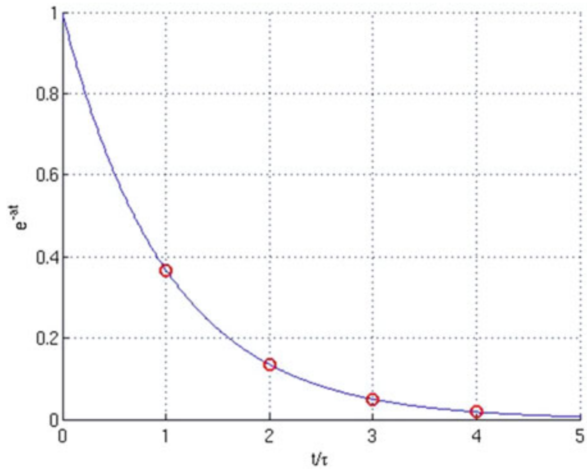
Solution We apply the definition of the Laplace transform directly: $\mathcal{L}[\mathrm{e}^{-at}] = \int_0^{\infty} \mathrm{e}^{-at}\mathrm{e}^{-st}dt = \int_0^{\infty} \mathrm{e}^{-(s+a)t}dt$. For the purposes of the integral, the quantity $s+a$ is a constant and, therefore, the integral is evaluated as follows.

$$\mathcal{L}[\mathrm{e}^{-at}] = \int_0^{\infty} \mathrm{e}^{-(s+a)t}dt = -\frac{1}{s+a}\left[\mathrm{e}^{-(s+a)t}\right]_0^{\infty} = -\frac{1}{s+a}(0-1) = \frac{1}{s+a}$$

This is a fundamental result in dynamic systems and is used throughout this book.

As we will see, a decaying exponential describes the behavior of many physical systems with surprising accuracy. For example, the exponential function describes the velocity of a mass moving under the influence of viscous friction. It also describes the discharge of a capacitor through a resistor. A potential limitation of this description is that a decaying exponential requires infinite time to decay to zero. We recognize that this model cannot be completely accurate because real moving masses will eventually reach zero velocity and real capacitors will eventually lose all charge (both within a finite time). Therefore, when describing real systems, we will see that we must decide when to declare that a model has reached its limit; we must decide when an exponential no longer provides an adequate description

Fig. 2.7 Plot of the exponential e^{-at}



of a system. To establish this limit, we use the time constant, τ, of the exponential $f(t) = e^{-at}$.

$$\tau = \frac{1}{a}. \tag{2.15}$$

The exponential is then plotted as a function of the number of time constants as shown in Fig. 2.7. After four time constants, 4τ, the exponential function has decayed to 0.018, less than 2% of its initial unit value. For most applications, this attenuation is sufficient to declare that the exponential model has fulfilled its purpose. Therefore, we will generally only plot our exponential functions for a time period of four time constants.

Example 2.8 As a final example for this section we find the Laplace transform of the cosine function $f(t) = \cos(\omega t)$, where ω is a constant describing the frequency of oscillation in units of radians per second (or simply inverse seconds).

Solution This transform can be found using integration by parts. An alternate approach is to apply Euler's formula (Eq. 2.9) and rewrite the function as a sum of complex exponents.

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

After substitution and application of the definition of the Laplace transform, we obtain $\mathcal{L}[\cos(\omega t)] = \frac{1}{2}\mathcal{L}[e^{j\omega t}] + \frac{1}{2}\mathcal{L}[e^{-j\omega t}]$. Now, employing the Laplace transform of an exponential (see Table 2.1, entry 6), we obtain the following result.

Table 2.1 Laplace transforms [2]

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $u(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \ldots$	$\frac{1}{s^n}$
5	$t^n, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	$t\mathrm{e}^{-at}$	$\frac{1}{(s+a)^2}$
8	$\frac{t^{n-1}}{(n-1)!}\mathrm{e}^{-at}, \quad n = 1, 2, 3, \ldots$	$\frac{1}{(s+a)^n}$
9	$t^n\mathrm{e}^{-at}, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
12	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1-\mathrm{e}^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(\mathrm{e}^{-at}-\mathrm{e}^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(b\mathrm{e}^{-bt}-a\mathrm{e}^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left(1+\frac{1}{a-b}(b\mathrm{e}^{-at}-a\mathrm{e}^{-bt})\right)$	$\frac{1}{s(s+a)(s+b)}$
18	$\frac{1}{a^2}(1-\mathrm{e}^{-at}-at\mathrm{e}^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at-1+\mathrm{e}^{-at})$	$\frac{1}{s^2(s+a)}$
20	$\mathrm{e}^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
21	$\mathrm{e}^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}}\mathrm{e}^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

(continued)

Table 2.1 (continued)

	$f(t)$	$F(s)$
23	$-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t-\phi\right)$ $\phi=\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$
24	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right)$ $\phi=\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$	$\frac{\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$
25	$1-\cos(\omega t)$	$\frac{\omega^2}{s(s^2+\omega^2)}$
26	$\omega t-\sin(\omega t)$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
27	$\sin(\omega t)-\omega t\cos(\omega t)$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
28	$\frac{1}{2\omega}t\sin(\omega t)$	$\frac{s}{(s^2+\omega^2)^2}$
29	$t\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
30	$\frac{1}{\omega_2^2-\omega_1^2}(\cos(\omega_1 t)-\cos(\omega_2 t)),\ \omega_1^2\neq\omega_2^2$	$\frac{s}{\left(s^2+\omega_1^2\right)\left(s^2+\omega_2^2\right)}$
31	$\frac{1}{2\omega}(\sin(\omega t)+\omega t\cos(\omega t))$	$\frac{s^2}{(s^2+\omega^2)^2}$

$$\mathcal{L}[\cos(\omega t)]=\frac{1}{2}\left(\frac{1}{s+j\omega}+\frac{1}{s-j\omega}\right)$$

We simplify by multiplying by the numerator and denominator of each term by the complex conjugate of the denominator.

$$\mathcal{L}[\cos(\omega t)]=\frac{1}{2}\left(\frac{1}{s+j\omega}+\frac{1}{s-j\omega}\right)=\frac{1}{2}\left(\frac{1}{s+j\omega}\frac{s-j\omega}{s-j\omega}+\frac{1}{s-j\omega}\frac{s+j\omega}{s+j\omega}\right)$$
$$=\frac{1}{2}\left(\frac{s-j\omega}{s^2+\omega^2}+\frac{s+j\omega}{s^2+\omega^2}\right)=\frac{1}{2}\left(\frac{2s}{s^2+\omega^2}\right)=\frac{s}{s^2+\omega^2}$$

This is entry 11 of Table 2.1. The Laplace transform of the sine function can be determined in a similar manner.

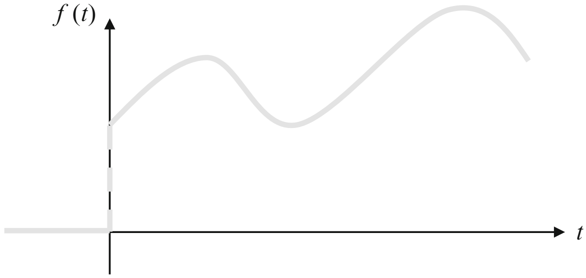


Fig. 2.8 A function, $f(t)$, that is zero before $t = 0$

2.6 Properties of the Laplace Transform

As you may have already noticed in Sect. 2.5 examples and Table 2.1, patterns appear in the Laplace transforms. In addition to Table 2.1, certain properties of the Laplace transform are important in their application.

2.6.1 Linearity

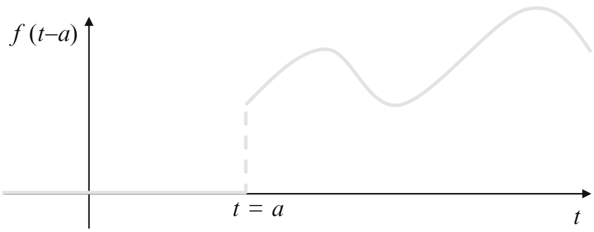
Since the Laplace transform is an integral, it follows all the principles of linearity. The most important ones are: (1) multiplication by a constant; and (2) sum of two functions. Equation (2.16) shows that the Laplace transform of a time function multiplied by a constant A yields the Laplace domain representation of the time function also multiplied by A. Additionally, we see that the Laplace transform of the sum of two time functions is the sum of the Laplace domain representation of the time functions.

$$\begin{aligned} \mathcal{L}[Af(t)] &= AF(s) \\ \mathcal{L}[f(t) + g(t)] &= F(s) + G(s) \end{aligned} \tag{2.16}$$

2.6.2 Laplace Transform of a Time-Delayed Function

In dynamic systems, it is common for the input to start at a delayed time (i.e., not $t = 0$). A simple example is the step input, $u(t - a)$, which could, for example, be an electric switch turned on at $t = a$ (later than $t = 0$). Alternately, a load might suddenly be applied at $t = a$. Some systems, such as the tool vibration in a machining operation, have inherent delays in the forcing function that can cause instability. Figure 2.8 shows a general function that begins at $t = 0$, while Fig. 2.9 shows the same function that is delayed until $t = a$. The goal is to determine the Laplace transform of the delayed function in terms of $F(s)$, the presumably known Laplace transform of the un-delayed function.

Fig. 2.9 A function, $f(t-a)$, that is zero before $t=a$



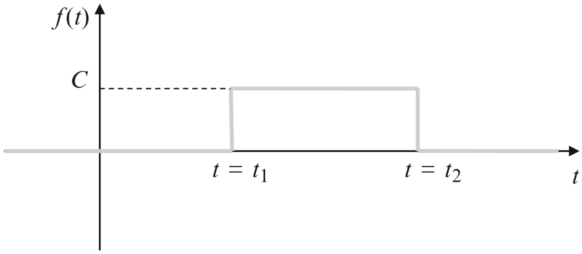


Fig. 2.10 A pulse function with amplitude C between $t=t_1$ and $t=t_2$ and zero at all other times

Applying the definition of the Laplace transform, we obtain $\mathcal{L}[f(t-a)] = \int_0^\infty f(t-a)e^{-st}dt$. Next, we substitute: $\rho = t - a$ and $d\rho = dt$ to obtain a modified version of the original integral.

$$\mathcal{L}[f(t-a)] = \int_{-a}^{\infty} f(\rho)e^{-s(\rho+a)}d\rho = e^{-sa}\int_{-a}^{\infty} f(\rho)e^{-s\rho}d\rho$$

Because $\rho = 0$ is equivalent to $t = a$, the integral limits can be redefined to be $\mathcal{L}[f(t-a)] = e^{-sa}\int_0^\infty f(\rho)e^{-s\rho}d\rho$. We see that the integral is simply $F(s)$, the Laplace transform of the un-delayed function. Substituting yields the final transform: $\mathcal{L}[f(t-a)] = e^{-sa}F(s)$.

In the Laplace domain, multiplying a function by e^{-sa} is equivalent to delaying the function by a in the time domain. For this reason, e^{-sa} is known as the delay operator and can be used to delay any function by an arbitrary time, a.

Example 2.9 Find the Laplace transform of the pulse function displayed in Fig. 2.10 that is initiated at $t=t_1$ with an amplitude of C and becomes zero again at $t=t_2$.

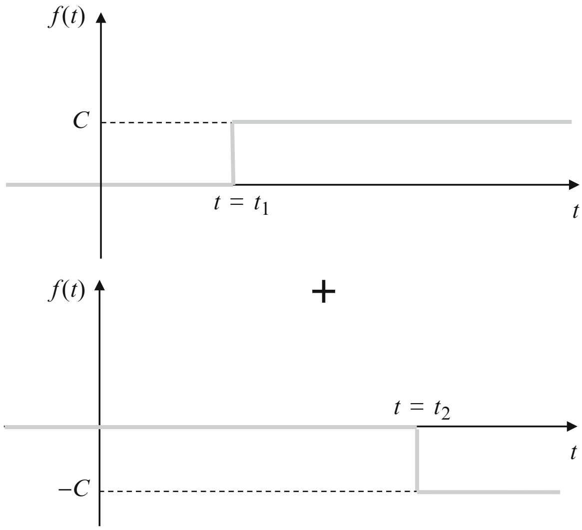


Fig. 2.11 The pulse function as a combination of two-step functions

Solution The function is the sum of the two-step functions shown in Fig. 2.11.

The straightforward solution is to sum the positive and negative steps, each the appropriate time delay, in the Laplace domain.

$$F(s) = \left(\mathrm{e}^{-st_1}\frac{C}{s}\right) + \left(\mathrm{e}^{-st_2}\frac{-C}{s}\right) = C\left(\frac{\mathrm{e}^{-st_1}}{s} - \frac{\mathrm{e}^{-st_2}}{s}\right)$$

2.6.3 Laplace Transform of a Time Derivative

In the solution of linear differential equations by Laplace transforms, it is necessary to calculate the Laplace transform of time derivatives. We use the typical "over dot" shorthand notation to represent a time derivative: $\frac{df}{dt} = \dot{f}(t)$. The Laplace transform of a time derivative proceeds as follows. Because the solution of differential equations requires knowledge of the initial conditions, we will also need to consider the inclusion of initial conditions in the Laplace transformation as we proceed.

$$\mathcal{L}\left[\dot{f}(t)\right] = \int_0^\infty \frac{df}{dt}\mathrm{e}^{-st}dt$$

We can solve this integral using integration by parts: $\int_b^b udv = uv|_a^b - \int_b^b vdu$. We define the integration by parts variables as shown.

$$u = \mathrm{e}^{-st} dv = \frac{df}{dt}dt$$
$$du = -s\mathrm{e}^{-st} v = f(t)$$

Substitution yields the following expression.

$$\mathcal{L}\left[\dot{f}(t)\right] = \int_0^\infty \frac{df}{dt}\mathrm{e}^{-st}dt = \mathrm{e}^{-st}f(t)|_0^\infty - \int_0^\infty f(t)s\mathrm{e}^{-st}dt = f(t)\mathrm{e}^{-st}\Big|_0^\infty + s\int_0^\infty f(t)\mathrm{e}^{-st}dt$$

If *f(t)* is well-behaved, then the first term can be evaluated, while the integral in the second term is just *F(s)*.

$$\mathcal{L}\left[\dot{f}(t)\right] = \int_0^\infty \frac{df}{dt}\mathrm{e}^{-st}dt = f(t)\mathrm{e}^{-st}|_0^\infty + s\int_0^\infty f(t)\mathrm{e}^{-st}dt = (f(t)\cdot 0 - f(0)\cdot 1) + sF(s)$$

The final result is written as:

$$\mathcal{L}\left[\dot{f}(t)\right] = sF(s) - f(0), \tag{2.17}$$

where $f(0)$ is interpreted as the initial value of the function (i.e., the value at $t = 0$). To determine the Laplace transforms of higher order derivatives, we apply a recursive procedure. For example, if we wish to find the Laplace transform of a second time derivative, we make the following substitutions.

$$g(t) = \dot{f}(t)$$
$$\dot{g}(t) = \ddot{f}(t)$$

Now we take the Laplace transform of $\dot{g}(t)$ using Eq. (2.17).

$$\mathcal{L}\left[\ddot{f}(t)\right] = L[\dot{g}(t)] = sG(s) - g(0)$$

Further, we recognize that the Laplace transform of $g(t) = \dot{f}(t)$ is also given by Eq. (2.16) and combine.

$$\mathcal{L}\left[\ddot{f}(t)\right] = L[\dot{g}(t)] = s\big(sF(s) - f\big(0\big) - g(0)$$
$$\mathcal{L}\left[\ddot{f}(t)\right] = s^2F(s) - sf(0) - \dot{f}(0)$$

This procedure can be repeated to obtain the Laplace transform of any order derivative of a function. As the order of the derivative increases, the number of initial conditions required for evaluation is naturally equal to the number of initial conditions needed to solve a differential equation of that same order. For example, the second derivative of position, or acceleration, appears in the second-order equation of motion for an oscillating mass attached to ground through a spring. To determine the solution for this system, we require two initial conditions (typically an initial position and velocity).

2.6.4 Initial and Final Value Theorems

As we will see later in the text, system design and analysis can often be completed directly in the Laplace domain without requiring observation of the corresponding time domain behavior. The initial value theorem and the final value theorem are important tools for system analysis in the Laplace domain. These enable conclusions about the time domain behavior to be drawn without explicitly determining the time domain response of the system. We state them without proof here and then provide examples that illustrate how to practically apply these theorems.

The *initial value theorem* enables $f(0)$ to be determined from $F(s)$ only.

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} (sF(s)) \tag{2.18}$$

The limit is taken as t approaches 0 from the positive side ($t > 0$) because we assume that our functions are zero until $t = 0$. This is also implicit in the selection of the Laplace transform lower limit.

The *final value theorem* enables the limiting value of $f(t)$ as t becomes very large (infinite) to be determined.

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} (sF(s)) \tag{2.19}$$

Notice that in these two theorems, as t becomes large, s becomes small and vice versa. Let us explore this for the initial value theorem. Consider Eq. (2.17), which is reproduced here.

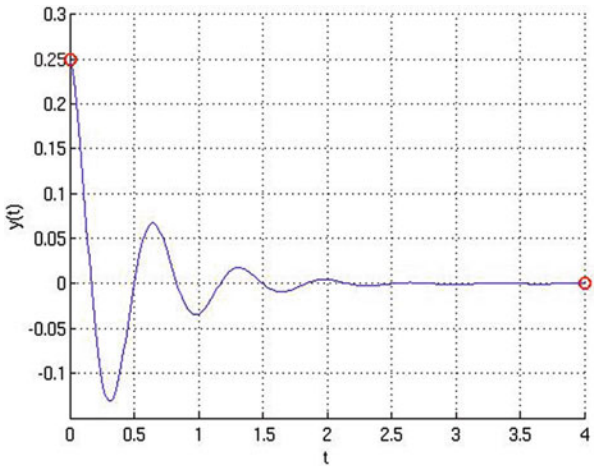
$$\mathcal{L}[\dot{f}(t)] = \mathcal{L}\left[\frac{d}{dt}f(t)\right] = \int_0^\infty \left[\frac{d}{dt}f(t)\right] \mathrm{e}^{-st}dt = sF(s) - f(0)$$

Now apply the limit to both sides of the equation as $s \to \infty$.

$$\lim_{s\to\infty} \int_0^\infty \left[\frac{d}{dt}f(t)\right] \mathrm{e}^{-st}dt = 0 = \lim_{s\to\infty} [sF(s) - f(0)]$$

The left-hand side becomes zero as $s \to \infty$ due to the exponential term. Rearranging, we obtain $\lim\limits_{s\to\infty} sF(s) = f(0)$.

Example 2.10 Consider the following function which represents the displacement of a vibrating structure: $y(t) = 0.25e^{-2t}\cos(3\pi t)$. The time domain response was plotted in MATLAB®. We observe that the starting value is 0.25 and the ending value approaches zero. Verify this result using the initial and final value theorems.



Solution To demonstrate the initial and final value theorems, we use entry 21 of Table 2.1 to find the function's Laplace transform.

$$Y(s) = 0.25\frac{s+2}{(s+2)^2 + (3\pi)^2}$$

We apply Eq. (2.18) to determine the initial value.

$$\lim_{t\to 0} y(t) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty}\left(s\frac{0.25(s+2)}{(s+2)^2+(3\pi)^2}\right)$$

We expand the numerator and denominator and divide through by the highest power in s for both. We then set all terms with s in the denominator equal to zero (as $s \to \infty$) in order to obtain the result.

$$\lim_{s\to\infty} sY(s) = \lim_{s\to\infty}\left(0.25\frac{(s^2+2s)}{s^2+4s+4+(3\pi)^2}\right) = \lim_{s\to\infty}\left(0.25\frac{(1+\frac{2}{s})}{1+\frac{4}{s}+\frac{4+(3\pi)^2}{s^2}}\right)$$

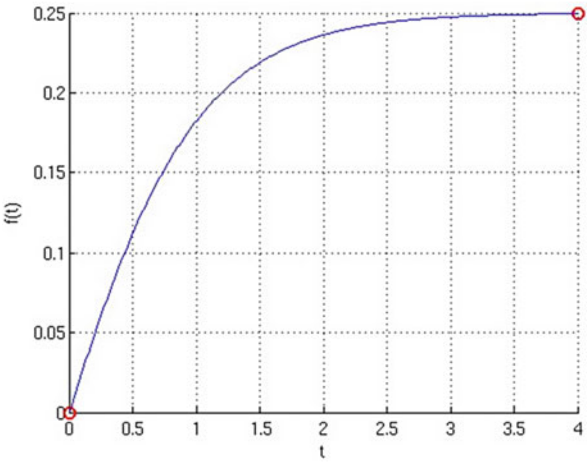
$$= 0.25\frac{1}{1} = 0.25$$

This matches the time domain graphical result. Next, we apply the final value theorem. In this case, we do not divide by the highest power in s. Rather, we simply set all the s terms equal to zero (as $s \to 0$).

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0}\left(0.25\frac{(s^2+2s)}{s^2+2s+4+(3\pi)^2}\right) = 0.25\frac{0}{4+(3\pi)^2} = 0$$

This also agrees with the time domain behavior of the function.

Example 2.11 Consider the function $f(t) = \frac{1}{4}(1 - e^{-2t} - 2te^{-2t})$. The MATLAB® time domain response is shown, where we see that the function is zero at $t = 0$ and approaches 0.25 as time gets large.



Suppose this was the solution for the model of a dynamic system. Sometimes it is important to know, in a control system for example, the initial and final values of the solution. However, we may only have the solution in the Laplace domain.

Solution We find the Laplace transform of the function using entry 18 of Table 2.1.

$$F(s) = \frac{1}{s(s+2)^2}$$

Applying Eq. (2.18), we find the initial value.

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\frac{1}{(s+2)^2} = \lim_{s\to\infty}\frac{1}{s^2+4s+4} = \lim_{s\to\infty}\frac{\frac{1}{s^2}}{1+\frac{4}{s}+\frac{4}{s^2}} = \frac{0}{1} = 0$$

Applying Eq. (2.19), we find the final value.

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0}\frac{1}{(s+2)^2} = \lim_{s\to 0}\frac{1}{s^2+4s+4} = \frac{1}{4}$$

These agree with the time domain plot of the function.

2.7 Inverting Laplace Transforms

In order to solve[7] a differential equation (that models a dynamic system) using Laplace transforms, we need to first understand how to apply the *forward Laplace transform* (Sect. 2.5, Table 2.1). We can then perform simple algebraic manipulations in the Laplace domain and finally apply the *inverse Laplace transform* to obtain the solution in the time domain. We represent the inverse Laplace transform as shown in Eq. (2.20).

$$\mathcal{L}^{-1}[F(s)] = f(t) \tag{2.20}$$

The inverse Laplace transform is an integral, so it also obeys the linearity properties listed in Eq. (2.16).

$$\begin{aligned} &\mathcal{L}^{-1}[AF(s)] = A\mathcal{L}^{-1}[F(s)] = Af(t) \\ &\mathcal{L}^{-1}[F(s)+G(s)] = \mathcal{L}^{-1}[F(s)] + \mathcal{L}^{-1}[G(s)] = f(t) + g(t) \end{aligned} \tag{2.21}$$

Typically, in system dynamics, the solution for a selected model will involve determining the inverse Laplace transform of a function with the form $F(s) = \frac{B(s)}{A(s)}$. The numerator and denominator of $F(s)$ are typically polynomials in s. That is, $A(s)$ and $B(s)$ can be described as sums of terms with descending powers of s.

$$\begin{aligned} A(s) &= c_n s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0 \\ B(s) &= d_m s^m + d_{m-1}s^{m-1} + \ldots + d_1 s + d_0 \end{aligned}$$

We say that $A(s)$ is a polynomial of order n and $B(s)$ is a polynomial of order m because these are the highest powers of s. The system can also be written in *pole-zero* form by factoring the numerator and denominator.

[7] By solve, we mean that we wish to determine the time domain solution of the differential equation.

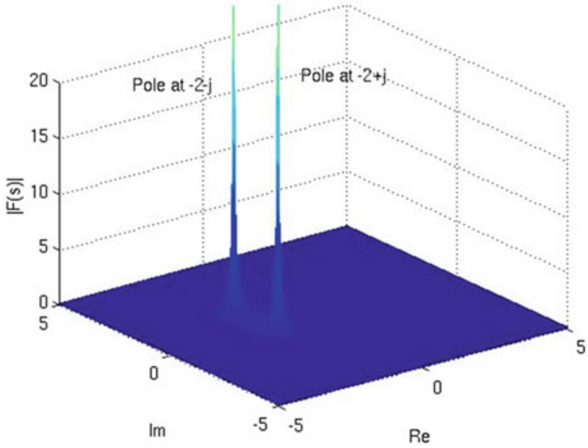


Fig. 2.12 The magnitude of $F(s)$ plotted as a function of s. The axes represent the real and imaginary parts of the complex variable s and the surface represents the magnitude of $F(s)$

$$F(s) = \frac{K(s - z_1)(s - z_2)(s - z_3) \ldots (s - z_m)}{(s - p_1)(s - p_2)(s - p_3) \ldots (s - p_n)}$$

$F(s)$ is zero when $s = z_1$, $s = z_2$, etc., so these values of s are called the *zeroes* of $F(s)$. $F(s)$ approaches infinity when $s = p_1$, $s = p_2$, etc. These are called the *poles* of $F(s)$. Note that both poles and zeroes can be real or complex (i.e., they include an imaginary part). Later we will see that the function $F(s)$ will represent a transfer function of a system. In that case the poles provide direct system properties.

We can visualize the effect of poles graphically if we plot the magnitude of $F(s)$ for a particular example.

$$F(s) = \frac{1}{s^2 + 4s + 5} = \frac{1}{(s - (-2 + j))(s - (-2 - j))}$$

For each complex value of s, the magnitude of $F(s)$ has a corresponding value. That value approaches infinity when s is equal to a pole of $F(s)$ as shown in Fig. 2.12, where the axes represent the real and imaginary parts of the Laplace variable s. The poles are $s = -2 + j$ and $s = -2 - j$.

The primary procedure we will apply to determine the inverse Laplace transform for engineering system functions is *partial fraction expansion*. Partial fractions are used to convert a function that is not in a form we can find in our Laplace transform table into a form that is available in the table. We will demonstrate this by examples. In these examples, it is assumed that the numerator and denominator polynomials have already been factored and like terms have been canceled from the two.

There are three primary types of partial fraction expansions; these provide "shortcuts" for inverting a particular Laplace transform. These types are based on the $F(s)$ poles (or roots of the denominator). A fourth case occurs frequently when a system is subjected to a step input.

1. Distinct real poles (or roots).
2. Complex poles.
3. Repeated real poles.
4. Special case that often occurs with step inputs to systems.

There is no "magic" to partial fractions. The central question to answer is this: Given a certain form of $F(s)$ that is not easily invertible (i.e., does not appear in our Laplace transform table), is it possible to write $F(s)$ in another form that is more easily invertible? There are typically multiple ways to invert a Laplace transform and even if the different results do not "look" the same, they must give the same answer. The Laplace transform and its inverse are unique, which means there cannot be two different functions in the time domain that represent the inverse of the same function in the Laplace domain.

2.7.1 Distinct Real Poles

In this case, when the denominator is factored, the poles are all *distinct real numbers*. This is illustrated by the following example. In this case, a useful partial fractions expansion is the sum of a series of fractions with the individual poles appearing in the denominators.

$$\begin{aligned} F(s) &= \frac{K(s-z_1)(s-z_2)(s-z_3)\ldots(s-z_m)}{(s-p_1)(s-p_2)(s-p_3)\ldots(s-p_n)} \\ &= \frac{a_1}{s-p_1} + \frac{a_2}{s-p_2} + \ldots + \frac{a_n}{s-p_n} \end{aligned} \tag{2.22}$$

Here the values $a_1, a_2 \ldots a_n$ are constants. We apply the linearity principle of the Laplace transform and its inverse to obtain Eq. (2.23).

$$\begin{aligned} f(t) = \mathcal{L}^{-1}[F(s)] &= \mathcal{L}^{-1}\left[\frac{a_n}{s+p_n}\right] + \ldots + \mathcal{L}^{-1}\left[\frac{a_2}{s+p_2}\right] + \mathcal{L}^{-1}\left[\frac{a_1}{s+p_1}\right] \\ &= a_n e^{-p_n t} + \ldots + a_2 e^{-p_2 t} + a_1 e^{-p_1 t} \end{aligned} \tag{2.23}$$

We illustrate this case by an example.

Example 2.12 Find the inverse Laplace transform of the Laplace domain function $F(s) = \frac{s-3}{s^2+6s+5}$.

Solution First, we factor the denominator.

$$F(s) = \frac{s-3}{(s+1)(s+5)}$$

If there were common terms between the numerator and denominator, we would cancel them at this stage. The method of partial fractions says that $F(s)$ can be expanded using Eq. (2.22).

$$F(s) = \frac{s-3}{(s+1)(s+5)} = \frac{a_1}{s+1} + \frac{a_2}{s+5}$$

To determine a_1 and a_2, we multiply both sides of the equation by the left-hand side denominator. We then group terms in powers of s.

$$\begin{aligned}
&s-3 = \frac{a_1(s+1)(s+5)}{s+1} + \frac{a_2(s+1)(s+5)}{s+5} \\
&s-3 = a_1(s+5) + a_2(s+1) \\
&s(a_1 + a_2 - 1) + (5a_1 + a_2 + 3) = 0
\end{aligned}$$

Because s is a variable and can take on any real or complex value, the only way for this expression to always be zero is for both the multiplier of s (in the first parenthesis) and the constant term (in the second parenthesis) to be equal to zero. To determine a_1 and a_2, we solve the two simultaneous linear equations.

$$a_1 + a_2 - 1 = 0$$

$$5a_1 + a_2 + 3 = 0$$

The only pair of values that satisfies both equations is $a_1 = -1$ and $a_2 = 2$.

There is an alternate way to find a_1 and a_2 that takes advantage of s being a variable that can take on any value, so that the expansion must be true no matter the value of s. To find a_1, we begin with $F(s) = \frac{s-3}{(s+1)(s+5)} = \frac{a_1}{s+1} + \frac{a_2}{s+5}$ and multiply both sides by $s+1$. We then let s approach the pole, $s = -1$, in the limit.

$$\begin{aligned}
&\lim_{s\to-1}\left[\frac{s-3}{(s+5)}\right] = \lim_{s\to-1}\left[a_1 + \frac{a_2(s+1)}{s+5}\right] \\
&\frac{-1-3}{(-1+5)} = a_1 + \frac{a_2(-1+1)}{-1+5} \\
&\frac{-4}{4} = a_1 + \frac{a_2(0)}{4}
\end{aligned}$$

This yields $a_1 = -1$ as before. We use a limit because we know that the original function becomes infinite when s is equal to -1.

To find a_2, we multiply both sides by $(s+5)$ and repeat the procedure.

$$\lim_{s\to-5}\left[\frac{s-3}{(s+1)}\right]=\lim_{s\to-5}\left[\frac{a_1(s+5)}{s+1}+a_2\right]$$
$$\frac{-5-3}{(-5+1)}=\frac{a_1(-5+5)}{-5+1}+a_2$$
$$\frac{-8}{-4}=\frac{a_1(0)}{-4}+a_2$$

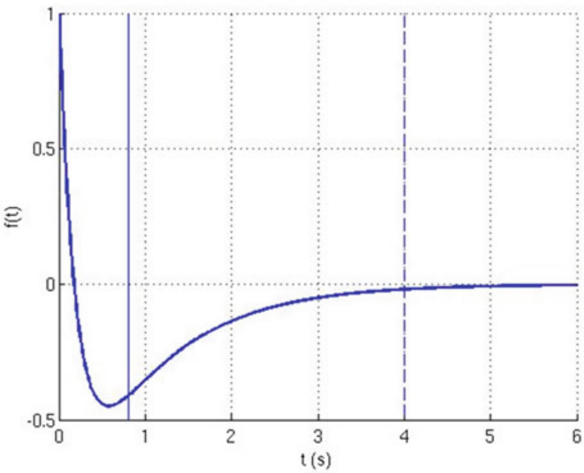
We see that $a_2=2$ as before. Substituting gives our final partial fraction expansion, where we subdivided our original function into two partial fractions.

$$F(s)=\frac{-1}{s+1}+\frac{2}{s+5}$$

calculating the inverse Laplace transform and making use of linearity, we find that the time domain solution for the selected Laplace function with real poles is a sum of exponential functions; this is the general case.

$$f(t)=\mathcal{L}^{-1}[F(s)]=\mathcal{L}^{-1}\left[\frac{-1}{s+1}\right]+\mathcal{L}^{-1}\left[\frac{2}{s+5}\right]=-\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]+2\mathcal{L}^{-1}\left[\frac{1}{s+5}\right]$$
$$f(t)=-\mathrm{e}^{-t}+2\mathrm{e}^{-5t}$$

A plot of this function is provided.



Notice the effect of the two exponentials. The first exponential has a time constant of 1 s and is relatively slow, while the second is faster with a shorter time constant of 0.2 s (1/5). Therefore, the second exponential changes rapidly over about four time constants, $4(0.2)=0.8$ s (solid vertical line), while the first exponential changes more slowly over an interval of about $4(1) = 4$ s (dashed vertical line). Notice also that the time constants are known as soon as we have found the poles (roots of the denominator); they are simply the inverse of the absolute value of p_i for the pole form $(s - p_i)$.

2.7.2 Complex Poles

In this case the denominator can still be factored to produce distinct poles/roots, but the poles/roots are complex numbers. More specifically, they appear as pairs of complex conjugates. In this situation, the most convenient approach is not the partial fractions expansion solution we demonstrated for real and distinct roots, although this can be done as long as you are familiar with manipulating complex numbers and adept at using Euler's formula. As an alternative, for a second-order system, we *complete the square* in the denominator and write the function $F(s)$ in a form that represents exponentially decaying sine and cosine terms (in the time domain) that are found in Table 2.1. This is illustrated by the following example.

Example 2.13 Determine the inverse Laplace transform of the function $F(s) = \frac{3}{s^2+4s+20}$.

Solution Using the quadratic equation, we find the roots to the denominator to be:

$$s_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4(1)20}}{2(1)} = -2 \pm \frac{\sqrt{-64}}{2} = -2 \pm j4.$$

Rewriting gives:

$$F(s) = \frac{3}{s^2 + 4s + 20} = \frac{3}{(s + 2 - j4)(s + 2 + j4)}.$$

To complete the square in the denominator, we take half of the constant multiplying s, square it, and then "borrow" that value from the constant term. In this case, the multiplier on s for the polynomial representation of the denominator is 4. Half of 4 squared is again 4. Borrowing 4 from 20 leaves 16.

$$F(s) = \frac{3}{s^2 + 4s + 4 + 16}$$

Because $s^2 + 4s + 4$ is a perfect square, we obtain the following.

$$F(s) = \frac{3}{(s + 2)^2 + 16} = \frac{3}{(s + 2)^2 + 4^2}$$

This can now be compared to entry 20 in Table 2.1 where $a = 2$ and $\omega = 4$. However, because use of this entry also requires ω (or 4) in the numerator we can manipulate the expression as follows.

$$F(s) = \frac{3}{4}\left(\frac{4}{(s+2)^2 + 4^2}\right)$$

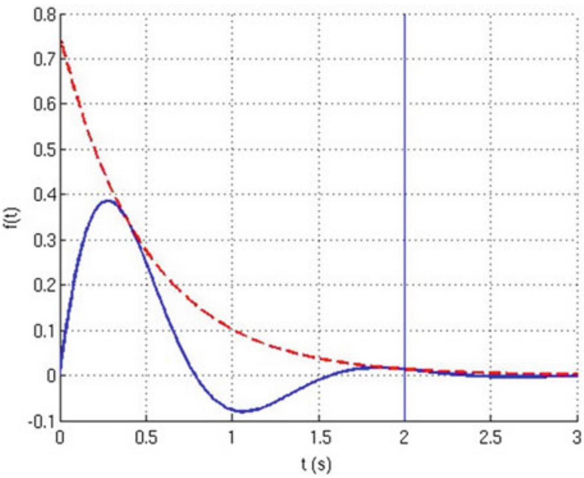
Now the linearity property is applied:

$$f(t) = \mathcal{L}^{-1}\left[\frac{3}{4}\left(\frac{4}{(s+2)^2 + 4^2}\right)\right] = \frac{3}{4}\mathcal{L}^{-1}\left[\frac{4}{(s+2)^2 + 4^2}\right]$$

and we obtain the final time domain function.

$$f(t) = \frac{3}{4}e^{-2t}\sin 4t$$

A plot of this function is provided (solid line), while the *exponential envelope*, $\frac{3}{4}e^{-2t}$, is shown as a dashed line. The function has decayed appreciably when the time reaches four time constants of the exponential envelope, or $4(0.5) = 2$ s (solid vertical line).



2.7.3 Repeated Real Poles

Sometimes, the poles are real but they are repeated. For example, the denominator might contain the factor $(s - 3)^2$ in which case the “strength” of the pole/root has been increased. Since true repeated roots require exact values for some of the physical constants in the system to make a perfect square, the most common/practical cause of repeated roots is a ramp input (Table 2.1, entry 3). In this case, the term s^2 appears in the denominator of the Laplace transform and there are two poles (roots in the denominator) at $s = 0$. For repeated roots, it is necessary to modify the partial fractions expansion for real roots. In particular, a single repeated root is treated using the partial fraction expansion in Eq. (2.24), where the root p_1 is repeated.

$$\begin{aligned} F(s) &= \frac{K(s - z_1)(s - z_2)(s - z_3)\ldots(s - z_m)}{(s - p_1)^2(s - p_3)\ldots(s - p_n)} \\ &= \frac{a_1}{s - p_1} + \frac{a_2}{(s - p_1)^2} + \frac{a_3}{s - p_3} + \ldots + \frac{a_n}{s - p_n} \end{aligned} \tag{2.24}$$

The inverse Laplace transform for this expansion is provided in Eq. (2.25).

$$f(t) = a_1 \mathrm{e}^{-p_1 t} + a_2 t\mathrm{e}^{-p_1 t} + a_3 \mathrm{e}^{-p_3 t} + \ldots + a_n \mathrm{e}^{-p_n t} \tag{2.25}$$

Example 2.14 Determine the inverse Laplace transform of the function $F(s) = \frac{2s+10}{(s+1)^2(s+4)}$ with the repeated pole $s = -1$.

Solution This function has two roots at –1 and one root at –4. According to Eq. (2.24), we expand as follows.

$$F(s) = \frac{2s + 10}{(s + 1)^2(s + 4)} = \frac{a_1}{s + 1} + \frac{a_2}{(s + 1)^2} + \frac{a_3}{(s + 4)}$$

To identify the constants, we multiply both sides by the left-hand side denominator and group terms by powers of s.

$$\begin{aligned} &2s + 10 = a_1(s + 1)(s + 4) + a_2(s + 4) + a_3(s + 1)^2 \\ &0 = (a_1 + a_3)s^2 + (5a_1 + a_2 + 2a_3 - 2)s + (4a_1 + 4a_2 + a_3 - 10) \end{aligned}$$

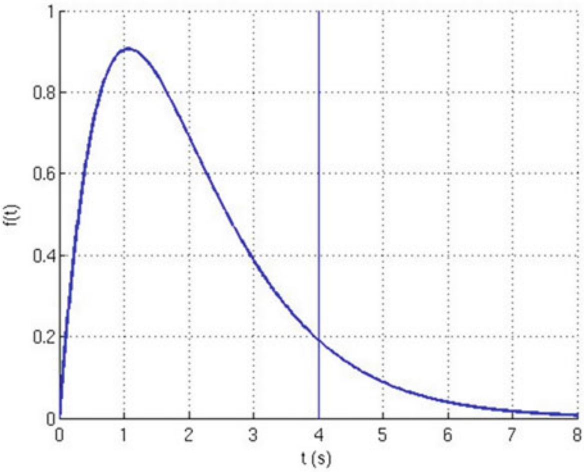
This gives us three simultaneous equations:

$$\begin{aligned} &a_1 + a_3 = 0 \\ &5a_1 + a_2 + 2a_3 - 2 = 0 \\ &4a_1 + 4a_2 + a_3 - 10 = 0, \end{aligned}$$

which are then solved for the constants $a_1 = -\frac{2}{9}$, $a_2 = \frac{8}{3}$, and $a_3 = \frac{2}{9}$. The inverse Laplace transform is then applied to determine the time domain solution.

$$f(t) = -\frac{2}{9}e^{-t} + \frac{8}{3}te^{-t} + \frac{2}{9}e^{-4t}$$

A plot of this function is provided.



Notice that, even though the two time constants for the exponentials are 0.25 and 1, the function still has a significant value at a time of 4(1) = 4 s (solid vertical line). This is the effect of the repeated root, which results in the term te^{-t}. For this term the multiplicative factor t reduces the rate of decay; therefore, our time constant estimation is not strictly applicable.

2.7.4 Special Case That Often Occurs with Step Inputs to Systems

Another common situation is Laplace domain functions with a third order in s; this occurs when a system is driven by a step input. In this case, the denominator can first be subdivided by the partial fraction expansion shown in Eq. (2.26). Here, $p(s)$ is a polynomial of order two or less.

$$F(s) = \frac{p(s)}{s(s^2 + bs + c)} = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2 + bs + c} \tag{2.26}$$

Example 2.15 Determine the inverse Laplace transform of the function $F(s) = \frac{s+3}{s(s^2+4s+20)}$.

Solution We begin by attempting the expansion suggested by Eq. (2.26).

$$\frac{s+3}{s(s^2+4s+20)} = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2+4s+20}$$

We multiply both sides by the left-hand side denominator, expand, and separate powers of s.

$$s+3 = a_1(s^2+4s+20) + (a_2 s + a_3)s$$
$$(a_1+a_2)s^2 + (4a_1 + a_3 - 1)s + (20a_1 - 3) = 0$$

Setting the coefficients of the different powers of s equal to zero and solving the three simultaneous equations results in $a_1 = \frac{3}{20}$, $a_2 = -\frac{3}{20}$, and $a_3 = \frac{2}{5}$. Substituting these coefficients into the expansion results in the following expression.

$$F(s) = \frac{3}{20}\left(\frac{1}{s} - \frac{s - \frac{8}{3}}{s^2+4s+20}\right)$$

Next, we recognize that the second term has two complex conjugate poles (see Example 2.13). Completing the square gives the following expression, where we have divided the numerator into two parts. In the final step, we extract the constant $\frac{7}{6}$ to leave 4 in the numerator for the third term. These modifications result in expressions that appear in Table 2.1.

$$F(s) = \frac{3}{20}\left(\frac{1}{s} - \frac{s+2-\frac{14}{3}}{(s+2)^2+4^2}\right) = \frac{3}{20}\left(\frac{1}{s} - \frac{s+2}{(s+2)^2+4^2} + \frac{\frac{14}{3}}{(s+2)^2+4^2}\right)$$
$$= \frac{3}{20}\left(\frac{1}{s} - \frac{s+2}{(s+2)^2+4^2} + \frac{7}{6}\frac{4}{(s+2)^2+4^2}\right)$$

Now we can invert the expression using Table 2.1 and determine the time domain result

$$f(t) = \frac{3}{20}\left(u(t) - \mathrm{e}^{-2t}\cos(4t) + \frac{7}{6}\mathrm{e}^{-2t}\sin(4t)\right)$$

Notice that even after the two exponential terms have decayed to zero, $f(t)$ retains a constant value of $\frac{3}{20}$. We confirm this result by applying the final value theorem.

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}(sF(s)) = \lim_{s\to 0}\left(\frac{3}{20}\left(\frac{s}{s} - \frac{s(s+2)}{(s+2)^2+4^2} + \frac{7}{6}\frac{4s}{(s+2)^2+4^2}\right)\right)$$
$$= \frac{3}{20}\left(1 - \frac{0(0+2)}{(0+2)^2+4^2} + \frac{7}{6}\frac{4\cdot 0}{(0+2)^2+4^2}\right) = \frac{3}{20}$$

2.8 Using MATLAB® to Find Laplace and Inverse Laplace Transforms

MATLAB® can also be used to obtain the Laplace and inverse Laplace transform of a function using the commands `laplace` and `ilaplace`. When we apply these commands, we first must tell MATLAB® that it is working with algebraic symbols and not numeric quantities. The command `syms` is used to do this. The quantities that follow the commands `syms` are treated as algebraic by MATLAB®. In the following example, the command `diff` is used to differentiate $x(t)$, the solution to Example 2.13, to obtain $\dot{x}(t)$ by applying the product rule.

```
>>syms x t
>>x = 3/4*exp(-2*t)*sin(4*t);
>>dx_dt = diff(x)
dx_dt =
3*cos(4*t)*exp(-2*t) - (3*sin(4*t)*exp(-2*t))/2
```

Next, we use the command `laplace` to find the Laplace transform of $x(t)$ and we see that this does indeed match with $F(s)$ from Example 2.13 after we completed the square.

```
>>syms x t X
>>x = 3/4*exp(-2*t)*sin(4*t);
>>X = laplace(x)
X =
3/((s + 2)^2 + 16)
```

Now consider Example 2.14. The inverse Laplace transform is easily determined using MATLAB®.

```
>>syms F s f t
>>F = (2*s+10)/((s+1)^2*(s+4));
>>f = ilaplace(F)
f =
(2*exp(-4*t))/9 - (2*exp(-t))/9 + (8*t*exp(-t))/3
```

We will now demonstrate how to use the symbolic capability of MATLAB® to obtain the solution of linear differential equations using Laplace transforms.

2.9 Solving Differential Equations Using Laplace Transforms

As we stated previously, many dynamic systems may be modeled using linear differential equations. Therefore, our primary motivation for learning Laplace transforms is to analyze the behavior of these models. In this section, we demonstrate the solution of linear differential equations using Laplace transforms.

Example 2.16 Solve the differential equation $3\ddot{x} + 12\dot{x} + 60x = 0$ assuming the initial conditions $x(0) = 0$ and $\dot{x}(0) = 3$.

Solution To solve, we convert from the time to Laplace domain using the Laplace transform.

$$3\left(s^2X(s) - sx(0) - \dot{x}(0)\right) + 12(sX(s) - x(0)) + 60X(s) = 0$$

Substituting the initial conditions and rearranging enables us to solve for $X(s)$.

$$\begin{aligned} &X(s)(3s^2 - 3s \cdot 0 + 12s + 60) - 3(3) - 12(0) = 0 \\ &X(s) = \frac{9}{3s^2 + 12s + 60} = \frac{3}{s^2 + 4s + 20} \end{aligned}$$

Converting back to the time domain yields the same result as Example 2.13.

$$x(t) = \frac{3}{4}\mathrm{e}^{-2t} \sin 4t$$

Example 2.17 Assuming $x(0) = 0$ and $\dot{x}(0) = 3$, solve the differential equation $3\ddot{x} + 18\dot{x} + 15x = 0$ for $x(t)$.

Solution We begin by eliminating the common factor of 3, converting to the Laplace domain using the Laplace transform, and applying the initial conditions.

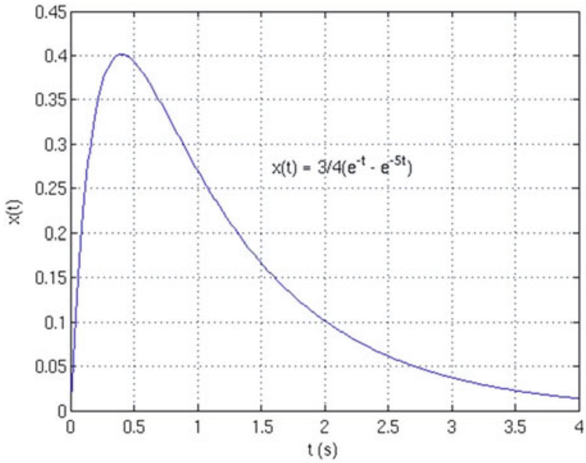
$$\begin{aligned} &s^2X(s) - sx(0) - \dot{x}(0) + 6(sX(s) - x(0)) + 5X(s) = 0 \\ &s^2X(s) - s \cdot 0 - 3 + 6sX(s) - 6 \cdot 0 + 5X(s) = 0 \end{aligned}$$

Finally, we solve for $X(s)$.

$$X(s) = \frac{3}{s^2 + 6s + 5}$$

```
>>syms X s x t
>>X = 3/(s^2+6*s+5);
>>x = ilaplace(X)
x =
(3*exp(-t))/4 - (3*exp(-5*t))/4
>>tau = 1;
>>t = [0:tau/100:4*tau];
>>plot(t, eval(x));
>>xlabel('t(s)')
>>ylabel('x(t)')
```

The symbolic result was: `x = (3*exp(-t))/4 - (3*exp(-5*t))/4`. In the `plot` command, we used `eval` to insert the value of $x(t)$ over a time span equal to four times the dominant (larger) time constant (`tau = 1`). We selected the time step to be `tau/100`.



Example 2.18 Solve the following differential equation $10\ddot{x} + 90\dot{x} + 200x = -400\, u(t)$ with zero initial conditions. Note the right-hand side input is a negative step with a magnitude of 400.

Solution First, we compute the Laplace transform and solve for $X(s)$. Recall that the Laplace transform of a unit step is $\frac{1}{s}$.

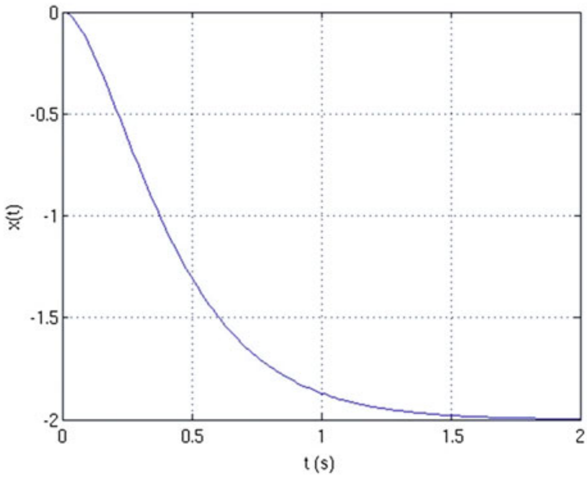
$$X(s) = \frac{-400}{s(10s^2 + 90s + 200)}$$

```
>>clear
>>syms X t s x
>>X = -400/(s*(10*s^2+90*s+200));
>>x = ilaplace(X)
x =
10/exp(4*t) - 8/exp(5*t) - 2
```

We can rearrange to obtain a form with which we are more familiar.

$$x(t) = 10e^{-4t} - 8e^{-5t} - 2$$

A plot of this solution is provided. Note that the final value approaches –2.



Problems

1. Solve the following equations and plot your answers in the complex plane with the real part along the horizontal axis and the imaginary part along the vertical axis.
   (a) $3s^2 + 21s + 36 = 0$
   (b) $s^2 + 4s + 29 = 0$
   (c) $3s^2 + 75 = 0$
   (d) $s^3 + 7s^2 + 32s + 60 = 0$

2. Solve the following equations and plot your answers in the complex plane with the real part along the horizontal axis and the imaginary part along the vertical axis.
   (a) $s^2 + 5s + 4 = 0$
   (b) $s^2 + 6s + 13 = 0$
   (c) $s^2 + 20 = 0$
   (d) $s^3 + 7s^2 + 24s + 18 = 0$

3. Type the following lines at the MATLAB® command prompt.

```
>> % f(t) = e^-(5t)*cos(20*t)
>> a = 5;
>> tau = 1/a;
>> w = 20*pi;
>> t = [0:tau/100:8*tau];
>> f = exp(-a*t).*cos(w*t);
>> plot(t, f)
>> xlabel('t(s)')
>> ylabel('f(t)')
```

Suppose the plot depicts position, x, versus time, t, of a mechanical system. The response in the plot represents a *damped oscillation*. What is the *frequency* in Hertz (cycles/second) of the oscillations you see in the plot? How long does it take the oscillations to decay to within approximately 2% of 0 (the final value)?

4. Use integration by parts to determine the Laplace transform of the following function.

$$f(t) = \begin{cases} 0 & t < 0 \\ t^2 e^{-4t} & t \geq 0 \end{cases}$$

5. Use integration by parts to determine the Laplace transform of the following function. Check your answer against the Laplace transform table.

$$f(t) = \begin{cases} 0 & t < 0 \\ \sin(2t) & t \geq 0 \end{cases}$$

6. Calculate the Laplace transform of the following functions using the Laplace transform table.
   (a) e^{-10t}
   (b) $e^{-10t}\cos(4t)$
   (c) $e^{-5t}e^{4t}$
   (d) $e^{-10t}\sin(2t)$

7. Calculate the Laplace transform of the following functions using the Laplace transform table.
   (a) e^{-2t}
   (b) $e^{-2t}\cos(16t)$
   (c) te^{-2t}
   (d) $e^{-2t}\sin(16t) + e^{-2t}\cos(16t)$

8. Calculate the inverse Laplace transforms of the following functions both analytically and using the MATLAB® command `ilaplace`.
   (a) $F_1(s) = \frac{s+1}{(s+3)(s+4)}$
   (b) $F_2(s) = \frac{s}{s^2+8s+52}$
   (c) $F_3(s) = \frac{52}{s(s^2+8s+52)}$
   (d) $F_4(s) = \frac{6}{(s+2)(s+1)^2}$

9. Solve the following differential equations using Laplace transforms. Substitute your answers into the original equations to verify them.
   (a) $\ddot{x} + 36x = 0,\quad x(0) = 2,\ \dot{x}(0) = 0$
   (b) $\ddot{x} + 10\dot{x} + 41x = \mathrm{u}(t)\quad x(0) = 0;\ \dot{x}(0) = 1$
   (c) $\dot{y} + 10y = t,\quad y(0) = 0$
   (d) $\ddot{y} + 4\dot{y} + 10y = 0,\quad y(0) = 2,\ \dot{y}(0) = 0$

10. Solve the following differential equations using Laplace transforms. Substitute your answers into the original equations to verify them.
    (a) $\dot{z} + 2z = \mathrm{e}^{-3t}$, $z(0) = 0$
    (b) $2\ddot{z} + 162z = 0$, $z(0) = 1,\ \dot{z}(0) = 0$
    (c) $3\ddot{x} + 12\dot{x} + 60x = \delta(t)$, $x(0) = 0;\ \dot{x}(0) = 0$
    (d) $\ddot{x} + 10\dot{x} + 25x = 0$, $x(0) = 1;\ \dot{x}(0) = 0$

11. Solve $\dot{w} + 10w = 10 \cdot \mathrm{u}(t)$ with initial condition $w(0) = 0$ for $w(t)$ using Laplace transforms. Also, apply the initial and final value theorems to $W(s)$ and show that the results match your solution.

12. Solve $\ddot{x} + 10\dot{x} + 25x = 0$ with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$ for $x(t)$ using Laplace transforms. Also, apply the initial and final value theorems to $X(s)$ and show that the results match your solution.

13. Solve $\ddot{x} + 10\dot{x} + 25x = 25 \cdot \mathrm{u}(t)$ with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ for $x(t)$ using Laplace transforms. Also, apply the initial and final value theorems to $X(s)$ and show that the results match your solution.

14. Solve $\ddot{x} + 4\dot{x} + 20x = 2 \cdot \delta(t)$ with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ for $x(t)$ using Laplace transforms. Also, apply the initial and final value theorems to $X(s)$. Explain the results.

References

1. Den Hartog JP (1956) Mechanical vibrations, 4th edn. McGraw-Hill Book Company, New York
2. Ogata K (1992) System dynamics, 2nd edn. Prentice Hall, Englewood Cliffs, NJ

Elements of Lumped Parameter Models 3

3.1 Introduction

In modeling systems for dynamic analysis, the modeler's goal is to determine a differential equation that adequately describes the system behavior without introducing unnecessary complication. A successful modeler identifies the critical elements of a system and then incorporates them into a *lumped parameter model*. For a mechanical example, consider an automobile chassis/body and its interactions with the road through its suspension. Figure 3.1a shows the front suspension for a 1924 Ford Model T. Although all elements of the dynamic system can deform elastically, have mass, and offer the potential to dissipate mechanical energy as heat, we recognize that certain elements are inherently more flexible while others have significantly more mass. We therefore *lump* the elements together into ideal masses/inertias, springs, and energy loss elements (dampers) so that we can realistically analyze the system using a simplified model. Through analysis of this model, we identify the most important system dynamics.

For example, we might model an automobile in a single plane (Fig. 3.1b shows a side view) allowing vertical translational and in-plane rotational motions (two degrees of freedom). For modern suspension systems, we may treat the suspension springs as ideal elastic elements and the shock absorbers as ideal dampers. The body of the car could be treated as an ideal inertia that can undergo both linear and angular acceleration. This model maintains the ability of the car body to move up and down in response to the road surface and pitch around an axis perpendicular to the direction of motion. A further simplification is shown in Fig. 3.1c where all of the springs and dampers have been combined together and we only examine the up/down motion of the car body in response to the road (single degree of freedom). Lumped parameter modeling is a powerful tool for examining the most relevant system dynamics (e.g., the bulk vibration behavior of an automobile suspension) while leaving some of the less critical behavior for a more detailed analysis.

M.A. Davies, T.L. Schmitz, *System Dynamics for Mechanical Engineers*,
DOI 10.1007/978-1-4614-9293-1_3

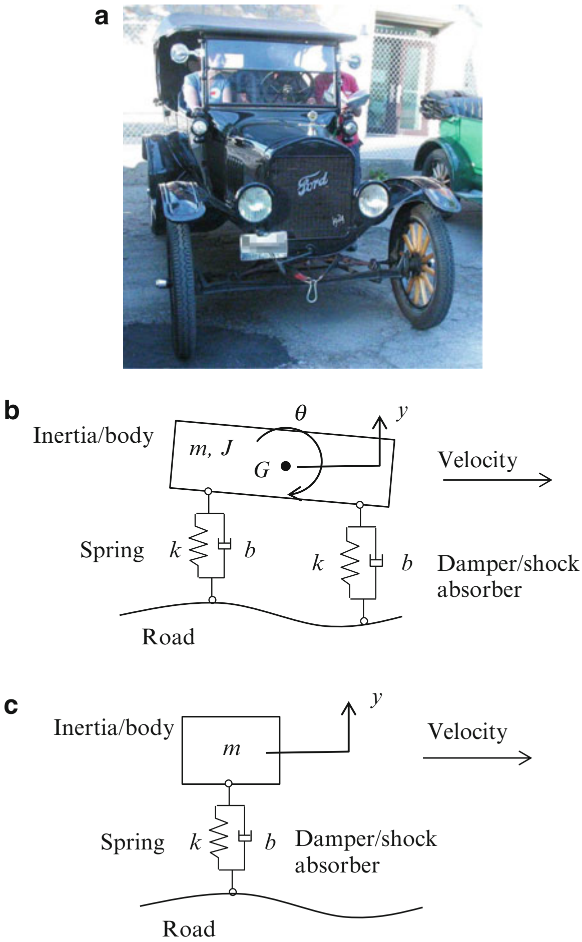


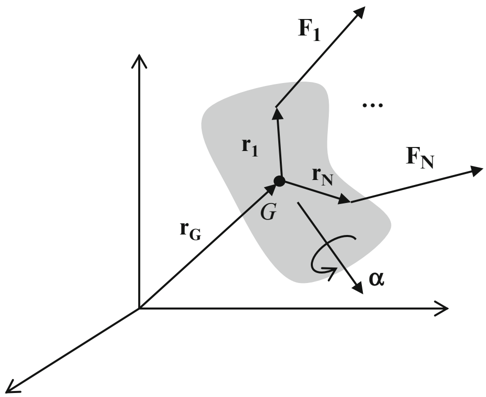
Fig. 3.1 (**a**) 1924 Ford Model T suspension [1] and (**b**, **c**) two simplified lumped-parameter models

3.2 Inertial Elements

The ideal inertial element is a rigid body. The ideal rigid body [2] shown in Fig. 3.2 has six degrees of freedom that are most often described by three independent orthogonal linear displacements of its center of mass, G, and three rotations (e.g., the Euler angles[1] [3]). When Newton's laws are applied to a rigid body using a coordinate system with its origin at G, the equations of motion simplify so that the center of mass acceleration can be described independently of the angular accelerations. In this coordinate frame, the linear acceleration of G is given by:

[1] Euler angles describe the three angles that decompose any three-dimensional rotation of a body.

Fig. 3.2 Rigid body moving under the influence of applied forces $\mathbf{F_1}$ through $\mathbf{F_N}$



$$\sum_{i=1}^{N} \mathbf{F_i} = m\mathbf{a} = m\frac{d^2\mathbf{r_G}}{dt^2} \tag{3.1}$$

where $\mathbf{r_G}$ is the vector position of the center of mass of the body, m is the mass, $\mathbf{a}$ is the acceleration vector, and $\mathbf{F_i}$ is a vector force applied to the body (N total forces). The angular acceleration, $\boldsymbol{\alpha}$, of the body is given by Eq. (3.2).

$$\sum \mathbf{M_G} = \sum_{i=1}^{N} \mathbf{r_i} \times \mathbf{F_i} = \mathbf{J_G}\boldsymbol{\alpha} \tag{3.2}$$

In Eq. (3.2), $\mathbf{J_G}$ is the mass moment of inertia taken about the center of mass, $\mathbf{r_i}$ is a vector from G to the location of the application of $\mathbf{F_i}$, and $\mathbf{M_G}$ is a vector moment calculated from the cross product of $\mathbf{r_i}$ and $\mathbf{F_i}$.

Equations (3.1) and (3.2) describe the general motion of any rigid body in space and may exhibit complex behavior [3] (e.g., gyroscope systems). However, for the purposes of this book, the degrees of freedom are typically restricted to one translational and/or one rotational degree of freedom. This is sufficient to describe many real systems as exemplified by Fig. 3.1 and, in practice, coordinate transformations often enable the treatment of individual degrees of freedom separately.

For the purpose of this book, we consider an ideal mass and an ideal rotational inertia as shown in Fig. 3.3. The mass, m, is restricted to move in the horizontal direction, x. Symbolically, this is represented by ideal "frictionless" wheels that restrict the mass to motion to the horizontal direction (visualize a piston moving in a cylinder). The direction of the motion may vary from system to system; for example, we will often consider systems that move vertically under the influence of gravity. The equation describing the motion of the mass with a single degree of freedom can be obtained from Newton's second law:

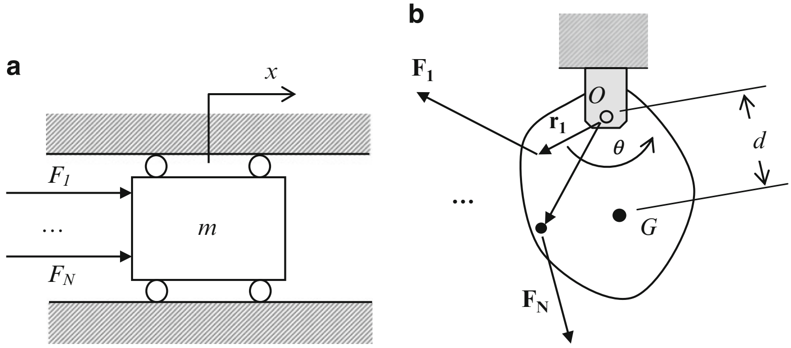


Fig. 3.3 Ideal (**a**) mass m and (**b**) rotational inertia J_O, each restricted to a single degree of freedom of motion

$$\sum_{i=1}^{N} F_i = ma = m\ddot{x} \tag{3.3}$$

where $\ddot{x} = \frac{d^2x}{dt^2}$ is the mass (translational) acceleration, a. Alternatively, we could define the velocity, $v = \dot{x}$, and rewrite Newton's law in terms of velocity. Depending on the other elements in the system, using position or velocity may be more appropriate or convenient as we will see in Chap. 4.

$$\sum_{i=1}^{N} F_i = m\dot{v} \tag{3.4}$$

The ideal rotational inertia is treated in a similar manner. The pinned connection in Fig. 3.2b with its center at point O specifies that the rigid body is restricted to rotate about an axis perpendicular to the page which passes through O. In this specific case of a fixed rotation point, Newton's equations describing the rotation of the rigid body reduce to:

$$\sum_{i=1}^{N} M_i = J_O\ddot{\theta} = J_O\alpha \tag{3.5}$$

where $\ddot{\theta} = \frac{d^2\theta}{dt^2}$ is the angular acceleration, also identified as α, of the inertia about the axis of rotation and J_O is the mass moment of inertia about the axis of rotation. If all of the forces and their locations are identified, then the sum of the moments can be defined using the cross product of the position vector for each force as shown in Eq. (3.2). If there are any "pure" moments (or couples) applied to the body, they are also included in the sum on the left side of Eq. (3.5). Typically, we may know the moment of inertia about G, rather than O. When this is the case, J_O can be determined from the parallel axis theorem:

$$J_O = J_G + md^2 \tag{3.6}$$

where m is the mass of the body and d is the perpendicular distance between parallel axes oriented perpendicular to the page that pass through O and G, respectively.

As with the ideal mass example, we can define the angular velocity, $\omega = \dot{\theta}$, and rewrite Newton's law in terms of the angular velocity:

$$\sum_{i=1}^{N} M_i = J_O \dot{\omega}, \tag{3.7}$$

which can provide a convenient variable in some problems.

The key idea for inertial elements is that they respond to either applied forces or moments with a proportional acceleration, or rate of change of the linear or angular velocity. As we develop models for mechanical systems in Chap. 4 and, as we develop analogous equations for other seemingly dissimilar types of dynamic systems such as electric circuits, this behavior will be shown to be very important.

Inertial elements store mechanical energy as kinetic energy. In the case of the rectilinear (translating) motion of a mass, the kinetic energy, KE, is:

$$KE = \frac{1}{2} m \dot{x}^2. \tag{3.8}$$

Note that the SI units are kg-(m/s)^2, or N-m (J). Similarly, for a rotational inertia, the kinetic energy is proportional to the angular velocity squared.

$$KE = \frac{1}{2} J_O \dot{\theta}^2 \tag{3.9}$$

The SI units for Eq. (3.9) are kg-m^2-(rad/s)^2, or N-m. Pure masses/rotational inertias are conservative; they do not dissipate the energy of mechanical motions.

3.3 Linear Spring Elements

A linear spring is an idealized elastic element that produces a resistance force that is proportional to relative linear displacement between its two ends (tension and compression). Figures 3.4a, b show a column of elastic material and a cantilever beam, each of which approximately produce a force linearly proportional to the displacement, x, as shown. The proportionality constant between the displacement and the force is the stiffness, k. The stiffness is a function of the other physical variables for the particular element as shown in the figure. For the column, it is proportional to the product of the elastic modulus (Young's modulus), E, and the cross-sectional area, A, and inversely proportional to the column length, L. For the cantilever, the stiffness is linearly proportional to the elastic modulus and the second moment of area, J_A, and inversely proportional to the cube of the length, L.

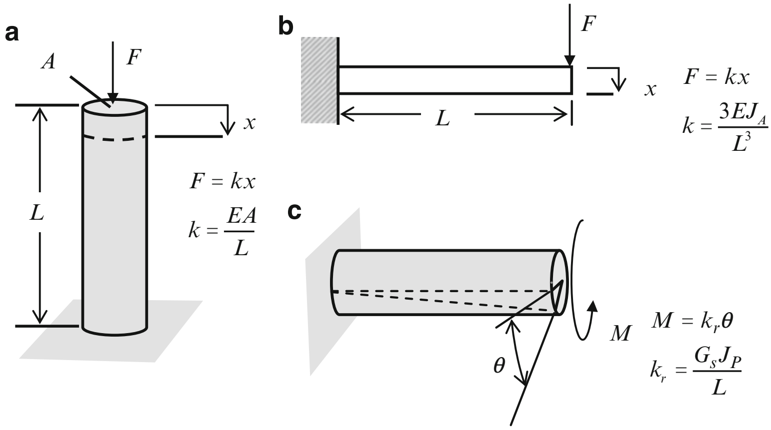


Fig. 3.4 Example spring elements

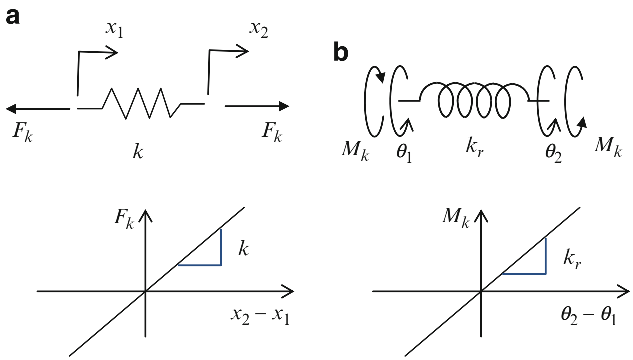


Fig. 3.5 Ideal springs: (**a**) linear and (**b**) torsional

A torsional spring is another idealized element that produces a resistance moment in response to some relative twist between its two ends. The constant of proportionality (stiffness) is denoted k_r. A common example is a cantilever shaft, as shown in Fig. 3.4c. In this case, k_r is proportional to the product of shear modulus for the shaft material, G_s, and the polar moment of inertia of the cross-section, J_P, and inversely proportional to its length.

In system models, we represent linear and torsional springs with the symbols shown in Fig. 3.5. Fig. 3.5a shows a linear spring and the linear dependence of the applied force on the relative displacement of the two ends.

$$F_k = k(x_2 - x_1) = k\Delta x \tag{3.10}$$

Similarly Fig. 3.5b shows a torsional spring and the linear relationship between the applied moment and the relative angular displacement between the two ends.

$$M_k = k_r(\theta_2 - \theta_1) = k_r\Delta\theta \tag{3.11}$$

The spring force in a dynamic system tends to restore the system to its equilibrium position. Linear springs store mechanical potential energy, U_e, according to Eq. (3.12), which is derived from the total work done to displace the spring.

$$U_e = \frac{1}{2}k(\Delta x)^2 \tag{3.12}$$

Similarly, torsional springs store mechanical energy according to Eq. (3.13).

$$U_e = \frac{1}{2}k_r(\Delta\theta)^2 \tag{3.13}$$

Springs are conservative; they do not remove energy from mechanical motions.

3.4 Linear Damping Elements

Masses/inertias and springs store energy, but do not dissipate it. Thus, if a spring and mass exclusively form a system, they will trade energy back and forth forever and the motion of the system will never stop. Our experience shows that this is not the case for physical systems. The mechanical energy is always dissipated, usually in the form of heat. The dissipation mechanism in a mechanical system is typically some form of friction: either dry friction or viscous friction. Dry friction is a complex nonlinear force that is not completely understood and is difficult to model. Fortunately, because we seek to minimize dry friction and the associated wear in the design of mechanical systems, viscous (velocity-dependent) friction is typically a good representation of reality.

Viscous friction is modeled by a velocity-, or rate-, dependent force. The rationale for this can be understood from Fig. 3.6, which depicts a block sliding over a surface with a thin viscous layer of fluid (such as oil or other lubricant) separating the two. If a no slip condition is assumed between the block and the fluid layer, and the fluid is assumed to be Newtonian and shears evenly as shown, then the resistance force encountered by the block as it slides is proportional to the velocity.

$$F = b\dot{x},\ b = \frac{\mu A}{l} \tag{3.14}$$

In Eq. (3.14), b is the damping factor. In this case, it is proportional to the fluid viscosity, μ, and the exposed area of the block, A, and inversely proportional to the fluid film thickness, l. The SI units of b are N-s/m.

Another element that produces a velocity-dependent force is a shock absorber; see Fig. 3.7a. A shock absorber typically contains a piston surrounded by oil in a chamber. As the piston moves, the oil is forced through small holes in the piston from one side to the other. The faster the piston is displaced, the greater the

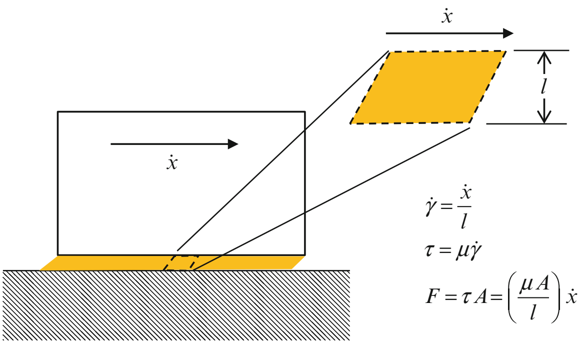


Fig. 3.6 A block sliding over a surface with a fluid film produces a resistance force, F, proportional to the block velocity, $\dot{x}$. In the figure, l is the film thickness, $\dot{\gamma}$ is the average strain rate in the fluid, τ is the shear stress, A is the cross-sectional area of the contact zone, and μ is the fluid viscosity

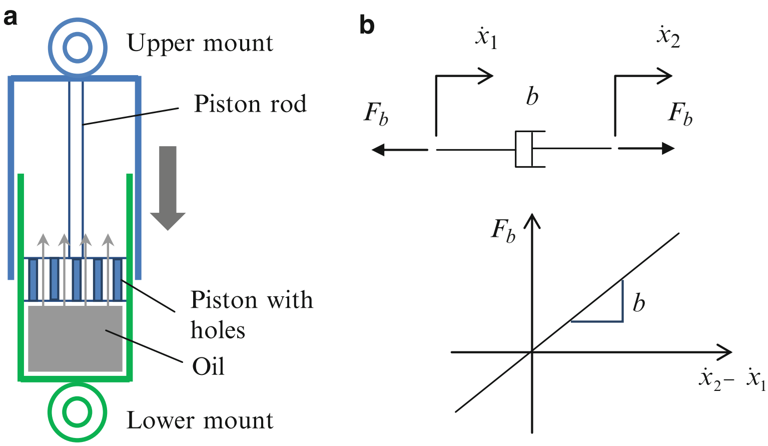


Fig. 3.7 (**a**) In a shock absorber, oil is forced through holes in the piston as it moves and produces a force proportional to the piston velocity. (**b**) The symbol used to represent an ideal linear damper is shown. The damping force is proportional to the relative velocity between the two ends

resistance force. Shock absorbers have some amount of nonlinearity. However, for small motion, this nonlinearity can often be ignored and the damping force can be approximated by a force that is linearly dependent on the piston velocity; see Eq. (3.15) and Fig. 3.7b.

$$F_b = b(\dot{x}_2 - \dot{x}_1) \tag{3.15}$$

The SI units of b are again N-s/m. While we have described the role of fluid layers in producing velocity-dependent forces, it is also possible to model energy loss in mechanical systems arising from other sources with an ideal linear damper.

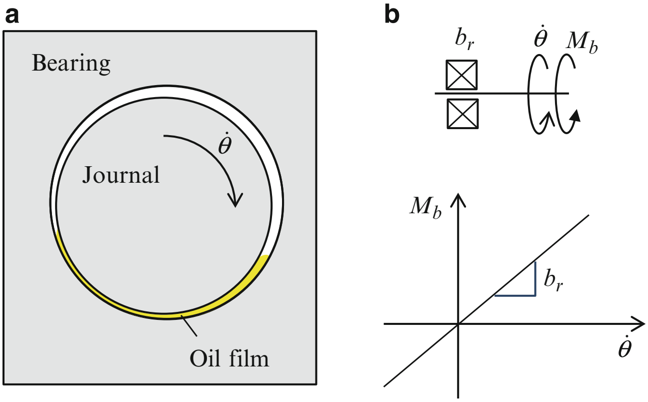


Fig. 3.8 (**a**) Journal bearing in which the journal rotates with an angular velocity, $\dot{\theta}$, within the bearing. The oil film separating the journal and bearing produces a shear stress that generates a resistance moment, M_b. (**b**) The symbol for an ideal bearing is shown. A plot of the applied moment, M_b, which is proportional to the angular speed, $\dot{\theta}$, using the proportionality constant, b_r, is also provided

Energy loss or damping in rotary systems occurs when there is a moment produced by motion that depends on the angular velocity. Physically, the mechanism is similar to linear damping. A journal bearing is shown in Fig. 3.8a. It consists of a cylindrical journal (shaft) rotating in a bearing housing. A hydrostatic oil film between the shaft and housing prevents solid/solid contact and the associated dry friction and wear. The shear strain rate in the film is proportional to the rotational velocity in a manner similar to Fig. 3.6. The shear stress increases with strain rate and leads to an overall moment on the shaft that is approximately proportional to the rotation rate. Figure 3.8b displays the symbol we will use in system diagrams for an ideal bearing and the linear relationship between the applied moment, M_b, and the angular rotation rate, $\dot{\theta}$ (see Eq. 3.16). The linearity of this relationship, while never exactly satisfied in real systems, often provides an acceptable approximation for analysis and design.

$$M_b = b_r\dot{\theta} \tag{3.16}$$

A fluid coupling, or torque converter, is another type of rotary element that generates a moment using fluid motion. As shown in Fig. 3.9a, rotation of the left (driving) impeller causes fluid coupling that drives the right (driven) impeller and transmits the moment without any direct mechanical connection. Similar to the bearing analysis, the moment transferred through the coupling is linearly proportional to the relative angular velocity of the driving and driven shafts. Because the moment is small for low rotational speeds, fluid couplings allow slip at low speeds. At high speeds, however, the coupling approximates a solid connection. The symbol and moment-speed relationship is displayed in Fig. 3.9b.

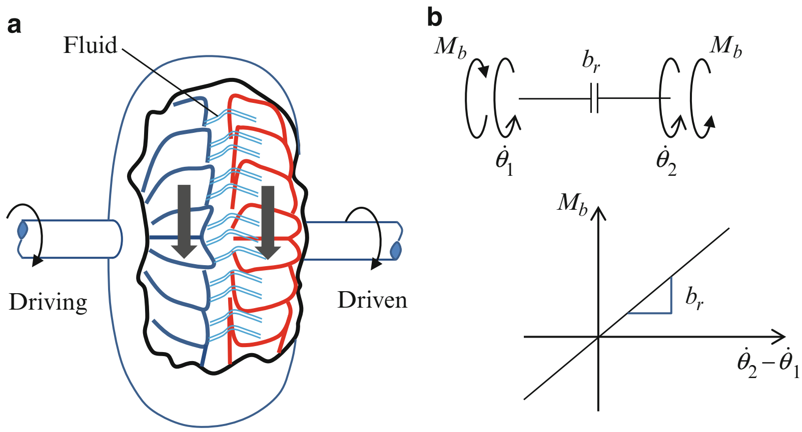


Fig. 3.9 (**a**) A fluid coupling with a driving impeller (*left*) that pumps the fluid which then couples to the driven impeller (*right*), thus transmitting a moment without any direct mechanical connection. (**b**) The symbol used for an ideal fluid coupling where the applied moment is proportional to the relative angular velocity between the two ends is shown

$$M_b = b_r\left(\dot{\theta}_2 - \dot{\theta}_1\right) \tag{3.17}$$

In the mechanical systems we will examine in Chap. 4, dynamic motions represent a trading of kinetic energy from a mass/inertia into potential energy of a spring with the damping elements dissipating the energy of motion as heat.

3.5 Combinations of Springs and Dampers

In many situations, several ideal spring or damper elements are needed to accurately model a dynamic system. For example, in an automotive suspension there are four main springs and dampers (shock absorbers). In these situations, it is often advantageous to combine the elements into a single, equivalent element. There are two ways springs and dampers may be combined, in parallel or in series, as shown in Fig. 3.10.

Each series or parallel arrangement can be combined to form an equivalent single element. The analysis begins by examining the free body diagram of the connection points and assuming that the system is in static equilibrium. For parallel arrangements, the relevant free body diagrams are provided in Fig. 3.11. Examining free body diagram in Fig. 3.11b, we apply Newton's second law of motion and assume the mass of the free body is negligible. The force summation is given by Eq. (3.18).

$$F_k - F_{k_1} - F_{k_2} = 0 \tag{3.18}$$

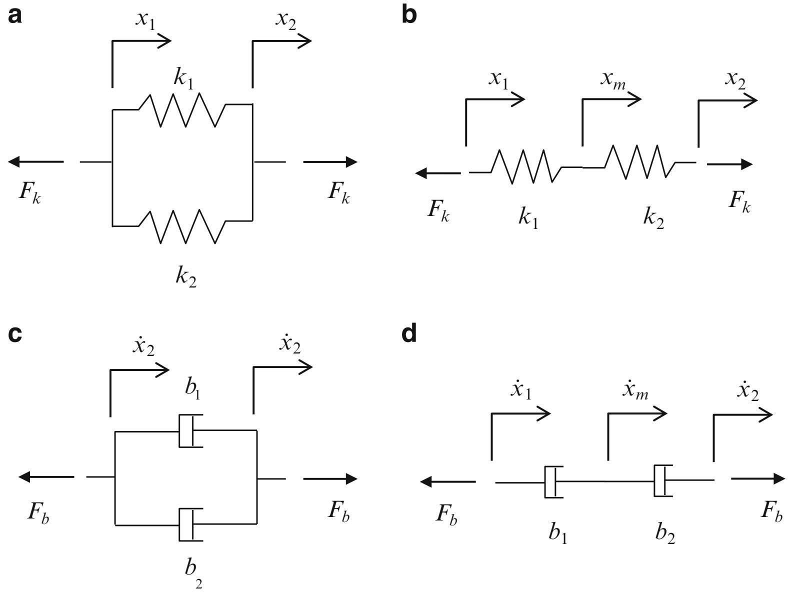


Fig. 3.10 (**a**) Springs in parallel; (**b**) springs in series; (**c**) dampers in parallel; and (**d**) dampers in series

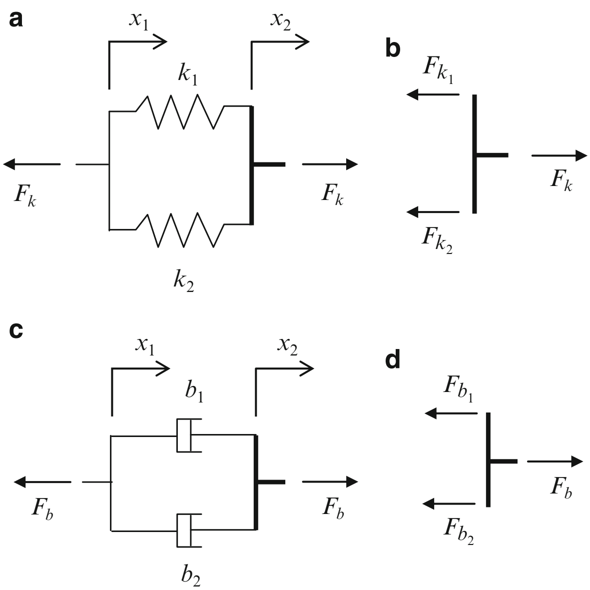


Fig. 3.11 Springs and dampers in parallel (*left* column) and the corresponding free body diagrams (*right* column)

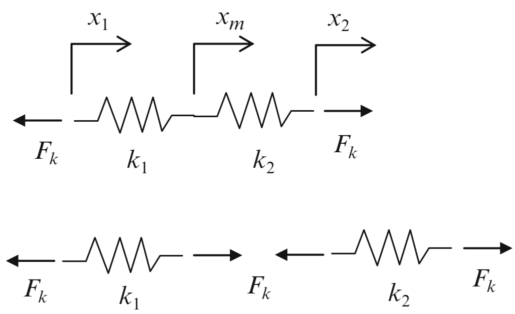


Fig. 3.12 When springs in series are separated, the force in each spring must be equal

This equation can be rearranged to determine the equivalent spring force, F_k.

$$F_k = F_{k_1} + F_{k_2} \tag{3.19}$$

Next, examining the two springs and the sign convention for the forces shown in the free body diagram, we notice that the springs will act in the direction shown when they are in tension. This occurs when $x_2 > x_1$ and, therefore, Eq. (3.19) can be rewritten in terms of the relevant coordinates.

$$F_k = k_1(x_2 - x_1) + k_2(x_2 - x_1) = (k_1 + k_2)(x_2 - x_1) \tag{3.20}$$

Comparing this result to Eq. (3.10), it is evident that the two parallel springs combine into an *equivalent* spring stiffness, k_{eq}, that is the sum of the individual stiffnesses. The same approach can be used to combine any number of springs in a parallel arrangement.

$$k_{eq} = k_1 + k_2 \tag{3.21}$$

A similar analysis can be performed for the free body diagram in Fig. 3.11d and the result is that two dampers in parallel may also be replaced by an equivalent damper with a damping constant equal to the sum of the two parallel dampers. Again, any number of parallel dampers can be combined in this way.

$$b_{eq} = b_1 + b_2 \tag{3.22}$$

The physical explanation for Eqs. (3.21) and (3.22) is that the springs or dampers act together to oppose an applied force and, therefore, the combined stiffness or damping is greater than that of one spring or damper alone. In fact, it is equal to the sum of the individual spring/damper constants.

Figure 3.12 shows two springs in series. The force must be equal in each spring when the two springs are (artificially) separated, so we obtain Eq. (3.23).

$$F_k = k_1(x_m - x_1) = k_2(x_2 - x_m) \tag{3.23}$$

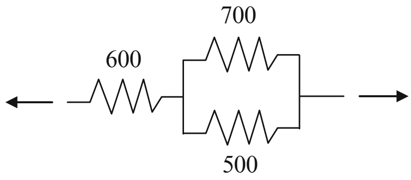


Fig. 3.13 Spring combination (the spring stiffnesses shown are in units of N/m)

This expression can be solved for x_m.

$$x_m = \frac{k_1}{k_1 + k_2} x_1 + \frac{k_2}{k_1 + k_2} x_2 \tag{3.24}$$

Equation (3.24) can then be combined with Eq. (3.23) to obtain an expression for the spring force in terms of the displacement difference between the two ends of the combination.

$$F_k = \left(\frac{k_1 k_2}{k_1 + k_2} \right) (x_2 - x_1) \tag{3.25}$$

From Eq. (3.25) we find that the equivalent spring stiffness for two springs in series is:

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2} = \left(\frac{1}{k_1} + \frac{1}{k_2} \right)^{-1}. \tag{3.26}$$

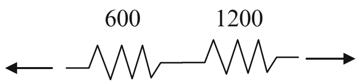
Using a similar analysis, the equivalent damping constant of two dampers in series is:

$$b_{eq} = \frac{b_1 b_2}{b_1 + b_2} = \left(\frac{1}{b_1} + \frac{1}{b_2} \right)^{-1}. \tag{3.27}$$

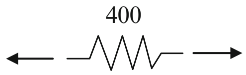
By application of these expressions, any arrangement of linear springs or linear dampers may be combined to determine the equivalent spring or damper that can replace them.

Example 3.1 A set of springs is arranged as shown in Fig. 3.13. Find the equivalent spring to replace the combination.

Solution First we combine the two parallel springs on the right side of the diagram by taking the sum of the individual stiffnesses (Eq. 3.21), $700 + 500 = 1200$ N/m.



Next, we combine the two springs in series using Eq. (3.26) to obtain the total equivalent linear stiffness. This gives an equivalent spring with a stiffness of $\frac{600 \cdot 1200}{600 + 1200} = 400\ \mathrm{N/m}$.



Note that in the series combination, the 1200 N/m and 600 N/m springs combine to provide a spring that is less stiff than either of the two original springs alone.

3.6 Transmission Elements

An ideal transmission element transforms one type of motion/force/moment[2] into another without a loss of power. That is, in an ideal transmission element, there is no loss of energy. The word transmission immediately brings to mind an automotive transmission which uses gears to change one angular rotation rate and moment into another angular rotation rate and moment with the goal of maximizing the transmission of power from the engine to the automobile motion. While gears represent one type of transmission element, the term is more broadly applicable and can include, for example:

- levers
- pulleys
- belts
- rollers or wheels
- rack and pinion systems
- leadscrews
- ballscrews.

Typically, a transmission element will have an input side and an output side and the motion from input to output is related, assuming no losses, in one of two ways.

1. Geometric constraint: For a transmission element, there will be a relationship between the geometry of the motion at the input to the geometry of the motion at the output.
2. Energy conservation constraint: For an ideal transmission element, the power input to the element will be equal to the power output from the element.

The geometric constraint must hold if the transmission element is operating properly. For example, the speed of the motion at the interface between two meshing gears must be the same or the teeth will be sheared off the gears. The energy conservation constraint holds if the transmission element does not have

[2] The terms "moment" and "torque" are often used interchangeably. We will use the term "moment" for consistency in notation.

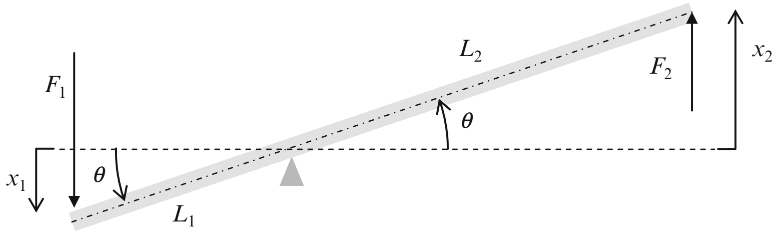


Fig. 3.14 Lever with input force and displacement, F_1 and x_1, and output force and displacement, F_2 and x_2. The lengths of the lever arms to the fulcrum are L_1 and L_2, respectively, and the angle of rotation is θ

significant energy dissipation. This is typically approximately true for a good transmission element because it is designed to transmit as much of the input power to the output as possible. Of course, neither constraint is strictly true in reality, but, if deviations are small, these can be incorporated into other ideal elements in the system model. For example, if a gear train is only 80% efficient, the energy loss can be incorporated into an ideal rotational damper in the system. We will now consider some typical transmission elements that we encounter in mechanical systems and discuss the constraints for each.

3.6.1 Levers

A lever is shown in Fig. 3.14. A force F_1 is applied to the left end of the lever and causes a displacement x_1. At the right end of the lever this produces a corresponding force F_2 and displacement x_2. The lever rotates counterclockwise by an angle θ. The geometric relationship for the lever is based on the sine of this angle.

$$\sin\theta = \frac{x_1}{L_1} = \frac{x_2}{L_2} \tag{3.28}$$

This can be rewritten as equal ratios of the displacements and lengths.

$$\frac{x_1}{x_2} = \frac{L_1}{L_2} \tag{3.29}$$

In Eq. (3.29), $\frac{L_1}{L_2}$ is the *lever ratio* or *transmission ratio*. This describes the geometric constraint for the lever and must be satisfied if the lever is to remain rigid. If we assume no losses (due to friction at the fulcrum, for example), the work[3] done on one end of the lever is equal to the work done on the other end; see Eq. (3.30).

[3] The lever work is the product of the force and the distance over which it acts.

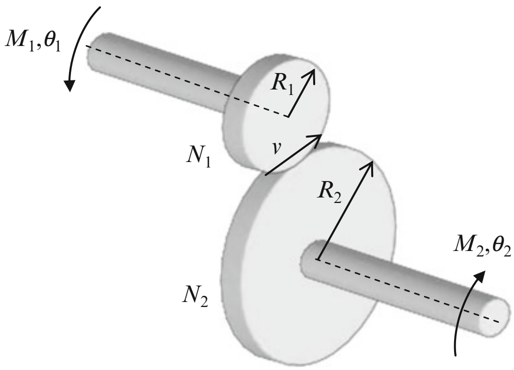


Fig. 3.15 Gears with input moment and angular displacement, M_1 and θ_1, and output moment and angular displacement, M_2 and θ_2. The input moment and rotation for the top gear are shown in the counterclockwise direction, so the bottom gear output moment and rotation must be in the clockwise direction. The radii of the gears are R_1 and R_2. The velocity of the gears at the point of contact is v. The teeth are not shown, but the number of teeth on the two gears are N_1 and N_2

$$F_1 x_1 = F_2 x_2 \tag{3.30}$$

Assuming constant force and taking a time derivative of Eq. (3.30), we can equate the power[4] input on one end of the lever, P_1, to the power output on the other end, P_2.

$$P_1 = F_1 \dot{x}_1 = F_2 \dot{x}_2 = P_2 \tag{3.31}$$

Rearranging Eq. (3.30) and using Eq. (3.29), we see that the *force ratio* is related to the inverse of the lever ratio.

$$\frac{F_1}{F_2} = \frac{x_2}{x_1} = \frac{L_2}{L_1} \tag{3.32}$$

The inversion of the lever ratio between Eqs (3.29) and (3.32) is important when we analyze the dynamics of systems and it is common to all transmission elements. For small angles, levers transform rectilinear motion into rectilinear motion such that the forces and displacements may change, but the energy is conserved.

3.6.2 Gears

Gears transmit moment, rotation, and power. To increase the ability to transmit high moment and power, gears have teeth to prevent slipping (as opposed to friction rollers, for example). Figure 3.15 shows two meshing gears. The input moment and

[4] The lever power is the product of the force and the velocity at which it is applied.

angular displacement are M_1 and θ_1, respectively, and output moment and angular displacement are M_2 and θ_2, respectively. When the gears rotate, the arc length on one gear that moves through the point of engagement must equal the arc length for the other gear. This is the essence of the geometric constraint imposed by gears and can be stated as shown in Eq. (3.33).

$$R_1\theta_1 = R_2\theta_2 \tag{3.33}$$

Differentiating Eq. (3.33) gives a more familiar form of the geometric constraint. Specifically, the velocity as seen on each gear at the point of engagement must be the same. If not, the teeth are slipping.

$$v = R_1\dot{\theta}_1 = R_2\dot{\theta}_2 \tag{3.34}$$

Equation (3.34) can be rearranged so that the ratio of the angular velocities of the two gears is equal to the ratio of the radii of the two gears. This relationship is referred to as the *gear ratio* or *transmission ratio*.

$$\frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{R_2}{R_1} \tag{3.35}$$

For an ideal gear pair, the power[5] input to the gears is equal to the power delivered by the gears. This power equality is listed in Eq. (3.36).

$$M_1\dot{\theta}_1 = M_2\dot{\theta}_2 \tag{3.36}$$

Equation (3.36) can be rearranged and combined with Eq. (3.35) to relate the *moment (or torque) ratio* to the gear ratio. Again, note the inversion of the gear ratio between Eqs. (3.35) and (3.37).

$$\frac{M_1}{M_2} = \frac{\dot{\theta}_2}{\dot{\theta}_1} = \frac{R_1}{R_2} \tag{3.37}$$

A common way to designate the gear ratio is to use the number of teeth rather than the gear radii. Since two meshing gears must have the same tooth spacing, Δ, then Eq. (3.35) can be rewritten in terms of the gear circumferences and, equivalently, as a function of the number of teeth on each gear. As seen in Eq. (3.38), the ratio of the gear radii is equal to the ratio of the number of teeth on each gear.

$$\frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{2\pi R_2}{2\pi R_1} = \frac{N_2\Delta}{N_1\Delta} = \frac{N_2}{N_1} \tag{3.38}$$

[5] For the rotating gears, the power is equal to the product of the moment and the angular velocity.

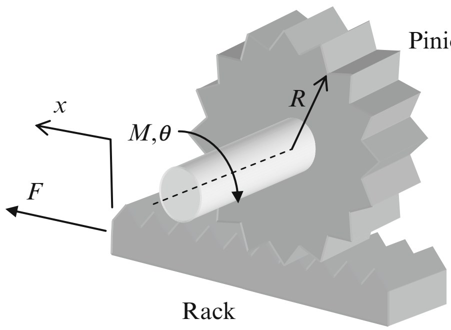


Fig. 3.16 Rack and pinion with input moment and angular displacement, M and θ, and output force and displacement, F and x. The pinion radius is R

Gears are the rotational analog of levers. Gears transform rotational motion into rotational motion such that the moments and angles may change, but the energy is conserved.

3.6.3 Rack and Pinion

A rack and pinion transforms rotational motion into rectilinear motion. The pinion is a rotary gear with teeth that engage the teeth on a linear rack. Geometrically, if the pinion rotates through an angle θ, then the resulting arc length, $R\theta$, must be equal to the linear movement of the rack, x.

$$R\theta = x \tag{3.39}$$

Differentiating Eq. (3.39) with respect to time and rearranging, we obtain an expression for the ratio of the angular and linear velocity.

$$\frac{\dot{x}}{\dot{\theta}} = R \tag{3.40}$$

In Eq. (3.40), the radius of the pinion is the *transmission ratio*. Equating the power delivered to the pinion to the power output by the rack, we obtain a second expression which relates the moment and force.

$$M\dot{\theta} = F\dot{x} \tag{3.41}$$

Rearranging and using Eq. (3.40), we obtain an expression that equates the ratio of force and moment to the inverse of the transmission ratio.

$$\frac{F}{M} = \frac{\dot{\theta}}{\dot{x}} = \frac{1}{R} \tag{3.42}$$

This gives the inversion of the transmission ratio as observed previously. Note that the relationships for a rack and pinion are the same as for a friction wheel driving a linear slide without slip. As with gears, the teeth enable the transmission of more power through the element than friction alone.

Example 3.2 A power of 10 kW is delivered to a gear with 90 teeth rotating at 2000 revolutions per minute (rpm). The first gear is engaged with a second gear having 15 teeth. Determine the (a) input moment; (b) output moment; and (c) output rotation rate (in rpm).

Solution First, we determine the input angular velocity in radians per second (rad/s).

$$\dot{\theta}_1 = 2000\,\frac{\text{rev}}{\text{min}}\cdot\frac{2\pi\,\text{rad}}{\text{rev}}\cdot\frac{1\text{ min}}{60\text{s}} = 209.4\,\text{rad/s}$$

Next, we find the input moment by dividing the input power by the angular speed.

$$M_1 = \frac{10000\ \ \text{W}}{209.4\ \text{rad/s}} = 47.7\ \text{N-m}$$

The output moment is determined by combining Eqs. (3.37) and (3.38) to find:

$$\frac{M_1}{M_2} = \frac{N_1}{N_2}$$

and rearranging.

$$M_2 = \frac{N_2}{N_1}M_1 = \frac{90}{15}(47.7) = 286.5\ \text{N-m}$$

Finally, we use Eq. (3.37) to find the output angular speed.

$$\dot{\theta}_2 = \frac{M_1\dot{\theta}_1}{M_2} = \frac{P_1}{M_2} = \frac{10000\ \ \text{W}}{286.5\ \ \text{N-m}} = 34.9\ \text{rad/s}$$

This is 333 rpm. Therefore, the gear train increases the moment, but delivers that moment at a decreased angular speed. This is a common arrangement for electric motors which typically develop their maximum power output at a relatively low moment, but high angular speed.

Example 3.3 It is desired to initially accelerate a mass of 2000 kg at 0.5 m/s^2. This is done through a rack and pinion drive. The pinion is attached to a motor with a maximum stall torque (or moment) of $M_{stall} = 50$ N-m, while the rack moves with the mass. Assuming there is no resistance to the motion of the rack/mass on the bearings (shown as two small wheels), what it the required pinion radius? (Fig. 3.17)

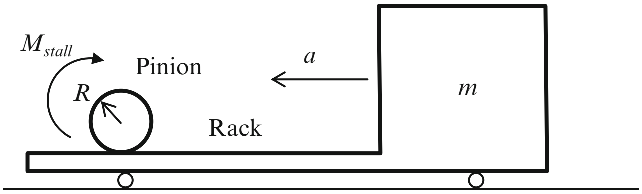


Fig. 3.17 Rack and pinion drive for a large mass, m. The pinion is fixed to a motor that provides the input moment, while the rack is attached to the mass and translates in the horizontal direction

Solution First, we determine the force required to give the desired initial acceleration of the mass using Eq. (3.3).

$$F = ma = (2000)(0.5) = 1000\,\mathrm{N}$$

Next, we use Eq. (3.42) and solve for the pinion radius assuming the stall torque (or moment) is applied by the motor initially.

$$R = \frac{M}{F} = \frac{50}{1000} = 0.05\,\mathrm{m}$$

Therefore, a pinion with a radius of 5 cm is required.

3.7 Summary

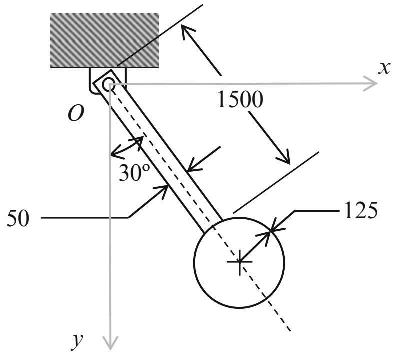
In this chapter, we introduced the basic lumped elements necessary to model the dynamics of mechanical systems. These include:

- linear and rotary inertia
- linear and rotary springs
- linear and rotary damping elements
- transmission elements.

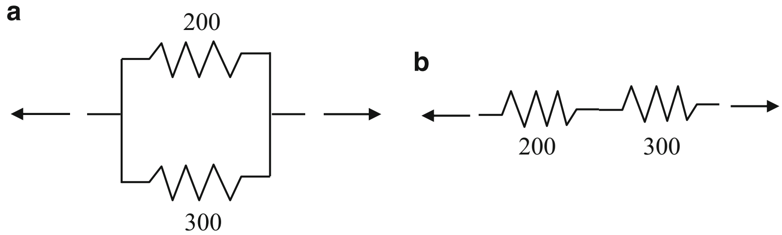
In Chaps. 4 and 5, we will show how to analyze the dynamics of combinations of these elements by first developing differential equations of motion for the systems and then solving and analyzing the solutions to these equations using the Laplace transform techniques developed in Chap. 2.

Problems

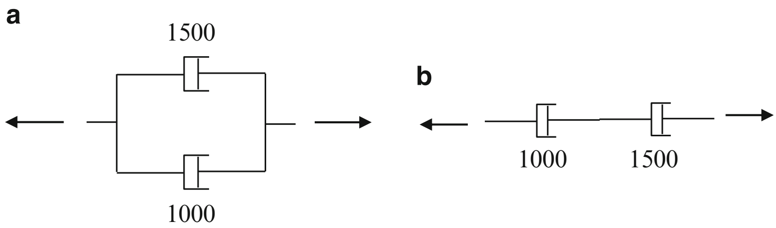
1. A brass pendulum (8500 kg/m^3 density) consists of a rectangular solid bar with a circular plate rigidly attached at its end. The assembly swings from a frictionless pinned connection, O. The rectangular bar has dimensions of 1500 mm in length by 50 mm in width by 10 mm in thickness as shown. The radius and thickness of the circular plate are 125 mm and 20 mm, respectively. The pivot, O, is located 25 mm from the end of the bar. Using the x–y coordinate system at O, determine: (a) the mass of the pendulum assembly and the vectoral location of its center of mass; and (b) the mass moment of inertia of the pendulum assembly about O.



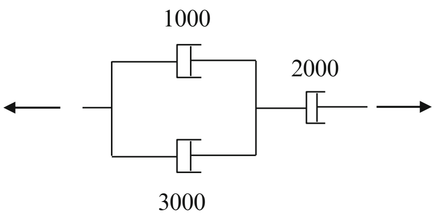
2. Consider the brass pendulum from Problem 1 in motion such that it makes an angle of 30° from the vertical. Find: (a) the moment acting about the pivot point, O; and (b) the instantaneous angular acceleration of the assembly at this position.
3. Calculate the equivalent stiffness of the following spring combinations. The spring constants shown are in units of N/m.



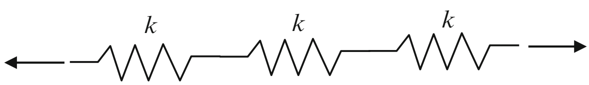
4. Calculate the equivalent damping for the following viscous damper combinations. The damping coefficients shown are units of N-s/m.



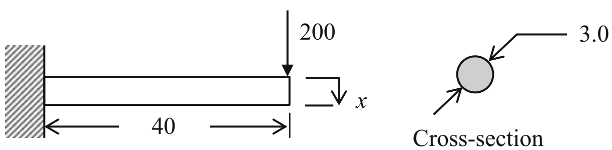
5. Calculate the equivalent damping for the following viscous damper combination. The damping coefficients shown are units of N-s/m.



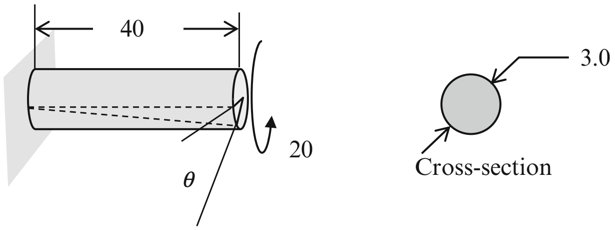
6. Show that the equivalent stiffness of three equal springs with stiffness, k, arranged in series is $\frac{k}{3}$.



7. A solid aluminum cantilever beam is 40 cm in length and has a circular cross section with a diameter of 3.0 cm. Calculate the lateral stiffness of the beam, k, in N/mm if subjected to a downward load as shown. Determine the displacement of the tip in mm when the load is 200 N.



8. A solid aluminum cantilever beam is 40 cm in length and has a circular cross section with a diameter of 3.0 cm. Calculate the torsional stiffness of the beam, k_r, in N-m/rad and the angular displacement at the end in deg when it is subjected to a moment of 20 N-m.



9. For Problems 7 and 8 calculate the elastic potential energy in J that is stored in the linear and torsional springs, respectively.
10. A power of 5 kW is delivered to a gear with 44 teeth rotating at 1500 rev/min (rpm). The first gear is engaged with a second gear having 11 teeth. Determine the: (a) input moment in N-m; (b) output moment in N-m; and (c) output rotation rate in rpm.

References

1. http://en.wikipedia.org/wiki/Suspension_(vehicle)
2. Meriam JL, Kraige LG (2012) Engineering mechanics: dynamics. Wiley, New York, NY
3. Greenwood DT (1987) Intermediate dynamics, 2nd edn. Prentice Hall, Englewood Cliffs, NJ

Transient Rectilinear Motion of Mechanical Systems

4

4.1 Introduction

You are already familiar with many types of mechanical systems that may undergo oscillations, or back and forth repetitive motions. They can occur in many mechanical systems including:

- the swinging pendulum on a grandfather clock
- the up and down motions of your car on its suspension as you drive over a speed bump
- a flag flying in the wind
- coffee in a moving cup
- a plucked guitar string.

In this chapter, we begin to model such systems and analyze their motions using the models we develop. The common features of these examples are that they each have some type of inertia/mass as well as some type of restoring force/spring. As stated in Chap. 3, both of these element types can store energy. The inertia/mass may alter its gravitational potential energy as it changes height and/or it may store kinetic energy during motion. A deflected spring stores energy as elastic potential energy. Oscillation is a trading of energy from the mass (kinetic energy) to the spring (elastic potential energy) and back again, over and over. Because real systems do not continue oscillating forever without some external energy input, models of oscillating mechanical systems typically include one or more energy dissipating elements (dampers).

In this chapter, we begin by examining a simple harmonic oscillator, which contains only a spring and a mass (no damper/energy loss). We then expand on this initial analysis by adding a damping element to the system. We restrict our analyses to *transient motions*, which occur in response to initial conditions or other input, such as a step function, which does not vary in a dynamic manner. Transient motions attenuate (or damp out) over time. The alternative to transient motions

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is sustained motions. These occur where the input is dynamic in nature and persists over time, such as a sinusoidal input. Sustained motions are discussed in Chap. 11.

4.2 Simple Harmonic Oscillator

A simple harmonic oscillator is modeled as a mass attached to a spring as shown in Fig. 4.1a. The mass is constrained to move horizontally on ideal "frictionless" rollers along a flat surface. It moves in response to initial conditions and/or an applied external force, $F(t)$.

The equation of motion for a simple harmonic oscillator is obtained by applying Newton's second law to the free body diagram in Fig. 4.1b. In the y-direction, the normal force, N, balances the weight, mg, of the mass and there is no acceleration. In the x-direction, Newton's second law is applied to determine the dependence of the mass acceleration on the spring force, F_k, and the applied force, $F(t)$; see Eq. (4.1). The spring force, F_k, depends linearly on the displacement, x, of the mass as described in Chap. 3; see Eq. (4.2).

$$F(t) + F_k = m\ddot{x} \tag{4.1}$$

$$F(t) - kx = m\ddot{x} \tag{4.2}$$

The sign of the spring force is negative because a positive displacement of the mass (to the right) results in a negative force (i.e., it acts to the left). A negative displacement of the mass (to the left) results in a positive force (now it acts to the right). Therefore, the spring is always acting to return the mass to the zero position. This equation is typically algebraically manipulated so that the unknown displacement and its derivatives are moved to the left-hand side and the driving force (should it exist) appears on the right.

$$m\ddot{x} + kx = F(t) \tag{4.3}$$

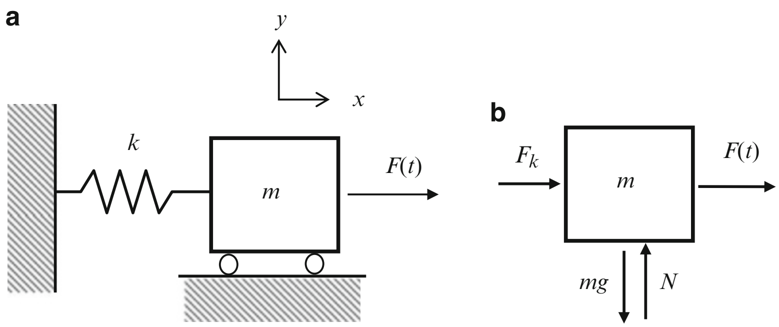


Fig. 4.1 (**a**) Simple harmonic oscillator with mass, m, stiffness, k, and externally applied force, $F(t)$; and (**b**) free body diagram for the moving mass

This is the differential *equation of motion* for a simple harmonic oscillator in the time domain. To understand the motion of the system, it is useful to consider some special cases.

4.2.1 Initial Velocity Only ($x(0) = 0$, $\dot{x}(0) = v_0$, $F(t) = 0$)

Consider Eq. (4.3) for the case with no input force, $F(t) = 0$, zero initial position, $x(0) = 0$, and a nonzero initial velocity, $\dot{x}(0) = v_0$. The Laplace transform of Eq. (4.3) for this case is:

$$m\left(s^2X(s) - sx(0) - \dot{x}(0)\right) + kX(s) = 0. \tag{4.4}$$

Substituting the initial conditions and solving for $X(s)$ gives:

$$X(s) = mv_0\left(\frac{1}{ms^2 + k}\right). \tag{4.5}$$

In Eq. (4.5), $X(s)$ describes the motion in the Laplace domain. To determine the motion in the time domain, we determine the inverse Laplace transform of Eq. (4.5). To do so, we use the linearity of the Laplace transform to remove a constant term and then look up the inverse transform in Table 2.1 (entry 10).

$$X(s) = \frac{mv_0}{k}\left(\frac{k}{ms^2 + k}\right) = \frac{v_0}{\frac{k}{m}}\left(\frac{\frac{k}{m}}{s^2 + \frac{k}{m}}\right) = \frac{v_0}{\omega_n^2}\left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right) = \frac{v_0}{\omega_n}\left(\frac{\omega_n}{s^2 + \omega_n^2}\right) \tag{4.6}$$

Equation (4.5) has been rewritten in Eq. (4.6) to include ω_n, the natural frequency of the system; see Eq. (4.7).

$$\omega_n = \sqrt{\frac{k}{m}} \tag{4.7}$$

The inverse Laplace transform of Eq. (4.6) gives the displacement as a function of time.

$$x(t) = \frac{v_0}{\omega_n}\sin\left(\omega_n t\right) \tag{4.8}$$

The frequency of the oscillations is ω_n (rad/s). The frequency in cycles per second, or Hertz (Hz), f_n, is determined by dividing the frequency in rad/s by 2π rad/cycle.

$$f_n = \frac{\omega_n}{2\pi} \tag{4.9}$$

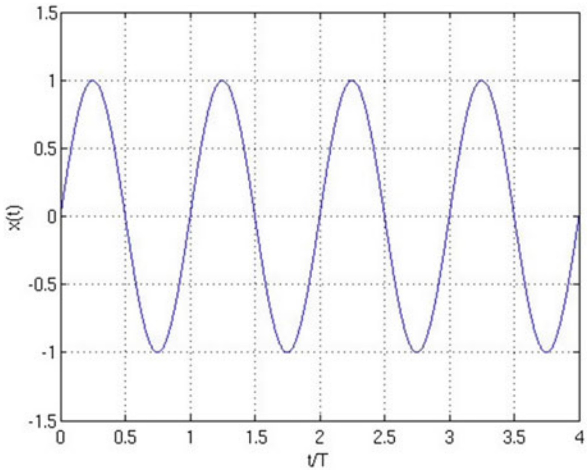


Fig. 4.2 Motion of a simple harmonic oscillator with an initial velocity. The horizontal axis is specified in increments of the period, T. One oscillation cycle is completed in each period

The period of the motion, or time to complete one cycle of oscillation, T, is $\frac{1}{f_n}(\mathrm{s})$. Figure 4.2 shows a plot of the motion as a function of time for four cycles of oscillation. The oscillations continue indefinitely since there is no energy loss mechanism. Two important conclusions can be drawn from Eq. (4.7): (1) the oscillating frequency increases when the stiffness is increased; and (2) the frequency decreases when the mass is increased.

4.2.2 Impulse Input ($x(0) = 0$, $\dot{x}(0) = 0$, $F(t) = P_0 \cdot \delta(t)$)

Consider the case where the force is an impulsive load with magnitude P_0 in units of momentum.[1] The impulse input, $\delta(t)$, unit is inverse seconds and, therefore, the units of $P_0 \cdot \delta(t)$ are momentum per second, or force. In SI, the corresponding units are (kg-m/s)-1/s, or N. The equation of motion for the simple harmonic oscillator with an impulse input is given by Eq. (4.10).

$$m\ddot{x} + kx = P_0 \cdot \delta(t) \tag{4.10}$$

With zero initial conditions and noting that the Laplace transform of $\delta(t)$ is 1 (Table 2.1, entry 1), the Laplace transform of Eq. (4.10) is:

$$X(s) = \frac{P_0}{ms^2 + k} = \frac{P_0}{k}\left(\frac{\frac{k}{m}}{s^2 + \frac{k}{m}}\right) = \frac{P_0}{m\frac{k}{m}}\left(\frac{\frac{k}{m}}{s^2 + \frac{k}{m}}\right) = \frac{P_0}{m\omega_n}\left(\frac{\omega_n}{s^2 + \omega_n^2}\right). \tag{4.11}$$

[1] The units of momentum are the product of the units of mass and velocity.

The inverse Laplace transform gives $x(t)$.

$$x(t) = \frac{P_0}{m\omega_n} \sin(\omega_n t) \tag{4.12}$$

This is the same as Eq. (4.8) if we apply the Eq. (4.13) equality.

$$P_0 = mv_0 \tag{4.13}$$

The physical meaning of Eq. (4.13) is that an impulsive load on a system gives rise to an instantaneous change in the system momentum. Therefore, an impulse load is equivalent to an initial velocity. Equation (4.12) is referred to as the time domain *impulse response* of the simple harmonic oscillator.

4.2.3 Step Input ($x(0) = 0$, $\dot{x}(0) = 0$, $F(t) = F_0 \cdot u(t)$)

A third case of interest is the response of the system to a step input force, $F(t) = F_0 \cdot u(t)$. The equation of motion for this case is given by Eq. (4.14).

$$m\ddot{x} + kx = F_0 \cdot u(t) \tag{4.14}$$

Given that the Laplace transform of a unit step function is $\frac{1}{s}$ (Table 2.1, entry 2), the Laplace transform of Eq. (4.14) is:

$$X(s) = \frac{F_0}{s(ms^2 + k)} = F_0 \frac{\frac{1}{m}}{s\left(s^2 + \frac{k}{m}\right)} = \frac{F_0}{m} \frac{1}{s\left(s^2 + \omega_n^2\right)}. \tag{4.15}$$

One method for inverting this Laplace transform is a partial fraction expansion of the form provided in Eq. (4.16).

$$\frac{1}{s\left(s^2 + \omega_n^2\right)} = \frac{1}{\omega_n^2}\left(\frac{1}{s} - \frac{s}{s^2 + \omega_n^2}\right) \tag{4.16}$$

Combining Eqs. (4.15) and (4.16) and using the definition of the natural frequency from Eq. (4.7), we apply entries 2 and 11 from Table 2.1 to obtain the time domain response shown in Eq. (4.17).

$$x(t) = \frac{F_0}{m} \frac{1}{\omega_n^2}(1 - \cos(\omega_n t)) = \frac{F_0}{k}(1 - \cos(\omega_n t)) \tag{4.17}$$

The constant term $\frac{F_0}{k}$ is the static equilibrium displacement of the mass under the application of a constant force F_0. The second term (in parenthesis) is an oscillation about that equilibrium displacement. A plot of the motion described by Eq. (4.17) is provided in Fig. 4.3. The system starts at zero displacement and then oscillates at

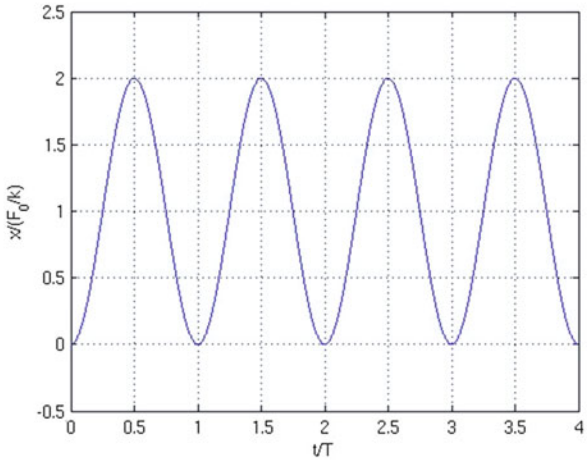


Fig. 4.3 Motion of a simple harmonic oscillator with a step input force

the natural frequency around $\frac{F_0}{k}$ with a maximum value of $\frac{2F_0}{k}$ and a minimum value of zero. Even this simple model provides useful information. When a dynamic load is applied to an undamped system, the maximum displacement is twice that predicted from a static analysis alone. This is important to consider when designing a structure so that it will not fail in response to dynamic loading.

4.3 Damped Harmonic Oscillator

A more realistic model for many mechanical dynamic systems is the damped harmonic oscillator shown in Fig. 4.4a. The equation of motion for the system can be obtained by considering the free body diagram in Fig. 4.4b.

$$F(t) + F_k + F_b = m\ddot{x} \tag{4.18}$$

The forces in the spring and damper act against the displacement and velocity directions, respectively. Rearranging the equation of motion, we obtain:

$$m\ddot{x} + b\dot{x} + kx = F(t). \tag{4.19}$$

This is the equation of motion for a damped harmonic oscillator in the time domain. It is instructive to consider the general solution to this equation of motion for the simple case of zero initial displacement, a nonzero initial velocity, v_0, and no applied force: $x(0) = 0$, $\dot{x}(0) = v_0$, $F(t) = 0$. For this case, the Laplace transform of Eq. (4.19) is:

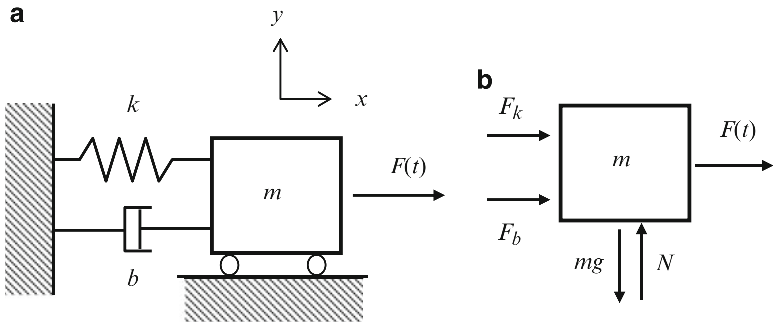


Fig. 4.4 (**a**) Damped harmonic oscillator with mass, m, stiffness, k, damping, b, and externally applied force $F(t)$; and (**b**) free body diagram of the moving mass

$$X(s) = \frac{mv_0}{ms^2 + bs + k}. \tag{4.20}$$

Rearranging Eq. (4.20) yields:

$$X(s) = mv_0\left(\frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}\right) = \frac{v_0}{\frac{k}{m}}\left(\frac{\frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}\right). \tag{4.21}$$

We now define the dimensionless damping ratio, ζ.

$$\zeta = \frac{b}{2\sqrt{km}} \tag{4.22}$$

Using this relationship, $\frac{b}{m} = 2\zeta\omega_n$ can be substituted into Eq. (4.21) and it can be rewritten to obtain:

$$X(s) = \frac{v_0}{\omega_n^2}\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right). \tag{4.23}$$

Completing the square in the denominator we obtain:

$$X(s) = \frac{v_0}{\omega_n^2}\left(\frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}\right). \tag{4.24}$$

Now we define another quantity, the damped natural frequency, ω_d.

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} \tag{4.25}$$

The natural frequency is the frequency at which the system oscillates when the damping is zero. The damped natural frequency is the frequency at which a system oscillates for the case where $0 < \zeta < 1$. Combining Eqs. (4.24) and (4.25), we obtain:

$$X(s) = \frac{v_0}{\omega_n^2}\left(\frac{\omega_n^2}{(s+\zeta\omega_n)^2 + \omega_d^2}\right), \tag{4.26}$$

which can be inverted using entry 22 in Table 2.1 (the format in Eq. (4.23) actually matches the table) to obtain the time domain behavior shown in Eq. (4.27).

$$x(t) = \frac{v_0}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) \tag{4.27}$$

The time constant, τ, of the exponential term (see Chap. 2) can now also be determined.

$$\tau = \frac{1}{\zeta\omega_n} \tag{4.28}$$

Equations (4.7) (ω_n), (4.22) (ζ), (4.25) (ω_d), (4.27) (time domain response of a single degree of freedom damped harmonic oscillator subject to a nonzero initial velocity) and (4.28) (τ) are some of the most important in all of engineering mechanics. If ζ is less than 1, ω_d is real and the sine term in Eq. (4.27) oscillates at the damped natural frequency. Additionally, the exponential term[2] decays to zero as the time gets large and eventually "damps out" the oscillation. As stated in Eq. (4.28), the time constant of the exponential is $(\zeta\omega_n)^{-1}$. If ζ is greater than 1, ω_d is imaginary. In this case, by applying the exponential definition of the sine function, Eq. (4.27) can be written as:

$$x(t) = \frac{v_0}{\omega_n\sqrt{\zeta^2 - 1}}\left(\frac{e^{-\left(\omega_n\left(\zeta - \sqrt{\zeta^2-1}\right)\right)t} - e^{-\left(\omega_n\left(\zeta + \sqrt{\zeta^2-1}\right)\right)t}}{2}\right). \tag{4.29}$$

Since $\zeta > \sqrt{\zeta^2 - 1}$, the parenthetical expression in Eq. (4.29) is a sum of two decaying exponentials and does not oscillate. Therefore, the value of the unitless damping ratio ζ determines the qualitative behavior of the system:

- $\zeta = 0$, undamped—oscillation with a constant magnitude
- $\zeta < 1$, underdamped—oscillation with an exponentially decaying magnitude
- $\zeta = 1$, critically damped—limiting value for no oscillation
- $\zeta > 1$, overdamped—no oscillation.

[2] We can think of the exponential terms as the "damping envelope" since it drives the vibration magnitude toward zero as time gets large.

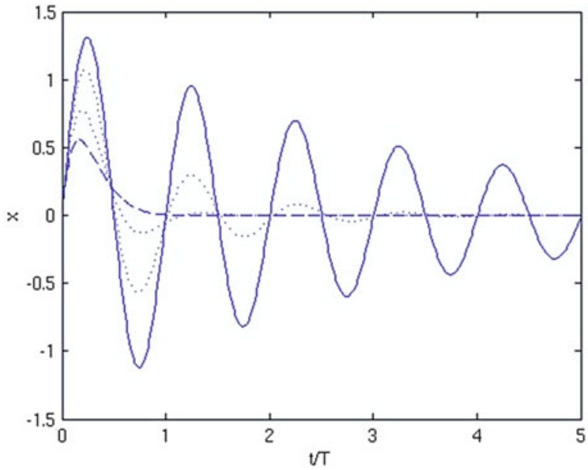


Fig. 4.5 Effect of the damping ratio on the oscillating behavior

The damping ratio is often expressed as a percent of the critical damping value. Therefore, if $\zeta = 0.1$ the system is considered to be 10% damped. The effect of changing the value of the damping ratio is summarized graphically in Fig. 4.5. The results for four damping ratios are provided: $\zeta = 0.05$ (solid line), $\zeta = 0.2$ (dotted), $\zeta = 0.5$ (dotted), and $\zeta = 0.9$ (dashed).

The natural frequency, damping ratio, and damped natural frequency determine the behavior of a system even in the presence of inputs and other initial conditions. Let's consider the following examples.

Example 4.1 Consider the simple harmonic oscillator shown in Fig. 4.1. Suppose the spring stiffness is 2500 N/m and the mass is 100 kg. Determine the natural frequency in both rad/s and Hz and then find the period of the oscillations. If the initial conditions are $x(0) = 2\,\mathrm{m}$, $\dot{x}(0) = 0$, and there is no externally applied force, use MATLAB® to find the time domain response by applying the inverse Laplace transform command (`ilaplace`) and then plot the motion for four periods.

Solution Substitution in Eq. (4.3) gives the equation of motion for the system.

$$100\ddot{x} + 2500x = 0$$

The natural frequency is:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{100}\frac{\mathrm{N/m}}{\mathrm{kg}}} = \sqrt{25\frac{1}{\mathrm{s}^2}} = 5\ \mathrm{rad/s}.$$

We convert to Hertz using Eq. (4.9).

$$f_n = \frac{\omega_n}{2\pi} = \frac{5}{2\pi} = 0.80 \text{ Hz}$$

The period of the oscillations is the inverse of f_n.

$$T = \frac{1}{f_n} = 1.26\,\text{s}$$

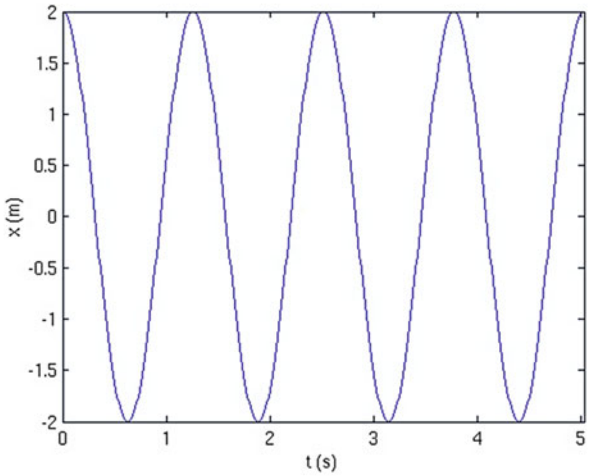
In order to find $x(t)$, we must first determine the Laplace transform of the equation of motion to identify $X(s)$. We divide by the mass and include the initial conditions x_0 and v_0.

$$\begin{aligned}
&\ddot{x} + 25x(t) = 0 \\
&s^2X(s) - sx_0 - v_0 + 25X(s) = 0 \\
&X(s) = \frac{sx_0 + v_0}{s^2 + 25}
\end{aligned}$$

Substituting for the generic initial condition placeholders, we obtain $X(s) = \frac{2s}{s^2+25}$. The inverse Laplace transform is selected from Table 2.1 (entry 11). We obtain $x(t) = 2\cos(5t)$. We can now use MATLAB® to calculate the inverse Laplace transform and plot the time domain response.

```
clear all           % clear all variables from the memory
clc                 % clear the screen
close all           % close all figure windows

% Parameters
k = 2500;           % N/m
m = 100;            % kg
wn = sqrt(k/m);     % rad/s
fn = wn/(2*pi);     % Hz
T = 1/fn;           % s

% Inverse Laplace transform
syms x X t s;
X = 2*s/(s^2+25);
x = ilaplace(X)

% Time vector four periods in length with 100 steps per period
t = [0:T/100:4*T];
xx = eval(x);
plot(t, xx)
xlabel('t (s)')
ylabel('x (m)')
axis([0 max(t) min(xx) max(xx)])
```

The corresponding figure is displayed.



Because there is no energy loss (damping) in the system, it begins at the initial position of 2 m and oscillates back and forth between 2 and −2 m indefinitely with a period of 1.26 s.

Example 4.2 Consider the damped harmonic oscillator shown in Fig. 4.4. Suppose the spring stiffness is 2500 N/m, the mass is 100 kg, and the viscous damping coefficient is 600 N-s/m. Determine the natural frequency (in both rad/s and Hz), the damping ratio, and the damped natural frequency. The initial conditions are $x(0) = 2\,\mathrm{m}$ and $\dot{x}(0) = 0$; there is no externally applied force. Use MATLAB® to find $x(t)$ and plot it for ten periods.

Solution Substitution in Eq. (4.19) gives the equation of motion.

$$100\ddot{x} + 600\dot{x} + 2500x = 0$$

The natural frequency and period are the same as in Example 4.1.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{100}\frac{\mathrm{N/m}}{\mathrm{kg}}} = \sqrt{25\frac{1}{\mathrm{s}^2}} = 5\,\mathrm{rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{5}{2\pi} = 0.80\,\mathrm{Hz}$$

$$T = \frac{1}{f_n} = 1.26\,\mathrm{s}$$

The damping ratio is given by Eq. (4.22) and the damped natural frequency by Eq. (4.25).

$$\zeta = \frac{b}{2\sqrt{km}} = \frac{600}{2\sqrt{2500 \times 100}} = 0.6$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4\,\mathrm{rad/s}$$

The time constant for the underdamped system is calculated using Eq. (4.28).

$$\tau = \frac{1}{\zeta\omega_n} = 0.33\,\mathrm{s}$$

Note that Eq. (4.28) provides a general expression for the time constant of an underdamped system even though Eq. (4.27) was derived using an initial velocity only. The initial conditions and the input do not affect the natural frequencies, damping ratios, and time constants associated with a given system.

To find $x(t)$, we calculate the Laplace transform of the equation of motion to find $X(s)$. We divide by the mass and include the initial conditions x_0 and v_0.

$$\ddot{x} + 6\dot{x} + 25x = 0$$
$$s^2X(s) - sx_0 - v_0 + 6(sX(s) - x_0) + 25X(s) = 0$$
$$X(s) = \frac{sx_0 + v_0 + 6x_0}{s^2 + 9s + 25}$$

Substituting the initial conditions yields $X(s) = \frac{2s+12}{s^2+6s+25}$. We use MATLAB® to calculate the inverse Laplace transform and plot the time domain response.

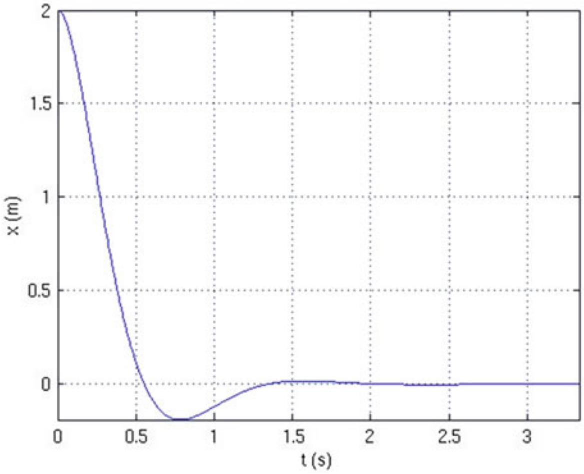
```
clear all
clc
close all

% Parameters
k = 2500;
b = 600;     % N-s/m
m = 100;
wn = sqrt(k/m);
fn = wn/(2*pi);
T = 1/fn;
zeta = b/(2*sqrt(k*m));
wd = wn*sqrt(1-zeta^2);
tau = 1/(zeta*wn);

% Inverse Laplace transform
syms x X t s;
X = (2*s+12)/(s^2+6*s+25);
x = ilaplace(X)

% Plot the response
t = [0:tau/100:10*tau];
```

```
xx = eval(x);
plot(t, xx)
xlabel('t (s)')
ylabel('x (m)');
axis([0 max(t) min(xx) max(xx)])
```

MATLAB® gives the inverse Laplace transform as:

$$x(t) = 2e^{-3t}\left(\cos(4t) + \frac{3}{4}\sin(4t)\right) \text{ m.}$$

The plot of the motion is shown in the figure.



Note that the system again begins at the initial displacement of 2 m, but the motion damps out in approximately one oscillation. The undamped system from Example 4.1 is now 60% damped.

Example 4.3 Next, consider a damped harmonic oscillator with $m = 10$ kg, $b = 20$ N-s/m, and $k = 500$ N/m. Suppose there is a step input force, $F(t) = 1000 \cdot u(t)$ N, and the initial conditions are zero. Determine the natural frequency (in both rad/s and Hz), the damping ratio, and the damped natural frequency. Use MATLAB® to find $x(t)$ and plot the motion for a total of 10 system time constants. Finally, apply the final value theorem to $X(s)$ and compare the result to the $x(t)$ plot. Explain the result physically.

Solution Equation (4.19) gives the equation of motion for the system after substitution.

$$10\ddot{x} + 20\dot{x} + 500x = 1000 \cdot u(t)$$

The natural frequency is:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{500}{10}\frac{\mathrm{N/m}}{\mathrm{kg}}} = \sqrt{50\frac{1}{\mathrm{s}^2}} = 7.1\,\mathrm{rad/s}.$$

We convert to Hertz using Eq. (4.9).

$$f_n = \frac{\omega_n}{2\pi} = \frac{7.1}{2\pi} = 1.1\,\mathrm{Hz}$$

The period is:

$$T = \frac{1}{f_n} = 0.89\,\mathrm{s}.$$

The damping ratio is given by Eq. (4.22) and the damped natural frequency by Eq. (4.25).

$$\zeta = \frac{b}{2\sqrt{km}} = \frac{20}{2\sqrt{500 \cdot 10}} = 0.14$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 7.0\,\mathrm{rad/s}$$

The time constant of an underdamped system calculated using Eq. (4.28).

$$\tau = \frac{1}{\zeta\omega_n} = 1.0\,\mathrm{s}$$

Now we determine the Laplace transform of the equation of motion to find $X(s)$.

$$\begin{aligned}
&\ddot{x} + 2\dot{x} + 50x = 100 \cdot u(t) \\
&(s^2 + 2s + 50)X(s) = \frac{100}{s} \\
&X(s) = \frac{100}{s(s^2 + 2s + 50)}
\end{aligned}$$

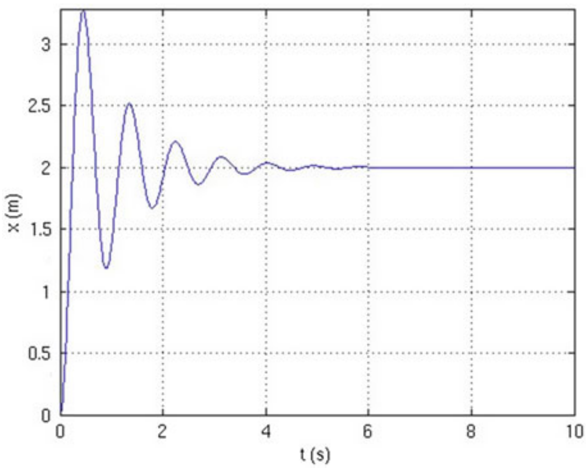
Applying the final value theorem (Eq. 2.19) to $X(s)$ shows that this system settles to a steady-state displacement of 2 m.

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0}(sX(s)) = \lim_{s\to 0}\left(\frac{100}{s^2 + 2s + 50}\right) = 2\,\mathrm{m}$$

The following MATLAB® code inverts $X(s)$ and plots the resulting motion.

```
clear all
clc
close all
```

```
% Parameters
k = 500;
b = 20;
m = 10;
wn = sqrt(k/m);
fn = wn/(2*pi);
T = 1/fn;
zeta = b/(2*sqrt(k*m));
wd = wn*sqrt(1-zeta^2);
tau = 1/(zeta*wn);

% Inverse Laplace transform
syms x X t s;
X = 100/(s*(s^2+2*s+50));
x = ilaplace(X);

% Plot the result
t = [0:tau/100:10*tau];
xx = eval(x);
plot(t, xx)
ylabel('x (m)')
xlabel('t (s)')
axis([0 max(t) min(xx) max(xx)])
```



The inverse Laplace transform of $X(s)$ given by MATLAB® is:

$$x(t) = 2 - 2e^{-t}\left(\cos(7t) + \frac{1}{7}\sin(7t)\right)\text{ m.}$$

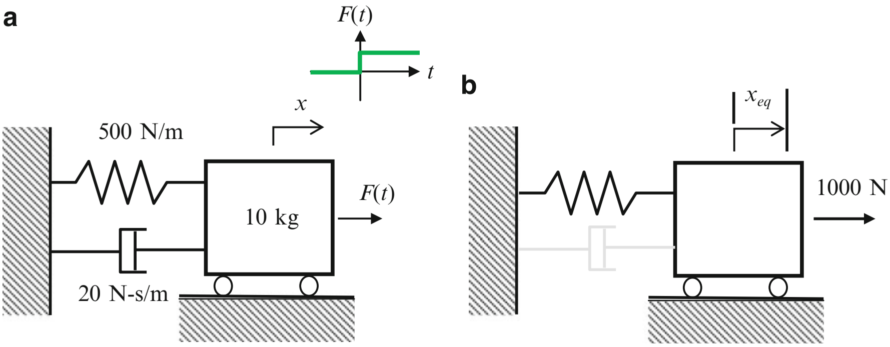


Fig. 4.6 Damped harmonic oscillator subjected to a step input. As time goes to infinity, the motion damps out and the damper ceases to produce a force

The leading term is a constant (2). Since the exponential drives the sine and cosine terms to zero as time approaches infinity, the value approached by $x(t)$ as t increases is the constant. This matches both the figure and the final value theorem result. To interpret this physically, consider the system schematic shown in Fig. 4.6. When the step input is first applied, the mass accelerates, the spring deflects, and the damper velocity changes. Both the damper and the spring exert forces on the mass. However, the motions of the mass attenuate over time and the mass reaches a new equilibrium position, x_{eq}. At this point, the mass has no velocity and, therefore, the damper applies no force. When the mass is no longer moving, the 1000 N of applied force is balanced by the spring force. The deflection of the spring is 2 m so that the spring applies a force of $F = kx = 500 \cdot 2 = 1000$ N, which exactly counteracts the applied step load.

4.4 The Effect of Gravitational Loads

In many instances, the systems we study are subjected to direct weight loading in the direction of the dynamic motions as shown in Fig. 4.7a. In the figure, two coordinate systems are defined. The coordinate y is zero when the spring is unstretched (i.e., it is at its free length). The coordinate x is zero at the stretched equilibrium position of the spring, which is a distance y_{eq} below $y = 0$. The position $x = 0$ is the location where the system would settle under the influence of the gravitational load alone.

To understand the effect of gravity on the system, we use the free body diagram shown in Fig. 4.7b. Let us first write the equations of motion in terms of y. Applying Newton's law to the free body diagram, we obtain:

$$-mg \cdot u(t) + F_k + F_b = m\ddot{y}. \tag{4.30}$$

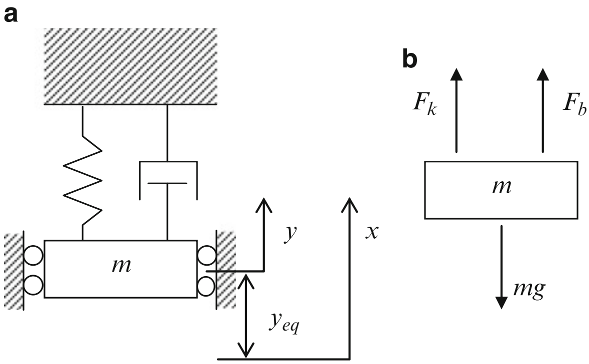


Fig. 4.7 (**a**) Damped harmonic oscillator moving vertically under the influence of gravity; and (**b**) free body diagram

The gravitational load is a step input because we support the weight until time $t = 0$ and then release it. Because we are working in the y coordinate system, the spring force acts to bring the position of the mass back to $y = 0$ and is therefore written as $F_k = -ky$. By the same argument, the damping force is $F_b = -b\dot{y}$ and we obtain the equation of motion shown in Eq. (4.31).

$$m\ddot{y} + b\dot{y} + ky = -mg \cdot u(t) \tag{4.31}$$

Computing the Laplace transform of this equation and assuming zero initial conditions, we find $Y(s)$.

$$Y(s) = \frac{-mg}{s(ms^2 + bs + k)} \tag{4.32}$$

Without inverting the Laplace transform, we apply the final value theorem to determine the steady-state, or equilibrium, position that results from the application of the gravitational load.

$$y_{eq} = \lim_{t \to \infty} y(t) = \lim_{s \to 0} (sF(s)) = \lim_{s \to 0} \left(\frac{-mg}{ms^2 + bs + k} \right) = \frac{-mg}{k} \tag{4.33}$$

As noted, y_{eq} is the position at which the spring force balances the gravitational load. The negative sign indicates that the equilibrium position is below the unstretched spring position. Using Eq. (4.33), we can identify the relationship between the x and y coordinate systems from Fig. 4.7a.

$$y = x - \frac{mg}{k} \tag{4.34}$$

Since the equilibrium displacement (mg/k) is a constant, taking the first two time derivatives of Eq. (4.34) results in Eq. (4.35).

$$\begin{aligned} \dot{y} &= \dot{x} \\ \ddot{y} &= \ddot{x} \end{aligned} \tag{4.35}$$

Substituting Eqs. (4.34) and (4.35) into Eq. (4.31), we obtain the equation of motion in terms of x only.

$$m\ddot{x} + b\dot{x} + k\left(x - \frac{mg}{k}\right) = -mg \cdot u(t) \tag{4.36}$$

We see that the weight terms now cancel on either side of the equation and we obtain the familiar equation of motion in x.

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{4.37}$$

This is identical to Eq. (4.19) with no externally applied force. All of the previous discussion therefore applies to the motion of the Fig. 4.7 system as well. Thus, when we consider the coordinate system to be located at the stretched (equilibrium) length of the spring, the behavior is the same as if there was no gravitational load at all. Physically, at $x = 0$, the equilibrium stretch in the spring balances the weight. Since the spring is linear, an upward motion in x causes a net downward force $-kx$. If the mass moves downward, the spring force is greater than mg, and there is a net upward force kx. This is only true for a linear spring, but it generalizes to any number of elements.

Example 4.4 Consider a mass of 10 kg suspended from a 200 N/m spring and a 40 N-s/m damper. Find the damping ratio, natural frequency, damped natural frequency, and time constant for the system. Then, if the mass is supported at $y = 0$ until time $t = 0$ and then released, determine $y(t)$ and plot the motion for a time interval of eight time constants using MATLAB®.

Solution The damping ratio, natural frequency, and damped natural frequency can be calculated from Eqs. (4.7), (4.22) and (4.25), respectively. The gravitational load does not change the fundamental system behavior because it is simply a specific input force.

$$\begin{aligned} \omega_n &= \sqrt{\frac{200}{10}\frac{\mathrm{N/m}}{\mathrm{kg}}} = 4.47\,\mathrm{rad/s} \\ \zeta &= \frac{b}{2\sqrt{km}} = \frac{40}{2\sqrt{(200)(10)}}\frac{\mathrm{N\text{-}s/m}}{\sqrt{\mathrm{N/m\text{-}kg}}} = 0.447 \\ \omega_d &= \omega_n\sqrt{1-\zeta^2} = 4\,\mathrm{rad/s} \\ \tau &= \frac{1}{\zeta\omega_n} = 0.5\,\mathrm{s} \end{aligned}$$

We substitute in Eq. (4.30) to identify the time domain equation of motion.

$$10\ddot{y} + 40\dot{y} + 200y = -98.1 \cdot u(t)$$

Applying the Laplace transform gives $Y(s)$.

$$Y(s) = \frac{-98.1}{s(10s^2 + 40s + 200)}$$

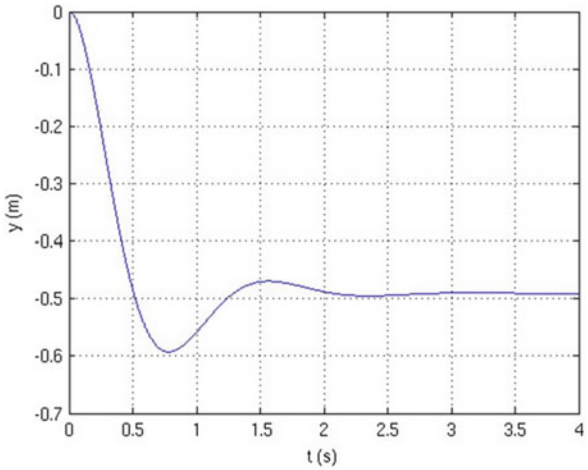
This can now be inverted and plotted using MATLAB®.

```
clear all
clc
close all

% Parameters
m = 10;
b = 40;
k = 200;
g = 9.81;

% Define and invert the Laplace transform
syms Y s y t;
Y = -m*g/(s*(m*s^2+b*s+k));
y = ilaplace(Y);
display(y)

% Find the natural frequency, damping ratio, and time constant
wn = sqrt(k/m);
zeta = b/(2*sqrt(k*m));
wd = wn*sqrt(1-zeta^2);
tau = 1/(zeta*wn);
display(wn)
display(zeta)
display(wd)

% Plot the response
figure(1)
t = [0:tau/1000:8*tau];
plot(t, eval(y))
xlabel('t (s)')
ylabel('y (m)')
```

MATLAB® indicates that $y(t)$ is given by the following expression.

$$y(t) = \frac{981}{2000}\left(e^{-2t}\left(\cos(4t) + \frac{1}{2}\sin(4t)\right) - 1\right)$$

A plot of the motion is provided.



The equilibrium position of the mass is determined from Eq. (4.33).

$$y_{eq} = -\frac{mg}{k} = -\frac{10 \cdot 9.81}{200} = -0.49\,\mathrm{m}$$

This matches the figure. The mass starts at the zero position and settles to a position of 0.49 m below that starting point. At this location the spring is sufficiently stretched to produce a force that balances the gravitational force.

Example 4.5 Consider the same system that was analyzed in Example 4.4, but now assume that it begins at its equilibrium position (rather than the unstretched spring position). For an initial velocity of 1 m/s upward, determine and plot the motion using MATLAB®.

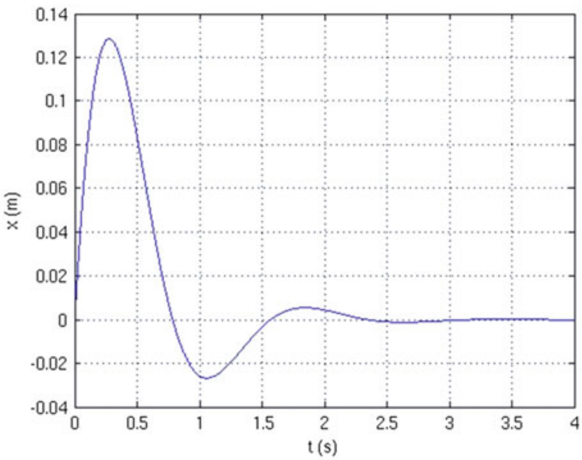
Solution Equation (4.37) now defines the equation of motion. We have already solved this equation for the case of an initial velocity; see Eq. (4.27).

$$x(t) = \frac{v_0}{\omega_d} e^{-\zeta\omega_n t} \sin\left(\omega_d t\right)$$

We code this in MATLAB® directly.

```
clear all
clc
close all

% Parameters
m = 10;
b = 40;
k = 200;
v0 = 1;      % m/s
wn = sqrt(k/m);
zeta = b/(2*sqrt(k*m));
```

```
wd = wn*sqrt(1-zeta^2);
tau = 1/(zeta*wn)

% Plot the response
figure(1)
t = [0:tau/1000:8*tau];
x = v0/wd*exp(-zeta*wn*t).*sin(wd*t);
plot(t, x)
xlabel('t (s)')
ylabel('x (m)')
```



The mass initially moves in the positive direction because of the upward initial velocity, but it finally settles back to the $x = 0$ position. Recall that, in this coordinate system, $x = 0$ is the equilibrium position of the mass with the spring stretched to support the weight.

4.5 Transfer Functions and the Characteristic Equation

When analyzing the mechanical system in Example 4.3, we implicitly assumed that there was an *input* force, $F(t)$, and a corresponding system response, or *output*, $x(t)$. In the absence of initial conditions, it is the input that causes the system motion, or output. To be more specific, consider the equation of motion of the damped harmonic oscillator shown in Fig. 4.4. Assuming zero initial conditions, calculating the Laplace transform of Eq. (4.19), and dividing by $F(s)$ gives Eq. (4.38).

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} \tag{4.38}$$

Equation (4.38) describes the *transfer function* of the damped harmonic oscillator. The transfer function is the ratio of the system output to the system input in the Laplace domain.

For general linear systems, the transfer function is a ratio of polynomials in the Laplace variable, s, which represent the output, $O(s)$, and input, $I(s)$.

$$\frac{O(s)}{I(s)} = \frac{b_m s^m \cdots + b_2 s^2 + b_1 s + b_0}{a_n s^n \cdots + a_2 s^2 + a_1 s + a_0} \tag{4.39}$$

Typically, $m < n$, so the order of the numerator is less than the order of the denominator. The denominator is called the *characteristic equation* of the system. The roots of the characteristic equation are found by solving Eq. (4.40).

$$a_n s^n \cdots + a_2 s^2 + a_1 s + a_0 = 0 \tag{4.40}$$

Recalling the methods for inverting Laplace transforms discussed in Chap. 2, we recognize that the roots of the characteristic equation determine the type of expansion required to conveniently invert the Laplace transform. The type of expansion then determines the mathematical form of the system motion. We can make the following general comments about the roots of the characteristic equation:

1. real negative roots lead to decaying exponential (nonoscillatory) parts of the inverse Laplace transform
2. complex roots with negative real parts lead to decaying oscillatory terms
3. real positive roots or complex roots with positive real parts lead to exponentially growing parts of the inverse Laplace transform, which indicate system instability[3]

If the characteristic equation is second order as shown in Eq. (4.23), case 1 corresponds to a damping ratio greater than one (overdamped), while case 2 corresponds to a damping ratio less than one (underdamped). The *order* of the system is the highest power of s in the denominator, n. The roots of the characteristic equation give us properties of the system independent of the type of input it experience.

To further explore the transfer function concept, consider the system displayed in Fig. 4.8. Analyzing the free body diagram shown in Fig. 4.8b, we obtain the following equation of motion.

[3] We can think of a stable system as one that tends to return to its equilibrium position when disturbed from that position and an unstable system as one that tends to move away from its equilibrium position and not return. A mental picture for stability is provided by a ball placed inside a bowl (against the concave inner surface), while a picture of instability is provided by a ball placed on top of an inverted bowl (against the outer convex surface).

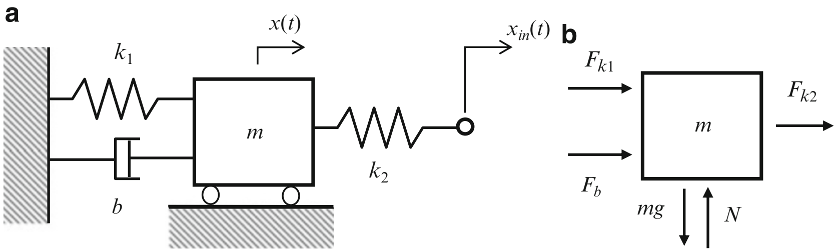


Fig. 4.8 (**a**) A system with two different springs, a mass, and a damper. The input is the displacement at the end of the second spring, $x_{in}(t)$, and the output is the mass displacement, $x(t)$; and (**b**) the free body diagram

$$F_{k1} + F_{k2} + F_b = m\ddot{x} \tag{4.41}$$

The spring force F_{k2} is positive if $x_{in} > x$ and is proportional to the deflection, $F_{k2} = k_2(x_{in} - x)$. The spring force F_{k1} is negative for a positive displacement of the mass and is given by $F_{k1} = -k_1 x$. The force in the damper is negative when the velocity of the mass is positive, $F_b = -b\dot{x}$. Substituting these expressions into the equation of motion and rearranging so that x_{in} is on the right-hand side gives the equation of motion for the system.

$$m\ddot{x} + b\dot{x} + (k_1 + k_2)x = k_2 x_{in} \tag{4.42}$$

To obtain the transfer function, we assume zero initial conditions and apply the Laplace transform. We then rearrange to obtain the ratio of the output to the input. This system is second order because the highest power of s in the denominator (characteristic equation) is 2.

$$\frac{X(s)}{X_{in}(s)} = \frac{k_2}{ms^2 + bs + k_1 + k_2} \tag{4.43}$$

Example 4.6 Assume that a step input, $x_{in}(t) = Au(t)$, is applied to the system in Fig. 4.8. Find the steady-state displacement of the mass and describe the result physically.

Solution Using the transfer function given by Eq. (4.43) and applying the step input, we obtain the following expression for $X(s)$.

$$X(s) = \left(\frac{k_2}{ms^2 + bs + k_1 + k_2}\right)\frac{A}{s}$$

Using the final value theorem (Eq. 2.19), we find the steady-state, or equilibrium, displacement of the mass.

$$x_{eq} = \lim_{t\to\infty} x(t) = \lim_{s\to0} (sX(s)) = \lim_{s\to0} \left(\left(\frac{k_2}{ms^2 + bs + k_1 + k_2} \right) A \right) = \left(\frac{k_2}{k_1 + k_2} \right) A$$

If we rearrange this equation, we obtain:

$$k_1 x_{eq} = k_2 \left(A - x_{eq} \right)$$

and we conclude that the equilibrium displacement is the displacement required for the forces in the two springs to be equal. This makes physical sense since, at equilibrium, the force of the damper is zero and the spring forces must balance.

MATLAB® offers several functions that can be used to find the response of a system when the transfer function is known. The primary commands are `step`, `impulse`, and `lsim`. These commands require that the system is first defined and then stored. This is completed with the aid of another function, `tf`.

Example 4.7 Consider the transfer function of a second-order system which relates the output displacement to an input force.

$$\frac{X(s)}{F(s)} = \frac{s+1}{s^2 + 4s + 20}$$

Find the response of the system to step inputs of 1 and 20 using the `step` command.

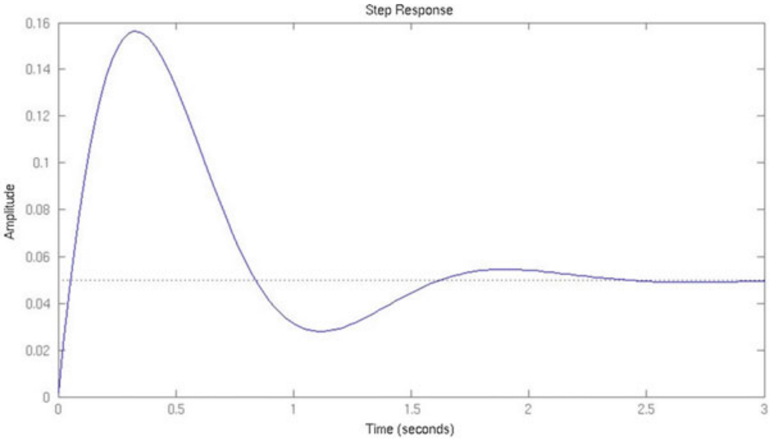
Solution In order to use any of the transfer function commands, MATLAB® requires that we first describe the system. To do this, we define two vectors, the first specifies the numerator coefficients of the descending powers of s (starting with the highest) and the second defines the denominator coefficients. Usually we use the variable names `num` and `den` for this purpose, but this is not required. For the transfer function used in this example, these are defined as `num = [1 1]` and `den = [1 4 20]`. The coefficient of the zeroth power of s (constant term) is the rightmost term in the vectors and then the coefficients of ascending powers proceed to the left. If a power of s is not present, a 0 must be entered for that coefficient at the appropriate position in the vector

We enter the following code at the MATLAB® command line to define the transfer function. Notice that MATLAB® displays the transfer function after the execution of the function `tf`.

```
>> num = [1 1];
>> den = [1 4 20];
>> sys = tf(num, den)
sys =
       s + 1
   --------------
   s^2 + 4 s + 20
Continuous-time transfer function.
```

The final statement tells us we have defined the correct transfer function. Next, if we want to observe the response of the system to a unit step input, we type the following.

```
>> step(sys)
```



Notice that `step(sys)` causes MATLAB® to calculate and plot the unit step response of the system. Thus, the input is $F(s) = 1/s$ and $X(s)$ is given by the following expression.

$$X(s) = \left(\frac{s+1}{s^2+4s+20}\right)\frac{1}{s}$$

MATLAB® also places a dotted line on the graph to show the steady-state value approached by the system. Applying the final value theorem to $X(s)$ gives:

$$\lim_{t\to 0} x(t) = \lim_{s\to 0}(sX(s)) = \lim_{s\to 0}\left(\frac{s+1}{s^2+4s+20}\right) = 0.05,$$

which agrees with the dotted line location. When we need to determine the response to a step input that has a magnitude different from 1, we use the system linearity. Suppose the system input is $F_0 \cdot u(t)$, then $X(s)$ becomes:

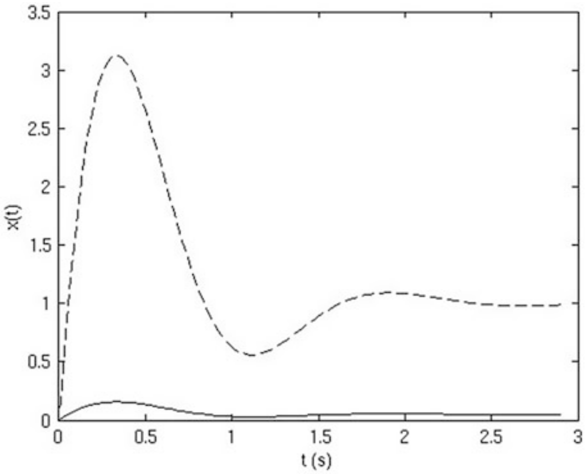
$$X(s) = \left(\frac{s+1}{s^2+4s+20}\right)\frac{F_0}{s} = F_0\left[\left(\frac{s+1}{s^2+4s+20}\right)\frac{1}{s}\right]$$

and the expression in the square brackets is the unit step response in the Laplace domain. Due to linearity, the response of the system is simply the unit step response multiplied by F_0. The `step` command can be used to store the values of the system response using:

```
>> [xu, t] = step(sys);
```

This assigns the step response and associated time to the vectors `xu` and `t`, respectively, for further processing. If we now want the response to a step input with a magnitude 20, we simply multiply `xu` by 20. We can then plot both the new response and the unit step response on the same graph.

```
>> x = 20*xu;
>> plot(t, xu, 'k-', t, x, 'k--')
>> xlabel('t (s)')
>> ylabel('x(t)')
```



The multiple entries in the `plot` command enable us to graph both responses on the same figure. Note that we specified the step response to be a solid (-) black (`k`) line and the magnitude 20 response to be a dashed (−−) black (`k`) line. The dashed line is simply scaled to be 20 times greater than the solid line at each instant in time.

Example 4.8 Consider the second-order transfer function relating the output displacement to an input force.

$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + 10s + 50}$$

Use MATLAB® to find and plot the response of the system to a unit impulse input, $\delta(t)$, by applying the `impulse` command.

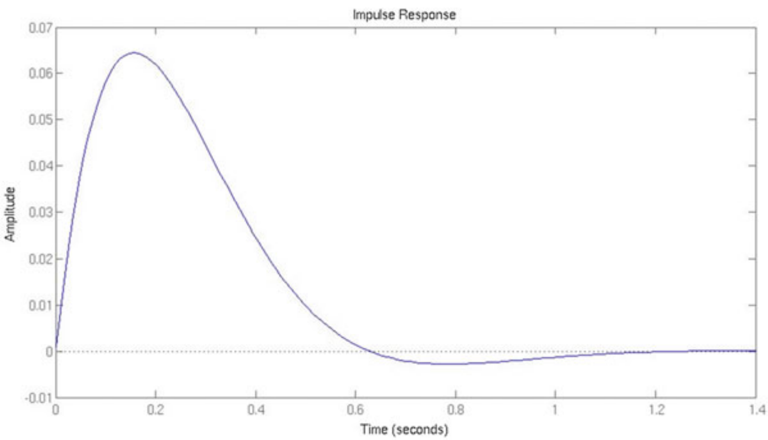
Solution The system is again defined using the vectors `num` and `den`. Then the `impulse` command is used to find and plot the response. The code is provided.

```
clear all
```

```
clc
close all
% System definition
num = [1];
den = [1 10 50];
sys = tf(num, den)

% Plot the response
impulse(sys);
```

MATLAB® displays the transfer function and produces the plot.

```
sys =
        1
   ---------------
   s^2 + 10 s + 50
Continuous-time transfer function.
```

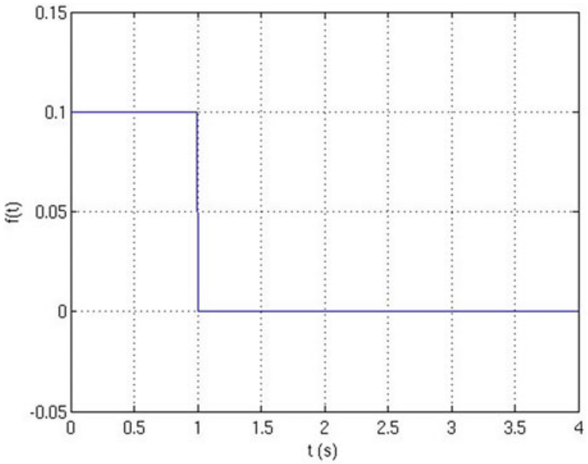


Notice that the system response to an impulse is qualitatively similar to the response to an initial velocity shown in Example 4.5. This is because an impulsive force results in an instantaneous change in momentum (velocity) of the system.

Example 4.9 Consider the following transfer function relating displacement to an input force.

$$\frac{X(s)}{F(s)} = \frac{2s + 104}{s^2 + 4s + 104}$$

Plot the response of the system to the terminated step input, $f(t)$, that is 0.1 for 1 s (starting from $t = 0$) and then returns to 0. Find the response for a total time of 4 s.



Solution To define the step input, the `find` command is a useful resource. The `find` command determines the indices of a vector that satisfy the specified criteria. For example, if we want to know the indices of the entries of a vector `A` that are greater than 4, we use the following code.

```
>> A = [1 2 8 9 0 10 3];
>> index = find(A>4)
index =
     3   4   6
```

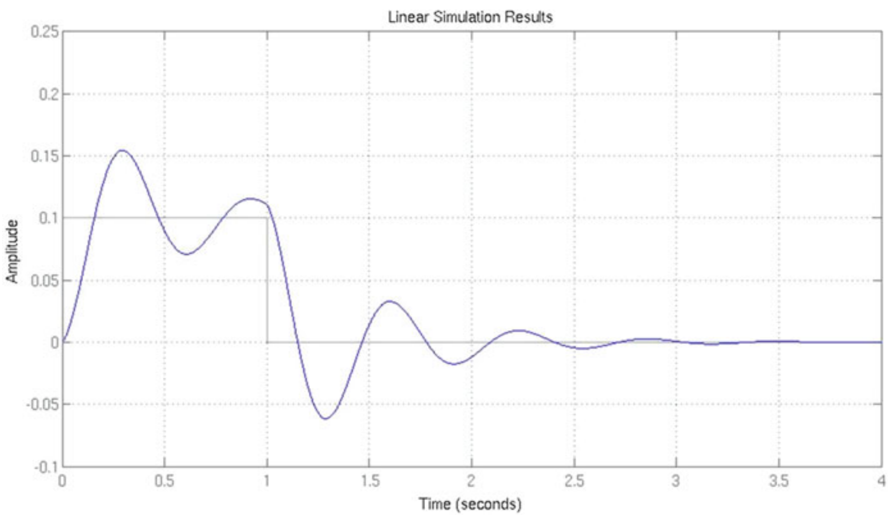
The `index` vector tells us that entries 3, 4, and 6 of the vector `A` are all greater than 4, which can be verified by examination.

We use the `find` command in a similar way to generate the force input. We first generate a time vector spanning the interval from 0 to 4. We then initialize $f(t)$ to zero. We next determine the indices of the time vector for which the time is less than 1. Finally, we assign a new value of 0.1 to all entries of $f(t)$ that correspond to those indices.

```
clear all
clc
close all

% Define the input function
t = [0:0.001:4];
f = zeros(1, length(t));
index = find(t<1);
f(index) = 0.1;

% Plot the input function
figure(1)
```

```
plot(t, f)
xlabel('t (s)')
ylabel('f(t)')
axis([0 4 -0.05 0.15])
grid

% Define the transfer function
num = [2 104];
den = [1 4 104];
sys = tf(num, den)

% Plot the input and the output response
figure(2)
lsim(sys, f, t);
grid
```

The result of the `lsim` command is shown.



Note that the system first responds with a step upward. If the step had persisted after 1 s, then we would have obtained a typical step response. However, when the step force is terminated at 1 s, the system oscillates and returns to zero.

Example 4.10 Find the transfer function for a vehicle driving over a bumpy road. The model is shown in Fig. 4.9. The coordinate x is defined to be zero at the loaded equilibrium position so the gravitational force does need not be considered (as described previously, the equilibrium spring force balances the vehicle weight). The vehicle mass is 750 kg and the suspension can be modeled by a single equivalent spring with a stiffness of 75000 N/m and a single equivalent damper with a coefficient of 9000 N-s/m. The tires are assumed to follow the road and provide the input, $x_{in}(t)$, to the bottom of the suspension. The output is the vehicle

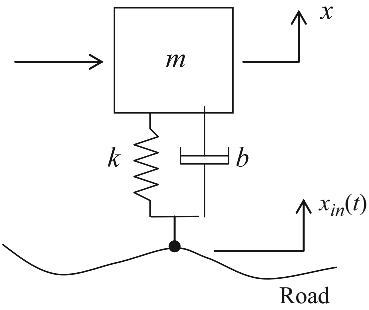
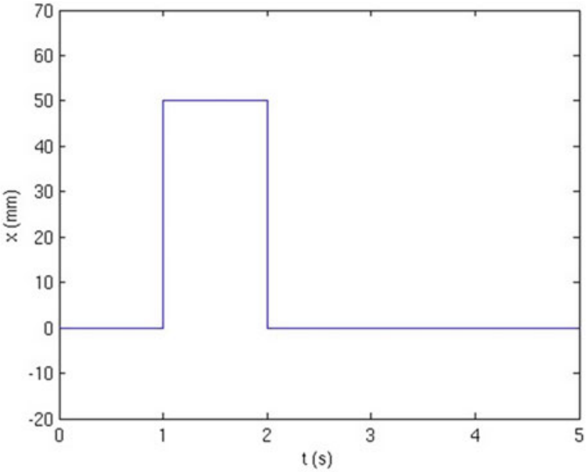
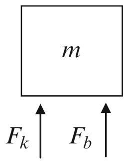


Fig. 4.9 Model of a vehicle where $x_{in}(t)$ is the input from the road displacing the vehicle tire and the output is the vehicle displacement, $x(t)$. The suspension is modeled as a single equivalent spring and damper

motion, $x(t)$. Determine the natural frequency, damping ratio, and the damped natural frequency for the model. Then find the response to a simulated 50 mm square "bump", simulate the motion for 5 s, and plot the response in units of millimeters.



Solution To model the system we begin with the vehicle free body diagram.



The equation of motion is obtained from Newton's second law.

$$F_k + F_b = m\ddot{x}$$

The spring force is positive when $x_{in} > x$ and the damper force is positive when $\dot{x}_{in} > \dot{x}$. Writing the forces in terms of the input and output gives the equation of motion in the time domain.

$$k(x_{in} - x) + b(\dot{x}_{in} - \dot{x}) = m\ddot{x}$$

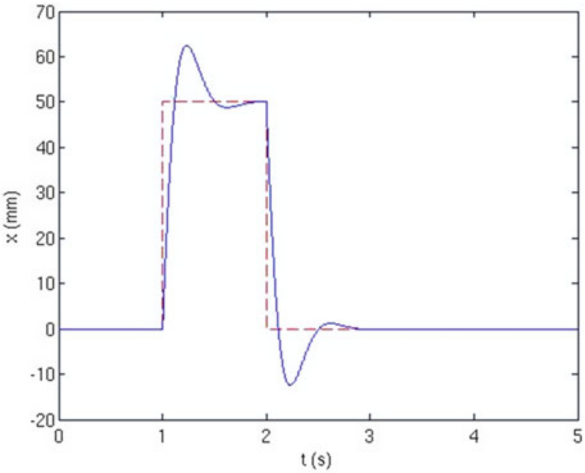
We rearrange this equation so the output variable (x) is on the left-hand side of the equation and the input variable (x_{in}) is on the right-hand side. We then transform this time domain equation into the Laplace domain, while assuming zero initial conditions. This gives the transfer function.

$$\frac{X(s)}{X_{in}(s)} = \frac{bs + k}{ms^2 + bs + k}$$

This system is second order. The natural frequency, damping ratio, and damped natural frequency are 10 rad/s (1.6 Hz), 0.6, and 8 rad/s (1.27 Hz), respectively. To define the input function, plot the input, and find and plot the response of the vehicle to this input, we use the following code.

```
clear all
clc
close all

% Parameters
m = 750;
k = 75000;
b = 9000;
wn = sqrt(k/m)
fn = wn/(2*pi)
zeta = b/(2*sqrt(k*m))
wd = wn*sqrt(1-zeta^2)
fd = wd/(2*pi)

% Define the input
t = [0:0.001:5];
xin = zeros(1, length(t));
index = find(t>1 & t<2);
xin(index) = 0.05;

figure(1)
plot(t, xin*1000)      % mm
xlabel('t (s)');
ylabel('x (mm)');
axis([0 5 -20 70])

% Define the system
num = [b k];
```

```
den = [m b k];
sys = tf(num, den);
[x, t] = lsim(sys, xin, t);

figure(2)
plot(t, 1000*xin, 'r-', t, 1000*x, 'b-')
xlabel('t (s)')
ylabel('x (mm)')
```



Note that the vehicle body overshoots the displacement generated by the "bump," oscillates at the damped natural frequency, and the motion quickly begins to attenuate. When the bump terminates, the vehicle body eventually returns to its zero position, but first overshoots (in the negative x-direction) as it compresses the suspension before the motion finally damps out.

4.6 Multiple Degree of Freedom Systems

In the previous examples, the systems were second order. The order of a system is typically set equal to the number of independent energy storage elements in the system. By "independent," we mean elements that cannot be combined into a single equivalent element as described in Chap. 3. Thus, a system with a single equivalent spring and a single mass is second order. A system with two masses and two equivalent springs is fourth order. A third-order system results when one of the masses in a two-mass system is negligible compared to the other mass. In this section, we examine how to develop the equations of motion and transfer functions for systems with an order greater than 2. We will also discuss how the behavior of these systems can be deduced from the roots of the characteristic equation.

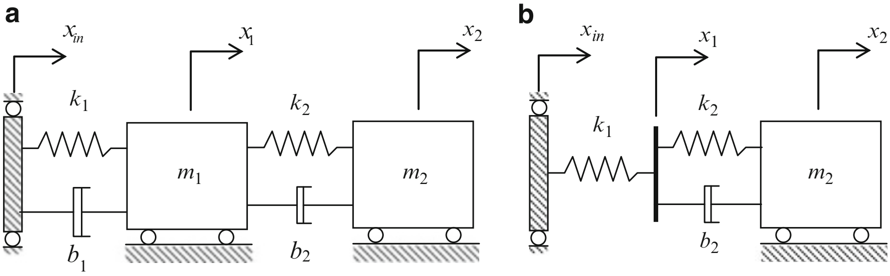
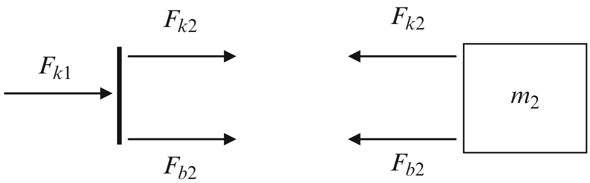


Fig. 4.10 (**a**) A fourth-order system containing two masses and two springs; and (**b**) a third-order system formed from (**a**) by setting m_1 and b_1 equal to zero. The input is the displacement of the "wall" to the left of the spring k_1

A fourth-order system is shown in Fig. 4.10a. It is fourth order because it contains four energy storage elements, two masses and two springs. The system becomes third order as shown in Fig. 4.10b if the mass m_1 is reduced to zero leaving only three energy storage elements. The damper b_1 is also zero for this example. The massless element ($m_1 = 0$) is indicated by the thick black line.

To develop the equations of motion for the third-order system depicted in Fig. 4.10b, we use the free body diagrams for the massless element and m_2.



Since the springs and dampers are assumed to be massless, they admit no force imbalance across them. Therefore, the forces F_{k2} and F_{b2} acting to the right on the massless element act to the left on the mass (i.e., the forces are equal and opposite). Applying Newton's second law to the massless element gives Eq. (4.44).

$$F_{k1} + F_{k2} + F_{b2} = 0 \tag{4.44}$$

The inertial term is zero and, therefore, the sum of the forces is zero. This is equivalent to saying that the forces acting on this element are in static equilibrium, but physically the system is not static. If the forces are out of balance on the massless element, it will accelerate instantaneously to eliminate that imbalance. This is approximately true in many practical systems where the mass of one part of the system is significantly less than the masses of all other elements.

The force directions on the free body diagrams are sign conventions only. They do not determine the directions in which the forces actually act. During a dynamic motion, the forces change direction periodically. The forces F_{k1}, F_{k2}, and F_{b2} act in the directions shown when $x_{in} > x_1$, $x_2 > x_1$, and $\dot{x}_2 > \dot{x}_1$, respectively. We can therefore rewrite Eq. (4.44) in terms of the positions and velocities of the elements.

$$k_1(x_{in} - x_1) + k_2(x_2 - x_1) + b_2(\dot{x}_2 - \dot{x}_1) = 0 \tag{4.45}$$

Applying Newton's second law to the free body diagram for mass m_2, we obtain Eq. (4.46):

$$-F_{k2} - F_{b2} = m_2\ddot{x}_2, \tag{4.46}$$

which can then be rewritten in terms of the model coordinates.

$$-k_2(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1) = m_2\ddot{x}_2 \tag{4.47}$$

We rearrange these equations algebraically to obtain Eqs. (4.48) (rewriting Eq. 4.47) and (4.49) (rewriting Eq. 4.45). These two equations represent coupled[4] first- and second-order differential equations of motion in the time domain.

$$m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2x_2 = k_2x_1 + b_2\dot{x}_1 \tag{4.48}$$

$$b_2\dot{x}_1 + (k_1 + k_2)x_1 = b_2\dot{x}_2 + k_2x_2 + k_1x_{eq} \tag{4.49}$$

We can consider the output of the system to be the motions x_1 or x_2. We write the transfer function to relate the output to the input, x_{in}. To find the transfer functions, we first determine the Laplace transform of Eq. (4.48).

$$\left(m_2s^2 + b_2s + k_2\right)X_2(s) = (b_2s + k_2)X_1(s) \tag{4.50}$$

Next, we write the Laplace transform of Eq. (4.49).

$$(b_2s + k_1 + k_2)X_1(s) = (b_2s + k_2)X_2(s) + k_1X_{in}(s) \tag{4.51}$$

Combining Eqs. (4.50) and (4.51), we can rearrange to obtain the transfer functions for the system.

$$\begin{aligned} \frac{X_1(s)}{X_{eq}(s)} &= \frac{m_2k_1s^2 + b_2k_1s + k_1k_2}{m_2b_2s^3 + m_2(k_1 + k_2)s^2 + b_2k_1s + k_1k_2} \\ \frac{X_2(s)}{X_{eq}(s)} &= \frac{b_2k_1s + k_1k_2}{m_2b_2s^3 + m_2(k_1 + k_2)s^2 + b_2k_1s + k_1k_2} \end{aligned} \tag{4.52}$$

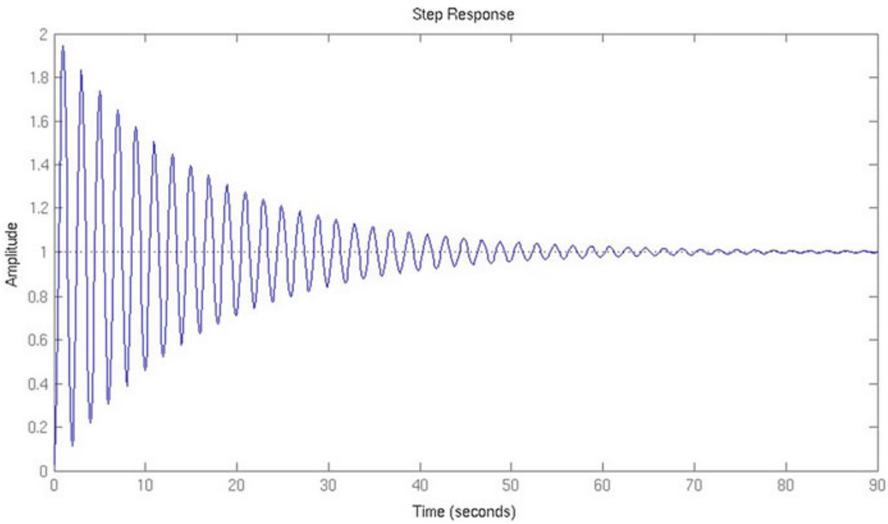
Example 4.11 Suppose the system shown in Fig. 4.10b is subjected to a unit step input. If $m_2 = 1$ kg, $b_2 = 80$ N-s/m, $k_1 = 10$ N/m, and $k_2 = 20$ N/m, find and plot the step response for the system $x_2(t)$. Relate the observed step response to the roots of the characteristic equation.

[4] Equations (4.48) and (4.49) are coupled because they both contain x_1 and x_2.

Solution We analyze Eq. (4.52) using the following MATLAB® code.

```
clear all
clc
close all

% Parameters
m2 = 1;
k1 = 10;
k2 = 20;
b2 = 80;

% Define the system
num = [b2*k1 k1*k2];
den = [m2*b2 m2*(k1+k2) b2*k1 k1*k2];
roots(den)
sys = tf(num, den);
step(sys)
```

The resulting step response is shown.

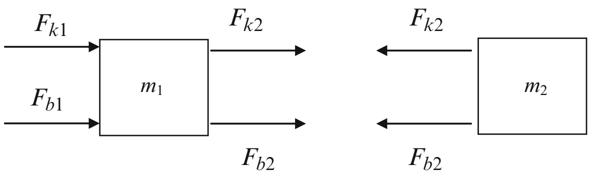


The roots of the characteristic equation determine the system behavior. The MATLAB® command `roots` can be used to interpret the graph. The roots of the denominator are given by the following.

```
>> roots(den)
ans =
   -0.0621 + 3.1567i
   -0.0621 - 3.1567i
   -0.2508 + 0.0000i
```

The roots include a complex conjugate pair and a real negative root. The complex conjugate pair gives an exponentially decaying oscillatory response, while the negative real root results in a decaying exponential. The frequency of the oscillatory motion corresponds to the imaginary part of the complex conjugate pair: 3.16 rad/s or 0.5 Hz. Approximately one oscillation can be seen in the first 2 s of the step response, which corresponds to the 0.5 Hz frequency. The time constant for the complex conjugate pair is determined from the inverse of the real part of the root, or 16.1 s. Therefore, it should take approximately four time constants, or 64.4 s, for these oscillations to damp out. This matches the graphical result, where the oscillations are seen to attenuate in approximately 65 s. Finally, we see that the response initially overshoots the final value of 1.0 and then exponentially decays back to that value over several cycles of oscillation. This trend corresponds to the real root of the system, which has a time constant of 3.99 s. The exponential trend (which is effectively masked by the large oscillations) approaches its steady-state value in approximately 16 s.

Next, consider the fourth-order system displayed in Fig. 4.10a. To identify the equations of motion for this system, we again use two free body diagrams.



Applying Newton's second law to mass m_1, we obtain Eq. (4.53).

$$F_{k1} + F_{b1} + F_{k2} + F_{b2} = m_1\ddot{x}_1 \tag{4.53}$$

Using similar reasoning to that described for the third-order system, we can write the forces in terms of the displacement variables and their derivatives. We rearrange algebraically to obtain Eq. (4.54).

$$m_1\ddot{x}_1 + (b_1 + b_2)\dot{x}_1 + (k_1 + k_2)x_1 = b_1\dot{x}_{in} + k_1x_{in} + b_2\dot{x}_2 + k_2x_2 \tag{4.54}$$

Considering the second free body diagram and applying Newton's second law, we again obtain Eq. (4.48) and, after performing the Laplace transform, Eq. (4.50). By transforming Eq. (4.54) into the Laplace domain and inserting the result into Eq. (4.50), we obtain the transfer function that relates $X_1(s)$ to $X_{in}(s)$.

$$\frac{X_1(s)}{X_{in}(s)} = \frac{m_2b_1s^3 + (b_1b_2 + m_2k_1)s^2 + (b_1k_2 + b_2k_1)s + k_1k_2}{m_1m_2s^4 + (m_1b_2 + m_2b_1 + m_2b_2)s^3 + (m_1k_2 + m_2k_1 + m_2k_2 + b_1b_2)s^2 + (b_1k_2 + b_2k_1)s + k_1k_2} \tag{4.55}$$

From Eq. (4.50) we obtain:

$$X_1(s) = \frac{b_2 s + k_2}{m_2 s^2 + b_2 s + k_2} X_2(s), \tag{4.56}$$

which can be substituted into Eq. (4.55) to obtain the transfer function that relates $X_2(s)$ to $X_{in}(s)$.

$$\frac{X_2(s)}{X_{in}(s)} = \frac{b_1 b_2 s^2 + (b_1 k_2 + b_2 k_1)s + k_1 k_2}{m_1 m_2 s^4 + (m_1 b_2 + m_2 b_1 + m_2 b_2)s^3 + (m_1 k_2 + m_2 k_1 + m_2 k_2 + b_1 b_2)s^2 + (b_1 k_2 + b_2 k_1)s + k_1 k_2} \tag{4.57}$$

We use Eqs. (4.55) and (4.57) to determine the motion of the fourth-order system shown in Fig. 4.10a in response to any input; we complete this action in MATLAB® using the `lsim` command. The characteristic equation for the two transfer functions is the same (i.e., the denominators are the same). This is expected because the characteristic equation describes the fundamental behavior of the system (natural frequencies and damping) and both transfer functions describe the motion of the same system.

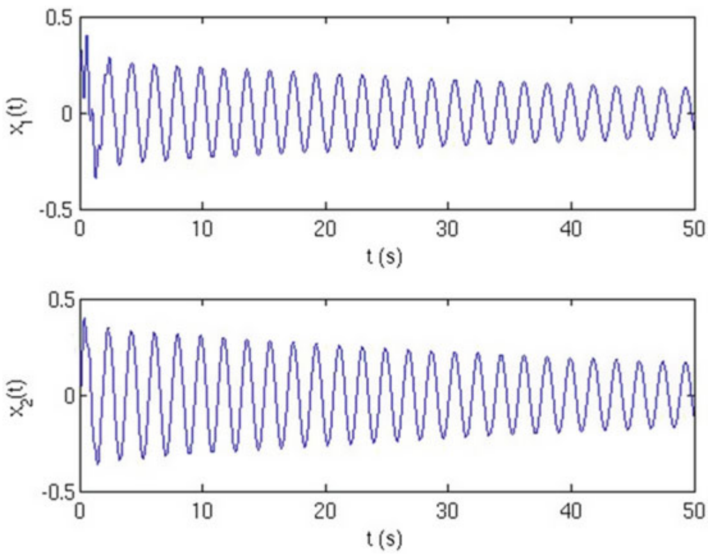
Example 4.12 Consider a system that can be modeled as shown in Fig. 4.10a. The values of the parameters are $m_1 = 10$ kg, $m_2 = 5$ kg, $k_1 = 500$ N/m, $k_2 = 200$ N/m, $b_1 = 5$ N-s/m, and $b_2 = 0.25$ N-s/m. Find the response of the system to an impulse with a magnitude of 500 N-s (or kg-m/s) and plot $x_1(t)$ and $x_2(t)$ on two subplots with one vertically positioned above the other. Use the `subplot` command in MATLAB®.

Solution The code used to plot the responses for the system is provided here.

```
clear all
clc
close all

% Parameters
m1 = 5;
m2 = 10;
k1 = 200;
k2 = 500;
b1 = 0.25;
b2 = 5;
P0 = 0.1;     % N/s

% Define the systems
num1 = [m2*b1 (b1*b2+m2*k1) (b1*k2+b2*k1) k1*k2];
den1 = [m1*m2 (m1*b2+m2*b1+m2*b2) (m1*k2+m2*k1+m2*k2+b1*b2)
(b1*k2+b2*k1) k1*k2];
sys1 = tf(num1, den1);
```

```
num2 = [b1*b2 (b1*k2+b2*k1) k1*k2];
den2 = den1;
sys2 = tf(num2, den2);

% Find and plot the response
[x1u, t] = impulse(sys1);
x1 = P0*x1u;
[x2u, t] = impulse(sys2, t);
x2 = P0*x2u;

figure(1)
subplot(211)
plot(t, x1)
xlabel('t (s)')
ylabel('x_1(t)')
subplot(212)
plot(t, x2)
xlabel('t (s)')
ylabel('x_2(t)')

roots(den1)
```

The plots provided here demonstrate an overall damped behavior, but with a different character than either the second- or third-order responses we have examined so far.



The roots of the characteristic equation describe the behavior of the system. Using MATLAB®, the roots are determined for the selected parameter values.

```
>> roots(den1)
ans =
   -0.7596 +13.3497i
   -0.7596 -13.3497i
   -0.0154 + 3.3445i
   -0.0154 - 3.3445i
```

The roots are grouped into two complex conjugate pairs, which indicated a damped oscillation associated with each. The frequency of the second set of roots is 3.3 rad/s (0.5 Hz); the time constant is 64.9 s. This is seen in the plots, which show approximately five oscillations in the first 10 s that damp out very slowly. The first set of roots has a frequency of 13.3 rad/s (2.1 Hz) with a time constant of 1.3 s. These oscillations are observed in approximately the first 4 s of the $x_1(t)$ plot. The two frequencies of 0.5 and 2.1 Hz are associated with the two *modes* of the system. Modes are characteristic shapes of oscillation in a system and are discussed further in Chap. 11.

4.7 Summary

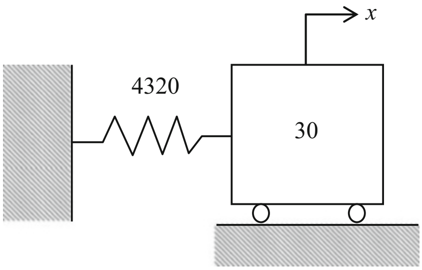
In this chapter, we discussed the following key elements.

- Mechanical oscillations are repetitive motions that result from a trading of energy between kinetic energy and elastic potential energy.
- Transient mechanical oscillations damp out over time as the system settles to an equilibrium position.
- Mechanical systems consisting of combinations of springs, masses, and dampers can be modeled by applying Newton's second law to obtain differential equations of motion in the time domain.
- Motions of mechanical systems can be determined by applying Laplace transform techniques to solve the differential equations of motion and these solutions show exponential (damped) and sinusoidal (oscillatory) behavior.
- A transfer function is the ratio of the output to input of a system in the Laplace domain assuming zero initial conditions.
- The denominator of a transfer function is known as the characteristic equation of a system.
- The order of a system is given by the highest power of s in the characteristic equation.
- Second-order systems have three important descriptors: the natural frequency, ω_n, the damping ratio, ζ, and the damped natural frequency, ω_d.
- The natural frequency of a second-order system is the characteristic frequency of oscillation of the system when the damping is zero; a simple harmonic oscillator that is excited by a transient input or initial conditions will oscillate at its natural frequency indefinitely.
- The damping ratio determines whether a system will oscillate or not.

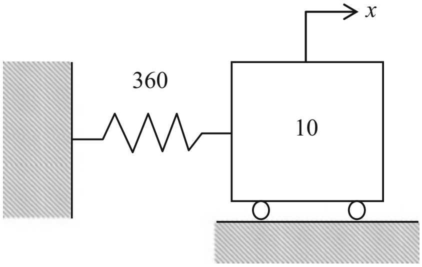
- If the damping ratio of a second-order system is less than 1, the system is underdamped and its transient motion will exhibit oscillations at its damped natural frequency.
- If the damping ratio of a second-order system is greater than 1, the system is overdamped and its transient response will be exponential and it will not oscillate.
- If the damping ratio of a system is equal to 1, the system is critically damped.
- The order of a system is determined by the number of energy storage elements in the system. For example, a system with two masses and two springs is fourth order.

Problems

1. Consider the following model for a mechanical dynamic system consisting of a linear spring with stiffness 4320 N/m that is connected between a rigid wall and a 30 kg moving mass (frictionless rollers).



The initial position is $x = 0$ and the initial velocity is 6 m/s. Complete the following.
   (a) Find the equation of motion and solve for $x(t)$ using Laplace transforms.
   (b) Check your answer for part (a) using the `ilaplace` command in MATLAB®.
   (c) Substitute your answer from part (a) into the original equation of motion to verify that it is correct.
   (d) Find the natural frequency ω_n and the period of one oscillation. Use these values to plot $x(t)$ in MATLAB® for five oscillation cycles. Use a vertical axis span from -2.0 m to 2.0 m.
2. Consider the following model for a mechanical dynamic system consisting of a linear spring with stiffness 360 N/m that is connected between a rigid wall and a 10 kg moving mass (frictionless rollers).



The initial position is $x = 0$ and the initial velocity is 2 m/s. Complete the following.

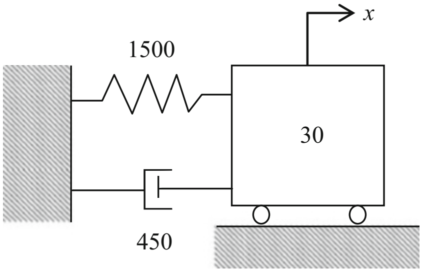
(a) Find the equation of motion and solve for $x(t)$ using Laplace transforms.

(b) Check your answer for part (a) using the `ilaplace` command in MATLAB®.

(c) Substitute your answer from part (a) into the original equation of motion to verify that it is correct.

(d) Find the natural frequency ω_n and the period of one oscillation. Use these values to plot $x(t)$ in MATLAB® for five oscillation cycles. Use a vertical axis span from -2.0 m to 2.0 m.

3. Consider the following model for a mechanical dynamic system consisting of a linear spring with stiffness 1500 N/m and a linear viscous damper with damping 450 N-s/m that are connected between a rigid wall and a 30 kg moving mass (frictionless rollers).



The initial position is $x = 0$ and the initial velocity is 1 m/s. Complete the following.

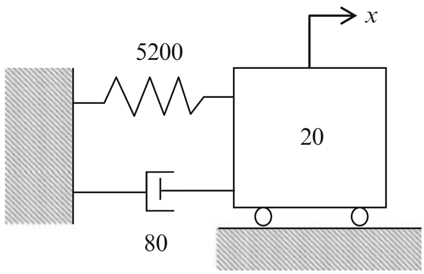
(a) Find the equation of motion and solve for $x(t)$ using Laplace transforms and check using the `ilaplace` command in MATLAB®.

(b) Calculate the natural frequency ω_n, the damping ratio ζ, and the dominant time constant for this system.

(c) Using the symbolic manipulation capability in MATLAB®, show that your answer satisfies the initial conditions and solves the original equation of motion.

(d) In MATLAB®, plot $x(t)$ for a duration equal to four times the dominant time constant.

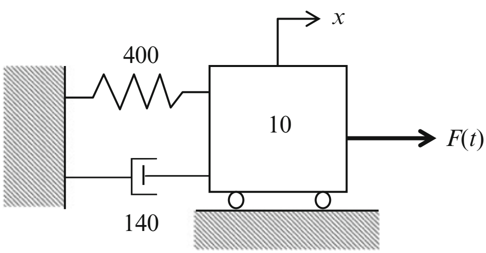
4. Consider the following model for a mechanical dynamic system consisting of a linear spring with stiffness 5200 N/m and a linear viscous damper with damping 80 N-s/m that are connected between a rigid wall and a 20 kg moving mass (frictionless rollers).



The initial position is $x = 1$ m and the initial velocity is 0 m/s. Complete the following.

(a) Find the equation of motion and solve for $x(t)$ using Laplace transforms and check using the `ilaplace` command in MATLAB®.
(b) Calculate the natural frequency ω_n, the damping ratio ζ, and the dominant time constant for this system.
(c) Using the symbolic manipulation capability in MATLAB®, show that your answer satisfies the initial conditions and solves the original equation of motion.
(d) In MATLAB®, plot $x(t)$ for a duration equal to four times the dominant time constant.

5. Consider the following model for a mechanical dynamic system consisting of a linear spring with stiffness 400 N/m and a linear viscous damper with damping 140 N-s/m that are connected between a rigid wall and a 10 kg moving mass (frictionless rollers).

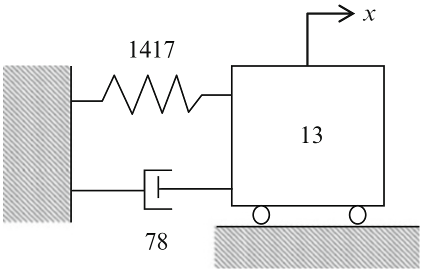


The initial position and velocity are zero. There is a nonzero input force.
$F(t) = 800 \cdot \mathrm{u}(t)$ N
Complete the following.

(a) Find the equation of motion and solve for $x(t)$ using Laplace transforms and check using the `ilaplace` command in MATLAB®.
(b) Calculate the natural frequency ω_n, the damping ratio ζ, and the dominant time constant for this system.

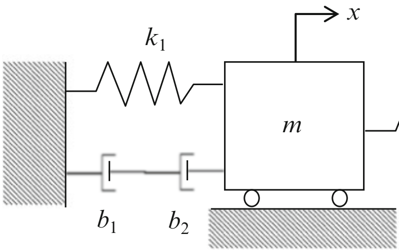
(c) Using the symbolic manipulation capability in MATLAB®, show that your answer satisfies the initial conditions and solves the original equation of motion.
(d) In MATLAB®, plot $x(t)$ for a duration equal to four times the dominant time constant.

6. Consider the following model for a mechanical dynamic system consisting of a two linear springs with stiffnesses 1417 N/m and a linear viscous damper with damping 78 N-s/m that are connected between a rigid wall and a 13 kg moving mass (frictionless rollers).



The initial position and velocity are zero. There is a nonzero input force.
$F(t) = 2834 \cdot u(t)\,\text{N}$
Complete the following.
(a) Find the equation of motion and solve for $x(t)$ using Laplace transforms and check using the `ilaplace` command in MATLAB®.
(b) Calculate the natural frequency ω_n, the damping ratio ζ, and the dominant time constant for this system.
(c) Using the symbolic manipulation capability in MATLAB®, show that your answer satisfies the initial conditions and solves the original equation of motion.
(d) In MATLAB®, plot $x(t)$ for a duration equal to four times the dominant time constant.

7. Consider the following model for a mechanical dynamic system. The system has two linear springs with stiffnesses k_1 and k_2, two linear viscous dampers with damping constants b_1 and b_2, and a mass m (frictionless rollers).

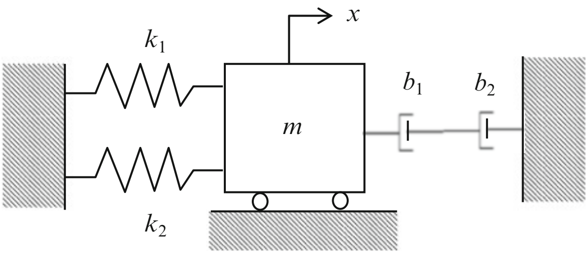


The values of the system constants are provided.

$$\begin{aligned} m &= 1\,\text{kg} \\ k_1 &= 300\,\text{N/m} \\ k_2 &= 350\,\text{N/m} \\ b_1 &= 12\,\text{N-s/m} \\ b_2 &= 60\,\text{N-s/m} \end{aligned}$$

The initial position is $x = 0$ and the initial velocity is 1 m/s. Find the equation of motion for the system. Solve for the motion $x(t)$ and plot the motion for a duration equal to four times the system time constant.

8. Consider the following model for a mechanical dynamic system. The system has two linear springs with stiffnesses k_1 and k_2, two linear viscous dampers with damping constants b_1 and b_2, and a mass m (frictionless rollers).



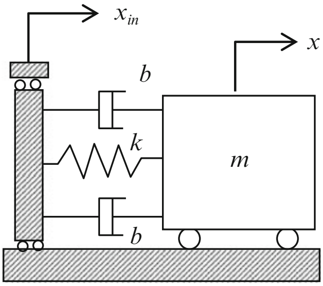
The values of the system constants are provided.

$$\begin{aligned} m &= 1\,\text{kg} \\ k_1 &= 200\,\text{N/m} \\ k_2 &= 450\,\text{N/m} \\ b_1 &= 30\,\text{N-s/m} \\ b_2 &= 15\,\text{N-s/m} \end{aligned}$$

The initial position is $x = 0$ and the initial velocity is 1 m/s. Find the equation of motion for the system. Solve for the motion $x(t)$ and plot the motion for a duration equal to four times the system time constant.

9. Consider the following model for a mechanical dynamic system. The system has a mass m, linear spring with stiffness k, and two identical viscous dampers

with damping constant b. The left wall generates an input motion $x_{in}(t)$ and this causes the mass to undergo a displacement $x(t)$ from its equilibrium position.



The values of the system constants are provided.

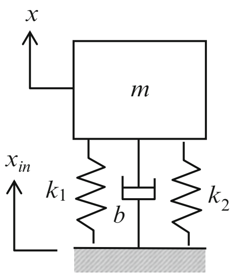
$$m = 1\,\text{kg}$$
$$k = 97\,\text{N/m}$$
$$b = 4\,\text{N-s/m}$$

The initial position and velocity are zero. The base motion is given by the following function.

$$x_{in}(t) = 2 \cdot u(t)\ \text{m}$$

Find the equation of motion for the system and take the Laplace transform of the equations to find $X(s)$. Using the `ilaplace` in MATLAB®, solve for the motion $x(t)$ and plot the motion for a duration equal to four times the system time constant.

10. Consider the following model for a mechanical dynamic system. The system has two linear springs with stiffnesses k_1 and k_2, a viscous damper with damping constant b, and a mass m. The base generates an input motion $x_{in}(t)$ and this causes the mass to undergo a displacement $x(t)$ from its equilibrium position under gravity load.



The values of the system constants are provided.

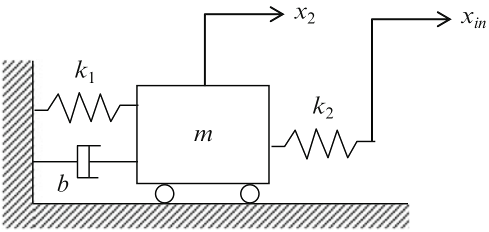
$$m = 1\,\text{kg}$$
$$k_1 = k_2 = 48.5\,\text{N/m}$$
$$b = 8\,\text{N-s/m}$$

The initial position and velocity are zero. The base motion is given by the following function.

$$x_{in}(t) = 2 \cdot u(t)\,\text{m}$$

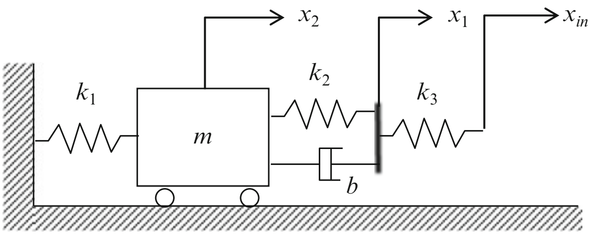
Find the equation of motion for the system. Solve for the motion *x(t)* and plot the motion for a duration equal to four times the system time constant.

11. Consider a system that can be modeled as shown. The input $x_{in}(t)$ is a prescribed motion at the right end of spring k_2. Find the system transfer function $\frac{X(s)}{X_{eq}(s)}$.



The values of the parameters are $m = 30$ kg, $k_1 = 700$ N/m, $k_2 = 1300$ N/m, and $b = 200$ N-s/m. Write a MATLAB® script file that: (a) calculates the natural frequency, damping ratio, and damped natural frequency for the system; and (b) uses the `impulse` command to find and plot the response of the system to a unit impulse input.

12. Consider a system that can be modeled as shown. The input $x_{in}(t)$ is a prescribed motion at the right end of spring k_3. Find the system transfer functions $\frac{X_1(s)}{X_{eq}(s)}$ and $\frac{X_2(s)}{X_{eq}(s)}$ in terms of the system parameters.



The values of the parameters are $m = 2$ kg, $k_1 = k_2 = k_3 = 20$ N/m, and $b = 30$ N-s/m. Assume the initial conditions are zero and use the `step` command in MATLAB® to find and plot the responses $x_1(t)$ and $x_2(t)$ to a unit step input.

Transient Rotational Motion of Mechanical Systems

5

5.1 Introduction

In Chap. 4, we focused on motions that occur along a linear path. Similar motions also occur in rotary systems where the variable is angular, rather than linear, displacement. Examples of rotational motions include:

- a swinging pendulum
- elastic twist in a rotating drive shaft
- roll, pitch, and yaw of an automobile on its suspension or a ship in the water
- shimmy of a shopping cart.

As with rectilinear motions, rotational motions also involve repeated trading of kinetic and potential energies. Energy loss occurs through rotational damping that extracts energy from the mechanical motions and causes them to attenuate over time. For viscous damping, which we often apply for mathematical convenience, the dissipative forces depend on the angular velocity. The amount of energy loss relative to the stored energy determines whether the motions are undamped,[1] underdamped, critically damped, or overdamped. The mathematical description of rotary motions is analogous to that for rectilinear motions. The definitions of natural frequency, damping ratio, and damped natural frequency are equivalent. This serves to highlight a critical aspect of the study of system dynamics: *modeling analogies exist between systems with very different physical structures*.

[1] An undamped system description, or model, will always be approximate, of course.

M.A. Davies, T.L. Schmitz, *System Dynamics for Mechanical Engineers*,
DOI 10.1007/978-1-4614-9293-1_5

5.2 The Simple Pendulum

The pendulum is a common physical system. The Foucault pendulum in Paris (1851) was used to first demonstrate that the Earth is rotating. Before the invention of the clock, Galileo developed sets of pendula to help early medical practitioners to measure the pulse of their patients. Shortly after, in 1656, Huygens invented the first pendulum-based clock, which was the most precise timekeeping device available until the 1930s. The pendulum is also a basic mechanical model for motions of many physical systems, including cranes, aircraft, and ships.

The simple pendulum is modeled as a mass at the end of a rigid bar with negligible mass. The rigid bar is connected to ground through a rotational joint as shown in Fig. 5.1a. The pendulum has a mass, m, and a length, L, and the rotational joint has (viscous) rotary damping, b_r, which produces a moment, M_b. The free body diagram for the pendulum is shown in Fig. 5.1b. It includes a dissipative moment induced at the bearing and the gravity force (weight), $F_g = mg$, acting on the mass. There may also be an applied input moment, $M(t)$, that drives the pendulum, but this is not required. Using Eq. (3.5) and summing moments about the joint center, O, we obtain Eq. (5.1).

$$\sum M_O = M(t) + M_b - F_g L \sin\theta = J_O \ddot{\theta} \tag{5.1}$$

The resistance moment from the bearing damping acts against the direction of the rotational velocity of the pendulum, $-b_r\dot{\theta}$. The (rotational) mass moment of inertia of the pendulum (i.e., the mass–bar combination) about O is mL^2. Substitution in Eq. (5.1) yields the time domain nonlinear equation of motion in Eq. (5.2).

$$mL^2\ddot{\theta} + b_r\dot{\theta} + mgL\sin\theta = M(t) \tag{5.2}$$

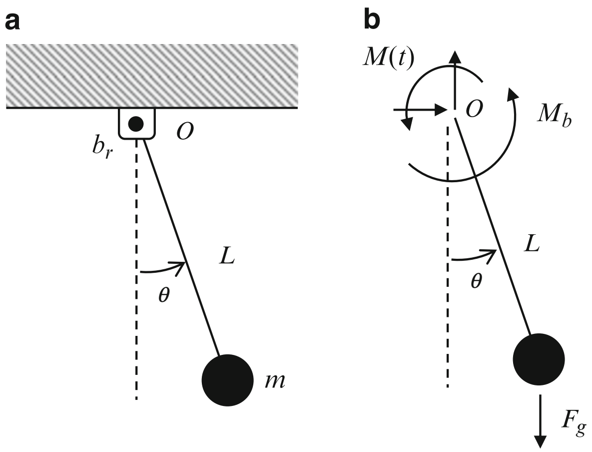


Fig. 5.1 (**a**) Simple pendulum model; and (**b**) free body diagram of the pendulum

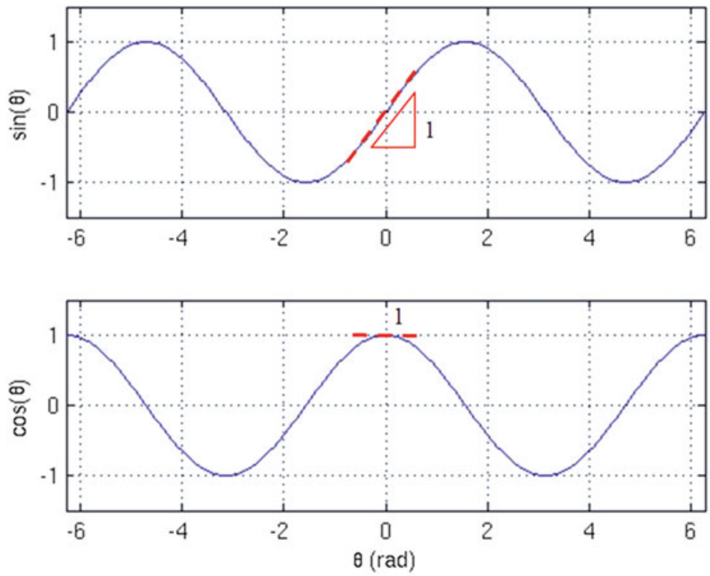


Fig. 5.2 Sine function (*top*) and cosine function (*bottom*) with their first-order approximations

The equation is *nonlinear* because the moment due to the weight of the pendulum depends on the sine of the angular displacement (and not just the displacement). The sine function is displayed in Fig. 5.2 (top). Graphically, the approximation of the sine function for a small angle, θ, is a line with a unit slope. This approximation has an error of less than 3% for angles as large as $\pi/6$ radians, or 30°. The Taylor series expansion of the sine function gives the same result.

$$\sin\theta|_{\theta=0} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \tag{5.3}$$

If θ is small, the first term in Eq. (5.3) dominates and the sine function can be approximated simply by θ (expressed in rad). Applying this approximation to Eq. (5.2), we obtain the linearized equation of motion for the pendulum; see Eq. (5.4).

$$mL^2\ddot{\theta} + b_r\dot{\theta} + mgL\theta = M(t) \tag{5.4}$$

This equation can be analyzed to determine the behavior of a pendulum under a various conditions. Notice that in this system, gravity assumes the role of the spring in Chap. 4. Gravity acts to restore the system to its zero position in the same way that the elastic spring restored the mass to its zero position for the simple harmonic oscillator. The pendulum motions involve a trading of rotational kinetic energy and gravitational potential energy, but, unlike the rectilinear harmonic oscillator, a single element, the mass, is storing both kinetic and potential energies. Thus, in effect, there are still two energy storage elements and the system is second order. The rotational damper removes energy from the mechanical motions and dissipates it as heat.

5.2.1 Simple Undamped Pendulum with Initial Velocity Only ($\theta(0) = 0$, $\dot{\theta}(0) = \omega_0$, $M(t) = 0$)

If the damping is zero and there is no input moment, the equation of motion simplifies to:

$$\ddot{\theta} + \omega_n^2 \theta = 0, \tag{5.5}$$

where we define the pendulum natural frequency as:

$$\omega_n = \sqrt{\frac{g}{L}}. \tag{5.6}$$

Calculating the Laplace transform of Eq. (5.5) for a zero initial angle and initial angular velocity of ω_0, we obtain:

$$\Theta(s) = \frac{\omega_0}{\omega_n}\left(\frac{\omega_n}{s^2 + \omega_n^2}\right). \tag{5.7}$$

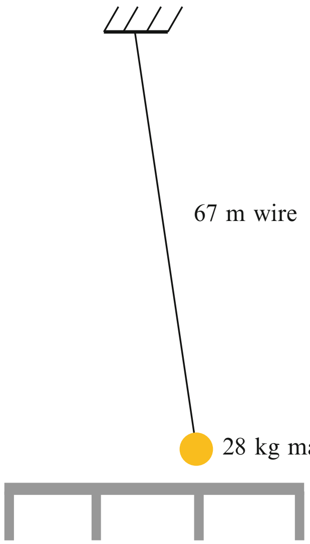
Note that Θ (capital θ), the Laplace domain representation of the pendulum's changing angle, is used to differentiate the Laplace domain variable from the time domain variable, θ. The inverse Laplace transform gives the angle in the time domain.

$$\theta(t) = \frac{\omega_0}{\omega_n} \sin(\omega_n t) \tag{5.8}$$

The pendulum oscillates continuously at its natural frequency, ω_n. The natural frequency is a function of the acceleration of gravity and the length of the pendulum, but is independent of the mass. The units of both angular velocity and natural frequency are rad/s. Thus, $\theta(t)$ is unitless (rad).

Example 5.1 A Foucault pendulum is a device originally conceived by the physicist Léon Foucault to demonstrate the Earth's rotation. The first demonstration took place in 1851 in the Paris Observatory and then the pendulum was rebuilt a few weeks later in the dome of the Panthéon in Paris, France. This pendulum consisted of a 28 kg brass-coated, lead bob suspended on a 67 m wire. An exact replica of the original is on display in the Panthéon today. The plane in which the pendulum swings rotates at 11°/h making a full rotation every 32.7 h. Given this information,

Fig. 5.3 Model of the Panthéon Foucault pendulum



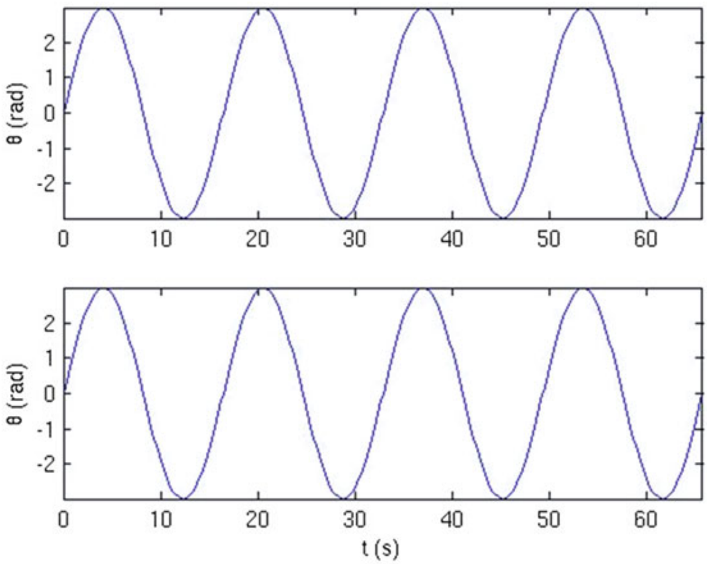
find the natural frequency of the pendulum. Assuming a zero initial position and an initial velocity of 0.02 rad/s, find $\theta(t)$. Use MATLAB® to find the inverse Laplace transform by applying `ilaplace`. Plot the angle in degrees and compare the result of `ilaplace` with Eq. (5.8) for a total of four oscillation cycles; place the two plots one above the other, using the `subplot` command.

Solution Using Eq. (5.8), we determine the natural frequency to be approximately 0.38 rad/s, or 0.06 Hz. Thus, the pendulum completes one oscillation every 16.4 s. Substituting the initial velocity and natural frequency into Eq. (5.8) gives the angle as a function of time, $\theta(t) = 0.052 \sin(0.38t)$. We can now use MATLAB® to verify the inverse Laplace transform and plot the time domain response.

```
clear all
clc
close all

% Parameters
L = 67;             % m
g = 9.81;           % m/s^2
wn = sqrt(g/L);     % rad/s
fn = wn/(2*pi);     % Hz
T = 1/fn;           % s
w0 = 0.02;          % rad/s

% Inverse Laplace transform
syms q Q t s;
Q = w0/wn*(wn/(s^2+wn^2));
q = ilaplace(Q)
```

```
% Time vector four periods long
t = [0:T/100:4*T];

% Evaluate the inverse Laplace transform
qq1 = eval(q);

% Use Equation 5.8 to define the motion
qq2 = w0/wn*sin(wn*t);

figure(1)
subplot(211)  % Upper plot
plot(t, qq1)
xlabel('t (s)')
ylabel('\theta (rad)')
axis([0 max(t) min(qq1) max(qq1)])
subplot(212)  % Lower plot
plot(t, qq2)
xlabel('t (s)')
ylabel('\theta (rad)')
axis([0 max(t) min(qq2) max(qq2)])
```

The corresponding plot is provided.



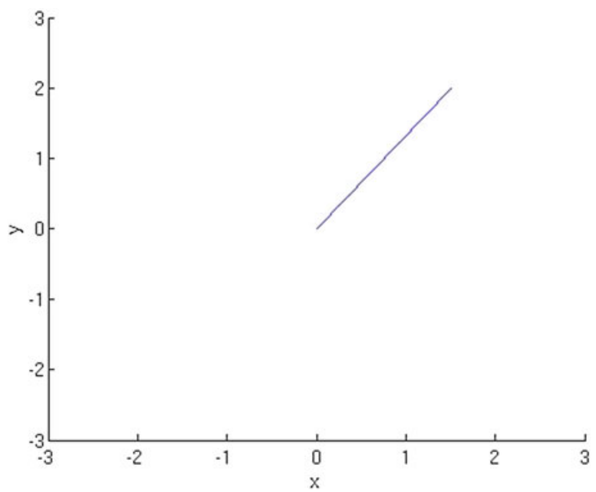
If you cannot visit Paris to view the Foucault pendulum in person, there are numerous YouTube videos available. It is instructive to view one and use a stopwatch to time the pendulum motion. The authors did this and obtained a measurement period of 17 ± 0.5 s, which agrees with the calculations.

Example 5.2 Rewrite the simulation from Example 5.1 using the MATLAB® `line` and `patch` commands to animate the pendulum motion. Use a line to represent the pendulum link and a patch to represent the bob.

Solution The `line` command draws a line from one point to another. The *x*-coordinates (horizontal) are defined by a vector of numbers, `xline`, and the *y*-coordinates (vertical) are defined by a second vector, `yline`. For example, the following code draws a line from the point (0,0) to the point (1.5,2) in a figure window. The line is displayed in a square figure window that spans from −3 to 3 in both the *x*- and *y*-directions.

```
clear all
clc
close all

% Parameters
xline = [0 1.5];  % Define the x-coordinates for both ends of the line
yline = [0 2];    % Define the y-coordinates for both ends of the line

% Draw the line
figure(1)
line(xline, yline)
axis([-3 3 -3 3])
xlabel('x')
ylabel('y')
```

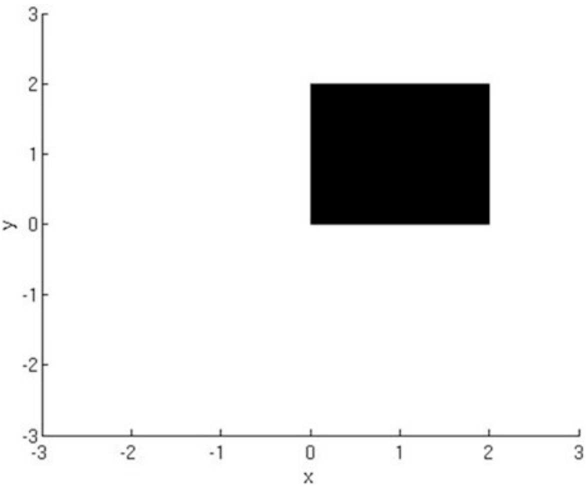
The result is provided.



The `patch(xpatch, xpatch, C)` command uses the vectors `xpatch` and `ypatch` to define the four corners of a filled solid which is then displayed on the screen with the color defined by `C` (e.g., black is denoted by `'k'`). The following code displays a black square patch in the center of the screen with its lower left corner at (0,0) and sides of length 2.

```
clear all
clc
close all

% Parameters
xpatch = [0 0 2 2]; % Define the x-coordinates for the patch corners
ypatch = [0 2 2 0]; % Define the y-coordinates for the patch corners

% Draw the patch
figure(1)
patch(xpatch, ypatch, 'k')
axis([-3 3 -3 3])
xlabel('x')
ylabel('y')
```

The result is provided.

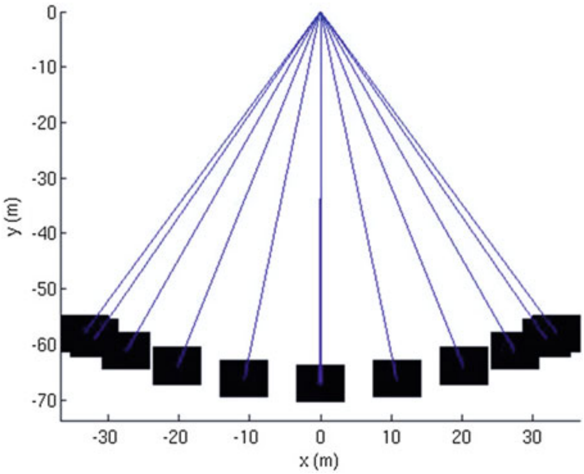


With these commands and a `for` loop, we can animate the pendulum motion simulated in Example 5.1. At each point we must define the start and endpoint of the line. We use (0,0) as the base point; the end point at any time is then ($-L\cos\theta$,

$L \sin \theta$). The patch (bob) is defined to be a square with sides that are $L/20$ in length. The MATLAB® code is provided.

```
clear all
clc
close all

% Parameters
L = 67;              % m
g = 9.81;            % m/s^2
wn = sqrt(g/L);      % rad/s
fn = wn/(2*pi);      % cyc/s
T = 1/fn;            % s
w0 = 0.2;            % rad/s

% Time vector four periods long
t = [0:T/50:4*T]; % coarse sampling for animation

% Use Equation 5.8 to define the motion
q = w0/wn*sin(wn*t);

% Draw a line from (0,0) to the pendulum end at each time instant
% After each display, erase the picture to obtain the animation
effect
for cnt = 1:length(t)
  clf  % Clear the figure after each display
  axis([-1.1*L/2 1.1*L/2 -1.1*L 0])
  xline = [0 L*sin(q(cnt))];
  yline = [0 -L*cos(q(cnt))];
  line(xline, yline)
  xlabel('x (m)')
  ylabel('y (m)')
  side = L/20;
  xpatch = [L*sin(q(cnt))+side L*sin(q(cnt))+side L*sin
           (q(cnt))-side L*sin(q(cnt))-side];
  ypatch = [-L*cos(q(cnt))+side -L*cos(q(cnt))-side -L*cos
          (q(cnt))-side -L*cos(q(cnt))+side];
  patch(xpatch, ypatch, 'k')
  pause(0.1) % Short pause ensures MATLAB will display the line
end
```

The results of the animations which are displayed dynamically by this code are shown superimposed in the following figure.



5.2.2 Simple Damped Pendulum with Initial Velocity ($\theta(0) = 0,\quad \dot{\theta}(0) = \omega_0,\quad M(t) = 0$)

Let's consider again Eq. (5.4), but now include damping. We will see that this system is analogous to the damped simple harmonic oscillator from Chap. 4. If there is no driving moment, after dividing each term by the mass moment of inertia, mL^2, Eq. (5.4) can be written as:

$$\ddot{\theta} + \frac{b_r}{mL^2}\dot{\theta} + \frac{g}{L}\theta = 0. \tag{5.9}$$

Following Chap. 4, we define the damping ratio for the pendulum.

$$\zeta = \frac{b_r}{2\omega_n mL^2} = \frac{b_r}{2mL^{3/2}\sqrt{g}} \tag{5.10}$$

We calculate the Laplace transform of Eq. (5.9) assuming an initial angular velocity, ω_0, and a zero initial angle; see Eq. (5.11). The equation is similar to Eq. (4.23).

$$\Theta(s) = \frac{\omega_0}{\omega_n^2}\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{5.11}$$

Because Eq. (5.11) is analogous to Eq. (4.23), all the mathematical development in Chap. 4 again applies. Specifically, the natural frequency is the frequency of oscillation that occurs in the absence of damping. If the damping ratio ζ is greater than zero but less than 1, the system is underdamped and will oscillate at its damped natural frequency, ω_d (see Eq. 4.25). The underdamped response is given by Eq. (5.12).

$$\theta(t) = \frac{\omega_0}{\omega_d} \mathrm{e}^{-\zeta\omega_n t} \sin(\omega_d t) \tag{5.12}$$

If the damping ratio is greater than 1, the system is overdamped and the motion is described by Equation 5.13.

$$\theta(t) = \frac{\omega_0}{\omega_n\sqrt{\zeta^2 - 1}} \left(\frac{e^{-\left(\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)\right)t} - e^{-\left(\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)\right)t}}{2} \right) \tag{5.13}$$

If the damping ratio is 1, Eq. (5.11) simplifies to:

$$\Theta(s) = \omega_0 \left(\frac{1}{s^2 + 2\omega_n s + \omega_n^2} \right) = \omega_0 \left(\frac{1}{(s + \omega_n)^2} \right), \tag{5.14}$$

where the denominator is a perfect square. Using entry 7 of Table 2.1, we see that the time domain solution is the product of a decaying exponential term and a linear time term.

$$\theta(t) = \omega_0 t \, \mathrm{e}^{-\omega_n t} \tag{5.15}$$

This is the case of critical damping. The decaying behavior can be mathematically verified using L'Hopital's rule. Physically, L'Hopital's rule tells us that the exponential function decays more quickly than the linearly growing term and, therefore, the entire function still decays to zero in the limit of infinite time.

Notice that by adding damping to the system, we have now linked that frequency of oscillation, ω_d, to the mass of the system through the damping ratio (see Eq. 5.10). This link is not present in the absence of damping.

Example 5.3 Consider a pendulum with a length of 1.5 m and a mass of 1.0 kg. Determine the damping value necessary to obtain damping ratios of: (a) 30%, (b) 100%, and (c) 200%. Assuming an initial velocity of 3.0 rad/s, find $\theta(t)$ for each case. Then use MATLAB® and to plot the motions over a length of time equal to four time constants from case (a), the most lightly damped example. Plot case (a) as a solid black line, case (b) as a dashed line, and case (c) as a dash-dot line. Comment on the validity of the solution for each case.

Solution First we calculate the natural frequency of the system $\omega_n = \sqrt{g/L}$. This is equal to approximately 2.6 rad/s, or 0.4 Hz. Next, rearranging Eq. (5.10) we obtain the expression for b_r.

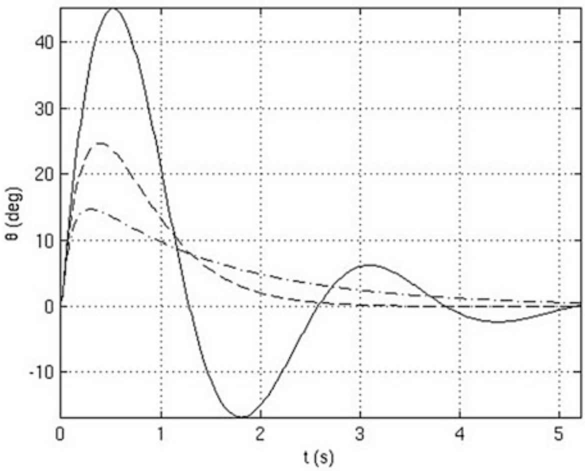
$$b_r = 2\omega_n m L^2 \zeta \tag{5.16}$$

For the three cases, we obtain values of approximately 3.5 N-m-s, 11.5 N-m-s, and 23.0 N-m-s, respectively. The following MATLAB® script file uses Eqs. (5.12), (5.13), and (5.15) to calculate the motion for each case. Notice that we can perform the conversion to degrees within the plot command; MATLAB® allows you to complete calculations within the argument of any function. Also, note that we have used the command `disp` to display the values of the rotational damping for each case in a readable format.

```
clear all
clc
close all

% Parameters
L = 1.5;          % m
m = 1.0;          % kg
g = 9.81;         % m/s^2
wn = sqrt(g/L);   % rad/s
fn = wn/(2*pi);   % cyc/s
T = 1/fn;         % s
w0 = 3.0;         % rad/s

% Case 1
zeta1 = 0.3;
br1 = 2*zeta1*wn*m*L^2;   % N-m-s
tau = 1/(zeta1*wn);       % s
t=[0:tau/100:4*tau];      % Time vector
wd1 = wn*sqrt(1-zeta1^2);
q1 = w0/wd1*exp(-zeta1*wn*t).*sin(wd1*t);

% Case 2
zeta2 = 1;
br2 = 2*zeta2*wn*m*L^2;
q2 = w0*t.*exp(-wn*t);

% Case 3
zeta3 = 2;
br3 = 2*zeta3*wn*m*L^2;
a1 = wn*(zeta3-sqrt(zeta3^2-1));
a2 = wn*(zeta3+sqrt(zeta3^2-1));
q3 = w0*1/(wn*sqrt(zeta3^2-1))*(exp(-a1*t)-exp(-a2*t))/2;

% Plot the solutions on a single graph
figure(1)
plot(t, q1*360/(2*pi), 'k-', t, q2*360/(2*pi), 'k-', t, q3*360/
(2*pi), 'k-.')
xlabel('t (s)')
ylabel('\theta (deg)')
axis([0 max(t) min(q1*360/(2*pi)) max(q1*360/(2*pi))])
grid
```

```
% Display the rotational damping value for each case
disp('case(a) br = ');
disp(br1)
disp('case(b) br = ');
disp(br2)
disp('case(c) br = ');
disp(br3)
```

The three results are shown in the figure.



The rotational damping values for each case are displayed in the command window.

```
case(a) br =
 3.4524
case(b) br =
 11.5080
case(c) br =
 23.0161
```

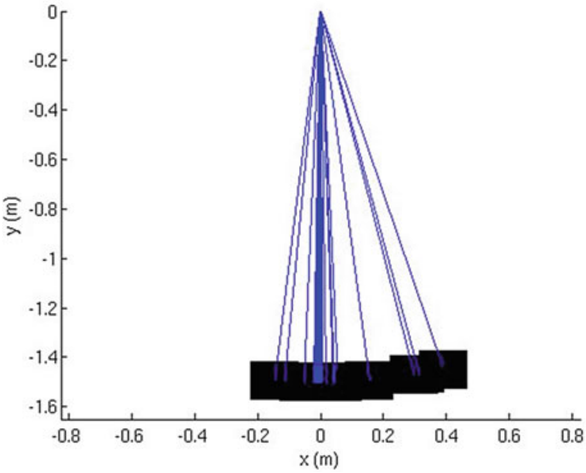
Notice that the maximum angle reached for case (a) exceeds 45°, while for cases (b) and (c) the angle reaches maxima of approximately 25° and 15°, respectively. Recall that the accuracy of the linearized approximation of the actual equation of motion degrades for larger θ values; see Fig. 5.2.

Example 5.4 Modify the program from Example 5.2 to animate the motion for case (a) from Example 5.3 using a reduced initial velocity of 1.0 rad/s.

Solution The MATLAB® code is provided.

```
clear all
clc
close all

% Parameters
L = 1.5;          % m
m = 1;            % kg
g = 9.81;         % m/s^2
wn = sqrt(g/L);   % rad/s
fn = wn/(2*pi);   % Hz
T = 1/fn;         % s
w0 = 1;           % rad/s

zeta = 0.3;
br = 2*zeta*wn*m*L^2;
tau = 1/(zeta*wn);
t = [0:tau/100:4*tau];  % time vector
wd = wn*sqrt(1-zeta^2);
q = w0/wd*exp(-zeta*wn*t).*sin(wd*t);

% Draw a line from (0,0) to the end of the pendulum at each time
% After each display, erase the picture to obtain the animation
effect
for cnt = 1:length(t)
  clf   % clear the figure after each display
  axis([-1.1*L/2 1.1*L/2 -1.1*L 0])
  xline=[0,L*sin(q(cnt))];
  yline=[0,-L*cos(q(cnt))];
  line(xline,yline)
  set(gca,'FontSize', 14)
  xlabel('x (m)')
  ylabel('y (m)')
  side = L/20;
  xpatch = [L*sin(q(cnt))+side L*sin(q(cnt))+side L*sin(q(cnt))-
          side L*sin(q(cnt))-side];
  ypatch = [-L*cos(q(cnt))+side -L*cos(q(cnt))-side -L*cos(q
            (cnt))- side -L*cos(q(cnt))+side];
  patch(xpatch, ypatch, 'k')
  pause(0.1) % short pause ensures MATLAB will display the line
end
```

The results, shown as a superposition of various pendulum positions, are displayed in the following figure. Notice that the motions are oscillatory, but settle back to the zero position.



5.2.3 Simple Damped Pendulum with Step Input ($\theta(0) = 0, \dot{\theta}(0) = 0, M(t) = M_O \cdot u(t)$)

As a final pendulum example, consider the case where the pendulum has zero initial conditions, but is subjected to a moment step input. For this, case Eq. (5.4) becomes:

$$mL^2\ddot{\theta} + b_r\dot{\theta} + mgL\theta = M_O \cdot u(t). \tag{5.17}$$

Dividing each term by the (rotational) mass moment of inertia, applying the definitions of the natural frequency and damping ratio, and calculating the Laplace transform, we obtain the Laplace domain solution shown in Eq. (5.18).

$$\Theta(s) = \frac{M_O}{mL^2}\left(\frac{1}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}\right) = \frac{M_O}{mgL}\left(\frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}\right) \tag{5.18}$$

Using entry 24 of Table 2.1, the time domain solution displayed in Eq. (5.19) is obtained.

$$\begin{aligned} \theta(t) &= \frac{M_O}{mgL}\left(1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right)\right) \\ \phi &= \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \end{aligned} \tag{5.19}$$

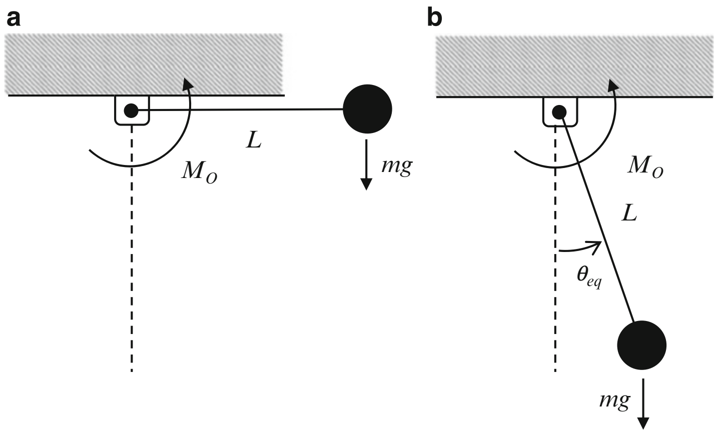


Fig. 5.4 (**a**) Pendulum in equilibrium at 90° as a result of an applied moment $M_O = mgL$; and (**b**) pendulum in equilibrium at an angle less than 90° such that $mgL \sin\theta_{eq} = M_O$

By applying the final value theorem (Eq. 2.19) to Eq. (5.18) or, equivalently, allowing time to approach infinity in Eq. (5.19), we obtain the steady-state, or equilibrium, angular displacement, θ_{eq}.

$$\theta_{eq} = \frac{M_O}{mgL} \tag{5.20}$$

Equation (5.20) is only applicable for small angles. It is evident from Fig. 5.4a that when the applied moment approaches mgL the equilibrium angular displacement should approach $\frac{\pi}{2}$ (90°) and not 1, as Eq. (5.20) suggests. This is the error associated with the approximation $\sin\theta \cong \theta$. From Fig. 5.4b, we see that for large angles less than 90°, the equilibrium displacement is given by Eq. (5.21).

$$\sin\theta_{eq} = \frac{M_O}{mgL} \tag{5.21}$$

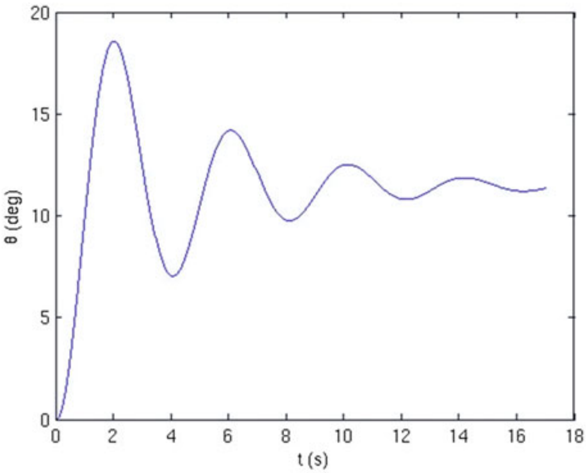
For small angles, Eq. (5.20) indicates that the equilibrium angular displacement is equal to the ratio of the applied moment M_O to the moment mgL that would be required to hold the pendulum at 90°.

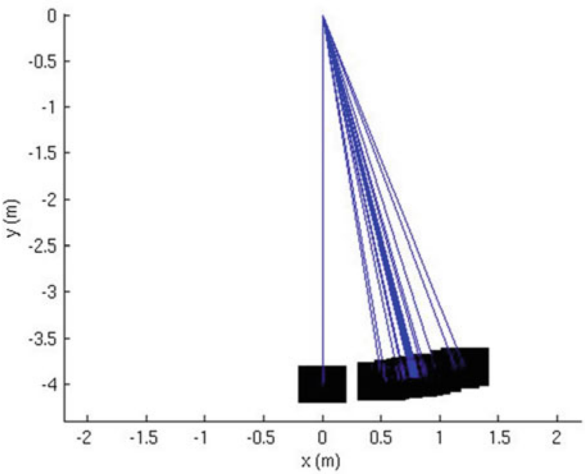
Example 5.5 Suppose a pendulum with a mass of 2 kg, a length of 4 m, and a damping ratio of 0.15 is subjected to a moment step input that is 20% of the moment required to hold the pendulum at 90°. Plot the motion of the pendulum in MATLAB® and animate it using the methods described previously.

Solution The following MATLAB® code plots the motion defined by Eq. (5.19) and then animates it.

```
clear all
clc
close all

% Parameters
L = 4;               % m
m = 2;               % kg
g = 9.81;            % m/s^2
M0 = 0.2*m*g*L;      % N-m
wn = sqrt(g/L);      % rad/s
zeta = 0.15;
tau = 1/(zeta*wn);   % s
wd = wn*sqrt(1-zeta^2);

% Determine the motion
t = [0:tau/100:4*tau];
phi = atan2(sqrt(1-zeta^2), zeta);
q = M0/(m*g*L)*(1-1/(sqrt(1-zeta^2))*exp(-zeta*wn*t).*sin(wd*t
+phi));

% Plot the results
figure(1)
plot(t, q*360/(2*pi))
xlabel('t (s)')
ylabel('\theta (deg)')

figure(2)
% Draw a line from (0,0) to the end of the pendulum at each time
% After each display, erase the picture to obtain the animation
effect
for cnt=1:length(t)
  clf  % clear the figure after each display
  axis([-1.1*L/2 1.1*L/2 -1.1*L 0])
  xline = [0, L*sin(q(cnt))];
  yline = [0, -L*cos(q(cnt))];
  line(xline, yline)
  set(gca,'FontSize', 14)
  xlabel('x (m)')
  ylabel('y (m)')
  side = L/20;
  xpatch = [L*sin(q(cnt))+side L*sin(q(cnt))+side L*sin(q(cnt))
           -side L*sin(q(cnt))-side];
  ypatch = [-L*cos(q(cnt))+side  -L*cos(q(cnt))-side  -L*cos
             (q(cnt))-side -L*cos(q(cnt))+side];
  patch(xpatch, ypatch, 'k')
  pause(0.1) % short pause ensures MATLAB will display the line
end
```

The results are provided for both the time domain solution and snapshots from the animation of the x–y pendulum coordinates during oscillation. Both provide a final angular position of approximately 11.5°.





5.3 Pendulum-Like Systems

Many systems in which a mass is offset from a center of rotation can be modeled as pendula. The pendulum element may be combined with other linear and rotational damping and stiffness elements. Inputs can include moments, forces, or displacements. An example is shown in Fig. 5.5. The mass-rod element has four

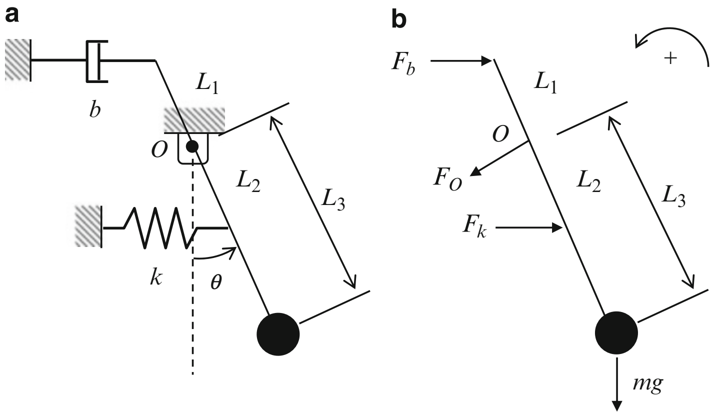


Fig. 5.5 (**a**) Mass suspended by a massless bar with a pin at O and connected to ground through a spring and a damper, and (**b**) free body diagram

forces acting on it, the weight, mg, the spring force, F_k, the damper force, F_b, and the pin force, F_O (no damping is associated with the pin at O). It is important to understand that the directions the forces are drawn on the free body diagram are sign conventions, that is, when the forces act in the directions drawn they have a positive value and when they act opposite the directions shown they are negative. At any particular time during the motion of the system, they may act in either the positive or negative direction.

Summing the moments about O (counter-clockwise assumed to be positive) gives the following equation of motion.

$$\sum M_O = F_k L_2 \cos\theta - F_b L_1 \cos\theta - mgL_3 \sin\theta = J_O \ddot{\theta} \tag{5.22}$$

The spring force F_k acts opposite the direction shown when the angular displacement θ is positive. The displacement at the end of the spring is $L_2 \sin\theta$, so the force is $-kL_2 \sin\theta$. The damper acts in the direction shown when the angular velocity of the end of the rod $\dot{\theta}$ is positive. This velocity is given by $\frac{d}{dt}(L_1 \sin\theta)$ which, by the chain rule, can be expressed as $\dot{\theta}L_1 \cos\theta$. If we approximate $\cos\theta$ as 1 for small angles (see Fig. 5.2), then the velocity simplifies to $\dot{\theta}L_1$. We recognize this as the velocity of a point moving at an angular velocity of $\dot{\theta}$ on a circle with a radius L_1. Thus, the damping force, F_b, is given by $bL_1\dot{\theta}$. Substituting these forces into Eq. (5.22) and assuming small angles, we obtain the linearized equation of motion for the system.

$$mL_3^2\ddot{\theta} + bL_1^2\dot{\theta} + \left(kL_2^2 + mgL_3\right)\theta = 0 \tag{5.23}$$

Dividing each term by the (rotational) mass moment of inertia, $mL_3{}^2$, we obtain Eq. (5.24), which enables us to identify the natural frequency and damping ratio.

$$\ddot{\theta} + \frac{bL_1^2}{mL_3^2}\dot{\theta} + \frac{\left(kL_2^2 + mgL_3\right)}{mL_3^2}\theta = \ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0 \tag{5.24}$$

We see that $\omega_n^2 = \frac{\left(kL_2^2+mgL_3\right)}{mL_3^2}$ so that $\omega_n = \sqrt{\frac{\left(kL_2^2+mgL_3\right)}{mL_3^2}}$ and $2\omega_n\zeta = \frac{bL_1^2}{mL_3^2}$ so that $\zeta = \frac{bL_1^2}{2m\omega_n L_3^2}$. Notice that we evaluate the dependence of the natural frequency and damping ratio on the system constants for each different system we study, but the meaning of these key parameters remains the same.

There are interesting subtleties to Eq. (5.24). First, Fig. 5.5 has three energy storage elements, the mass, which stores both gravitational potential energy and kinetic energy, and the spring, which stores elastic potential energy. Given the discussions from Chap. 4, we would therefore expect a third-order equation of motion. So, why is Eq. (5.24) second order? Notice that the effect of gravity and the spring deflection are combined in the third term of the equation to form an effective rotational spring constant, ${kL_2}^2 + mgL_3$. It is not possible to store gravitational potential energy independently without also storing potential energy in the elastic deflection of the spring. In effect there are two parallel springs, both acting between the mass and ground to restore the system to the equilibrium position of $\theta = 0$. The stiffness of one rotational spring is ${kL_2}^2$ and the stiffness of the second is mgL_3. Both have units of N-m, or N-m/rad, and their sum is the total stiffness (these are parallel springs). The term ${kL_2}^2$ is known as the *reflected rotational stiffness* of the linear spring, k. We will explore this reflected quantity concept in greater detail in Chap. 6 when we consider transmission elements.

Example 5.6 Suppose we have the system depicted in Fig. 5.5 with L_1, L_2, and L_3 equal to 1 m, 2 m, and 4 m, respectively. The mass is 2 kg and the linear damping constant, b, is 50 N-s/m. Use MATLAB® to plot the damped natural frequency, f_d, in Hz and the damping ratio, ζ, as a function of the spring stiffness, k, from 0 to 1000 N/m. Choose k such that the damped natural frequency is 1 Hz and determine the associated damping ratio.

Solution The equations for the natural frequency and damping ratio for this system have previously been determined.

$$\omega_n = \sqrt{\frac{\left(kL_2^2 + mgL_3\right)}{mL_3^2}}, \qquad \zeta = \frac{bL_1^2}{2m\omega_n L_3^2}$$

The natural frequency reduces to $\sqrt{\frac{g}{L_3}}$ for the simple pendulum ($k = 0$), as expected. The frequency increases with increased spring stiffness (as we observed for the systems in Chap. 4). Because the natural frequency increases with increased stiffness, the damping ratio necessarily decreases. This makes sense physically because the damping ratio is fundamentally a ratio of energy loss mechanisms to energy storage mechanisms, so, as the stiffness increases, the energy storage capability of the system increases relative to the energy loss capability. The damped natural frequency is found using Eq. (4.25), $\omega_d = \omega_n\sqrt{1-\zeta^2}$. If we combine this equation with the equations for this system, we can determine the damped natural frequency as a function of k.

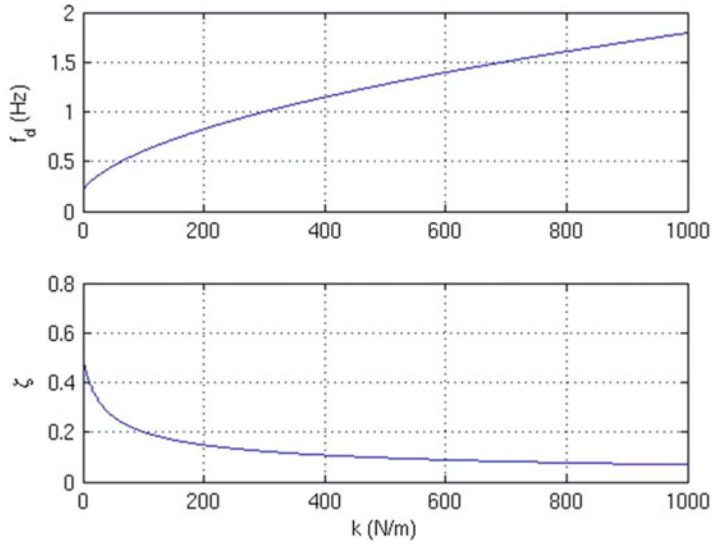
$$\omega_d = \sqrt{\frac{\left(kL_2^2 + mgL_3\right)}{mL_3^2}\left(1 - \frac{b^2L_1^4}{4m\left(kL_2^2 + mgL_3\right)L_3^2}\right)}$$

Solving this equation for the stiffness value to give a certain damped natural frequency is not possible directly because the equation is transcendental. However, we can use the find function to determine the solution. The following MATLAB® code calculates the damped natural frequency and damping ratio as a function of stiffness. Note the need to use the `.*` and `./` operations when multiplying and dividing the vectors.

```
clear all
clc
close all

% Parameters
L1 = 1;          % m
L2 = 2;          % m
L3 = 4;          % m
m = 2;           % kg
g = 9.81;        % m/s^2
b = 50;          % N-s/m

% Stiffness vector
k = [0:0.1:1000];    % N/m

% Calculate the natural frequency and damping ratio
wn = sqrt((k*L2^2+m*g*L3)/(m*L3^2));
fn = wn/(2*pi);
zeta = b*L1^2./(2*m*wn*L3^2); % ./ is required because we are
     dividing by the vector wn

% Determine the damped natural frequency
wd = wn.*sqrt(1-zeta.^2);   % .* is required becuase wn and zeta are
     vectors
fd = wd/(2*pi);

% Plot the damped natural frequency and damping ratio versus k
figure(1)
subplot(211)
plot(k, fd)
ylabel('f_d (Hz)')
grid
```

```
subplot(212)
plot(k, zeta)
xlabel('k (N/m)')
ylabel('\zeta')
grid

index = find(fd >= 1);
fd(index(1))
k(index(1))
zeta(index(1))
```

The resulting plots are provided. Using the `find` function in MATLAB®, we determine the stiffness required for a damped natural frequency of 1 Hz and the damping ratio. These are 301.1 N/m and 0.1234, respectively.



Example 5.7 For the parameters identified in Example 5.6, use MATLAB® to find and plot the motion of the system given an initial angle of 0.5 rad and zero initial velocity. Plot the motion for eight time constants.

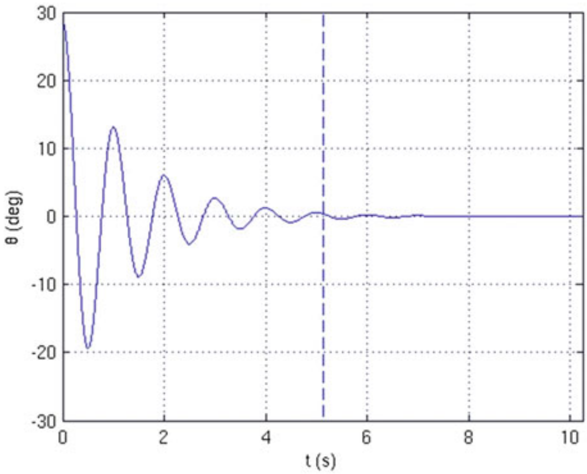
Solution Using the definitions of natural frequency and damping ratio, we calculate the Laplace transform of Eq. (5.24) to obtain Eq. (5.25).

$$\Theta(s) = \frac{s\theta_0 + 2\zeta\omega_n\theta_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.25}$$

This is implemented in the following MATLAB® code.

```
clear all
clc
close all

% Parameters
L1 = 1;         % m
L2 = 2;         % m
L3 = 4;         % m
m = 2;          % kg
g = 9.81;       % m/s^2
b = 50;         % N-s/m
k = 301.1;      % N/m
q0 = 0.5;       % rad

% Calculate the natural frequency and damping ratio
wn = sqrt((k*L2^2+m*g*L3)/(m*L3^2));
fn = wn/(2*pi);
zeta = b*L1^2/(2*m*wn*L3^2);
tau = 1/(zeta*wn);

% Find the time domain solution
syms q Q s t
Q = (s*q0+2*zeta*wn*q0)/(s^2+2*zeta*wn*s+wn^2);
q = ilaplace(Q)

% Plot the result
t = [0:tau/1000:8*tau];
qq = eval(q);
plot(t, qq*360/(2*pi))
grid
xlabel('t (s)')
ylabel('\theta (deg)')
axis([0 max(t) -30 30])

% Draw a line at four time constants
line([4*tau 4*tau],[-30 30])
```

The resulting plot is provided. We see that there is one oscillation per second (1 Hz damped natural frequency) as desired and the time constant, $\frac{1}{\zeta\omega_n}$, is 1.28 s. Four time constants are identified on the plot as a dashed vertical line; the motion has essentially damped to zero after this time.



5.4 Rotational Drive Systems

Rotational oscillations also occur in rotary drive systems. Direct rotational drives are prevalent and occur in many physical systems including:

- robots
- automobiles
- machine tools
- cranes
- jet engines
- power generators
- motors.

The most basic model for a rotational system with a positional drive input is shown in Fig. 5.6a. An inertia, J, is driven through a shaft with stiffness, k, and a prescribed input angle, $\theta_{in}(t)$. Figure 5.6b shows the free body diagram of the inertia. Applying Newton's second law about the z-axis gives:

$$M_b + M_b + M_k = J\ddot{\theta}. \tag{5.26}$$

We assume that the bearing supports are close to the inertia so that the angular velocity seen at each bearing is the same as the angular velocity of the inertia, $\dot{\theta}$. In this case, each bearing moment acts against the direction of rotation giving a moment, $-b_r\dot{\theta}$, for each bearing. The moment is negative when the velocity is in the positive direction according to the sign convention for θ. If the velocity is negative the moment becomes positive so it is always acting against the direction of the

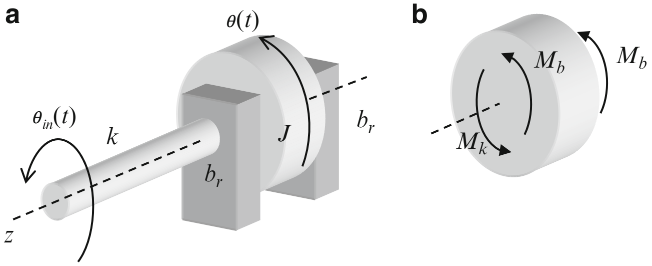


Fig. 5.6 (**a**) The inertia, J, is driven though a flexible input shaft with stiffness, k. The shaft is supported by two bearings, each with a rotational damping constant b_r; and (**b**) free-body diagram of J. The input angle is $\theta_{in}(t)$ and the angle of rotation for the inertia is $\theta(t)$

angular velocity. The moment from the shaft (spring), M_k, is positive when $\theta_{in} > \theta$ and is therefore given by $k_r(\theta_{in} - \theta)$. Substituting these moment expressions in Eq. (5.26) and rearranging, we obtain the following equation of motion.

$$J\ddot{\theta} + 2b_r\dot{\theta} + k_r\theta = k_r\theta_{in} \tag{5.27}$$

This is the basic equation for a position drive on a rotational system. We will now examine its behavior for various circumstances.

5.4.1 No Input Angle ($\theta_{in} = 0$)

Where there is no input, Eq. (5.27) reduces to Eq. (5.28) after dividing by J.

$$\ddot{\theta} + \frac{2b_r}{J}\dot{\theta} + \frac{k_r}{J}\theta = 0 \tag{5.28}$$

We can now define the natural frequency and damping ratio of the system by inspection.

$$\begin{aligned} \omega_n &= \sqrt{\frac{k_r}{J}} \\ \zeta &= \frac{b_r}{J\omega_n} \end{aligned} \tag{5.29}$$

Assuming an initial angle, θ_0, and an initial angular velocity, ω_0, the Laplace transform of Eq. (5.28) is:

$$\Theta(s) = \frac{s\theta_0 + \omega_0 + 2\zeta\omega_n\theta_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{5.30}$$

Equation (5.30) can be inverse transformed to find $\theta(t)$.

Example 5.8 Consider a hollow steel shaft 0.5 m long with an outer diameter of 5 cm and an inner diameter of 2.5 cm that is driving a solid cylindrical steel wheel having a diameter of 0.3 m and a thickness of 0.1 m. Find the natural frequency of the system in Hz. Next, if the damping ratio of the system is 5% due to the bearings and other components, determine the damping constant, b_r, and the damped natural frequency of the system. If the inertia has an initial angle of 100 μrad, find and plot its motion as a function of time. Finally, compare the (rotational) mass moment of inertia of the shaft to the wheel and comment on the use of a lumped parameter model (such as Fig. 5.6) to describe this system.

Solution The rotational stiffness of a shaft is a function of the shear modulus, the second moment of area of the shaft about the z-axis (i.e., polar moment) and the length of the shaft (Fig. 3.4).

$$k_r = \frac{G_s J_P}{L}$$

The second moment of area for a hollow shaft is:

$$J_P = \frac{\pi}{4}\left(R_o^4 - R_i^4\right),$$

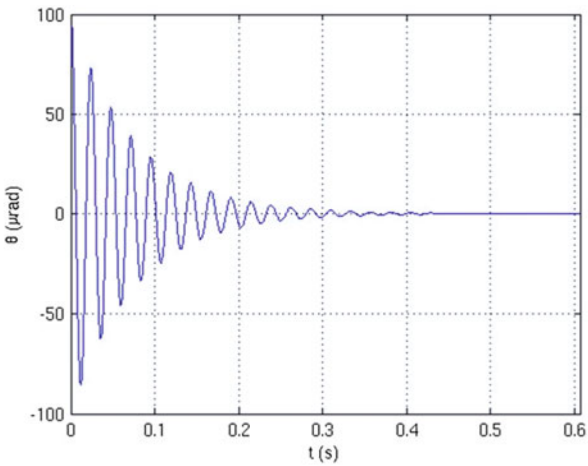
where R_i and R_o are the inner and outer radii of the shaft, respectively. The shear modulus for steel, G_s, is approximately 75 GPa. Substituting the numerical values for this shaft, we obtain a stiffness of approximately 43.1 kN-m/rad. The mass moment of inertia for a solid cylindrical wheel is $\frac{mR_w^2}{2}$, where m is the mass of the wheel and R_w is the radius. Using 7800 kg/m^3 as the density of steel, we obtain a mass moment of inertia of 0.62 kg-m^2. Using Eqs. (5.29), we calculate a natural frequency of 263 rad/s (42 Hz) and a required rotational damping of 8.2 N-m-s/rad to produce a damping ratio of 0.05.

The following MATLAB® code completes the desired calculations and plots the response.

```
clear all
clc
close all

% Parameters
Ls = 0.5;         % m
Rso = 0.05/2;     % m
Rsi = 0.025/2;    % m
Lw = 0.1;         % m
Rw = 0.3/2;       % m
rho = 7800;       % kg/m^3
Gs = 75e9;        % Pa
JP = pi/4*(Rso^4-Rsi^4);   % m^4
q_0 = 100e-6;              % rad
w_0 = 0;                   % rad/s
```

```
% Derived parameters
kr = Gs*JP/Ls;              % N-m or N-m/rad
Vw = pi*Rw^2*Lw;            % m^3
mw = rho*pi*Rw^2*Lw;        % kg
J = mw*Rw^2/2;              % kg-m^2
wn = sqrt(kr/J);            % rad/s
fn = wn/(2*pi);             % Hz
zeta = 0.05;
br = J*zeta*wn;
Vs = pi*(Rso^2-Rsi^2)*Ls;
ms = rho*Vs;
Js = ms/2*(Rsi^2+Rso^2);

% Equation of motion
syms Q q s t
Q=(s*q_0+w_0+2*zeta*wn*q_0)/(s^2+2*zeta*wn*s+wn^2);
q=ilaplace(Q);

tau = 1/(zeta*wn);
t = [0:tau/1000:8*tau];
qq = eval(q);

figure(1)
plot(t, qq*1e6)
grid
xlabel('t (s)')
ylabel('\theta (\murad)')
axis([0 max(t) -100 100])
```

The resulting plot is provided.



The mass moment of inertia for a hollow cylinder is:

$$J_P = \frac{1}{2} m \left(R_o^2 - R_i^2 \right),$$

which gives $2.4 \cdot 10^{-3}$ kg-m^2 for this shaft. Thus, the mass moment of inertia of the wheel is approximately 260 times greater than that for the shaft. The error from considering all of the inertia to be concentrated in the wheel is negligible and a lumped parameter approach is justified.

5.4.2 Nonzero Input Angle ($\theta_{in} \neq 0$)

While the previous section provided the basic vibrational characteristics for the shaft–wheel–bearing combination, a more common situation is for the inertia to be the rotor of a spindle or a robot arm that is commanded to move to a certain angle, $\theta_{in}(t)$. In that case, if we assume zero initial conditions, we can calculate the Laplace transform of Eq. (5.27) to obtain the transfer function for the system; see Eq. (5.31). This transfer function can be used to examine the system behavior for arbitrary inputs (recall the MATLAB® functions `step`, `impulse`, and `lsim`).

$$\frac{\Theta(s)}{\Theta_{\text{in}}(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.31}$$

Example 5.9 Suppose a counter-balanced section of a robot arm, modeled as shown in Fig. 5.7, has a rotational inertia of 1.0 kg-m^2 and a damping ratio from all velocity-dependent sources of 10%. It is driven through a shaft and associated drive components with a combined rotational stiffness of 5 kN-m/rad. Determine the total damping in the system to achieve this damping ratio, the natural frequency of the system, and the damped natural frequency of the system. Next, find the step response for a commanded step angular rotation of 5° and determine the maximum angular overshoot (beyond the commanded 5°) in degrees using the `max` command in MATLAB®. Assume zero initial conditions.

Solution For this case all of the damping is combined into one velocity-dependent moment; there are not two separate dampers for this model. Therefore, the equation of motion becomes:

$$\ddot{\theta} + \frac{b_r}{J}\dot{\theta} + \frac{k_r}{J}\theta = \frac{k_r}{J}\theta_{in} \tag{5.32}$$

and the natural frequency and damping ratio are given by Eq. (5.33). These are analogous to Eqs. (4.7) and (4.22).

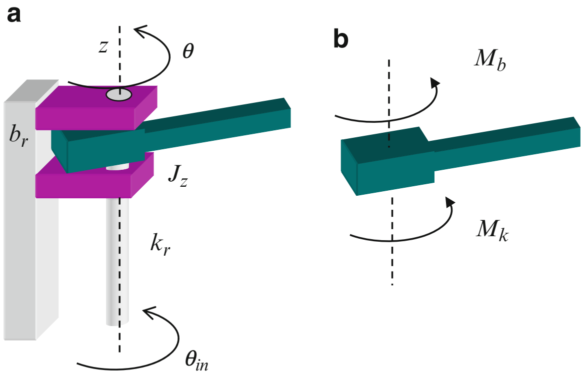


Fig. 5.7 (**a**) A robot arm with a (rotational) mass moment of inertia, J_z, is driven about the z-axis. The arm is balanced about the z-axis so the weight does not cause a moment on the bearings. It is driven though a flexible input shaft with stiffness, k_r, and is supported by two bearings. The system has a total rotational damping constant, b_r, from all sources. The input is $\theta_{in}(t)$ and the angle of rotation of the inertia is $\theta(t)$; and (**b**) free body diagram of the arm

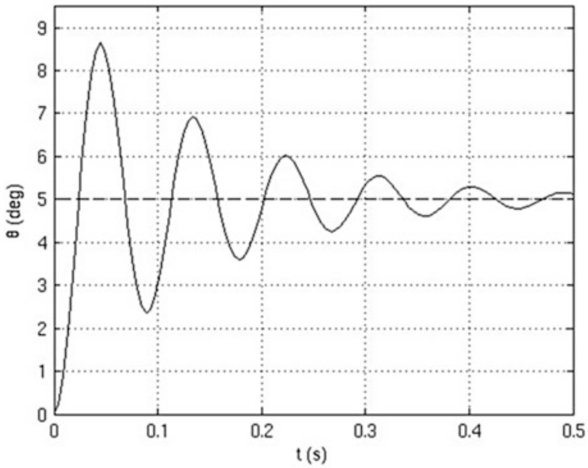
$$\begin{aligned}\omega_n &= \sqrt{\frac{k_r}{J}}\\ \zeta &= \frac{b_r}{2J\omega_n} = \frac{b_r}{2\sqrt{k_r J}}\end{aligned} \tag{5.33}$$

We use Eq. (5.31) to define the transfer function in MATLAB® and write the code to simulate the step response of the system. The MATLAB® code follows.

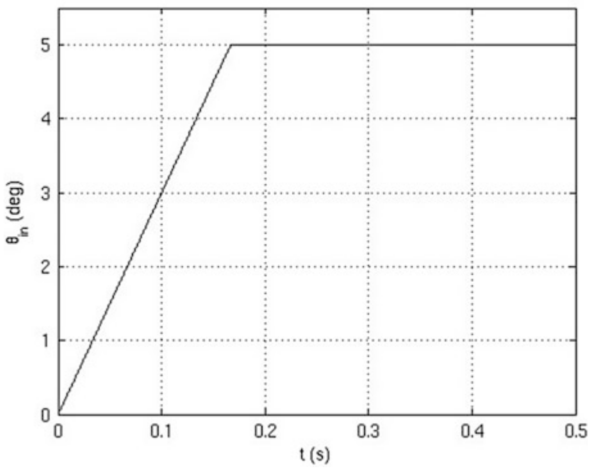
```
clear all
clc
close all

% Parameters
kr = 5e3;              % N-m/rad
J = 1.0;               % kg-m^2
wn = sqrt(kr/J);       % rad/s
zeta = 0.1;
br = 2*zeta*wn*J;      % N-m-s
q_max = 5;             % deg

% Define the system transfer function
num = [wn^2];
den = [1 2*zeta*wn wn^2];
sys = tf(num,den);

% Find the unit step response
[qu, t] = step(sys);

% Scale the unit step response by the input size
q = q_max*qu;
```

```
% Define a step input vector for comparison to the response
qin = q_max*ones(1, length(t));

% Plot the results
figure(1)
plot(t, qin, 'k-', t, q, 'k-')
grid
xlabel('t (s)');
ylabel('\theta (deg)');
axis([0 0.5 0 1.1*max(q)])

% Find the maximum overshoot in deg
q_os = max(q)-q_max;
display(q_os)
```

The corresponding plot is provided. We observe that the maximum overshoot of 3.65° (i.e., the difference between the maximum value of 8.65° and the commanded angle of 5°) occurs at 0.044 s. This could certainly be a concern depending on the required accuracy of the operation being performed and could cause damage to the components either immediately or due to repeated shock loads.



Example 5.10 One reason for the large overshoot in the previous example is that a step input produces a significant shock load on the system. In reality, most input drives would not be able to realize a true step input even if commanded to do so. Therefore, to produce a more controlled motion, a more typical input is a truncated ramp as shown.



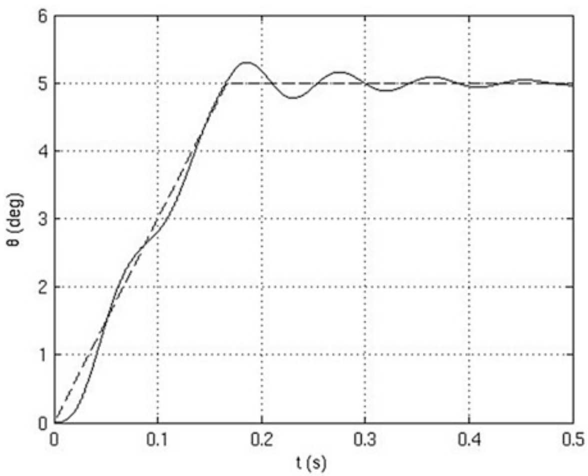
This θ_{in} input profile commands the arm to move at 30°/s to a final angle of 5° and then stops. Use the MATLAB® command `lsim` to find and plot the response of the robot arm detailed in Example 5.9. Find the maximum overshoot for this case and compare it to Example 5.9.

Solution The MATLAB® code first generates the input profile and then uses that profile to drive the system and find the response. To generate the input profile, the `find` command is again useful. The MATLAB® code is provided.

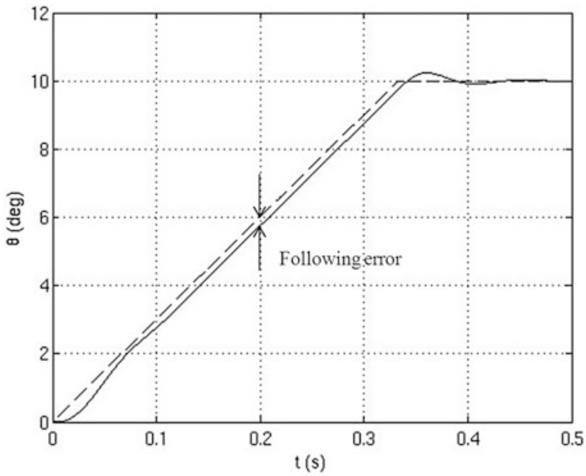
```
clear all
clc
close all

% Parameters
kr = 5e3;            % N-m/rad
J = 1.0;             % kg-m^2
wn = sqrt(kr/J);     % rad/s
zeta = 0.1;
br = 2*zeta*wn*J;    % N-m-s
q_max = 5;           % step, deg
q_rate = 30;         % ramp rate, deg/s

% Define the system transfer function
num = [wn^2];
den = [1 2*zeta*wn wn^2];
sys = tf(num, den);

% Define the truncated ramp input
t = [0:0.0001:0.5];
qin = q_rate*t;
index = find(qin > q_max);
qin(index) = q_max;
```

```
figure(1)
plot(t, qin, 'k-')
set(gca,'FontSize', 14)
xlabel('t (s)')
ylabel('\theta_{in} (deg)')
grid
axis([0 max(t) 0 1.1*q_max])

% Use the lisim command to find and plot the system response
figure(2)
[q, t] = lsim(sys, qin, t);
set(gca,'FontSize', 14)
plot(t, qin, 'k-', t, q, 'k-')
grid
xlabel('t (s)');
ylabel('\theta (deg)');

% Maximum overshoot
q_os = max(q)-q_max;
display(q_os)
```

The resulting output plot is shown.



Comparing this figure to the result from Example 5.9, we observe that the maximum overshoot is reduced from the large 3.6° to a more reasonable 0.31°. As an exercise, verify that by decreasing the ramp rate from 30 to 10°/s, the maximum overshoot is again reduced to 0.12°. Also, notice that due to the low damping, the angle does not exactly track the ramp. The transients persist throughout the ramp time. However, if the ramp is longer and/or the damping is larger, the transient damps out and the system does track the ramp, but with a lag. For example, if the truncated ramp ends at 10° and the damping ratio is increased to 30%, the response changes.



The vertical offset, or lag, in following the ramp is known as the *following error*. It is very important in the design of equipment, such as machine tools and robots, that must accurately follow a prescribed profile. In machine tools, the following error can lead to an error in the shape of a machined component for multi-axis motions if the following error is not identical for all axes. In robots, a following error could lead to a collision between the robot and some other component in the workspace.

5.5 Multiple Degree of Freedom Rotational Systems

Suppose that we now have two rotational inertias driven by flexible shafts as shown in Fig. 5.8. Bearings support the inertias and supply damping. Free body diagrams for the two inertias are provided in Fig. 5.9 with the positive sign convention chosen to be in the direction of positive angle (counter-clockwise about the z-axis as seen from the left). Notice that if the spring moment, M_{k_2}, is assumed to act in the positive direction on inertia J_1, it must act with an equal magnitude, but opposite direction on inertia J_2 if the spring element is assume to have zero inertia. Applying Newton's second law to each inertia, we obtain the equations of motion.

$$\begin{aligned} M_{k1} + M_{b1} + M_{k2} &= J_1\ddot{\theta}_1 \\ M_{b2} - M_{k2} &= J_2\ddot{\theta}_2 \end{aligned} \tag{5.34}$$

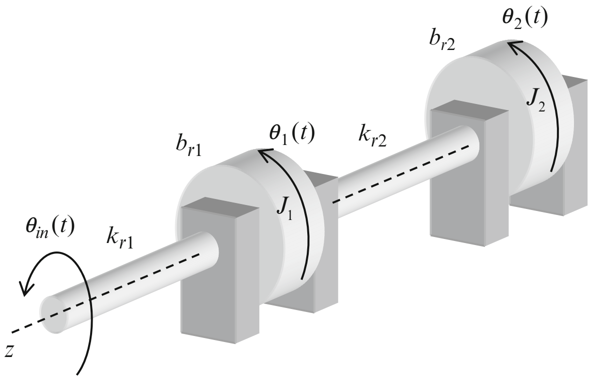


Fig. 5.8 Multiple degree of freedom rotational system with two rotational inertias, J_1 and J_2, being driven through compliant shafts with stiffnesses k_{r1} and k_{r2}. Each bearing set has a total bearing constant of b_{r1} and b_{r2}, respectively. The system is driven by the input angle $\theta_{in}(t)$

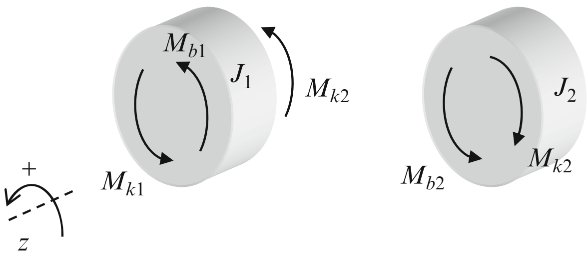


Fig. 5.9 Free body diagrams for inertias J_1 and J_2 from Fig. 5.8

Writing the moments in terms of the angular coordinates, we obtain:

$$\begin{aligned} k_{r1}(\theta_{in} - \theta_1) - b_{r1}\dot{\theta}_1 + k_{r2}(\theta_2 - \theta_1) = J_1\ddot{\theta}_1 \\ -b_{r2}\dot{\theta}_2 - k_{r2}(\theta_2 - \theta_1) = J_2\ddot{\theta}_2, \end{aligned} \tag{5.35}$$

which can be rearranged to produce Eq. (5.36).

$$\begin{aligned} J_1\ddot{\theta}_1 + b_{r1}\dot{\theta}_1 + (k_{r1} + k_{r2})\theta_1 = k_{r1}\theta_{in} + k_{r2}\theta_2 \\ J_2\ddot{\theta}_2 + b_{r2}\dot{\theta}_2 + k_{r2}\theta_2 = k_{r2}\theta_1 \end{aligned} \tag{5.36}$$

These equations are similar to Eqs. (4.48) and (4.54) with m replaced by J, b replaced by b_r, and k replaced by k_r. The difference between Eq. (5.35) and Eqs. (4.48) and (4.54) is that the dampers do not act between the inertias, but instead act between the inertias and ground. To obtain the transfer functions, we determine the Laplace transform of the second of Eq. (5.36) to obtain a relationship between $\Theta_1(s)$ and $\Theta_2(s)$.

$$\Theta_2(s) = \left(\frac{k_{r2}}{J_2 s^2 + b_{r2}s + k_{r2}}\right)\Theta_1(s) \tag{5.37}$$

We then take the Laplace transform of the first of Eq. (5.36) and substitute for $\Theta_2(s)$ to obtain the transfer function between $\Theta_1(s)$ and the input angle.

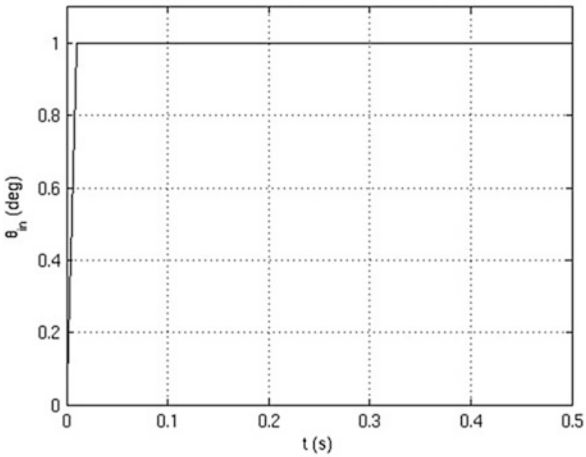
$$\frac{\Theta_1(s)}{\Theta_{\text{in}}(s)} = \frac{J_2 k_{r1} s^2 + b_{r2} k_{r1} s + k_{r2} k_{r1}}{\begin{array}{l} J_1 J_2 s^4 + (J_1 b_{r2} + J_2 b_{r1}) s^3 + (J_1 k_{r2} + J_2 k_{r1} + J_2 k_{r2} + b_{r1} b_{r2}) s^2 \\ + (b_{r1} k_{r2} + b_{r2} k_{r1} + b_{r2} k_{r2}) s + k_{r1} k_{r2} \end{array}} \tag{5.38}$$

Inverting Eq. (5.37) and inserting into Eq. (5.38), we obtain the transfer function for $\Theta_2(s)$.

$$\frac{\Theta_2(s)}{\Theta_{\text{in}}(s)} = \frac{k_{r1} k_{r2}}{\begin{array}{l} J_1 J_2 s^4 + (J_1 b_{r2} + J_2 b_{r1}) s^3 + (J_1 k_{r2} + J_2 k_{r1} + J_2 k_{r2} + b_{r1} b_{r2}) s^2 \\ + (b_{r1} k_{r2} + b_{r2} k_{r1} + b_{r2} k_{r2}) s + k_{r1} k_{r2} \end{array}} \tag{5.39}$$

Equations (5.38) and (5.39) are analogous to Eqs. (4.55) and (4.57).

Example 5.11 A dynamic system can be modeled as shown in Fig. 5.8. Suppose the system parameters are: $J_1 = 0.5$ kg-m^2, $J_2 = 0.5$ kg-m^2, $k_{r1} = 10000$ N-m/rad, $k_{r2} = 10000$ N-m/rad, $b_{r1} = 5$ N-m-s, and $b_{r2} = 5$ N-m-s. If the system is subjected to a ramped input angle command with a rate of 100°/s up to a final angle of 1°, find and plot the responses $\theta_1(t)$ and $\theta_2(t)$.



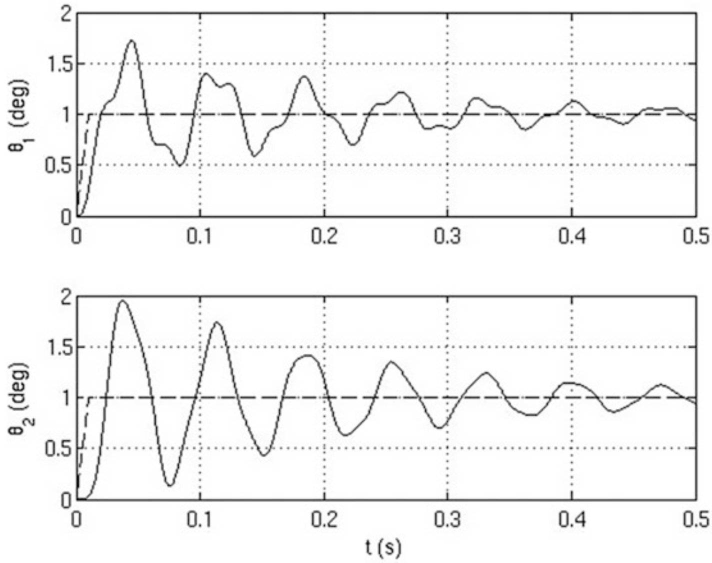
Solution The MATLAB® code used to simulate the system response is provided.

```
clear all
clc
close all
```

```
% Parameters
kr1 = 10e3;      % N-m/rad
kr2 = 10e3;      % N-m/rad
J1 = 0.5;        % kg-m^2
J2 = 0.5;        % kg-m^2
br1 = 5;         % N-m-s
br2 = 5;         % N-m-s
q_max = 1;       % step, deg
q_rate = 100;    % ramp rate, deg/s

% Define the system transfer functions
num1 = [J2*kr1 br2*kr1 kr2*kr1];
den1 = [J1*J2 (J1*br2+J2*br1) (J1*kr2+J2*kr1+J2*kr2+br1*br2)
(br1*kr2+br2*kr1+br2*kr2) kr1*kr2];
sys1 = tf(num1, den1);

num2 = [kr1*kr2];
den2 = den1;
sys2 = tf(num2, den2);

% Define the truncated ramp input
t = [0:0.0001:0.5];
qin = q_rate*t;
index = find(qin > q_max);
qin(index) = q_max;

figure(1)
plot(t, qin, 'k-')
set(gca,'FontSize', 14)
xlabel('t (s)')
ylabel('\theta_{in} (deg)')
grid
axis([0 max(t) 0 1.1*q_max])

% Determine the response using lsim
[q1, t] = lsim(sys1, qin, t);
[q2, t] = lsim(sys2, qin, t);

% Plot the results
figure(2)
subplot(211)
plot(t, qin, 'k-', t, q1, 'k-')
grid
ylabel('\theta_1 (deg)')
subplot(212)
```

```
plot(t, qin, 'k-', t, q2, 'k-')
grid
xlabel('t (s)')
ylabel('\theta_2 (deg)')
```

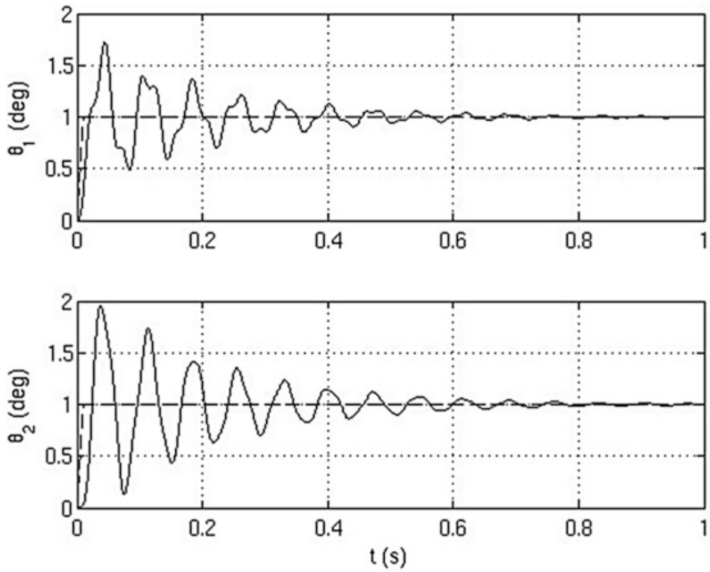
The responses are shown.



Notice that, as in Example 4.12, there are two frequencies in the motion. The frequencies are determined by examining the roots of the transfer function denominator, or characteristic equation.

```
>> roots(den1)

ans =

  1.0e+02 *

 -0.0500 + 2.2877i
 -0.0500 - 2.2877i
 -0.0500 + 0.8726i
 -0.0500 - 0.8726i
```

As described in Chap. 4, the roots are complex conjugate pairs. The imaginary parts of the roots define the damped natural frequencies of the system. In this case, they are 228.8 rad/s (36.4 Hz) and 87.3 rad/s (13.9 Hz). We see that both angles undergo approximately seven cycles of primary oscillation in 0.5 s, or about 14 Hz. In the plot of $\theta_1(t)$, a second oscillation at approximately 2.5 times this frequency is superimposed on the 14 Hz oscillation. This motion corresponds to the 36.4 Hz damped natural frequency. The time constant for each pair of roots is $1/5 = 0.2$ s and, therefore, the oscillations should attenuate in approximately 0.8 s. An expanded time view is provided to demonstrate the damped behavior. The roots of the characteristic equation provide system properties: frequencies of oscillation and time constants. These system properties are manifested in the signals produced for the specific input and are plotted below.



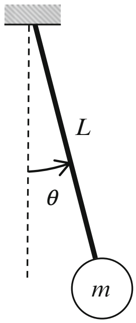
5.6 Summary

In this chapter, we discussed the following key concepts:

- Rotational systems are modeled by equations that are similar to those that describe rectilinear systems (Chap. 4) and, therefore, both systems undergo similar types of motions, where the angle $\theta(t)$ is analogous to the displacement $x(t)$.
- A pendulum is a common rotational system and pendulum-like models describe a wide range of systems, such as cranes, ships rocking on the water, and aircraft.
- Simple pendulum oscillations involve a trading of energy from kinetic to gravitational potential energy of the mass.
- Similar to rectilinear systems, second-order rotational systems have a natural frequency, ω_n, damping ratio, ζ, and damped natural frequency, ω_d. The physical interpretation of these parameters is the same as was discussed in Chap. 4.

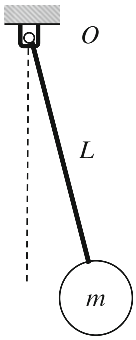
Problems

1. A simple pendulum in a grandfather clock is 1.5 m in length and has an end mass of 3 kg. The rod can be considered to be massless.



Complete the following.
(a) Determine the equations of motion for the pendulum and linearize using the small angle approximation.
(b) Calculate the natural frequency in cycles per second (Hz).
(c) The initial angle of the pendulum is zero and the initial angular velocity is 0.5 rad/s. Find the expression for $\theta(t)$ using Laplace transforms.
(d) Using MATLAB®, plot $\theta(t)$ for a time interval equal to four full oscillations.

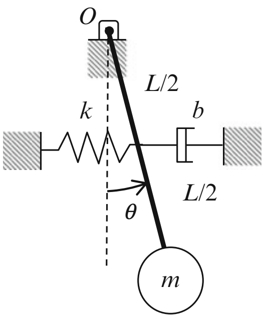
2. A wrecking ball with mass m hangs from a crane by a cable with length L. It can be modeled as a simple pendulum as shown in the figure. There is a connection at point O with viscous rotational damping b (N-m-s).



Complete the following.
(a) Determine the equation of motion for the wrecking ball and cable and linearize using the small angle approximation.
(b) Find equations for the natural frequency, ω_n, and the damping ratio, ζ, in terms of m, L, g, and b.

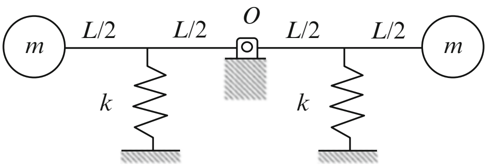
(c) The cable has a length, L, of 40 m and the wrecking ball has a mass m of 20 kg. Determine the value of b required to obtain a damping ratio of 10%.
(d) The ball is released from an initial angle of 20° from the vertical with an initial velocity of -0.05 rad/s. Using a MATLAB® script file, determine $\theta(t)$ using the `ilaplace` command and then plot the motion over one cycle of oscillation.

3. Consider the pendulum system model shown in the figure. A spring and damper are attached at the midpoint of the pendulum arm as shown. The spring is unstretched when $\theta = 0$. Assume no energy loss in the bearing at O.



Complete the following.
(a) Determine the equation of motion for the system in terms of m, L, k, b, and θ and linearize using the small angle approximation.
(b) Find equations for the natural frequency, ω_n, and the damping ratio, ζ, for this system in terms of m, L, k, and b.
(c) If the pendulum mass, m, is 2 kg, the length, L, is 1 m, and the stiffness, k, is 122 N/m, calculate ω_n and then determine the value of b required to obtain 25% damping
(d) Suppose the pendulum has an initial angle of 0.25 rad and an initial angular velocity of -2 rad/s, use a MATLAB® script file to calculate $\theta(t)$ and plot the motion for four time constants of the system.

4. Consider the platform model shown in the figure. It is hinged at the point O with two masses at the ends and two springs placed halfway between the masses and the hinge point.



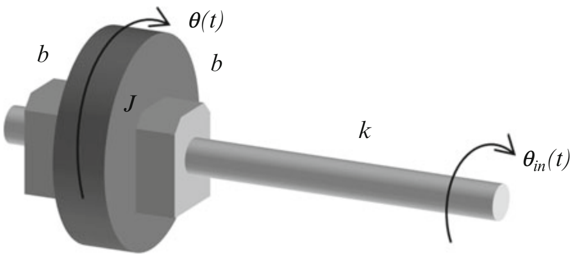
Complete the following.

(a) Determine the equation of motion for the system and linearize using the small angle approximation.

(b) If the mass, m, is 2 kg, the length, L, is 4 m, and the stiffness k is 2000 N/m, calculate the natural frequency of the system. Explain why this is independent of the length L.

(c) If the initial angle of the platform is zero (horizontal) and the initial angular velocity is 1 rad/s, use a MATLAB® script file to calculate $\theta(t)$ and plot the motion for six oscillation cycles.

5. A model of a rotational inertia, J, being driven by a motor is displayed in the figure. The motor produces an input angle $\theta_{in}(t)$ at the end of the shaft. The rotation angle of the inertia is $\theta(t)$. There are two bearings fitting close to the inertia as shown. The rotational stiffness of the shaft is k_r and the bearings each have rotational viscous damping b.



Complete the following.

(a) Determine the equation of motion for the system.

(b) Determine the damping ratio and natural frequency in terms of the model parameters.

(c) Show that the damping ratio, ζ, is unitless and that the units of the natural frequency, ω_n, are rad/s.

(d) If the mass moment of inertia, J, is 0.025 kg-m^2 and the rotational stiffness, k_r, is 9000 N-m/rad, calculate b_r such that the system is 10% damped.

(e) If the input is a 0.01 rad step, use a MATLAB® script file to calculate $\theta(t)$ and plot the motion for a period of four system time constants. Also, identify the maximum angular displacement using the command `max`.

Combined Rectilinear and Rotational Motions: Transmission Elements

6

6.1 Introduction

Chapters 4 and 5 examined rectilinear and rotational motions of systems separately. However, for the majority of mechanical systems rotational and linear motions occur simultaneously. Familiar examples include:

- rectilinear motion of the pistons in an engine cause a rotation of the crank shaft
- rotational motion of a motor drives a leadscrew which produces rectilinear motion of the tray in a DVD player
- linear motion of a cable yields rotational motion of a pulley system
- rotational motion of a pinion produces linear motion along a rack
- rotation of automobile wheels provides linear motion along the road.

In other cases, rectilinear motion may lead to other rectilinear motion (e.g., a lever) or rotational motion may lead to other rotational motion (e.g., gears), but the motion/force/moment magnitude is changed. In each case, the modification of one type of motion to another involves a transmission element: slider crank, leadscrew, pulley, rack and pinion, rolling wheel, gears, and levers. When we model these systems, we find that the equations of motion have the same character as those in Chaps. 4 and 5 and, therefore, the analysis techniques we already developed can be applied again. We also find that a rotational element may appear to be an equivalent, or *reflected*, rectilinear element or vice versa. These concepts can be somewhat counterintuitive at first and, therefore, require practice before they can be mastered and then used in dynamic system design.

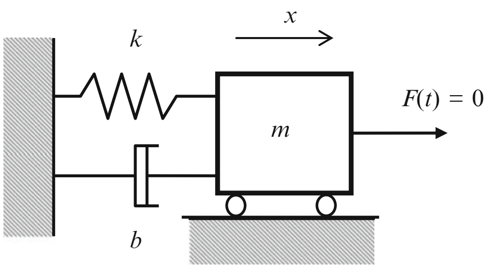
M.A. Davies, T.L. Schmitz, *System Dynamics for Mechanical Engineers*,
DOI 10.1007/978-1-4614-9293-1_6

6.2 System Analysis

Transmission elements add complexity to the analysis of dynamic systems. To accommodate this increased complexity, we will implement a systematic, stepwise approach. In Chaps. 4 and 5, a systematic approach was developed, but not formalized. In each example, the following steps were completed:

1. Define coordinate systems for each independently moving elements and inputs.
2. Draw free body diagrams for all elements with independent coordinates.
3. Apply Newton's second law to each free body diagram.
4. Write forces/moments in terms of the independent variables to identify the time domain equation(s) of motion.
5. Compute the Laplace transform of the time domain equations of motion to find the Laplace domain equation(s) of motion.
6. Solve using appropriate methods (analytical and MATLAB® techniques were demonstrated).

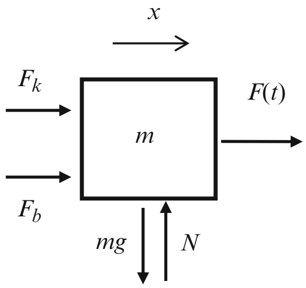
To illustrate, consider Example 4.2 again. The goal was to determine the response of an undriven damped harmonic oscillator (i.e., $F(t) = 0$) with a stiffness, k, of 2500 N/m, a mass, m, of 100 kg, and a viscous damping coefficient, b, of 600 N-s/m. The initial conditions were $x(0) = 2$ m and $\dot{x}(0) = 0$.



Let's complete the steps we just identified.

Step 1: Assign coordinates to all independently moving elements in the system. In this case, we define the coordinate, x, to be the horizontal mass motion (x is zero at its equilibrium position). For more complex problems, we may need to assign multiple coordinates. For example, we assign coordinates x_1 and x_2 to the independently moving (massless) bar and inertial mass in Fig. 4.10b.

Step 2: Draw a free body diagram of the mass. For the problem in Fig. 4.10, we would require free body diagrams for both the bar and the mass.



Step 3: Apply Newton's second law to each free body diagram. For this example, we only have the single moving mass. As discussed in Chap. 4, the vertical forces are in equilibrium, while the imbalance in the horizontal forces causes an acceleration of the mass: $F_k + F_b = m\ddot{x}$.

Step 4: This system has no transmission element.

Step 5: Write the forces in terms of the independent variable, x, and its derivatives to identify the time domain equation of motion: $m\ddot{x} + b\dot{x} + kx = 0$.

Step 6: Calculate the Laplace transform of the time domain equation of motion and substitute the numerical values to obtain the Laplace domain equation of motion: $X(s) = \frac{2s+12}{s^2+6s+25}$. Compute the inverse Laplace transform to obtain the time domain mass motion: $x(t) = 2\mathrm{e}^{-3t}\left(\cos(4t) + \frac{3}{4}\sin(4t)\right)$ m.

We will now implement these steps to analyze systems that include transmission elements.

6.3 Systems with Transmission Elements

As described in Chap. 3, transmission elements provide two main constraints which must be considered in dynamic analysis: *geometric* constraints and *force/moment* constraints. When analyzing a dynamic system with a transmission element, the same six steps are applied, but we add a seventh step where we enforce the transmission element constraints. The new steps are:

1. Define the coordinate systems.
2. Draw free body diagrams for all inertias in the system.
3. Apply Newton's second law to each free body diagram.
4. Develop the geometric and force/moment relationships for the transmission element(s).
5. Combine steps 3 and 4 to write the time domain equations of motion.
6. Compute the Laplace transform of the time domain equations of motion to find the Laplace domain equation of motion.
7. Solve using appropriate methods.

Step 5 produces a set of equations that are either written in terms of the input or output side of the transmission element. Elements on the opposite side of the transmission element appear in the equations of motion and their values are modified by the square of the transmission ratio. These are *reflected* quantities. Because transmission ratios are typically quite large or small, the reflected quantity is often significantly different in magnitude than the actual quantity. For example, a mass on the opposite side of a lever will appear as a reflected mass that may be much larger or smaller than the actual mass. A common lever use is to magnify a force in order to move a large mass. The reflected mass appears to be much smaller than the actual mass when viewed from the input side of the lever. Another familiar use of this concept is to use gears to "gear up" or "gear down" a system so that the reflected inertia is much smaller or larger than the actual driven inertia. This is common practice when using a high speed, low torque motor to slowly accelerate a large rotational inertia. The reverse is common when using a low speed, high torque turbine (such as a windmill) to drive a generator that operates most efficiently at high speed and lower torque. As we will see in the following sections, the dynamic quantities are all modified to obtain reflected stiffness and damping in the equations of motion.

6.4 Levers in Dynamic Systems

A lever is shown in Fig. 3.14 and the constraint equations are given by Eqs. (3.29) and (3.32). To see how we use these in dynamic analysis, consider the model for a platform depicted in Fig. 6.1a. The platform is a uniform bar of length L and mass m. It is supported at its center (and center of mass) and the spring k is unstretched when $\theta = 0$. Therefore, the equilibrium position of the bar is also at $\theta = 0$ and the spring will act to return it to this position. In analyzing the problem, step 1 is the coordinate definition. The coordinate θ describes the rotational position of the bar (counterclockwise rotation from the equilibrium position). Step 2 is to draw the free body diagram of the bar; see Fig. 6.1b. Both forces have been declared to be positive when acting upward and this sign convention leads them to generate opposite moments.

In step 3, we apply Newton's second law for rotation and sum the moments about the pivot O.

$$\sum M_O = -F_k L_2 \cos\theta + F_b L_1 \cos\theta = J_O \ddot{\theta} \tag{6.1}$$

The moments are given by the product of the forces and associated moment arms. For example, the moment due to the spring is the product of the spring force, F_k, and the distance between the pivot O and the spring, $L_2 \cos\theta$. The moment is negative because a positive spring force (upward) produces a clockwise moment. The moment from the damper is found in a similar manner. To determine the equation of motion in terms of the independent coordinate, θ, we need to relate F_k and F_b to θ and its derivatives. When θ is positive, F_k acts in the direction shown and is therefore positive; it is equal to the spring constant multiplied by the spring

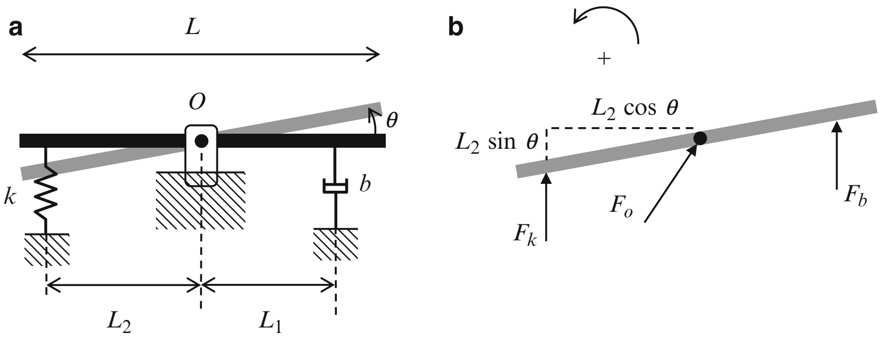


Fig. 6.1 Levered suspension system for a uniform platform of mass m and length L

deflection, $L_2 \sin\theta$. When θ is positive, the damper force, F_b, acts opposite to the direction shown. The velocity at the top end of the damper is $\frac{\mathrm{d}}{\mathrm{d}t}(L_1 \sin\theta)$, or $L_1\dot{\theta}\cos\theta$, and the force is $-bL_1\dot{\theta}\cos\theta$. Substitution in Eq. (6.1) yields the equation of motion.

$$-(k(L_2 \sin\theta))L_2 \cos\theta + \left(-bL_1\dot{\theta}\cos\theta\right)L_1 \cos\theta = J_O\ddot{\theta} \tag{6.2}$$

The (rotational) moment of inertia for a uniform bar is $\frac{mL^2}{12}$. Using this expression and applying the small angle approximation, the equation of motion is linearized as shown in Eq. (6.3).

$$\frac{mL^2}{12}\ddot{\theta} + bL_1^2\dot{\theta} + kL_2^2\theta = 0 \tag{6.3}$$

This is rearranged into a form that enables us to identify the natural frequency and damping ratio as seen previously in Chaps. 4 and 5.

$$\ddot{\theta} + 12\frac{bL_1^2}{mL^2}\dot{\theta} + 12\frac{kL_2^2}{mL^2}\theta = 0 \tag{6.4}$$

For this system, the natural frequency and damping ratio are given by Eq. (6.5).

$$\begin{aligned} \omega_n &= \sqrt{12\frac{kL_2^2}{mL^2}} \\ \zeta &= 6\frac{bL_1^2}{mL^2\omega_n} \end{aligned} \tag{6.5}$$

Notice that Eq. (6.5) is comparable to Eqs. (4.7) and (4.22) except for the leading constant 12. The reflected stiffness, k_{ref}, and damping, b_{ref}, are shown in Eq. (6.6).

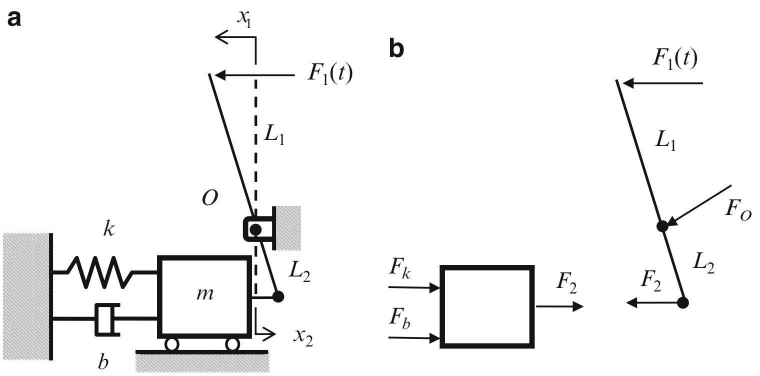


Fig. 6.2 Levered drive of a spring mass damper

$$\begin{aligned} k_{ref} &= k\frac{L_2^2}{L^2} \\ b_{ref} &= b\frac{L_1^2}{L^2} \end{aligned} \tag{6.6}$$

The ratios, $\frac{L_2}{L}$ and $\frac{L_1}{L}$, are the *lever ratios* or *lever transmission ratios* for the spring and the damper, respectively. Notice that if L_1 or L_2 or zero the effect of the damper and spring are removed because neither force has a moment arm about point O. Also, note that the square of the transmission ratios modify the physical quantities; we will see this pattern for other systems as well. In this problem we did not complete step 4 because the problem does not have an explicit input. In the next example, we will see how step 4 is completed.

Example 6.1 Consider a damped harmonic oscillation that is driven by a force through a lever as shown in Fig. 6.2. The lever is assumed to have zero mass. Derive the equations of motion in terms of x_1 if the stiffness is 2500 N/m, the mass is 100 kg, and the viscous damping coefficient is 600 N-s/m. The length L_1 is equal to 1.5 m and the length L_2 is equal to 0.5 m. Show that the natural frequency and damping ratio do not depend on the transmission ratio and then identify them. Finally, determine the required step input force (applied to the lever) to cause a 5 mm steady state mass displacement.

Solution Step 1 is to declare the coordinates. We define x_1 and x_2 as the deflections on the two sides of the lever as shown in Fig. 6.2a. The forces on either side of the lever are F_1 and F_2. In step 2, we draw the free body diagrams for the lever and mass as shown in Fig. 6.2b. In step 3, we apply Newton's second law to each free body. First, the lever is assumed to have zero mass (rotational inertia), so the sum of the moments is zero.

$$F_1L_1 - F_2L_2 = 0 \tag{6.7}$$

The equation of motion for the spring mass damper is developed in the same manner as described in Chap. 4.

$$m\ddot{x}_2 + b\dot{x}_2 + kx_2 = F_2 \tag{6.8}$$

Step 4 is to identify the force and displacement relationships for the lever; see Eqs. (3.29) and (3.32). Using these equations together with Eq. (6.8), we can write the time domain equations of motion for either the input (side 1) or the output (side 2). Writing Eq. (6.8) in terms of the input, we obtain Eq. (6.9).

$$m\left(\frac{L_2}{L_1}\right)^2\ddot{x}_1 + b\left(\frac{L_2}{L_1}\right)^2\dot{x}_1 + k\left(\frac{L_2}{L_1}\right)^2 x_1 = F_1 \tag{6.9}$$

This is equivalent to Eq. (4.19), but now with reflected mass, stiffness, and damping terms.

$$\begin{aligned} m_{ref} &= m\left(\tfrac{L_2}{L_1}\right)^2 \\ b_{ref} &= b\left(\tfrac{L_2}{L_1}\right)^2 \\ k_{ref} &= k\left(\tfrac{L_2}{L_1}\right)^2 \end{aligned} \tag{6.10}$$

As we noted previously, the reflected quantities are modified by the square of the transmission ratio. The natural frequency and damping ratio can be defined in terms of the reflected quantities.

$$\begin{aligned} \omega_n &= \sqrt{\frac{k_{ref}}{m_{ref}}} = \sqrt{\frac{k}{m}} \\ \zeta &= \frac{b_{ref}}{2\sqrt{k_{ref}m_{ref}}} = \frac{b\left(\tfrac{L_2}{L_1}\right)^2}{2\sqrt{k\left(\tfrac{L_2}{L_1}\right)^2 m\left(\tfrac{L_2}{L_1}\right)^2}} = \frac{b}{2\sqrt{km}} \end{aligned} \tag{6.11}$$

The natural frequency and damping ratio are the same when seen through the lever as they are when the lever is not present. This makes physical sense—the frequency of oscillation of the lever will match that of the spring mass damper and the damping behavior will be the same as well.

Assuming a step input, $F_1 = F_o \cdot u(t)$, substituting this force into Eq. (6.9), and calculating the Laplace transform, we obtain Eq. (6.12).

$$X_1(s) = \frac{F_o}{s\left(m\left(\tfrac{L_2}{L_1}\right)^2 s^2 + b\left(\tfrac{L_2}{L_1}\right)^2 s + k\left(\tfrac{L_2}{L_1}\right)^2\right)} \tag{6.12}$$

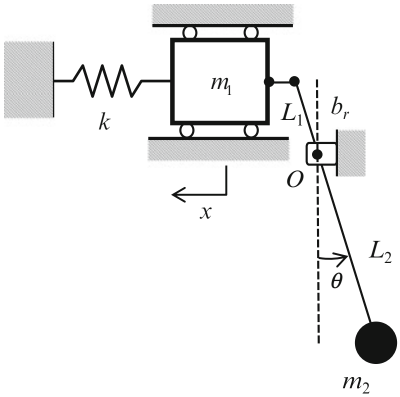


Fig. 6.3 Pendulum attached to a simple harmonic oscillator through a lever

Applying the final value theorem gives the steady state displacement at the input end of the lever, x_{1eq}.

$$x_{1eq} = \frac{F_o}{k\left(\frac{L_2}{L_1}\right)^2} \tag{6.13}$$

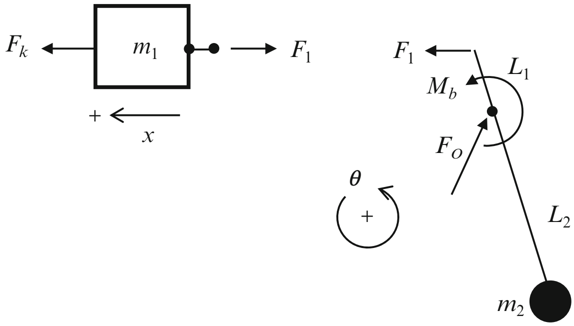
In order for the equilibrium displacement of the mass to be 5 mm, the equilibrium displacement x_{1eq} must be three times as large, or 15 mm, because L_1 is three times L_2. Substitution into Eq. (6.13) gives an F_o value of 4.17 N.

Example 6.2 Consider a pendulum (lever) attached to a simple harmonic oscillator constrained to move along the horizontal path as shown in Fig. 6.3. Find the equations of motion in terms of the angular variable, θ, and also in terms of the linear variable, x. Identify the reflected quantities in both forms of the equation and find the natural frequency and damping ratio for the system. Assume the following values: m_1 and m_2 are 2 and 4 kg, respectively, L_1 and L_2 are 1 and 2 m, respectively, b_r is 10 N-m-s, and k is 200 N/m. In the rotational coordinate system, calculate the ratio of the reflected inertia to the pendulum inertia and calculate the ratio of the reflected rotational stiffness to the pendulum stiffness. Finally, find the natural frequency and damped natural frequency and plot the motion for an initial angular velocity, $\dot{\theta}_0 = \omega_0 = 0.25\,\mathrm{rad/s}$. Compare $\theta(t)$ and $x(t)$.

Solution

Step 1: Declare the coordinates. We arbitrarily select the angular variable θ to be positive in the counterclockwise direction and the linear variable x to be positive in the left direction, so that a positive angular displacement corresponds to a positive linear displacement.

Step 2: Draw a free body diagram for each moving element: the mass and the pendulum. Notice that the force F_1 between the mass and the pendulum must be equal and opposite by Newton's third law. Remember, the direction of the forces on the free body diagram is merely a sign convention—it does not imply that the forces actually act in the directions shown. In fact, as the system moves, the force directions change dynamically.



Step 3: Apply Newton's second law to each free body diagram—use the rectilinear version for the mass and the rotational version for the pendulum.

For m_1, applying Newton's second law gives:

$$F_k - F_1 = m_1\ddot{x}$$

and, after writing the spring force in terms of x, we obtain Eq. (6.14).

$$m_1\ddot{x} + kx = F_1 \tag{6.14}$$

Next, we sum moments about the pinned point O and apply Newton's second law to obtain: $M_b - m_2gL_2\sin\theta + F_1L_1\cos\theta = J\ddot{\theta}$. After writing the bearing moment (due to viscous damping) in terms of $\dot{\theta}$ and linearizing the sine and cosine terms, we obtain the equation of motion for the pendulum.

$$J\ddot{\theta} + b_r\dot{\theta} + m_2gL_2\theta = -F_1L_1 \tag{6.15}$$

We know that the rotational moment of inertia for m_2 about the pivot is $J = m_2{L_2}^2$, but will leave it as simply J for now so that we may more clearly interpret the results.

Step 4: Using Eq. (3.28) and the small angle approximation, we find the following relationship between the mass displacement and the pendulum angle.

$$x = L_1\theta \tag{6.16}$$

We differentiate Eq. (6.16) twice to relate the linear and angular velocities and accelerations.

$$\begin{aligned} \dot{x} &= L_1\dot{\theta} \\ \ddot{x} &= L_1\ddot{\theta} \end{aligned} \tag{6.17}$$

Step 5: Combining Eqs. (6.14) and (6.15) produces an equation that includes both of the dynamic variables x and θ.

$$J\ddot{\theta} + b_r\dot{\theta} + m_2 g L_2 \theta = -(m_1\ddot{x} + kx)L_1 \tag{6.18}$$

Finally, using Eqs. (6.16) and (6.17) we can write Eq. (6.18) in terms of θ only.

$$\left(J + m_1 L_1^2\right)\ddot{\theta} + b_r\dot{\theta} + \left(m_2 g L_2 + kL_1^2\right)\theta = 0 \tag{6.19}$$

The first term includes the rotational moment of inertia for the pendulum, J, and the reflected inertia for the mass, $J_{ref} = m_1 {L_1}^2$, which is the moment of inertia of a mass m_1 at a distance L_1 from its center of rotation. The second (velocity) term includes only the rotational damping of the bearing. The third term includes the pendulum "stiffness" term, $m_2 g L_2$, and the reflected rotational stiffness of the rectilinear spring, $k_{ref} = k{L_1}^2$. Notice that since the units of k are N/m, the units of the reflected term are N-m or N-m/rad, as expected. Alternatively, we can use Eqs. (6.16) and (6.17) to write the equations of motion in terms of x only.

$$\left(\frac{J}{L_1^2} + m_1\right)\ddot{x} + \frac{b_r}{L_1^2}\dot{x} + \left(\frac{m_2 g L_2}{L_1^2} + k\right)x = 0 \tag{6.20}$$

We divided through by the length L_1 so that the units of the term multiplying acceleration are converted to mass and, therefore, the equation may be properly interpreted by Newton's second law. Namely, we notice that the mass term now includes m_1 and the reflected mass of the pendulum, $m_{ref} = \frac{J}{L_1^2}$. Similarly, the damping and stiffness terms include reflected values: $b_{ref} = \frac{b_r}{L_1^2}$ and $k_{ref} = \frac{m_2 g L_2}{L_1^2}$. Notice that the units of these two terms are N-s/m and N/m, as expected. Equations (6.18) and (6.19) describe the *same motion in different coordinates* because Eqs. (6.15) and (6.16) constrain x and θ to move together.

Step 6: Calculating the Laplace transform of Eq. (6.19) and assuming a nonzero initial velocity, ω_0, we obtain the Laplace domain equation of motion.

$$\Theta(s) = \frac{\omega_0}{s^2 + \left(\frac{b_r}{J + m_1 L_1^2}\right)s + \frac{m_2 g L_2 + kL_1^2}{J + m_1 L_1^2}} \tag{6.21}$$

From Eq. (6.21), the natural frequency and damping ratio can be determined.

$$\omega_n = \sqrt{\frac{m_2 g L_2 + kL_1^2}{J + m_1 L_1^2}} \tag{6.22}$$

$$\zeta = \frac{1}{2\omega_n}\left(\frac{b_r}{J + m_1 L_1^2}\right) \tag{6.23}$$

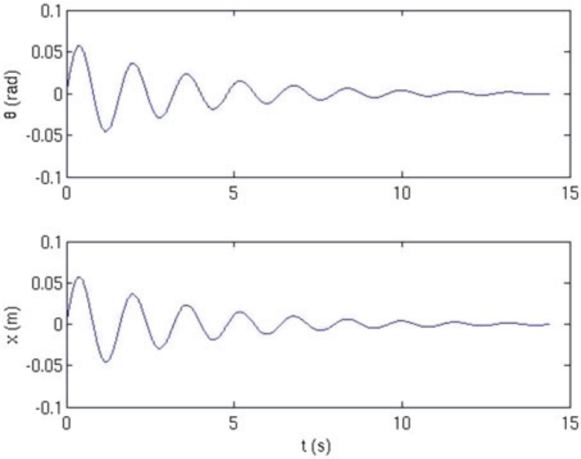
Substituting these definitions, Eq. (6.21) becomes identical to Eq. (5.11) for the damped pendulum. Subsequently, the solution for the motion is given by Eqs. (5.12), (5.13), and (5.15), depending on the damping ratio. As seen in the next step, this system is underdamped, so the motion is given by $\theta(t) = \frac{\omega_0}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t)$. The only mathematical differences between this example and the damped pendulum shown in Fig. 5.1 are the natural frequency and damping ratio definitions. Again, we see that there is a mathematical and physical analogy between the apparently more complex system shown in Fig. 6.3 and the simpler system shown in Fig. 5.1.

Step 7: By inserting the numerical values for this example, we determine the natural frequency and damping ratio and then plot the motions using MATLAB®. Note that once $\theta(t)$ is determined, $x(t)$ is found from Eq. (6.16).

The corresponding plot is provided. It is observed that the motions are linked by the constraint.

```
clear all
clc
close all

% Parameters
g = 9.81;    % m/s^2
m1 = 2;      % kg
m2 = 4;      % kg
br = 10;     % N-m-s
k = 200;     % N/m
L1 = 1;      % m
L2 = 2;      % m
w0 = 0.25;   % rad/s

% Natural frequency and damping ratio
J = m2*L2^2;
J_ref = m1*L1^2;
k_ref = k*L1^2;
display(J/J_ref)
display(m2*g*L2/k_ref)
wn = sqrt((m2*g*L2 + k*L1^2)/(J + m1*L1^2));
zeta = 1/(2*wn)*(br/(J+m1*L1^2));
wd = wn*sqrt(1-zeta^2);
display(wn)
display(zeta)

% Calculate and plot the motions
tau = 1/(zeta*wn);
t = [0:tau/1000:4*tau];
```

```
q = w0/wn*exp(-zeta*wn*t).*sin(wd*t);
x = L1*q;

figure(1)
subplot(211)
plot(t, q)
ylabel('\theta (rad)')
subplot(212)
plot(t, x)
xlabel('t (s)')
ylabel('x (m)')
```



6.5 Gears in Dynamic Systems

Gears are the rotational analogy of levers. The constraint relationships for gears were developed in Chap. 3 and are summarized by Eq. (3.37). Like other transmission elements, gears also generate reflected quantities in the equations of motion. They are useful in matching the characteristics of a driving actuator, such as an electric motor, to the system being driven and enable the driving actuator to operate at conditions under which it is most efficient.

To demonstrate how gears are analyzed in dynamic systems, consider the system first shown in Fig. 5.6, but let us now add a gear drive. The new system is shown schematically in Fig. 6.4.

For systems such as this, it is important to carefully define the coordinates (step 1). Since the gears can move independently, they each have an independent angle. The free body diagram of the inertia (step 2) is the same as shown in Fig. 5.6b. Newton's laws (step 3) give Eq. (5.26). At this point, however, we note

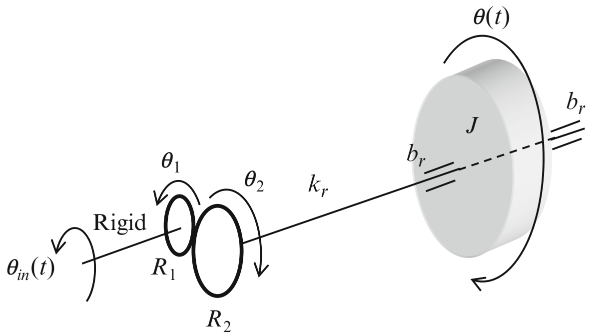


Fig. 6.4 Inertia supported on bearings and driven through a flexible shaft and gear train with a prescribed input angle, $\theta_{in}(t)$. The shaft on the input side of the gears is assumed to be rigid

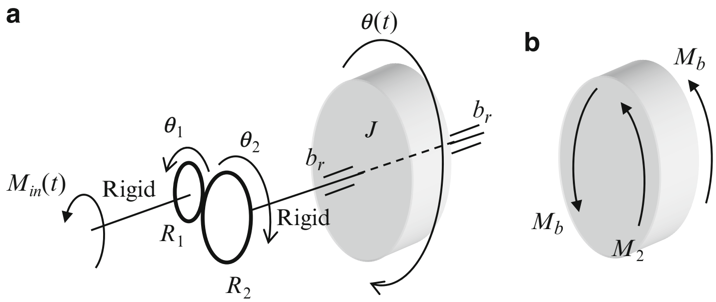


Fig. 6.5 Inertia supported on bearings and driven through a flexible shaft and a gear train with a prescribed input torque $M_{in}(t)$. The shafts on both sides of the gears are assumed to be rigid

that the rotational spring moment depends on the angle of the second gear, θ_2, and not directly on the input angle. This yields an equation of motion that differs slightly from Eq. (5.27).

$$J\ddot{\theta} + 2b_r\dot{\theta} + k_r\theta = k_r\theta_2 \tag{6.24}$$

Next, we implement the gear constraint from Eq. (3.33) and let $\theta_1 = \theta_{in}$ (due to the rigid shaft assumption) to obtain the final equation of motion. Note that if $R_1 < R_2$, a large angular change at the input is transformed into a small angular change at the output and, if $R_2 < R_1$, the reverse is true.

$$J\ddot{\theta} + 2b_r\dot{\theta} + k_r\theta = k_r\frac{R_1}{R_2}\theta_{in}(t) \tag{6.25}$$

To understand the concept of reflected quantities, consider the variation on Fig. 6.4 shown in Fig. 6.5a. In this case, the input moment (torque), $M_{in}(t)$, is controlled, but the angle on the input side, $\theta_1(t)$, is not. Both the input and output shafts are assumed to be rigid. The coordinate definitions are the same as the previous example. The moment $M_2(t)$ depends on the gears. Applying Newton's second law to the free body diagram displayed in Fig. 6.5b and recognizing that the moments oppose the angular velocity, we obtain the equation of motion.

$$J\ddot{\theta} + 2b_r\dot{\theta} = M_2(t) \tag{6.26}$$

Transforming the moment using Eq. (3.37), the angle using Eq. (3.35) and its derivatives, and recognizing that $M_1(t) = M_{in}(t)$, we obtain the equation of motion for the input side of the gears.

$$J\left(\frac{R_1}{R_2}\right)^2\ddot{\theta}_1 + 2b_r\left(\frac{R_1}{R_2}\right)^2\dot{\theta}_1 = M_{in}(t) \tag{6.27}$$

This is a particularly useful formulation of the equation of motion because it describes what the system drive "sees" from the combined system. The reflected inertia of the system is $J_{ref} = J\left(\frac{R_1}{R_2}\right)^2$ and the reflected rotational damping is $b_{ref} = b_r\left(\frac{R_1}{R_2}\right)^2$. Both physical quantities are modified by the square of the gear ratio. So, for example, if the radius of gear 1 is one-tenth the radius of gear 2, both the reflected inertia and the reflected damping are 100 times smaller than the actual quantities. When designing systems, this "gearing down" of the reflected quantities is useful when it is desired that the drive (a motor, for example) behaves as if it is not attached to the system. It also makes the drive insensitive to changes in the system, such as the change in inertia that results when a robot arm picks up a payload. The reverse phenomenon of "gearing up" the dynamic quantities can make a compact energy storage element (such as a flywheel) behave as a much larger one.

Example 6.3 In a toy car, a stainless steel flywheel measuring 30 mm in diameter and 3 mm thick is used to store kinetic energy. The flywheel is "spun up" by rolling the car's wheel along the ground. If the arrangement is the same as shown in Fig. 6.5 and the damping can be assumed to be zero, determine the gear ratio required for the flywheel to store 160 mJ of kinetic energy when the input angular velocity is 1 revolution per second.

Solution The inertia of a solid cylindrical flywheel is:

$$J = \frac{mR^2}{2}.$$

If the flywheel density is 7800 kg/m^3, its mass 16.5 g and the mass moment of inertia is 1.86×10^{-6} kg-m^2. The kinetic energy of the flywheel is:

$$\mathrm{KE} = \frac{1}{2}J\omega_2^2$$

and, to obtain 160 mJ of energy, ω_2 must be approximately 411 rad/s. The input speed, ω_1, was given as 6.3 rad/s. Therefore, the gear ratio must be at least $\frac{\omega_2}{\omega_1}$, or 66 (rounding to the nearest larger whole number).

6.6 Other Transmission Elements

There are many other transmission elements in addition to gears and levers. Rectilinear examples include mechanisms and flexures which are typically combinations of levers. Rotary examples include belt and pulley combinations and rollers. Many transmission elements transform rotational motion into linear motion or vice versa. An example detailed in Chap. 3 is the rack and pinion. Other include leadscrews, ballscrews, or a simple wheel or roller. We must be able to analyze the effects of these elements on system behavior as well. The approach we use when analyzing a system with these transmission elements includes the same steps we demonstrated for gears and levers.

Example 6.4 Consider the rack and pinion drive system shown in Fig. 6.6. The rotary inertia, J, represents the rotor of the drive motor and the input moment, $M_{in}(t)$, is the applied motor torque (moment). The motor is assumed to have damping, b_r, and there is rectilinear damping, b, in the guideways that support the mass, m. The drive shaft from the motor to the pinion is assumed to be rigid.

Solution

We define in step 1 θ as the angle of the motor rotor and x as the mass position. Free body diagrams of the moving components are shown in Fig. 6.7 (step 2). Notice that the contact force, F, acting on the pinion from the rack is equal and opposite to the

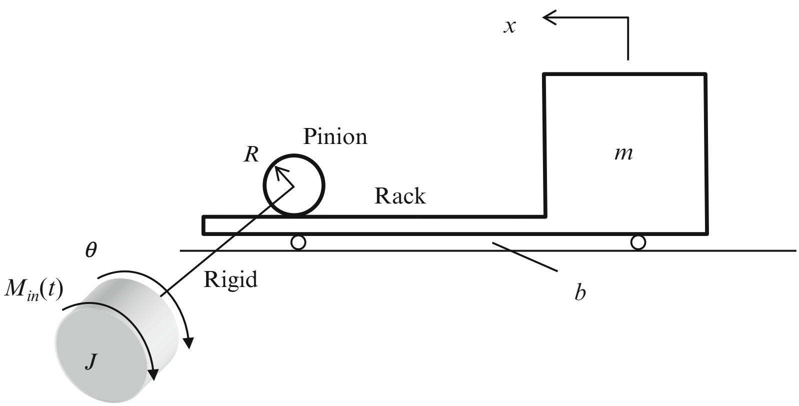


Fig. 6.6 Motor drive for rack and pinion system

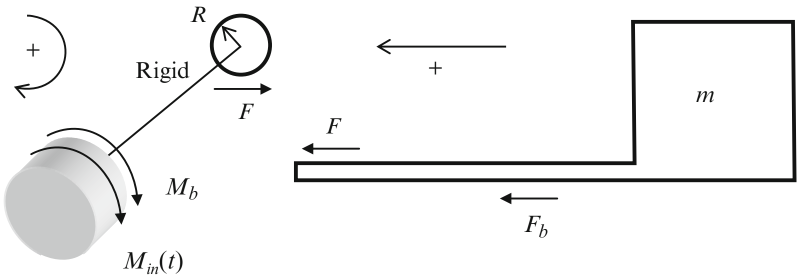


Fig. 6.7 Free body diagrams of components displayed in Fig. 6.6

contact force acting on the rack from the pinion. In step 3, we apply Newton's second law to each free body diagram. For the rotational side, we obtain Eq. (6.28).

$$M_{in}(t) + M_b - FR = J\ddot{\theta} \tag{6.28}$$

After substituting $M_b = -b_r\dot{\theta}$, we obtain the differential equation of motion shown in Eq. (6.29).

$$J\ddot{\theta} + b\dot{\theta} = M_{in}(t) - FR \tag{6.29}$$

Next, for the rectilinear side, we obtain a second (related) equation of motion.

$$F + F_b = m\ddot{x} \tag{6.30}$$

As previously, we recognize that the damping force acts opposite to the velocity and, therefore, we obtain Eq. (6.31).

$$m\ddot{x} + b\dot{x} = F \tag{6.31}$$

Notice that if we substitute the time derivatives from Eqs. (6.29) and (6.31) with velocities:

$$\begin{aligned} v &= \dot{x} \\ \omega &= \dot{\theta}, \end{aligned} \tag{6.32}$$

then we obtain the first-order equations provided in Eq. (6.33).

$$\begin{aligned} &J\dot{\omega} + b_r\omega = M_{in}(t) - FR \\ &m\dot{v} + bv = F \end{aligned} \tag{6.33}$$

Notice Eq. (6.33) contains both the rotational variable ω and the rectilinear variable v. In step 4, we define the constraints for the rack and pinion transmission element which enables us to connect the two equations of motion and combine them into a single equation. These constraints are given by Eqs. (3.40) and (3.42) which, when written using the velocity variables, reduce to Eq. (6.34).

$$\begin{aligned} v &= \omega R \\ M\omega &= Fv \end{aligned} \tag{6.34}$$

The second of these equations is the energy/power balance for the rack and pinion, where M is the moment applied directly to the pinion. This quantity is not shown explicitly in our combined free body diagrams; however, when we combine the two expressions in Eq. (6.34), we obtain $M = FR$. This is the result obtained if we apply Newton's second law to the pinion alone assuming that is has no appreciable inertia. As stated previously, we have neglected any frictional forces between the pinion and the rack; however, in a lumped parameter approach such losses can often be included in b_r and/or b.

Next, we combine Eqs. (6.33) and (6.34) (step 5). First, we use the second of Eq. (6.33) to replace F in the first equation. This gives a mixed formulation.

$$J\dot{\omega} = M_{in}(t) - (m\dot{v} + bv)R \tag{6.35}$$

Next, using the first of Eq. (6.34), we replace the rectilinear velocity with the angular velocity to obtain the equation of motion as seen from the rotational side.

$$\left(J + mR^2\right)\dot{\omega} + \left(b_r + bR^2\right)\omega = M_{in}(t) \tag{6.36}$$

Using $\omega = \frac{v}{R}$ from Eq. (6.34) and dividing each term by R, we obtain the equations of motion as seen from the rectilinear side.

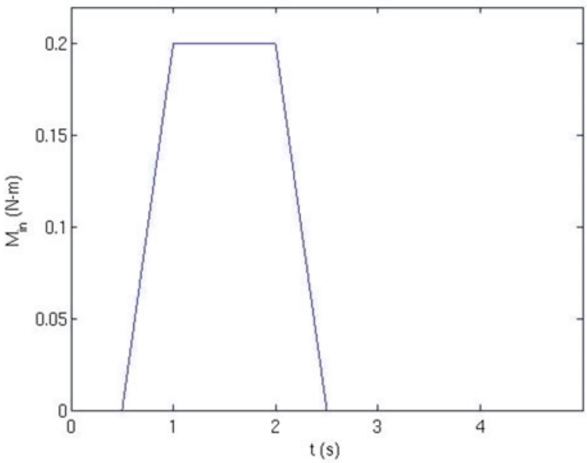
$$\left(m + \frac{J}{R^2}\right)\dot{v} + \left(b + \frac{b_r}{R^2}\right)v = \frac{M_{in}(t)}{R} \tag{6.37}$$

The purpose of dividing by R was to obtain units of Newtons on the right-hand side of the equation. The quantity $\frac{M_{in}(t)}{R}$ appears as an equivalent force input on the rectilinear side.

Equation (6.36) is used when it is desired to select a motor drive for a rack and pinion system. The reflected inertia of the mass attached to the rack is mR^2 and the reflected damping of the support rails on the mass/rack is bR^2. The motor providing the input moment, $M_{in}(t)$, must not only drive the actual motor inertia, J, and damping, b_r, but also the reflected inertia and damping. Note that the reflected quantities are modified by the square of the transmission ratio which, for a rack and pinion, reduces to the pinion radius, R.

Equation (6.37) is used to examine the mass motion. For example, the Fig. 6.6 arrangement is a common design for a large machine tool axis when the component being machined is mounted on the machine table and the total mass of the combination is m. In this case, it is important that the velocity and position of the table be controlled with sufficient accuracy (i.e., better than the required accuracy for the machined part). For example, a table with a total motion range of 10 m might have a desired positioning accuracy on the order of 100 μm, or one part in 10^5. This can only be accomplished with careful mechanical design/construction and accurate motor control. We will elaborate with examples.

Example 6.5 A rack and pinion positioning system is arranged as shown in Fig. 6.6. The moving mass m is 1.5 kg, the pinion radius, R, is 25 mm and the mass moves on linear guideways with damping, b, of 0.5 N-s/m. The inertia of the motor rotor, shaft, and pinion, J, is 0.0025 kg-m^2 and the rotational damping of the motor and pinion arrangement, b_r, is 0.007 N-m-s. The input moment ramp linearly from zero to 0.2 N-m in 0.5 s and then returns to zero in 0.5 s after a 1 s pause.



Find and plot the system responses, $v(t)$ and $\omega(t)$, using the `lsim` command in MATLAB®. Also, find and plot the position, $x(t)$. Finally, determine the time constant of the system and comment on its relation to the system response graphs.

Solution The equations of motion are given by Eqs. (6.36) and (6.37). The rotational equation of motion is:

$$\left(J + mR^2\right)\dot{\omega} + \left(b_r + bR^2\right)\omega = M_{in}(t).$$

Once $\omega(t)$ is determined, the velocity is obtained by applying the constraint $v(t) = R\omega(t)$. Using this constraint, we can also solve for the angular and linear velocity transfer functions. The angular velocity transfer function is found by calculating the Laplace transform of the rotational equation of motion.

$$\frac{\Omega(s)}{M_{in}(s)} = \frac{1}{\left(J + mR^2\right)s + \left(b_r + bR^2\right)}$$

Comparing the right-hand side of the transfer function to $\frac{1}{s+a}$, we see that $a = \frac{b_r + bR^2}{J + mR^2}$ and, therefore, the time constant is $\tau = \frac{J + mR^2}{b_r + bR^2}$. The linear velocity transfer function is found by multiplying the angular velocity transfer function by the pinion radius.

$$\frac{V(s)}{M_{in}(s)} = \frac{R}{\left(J + mR^2\right)s + \left(b_r + bR^2\right)}$$

This is equivalent to the result obtained by taking the Laplace transform of Eq. (6.37). To determine the position as a function of time, we must integrate the velocity over time. In the Laplace domain, this integration is achieved using: $X(s) = \frac{1}{s}V(s)$; see Chap. 2. The position transfer function is:

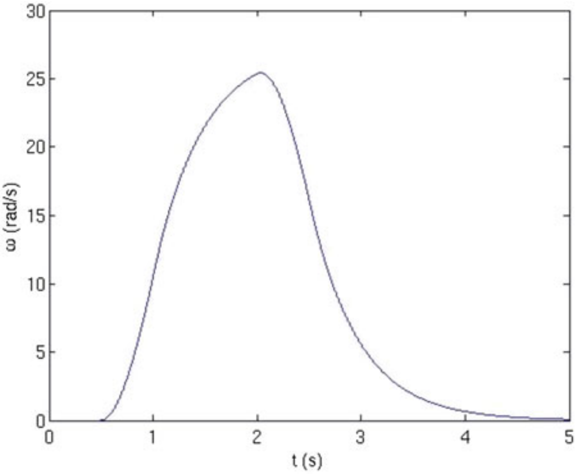
$$\frac{X(s)}{M_{in}(s)} = \frac{R}{\left(J + mR^2\right)s^2 + \left(b_r + bR^2\right)s}.$$

The MATLAB® code used to analyze this system is provided.

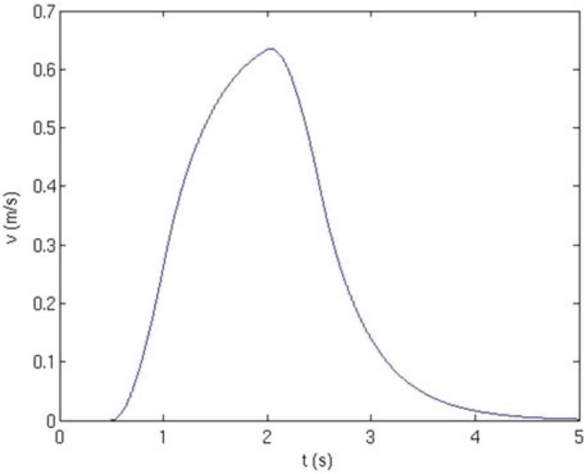
```
clear all
clc
close all

% Parameters
R = 25e-3;     % pinion radius, m
J = 0.0025;    % moment of inertia, kg-m^2
m = 1.5;       % mass of moving component, kg
br = 0.007;    % rotational damping, N-m-s
b = 0.5;       % linear damping, N-s/m
tau = (J+m*R^2)/(br+b*R^2); % time constant, s
display(tau)

% Define input moment
M_max = 0.2;   % peak moment, N-m
t0 = 0.5;      % input start time, s
t1 = 1;        % ramp up end time, s
t2 = 2;        % end of constant moment, s
t3 = 2.5;      % ramp down end time, s

% Define time vector
t = [0:tau/1000:2*t3];
% Initialize input moment to zero
M_in = zeros(1, length(t));
% Ramp up
index = find(t>t0 & t<t1);
m1 = M_max/(t1-t0);          % ramp up slope
M_in(index) = m1*(t(index)-t0);
% Constant moment
```

```
index = find(t>t1 & t<t2);
M_in(index) = M_max;
% Ramp down
index = find(t>t2 & t<t3);
m3 = -M_max/(t3-t2);        % ramp down slope
M_in(index) = M_max + m3*(t(index)-t2);

% Plot input moment
figure(1)
plot(t, M_in)
xlabel('t (s)')
ylabel('M_{in} (N-m)')
axis([0 max(t) 0 1.1*M_max])

% Determine system response (angular velocity)
num1 = [1];
den1 = [J+m*R^2 br+b*R^2];
sys1 = tf(num1, den1);
[w, t] = lsim(sys1, M_in, t);
figure(2)
plot(t, w)
xlabel('t (s)')
ylabel('\omega (rad/s)')

% Determine system response (linear velocity)
v = R*w;   % rack and pinion constraint
figure(3)
plot(t, v)
xlabel('t (s)')
ylabel('v (m/s)')

% Determine system response (linear position)
num2 = [R];
den2 = [J+m*R^2 br+b*R^2 0];
sys2 = tf(num2, den2);
[x, t] = lsim(sys2, M_in, t);
figure(4)
plot(t, x)
xlabel('t (s)')
ylabel('x (m)')
```

The time constant for the system is 0.47 s. The system response plots are provided. The time-dependent angular velocity begins to increase first following the ramp input. Before it reaches steady state behavior, the input moment is removed and the angular velocity decays back to zero.



The linear velocity follows the same profile because it is calculated from the product of the angular velocity and the pinion radius.



The velocity peaks at 0.635 m/s before decaying back to zero. The position is the integral of the velocity.

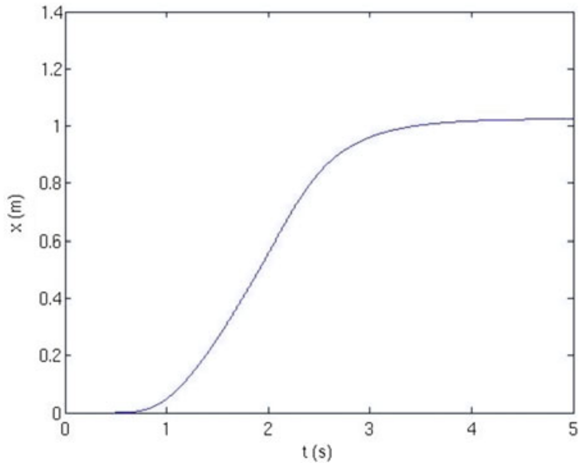
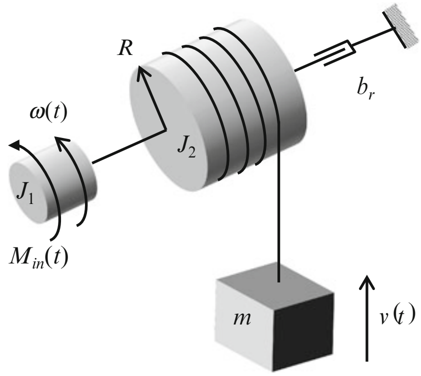


Fig. 6.8 Winch system for lifting a mass



Notice that the final position of the moving mass changes by slightly more than 1 m. When the moment is removed, the position remains at this new value.

Example 6.6 A large winch system (Fig. 6.8) is driven by a motor having a rotational inertia J_1 and an input motor moment $M_{in}(t)$. The winch consists of a pulley and cable that we assume always remains taught. The rotating pulley has radius, R, and inertia, J_2. The pulley and motor are rigidly attached. The total rotational damping from all bearings is b_r. A mass, m, is lifted by the cable wrapped around the drum. Determine the equations of motion for the system in terms of $\omega(t)$ and $v(t)$, and identify the reflected quantities in each equation. Given that J_1 is 0.05 kg-m^2, J_2 is 0.5 kg-m^2, m is 2 kg, b_r is 1.5 N-m-s, and the pulley radius, R, is 0.25 m, determine the magnitude of an input step moment required for the steady state velocity of the mass to be 2 m/s. For this input moment, plot the corresponding response, $v(t)$, in MATLAB®.

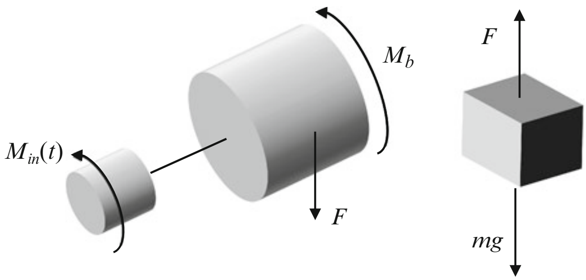


Fig. 6.9 Free body diagrams of the system elements from Fig. 6.8

Solution First (step 1), we define the coordinates. In this case, we have the velocities, ω and v, as shown. The next step (step 2) is to draw free body diagrams of each inertia/mass in the system; these are displayed in Fig. 6.9.

Notice that there is an equal and opposite force from the cable, F, acting upward on the mass and downward on the pulley. Next, we apply Newton's second law to each moving element. For the rotational inertia, we obtain Eq. (6.38) after substituting $M_b = -b\omega$.

$$(J_1 + J_2)\dot{\omega} + b_r\omega = M_{in}(t) - FR \tag{6.38}$$

For the mass, we have:

$$m\dot{v} = F - mg, \tag{6.39}$$

where the unknown force in the cable F appears in both Eqs. (6.38) and (6.39). In step 4, we determine the constraints for the transmission element. In this case, the transmission element is the pulley/cable combination. Geometrically, if the cable is to remain in tension, the rate at which the cable winds around the drum must be equal to the velocity of the mass.

$$v = \omega R \tag{6.40}$$

The power delivered by the cable to the mass must also equal the power delivered by the pulley to the cable.

$$Fv = FR\omega \tag{6.41}$$

After canceling the cable force, Eq. (6.41) reduces to Eq. (6.40) and we obtain no new information.

The fifth step is to combine the results of steps 3 and 4 to obtain the equations of motion for the system. Solving Eq. (6.39) for F and substituting into Eq. (6.38), we obtain the mixed formulation: $(J_1 + J_2)\dot{\omega} + b_r\omega = M_{in}(t) - m\dot{v}R - mgR$. By applying Eq. (6.40) and its derivative, we obtain the rotational equation of motion:

$$\left(J_1 + J_2 + mR^2\right)\dot{\omega} + b_r\omega = M_{in}(t) - mgR \tag{6.42}$$

and the rectilinear equation of motion:

$$\left(\frac{J_1+J_2}{R^2}+m\right)\dot{v}+\frac{b_r}{R^2}v=\frac{M_{in}(t)}{R}-mg. \tag{6.43}$$

We divided both sides of Eq. (6.43) by R a second time to obtain units of force on the driving (right-hand) side of the equation. We interpret mR^2 in the first term of Eq. (6.42) as the reflected inertia of the mass on the rotational side. Similarly, $\frac{J_1+J_2}{R^2}$ and $\frac{b_r}{R^2}$ are the reflected mass and reflected linear damping corresponding to the rotational inertias and rotational damper on the rectilinear side.

For a step input moment, $M_{in}(t)=M_0\cdot u(t)$, and assuming that the gravitational force is not present until time zero so that the gravitational force also appears as a step, we calculate the Laplace transform of Eq. (6.43).

$$V(s)=\frac{M_0}{R}\frac{1}{s\left(\left(\frac{J_1+J_2}{R^2}+m\right)s+\frac{b_r}{R^2}\right)}-\frac{mg}{s\left(\left(\frac{J_1+J_2}{R^2}+m\right)s+\frac{b_r}{R^2}\right)} \tag{6.44}$$

Note that if we do not assume the weight to be applied at the same time as the input moment, the system would be subjected to the weight with nothing to counter it and the system would disassemble. Assuming the weight is a step input is consistent with considering the mass to be picked up off a supporting surface at time zero.

The steady state velocity is determined by applying the final value theorem to Eq. (6.44).

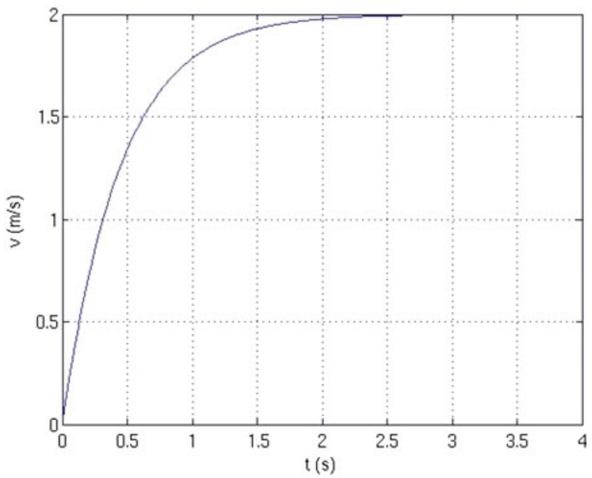
$$v_{ss}=\lim_{s\to 0}(sV(s))=\frac{M_0R}{b_r}-\frac{mgR^2}{b_r} \tag{6.45}$$

Since the units of moment divided by the rotational damping constant are rad/s, we see that both terms on the right-hand side of Eq. (6.45) are m/s since the units of R is meters. Solving for the applied moment in terms of the steady state velocity, we obtain: $M_0=b_r\frac{v_{ss}}{R}+mgR$. This indicates that the applied moment must cancel the sum of the rotational damping moment, $b_r\omega_{ss}$, and the moment due to the gravitational load, mgR. The associated MATLAB® code is provided.

```
clear all
clc
close all

% Parameters
m = 2;
br = 1.5;
J1 = 0.05;
J2 = 0.5;
```

```
R = 0.25;
v_ss = 2;
g = 9.81;

% Steady state moment
M0 = (br*v_ss + m*g*R^2)/R;
tau = (J1 + J2 + m*R^2)/br;
display(M0)
display(tau)

% Plot velocity as a function of time
t = [0:tau/100:8*tau];
syms V s v den
den = ((J1+J2)/R^2 + m)*s + br/R^2;
V = M0/(R*s*den) - m*g/(s*den);
v = ilaplace(V);
vv = eval(v);
figure(1)
plot(t, vv)
grid
xlabel('t (s)')
ylabel('v (m/s)')
```

The moment required to produce a steady state upward velocity of 2 m/s is 16.9 N-m and the time constant for the system is 0.45 s. The corresponding plot is shown.



Example 6.7 Leadscrews and ballscrews are transmission elements that transform rotary motion into linear motion. The leadscrew functions just like a nut and bolt,

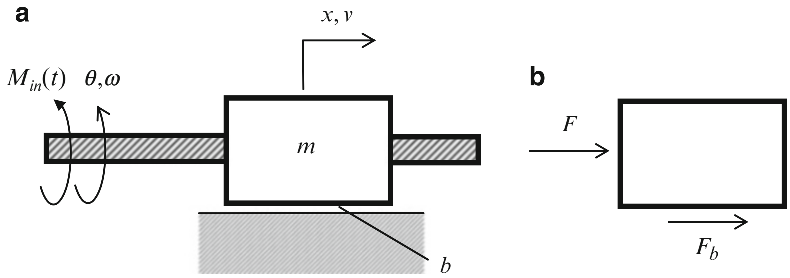
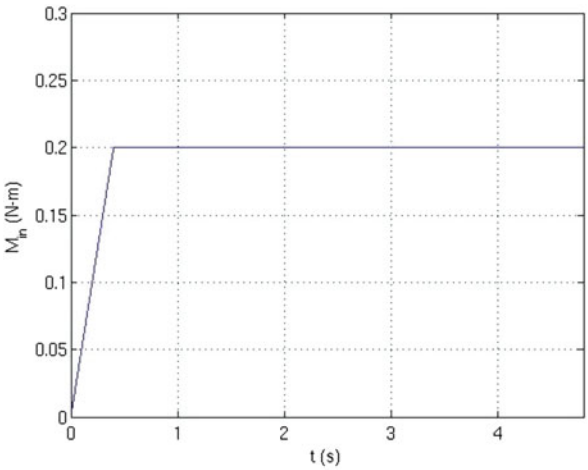


Fig. 6.10 (**a**) A simple leadscrew arrangement driving a mass across a slide with linear damping, *b*; and (**b**) free body diagram for the moving mass

except that the screw turns and that causes linear motion of the nut. The nut is attached to a carriage that moves on (linear) guideways. Guideways can be stacked on top of one another to produce simultaneous motions in orthogonal directions. A simple leadscrew arrangement is shown in Fig. 6.10. The screw has a pitch of 2 mm/rev and the mass is 4 kg. The damping constant is 5 N-s/m. Determine the equations of motion on the rotational side, the reflected inertia and rotational damping seen at the drive of the screw, and then determine the steady state velocity for the specified input moment profile; the ramp time is one quarter of the system time constant. Find and plot $\omega(t)$ in rpm using MATLAB®.



The coordinates θ and x are shown in Fig. 6.10a (step 1) and the free body diagram (step 2) is shown in Fig. 6.10b. The equation of motion for the mass is determined by applying Newton's second law to the free body diagram (step 3). After substituting $F_b = -bv$, we have:

$$m\dot{v} + bv = F, \tag{6.46}$$

where F is the force exerted by the screw on the mass. Step 4 is to determine the constraints imposed by the transmission element—the leadscrew. If p is the leadscrew pitch in m/rad, then the position is given by:

$$x = p\theta. \tag{6.47}$$

Equation (6.47) can be differentiated to find the relationships between angular and linear velocities, $v = p\omega$, and the angular and linear accelerations, $\dot{v} = p\dot{\omega}$. Further, for a lossless leadscrew, the power into the leadscrew is equal to the power delivered to the mass.

$$M_{in}\omega = Fv \tag{6.48}$$

When Eq. (6.48) is combined with the geometric constraint, we obtain a relationship in terms of the leadscrew pitch.

$$M_{in} = Fp \tag{6.49}$$

By substituting these relationships into Eq. (6.46) (step 5), we obtain the equation of motion in terms of the rotational variable.

$$mp^2\dot{\omega} + bp^2\omega = M_{in}(t) \tag{6.50}$$

The reflected inertia is mp^2 and the reflected rotational damping is bp^2. Calculating the Laplace transform of Eq. (6.50), we obtain the system transfer function which can be used to simulate the response in MATLAB®.

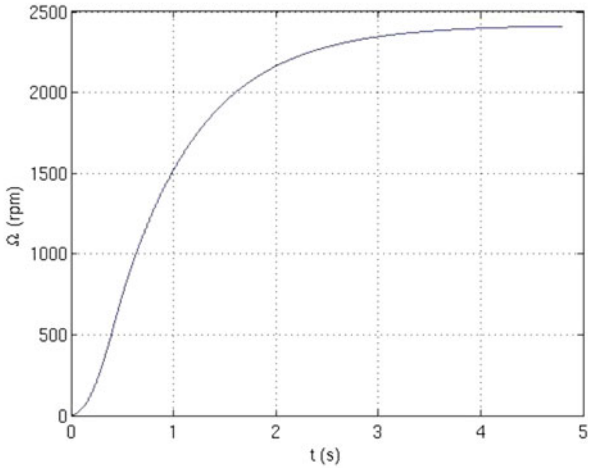
$$\frac{\Omega(s)}{M_{in}(s)} = \frac{1}{mp^2 s + bp^2} \tag{6.51}$$

From Eq. (6.51), we see that the time constant is $\frac{m}{b}$, which could also have been obtained from Eq. (6.46). The associated MATLAB® code is provided.

```
clear all
clc
close all

% Parameters
m = 4;
b = 5;
p = 0.002*2*pi;   % m/rad
M0 = 0.2;
tau = m/b;
t = [0:tau/1000:6*tau];
```

```
% Input profile
M_in = zeros(1, length(t));
t1 = tau/2;
index = find(t < t1);
m1 = M0/t1;
M_in(index) = m1*t(index);
index = find(t >= t1);
M_in(index) = M0;
figure(1)
plot(t, M_in)
grid
xlabel('t (s)')
ylabel('M_{in} (N-m)')
axis([0 max(t) 0 1.5*M0])

% System response
num = [1];
den = [m*p^2 b*p^2];
sys = tf(num, den);
[w, t] = lsim(sys, M_in, t);
figure(2)
W = w/(2*pi)*60;    % convert to rpm
plot(t, W)
grid
xlabel('t (s)')
ylabel('\Omega (rpm)')
```

The resulting plot is shown.



Because the moment ramps to a steady value, the steady state velocity is the same as would be obtained for a step input, $M_0 \cdot u(t)$. Substituting the step input into the transfer function and applying the final value theorem, we obtain Eq. (6.52).

$$\omega_{ss} = \frac{M_0}{bp^2} \tag{6.52}$$

Inserting the values given for this problem, we obtain a steady state angular velocity of 253 rad/s, or 2416 rpm, which matches the plot.

6.7 Higher Degree of Freedom Systems and Transfer Functions

It is often necessary to model multiple degree of freedom systems that include transmission elements. Consider the system shown in Fig. 6.11 that incorporates drive train flexibility. The motor rotor has rotational inertia J_1 with an input moment of $M_{in}(t)$. The motor is attached through a rigid shaft to a set of gears with a gear ratio $N = \frac{R_2}{R_1}$. The gears drive a second shaft with significant flexibility, k_{r2}, that drives a second inertia, J_2. This inertia is supported on bearings with rotational damping, b_{r2}. For now, we consider the motor side damping b_{r1} to be negligible, i.e., $b_{r1} = 0$. Our task is to determine the equation of motion and transfer function for the system, $\frac{\Phi(s)}{M_{in}(s)}$. Because the system has three energy storage elements, two inertias coupled through a spring, we expect that the system will be third order. Using this example, we will identify concepts that will enable us to analyze even higher order, more complex systems.

In step 1, we define the coordinate θ for the angle of motor inertia J_1, ϕ for the angle of the driven inertia J_2, and θ_1 and θ_2 for the gear angles. Free body diagrams for inertias J_1 and J_2 and each gear are shown in Fig. 6.12 (step 2). Note that with our sign conventions, a negative moment M_1 is applied to J_1, which is transmitted by the rigid shaft to gear 1. Gear 1 has an equal and opposite moment applied to it by the rigid shaft. Gears 1 and 2 also have equal and opposite contact forces F_g acting on them. Gear 2 has a moment M_2 applied to it by the flexible shaft and this moment is of equal magnitude to the moment applied to J_2 by the flexible shaft, which we have denoted M_{k2}. Thus, the power *delivered to* gear 1 is $M_1\dot{\theta}_1$ and the power *delivered by* gear 2 is $M_{k2}\dot{\theta}_2$. This will be important in step 4.

Now we proceed with step 3. Applying Newton's second law to the first inertia, we obtain Eq. (6.53).

$$J_1\ddot{\theta} = M_{in} - M_1 \tag{6.53}$$

For the second inertia, we obtain Eq. (6.54). We do not make the variable substitutions until step 5.

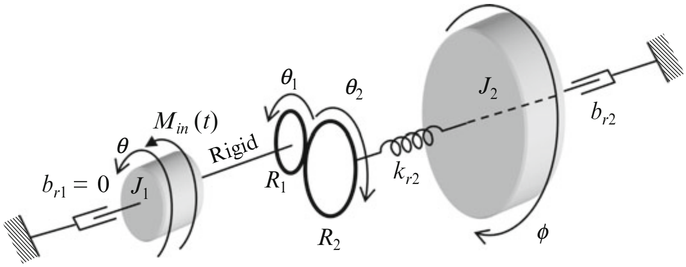


Fig. 6.11 Model for a motor driving an inertia through a gear train and a flexible shaft

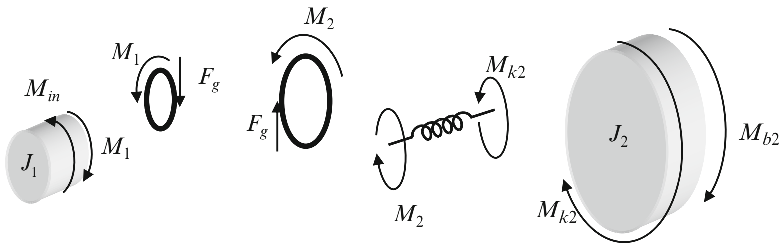


Fig. 6.12 Free body diagrams for the system elements in Fig. 6.11

$$J_2\ddot{\phi} = M_{b2} + M_{k2} \tag{6.54}$$

In step 4, we develop the constraints for the gears which were discussed in Chap. 3. The geometric constraint for the gears is $R_1\theta_1 = R_2\theta_2$ and, because the first shaft is rigid, $\theta_1 = \theta$. This can be differentiated any number of times to relate angular velocities and accelerations. The power constraint for the gears is an important concept in this problem. Namely, we must constrain the instantaneous power delivered to gear 1 to be equal to the instantaneous power delivered by gear 2 (if the gears are to be assumed lossless); see Eq. (6.55).

$$M_1\dot{\theta}_1 = M_{k2}\dot{\theta}_2 \tag{6.55}$$

Therefore, $M_1 = \frac{M_{k2}}{N}$.

In step 5, we combine the knowledge from steps 3 and 4 in order to obtain the equations of motion in terms of the independent variables θ and ϕ. Equation (6.53) becomes $J_1\ddot{\theta} = M_{in} - \frac{M_{k2}}{N}$ and, recognizing that $M_{k2} = k_{r2}(\theta_2 - \phi)$, we obtain Eq. (6.56).

$$NJ_1\ddot{\theta} = NM_{in} - k_{r2}(\theta_2 - \phi) \tag{6.56}$$

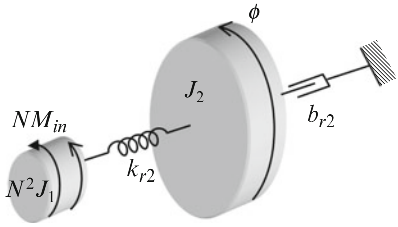


Fig. 6.13 System equivalent to Fig. 6.11

Now, using the geometric constraint, we set $\theta = \theta_1 = N\theta_2$ and obtain:

$$N^2 J_1 \ddot{\theta}_2 = NM_{in} - k_{r2}(\theta_2 - \phi). \tag{6.57}$$

Equation (6.54) becomes:

$$J_2 \ddot{\phi} = -b_{r2}\dot{\phi} + k_{r2}(\theta_2 - \phi). \tag{6.58}$$

Equations (6.57) and (6.58) are two coupled second-order differential equations in the variables θ_2 and ϕ. These equations are the same as would be obtained for the simplified system shown in Fig. 6.13. The inertia J_1 with angular displacement coordinate θ has been reflected through the gears as $N^2 J_1$ with displacement coordinate θ_2 and the input moment has been reflected through the gears as NM_{in}. The dynamic quantities get reflected by the transmission ratio squared, while the force and moment inputs get reflected by the transmission ratio alone. The coordinate simultaneously gets transformed to the coordinate on the opposite side of the transmission element. These are "rules" we will be able to use for more complex systems to quickly determine equations of motion without always repeating the procedure followed here.

In step 6, we use the Laplace transform to isolate the output variable of interest, ϕ, and the input variable, M_{in}. Calculating the Laplace transform of Eq. (6.57), we obtain $\Theta_2(s)$ in terms of $\Phi(s)$ and $M_{in}(s)$.

$$\Theta_2(s) = \frac{NM_{in}(s) + k_{r2}\Phi(s)}{N^2 J_1 s^2 + k_{r2}} \tag{6.59}$$

The Laplace transform of Eq. (6.58) yields:

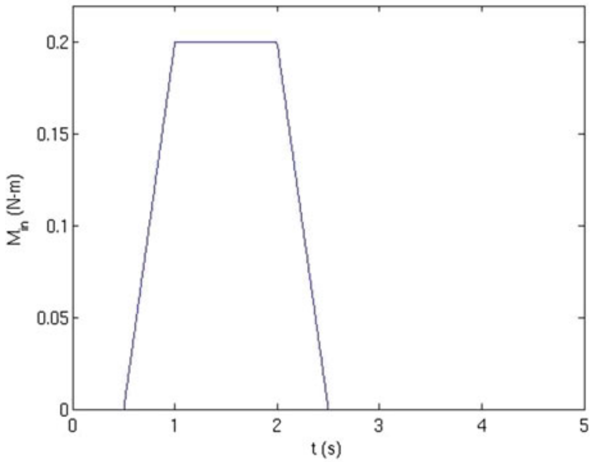
$$\Phi(s)\left(J_2 s^2 + b_{r2}s + k_{r2}\right) = k_{r2}\Theta_2(s). \tag{6.60}$$

Combining Eqs. (6.59) and (6.60), we obtain the desired transfer function.

$$\frac{\Phi(s)}{M_{in}(s)} = \frac{k_{r2}N}{s\left(N^2 J_2 J_1 s^3 + N^2 J_1 b_{r2} s^2 + \left(J_2 + N^2 J_1\right)k_{r2}s + b_{r2}k_{r2}\right)} \tag{6.61}$$

Note that this system is third order, as expected, because it can be written in terms of an angular velocity, $s\Phi(s)$, which is $\dot{\phi}$ in the time domain.

Example 6.8 Consider Fig. 6.11 wher the same motor from Example 6.5 with J_1 equal to 0.0025 kg-m^2 is used to drive an inertia of 1 kg-m^2 through a gear train with a ratio of $N = 10$. The shaft stiffness is 40 kN-m/rad and the damping is 50 N-m-s. The input profile is the same as described in Example 6.5. Assume the motor damping is zero. Find and plot the response of the driven inertia using MATLAB®.

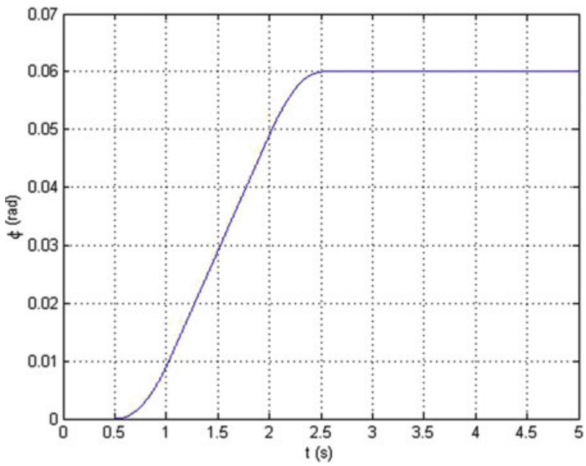


Solution We use Eq. (6.61) directly in the MATLAB® code provided here.

```
clear all
clc
close all

% Define input moment
M_max = 0.2;
t0 = 0.5;
t1 = 1;
t2 = 2;
t3 = 2.5;
t = [0:t3/1000:2*t3];
M_in = zeros(1, length(t)); % initialize input moment to zero
% Ramp up
index = find(t>t0 & t<=t1);
m1 = M_max/(t1-t0);
M_in(index) = m1*(t(index)-t0);
% Constant moment
index = find(t>t1 & t<=t2);
M_in(index) = M_max;
% Ramp down
index = find(t>t2 & t<=t3);
```

```
m3 = -M_max/(t3-t2);
M_in(index) = M_max+m3*(t(index)-t2);

% Plot input moment
figure(1)
plot(t, M_in)
xlabel('t (s)')
ylabel('M_{in} (N-m)')
axis([0 max(t) 0 1.1*M_max])

% Parameters
J1 = 0.0025;
N = 10;
J2 = 1;
br2 = 50;
kr2 = 40000;

% System response
num = [N^2*kr2];
den = [N^2*J1*J2 N^2*J1*br2 (J2+N^2*J1)*kr2 br2*kr2 0];
sys = tf(num, den);
[phi, t] = lsim(sys, M_in, t);
figure(2)
plot(t,phi)
grid
xlabel('t (s)')
ylabel('\phi (rad)')
```

The results are shown.



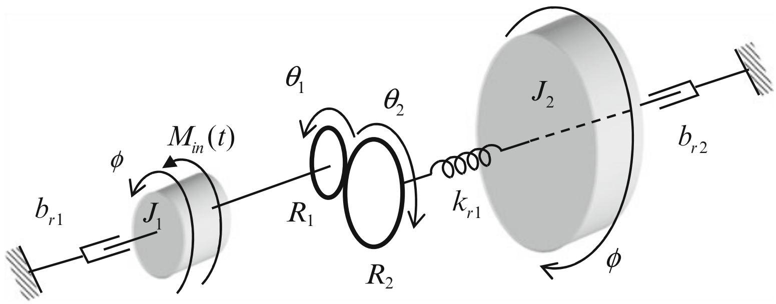


Fig. 6.14 Model for a motor driving an inertia through a gear train and a flexible shaft

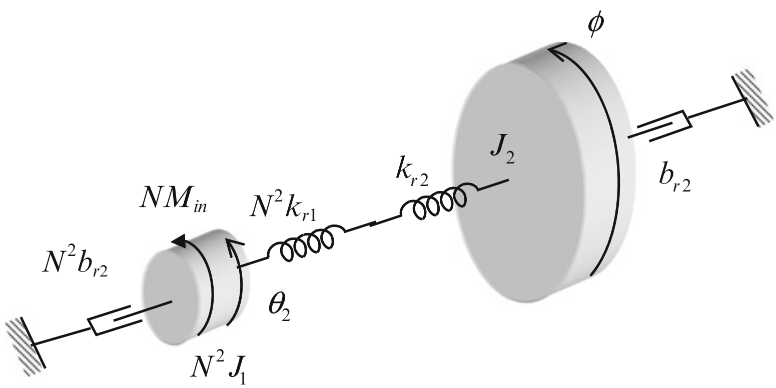


Fig. 6.15 System equivalent to model shown in Fig. 6.14

The roots of the denominator are provided here.

```
>> roots(den)

ans =

  1.0e+02 *

    0

  -0.0496 + 4.4674i
  -0.0496 - 4.4674i
  -0.4008
```

The zero root in the denominator occurs because there is nothing constraining the inertias to an equilibrium position. Thus a constant moment input will lead to a continuously changing angle. The other roots indicate that the time constants of this system are approximately 0.2 and 0.02 s, respectively, and the frequency of oscillation is 447 rad/s (71.1 Hz). Since the ramp time is 0.5 s, it does not excite the oscillations and the response of the system is parabolic during the ramp up and ramp down portions of the input with a linear increase in angle during the constant moment portion. There is a total angular displacement of 0.06 rad or 3.4 deg.

Consider the more complicated system shown in Fig. 6.14. Find the equations of motion using the concept of reflected quantities to develop an equivalent system without a gear train.

The equivalent system without a gear train is displayed in Fig. 6.15.

The equations of motion for this system are similar to those previously developed. For the left inertia:

$$N^2 J_1 \ddot{\theta}_2 = N M_{in} - k_{req}(\theta_2 - \phi) - N^2 b_{r1} \dot{\theta}_2 \tag{6.62}$$

and for the right:

$$J_2 \ddot{\phi} = -b_{r2} \dot{\phi} + k_{req}(\theta_2 - \phi), \tag{6.63}$$

where the equivalent spring stiffness is given by the series combination of the two shafts.

$$k_{req} = \frac{N^2 k_{r1} k_{r2}}{N^2 k_{r1} + k_{r2}} \tag{6.64}$$

6.8 Summary

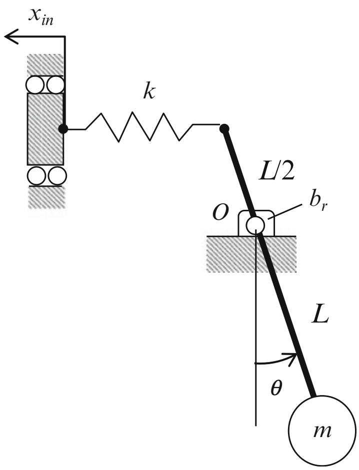
In this chapter, we analyzed dynamic systems with transmission elements. We developed a stepwise approach, which we will continue to use throughout the text. An additional step (step 4) was added to our previous analyses to incorporate the geometric and power constraints for the transmission elements. To summarize, we discussed the following concepts:

- rectilinear, rotary, and mixed rotational/rectilinear transmission elements in dynamic systems
- levered systems that translate a rectilinear displacement/force into an alternate displacement/force using appropriate constraints
- geared systems that translate a rotational torque/angle into an alternate moment/angle using appropriate constraints
- mixed systems, such as a rack and pinion, that translate moment/angle into rectilinear displacement/force using appropriate constraints
- reflected quantities in systems with transmission elements
- a dynamic quantity, A (i.e., inertia, stiffness, damping), on one side of a transmission element appears on the other side of the transmission element as the reflected quantity $\alpha^2 A$, where α is the transmission ratio of the element
- a dynamic quantity, B (i.e., force, moment), on one side of a transmission element appears on the other side of the transmission element as the reflected quantity αB, where α is the transmission ratio of the element.

In the following chapters, we will study physical systems that seem very different (e.g., electrical circuits), but observe that they have direct analogies to the mechanical systems we have studied so far.

Problems

1. A positioning system is designed with a lever arrangement as shown in the figure. It is controlled by a displacement, x_{in}, applied through a spring. There is rotational viscous damping, b_r, in the pivot bearing at O.



Complete the following.

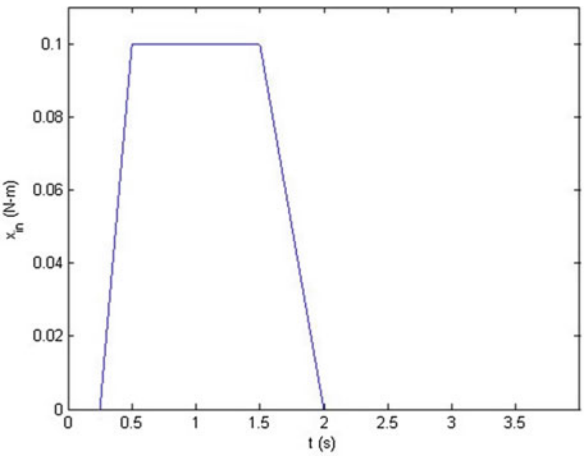
(a) Determine the equation of motion for the system in terms of θ and linearize using the small angle approximation.

(b) Find the system transfer function $\frac{\Theta(s)}{X_{in}(s)}$.

(c) Determine the expression for the reflected rotational stiffness of the spring, k, and the reflected inertia of the mass, m.

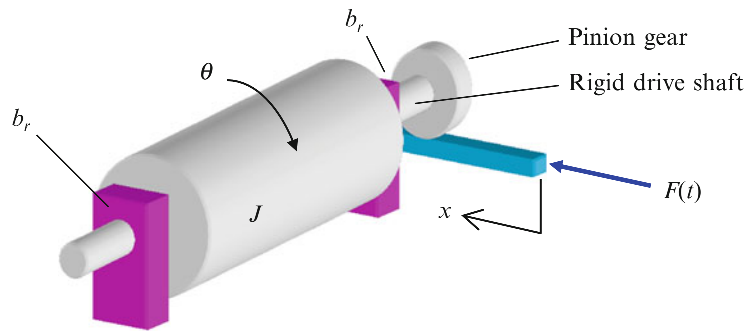
(d) Determine expressions for the natural frequency, ω_n, and damping ratio, ζ.

(e) The values of L, m, k, and b_r are 1.5 m, 2 kg, 500 N/m, and 10 N-m-s respectively. Calculate the natural frequency ω_n and damping ratio ζ. Using a MATLAB® script file and the `lsim` command, simulate the response to the input provided and plot the response. The ramp up begins at 0.25 s and ends at 0.5 s. The ramp down begins at 1.5 s and ends at 2.0 s. The total time interval is 4.0 s.



(f) Repeat your simulation for a rotational damping, b_r, of 50 N-m-s. What is the new damping ratio? What is the effect on the response?

2. A rack and pinion drives system for a cylindrical drum in a chemical mixing system is depicted in the figure. The input is the force, $F(t)$, which produces the motion, x, and rotates the pinion gear. There is viscous rotational damping, b_r, in each of the two bearings and the rotating drum has an inertia, J. The radius of the pinion gear is R and the shaft connecting the pinion to the drum is rigid.



Complete the following.

(a) Determine the equation of motion in terms of v and $\dot{v}$, where $v = \dot{x}$.

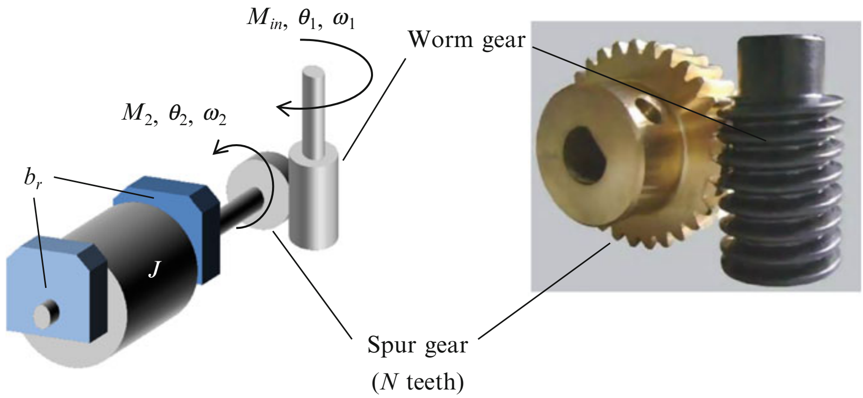
(b) Determine expressions for the reflected inertia of the drum and the reflected damping of the bearings.

(c) Determine the equation of motion in terms of ω and $\dot{\omega}$, where $\omega = \dot{\theta}$.

(d) Find the transfer function $\frac{\Omega(s)}{F(s)}$ and the time constant for the system.

(e) The values of J, R, and b_r are 2.5 kg-m^2, 0.1 m, and 5 N-m-s, respectively. Using a MATLAB® script file and the `step` command, simulate and plot the response $\omega(t)$ to an input force $F(t) = F_0 \cdot u(t)$, where F_0 is 500 N for a time interval of four time constants. Apply the final value theorem and initial value theorem to the solution in the Laplace domain and compare the results to your plot.

3. Consider the positioning system shown in the figure; it describes a rotary inertia, J, driven by a worm gear with an input moment, $M_{in}(t)$. The worm gear drives a spur gear with N teeth so that $\theta_2 = \frac{\theta_1}{N}$. The total rotational damping of the bearings is b_r and the shafts can be assumed to be rigid.



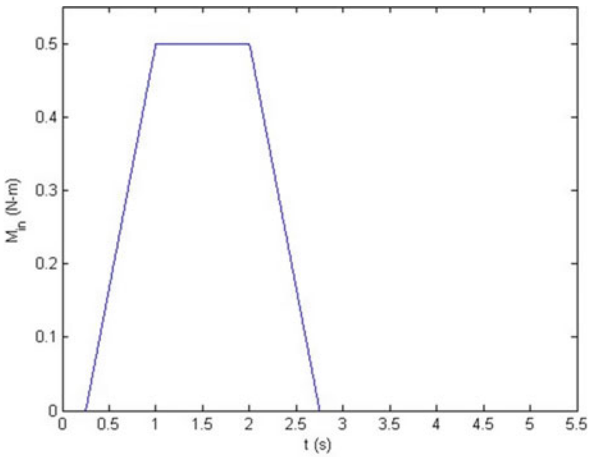
Complete the following.

(a) Determine the equation of motion in terms of ω_1 and $\dot{\omega}_1$.

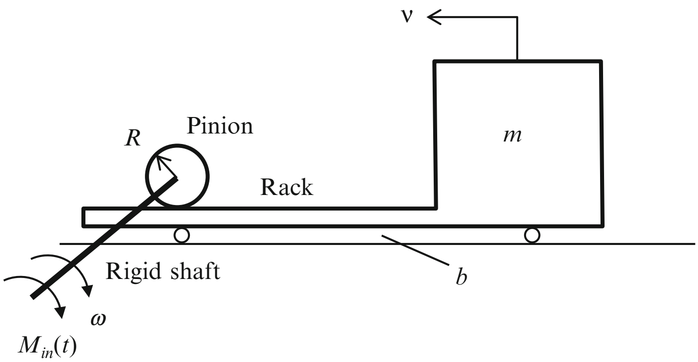
(b) Determine and find expressions for the reflected inertia of J and the reflected damping of b_r at the worm gear input.

(c) Calculate the transfer functions $\frac{\Omega_1(s)}{M_{in}(s)}$ and $\frac{\Omega_2(s)}{M_{in}(s)}$ and determine an expression for the system time constant.

(d) The values of J, N, and b_r are 1.2 kg-m^2, 100 teeth, and 5 N-m-s, respectively. Using a MATLAB® script file and the `lsim` command, simulate and plot the response $\omega_2(t)$ to the input moment provided. The initial ramp up begins at 0.25 s and ends at 1.0 s. The ramp down begins at 2.0 s and ends at 2.75 s. The total time interval is 5.5 s.



4. A motor on high-speed gantry machine tool drives a large mass with a moment, $M_{in}(t)$, applied through a rack and pinion. The pinion radius is R and the mass is m. There is linear damping, b, acting against the motion of the mass from the oil slideways (linear bearings).



Complete the following.

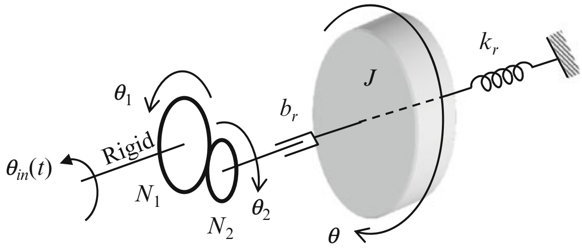
(a) Write the equations of motion in terms of, first, ω and $M_{in}(t)$ and then v and $M_{in}(t)$. Using the first equation, find an expression for the equivalent rotational inertia of the mass, m, as seen by the motor.

(b) The values of the parameters m, b, and R are 2000 kg, 400 N-s/m, and 0.2 m, respectively. The input moment is a step $M_{in}(t) = M_0 \cdot u(t)$, where M_0 is 10 N-m. Find the steady state velocity and time constant for the system.

(c) Find the velocity $v(t)$ by solving the equation of motion using Laplace transforms.

(d) Use a MATLAB® script file to plot the system response, $v(t)$, to the step input for a duration of four time constants.

5. A disk with inertia, J, is attached to ground through a flexible shaft with rotational stiffness, k_r. It is driven through a fluid coupling with resistance (rotational damping), b_r, that connects gear 2 with N_2 teeth to the inertia J. Gear 2 is driven by gear 1 with N_1 teeth. The input is a prescribed angular displacement, $\theta_{in}(t)$, of gear 1 and the output is the angle $\theta(t)$ of the disk.



Complete the following.

(a) Determine the equation of motion for the system in terms of θ and $\theta_{in}(t)$.

(b) The values of the parameters J and k_r are 0.05 kg-m^2 and 9000 N-m/rad, respectively. The gear ratio is 20:1 ($N_1 = 200$ and $N_2 = 10$). Calculate the natural frequency, ω_n, and determine the value of b_r such that the system is 40% damped.

(c) Using a MATLAB® script file and the `ilaplace` command, find $\theta(t)$ in response to a ramp input $\theta_{in}(t) = 2t$ rad and plot the response for eight time constants. Apply the initial value theorem and final value theorem to the solution in the Laplace domain. Why does $\theta(t)$ reach a nonzero steady state value?

7 Electric Circuits

7.1 Introduction

In the previous chapters, we examined various mechanical systems. In this chapter, we will study what appears to be a very different type of system—electrical circuits, which have three primary passive elements: resistors, capacitors, and inductors. These elements may seem quite different than the mechanical elements we studied previously: dampers, springs, and masses. However, we will see that there are mathematical analogies between the electrical and mechanical elements that make their behavior nearly identical. Therefore, all of the mathematical tools that we have developed for mechanical systems analysis are directly applicable to electrical systems.

For the mechanical engineer, it is important to understand electric circuit behavior and be able to use electrical elements in electro-mechanical system design. In modern engineering, very few systems are purely mechanical, but instead they have both electrical and mechanical, as well as other, elements combined. For example, we often use sensors to measure system behavior, such as strain gages, accelerometers, tachometers, and optical encoders. The sensor outputs (typically in the form of a voltage) require pre- and postprocessing. We use combinations of electrical elements to function as filters and amplifiers for the signals. Actuators, including motors, linear motors, and hydraulic or pneumatic actuators, affect system behavior by providing the input force or moment/torque. These actuators are typically controlled by electrical inputs, so knowledge of electrical elements and their behavior in systems is again required. By learning to analyze electrical systems, we will be building the framework necessary to analyze electro-mechanical systems in chapter 8.

Let's explore the analogous behavior of electrical and mechanical elements.

- A mass and a spring store energy as either kinetic or potential energy as we discussed previously. A capacitor and an inductor also store energy. Specifically,

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a capacitor stores its energy in an electric field and an inductor stores its energy in a magnetic field.
- A damper dissipates mechanical energy as heat. Similarly, a resistor dissipates electrical energy as heat.
- A mass resists sudden changes in velocity, while an inductor resists sudden changes in electric current, effectively acting as an "electrical inertia."
- When charge builds up on a capacitor it pushes back more forcefully against additional charge build up; this is similar to the spring force growth when its displacement is increased.

Clearly, there are similarities between the behavior of electrical and mechanical elements. In fact, before the invention of digital computers, many mechanical systems were "simulated" using analog computers, which are large reconfigurable circuit boards [1].

7.2 Electrical Element Input/Output Relationships

There are two input types (sources) in electric circuits: (1) voltage and (2) current sources. A voltage source is represented by a circular symbol with the voltage, $e(t)$, located inside the circle as in Fig. 7.1. Positive and negative signs indicate the voltage sign convention. A voltage source is an idealized element that provides the desired voltage independent of the circuit to which it is attached. A current source is also represented as a circular element, but with an arrow drawn in the direction of the positive current (according to the sign convention). Ground is represented as a triangle; ground is a reference voltage which can physically be realized by driving a metal stake into the ground with the ground forming an infinite sink for electrical charge. In a circuit, the ground voltage is taken as zero volts. As noted in Sect. 7.1, there are three primary passive electrical elements used in analog electric circuits:

- resistors
- capacitors
- inductors.

For mechanical elements, we determined the input/output relationships between force and position (or its time derivatives). For electrical elements, we determine the input/output relationship between voltage and current (or their time derivatives). Current, $i(t)$, is the rate of charge flow measured in Coulombs/second (C/s), or Amperes (A), and voltage, $e(t)$, is the potential energy per unit charge measured in Joules/Coulomb (J/C), or Volts (V).

A resistor impedes the flow of electrical current when a voltage is applied across it. Resistors are commonly used in electronic circuits to reduce the current or voltage inputs to other circuit elements so that they will work properly. A schematic diagram of a resistor is shown in Fig. 7.1a. The current that flows in the resistor,

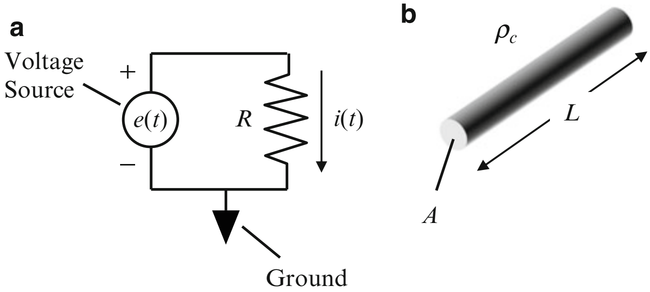


Fig. 7.1 (**a**) The symbol for a resistor with resistance, R, is a saw tooth line. When a voltage, $e(t)$, is applied across it, a current $i(t)$ flows through it. (**b**) A resistor is comprised of a conducting material such as a wire with a length, L, a cross-sectional area, A, and a charge density, ρ_c

denoted $i(t)$, is proportional to the voltage, $e(t)$, applied across it using the proportionality constant, R:

$$e(t) = i(t)R. \tag{7.1}$$

Equation (7.1) is known as *Ohm's law*. The units of resistance are V/A, or Ohms (Ω). As shown in Fig. 7.1b, a conductor with length, L, cross-sectional area, A, and charge density, ρ_c, has a *resistance* given by Eq. (7.2).

$$R = \frac{\rho_c L}{A} \tag{7.2}$$

Therefore, the longer the conducting element, the greater its resistance and the larger the cross-sectional area, the smaller its resistance. A resistor is a static element because it is assumed that a current will flow the instant a voltage is applied, so the output (current) depends only on the value of the applied voltage (input) at the current time. The Laplace transform of Eq. (7.1) gives the Laplace domain voltage–current relationship for the resistor.

$$V(s) = I(s)R \tag{7.3}$$

The ratio of voltage $V(s)$ to current $I(s)$ in the Laplace domain is known as the *impedance*, Z. Because the resistor is a static element, its impedance is the same in both the time and Laplace domains, i.e., $Z = R$.

A capacitor stores energy in an electric field. A charge is stored in the capacitor when a voltage is applied across it. A capacitor consists of two conductive plates separated by a dielectric material with electrical permittivity, ε, as shown in Fig. 7.2a. When a voltage, $e(t)$, is applied to one plate of the capacitor and the other is grounded, a charge, $\pm Q$, measured in Coulombs is stored on the two plates as electrons are removed from the positive plate and build up on the negative plate. The amount of charge stored for a given voltage is defined as the *capacitance*, C, of the capacitor.

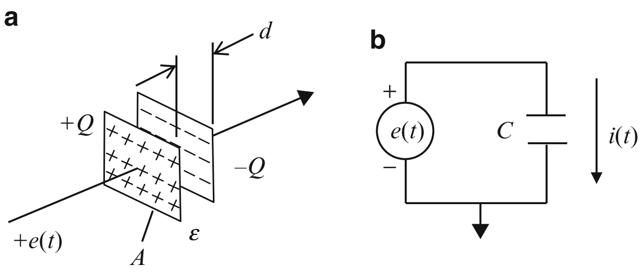


Fig. 7.2 (**a**) A capacitor consists of two conducting plates with cross-sectional area, A, separated by a distance, d, with the gap filled by a dielectric material with electrical permittivity, ε. (**b**) The capacitor symbol is a pair of parallel lines. When a voltage, $e(t)$, is applied across it, a current, $i(t)$, flows through it until a charge, $\pm Q$, is stored on the two plates

$$Q(t) = Cv(t) \tag{7.4}$$

The capacitance is related to the properties of the capacitor by Eq. (7.5). As the area of the plates is increased or the distance between the two plates is decreased, the capacitance increases.

$$C = \frac{\varepsilon A}{d} \tag{7.5}$$

Let us take the time derivative of Eq. (7.4). Because $\frac{dQ}{dt}$ is equal to current, we obtain Eq. (7.6) after substitution.

$$i(t) = C\frac{dv}{dt} \tag{7.6}$$

The units of capacitance are Ampere-seconds/Volt, or Farads; alternatively, the units may be expressed as Coulombs/Volt as can be readily seen from Eq. (7.4). Taking the Laplace transform of Eq. (7.6) gives $I(s) = C(sV(s) - v(0))$. By assuming zero initial voltage (i.e., $v(0) = 0$) and rearranging, we obtain $V(s) = I(s)\left(\frac{1}{Cs}\right)$.

The term $\frac{1}{Cs}$ is the impedance, Z, of the capacitor. Comparing this expression to Eq. (7.3), we see that $\frac{1}{Cs}$ plays the same role for the capacitor as R does for the resistor in the Laplace domain. Note that since s is the Laplace transform of a time derivative, its units are 1/seconds (or rad/s), and, therefore, the units of the capacitor's impedance are V/C/s, V/A, or Ohms (Ω). The conclusion is that impedance can be thought of as a time-varying resistance. Initially, an uncharged capacitor will charge freely with no impedance to current flow. However, as the charge builds up, the impedance increases as it becomes more and more difficult to add charge to the already charged plates (Fig. 7.2).

An inductor stores energy in a magnetic field. An inductor consists of an electrically conductive coil wound around a magnetic core. The magnetic field in the inductor depends on the current flowing in the conducting coil. By Faraday's

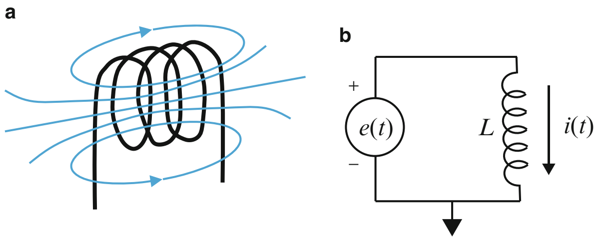


Fig. 7.3 (**a**) A coil of wire, often with a ferromagnetic core, can act as an inductor. When a current runs through the coil, a magnetic field forms around it as shown by the closed curves encircling the wires. The magnetic field expands as the current begins to flow and a back voltage proportional to the rate of change of the current is generated. (**b**) The inductor is represented as a coiled wire symbol in a circuit

law of electromagnetic induction, when the current in the inductor changes the magnetic field is modified and this varying magnetic field causes a back voltage which opposes the change. Therefore, the voltage drop across the inductor is proportional to the rate of change of current through the inductor. The inductance, L, is the constant of proportionality.

$$e(t) = L\frac{di}{dt} \tag{7.7}$$

The units of inductance are Henries or V-s/A. By calculating the Laplace transform of Eq. (7.7), we obtain Eq. (7.8).

$$V(s) = I(s)(Ls) \tag{7.8}$$

The Ls term is the inductor's impedance. Again taking into account the units of the Laplace variable, we find that the unit of impedance is Ohms (Ω). As with the capacitor, the inductor presents a time-varying resistance to current flow. When a voltage is initially applied to an inductor, it has high impedance. Current begins to flow in the coils causing a rapidly changing magnetic field and a large voltage drop, which serves to impede additional current flow. As the current reaches equilibrium, the magnetic field becomes static and the impedance to current flow is reduced to near zero.

As with the mechanical system elements, we assume lumped parameter a models for the passive circuit elements. Specifically, each element in a system representation is considered to be a pure resistance, capacitance, or inductance. A resistor is assumed to have no capacitance or inductance, an inductor is assumed to have no capacitance or resistance, and a capacitor is assumed to have no resistance or inductance. In reality, these assumptions are not true. For example, an inductor is a coil of wire and may have non-negligible electrical resistance compared to the other elements in the circuit.

7.3 Impedances in Series and Parallel

Like springs and dampers, multiple impedances can be combined to form an equivalent impedance. The combined impedance depends upon whether the individual impedances are in series or parallel with each other. Figure 7.4 shows the two possible configurations. For the series arrangement in Fig. 7.4a, the applied voltage, E_{in}, is equal to the sum of the voltage drop across the first impedance and the voltage drop across the second impedance.

$$E_{in} = IZ_1 + IZ_2 = I(Z_1 + Z_2) \tag{7.9}$$

Because the current must be the same through each impedance, the equivalent impedance is the sum of the two individual impedances.

In the parallel arrangement in Fig. 7.4b, the voltage drop across each impedance must be the same. The current through each element is now not necessarily the same.

$$E_{in} = I_1 Z_1 = I_2 Z_2 \tag{7.10}$$

Further, the total current must be the sum of the currents flowing in the individual branches (conservation of change):

$$I = I_1 + I_2. \tag{7.11}$$

Solving Eq. (7.10) for I_1 and I_2 and substituting in Eq. (7.11) yields:

$$E_{in} = \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) I = \left(\frac{Z_1 Z_2}{Z_1 + Z_2}\right) I. \tag{7.12}$$

From Eq. (7.12), we see that the equivalent impedance for a parallel arrangement is the product of the two impedances divided by their sum. Impedances can be combined to simplify circuit analysis.

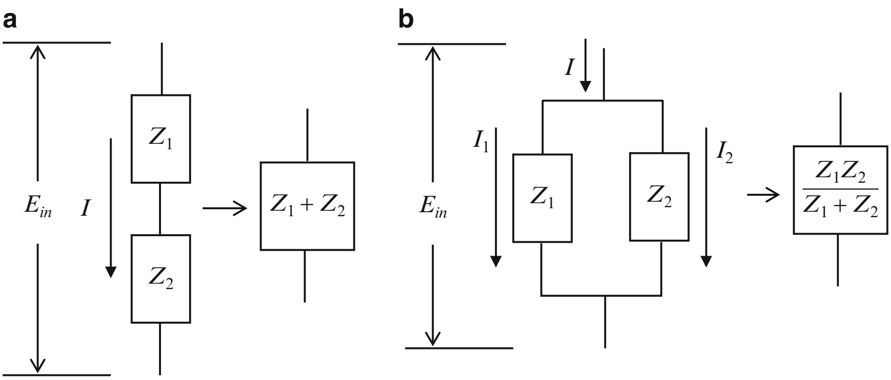


Fig. 7.4 Impedances in (**a**) series and (**b**) parallel

There is a useful physical interpretation of Eqs. (7.9) and (7.12). Two resistors in series (Eq. 7.9) both impede current flow and, therefore, if one impedance becomes very large, the entire impedance becomes large because we sum the two impedances to find the equivalent value. On the other hand, when two impedances are in parallel, if one impedance gets very large, all of the current flows through the other impedance. If Z_1 becomes very large in Eq. (7.12), we see that the total impedance approaches Z_2; all the current flows through the second impedance, Z_2.

7.4 Kirchhoff's Laws for Circuit Analysis

Kirchhoff's laws are an approximate method for analyzing the voltages and currents in electric circuits in the low-frequency limit. Circuits consist of collections of elements and *nodes* as demonstrated in Fig. 7.5. The passive elements in the circuit are the resistor, inductor, and capacitor and a circuit with these three elements is typically called an R–L–C circuit. The nodes are the black dots. There are also *loops* in the circuit as indicated by the circular arrows; there is a third loop that goes around the outside of the circuit (not shown). However, analyzing only two of the loops will generate two linearly independent equations. The two loops shown are known as *essential meshes*. An essential mesh is a loop that does not enclose any other loop.

For each node, *Kirchhoff's first law* states that the sum of all the currents flowing into a node is zero:

$$\sum_{k=1}^{n} i_k = 0, \tag{7.13}$$

where n is the number of currents. This expression says that the total charge in the circuit is conserved and the sign convention is that a current entering a node is positive and a current exiting a node is negative. Therefore, for the upper node,

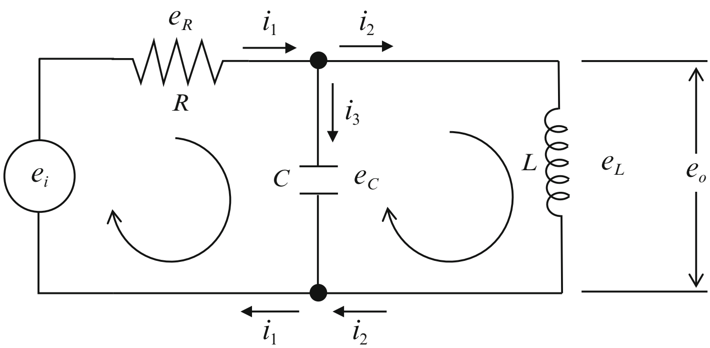


Fig. 7.5 A circuit with a resistor, a capacitor, and an inductor

we have: $i_1 - i_2 - i_3 = 0$. This can be rewritten as: $i_1 = i_2 + i_3$, which states that the current entering the node is equal to the current leaving the node; no charge can build up at the node itself.

Kirchhoff's second law states that the total voltage drop around a loop in the circuit is zero.

$$\sum_{k=1}^{n} e_k = 0 \tag{7.14}$$

It is based on the Maxwell–Faraday law of induction and is true as long as the loop considered is small and/or the magnetic flux through the loop is negligible and changing slowly. Equations (7.13) and (7.14) are used to determine circuit behavior.

7.5 Differential Equation Methods

Like other dynamic systems, a step-by-step procedure can be established for analyzing circuits. There are two methods commonly used for analyzing analog electric circuits: (1) the differential equation method and (2) the impedance method. We will first consider the differential equation method, which is required for analyzing circuits with nonzero initial conditions (i.e., nonzero current or voltage at the analysis starting time). In this method, we use Kirchhoff's laws along with the time domain relationships for the electric circuit elements. After manipulation, a differential equation that describes the behavior of the variable of interest (e.g., output voltage, current, or charge on a capacitor) is developed and then solved using Laplace transforms. This method parallels the development of the equations of motion for mechanical systems used in previous chapters. We will also study impedance methods because they are sometimes more convenient to apply.

For electrical circuit analysis, we follow steps similar to those detailed in Chap. 6 for mechanical systems.

1. Identify all currents and voltages in the circuit.
2. Declare sign conventions including positive direction for currents and recognize that a current flowing through an element will lead to a voltage drop in the direction of positive current.
3. Apply Kirchhoff's laws to the circuit.
4. Write the voltage drops in terms of the independent variables to identify the time domain equation(s) describing the circuit behavior.
5. Compute the Laplace transform of the time domain equations of motion to find the Laplace domain equation(s) of motion.
6. Solve using appropriate methods (analytical and MATLAB® techniques).

Application of these steps will now be demonstrated for the circuit shown in Fig. 7.5.

Example 7.1 Consider the circuit shown in Fig. 7.5. Suppose the input voltage is zero, but the capacitor has an initial charge $Q(0) = Q_0 = 20\ \mu\text{C}$ with a zero initial rate of charge change, $\dot{Q}(0) = \dot{Q}_0 = 0$. The resistance is 10 Ω, the capacitance is 10 μF, and the inductance is 2 mH. Find the charge on the capacitor, $Q(t)$, and plot it as a function of time using MATLAB®.

Solution We follow the circuit analysis steps that we previously identified.

Step 1: The currents and voltage drops are identified in Fig. 7.5.

Step 2: They are considered to be positive in the direction of the arrows. There is a voltage drop in the resistor from left to right, a voltage drop in the capacitor from top to bottom, and a voltage drop in the inductor from top to bottom.

Step 3: Applying Kirchhoff's node law to the upper node we find that:

$$i_1 = i_2 + i_3. \tag{7.15}$$

Applying Kirchhoff's voltage law to the left loop, we obtain:

$$0 - e_R - e_C = 0 \tag{7.16}$$

Applying Kirchhoff's voltage law to the right loop, we obtain:

$$-e_L + e_C = 0. \tag{7.17}$$

Note that the second voltage is $+e_C$ because the loop is in the direction opposite the positive current direction when we traverse the capacitor. Finally, we also note that the output voltage is the drop across the inductor: $e_o(t) = e_L$.

Step 4: Next, we write the voltage drops in terms of the system variables (currents, charges, and voltages) and the parameters. Equation (7.16) is rewritten to be:

$$-i_1 R - \frac{Q}{C} = 0. \tag{7.18}$$

Equation (7.17) becomes:

$$-L\frac{di_2}{dt} + \frac{Q}{C} = 0. \tag{7.19}$$

Finally, we note that $i_3 = \frac{dQ}{dt} = \dot{Q}$, where $\dot{Q}$ is the rate of charge build-up on the capacitor.

Our goal is to combine all of these equations to obtain a single differential equation describing the charge on the capacitor. Combining Eqs. (7.15), ($i_2 = i_1 - i_3$) and (7.19), we obtain:

$$-L\frac{d(i_1 - i_3)}{dt} + \frac{Q}{C} = -L\frac{di_1}{dt} + L\frac{di_3}{dt} + \frac{Q}{C} = 0. \tag{7.20}$$

Recognizing that $\frac{di_3}{dt} = \ddot{Q}$ and that the time derivative of Eq. (7.18) gives $\frac{di_1}{dt} = -\frac{\dot{Q}}{RC}$, Eq. (7.20) yields one differential equation in terms of the charge Q and its derivatives. After dividing each term by L we obtain:

$$\ddot{Q} + \left(\frac{1}{RC}\right)\dot{Q} + \left(\frac{1}{LC}\right)Q = 0, \tag{7.21}$$

which is similar to Eqs. (4.19) (with no force) and (5.9). We again define a natural frequency and damping ratio.

$$\begin{aligned} \omega_n &= \sqrt{\frac{1}{LC}} \\ \zeta &= \frac{1}{2\omega_n RC} \end{aligned} \tag{7.22}$$

Just as in previous systems, the natural frequency is the frequency of oscillation when there is no viscous damping. A damping ratio less than one indicates an underdamped system, whereas a damping ratio greater than one indicates an overdamped system. Since the units of L are V-s/A and the units of C are C/V, the LC term has units of s^2 and, therefore, ω_n has units of s^{-1} or rad/s. Further, since R has units of V/A and C has units of C/V, then ζ is unitless. The circuit is second order because it has two independent energy storage elements (C and L).

Step 5: Rewriting Eq. (7.21) in terms of the natural frequency and the damping ratio and taking the Laplace transform, we obtain $Q(s)$ in terms of the initial conditions.

$$Q(s) = \left(\frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)Q_0 + \left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)\dot{Q}_0 \tag{7.23}$$

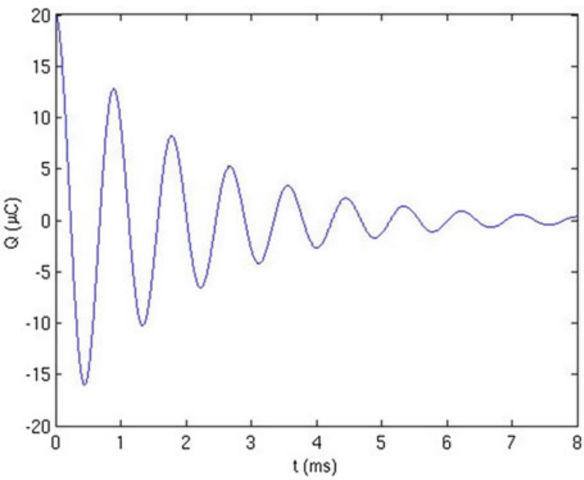
This can be inverted in MATLAB® and the results plotted.

Step 6: The MATLAB® code that solves this problem is provided. Recall that the initial rate of charge change for the capacitor was specified to be zero, $\dot{Q}_0 = 0$.

```
clear all
clc
close all
```

```
% Parameters
Q0 = 20e-6;    % Coulombs
R = 100;       % Ohms
C = 10e-6;     % Farads
L = 2e-3;      % Henries

% Natural frequency and damping ratio
wn = sqrt(1/(L*C));    % rad/s
zeta = 1/(2*wn*R*C);   % unitless

% Find the charge as a function of time
syms Q q s t;
Q = Q0*(s + 2*zeta*wn)/(s^2 + 2*zeta*wn*s + wn^2);
q = ilaplace(Q);

% Time response
tau = 1/(zeta*wn);
t = [0:tau/1000:4*tau];
qq = eval(q);
figure(1)
plot(t*1000, qq*10^6)
xlabel('t (ms)')
ylabel('Q (\muC)')
```

The time domain response is shown.



The natural frequency of this system is 1125 Hz and the damping ratio is 0.071. This gives approximately one oscillation per millisecond for the lightly damped system. In each oscillation, the charge stored in the capacitor flows into the circuit as a current. While the current is flowing, some energy is stored in the magnetic field of the inductor and some is dissipated in the resistor. The oscillations indicate that there is a reversal of charge on the capacitor plates. The decrease in oscillation amplitude with time shows that the maximum charge decreases over time as energy is lost to heat in the resistor.

Example 7.2 Consider again the circuit shown in Fig. 7.5. Suppose the input voltage is a step function: $e_i(t) = E_o \cdot u(t)$, where $E_0 = 5$ V. The resistance is 10 Ω, the capacitance is 10 μF, and the inductance is 2 mH. Find the output voltage, $e_o(t)$, and plot it as a function of time using MATLAB®.

Solution The development is similar to Example 7.1.
Steps 1 and 2 are the same. In Step 3, we again obtain Eqs. (7.15) and (7.17), but Eq. (7.16) is modified to include the input voltage.

$$e_i(t) - e_R - e_C = 0 \tag{7.24}$$

In Step 4, we rewrite Eq. (7.24) as:

$$e_i(t) - i_1 R - \frac{Q}{C} = 0. \tag{7.25}$$

Noting that $L\frac{di_2}{dt} = e_o$ and combining this with Eq. (7.19) we find that $e_o = \frac{Q}{C}$. Therefore, Eq. (7.24) becomes:

$$e_i(t) - i_1 R - e_o = 0 \tag{7.26}$$

and combining this with Eq. (7.15), we obtain:

$$e_i(t) - (i_2 + i_3)R - e_o = 0. \tag{7.27}$$

Taking the time derivative of this equation and using $\frac{di_2}{dt} = \frac{e_o}{L}$, $\frac{di_3}{dt} = \ddot{Q}$ and $\ddot{Q} = C\ddot{e}_C = C\ddot{e}_L = C\ddot{e}_o$, we obtain a differential equation for $e_o(t)$.

$$\ddot{e}_o + \frac{1}{RC}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{RC}\dot{e}_i \tag{7.28}$$

In Step 5, we calculate the Laplace transform of this equation to obtain the transfer function for the circuit (we have assumed zero initial conditions).

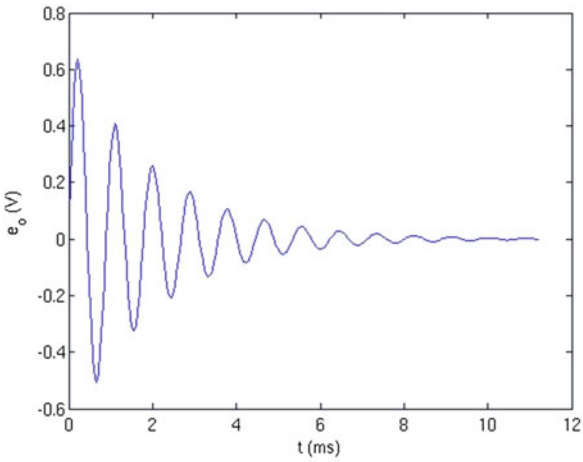
$$\frac{E_o(s)}{E_i(s)} = \frac{1}{RC}\left(\frac{s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}\right) = \frac{Ls}{RLCs^2 + Ls + R} \tag{7.29}$$

The natural frequency and damping ratio are the same as in the previous problem and are given by Eq. (7.22). Again, the system is second order because it has two independent energy storage elements (C and L).

In Step 6, we utilize the `step(sys)` command in MATLAB® to determine the step response of the system. The code is provided.

```
clear all
clc
close all

% Parameters
R = 100;        % Ohms
C = 10e-6;      % F
L = 2e-3;       % H
Ei = 5;         % V

% Natural frequency and damping ratio
wn = sqrt(1/(L*C));     % rad/s
zeta = 1/(2*wn*R*C);    % unitless

% Find the voltage as a function time
num = [L 0];
den = [R*L*C L R];
sys = tf(num,den);
[eo_u,t] = step(sys);
eo = eo_u*Ei;

figure(1)
plot(t*1000, eo)
set(gca,'FontSize', 14)
xlabel('t (ms)')
ylabel('e_o (V)')
```

The results are provided.



Again the frequency of oscillation is 1125 Hz and the damping ratio is 0.071. Substituting the Laplace transform of the voltage input step into Eq. (7.29), we obtain the Laplace domain expression for the output voltage:

$$E_o(s) = \frac{1}{RC}\left(\frac{1}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}\right)E_0. \tag{7.30}$$

Applying the initial value theorem and the final value theorem to Eq. (7.30), we find that both the initial and final values of the output voltage are zero, which agrees with our plot. If we examine Eq. (7.28), we note that the derivative of the input voltage drives the circuit. The time derivative of a step input is an impulse. This is consistent with an initial impulsive excitation of the output voltage oscillations in the circuit followed by a settling back to zero volts.

7.6 Impedance Method

The impedance method for analyzing electric circuits enables the circuit transfer functions to be calculated using only algebra. In the impedance method, we use the Laplace transform of the circuit element behavior (i.e., the element's impedance), which allows us to treat each element using Ohm's law. Therefore, the circuit can be analyzed by algebraic manipulation of Kirchhoff's laws. The impedance method assumes that the circuit's initial conditions are zero.

Before moving on, we introduce *mesh analysis*, a simplified approach for planar circuits that eliminates the need to directly apply Kirchhoff's current law. We demonstrate the method using the circuit we already analyzed; Fig. 7.6 shows the circuit from Fig. 7.5. In mesh analysis, we first identify the *essential meshes* in the circuit. There are two, labeled 1 and 2 in the figure. These are assigned mesh currents I_1 and I_2, where the capital represents the Laplace transform of i_1 and i_2.

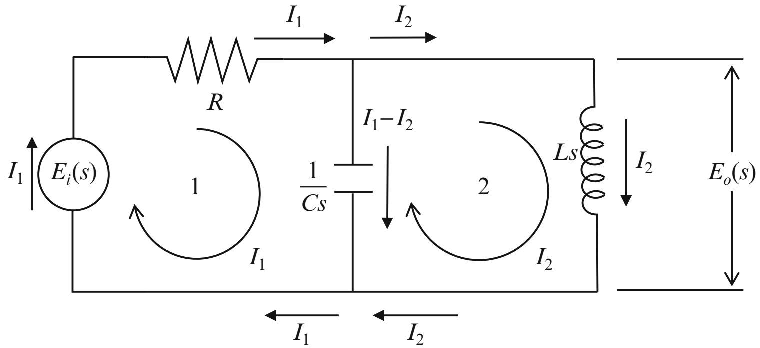


Fig. 7.6 Impedance approach to the circuit in Fig. 7.5 using mesh analysis

Using the mesh currents, we see that the currents in the individual branches of the circuit are identified as shown (I_1, I_2, and $I_1 - I_2$). With this definition, the current laws at the nodes are expressed using two independent variables instead of three, which simplifies the equations.

The variables in the circuit have all been converted to the Laplace domain in Fig. 7.6. Consequently, the impedances of the inductor and capacitor are shown and are treated as if they were simple resistors in a static circuit. The dynamics are captured by the Laplace transform. Because we are using the impedance method, we can no longer directly account for nonzero initial conditions, but we can quickly identify the transfer function for the circuit. Mesh analysis applies the following modified steps.

1. Identify the circuit element impedances (Laplace domain representation) and combine any impedances (in parallel or series) to form an equivalent circuit.
2. Identify all mesh currents and voltages in the circuit.
3. Declare sign conventions including positive direction for currents.
4. Apply Kirchhoff's voltage law to each essential mesh.
5. Manipulate the equations algebraically to determine the circuit transfer function.
6. Solve using appropriate methods (analytical and MATLAB® techniques).

For the circuit in Fig. 7.6, we will skip Step 1 for now in order to illustrate the mesh approach. Steps 2 and 3 have already been completed. Next, in Step 4, we apply the Kirchhoff's voltage law to each essential mesh. For mesh 1, we have:

$$E_i(s) - I_1 R - (I_1 - I_2)\frac{1}{Cs} = 0. \tag{7.31}$$

For mesh 2, Kirchhoff's voltage law gives:

$$-I_2 Ls + (I_1 - I_2)\frac{1}{Cs} = 0, \tag{7.32}$$

where we notice the voltage drop across the capacitor in Eq. (7.31) is equal and opposite to the voltage drop across the capacitor in Eq. (7.32). A third equation is obtained by determining the voltage drop across the inductor, which relates the output voltage and current I_2.

$$E_o(s) = I_2 Ls \tag{7.33}$$

In Step 5, we algebraically manipulate Eqs. (7.31) through (7.33) to obtain the circuit transfer function, $\frac{E_o(s)}{E_i(s)}$. There are numerous approaches to completing the algebra. One method is to recognize that we can add Eqs. (7.31) and (7.32) to obtain:

$$E_i(s) - I_1 R - I_2 Ls = 0. \tag{7.34}$$

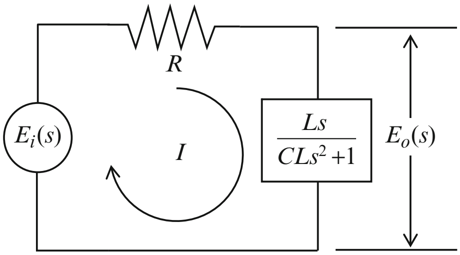


Fig. 7.7 Combine the impedances for the circuit in Fig. 7.6 to simplify the mesh analysis

Next, we recognize from Eq. (7.33) that $I_2 = \frac{E_o(s)}{Ls}$ and we combine this with Eq. (7.32) to obtain an expression for I_1 in terms of the output voltage.

$$I_1 = \left(\frac{CLs^2 + 1}{Ls}\right) E_o(s) \tag{7.35}$$

Next we insert our expressions for I_1 and I_2 into Eq. (7.34) and solve for the transfer function.

$$\frac{E_o(s)}{E_i(s)} = \frac{Ls}{RCLs^2 + Ls + R} \tag{7.36}$$

This is the same result we obtained using the differential equation method; see Eq. (7.29).

A third method for approaching this problem is to recognize (in Step 1) that the impedances of the capacitor and the inductor can be combined before the circuit is analyzed. The circuit in Fig. 7.6 can then be replaced by the equivalent circuit displayed in Fig. 7.7. Applying Kirchhoff's voltage law to this circuit gives:

$$E_i(s) - IR - E_o(s) = 0. \tag{7.37}$$

The current flowing in the circuit is found by first finding the total circuit impedance, Z_{TOT}.

$$Z_{TOT} = R + \frac{Ls}{CLs^2 + 1} = \frac{RCLs^2 + Ls + R}{CLs^2 + 1} \tag{7.38}$$

The current is the input voltage divided by the total impedance.

$$I = \frac{E_i(s)}{Z_{TOT}} = E_i(s)\left(\frac{CLs^2 + 1}{RCLs^2 + Ls + R}\right) \tag{7.39}$$

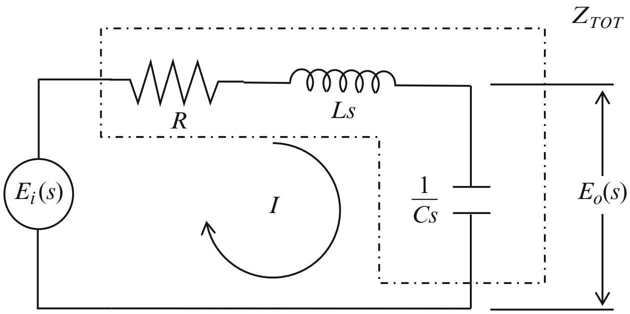


Fig. 7.8 Example 7.3 R–L–C circuit

This current multiplied by the appropriate impedance is the output voltage drop.

$$E_o(s) = I\left(\frac{Ls}{CLs^2 + 1}\right) = \left(\frac{Ls}{RCLs^2 + Ls + R}\right)E_i(s) \tag{7.40}$$

Note that the transfer function is the same as shown in Eq. (7.36).

Example 7.3 Consider the R–L–C circuit configuration shown in Fig. 7.8. Find the transfer function of this circuit, $\frac{E_o(s)}{E_i(s)}$. Next, determine the natural frequency, damping ratio, and damped natural frequency of the circuit if the capacitance is 1 μF, the inductance is 120 mH, and the resistance is 100 Ω. Finally, use MATLAB® to find and plot $e_o(t)$ for the case where the input is a 2 V step; interpret the results physically.

Solution

Step 1: The total impedance of the three series elements is given by the sum of the three elements:

$$Z_{TOT} = R + Ls + \frac{1}{Cs} = \frac{LCs^2 + RCs + 1}{Cs}. \tag{7.41}$$

Step 2: In the Fig. 7.8 circuit, there is one current, an input voltage, and an output voltage. There are also voltage drops across each circuit element. To find these voltage drops, we use the respective impedances and Ohm's law.

Step 3: The current is considered positive when it flows in the clockwise direction and leads to voltage drops in that direction across each element.

Steps 4 and 5: We can apply Kirchhoff's law to this circuit or we can recognize that the current flowing in the circuit is the applied voltage divided by the total circuit impedance.

$$I = \left(\frac{Cs}{LCs^2 + RCs + 1}\right)E_i(s) \tag{7.42}$$

The output voltage is the current multiplied by the capacitor impedance. The numerator term, Cs, cancels and we find the transfer function, $\dfrac{E_o(s)}{E_i(s)} = \dfrac{1}{LCs^2 + RCs + 1}$.

Step 6: If we divide each term in the denominator of the transfer function by LC, we can immediately identify the natural frequency and damping ratio for the circuit, $s^2 + \frac{R}{L}s + \frac{1}{LC} = s^2 + 2\zeta\omega_n s + \omega_n^2$.

$$\begin{aligned} \omega_n &= \sqrt{\frac{1}{LC}} \\ \zeta &= \frac{R}{2\omega_n L} \end{aligned} \tag{7.43}$$

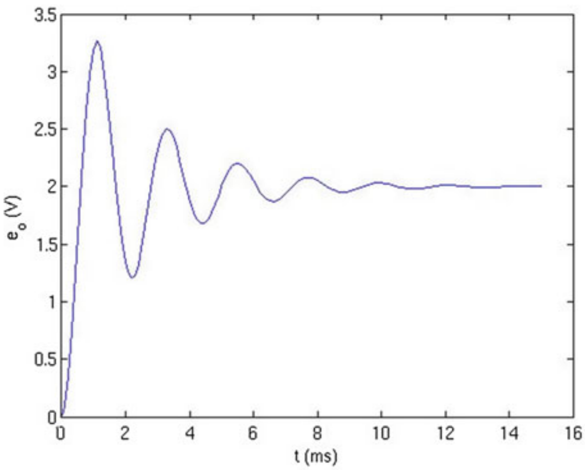
The system is second order because it has two energy storage elements (C and L). Note that the natural frequency is the same as it was for the R-L-C circuit in Fig. 7.6 the time scale over which energy is traded between the capacitor and inductor determines this frequency. However, notice that the damping ratio in Example 7.2 (Eq. 7.22) decreases with larger R, while the damping ratio for this example increases with larger R. Because the resistor is a dissipative element, it is intuitive that the damping ratio (i.e., the rate of energy loss) should increase with R. For Example 7.2 the inductor and capacitor are in parallel and there is a path through which they can trade energy without it passing through the resistor. Therefore, if the resistor is large, any energy in the inductor and capacitor will oscillate back and forth between them with minimal current flow through the resistor. In this case, the oscillations will continue indefinitely (i.e., low damping for high R). For the circuit in Fig. 7.8, energy flowing between the capacitor and inductor must also flow through the resistor. Increasing the resistor's value will therefore yield greater damping. For the values provided in this problem, the natural frequency is 2889 rad/s or 459 Hz. The damping ratio is 0.144, or approximately 14% damped, and the damped natural frequency is 2857 rad/s or 455 Hz. The damped natural frequency is close to the natural frequency because the damping ratio is small.

The MATLAB® code to solve the problem and plot the result is provided.

```
clear all
clc
close all

% Parameters
R = 100;        % Ohms
C = 1e-6;       % F
L = 120e-3;     % H
EI = 2;         % V

% Natural frequency and damping ratio
wn = sqrt(1/(L*C));        % rad/s
zeta = R/(2*wn*L);         % unitless
```

```
wd = wn*sqrt(1-zeta^2); %  rad/s

% Find the voltage as a function time
num = [1];
den = [L*C R*C 1];
sys = tf(num, den);
[eo_u, t] = step(sys);
eo = eo_u*EI;

figure(1)
plot(t*1000, eo)
xlabel('t (ms)')
ylabel('e_o (V)')
```

The results are given.



Approximately five oscillations are observed in the first 10 ms corresponding to the damped natural frequency of nearly 500 Hz (500 oscillations per second, or 5 per 10 ms). The starting voltage is zero because the capacitor is initially uncharged and therefore presents no impedance to current flow and has no voltage drop across it. The final voltage is 2 V, which is equal to the step input size. This is expected since the capacitor presents an infinite impedance when it is charged and, therefore, all of the voltage drop is across the capacitor. These conclusions can be verified by applying the initial and final value theorems to the system transfer function with a 2 V step input.

Example 7.4 Consider the circuit shown in Fig. 7.9. Find the circuit transfer function and associated time constant. If the capacitance is 5 mF and the three resistors are 100 Ω, plot the response to a 5 V step input and explain the results with physical arguments.

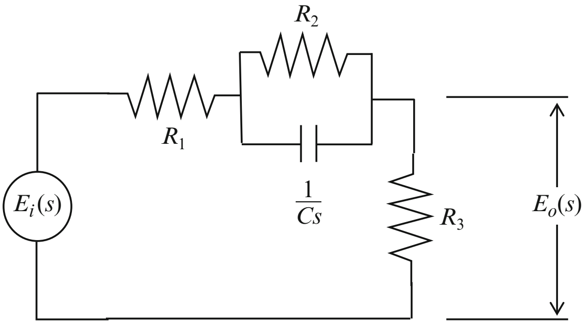


Fig. 7.9 A dynamic voltage divider circuit

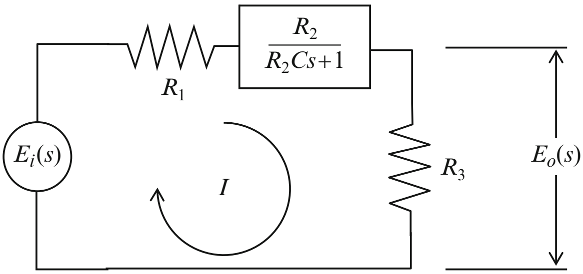


Fig. 7.10 The circuit form Fig. 7.9 is replaced by an equivalent circuit after combining the two parallel elements

Solution

Step 1: We combine the parallel resistor, R_2, and capacitor to form the equivalent circuit shown in Fig. 7.10.

Step 2: There is a single essential mesh.

Step 3: The current is assumed to be positive in the clockwise direction and voltage drops across the elements are assumed to occur in the direction of positive current flow.

Step 4: Apply Kirchhoff's voltage law to the circuit.

$$E_i(s) - IR_1 - I\left(\frac{R_2}{R_2Cs + 1}\right) - IR_3 = 0 \tag{7.44}$$

Equation (7.44) can be rearranged to find the current flowing in the loop, I.

$$I = \frac{E_i(s)}{R_1 + \left(\frac{R_2}{R_2Cs+1}\right) + R_3} = E_i(s)\left(\frac{R_2Cs + 1}{R_2(R_1 + R_3)Cs + R_1 + R_2 + R_3}\right) \tag{7.45}$$

Again, the loop current is the applied voltage divided by the total circuit impedance.

Step 5: The output voltage is the current multiplied by the resistance, R_3. We can now determine the transfer function:

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2R_3Cs + R_3}{R_2(R_1 + R_3)Cs + R_1 + R_2 + R_3}. \tag{7.46}$$

Step 6: This is a first-order system because it only has one independent energy storage element (C). The time constant can be found by comparing Eq. (7.46) to the form $\frac{1}{s+a}$; it is given by $\tau = \frac{1}{a} = \frac{R_2(R_1+R_3)C}{R_1+R_2+R_3}$. The first-order system has an exponential (nonoscillatory) response. Using the supplied values, the time constant is 0.33 ms.

The MATLAB® code used to solve the problem and plot the result is provided.

```
clear all
clc
close all

% Parameters
R1 = 100;      %  Ohms
R2 = 100;      %  Ohms
R3 = 100;      %  Ohms
C = 5e-6;      %  F
EI = 5;        %  V

% Time constant
tau = R2*(R1 + R3)*C/(R1 + R2 + R3);

% Find the voltage as a function time
num = [R2*R3*C R3];
den = [R1*(R2+R3)*C R1+R2+R3];
sys = tf(num, den);
[eo_u, t] = step(sys);
eo = eo_u*EI;

figure(1)
plot(t*1000,eo)
set(gca,'FontSize', 14);
xlabel('t (ms)')
ylabel('e_o (V)')
axis([0 2 0 3])
```

The corresponding plot is shown.

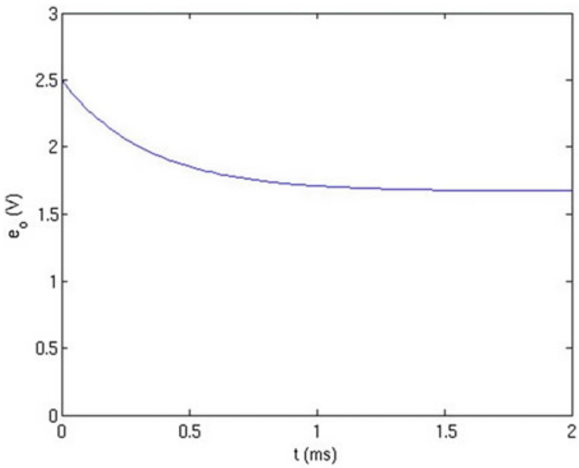
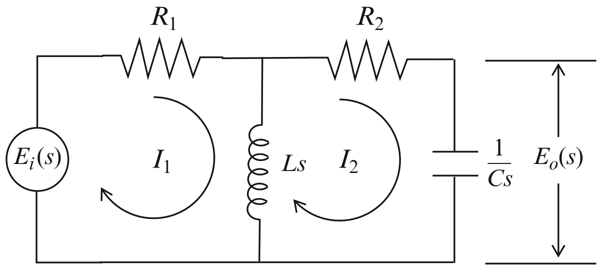


Fig. 7.11 R–L–C circuit with two resistors



Initially, the capacitor is uncharged and provides no impedance to current flow. The voltage drop is divided equally between R_1 and R_3. Therefore, the output voltage is initially 2.5 V, half the value of the input step. When the capacitor becomes charged after approximately four time constants (1.33 ms), the capacitor has infinite impedance and the input voltage drop is divided between the three equal resistors. At this time, the output voltage is one-third of 5 or 1.66 V. This is verified by applying the initial value theorem and the final value theorem to Eq. (7.47) with a 5 V step input. This type of circuit is sometimes called a voltage divider because it "divides" the input voltage. In this case, the division is time-dependent.

Example 7.5 Determine the transfer function for the electric circuit shown in Fig. 7.11. If $L = 0.75$ mH, $C = 0.5$ μF, $R_1 = 100\ \Omega$, and $R_2 = 100\ \Omega$, find the natural frequency and the damping ratio for the system. Then use MATLAB® to plot the system response to a 2 V step input.

Solution

Step 1: There are four impedances in this circuit. While it is possible to combine them together, we recognize that we must maintain the ability to calculate I_2 in order to find the output voltage drop across the capacitor.

Step 2: There are two essential meshes with the current loops shown in Fig. 7.11.
Step 3: Assume the current to be positive in the clockwise direction.
Step 4: Write Kirchhoff's voltage law for both circuit loops. For loop 1, we obtain:

$$E_i(s) - I_1 R_1 - (I_1 - I_2)Ls = 0. \tag{7.47}$$

For loop 2, the equation is:

$$(I_1 - I_2)Ls - I_2 R_2 - E_o(s) = 0. \tag{7.48}$$

For the capacitor we also know that $E_o(s) = I_2 \frac{1}{Cs}$, which gives an expression for I_2.

$$I_2 = CsE_o(s) \tag{7.49}$$

Step 5: We begin by combining Eqs. (7.47) and (7.48) to obtain an expression for I_1.

$$I_1 = \frac{CLs^2 + CR_2 s + 1}{Ls} E_o(s) \tag{7.50}$$

Now we substitute I_1 from Eq. (7.50) and I_2 from Eq. (7.49) into Eq. (7.47) to obtain the circuit transfer function.

$$\frac{E_o(s)}{E_i(s)} = \frac{Ls}{CL(R_1 + R_2)s^2 + (CR_1 R_2 + L)s + R_1} \tag{7.51}$$

Substantial algebraic manipulation is required to obtain Eq. (7.51). A common approach is to expand, group terms for $E_o(s)$ and $E_i(s)$, and then rearrange to find the transfer function. Notice that this is a second-order system and that all the denominator terms are positive. As with mechanical systems, the positive terms in the denominator are necessary to ensure system stability.

Step 6: By rewriting the denominator to isolate the s^2 term, we directly identify the natural frequency and damping ratio.

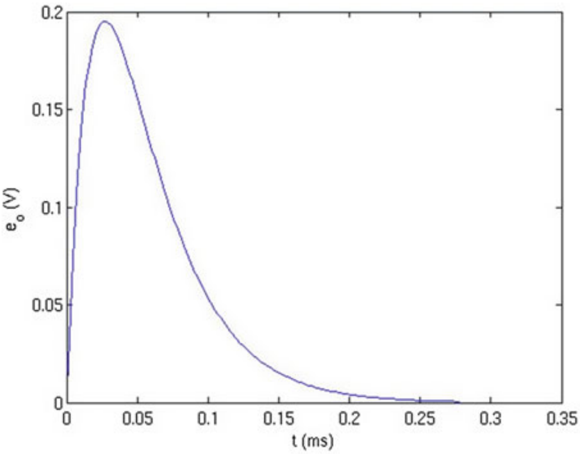
$$\begin{aligned} \omega_n &= \sqrt{\frac{R_1}{CL(R_1 + R_2)}} \\ \xi &= \frac{CR_1 R_2 + L}{2\omega_n CL(R_1 + R_2)} \end{aligned} \tag{7.52}$$

Notice that no matter what form the second-order transfer function takes, or how complex it may be, we can immediately identify the natural frequency and damping ratio by comparing the denominator to the form $s^2 + 2\zeta\omega_n s + \omega_n{}^2$. For the given parameter values, the natural frequency is 35515 rad/s or 5812 Hz. The damping ratio is 1.05 so the system is overdamped and will not oscillate.

The MATLAB® code used to solve the problem and plot the result is provided.

```
clear all
clc
close all

% Parameters
L = 0.75e-3;
C = 0.5e-6;
R1 = 100;
R2 = 100;
EI = 2;

% Natural frequency and damping ratio
wn = sqrt(R1/(C*L*(R1 + R2)));
zeta=(C*R1*R2 + L)/(2*wn*C*L*(R1 + R2));

% System step response
num = [L 0];
den = [C*L*(R1 + R2) C*R1*R2 + L R1];
sys = tf(num, den);
[eo_u, t] = step(sys);
eo = eo_u*EI;

figure(1)
plot(t*1000, eo)
xlabel('t (ms)')
ylabel('e_o (V)')
```

The step response is shown.



The initial output voltage drop is zero because the capacitor is uncharged and presents zero impedance. The final voltage drop is zero because the inductor

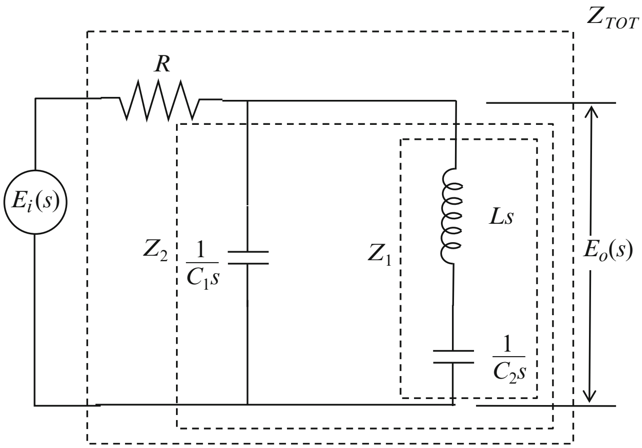


Fig. 7.12 Third-order R–L–C circuit

reaches zero impedance and, therefore, all of the current bypasses the loop containing the output voltage. Again this behavior can be verified with the initial and final value theorems.

Example 7.6 Consider the circuit shown in Fig. 7.12. This is a third-order circuit because it has three energy storage elements: two capacitors and an inductor. Find the transfer function of the circuit using the impedance method and then plot the response to a 5 V step input if L is 10 mH, C_1 is 5 μF, C_2 is 10 μF, and R is 150 Ω. Qualitatively explain the results.

Solution

Step 1: Three impedances in this circuit may be combined. First, combining the series combination of the inductor, L, and capacitor, C_2, gives Z_1.

$$Z_1 = Ls + \frac{1}{C_2 s} = \frac{LC_2 s^2 + 1}{C_2 s} \tag{7.53}$$

Next, combining the parallel combination of Z_1 and C_1 gives Z_2.

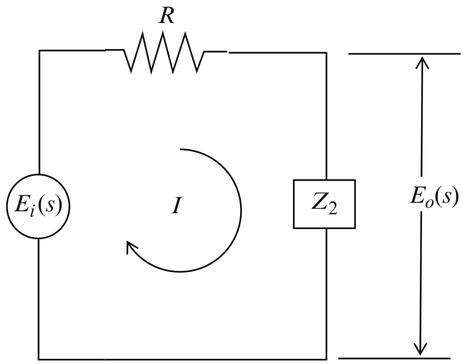
$$Z_2 = \frac{Z_1 \frac{1}{C_1 s}}{Z_1 + \frac{1}{C_1 s}} = \frac{LC_2 s^2 + 1}{LC_1 C_2 s^3 + (C_1 + C_2)s} \tag{7.54}$$

The equivalent circuit is shown in Fig. 7.13.

Step 2: There is now a single essential mesh in the circuit, which greatly simplifies the analysis.

Step 3: Assume the positive current direction to be clockwise.

Fig. 7.13 Simplified third-order R–L–C circuit



Step 4: Apply Kirchhoff's voltage law to the single circuit loop. We obtain:

$$E_i(s) - IR - IZ_2 = 0, \tag{7.55}$$

where we notice that $R + Z_2$ is the total circuit impedance, Z_{TOT}. Rearranging Eq. (7.55), we find that the current is the input voltage divided by the total impedance of the circuit as expected.

$$I = \frac{E_i(s)}{R + Z_2} = \frac{LC_1C_2s^3 + (C_1 + C_2)s}{RLC_1C_2s^3 + LC_2s^2 + R(C_1 + C_2)s + 1}E_i(s) \tag{7.56}$$

Step 5: Finally, the output voltage drop is:

$$E_o(s) = IZ_2 = \frac{LC_2s^2 + 1}{RLC_1C_2s^3 + LC_2s^2 + R(C_1 + C_2)s + 1}E_i(s) \tag{7.57}$$

and the transfer function is:

$$\frac{E_o(s)}{E_i(s)} = \frac{LC_2s^2 + 1}{RLC_1C_2s^3 + LC_2s^2 + R(C_1 + C_2)s + 1}. \tag{7.58}$$

Step 6: The transfer function is third order as we expected. When working complex transfer functions, it is important to ensure that they make physical sense. Since this is a ratio of voltages, the transfer function should be unitless. The "1" in the numerator and denominator is unitless, so the other terms must be unitless as well. For example, the leading term in the denominator can be rewritten to be:

$$RLC_2C_1s^3 = \frac{R \cdot Ls}{\left(\frac{1}{C_1s}\right) \cdot \left(\frac{1}{C_2s}\right)}. \tag{7.59}$$

This is recognized as Ω^2 over Ω^2 and is, as required, unitless. Assuming a step input $E_i(s) = \frac{E_I}{s}$, the output voltage is:

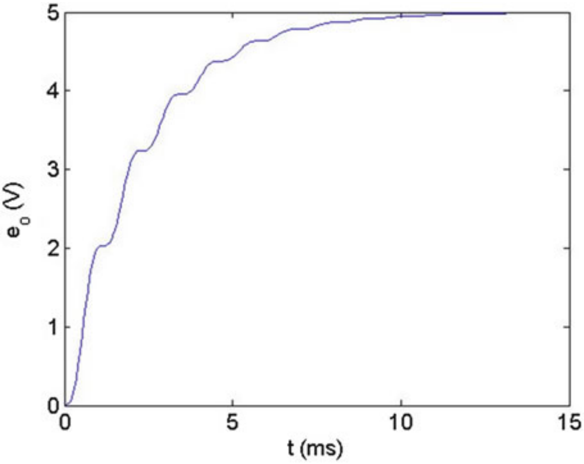
$$E_o(s) = \left(\frac{LC_2 s^2 + 1}{RLC_1C_2s^3 + LC_2s^2 + R(C_1 + C_2)s + 1} \right) \frac{E_I}{s}. \tag{7.60}$$

Applying the initial value theorem, we find an initial voltage drop of zero. This is to be expected because the uncharged capacitor, C_1, exhibits zero impedance (short circuit). The final value theorem shows that the final voltage drop at the output will be E_I; this is also to be expected because both capacitors will have infinite impedance (broken circuit). The dynamic behavior of the voltage as it makes this transition (oscillatory to nonoscillatory) depends on the roots of the transfer function denominator.

To simulate the behavior for the two cases, the following MATLAB® code is used.

```
clear all
clc
close all

% Parameters
L = 10e-3;
C1 = 5e-6;
C2 = 10e-6;
R = 150;
EI = 5;

% System step response
num = [L*C2 1];
den = [R*L*C1*C2 L*C2 R*(C1+C2) 1];
sys = tf(num, den);
[eo_u, t] = step(sys);
eo = eo_u*EI;

figure(1)
plot(t*1000, eo)
xlabel('t (ms)')
ylabel('e_o (V)')
```

The resulting step response is shown.



As with other third-order systems, we notice an oscillatory behavior superimposed on an exponential response. Let us examine the roots of the denominator using MATLAB®.

```
>> roots(den)

ans =

  1.0e+03 *
 -0.4415 + 5.4229i
 -0.4415 - 5.4229i
 -0.4504 + 0.0000i
```

The two complex conjugate roots yield an exponential response (i.e., the real part of the root) of $e^{-441{,}5t}$ and an oscillatory part (i.e., the imaginary part of root) having a frequency or 5423 rad/s or 863 Hz. We expect to see oscillations with a period of 1.2 ms (1/863) that decay with increasing time. This is verified by examining the “waves riding on the exponential” in the plot. The real root gives an exponential with a time constant of 2.2 ms, which explains the overall shape of the response.

7.7 Operational Amplifiers

An operational amplifier (op-amp) is an active circuit element (i.e., it has an external power input) commonly used by mechanical engineers to condition signals from sensors, drive other circuit elements, and/or power devices such as motors or actuators. Signal conditioning typically refers to amplification and filtering of small voltages so that they may be measured/recorded or used as feedback signals.

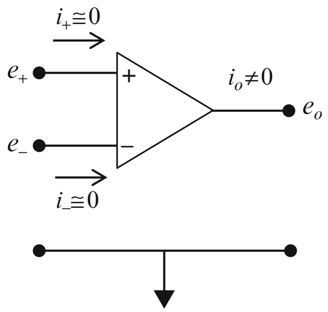


Fig. 7.14 Schematic representation of an operational amplifier

The symbol for an op-amp is shown in Fig. 7.14. An op-amp has a high impedance input so that little current flows into the op-amp at the + and – terminals. It has a low impedance output and a substantial current may flow at the output terminal. An op-amp amplifies the difference between e_+ and e_- to give an output voltage.

$$e_o = K(e_+ - e_-) \tag{7.61}$$

Here the constant, K, is very large (on the order 10^5 to 10^6), but it has a large uncertainty. Consequently, op-amps are used in circuits with configurations that do not use K as a critical parameter. Rearranging Eq. (7.61), we see that another way to view the op-amp behavior is that it forces the voltage difference at the two input terminals to be nearly equal:

$$e_+ - e_- = \frac{e_o}{K} \cong 0, \tag{7.62}$$

which implies that:

$$e_+ = e_-. \tag{7.63}$$

In order to enforce Eq. (7.63), the op-amp uses its output current. The circuit is arranged so that the output current can affect the input voltages; it has a built-in feedback loop. We will analyze two common op-amp circuits using the impedance approach: (1) inverting op-amp and (2) the noninverting op-amp.

7.7.1 Inverting Op-Amp

An inverting op-amp is shown in Fig. 7.15. Let's calculate the transfer function for the circuit, $\frac{E_o(s)}{E_i(s)}$. The steps for analyzing the op-amp circuit summarized here.

1. Convert the circuit elements to impedances and combine those arranged in series and/or parallel configurations.
2. Recognize that the high input impedance breaks the circuit into separate circuits.
3. Compute E_+ in terms of the input and output voltages.
4. Compute E_- in terms of the input and output voltages.

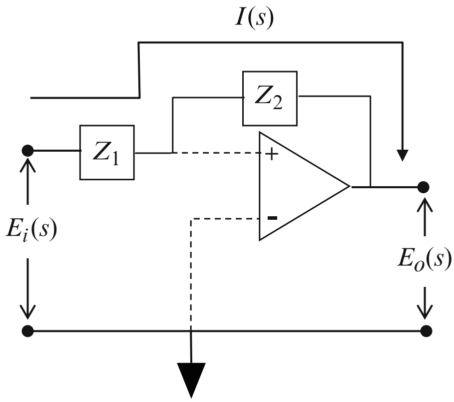


Fig. 7.15 Inverting op-amp configuration

5. Set $E_+ = E_-$ and solve for the transfer function.
6. Use analytical methods or MATLAB® to determine the circuit behavior.

For the circuit in Fig. 7.15, we first assume that any circuit elements in the $Z_1(s)$ and $Z_2(s)$ locations have already been combined. We will analyze different impedance types in the examples that follow. The dotted lines indicate that there is no current input to the op-amp. Because of this, a current, $I(s)$, flows in the upper branch of the circuit:

$$I(s) = \frac{E_i(s) - E_o(s)}{Z_1 + Z_2}. \tag{7.64}$$

The current is the voltage difference between the two points divided by the total impedance between them (Ohm's law). Using this current, we can determine the voltage at the positive input to the op-amp.

$$E_+ = E_i(s) - I(s)Z_1 \tag{7.65}$$

The negative terminal is grounded and so has a zero voltage.

$$E_- = 0 \tag{7.66}$$

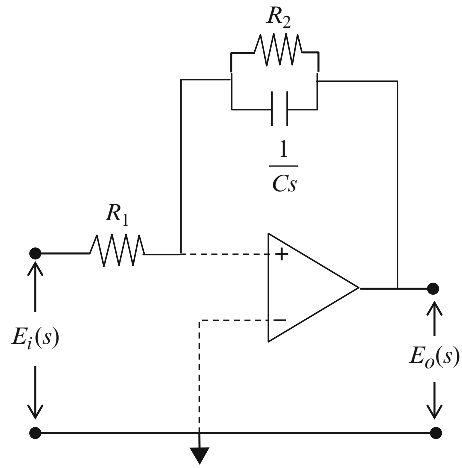
Setting Eq. (7.65) equal to Eq. (7.66), replacing the current using Eq. (7.64), and multiplying through by $Z_1 + Z_2$, we obtain: $E_i(s)(Z_1 + Z_2) - (E_i(s) - E_o(s))Z_1 = 0$. This is rewritten to identify the transfer function.

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2}{Z_1} \tag{7.67}$$

This is the "standard" inverting op-amp equation.

Example 7.7 Suppose an inverting op-amp has simple resistors as the impedances: $Z_1 = R_1$ and $Z_2 = R_2$. Find the response to a step input of 5 mV given $R_2 = 10\,\mathrm{k\Omega}$ and $R_1 = 1\,\mathrm{k\Omega}$. Also, find the output voltage.

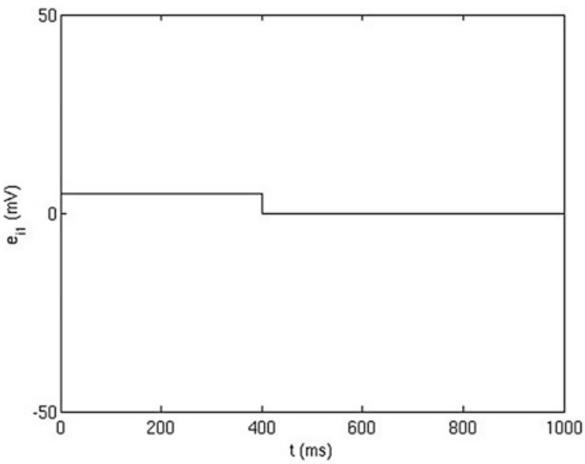
Fig. 7.16 Low pass filter/ amplifier circuit



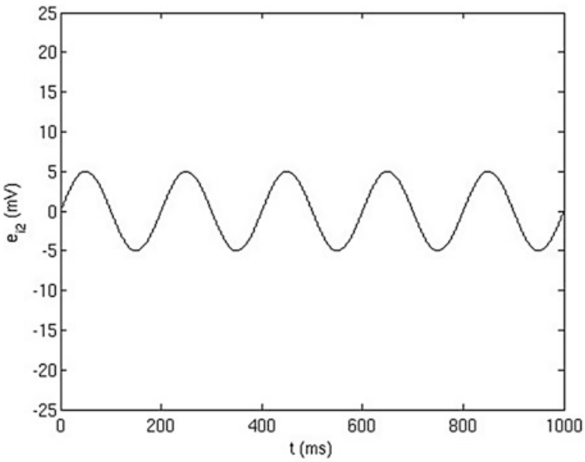
Solution Because we have already developed Eq. (7.67), we may use it to find: $\frac{E_o(s)}{E_i(s)} = -\frac{R_2}{R_1} = -10$. In this case, the transfer function is a constant. This is called the *gain* of the circuit. Therefore, a 5 mV step will be *amplified* by the gain and its sign will be changed (inverted) to give an output of −50 mV.

Example 7.8 Find the transfer function for the op-amp configuration shown in Fig. 7.16. Given that $R_2 = 10\,\text{k}\Omega$, $R_1 = 1\,\text{k}\Omega$, and $C = 10\ \mu\text{F}$, find the time constant for the circuit. Next, use the `lsim` command in MATLAB® to calculate the response to the following input voltages: (a) a 5 mV step with a duration of 4τ; (b) a 5 mV sine wave with a period of 2τ; and (c) a 5 mV sine wave with a period of 0.25τ. Explain the output response using the time constant. The input voltage profiles are shown.

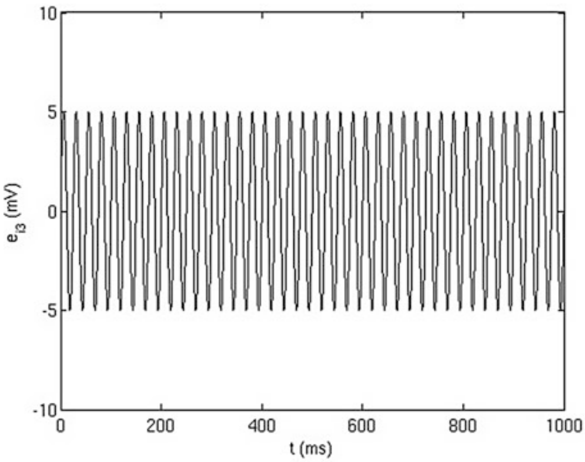
(a) 5 mV step with duration of 4τ



(b) 5 mV sine wave with a period of 2τ



(c) 5 mV sine wave with a period of 0.25τ



Solution The combined impedances are $Z_1 = R_1$ and $Z_2 = \frac{R_2}{R_2Cs+1}$, so the transfer function is:

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2}{R_1R_2Cs + R_1}. \tag{7.68}$$

By rearranging the denominator into the first order $s + a$ form, we find that $a = \frac{1}{R_2C}$. The circuit time constant is therefore $\tau = \frac{1}{a} = R_2C$ or 100 ms. The MATLAB® code used to determine the system response is listed.

```
clear all
clc
close all

% Parameters
R1 = 1000;
R2 = 10000;
C = 10e-6;
EI = 0.005;
tau = R2*C;

% Time vector
t = [0:tau/1000:10*tau];

% Input (a)
ei1 = zeros(1, length(t));
index = find(t < 4*tau);
ei1(index) = EI;
figure(1)
plot(t*1000, 1000*ei1, 'k-')
xlabel('t (ms)')
ylabel('e_{i1} (mV)')
axis([0 1000 -50 50])
hold on

% Input (b)
T1 = 2*tau;
f1 = 1/T1;
w1 = 2*pi*f1;
ei2 = EI*sin(t*w1);
figure(2)
plot(t*1000, 1000*ei2, 'k-')
xlabel('t (ms)')
ylabel('e_{i2} (mV)')
axis([0 1000 -25 25])
hold on

% Input (c)
T3 = 0.25*tau;
f3 = 1/T3;
w3 = 2*pi*f3;
ei3 = EI*sin(t*w3);
figure(3)
plot(t*1000, 1000*ei3, 'k-')
xlabel('t (ms)')
ylabel('e_{i3} (mV)')
axis([0 1000 -10 10])
hold on
```

```
% System step response
num = [-R2];
den = [R1*R2*C R1];
sys = tf(num, den);

% Response to input 1
figure(1)
[eo1, t] = lsim(sys, ei1, t);
plot(1000*t, 1000*eo1, 'k-')

figure(2)
[eo2, t] = lsim(sys, ei2, t);
plot(1000*t, 1000*eo2, 'k-')

figure(3)
[eo3, t] = lsim(sys, ei3, t);
plot(1000*t, 1000*eo3, 'k-')
```

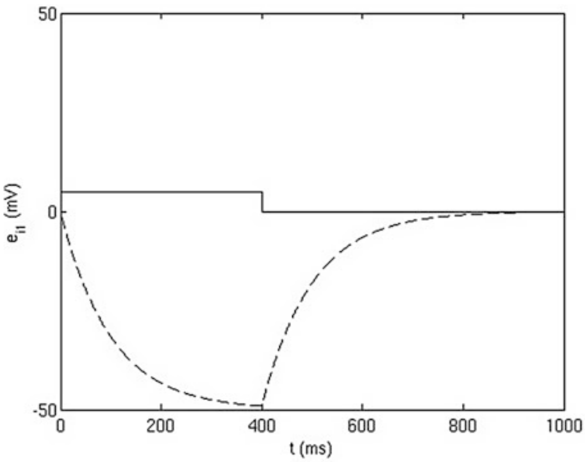
(a) Because the step input persists for 4τ, the response nearly reaches a steady state. The system response to a step input of amplitude E_{I} is given by:

$$E_o(s) = -\left(\frac{R_2}{R_1R_2Cs + R_1}\right)\frac{E_{\mathrm{I}}}{s}. \tag{7.69}$$

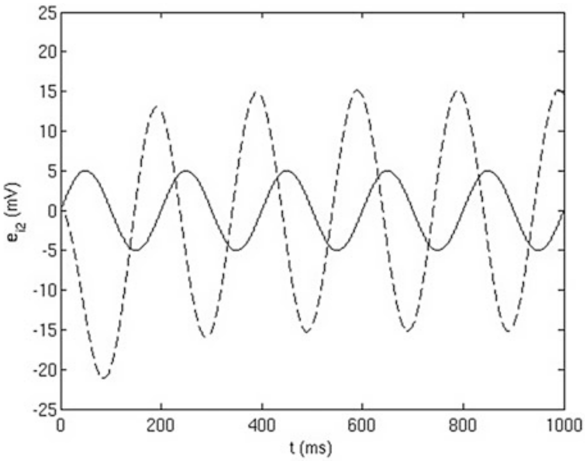
By applying the final value theorem, we obtain the steady state response.

$$\lim_{t\to\infty} e_o(t) = -\left(\frac{R_2}{R_1}\right)E_{\mathrm{I}} \tag{7.70}$$

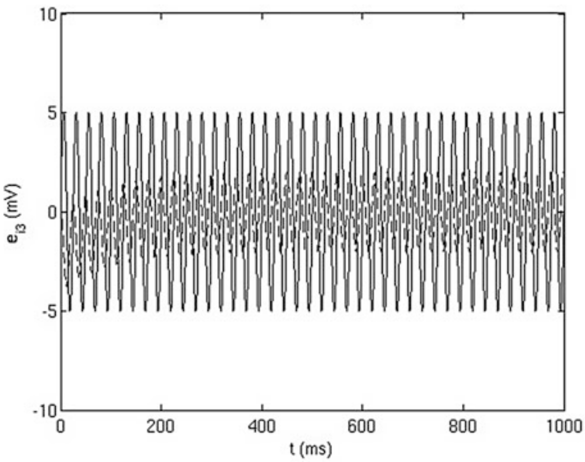
At steady state, the circuit behaves the same as the circuit in Example 7.7 with a gain of -10. Physically, this occurs because the capacitor impedance becomes infinite when it charges and leaves the resistors as the only elements in the circuit. The response to the truncated step input is shown. It reaches a value of nearly -50 mV after 4τ and then settles back to zero, after an additional 4τ s.



(b) Because this sine wave oscillates with a period greater than the time constant of the circuit, the output sine wave amplitude is increased by a fraction of the circuit gain and inverted. The output sine wave is *amplified*. If the sine wave varied with an even greater period, the output wave would approach an amplitude of 50 mV.



(c) Because this sine wave varies with a period less than the circuit time constant, the circuit cannot fully respond and the amplitude of the output wave is less than the input amplitude. The output is *attenuated*.



The type of circuit shown in Fig. 7.16 is known as a low pass filter/amplifier since low frequency signals pass through the circuit with amplification, while high frequency signals are attenuated (they do not pass through the circuit at their original amplitude).

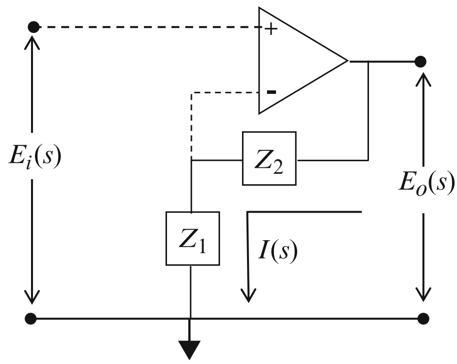


Fig. 7.17 Noninverting op-amp configuration

7.7.2 Noninverting Op-Amp Configuration

While the circuit in Fig. 7.15 is an inverting op-amp due to the negative sign in the transfer function, the circuit shown in Fig. 7.17 is a noninverting op-amp configuration. The dotted lines again indicate the high impedance inputs. Since it is attached directly to the input, the positive voltage is:

$$E_+ = E_i(s). \tag{7.71}$$

We see that a current, $I(s)$, flows in the lower branch of the circuit:

$$I(s) = \frac{E_o(s)}{Z_1 + Z_2}. \tag{7.72}$$

Using this current, we can calculate the voltage at the negative terminal.

$$E_- = E_o(s) - I(s)Z_2 \tag{7.73}$$

Setting $E_+ = E_-$, substituting for I, and multiplying through by $Z_1 + Z_2$, we obtain the transfer function.

$$\frac{E_o(s)}{E_i(s)} = 1 + \frac{Z_2}{Z_1} \tag{7.74}$$

The op-amp is called noninverting because the right-hand side of Eq. (7.74) is positive.

Example 7.9 For a noninverting op-amp with resistors as impedances: $Z_1 = R_1$ and $Z_2 = R_2$, find the response to a step input of 5 mV if $R_2 = 10\,\mathrm{k\Omega}$ and $R_1 = 1\,\mathrm{k\Omega}$. Also, find the output voltage.

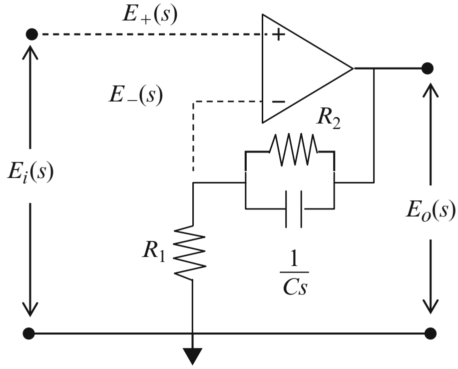


Fig. 7.18 Noninverting filter/amplifier circuit

Solution Using Eq. (7.74), we find that the transfer function for this circuit is again a constant: $\frac{E_o(s)}{E_i(s)} = 1 + \frac{R_2}{R_1} = 11$. The 5 mV step input is amplified to give an output of 55 mV. Notice that as the ratio of the resistors becomes larger, the "1" becomes negligible and the circuit gain is approximately $\frac{R_2}{R_1}$.

Example 7.10 Consider the noninverting op-amp circuit shown in Fig. 7.18. Calculate the transfer function of the circuit and, given $R_2 = 10\,\mathrm{k\Omega}$, $R_1 = 1\,\mathrm{k\Omega}$, and $C = 10\,\mu\mathrm{F}$, find the time constant of the circuit. Use MATLAB® to calculate the response to the same input voltage profiles given in Example 7.8.

Solution We find the transfer function using Eq. (7.74).

$$\frac{E_o(s)}{E_i(s)} = \frac{R_1 R_2 Cs + R_1 + R_2}{R_1 R_2 Cs + R_1} \tag{7.75}$$

Since the denominator is the same as that in Example 7.8, the time constant is again 100 ms. For this circuit, when a step input E_{I} is applied, the steady state response is:

$$\lim_{t\to\infty} e_o(t) = \left(1 + \frac{R_2}{R_1}\right) E_{\mathrm{I}}. \tag{7.76}$$

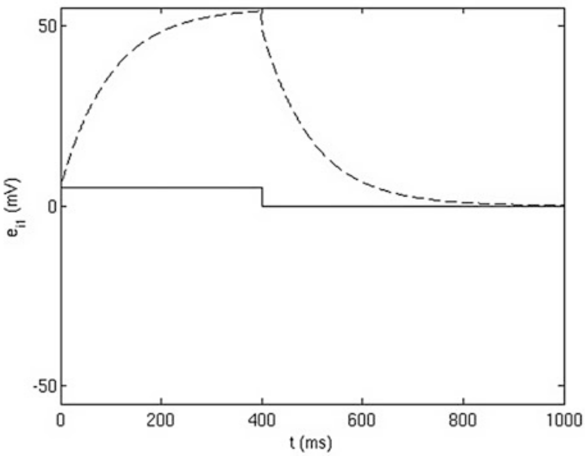
The gain approaches the value obtained in Example 7.9 for steady state behavior. The MATLAB® code used to determine the system response is provided.

```
clear all
clc
close all

% Parameters
R1 = 1000;
R2 = 10000;
C = 10e-6;
```

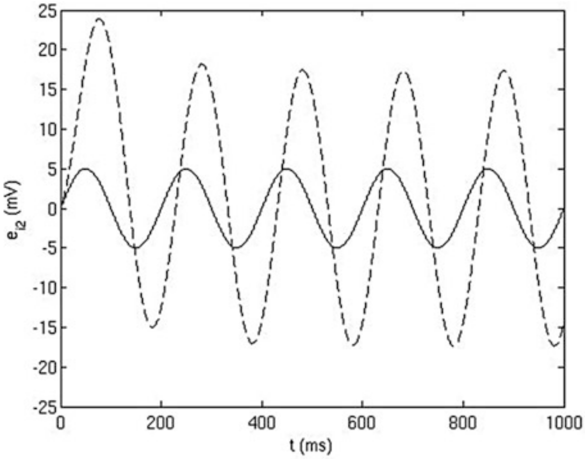
```
EI = 0.005;
tau = R2*C;

% Time vector
t = [0:tau/1000:10*tau];

% Input (a)
ei1 = zeros(1, length(t));
index = find(t < 4*tau);
ei1(index) = EI;
figure(1)
plot(t*1000, 1000*ei1, 'k-')
xlabel('t (ms)')
ylabel('e_{i1} (mV)')
axis([0 1000 -55 55])
hold on

% Input (b)
T1 = 2*tau;
f1 = 1/T1;
w1 = 2*pi*f1;
ei2 = EI*sin(t*w1);
figure(2)
plot(t*1000, 1000*ei2, 'k-')
xlabel('t (ms)')
ylabel('e_{i2} (mV)')
axis([0 1000 -25 25])
hold on

% Input (c)
T3 = 0.25*tau;
f3 = 1/T3;
w3 = 2*pi*f3;
ei3 = EI*sin(t*w3);
figure(3)
plot(t*1000, 1000*ei3, 'k-')
xlabel('t (ms)')
ylabel('e_{i3} (mV)')
axis([0 1000 -10 10])
hold on

% System step response
num = [R1*R2*C R1 + R2];
den = [R1*R2*C R1];
sys = tf(num, den);

% Response to input 1
figure(1)
```

```
[eo1, t] = lsim(sys, ei1, t);
plot(1000*t, 1000*eo1, 'k-')

figure(2)
[eo2, t] = lsim(sys, ei2, t);
plot(1000*t, 1000*eo2, 'k-')

figure(3)
[eo3, t] = lsim(sys, ei3, t);
plot(1000*t, 1000*eo3, 'k-')
```

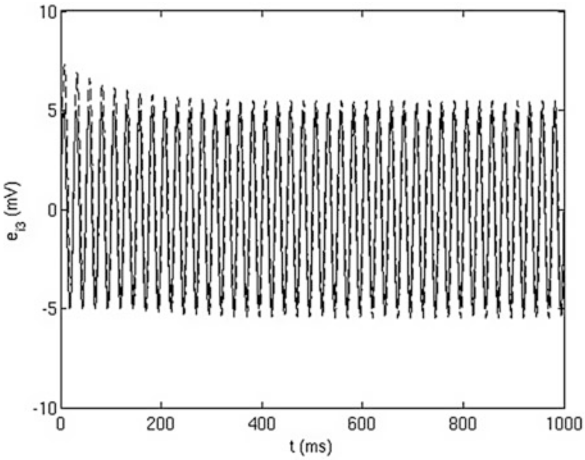
The response for each input voltage is shown.
(a) We see that the response is not inverted and approaches a value of 55 mV before the input ends.



(b) The low frequency response is not inverted and has a significant gain.



(c) The higher frequency response has a gain of approximately 1 and reproduces the input signal.



We see that the circuit behaves as a low pass filter/amplifier.

7.7.3 Other Configurations

For op-amp configurations that are different than the inverting and noninverting types, we simply follow the six steps described previously to determine the transfer function. We illustrate by example.

Example 7.11 Find the transfer function for the op-amp circuit shown in Fig. 7.19. Given that $R_1 = 10\,\text{k}\Omega$, $R_2 = 20\,\text{k}\Omega$, $R_3 = 100\,\text{k}\Omega$, and $C = 20\,\mu\text{F}$, find the time constant of the circuit and use MATLAB® to determine the response to a 5 mV step input.

Solution

Step 1: The circuit impedances are shown in Fig. 7.19.

Step 2: The high impedance inputs break the circuit into two independent parts, one with a current $I_1(s)$ and the other with current $I_2(s)$.

Step 3: We calculate E_+ by first determining $I_1(s)$. The impedance between $E_i(s)$ and ground is $R_1 + \frac{1}{Cs}$. The current is the voltage drop divided by the impedance.

$$I_1(s) = \left(\frac{Cs}{R_1Cs + 1}\right)E_i(s) \tag{7.77}$$

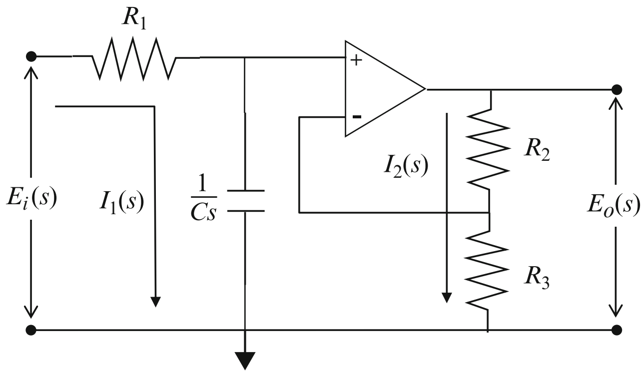


Fig. 7.19 More complex op-amp circuit

The voltage E_+ is calculated by starting at ground (0 V) and calculating the voltage rise from ground across the capacitor, $I_1(s)\frac{1}{Cs}$. Substituting for I_1 gives:

$$E_+(s) = \left(\frac{1}{R_1Cs + 1}\right)E_i(s). \tag{7.78}$$

Step 4: Similarly, E_- is determined by first finding $I_2(s)$ and then finding the rise from ground across the resistor R_3.

$$E_- = \left(\frac{R_3}{R_2 + R_3}\right)E_o(s) \tag{7.79}$$

Step 5: Setting $E_+ = E_-$ and rearranging we obtain the circuit transfer function.

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2 + R_3}{CR_1R_3s + R_3} \tag{7.80}$$

Step 6: The time constant of the circuit is CR_1 or 0.2 s. For a step input $\frac{E_\mathrm{I}}{s}$, the circuit response is given by:

$$E_o(s) = \left(\frac{R_2 + R_3}{CR_1R_3s + R_3}\right)\frac{E_\mathrm{I}}{s}. \tag{7.81}$$

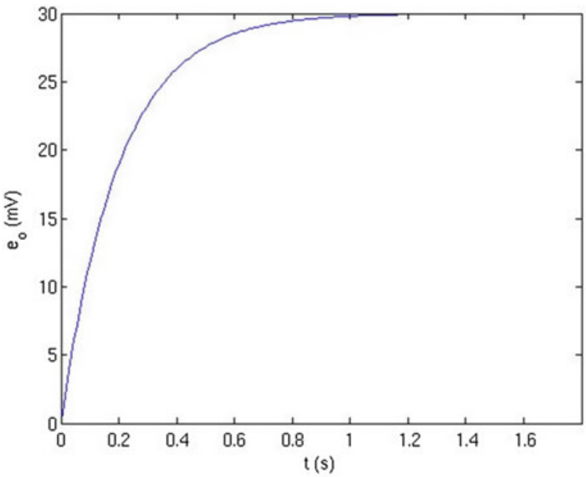
Applying the final value theorem, we obtain the steady state voltage.

$$\lim_{t\to\infty} e_o(t) = \left(1 + \frac{R_2}{R_3}\right)E_\mathrm{I} \tag{7.82}$$

The MATLAB® code used to determine the system response is given.

```
clear all
clc
close all
% Parameters
R1 = 10000;    %  Ohms
R2 = 100000;   %  Ohms
R3 = 20000;    %  Ohms
C = 20e-6;     %  F
EI = 0.005;    %  V
tau = R1*C;    %  s

% Transfer function
num = [R2 + R3];
den = [C*R1*R3 R3];
sys = tf(num, den);
[eo_u, t] = step(sys);
eo = eo_u*EI;
figure(1)
plot(t, 1000*eo)
xlabel('t (s)')
ylabel('e_o (mV)')
```

The results are given.



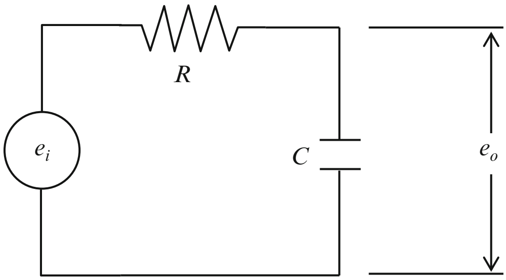
7.8 Summary

In this chapter, we described the analysis of electric circuits. We showed that the passive electrical elements are analogous to mechanical elements. The key points of the chapter are summarized.

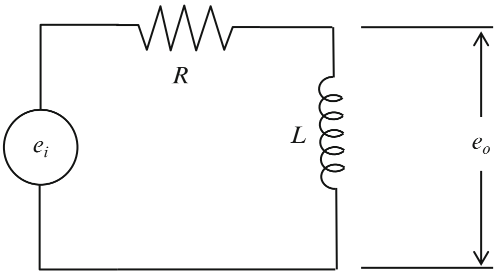
- The resistor, capacitor, and inductor are the three primary passive electrical elements.
- A resistor dissipates electrical energy as heat and is analogous to the mechanical damper.
- A capacitor stores electric charge/energy in an electric field and is analogous to a mechanical spring.
- An inductor stores energy in a magnetic field, resists rapid changes in current, and is analogous to a mechanical inertia.
- Circuits are analyzed using Kirchhoff's current and voltage laws.
- Kirchhoff's current law can be simplified using the mesh approach to circuit analysis.
- The order of a passive electric circuit is equal to the number of independent energy storage elements.
- Circuits can be analyzed using either the differential equation method or the impedance method, which uses the Laplace transform of the input/output relationships for the circuit elements.
- A first-order electrical system consists of either a capacitor and a resistor or an inductor and a resistor; the time constant of the circuit has the same physical interpretation as the time constant for a first-order mechanical system.
- An R–L–C circuit is a second-order system.
- In an R–L–C circuit, the natural frequency and damping ratio are given by comparing the denominator of the transfer function to $s^2 + 2\zeta\omega_n + \omega_n^{\ 2}$.
- Electric circuits are important to the mechanical engineer for: (1) conditioning of signals from sensors; and (2) electrical input signals for actuators. Most modern mechanical systems have both electrical and mechanical elements.

Problems

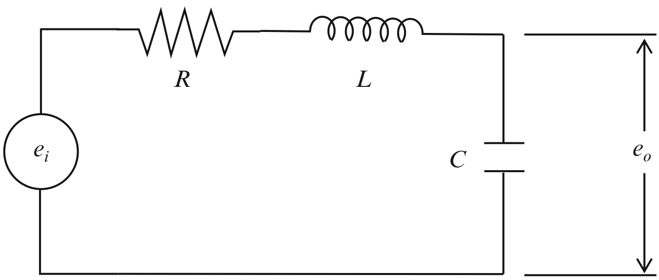
1. Consider the R-C circuit shown in the figure. Suppose the input voltage is a step function: $e_i(t) = E_0 \cdot u(t)$, where $E_0 = 5\,\text{V}$. The resistance is 100 Ω and the capacitance is 10 μF. Find the time domain differential equation describing the output voltage, $e_o(t)$. Calculate the time constant of the system. Solve the equation for $e_o(t)$ using Laplace transforms assuming the initial charge in the capacitor is zero. Plot $e_o(t)$ using MATLAB® for four time constants of the system; use units of ms for the time axis. Apply the final value theorem and initial value theorem to the solution in the Laplace domain and show that the results agree with your plot.



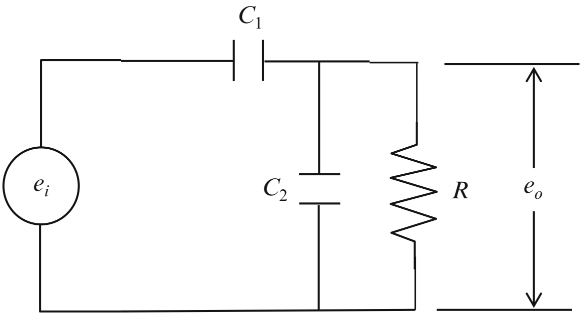
2. Consider the R-L circuit shown in the figure. Suppose the input voltage is a step function: $e_i(t) = E_0 \cdot u(t)$, where $E_0 = 5\,\text{V}$. The resistance is 100 Ω and the inductance is 10 mH. Find the time domain differential equation describing the output voltage, $e_o(t)$. Calculate the time constant for the system. Solve the equation for $e_o(t)$ using Laplace transforms assuming the initial voltage drop across the inductor is zero. Plot $e_o(t)$ using MATLAB® for four time constants of the system; use units of ms for the time axis. Apply the final value theorem and initial value theorem to the solution in the Laplace domain and show that the results agree with your plot.



3. Using the fact that impedance is in units of ohms, prove that the time constants you calculated in Problems 1 and 2 have units of seconds. (Hint: What are the units of the Laplace variable s?)
4. Consider the R-L-C circuit shown in the figure. Suppose the input voltage is a step function: $e_i(t) = E_0 \cdot u(t)$, where $E_0 = 5\,\text{V}$. The initial conditions are zero. The resistance is 50 Ω, the capacitance is 5 μF, and the inductance is 100 mH. Find the time domain differential equation describing the output voltage, $e_o(t)$. Calculate the natural frequency, ω_n, and damping ratio, ζ, for the system. Using a script file in MATLAB®, find $e_o(t)$ using the `ilpalace` command and plot $e_o(t)$ for four time constants of the system; use units of ms for the time axis.



5. Consider the circuit shown in the figure. Using impedance methods, find the transfer function $\frac{E_o(s)}{E_i(s)}$.

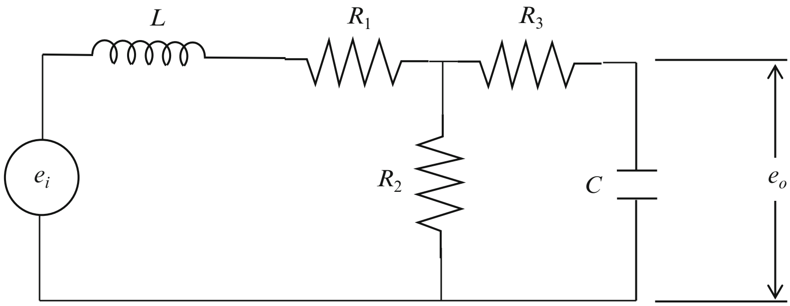


The values of the system constants are provided.

$$C_1 = 0.01\,\mathrm{F}$$
$$C_2 = 0.01\,\mathrm{F}$$
$$R = 100\,\Omega$$

Using a script file in MATLAB® and the `step` command, find and plot the unit step response $e_o(t)$. Apply the final value theorem and initial value theorem to the solution in the Laplace domain and show that the results agree with your plot.

6. Consider the circuit shown in the figure. Using impedance methods, find the transfer function $\frac{E_o(s)}{E_i(s)}$.

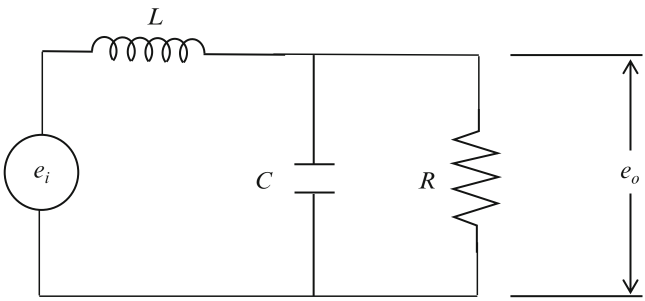


The values of the system constants are provided.

$$
\begin{aligned}
L &= 0.1\,\mathrm{H} \\
C &= 1\,\mathrm{F} \\
R_1 &= 100\,\Omega \\
R_2 &= 100\,\Omega \\
R_3 &= 150\,\Omega
\end{aligned}
$$

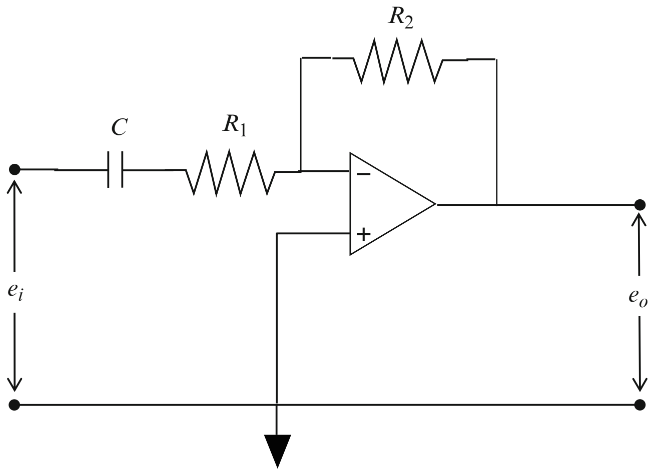
Using a script file in MATLAB® and the command `step`, find and plot the unit step response $e_o(t)$. Apply the final value theorem and initial value theorem to the solution in the Laplace domain and show that the results agree with your plot.

7. Consider the circuit shown in the figure. Using impedance methods, find the transfer function $\frac{E_o(s)}{E_i(s)}$.



Choose values for R, L, and C that ensure the system will oscillate. Using a script file in MATLAB® and the command `step`, find and plot the unit step response $e_o(t)$ using your values. Apply the final value theorem and initial value theorem to the solution in the Laplace domain and show that the results agree with your plot.

8. Consider the op-amp circuit shown in the figure. Using impedance methods, find the transfer function $\frac{E_o(s)}{E_i(s)}$.

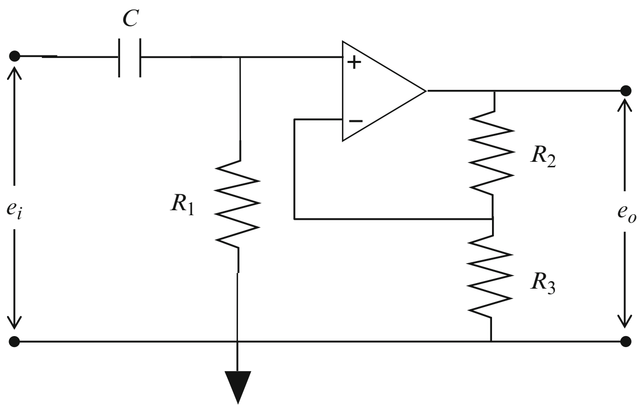


The values of the system constants are provided.

$$C = 20\,\mu\text{F}$$
$$R_1 = 100\,\Omega$$
$$R_2 = 200\,\Omega$$

Using a script file in MATLAB® and the command `step`, find and plot the unit step response $e_o(t)$. Apply the final value theorem and initial value theorem to the solution in the Laplace domain and show that the results agree with your plot.

9. Consider the op-amp circuit shown in the figure. Using impedance methods, find the transfer function $\frac{E_o(s)}{E_i(s)}$.



The values of the system constants are provided.

$$C = 20\ \mu\text{F}$$
$$R_1 = 100\ \Omega$$
$$R_2 = 100\ \Omega$$
$$R_3 = 100\ \Omega$$

Using a script file in MATLAB® and the command `step`, find and plot the unit step response $e_o(t)$. Apply the final value theorem and initial value theorem to the solution in the Laplace domain and show that the results agree with your plot.

Reference

1. Weyrick RC (1970) Fundamentals of analog computers. Prentice Hall, Englewood Cliffs, NJ

Electromechanical Systems

8

8.1 Introduction

So far, we have discussed mechanical and electrical systems separately. However, modern systems typically include both mechanical and electrical components; these are referred to as electromechanical systems. For mechanical engineers, it is critical to understand how to model systems with electrical, electronic, and, naturally, mechanical elements. Examples that will be treated in this chapter include electric motors, generators, and others. Of course, many systems such as robots, machine tools, and conveyer belts combine multiple elements into a more complex electromechanical system. We will demonstrate the basic analysis pattern for simpler electromechanical systems, which can be applied to more complicated systems as well.

8.2 Permanent Magnet Direct Current Motors

An electric motor converts electrical energy into mechanical energy. In this section, we develop a model for a permanent magnet direct current (DC) motor. A permanent magnet DC motor consists of a rotor with wire windings and a stator that includes permanent magnets. The operating principle of a DC motor can be understood by examining the rotation of a single current-carrying wire in a magnetic field as shown in Fig. 8.1a, where the magnetic field is generated by two nonrotating permanent magnets. The magnetic field vectors are nearly parallel between the magnets and point from the north (N) pole to the south (S) pole of the magnet. In electromagnetics, the Lorentz force, **F**, on a charge, q, moving with a velocity, **v**, in the presence of an electric field, **E**, and a magnetic field, **B**, can be expressed as shown in Eq. (8.1), where $\times$ represents the vector cross product.

M.A. Davies, T.L. Schmitz, *System Dynamics for Mechanical Engineers*,
DOI 10.1007/978-1-4614-9293-1_8

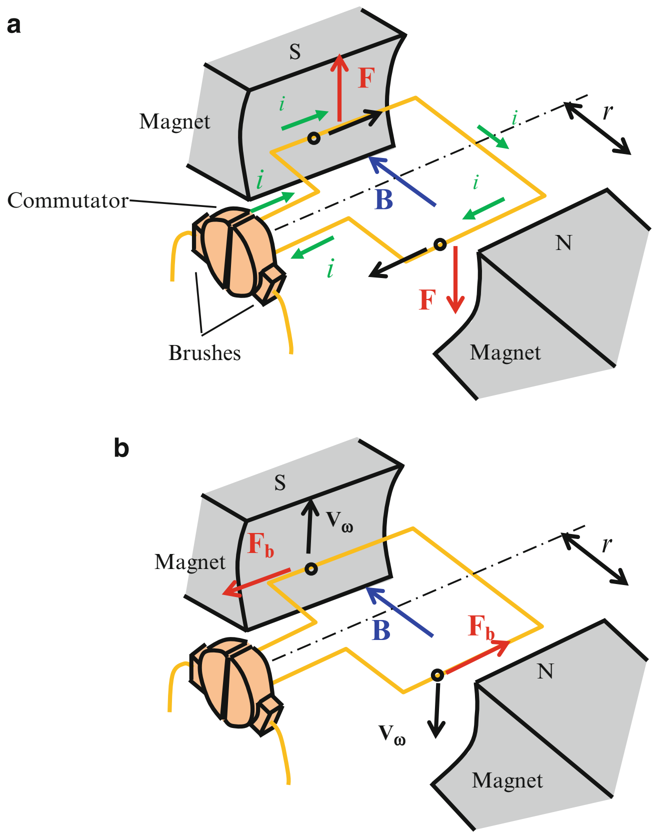


Fig. 8.1 A single current-carrying loop in a magnetic field, where i is the current, $\mathbf{B}$ is the magnetic field vector, and r is the wire loop radius. The forces developed in the loop, $\mathbf{F}$, are shown in (**a**) and the source of back-emf, $\mathbf{F}_b$, is shown in (**b**)

$$\mathbf{F} = q\mathbf{E} + q(\mathbf{v}\times\mathbf{B}) \tag{8.1}$$

The electric field in the vicinity of the current loop is negligible, so the force is derived from interactions of charge with the magnetic field. Positive charge motion is in the direction of positive current flow (in reality, electrons are moving in the opposite direction). Thus, in the position shown, a positive charge on the left branch of the loop will experience a force $q(\mathbf{v} \times \mathbf{B})$ in the upward direction. A positive charge moving in the right branch of the loop will similarly experience a force in the downward direction. In the other loop branches, the velocity of the charges is parallel to the magnetic field, so the charges experience no force. When the charges are integrated (added), the forces on the left and right branches of the loop have a magnitude iLB, where L is the length of the branch. In total, the current, i, flowing in the loop will result in a moment:

$$M = 2LBri, \tag{8.2}$$

where r is the radius of the current loop.

The moment applied to the current loop causes an angular acceleration and, subsequently an angular velocity, $\boldsymbol{\omega}$. Once the loop begins to rotate, the charges moving in the wire develop another velocity component, $\mathbf{v}_\omega$, perpendicular to the current direction as shown in Fig. 8.1b. This velocity component causes a force, $\mathbf{F}_b$, that opposes the direction of current flow. This force is referred to as the back electromotive force, or back-emf. This produces a dissipative voltage, e_b. The back-emf is proportional to the rotational velocity of the loop.

As the current loop begins to rotate, the forces cause the current loop to reorient from the horizontal plane to the vertical plane. If the current direction remained the same, direction the loop would stop rotating when the moment arm of the forces approached zero and the loop would then settle at a constant orientation. However, to produce continuous rotation, the brushes and commutator reverse the direction of current when the loop reaches the vertical orientation. This changes the force direction. Angular momentum carries the loop through the vertical position, but with the reversal in force direction the loop continues to rotate. This is the basic operating principle for a permanent magnet DC motor. However, if there was only a single current loop, the moment would vary periodically with the rotation angle.

Therefore, to smooth the moment and increase its magnitude, many windings of wire are added at different angles; each winding then becomes a pole of the motor. A representation of a permanent magnet DC motor with several motor windings is shown in Fig. 8.2a. When each winding is energized by the commutator, a torque is generated with a tendency to align its north/south pole with the south/north pole of the permanent magnets. Before this alignment is complete, the next winding is energized continuing the process as shown in Fig. 8.2b. In this case, the variations in moment (referred to as torque ripple) are minimized and the moment applied to the rotor is proportional to the current flowing in the windings.

$$M = Ki \tag{8.3}$$

In Eq. (8.3), K is the *motor torque constant* and is a function of the motor design, including the strength of the magnetic field, the length of the wire in the windings, the number of windings, and the radius of the rotor. Similarly, the back-emf voltage developed in the motor is proportional to the rotational speed:

$$e_b = K_b\omega, \tag{8.4}$$

where K_b is the *motor back-emf constant* that is also a function of the motor design parameters.

We next analyze the DC motor electromechanical system by considering the electric circuit and the mechanical elements. We use the constitutive equations,

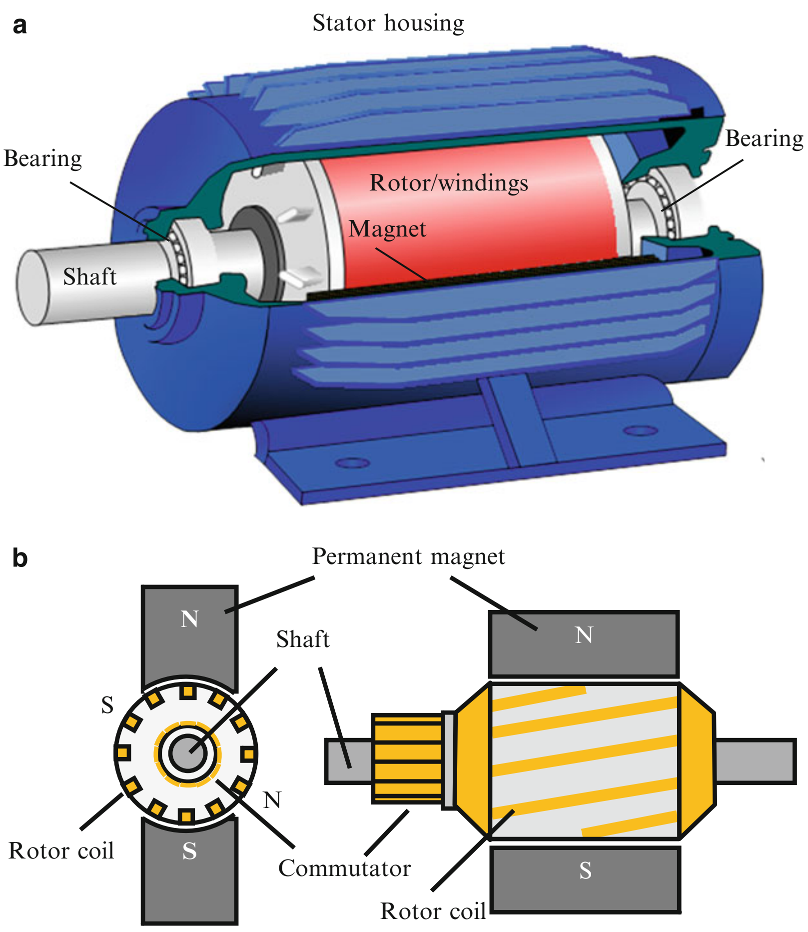


Fig. 8.2 Permanent magnet DC motor shown in (**a**) cutaway and (**b**) cross section, (Fig. 8.2 courtesy of G. Hodgins)

including Eqs. (8.3) and (8.4), to determine a transfer function for the system. In this case, a useful transfer function, $\frac{\Omega(s)}{E_{in}(s)}$, relates the motor's input voltage, e_{in}, to the output rotational speed, ω.

8.2.1 Motor Transfer Function

The electric circuit for a motor is displayed in Fig. 8.3. The motor windings have an inductance, L, and a resistance, R. Applying Kirchhoff's voltage law to the circuit, we obtain the time domain equation:

$$e_{in}(t) - iR - L\frac{di}{dt} - e_b(t) = 0. \tag{8.5}$$

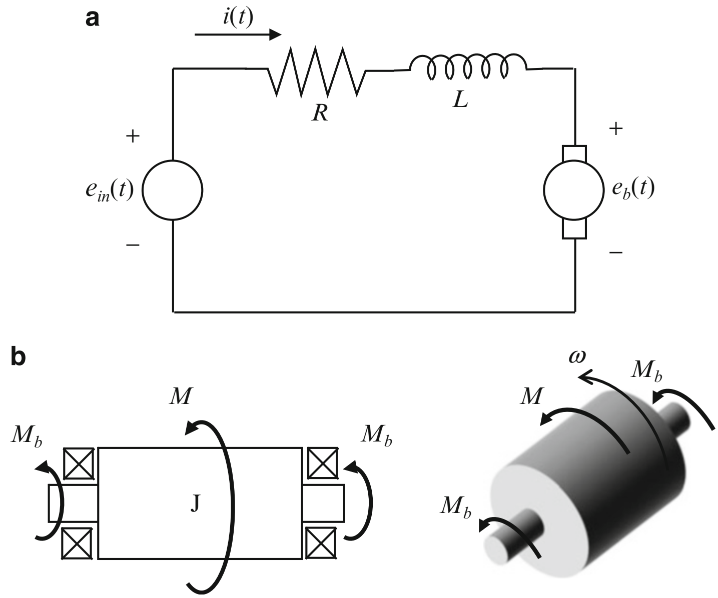


Fig. 8.3 (**a**) Motor circuit and (**b**) rotor mechanics

Assuming zero initial conditions and calculating the Laplace transform, we obtain:

$$E_{in}(s) - I(s)R - LsI(s) - E_b(s) = 0. \tag{8.6}$$

Next, we examine the rotor dynamics; see Fig. 8.3b. Applying Newton's second law for rotation, Eq. (3.5), we obtain the rotor's equation of motion.

$$\sum M = M + M_b + M_b = J\dot{\omega} \tag{8.7}$$

Combining both dissipative bearing moments, M_b, into a single rotational damper for the system, we obtain an equation of motion in terms of the rotor's angular velocity. The Laplace transform of Eq. (8.8) is provided in Eq. (8.9).

$$J\dot{\omega} + b_r\omega = M \tag{8.8}$$

$$(Js + b_r)\Omega(s) = M(s) \tag{8.9}$$

Similarly, the Laplace transforms of Eqs. (8.3) and (8.4) are provided in Eqs. (8.10) and (8.11).

$$M(s) = KI(s) \tag{8.10}$$

$$E_b(s) = K_b\Omega(s) \tag{8.11}$$

Now, we combine Eqs. (8.10), (8.11), (8.6), and (8.9) to obtain the motor's transfer function. First, we combine Eqs. (8.9) and (8.10).

$$(Js + b_r)\Omega(s) = KI(s) \tag{8.12}$$

Next, we combine Eq. (8.11) with Eq. (8.6) and solve for $I(s)$.

$$I(s) = \frac{E_{in}(s) - K_b\Omega(s)}{Ls + R} \tag{8.13}$$

Now we combine Eqs. (8.12) and (8.13) and solve for the transfer function of a permanent magnet DC motor.

$$\frac{\Omega(s)}{E_{in}(s)} = \frac{K}{JLs^2 + (JR + b_rL)s + b_rR + KK_b} \tag{8.14}$$

The denominator of Eq. (8.14) is second order. There are two energy storage elements in the system, the inductor created by the motor windings, L, which stores energy in a magnetic field and the rotor inertia, J, which stores kinetic energy in rotation. Energy loss occurs mechanically in the bearings, represented by the damping coefficient, b_r, and electrically through the resistance, R.

8.2.2 Dynamic Motor Response

Because the system is second order, it is theoretically possible to have oscillatory behavior during which energy is periodically traded back and forth between the inductor and the rotational inertia. However, in practice electric motors are overdamped. In this case, the denominator has negative, real roots and exhibits a nonoscillatory, exponential response. From the denominator of Eq. (8.14), the damping ratio of an electric motor is determined.

$$\zeta = \frac{b_rL + JR}{2\sqrt{JL(b_rR + KK_b)}} \tag{8.15}$$

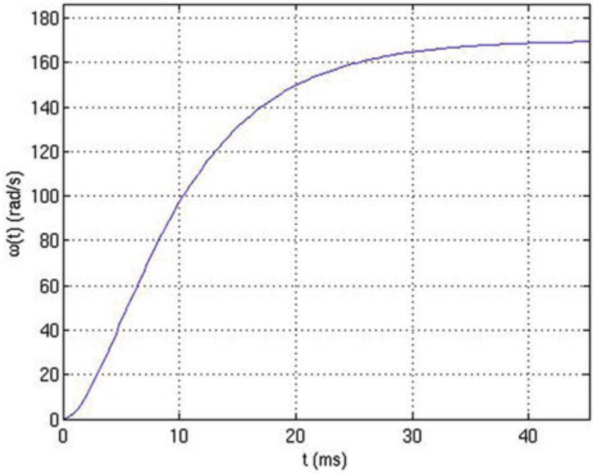
Consider the parameters for a small commercially available motor (12 V, 1800 rpm, 110 W): $K = 0.05$ N-m/A, $K_b = 0.05$ V-s, $R = 0.15$ Ω, $L = 0.1$ mH, $J = 3.883 \times 10^{-4}$ kg-m^2, $b_r = 0.03$ N-m/krpm (2.865×10^{-4} N-m-s). For these parameters, $\zeta = 2.93$, which gives overdamped behavior. The characteristic equation will have negative, real roots and the response will be a combination of exponentials with two time constants.

Example 8.1 Consider a 120 W DC motor with the constants: $K = 0.42$ N-m/A, $K_b = 0.42$ V-s, $R = 4.9\ \Omega$, $L = 11.2$ mH, $J = 3.883 \times 10^{-4}$ kg-m^2, $b_r = 0.03$ N-m/krpm (2.865×10^{-5} N-m-s). Find and plot the motor speed, $\omega(t)$, in response to a step voltage input $e_{in}(t) = E_0 \cdot u(t)$, where $E_0 = 72$ V using MATLAB®. Derive Eq. (8.15) from Eq. (8.14) and use MATLAB® to calculate the damping ratio for this motor. Determine the roots of the motor's characteristic equation and use them to describe the behavior observed in the step response. Finally, use the final value theorem to determine the steady state speed of the motor and compare it to the plot. Analytically show that for small R and b_r, the steady state speed is E_0/K_b.

Solution We use the transfer function defined in Eq. (8.14). The MATLAB® code for the DC motor behavior is provided.

```
clear all
clc
close all

% Parameters
K = 0.42;           %  N-m/A
Kb = 0.42;          %  V/rad/s
R = 4.9;            %  Ohms
L = 11.2e-03;       %  H
J = 3.883e-04;      %  kg-m^2
br = 0.03/1000*60/(2*pi); % N-m-s
E0 = 72;            % 72 V step input
zeta = (br*L+J*R)/(2*sqrt(J*L*(br*R+K*Kb)));

% Transfer function
num = [K];
den = [J*L (J*R+br*L) br*R+K*Kb];
r = roots(den);

sys = tf(num,den);
[w_u, t] = step(sys);
w = w_u*E0;

figure(1)
plot(1000*t, w)
xlabel('t (ms)')
ylabel('\omega(t) (rad/s)')
axis([0 max(1000*t) 0 1.1*max(w)])

% Display damping ratio and roots
display(zeta)
display(r)
```

The time domain response is shown.



The damping ratio for any second-order characteristic equation is determined by writing it in standard form. From Eq. (8.14), the characteristic equation of a DC motor is:

$$JLs^2 + (b_rL + JR)s + b_rR + KK_b.$$

To write it in the required form, $s^2 + 2\zeta\omega_n s + \omega_n{}^2$, we factor the constant multiplying s^2 out of the equation:

$$JL\left(s^2 + \frac{(b_rL + JR)}{JL}s + \frac{b_rR + KK_b}{JL}\right).$$

From this expressions, we identify the natural frequency as the square of the third term:

$$\omega_n = \sqrt{\frac{b_rR + KK_b}{JL}}. \tag{8.16}$$

We can now use the middle term to identify the damping ratio:

$$2\zeta\omega_n = \frac{b_rL + JR}{JL}.$$

Rewriting gives:

$$\zeta = \frac{b_rL + JR}{2\omega_n JL}.$$

By substituting for the natural frequency into this expression and simplifying, we obtain Eq. (8.15).

$$\zeta = \frac{b_r L + JR}{2\sqrt{\frac{b_r R + KK_b}{JL}}JL} = \frac{b_r L + JR}{2\sqrt{JL(b_r R + KK_b)}}$$

The damping ratio and characteristic equation roots from MATLAB® follow.

```
zeta =

 1.0837

r =

 -303.5516

 -134.6861
```

As discussed previously, the motor is overdamped and, therefore, the roots of the characteristic equation are both negative and real. The two time constants associated with these roots are: $\tau_1 = (1/303.55)$ s = 3.3 ms and $\tau_2 = (1/134.69)$ s = 7.4 ms. The second time constant is dominant. In the graph, we see that τ_1 is active for the initial exponential build up in speed ($4\tau_1$ is approximately 13.2 ms) and τ_2 determines the time to reach the steady state speed ($4\tau_2$ is approximately 29.6 ms). We estimate the steady state speed by applying the final value theorem to Eq. (8.14) with a step input voltage.

$$\Omega(s) = \left(\frac{K}{JLs^2 + (JR + b_r L)s + b_r R + KK_b}\right)\frac{E_0}{s}$$

Implementing the final value theorem gives:

$$\omega_{ss} = \lim_{t\to\infty}(\omega(t)) = \lim_{s\to 0}(s\Omega(s)) = \frac{KE_0}{b_r R + KK_b}, \tag{8.17}$$

which can be simplified to Eq. (8.18) provided that $b_r R << KK_b$.

$$\omega_{ss} \cong \frac{E_0}{K_b} \tag{8.18}$$

Substituting in Eq. (8.17) gives 170.1 rad/s, while Eq. (8.18) yields a steady state speed of 171.4 rad/s. Note that Eq. (8.18) implies that the motor will continue to speed up until the back-emf cancels the input voltage. Examining Fig. 8.3a, this is expected since an inductor allows the maximum current to pass at steady state (i.e., it becomes a short circuit) and, if the resistor is small, the only voltage countering the input voltage in the circuit is the back-emf.

8.2.3 Motor Time Constants and Approximate First-Order Behavior

As discussed previously, the roots of the characteristic equation (the denominator of the Eq. (8.14) transfer function) determine the behavior of the system. We have already seen that most motors are overdamped which gives negative real roots of the characteristic equation and gives exponential, nonoscillatory behavior. The two roots provide two time constants for DC motors. The larger, dominant time constant depends on the motor inertia and is, therefore, called the mechanical time constant, τ_{mech}. The smaller time constant depends only on the motor inductance and resistance and is referred to as the electrical time constant, τ_{elec}. In some cases, the electrical time constant can be ignored and the motor may be modeled as a first-order system. If high acceleration, high accuracy motion control is required, however, the full second-order model must be used.

Example 8.2 Consider a 120 W electric motor with the constants: $K = 0.11$ N-m/A, $K_b = 0.11$ V-s, $R = 0.60\ \Omega$, $L = 0.72$ mH, $J = 3.883 \times 10^{-4}$ kg-m^2, $b = 0.03$ N-m/krpm (2.865×10^{-4} N-m-s). Use MATLAB® to find the damping ratio and the roots of the characteristic equation. Write the characteristic equation and identify the order of magnitude for each term to determine those that might be negligible and those that are important.

Solution The MATLAB® code is provided.

```
clear all
clc
close all

% Parameters
K = 0.11;          %  N-m/A
Kb = 0.11;         %  V/rad/s
R = 0.60;          %  Ohms
L=0.72e-03;        %  H
J = 3.883e-04;     %  kg-m^2
br = 0.03/1000*60/(2*pi);  % N-m-s
zeta = (br*L+J*R)/(2*sqrt(J*L*(br*R+K*Kb)));

fprintf('zeta = %g \n', zeta);
fprintf('JL = %g \n', J*L);
fprintf('JR = %g \n', J*R);
fprintf('b_rL = %g \n', br*L);
fprintf('b_rR = %g \n', br*R);
fprintf('KK_b = %g \n', K*Kb);

den = [J*L J*R+br*L br*R+K*Kb];
r = roots(den);
tau = 1./r;
```

```
fprintf('The roots of the characteristic equation are %d and %d.
\n', r(1), r(2))
fprintf('The time constants of the system are %d ms and %d ms. \n',
-tau(1)*1e3, -tau(2)*1e3)
```

By executing the program, the following results are obtained.

```
zeta = 1.99052
JL = 2.79576e-07
JR = 0.00023298
b_rL = 2.06265e-07
b_rR = 0.000171887
KK_b = 0.0121
The roots of the characteristic equation are -7.776240e+02 and
-5.644712e+01.
The time constants of the system are 1.285969e0 ms and 1.771569e
+01 ms.
```

Substituting the example constants in the characteristic equation gives:

$$JLs^2 + (b_rL + JR)s + b_rR + KK_b = 0$$
$$\left(2.80 \times 10^{-7}\right)s^2 + \left(2.06 \times 10^{-7} + 2.33 \times 10^{-4}\right)s + 1.72 \times 10^{-4} + 1.21 \times 10^{-2} = 0$$

It is evident that in this example, $b_rL << JR$ and $b_rR << KK_b$. This is generally true for electric motors and this allows us to make mathematical simplifications that clarify the physical behavior of the motor.

Using the results from the typical system modeled in Example 8.2, we can ignore the small terms. The simplified characteristic equation is given by Eq. (8.19).

$$s^2 + \frac{R}{L}s + \frac{KK_b}{JL} = 0 \tag{8.19}$$

Now assume the equation can be factored as:

$$\left(s + \frac{R}{L}\right)\left(s + \frac{KK_b}{RJ}\right) = 0. \tag{8.20}$$

The error introduced by Eq. (8.20) is small when $\frac{R}{L} \ll \frac{KK_b}{RJ}$.

Equation (8.20) yields two time constants.

$$\tau_{elec} = \frac{L}{R} \tag{8.21}$$

$$\tau_{mech} = \frac{RJ}{KK_b} \tag{8.22}$$

We now recognize that Eq. (8.20) is a good approximation of the DC motor's characteristic equation when $\tau_{mech} << \tau_{elec}$, which is true for most permanent magnet DC motors.

The electrical time constant is equal to the time constant for an R–L circuit. This is the time required to establish a steady magnetic field in the motor coils. The mechanical time constant depends on the rotor inertia and the coil resistance. It increases if either is increased. Greater inertia decreases the acceleration, so the time to reach a steady state speed increases. Greater coil resistance decreases the torque (for a given input voltage) and, therefore, increases the time to reach a steady state speed. The mechanical time constant decreases if the torque constant or the back-emf constant is increased. When the torque constant is increased the torque is increased and the motor accelerates to steady state more quickly. A greater back-emf constant increases the back-emf for a given speed and, therefore, decreases the time to reach a speed that counters the applied voltage (see Eq. 8.18).

Applying the assumptions we have identified, the motor transfer function can be approximated as:

$$\frac{\Omega(s)}{E_{in}(s)} = \frac{1/K_b}{(\tau_{elec}s+1)(\tau_{mech}s+1)}. \tag{8.23}$$

In many cases, the electrical time constant is small enough that it is justified to assume it is zero. The resulting transfer function is first order.

$$\frac{\Omega(s)}{E_{in}(s)} = \frac{1/K_b}{\tau_{mech}s+1} \tag{8.24}$$

Whether Eqs. (8.14), (8.23), or (8.24) is used to model motor behavior is dependent on the requirements of the application and the motor characteristics. We examine the differences between the different levels of approximation in Example 8.3.

Example 8.3 Consider two motors:

1. The 110 W electric motor from the paragraph preceding Example 8.1 with the constants: $K = 0.05$ N-m/A, $K_b = 0.05$ V-s, $R = 0.15$ Ω, $L = 0.18$ mH, $J = 3.883 \times 10^4$ kg-m^2, $b_r = 0.03$ N-m/krpm (2.865×10^{-4} N-m-s).
2. The 120 W electric motor from Example 8.1 with the constants: $K = 0.42$ N-m/A, $K_b = 0.42$ V-s, $R = 4.9$ Ω, $L = 11.2$ mH, $J = 3.883 \times 10^{-4}$ kg-m^2, $b_r = 0.03$ N-m/krpm (2.865×10^{-4} N-m-s).

Use MATLAB® to: (a) find the damping ratio; (b) find the electrical and mechanical time constants; (c) find the steady state speed for a 35 V step input; and (d) plot the step response of both motors using the transfer function given by Eq. (8.14) and the approximate transfer functions given by Eqs. (8.23) and (8.24). For each motor calculate the error in predicted speed between Eqs. (8.24) and (8.14) and plot it as a function of time. Compare and comment on the suitability of the models for the two different motors.

Solution The MATLAB® code is provided.

```
clear all
clc
close all

% Motor 1 analysis
% Parameters
K = 0.05;                      % N-m/A
Kb = 0.05;                     % V/rad/s
R = 0.15;                      % Ohms
L = 0.18e-03;                  % H
J = 3.883e-04;                 % kg-m^2
br = 0.03/1000*60/(2*pi);      % N-m-s
E0 = 35;                       % 35 V step input

% Calculated parameters
tau_elec = L/R;
tau_mech = J*R/(K*Kb);
zeta = (br*L+J*R)/(2*sqrt(J*L*(br*R+K*Kb)));
w_ss = E0/Kb;

% Display calculated results
fprintf('MOTOR 1\n')
fprintf('———————\n')
fprintf('The electrical time constant of the motor is %4.3 g ms.
\n', 1000*tau_elec)
fprintf('The mechanical time constant of the motor is %4.3 g ms.
\n', 1000*tau_mech)
fprintf('The steady state speed of the motor is %4.3 g rad/s. \n',
w_ss)

% System transfer functions
num1 = [K];
den1 = [J*L (J*R+br*L) br*R+K*Kb];
sys1 = tf(num1, den1);

num2 = [1/Kb];
den2 = conv([tau_mech 1],[tau_elec 1]);
sys2 = tf(num2, den2);

num3 = [1/Kb];
den3 = [tau_mech 1];
sys3 = tf(num3, den3);
```

```
% System responses (use common time vector)
t = [0:tau_mech/1000:6*tau_mech];
[w1_u, t1] = step(sys1, t);
w1 = w1_u*E0;
[w2_u, t2] = step(sys2, t);
w2 = w2_u*E0;
[w3_u, t3] = step(sys3, t);
w3 = w3_u*E0;

% Calculate percent error
Err1 = w3 - w1;

% Plot the results
figure(1)
subplot(211)
set(gca, 'FontSize', 14);
plot(1000*t1, w1, 'k-', 1000*t2, w2, 'k-', 1000*t3, w3, 'k-.')
grid
ylabel('\omega(t) (rad/s)')
axis([0 max(1000*t1) 0 1.1*max(w1)])
legend('Equation 8.14', 'Equation 8.23', 'Equation 8.24')

figure(2)
subplot(211)
set(gca, 'FontSize', 14);
plot(1000*t, Err1)
grid
ylabel('Error in \omega(t) (rad/s)')
axis([0 max(1000*t1) 1.1*min(Err1) 1.1*max(Err1)])

% Motor 2 analysis
clear all

% Parameters
K = 0.42;                        % N-m/A
Kb = 0.42;                       % V/rad/s
R = 4.9;                         % Ohms
L = 11.2e-03;                    % H
J = 3.883e-04;                   % kg-m^2
br = 0.03/1000*60/(2*pi);        % N-m-s
E0 =35;                          % 35 Volt step input

% Calculated parameters
tau_elec = L/R;
tau_mech = J*R/(K*Kb);
zeta = (br*L+J*R)/(2*sqrt(J*L*(br*R+K*Kb)));
w_ss = E0/Kb;
```

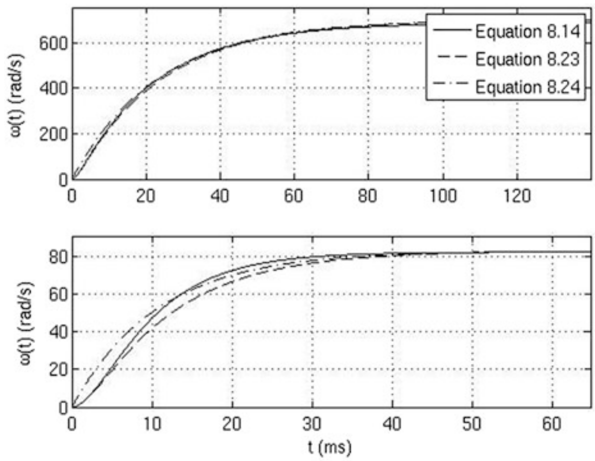
```
% Display calculated results
fprintf('\n \nMOTOR 2\n')
fprintf('———————\n')

fprintf('The electrical time constant of the motor is %4.3 g ms.
\n', 1000*tau_elec)
fprintf('The mechanical time constant of the motor is %4.3 g ms.
\n', 1000*tau_mech)
fprintf('The steady state speed of the motor is %4.3 g rad/s. \n',
w_ss)

% System transfer functions
num1 = [K];
den1 = [J*L (J*R+br*L) br*R+K*Kb];
sys1 = tf(num1, den1);

num2 = [1/Kb];
den2 = conv([tau_mech 1], [tau_elec 1]);
sys2 = tf(num2, den2);

num3 = [1/Kb];
den3 = [tau_mech 1];
sys3 = tf(num3, den3);

% System responses (use common time vector)
t = [0:tau_mech/1000:6*tau_mech];
[w1_u, t1] = step(sys1, t);
w1 = w1_u*E0;
[w2_u, t2] = step(sys2, t);
w2 = w2_u*E0;
[w3_u, t3] = step(sys3, t);
w3 = w3_u*E0;

% Calculate percent error
Err2 = w3 - w1;

% Plot the results
figure(1)
subplot(212)
set(gca, 'FontSize', 14);
plot(1000*t1, w1, 'k-', 1000*t2, w2, 'k-', 1000*t3, w3, 'k-.')
grid
xlabel('t (ms)')
ylabel('\omega(t) (rad/s)')
axis([0 max(1000*t1) 0 1.1*max(w1)])

figure(2)
subplot(212)
set(gca, 'FontSize', 14);
```

```
plot(1000*t, Err2)
grid
xlabel('t (ms)')
ylabel('Error in \omega(t) (rad/s)')
axis([0 max(1000*t1) 1.1*min(Err2) 1.1*max(Err2)])
```

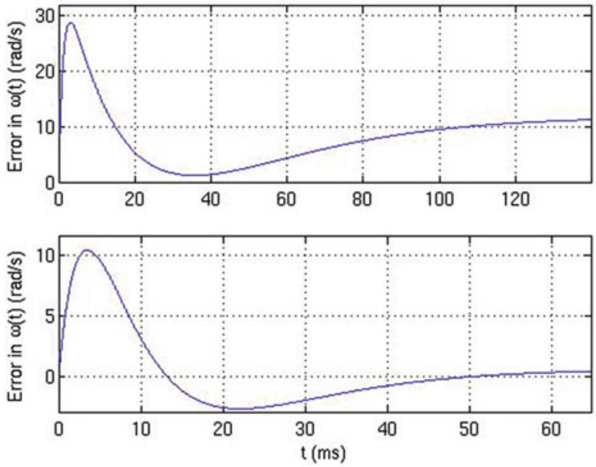
When the code is executed, the following results are obtained.

```
MOTOR 1
______
The electrical time constant of the motor is 1.2 ms.
The mechanical time constant of the motor is 23.3 ms.
The steady state speed of the motor is 700 rad/s.

MOTOR 2
______
The electrical time constant of the motor is 2.29 ms.
The mechanical time constant of the motor is 10.8 ms.
The steady state speed of the motor is 83.3 rad/s.
```

The angular velocity as a function of time is displayed.



Visual comparison indicates that Eqs. (8.23) and (8.24) better approximate the behavior of the more detailed model Eq. (8.14) for the first motor (top panel). This is expected since the ratio of the mechanical time constant to the electrical time constant for Motor 1 is much larger (nearly 20:1) than for Motor 2 (less than 5:1). The errors between the two models are also displayed.



Both motors show that the error increases rapidly while the inductor and resistor are coming to equilibrium and then decreases. The error actually becomes negative for Motor 2 when the curves cross one another. The error damps to a final steady state value. Numerically, Motor 1 has larger maximum speed error, but when this is normalized to the steady state speed, the maximum percentage error for Motor 1 is approximately 4%, while for Motor 2 it is approximately 12.5%.

The suitability of a model depends on the application. If an application demands precise speed or position control, the speed or position would be measured and closed loop control would be implemented to obtain a *DC servomotor* (see Chap. 10). The level of approximation suitable for designing the servomotor depends on the desired accuracy and robustness of the final motor system.

8.2.4 Steady State Motor Behavior in Response to External Load

In many designs, transmission elements are used so that the external mechanical loads seen by a motor are minimized. In this case, the free-running behavior of the motor described in the previous section is relevant. However, such designs are conservative and limit the accelerations and velocities that can be achieved. When the external loads are not negligible, their effect on motor behavior must be considered. Figure 8.4 illustrates the motor behavior in the presence of an externally applied moment, $M_l(t)$, that opposes the motor moment, $M(t)$. In this case, Eq. (8.9) is modified to be:

$$(Js + b_r)\Omega(s) = M(s) - M_l(s). \tag{8.25}$$

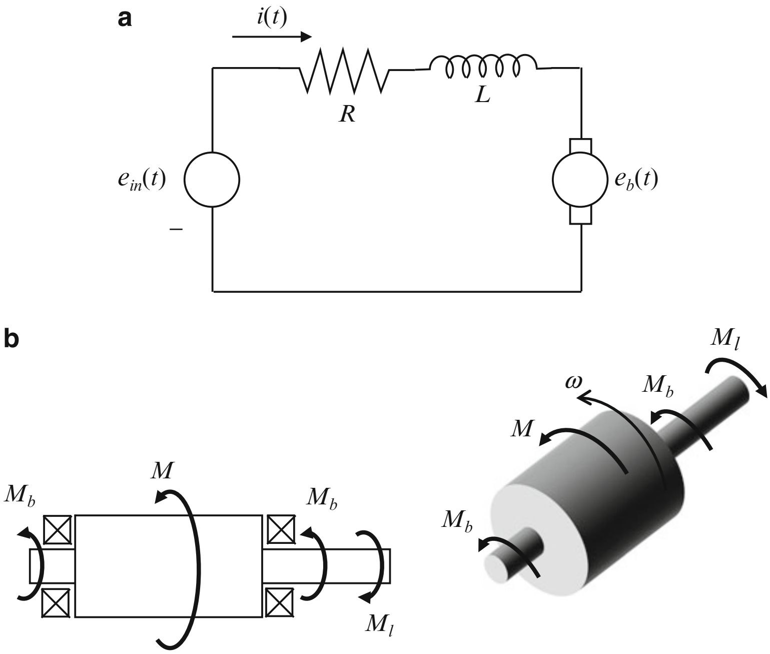


Fig. 8.4 (**a**) Motor circuit and (**b**) rotor dynamics model with an applied external load

By combining Eq. (8.25) with Eqs. (8.6), (8.10), and (8.11), the motor speed in the Laplace domain is found. Note that now both the voltage and the external load moment are inputs in Eq. (8.26).

$$\Omega(s) = \left[\frac{K}{JLs^2 + (JR + Lb_r)s + b_rR + KK_b}\right]E_{in}(s) - \left[\frac{R + Ls}{JLs^2 + (JR + Lb_r)s + b_rR + KK_b}\right]M_l(s) \tag{8.26}$$

The first expression in brackets is the transfer function from motor voltage to speed which we derived previously. The second expression in brackets is the transfer function from the externally applied load to speed.

The ability of the motor to deliver energy to the external environment is particularly important. Suppose that a step voltage input, $E_{in}(s) = E_0/s$, is applied to the motor and simultaneously, a step input in external load is applied, $M_l(s) = M_0/s$. In response, the motor accelerates to a steady state speed less than the unloaded steady state speed determined from Eqs. (8.17) and (8.18).

Assuming step inputs in Eq. (8.26) and applying the final value theorem to determine the steady state speed, we obtain:

$$\omega_{ss_l} = \frac{K}{b_r R + KK_b} E_0 - \frac{R}{b_r R + KK_b} M_0. \tag{8.27}$$

For a given voltage input, the loaded steady state speed decreases with increased load moment until it reaches zero. At this point, the motor stalls. Setting the steady state speed equal to zero in Eq. (8.27) gives the moment required to stall the motor.

$$M_{stall} = K\frac{E_0}{R} \tag{8.28}$$

The term E_0/R is the steady state current flowing in the motor with no back-emf (because the motor is not rotating). The product of this current and the motor constant gives the stall moment, often called the stall torque. When stalled, the electrical power input, $E_0 I = E_0 \frac{E_0}{R} = \frac{E_0{}^2}{R}$, is dissipated in the motor windings as heat and no power is delivered to the external load. Equation (8.27) is often rearranged to give the externally delivered moment as a function of speed.

$$M_0 = \frac{K}{R} E_0 - \left(\frac{b_r + KK_b}{R}\right) \omega_{ss_l} \tag{8.29}$$

Plots of M_0 as a function of the steady state speed are called *motor torque curves*. The mechanical power delivered to the external load in steady state is the product of the moment and the angular speed.

$$P = \frac{K}{R} E_0 \omega_{ss_l} - \left(\frac{b_r + KK_b}{R}\right) \omega_{ss_l}^2 \tag{8.30}$$

Plots of the Eq. (8.30) power versus steady state speed are called *motor power curves*. The maximum power is determined from:

$$P_{max} = \frac{M_{stall}\omega_{ss_l}}{2 \quad 2} \cong \frac{1}{4}\frac{K}{K_b}\frac{E_0^2}{R}. \tag{8.31}$$

Equation (8.31) can be derived by: (1) differentiating Eq. (8.30) with respect to rotation speed, setting the result equal to zero, and solving for the speed at maximum power; and (2) multiplying this speed by half the stall moment. Alternately, the symmetry of the motor power curve can be considered as discussed in Example 8.4. Note that since the ratio K/K_b is close to 1 for most motors, the maximum power delivered is approximately 25% of the electrical power dissipated as heat at stall.

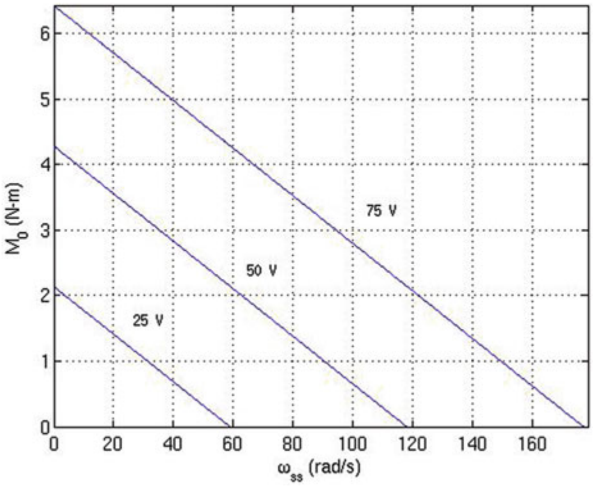
Example 8.4 Consider a motor with the Example 8.1 parameters: $K = 0.42$ N-m/A, $K_b = 0.42$ V-s, $R = 4.9\ \Omega$, $L = 11.2$ mH, $J = 3.883 \times 10^{-4}$ kg-m^2, $b_r = 0.03$ N-m/krpm (2.865×10^{-4} N-m-s). Find and plot the motor power and torque curves for three different step input voltages, E_0: 25 V, 50 V, and 75 V.

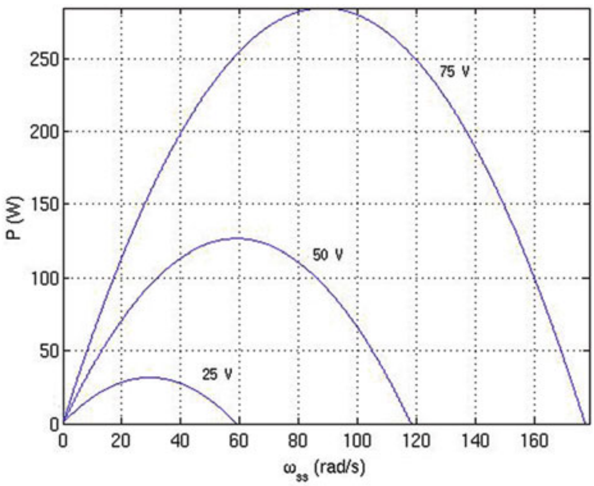
Solution The MATLAB® code is provided.

```
clear all
clc
close all

% Parameters
K = 0.42;                        % N-m/A
Kb = 0.42;                       % V/rad/s
R = 4.9;                         % Ohms
L = 11.2e-03;                    % H
J = 3.883e-04;                   % kg-m^2
br = 0.03/1000*60/(2*pi);        % N-m-s
E0 = [25 50 75];                 % step input voltage

for cnt = 1:length(E0)
  V0 = E0(cnt);
  % Angular speed vector
  w_max = V0/Kb;
  w_ss = [0:w_max/1000:w_max]; % zero to maximum speed, rad/s

  % Calculate torque as a function of speed
  M0 = K*V0/R - ((br*R+K*Kb)/R)*w_ss;

  % Plot the torque curve
  figure(1)
  set(gca, 'FontSize', 14);
  plot(w_ss, M0)
  grid
  xlabel('\omega_{ss} (rad/s)')
  ylabel('M_0 (N-m)')
  axis([0 max(w_ss) 0 max(Tl)])
  hold on

  % Calculate power as a function of speed
  P = M0.*w_ss;

  % Plot the power curve
  figure(2)
  set(gca, 'FontSize', 14);
  plot(w_ss, P)
  grid
  xlabel('\omega_{ss} (rad/s)')
  ylabel('P (W)')
```

```
    axis([0 max(w_ss) 0 max(P)]);
    hold on

    % Calculate the maximum motor power output
    P_max = 1/4*(K*V0^2/(Kb*R));
    fprintf('For input voltage %2.0f V, the maximum power is %3.0f W
    \n', V0, P_max);
end
```

The results are provided.





```
For input voltage 25 V, the maximum power is 32 W
For input voltage 50 V, the maximum power is 128 W
For input voltage 75 V, the maximum power is 287 W
```

Choosing the 75 V curves as an example, the maximum steady angular speed of the motor is approximately 179 rad/s. At this speed, the motor cannot deliver any moment (without slowing down) and, therefore, no external mechanical power is delivered. The moment at stall is 6.42 N-m, but the angular speed is zero and, again, no external mechanical power can be delivered. Since the power curve is symmetric, the maximum power is delivered at half the maximum speed, 89.5 rad/s. At this speed, the moment is 3.21 N-m (this is half the maximum value since the torque curve is linear) and the power delivered is approximately 287 W.

8.3 Electric Generators

An electric generator converts mechanical energy into electrical energy. The mechanical energy can come from many sources such as an internal combustion engine, a gas turbine, a steam turbine, or a water turbine. There are many dynamic phenomena associated with electric generators. Since the input moment is often not smooth, such as that obtained from a reciprocating internal combustion engine, the response of the generator to a time varying input moment is critical to predict the output voltage and power, as well as the load on the generator components and their corresponding lifetime.

For large power generators that feed the power grid, the load may vary dynamically. In this case, the generator must be equipped to handle the switching to other generators on the grid and/or changes in power usage on the grid. Start-up of the grid after a partial or complete power outage is an example. A related phenomenon, often called generator run-away, occurs when the load on the generator is instantaneously reduced to zero due to a power outage. When this occurs, all electrical damping in the system is instantaneously lost and only mechanical damping in the generator limits its angular velocity. If the situation is not immediately remedied by additional breaking or rapid shut down, the large angular speed damages the generator and associated components. To determine the dynamic response of the generator, it is first necessary to know its transfer function.

A generator is similar to an electric motor as shown in Fig. 8.5. The rotor is driven by an input moment $M_{in}(t)$ on the rotor and the rotation generates a voltage, $e_b(t)$ (analogous to the back-emf), which causes a current in the circuit. Power is then delivered to a load, modeled as a resistance R_L with a voltage drop, $e_0(t)$.

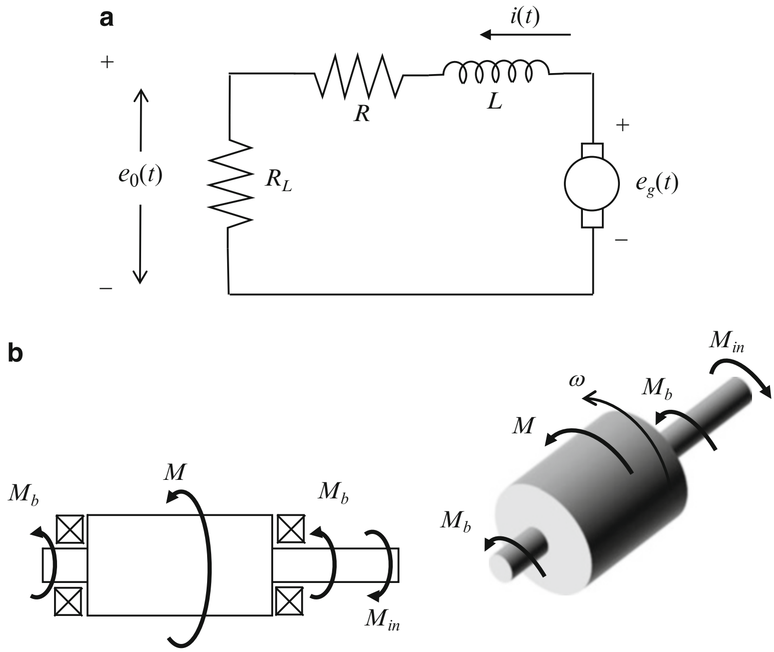


Fig. 8.5 (**a**) Electric generator circuit and (**b**) rotor mechanics

The generator transfer function relates the output voltage to the input moment, $E_0(s)/M_{in}(s)$. Applying Kirchhoff's voltage law to the circuit, we obtain:

$$E_g(s) - I(s)R - I(s)Ls - I(s)R_L = 0. \tag{8.32}$$

The output voltage, or *on-load voltage* of the generator, is the voltage drop across the load: $E_0(s) = I(s)R_L$. The generator *open-circuit voltage*, $E_g(s)$, is proportional to the angular rotation rate:

$$E_g(s) = K_g\Omega(s), \tag{8.33}$$

where K_g is the *generator constant*. Applying Newton's second law to the rotor and calculating the Laplace transform gives:

$$(Js + b_r)\Omega(s) = M_{in}(s) - M(s). \tag{8.34}$$

The generator moment, $M(s)$, opposes the rotation direction and is given by Eq. (8.35), where K is the *generator torque constant*.

$$M(s) = KI(s) \tag{8.35}$$

Equations (8.32)–(8.35) can be rearranged algebraically to find the transfer function. Combining Eqs. (8.33) with (8.32), we obtain the current:

$$I(s) = \left(\frac{K_g}{Ls + R + R_L}\right)\Omega(s). \tag{8.36}$$

Next, Eqs. (8.35) and (8.36) are combined with Eq. (8.34) to obtain the transfer function relating the angular speed and input moment.

$$\frac{\Omega(s)}{M_{in}(s)} = \frac{Ls + R + R_L}{JLs^2 + (b_r L + J(R + R_L))s + b_r(R + R_L) + KK_g} \tag{8.37}$$

From Eq. (8.37), the current in the generator circuit is written as a function of input moment.

$$I(s) = \left(\frac{K_g}{JLs^2 + (b_r L + J(R + R_L))s + b_r(R + R_L) + KK_g}\right)M_{in}(s) \tag{8.38}$$

Using the current, the voltage drop across the load and, therefore, the final transfer function is determined.

$$\frac{E_0(s)}{M_{in}(s)} = \frac{K_g R_L}{JLs^2 + (b_r L + J(R + R_L))s + b_r(R + R_L) + KK_g} \tag{8.39}$$

Example 8.5 Suppose we have a small generator with the following parameters: $K = 0.5$ N-m/A, $K_g = 0.5$ V-s, $R = 12\ \Omega$, $L = 20$ mH, $J = 2.5 \times 10^{-4}$ kg-m^2, $b_r = 5 \times 10^{-5}$ N-m-s. Suppose it drives a small light bulb with a resistance of 20 Ω and the input torque through a gear train is 0.5 N-m. Using MATLAB® find and plot the time-dependent angular speed of the generator as well as the current and voltage applied to the load (bulb). Identify the characteristics of the curves and the time constants. Finally, determine the steady state voltage and power flowing to the bulb.

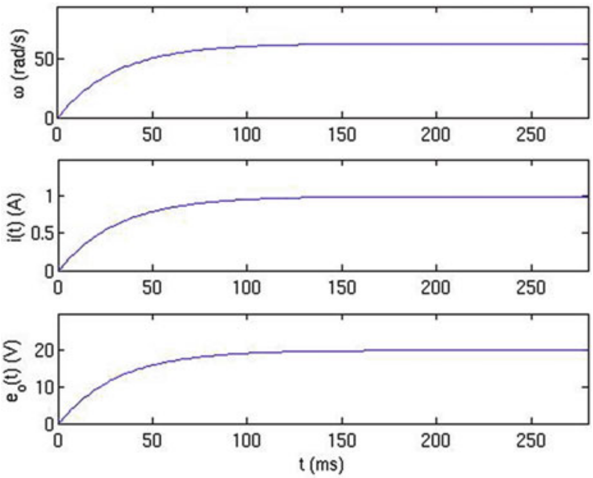
Solution We use the following MATLAB® code to determine the system transfer functions and plot the responses.

```
clear all
clc
close all

% Parameters
K = 0.5;        % N-m/A
Kg = 0.5;       % V/rad/s
R = 12.0;       % Ohms
```

```
L = 20e-03;   % H
J = 2.5e-04;  % kg-m^2
br = 5e-05;   % N-m-s
RL = 20;      % Ohms
M0 = 0.5;     % N-m

% Transfer Functions
num1 = [L R+RL];
den1 = [J*L br*L+J*(R+RL) br*(R+RL)+K*Kg];
num2 = [Kg];
num3 = [Kg*RL];
den2 = den1;
den3 = den1;
sys1 = tf(num1, den1);
sys2 = tf(num2, den2);
sys3 = tf(num3, den3);

% Find the responses
[w_u, t] = step(sys1);
w = w_u*M0;
[i_u, t] = step(sys2, t);
i = i_u*M0;
[eo_u, t] = step(sys3, t);
eo = eo_u*M0;

% Plot the responses
figure(1)
subplot(311)
plot(t*1000, w)
ylabel('\omega (rad/s)')
axis([0 max(1000*t) 0 1.5*max(w)]);
subplot(312)

plot(t*1000, i)
ylabel('i(t) (A)')
axis([0 max(1000*t) 0 1.5*max(i)]);

subplot(313)
plot(t*1000, eo)
xlabel('t (ms)')
ylabel('e_o(t) (V)')
axis([0 max(1000*t) 0 1.5*max(eo)]);

% Steady state voltage and current
eo_ss = M0*Kg*RL/(br*(R+RL)+K*Kg);
iss = eo_ss/RL;
Pss = eo_ss*iss;
```

```
fprintf('The steady state voltage is %4.1f V\n',eo_ss)
fprintf('The steady state current is %4.1f A\n',iss)
fprintf('The steady state power is %4.1f W\n',Pss)
```

The results are displayed in the figure.



The steady state voltage for a step input moment $\frac{M_0}{s}$ is determined by applying the final value theorem.

$$\lim_{t\to\infty}(e_0(t)) = e_{0_{ss}} = \lim_{s\to 0}(sE_0(s)) = \frac{M_0K_gR_L}{b_r(R+R_L)+KK_g} \tag{8.40}$$

The steady state current is then the steady state voltage divided by the load resistance

$$i_{ss} = \frac{M_0K_g}{b_r(R+R_L)+KK_g} \tag{8.41}$$

The MATLAB® calculations match the plots. We see we have a 20 W light bulb.

```
The steady state voltage is 19.9 V
The steady state current is 1.0 A
The steady state power is 19.7 W
```

8.4 Other Electromechanical Devices: Acoustic Speaker/Voice Coil

A speaker is another common electromechanical device that converts electrical energy into mechanical energy. In this case, electrical energy is transformed into mechanical motion of the air/sound waves. A speaker cross-section is shown in Fig. 8.6 and a simplified electromechanical schematic is presented in Fig. 8.7. A speaker is an electromechanical system where a force is applied to a moving mass by a voice coil actuator. Current passes through coils of wire in a magnetic field that is typically generated by permanent magnets. This produces a Lorentz force on the coil. With the current flowing in the direction indicated in Fig. 8.7 and the magnetic field lines flowing from the north (N) to south (S) poles of the permanent magnets, a force to the right will be generated on the coil. The cone of the speaker moves, which then generates a sound wave in the surrounding air. The mechanical portion of the speaker is a spring-mass-damper. The mass, m, is the sum of the moving mass of the speaker, m_s, and the equivalent moving mass of the air, m_{air}, being driven to create acoustic waves. Similarly, the total stiffness of

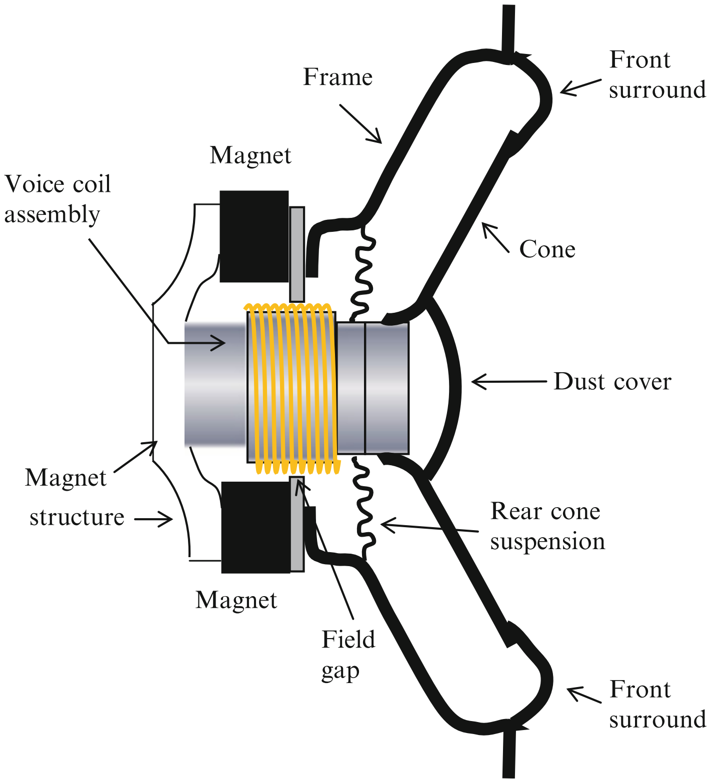


Fig. 8.6 Speaker cross-section

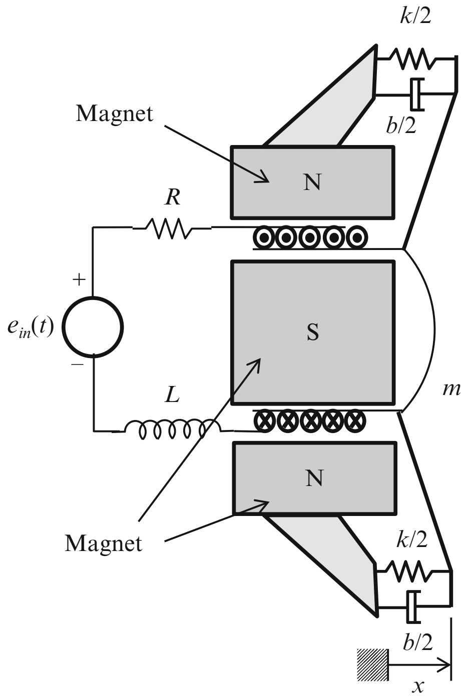


Fig. 8.7 Simplified electromechanical schematic for a speaker

the speaker, k, is the stiffness of the speaker suspension cone, k_{susp}, and the equivalent stiffness of the air being moved, k_{air}. The system damping, b, is the sum of the suspension cone damping, b_{susp}, and the damping associated with the fluid motion of the air, b_{air}.

The input to the speaker is the voltage $e_{in}(t)$ and the output is the speaker motion $x(t)$. The transfer function, $X(s)/E_{in}(s)$, is derived by considering the mechanical and electrical behavior separately. The mechanics of the speaker simplify to a spring–mass–damper as shown in Fig. 8.8a. The electrical portion of the speaker is a circuit containing the coil inductance, L, the coil resistance, R, and the back-emf of the moving coil as displayed in Fig. 8.8b. Examining Fig. 8.8a first, the force on the moving mass generated by the voice coil is given by:

$$F_s(t) = lBi(t), \tag{8.41}$$

where l is the length of the wire and B is the magnetic field strength. Applying Newton's law to the speaker's mechanical model, we obtain:

$$m\ddot{x} + b\dot{x} + kx = F_s(t). \tag{8.42}$$

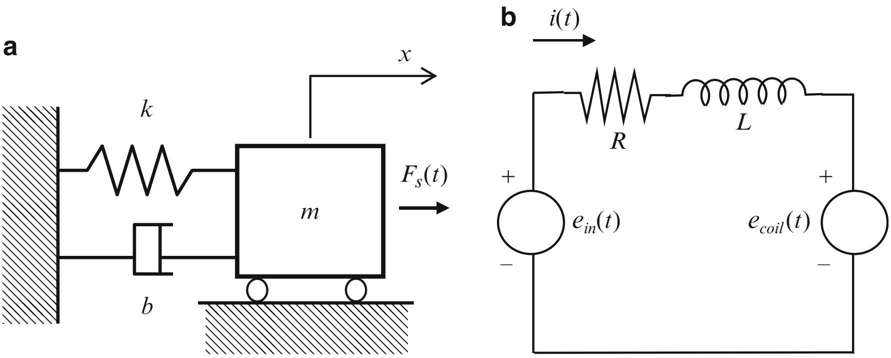


Fig. 8.8 Speaker (**a**) mechanical and (**b**) electrical models

Next, examining the electrical circuit in Fig. 8.8b, the time domain equation from Kirchhoff's voltage law is:

$$e_{in}(t) - i(t)R - L\frac{di}{dt} - e_{coil}(t) = 0. \tag{8.43}$$

There is a voltage drop across the coil (back-emf) that is proportional to the coil velocity.

$$e_{coil}(t) = Bl\dot{x} \tag{8.44}$$

To obtain the speaker's transfer function, we calculate the Laplace transform of Eqs. (8.41)–(8.44) to obtain Eqs. (8.45)–(8.48).

$$\left(ms^2 + bs + k\right)X(s) = F_s(s) \tag{8.45}$$

$$E_{in}(s) - I(s)R - LsI(s) - E_{coil}(s) = 0 \tag{8.46}$$

$$F_s(s) = lBI(s) \tag{8.47}$$

$$E_{coil}(s) = BlsX(s) \tag{8.48}$$

The current is found by combining Eq. (8.48) with Eq. (8.46).

$$I(s) = \frac{BlsX(s) - E_{in}(s)}{Ls + R} \tag{8.49}$$

Combining Eq. (8.47) with Eq. (8.45) and then using Eq. (8.49) to replace the current $I(s)$ gives the final transfer function relating input voltage to displacement.

$$\frac{X(s)}{E_{in}(s)} = \frac{Bl}{mLs^3 + (mR + bL)s^2 + \left(kL + bR + B^2l^2\right)s + kR} \tag{8.50}$$

Note that the pattern of solution for this problem is the same as that for the permanent magnet DC motor. Equation (8.45) describes the mechanical behavior of the speaker moving element. Equation (8.46) describes the electric behavior of the system. Equations (8.47) and (8.48) describe the electromechanical coupling. Combining the four equations algebraically gives the transfer function between the speaker movement and the input voltage. The final system is third order and, therefore, the characteristic equation has three roots. As discussed previously, real roots represent exponential behavior and complex roots represent damped oscillations. The particular behavior of a given speaker depends on the values of the speaker parameters. This is examined using Example 8.6.

Example 8.6 Consider a speaker that can be modeled by the transfer function provided in Eq. (8.50). The mass of the moving speaker element, m_s, is 30 g and the equivalent mass of the moving air, m_{air}, is 0.675 g. The stiffness of the speaker suspension, k_{susp}, is 850 N/m and the equivalent stiffness of the air, k_{air}, is 2500 N/m. The system damping ratio is 23%. The electrical resistance of the speaker coils, R, is 7.5 Ω and the inductance, L, is 1.1 mH. The product of the magnetic field and the coil length, Bl, is 8.5 N/A. Calculate the roots of the characteristic equation for the speaker and determine the corresponding natural frequency and time constant. Find the steady displacement produced in the speaker for a 50 V input. Finally, use the MATLAB® command lsim to determine the response of the speaker to a square wave input of magnitude 50 V and periods of 100, 10, and 1 ms.

Solution The MATLAB® code is provided.

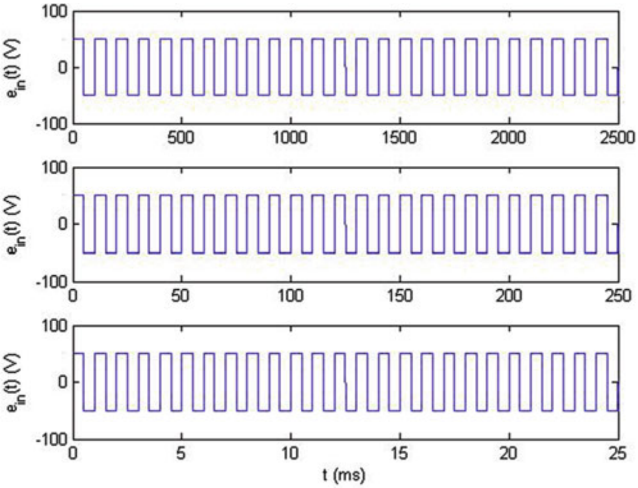
```
clear all
clc
close all

% Parameters
R = 7.5;             % Ohms
L = 1.1e-03;         % H
m = 30.675e-03;      % kg
k = 2500 + 850;      % N/m
wn = sqrt(k/m);      % rad/s
zeta = 0.23;
Bl = 8.5;               % N/A
b = 2*zeta*wn*m;        % N-s/m
E0 = 50;                % V
T = [100 10 1]*1e-03;   % s
```

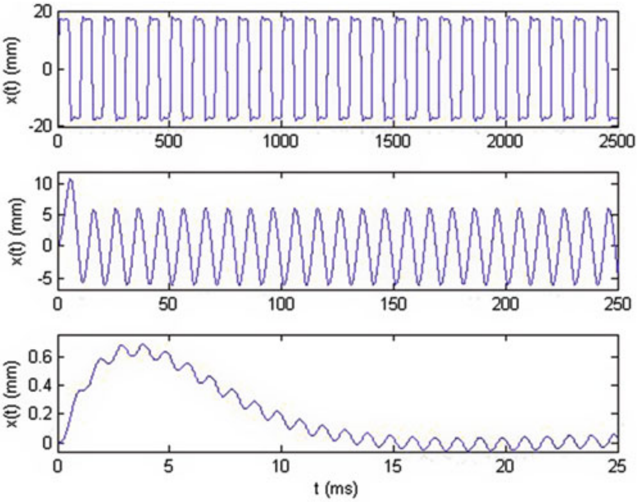
```
% Speaker model
num = [Bl];
den = [m*L m*R+b*L k*L+b*R+Bl^2 k*R];
sys = tf(num, den);
r = roots(den);
fprintf('The roots of the characteristic equation are:\n')
r

for cnt = 1:length(T)
 f = 1/T(cnt);                      % Frequency, Hz
 t = [0:T(cnt)/1000:25*T(cnt)];     % Time, s
 ein = E0*square(2*pi*f*t);         % Square wave (V)

 figure(1)
 subplot(3,1,cnt)
 plot(t*1000, ein)
 ylabel('e_{in}(t) (V)')
 axis([0 1000*max(t) -100 100])
 hold on

 % System response
 [x, t] = lsim(sys, ein, t);
 figure(2)
 subplot(3,1,cnt)
 plot(t*1000, x*1e03);
 ylabel('x(t)(mm)')
 axis([0 1000*max(t) 1.1*min(1e03*x) 1.1*max(1e03*x)])
 hold on

end
figure(1)
xlabel('t (ms)')
figure(2)
xlabel('t (ms)')
```

```
  The roots of the characteristic equation are:
r =

 1.0e+03 *
 -6.4807
 -0.2447 + 0.2345i
 -0.2447 - 0.2345i
```

The first root is real and therefore corresponds to an exponential response with a time constant of 0.15 ms; this is the electrical time constant of the system, L/R. The second pair of complex conjugate roots correspond to a damped sinusoid with frequency of 234.5 rad/s (37.3 Hz), a 26.8 ms period, and a time constant of 4.1 ms.

The inputs and corresponding responses are provided.





Since the period of the first wave is much longer than the two system time constants, it appears as a train of step responses, i.e., the speaker can track the input wave. The magnitude of the response is estimated from the final value theorem applied to the system for a step input E_0/s.

$$x_{ss} = \lim_{s \to 0} (sX(s)) = \frac{Bl}{k}\frac{E_0}{R} \tag{8.51}$$

The term E_0/R is the current when there is no back-emf (constant displacement). This current multiplied by Bl is the steady state force. The ratio of the force to the stiffness then gives the steady state displacement. Using Eq. (8.51) and a 50 V step input, the steady state displacement is 17 mm.

This matches the displacement in the first response. The second input has a period that is only about 2.5 times the dominant time constant of the system and, therefore, the system cannot track the square wave. It responds at the same frequency, but does not reach the same amplitude as the first wave and the phase of the response lags the input wave. The third response begins with an exponential increase and then decays back to a mean zero response with an amplitude that is much less than the second wave.

Rotary motors and speakers/voice coil systems are common electromechanical systems encountered by the mechanical engineer. Other systems of interest that can be solved using a similar approach to that presented here are sensors such as accelerometers, actuators such as piezoelectric drive elements, and many micro-electromechanical systems (MEMs) [1].

8.5 Summary

In this chapter, we analyzed electromechanical systems using the permanent magnet DC motor and voice coil as examples. The key points of the chapter are summarized.

- Electromechanical systems contain energy storage elements such as masses, rotary inertias, springs, and inductors.
- The order of an electromechanical system is equal to the number of independent energy storage elements: a permanent magnet DC motor is second order (J and L) and a speaker is third order (m, k, and L).
- The transfer function for an electromechanical system typically involves a coupled mechanical analysis (Newton's laws) and electric circuit analysis (Kirchhoff's voltage law); combining the equations from these analyses with constitutive equations for the particular system enable algebraic manipulation to determine the transfer function.
- An electromechanical system will have behavior associated with the electrical side and the mechanical side and often these behaviors can be separated. For example, the electric time constant, τ_{elec}, and mechanical time constant, τ_{mech}, can typically be identified.
- If the electrical and mechanical behaviors occur on different time scales, the system may be approximately separated into two systems or, as in the case of many electric motors, approximated with a first-order system.

Problems

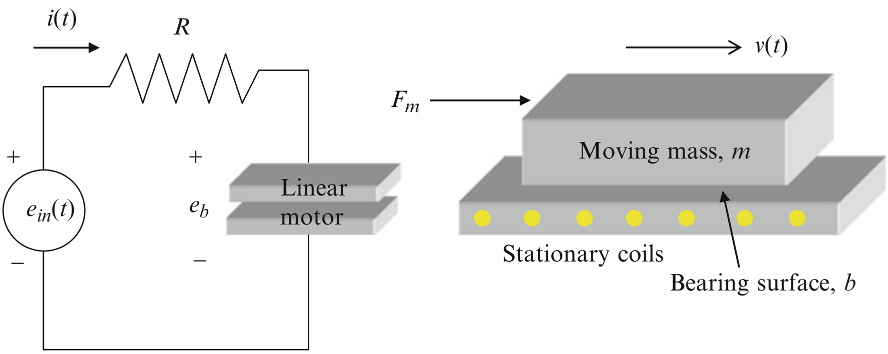
1. Consider a 122 W DC motor with the constants: $K = 0.11$ N-m/A, $K_b = 0.11$ V-s, $R = 0.8\ \Omega$, $L = 0.72$ mH, $J = 3.883 \times 10^{-4}$ kg-m^2, and $b_r = 0.03$ N-m/krpm. Find and plot the motor speed, $\omega(t)$, in response to a step voltage input, $e_{in}(t) = E_0 \cdot u(t)$, where $E_0 = 40$ V using MATLAB®. Determine the roots of the motor's characteristic equation and use them to describe the behavior observed in the step response. Compare the roots with the approximations of the electrical and mechanical time constants given by Eqs. (8.21) and (8.22). Finally, use the final value theorem to determine the steady state speed of the motor and compare it to the plot.

2. For the motor considered in problem 1, calculate and plot the power and torque curves using MATLAB® for input voltages of 10 V, 20 V, and 40 V. Display the curves on the same graph.

3. Consider a 250 W DC motor with the constants: $K = 0.06$ N-m/A, $K_b = 0.06$ V-s, $R = 0.14\ \Omega$, $L = 0.24$ mH, $J = 2.965 \times 10^{-4}$ kg-m^2, and $b_r = 0.02$ N-m/krpm. Find and plot the motor speed, $\omega(t)$, in response to a step voltage input, $e_{in}(t) = E_0 \cdot u(t)$, where $E_0 = 24$ V using MATLAB®. Determine the roots of the motor's characteristic equation and use them to describe the behavior observed in the step response. Compare the roots with the approximations of the electrical and mechanical time constants given by Eqs. (8.21) and (8.22). Finally, use the final value theorem to determine the steady state speed of the motor and compare it to the plot.

4. For the motor considered in problem 3, calculate and plot the power and torque curves using MATLAB® for input voltages of 6 V, 12 V, and 24 V. Display the curves on the same graph.

5. A linear motor is the same as a rotational DC motor except that its rotor and stator are "unwrapped" so that it produces a direct force and velocity rather than a torque and rotational velocity. The electrical and mechanical schematics of the motor are provided. You may neglect the motor's inductance, but you must consider its electrical resistance, R, and the mechanical damping, b, of the linear guide ways for an accurate model.



The back-emf is proportional to the table velocity with a proportionality constant, A (units are V-s/m).

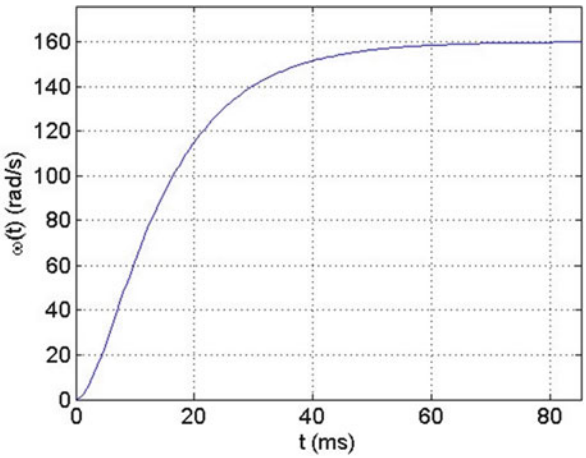
$$e_b(t) = Av(t)$$

The linear motor force is proportional to the motor current with a proportionality constant, B (units are N/A).

$$F_m(t) = Bi(t)$$

Determine the transfer function $\frac{V(s)}{E_{in}(s)}$ for the linear motor and write the expression for the time constant.

6. The output speed response, $\omega(t)$, of an electric motor to a step input of 80 V with no load torque applied is displayed in the figure.



   (a) Determine the motor constant, K.
   (b) If $J = 0.002$ kg-m^2 and $L = 6$ mH, estimate K_b and R for the motor.

7. A generator has the following parameters: $K = 2$ N-m/A, $K_g = 2$ V-s, $R = 200\ \Omega$, $L = 0.5$ H, $J = 5 \times 10^{-3}$ kg-m^2, and $b_r = 3 \times 10^{-4}$ N-m-s. Suppose it drives a small light bulb with a resistance of 1 kΩ and the input torque to the generator is 5 N-m · u(t). Using MATLAB®, find and plot the time-dependent angular speed, current, and voltage applied to the load. Determine the steady state voltage and power flowing to the load.

Reference

1. Nicolae Lobontiu (2010) System dynamics for engineering students: concepts and applications, 1st edn. Academic Press, ISBN-10: 0240811283, ISBN-13: 978-0240811284

Thermal Systems

9

9.1 Introduction

Thermal systems can be modeled using the techniques developed in the previous chapters for mechanical, electrical, and electromachanical systems. Because we are able to describe thermal systems using similar differential equations, the same mathematical solutions are applicable. Thermal systems are ubiquitous in mechanical engineering: the heating/cooling of electronics in a computer, temperature control in a house or building, the radiator in an automobile, and temperature control for a swimming pool, to name a few. There are direct electrical and mechanical analogies to the elements in a thermal system.

In this chapter, we will examine the behavior of the basic lumped parameter thermal elements: the *thermal mass/capacitance* and the *thermal damper/resistance*. To do so, we require some simplifying assumptions: (1) the details of heat transfer such as conduction and convective heat exchange can be approximated with a single parameter; and (2) material mass in a system is assumed to have uniform temperature. These assumptions may appear aggressive, but they enable the approximate description of many engineering systems, with sufficient accuracy for most cases. More detailed treatments, when necessary, can be found in a course on heat transfer.

After we have developed and understood the thermal mass and thermal resistance concepts, we will then add a basic measurement control loop that introduces a *thermal spring* into the system. We then show that this system has analogous behavior to a mechanical spring–mass–damper and an R–L–C circuit. All the concepts previously developed, including natural frequency, damping ratio, damped natural frequency, and time constant, are then directly applicable and have the same physical meaning. We will see that a thermal controller, e.g., on a heated swimming pool, can exhibit overdamped or underdamped behavior in the same way as a spring–mass–damper system. The time scales are usually very different, but the fundamental behavior is the same. The introduction of a control loop will be discussed in more detail in Chap. 10.

M.A. Davies, T.L. Schmitz, *System Dynamics for Mechanical Engineers*,
DOI 10.1007/978-1-4614-9293-1_9

9.2 Basic Lumped Parameter Thermal Elements

9.2.1 Thermal Mass

A lumped thermal element is a good approximation for a material at a uniform temperature. A mass of well-mixed fluid, like the water in a swimming pool or coffee in a cup, are good examples. However, the approximation is also applicable to large masses where the conduction within the material occurs much faster than any heat exchange with the environment, including large metallic masses such as engine blocks or machine tool structures can often be considered as bodies at a constant temperature.

To quantify a thermal mass, consider Fig. 9.1 which shows a mass, m, of material with specific heat capacity, c, and temperature, $T(t)$, which changes in time, but does not vary spatially within the material. The specific heat capacity of a material defines the temperature change, ΔT, in the mass given a heat input, Q. Assuming that the specific heat remains constant during the heat addition, the temperature change is:

$$\Delta T = \frac{Q}{mc} \tag{9.1}$$

The units of the specific heat are J/kg-K or, equivalently, J/kg-°C so that the quantity $(mc)^{-1}$ has units of °C/J. To gain a physical perspective, consider that the specific heat capacity of steel is approximately 460 J/kg-°C, while that of water

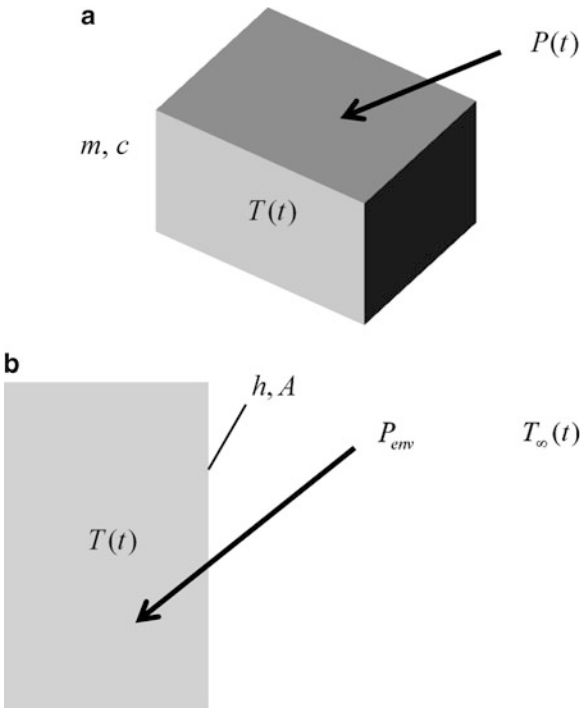


Fig. 9.1 Basic lumped parameter thermal elements: (**a**) thermal mass and (**b**) thermal resistance

is nearly ten times greater, 4180 J/kg-°C. Therefore, 460 J of energy are required to increase the temperature of 1 kg of steel by 1 °C, while 4180 J are required to increase the temperature of 1 kg of water by the same amount. Water offers a greater thermal mass than steel.

To describe the dynamic change in temperature of a thermal mass, we can imagine adding small amounts of energy, ΔQ, over short time intervals, Δt. In the limit, as the time interval becomes infinitesimally small, Eq. (9.1) becomes a differential equation, where the input power is described by $P(t) = \frac{\Delta Q}{\Delta t} = \dot{Q}$.

$$\dot{T}(t) = \frac{1}{mc}P(t) \tag{9.2}$$

As Eq. (9.2) shows, the rate of increase in temperature is proportional to the power input. Note that a sign convention has been assumed: heat flow is positive when directed into the thermal mass. Equation (9.2) is the basis for bulk heat transfer. If the temperature is considered to be analogous to the velocity of a moving mass, Eq. (9.2) can be rewritten as $P(t) = mc\dot{T}(t)$. This is similar to Newton's second law, where $P(t)$ is the force and the thermal mass, mc, is the inertial mass, m. Similarly, if $T(t)$ is the equivalent of voltage, then $P(t)$ is the current and mc is the capacitance, C. Physically, the thermal mass stores energy as heat just as:

- a capacitor stores energy as an electric field
- an inertial mass stores energy as velocity.

9.2.2 Thermal Resistance: Heat Exchange with the Environment

Consider a thermal mass with temperature $T(t)$ that is exposed to an environment at a different temperature $T_\infty(t)$, where the ∞ subscript indicates a temperature far from the mass; see Fig. 9.1b. Due to the temperature difference, the thermal mass will exchange heat with the environment. The heat exchange is often quite complex involving thermal gradients, conduction, and convection, or heat exchange by material transport. However, in many cases the entire exchange can be lumped in a single coefficient, h, the *heat transfer coefficient*. It is then reasonable to assume that the power is linearly proportional to the temperature difference between the body and its environment:

$$P_{env} = hA(T_\infty(t) - T(t)), \tag{9.3}$$

where A is the exposed surface area of the thermal mass. The units of h are W/m^2-°C and, therefore, the units of the product hA are W/°C. Again, making the analogy of temperature to voltage or velocity and heat flux to current or force, we see that Eq. (9.3) describes a resistor with a voltage drop across it or a damper with a velocity difference between its two ends. Subsequently, hA is analogous to damping and $(hA)^{-1}$ is analogous to resistance. When hA is large, the heat flux is large for a given temperature difference; when the damping coefficient b is large, the force is large for a given velocity difference. Similarly greater values of $(hA)^{-1}$ (smaller hA) will produce smaller heat flow for a given temperature difference and thus $(hA)^{-1}$, in units of °C/W, is often called the thermal resistance of an interface or element. Multiple interfaces arranged in series add as resistors in series following the electrical analogy.

9.3 Thermal Mass Subject to a Constant Heat Input

A common situation is the constant heating of a thermal mass that is insulated from its environment; see Fig. 9.2. The thermal mass does not have to be fully insulated for this to be the case. If the input heat flux is so great that heat loss can be ignored, the situation is similar. Examples include the initial heating of a bowl of water in a microwave and the induction heating of a mass of metal. To model this situation, we consider a step power input of magnitude P_0 to a thermal mass:

$$\dot{T}(t) = \frac{1}{mc} P_0 \cdot u(t). \tag{9.4}$$

Equation (9.4) describes a situation in which the time rate of change of temperature is constant; we observe this physically as a linear change in temperature with time.

To solve Eq. (9.4), we apply the Laplace transform to obtain:

$$T(s) = \frac{P_0}{mc} \frac{1}{s^2} + T_0 \frac{1}{s} \tag{9.5}$$

where T_0 is the initial material temperature. Computing the inverse Laplace transform of Eq. (9.5) gives the temperature as a function of time:

$$T(t) = \frac{P_0}{mc} t + T_0. \tag{9.6}$$

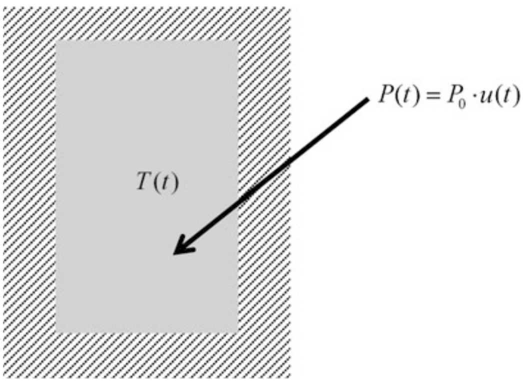


Fig. 9.2 An insulated thermal mass subjected to a constant heat input

This shows that the temperature is initially T_0 and changes linearly with time at a rate of $\frac{P_0}{mc}$. If the power input P_0 is positive, the temperature increases. If it is negative (energy is removed from the body), the temperature decreases. The relative rate of increase/decrease is scaled by the thermal mass, mc. A body with a greater thermal mass tends to be less sensitive to heat exchange. One function of a large thermal mass, such as a large cast iron stove, is to maintain a constant temperature in the presence of thermal disturbances. This is similar to the role of a capacitive surge protector for smoothing voltage fluctuations.

Example 9.1 An 80 mm diameter, 85 mm tall cup is filled with water. The cup is placed in a 1100 W microwave oven which produces a heat input to the water of approximately 400 W. Assuming constant heating and no heat exchange with the environment, determine the temperature as a function of time $T(t)$ and find the time required for the temperature of the water to rise from room temperature, 20 °C, to the boiling temperature of 100 °C. Assume the water has a density of 1000 kg/m^3 and a specific heat of 4183 J/kg-°C. Use MATLAB® to plot $T(t)$ over the time interval.

Solution

Step 1: Calculate the volume, mass, and thermal mass of the material. The cylindrical volume is:

$$V = \pi R^2 H = 4.273 \times 10^{-4}\,\mathrm{m}^3,$$

where R is the radius and H is the height. The mass is the product of the density and volume: 0.427 kg. The thermal mass mc is 1787 J/°C.

Step 2: Determine the rate of heating, P_0/mc. Substitution yields 0.224 °C/s. Therefore, it should take 357.4 s to raise the temperature by 80 °C using $\frac{\Delta T}{\Delta t} = \frac{P_0}{mc}$.

Step 3: Determine the heating time using Eq. (9.6). Denoting the final temperature, T_f, and the time required to reach this temperature, t_f, we obtain:

$$t_f = \frac{mc}{P_0}\left(T_f - T_0\right).$$

Using the problem parameters, we obtain a time of 357.4 s, or approximately 6 min. This matches the previous calculation and is reasonable based on experience.

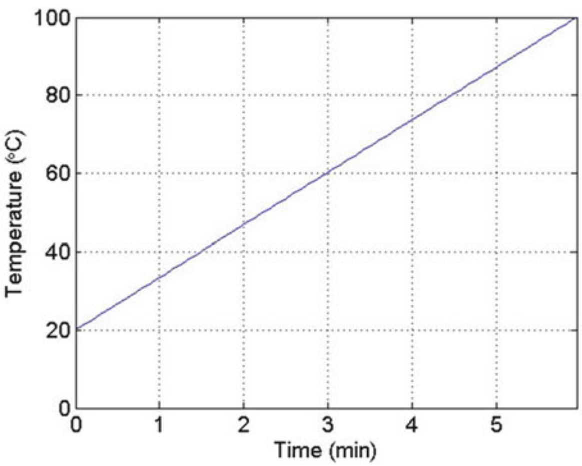
Step 4: Write the MATLAB® code to solve the problem and display the temperature as a function of time.

```
clear all
clc
close all

% Parameters
P0 = 400;        % power input, W
D = 80e-03;      % diameter, m
```

```
H = 85e-03;      % height, m
c = 4183;        % specific heat, J/kg-C
rho = 1000;      % density, kg/m^3
T0 = 20;         % initial temperature, C
Tf = 100;        % final temperature, C

% Calculated parameters
R = D/2;
V = pi*R^2*H;
m = rho*V;
tf = m*c/P0*(Tf - T0);

% Display results
fprintf('The time required for the %5.2f g of water to reach
100 degrees Celsius is %5.2f minutes. \n', 1000*m, tf/60);

% Time vector and temperature versus time
t = [0:tf/100:tf];    % s
T = P0/(m*c)*t + T0;  % C

% Plot the temperature versus time
figure(1)
plot(t/60, T)
grid
xlabel('Time (min)')
ylabel('Temperature (\circC)')
axis([0 max(t/60) 0 max(T)])
```

The results are provided.

```
The time required for the 427.26 grams of water to reach 100 degrees
Celsius is 5.96 minutes.
```



Example 9.2 Radio frequency (RF) induction heating is an electromagnetic process where an oscillating magnetic field generated by a coil of wire induces eddy currents in a metal. The eddy currents are dissipated as heat due to the metal's resistance. Induction heating is often used when heat treating metals. The process is so fast that it is reasonable to assume that the heat loss to the environment during heating can be neglected. Steel bars measuring 2 cm in diameter and 6 cm in length are RF induction heated from room temperature, 20 °C, to 800 °C. The steel has a density of 7800 kg/m^3 and a specific heat of 486 J/kg-°C. The input power from the heater is 15 kW. Plot $T(t)$ over a time interval of 4.5 s. Use the MATLAB® `find` command to determine the time at which the bar temperature reaches 800 °C.

Step 1: Calculate the volume, mass, and thermal mass of the material. The cylindrical volume is:

$$V = \pi R^2 H = 1.885 \times 10^{-5}\,\mathrm{m}^3.$$

The corresponding mass is 0.147 kg and the thermal mass, mc, is 71.5 J/°C.

Step 2: Determine the rate of heating, P_0/mc. Substitution gives a heating rate of 209.9 °C/s. Therefore, it requires only 3.72 s to heat the steel by 780 °C.

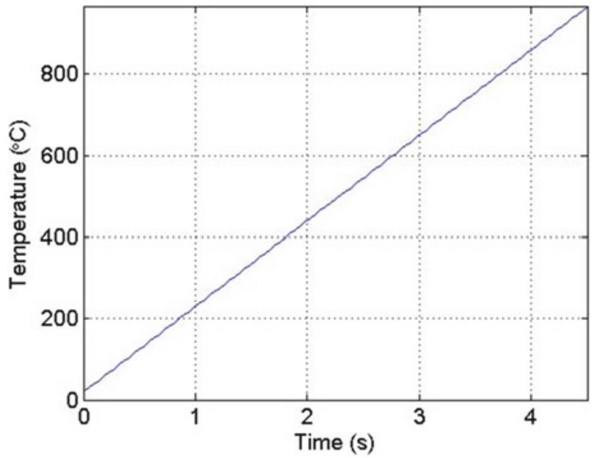
Step 3: The MATLAB® code used to solve the problem and display the temperature as a function of time is provided. The `find` command is applied to determine the heating time and display it.

```
clear all
clc
close all

% Parameters
P0 = 15000;   % power input, W
D = 2e-02;    % diameter, m
H = 6e-02;    % height, m
c = 486;      % specific heat, J/kg-C
rho = 7800;   % density, kg/m^3
T0 = 20;      % initial temperature, C
Tf = 800;     % final temperature, C
tf = 4.5;     % final time, s

% Calculated parameters
R = D/2;
V = pi*R^2*H;
m = rho*V;

% Time vector and temperature versus time
t = [0:tf/100:tf];     % s
T = P0/(m*c)*t + T0;   % C

% Plot the temperature versus time
figure(1)
plot(t, T)
```

```
grid
xlabel('Time (s)')
ylabel('Temperature (\circC)')
axis([0 max(t) 0 max(T)])

% Find the heating time and display it
index = find(T > Tf);
t_heat = t(min(index));
fprintf('The time required for the %5.2f grams of steel to reach
800 degrees Celsius is %5.2f seconds. \n', 1000*m, t_heat);
```

The results are provided here.

```
The time required for the 147.03 grams of steel to reach 800 degrees
Celsius is 3.73 seconds.
```



9.4 Thermal Mass Subjected to Heat Exchange with the Environment

Next, we examine a common situation. An object is initially at temperature T_0 and is then subjected to heat exchange with the environment at temperature T_{env} beginning at time $t = 0$. This approximates the situation that the thermal mass of the environment is much larger than that of the object, so that any heat exchange between the two does not affect the temperature of the environment. This is applicable, for example, if the environmental temperature is being held constant by an active controller. This is realized when a building exchanges heat with the atmosphere or when a can of soda exchanges heat with the inside of a refrigerator.

For this case we begin with Eq. (9.2) and recognize that the power (heat) input is defined by Eq. (9.3).

$$mc\dot{T} = hA(T_{\infty}(t) - T(t)) \tag{9.7}$$

Rearranging, we obtain a differential equation describing the time-dependent temperature.

$$mc\dot{T} + hAT(t) = hAT_{\infty}(t) \tag{9.8}$$

For this case, the environmental input is a step function.

$$mc\dot{T} + hAT(t) = hAT_{env} \cdot u(t) \tag{9.9}$$

Computing the Laplace transform and rearranging, we obtain:

$$T(s) = aT_{env}\frac{1}{s(s+a)} + T_0\frac{1}{s+a}, \quad \text{where} \tag{9.10}$$

$$a = \frac{hA}{mc}. \tag{9.11}$$

By calculating the inverse Laplace transform of Eq. (9.10), we obtain:

$$T(t) = T_{env}(1 - \mathrm{e}^{-at}) + T_0\mathrm{e}^{-at}. \tag{9.12}$$

Equation (9.12) shows that $T(0) = T_0$ and, as the time becomes very large, $\lim_{t\to\infty} T(t) = T_{env}$. Therefore, the temperature begins at the initial temperature and exponentially approaches the environmental temperature. The time required for the exponentials to adjust is the *thermal time constant*, and is given by:

$$\tau_{thermal} = \frac{1}{a} = \frac{mc}{hA}. \tag{9.13}$$

As discussed previously, $4\tau_{thermal}$ is the time required for the exponentials to be less than 2% and this value approximates the time for the system to reach thermal equilibrium.

Example 9.3 Consider a 71 mm diameter, 54 mm tall cup filled with water. The water is heated in a microwave oven to an initial temperature of 41.6 °C and then exposed to the environment at a temperature of 21 °C. Assume that:

- the entire surface of the water is exposed to the environment and that the heat transfer coefficient between the water and the environment is 19 W/m^2-°C
- the water has a density of 1000 kg/m^3 and a specific heat of 4183 J/kg-°C.

Data showing temperature versus time for the water are provided in the following table. Data were collected using a stopwatch and a digital kitchen thermometer with a resolution of 0.1 °C.

Time (min)	Temperature (°C)
0	41.6
5	38.9
10	36.6
15	34.7
20	33.1
25	31.7
30	30.4
35	29.4
40	28.4
45	27.6
50	26.9
55	26.2
60	25.6
65	25

Using MATLAB®, find the time constant for the cup of water. Plot the temperature over four time constants and compare it to the data. Plot the data on the same graph as the prediction using a black "+" to represent each data point. Discuss the results. What is the sensitivity to h?

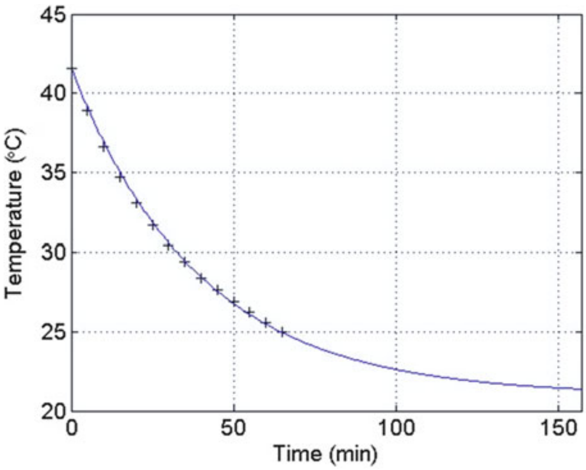
Solution The MATLAB® code is provided.

```
clear all
clc
close all

% Data
T_exp = [41.6 38.9 36.6 34.7 33.1 31.7 30.4 29.4 28.4 27.6 26.9 26.2
25.6 25.0];  % C
t_exp = 60*[0 5 10 15 20 25 30 35 40 45 50 55 60 65];  % s

% Parameters
h = 19;        % W/(m^2-C)
D = 71e-03;    % diameter, m
H = 54e-03;    % height, m
rho = 1000;    % density, kg/m^3
c = 4183;      % specific heat, J/(kg-C)
T0 = 41.6;     % initial temperature, C
T_env = 21;    % environmental temperature, C

% Calculated parameters
R = D/2;
V = pi*R^2*H;
A = 2*pi*R*H + 2*pi*R^2;     % surface area, m^2
m = rho*V;
a = h*A/(m*c);     % 1/s
```

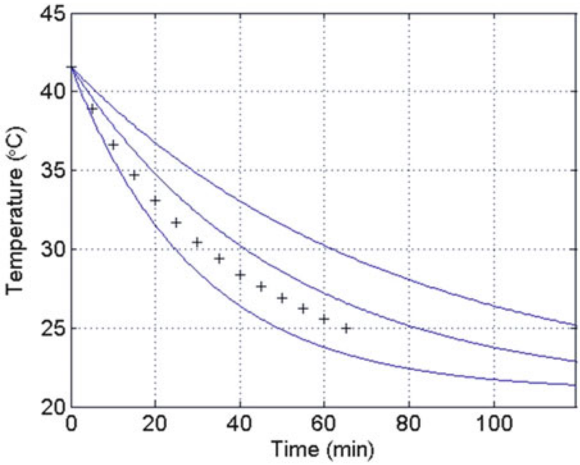
```
tau = 1/a;     % time constant, s
fprintf('The time constant of the coffee cup is %3.0f minutes.\n',
tau/60)

% Analytical temperature prediction
t = [0:tau/100:4*tau];                      % time, s
T = T_env*(1-exp(-a*t))+T0*exp(-a*t);   % temperature, C

% Comparison plot
figure(1)
plot(t_exp/60, T_exp, 'k+', t/60, T, 'b-')
grid
xlabel('Time (min)')
ylabel('Temperature (\circC)')
axis([0 max(t)/60 20 45])
```

The results are provided.

```
The time constant of the coffee cup is 39 minutes.
```



The time constant of the coffee cup of water is quite long due to the high specific heat of water. The thermal mass, *mc*, of the water is 894 J/°C. The fit of the prediction to the data is highly dependent upon the heat transfer coefficient, *h*. While 19 W/m^2-°C is a reasonable value for this situation, it was chosen to obtain agreement between theory and experiment. Heat transfer values of {10, 15, and 25} W/m^2-°C are shown in comparison to the experimental data in the following figure.



The value of 19 W/m^2-°C clearly provides a better fit to the experimental data than the other values. However, two observations can be made: (1) the experimental data do approximately follow an exponential curve of the form predicted; and (2) the entire experiment can be viewed as a method for measuring the heat transfer coefficient.

Example 9.4 The exposed upper surface area of an indoor swimming pool is 625 m^2 and it is 1.25 m deep. The pool is insulated and primarily exchanges heat with the environment through its exposed surface. At this surface, the heat transfer coefficient is 15 W/m^2-°C. Find the thermal mass of the swimming pool in J/°C and the thermal time constant. In the absence of any sensing or thermal control, estimate the time required for the pool to come to thermal equilibrium with its environment. The water has a density of 1000 kg/m^3 and a specific heat of 4183 J/kg-°C.

Solution The time required for a system to reach thermal equilibrium is approximately four time constants. Equation (9.13) may be used to estimate the time for a system to reach thermal equilibrium and is amazingly versatile; it is applicable to microscopic systems, such as cells, as well as very large systems, such as buildings and lakes. The following MATLAB® code completes the required calculations and displays the results. Note that using the format `%0.5 g` in the `fprintf` command causes MATLAB® to display the thermal mass in scientific notation and to display only three significant figures.

```
clear all
clc
close all

% Parameters
D = 1.25;     % depth, m
A = 625;      % area, m^2
```

```
rho = 1000;  % density, kg/m^3
c = 4183;    % specific heat, J/(kg-C)
h = 15;      % heat transfer coefficient, W/(m^2-C)

% Calculated parameters
V = A*D;     % volume, m^3
m = rho*V;   % mass, kg
% Time constant
tau = m*c/(h*A);                 % time constant, s
tau_days = tau/(3600)*1/24;      % time constant, days

% Display results
fprintf('The thermal mass of the water is %0.5 g J/C.\n', round
(m*c));
fprintf('The time constant of the pool is %2.0f days.\n',
tau_days);
fprintf('The pool will require approximately %2.0f days to reach
equilibrium temperature.\n', 4*tau_days);
```

The results are provided.

```
The thermal mass of the water is 3.268e+09 J/C.
The time constant of the pool is 4 days.
The pool will require approximately 16 days to reach equilibrium
temperature.
```

The thermal time constant of large masses of water is very long. This is the reason large masses of water, such as large lakes, tend to regulate the climate temperature in the land surrounding them. A pool will also regulate the temperature of the surrounding room in a similar manner.

9.5 Thermal Mass Subjected to Heat Exchange with the Environment and Power Input

Let us next consider a system subjected to external heating that is also allowed to exchange heat with the environment. An example is the swimming pool in Example 9.4 with a constant water heating (power input). In this case, the equation describing the system behavior is a combination of the heat exchange modeled in Sects. 9.3 and 9.4. The differential equation describing the new behavior is:

$$mc\dot{T} + hAT(t) = hAT_{\infty}(t) + P(t), \tag{9.14}$$

which is a modification of Eq. (9.8) that includes the additional power source. Consider the following case: at time $t = 0$ the system is at an initial temperature T_0

and is simultaneously subjected to a step input power $P(t) = P_0 \cdot u(t)$ and a step environmental temperature input $T_\infty(t) = T_{env} \cdot u(t)$. After some rearrangement, Eq. (9.14) may be rewritten as:

$$\dot{T} + aT(t) = aT_{env} \cdot u(t) + \frac{P_0}{mc} \cdot u(t). \tag{9.15}$$

We recognize this as a linear combination of Eqs. (9.4) and (9.8), where a is again equal to the inverse of the thermal time constant. Calculating the Laplace transform of Eq. (9.15) gives the temperature in the Laplace domain:

$$T(s) = aT_{env}\frac{1}{s(s+a)} + \frac{P_0}{mc}\frac{1}{s(s+a)} + T_0\frac{1}{s+a}. \tag{9.16}$$

Calculating the inverse Laplace transform on Eq. (9.16) gives the time domain temperature. Before inverting Eq. (9.16), it is useful to discuss its behavior. This can be determined using the initial and final value theorems as discussed previously. Applying the initial value theorem to Eq. (9.16) gives:

$$\lim_{t\to 0}(T(t)) = \lim_{s\to\infty}(sT(s)) = T_0. \tag{9.17}$$

The initial temperature is the initial condition as expected. This is not a surprising result, but still a useful error check for more complex systems where the Laplace transforms can be mathematically complicated. It is verified that the limit does produce the expected result by recognizing the following: (1) $\lim_{s\to\infty}\left(s\frac{1}{s(s+a)}\right) = \lim_{s\to\infty}\left(\frac{1/s}{1+a/s}\right) = \frac{0}{1} = 0$; and (2) $\lim_{s\to\infty}\left(\frac{s}{s+a}\right) = \lim_{s\to\infty}\left(\frac{1}{1+a/s}\right) = \frac{1}{1} = 1$. Applying the final value theorem yields a more interesting result. The steady state temperature, T_{ss}, reached by the system is:

$$T_{ss} = \lim_{t\to\infty}(T(t)) = \lim_{s\to 0}(sT(s)) = T_{env} + \frac{P_0}{hA}. \tag{9.18}$$

Rearranging Eq. (9.18) gives:

$$P_0 = hA(T_{ss} - T_{env}). \tag{9.19}$$

Steady state is reached at a temperature such that the heat loss to the environment, $hA(T_{ss} - T_{env})$, balances the power input, P_0. This is a statement of energy conservation at equilibrium.

While the initial and final value theorems provide us with useful information about the system behavior, calculating the inverse Laplace trasform of Eq. (9.16) provides a full time domain solution to the problem.

$$T(t) = T_{env}(1 - e^{-at}) + \frac{P_0}{hA}(1 - e^{-at}) + T_0 e^{-at} \tag{9.20}$$

Note that allowing the exponentials to go to zero in the limit of infinite time produces Eq. (9.19). The system behavior described by Eq. (9.20) is exponential and, therefore, it reaches equilibrium in approximately four thermal time constants; see Eq. (9.13). The "rule of thumb" given by Eq. (9.13) remains useful for estimating the time for a system to reach thermal equilibrium for the case of constant external heating. Note that the third term in Eq. (9.20) goes to zero in the limit as time goes to infinity. The initial temperature does not have any effect on the system equilibrium behavior. The system has "no memory" of its initial condition.

Example 9.5 Consider the indoor swimming pool from Example 9.4. Heat exchange is through the exposed upper surface of the water with a heat transfer coefficient of 15 W/m^2-°C. The water has a density of 1000 kg/m^3 and a specific heat of 4183 J/kg-°C. If the environmental temperature is 22 °C, determine the constant heat input required to maintain the pool temperature at 30 °C. Next, assume that after filling the pool is at an initial temperature of 18 °C when the heater is turned on. Use MATLAB® to determine the temperature as a function of time and plot it over a time interval of eight thermal time constants. Report your results in °C and days.

Solution

Step 1: Assuming a steady state temperature of 30 °C and an environmental temperature of 22 °C, we can calculate the required power input using Eq. (9.19). The cross-sectional area of the top of the pool is 625 m^2 and therefore we obtain:

$$P_0 = \left(15\,\mathrm{W/m^2\text{-}°C}\right)\left(625\,\mathrm{m^2}\right)(30\ °\mathrm{C} - 22\,°\mathrm{C}) = 75\,\mathrm{kW}.$$

Typical pool heaters are specified in British thermal units (BTU) per hour. This power input of 175 kW is equal to approximately 256000 BTU/h. Pool heaters of up to 400000 BTU/h are readily available commercially. Most of these use propane or natural gas to maintain this high energy output.

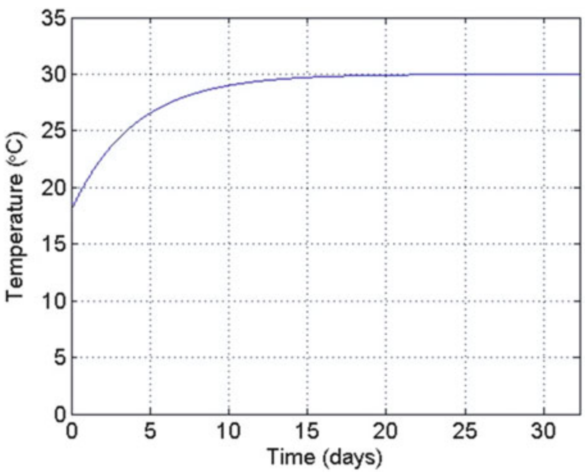
Step 2: The following MATLAB® code calculates and plots the pool temperature as a function of time.

```
clear all
clc
close all

% Parameters
D = 1.25;      % depth, m
A = 625;       % area, m^2
rho = 1000;    % density, kg/m^3
c = 4183;      % specific heat, J/(kg-C)
```

```
h = 15;         % heat transfer coefficient, W/(m^2-C)
T_ss = 30;      % equilibrium temperature, C
T_env = 22;     % environmental temperature, C
T0 = 18;        % initial temperature, C

% Calculated parameters
V = A*D;        % volume, m^3
m = rho*V;      % mass, kg

% Time constant
tau = m*c/(h*A);    % Time constant, s
a = 1/tau;

% Power input
P0 = h*A*(T_ss - T_env);

% Display the results
fprintf('The power input required to hold the temperature at %0.5 g
C is %3.0f kW. \n', T_ss, P0/1000);

% Time vector
t = [0:tau/1000:8*tau];     % s

% Temperature as a function of time
T = T_env*(1-exp(-a*t))+P0/(h*A)*(1-exp(-a*t))+T0*exp(-a*t);

figure(1)
plot(t/(3600*24), T)
grid
xlabel('Time (days)')
ylabel('Temperature (\circC)')
axis([0 max(t)/(3600*24) 0 35])
```

The results are provided.

```
The power input required to hold the temperature at 30 C is 75 kW.
```



The pool reaches the desired equilibrium temperature over a time period of 2 weeks to 1 month. The thermal time constant of 4 days is the relevant quantity for determining the response time, a quantity that could easily be estimated for many objects including a swimming pool.

9.6 A Proportional Integral Derivative (PID) Thermal Control System

As a final case, we consider a thermal mass with active *feedback control*. This is a precursor to the next chapter where we will show how to represent all systems in this book using simplified pictorial descriptions referred to as *block diagrams*. Consider a thermal mass where the temperature, $T(t)$, is measured by a *sensor* and a *controller* is used in conjunction with an *actuator* (heating/cooling system) to drive the temperature to a desired (set) value, $T_{set}(t)$, as shown in Fig. 9.3. This is accomplished by regulating the power input to the system, $P_{act}(t)$, which may take on a positive or negative value depending on whether heat is being added or removed from the system, respectively. *Disturbances* arise that drive the temperature away from the desired temperature. In this case, we consider the heat exchange with the environment as the system disturbance.

Thermal control is a common situation. The heating, ventilation, and air conditioning (HVAC) system in a house or apartment is used to control the temperature to a set value, $T_{set}(t)$, using the thermostat. The thermostat also has a temperature sensor, typically a resistance thermometer or thermistor, to measure the current room temperature. The HVAC system is actuated when the current temperature is different from the set temperature. The temperature is disturbed from the desired temperature by external factors. In this case, the most common factors are changes in the external temperature, $T_\infty(t)$, on a daily cycle or heat sources within the living space, such as turning on the oven or having an event with many people. Typically,

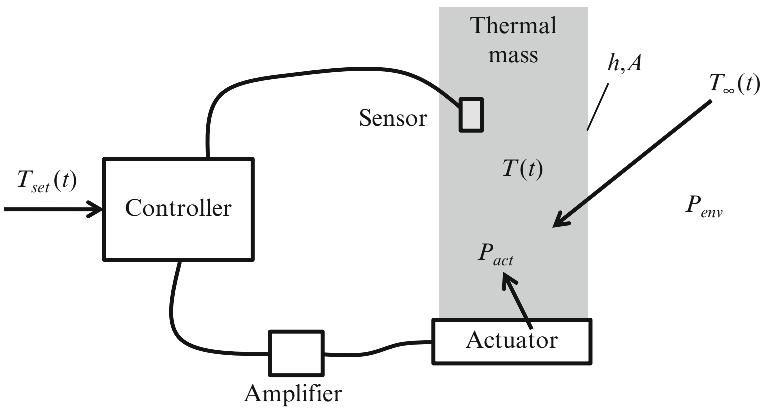


Fig. 9.3 Thermal control system

HVAC systems are relatively unsophisticated; they only actuate the heating or the air conditioning (cooling), not both, depending on a user setting. Further, standard HVAC systems are typically either on or off and do not have ability to produce variable levels of heating/cooling depending on the mismatch between the current temperature and the desired temperature. However, precision temperature control systems capable of up- or down-regulating the temperature continuously do exist—an example application is temperature-controlled laboratory space. We will first consider the general problem of controlling the temperature of a thermal mass and then show by example that the system can exhibit both oscillatory and nonoscillatory behavior similar to the mechanical and electrical systems we have discussed.

Referring to Fig. 9.3, the controller accepts input from the user, $T_{set}(t)$. A sensor measures the temperature, $T(t)$, which is electronically relayed to the controller, typically as a voltage signal. The controller computes the temperature error, $\varepsilon(t)$, using Eq. (9.21).

$$\varepsilon(t) = T_{set}(t) - T(t) \tag{9.21}$$

The error is then used to generate a voltage to control the actuator output. Most often this output voltage must be amplified to a power level suitable to drive the actuator. If the error is greater than zero, heat must be added to the system ($P_{act} > 0$). If the error is less than zero, heat must be removed ($P_{act} < 0$).

As discussed in Chap. 10, the design of a control system depends on defining the actuator control signal from the error. One of the simplest approaches is to make the actuator output proportional to the error.

$$P_{act}(t) = K_P \varepsilon(t) \tag{9.22}$$

Here, K_P is a proportionality constant and Eq. (9.22) describes *proportional control*. Examining Eq. (9.22), the actuator output will be positive when the error is positive, $T_{set}(t) < T(t)$, and negative when the error is negative, $T_{set}(t) > T(t)$. This will serve to drive the temperature toward the set value. However, when the error is zero, the actuator is not active and the system will then change temperature due to external influences. Therefore, countering external disturbances requires a temperature error. If the external disturbance is constant, the error will reach a steady state value at which the disturbance is countered by the actuator. This equilibrium error is called *steady state error*. If K_P is large, the error may be small, but still undesirable.

Steady state errors are corrected by integrating the error over time so that even small steady state errors occurring over long time periods will be "added up" to produce an actuator signal. Mathematically, this is described by:

$$P_{act}(t) = K_I \int_0^t \varepsilon(t) dt, \tag{9.23}$$

where K_I is the integrator constant and Eq. (9.23) describes *integral control*. In most situations, proportional control adds "stiffness" to a system and integral control raises the order of a system by one: a first-order system with an integral controller becomes second order, a second-order system becomes third order, and so on.

It is also desirable to affect the "damping" of the system. This is done with a feedback signal proportional to the derivative of the error.

$$P_{act}(t) = K_D \dot{\varepsilon}(t) \tag{9.24}$$

Here, K_D is the differentiator constant and Eq. (9.24) describes *derivative control*. A linear combination of Eqs. (9.22), (9.23), and (9.24) gives the following actuator output equation:

$$P_{act}(t) = K_P \varepsilon(t) + K_I \int_0^t \varepsilon(t)dt + K_D \dot{\varepsilon}(t), \tag{9.25}$$

which represents proportion-integral-derivative, or PID, control. PID controllers are relatively simple and easy to implement and can be tuned, i.e., the constants can be selected to provide good performance. As a result, they are quite common in industrial applications.

To see how a PID thermal controller will behave in practice, we apply an energy balance to the thermal mass in Fig. 9.3. This gives:

$$mc\dot{T} + hAT(t) = hAT_\infty(t) + P_{act}(t). \tag{9.26}$$

Notice that the right-hand side of the equation, which represents the system input, now depends on the environment and the actuator. Combining Eqs. (9.25) and (9.26) provides the governing differential equation for the temperature of the thermal mass under the influence of the controller.

$$\dot{T} + aT(t) = aT_\infty(t) + \frac{1}{mc}\left(K_P \varepsilon(t) + K_I \int_0^t \varepsilon(t)dt + K_D \dot{\varepsilon}(t)\right) \tag{9.27}$$

Combining Eqs. (9.27) and (9.21) and computing the Laplace transform, we obtain the Laplace domain temperature equation.

$$\begin{aligned} T(s)\left(s^2(k_d+1) + \left(k_p+1\right)s + k_i\right) &= (as)T_\infty(s) + \left(k_d s^2 + k_p s + k_i\right)T_{set}(s) \\ &+ (s(k_d+1))T_0 + sk_d T_{set_0} \end{aligned} \tag{9.28}$$

Here, $k_p = \frac{K_P}{mc}$, $k_d = \frac{K_D}{mc}$, $k_i = \frac{K_I}{mc}$, and T_{set_0} is the initial value of the set temperature. We also made use of the property that the Laplace transform of a time integral is $\frac{1}{s}$.

(Consider that the Laplace transform of a time derivative is s. In the time domain, the integral of a time derivative returns the original variable $\int \dot{x} dt = x$. In the Laplace domain, the integral of a time derivative gives rise to an algebraic cancellation.) Rearranging Eq. (9.28), we see that the temperature, $T(s)$, depends on two input variables, $T_\infty(s)$ and $T_{set}(s)$.

$$\begin{aligned} T(s) = &\left[\frac{as}{(1+k_d)s^2+(a+k_p)s+k_i}\right]T_\infty(s) \\ &+\left[\frac{k_d s^2+k_p s+k_i}{(1+k_d)s^2+(a+k_p)s+k_i}\right]T_{set}(s) \\ &+\left[\frac{(1+k_d)s}{(1+k_d)s^2+(a+k_p)s+k_i}\right]T_0 \\ &+\left[\frac{k_d s}{(1+k_d)s^2+(a+k_p)s+k_i}\right]T_{set_0} \end{aligned} \tag{9.29}$$

The first term on the right-hand side is the transfer function $\frac{T(s)}{T_\infty(s)}$ and the second term is the transfer function $\frac{T(s)}{T_{set}(s)}$. Conceptually, this is similar to Eq. (8.26). Note also that when there is no controller, $k_p = k_d = k_i = 0$, the equation reduces to Eq. (9.10) as expected.

Consider the situation in which the thermal control system represented by Eq. (9.29) is subjected to a step input in environmental (∞) temperature, $\frac{T_{env}}{s}$, and a step input in set temperature, $\frac{T_{des}}{s}$, at time $t = 0$. The final value theorem gives:

$$T_{ss} = \lim_{s\to 0}(s\mathrm{T}(s)) = 0 + \frac{K_I}{K_I}T_{des} + 0 = T_{des} \tag{9.30}$$

The temperature converges to the desired temperature, T_{des}, due to the presence of the integrator.

Example 9.6 Consider again the indoor swimming pool from Example 9.4; it has 625 m^2 surface area and is 1.25 m deep. Heat exchange occurs through the exposed water surface with a heat transfer coefficient of 25 W/m^2-°C. The water has a density of 1000 kg/m^3 and a specific heat of 4183 J/kg-°C. The environmental temperature is 27 °C. The pool temperature is maintained by a PID controller with a set temperature of $T_{des} = 30$ °C. The initial pool temperature is 22 °C. Use the MATLAB® function `ilaplace` to determine the temperature as a function of time and plot it for two cases: (1) $K_P = 10000, K_D = 10, K_I = 0$; and (2) $K_P = 10000$, $K_D = 10, K_I = 1$. Explain your results using the final value theorem and the roots of the transfer function's denominator.

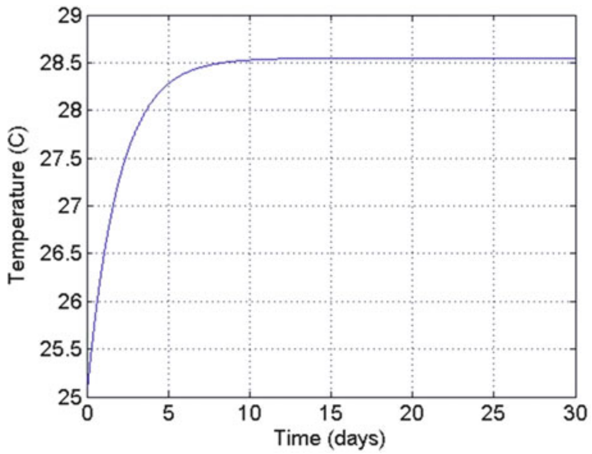
Solution The MATLAB® code used to solve this problem is given for case (1). For case (2), the values of the controller constants only need to be changed.

```
clear all
clc
close all

% Parameters
D = 1.25;          % depth, m
A = 625;           % area, m^2
rho = 1000;        % density, kg/m^3
c = 4183;          % specific heat, J/(kg-C)
h = 15;            % heat transfer coefficient, W/(m^2-C)
T_des = 30;        % equilibrium temperature, C
T_env = 27;        % environmental temperature, C
T0 = 25;           % initial temperature, C
Tset0 = T_des;     % initial set temperature, C

% Calculated parameters
V = A*D;             % volume, m^3
m = rho*V;           % mass, kg
tau = m*c/(h*A);     % time constant, s
a = 1/tau;

% Controller parameters
KP = 10000;
KD = 10;
KI = 0;
kp = KP/(m*c);
kd = KD/(m*c);
ki = KI/(m*c);

% Transfer function in Laplace domain
syms TT s T t
den = (kd + 1)*s^2+(a + kp)*s + ki;
A1 = (a*s/den)*T_env/s;
A2 = ((kd*s^2 + kp*s + ki)/den)*T_des/s;
A3 = s*(kd + 1)/den*T0;
A4 = s*kd/den*Tset0;
TT = A1 + A2 + A3 + A4;
T = ilaplace(TT);

% Time vector, s
t = [0:0.001:30]*3600*24;
TTT = eval(T);
set(gca, 'FontSize', 14)
plot(t/(3600*24), TTT)
grid
```

```
xlabel('Time (days)')
ylabel('Temperature (C)')

% Check the denominator roots
den_check = [(kd + 1), (a + kp), ki];
roots(den_check)
```

The results for case 1 are provided.

```
The roots of the characteristic equation are:
ans =
 1.0e-05 *
 0
 -0.5929
```



This indicates that the system is first order and has a time constant of 168700 s or 1.95 days. Instead of 4 days, the implementation of the proportional-derivative control reduces the time constant to approximately 2 days. However, the water temperature does not reach the desired 30 °C, but reaches a temperature only slightly greater than 28.5 °C. This steady state error is present because there is no integral control. The steady state error can be determined by applying the final value theorem to Eq. (9.29) with $k_i = 0$. In this case, Eq. (9.29) becomes first order.

$$T(s) = \left[\frac{a}{(1+k_d)s + \left(a + k_p\right)}\right]\frac{T_{env}}{s} + \left[\frac{k_d s + k_p}{(1+k_d)s + \left(a + k_p\right)}\right]\frac{T_{des}}{s} + \left[\frac{(1+k_d)}{(1+k_d)s + \left(a + k_p\right)}\right]T_0 + \left[\frac{k_d}{(1+k_d)s + \left(a + k_p\right)}\right]T_{set_0} \tag{9.31}$$

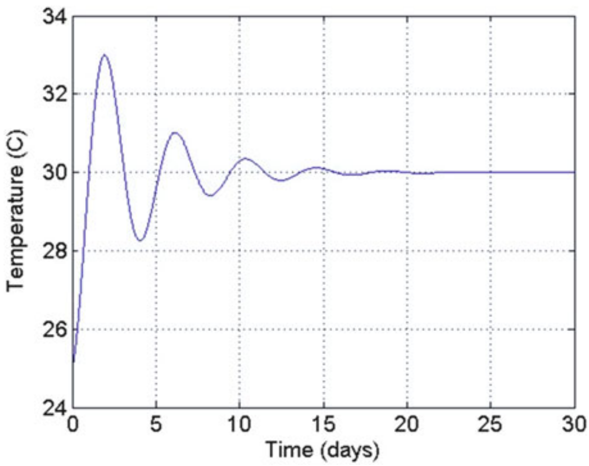
Applying the final value theorem to Eq. (9.31) gives the steady state temperature.

$$T_{ss} = \left[\frac{a}{a + k_p}\right] T_{env} + \left[\frac{k_p}{a + k_p}\right] T_{des} \tag{9.32}$$

Substitution yields $T_{ss} = 28.54\,°\mathrm{C}$, which matches the figure. Physically, the error is set when the power input from the controller is equal to the heat loss to the environment; this result is obtained by rearranging Eq. (9.32).

The result for case (2) is provided next.

```
The roots of the characteristic equation are:
ans =
 1.0e-04 *
 -0.0296 + 0.1724i
 -0.0296 - 0.1724i
```



The complex roots result in oscillation at a frequency of $\omega_d = 0.172 \times 10^{-4}$ rad/s or 0.24 cycles/day and a time constant of 337340 s or 3.9 days. There is approximately three cycles in 12.5 days which is consistent with the natural frequency. The oscillations attenuate in 15–20 days, or approximately four time constants. The oscillatory system is second order and the natural frequency is calculated using $\omega_d = \sqrt{\frac{k_i}{1+k_d}}$. The controller has added a *thermal stiffness* to the problem. The advantage is that the temperature now reaches the desired value. The potential disadvantage is that the temperature can now oscillate.

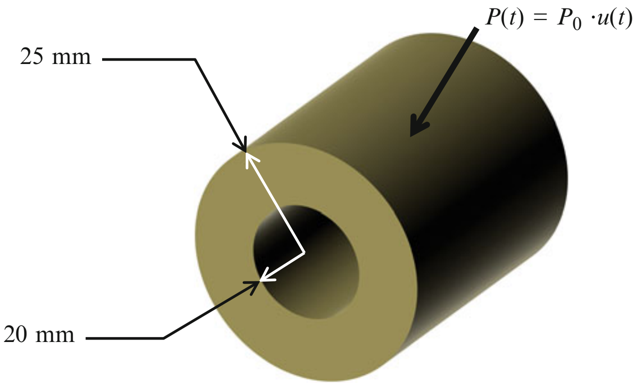
9.7 Conclusions

In this chapter, we examined several aspects of thermal systems.

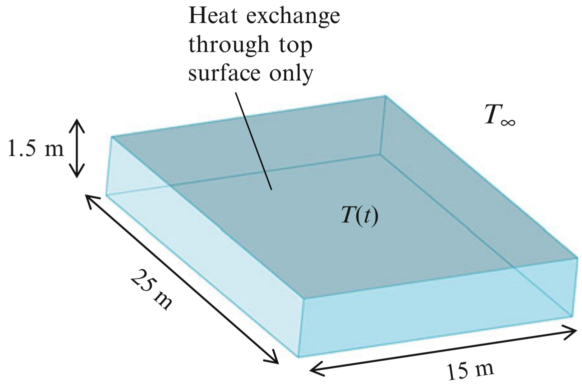
- Thermal systems can often be modeled with lumped parameters, including thermal masses and thermal dampers.
- Thermal systems are generally first order and the thermal time constant is $\frac{mc}{hA}$.
- When a PID control system is added to a thermal system, it becomes second order and may exhibit oscillatory behavior indicating the presence of a thermal mass, stiffness, and damping.

Problems

1. A motor coil can be modeled as a hollow copper cylinder with an outer diameter of 50 mm, an inner diameter of 40 mm, and a length of 60 mm. The motor stalls at time $t = 0$ and, at this point, the copper experiences a step input in electrical heating of $P_0 \cdot u(t)$, where P_0 is 350 W. Assume that there is little time for significant heat loss to the environment so that the copper cylinder can be considered to be insulated. If the initial temperature is 30 °C, find the temperature as a function of time, $T(t)$, analytically using Laplace transforms. Calculate the time required to reach the breakdown temperature of the wire insulation, which is approximately 250 °C. Write a script file in MATLAB® to: (a) determine the time for the copper to reach 250 °C using the command `find`; and (b) plot $T(t)$ over this time interval. Copper has a density of 8960 kg/m^3 and a specific heat of 390 J/kg-°C.



2. A rectangular swimming pool is 1.5 m deep, 15 m wide, and 25 m long. Assume the bottom and sides of the swimming pool are insulated so that the pool exchanges heat with its environment through the top surface only. The density of water is 1000 kg/m^3, the specific heat is 4180 J/kg-°C, and the heat transfer coefficient is 10 W/m^2-°C.



Complete the following.

(a) Determine the thermal time constant of the pool and express your answer in days.

(b) The swimming pool is being held at $T_0 = 30$ °C when the power fails and external heating stops. The air temperature around the pool (T_∞) drops quickly and can be assumed to be 10 °C at time $t = 0$. Write a script file in MATLAB® that plots the pool temperature for a duration of four thermal time constants.

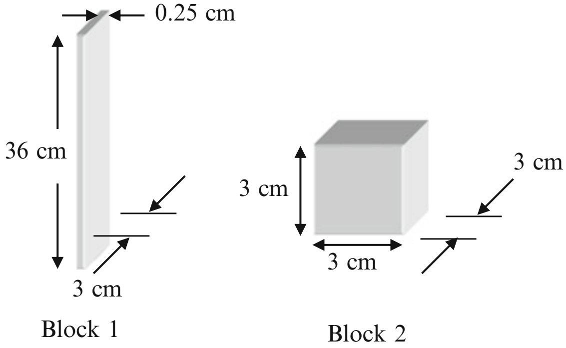
3. Consider again the swimming pool from Problem 2. Suppose the pool has chilled to $T_0 = 10$ °C when the heat comes back on. The air temperature quickly warms up so that $T_\infty = T_{\text{env}} \cdot u(t)$, where $T_{env} = 22$ °C. The water heater comes back on at the same time providing a step input in power to the pool, $P_0 \cdot u(t)$.

Complete the following.

(a) Applying the final value theorem to the Laplace domain solution for $T(s)$, determine the value of P_0 required to bring the pool back to an equilibrium temperature of 30 °C and hold it at that temperature.

(b) Using the value of P_0 determined in part (a), write a script file in MATLAB® that plots the pool temperature for a duration of four thermal time constants.

4. Two aluminum blocks with different dimensions, but the same mass are shown in the figure. The dimensions of block 1 are 0.25 cm in thickness, 3 cm in width, and 36 cm in length. The dimensions of block 2 are 3 cm for each side of the cube. Aluminum has a density of 2700 kg/m^3 and a specific heat capacity of 900 J/kg-°C. The value of the heat transfer coefficient to the surrounding air is 10 W/m^2-°C.



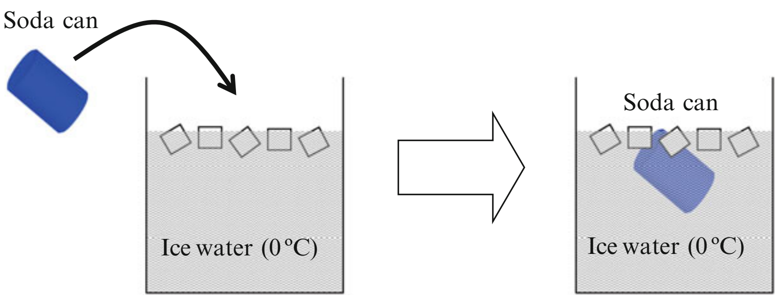
Complete the following.

(a) Calculate thermal time constant (in min) for both blocks.

(b) Each block is heated to an initial temperature 120 °C and then allowed to cool in the surrounding air which is at 20 °C. Write a script file in MATLAB® that plots the temperature as a function of time, $T(t)$, for both blocks on the same graph. Plot the temperature of block 1 using a solid black line and the temperature of block 2 using a dotted black line. Plot for a duration equal to four of the longer time constant between blocks 1 and 2. The time axis should be in units of minutes.

(c) Investigate cooling fins such as those used in an automobile radiator or an air conditioner and comment on cooling fin design based on the results of this problem.

5. In order to cool soda cans quickly for a cookout, you place them in a bath of ice water at 0 °C. The initial temperature of the soda is 30 °C. The cans are cylindrical with a height of 160 mm and a diameter of 64 mm. The soda can be considered to be mostly water and so has a density of 1000 kg/m^3 and specific heat of 4180 J/kg-°C. The heat transfer coefficient between the ice water and the can is 320 W/m^2-°C.



Complete the following.

(a) Calculate the thermal time constant for each can in minutes.

(b) Write a script file in MATLAB® that plots the temperature as a function of time, $T(t)$, for the cans. The time axis should be in units of minutes. Using the `find` command, determine the time in minutes required for a can to cool to 5 °C.

Block Diagrams and Introduction to Control Systems

10

10.1 Introduction

We have studied several dynamic systems in previous chapters. While they have appeared to be physically distinct, we have seen that lumped parameter models of these mechanical, electrical, electromechanical, and thermal systems all exhibit the same fundamental behavior. *Block diagrams* provide a Laplace domain, visual description that enables physically distinct systems to be represented using a common structure. In a block diagram, transfer functions of the system elements are represented by individual blocks. Inputs and outputs, or *signals*, that flow to and from the system elements are represented by lines or arrows and their terminations define the manner in which different parts of the system interact. Block diagrams are used extensively to analyze *control systems*, where they provide a compact and intuitive representation of feedback control loops.

In this chapter, we will study the basic elements of a block diagram and describe how to find the overall transfer function for block diagrams that include feedback loops. We will then demonstrate the use of block diagrams to better understand the concepts described in this book.

10.2 Block Diagram Algebra

Two elements, the *block* and *summation circle*, that appear in most block diagrams are shown in Fig. 10.1. A block indicates a multiplication between the input signal (with the arrow pointing toward the block), $I(s)$, and the transfer function included inside the block, $G(s)$. As depicted in Fig. 10.1a, the output, $O(s)$ is equal to their product.

M.A. Davies, T.L. Schmitz, *System Dynamics for Mechanical Engineers*,
DOI 10.1007/978-1-4614-9293-1_10

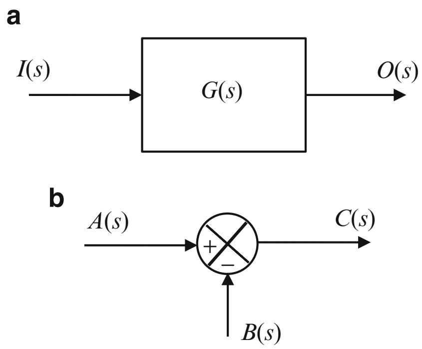


Fig. 10.1 Elements of a block diagram: (**a**) block; and (**b**) summation circle

$$O(s) = G(s)I(s) \tag{10.1}$$

Equation (10.1) can be rearranged as $O(s)/I(s) = G(s)$. In this form $G(s)$ is recognized as a transfer function with the input, $I(s)$, and the output, $O(s)$.

Figure 10.1b displays a summation circle. The inputs (again recognized by the arrow directions into the circle) are summed using the included signs to give the output (denoted by the arrow pointing away from the circle). Because $A(s)$ has a (+) sign and $B(s)$ has a (−) sign, the mathematical representation is:

$$C(s) = A(s) - B(s). \tag{10.2}$$

If $B(s)$ is a control signal used to modify the desired input, $A(s)$, this is referred to as negative feedback. For positive feedback, $A(s)$ and $B(s)$ are added.

10.3 Feedback Loops with Proportional Gain

Systems are designed to accomplish one or more tasks. Examples displayed in Fig. 10.2 include: (a) moving a machine table to a selected position at a given rate by commanding a motor to rotate and produce linear motion using a leadscrew; (b) moving a two rotary link robot arm to a desired position by commanding its rotary joints to move to prescribed angles; and (c) changing the temperature of a swimming pool to a selected value. In order to accomplish the desired task, the system can be operated in an *open loop* or *closed loop* mode. The desired task for many systems is to follow a commanded path with minimal error. To analyze this situation, we must first discuss the structure of open loop and closed loop systems, then describe how a command is applied, and finally determine whether the system response follows the command.

In the open loop system shown in Fig. 10.3a, an input, $I(s)$, is defined and used to produce the output, $O(s)$. However, the output is not monitored to ensure that it reaches or remains at the specified value. Identifying and improving the output accuracy can be accomplished by trial and error, modeling, calibration, or a combination of these. The system can be modeled in the Laplace domain by a

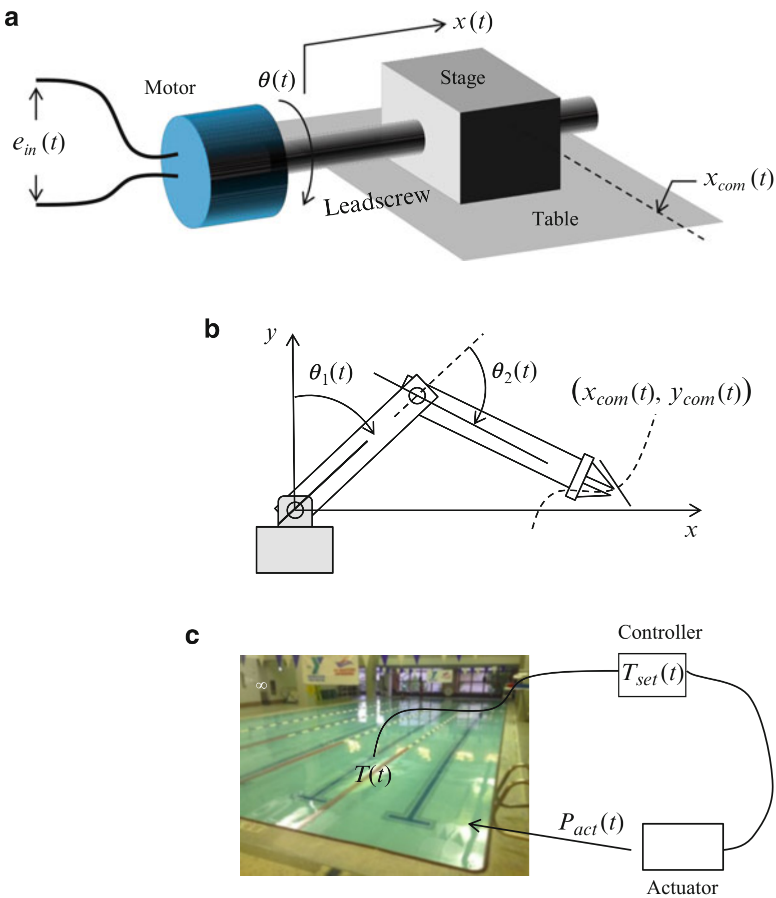


Fig. 10.2 Systems accepting a commanded input and responding to meet that command: (**a**) a leadscrew attached to a motor that drives a machine table; (**b**) a two link robot arm commanded to move to a given position; (**c**) a swimming pool commanded to reach and maintain a prescribed temperature

transfer function, $G_1(s)$. In system dynamics and control systems analyses, this model is often referred to as the *plant* (think power plant, not flower). An advantage of open loop operation is that it is inexpensive to implement. A disadvantage is that the system does not have any information about its state and, therefore, any changes to the system or the introduction of unknown or unmodeled disturbances will cause the system output to differ from the desired value. For example, we could model the swimming pool in Fig. 10.2c as a thermal mass with heat loss to the environment and then calculate the heat input needed to maintain the pool at the commanded temperature, $T_{com}(t)$. Next, we would purchase an adequate heater to apply this input and measure the response of the pool. We could then modify the heat input to compensate for any errors in our modeling until the pool maintains the desired

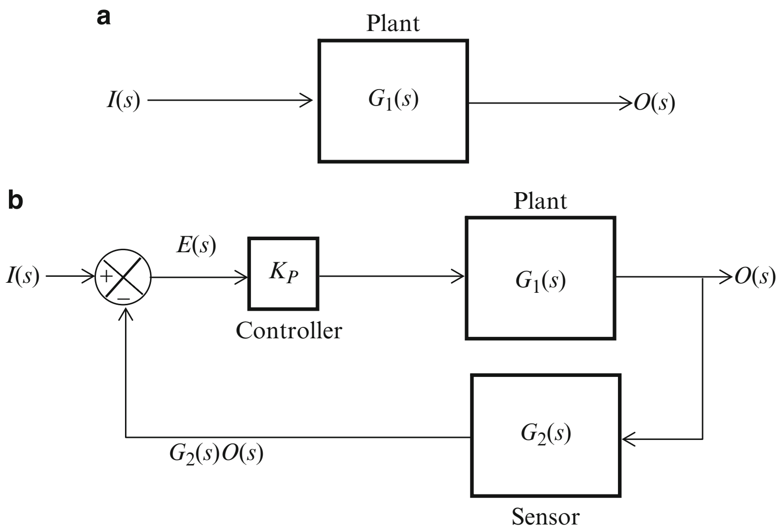


Fig. 10.3 Block diagrams for two system architectures: (**a**) open loop; and (**b**) closed loop

temperature within an allowable error. The problem arises when something happens that we did not anticipate. For example, the air temperature in the pool area might be particularly high for a long period of time (due to unexpected weather conditions) and this could cause the pool temperature to rise. The open loop system cannot compensate for the unanticipated thermal disturbance. The system behavior is not *robust* to changes in the environment.

In a closed loop system, a sensor is added to measure the value of the output as shown in Fig. 10.3b. The sensor output is compared to the input; this feedback may be positive or negative. The input, $I(s)$, is now interpreted as the desired value of the output, $O(s)$. The feedback loop serves two main purposes: (1) to cause the output to approach the input; and (2) to change the dynamic characteristics of the original system in a beneficial way. Let us calculate the new transfer function for the closed loop system and compare its dynamic behavior to that of the original, open loop system.

The sensor dynamics are modeled by a second transfer function, $G_2(s)$. The *feedback loop* is added to the control system as shown in Fig. 10.3b. The feedback loop, sometimes called a backward loop, shows that the output, $O(s)$, is measured and a new signal, $G_2(s)O(s)$, is generated. This feedback signal is compared to the input, $I(s)$. In this case, their difference generates an error signal, $E(s)$.

$$E(s) = I(s) - G_2(s)O(s) \tag{10.3}$$

The error signal is the input to the system controller. Here, the controller is simply a proportional gain, K_P, that is multiplied by the error to produce the plant input.

If the error is zero, the system is at the desired state, there is no plant input, and the output, therefore, does not change. If the error is nonzero, however, there is a non zero input acting to comect the error in the output.

To determine the closed loop transfer function, we recognize that the error signal is multiplied by the proportional gain, K_P, to produce the plant input signal, $K_P E(s)$. This input is multiplied by the plant's transfer function to produce the (new) system output.

$$K_P E(s) G_2(s) = O(s) \tag{10.4}$$

Combining Eqs. (10.3) and (10.4) and algebraically rearranging, we obtain the final transfer function.

$$\frac{O(s)}{I(s)} = \frac{K_P G_1(s)}{1 + K_P G_1(s) G_2(s)} \tag{10.5}$$

Clearly, this transfer function differs from the open loop transfer function, $G_1(s)$. Its denominator defines the dynamic characteristics of the closed loop system.

Let's explore this feedback process using the swimming pool example. The output is the temperature, $T(s)$, and the feedback signal $G_2(s)T(s)$ is measured by a resistance thermometer that produces a voltage. The input voltage, $V_{set}(s)$, is the voltage that is produced by the sensor when the temperature is at the desired value. The error in temperature is therefore proportional to the error in voltage. This voltage error is amplified by a gain, K, and provides the input to the system actuator (in this case a heater/cooler): $K(V_{set}(s) - G_2(s)T(s))$. This is the proportional control discussed in Chap. 9. If the temperature is lower than the desired temperature, heat is added. If the temperature is higher than the desired value, heat is removed.

To further analyze the behavior of feedback loops, assume a step input, $I(s) = \frac{O_{des}}{s}$, is applied to a negative feedback system. Substituting into Eq. (10.5) gives:

$$O(s) = \left(\frac{K_P G_1(s)}{1 + K_P G_1(s) G_2(s)}\right) \frac{O_{des}}{s}. \tag{10.6}$$

The goal of the feedback loop is to drive the output toward the desired value. To assess the system performance, we apply the final value theorem to determine the steady state value, o_{ss}.

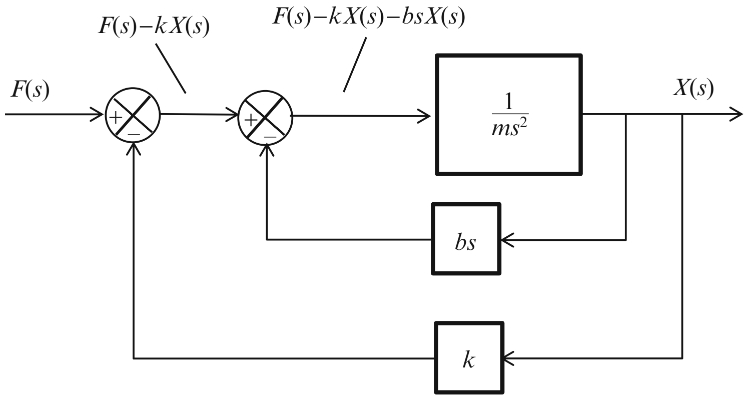
$$o_{ss} = \lim_{t \to \infty}(o(t)) = \lim_{s \to 0}(sO(s)) = \left(\frac{K_P G_1(0)}{1 + K_P G_1(0) G_2(0)}\right) O_{des} \tag{10.7}$$

When the expression in parentheses is not unity, there will be a steady state error in the system, and the value will not approach the desired value. Often, the sensor dynamic is negligible and the gain in the sensor loop can be set to unity, $G_2(0) = 1$. In this case, the steady state error, e_{ss}, is:

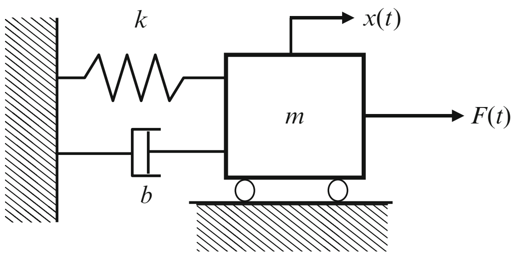
$$e_{ss} = O_{des} - o_{ss} = O_{des}\left(1 - \frac{K_P G_1(0)}{1 + K_P G_1(0)}\right) \tag{10.8}$$

The plant, $G_1(s)$, is typically not under our control; it is defined by the physical system. However, the gain, K_P, is a parameter that we choose. Equation (10.8) shows that as the gain becomes large, the steady state error approaches zero. Physically, this is because we amplify very small errors to produce large plant inputs. The steady state error is decreased by increasing the proportional gain in a system with a feedback loop.

Example 10.1 Consider the following block diagram.



Calculate the closed loop transfer function, $\frac{X(s)}{F(s)}$. Show that this is the same transfer function obtained for the spring–mass–damper system.



Solution
Step 1: Find the signal after the first (left) summation circle.

$$F(s) - kX(s)$$

Step 2: Find the signal after the second (right) summation circle.

$$F(s) - kX(s) - bsX(s)$$

Step 3: Multiply this input signal by the transfer function inside the box and set the result equal to the output.

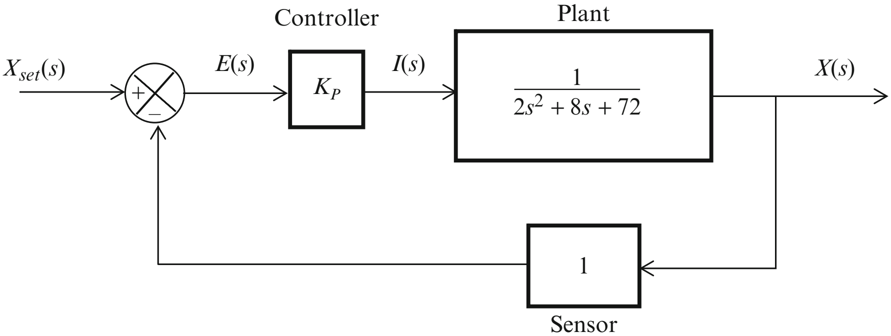
$$(F(s) - kX(s) - bsX(s))\frac{1}{ms^2} = X(s)$$

Step 4: Algebraically rearrange to solve for the transfer function.

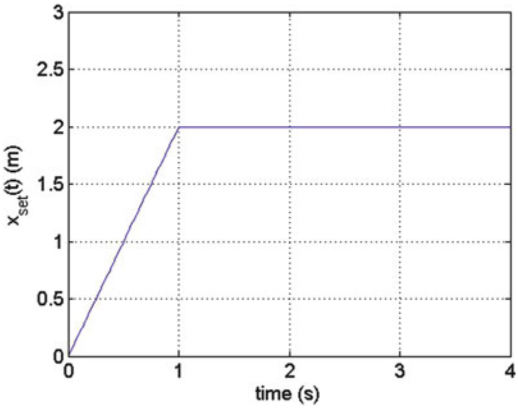
$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

The equation of motion for the spring–mass–damper system yields the same transfer function. This is the physical interpretation. The transfer function of a lumped mass is $\frac{X(s)}{F(s)} = \frac{1}{ms^2}$. This is the plant in the block diagram. The spring is a device that "measures" displacement and produces a force proportional to the displacement using the spring constant, k; this provides proportional feedback. Similarly, a damper is a device that "measures" a velocity (the time derivative of position) and produces a force proportional to the velocity using the damping coefficient, b. This is derivative feedback. Therefore, a spring–mass–damper system can be described as a mass with two force feedback loops: a spring (proportional, P) that returns the mass to its equilibrium position using position feedback and a damper (derivative, D) that dissipates energy using velocity feedback. A spring-mass-damper is an example of a PD system.

Example 10.2 The following block diagram represents a positioning system modeled as a spring-mass-damper with a feedback loop. The input is a desired position value, $X_{set}(s)$, and the output is the actual position of the system, $X(s)$. The sensor gain in the feedback loop is unity.



Determine: (a) the transfer function, $X(s)/X_{\text{set}}(s)$, in terms of the proportional gain, K_P; (b) the natural frequency of the plant alone (open loop) and the closed loop system in terms of K_P; (c) the value of K_P required to obtain a damped natural frequency of 10 rad/s; (d) the damping ratio and time constant of the closed loop system for the value of K_P from part (c); (e) the response of the closed loop system to the following input using the MATLAB® `lsim` command; and (f) the new damping ratio and natural frequency if K_P is increased by a factor of 10.



Solution Part (a)

Step 1: Determine the error, $E(s)$.

$$E(s) = X_{set}(s) - X(s)$$

Step 2: Define the plant input, $I(s)$.

$$I(s) = K_P(X_{set}(s) - X(s))$$

Step 3: Multiply the input by the plant transfer function and set it equal to the output $X(s)$.

$$K_P(X_{set}(s) - X(s))\left(\frac{1}{2s^2 + 8s + 72}\right) = X(s)$$

Step 4: Solve for the transfer function.

$$\frac{X(s)}{X_{set}(s)} = \frac{K_P}{2s^2 + 8s + 72 + K_P} \tag{10.9}$$

Part (b)
The open loop transfer function for the plant alone is $\frac{1}{2s^2+8s+72}$. Rearranging the denominator into the standard form gives $\frac{1}{2(s^2+4s+36)}$, we find the natural frequency.

$$\omega_n = \sqrt{36} = 6\,\text{rad/s}.$$

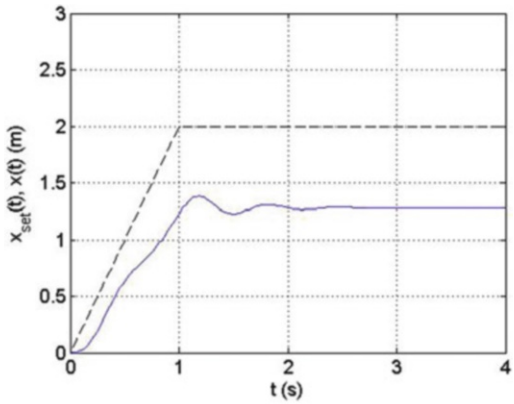
Part (c)
Rearranging the denominator of the closed loop system, we obtain the natural frequency $\omega_n = \sqrt{36 + \frac{K_P}{2}}$. Setting this equal to 10 rad/s we obtain $\sqrt{36 + \frac{K_P}{2}} = 10$ and the proportional gain is $K_P = 128$.

Part (d)
The damping ratio is given by $2\zeta\omega_n = 4$, or $\zeta = 0.2$. The time constant for the second-order underdamped system is $\tau = \frac{1}{\zeta\omega_n} = \frac{1}{0.2 \cdot 10} = 0.5\,\text{s}$.
Part (e)
The MATLAB® code used to plot the input and output is provided.

```
clear all
clc
close all

% Input parameters
KP = 128;                  % proportional gain
x_set = 2;                 % step input, m
ramp_rate = 2;             % ramp rate to step, m/s

% System
num = [KP];
den = [2 8 72 + KP];
sys = tf(num, den);

% Derived parameters
wn = sqrt(36 + KP/2);      % natural frequency, rad/s
zeta = 4/(2*wn);           % damping ratio
tau = 1/(zeta*wn);         % time constant, s
t = [0:2/100:4];           % time, s

% Ramp and hold input
u = x_set*ones(1, length(t));
index = find(t < 1);
u(index) = ramp_rate*t(index);

figure(1)
plot(t, u)
set(gca, 'FontSize', 14)
xlabel('t (s)')
ylabel('x_{set}(t) (m)')
grid
```

```
axis([0 max(t) 0 1.5*x_des])

% Response of system to input
[x, t] = lsim(sys, u, t);

figure(2)
plot(t, u, 'k-', t, x)
set(gca, 'FontSize', 14)
xlabel('t (s)')
ylabel('x_{set}(t), x(t) (m)')
axis([0 max(t) 0 1.5*x_set])
grid
```

The results are displayed.



The system does not follow the desired initial ramp and reaches a steady state value of 1.28 m (determined by evaluating `x(length(t))` in MATLAB®). When compared to the commanded value of 2 m, the steady state error is 0.72 m. Applying the final value theorem to Eq. (10.9), we calculate the steady state error for a step input: $e_{ss} = X_{set} - x_{ss} = 2\left(1 - \frac{128}{72+128}\right) = 0.72$.

Part (f)
Increasing K_P to 1280 increases the natural frequency and decreases the damping ratio for the closed loop system. The MATLAB® code used to calculate the new natural frequency and damping ratio and display the new system response is given.

```
clear all
clc
close all

% Input parameters
KP = 1280;                    % proportional gain
```

```
x_set = 2;                % step input, m
ramp_rate = 2;            % ramp rate to step, m/s

% System
num = [KP];
den = [2 8 72 + KP];
sys = tf(num, den);

% Derived parameters
wn = sqrt(36 + KP/2);     % natural frequency, rad/s
zeta = 4/(2*wn);          % damping ratio
tau = 1/(zeta*wn);        % time constant, s
fprintf('The natural frequency of the system is %4.1f rad/s and the
damping ratio is %4.2 f.\n', wn, zeta)
t = [0:2/100:4];          % time, s

% Ramp and hold input
u = x_set*ones(1, length(t));
index = find(t < 1);
u(index) = ramp_rate*t(index);

figure(1)
plot(t, u)
set(gca, 'FontSize', 14)
xlabel('t (s)')
ylabel('x_{set}(t) (m)')
grid
axis([0 max(t) 0 1.5*x_des])

% Response of system to input
[x, t] = lsim(sys, u, t);

figure(2)
plot(t, u, 'k-', t, x)
set(gca, 'FontSize', 14)
xlabel('t (s)')
ylabel('x_{set}(t), x(t) (m)')
axis([0 max(t) 0 1.5*x_set])
grid
```

The results are shown.

```
The natural frequency of the system is 26.0 rad/s and the damping
ratio is 0.08.
```

Figure 10.4 shows that the system now follows the commanded motion more closely and reaches a final value of 1.89 with a corresponding error of 0.11. This matches the result obtained by substituting into Eq. (10.8). Note that the natural frequency increased from 10 to 26 rad/s (4.1 Hz) and the damping ratio decreased

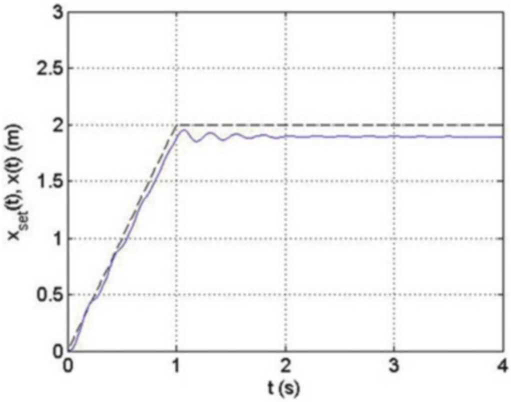
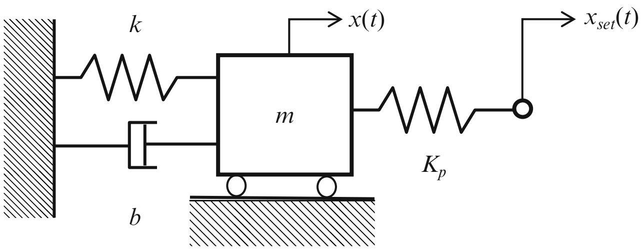


Fig. 10.4 Commanded system motion, $x_{set}(t)$ (dotted line), and actual system motion, $x(t)$ (solid line), plotted as a function of time. Since $x_{set}(t) > x(t)$ for all time, the error $e(t) = x_{set}(t) - x(t)$ is always positive

from 0.2 to 0.08 with the increase K_P. The more lightly damped oscillations are evident in the response plot. To obtain a response that more accurately follows the input, we sacrificed damping. Notice also in Fig. 10.4 that the system motion, $x(t)$, lags the commanded input so that the error is biased in the positive direction. This will be discussed further.

There is a physical explanation for the results obtained in this example. The transfer function given in Eq. (10.9) describes the modified spring-mass-damper system shown here with $m = 2$ kg, $k = 72$ N/m, and $b = 8$ N-m/s.



The closed loop controller produced by systems with proportional gain and a force actuator is equivalent to adding a mechanical spring with stiffness, K_P, and commanding the input, $x_{set}(t)$, to act through this spring. When the input, $x_{set}(t)$, reaches a steady value, x_{des}, the system arrives at a new static equilibrium, x_{ss}, where the forces in the two springs are balanced.

$$kx_{ss} = K_P(x_{des} - x_{ss}) \tag{10.10}$$

Solving Eq. (10.10) for the steady state position, x_{ss}, and subtracting this value from the desired position, we obtain the error in position:

$$e_{ss} = x_{des} - x_{ss} = x_{des}\left(1 - \frac{\frac{K_P}{k}}{1 + \frac{K_P}{k}}\right). \tag{10.11}$$

Equation (10.11) is equivalent to Eq. (10.8). Substituting K_P=1280 and k=72 N/m from Example 10.2(f) in Eq. 10.11 gives a result of 0.11 m which agress with the results displayed in Fig. 10.4.

If there is a steady state error, why did we implement a feedback loop at all? We could have operated the system in an open loop configuration. The open loop system in this example has a stiffness of 72 N/m. Therefore, we could have applied a step input force of 144 N and the system would have responded by moving to $x_{ss} = 2\,\text{m}$, which is the desired position. However, open loop control presumes perfect knowledge of the system. There are two primary reasons for implementing closed loop control:

- imperfect knowledge of the system.
- imperfect knowledge of external disturbances.

In order to prescribe the required input force for the open loop system, the spring stiffness must be known perfectly. If the stiffness was 100 N/m, rather than 72 N/m, then a force input of a 144 N step would produce a displacement of only 1.44 m and the error would be 0.56 m. For the closed loop case with a proportional gain of 1280, the error would only be 0.14 m for a spring stiffness of 100 N/m (recall that the error was 0.11 m for the specified stiffness of 72 N/m). The error for the closed loop system remains small and nearly the same when k is not accurately known. We conclude that the closed loop system is more robust to imperfect knowledge of the plant and is also robust to changes in the plant or operating conditions over time. As we will discuss, closed loop control is also more robust to changes in external disturbances which are typically unknown.

Minimizing the steady state error is often the primary goal of the feedback loop. Increasing the proportional gain, K_P, will drive the steady state error toward zero and achieve this objective. However, it will also decrease the damping, so the system will oscillate with larger amplitude and require a greater number of oscillations to reach steady state. The introduction of integral (I) and derivative (D) terms in the controller, referred to collectively as PID, also reduces the steady state error, but without sacrificing damping.

10.4 Feedback Loops with Proportional, Integral, and Derivative Gains (PID Control)

Consider now a modification of Fig. 10.3b in which the sensor transfer function is set to unity, but the controller is modified to include the proportional-integral-derivative gain, $K_P + \frac{K_I}{s} + K_D s$. This is displayed in Fig. 10.5.

The closed loop transfer function is determined as demonstrated previously. The error is multiplied by the controller gain and then used as input to the plant which subsequently produces an equation for the output.

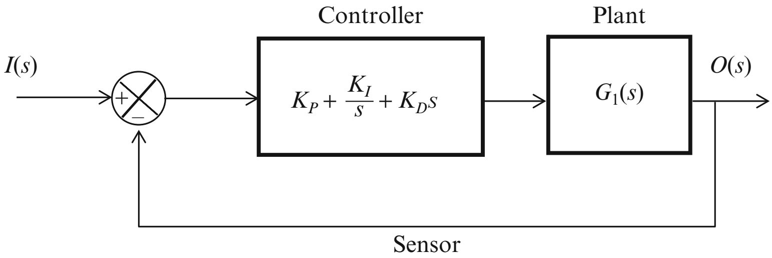


Fig. 10.5 Block diagrams of a PID control system architecture

$$(I(s) - O(s))\left(K_P + \frac{K_I}{s} + K_D s\right)G_1(s) = O(s) \tag{10.12}$$

Equation (10.12) is rearranged algebraically to identify the closed loop transfer function.

$$\frac{O(s)}{I(s)} = \frac{(K_D s^2 + K_P s + K_I)G_1(s)}{K_D s^2 G_1(s) + (1 + K_P G_1(s))s + K_I G_1(s)} \tag{10.13}$$

We now examine the behavior of this new transfer function in more detail.

As stated previously, the goal of the closed loop system is to follow a commanded input. It is desired that the output, $O(s)$, follows the input with an error, $E(s) = I(s) - O(s)$, that satisfies the system specifications. For a swimming pool, a 0.5 °C error (or more) may be acceptable. This error level may not be adequate for a precision metrology laboratory, however. For a position control system on a painting robot, 0.1 mm might be an acceptable maximum error. For an ultra-precision machine tool, on the other hand, 0.01 μm is a typical target. The controller gains, K_P, K_I, and K_D, are selected to meet the application requirements.

Let us examine the steady state response of the system with PID control to a step input command, $I(s) = \frac{O_{des}}{s}$. Again, the final value theorem is applied to determine the error between the commanded step input and steady state output.

$$e_{ss} = O_{des} - o_{ss} = O_{des} - \lim_{s \to 0}(sO(s)) = O_{des} - \frac{K_1 G_1(0)}{K_1 G_1(0)} O_{des} = 0 \tag{10.14}$$

We see that the addition of the integral term, $\frac{K_I}{s}$, leads to steady state step response that exactly matches the commanded/input step; this is theoretically true for any value of K_I. Compare this result to Eq. (10.8), where the steady state error can only be driven toward zero by making the proportional gain infinitely large. As we saw in Example 10.2, this approach will cause the damping ratio of the closed loop system to decrease, so we trade a more accurate steady state step response for more oscillatory behavior and decreased stability. Adding the integrator does not require this tradeoff.

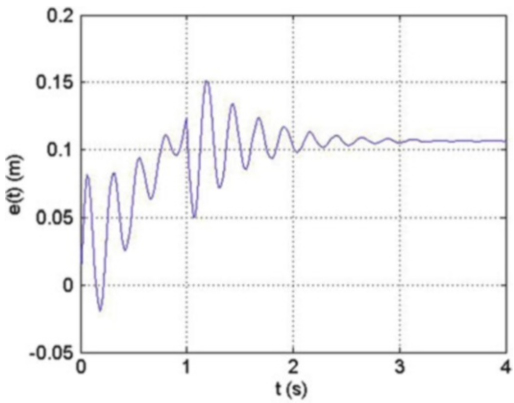


Fig. 10.6 The position error for the Example 10.2 system with a proportional gain of 1280

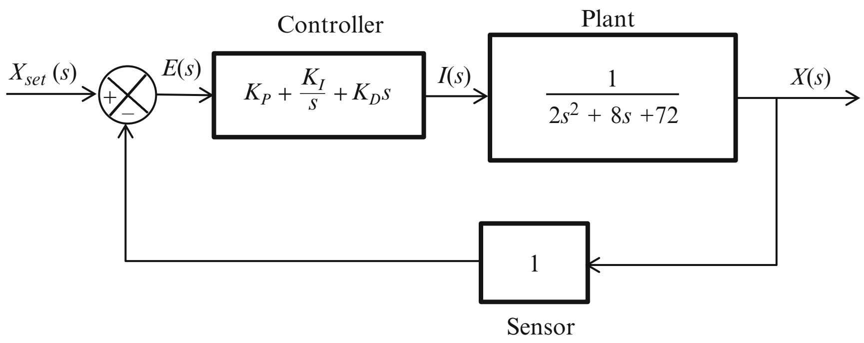
Why does the addition of integral feedback have this effect? Figure 10.4 demonstrates the physical answer. The system output perpetually lags the commanded input so that the error is biased in the positive direction, $e(t) > 0$. To observe this more clearly, the following lines are added to the code provide to solve Example 10.2(f).

```
e = u' - x;
figure(3)
plot(t, e)
set(gca, 'FontSize', 14)
xlabel('t (s)')
ylabel('e(t) (m)')
grid
```

The result is displayed in Fig. 10.6.

We have already noted that proportional control enables this steady state error to approach a value of 0.11 m. An equilibrium is reached in which input control signal is balanced by the natural restoring force applied by the spring stiffness. When an integrator is added, the control signal includes a term that is the integral of the error: $K_I \int e(t)dt$. This feedback is no longer proportional to the error, $e(t)$, but rather is proportional to the area under the error versus time curve. Therefore, even a small positive steady state error will "add up" over time to produce a correcting control signal. Unlike the proportional error signal, the integrator in the control loop eliminates presisting steady state error. The rate at which the system responds to such small errors depends on the value of the constant, K_I.

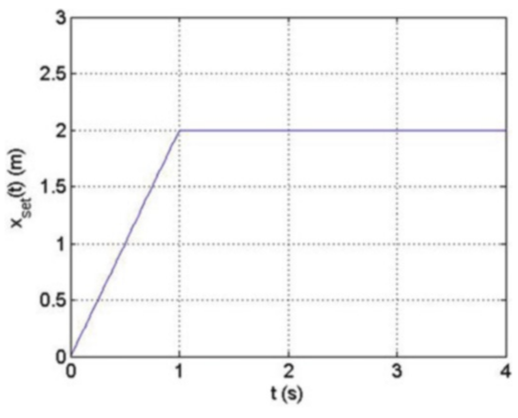
Example 10.3 Consider the plant in Example 10.2, but now with a PID position control system. The input is a desired position value, $X_{set}(s)$, and the output is the actual position of the system, $X(s)$. The sensor gain is unity.



(a) Find the transfer function, $X(s)/X_{\mathrm{set}}(s)$, in terms of the controller gains.

(b) Calculate the roots of the characteristic equation (denominator) and interpret the results for three cases: Case 1: $K_P = 128$, $K_I = 0$, $K_D = 0$; Case 2: $K_P = 128$, $K_I = 10$, $K_D = 10$; Case 3: $K_P = 128$, $K_I = 300$, $K_D = 10$.

(c) Use MATLAB® to find the response of the closed loop system to the following input for Cases 1 and 2.



Solution

Part (a)

We could use Eq. (10.13) with the plant transfer function, $G_1(s) = \frac{1}{2s^2+8s+72}$, to identify the closed loop transfer function directly. However, we will complete the block diagram algebra to demonstrate the process for a PID system.

Step 1: Calculate $E(s)$.

$$E(s) = X_{set}(s) - X(s)$$

Step 2: Calculate $I(s)$.

$$I(s) = \left(K_P + \frac{K_I}{s} + K_D s\right)(X_{set}(s) - X(s))$$

Step 3: Set the plant output equal to $X(s)$.

$$\left(K_P + \frac{K_I}{s} + K_D s\right)(X_{set}(s) - X(s))\left(\frac{1}{2s^2 + 8s + 72}\right) = X(s)$$

Step 4: Solve for the transfer function. We show some of the algebra steps as a recommended procedure. Elimination of the denominators to produce a single polynomial multiplier on either side of the equation, followed by cross multiplication, is an efficient approach that avoids many algebraic errors.

$$\left(K_P + \frac{K_\mathrm{I}}{s} + K_D s\right)(X_{set}(s) - X(s)) = X(s)(2s^2 + 8s + 72)$$
$$(sK_P + K_\mathrm{I} + K_D s^2)(X_{set}(s) - X(s)) = X(s)(2s^3 + 8s^2 + 72s)$$
$$(sK_P + K_\mathrm{I} + K_D s^2)X_{set}(s) = X(s)(2s^3 + (8 + K_D)s^2 + (72 + K_P)s + K_\mathrm{I})$$
$$\frac{X(s)}{X_{set}(s)} = \frac{K_D s^2 + K_P s + K_I}{2s^3 + (8 + K_D)s^2 + (72 + K_P)s + K_I}$$

Notice that the introduction of the PID controller raises the order of the system by one from second to third.

Part (b)
Use the `roots` command in MATLAB®.

```
clear all
clc
close all

% Parameters
% Case 1
KP = 128;          % proportional gain
KI = 0;            % integral gain
KD = 0;            % derivative gain
den1 = [2 8+KD 72+KP KI];
roots(den1)
```

```
% Case 2
KP = 128;
KI = 10;
KD = 10;
den2 = [2 8+KD 72+KP KI];
roots(den2)

% Case 3
KP = 128;       % proportional gain
KI = 0;         % integral gain
KD = 0;         % derivative gain
den3 = [2 8+KD 72+KP KI];
roots(den3)
```

The results are shown.

```
ans =
  0.0000 + 0.0000i
 -2.0000 + 9.7980i
 -2.0000 - 9.7980i
ans =
 -4.4749 + 8.9177i
 -4.4749 - 8.9177i
 -0.0502 + 0.0000i
ans =
 -3.6430 + 8.6163i
 -3.6430 - 8.6163i
 -1.7141 + 0.0000i
```

Case 1 is the same as Example 10.2. There is an exponential response with a time constant of 0.5 s and a natural frequency of 10 rad/s. The damping ratio is 0.2 so the damped natural frequency is 9.80 rad/s, which matches the imaginary part of the roots. Adding the proportional and derivative gain in case 2 makes the system third order. The complex conjugate pair corresponds to a damped oscillatory response with a time constant of 0.22 s and a frequency of 8.92 rad/s. The third (real) root corresponds to an exponential response with a time constant of 19.92 s, much longer than that for the complex conjugate roots. The increase in the integral gain to 300 in case 3 brings the time constants of the three roots closer together. The complex conjugate pair has a time constant of 0.27 s and an oscillation frequency of 8.61 rad/s, while the real root has a time constant of 0.58 s. We label these two time constants τ_1 and τ_2, respectively, for later discussion.

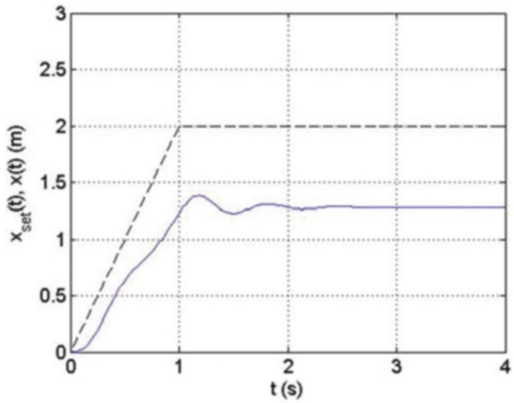
Part (c)

The MATLAB® code used to generate the response of the system for case 3 is provided. Changing the values of the gains gives the solutions for the other two cases.

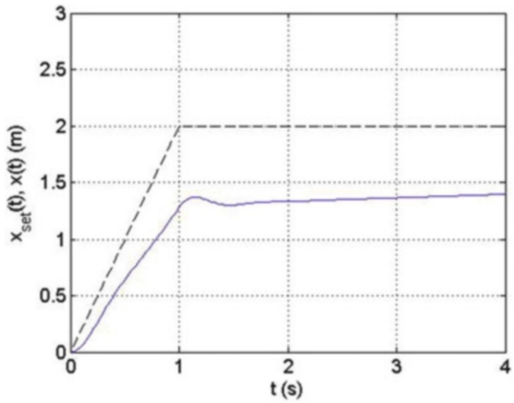
```
clear all
clc
close all

% Controller parameters
% Case 3
KP = 128;      % proportional gain
KI = 300;      % integral gain
KD = 10;       % derivative gain

% Input parameters
x_des = 2;         % size of step input, m
ramp_rate = 2;     % rate of ramp, m/s

% System
num = [KD KP KI];
den = [2 8+KD 72+KP KI];
sys = tf(num,den);

% Time vector
t = [0:2/100:4];   % s
% Ramp and hold input
u = x_des*ones(1, length(t));
index = find(t < 1);
u(index) = ramp_rate*t(index);

figure(1)
plot(t, u)
set(gca, 'FontSize', 14)
xlabel('t (s)')
ylabel('x_{set}(t) (m)')
grid
axis([0 max(t) 0 1.5*x_des])

% Response of system to input
[x, t] = lsim(sys, u, t);

figure(2)
plot(t, u, 'k-', t, x)
set(gca, 'FontSize', 14)
xlabel('t (s)')
ylabel('x_{set}(t), x(t) (m)')
axis([0 max(t) 0 1.5*x_des])
grid
```

The results for the three cases are provided.

Case 1 shows the same response as Example 10.2. A large steady state error is present as discussed previously.



Case 2 shows a response similar to Example 10.2, but the solution slowly converges toward the commanded position.



As shown in Fig. 10.7, case 3 tracks the command more closely and approaches zero steady state error. An oscillation is superimposed onto an exponential response, which is consistent with our evaluation of the system roots. The oscillations associated with the complex roots and the exponential response associated with the real root are excited by the input. The oscillations alternate approximately 1.08 s ($4\tau_1$) after the initial ramp up ends at 1 s. The exponential response to the commanded steady state value of 2.0 alternates approximately 2.32 s ($4\tau_2$) after the initial ramp is completed.

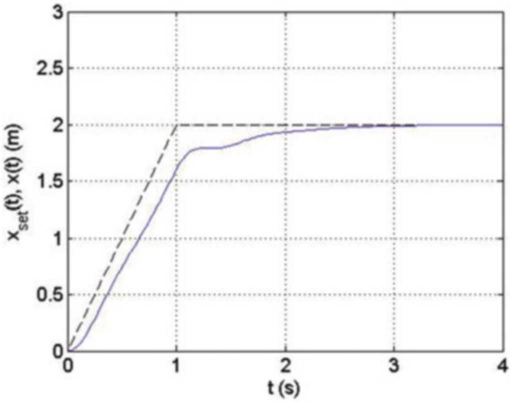


Fig. 10.7 Commanded system motion, $x_{set}(t)$ (dotted line), and actual system motion, $x(t)$ (solid line), plotted as a function of time for PID controller Case 3

It is clear that the case 3 response reaches the commanded value with no error in the shortest time. One practical objective that must be considered is the capability of the actuator to accept the input signal, $I(s)$. Considering the swimming pool from Chap. 9, the speed of the response is increased by the addition of the PID controller. However, the improved response is only possible if the heater does not exceed its maximum power output. "Tuning" the PID controller to obtain the desired/optimal behavior while considering the actuator limitations can be completed. These techniques are best described in a control systems course and are not detailed here.

10.5 Block Diagram Representation of a Permanent Magnet DC Motor

Consider again Eq. (8.26) which provides the approximate second-order response of a permanent magnet DC electric motor to an input voltage and a load torque.

$$\Omega(s) = \left[\frac{K}{JLs^2 + (JR + Lb_r)s + b_rR + KK_b}\right]E_{in}(s) - \left[\frac{R + Ls}{JLs^2 + (JR + Lb_r)s + b_rR + KK_b}\right]M_l(s)$$

This can be represented by Fig. 10.8.

The electrical and mechanical parts of the motor response are first order. The back-emf is interpreted as a feedback loop taking a "measurement" of the angular velocity and producing a voltage, $K_b\Omega(s)$. In fact, one tachometer design is a small

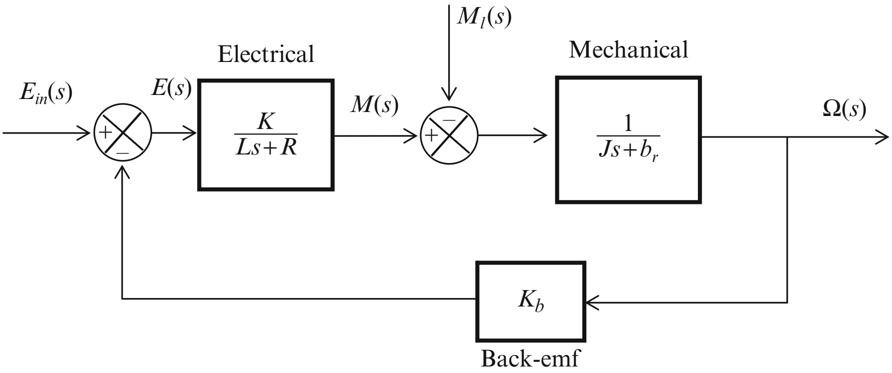


Fig. 10.8 Block diagram for a permanent magnet DC electric motor

electric generator and, therefore, calling this feedback loop a measurement makes practical and physical sense. Subtracting the back-emf from the input voltage at the first summation circle yields the error:

$$E(s) = E_{in}(s) - K_b\Omega(s). \tag{10.15}$$

This error voltage is applied to the electrical portion of the motor to produce a current and, subsequently, a motor torque/moment, $M(s)$:

$$M(s) = K\left(\frac{E(s)}{Ls + R}\right) = KI(s), \tag{10.16}$$

where $I(s)$ was defined in Eq. (8.13). The load torque/moment is then subtracted from the motor torque/moment and the difference is applied to the mechanical part of the motor to produce the angular speed.

$$\frac{M(s) - M_l(s)}{Js + b} = \Omega(s) \tag{10.17}$$

Equation (10.17) is equivalent to Eq. (8.25). Combining Eqs. (10.15), (10.16), and (10.17) gives Eq. (8.26). The block diagram shown in Fig. 10.8 is a pictorial representation of a permanent magnet DC motor and we have gained additional understanding of the motor, particularly the back-emf feedback loop from this representation.

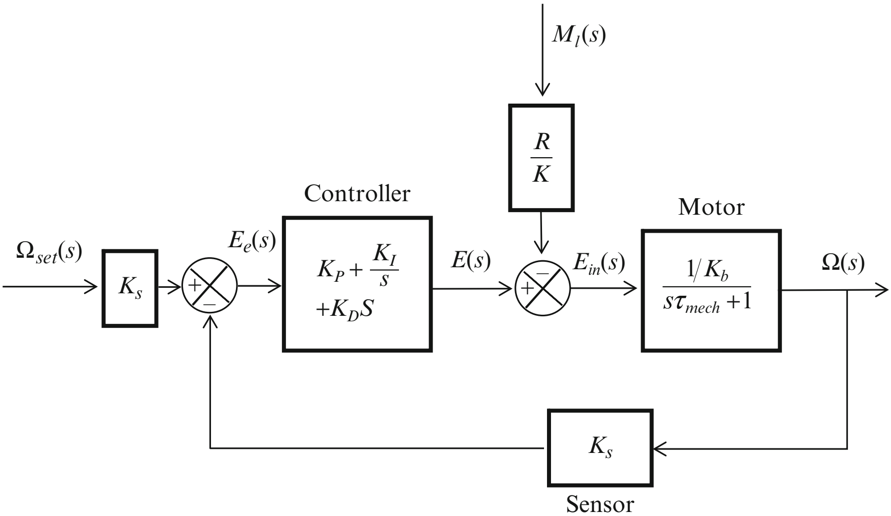


Fig. 10.9 Block diagram for a servomotor with a load torque/disturbance

10.6 Application of Block Diagrams to Servomotor Control

A DC servomotor is a DC motor with additional velocity feedback so that the motor will reach the commanded speed, $\Omega_{set}(s)$. The block diagram of a DC servomotor with PID control is shown in Fig. 10.9. The motor is modeled as a first-order system. The load moment torque is modeled as a disturbance that gets converted to a disturbance voltage which is then subtracted from the voltage out of the controller. This load moment can be considered to be a known load or as an unknown disturbance (or both) and the purpose of the control loop is to maintain the motor speed in the presence of the disturbance.

The goal for a servomotor is that the actual angular speed tracks the commanded angular speed, $\omega_{set}(t)$. To analyze the system behavior, we calculate $\Omega(s)$ using block diagram algebra. We first find $E_e(s)$.

$$E_e(s) = K_s(\Omega_{set}(s) - \Omega(s)) \tag{10.18}$$

The error voltage is the difference between the desired and actual speeds multiplied by the sensor gain. Next we find $E(s)$, the voltage out of the controller.

$$E(s) = E_e(s)\left(K_P + \frac{K_I}{s} + K_D s\right) \tag{10.19}$$

The voltage into the motor, $E_{in}(s)$, is the difference between the voltage out of the controller and the disturbance voltage:

$$E_{in}(s) = E(s) - \frac{R}{K}M_l(s). \tag{10.20}$$

Finally, we find $\Omega(s)$ in terms of the input voltage.

$$E_{in}(s)\left(\frac{1/K_b}{\tau_{mech}s + 1}\right) = \Omega(s) \tag{10.21}$$

Combining Equations 10.20 and 10.21:

$$\left(\frac{1/K_b}{\tau_{mech}s + 1}\right)E(s) - \left(\frac{1/K_b}{\tau_{mech}s + 1}\right)\left(\frac{R}{K}M_l(s)\right) = \Omega(s). \tag{10.22}$$

Note that Eq. (10.22) is the equation of motion for a first-order motor model subjected to a load/disturbance moment, $M_l(s)$. The expression $\frac{R}{K}M_l(s)$ has units of voltage, which emphasizes that it is an additional load voltage caused by the external load/disturbance. Next, we combine Eq. (10.22) with Eq. (10.19).

$$E_e(s)\left(K_D s^2 + sK_P + K_I\right) - \frac{R}{K}M_l(s) = K_b(\tau_{mech}s + 1)\Omega(s) \tag{10.23}$$

Finally, we substitute for $E_e(s)$ using Eq. (10.18) and algebraically rearrange.

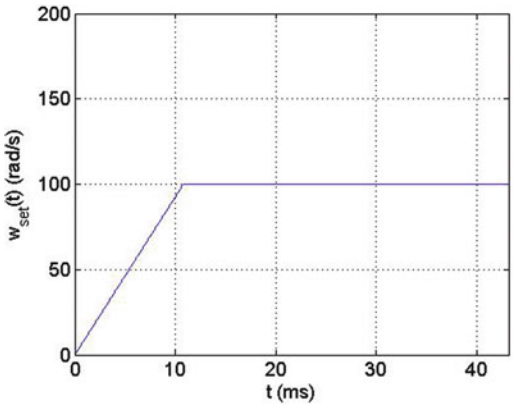
$$\begin{aligned} &\Omega_{set}(s)(K_sK_Ds^2 + K_sK_Ps + K_sK_I) - \left(\frac{Rs}{K}M_l(s)\right) \\ &= ((K_sK_D + K_b\tau_{mech})s^2 + (K_sK_P + K_b)s + K_sK_I)\Omega(s) \end{aligned} \tag{10.24}$$

Because the algebra is involved even for this relatively simple controller, it is useful to check the units of the individual terms to identify errors. For the controller to produce a voltage with a voltage input, K_P must be unitless, K_D must have units of seconds, and K_I must have units of inverse seconds. Taking this into consideration, we can show that each term in Eq. (10.24) has units of V/s. Finally, the expression for the angular velocity in the Laplace domain is determined.

$$\begin{aligned} \Omega(s) = &\left(\frac{K_sK_Ds^2 + K_sK_Ps + K_sK_I}{(K_sK_D + K_b\tau_{mech})s^2 + (K_sK_P + K_b)s + K_sK_I}\right)\Omega_{set}(s) \\ &+\left(\frac{Rs/K}{(K_sK_D + K_b\tau_{mech})s^2 + (K_sK_P + K_b)s + K_sK_I}\right)M_l(s) \end{aligned} \tag{10.25}$$

The denominator of each term in Eq. (10.25) is the characteristic equation of the closed loop system. The addition of the PID controller raises the order of the system by one from first to second order. Since the characteristic equation is second order, the system can be characterized by a damping ratio and natural frequency as we discuss in Example 10.4.

Example 10.4 Consider motor 2 from Example 8.3 with the constants: $K = 0.42$ N-m/A, $K_b = 0.42$ V-s, $R = 4.9$ Ω, $L = 11.2$ mH, $J = 3.883 \times 10^{-4}$ kg-m^2, $b_r = 0.03$ N-m/krpm (2.865×10^{-4} N-m-s). Thus, the mechanical time constant of the motor is 10.8 ms and the electrical time constant is 2.3 ms, but is neglected. Apply closed loop velocity control as shown in Fig. 10.8 so that the motor becomes a servomotor. The sensor constant, K_s, is 1 and we assume there is no disturbance moment ($M_l = 0$). Consider an input that ramps up to 100 rad/s in one mechanical time constant as shown.



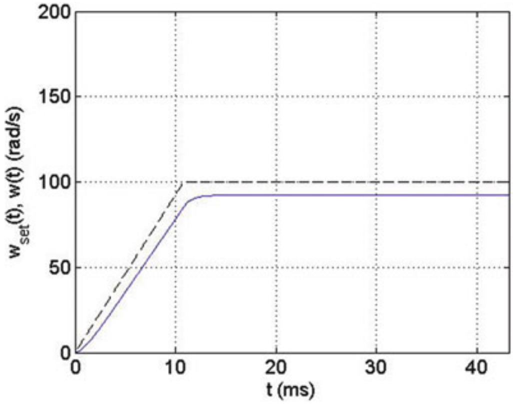
Use MATLAB® to find and plot the system response for two cases: (1) $K_P = 5$, $K_I = 0$, $K_D = 0$; and (2) $K_P = 5$, $K_I = 500$, and $K_D = 0$.

Solution The MATLAB® code is given for case 1. For case 2, we change the controller parameters.

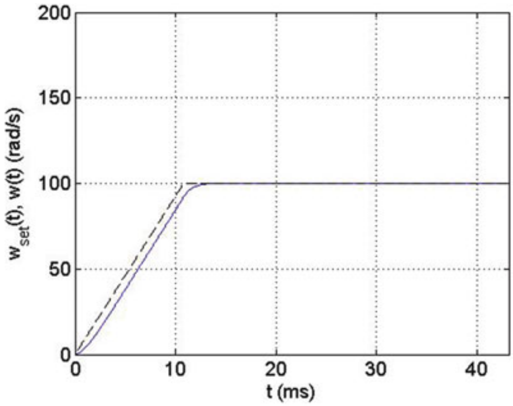
```
clear all
clc
close all

% Parameters
K = 0.42;                % N-m/A
Kb = 0.42;               % V/rad/s
R = 4.9;                 % Ohms
L = 11.2e-03;            % H
J=3.883e-04;             % kg-m^2
```

```
br = 0.03/1000/(2*pi)*60;  % N-m-s

% Controller parameters
Ks = 1;
KP = 5;
KI = 0;
KD = 0;

% Calculated parameters
tau_elec = L/R;             % neglected
tau_mech = J*R/(K*Kb);      % s
wn = sqrt(Ks*KI/(Ks*KD+Kb*tau_mech))
zeta = 1/(2*wn)*(Ks*KP+Kb/(Ks*KD+Kb*tau_mech))

% Input parameters
w_des = 100;                % input voltage amplitude
maxt = tau_mech;
ramp_rate = 100/maxt;       % ramp rate
t = [0:maxt/1000:4*maxt];
w_set = w_des*ones(1,length(t));
index = find(t < maxt);
w_set(index) = ramp_rate*t(index);

figure(1)
plot(t*1000, w_set)
set(gca, 'FontSize', 14)
xlabel('t (ms)')
ylabel('w_{set}(t) (rad/s)')
axis([0 max(1000*t) 0 200])
grid

% Define the system
num = [Ks*KD Ks*KP Ks*KI];
den = [(Ks*KD+Kb*tau_mech) (Ks*KP+Kb) Ks*KI];
roots(den)
sys = tf(num, den);

% Find the system response
[w,t] = lsim(sys, w_set, t);

figure(2)
plot(t*1000, w, t*1000, w_set, 'k-')
set(gca, 'FontSize', 14)
xlabel('t (ms)')
ylabel('w_{set}(t), w(t) (rad/s)')
axis([0 max(1000*t) 0 200])
grid
```

The results are given for case 1.



The results are also provided for case 2.



For case 1 with just proportional gain, a large steady state error is present. Adding integral gain for case 2 eliminates the steady state error.

10.7 Summary

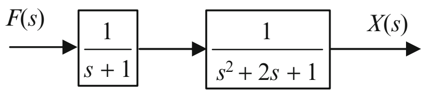
In this chapter, we discussed the following key concepts.

- Block diagrams provide a symbolic language that enables systems of different physical nature to be represented by a common framework.

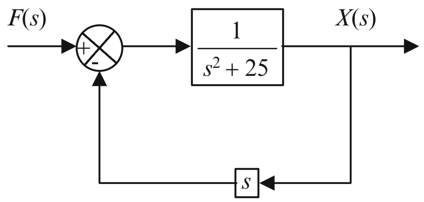
- Block diagrams contain summation circles which represent addition and subtraction operations and blocks which represent multiplication.
- In a block diagram, system components having different physical behavior (for example, mechanical and electrical components) can be represented together and the interconnections between the system elements are made clear.
- Block diagrams are the fundamental language of control systems and feedback control loops can be represented by connections between inputs and outputs in the block diagram.
- Basic feedback control "closes the loop" on a system so that a commanded behavior becomes the input to the system, the output is measured and compared to the commanded behavior, and the error between the commanded and actual behavior is used to cause the system output to follow the commanded input.
- In proportional-integral-derivative (PID) control, the error is used to produce a corrective system input signal that is a superposition of a signal proportional to the error, a signal proportional to the time derivative of the error, and a signal proportional to the integral of the error.
- While the final result depends on where the terms appear in the characteristic equation, in PID control, proportional feedback generally changes the system stiffness and derivative feedback generally changes the system damping.
- In PID control, the integrator raises the order of the original system by one and is used to drive steady state errors to zero.

Problems

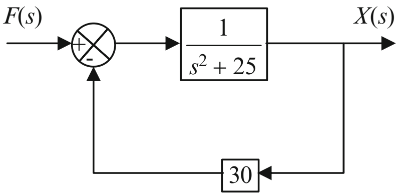
1. Find the transfer function $\frac{X(s)}{F(s)}$ for the block diagram shown in the figure. Write a MATLAB® script file that plots the response of the system to a unit step input.



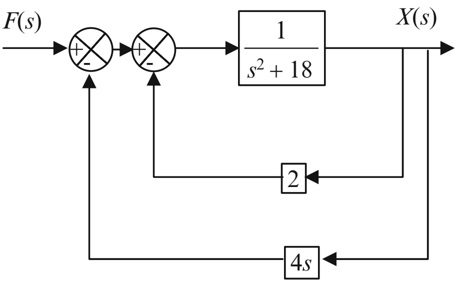
2. Find the transfer function $\frac{X(s)}{F(s)}$ for the block diagram shown in the figure. Write a MATLAB® script file that plots the response to a unit step input.



3. Find the tranfer function $\frac{X(s)}{F(s)}$ for the block diagram shown in the figure. Write a MATLAB® script file that plots the response to a step input $F(t) = 20 \cdot u(t)$.

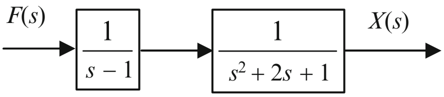


4. Find the transfer function $\frac{X(s)}{F(s)}$ for the block diagram shown in the figure. Write a MATLAB® script file that plots the response to an impulse input $F(t) = 20 \cdot \delta(t)$.

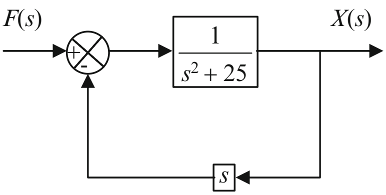


5. Find the transfer functions for the following block diagrams. Use the `roots` command in a MATLAB® to determine the roots of the characteristic equation and classify the system as **stable**, **borderline stable**, or **unstable** based on the result. For (b) and (c), also calculate the damping ratio, ζ, and natural frequency, ω_n.

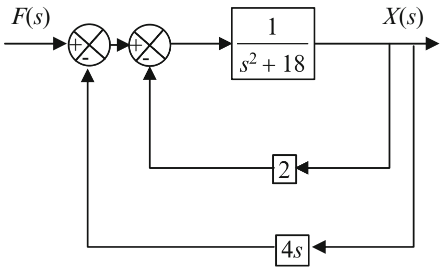
a



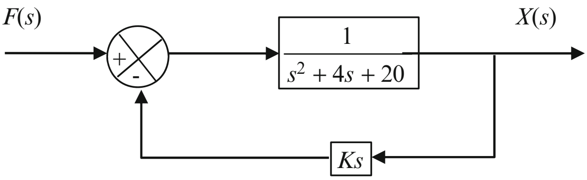
b



c



6. Consider the following block diagram model of a dynamic system where K is a constant feedback parameter.



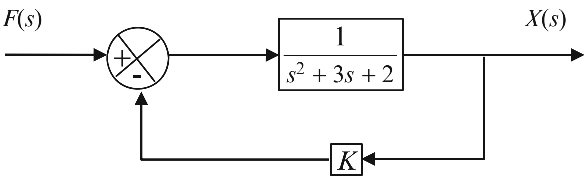
Complete the following.

(a) Calculate the transfer function $\frac{X(s)}{F(s)}$ in terms of the feedback parameter K.

(b) For the case where $K = 0$ and the input, $f(t)$, is a unit delta function, $\delta(t)$, calculate $x(t)$ using Laplace transforms.

(c) Calculate the value of K that is required for the system to be critically damped ($\zeta = 1$).

7. Consider the following block diagram model of a dynamic system where K is a constant feedback parameter.

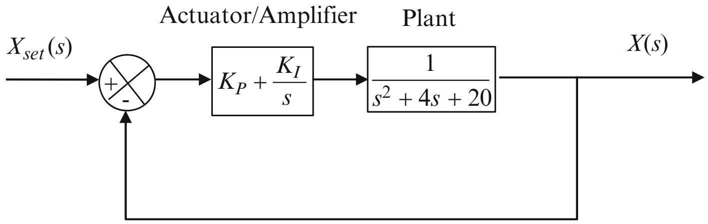


Complete the following.

(a) Calculate the transfer function $\frac{X(s)}{F(s)}$ in terms of the feedback parameter K.

(b) Calculate $x(t)$ using Laplace transforms if $F(t) = \delta(t)$ and $K = 0$ and, then, $K = 10$.

8. The following block diagram shows the closed loop dynamics of a positioning system consisting of a second-order plant, a feedback loop, and an actuator/amplifier with proportional integral gain.



Complete the following.

(a) Calculate the natural frequency, ω_n, and damping ratio, ζ, for the plant.

(b) Calculate the transfer function $\frac{X(s)}{X_{set}(s)}$.

(c) If the integral gain constant, K_I, is set to zero, calculate the value of K_P required for the natural frequency of the closed loop system to be 6 rad/s.

(d) If the proportional gain constant, K_P, is equal to 20 and the commanded position is a step input $x_{set} = 3 \cdot u(t)$, calculate the steady state error where the integral gain constant, K_I, is equal to zero.

(e) Repeat part (d) when the integral gain constant, K_I, is equal to 1.

Frequency Domain Analysis 11

11.1 Introduction

So far, we have mostly examined the response of dynamic systems to relatively simple inputs: impulses, steps, and ramps. While we have used MATLAB® to calculate the response of systems to more complex inputs numerically, there are also analytical methods that can be used to identify the response of systems to inputs of a general form. The most common analytical method is *frequency domain analysis*: the analysis of the steady state response for a linear system to *sinusoidal* (harmonic) *inputs*.

Frequency domain analysis is powerful. First, it enables us to determine the response of a system to a sinusoidal input function, $f(t)$, with a selected frequency, ω, amplitude, A, and phase, ϕ.

$$f(t) = A \sin(\omega t + \phi) \tag{11.1}$$

Many practical inputs closely resemble sinusoids, including a rotating imbalanced mass, the output from a generator, the motions of a reciprocating engine, and vortices shedding from the edges of a bluff body (obstruction). Since these inputs usually persist over a long time, we are able to analyze the steady state system response to a steady sinusoidal input.

A second important reason to apply frequency domain analysis is the availability of *Fourier methods*. *Fourier series* are sums of sinusoidal functions that can be used to construct other repeating (or periodic) signals. Joseph Fourier (1768–1830) showed that any periodic signal could be constructed from a series of sine and cosine functions with appropriate amplitudes. Therefore, any system input, $f(t)$, that satisfies the following condition (i.e., it is periodic):

$$f(t+T) = f(t), \tag{11.2}$$

M.A. Davies, T.L. Schmitz, *System Dynamics for Mechanical Engineers*,
DOI 10.1007/978-1-4614-9293-1_11

where T is the period, can be written as a sum of pure sinusoids. For linear systems, a periodic input will produce an output that is a superposition of the system response to a set of sinusoidal inputs. *Fourier integrals* extend the analysis, enabling nonperiodic functions to be represented by combinations of sinusoids. Consequently, knowing the response of a system to a sinusoidal input allows us to determine the response of that same system to nearly any input.

The objective of this chapter is to find the response of linear dynamic systems to sinusoidal inputs by building on the techniques we have already developed. This then enables us to determine the system response to any periodic input based on Fourier series techniques. It also introduces the concepts of *filtering* and *resonance*, which are critical to dynamic system design and analysis. We will show that the Laplace transform techniques we have already developed conveniently lend themselves to frequency domain analysis.

11.2 Response of Spring–Mass System to a Periodic Input

To demonstrate the approach, we first consider a spring–mass system subjected to a sinusoidal input force with frequency, ω, and amplitude, F_0. The equation of motion for the system is:

$$m\ddot{x} + kx = F_0 \sin(\omega t). \tag{11.3}$$

Rewriting in standard notation, we obtain:

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m} \sin(\omega t), \tag{11.4}$$

where $\omega_n = \sqrt{\frac{k}{m}}$. We see that there are two frequencies in the system, the natural frequency, ω_n, and the sinusoidal force frequency, ω, often called the *driving frequency*.

Calculating the Laplace transform of Eq. (11.4) and assuming zero initial conditions we find $X(s)$.

$$X(s) = \frac{F_0}{m}\left(\frac{\omega}{s^2 + \omega^2}\right)\left(\frac{1}{s^2 + \omega_n^2}\right) \tag{11.5}$$

To solve Eq. (11.5), we use a partial fractions expansion:

$$\left(\frac{\omega}{s^2 + \omega^2}\right)\left(\frac{1}{s^2 + \omega_n^2}\right) = \frac{A}{s^2 + \omega^2} + \frac{B}{s^2 + \omega_n^2}. \tag{11.6}$$

The constants A and B are given by:

$$A = \frac{\omega}{\omega_n^2 - \omega^2}, \qquad B = -\frac{\omega}{\omega_n^2 - \omega^2}. \tag{11.7}$$

Substitution yields:

$$x(t) = \frac{F_0}{m}\left(\frac{1}{\omega_n^2 - \omega^2}\right)\sin(\omega t) + \frac{F_0}{m}\left(\frac{\omega}{\omega_n}\right)\left(\frac{1}{\omega_n^2 - \omega^2}\right)\sin(\omega_n t). \tag{11.8}$$

Equation (11.8) has two terms. The first oscillates at the driving frequency, ω, and the second at the natural frequency, ω_n. Both change magnitude with the driving frequency.

The form of Eq. (11.8) demonstrates several important concepts. First, when the driving frequency approaches zero, the amplitude of the first term becomes $\frac{F_0}{m\omega_n^2}$, or $\frac{F_0}{k}$, and the amplitude of the second term becomes zero. The first term gives the static equilibrium displacement of the spring subjected to an applied force with magnitude F_0. This is expected for a slowly varying force ($\omega \rightarrow 0$). A second observation is that the amplitude of both terms become infinite when the denominator, ${\omega_n}^2 - \omega^2$, becomes zero. This occurs when the driving frequency matches the natural frequency; this condition is known as *resonance*. Resonance occurs when the amplitude of the response of a system to a sinusoidal input reaches a maximum at a particular frequency, called the *resonant frequency*. A second-order underdamped system will exhibit resonant behavior. Higher-order systems may have several resonant frequencies, each one associated with a separate *vibration mode* of the system. An undamped system resonates at its natural frequency as we have just shown. Systems with low damping will resonate near the natural frequency. When the driving frequency is much larger than the natural frequency, both of the denominator terms become large and the combined amplitudes approach zero.

Example 11.1 Consider an undamped oscillator of the form shown in Fig. 11.1. The mass and stiffness are 2 kg and 20000 N/m, respectively. Find the response of

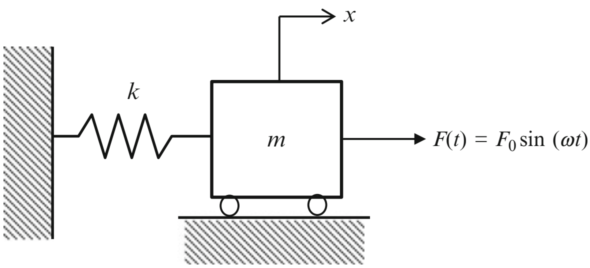


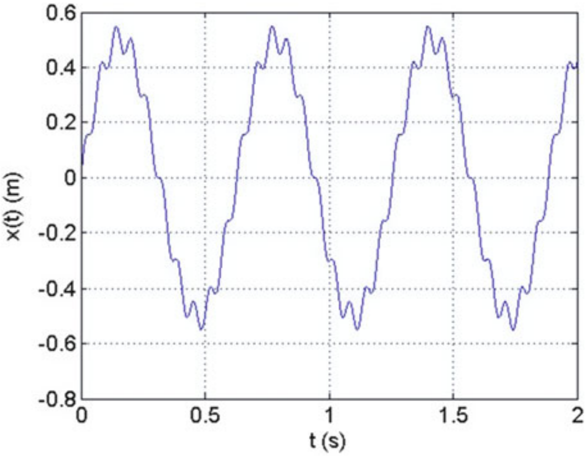
Fig. 11.1 Spring–mass system subjected to a periodic input force with frequency, ω

the system to a sinusoidal input $F_0 \sin(\omega t)$, where $F_0 = 10000$ N for three different driving frequencies of 10, 90, and 300 rad/s. Plot the responses using MATLAB® and comment on the results.

Solution The MATLAB® code is provided.

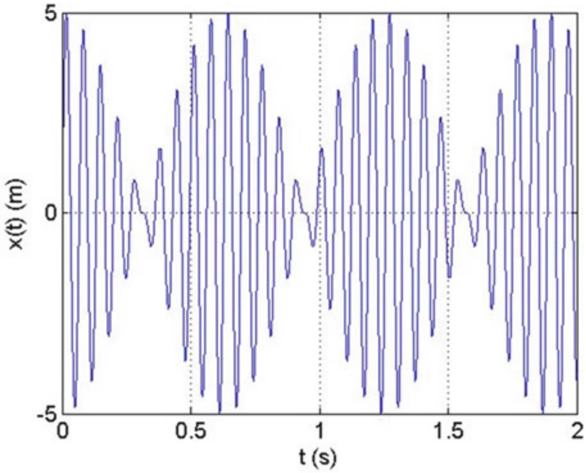
```
clear all
clc
close all

% Parameters
m = 2;              % kg
k = 20000;          % N/m
F0 = 10000;         % N
w = [10 90 300];    % rad/s

% Calculated parameters
wn = sqrt(k/m);     % rad/s
Tn = 1/wn;          % s
t = [0:Tn/100:200*Tn]; % time vector, s

% Response
for cnt = 1:length(w)
  x = F0/m*(1./(wn^2-w(cnt)^2))*sin(w(cnt)*t) +
  F0/m*(w(cnt)/wn)*(1./(wn^2-w(cnt)^2))*sin(wn*t);

  figure(cnt)
  plot(t, x)
  set(gca, 'FontSize', 14);
  xlabel('t (s)')
  ylabel('x(t) (m)')
  grid

  figure(length(w)+1)
  subplot(3, 1, cnt)
  plot(t, x)
  set(gca, 'FontSize', 14);
  if cnt == length(w)
    xlabel('t (s)')
  end
  ylabel('x(t) (m)');
  axis([0 max(t) -5 5])
  grid
  hold on
end
```

The result for the driving frequency of 10 rad/s is displayed.



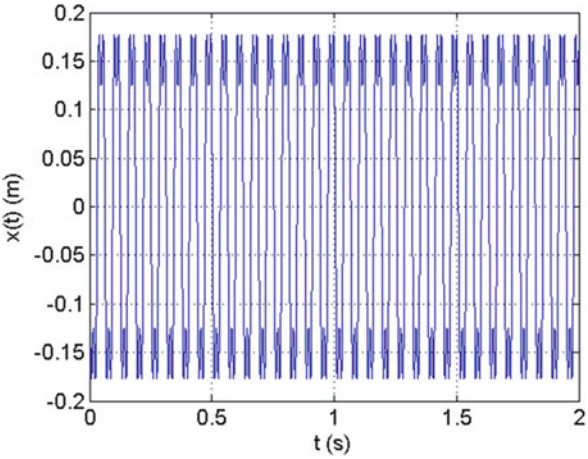
We observe that the low frequency component corresponding to the driving frequency dominates. The higher frequency component at the natural frequency of 100 rad/s is superimposed on the 10 rad/s component. The maximum displacement is only slightly greater than the static equilibrium displacement of 0.5 m.

The result for the driving frequency of 90 rad/s is shown next.

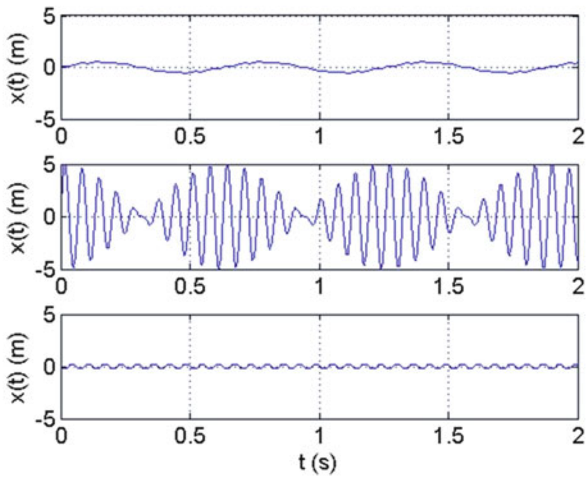


Since the driving frequency of 90 rad/s is close to the natural frequency of 100 rad/s, the amplitude of the response is larger, reaching a value of 5 m. Note the oscillation envelope—the amplitude is modulated over several higher frequency oscillations at approximately 100 rad/s (15.9 Hz). This modulation is called a *beating oscillation* or simply *beating*. From the figure, we see that the modulation period is approximately 0.6 s. The corresponding *beat frequency* is equal to the difference between the interacting signals: $100-90=10$ rad/s (1.6 Hz). The exact period is therefore $1/1.6=0.63$ s.

The response for the 300 rad/s driving frequency is now displayed. The maximum amplitude is equal to only 0.15 m. Since the driving frequency exceeds the resonant frequency, the amplitude is reduced.



To compare the three responses visually, all three are included in a single plot.



The combined plot shows a small amplitude at low frequency, a large amplitude near resonance, and a small amplitude at a frequency beyond resonance. This system demonstrates the concept of resonance, but is undamped. For this reason, the steady state response includes two frequencies, the driving frequency and the natural frequency. Damping collapses the steady state response to a single sinusoid at the driving frequency. The response changes both amplitude and phase (relative to the driving sinusoid) as the driving frequency is varied. These concepts will be demonstrated using the *frequency response function*.

11.3 Frequency Response Functions

The steady state response for damped systems has the same frequency as the driving input, but its amplitude and phase change as the driving frequency is modified. This information is captured by the frequency response function. The frequency response function is obtained from the transfer function of a system, $G(s)$, by replacing the variable, s, with the complex number, $j\omega$, where ω is the frequency of the sinusoidal driving term. The frequency response function, $G(j\omega)$, is a complex number whose magnitude $|G(j\omega)|$ is the magnitude of the sinusoidal system response. The complex phase, ϕ, is given by:

$$\phi = \tan^{-1}(\mathrm{Im}(G(j\omega))/\mathrm{Re}(G(j\omega))). \tag{11.9}$$

The complex frequency response function is typically displayed as the magnitude and phase, but can also be described in terms of its real and imaginary components:

$$G_{\mathrm{Re}}(j\omega) = \mathrm{Re}(G(j\omega)) \tag{11.10}$$

$$G_{\mathrm{Im}}(j\omega) = \mathrm{Im}(G(j\omega)). \tag{11.11}$$

Both representations have useful physical interpretations.

Replacing the Laplace variable, s, with the complex number, $j\omega$, to obtain a complex function that describes the system response to a steady sinusoidal input is proven mathematically as follows [1]. Consider a linear, stable, time-invariant (i.e., the parameters do not vary in time) system with a transfer function, $G(s)$. The input to the system is:

$$f(t) = F_0 \sin(\omega t) \tag{11.12}$$

with the Laplace transform, $F(s)$, that can be factored to have two imaginary roots.

$$F(s) = F_0 \frac{\omega}{s^2 + \omega^2} = F_0 \frac{\omega}{(s + j\omega)(s - j\omega)} \tag{11.13}$$

The response of the system to the input is then given by $X(s)$.

$$X(s) = G(s)F(s) \tag{11.14}$$

The transfer function of the system can be factored in the usual way to produce n poles, $-p_1, -p_2, \ldots, -p_n$, and m zeros, $-z_1, -z_2, \ldots, -z_m$, based on the orders of the numerator and denominator.

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)} \tag{11.15}$$

If the denominator poles are distinct, then Eq. (11.15) enables a partial fractions expansion as described in Chap. 2 and the combination of Eqs. (11.13)–(11.15) produces the following result.

$$X(s) = G(s)\frac{F_0\omega}{s^2+\omega^2} = \frac{a}{s+j\omega} + \frac{\overline{a}}{s-j\omega} + \frac{b_1}{s+p_1} + \frac{b_2}{s+p_2} + \cdots + \frac{b_n}{s+p_n} \tag{11.16}$$

Here a and $\overline{a}$ are complex conjugates and $b_1, b_2, \ldots, b_n$ are constants determined from the expansion. Inverting Eq. (11.16) gives the time domain response.

$$x(t) = a\,\mathrm{e}^{-j\omega t} + \overline{a}\,\mathrm{e}^{j\omega t} + b_1\,\mathrm{e}^{-p_1 t} + b_2\,\mathrm{e}^{-p_2 t} + \cdots + b_n\,\mathrm{e}^{-p_n t} \tag{11.17}$$

If the system is stable, the terms $b_1\,\mathrm{e}^{-p_1 t}, b_2\,\mathrm{e}^{-p_2 t} \ldots b_n\,\mathrm{e}^{-p_n t}$ decay to zero leaving only the first two terms as the steady state response.

$$x(t) = a\,\mathrm{e}^{-j\omega t} + \overline{a}\,\mathrm{e}^{j\omega t} \tag{11.18}$$

The constants a and $\overline{a}$ are determined from Eq. (11.16).

$$\begin{aligned} a &= G(s)\tfrac{F_0\omega}{s^2+\omega^2}(s+j\omega)\Big|_{s=-j\omega} \\ \overline{a} &= G(s)\tfrac{F_0\omega}{s^2+\omega^2}(s-j\omega)\Big|_{s=j\omega} \end{aligned} \tag{11.19}$$

Factoring and substituting the values for s gives:

$$\begin{aligned} a &= G(s)\tfrac{F_0\omega}{s-j\omega}\Big|_{s=-j\omega} = -G(-j\omega)\frac{F_0}{2j} \\ \overline{a} &= G(s)\tfrac{F_0\omega}{s+j\omega}\Big|_{s=j\omega} = G(j\omega)\frac{F_0}{2j}. \end{aligned} \tag{11.20}$$

Finally, we introduce the magnitude and phase representation of $G(j\omega)$.

$$\begin{aligned} G(j\omega) &= G_{\mathrm{Re}}(j\omega) + jG_{\mathrm{Im}}(j\omega) = |G(j\omega)|\mathrm{e}^{j\phi} \\ G(-j\omega) &= |G(-j\omega)|\mathrm{e}^{-j\phi} = |G(j\omega)|\mathrm{e}^{-j\phi} \end{aligned} \tag{11.21}$$

The second expression is obtained because changing the sign on the imaginary part of a complex number does not change its magnitude. Combining Eqs. (11.18), (11.20), and (11.21) and rearranging yields:

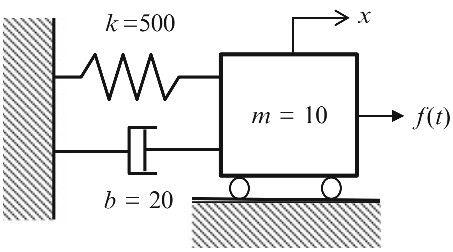
$$\begin{aligned} x(t) &= a\,\mathrm{e}^{-j\omega t} + \overline{a}\,\mathrm{e}^{j\omega t} \\ &= -G(-j\omega)\frac{F_0}{2j}\mathrm{e}^{-j\omega t} + G(j\omega)\frac{F_0}{2j}\mathrm{e}^{j\omega t} \\ &= -|G(j\omega)|\mathrm{e}^{-j\phi}\frac{F_0}{2j}\mathrm{e}^{-j\omega t} + |G(j\omega)|\mathrm{e}^{j\phi}\frac{F_0}{2j}\mathrm{e}^{j\omega t} \\ &= |G(j\omega)|F_0\left(\frac{\mathrm{e}^{j(\omega t+\phi)} - \mathrm{e}^{-j(\omega t+\phi)}}{2j}\right). \end{aligned} \tag{11.22}$$

Applying Euler's formula produces the final result.

$$x(t) = F_0|G(j\omega)|\sin(\omega t + \phi) \tag{11.23}$$

Therefore, the magnitude and phase of the frequency response function, $G(j\omega)$, specifies the magnitude and phase of the steady state response of a stable, time-invariant, linear system to a sinusoidal input.

Example 11.2 Consider again the damped harmonic oscillator examined in Example 4.3 with $m = 10$ kg, $b = 20$ N-s/m, and $k = 500$ N/m, but now with a harmonic input force, $f(t)$.



Write the transfer function, $G(s) = \frac{X(s)}{F(s)}$, by inspection and then express the corresponding frequency response function, $G(j\omega)$, analytically. Calculate the natural frequency, ω_n, the damping ratio, ζ, and the damped natural frequency, ω_d. Use a MATLAB® frequency vector ranging from 0 to $2\omega_n$ and plot the frequency response function in terms of its magnitude and phase and its real and imaginary components. Show that the steady state gain, $G(0)$, is equal to $1/k$ and also find and display the frequency at which the maximum of $|G(j\omega)|$ occurs. Compare this resonant frequency, ω_r, with ω_n and ω_d. Take advantage of the complex number capability of MATLAB® and the relation between the frequency response function and the transfer function.

Solution The transfer function for the system is: $G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2+bs+k}$. The frequency response function is obtained by replacing s with $j\omega$:

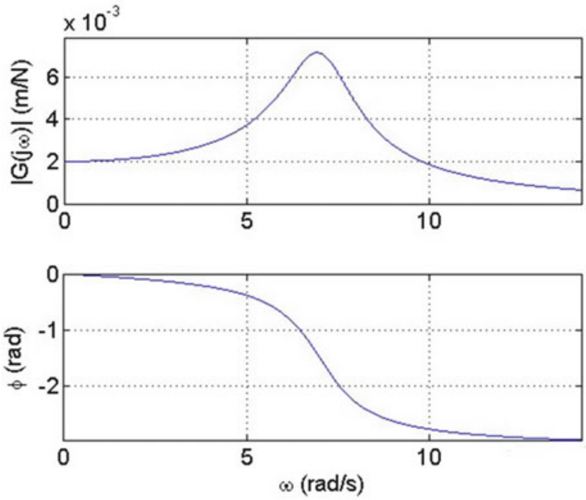
$$G(j\omega) = \frac{1}{-m\omega^2 + k + b\omega j} = \frac{1/m}{\omega_n^2 - \omega^2 + 2\zeta\omega\omega_n j}.$$

The MATLAB® code is provided.

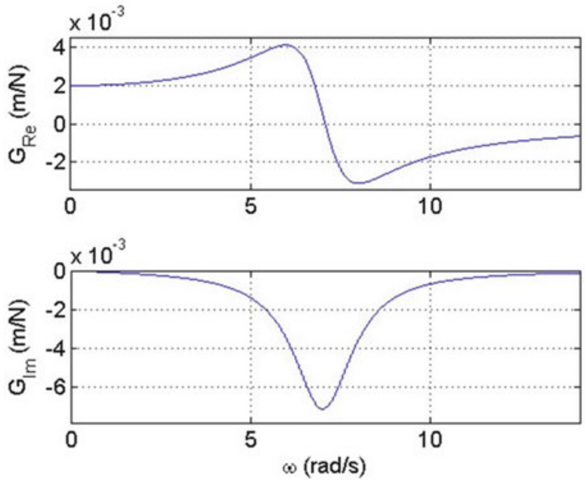
```
clear all
clc
close all

% Parameters
k = 500;     % N/m
b = 20;      % N-s/m
m = 10;      % kg

% Calculated parameters
wn = sqrt(k/m);        % rad/s
fn = wn/(2*pi);        % Hz
T = 1/fn;           % s
zeta = b/(2*sqrt(k*m));
wd = wn*sqrt(1-zeta^2);    % rad/s
tau = 1/(zeta*wn);       % s
fprintf('The natural frequency is %4.2f rad/s.\n', wn)
fprintf('The damping ratio is %4.2 f.\n', zeta)
fprintf('The damped natural frequency is %4.2f rad/s.\n', wd)

% Frequency vector
w = [0:2*wn/10000:2*wn];     % rad/s

% Frequency response function
G = 1./(-m*w.^2 + k + b*w*1j); % m/N
G_Re = real(G);    % m/N
G_Im = imag(G);    % m/N
G_mag = abs(G);    % m/N
phi = angle(G);    % rad
```

```
% Find and display the resonant frequency
index = find(G_mag == max(G_mag));
wr = w(index);
fprintf('The resonant frequency is %4.2 f.\n', wr)

% Magnitude and phase
figure(1)
subplot(211)
plot(w, G_mag);
set(gca, 'FontSize', 14);
ylabel('|G(j\omega)| (m/N)')
axis([min(w) max(w) 0 1.1*max(G_mag)])
grid
subplot(212)
plot(w, phi)
set(gca, 'FontSize', 14);
xlabel('\omega (rad/s)')
ylabel('\phi (rad)')
axis([min(w) max(w) min(phi) max(phi)]);
grid

% Find and display the zero-frequency magnitude
fprintf('The magnitude of the response at zero frequency is %4.2 g.
\n', G_mag(1))
fprintf('The value of 1/k is  %4.2 g.\n',1/k)

% Real and imaginary components
figure(2)
subplot(211)
plot(w, G_Re)
set(gca,'FontSize',14);
ylabel('G_{Re} (m/N)')
axis([min(w) max(w) 1.1*min(G_Re) 1.1*max(G_Re)]);
grid
subplot(212)
plot(w, G_Im)
set(gca,'FontSize',14);
xlabel('\omega (rad/s)')
ylabel('G_{Im} (m/N)')
axis([min(w) max(w) 1.1*min(G_Im) 1.1*max(G_Im)]);
grid
```

The magnitude and phase are shown.



The real and imaginary parts of the response function are also displayed.



The results of the calculations are provided.

```
The natural frequency is 7.07 rad/s.
The damping ratio is 0.14.
The damped natural frequency is 7.00 rad/s.
The resonant frequency is 6.93.
```

```
The magnitude of the response at zero frequency is 0.002.
The value of 1/k is 0.002.
```

The natural frequency, damped natural frequency and resonant frequency are not the same. The magnitude at zero frequency is equal to $\frac{1}{k}$.

Example 11.3 For the same system as in Example 11.2, use the `lsim` command in MATLAB® to simulate the response for a harmonic input with a 50 N magnitude and driving frequencies of $\frac{\omega_n}{2}$, ω_n, and $2\omega_n$. Display the input and output on subplots so that their phases can be compared. Reconcile your results with the magnitude and phase plots from Example 11.2.

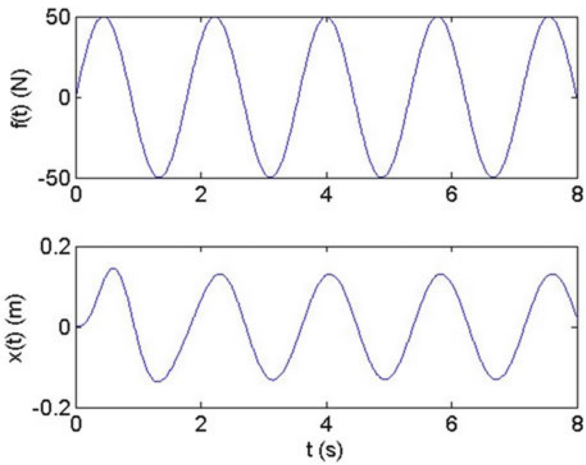
Solution The MATLAB® code used to produce the desired results is given.

```
clear all
clc
close all

% Parameters
k = 500;     % N/m
b = 20;      % N-s/m
m = 10;      % kg
F0 = 50;     % N

% Calculated parameters
wn = sqrt(k/m);         % rad/s
fn = wn/(2*pi);         % Hz
T = 1/fn;            % s
zeta = b/(2*sqrt(k*m));
wd = wn*sqrt(1-zeta^2);    % rad/s
tau = 1/(zeta*wn);      % s

% Time vector
t = [0:tau/100:8*tau];    % s

% Driving frequencies
w = [wn/2 wn 2*wn];      % rad/s

% Define the system
num = [1];
den = [m b k];
sys = tf(num, den);

% Execute the simulation and display the results
for cnt = 1:length(w)
  f = F0*sin(w(cnt)*t);
  figure(cnt)
  [x, t] = lsim(sys, f, t);
```

```
  subplot(211)
  plot(t, f)
  set(gca, 'FontSize', 14);
  ylabel('f(t) (N)')
  subplot(212)
  plot(t, x)
  set(gca, 'FontSize', 14);
  xlabel('t (s)')
  ylabel('x(t) (m)')
end
```

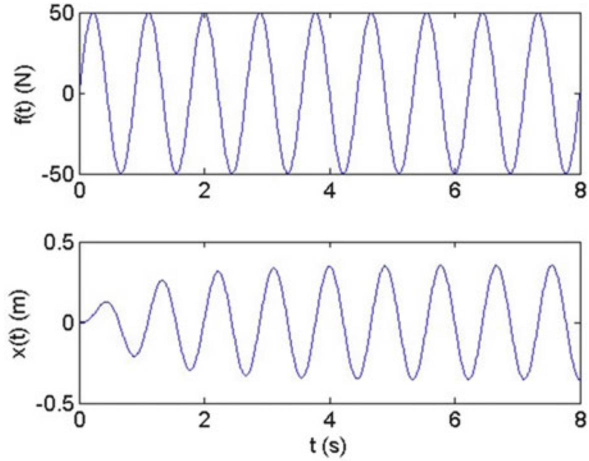
The results are displayed.

The results for a driving frequency, $\omega = \frac{\omega_n}{2} = 3.54\,\text{rad/s}$, are displayed.



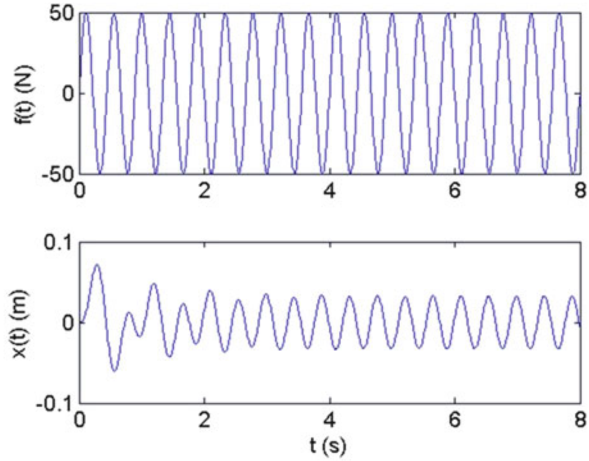
The maximum response amplitude is approximately 0.13 m and the phase lags the input by approximately 10% of one cycle or 0.6 rad. From the Example 11.2 magnitude plot, we see that the frequency response function magnitude value is 0.0027 m/N at 3.54 rad/s. If we multiply this value by the forcing function magnitude of 50 N, we obtain a response magnitude of 0.135 m, which is consistent with the previous plot. The phase shift at this frequency is approximately -0.6 rad, which is consistent with the 0.6 rad lag that we observe.

The results for a driving frequency, $\omega = \omega_n = 7.07$ rad/s are displayed.



The amplitude at this frequency is approximately 0.35 m. Examining the magnitude plot from Example 11.2, we estimate 0.007 m/N for 7.07 rad/s. For a 50 N input, we calculate a response magnitude of 0.35 m. The response lags the input by approximately 25% of one cycle, or $\frac{\pi}{2}$rad, from the figure. We will see that the response does indeed lag the input by $\frac{\pi}{2}$rad at resonance.

The results for a driving frequency, $\omega = 2\omega_n = 14.14$ rad/s are displayed.



After initial transients, the amplitude of the response is approximately 0.03 m. The magnitude at this frequency is approximately 7×10^{-4} m/N, so the 50 N input gives 0.035 m. The response lags the input by π rad, which is consistent with the frequency response function phase plot from Example 11.2.

Example 11.4 Review the R–L–C circuit in Example 7.3. The transfer function was:

$$G(s) = \frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}.$$

For C, L, and R values of 1 μF, 120 mH, and 100 Ω, respectively, use MATLAB® to find the frequency response function and plot it in terms of the magnitude and phase and real and imaginary components. Determine the DC (zero frequency) value: $|G(0)|$.

Solution The frequency response function is identified by replacing s with $j\omega$ in the transfer function.

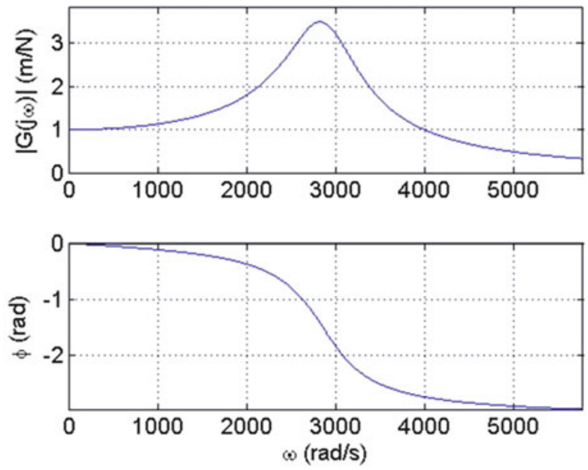
$$G(j\omega) = \frac{1}{-LC\omega^2 + 1 + RC\omega j} = \frac{1/(LC)}{\omega_n^2 - \omega^2 + 2\zeta\omega\omega_n j}$$

The MATLAB® code is provided.

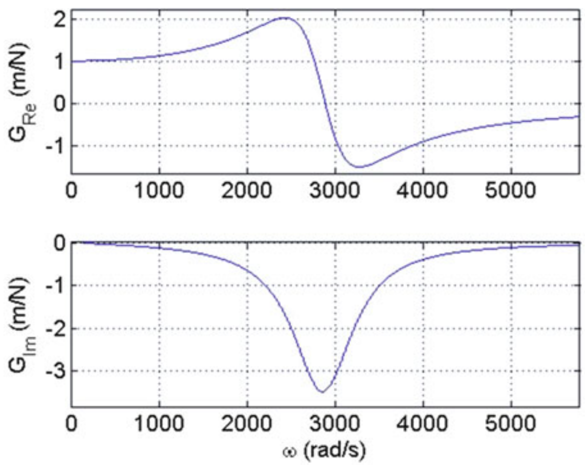
```
clear all
clc
close all

% Parameters
R = 100;       % Ohms
C = 1e-6;      % F
L = 120e-3;    % H

% Natural frequency and damping ratio
wn = sqrt(1/(L*C));       % rad/s
zeta = R/(2*wn*L);        % unitless
wd = wn*sqrt(1-zeta^2); % rad/s
fn = wn/(2*pi);           % Hz
T = 1/fn;                  % s
tau = 1/(zeta*wn);        % s
fprintf('The natural frequency is %4.1f rad/s.\n', wn)
fprintf('The damping ratio is %4.2 f.\n', zeta)
fprintf('The damped natural frequency is %4.1f rad/s.\n', wd)

% Frequency vector
w = [0:2*wn/10000:2*wn];     % rad/s

% Frequency response function
G = 1./(-L*C*w.^2 + 1 + R*C*w*1j);
G_Re = real(G);
```

```
G_Im = imag(G);
G_mag = abs(G);
phi = angle(G);

% Find and display the resonant frequency
index = find(G_mag == max(G_mag));
wr = w(index);
fprintf('The resonant frequency is %4.1f rad/s.\n', wr)

% Magnitude and phase
figure(1)
subplot(211)
plot(w, G_mag);
set(gca, 'FontSize', 14);
ylabel('|G(j\omega)| (m/N)')
axis([min(w) max(w) 0 1.1*max(G_mag)])
grid

subplot(212)
plot(w, phi)
set(gca, 'FontSize', 14);
xlabel('\omega (rad/s)')
ylabel('\phi (rad)')
axis([min(w) max(w) min(phi) max(phi)]);
grid

% Find and display the zero-frequency magnitude
fprintf('The magnitude of the response at zero frequency is %1.2 g.
\n', G_mag(1))

% Real and imaginary components
figure(2)
subplot(211)
plot(w, G_Re)
set(gca,'FontSize',14);
ylabel('G_{Re} (m/N)')
axis([min(w) max(w) 1.1*min(G_Re) 1.1*max(G_Re)]);
grid

subplot(212)
plot(w, G_Im)
set(gca,'FontSize',14);
xlabel('\omega (rad/s)')
ylabel('G_{Im} (m/N)')
axis([min(w) max(w) 1.1*min(G_Im) 1.1*max(G_Im)]);
grid
```

The magnitude and phase plots are shown.



The plots of the real and imaginary components are displayed.



The numerical results are given.

```
The natural frequency is 2886.8 rad/s.
The damping ratio is 0.14.
The damped natural frequency is 2856.5 rad/s.
The resonant frequency is 2826.1 rad/s.
The magnitude of the response at zero frequency is 1.
```

Notice that the damping ratios for both this example and Example 11.3 are identical and, consequently, the shapes of the frequency response curves are the same except for amplitude and frequency scaling. This reinforces the analogies between electrical and mechanical systems. Again, the natural frequency, damped natural frequency, and resonant frequency are not identical. The amplitude at zero frequency (i.e., the DC response) is 1. A constant input voltage of 1 V yields a 1 V steady state output; the entire voltage drop is across the capacitor (see Fig. 7.8).

11.4 Properties of the Frequency Response Function

Considering the mechanical case definition in Example 11.2, we have already shown that the frequency response function can be written as:

$$G(j\omega) = \frac{1/m}{\omega_n^2 - \omega^2 + 2\zeta\omega\omega_n j}. \tag{11.24}$$

Except for the constant factor, all second-order frequency response functions are the same. If we divide the numerator and denominator by ω_n, the frequency response function can be written in terms of a frequency ratio, $r = \frac{\omega}{\omega_n}$.

$$G(j\omega) = \frac{1}{k}\left(\frac{1}{(1 - r^2) + 2\zeta r j}\right) \tag{11.25}$$

To find the real and imaginary components, as well as the magnitude and phase, we rationalize Eq. (11.25) to obtain:

$$G(j\omega) = \frac{1}{k}\left(\frac{(1 - r^2) - 2\zeta r j}{(1 - r^2)^2 + (2\zeta r)^2}\right). \tag{11.26}$$

Equation (11.26) can be separated into the real and imaginary parts.

$$G_{\mathrm{Re}}(j\omega) = \frac{1}{k}\left(\frac{(1 - r^2)}{(1 - r^2)^2 + (2\zeta r)^2}\right) \tag{11.27}$$

$$G_{\mathrm{Im}}(j\omega) = \frac{1}{k}\left(\frac{-2\zeta r}{(1 - r^2)^2 + (2\zeta r)^2}\right) \tag{11.28}$$

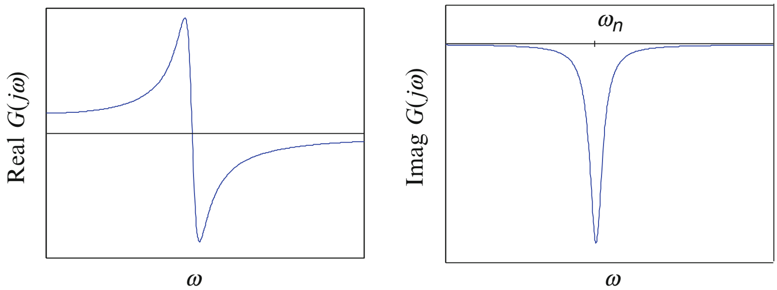


Fig. 11.2 The real (*left*) and imaginary (*right*) components of the frequency response function

The magnitude is determined by calculating the square root of the sum of the squares of the real and imaginary parts.

$$|G(j\omega)| = \frac{1}{k}\left(\frac{1}{(1-r^2)^2 + (2\zeta r)^2}\right)^{\frac{1}{2}} \tag{11.29}$$

The phase is the inverse tangent of the imaginary part divided by the real part.

$$\phi(G(j\omega)) = \tan^{-1}\left(\frac{-2\zeta r}{1-r^2}\right) \tag{11.30}$$

Next, we will explore the behavior of these functions and establish the physical interpretation.

The real part of the frequency response function, defined in Eq. (11.27), is positive for $\omega < \omega_n$, zero when $\omega = \omega_n$, and negative when $\omega > \omega_n$. The imaginary part (Eq. 11.28) is negative for all ω and is most negative when $\omega = \omega_n$. The character of the real and imaginary components of a second-order frequency response function is shown in Fig. 11.2. They are similar to the plots obtained in the previous examples.

We have already seen that for systems with nonzero damping, the natural frequency, the damped natural frequency, and the resonant frequency are not equal. The resonant frequency, ω_r, is defined as the frequency at which the magnitude of the response is maximum.

We identify the maximum by differentiating Eq. (11.29) it with respect to r and setting the result equal to zero.

$$\frac{\mathrm{d}}{\mathrm{d}r}|G(r)| = \frac{1}{2k}\left[\sqrt{(1-r^2)^2 + (2\zeta r)^2}\right]\left[-4r\left(1-r^2\right) + 8\zeta^2 r\right] = 0 \tag{11.31}$$

The first term is never zero (for a nonzero damping ratio) and, therefore, the solution is obtained by setting the second term equal to zero. After canceling common terms this gives: $r^2 - 1 + 2\zeta^2 = 0$. The expression for the resonant frequency is then:

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}. \tag{11.32}$$

The value is less than both the natural frequency and the damped natural frequency; we observed this result in the previous examples. Substituting Eq. (11.32) into Eq. (11.29) gives the magnitude at resonance.

$$|G(j\omega)|_{res} = \frac{1}{2k\zeta\sqrt{1 - \zeta^2}} \tag{11.33}$$

The amplitude at resonance approaches infinity as the damping approaches zero. The phase at resonance is:

$$\varphi_r = \tan^{-1}\left(\frac{-\sqrt{1 - 2\zeta^2}}{\zeta}\right). \tag{11.34}$$

For small damping ratios, the term in the parentheses becomes negative and large and, therefore, the phase at resonance approaches $-\frac{\pi}{2}$. We also observed this result in the examples. To summarize, as the driving frequency is increased so that a system passes through resonance, the amplitude of the oscillations is maximized and the phase passes through $-\frac{\pi}{2}$; it approaches $-\pi$ as the driving frequency is increased to values significantly larger than the resonant frequency.

Example 11.5 For the system examined in Example 11.2, determine the magnitude and phase of the response at resonance and compare to the derived results.

Solution We use MATLAB® to solve the problem and display the results. The code is provided.

```
clear all
clc
close all

% Parameters
k = 500;     % N/m
b = 20;      % N-s/m
m = 10;      % kg

% Calculated parameters
wn = sqrt(k/m);               % rad/s
zeta = b/(2*sqrt(k*m));       % unitless
wd = wn*sqrt(1-zeta^2);       % rad/s
```

```
wr = wn*sqrt(1-2*zeta^2);          % rad/s
G_mag_r = 1/(2*k*zeta*sqrt(1-zeta^2)); % m/N
phi_r = atan2(-sqrt(1-2*zeta^2), zeta); % rad

% Display results
fprintf('The natural frequency is %4.2f rad/s.\n', wn)
fprintf('The damping ratio is %4.2 f.\n', zeta)
fprintf('The damped natural frequency is %4.2f rad/s.\n', wd)
fprintf('The resonant frequency is %4.2f rad/s.\n', wr)
fprintf('The amplitude at resonance is %5.2 g m/N.\n', G_mag_r)
fprintf('The phase at resonance is %5.2f rad.\n', phi_r)
```

The results are displayed and agree with Example 11.2.

```
The natural frequency is 7.07 rad/s.
The damping ratio is 0.14.
The damped natural frequency is 7.00 rad/s.
The resonant frequency is 6.93 rad/s.
The amplitude at resonance is 0.0071 m/N.
The phase at resonance is -1.43 rad.
```

11.5 Multiple Degree-of-Freedom Systems

The analysis of forced sinusoidal motions is not limited to single degree-of-freedom systems. The analysis is applicable to any system for which we can determine a transfer function. Consider again the two degree-of-freedom system depicted in Fig. 4.11a, a two-mass system driven by motions of the base. The transfer functions were given in Eqs. (4.55) and (4.57) and are reproduced here.

$$G_1(s) = \frac{X_1(s)}{X_{in}(s)} = \frac{m_2 b_1 s^3 + (b_1 b_2 + m_2 k_1)s^2 + (b_1 k_2 + b_2 k_1)s + k_1 k_2}{m_1 m_2 s^4 + (m_1 b_2 + m_2 b_1 + m_2 b_2)s^3 + (m_1 k_2 + m_2 k_1 + m_2 k_2 + b_1 b_2)s^2 + (b_1 k_2 + b_2 k_1)s + k_1 k_2}$$

$$G_2(s) = \frac{X_2(s)}{X_{in}(s)} = \frac{b_1 b_2 s^2 + (b_1 k_2 + b_2 k_1)s + k_1 k_2}{m_1 m_2 s^4 + (m_1 b_2 + m_2 b_1 + m_2 b_2)s^3 + (m_1 k_2 + m_2 k_1 + m_2 k_2 + b_1 b_2)s^2 + (b_1 k_2 + b_2 k_1)s + k_1 k_2}$$

The corresponding frequency response functions are determined by replacing the Laplace variable s with $j\omega$ to obtain:

$$G_1(j\omega) = \frac{k_1k_2 - (b_1b_2 + m_2k_1)\omega^2 + j((b_1k_2 + b_2k_1)\omega - m_2b_1\omega^3)}{m_1m_2\omega^4 - (m_1k_2 + m_2k_1 + m_2k_2 + b_1b_2)\omega^2 + k_1k_2 \\ -j((m_1b_2 + m_2b_1 + m_2b_2)\omega^3 - (b_1k_2 + b_2k_1)\omega)}$$
$$G_2(j\omega) = \frac{k_1k_2 - b_1b_2\omega^2 + j(b_1k_2 + b_2k_1)\omega}{m_1m_2\omega^4 - (m_1k_2 + m_2k_1 + m_2k_2 + b_1b_2)\omega^2 + k_1k_2 \\ -j((m_1b_2 + m_2b_1 + m_2b_2)\omega^3 - (b_1k_2 + b_2k_1)\omega)}. \tag{11.35}$$

Following our discussions in Chap. 4, we note that functions of the form described by $G_1(j\omega)$ and $G_2(j\omega)$ have two resonant frequencies corresponding to two modes of oscillation. We will explore this further using an extension of Example 4.12.

Example 11.6 Consider again the system examined in Example 4.12 with parameters: $m_1 = 10$ kg, $m_2 = 5$ kg, $k_1 = 500$ N/m, $k_2 = 200$ N/m, $b_1 = 5$ N-s/m, and $b_2 = 0.25$ N-s/m. As described in Example 4.12, this system has two modes of oscillation. These were identified from the roots of the characteristic equation which are given again here.

```
-0.2150 + 9.3230i
-0.2150 - 9.3230i
-0.0725 + 4.7951i
-0.0725 - 4.7951i
```

The two modes are associated with the frequencies 4.80 and 9.32 rad/s and have time constants of 13.8 s and 4.7 s, respectively. For this system, plot the frequency response functions, $G_1(j\omega)$ and $G_2(j\omega)$, in terms of both magnitude and phase and real and imaginary components.

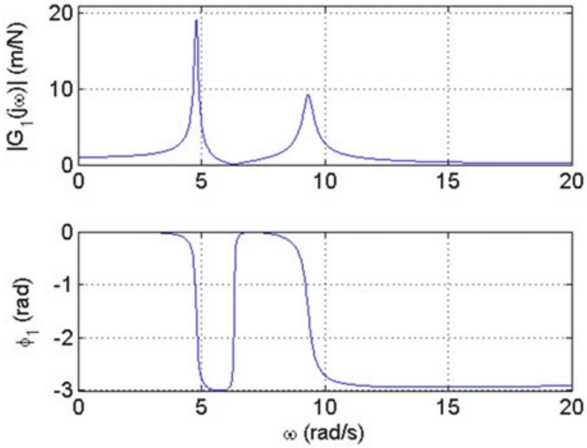
Solution The MATLAB® code used to complete this problem is provided.

```
clear all
clc
close all

% Parameters
m1 = 10;       % kg
m2 = 5;
k1 = 500;      % N/m
k2 = 200;
b1 = 5;        % N-s/m
b2 = 0.25;
```

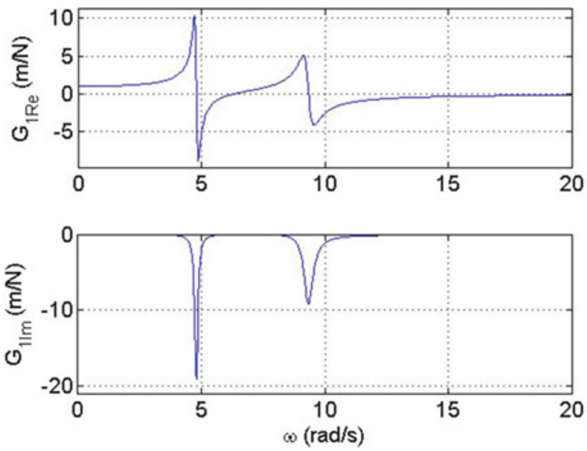
```
% Frequency vector
w = [0:20/1000:20];   % rad/s

% Frequency response function G1
num1 = k1*k2 - (b1*b2+m2*k1)*w.^2 + 1j*((b1*k2+b2*k1)*w-m2*b1*w.
^3);
den1 = m1*m2*w.^4 - (m1*k2+m2*k1+m2*k2+b1*b2)*w.^2 + k1*k2 - 1j*
((m1*b2+m2*b1+m2*b2)*w.^3-(b1*k2+b2*k1)*w);

G1 = num1./den1;
G1_Re = real(G1);
G1_Im = imag(G1);
G1_mag = abs(G1);
phi1 = unwrap(angle(G1));

% Magnitude and phase
figure(1)
subplot(211)
plot(w, G1_mag)
set(gca, 'FontSize', 14);
ylabel('|G_1(j\omega)| (m/N)')
axis([min(w) max(w) 0 1.1*max(G1_mag)])
grid
subplot(212)
plot(w, phi1)
set(gca, 'FontSize', 14);
xlabel('\omega (rad/s)')
ylabel('\phi_1 (rad)')
axis([min(w) max(w) min(phi1) max(phi1)])
grid

% Real and imaginary components
figure(2)
subplot(211)
plot(w, G1_Re)
set(gca, 'FontSize', 14);
ylabel('G_{1Re} (m/N)')
axis([min(w) max(w) 1.1*min(G1_Re) 1.1*max(G1_Re)])
grid

subplot(212)
plot(w, G1_Im)
set(gca, 'FontSize', 14);
xlabel('\omega (rad/s)')
ylabel('G_{1Im} (m/N)')
axis([min(w) max(w) 1.1*min(G1_Im) 1.1*max(G1_Im)])
grid
```

```
% Frequency Response Function G2
num2 = k1*k2 - b1*b2*w.^2 + 1j*(b1*k2+b2*k1)*w;
den2 = m1*m2*w.^4 - (m1*k2+m2*k1+m2*k2+b1*b2)*w.^2 + k1*k2 - 1j*
((m1*b2+m2*b1+m2*b2)*w.^3 - (b1*k2+b2*k1)*w);
G2 = num2./den2;
G2_Re = real(G2);
G2_Im = imag(G2);
G2_mag = abs(G2);
phi2 = unwrap(angle(G2));

% Magnitude and phase
figure(3)
subplot(211)
plot(w, G2_mag)
set(gca, ’FontSize’, 14);
ylabel(’|G_2(j\omega)| (m/N)’)
axis([min(w) max(w) 0 1.1*max(G2_mag)])
grid

subplot(212)
plot(w, phi2)
set(gca, ’FontSize’, 14);
xlabel(’\omega (rad/s)’)
ylabel(’\phi_2 (rad)’)
axis([min(w) max(w) min(phi2) max(phi2)])
grid

% Real and imaginary components
figure(4)
subplot(211)
plot(w, G2_Re)
set(gca, ’FontSize’, 14);
ylabel(’G_{2Re} (m/N)’)
axis([min(w) max(w) 1.1*min(G2_Re) 1.1*max(G2_Re)]);
grid

subplot(212)
plot(w, G2_Im)
set(gca, ’FontSize’, 14);
xlabel(’\omega (rad/s)’)
ylabel(’G_{2Im} (m/N)’)
axis([min(w) max(w) 1.1*min(G2_Im) 1.1*max(G2_Im)])
grid
```

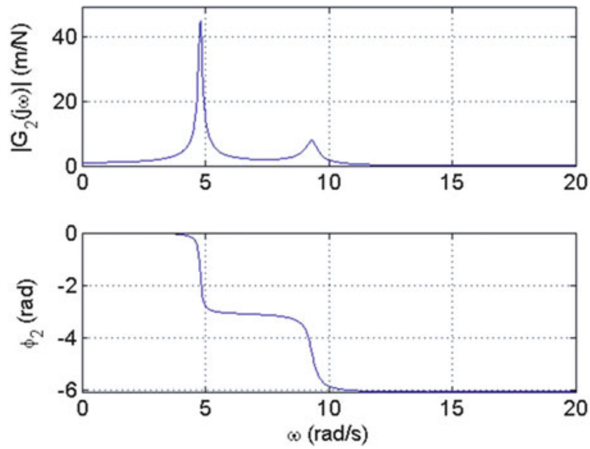
The magnitude and phase of $G_1(j\omega)$ are shown.



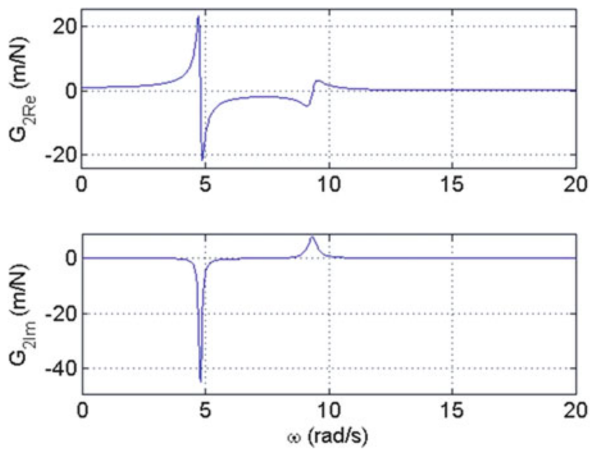
The real and imaginary parts of $G_1(j\omega)$ are shown.



The magnitude and phase of $G_2(j\omega)$ are shown.



The real and imaginary parts of $G_2(j\omega)$ are shown.



The resonant peaks in both sets of curves correspond to the frequencies identified using the characteristic equation.

11.6 Tuned-Mass Absorber Example

One important application of frequency domain analysis is the *tuned-mass absorber*. Consider the system shown in Fig. 11.3a. A mass, m_1, is being driven harmonically by motion of the wall at a forcing frequency, ω_f, through a spring, k_1.

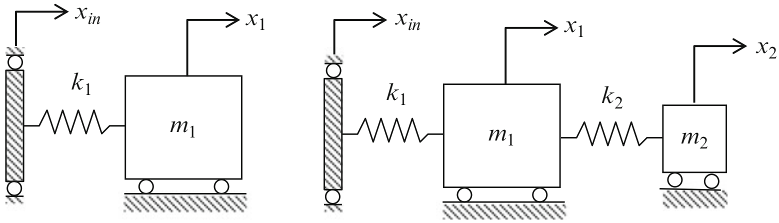


Fig. 11.3 (*Left*) A mass, m_1, is driven by motion of a wall at the forcing frequency, ω_f, and (*right*) a second mass, m_2, is added to the first mass to reduce the vibrations of m_1

If the mass motion is too large, it may cause mechanical failure, passenger discomfort in an automobile, or excessive vibration in a machine tool. To reduce the vibrations a second mass, m_2, is attached to m_1 through a spring k_2 with the objective of reducing or eliminating the vibration of m_1. The mass, m_2, is typically smaller than m_1; the combination of the added spring and mass is referred to as a tuned-mass absorber.

The analysis is completed assuming no damping to provide a base solution. In the presence of damping, the parameters can be tuned to optimize the system. For the system shown in Fig. 11.3, the frequency response function at m_1 is obtained by setting the damping terms equal to zero in Eq. (11.35).

$$G_1\left(j\omega_f\right) = \frac{k_1k_2 - m_2k_1\omega_f^2}{m_1m_2\omega_f^4 - (m_1k_2 + m_2k_1 + m_2k_2)\omega_f^2 + k_1k_2} \tag{11.36}$$

We desire to make this frequency response equal to zero at the driving frequency, ω_f. Setting the numerator equal to zero, we obtain:

$$k_1k_2 - m_2k_1\omega_f^2 = 0. \tag{11.37}$$

Rearranging Eq. (11.37), we obtain Eq. (11.38), which shows that we set the absorber natural frequency (if considered independently) equal to the driving frequency in order to force the response at m_1 to be zero at the driving frequency. Equation (11.38) is the fundamental equation of a tuned-mass absorber. We now demonstrate tuned-mass absorber design using Example 11.7.

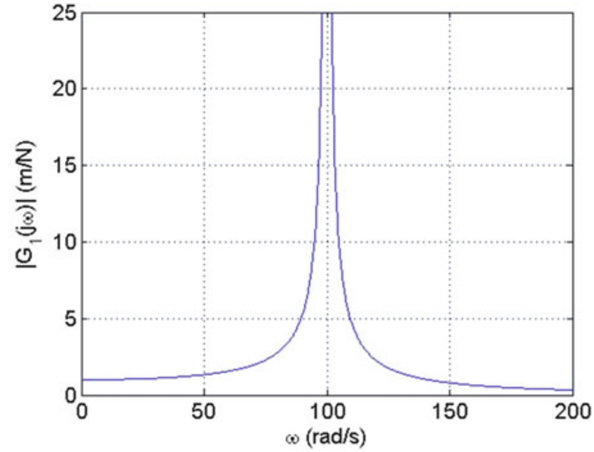
$$\omega_f = \sqrt{\frac{k_2}{m_2}} \tag{11.38}$$

Example 11.7 Suppose a single degree-of-freedom system has a mass, m_1, of 1 kg and a stiffness, k_1, of 10000 N/m. Plot the frequency response and determine the magnitude if it is driven at 100 rad/s. Design a tuned-mass absorber with one-tenth the mass of the original system (0.1 kg) and plot the modified frequency response at m_1 with the absorber added. Note that the new system has two degrees-of-freedom.

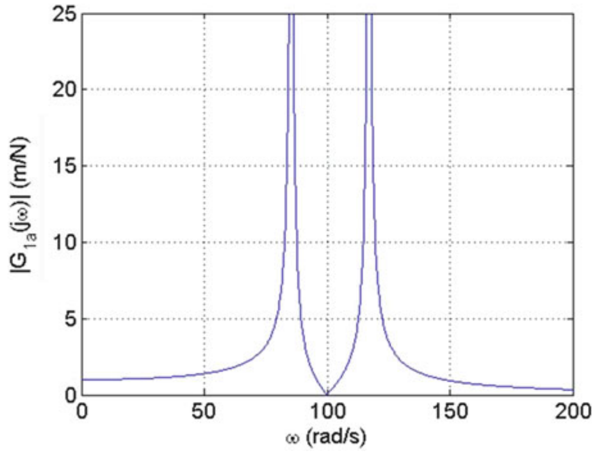
Solution The MATLAB® code used to complete this example is provided.

```
clear all
clc
close all

% Parameters
m1 = 1;
k1 = 10000;
wf = 100;

% Frequency vector
w = [0:200/1000:200];

% Frequency response of original system
G1 = k1./(-m1*w.^2 + k1);
G1_Re = real(G1);
G1_Im = imag(G1);
G1_mag = abs(G1);

% Plot the response of the unaltered system
figure(1)
plot(w, G1_mag)
set(gca, 'FontSize', 14);
xlabel('\omega (rad/s)')
ylabel('|G_1(j\omega)| (m/N)')
axis([min(w) max(w) 0 25])
grid

% Design the tuned-mass absorber
m2 = 0.1;
k2 = wf^2*m2;

% Frequency response function G1
num1a = k1*k2 - m2*k1*w.^2;
den1a = m1*m2*w.^4 - (m1*k2+m2*k1+m2*k2)*w.^2 + k1*k2;
G1a = num1a./den1a;
G1a_Re = real(G1a);
G1a_Im = imag(G1a);
G1a_mag = abs(G1a);

% Plot the response of the new system
figure(2)
plot(w, G1a_mag)
set(gca,'FontSize',14);
xlabel('\omega (rad/s)')
ylabel('|G_{1a}(j\omega)| (m/N)')
axis([min(w) max(w) 0 25]);
grid
```

The frequency response of the original system is shown.



The magnitude of the response at the natural frequency of 100 rad/s is infinite because there is no damping. The response after the addition of the tuned-mass absorber with a spring stiffness of $k_2 = \omega_f^2 m_2$ or 1000 N/m is also displayed.



The single mode has been split into two modes spaced around the driving frequency of 100 rad/s such that the response at 100 rad/s has been reduced to zero, as desired.

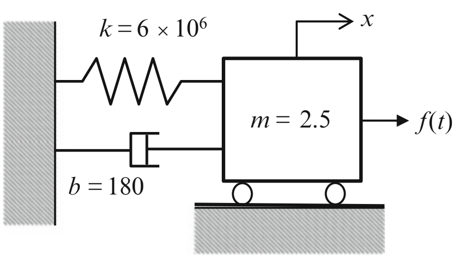
11.7 Summary

In this chapter, we discussed the following key elements:

- The response of a system to a sinusoidal input of frequency, ω, can be determined by replacing s in the transfer function, $G(s)$, with the complex number $j\omega$ to form the frequency response function, $G(j\omega)$.
- The magnitude, $|G(j\omega)|$, and phase, $\phi(\omega)$, of the complex frequency response function, $G(j\omega)$, give the corresponding amplitude and phase of the sinusoidal response of the system to the sinusoidal input.
- For a second-order system, the magnitude of the frequency response function is maximum at the resonant frequency, ω_r, and the phase of the output lags the input by approximately $\frac{\pi}{2}$rad at resonance.
- The frequency response function, $G(j\omega)$, can also be represented by its real and imaginary components.
- For a second-order system, the real part of $G(j\omega)$ is zero at the natural frequency, ω_n, and the imaginary part reaches its most negative value.
- For higher-order systems, the frequency response function shows resonances and large phase shifts corresponding to the natural frequencies associated with the vibration modes.
- A tuned-mass absorber can be added to a single degree-of-freedom system to eliminate the vibrations at a particular forcing frequency, ω_f.

Problems

1. A single degree of freedom spring-mass-damper system is shown with $m = 2.5$ kg, $k = 6 \times 10^6$ N/m, and $b = 180$ N-s/m. A force harmonic $f(t)$ is applied to the mass.

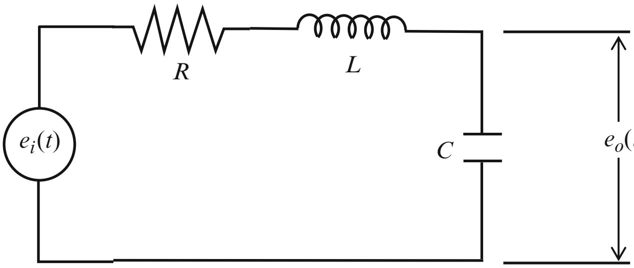


Complete the following.

(a) Calculate the natural frequency ω_n (in rad/s), the damping ratio ζ, the damped natural frequency ω_d (in rad/s), and the resonant frequency ω_r (in rad/s).

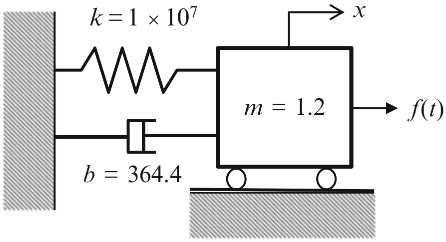
(b) Find the transfer function $G(s) = \frac{X(s)}{F(s)}$ for the system and then, by replacing s with $j\omega$, find the FRF for the system, $G(j\omega)$.
(c) Write a MATLAB® script file to plot the magnitude (in m/N), phase (in deg), and real and imaginary parts (in m/N) of the FRF.
(d) Identify the frequency (in Hz) and amplitude (in m/N) for the key features from the plots.
(e) Determine the value of the magnitude of the FRF for this system at a forcing frequency of 1500 rad/s by combining the `find` and the `min` or `max` commands in MATLAB®. If the harmonic force magnitude is 250 N, determine the amplitude of the steady state response (in mm) at this frequency.

2. In the R-L-C circuit shown, the C, L, and R values are 10 μF, 250 mH, and 50 Ω, respectively. The circuit is subjected to a harmonic forcing voltage, $e_i(t)$.



Complete the following.
(a) Calculate the natural frequency ω_n (in rad/s), the damping ratio ζ, the damped natural frequency ω_d (in rad/s), and the resonant frequency ω_r (in rad/s).
(b) Find the transfer function $G(s) = \frac{E_o(s)}{E_{in}(s)}$ for the system and then, by replacing s with $j\omega$, find the FRF of the system, $G(j\omega)$.
(c) Write a MATLAB® script file to plot the magnitude, phase (in deg), and real and imaginary parts of the FRF.
(d) Identify the frequency (in Hz) and amplitude for the key features from the plots.
(e) Determine the value of the magnitude of the FRF for this system at a forcing frequency of 600 rad/s by combining the `find` and the `min` or `max` commands in MATLAB®. If the harmonic voltage magnitude is 5 V, determine the amplitude of the steady state response (in V) at this frequency.

3. A single degree of freedom lumped parameter system has mass, stiffness, and damping values of 1.2 kg, 1×10^7 N/m, and 364.4 N-s/m, respectively.

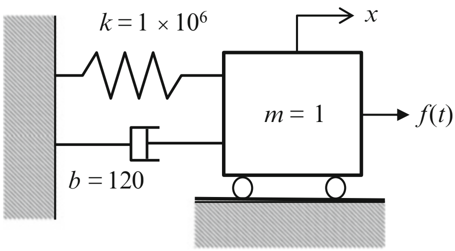


Complete the following.

(a) Plot the magnitude (m/N) vs. frequency (Hz) and phase (deg) vs. frequency (Hz) of the FRF.

(b) Plot the real part (m/N) vs. frequency (Hz) and imaginary part (m/N) vs. frequency (Hz) of the FRF.

4. A single degree of freedom spring-mass-damper system with $m = 1$ kg, $k = 1 \times 10^6$ N/m, and $b = 120$ N-s/m is subjected to forced harmonic vibration.



Complete the following.

(a) Calculate the natural frequency ω_n (in rad/s), the damping ratio ζ, the damped natural frequency ω_d (in rad/s), and the resonant frequency ω_r (in rad/s).

(b) Write expressions for the real part, imaginary part, magnitude, and phase of the system frequency response function (FRF). These expressions should be written as a function of the frequency ratio, $r = \frac{\omega}{\omega_n}$, stiffness, k, and damping ratio, ζ.

(c) Plot the real part (in m/N), imaginary part (in m/N), magnitude (in m/N), and phase (in deg) of the system FRF as a function of the frequency ratio, r. Use a range of 0 to 2 for r (note that $r = 1$ is near the resonant frequency).

5. A single degree of freedom spring-mass-damper system with $m = 1.2$ kg, $k = 1 \times 10^7$ N/m, and $b = 364.4$ N-s/m is subjected to a forcing function $f(t) = 15\sin(\omega_n t)$ N, where ω_n is the system's natural frequency. Determine the steady-state magnitude (in μm) and phase (in deg) of the vibration due to this harmonic force.

Reference

1. Ogata K (2004) System dynamics, 4th edn. Pearson Prentice Hall, Englewood Cliffs

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M.A. Davies, T.L. Schmitz, *System Dynamics for Mechanical Engineers*,
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